On the Logics of Algebra

by

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2008

Submitted in fulfilment of the
requirements for the degree of
Doctor of Philosophy
in the
School of Mathematical Sciences
University of KwaZulu-Natal
Westville

As the candidates supervisor, I have approved
this dissertation for submission

Signed: Name: Date:

Abstract

We present and consider a number of logics that arise naturally from universal algebraic considerations, but which are 'inherently unalgebraizable' in the sense of [BP89a], essentially because they have no theorems. Of particular interest is the *membership logic* of a quasivariety, which is determined by its theorems, which are the relative congruence classes of the term algebra together with the empty-set in the case that the quasivariety is non-trivial. The membership logic arises by a more general technique developed in this text, for inducing deductive systems from closed systems on the free algebras of quasivarieties. In order to formalize this technique, we develop a theory of logics over constructs, where constructs are concrete categories. With this theory in place, we are able to view a closed system over an algebra as a logic, and in particular a structural logic, structural with respect to a suitable construct, typically the construct consisting of all algebras in a quasivariety and all algebra homomorphisms between these algebras. Of course, in such a case, none of these logics are generally sentential (i.e., structural and finitary deductive systems in the sense of [BP89a]), since the formulae of sentential logics arise from the terms of the absolutely free term algebra, which is generally not a member of the quasivariety under interest. In such cases, where the term algebra is not a member of a quasivariety, the free algebra of the quasivariety on denumerably countable free generators takes on the role played by the term algebra in sentential logics. Many of the logics that we encounter in this text arise most naturally as finitary logics on this free algebra of the quasivariety and generally are structural with respect to the quasivariety. We call such logics *canons*, and show how such structural canons induce *sentential calculi*, which we call the induced *ideal*; the filters of the ideal on the free algebra are precisely the theories of the canon. The membership logic is the ideal of the cannon whose theories are the relative congruence classes on the free algebra.

The primary aim of this thesis is to provide a unifying framework for logics of this type which extends the Blok-Pigozzi theory of elementarily algebraizable (and protoalgebraic) deductive systems. In this extension there are two parameters: a set of formulae and a variable. When the former is empty or consists of theorems, the Blok-Pigozzi theory is recovered, and the variable is redundant. For the membership logic, the appropriate variant of equivalent algebraic semantics encompasses the relatively congruence regular quasivarieties. These results have appeared in [BR03].

The secondary aim of this thesis is to analyse our theory of parameterized algebraization from a non-parameterized perspective. To this end, we develop a theory of protoalgebraic logics over constructs and equivalence between logics from different constructs, which we then use to explain the results we obtained in our *parameterized* theories of protoalgebraicity, algebraic semantics and equivalent algebraic semantics. We relate this theory to the theory of deductively equivalent π-institutions [Vou03], and as a consequence obtain a number of improved and new results in the field of categorical abstract algebraic logic. We also use our theory of protoalgebraic logics over constructs to obtain a new and simpler characterization of structural finitary n-deductive systems, which we then use to close the program begun in [BR99], by extending those results for 1-deductive systems to n-deductive systems, and in particular characterizing the protoalgebraicity of the sentential n-deductive system $S^n(\mathcal{K}, \mathfrak{N})$, which is the natural extension of the 1-deductive system $S(\mathcal{K}, \tau)$ introduce in [BR99], in terms of the quasivariety $\mathcal{K}$ having $\langle \mathcal{K}, \mathfrak{N} \rangle$-coherent $\mathfrak{N}$-classes (we cannot see how to obtain this result from the standard characterization of protoalgebraic n-deductive systems of [Pal03], which is very complex). With respect to this program of completing [BR99], we also show that a quasivariety $\mathcal{K}$ is an equivalent algebraic semantics for a n-deductive system with defining equations $\mathfrak{N}$ iff $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N} \rangle$-regular; a notion of regularity that we introduce and characterize by a quasi-Mal'cev condition.

The third aim of this text is to unify as many disparate arguments and notions in algebraic logic under the banner of continuous translations between closed systems, where our use of the term continuous is in

the topological sense rather than in the order-theoretic sense, and, where possible, to give elementary, i.e. first order, definitions and proofs. To this end, we show that closed systems, closure operators and consequence relations can all be characterized elementarily over orders, and put into one-to-one correspondence that reflects exactly, the standard correspondences between the well-known concrete notions with the same name. We show that when the order is the complete power order over a set, then these elementary structures coincide with their well-known counterparts with the same name. We also introduce two other elementary structures over orders, namely the closed equivalence relation and something we term the proto-Leibniz relation; these elementary structures are also in one-to-one correspondence with the earlier mentioned structures; we have not seen concrete versions of these structures. We then characterize the structure homomorphisms between these structures, as well as considering galois relations between them; galois relations are pairs of order-preserving function in opposite directions; we call these translations, and they are elementary notions. We demonstrate how notions as disparate as structurality, semantics, algebraic semantics, the filter correspondence property, filters, models, semantic consequence, protoalgebraicity and even the logic $S(\mathcal{K}, \tau)$ of [BR99] and our logic $S^n(\mathcal{K}, \mathfrak{N})$, all fall within this framework, as does much of our parameterized theory and much of the theory of π-institutions.

A brief summary of the standard theory of deductive systems and their algebraization is provided for the reader unfamiliar with algebraic logics, as well as the necessary background material, including construct and category theory, the theory of structures and algebras, and the model theory of structures with and without equality.

Dedicated

to

Adrian

who is finding her place in this world.

In Memorium

Hambe Kahle
Frankie 'Gobbos' Sometsu

Declaration

This study represents original work by the author. It has not been submitted in any other form to another University. Where use was made of the work of others it has been duly acknowledged in the text.

Graham Barbour
June 2007

Acknowledgements

The author would like to thank and acknowledge the following for their support in various ways leading to the completion of this thesis:

- Dr G Amery. Thanks for the help and advice in bringing this work to a closure.
- Prof J Raftery. Thanks for the help with the paper and for your careful reading of this text.
- Prof S Maharaj. Thanks for applying the necessary doses of ‘wakka’.
- Teo Sturm. Thanks for being my teacher.
- Clint van Alten for his Masters thesis surveying AAL which I used extensively.
- J13 for providing the soundtrack.

Contents

‘But please do not come to the Learnèd English Dog if it’s religious Comfort you’re after. I may be præternatural, but I am not supernatural. ’Tis the Age of Reason, rrrf? There is ever an Explanation at hand, and no such thing as a Talking Dog, – Talking Dogs belong with Dragons and Unicorns. What there are, however, are Provisions for Survival in a World less fantastick.’

Mason & Dixon, T Pynchon

Introduction

The title of this thesis 'On the logics of algebra' is an attempt to redress the historical logical bias in the field of 'algebraic logic': if one begins from a universal algebraic perspective, one is inevitably led to consider logics that, while arising naturally from algebras, are 'inherently unalgebraizable' in the sense of [BP89a]. It is these logics that are the primary focus of this work.

With any quasivariety of algebras, it is possible to associate a deductive system, which we term the *membership logic* of the quasivariety, which is most easily defined as the logic whose theories are the relative congruence classes on the term algebra, together with the empty-set. The logic defined this way is both finitary and structural, and consequently a 1-deductive system in the sense of [BP89a]. Generally, the membership logic encodes all of the equational theory of its determining quasivariety, and much of the quasi-equational theory. In the case that the quasivariety is relatively regular, that is, relative congruences are uniquely determined by their congruence class at any point, the membership logic encodes all the quasi-equational theory of its determining quasivariety. The membership logic, however, is 'inherently unalgebraizable' in the sense of [BP89a]. Except for trivial quasivarieties, it is never protoalgebraic and hence, generally, cannot have an equivalent algebraic semantics, even in the case that the quasivariety is relatively regular.

The key reason for the membership logic's failure to be protoalgebraic stems from the fact that it has no theorems. A well-known characterization of protoalgebraicity is that the Leibniz operator preserve the inclusion-ordering of the theories of a logic [BP86]. Since the Leibniz operator always maps the empty-set of formulae to the largest congruence on the term algebra, except for the case of almost-trivial logics (i.e., logics whose only theories are the empty-set and the set of all formulae), logics without theorems can never be protoalgebraic.

The *primary focus* of this text is the presentation of a technique for inducing sentential calculi from algebraic closed systems on the free algebra of a quasivariety (these logics are generally without theorems and hence unalgebraizable), and to provide a framework in which logics without theorems may be 'algebraized'. We shall present a number of sentential calculi that arise via this technique, including, amongst others, the *membership logic* of a quasivariety, the *logics of subuniverses* of a quasivariety, the *logics of identified membership* determined by a quasivariety and a unary term, the *logics of solutions to separable equations* in two variables and the *logics of lattice ideals, filters and convexities*. All of these logics generally fail to have theorems, and by the above argument, are all 'inherently unalgebraizable'.

In order to effectively describe the technique that we have been using to induce sentential calculi from closed systems on free algebras, we have developed a theory of *logics over constructs*, where by a construct we mean a *concrete category*. This permits us to treat a quasivariety as a signature of logics, in which case a closed system on the free algebra on denumerably infinite generators may be viewed as a logic and more importantly, as a *structural* logic; structural with respect to all *relative endomorphisms*, i.e., endomorphisms whose kernels are relative congruences modulo the quasivariety. We then view the quasivariety as a *subconstruct* of the variety of *all* algebras (of some fixed type) and call a logic on the free algebra a *canon.* We show how a canon induces a sentential calculus (a deductive system on the absolutely free term algebra), which we call the *ideal*, that reflects both the *deductive truth* of the canon and the *equational truth* encoded in the *points* of the free algebra (which may be viewed as sets of terms). For example, the membership logic arises as the ideal of the canon whose theories are all relative congruence classes together with the empty-set. We prove that the theories of the canon are the *filters* of its ideal, precisely when the canon is *structural.* Many of the closed systems arising from universal algebras are definable on all algebras of some class, not just the free algebra. For example, the relative congruence classes (together with the empty-set) form a structural logic on each algebra of a given quasivariety (structural modulo relative endomorphisms). Further, it is often the case that these logics are 'structural' with respect to homomorphisms between their underlying algebras. We call such a class of logics an *archology*, and explore the relationship between the archology, its canon (which is a member of the archology) and the filters of the induced ideal. While an archology may be viewed as a π-institution [FS88], i.e., a multi-signature logic, the term 'archology' has been chosen to reflect its *prototypical* nature with respect to its induced ideal sentential calculi, and not its multi-signature nature.

With these examples in hand, we then turn to the question of 'algebraizing' these logics, given that they tend to have no theorems and as such, cannot be protoalgebraic, and hence are 'inherently unalgebraizable' in the sense of [BP89a]. The theory of 'algebraization' that we develop is, in a sense, a theory of 'relative algebraization', the relativization being expressed by a parameter X, which is a set of formulae. The parameter X describes necessary truths modulo which the logic is 'algebraizable'. For example, the parameterized notion of protoalgebraicity demands that the Leibniz operator be order preserving with respect to all theories containing X, and the parameterized version of algebraic semantics demands that all deductions of the form $X, \Gamma \vdash \phi$ be suitably interpreted in the quasi-equational theory of some quasivariety of algebras. The parameterized theory 'works best' when the parameter X is in a sense 'free' or 'variable like'; in essence, the parameter behaves like an arbitrarily chosen point, and in which case the parameter is determined, up to theory closure, by a variable z. In this case, the theory has many of the aspects of a topological localization theory: given an arbitrary point, the logic is locally algebraizable or local protoalgebraic 'around' or 'modulo' that point. One of the forms of expressing this freedom is the requirement that the theory generated by X is invariant with respect to substitutions that fix z. The theory is constructed in a such a way that our parametrized theory generalizes the standard theory of algebraization, which obtains as the special case that the parameter is the empty-set. It is worth noting that this extra parameter expresses itself by increasing by one, the arity of terms involved in notions such as defining equations (parameterized defining equations are sets of pairs of *binary* terms) and equivalence formulae (parameterized equivalence formulae are sets of *ternary* terms), and that this increase in arity is typical of the relationship between Mal'cev conditions for 'point-conditions' (for example, point-regularity) and Mal'cev conditions for the analogous 'full-condition' (in this example, full-regularity). In fact, the 'full-condition' can be seen as a relativization of the 'point-condition' in precisely the same manner that parameterized algebraization is a relativization of algebraization. As such, our theory gives logical meaning to algebraic notions generalizing full-regularity and full-coherence, just as the standard theory gives logical meaning to notions generalizing point-regularity and point-coherence. For example, we shall show that a quasivariety is *relatively congruence regular* precisely when its *membership logic is* $\langle\{z\}, z\rangle$*-algebraizable*, just as a quasivariety is *relatively point regular* precisely when its *assertional logic is algebraizable* [BR99]. The importance of this result, lies in the fact that *full congruence regularity*, unlike *point regularity*, falls outside of the domain of the standard theory of *algebraizable logics*, since, as we shall demonstrate, the fully congruence regular variety of *quasigroups* is the equivalent algebraic semantics for *no* structural and finitary 1-deductive system (called sentential 1-calculi in this text) and further, contains a non-trivial fully congruence regular subvariety that is the *algebraic semantics* of *no* non-trivial sentential 1-calculus. By characterizing the *purely universal algebraic* condition of relative congruence regularity in terms of the parameterized algebraizability of its membership logic, which in turn is characterizable in *purely logical* terms, we have succeeded in bringing relative congruence regularity (and hence full congruence regularity) into the domain of algebraic logic. The phrasing and establishment of *this* result was the primary motivation for our development of our theory of parameterized algebraization. In parts II, III and IV, we introduce a number of sentential 1-calculi that arise from universal algebraic considerations, obtained via general techniques developed in these parts, but which are generally unalgebraizable within the standard theory. In Part V we present our theory of parameterized algebraization, which we apply to these examples.

The *secondary aim* of this text, developed in Part VI, is to analyse our theory of parameterized algebraization, which we do by deriving some of the parameterized results from alternative non-parameterized theories. As noted earlier, relative protoalgebraicity requires that the Leibniz operator be order-preserving with respect to theories containing X (with the assumption that the theory generated by X is invariant with respect to substitutions that fix z). While the theories containing X form an algebraic closed system over the term algebra, they do not constitute the theories of a sentential calculus: structurality fails since the homomorphic pre-image of such a theory, while itself being a theory, need not contain X. When one considers the construct consisting of the term algebra and *only* those endomorphisms that fix z, the theories containing X are the theories of a *structural* and finitary logic. We develop a *non-parameterized* theory of protoalgebraic logics over constructs and equivalence between logics in different constructs, which we then apply to the logic described above, from which we are able to obtain many of the parameterized results. Of course, the parameter has been 'slipped into' the definition of the construct.

When we initiated this program, the theory of deductively equivalent π-institutions [Vou03] was not in the public domain. While we developed the theory of models, semantics and equivalence between logics over constructs independently, where our results specializes those in the literature we have duely referenced. In attempting to reconcile our theory with that of [Vou03], we discovered that the two theories coincide only for *term* π-institutions (which analogue, in our case, logics over a free object) and what we term *natural* translations. Further, there is a clear dichotomy in [Vou03], where in one direction the theory 'works' generally, while in the other direction, the theory 'works' only for *term* π-institutions. In analyzing why our results hold generally and those of [Vou03] do not, it became clear that the problem lies in the notion of naturality, which is implicit in the definition of a translation between π-institutions (implicit in the demand for a categorical natural transformation). This notion of naturality is purely *syntactic*, that is, it depends only on the signatures of the institutions and not the *logic* (encoded as closure operators). It is this *syntactic naturality* that is lost in the move from a translation to a theory-functor, and it is for this reason that a (*natural*) translation cannot be recovered from a theory-functor except in the *term* case. We have taken the opportunity to develop (up to interpretation) and suggest (in the case of deductive equivalence) a more general theory of equivalence between π-institutions, one which is based on a notion of *logical naturality* rather than *syntactic naturality*. Further, we have paid particular interest to the notion that a logic from one construct *model* a logic from another construct, a notion that is weaker than semantics (called *interpretablity* in [Vou03]). While this notion has appeared in the literature of π-institutions [Vou05], called semi-interpretation in that text, the direction in which it has been analysed is not as a weak form of interpretation, but rather as a precursor to the development of a model theory in the spirit of the matrix model-theory of sentential calculi [Vou07b]. We characterize this notion in the spirit of [Vou03] and as a consequence, characterize the property that a translation between sentential calculi *commutes with substitutions*, independently of any other conditions. Recently, a notion of protoalgebraic π-institutions has appeared in the literature [Vou07b], [Vou07c]. While our notion of protoalgebraic logics over constructs and the notion of a protoalgebraic π-institution begin from very different starting points, it is almost certain that much of our theory specializes theirs, although we have not yet analysed those texts deeply. As such, we have tended to reference when in doubt.

The *final aim* of this text is to *unify* as many of the closure operator arguments in the literature of algebraic logics, from both the standard theory and our parameterized theory, and where possible, establish these arguments from an *elementary* (i.e., first-order) perspective. Because so much of the sequel depends on these arguments, this theory is presented early in the text. We introduce five elementary classes of structures, all over orders, namely, *elementary closure operators* (having a unary operation symbol $\|\cdot\|$) , *elementary closed systems* (with a unary relation symbol cl), *elementary consequence relations* (with a binary relation symbol $\cdot \vdash \cdot$), *elementary closed-equivalence relations* (having a binary relation symbol $\cdot \dashv\vdash \cdot$) and *elementary proto-Leibniz relations* (with a ternary relation symbol $\cdot \Bumpeq(\cdot) \cdot$), and prove that the structures in these classes are in one-to-one correspondence; in the case of the first three classes, these correspondences match the well-known correspondences between closure operators, closed systems and consequence relations, which we show to coincide with their elementary counterparts in the case that the underlying order is the inclusion order on a power-set. Of these five elementary classes, we have only seen *elementary closure operators* in the literature (possibly because the elementary realization of closure operators is immediate). In the case of *elementary closed-equivalence relations* and *elementary proto-Leibniz relations*, we have not seen 'concrete' analogues of these notions either, although it is our intuition that the 'concrete' proto-Leibniz relation correspondence to the actual (non-elementary) relation of Leibniz that the (elementary) Leibniz relation from the theory of algebraic logics seeks to approximate (under the condition of protoalgebraicity).

While these elementary classes are in one-to-one correspondence, each class admits different structure homomorphisms (all of which are order-preserving functions, which we call *weak-translations*), so while we conflate them in the discourse, tending to speak only of elementary closed systems, we distinguish between the different types of homomorphism, speaking of $\|\cdot\|$-

homomorphisms, cl-homomorphisms, $\vdash$-homomorphisms, etc. We characterize these homomorphisms, as well as their strict and reflecting variants, where appropriate. As an example, we show that structurality in logics is definable as the requirement that all substitutions be $\vdash$-homomorphisms, and derive many of the standard characterizations of structurality. Next we consider *translations*, which are pairs of weak translations that constitute *galois relations* between the underlying orders, characterizing $\vdash$-translations, $\vdash$-reflecting translations and strict $\vdash$-translations; the first and last of which we refer to as *c-continuous* translations and *strictly c-continuous* translations respectively (given their natural generalization of continuous functions between topological spaces). As examples, we show that homomorphisms between algebras constitute c-continuous translations between the closed sets of filters of some sentential calculus, and show how the filter correspondence property may be characterized in terms of the strict c-continuity of reductive matrix homomorphisms, and as a consequence derive some of the standard results in algebraic logic, as well as some new characterizations of protoalgebraicity. The notion and theory of the *product by a translation* to a closed system is developed. We show how this construction gives rise to the class of sentential 1-calculi $S(\mathcal{K}, \tau)$ [BR99], which include all algebraizable sentential 1-calculi; later in the text we extend this construction to n-calculi. The last of the elementary theory concerns *c-iseomorphisms*. We are able to prove 'one direction' of the theory of equivalent logics in this elementary setting; that is we show that c-iseomorphic translations imply that the suborders of closed points (elementary points coincide with concrete sets) must be isomorphic.

Not all the closure related arguments from logic can be given an elementary footing. To this end, we consider 'concrete' closed systems and 'concrete' translations between them, which are grounded binary relationships between their universes (the set over which the power-order is taken); 'concrete' translations may alternatively be viewed as multi-maps, i.e., functions from a set to a power-set; these are precisely the translations between 'concrete' closed systems. Stronger characterizations of c-continuous and strictly c-continuous 'concrete' translations are obtained; given the earlier mentioned examples, more well-known results from algebraic logic obtain. Further, in the case of 'concrete' *c-iseomorphisms*, we are able to establish that isomorphisms between closed orders give rise to c-iseomorphic 'concrete' translations. The product of a source of 'concrete' translations (multiple translations from one set to multiple closed systems) is considered, and we show how the *semantic consequence* relation determined by a matrix may be realized as the product of a source. Products of sources are used later in the text in our theory of parameterized algebraization. We also develop a theory of the quotient by a sink of translations from a closed system, and we show how the *filters* of sentential calculi arise as the quotient of a sink. In [BJ06], a theory of *transformers between closure operators*, was published, as part of a generalization of the theory of algebraic logic that Blok and Jónsson where developing shortly before the death of the former. Transformers are special functions between the lattices of closed sets of two closure operators. We show that transformers between closure operators and *strictly* continuous relationships between spaces are in essence the same notion. We explicate, and duly reference, the relationships between our (independently obtained) notions and theirs.

We shall now briefly outline the structure and contents of this text, which consists of six parts.

In Part I, titled 'A Survey of Algebraic Logic', we present the *standard theory of algebraizable logics* as well as preliminary definitions and results required for such a presentation. A motivation of the problem is provided in $\oint 3$ of Part I.

In Part II, titled 'Elementary and Concrete Closure', we develop the theory of elementary closed system and c-continuous translations between them, as well as the 'concrete' theory where necessary. Examples demonstrating how this theory unifies disparate arguments from algebraic logics are presented. Numerous examples of closed systems over the universes of algebras are also given. These examples lead to the 'inherently unalgebraizable' logics to which our parameterized theory of algebraization will apply.

Part III, titled 'Constructural Abstract Logic', is dedicated to the theory of *solutions to equations* over algebras, and *regularity* and *coherence* in algebra. In the course of this presentation, we complete some outstanding issues, extending results of [BR99], pertaining to 1-deductive systems, to n-deductive systems more generally. We show that every algebraizable logic is equivalent to a logic arising from solutions to equations modulo a quasivariety, and that a quasivariety is the equivalent algebraic semantics of some logic precisely when relative congruences of that quasivariety are uniquely determined by solutions to equations modulo relative congruences. Consequently, logics arising from solutions to equations are central to algebraic logics. These logics are protoalgebraic precisely when the solutions to equations modulo the quasivariety are in some sense coherent. We demonstrate that full-regularity and full-coherence falls outside of the domain of standard algebraic logic, and introduce the membership logic as a remedy to this problem.

In Part IV, titled 'Regularity, Coherence and the Logics of Solutions to Equations', we develop a theory of logics over (objects of) constructs, where by a construct we mean a concrete category, as a means of explicating the technique we have been using to induce sentential calculi from closed systems on universal algebras, since it permits us to treat a quasivariety as a signature of logics, in which case a closed system on the free algebra on denumerably infinite generators may be viewed as a logic and more importantly, as a structural logic. We develop a model theory for such logics. This model theory differs from the standard model theories in that we treat logics as models of logics, an approach which permits us to draw more naturally on the theory of continuous functions developed in Part II, although we do consider matrices as models, which by do by viewing matrices as 'small' logics with only their designated sets and the universe as theorems, an approach which permits us to immediately induce the standard results of matrix model theory (at the level of discourse of matrices over objects) from the theory of logics as models of logics. Particular attention is given to the property of structurality, which in our theory is a condition and not a fact, showing that a logic is structural precisely when it has a semantics. We also describe the technique we have been using to induce sentential calculi from logics on the free algebra of a quasivariety, and from logics on all the algebras of a quasivariety. A number of examples are developed in Part IV that are mostly 'inherently unalgebraizable' but for which the our parametrized theory of algebraization pertains.

Part V, titled 'Parameterized Algebraization', is dedicated to our theory of parameterized algebraization. *Parametrized* notions of protoalgebraicity, algebraic semantics and equivalent algebraic semantics are developed, and all these conditions characterized in a manner such that the standard results in the literature obtain in the case that the parameter is taken to be the empty-set. The theory is applied to the examples introduced in the earlier parts of this document for which the standard theory is inapplicable. New characterization of the various universal algebraic notions of regularity and coherence are obtained. Most of this material has been published in [BR03].

In Part VI, titled 'Protoalgeabricity and Equivalence in Constructural Abstract Logic', we develop the theory of equivalent logics across different constructs (in terms of a category isomorphism between the constructs) and a theory of filter correspondence, Leibniz equivalence and protoalgebraicity. Unlike the rest of the theory of logics over constructs, the theory of protoalgebraicity that we develop requires that an additional property be satisfied by the construct (or at least a logic over the construct). We term this property *Leibniz analyzability*. We have obtained a new characterization of protoalgebraic n-deductive systems that is simpler than that of [Pal03], but which is closer in spirit to the analogous characterization of protoalgebraic 1-deductive systems given in [BP89a]. We show how much of our results from the theory of parameterized algebraization obtain from the results in this part.

Part VII, titled 'Interior and Space', is included for the sake of completeness. Much of the theory of open systems and spaces is given an elementary footing.

Part VIII, titled 'Open Problems', contains open problems beyond those contained in the body of this text; the latter problem relate directly to the material at hand. The open problems of Part VIII are more general.

In Appendix A we justify our usage of the term 'continuous', and in Appendix B we develop an elementary theory of open systems and spaces.

The verbosity of this presentation results from it having been necessarily written so as to be accessible to a non-expert reader. Workers in the field are requested to be patient in this regard. The reader familiar with algebraic logic is encouraged to start reading from $\oint 3$, with the proviso that they consider the examples of $\oint 2.4$, $\oint 2.6.1$ and $\oint 2.7.1$; they are also encouraged to read the introductory narrative of $\oint 1$ and $\oint 2$. A comprehensive glossary and index is provided.

Part I

A Survey of Algebraic Logic

Part I of this text presents the preliminary material required in the sequel ($\oint 1$), provides a survey of the standard theory of algebraic logic ($\oint 2$) and motivate the underlying problem to be tackled in this text $\oint 3$. We take the opportunity, in the course of this presentation, to highlight some topics of special interest or importance to the theory of algebraizable logics, in particular, we emphasise *matrix theory* (also called *M-theory*) and provide an up-to-date presentation of the theory of *elementary structures*.

In $\oint 1.1$, we introduce our *set theoretic* preliminaries. Attention is given to *binary relationships* between collections, and in particular the various notions of *image* under a binary relationship. *Matrices*, both unary and more generally, but *without* an underlying algebra are also considered. *Order* theoretic preliminaries are presented in $\oint 1.2$ and *systems* (such as *closed systems*) are briefly considered in $\oint 1.3$. In $\oint 1.4$. we consider *constructs* and *categories*, where by construct we mean a *concrete* category. In Part III, we shall consider the theory of logics over constructs, more general than the usual logics over algebras, and this section provides the necessary definitions and results for that programme. In order to consider equivalences between logics over constructs, we require the notion of a category and, in particular, a *category isomorphism* between constructs; consequently, a few definitions and results from category theory are also presented. Finally, in $\oint 1.5$, we present the standard theory of *elementary structures* and *universal algebras*. We pay special attention to *congruences on structures* and the *Leibniz equivalence relation* on a structure; these results draw largely from [Elg97] and [Elg98]. Most readers will be familiar with much of the material of $\oint 1.1$. We have endeavoured to reference back to this chapter whenever a notion is used for the first time in the later text, and a glossary of symbolisms has also been provided. As such, most readers may omit this chapter and refer back to it as needed. This notwithstanding, we would advise all readers to give at least a cursory glance at $\oint 1.1.2$ on binary relationships and their images, as well as to note our non-standard notions for the join and meet given in $\oint 1.2.3$ (we write $\blacktriangledown$ for $\bigvee$ and $\blacktriangle$ for $\bigwedge$).

The survey of the standard theory of algebraic logic given in $\oint 2$ is provided as a convenience, and may also be omitted by most readers, although we urge all readers to read the introductory discourse of the chapter, where we note a few non-standard terms that we use. In particular, we call an n-deductive system a *sentential n-calculus* so as to avoid conflict with our nomenclature pertaining to logics over constructs. The reader should also note the examples given in that chapter, in particular the notations that we use to denote these example logics, since we refer to them often in the sequel.

Chapter 1

Preliminaries

In this chapter we introduce our conventions and notations. In $\oint$1.1 we provide our set theoretic preliminaries, with a particular focus on binary *relationships* and *matrices*. Orders are considered in $\oint$1.2 and systems in $\oint$1.3, although the latter section is very brief given that closed systems are considered in depth in Part II. In $\oint$1.4 we present our *categorical* and *constructural* preliminaries, where a construct is our non-standard term for a *concrete category*; constructs are usually called structures [Ada83], but we use the term *structure* for an elementary structure. The theory of *elementary structures* and *universal algebras* is provided in $\oint$1.5.

Convention 1.1 (Symbol Overloading) We often overload symbols, by which we mean that the same symbol may be used in different contexts, the context being determined by the parameters to the symbol. The types of the parameters uniquely determine which particular case of the overloaded symbolism is being refered to. For example, we have many uses of the symbol '$\vdash_\cdot$', the $\cdot$ indicates where the parameter goes. If the parameter is a closed system $\mathbb{C}$, then $\vdash_{\mathbb{C}}$ is the consequence relation associated with the closed system. Alternatively, if the parameter is a sentential calculus $\mathcal{S}$, then $\vdash_{\mathcal{S}}$ is the syntactic consequence relation determined by the sentential calculus. Since our meta-language is strongly type checked, every term has a unique type, and so overloaded operators are unambiguously resolved. Computer scientists are well aware of this protocol. It has the advantage of reducing the number of unique symbols required; no matter how much of a TEXpert one is, symbols are always in short supply. The various meanings of an overloaded symbol should be in some way related.

1.1 Relationships, Relations, Operations and Matrices

For our (meta-theory) of sets, classes and collections, we shall be using **elementary collection theory** and the **basic set theory** (denoted by **GA**), as described in [Pot90]. We are also guided by [Lev79] and [FBHL73], in particular regarding use of the **axiom-schema of replacement**. Our choice is justified on ontological grounds and by its facilitation of the ease of use of the language.

While this chapter is intended to present the standard set-theoretic notations required for the sequel, during the presentation we focus on two issues pertinent to the rest of the text.

Firstly, we pay special attention to *binary relationships* between collections, and, in particular, consider various notions of *image*, in particular the *standard* image and the *reduced*-image. Binary relationships between collections play an important role in the theory of algebraizable logic, since *translations* between logics (in the sense of [BP89a]) are binary relationships. Secondly, we focus on *matrices* in the sense of *M-theory*, which, in this chapter, we do without assuming an underlying algebra. We consider both unary matrices, n-ary matrices, and a more general notion, which we call a *realm*. Attention is paid to matrix-homomorphisms and reduced matrix homomorphisms, as well as the Leibniz relation.

In ∮1.1.1 we present our notations for *sets*, *classes* and *collections*, as well as notations for *pairs*, *unions*, etc. This section is very brief. For a deeper insight, the reader is urged to see [Pot90]. ∮1.1.2 is concerned with *binary relationships* and their various *images*, as well as *functions*. In ∮1.1.3 we consider *binary relations*, *operators*, *equivalence relations*, *quotients*, *partitions* and *kernels*. Particular attention is paid to various notions of *quotient*, and the related notion of *compatibility*. A number of results are introduced that are repeatedly used later.

Unary matrices, without underlying algebras, are considered in ∮1.1.4, as well as *matrix-homomorphisms*, *reductions*, and the *Leibniz equivalence relation*. Many of the results for matrices more generally, can be derived from results for unary matrices. Notations and basic results for *families*, *vectors* and *tuples* are introduced in ∮1.1.5, and *products* are introduced ∮1.1.6. *Matrices* more generally, but still without underlying algebras, are considered in ∮1.1.8, as are *relations*, which are essentially n-ary matrices with a 'lighter' syntax. *Operations* are defined in ∮1.1.9.

1.1.1 Basic Specifications

Ontologically, the cumulative hierarchy of *collections* proceeds beyond the *universal class* of all *sets*, allowing one to easily, and sensibly, speak of *things* like: a partial order on the class of all groups; closure under a class operator; or, a mapping from Boolean algebras to Boolean rings, without continually 'bumping one's head'. While in this sense the theory of collections is 'radical', the axiomatization of collection theory, as opposed to the basic set theory, is 'conservative' in that the axioms of replacement, choice and infinity are not assumed. The apparent 'radical' nature of this theory, for example, quantifying over classes, should be tempered with the observation that many standard set theories (see [FBHL73, 23-25]) permit sets to contain *individuals*, and so do not disallow the formation of *sets* containing, for example, all groups or all sets (or, equivalently, all sets not containing themselves), provided these elements are not treated as sets, but as individuals.

Convention 1.2 (Specification Language) For definitional purposes, it is convenient, in this subsection and this subsection only, to access the **specification language**. Arbitrary terms are denoted by $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$, and arbitrary formulae are denoted by $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. These notions are to be taken as 'intuitive'.

Convention 1.3 (Basic Language) We denote **membership** by '$\in$' and **equality** by '$=$'. We provide no publicly visible symbols for either the **description operator** or **definite existential quantifier**. The logical connectives symbols are ' and ', ' or ', ' not ', ' implies ' and ' iff '. At times, we write 'if $\boldsymbol{\phi}$ then $\boldsymbol{\psi}$' for '$\boldsymbol{\phi}$ implies $\boldsymbol{\psi}$'. We may also write '$\rightarrow$' for ' implies ' and '$\leftrightarrow$' for ' iff '. We denote the universal quantifier by '$\forall\,[\cdot]\,*$' and the (standard) existential quantifier by '$\exists\,[\cdot]\,*$', capitalizing where appropriate. Additional commas and brackets are used to aid the eye.

We tend to denote arbitrary **entities** (or **points**) (e.g., members of a collection) by lower-case Roman letters, such as a, b and c, and compounds such as a_1, a_2, a' and a'.

We write '$\{a : \boldsymbol{\phi}(a)\}$' for the **collection of all** a **such that** $\boldsymbol{\phi}(a)$, and tend to denote arbitrary collections by upper-case Roman letters (and compounds) such as A, B, C, A_1, A_2, A'' and A'. No public symbols for **individuals** are provided.

Convention 1.4 (Basic Formulae and Terms) We denote **containment** by $\subseteq$, and **proper containment** by $\subsetneq$. The **empty-collection**, **intersection of** A, **intersection of** A **and** B, **relative complement of** B **in** A, **union of** A, **union of** A **and** B, **power of** A, **singleton of** a and **unordered-pair containing** a **and** b, are denoted by $\emptyset$, $\bigcap A$, $A \cap B$, $A - B$, $\bigcup A$, $A \cup B$, $\mathfrak{P}(A)$, $\{a\}$ and $\{a, b\}$, respectively. We also write $\mathfrak{P}^1(A)$ for $\mathfrak{P}(A)$ and $\mathfrak{P}^0(A)$ for A. For a collection A and integer $n > 2$, let $\mathfrak{P}^n(A)$ be recursively defined by $\mathfrak{P}^n(A) = \mathfrak{P}(\mathfrak{P}^{n-1}(A))$. We write $\overset{A}{\neg}B$ for $A - B$, and define $\bigcap^A \emptyset = A$ and $\bigcap^A B = \bigcap B$ for non-empty B. We call this 'operation' $\bigcap^A$ the **relative intersection**.

Convention 1.5 [Pot90, 25] We often write $\{a \in A : \boldsymbol{\phi}(a)\}$ for $\{a : a \in A \text{ and } \boldsymbol{\phi}(a)\}$. If $\boldsymbol{\tau}(a_1, \ldots, a_n)$ is a term depending on the variables $a_1, \ldots, a_n$, and $\boldsymbol{\phi}(a_1, \ldots, a_n)$ is a formula, then we abbreviate $\{b : \exists\, [a_1, \ldots, a_n]\; b = \boldsymbol{\tau}(a_1, \ldots, a_n) \text{ and } \boldsymbol{\phi}(a_1, \ldots, a_n)\}$ by $\{\boldsymbol{\tau}(a_1, \ldots, a_n) : \boldsymbol{\phi}(a_1, \ldots, a_n)\}$, and further, if 'symbol(a)', say, is a symbol at our disposal that denotes a term depending on a, then the term denoted by the expression 'symbol($\{\boldsymbol{\tau}(a_1, \ldots, a_n) : \boldsymbol{\phi}(a_1, \ldots, a_n)\}$)' may also be denoted 'symbol$_{\boldsymbol{\phi}(a_1,\ldots,a_n)}(\boldsymbol{\tau}(a_1, \ldots, a_n))$'.

Convention 1.6 ($\doteq$) It is convenient, on occasion, to *implicitly* introduce a definition by means of the symbol '$\doteq$', as a means to simplify the discourse, typically within proofs, but occasionally in other contexts. For example, '$\boldsymbol{\tau} \doteq \boldsymbol{\rho}$' means that within some subsequent scope, clear from the context, $\boldsymbol{\tau}$ is defined to be equal to $\boldsymbol{\rho}$. We also use this symbol in definitions that include an explicated characterization. For example, we often say 'We define $\boldsymbol{\tau} \doteq \boldsymbol{\rho} = \boldsymbol{\rho}'$', which is to be taken as defining $\boldsymbol{\tau}$ to be $\boldsymbol{\rho}$ and that $\boldsymbol{\rho}$ is equal to $\boldsymbol{\rho}'$; typically $\boldsymbol{\rho}'$ is a trivial reformulation of $\boldsymbol{\rho}$ from another perspective.

Definition 1.7 (Classes and Sets) We introduce sets by introducing a constant symbol Set, which denotes a collection called the **universal class**. The members of Set are said to be **small**. Small collections are called **sets**. Subcollections of Set are called **classes**. Classes which are not sets are called **proper classes**. We tend to denote arbitrary **sets** with the same symbols that denote arbitrary collections. When such a symbol is used, without some explicit mention that it denotes a set, then it is always taken to be a collection. □

Remark 1.8 Speaking of a *class* of things implies that these things are *small*, and that collections of *small* things must be *classes*.

Convention 1.9 (Pairs and Cartesian Products) The **ordered pair** with first-coordinate a and second-coordinate b is denoted by $\langle a, b\rangle$. For an ordered pair $\langle a, b\rangle$, we write $\langle a, b\rangle_{(0)}$ for a and $\langle a, b\rangle_{(1)}$ for b. We adopt the convention that pairs are treated as individuals, and consequently expressions with a pair to the right of the $\in$ relation is a *syntax error*. When we say that a **is**

an ordered-pair, we mean that, $\exists\,[b \text{ and } c]\; a = \langle b, c\rangle$, *and* that the aforementioned convention applies. We tend to denote an arbitrary pair by (typewriter-font) $\mathtt{p}$.

We write $A \times B$ for $\{\langle a, b\rangle \,:\, a \in A \text{ and } b \in B\}$, which we call the **Cartesian product** of A by B, and we write A^2 for $A \times A$, which we call A **squared** or the **square of** A. For $r \subseteq A \times A$, we write $r_{[0]}$ for $\{\mathtt{p}_{(0)} \,:\, \mathtt{p} \in r\}$ and $r_{[1]}$ for $\{\mathtt{p}_{(1)} \,:\, \mathtt{p} \in r\}$.

Convention 1.10 (Omitting Universal Quantifiers) When presenting formulae, we often omit universal quantifiers and, more importantly, we often omit explicit restrictions on variables. Omitting universal quantifiers is well-defined in model theory; simply prepend universal quantifiers for each free variable. In the case of omissions of restrictions, these are to be taken to the least restrictive restrictions for which the formulae is context sensible.

We introduce a convention for naming entities and naming collections of entities.

Convention 1.11 (Naming) When the name of a term begins with a *lower-case* letter, the term is to viewed as an entity and the name describes that entity. For example, $\mathsf{do}(r)$, defined below, names a single entity, in this case, the domain of a binary relationship. When the name of a term begins with an *upper-case* letter, the term is to viewed as a collection of entities and the name describes the entities within that collection. In this case the name is best read by lowering the case of the initial letter and introducing the appropriate pluralisation. For example, $\mathsf{Bship}(A, B)$, defined below, names the collection of all binary relationships from A to B, and is read 'binary relationships from A to B'. Occasionally we may break this convention.

We assume the reader to be familiar with ordinals and cardinals, and with ordinal and cardinal arithmetic.

Convention 1.12 (Ordinals and Cardinals) We usually denote ordinals by lower case Greek letters $\zeta, \eta, \xi, \ldots$ and cardinals by $\mathbf{m}, \mathbf{n}, \ldots$. We use the symbol $\oplus$ for the operation of ordinal addition; in particular, $\zeta \oplus 1$ is the ordinal successor of an ordinal ζ. We use $+$ and $\cdot$ (or juxtaposition) to denote the operations of cardinal addition and multiplication, respectively. The cardinality of a set A shall be denoted by $\mathsf{card}(A)$. We write $A \subseteq_{\mathsf{f}} B$ iff A is a finite subcollection of B. The least **infinite** ordinal is denoted by ω, which we also call the set of *natural numbers*; ω is also the least infinite cardinal.

1.1.2 Binary Relationships and their Images

1.1.2.1 Binary Relationships and Functions

Our notion of a binary relationship from one collection to another 'strongly types' the domain and codomain, as is typical for functions.

Definition 1.13 (Binary Relationships) A **binary relationship** r is determined by a collection $\mathsf{do}(r)$ called the **domain**, a collection $\mathsf{co}(r)$ called the **co-domain**, and a member D_r of $\mathfrak{P}(\mathsf{do}(r) \times \mathsf{co}(r))$, called the **designator**, members of which are called **designated pairs**. We usually write $a \;\; r \;\; b$ for $\langle a, b\rangle \in \mathsf{D}_r$. A binary relationship **from** A is a binary relationship with domain A and a binary relationship **to** B is a binary relationship with codomain B. The collection of all binary relationships from A to B, is denoted by $\mathsf{Bship}(A, B)$. We say that binary relationship s **extends** binary relationship r or that r **retracts** s, if $\mathsf{co}(r) = \mathsf{co}(s)$, $\mathsf{do}(r) \subseteq \mathsf{do}(s)$ and $r \subseteq s$. We write $\mathsf{gr}(r)$ and $\mathsf{rg}(r)$ for $(\mathsf{D}_r)_{[0]}$ and $(\mathsf{D}_r)_{[1]}$, which we call the **ground** and **range** of binary relationship r respectively. $\square$

Convention 1.14 (Conflating Binary Relationships with their Designated Pairs) For a binary relationship r, we often conflate the collection D_r of designated-pairs, with the binary relationship r itself. Conversely, we often conflate a subcollection $r \subseteq A \times B$ with the binary relationship with domain A, codomain B and designated-pairs r, provided the domain and codomain are unambiguously (or sometimes just naturally) context determinable. For example, by this convention, $\mathsf{Bship}(A, B) = \mathfrak{P}(A \times B)$ is syntactically well-defined and semantically valid. Similarly, $r_{[0]} = \mathsf{gr}(r)$ and $r_{[1]} = \mathsf{rg}(r)$.

Definition 1.15 (Special Binary Relationships) We call a binary relationship **empty** (or **void**) if it contains no pairs and **full** if it is precisely the Cartesian product of its domain with its codomain. The (unique) full and empty binary relationships from A to B are denoted by ${}_A\blacksquare_B$ and ${}_A\emptyset_B$ respectively. We call a binary relationship **ranged** if its co-domain and range coincide, **grounded** if its ground and domain coincide, and **compulsory** if it is ranged and grounded. We call a binary relationship r **contracting** (or **one-many**) if $a\ r\ b_1$ and $a\ r\ b_2$ implies $b_1 = b_2$; **expanding** (or **many-one**) if $a_1\ r\ b$ and $a_2\ r\ b$ implies $a_1 = a_2$; and **one-one** if it is contracting and expanding. $\square$

Remark 1.16 In a contracting relationship, an entity from the domain may be associated with *at most* one entity of the co-domain, while an entity from the co-domain may be associated without restriction to entities of the domain (hence the synonym 'one-many'). Non participation *is* permitted. The converse is true of expanding relationships.

Remark 1.17 Clearly, r is ranged iff $\forall\,[a \in \mathsf{do}(r)]\ \exists\,[b \in \mathsf{co}(r)]\ a\ r\ b$. Dually, r is grounded iff $\forall\,[b \in \mathsf{co}(r)]\ \exists\,[a \in \mathsf{do}(r)]\ a\ r\ b$.

Definition 1.18 (Operations on Binary Relationships) Let r and s be two binary relationships. The **filtration of** r **by** s is the binary relationship $r_{:s}$ with the same domain and codomain as r and with designator $\mathsf{D}_{r_{:s}} = \mathsf{D}_r \cap \mathsf{D}_s$. The **confinement of** r **to** s, where $\mathsf{D}_s \subseteq \mathsf{do}(r) \times \mathsf{co}(r)$, is the binary relationship $r_{|s|}$ with $\mathsf{do}(r_{|s|}) = \mathsf{gr}(s)$, $\mathsf{co}(r_{|s|}) = \mathsf{rg}(s)$ and $\mathsf{D}_{r_{|s|}} = \mathsf{D}_r \cap \mathsf{D}_s$. For $C \subseteq \mathsf{do}(r)$, we write $r_{|C}$ for $r_{|C \times \mathsf{co}(r)|}$, which we call the **restriction of** r **to** C, and for $D \subseteq \mathsf{co}(r)$, write $r_{D|}$ for $r_{|\mathsf{do}(r) \times D|}$, which we call the **co-restriction of** r **into** D.

The **relational-composition** of r by s, denoted $r \circ s$, is defined, iff $\mathsf{co}(r) \subseteq \mathsf{do}(s)$, in which case, it is the binary relationship with $\mathsf{do}(r \circ s) = \mathsf{do}(r)$, $\mathsf{co}(r \circ s) = \mathsf{co}(s)$ and $\mathsf{D}_{r \circ s} = \{\langle a, c\rangle\ :\ \exists\,[b \in \mathsf{co}(r)]\ a\ r\ b, b\ s\ c\}$. The **reverse** of r, denoted by $\overleftarrow{r}$, is the binary relationship with $\mathsf{do}(\overleftarrow{r}) = \mathsf{co}(r)$, $\mathsf{co}(\overleftarrow{r}) = \mathsf{do}(r)$ and $\mathsf{D}_{\overleftarrow{r}} = \{\langle b, a\rangle\ :\ \langle a, b\rangle \in r\}$. Sometimes the reverse is denoted by mirroring the symbol representing the relationship 'about the y-axis', e.g., $\geq$ for $\overleftarrow{\leq}$. The **negation** of r, denoted $\not r$, is the binary-relationship with the same domain and codomain as r and designated pairs $\overset{\mathsf{do}(r) \times \mathsf{co}(r)}{\neg} r$. $\square$

Remark 1.19 Note, that if either (or both) relationships are empty, then the filtration is empty.

$r[\cdot]$ is $\subseteq$ -preserving. (1.1)	$\overleftarrow{r}[\cdot]$ is $\subseteq$ -preserving. (1.2)
$r[\emptyset] = \emptyset.$ (1.3)	$\overleftarrow{r}[\emptyset] = \emptyset.$ (1.4)
$r[\mathsf{do}(r)] = r[\mathsf{gr}(r)] = \mathsf{rg}(r).$ (1.5)	$\overleftarrow{r}[\mathsf{co}(r)] = \overleftarrow{r}[\mathsf{rg}(r)] = \mathsf{gr}(r).$ (1.6)
$r[A] = \bigcup_{a\in A} r[\![a]\!].$ (1.7)	$\overleftarrow{r}[A] = \bigcup_{a\in A} \overleftarrow{r}[\![a]\!].$ (1.8)
$r\left[\bigcup \mathcal{A}\right] = \bigcup_{A\in\mathcal{A}} r[A].$ (1.9)	$\overleftarrow{r}\left[\bigcup \mathcal{B}\right] = \bigcup_{B\in\mathcal{B}} \overleftarrow{r}[B].$ (1.10)
$r\left[\bigcap \mathcal{A}\right] \subseteq \bigcap_{A\in\mathcal{A}} r[A].$ (1.11)	$\overleftarrow{r}\left[\bigcap \mathcal{B}\right] \subseteq \bigcap_{B\in\mathcal{B}} \overleftarrow{r}[B].$ (1.12)
$r\left[\bigcap \mathcal{A}\right] = \bigcap_{A\in\mathcal{A}} r[A]$ iff r is expanding. (1.13)	$\overleftarrow{r}\left[\bigcap \mathcal{B}\right] = \bigcap_{B\in\mathcal{B}} \overleftarrow{r}[B]$ iff r is contracting. (1.14)
$A \cap \mathsf{gr}(r) \subseteq \overleftarrow{r}[r[A]].$ (1.15)	$B \cap \mathsf{rg}(r) \subseteq r\left[\overleftarrow{r}[B]\right].$ (1.16)
$A \subseteq \overleftarrow{r}[r[A]]$ iff r is grounded. (1.17)	$B \subseteq r\left[\overleftarrow{r}[B]\right]$ iff r is ranged. (1.18)
$A \supseteq \overleftarrow{r}[r[A]]$ iff r is expanding. (1.19)	$B \supseteq r\left[\overleftarrow{r}[B]\right]$ iff r is contracting. (1.20)
$A = \overleftarrow{r}[r[A]]$ iff r is grounded and expanding. (1.21)	$B = r\left[\overleftarrow{r}[B]\right]$ iff r is ranged and contracting. (1.22)
$\overleftarrow{r}[B] \subseteq \overleftarrow{r}\left[r\left[\overleftarrow{r}[B]\right]\right].$ (1.23)	$r[A] \subseteq r\left[\overleftarrow{r}[r[A]]\right].$ (1.24)
$\overleftarrow{r}[B] = \overleftarrow{r}\left[r\left[\overleftarrow{r}[B]\right]\right]$ iff r is expanding. (1.25)	$r[A] = r\left[\overleftarrow{r}[r[A]]\right]$ iff r is contracting. (1.26)
$\overset{\mathsf{co}(r)}{\neg}(r[A]) \cap \mathsf{rg}(r) \subseteq r\left\lfloor \overset{\mathsf{do}(r)}{\neg} A \right\rfloor.$ (1.27)	$\overset{\mathsf{do}(r)}{\neg}(\overleftarrow{r}[B]) \cap \mathsf{gr}(r) \subseteq \overleftarrow{r}\left\lfloor \overset{\mathsf{co}(r)}{\neg} B \right\rfloor.$ (1.28)
$\overset{\mathsf{co}(r)}{\neg}(r[A]) \subseteq r\left[\overset{\mathsf{do}(r)}{\neg} A\right]$ iff r is ranged. (1.29)	$\overset{\mathsf{do}(r)}{\neg}(\overleftarrow{r}[B]) \subseteq \overleftarrow{r}\left[\overset{\mathsf{co}(r)}{\neg} B\right]$ iff r is grounded. (1.30)
$\overset{\mathsf{co}(r)}{\neg}(r[A]) = r\left[\overset{\mathsf{do}(r)}{\neg} A\right]$ iff r is ranged and expanding. (1.31)	$\overset{\mathsf{do}(r)}{\neg}(\overleftarrow{r}[B]) = \overleftarrow{r}\left[\overset{\mathsf{co}(r)}{\neg} B\right]$ iff r is grounded and contracting. (1.32)

Table 1.1: Fundamental properties of images and preimages. (See Proposition 1.33 on page 20.)

Remark 1.20 Note that composition is '**associative**', i.e., $(r \circ s)\circ t = r \circ (s \circ t)$, whenever either side of the equality is well-defined.

Remark 1.21 When either side is well-defined, $\overleftarrow{r \circ s} = \overleftarrow{s} \circ \overleftarrow{r}$.

Definition 1.22 (Functions) A **function** f is a binary relationship, such that, for each point $a \in \mathsf{do}(f)$, there exists a unique point in $\mathsf{co}(f)$, denoted by $f(a)$, with $a\ f\ f(a)$. We denote the collection of all functions from collection A into collection B by $A \to B$, and by B^A. For any collection $\mathcal{F}$ of functions, we write $f : \mathcal{F}$ for $f \in \mathcal{F}$, and so write $f : A \to B$ for $f \in A \to B$. A *function* is called a **surjection** (or **onto**) if it is ranged, an **injection** (or **one-to-one**) if it is expanding, and a **bijection** if it is a surjection and an injection.

We shall often describe a function by describing how each point a of the domain maps to the corresponding point b of the codomain with the notation $a \mapsto b$, which is read 'a maps to b'. When we say that function g **follows** function f, we mean that $\mathsf{co}(f) \subseteq \mathsf{do}(g)$.

For $B \subseteq A$, let ${}_B\!\hookrightarrow_A$ denote the function from B into A mapping $b \mapsto b$, which we call the **inclusion function** or **inclusion map**. □

Remark 1.23 Functions are precisely the grounded contracting binary relationships.

Remark 1.24 Functions are contracting, but contracting relationships need not be functions.

Remark 1.25 The *functions* (but not relationships) having functions as reverses are precisely the the compulsory (grounded and ranged) one-one (contracting and expanding) binary relationships.

Remark 1.26 For a function f, the following conditions are equivalent.

1. $f = \emptyset$.
2. $\mathsf{do}(f) = \emptyset$.
3. $\mathsf{gr}(f) = \emptyset$.
4. $\mathsf{rg}(f) = \emptyset$.

Remark 1.27 If $\mathsf{co}(f) = \emptyset$ then $f = \emptyset$, but the converse is not generally true, since $\emptyset : \emptyset \to A$ for *any* collection A.

Definition 1.28 (Operations on Functions) If the reverse $\overleftarrow{f}$ of function f is itself a function, we shall denote it by f^{-1}, which we call the **inverse of** f. If a function f has an inverse, we call f **invertible**. For $f : A \to B$ and $g : B \to C$, then the relational composition $f \circ g$ is a function from A into C, which we denote by gf and call the **functional product**. For a function f, let $\underset{\to[2]}{f}$ denote the function from $(\mathsf{do}(f))^2$ into $(\mathsf{co}(f))^2$ mapping $\langle a, b\rangle \mapsto \langle f(a), f(b)\rangle$, which we call the **binary promotion of** f; typically we drop the subscript '[2]' when unambiguous. □

Remark 1.29 The restriction $f_{|C}$ of function f to $C \subseteq \mathsf{do}(f)$ and the co-restriction $f_{D|}$ of function f into $D \subseteq \mathsf{co}(f)$ are both functions.

Remark 1.30 $(gf)_{|A} = g(f_{|A})$.

1.1.2.2 Poles and Images

Definition 1.31 (Poles and Images) Let r be a binary relationship. For $a \in \mathsf{do}(r)$, let

$$r[\![a]\!] = \{b \in \mathsf{co}(r) \ : \ a \ r \ b\},$$

which we call the **pole** of a under/by r or the **pole** of r at a. The *function* $r[\![\cdot]\!] : \mathsf{do}(r) \to \mathfrak{P}(\mathsf{co}(r))$ is denoted by $r_{[\![]\!]}(\cdot)$ and called the **pole function**. For $A \subseteq \mathsf{do}(r)$, we define

$$r\,[A] = \{b \in \mathsf{co}(r) \ : \ \exists\,[a \in A]\; a \ r \ b\}.$$

which we call the **image** of A under r. The *function* $r\,[\cdot]$ from $\mathfrak{P}(\mathsf{do}(r))$ into $\mathfrak{P}(\mathsf{co}(r))$, is denoted by $r_{[]}(\cdot)$, and is called the **image function**. The collection $r_{[\![]\!]}\,[\mathsf{do}(r)]$ of all poles of r is also denoted by $\mathsf{Pole}(r)$, and the collection $r_{[]}\,[\mathfrak{P}(\mathsf{do}(r))]$, of all images under r, is also denoted by $\mathsf{Image}(r)$. The poles and images of $\overleftarrow{r}$, are called the **pre-poles** and **pre-images** of r, respectively. We write $\mathsf{Prepole}(r)$ and $\mathsf{PreIm}(r)$ for $\mathsf{Pole}(\overleftarrow{r})$ and $\mathsf{Image}(\overleftarrow{r})$, respectively. □

Remark 1.32 Of course, any result pertaining to images, trivially yields a result pertaining to pre-images, etc. We explicate the 'dual' formulations only when instructive or consequential.

Proposition 1.33 The formulae of Table 1.1 are all valid. $\square$

Remark 1.34 Images of compositions coincide with composition of images.

Remark 1.35 So the collection $\mathsf{Image}(r)$ (resp. $\mathsf{PreIm}(r)$) of images (resp. pre-images) is closed under arbitrary unions, contains $\emptyset$, and contains $\mathsf{rg}(r)$ (resp. $\mathsf{gr}(r)$).

Definition 1.36 (Functional Pre-image) For a function f, we write $f^{-1}[\cdot]$ for $\overleftarrow{f}[\cdot]$. $\square$

Proposition 1.37 The formulae of Table 1.2 on page 21 are all valid.

Lemma 1.38 For a function f, $A \cup B \subseteq \mathsf{do}(f)$ and $C \subseteq \mathsf{co}(f)$, the following conditions are equivalent.

1. $A \subseteq f^{-1}[C]$ implies $B \subseteq f^{-1}[C]$.
2. $f[A] \subseteq C$ implies $f[B] \subseteq C$.

$\square$

The following result follows directly from Proposition 1.33 and definitions.

Corollary 1.39 Let f be a function.

1. The image $f[\cdot]$ preserves arbitrary unions.
2. The pre-image $f^{-1}[\cdot]$ preserves arbitrary unions and arbitrary intersections.
3. The image $f[\cdot]$ preserves arbitrary intersections iff f is an injection.

Remark 1.40 Let f be a *surjective* function and $B_1 \cup B_2 \subseteq \mathsf{rg}(f)$. If $f^{-1}[B_1] \subseteq f^{-1}[B_2]$, then $B_1 \subseteq B_2$. $\square$

1.1.3 Binary Relations and Quotients

1.1.3.1 Binary Relations

Definition 1.41 (Binary Relations) A **binary relation** is a binary relationship in which the domain and co-domain coincide. We speak of a *binary relation on* A, by which we mean a binary relation with domain (and hence co-domain) equal to A, and write $\mathsf{BRel}(A)$ for $\mathsf{Bship}(A, A)$. Rather than speaking of the domain or codomain of a binary relation, given that they are equal, we speak instead of the **universe** of a binary relation α, which we denote by $\mathsf{uni}(\alpha)$. We maintain the convention of denoting the ground $\mathsf{gr}(\alpha)$ by $\alpha_{(0)}$, which we call the **zeroth-projection**, and denoting the range $\mathsf{rg}(\alpha)$ by $\alpha_{(1)}$, which we call the **first-projection**. We write $\blacksquare_A$ for ${}_A\blacksquare_A$, which we also call the **square relation** on A, and write $\boldsymbol{\emptyset}_A$ for ${}_A\emptyset_A$. The **equality relation** (or **diagonal relation**) $=_A$ on A is the relation $\{\langle a, a\rangle \,:\, a \in A\}$. $\square$

$f[\cdot]$ is $\subseteq$-preserving. (1.33)	$f^{-1}[\cdot]$ is $\subseteq$-preserving. (1.34)
$f[\emptyset] = \emptyset.$ (1.35)	$f^{-1}[\emptyset] = \emptyset.$ (1.36)
$f[\mathsf{do}(f)] = \mathsf{rg}(f).$ (1.37)	$f^{-1}[\mathsf{co}(f)] = f^{-1}[\mathsf{rg}(f)] = \mathsf{do}(f) = \mathsf{gr}(f).$ (1.38)
$f[A] = \bigcup_{a \in A} f[\![a]\!].$ (1.39)	$f^{-1}[A] = \bigcup_{a \in A} \overleftarrow{f}[\![a]\!].$ (1.40)
$f\left[\bigcup \mathcal{A}\right] = \bigcup_{A \in \mathcal{A}} f[A].$ (1.41)	$f^{-1}\left[\bigcup \mathcal{B}\right] = \bigcup_{B \in \mathcal{B}} f^{-1}[B].$ (1.42)
$\forall\, [\emptyset \neq \mathcal{A} \subseteq \mathfrak{P}(\mathsf{do}(f))]\ f\left[\bigcap \mathcal{A}\right] \subseteq \bigcap_{A \in \mathcal{A}} f[A].$ (1.43)	If $\mathcal{B} \neq \emptyset$ then $f^{-1}\left[\bigcap \mathcal{B}\right] = \bigcap_{B \in \mathcal{B}} f^{-1}[B].$ (1.44)
f is 1-1 iff $\forall\, [\emptyset \neq \mathcal{A} \subseteq \mathfrak{P}(\mathsf{do}(f))]\ f\left[\bigcap \mathcal{A}\right] = \bigcap_{A \in \mathcal{A}} f[A].$ (1.45)	
$A \subseteq f^{-1}[f[A]].$ (1.46)	$B \cap \mathsf{rg}(f) = f\left[f^{-1}[B]\right].$ (1.47)
$A = f^{-1}[f[A]]$ iff f is one-to-one. (1.48)	$B = f\left[f^{-1}[B]\right]$ iff f is surjective. (1.49)
$f^{-1}[B] \subseteq f^{-1}\left[f\left[f^{-1}[B]\right]\right].$ (1.50)	
$f^{-1}[B] = f^{-1}\left[f\left[f^{-1}[B]\right]\right]$ iff f is one-to-one. (1.51)	$f[A] = f\left[f^{-1}[f[A]]\right].$ (1.52)
$\overset{\mathsf{co}(f)}{\neg}(f[A]) \cap \mathsf{rg}(f) \subseteq f\left[\overset{\mathsf{do}(f)}{\neg} A\right].$ (1.53)	
$\overset{\mathsf{co}(f)}{\neg}(f[A]) \subseteq f\left[\overset{\mathsf{do}(f)}{\neg} A\right]$ iff f is ranged. (1.54)	
$\overset{\mathsf{co}(f)}{\neg}(f[A]) \supseteq f\left[\overset{\mathsf{do}(f)}{\neg} A\right]$ iff f is one-to-one. (1.55)	
$\overset{\mathsf{co}(f)}{\neg}(f[A]) = f\left[\overset{\mathsf{do}(f)}{\neg} A\right]$ iff f is a bijection. (1.56)	$\overset{\mathsf{do}(f)}{\neg}(f^{-1}[B]) = f^{-1}\left[\overset{\mathsf{co}(f)}{\neg} B\right] = f^{-1}\left[\overset{\mathsf{rg}(f)}{\neg} B\right].$ (1.57)

Table 1.2: Fundamental properties of functional images and preimages (see Proposition 1.37).

Definition 1.42 (Operations on Binary Relations) If α is a binary relation and $C \subseteq \mathsf{uni}(\alpha)$, then we write $\alpha_{|C|}$ for $\alpha_{|C^2|}$, which we call the **restriction** of α **within** C. Inductively, we define the n**-th relational product** $\alpha \circ^n \beta$ by

$$\alpha \circ^1 \beta = \alpha,$$
$$\alpha \circ^{n+1} \beta = \begin{cases} (\alpha \circ^n \beta) \circ \beta & ; \quad \text{if } n \text{ is odd,} \\ (\alpha \circ^n \alpha) \circ \beta & ; \quad \text{otherwise.} \end{cases}$$

$\square$

Remark 1.43 Clearly, $=_A = \overleftarrow{=_A}$, and so $=_A[\cdot] = \overleftarrow{=_A}[\cdot]$.

Definition 1.44 (Special Binary Relations) We call a binary relation α **discrete** if $a\ \alpha\ b$ implies $a = b$, **left-classical** if $a\ \alpha\ b$ implies $a\ \alpha\ a$, **right-classical** if $a\ \alpha\ b$ implies $b\ \alpha\ b$, **classical** if left and right classical (i.e., $a\ \alpha\ b$ implies $a\ \alpha\ a\ \mathsf{and}\ b\ \alpha\ b$), **reflexive** if $\forall\,[a]\ a\ \alpha\ a$, **irreflexive** if $\forall\,[a]\ \neg(a\ \alpha\ a)$, **symmetric** if $a\ \alpha\ b$ implies $b\ \alpha\ a$, **anti-symmetric** if $(a\ \alpha\ b\ \mathsf{and}\ b\ \alpha\ a)$ implies $a = b$, a **trichotomy** if $\forall\,[a, b]\ a\ \alpha\ b$ or $a = b$ or $b\ \alpha\ a$, **transitive** if $(a\ \alpha\ b\ \mathsf{and}\ b\ \alpha\ c)$ implies $a\ \alpha\ c$, **completes right parallelograms** if $(a\ \alpha\ c\ \mathsf{and}\ b\ \alpha\ c\ \mathsf{and}\ a\ \alpha\ d)$ implies $b\ \alpha\ d$, a **tolerance** if reflexive and symmetric, a **quasi-order** if reflexive and transitive, a **strict-partial-order** if irreflexive and transitive, a **local-equivalence relation** if symmetric and transitive, an **equivalence relation** if a symmetric quasi-order and an **order** if anti-symmetric and a quasi-order. $\square$

Remark 1.45 Reflexive binary relations are both grounded and ranged. A symmetric binary relation is grounded iff it is ranged. A local-equivalence relation must be classical, yet need not be reflexive. Hence a local-equivalence relation is an equivalence relation iff it is grounded iff it is ranged iff it is reflexive. $\square$

Definition 1.46 (Functional-Images of Binary Relations) For binary relation α and function f with $\mathsf{do}(f) = \mathsf{uni}(\alpha)$, let $\underset{\longrightarrow}{f}[\alpha]$ denote the binary relationship with universe $\mathsf{co}(f)$ and designator $\underset{\longrightarrow_{[2]}}{f}[\mathsf{D}_\alpha]$, which we call the **image** of α under f. $\square$

Remark 1.47 $\underset{\longrightarrow}{f}[\alpha]_{[0]} = f\left[\alpha_{[0]}\right]$ and $\underset{\longrightarrow}{f}[\alpha]_{[1]} = f\left[\alpha_{[1]}\right]$, by Remark 1.30 on page 19.

Remark 1.48 (Images of Binary Relations) Let f be a function from A into B, and let α be a binary relation on A.

1. $\mathsf{gr}(\underset{\longrightarrow}{f}[\alpha]) = f[\mathsf{gr}(\alpha)]$ and $\mathsf{rg}(\underset{\longrightarrow}{f}[\alpha]) = f[\mathsf{rg}(\alpha)]$.
2. $\underset{\longrightarrow}{f}[\alpha][\![f(a)]\!] = f[\alpha[\![a]\!]]$, for each $a \in A$, and consequently, $\mathsf{Pole}(\underset{\longrightarrow}{f}[\alpha]) = f\{\mathsf{Pole}(\alpha)\}$.

$\square$

Definition 1.49 (Functional Images of Power-Collections) Let f be a function from A into B. For each *power-collection* $\mathcal{A} \subseteq \mathfrak{P}(A)$, let $f\{\mathcal{A}\} = \{f[A] : A \in \mathcal{A}\}$. $\square$

1.1.3.2 Operators

Definition 1.50 (Operators) An **operator** *over* collection A, is a function from A into itself. Operators are also called **unary operations**. We denote the *collection* of all operators over A by $\mathsf{Op}(A)$, which is a set whenever A is a set. A bijective operator is called a **permutation**. Let u be an operator on A and $B \subseteq A$. We say that an element $a \in A$ is **u-fixed** (or a **fixed point**), if $\mathsf{u}(a) = a$, and is **u-idempotent** (or an **idempotent point**), if $\mathsf{u}(\mathsf{u}(a)) = \mathsf{u}(a)$. The collection of all u-fixed points of B, is denoted by $\mathsf{fixed\text{-}points}_{\mathsf{u}}B$, and collection of all u-idempotent points of B, is denoted by $\mathsf{idempotent}_{\mathsf{u}}(B)$. We call u **fixed on** B (**idempotent on** B), if $\mathsf{fixed\text{-}points}_{\mathsf{u}}B = B$ ($\mathsf{idempotent}_{\mathsf{u}}(B) = B$), and **fixed** (**idempotent**) if fixed (idempotent) on A. Fixed operators are also said to be **constant**. The collection of all idempotent operations and permutations on a class A are denoted by $\mathsf{idempotent\text{-}operations}(A)$ and $\mathsf{permutations}(\mathrm{A})$ respectively. □

Convention 1.51 (Operators as Binary Relations) Of course, operators are precisely the grounded functional binary relations. More formally, for an operator u, let $\underset{\rightarrow}{\mathsf{u}}$ denote the binary relation with $\mathsf{uni}(\underset{\rightarrow}{\mathsf{u}}) = \mathsf{uni}(\mathsf{u})$, defined by $a_0 \underset{\rightarrow}{\mathsf{u}} a_1$ iff $a_1 = \mathsf{u}(a_0)$. The distinction between u and $\underset{\rightarrow}{\mathsf{u}}$ is essentially syntactic.

Convention 1.52 (Unary Infix Notation) We shall often express (unary) operators in **unary infix notation**, whereby we prefix the operand with a symbol and suffix the operand with a symbol. For example, we could say that $(\,\cdot\,]$ is a unary operator, in which case we would write $(a\,]$ for $(\,\cdot\,](a)$.

Convention 1.53 When we speak of an operator *on* some entity, we mean that the entity is *in* the domain of the operator. We operate *on* an entity, thereby obtaining another entity .

Remark 1.54 The equality relation $=_A$ on A is an operator on A. In fact it is the (unique) operator with $=_A(a) = a$ for all $a \in A$. When adopting the functional stance, we speak of the **identity operator** over A, which we tend to denote by id_A.

1.1.3.3 Equivalence Classes, Quotients, Partitions and Kernels

Recall the definition of an equivalence relation given in Definition 1.44.

Definition 1.55 (Equivalence Classes and Quotients) We denote the collection of all equivalence relations on A by $\mathsf{ER}(A)$. If α is an equivalence relation and $a \in \mathsf{uni}(\alpha)$, then we call the *collection* $\alpha[\![a]\!]$ the **equivalence-class of a modulo α** or the **equivalence-class of α containing** a or the **equivalence-class of α at** a. For $B \subseteq \mathsf{uni}(\alpha)$, the collection $\alpha_{[\![]\!]}[B]$, of all α-equivalence-classes of points in B, is denoted by $\mathsf{EC}_B(\alpha)$. We write $\mathsf{EC}(\alpha)$ or $\coprod \alpha$ for $\mathsf{EC}_{\mathsf{uni}(\alpha)}(\alpha)$, as is most context appropriate. When we speak of the **quotient** or **decomposition** $A/\,\alpha$ of A by α, we mean that $A/\,\alpha = \coprod \alpha$ and that $A = \mathsf{uni}(\alpha)$. Let $\mathsf{q}_\alpha(\cdot)$ denote the function from $\mathsf{uni}(\alpha)$ onto $\coprod \alpha$ mapping a to $\alpha[\![a]\!]$, which we call the **canonical quotient mapping**. □

Remark 1.56 For equivalence relation α and $a \in \mathsf{do}(\alpha)$, $\alpha[\![a]\!] = \overleftarrow{\alpha}[\![a]\!]$ (by symmetry).

Remark 1.57 [Pot90, 41] We should really speak of the equivalence *collection* of a modulo α.

Remark 1.58 $\coprod \alpha$ is a *class* iff every equivalence-class is a set, and is a *set* whenever $\mathsf{do}(\alpha)$ is a set.

Remark 1.59 [Pot90, R 2] $a\ \alpha\ b$ iff $\alpha[\![a]\!] = \alpha[\![b]\!]$.

Remark 1.60 If f is a function and α an equivalence relation on $\mathsf{co}(f)$, then $\underset{\rightarrow}{f}^{-1}[\alpha]$ is an equivalence relation on $\mathsf{do}(f)$ and $f^{-1}[\alpha[\![f(a)]\!]] = (\underset{\rightarrow}{f}^{-1}[\alpha])[\![a]\!]$.

Proof. It is simple to show that $\underset{\rightarrow}{f}^{-1}[\alpha]$ is an equivalence relation on $\mathsf{do}(f)$. Further, $f^{-1}[\alpha[\![f(a)]\!]] = \{a' : f(a') \in \alpha[\![f(a)]\!]\} = \{a' : f(a')\ \alpha\ f(a)\} = \{a' : \langle a', a\rangle \in \underset{\rightarrow}{f}^{-1}[\alpha]\} = (\underset{\rightarrow}{f}^{-1}[\alpha])[\![a]\!]$. $\diamond$

Definition 1.61 (Partitions) We call $\kappa \subseteq \mathfrak{P}(A)$ a **partition of** A if: $\emptyset \notin \kappa$; ($a \in \kappa$ and $b \in \kappa$) implies ($a = b$ or $a \cap b = \emptyset$); and $\bigcup \kappa = A$. If κ is a partition, we write $\mathsf{uni}(\kappa)$ for $\bigcup \kappa$, which we call the universe of κ. The members of a partition are called **parts**. With every partition κ of A, we associate the *equivalence relation* $\equiv_\kappa = \{\langle a, b\rangle \in A^2 : a, b \in \mathfrak{u}$ for some $\mathfrak{u} \in \kappa\}$ on A. We write q_κ for $\mathsf{q}_{\equiv_\kappa}$ (see Definition 1.55). $\square$

The proof that $\equiv_\kappa$ is indeed an equivalence relation is well-known.

Remark 1.62 Every partition of a set is a set.

Remark 1.63 $\mathsf{q}_\kappa(a) = \mathsf{q}_\kappa(b)$ iff a and b lie in the *same part* of κ.

Starting from an equivalence relation α, we obtain the partition $\coprod \alpha$ of $\mathsf{uni}(\alpha)$. Conversely, given a partition κ, $\equiv_\kappa$ is an equivalence relation on $\mathsf{uni}(\kappa)$.

Remark 1.64 $\coprod(\equiv_\kappa) = \kappa$ and $\equiv_{(\coprod \alpha)} = \alpha$. $\square$

So the partitions of, and the equivalence relations on, some collection are in natural one-to-one correspondence. Consequently some texts factor by equivalence relations while others factor by partitions, typically called *decomposition* in the latter context. We *factor* by equivalence relations thereby obtaining the *quotient* or *decomposition*.

Definition 1.65 (Compatibility) We say that a binary relation α is **compatible** with subcollection $B \subseteq \mathsf{uni}(\alpha)$ (or say that B is α-**closed** or that B is **closed** under α) if $a \in B, a\ \alpha\ b \to b \in B$. $\square$

Remark 1.66 A binary relation α is compatible with $A \subseteq \mathsf{uni}(\alpha)$ iff $\alpha[A] \subseteq A$.

Remark 1.67 If α is an equivalence relation, α is compatible with A iff $\alpha[A] = A$ iff $\alpha[A] \subseteq A$.

Definition 1.68 (The Kernel of a Function) [Pot90, 42] With each function f, define $\equiv_f = \{\langle a, b\rangle \in \mathsf{do}(f)^2 : f(a) = f(b)\}$, which is an equivalence relation on $\mathsf{do}(f)$, called the **kernel** of f. We write $\coprod f$ for $\coprod \equiv_f = \mathsf{do}(f)/\equiv_f$, and denote the quotient surjection $\mathsf{q}_{\equiv_f}$, from $\mathsf{do}(f)$ onto $\coprod f$, by q_f. $\square$

Remark 1.69 Function f is **injective** iff $\equiv_f = =_{\mathsf{do}(f)}$.

Remark 1.70 $a \equiv_f b$ iff $\mathsf{q}_f(a) = \mathsf{q}_f(b)$.

Remark 1.71 [Pot90, P 2.2.4] A function f is surjective iff for every function g following f and with $\equiv_f \subseteq \equiv_g$, there exists a unique function $h : \mathsf{co}(f) \to \mathsf{co}(g)$ such that $h = gf$.

Remark 1.72 If f is a function and $A \subseteq \mathsf{do}(f)$, $\equiv_f$ is compatible with A iff $f^{-1}[f[A]] = A$. $\square$

Beginning with an equivalence α, we obtain the quotient $\coprod \alpha$ of $\mathsf{uni}(\alpha)$, and the quotient surjection q_α of $\mathsf{uni}(\alpha)$ onto the quotient $\coprod \alpha$. From the quotient surjection q_α, we obtain the kernel $\equiv_{\mathsf{q}_\alpha}$ which is an equivalence on $\mathsf{uni}(\alpha)$.

Remark 1.73 [Pot90, P 2.4.1] If α is an equivalence relation, then $\alpha = \equiv_{\mathsf{q}_\alpha}$.

Beginning with a function f, we obtain the kernel $\equiv_f$, the quotient $\coprod f$ and the associated quotient surjection q_f of $\mathsf{do}(f)$ onto the quotient $\coprod f$.

Remark 1.74 $\equiv_{\mathsf{q}_f} = \equiv_f$ (by the Remark 1.73 on page 25).

Remark 1.75 If $f : A \to B$ then $\equiv_f = \underset{\to}{f}^{-1}[=_B]$.

Remark 1.76 If $f : A \to B$ and $\alpha \in \mathsf{ER}(B)$, then $\equiv_f \subseteq \underset{\to}{f}^{-1}[\alpha]$.

1.1.4 Unary Matrices

Definition 1.77 (Unary Matrices) A **unary matrix** (or **unary relation**) M is determined by its **universe** $\mathsf{uni}(\mathsf{M})$ and its **designator** $\mathsf{D_M}$, where $\mathsf{D_M} \subseteq \mathsf{uni}(\mathsf{M})$. By a unary A-matrix, we mean a unary matrix with universe A. We may present a unary matrix M by $\langle \mathsf{uni}(\mathsf{M}), \mathsf{D_M} \rangle$, and even just by $\langle \mathsf{D_M} \rangle$, when the universe is unambiguously context determinable. In such presentations, we may write $\langle A, \mathsf{a} \rangle$ for $\langle A, \{\mathsf{a}\} \rangle$, etc., where context unambiguous.

We call unary matrix M a **submatrix** of unary matrix N, denoted $\mathsf{M} \lhd_\mathsf{D} \mathsf{N}$ (or $\mathsf{M} \lhd \mathsf{N}$ where unambiguous), iff $\mathsf{uni}(\mathsf{M}) \subseteq \mathsf{uni}(\mathsf{N})$ and $\mathsf{D_M} = \mathsf{D_N} \cap \mathsf{uni}(\mathsf{N})$. For a unary matrix M and $A \subseteq \mathsf{uni}(\mathsf{M})$, we denote the submatrix $\langle A, \mathsf{D_M} \cap A \rangle$ by $\mathsf{M}_{|_A}$. $\square$

Definition 1.78 (Unary Matrix Homomorphisms, Isomorphisms and Reductions) A function f from $\mathsf{uni}(\mathsf{M})$ into $\mathsf{uni}(\mathsf{N})$ is called a **unary matrix homomorphism** (or just a **matrix homomorphism** or even a **homomorphism**) from M into N if $f[\mathsf{D_M}] \subseteq \mathsf{D_N}$. A **unary matrix isomorphism** (or just a **matrix isomorphism** or even an **isomorphism**) f from M onto N is a *bijective* unary matrix homomorphism from M onto N that additionally satisfies $f[\mathsf{D_M}] \supseteq \mathsf{D_N}$. A **unary matrix embedding** of M into N is a matrix isomorphism onto a submatrix of N. A unary matrix homomorphism f from M into N is called **reductive** if f is *surjective* and $\mathsf{D_M} = f^{-1}[\mathsf{D_N}]$, in which case N is called a **reduction** of M and M is called an **expansion** of N. The collection of all reduced matrix homomorphisms from M onto N is denoted by $\mathsf{M} \twoheadrightarrow^{\mathsf{r}} \mathsf{N}$. $\square$

Proposition 1.79 If f is a reductive unary matrix homomorphism from M onto N, then $\mathsf{D_M} = f^{-1}[\mathsf{D_N}]$ and $a \in \mathsf{D_M} \leftrightarrow f(a) \in \mathsf{D_N}$.

Corollary 1.80 If $f : \mathsf{M} \twoheadrightarrow^{\mathsf{r}} \mathsf{N}$, then, for all $\mathsf{D_N} \subseteq B \subseteq \mathsf{uni}(\mathsf{N})$, $\mathsf{D_M} \subseteq f^{-1}[B]$.

Corollary 1.81 For a *bijective* function f, the following conditions are equivalent.

1. f is a unary matrix isomorphism from M onto N.
2. $f[\mathsf{D}_\mathsf{M}] = \mathsf{D}_\mathsf{N}$.
3. $a \in \mathsf{D}_\mathsf{M} \leftrightarrow f(a) \in \mathsf{D}_\mathsf{N}$.
4. f is a unary matrix homomorphism from M onto N and $f(a) \in \mathsf{D}_\mathsf{N} \rightarrow a \in \mathsf{D}_\mathsf{M}$.

Proposition 1.82 [BP92, P 5.1][vA95, P 1.8.2 (ii)] Let f be a reductive matrix-homomorphism from M onto N, let A be any collection and let $g : A \rightarrow \mathsf{uni}(\mathsf{M})$. Then, $\forall\,[a \in A]\; g(a) \in \mathsf{D}_\mathsf{M} \leftrightarrow f(g(a)) \in \mathsf{D}_\mathsf{N}$.

Definition 1.83 (Quotients) Let M be a unary matrix. For an equivalence relation α on $\mathsf{uni}(\mathsf{M})$, we denote the matrix $\langle \coprod \alpha, \mathsf{q}_\alpha[\mathsf{D}_\mathsf{M}]\rangle$ by $\mathsf{M}/\,\alpha$, which we call the **quotient** or **decomposition** of M by α. □

Recall the notion that an equivalence relation α be *compatible* with a subcollection B of its universe (see Definition 1.65 on page 24). Essentially, B must be *closed under* α-equivalent points, alternatively phrased, B must be *a union of* α equivalence-classes, or alternatively phrased, B must be *partitioned by* α equivalence-classes.

Definition 1.84 (Compatible Equivalence Relations) We say that α is **compatible** with M, if D_M is compatible with α. □

1.1.5 Families, Vectors and Tuples

Consider a function f and $a \in \mathsf{do}(f)$. From the 'local' perspective, $f(a)$ is viewed as a *point* in $\mathsf{co}(f)$; it has *individual* semantics. From the 'global', i.e., set-theoretic, perspective, $f(a)$ is a *collection* (and this collection is a member of $\mathsf{co}(f)$). *Family notation* is merely an alternative convention to *functional notation*, aimed to syntactically enable an easy to use 'local' perspective for the product. Families and functions are the same thing. Set theoretically, the notions of *family* and *function* coincide. By calling something a family, we simply imply that certain syntactic conventions are in use. The syntax of families is convenient when specifying a codomain is awkward or undesired. Families are also called *vectors* or **systems**.

Convention 1.85 (Family/Vector Notation) A **vector** a is function from a collection $\mathsf{idx}(\mathsf{a})$, called the **index**. We write $\mathsf{a}_{(i)}$ for $\mathsf{a}(i)$, which we call **the element at the i-th coordinate**. An **I-vector over** A (or **I-indexed vector over** A) is a vector with codomain A and index I. We often say that $\langle a_i : i \in I\rangle$ is a vector over A, by which we mean that $\mathsf{co}(\langle a_i : i \in I\rangle) = A$, $\mathsf{idx}(\langle a_i : i \in I\rangle) = I$, and that $a_i = \langle a_i : i \in I\rangle_{(i)}$. We also write $\langle a_i\rangle_I$ for $\langle a_i : i \in I\rangle$ when context unambiguous. The unique vector in $\{a\}^I$ is denoted by $\langle \frac{I}{a}\rangle$. Arbitrary collections of vectors are denoted by $\mathtt{A}$ and $\mathtt{B}$. We use the word **family** in place of vector when the coordinates are to be viewed as collections. For example we speak of a family $\langle A_i : i \in I\rangle$. Arbitrary families are denoted by $\boldsymbol{F}$. Definitions and conventions pertaining to vectors apply to families, but not conversely. We tend to denote arbitrary vectors by (typewriter-font) a, b, c and d and arbitrary collections of vectors by (typewriter-font) $\mathtt{A}$ and $\mathtt{B}$; the usual adornments apply. We may denote arbitrary members of A^ω by $\tilde{a}$, $\tilde{b}$, etc.

The collection of all I-families over A is *already* denoted by A^I and also by $I \to A$ (see Definition 1.22 on page 18).

Definition 1.86 (Operations on Vectors) Let a be a vector and $J \subseteq \mathsf{idx}(\mathsf{a})$. For $b \in \mathsf{co}(\mathsf{a})$, let $[\frac{J}{b}](\mathsf{a})$ denote the $\mathsf{idx}(\mathsf{a})$-vector over $\mathsf{co}(\mathsf{a})$, with $([\frac{J}{b}](\mathsf{a}))_{(j)} = b$, for $j \in J$, and $([\frac{J}{b}](\mathsf{a}))_{(j)} = \mathsf{a}_{(j)}$, otherwise; mapping all indicies in J to b, and leaving the others as they were. Let $\mathsf{a}_{|J} = \langle \mathsf{a}_{(j)} : j \in J \rangle$. □

Definition 1.87 (Comparing Vectors) We say that an I-vector a is **coordinate-distinct**, if, for any $i, j \in I$, $\mathsf{a}_{(i)} = \mathsf{a}_{(j)}$ implies $i = j$ (i.e., $i \neq j$ implies $\mathsf{a}_{(i)} \neq \mathsf{a}_{(j)}$), and say that a is **constant** if, for all $i, j \in I$, $\mathsf{a}_{(i)} = \mathsf{a}_{(j)}$. With collections I and A, we associate a function $[\cdot = \bullet]_{\mathrm{A^I}} : A^I \times A^I \to \mathfrak{P}(I)$, called the **equalizer**, and which is defined by $[\mathsf{a} = \mathsf{b}]_{\mathrm{A^I}} = \{i \in I : \mathsf{a}_{(i)} = \mathsf{b}_{(i)}\}$. Typically we drop the subscript 'A^I', since this parameter is determined by the arguments of the function. We shall, without further explication, invoke the equalizer on any two functions with the same domain and codomain, considered as families. □

Remark 1.88 For $\mathsf{a}, \mathsf{b}, \mathsf{c} \in A^I$, $[\mathsf{a} = \mathsf{a}] = I$, $[\mathsf{a} = \mathsf{b}] = [\mathsf{b} = \mathsf{a}]$ and $[\mathsf{a} = \mathsf{c}] \supseteq [\mathsf{a} = \mathsf{b}] \cap [\mathsf{b} = \mathsf{c}]$.

Definition 1.89 (Promoting Binary Relations) With each binary relationship r and each collection I, we associate the binary relationship $\underset{\to[I]}{r}$, from $(\mathsf{do}(r))^I$ to $(\mathsf{co}(r))^I$, defined by $\mathsf{a} \underset{\to[I]}{r} \mathsf{b}$ iff $\forall\, [i \in I]\ \mathsf{a}_{(i)}\ r\ \mathsf{b}_{(i)}$. We typically drop the subscript '$[I]$' from this notation whenever context unambiguous (or 'naturally' context determined). □

Remark 1.90 If f is a function then so is $\underset{\to[I]}{f}$.

Convention 1.91 (Promoting Functions) Consequent to the previous remark, when f is a function, then we shall adopt *functional notation* for $\underset{\to[I]}{f}$.

Remark 1.92 Let f be a function and a an I-indexed vector over A.

1. $(\underset{\to[I]}{f}(\mathsf{a}))_{(i)} = f(\mathsf{a}_{(i)})$.
2. f is surjective (injective, bijective) iff $\underset{\to[I]}{f}$ is surjective (resp. injective, bijective).
3. $\underset{\to[I]}{\mathsf{id}_A} = \mathsf{id}_{(A^I)}$.
4. $\underset{\to[I]}{fg} = \underset{\to[I]}{f}\,\underset{\to[I]}{g}$.

Convention 1.93 (Tuples) For natural n, n-vectors are also called n**-tuples**, often denoted $\langle a_1, \ldots, a_n \rangle$. By a **tuple** or **finite sequence**, we mean an n-tuple for some natural n. For a function f from A^n, we may write $f(a_0, \ldots, a_{n-1})$ for $f(\langle a_0, \ldots, a_{n-1} \rangle)$. Our notation has been arranged so that we may unambiguously confuse the 2-tuple $\langle a_i : i \in 2 \rangle$ with the pair $\langle a_0, a_1 \rangle$, thereby resolving the potential ambiguity that arises between the collection of pairs $A \times A$ and the collection of 2-tuples A^2. We conflate the 1-tuple $\langle a_i : i \in 1 \rangle$ with the element a_0 (and *not* the singleton $\{a_0\}$), thereby equating A^1 with A. At times, we present an n-tuple with an index other than n. In particular, we often write $\langle a_1, \ldots, a_n \rangle \in A^n$ which simply means that $\langle a_1, \ldots, a_n \rangle_{(i)} = a_{i+1}$. We may denote an arbitrary n-tuple, for arbitrary n, by $\vec{a}$, $\vec{b}$, etc.

Remark 1.94 $\underset{\rightarrow [n]}{f}(a_0, \ldots, a_{n-1}) = \langle f(a_0), \ldots, f(a_{n-1}) \rangle$, if f is a function and n is a non-zero natural.

1.1.6 Products

Definition 1.95 (Products) The **product** of a function f, is the collection $\prod f$ of functions $g : \mathsf{do}(f) \rightarrow \bigcup \mathsf{rg}(f)$ satisfying $g(a) \in f(a)$, for all $a \in \mathsf{do}(f)$. □

Remark 1.96 The following conditions are equivalent.

1. $f = \emptyset$.
2. $\prod f = \{\emptyset\}$.
3. $\emptyset \in \prod f$.

□

Remark 1.97 Under the axiom-of-choice, $\prod f \neq \emptyset$.

Convention 1.98 (Products of Families) Products are often presented via families, i.e., given an I-family $\boldsymbol{F} = \langle A_i : i \in I \rangle$, we then tend to denote the product $\prod \boldsymbol{F}$ by $\prod_{i \in I} A_i$, which is (viewed as) the collection of all I-vectors a in $\bigcup_{i \in I} A_i = \bigcup \mathsf{uni}(\boldsymbol{F}))$ with $\mathsf{a}_{(i)} \in A_i$ for all $i \in I$. For each $i \in I$, $\mathsf{a} \mapsto \mathsf{a}_{(i)}$ defines a function from $\prod_{i \in I} A_i$ into A_i, which we call the i-th **projection map** and denote by $\boldsymbol{\pi}_i(\cdot)$.

Remark 1.99 $\prod_{i \in I} A_i \subseteq \left(\bigcup_{i \in I} A_i\right)^I$. □

1.1.7 Reduced Products

Definition 1.100 (The Relation $\mho^{\boldsymbol{F}}_{\mathcal{A}}$) Let $\boldsymbol{F}$ be a family and $\mathcal{A} \subseteq \mathfrak{P}(\mathsf{idx}(\boldsymbol{F}))$. Define a binary relation $\mho^{\boldsymbol{F}}_{\mathcal{A}}$ on $\prod \boldsymbol{F}$ by $\langle \mathsf{a}, \mathsf{b} \rangle \in \mho^{\boldsymbol{F}}_{\mathcal{A}}$ iff $[\mathsf{a} = \mathsf{b}] \in \mathcal{A}$. □

Remark 1.101 [BS81] If $\mathcal{F} \in \mathsf{Filter}(\mathsf{idx}(\boldsymbol{F}))$ then $\mho^{\boldsymbol{F}}_{\mathcal{F}}$ is an equivalence relation on $\prod \boldsymbol{F}$.

Remark 1.102 If, for each $i \in \mathsf{idx}(\boldsymbol{F})$, $\mathsf{card}(\boldsymbol{F}_{(i)}) > 1$, then $\mho^{\boldsymbol{F}}_{\mathcal{A}}$ is an equivalence relation on $\prod \boldsymbol{F}$ iff $\mathcal{A} \in \mathsf{Filter}(\mathsf{idx}(\boldsymbol{F}))$.

Definition 1.103 (Reduced Products and Ultraproducts) Let $\boldsymbol{F}$ be a family and $\mathcal{F} \in \mathsf{Filter}(\mathsf{idx}(\boldsymbol{F}))$. The quotient $(\prod \boldsymbol{F})/\mho^{\boldsymbol{F}}_{\mathcal{F}}$ is called a **reduced product** of $\boldsymbol{F}$, and is called an **ultraproduct** in the case that $\mathcal{F}$ is an ultrafilter. □

1.1.8 Realms, Matrices and Relations

One of the primary focus of this preliminary chapter is matrices in the sense of M-theory, but without an underlying algebra. In ∮1.1.4, we considered unary matrices. In doing so we considered the feature of a *designator*, but in a purely *scalar* manner. In this section, we consider the *vector* nature of matrices. Many of the results pertaining to matrices with *dimension* follow directly from results already established for unary matrices.

1.1.8.1 Realms, Matrices and Relations

We begin by introducing a more general notion than an n-*matrix*, namely that of a *realm*, where role of the dimension n is replaced by an arbitrary indexing collection.

Definition 1.104 (Realms) A **realm** (or **arbitrary-relation**) $\boldsymbol{R}$ is determined by a collection $\mathsf{uni}(\boldsymbol{R})$, called the **(scalar) universe**, a collection $\mathsf{idx}(\boldsymbol{R})$, called the **index**, and a collection $\mathsf{D}_{\boldsymbol{R}} \subseteq (\mathsf{uni}(\boldsymbol{R}))^{\mathsf{idx}(\boldsymbol{R})}$, called the **designator**. We denote the product $(\mathsf{uni}(\boldsymbol{R}))^{\mathsf{idx}(\boldsymbol{R})}$ by $\underrightarrow{\mathsf{uni}}(\boldsymbol{R})$, which we call the **vector-universe** of $\boldsymbol{R}$, the members of which are called **vectors**. We often present a realm by $\langle I, A, \mathtt{A}\rangle$, where $\mathtt{A} \subseteq A^I$, by which we mean the realm with index I, (scalar) universe A and designator $\mathtt{A}$. By an (I, A)**-realm** we mean a realm $\boldsymbol{R}$ with $\mathsf{uni}(\boldsymbol{R}) = A$ and $\mathsf{idx}(\boldsymbol{R}) = I$. The collection of all (I, A)-realms is denoted $\mathsf{Rlm}_I(A)$. For a collection A, an A-realm is a realm with (*scalar*) universe A. When the particular index is clear from the context, we may present a realm $\boldsymbol{R}$ by $\langle \mathsf{uni}(\boldsymbol{R}), \mathsf{D}_{\boldsymbol{R}}\rangle$; the index is often determinable from the definition of the second member of this pair. It is convenient, at times, to treat a subset $\mathtt{A} \subseteq A^I$ as the I-realm $\langle A, \mathtt{A}\rangle$, and to conflate a realm with its designator. Usage of these conventions is typically confined to proofs. □

Definition 1.105 (Matrices) A **matrix** M is a realm where $\mathsf{idx}(\mathsf{M})$ is a non-zero natural, which we call the **dimension** or **arity**, writing $\mathsf{dim}(\mathsf{M})$ or $\mathsf{ar}(\mathsf{M})$ for $\mathsf{idx}(\mathsf{M})$. A matrix is a matrix of dimension n, for some n. For a collection A, an A-matrix is a matrix with *scalars* A. When the particular dimension is clear from the context, we may present an A-matrix M by $\langle \mathsf{uni}(\mathsf{M}), \mathsf{D}_{\mathsf{M}}\rangle$. Further to our convention of conflating $\langle a\rangle$ and a, we conflate unary matrices and matrices of arity one. A **binary matrix** is a matrix of arity two. □

Convention 1.106 (Relations) A relation R is simply a matrix for which special syntax is to be applied. When we call something a relation, we shall confuse it with its designator, i.e., if R is a relation then we may write $\mathsf{a} \in R$ for $\mathsf{a} \in \mathsf{D}_R$. Typically, we use the term 'relation' when we do not want to view the relation as a matrix (for example, see Definition 1.108 on page 30). We may call $R \subseteq A^n$ a relation. By an n**-ary relation** we mean a relation of arity n. A *unary relation* is a relation of arity 1. By our convention of conflating $\langle a\rangle$ and a, we tend to treat a unary relation as a subcollection of its universe. We conflate 2-ary relations and binary relations as defined in ∮1.1.3.

Convention 1.107 (Products as Realms) When we call $\prod f$ a realm, we are speaking of the realm $\langle \mathsf{do}(f), \mathsf{co}(f), \prod f\rangle$. When we call A^n a matrix, we are speaking of the matrix $\langle A, A^n\rangle$ of dimension n.

Definition 1.108 (Promoting Relations) Let $\boldsymbol{R}$ be a realm. With each (scalar) m-ary relation R on $\mathsf{uni}(\boldsymbol{R})$, we associate the (vector) m-ary relation $\underset{\longrightarrow}{R}_{[\mathsf{idx}(\boldsymbol{R})]}$ on $\underset{\longrightarrow}{\mathsf{uni}}(\boldsymbol{R})$, defined by $\langle \mathsf{a}_0, \ldots, \mathsf{a}_{m-1}\rangle \in \underset{\longrightarrow}{R}_{[\mathsf{idx}(\boldsymbol{R})]}$ iff $\forall\, [i \in \mathsf{idx}(\boldsymbol{R})]\ \langle \mathsf{a}_{0(i)}, \ldots, \mathsf{a}_{m-1(i)}\rangle \in R$. We typically drop the subscript $[\mathsf{idx}(\boldsymbol{R})]$ from this notation whenever context unambiguous. $\square$

Remark 1.109 Conventionally, for $A \subseteq \mathsf{uni}(\boldsymbol{R})$, the notion $\underset{\longrightarrow}{A}$ is well-defined; it is a unary relation on $\underset{\longrightarrow}{\mathsf{uni}}(\boldsymbol{R})$.

Definition 1.110 (Treating Realms as Unary Matrices) With each realm $\boldsymbol{R}$, we denote the unary matrix $\langle \underset{\longrightarrow}{\mathsf{uni}}(\boldsymbol{R}), \mathsf{D}_{\boldsymbol{R}}\rangle$ by $\boldsymbol{R}_{\mathsf{u}}$. $\square$

Definition 1.111 (Subrealms and Submatrices) We call realm $\boldsymbol{R}$ a **subrealm** of realm $\boldsymbol{Q}$, denoted $\boldsymbol{R} \lhd_{\mathsf{D}} \boldsymbol{Q}$, iff $\mathsf{idx}(\boldsymbol{R}) = \mathsf{idx}(\boldsymbol{Q})$, $\mathsf{uni}(\boldsymbol{R}) \subseteq \mathsf{uni}(\boldsymbol{Q})$ and $\mathsf{D}_{\boldsymbol{R}} = \mathsf{D}_{\boldsymbol{Q}} \cap \underset{\longrightarrow}{\mathsf{uni}}(\boldsymbol{R})$. For a realm $\boldsymbol{R}$ and $A \subseteq \mathsf{uni}(\boldsymbol{R})$, we denote the subrealm $\langle \mathsf{idx}(\boldsymbol{R}), A, \mathsf{D}_{\boldsymbol{R}} \cap A^{\mathsf{dim}(\boldsymbol{R})}\rangle$ by $\boldsymbol{R}_{|_A}$. Subrealms of matrices are called *submatrices*. $\square$

1.1.8.2 Homomorphisms

We now consider homomorphisms between realms. We begin by considering the notion of a vector-homomorphism between $\boldsymbol{R}$ and $\boldsymbol{Q}$; vector-homomorphisms are precisely the matrix-homomorphisms between the unary matrices $\boldsymbol{R}_{\mathsf{u}}$ and $\boldsymbol{Q}_{\mathsf{u}}$ (see Definition 1.110). We then turn to realm-homomorphisms between realms $\boldsymbol{R}$ and $\boldsymbol{Q}$; these are functions between $\mathsf{uni}(\boldsymbol{R})$ and $\mathsf{uni}(\boldsymbol{Q})$ that when promoted, 'preserve the designator'. The reason for this two stage approach, is that the theory of vector-morphisms bootstraps from the theory of unary-matrix-homomorphisms, and the realm-homomorphisms become special cases of vector-homomorphisms. Consequently, no new proofs are required.

The term '*vector*' in 'vector function', as used in the following definition, derives from the fact that the function under consideration is from $\underset{\longrightarrow}{\mathsf{uni}}(\boldsymbol{R})$ into $\underset{\longrightarrow}{\mathsf{uni}}(\boldsymbol{Q})$, as opposed to a function from $\mathsf{uni}(\boldsymbol{R})$ into $\mathsf{uni}(\boldsymbol{Q})$, which would be called a '*scalar* function'. While in M-theory, one is not concerned with vector functions generally, but rather, only those vector functions arising from *promoted scalar functions* (discussed shortly), some notions and results are most easily phrased and proved at this level of discourse.

Definition 1.112 (Vector Homomorphisms, Isomorphisms and Reductions)
A *vector* function $\boldsymbol{f}$ from $\underset{\longrightarrow}{\mathsf{uni}}(\boldsymbol{R})$ into $\underset{\longrightarrow}{\mathsf{uni}}(\boldsymbol{Q})$ is called a **vector-homomorphism** from $\boldsymbol{R}$ into $\boldsymbol{Q}$ if $\mathsf{idx}(\boldsymbol{R}) = \mathsf{idx}(\boldsymbol{Q})$ and $\boldsymbol{f}[\mathsf{D}_{\boldsymbol{R}}] \subseteq \mathsf{D}_{\boldsymbol{Q}}$. A **vector-isomorphism** $\boldsymbol{f}$ from $\boldsymbol{R}$ onto $\boldsymbol{Q}$ is a *bijective* vector-homomorphism from $\boldsymbol{R}$ onto $\boldsymbol{Q}$ that additionally satisfies $\boldsymbol{f}[\mathsf{D}_{\boldsymbol{R}}] \supseteq \mathsf{D}_{\boldsymbol{Q}}$. A vector-homomorphism $\boldsymbol{f}$ from $\boldsymbol{R}$ into $\boldsymbol{Q}$ is called **reductive** if $\boldsymbol{f}$ is *surjective* and $\mathsf{D}_{\boldsymbol{R}} = \boldsymbol{f}^{-1}[\mathsf{D}_{\boldsymbol{Q}}]$. $\square$

The following proposition and corollary follows from Proposition 1.79 and Corollary 1.80 by considering the unary matrices $\boldsymbol{R}_{\mathsf{u}}$ and $\boldsymbol{Q}_{\mathsf{u}}$.

Proposition 1.113 If $\boldsymbol{f}$ is a reductive vector-homomorphism from $\boldsymbol{R}$ onto $\boldsymbol{Q}$, then $\boldsymbol{f}[\mathsf{D}_{\boldsymbol{R}}] = \mathsf{D}_{\boldsymbol{Q}}$ and $\mathsf{a} \in \mathsf{D}_{\boldsymbol{R}} \leftrightarrow \boldsymbol{f}(\mathsf{a}) \in \mathsf{D}_{\boldsymbol{Q}}$.

Corollary 1.114 For a *bijective* function $\boldsymbol{f}$, the following conditions are equivalent.

1. $\boldsymbol{f}$ is a vector-isomorphism from $\boldsymbol{R}$ onto $\boldsymbol{Q}$.

2. $\boldsymbol{f}[\mathsf{D}_{\boldsymbol{R}}] = \mathsf{D}_{\boldsymbol{Q}}$.

3. $\mathsf{a} \in \mathsf{D}_{\boldsymbol{R}} \leftrightarrow \boldsymbol{f}(\mathsf{a}) \in \mathsf{D}_{\boldsymbol{Q}}$.

4. $\boldsymbol{f}$ is a vector-homomorphism from $\boldsymbol{R}$ onto $\boldsymbol{Q}$ and $\boldsymbol{f}(\mathsf{a}) \in \mathsf{D}_{\boldsymbol{Q}} \rightarrow \mathsf{a} \in \mathsf{D}_{\boldsymbol{R}}$.

□

The following result is an immediate corollary to Proposition 1.82

Corollary 1.115 [BP92, P 5.1] Let $\boldsymbol{f}$ be a reductive vector-homomorphism from $\boldsymbol{R}$ onto $\boldsymbol{Q}$, let A be any collection and let $\boldsymbol{g} : A \rightarrow \mathsf{uni}(\boldsymbol{R})$. Then, $\forall\, [\mathsf{a} \in \underset{\rightarrow}{A}]\; \boldsymbol{g}(\mathsf{a}) \in \mathsf{D}_{\boldsymbol{R}} \leftrightarrow \boldsymbol{f}(\boldsymbol{g}(\mathsf{a})) \in \mathsf{D}_{\boldsymbol{Q}}$. □

In matrix-theory or M-theory, one is not really concerned with vector functions, but rather with *promoted* scalar functions. Recall that by Remark 1.92, f is surjective (resp. injective, bijective) iff $\underset{\rightarrow}{f}$ is surjective (resp. injective, bijective).

Definition 1.116 (Realm-Homomorphisms, Realm-Isomorphisms and Reductions) Let f be a *scalar* function from $\mathsf{uni}(\boldsymbol{R})$ into $\mathsf{uni}(\boldsymbol{Q})$. We call f a **realm-homomorphism** from $\boldsymbol{R}$ into $\boldsymbol{Q}$ if $\mathsf{idx}(\boldsymbol{R}) = \mathsf{idx}(\boldsymbol{Q})$ and $\underset{\rightarrow}{f}$ is a vector-homomorphism from $\boldsymbol{R}$ into $\boldsymbol{Q}$. We call a f a **realm-isomorphism** from $\boldsymbol{R}$ onto $\boldsymbol{Q}$ if $\underset{\rightarrow}{f}$ is a vector isomorphism from $\boldsymbol{R}$ onto $\boldsymbol{Q}$. A realm-homomorphism f from $\boldsymbol{R}$ into $\boldsymbol{Q}$ is called **reductive** if $\underset{\rightarrow}{f}$ is reductive, in which case $\boldsymbol{Q}$ is called a **reduction** of $\boldsymbol{R}$ and $\boldsymbol{R}$ is called an **expansion** $\boldsymbol{Q}$. In the case of matrices, we speak of **matrix-homomorphisms** and **matrix-isomorphisms**. □

The following remarks follow from Proposition 1.113 and Corollary 1.114

Remark 1.117 f is a realm-isomorphism iff f is bijective and $\underset{\rightarrow}{f}[\mathsf{D}_{\boldsymbol{R}}] = \mathsf{D}_{\boldsymbol{Q}}$.

Remark 1.118 A realm homomorphism f is reductive iff it is surjective and $\mathsf{D}_{\boldsymbol{R}} = \underset{\rightarrow}{f}^{-1}[\mathsf{D}_{\boldsymbol{Q}}]$.
□

The following result follows from Corollary 1.115

Proposition 1.119 [BP92, P 5.1] Let f be a reductive realm-homomorphism from $\boldsymbol{R}$ onto $\boldsymbol{Q}$, let A be any collection and let $g : A \rightarrow \mathsf{uni}(\boldsymbol{R})$. Then, $\forall\, [\mathsf{a} \in \underset{\rightarrow}{A}]\; \underset{\rightarrow}{g}(\mathsf{a}) \in \mathsf{D}_{\boldsymbol{R}} \leftrightarrow \underset{\rightarrow}{f}(\underset{\rightarrow}{g}(\mathsf{a})) \in \mathsf{D}_{\boldsymbol{Q}}$. □

1.1.8.3 Quotients

While we are able to define a vector-homomorphism from which the scalar notion of a realm-homomorphism is a special case, this is not generally possible for quotients; factoring must be by a scalar equivalence relation α on $\mathsf{uni}(\boldsymbol{R})$, and *not* a vector equivalence relation $\boldsymbol{\alpha}$ on $\underset{\rightarrow}{\mathsf{uni}}(\boldsymbol{R})$, since we need to posit a realm (the factor realm), and while the designator is naturally describable, the scalar universe is not: 'promotion' is natural, while 'demotion' is not.

Definition 1.120 (Quotient Realms) For a realm $\boldsymbol{R}$ and an equivalence relation α on $\mathsf{uni}(\boldsymbol{R})$, we denote the matrix $\langle \alpha\,[\mathsf{uni}(\boldsymbol{R})]\,, \underset{\rightarrow}{\alpha}\,[\mathsf{D}_{\boldsymbol{R}}]\rangle$ by $\boldsymbol{R}/\,\alpha$. □

Remark 1.121 If $\boldsymbol{Q}$ is a subrealm of $\boldsymbol{R}$ and α is an equivalence relation on $\mathsf{uni}(\boldsymbol{R})$ then, by definition, $\langle \alpha [\mathsf{uni}(\boldsymbol{Q})], \underset{\rightarrow}{\alpha} [\mathsf{D}_{\boldsymbol{R}}] \cap \alpha [\mathsf{uni}(\boldsymbol{Q})]\rangle$ is a subrealm of $\boldsymbol{R}/\ \alpha$. □

The reader is urged to recall Definition 1.65 and Remark 1.67.

Definition 1.122 (Compatibility) We say that a binary relation α on $\mathsf{uni}(\boldsymbol{R})$ is **compatible** with $\mathtt{A} \subseteq \underset{\rightarrow}{\mathsf{uni}}(\boldsymbol{R})$ if $\mathtt{a} \in \mathtt{A}, \mathtt{a} \underset{\rightarrow}{\alpha} \mathtt{b} \to \mathtt{b} \in \mathtt{A}$. □

Remark 1.123 A binary relation α on $\mathsf{uni}(\boldsymbol{R})$ is compatible with $\mathtt{A} \subseteq \underset{\rightarrow}{\mathsf{uni}}(\boldsymbol{R})$ iff $\underset{\rightarrow}{\alpha} [\mathtt{A}] \subseteq \mathtt{A}$.

1.1.9 Operations

Definition 1.124 (Operations) An n**-ary operation** on A, where n is a non-negative integer, is any function $O : A^n \to A$; we say that n is the **arity** or **rank** of the operation O. An n-ary operation, for some n, is called a **finitary** operation. The image of $\langle a_1, \ldots, a_n\rangle$ under an n-ary operation O is denoted by $O(a_1, \ldots, a_n)$, or by $O(\langle a_1, \ldots, a_n\rangle)$ when convenient. An operation O on A is called a **nullary** operation or **constant** if its arity is zero; it is completely determined by the image $O(\emptyset)$ in A of the element $\emptyset$ in A^0, and it is convenient to identify it with the element $O(\emptyset)$. An operation O on A is called **unary**, **binary**, **ternary**, or **quaternary** if its arity is one, two, three or four respectively. We denote the collection of all n-ary operations on A by $\mathsf{Op}_{[n]}(A)$, and we write $\mathsf{BOp}(A)$ for $\mathsf{Op}_{[2]}(A)$. □

Remark 1.125 The notion of *unary* operations and *operators* coincide; $\mathsf{Op}_{[1]}(A) = \mathsf{Op}(A)$.

Convention 1.126 (Operations as Relations) Extending Convention 1.51 on page 23, with each operation O, let $\underset{\rightarrow}{O}$ denote the $\mathsf{ar}(O) + 1$-ary relation with universe $\mathsf{uni}(O)$, defined by

$$\langle a_0, \ldots, a_{\mathsf{ar}(O)}, a_{\mathsf{ar}(O)+1}\rangle \in \underset{\rightarrow}{O} \text{ iff } a_{\mathsf{ar}(O)+1} = O(a_0, \ldots, a_{\mathsf{ar}(O)}).$$

Definition 1.127 (Binary Operations) A **binary operation** on A, is a function from $A \times A$ into A. Let $\boxdot$ be a **binary operation** on A. We usually write $a \boxdot b$ for $\boxdot(a, b)$. We call $\boxdot$ **commutative** if $a \boxdot b = b \boxdot a$, for all $a, b \in A$, **associative** if $a \boxdot (b \boxdot c) = (a \boxdot b) \boxdot c$, for all $a, b, c \in A$, and **idempotent** if $a \boxdot a = a$, for all $a \in A$. □

1.2 Order

In $\oint$1.2 we consider *quasi-orders*, *directions*, *orders* and *semilattice* and *lattice* orders, as well as the related notions of *compactness*, *ideals* and *filters*, and order preserving functions. The theory of algebraic logic makes deep use of *complete* and *algebraic* lattices, and does so in a verbose manner. For example, one often considers a function that maps an algebraic lattice isomorphically onto a join-complete subsemilattice $\mathbf{P}$ of a complete lattice $\mathbf{Q}$ such that $\mathbf{P}$ is compact in $\mathbf{Q}$. In this section, we have attempted to introduce all the notions necessary in as 'compact' a manner as possible, and have attempted to 'tone down' such statements as the one mentioned earlier, by means of a visual notation.

1.2.1 Quasi-Orders and Directions

Recall that by Definition 1.44, a *quasi-order* is a reflexive and transitive binary relation. The importance of quasi-orders lies in the fact that they are almost orders and almost equivalence relations.

Definition 1.128 (Quasi-Order Collections) A **quasi-ordered set Q** is determined by its universe $\mathsf{uni}(\mathbf{Q})$ and a binary relation $\sqsubseteq^{\mathbf{Q}}$ that is a quasi-order on $\mathsf{uni}(\mathbf{Q})$. We often describe a quasi-order $\mathbf{Q}$ by the notation $\langle \mathsf{uni}(\mathbf{Q}); \sqsubseteq^{\mathbf{Q}} \rangle$. Conventionally, we may conflate quasi-ordered sets and quasi-orders, for example, calling $\mathbf{Q}$ a quasi-order. We call $\mathbf{Q}$ a quasi-order in A if $\mathbf{Q}$ is a quasi-order and $\mathsf{do}(\mathbf{Q}) \subseteq A$. $\square$

Quasi-orders may be used to provide a *direction* to a set.

Definition 1.129 (Directions) A **$\sqsubseteq$-direction** on A, is a quasi-order on A such that, for all a and b in A, there exists c in A with $a \sqsubseteq c$ and $b \sqsubseteq c$. We speak of a **directed** quasi-order. $\square$

1.2.2 Orders

Recall that by Definition 1.44, an *order* is an anti-symmetric quasi-order.

Definition 1.130 (Orders) [Pot90, S 2.6] An ordered set $\mathbf{P}$ is determined by its universe $\mathsf{uni}(\mathbf{P})$ and an order $\leq^{\mathbf{P}}$ on $\mathsf{uni}(\mathbf{P})$. We often describe an order $\mathbf{P}$ by the notation $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}} \rangle$, and tend to conflate ordered sets and orders. Orders are sometimes referred to as **partial orders**, although not in this text.

An order $\mathbf{P}$ is called a **linear order** or **chain**, if, for all $a, b \in \mathsf{uni}(\mathbf{P})$, either $a \leq^{\mathbf{P}} b$ or $b \leq^{\mathbf{P}} a$. Let $\mathbf{P}$ be an order and let $a, b \in \mathsf{uni}(\mathbf{P})$. We write $a <^{\mathbf{P}} b$ precisely when $a \leq^{\mathbf{P}} b$ and $a \neq b$, and say that b **covers** a, denoted $a \dashv^{\mathbf{P}} b$, if $a <^{\mathbf{P}} b$ and, whenever $c \in \mathsf{uni}(\mathbf{P})$ with $a \leq^{\mathbf{P}} c \leq^{\mathbf{P}} b$, we have $a = c$ or $b = c$. If $a \dashv^{\mathbf{P}} b$ or $b \dashv^{\mathbf{P}} a$ then we will say that elements a and b are **adjacent**. We write $a \not\leq^{\mathbf{P}} b$ if $\langle a, b \rangle \notin \leq^{\mathbf{P}}$, $a \geq^{\mathbf{P}} b$ if $b \leq^{\mathbf{P}} a$, and say that a and b are **incomparable** if $a \not\leq^{\mathbf{P}} b$ and $b \not\leq^{\mathbf{P}} a$. We drop superscripts $\mathbf{P}$ from our notations wherever unambiguously possible. With each order $\mathbf{P}$, we associate the order $\mathbf{P}^{\mathsf{d}}$, called the **dual of P**, defined by $a \leq^{\mathbf{P}^{\mathsf{d}}} b$ iff $b \leq^{\mathbf{P}} a$. $\square$

Remark 1.131 For order $\mathbf{P}$, $a \geq^{\mathbf{P}} b$ iff $a \leq^{\mathbf{P}^{\mathsf{d}}} b$. $\square$

We shall often use the following simple observation.

Remark 1.132 Let $\mathbf{P}$ be an order and $a, b \in \mathsf{uni}(\mathbf{P})$.

$$a = b \text{ iff } \forall\, [c \in \mathsf{uni}(\mathbf{P})]\ (c \leq a \leftrightarrow c \leq b). \tag{1.58}$$

$\square$

Definition 1.133 (Suborders) We call order $\mathbf{P}$ a **suborder** of order $\mathbf{Q}$, denoted $\mathbf{P} \triangleleft \mathbf{Q}$, if $\mathsf{uni}(\mathbf{P}) \subseteq \mathsf{uni}(\mathbf{Q})$ and $\leq^{\mathbf{P}} \,=\, \leq^{\mathbf{Q}} \cap (\mathsf{uni}(\mathbf{P}))^2$. For order $\mathbf{P}$ and $B \subseteq \mathsf{uni}(\mathbf{P})$, let $\mathbf{P}_{|_B}$ denote the binary relation with universe B and $\leq^{\mathbf{P}_{|_B}} \,=\, \leq^{\mathbf{P}} \cap B^2$. $\square$

Remark 1.134 $\mathbf{P}_{|_B}$ is the unique suborder of $\mathbf{P}$ with universe $B \subseteq \mathsf{uni}(\mathbf{P})$.

Example 1.135 ($\mathfrak{P}$-Concrete Orders)

If $\mathcal{A} \subseteq \mathfrak{P}(A)$, then $\langle \mathcal{A}, \subseteq \rangle$ is an order. Such orders are said to be **inclusion-ordered over** A or **$\mathfrak{P}$-concrete over** A. We speak synonymously of **inclusion-orders**, **$\subseteq$-orders** and **$\mathfrak{P}$-concrete orders**, by which we mean an order that is $\mathfrak{P}$-concrete over some A. If $\mathbf{P}$ is $\mathfrak{P}$-concrete over some A, then it is also $\mathfrak{P}$-concrete over $\bigcup \mathsf{uni}(\mathbf{P})$, and $\bigcup \mathsf{uni}(\mathbf{P})$ is the least such set with this property.

In particular, $\langle \mathfrak{P}(A), \subseteq \rangle$ is an inclusion-order, which we denote by $\boldsymbol{\mathfrak{P}}(A)$ and call the **power order over** A. The inclusion-orders over A are precisely the suborders of $\boldsymbol{\mathfrak{P}}(A)$.

□

Definition 1.136 (Bounded Collections) An **upper bound** (resp. **lower bound**) of $X \subseteq \mathsf{uni}(\mathbf{P})$ in $\mathbf{P}$ is an element $b \in \mathsf{uni}(\mathbf{P})$, such that $a \leq b$ (resp. $b \leq a$), for all $a \in X$. We denote the set of all upper (lower) bounds of X in $\mathbf{P}$ by $\mathsf{upper}^{\mathbf{P}}(X)$ and $\mathsf{lower}^{\mathbf{P}}(X)$.

An upper (resp. lower) bound b of X is called a **least upper bound** (resp. **greatest lower bound**) or **supremum** (resp. **infimum**) of X if, for all upper (resp. lower) bounds c of X, $b \leq c$ (resp. $c \leq b$). If a least upper bound (resp. greatest lower bound) of X exists, then it is unique, and we denote it by $\blacktriangledown^{\mathbf{P}} X$ (resp. $\blacktriangle^{\mathbf{P}} X$). We write $a \vee^{\mathbf{P}} b$ (resp. $a \wedge^{\mathbf{P}} b$) for $\blacktriangledown^{\mathbf{P}}\{a, b\}$ (resp. $\blacktriangle^{\mathbf{P}}\{a, b\}$), where $a, b \in \mathsf{uni}(\mathbf{P})$ and $\blacktriangledown^{\mathbf{P}}\{a, b\}$ (resp. $\blacktriangle^{\mathbf{P}}\{a, b\}$) exists. X is said to have a **greatest element** (resp. **least element**) if $\blacktriangledown^{\mathbf{P}} X$ (resp. $\blacktriangle^{\mathbf{P}} X$) exists and is contained in X. We write $a = \blacktriangledown^{\mathbf{P}} X$ (resp. $a = \blacktriangle^{\mathbf{P}} X$) iff the join of X (resp. meet of X) exists and is equal to a. An order $\mathbf{P}$ is called **well ordered** if every non-empty subset has a least element.

If a least (resp. greatest) element exists in an order, it is often denoted by $0^{\mathbf{P}}$ (resp. $1^{\mathbf{P}}$). An order with a least (greatest) element is called **lower bounded** (resp. **upper bounded**) or an order **with** 0 (**with** 1). An order with a least and a greatest element is called **bounded**. An element a is said to be **least** (resp. **greatest**) if, for every $b \in \mathsf{uni}(\mathbf{P})$, $a \leq b$ (resp. $b \leq a$). □

Note 1.137 (Non-Standard Notations) The symbols '$\blacktriangledown$' and '$\blacktriangle$' are non-standard. Most authors write $\bigvee$ and $\bigwedge$ instead. We find that the large size of these symbols, together with typographic baselining, leads to an aesthetically unappealing, disrupted and *lengthy* layout. □

Remark 1.138 An order $\mathbf{P}$ is lower bounded (resp. upper bounded) iff $\blacktriangledown^{\mathbf{P}} \emptyset$ exists (resp. $\blacktriangle^{\mathbf{P}} \emptyset$ exists), in which case $\blacktriangledown^{\mathbf{P}} \emptyset = 0^{\mathbf{P}}$ (resp. $\blacktriangle^{\mathbf{P}} \emptyset = 1^{\mathbf{P}}$).

Definition 1.139 (Directed Collections) [DP90, 51] A subset $X \subseteq \mathsf{uni}(\mathbf{P})$ is said to be **$\leq^{\mathbf{P}}$-directed** (**$\geq^{\mathbf{P}}$-directed**) if every finite subset of X has an upper bound (resp. lower bound) in X. We also call $\leq^{\mathbf{P}}$-directed ($\geq^{\mathbf{P}}$-directed) sets **upwardly-directed** (resp. **downwardly-directed**). □

Remark 1.140 The *empty* set cannot be directed under this definition.

Remark 1.141 It is easily shown that a *non-empty* set X is upwardly-directed iff, for all $a, b \in X$, $\{a, b\}$ has an upper bound in X.

Remark 1.142 If $\mathbf{P}$ is upper bounded, then $\mathsf{uni}(\mathbf{P})$ is upwardly directed in $\mathbf{P}$.

1.2.3 Semilattice and Lattice Orders

In the theory of algebraizable logics, we are required to speak of 'join-semilattice isomorphisms from a complete lattice into a join-complete subsemilattice of a complete lattice' and other such tongue-twisters. We shall attempt to introduce the discourse of lattices and semilattices in a compact fashion, and to symbolise notations so as to aid the eye over technical phrases.

Definition 1.143 (Semilattice and Lattice Orders) We define the following abbreviations; the meanings of the symbols not yet encountered, such as $\dot{\blacktriangledown}$ and $\sqcup$ will become clear as we proceed.

$$
\begin{array}{lclcl}
\phi_0(Y) & \doteq & \phi_1(Y) & \doteq & \mathsf{card}(Y) = 0, \\
\phi_\vee(Y) & \doteq & \phi_\wedge(Y) & \doteq & \mathsf{card}(Y) = 2, \\
\phi_{\dot{\blacktriangledown}}(Y) & \doteq & \phi_{\dot{\blacktriangle}}(Y) & \doteq & \mathsf{card}(Y) > 0, \\
\phi_\sqcup(Y) & & & \doteq & Y \text{ is upwardly-directed}, \\
\phi_\sqcap(Y) & & & \doteq & Y \text{ is downwardly-directed}, \\
\phi_\blacktriangledown(Y) & \doteq & \phi_\blacktriangle(Y) & \doteq & \mathsf{true}.
\end{array}
$$

Let $\mathbf{P}$ be an order and X a subset of $\mathsf{uni}(\mathbf{P})$. Let $\Box$ be a symbol from amongst $\{0, \vee, \dot{\blacktriangledown}, \sqcup, \blacktriangledown\}$. We call X **$\Box$-consistent** in $\mathbf{P}$, if, $\forall(Y \subseteq X)$ such that $\phi_\Box(Y)$, $\blacktriangledown^{\mathbf{P}} Y$ exists iff $\blacktriangledown^{\mathbf{P}|_X} Y$ exists and, when either exists, they are equal. We call X **$\Box$-closed** in $\mathbf{P}$ if, $\forall(Y \subseteq X)$ such that $\phi_\Box(Y)$, $\blacktriangledown^{\mathbf{P}} Y$ exists and $\blacktriangledown^{\mathbf{P}} Y \in X$. For each symbol $\Box$ from amongst $\{1, \wedge, \dot{\blacktriangle}, \sqcap, \blacktriangle\}$, the notions that X be **$\Box$-consistent** in $\mathbf{P}$ and **$\Box$-closed** in $\mathbf{P}$, are defined dually.

For any finite sequence of symbols $\Box_1 \ldots \Box_n$ over $\{0, \vee, \dot{\blacktriangledown}, \sqcup, \blacktriangledown, 1, \wedge, \dot{\blacktriangle}, \sqcap, \blacktriangle\}$, we call X **$\Box_1 \ldots \Box_n$-consistent** in $\mathbf{P}$ (**$\Box_1 \ldots \Box_n$-closed**) in $\mathbf{P}$, if, for $1 \leq i \leq n$, X is $\Box_i$-consistent (resp. $\Box_i$-closed) in $\mathbf{P}$. We abbreviate $\vee \wedge$ by $\Diamond$, $\blacktriangledown \blacktriangle$ by $\blacklozenge$, $\Box 0$ by $\Box_0$ and $\Box 1$ by $\Box_1$. Let $\vec{\Box}$ be a sequence of symbols from amongst $\{0, 1, \vee, \wedge, \Diamond, \blacktriangledown, \blacktriangle, \blacklozenge, \dot{\blacktriangledown}, \dot{\blacktriangle}, \sqcup, \sqcap\}$. The set of all $\vec{\Box}$-closed subsets of $\mathbf{P}$ is denoted by $\mathsf{closed}^{\vec{\Box}}(\mathbf{P})$.

We say that an order $\mathbf{P}$ is **$\vec{\Box}$-ordered** (or a **$\vec{\Box}$-order**) if $\mathsf{uni}(\mathbf{P})$ is a $\vec{\Box}$-closed subset of $\mathbf{P}$. Let $\mathcal{A}$ be a set of orders. The subset of all $\vec{\Box}$-ordered elements of $\mathcal{A}$ is denoted by $\vec{\Box}(\mathcal{A})$. For orders $\mathbf{P}$ and $\mathbf{Q}$, $\mathbf{Q}$ is called a **$\vec{\Box}$-suborder** of $\mathbf{P}$, written $\mathbf{Q} \lhd_{\vec{\Box}} \mathbf{P}$, if $\mathbf{Q} \lhd \mathbf{P}$ and $\mathsf{uni}(\mathbf{Q})$ is a $\vec{\Box}$-consistent subset of $\mathbf{P}$. $\Box$

Remark 1.144 Figure 1.1 on page 36 describes some of the relationships between the various types of semilattice orders. Stricter conditions lie below weaker ones linked by a single line. Double lines indicate equivalence.

Convention 1.145 (Standard Lattice Notations) We have opted for a slightly non-standard nomenclature, in an attempt to simplify the discourse. In keeping with standard convention, we shall also call 0-orders **lower bounded orders**, 1-orders **upper bounded orders**, 01-orders **bounded orders**, $\vee$-orders **join-semilattice orders**, $\wedge$-orders **meet-semilattice orders**, $\Diamond$-orders **lattice orders**, $\dot{\blacktriangledown}$-orders **unbottomed complete join-semilattice orders**, $\dot{\blacktriangle}$-orders **untopped complete meet-semilattice orders**, $\blacktriangledown$-orders **complete join-semilattice orders**, $\blacktriangle$-orders **complete meet-semilattice orders**, $\blacklozenge$-orders **complete lattice orders**, and, $\sqcup$-orders **pre-complete orders**. We often drop the word 'order', speaking, for example, of a **complete join-semilattice**. Many authors fail to distinguish meet from join semilattices, in which case one speaks just of **semilattices** and **complete semilattices**.

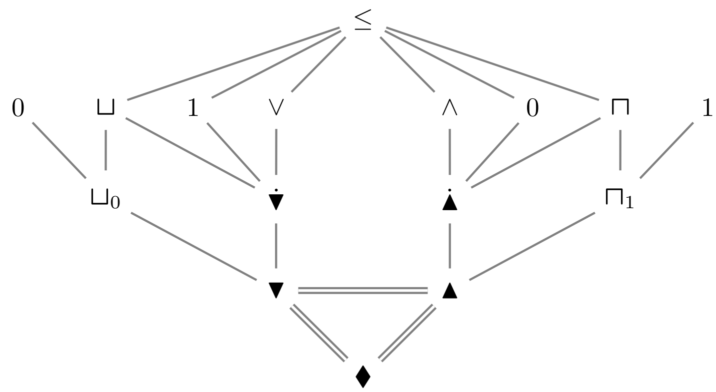


Figure 1.1: Various Notions of Semilattice and Lattice Orders (see Remark 1.144 on page 35)

We call $\vee$-suborders **join-subsemilattices**, $\wedge$-suborders **meet-subsemilattices**, $\Diamond$-suborders **sublattices**, $\blacktriangledown$-suborders **join-complete subsemilattices**, $\blacktriangle$-suborders **meet-complete subsemilattices** and $\blacklozenge$-suborders **complete sublattices**.

Note 1.146 (Consistency) The notion of 'consistency', which we have found in logical arguments, does not appear in lattice textbooks. It is similar in nature to the logical notion of sound-and-adequate [BM75]. $\square$

Remark 1.147 [BS81] It is easily shown that $\blacklozenge$-orders, $\blacktriangledown$-orders and $\blacktriangle$-orders 'coincide', in the sense that, for any order $\mathbf{P}$, the following conditions are equivalent: (i) $\mathbf{P}$ is a $\blacklozenge$-order; (ii) $\mathbf{P}$ is a $\blacktriangle$-order; (iii) $\mathbf{P}$ is an upper-bounded $\dot{\blacktriangle}$-semilattice; (iv) $\mathbf{P}$ is a $\blacktriangledown$-order; (v) $\mathbf{P}$ is a lower-bounded $\dot{\blacktriangledown}$-semilattice. These three notions of complete lattices admit different morphisms, however, and consequently may exhibit different constructural properties.

Example 1.148 ($\mathfrak{P}(A)$) [BS81, 17]

For any set A, $\mathfrak{P}(A)$ is an complete lattice order, with $\bigcup$ for $\blacktriangledown$ and $\bigcap$ as $\blacktriangle$.

$\square$

Join-complete subsemilattices of complete lattices play an important role in the theory of algebraizable logics. Let $\mathbf{P}$ and $\mathbf{Q}$ be $\blacklozenge$-lattices with $\mathsf{uni}(\mathbf{P}) \subseteq \mathsf{uni}(\mathbf{Q})$. In [BP89a], $\mathbf{P}$ is called a *join-complete subsemilattice* of $\mathbf{Q}$ if $\blacktriangledown^{\mathbf{P}} A = \blacktriangledown^{\mathbf{Q}} A$, for all $A \subseteq \mathsf{uni}(\mathbf{P})$. The following simple result demonstrates that $\mathbf{P} \lhd_{\blacktriangledown} \mathbf{Q}$ iff $\mathbf{P}$ is a join-complete subsemilattice of $\mathbf{Q}$ in the sense of [BP89a].

Proposition 1.149 If $\mathbf{P}$ and $\mathbf{Q}$ are $\blacklozenge$-lattices and $\mathsf{uni}(\mathbf{P}) \subseteq \mathsf{uni}(\mathbf{Q})$, then $\mathbf{P} \lhd_{\blacktriangledown} \mathbf{Q}$ iff $\blacktriangledown^{\mathbf{P}} A = \blacktriangledown^{\mathbf{Q}} A$, for all $A \subseteq \mathsf{uni}(\mathbf{P})$.

Proof. $\boxed{\Rightarrow}$ Since, by assumption, $\mathsf{uni}(\mathbf{P})$ is $\blacktriangledown$-consistent in $\mathbf{Q}$ and $\blacktriangledown^{\mathbf{Q}} A$ exists, $\blacktriangledown^{\mathbf{Q}} A = \blacktriangledown^{\mathbf{Q}|_{\mathsf{uni}(\mathbf{P})}} A$. Since $\mathbf{P} \lhd \mathbf{Q}$, $\mathbf{Q}|_{\mathsf{uni}(\mathbf{P})} = \mathbf{P}$. So $\blacktriangledown^{\mathbf{P}} A = \blacktriangledown^{\mathbf{Q}|_{\mathsf{uni}(\mathbf{P})}} A = \blacktriangledown^{\mathbf{Q}} A$. $\boxed{\Leftarrow}$ $a \leq^{\mathbf{P}} b$ iff $b = a \vee^{\mathbf{P}} b$ iff $b = a \vee^{\mathbf{Q}} b$ iff $a \leq^{\mathbf{Q}} b$. So $\mathbf{P} \lhd \mathbf{Q}$. Hence $\mathbf{Q}|_{\mathsf{uni}(\mathbf{P})} = \mathbf{P}$. Let $A \subseteq \mathsf{uni}(\mathbf{P})$. Since $\mathbf{P}$ and $\mathbf{Q}$ are $\blacktriangledown$-lattices, $\blacktriangledown^{\mathbf{Q}} A = \blacktriangledown^{\mathbf{Q}|_{\mathsf{uni}(\mathbf{P})}} A$ exists and $\blacktriangledown^{\mathbf{P}} A$ exists. (*We must show that* $\blacktriangledown^{\mathbf{Q}|_{\mathsf{uni}(\mathbf{P})}} A = \blacktriangledown^{\mathbf{P}} A$.) Then $\blacktriangledown^{\mathbf{Q}|_{\mathsf{uni}(\mathbf{P})}} A = \blacktriangledown^{\mathbf{Q}} A = \blacktriangledown^{\mathbf{P}} A$, by assumption. $\diamond$

Remark 1.150 Any $\vee$-consistent (etc.) subset of a $\vee$-closed (etc.) subset of $\mathbf{P}$, is itself a $\vee$-closed (etc.) subset of $\mathbf{P}$.

Remark 1.151 If $\mathbf{P} \lhd \mathbf{Q}$ and A is a subset of $\mathsf{uni}(\mathbf{P})$ such that $\blacktriangledown^{\mathbf{Q}} A$ exists and is contained in $\mathsf{uni}(\mathbf{P})$, then $\blacktriangledown^{\mathbf{Q}} A = \blacktriangledown^{\mathbf{P}} A$.

1.2.4 Compactness and Algebraic Lattice Orders

Definition 1.152 (Compactness) Let $\mathbf{P}$ be a complete-lattice. A **$\blacktriangledown$-base** of $\mathbf{P}$, is a subset B of $\mathsf{uni}(\mathbf{P})$, such that, for all $a \in \mathsf{uni}(\mathbf{P})$, there exists $B_a \subseteq B$, with $a = \blacktriangledown^{\mathbf{P}} B_a$. We define **$\blacktriangle$-bases** dually. Let $a \in \mathsf{uni}(\mathbf{P})$ and let $\mathbf{m}$ be a cardinal. We say that a has **$\blacktriangledown$-degree $\mathbf{m}$ in $\mathbf{P}$** (**$\blacktriangle$-degree $\mathbf{m}$ in $\mathbf{P}$**), if, for all $X \subseteq \mathsf{uni}(\mathbf{P})$, if $a \leq \blacktriangledown^{\mathbf{P}} X$ ($a \geq \blacktriangle^{\mathbf{P}} X$) then $a \leq \blacktriangledown^{\mathbf{P}} Y$ ($a \geq \blacktriangle^{\mathbf{P}} Y$), for some subset Y of X with $\mathsf{card}(Y) < \mathbf{m}$.

An element a of $\mathbf{P}$ is called **$\blacktriangledown$-compact** (**$\blacktriangle$-compact**) if it has $\blacktriangledown$-degree ω (resp. has $\blacktriangle$-degree ω). The set of all $\blacktriangledown$-compact ($\blacktriangle$-compact) elements of $\mathbf{P}$ is denoted $\mathsf{Cmp}_{\blacktriangledown}(\mathbf{P})$ ($\mathsf{Cmp}_{\blacktriangle}(\mathbf{P})$). We say that $\mathbf{P}$ is **$\blacktriangledown$-compactly generated** (**$\blacktriangle$-compactly generated**) if $\mathsf{Cmp}_{\blacktriangledown}(\mathbf{P})$ is a $\blacktriangledown$-base (resp. $\mathsf{Cmp}_{\blacktriangle}(\mathbf{P})$ is a $\blacktriangle$-base) for $\mathbf{P}$. We say that $\mathbf{P}$ is **$\blacktriangledown$-compact in** (**$\blacktriangle$-compact in**) complete-lattice $\mathbf{Q}$ if, $\mathbf{P} \lhd \mathbf{Q}$ and $\mathsf{Cmp}_{\blacktriangledown}(\mathbf{P}) = \mathsf{Cmp}_{\blacktriangledown}(\mathbf{Q}) \cap \mathsf{uni}(\mathbf{P})$ ($\mathsf{Cmp}_{\blacktriangle}(\mathbf{P}) = \mathsf{Cmp}_{\blacktriangle}(\mathbf{Q}) \cap \mathsf{uni}(\mathbf{P})$). We tend to drop the prefix $\blacktriangledown$, speaking, for example, of **compact** elements, etc. $\square$

Remark 1.153 It is not hard to show that B is a $\blacktriangledown$-base of a complete-lattice $\mathbf{P}$ iff, for all $a \in \mathsf{uni}(\mathbf{P})$, $a = \blacktriangledown^{\mathbf{P}} (\{b \in \mathsf{uni}(\mathbf{P}) \ : \ b \leq a\} \cap B)$, and dually for $\blacktriangle$-bases.

Remark 1.154 [BP89a] If $\mathbf{P}$ and $\mathbf{Q}$ are complete-lattices and $\mathbf{P} \lhd_{\blacktriangledown} \mathbf{Q}$, then $\mathsf{Cmp}_{\blacktriangledown}(\mathbf{Q}) \cap \mathsf{uni}(\mathbf{P}) \subseteq \mathsf{Cmp}_{\blacktriangledown}(\mathbf{P})$, although the converse inclusion is not generally valid.

Definition 1.155 (Algebraic Lattices) A $\blacktriangledown$-compactly generated ($\blacktriangle$-compactly generated) complete lattice is called **algebraic** (**co-algebraic**). $\square$

Example 1.156 (Familiar Algebraic Lattices)

> For any set A, $\mathfrak{P}(A)$ is an algebraic lattice, where the $\bigcup$-compact elements are precisely the finite subsets of A [BS81, 17]. The set of all equivalence relations on a set forms an inclusion ordered algebraic lattice. The subgroups of a group form an inclusion ordered algebraic lattice over the universe of the group.

$\square$

1.2.5 Ideals and Filters

Definition 1.157 (Intervals) For $\emptyset \neq A \subseteq \mathsf{uni}(\mathbf{P})$ and $\emptyset \neq B \subseteq \mathsf{uni}(\mathbf{P})$, we define

$$\begin{aligned}
[A, B]_{\mathbf{P}} &= \{c : \forall [a \in A, b \in B]\ a \leq c \leq b\}, \\
(A, B]_{\mathbf{P}} &= \{c : \forall [a \in A, b \in B]\ a < c \leq b\}, \\
[A, B)_{\mathbf{P}} &= \{c : \forall [a \in A, b \in B]\ a \leq c < b\}, \\
(A, B)_{\mathbf{P}} &= \{c : \forall [a \in A, b \in B]\ a < c < b\}, \\
[A\rangle_{\mathbf{P}} &= \{b : \forall [a \in A]\ a \leq b\}, \\
(A\rangle_{\mathbf{P}} &= \{b : \forall [a \in A]\ a < b\}, \\
\langle A]_{\mathbf{P}} &= \{b : \forall [a \in A]\ a \geq b\} \qquad\qquad \text{and} \\
\langle A)_{\mathbf{P}} &= \{b : \forall [a \in A]\ a > b\}.
\end{aligned}$$

For $a, b \in \mathsf{uni}(\mathbf{P})$, we write $[a, b]_{\mathbf{P}}$ for $[\{a\}, \{b\}]_{\mathbf{P}}$ and $[a\rangle_{\mathbf{P}}$ for $[\{a\}\rangle_{\mathbf{P}}$, etc. We may drop the subscript $\mathbf{P}$ from these notations when context unambiguous. □

Definition 1.158 (Downsets, Upsets and Convexities) Let $\mathbf{P}$ be an order. A (possibly-empty) subset A of $\mathsf{uni}(\mathbf{P})$ is called a **downset** if $\forall [a, b]\ a \in A$ and $b \leq a$ implies $b \in A$, an **upset** if $\forall [a, b]\ a \in A$ and $b \geq a$ implies $b \in A$ and a **convexity** (or **convex**) $\forall [a, b, c]\ a, c \in A$ and $a \leq b \leq c$ implies $b \in A$. The set of all downsets (resp. upsets, convexities) of $\mathbf{P}$ is denoted by $\mathsf{Dn}(\mathbf{P})$ (resp. $\mathsf{Up}(\mathbf{P})$, $\mathsf{Cx}(\mathbf{P})$). □

Remark 1.159 Each of the empty set and the universe of $\mathbf{P}$ are downsets, upsets and convexities.

Remark 1.160 $[A, B]_{\mathbf{P}}$, $(A, B]_{\mathbf{P}}$, $[A, B)_{\mathbf{P}}$ and $(A, B)_{\mathbf{P}}$ are convexities, $[A\rangle_{\mathbf{P}}$ and $(A\rangle_{\mathbf{P}}$ are upsets, and $\langle A]_{\mathbf{P}}$ and $\langle A)_{\mathbf{P}}$ are downsets.

Remark 1.161 Upsets and downsets are convex.

Remark 1.162 The intersection of an upset with a downset is a convex set, since both are convex, and every convex set arises in this manner.

Remark 1.163 [DP90, 14] X is a downset of $\mathbf{P}$ iff $\overset{\mathsf{uni}(\mathbf{P})}{\neg} X$ is an upset.

Remark 1.164 [BS81] If $\mathbf{P}$ is a $\blacktriangledown$-order, then

$$\overset{\mathbf{P}}{\blacktriangle} X = \overset{\mathbf{P}}{\blacktriangledown}\{a \in \mathsf{uni}(\mathbf{P}) : X \subseteq [a\rangle_{\mathbf{P}}\}, \tag{1.59}$$

and if $\mathbf{P}$ is a $\blacktriangle$-order, then

$$\overset{\mathbf{P}}{\blacktriangledown} X = \overset{\mathbf{P}}{\blacktriangle}\{a \in \mathsf{uni}(\mathbf{P}) : X \subseteq \langle a]_{\mathbf{P}}\}. \tag{1.60}$$

Remark 1.165 Let $\mathbf{P}$ be an order and $X, Y \subseteq \mathsf{uni}(\mathbf{P})$. Then $\langle X]_{\mathbf{P}}$ (resp. $[X\rangle_{\mathbf{P}}$, $[X, Y]_{\mathbf{P}}$) is a $\blacktriangledown$-consistent (resp. $\blacktriangle$-consistent, $\blacklozenge$-consistent) subset of $\mathbf{P}$, but generally, downsets (resp. upsets, convexities) are not even $\vee$-consistent (resp. $\wedge$-consistent, $\vee$-consistent nor $\wedge$-consistent).

Remark 1.166 [DP90, 27] The transitivity of $\leq$ ensures that the set $\mathsf{upper}^{\mathbf{P}}(X)$ ($\mathsf{lower}^{\mathbf{P}}(X)$) of all upper (resp. lower) bounds is an upset (resp. downset).

Definition 1.167 (Principle Downsets, Upsets and Convexities) We call $[a, b]_{\mathbf{P}}$, $[a\rangle_{\mathbf{P}}$ and $\langle a]_{\mathbf{P}}$, the **principle convexity from** a **to** b, the **principle upset determined by** a and the **principle downset determined by** a, respectively. □

Remark 1.168 Principal upsets (resp. downsets) are upwardly directed (resp. downwardly directed).

Definition 1.169 (Lattice-Order Ideals and Filters) Let $\mathbf{P}$ be a *lattice-order*. A *non-empty* subset A of $\mathsf{uni}(\mathbf{P})$ is called an **ideal** if it is a non-empty downset and $a, b \in A$ implies $a \vee b \in A$, and is called a **filter** if it is a non-empty upset and $a, b \in A$ implies $a \wedge b \in A$. Principal downsets (resp. upsets) of lattices are called **principal ideals** (resp. **principal filters**). □

Warning 1.170 In keeping with [DP90], we permit empty downsets and upsets but proscribe empty ideals and filters.

Remark 1.171 If $\mathbf{P}$ is a lattice, then the filters and ideals of $\mathbf{P}$ are all sublattices of $\mathbf{P}$. While convex-sets of lattices are not generally sublattices, convex-sets of the form $[X, Y]_{\mathbf{P}}$ do form sublattices of lattice $\mathbf{P}$.

Example 1.172 (Filters and Ultrafilters of Collections)

> $\mathcal{F} \subseteq \mathfrak{P}(I)$ is called a **filter over** I if $\mathcal{F}$ is a lattice filter of the power-order $\boldsymbol{\mathfrak{P}}(I)$ over I. We denote the set of all filters over a set I by $\mathsf{Filter}(I)$. A filter $\mathcal{F}$ over I is called a **proper filter over** I if $\mathcal{F} \neq \mathfrak{P}(I)$, and is called an **ultrafilter over** I if $\mathcal{F}$ is a proper filter over I that is maximal (with respect to inclusion) among all proper filters of over I; i.e., whenever $\mathcal{F}'$ is a proper filter over I and $\mathcal{F} \subseteq \mathcal{F}'$, then $\mathcal{F} = \mathcal{F}'$.
>
> The filters of the complete lattice $\boldsymbol{\mathfrak{P}}(I)$ play an important role in the model theory of elementary structures (see $\oint$1.5.9 for details).

□

1.2.6 Order Preserving and Reflecting Functions

Definition 1.173 (Order Preserving Functions, Order Isomorphisms) Let $\mathbf{P}$ and $\mathbf{Q}$ be orders and let $f : \mathsf{uni}(\mathbf{P}) \rightarrow \mathsf{uni}(\mathbf{Q})$. We call f **order-preserving** if $a \leq^{\mathbf{P}} b \rightarrow f(a) \leq^{\mathbf{Q}} f(b)$. We call f an **order-isomorphism** (or just an **isomorphism**) if it is bijective, order-preserving *and* f^{-1} is also order preserving, in which case we call $\mathbf{P}$ and $\mathbf{Q}$ **order-isomorphic** (or just **isomorphic**). The set of all order-preserving functions (order-isomorphisms) from $\mathbf{P}$ into $\mathbf{Q}$ is denoted by $\mathbf{P} \rightarrow_{\leq} \mathbf{Q}$ (resp. $\mathbf{P} \cong_{\leq} \mathbf{Q}$). An order is called **self-dual** if it is isomorphic to its dual. We say that $\mathbf{P}$ is **dual-isomorphic** to $\mathbf{Q}$ if $\mathbf{P}$ is isomorphic to the dual of $\mathbf{Q}$.

We call f **order-reflecting** if $f(a) \leq^{\mathbf{Q}} f(b) \rightarrow a \leq^{\mathbf{P}} b$. A **strictly order-preserving** function (or **order-embedding**) is order-preserving and order-reflecting. We denote the set of all strictly-order-preserving functions from $\mathbf{P}$ into $\mathbf{Q}$ by $\mathbf{P} \rightarrow_{\leq^*} \mathbf{Q}$. □

Remark 1.174 Note that *any* order reflecting function is always injective, and that the order-isomorphisms are precisely the surjective strictly order preserving functions.

Remark 1.175 [DP90, 23] Note further that orders $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic iff there exist order-preserving functions $f : \mathsf{uni}(\mathbf{P}) \to \mathsf{uni}(\mathbf{Q})$ and $g : \mathsf{uni}(\mathbf{Q}) \to \mathsf{uni}(\mathbf{P})$ with $gf = \mathsf{id}_{\mathsf{uni}(\mathbf{P})}$ and $fg = \mathsf{id}_{\mathsf{uni}(\mathbf{Q})}$.

Remark 1.176 If $f : \mathbf{P} \cong \mathbf{Q}$, then $\mathbf{P}$ has 0 (has 1) iff $\mathbf{Q}$ has 0 (resp. has 1), in which case $f(0^{\mathbf{P}}) = 0^{\mathbf{Q}}$ (resp. $f(1^{\mathbf{P}}) = 1^{\mathbf{Q}}$).

Definition 1.177 (Images and Preimages) Let f be *any* function from $\mathbf{P}$ into $\mathbf{Q}$. We denote the suborder $\mathbf{Q}|_{f[\mathsf{uni}(\mathbf{P})]}$ (of $\mathbf{P}$) by $f\,[\mathbf{P}]$, and denote the suborder $\mathbf{P}|_{f^{-1}[\mathsf{uni}(\mathbf{Q})]}$ (of $\mathbf{P}$) by $f^{-1}\,[\mathbf{Q}]$. $\Box$

Remark 1.178 Note that order preserving functions preserve upper and lower bounds; that is, if $f : \mathbf{P} \to \mathbf{Q}$ and a is an upper (lower) bound of $X \subseteq \mathsf{uni}(\mathbf{P})$, then $f(a)$ is an upper (resp. lower) bound of $f[X]$ in $\mathbf{Q}$. Conversely, any upper bound preserving or lower bound preserving function must be order preserving. Order preserving functions need *not*, however, preserve *least* upper bounds nor *greatest* lower bounds, and not even when considered as function onto the image suborder.

Remark 1.179 Upwardly directed and downwardly directed sets are preserved under the image of order preserving functions.

Definition 1.180 (Bound Preserving Functions) Let $\mathbf{P}$ and $\mathbf{Q}$ be orders and $h : \mathsf{uni}(\mathbf{P}) \to \mathsf{uni}(\mathbf{Q})$. We call h a **0-preserving function** (resp. **1-preserving function**), if it is order-preserving and, whenever $\mathbf{P}$ has 0 (resp. has 1), so has $\mathbf{Q}$, and $h(0^{\mathbf{P}}) = 0^{\mathbf{Q}}$ (resp. $h(1^{\mathbf{P}}) = 1^{\mathbf{Q}}$). Let $\Box$ be a symbol from amongst $\{\vee, \dot{\blacktriangledown}, \sqcup, \blacktriangledown\}$. We call h a **$\Box$-preserving function**, if, for all $Y \subseteq \mathsf{uni}(\mathbf{P})$ with $\phi_\Box(Y)$, whenever $\blacktriangledown^{\mathbf{P}} Y$ exists, then $\blacktriangledown^{\mathbf{Q}} h[Y]$ exists, and $h(\blacktriangledown^{\mathbf{P}} Y) = \blacktriangledown^{\mathbf{Q}} h[Y]$. For each symbol $\Box$ from amongst $\{\wedge, \dot{\blacktriangle}, \sqcap, \blacktriangle\}$, the notion that h be a **$\Box$-preserving function** is defined dually. For any non-empty finite sequence of symbols $\Box_1 \ldots \Box_n$ over $\{0, \vee, \dot{\blacktriangledown}, \sqcup, \blacktriangledown, 1, \wedge, \dot{\blacktriangle}, \sqcap, \blacktriangle\}$, we call h a **$\Box_1 \ldots \Box_n$-preserving function**, if, for $1 \leq i \leq n$, h is a $\Box_i$-preserving function. The set of all $\vec{\Box}$-preserving functions (surjections, injections, bijections) from $\mathbf{P}$ into $\mathbf{Q}$ is denoted by $\mathbf{P} \to_{\vec{\Box}} \mathbf{Q}$ (resp. $\mathbf{P} \twoheadrightarrow_{\vec{\Box}} \mathbf{Q}$, $\mathbf{P} \rightarrowtail_{\vec{\Box}} \mathbf{Q}$, $\mathbf{P} \cong_{\vec{\Box}} \mathbf{Q}$). $\Box$

Warning 1.181 (Continuity) Many texts, and in particular, texts on algebraic logic, call $\sqcup$-preserving functions (between algebraic lattices) *continuous*. We avoid this usage, since we shall be considering *continuous functions* (and relationships) in a sense analogous to that of continuous functions between topological closed systems (see Part II).

Remark 1.182 For any non-empty finite sequence of symbols $\vec{\Box}$ over $\{\vee, \wedge, \lozenge, \blacktriangledown, \blacktriangle, \blacklozenge, \dot{\blacktriangledown}, \dot{\blacktriangle}, \sqcup, \sqcap\}$, a $\vec{\Box}$-preserving function between orders is order-preserving.

Remark 1.183 If f is a $\vec{\Box}$-preserving function from order $\mathbf{P}$ into $\mathbf{Q}$, then f is a $\vec{\Box}$-preserving function from $\mathbf{P}$ onto the image order.

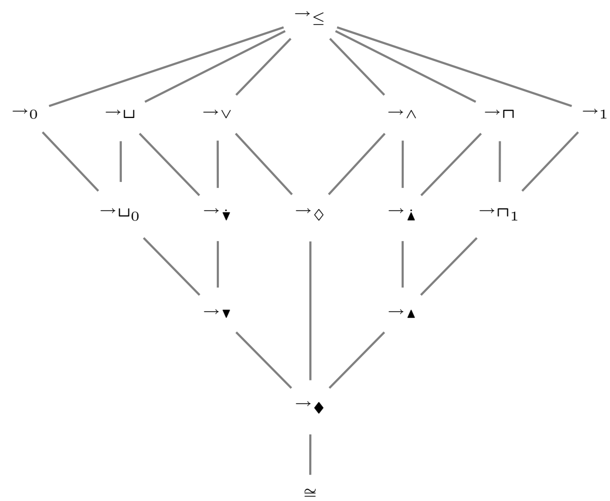


Figure 1.2: Order Preserving Functions (see Remark 1.184)

Remark 1.184 Figure 1.2 depicts (the Hasse diagram of) the inclusion ordering of the sets $\mathbf{P} \to_\Box \mathbf{Q}$, for arbitrary orders $\mathbf{P}$ and $\mathbf{Q}$. 'Stricter functions' lie below 'weaker functions', linked by a line. Note that $\blacktriangledown$-preserving functions (resp. $\blacktriangle$-preserving functions) between orders are 0-preserving (resp. 1-preserving), but that this is *not* generally true for $\dot{\blacktriangledown}$-preserving functions (resp. $\dot{\blacktriangle}$-preserving functions); not even between $\blacklozenge$-orders. Consequently, $\blacklozenge$-preserving functions between orders are 01-preserving. While $\blacktriangledown$-preserving functions (resp. $\blacktriangle$-preserving functions) between orders are *not* generally 1-preserving (resp. 0-preserving), not even when both orders are $\blacklozenge$-orders and the function is injective, even $\sqcup$-preserving functions and $\vee$-preserving functions (resp. $\sqcap$-preserving functions and $\wedge$-preserving functions) are 1-preserving (resp. 0-preserving) *onto* their *image* orders, and consequently, *surjective* $\sqcup$-preserving functions and $\vee$-preserving functions (resp. $\sqcap$-preserving functions and $\wedge$-preserving functions) are 1-preserving (resp. 0-preserving). Order isomorphisms are $\blacklozenge$-preserving.

Remark 1.185 An order isomorphism is a $\blacklozenge$-preserving function, and so the notion of a $\Box$-isomorphism is unnecessary, except for the (pathological) cases that $\Box \in \{0, 1, 01\}$. Consequently, Figure 1.2 collapses to a singleton when all $\to$ are replaced by $\Rightarrow$.

Remark 1.186 [BS81] Two $\Diamond$-orders $\mathbf{P}$ and $\mathbf{Q}$ are $\Diamond$-isomorphic iff there exists a bijection $f : \mathsf{uni}(\mathbf{P}) \to \mathsf{uni}(\mathbf{Q})$ that is both order preserving and order reflecting.

Remark 1.187 Let $\mathbf{P}$ and $\mathbf{Q}$ be orders and $f : \mathsf{uni}(\mathbf{P}) \to \mathsf{uni}(\mathbf{Q})$. The following conditions are equivalent.

1. $f : \mathbf{P} \to \mathbf{Q}$.
2. $f^{-1}[\cdot]$ defines a function from $\mathsf{Dn}(\mathbf{Q})$ into $\mathsf{Dn}(\mathbf{P})$.
3. $f^{-1}[\cdot]$ defines a function from $\mathsf{Up}(\mathbf{Q})$ into $\mathsf{Up}(\mathbf{P})$.

The following conditions are equivalent.

1. f is order-preserving and injective (surjective).
2. $f^{-1}[\cdot]$ defines a surjective (injective) function from $\mathsf{Dn}(\mathbf{Q})$ onto (into) $\mathsf{Dn}(\mathbf{P})$.
3. $f^{-1}[\cdot]$ defines a surjective (injective) function from $\mathsf{Up}(\mathbf{Q})$ onto (into) $\mathsf{Up}(\mathbf{P})$.

(See [DP90, Ex 1.24] for the downset conditions.) Notice the 'continuity like' flavour of these results. We shall be making much more of such conditions in the sequel.

Definition 1.188 (Lattice Homomorphisms) For $\square \in \{\vee, \blacktriangledown, \wedge, \blacktriangle\}$, a $\square$-preserving function between $\square$-semilattices is called a **$\square$-homomorphism** or a **$\square$-semilattice homomorphism**. For $\square \in \{\lozenge, \blacklozenge\}$, a $\square$-preserving function between $\square$-lattices is called a **$\square$-homomorphism** or a **$\square$-lattice homomorphism**. For $\square \in \{\blacktriangledown, \blacktriangle\}$, a **$\square$-semilattice homomorphism** between complete-lattices is also called a **$\square$-complete semilattice homomorphism**. $\square$

Remark 1.189 [DP90, P2.19] Injective semilattice and lattice homomorphisms are order-reflecting, and hence order-embeddings, and consequently are order-isomorphisms whenever bijective. $\square$

We following remarks play an important role in the theory of algebraic logics.

Remark 1.190 [BP89a] If $\mathbf{P}$ and $\mathbf{Q}$ are complete-lattices and $f : \mathbf{P} \rightarrowtail_{\blacktriangledown} \mathbf{Q}$ then $f[\mathbf{P}] \lhd_{\blacktriangledown} \mathbf{Q}$, and consequently $\mathsf{Cmp}_{\blacktriangledown}(\mathbf{Q}) \cap f[\mathsf{uni}(\mathbf{P})] \subseteq \mathsf{Cmp}_{\blacktriangledown}(f[\mathbf{P}])$ (see Remark 1.154).

Remark 1.191 [BP89a] Isomorphisms between complete-lattices map compact elements to compact elements.

1.3 Systems

Systems are generalizations of closed systems and topological closed systems. The presentation in this section is very brief, since closed systems are considered in far more depth in Part II.

Definition 1.192 (Systems) A **power system** (or just **system**) $\mathfrak{X}$ is determined by a set $\mathsf{uni}(\mathfrak{X})$ called the **universe**, elements of which are called **points**, and its **modules** $\mathsf{Module}(\mathfrak{X})$, where $\mathsf{Module}(\mathfrak{X}) \subseteq \mathfrak{P}(\mathsf{uni}(\mathfrak{X}))$. We speak of a **power system over** A, or an **A-system**, or even just a **system over** A, by which we mean a power system with universe A. Let $\mathsf{Sys}(A)$ denote the set of A-systems. When we say that $\langle A, \mathcal{A}\rangle$ is a system, we mean the system with universe A and modules $\mathcal{A}$. We call system $\mathfrak{Y}$ a **subsystem** of system $\mathfrak{X}$ if $\mathsf{uni}(\mathfrak{Y}) = \mathsf{uni}(\mathfrak{X})$ and $\mathsf{Module}(\mathfrak{Y}) \subseteq \mathsf{Module}(\mathfrak{X})$. For a power system $\mathfrak{X}$, the inclusion ordered set $\langle \mathsf{Module}(\mathfrak{X}), \subseteq\rangle$ is denoted by $\mathbf{Module}(\mathcal{A})$. It is convenient to conflate a system $\mathfrak{X}$ with its modules $\mathsf{Module}(\mathfrak{X})$, i.e., by this convention, syntactically $\mathsf{Module}(\mathfrak{X}) = \mathfrak{X}$. We may simply call $\mathcal{A}$ an A-system. For some *special systems* such as closed systems, we shall specifically *revoke* this convention, since it leads to confusion.

For any system $\mathfrak{X}$ over A, let $\mathfrak{X}' = \{B \in \mathfrak{P}(A) : \overset{A}{\neg} B \in \mathfrak{X}\}$ and $\neg\mathfrak{X} = \mathfrak{P}(A) - \mathfrak{X}$. We call $\cdot'$ the **inner-complementation operator** and speak of the **inner complement of a system**, and we call $\neg\cdot$ the **outer-complementation operator** and speak of the **outer complement of a system**.

Let $\mathfrak{X}$ be an A-system. We call $\mathfrak{X}$ a **$\top$-system** (or **topped**) if $A \in \mathfrak{X}$, and call $\mathfrak{X}$ a **$\bot$-system** (or **bottomed**) if $\emptyset \in \mathfrak{X}$. We call $\mathfrak{X}$ a **$\cup$-system**, a **$\cap$-system**, or a **$\neg$-system**, if $\mathfrak{X}$ is $\cup$-closed, $\cap$-closed, or, $\overset{A}{\neg}$-closed, respectively. We call $\mathfrak{X}$ a **$\bigcup$-system** (or **$\bigcup$-complete**), if $\mathfrak{X}$ is closed under arbitrary non-empty unions, and call $\mathfrak{X}$ a **$\bigcap$-system** (or **$\bigcap$-complete**), if $\mathfrak{X}$ is closed under arbitrary non-empty intersections,

System $\mathfrak{X}$ is called **upwardly-directed** or **$\subseteq$-directed** (resp. **downwardly-directed** or **$\supseteq$-directed**) if, for all $A, B \in \mathfrak{X}$, there exists $C \in \mathfrak{X}$ such that $A \cup B \subseteq C$ (resp. $C \subseteq A \cap B$). We speak of **upwardly-directed subsets** and **downwardly-directed subsets** of $\mathfrak{P}(X)$, by which we mean that the determined systems are upwardly-directed and downwardly-directed respectively. We call $\mathfrak{X}$ a $\sqcup$-system ($\sqcap$-system) if, for every non-empty $\subseteq$-directed (resp. $\supseteq$-directed) subsystem $\mathfrak{Y}$, $\bigcup \mathfrak{Y} \in \mathfrak{X}$ (resp. $\bigcap \mathfrak{Y} \in \mathfrak{X}$).

By a **$\star_1, \ldots, \star_n$-system**, for integer n and symbols $\star_i$ amongst the symbols $\top$, $\bot$, $\cup$, $\cap$, $\bigcup$, $\bigcap$, $\sqcup$, $\sqcap$ and $\neg$, we mean a system that is a $\star_i$-system, for each $0 < i \leq n$. $\square$

Warning 1.193 [DP90, D2.33] While the use of the terms 'topped' and 'bottomed' are potentially misleading, in that they specify a *particular* top (the universe) and a *particular* bottom (the empty subset) rather than just *some* top or bottom, this usage is standard in the literature.

Remark 1.194 Many texts require systems like ours to contain at least one module [DP90, D2.33]. The systems that we shall be concerned with in this text, are either $\top$-systems or $\bot$-systems (or both), and hence are non-empty by definition.

Remark 1.195 In the definition of directed systems, we may just as well have replaced the binary unions and intersections with arbitrary *finite* unions and intersections respectively.

Definition 1.196 (Closed Systems) [BS81] A **closed system** $\mathbb{C}$ is a $\top\bigcap$-system over a *non-empty* universe; we write $\mathsf{cl}_{\mathbb{C}}$ for $\mathsf{Module}(\mathbb{C})$. With each closed system $\mathbb{C}$ we associate the operator $\|\cdot\|_{\mathbb{C}}$ on $\mathsf{uni}(\mathbb{C})$, which we call the associated closure operator, defined by $\|A\|_{\mathbb{C}} = \bigcap\{G \in \mathsf{cl}_{\mathbb{C}} : A \subseteq G\}$. $\square$

Warning 1.197 We have proscribed closed systems over the empty-set, so as to ease the burden of dealing with this special case in proofs, given that all the examples that we shall encounter have non-empty universes.

Definition 1.198 (Algebraic Closed Systems) [BS81] A (concrete) closed system $\mathbb{C}$ is called **algebraic** (or **finitary**) if, for all non-empty $\subseteq$-directed set $\mathcal{A}$ of closed sets, $\bigcup \mathcal{A}$ is closed. We speak of algebraic *closure operators*, etc. $\square$

The reader unfamiliar with algebraic logic and algebraic closed systems should be aware that these two usages of the word 'algebraic' are very different; in the latter case, 'algebraic' reflects the fact that algebraic closed systems are typical of the closed systems encountered in universal algebra (and *elementary* logic more generally), in contrast with the closed systems encountered in topology. We prefer the alternative term 'finitary', which we tend to use later in the text so as to avoid potential confusion with other usages of the term 'algebraic'.

1.4 Constructs and Categories

In order for us to explain how 'structural' closed systems over the free algebra of a quasivariety induce sentential calculi, explicated in our theory of canons and archologies (see ∮8), we have needed to develop a theory of logics over constructs, where a construct is a concrete category, and in order to explain our theory of parameterized algebraization from a non-parameterized theory of equivalence, we have needed to develop a theory of equivalence between logics lying in different constructs; this requires the notion of a categorical isomorphism. Further, in Part VI of this text we work with *π-institutions* [FS88], which are categorical abstraction of multi-signature logics. In this section we introduce the definitions and results that we shall require from construct theory and category theory. In ∮1.4.1 constructs are considered; categories are considered in ∮1.4.2.

1.4.1 Constructs

1.4.1.1 Objects and Constructs

Definition 1.199 (Objects) [Ada83, D2] An **object $\mathbf{A}$** is determined by its **universe** $\mathsf{uni}(\mathbf{A})$ and its **character** $\mathsf{char}(\mathbf{A})$, both of which are sets. The elements of $\mathsf{uni}(\mathbf{A})$ are called **$\mathbf{A}$-tokens** (or just **tokens**) and the elements of $\mathsf{char}(\mathbf{A})$ are called **$\mathbf{A}$-characteristics** (or just **characteristics**). For each set A, let $\mathsf{Obj}(A)$ denote the class of all objects $\mathbf{A}$ with $\mathsf{uni}(\mathbf{A}) = A$. □

Warning 1.200 In the literature, the characteristics of an object is usually called the structure of that object. We avoid this usage, however, due to the primary role of (elementary) structures in the sequel. □

Definition 1.201 (Constructs and Morphisms) [Ada83, D 2] A **construct $\mathfrak{s}$** is determined by its **objects** $\mathsf{Obj}_{\mathfrak{s}}$ and its **morphisms** $\mathsf{Mph}_{\mathfrak{s}}$, where $\mathsf{Obj}_{\mathfrak{s}} = \langle \mathsf{Obj}_{\mathfrak{s}}(A) \ : \ A \in \mathsf{Set}\rangle$ is a Set-indexed family, $\mathsf{Obj}_{\mathfrak{s}}(A)$ is a *class* of objects with universe A, and $\mathsf{Mph}_{\mathfrak{s}}$ is a $\mathsf{Obj}_{\mathfrak{s}}^2$-indexed family $\{\mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B} \ : \ \mathbf{A}, \mathbf{B} \in \mathsf{Obj}_{\mathfrak{s}}\}$, the elements of which are called **$\mathfrak{s}$-morphisms (of $\mathbf{A}$ into $\mathbf{B}$)**, such that, for all $\mathfrak{s}$-objects $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B} \subseteq \mathsf{uni}(\mathbf{A}) \rightarrow \mathsf{uni}(\mathbf{B})$, $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{A}$, and, if $f : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$ and $g : \mathbf{B} \rightarrow_{\mathfrak{s}} \mathbf{C}$, then $gf : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{C}$. The members of $\mathsf{Obj}_{\mathfrak{s}}(A)$ are called $\mathfrak{s}$-objects on A. This class may be empty, in which case we say that **no $\mathfrak{s}$-objects are definable on** A. We shall often omit the construct $\mathfrak{s}$ from these and subsequent notations when ambiguity does not arise. □

Example 1.202 (The Construct of Sets)

> Let Set denote the *construct of sets* whose objects are all sets and whose morphisms are functions.

□

Definition 1.203 (Object-Small and Object-Discrete Constructs) A construct is called **object-small** (**object-discrete**) if, for each set A, $\mathsf{Obj}_{\mathfrak{s}}(A) \in \mathsf{Set}$ (resp. $\mathsf{card}(\mathsf{Obj}_{\mathfrak{s}}(A)) \leq 1$). □

Definition 1.204 (Special Morphisms) [Ada83, 1C] Injective, surjective and bijective $\mathfrak{s}$-morphisms are called **$\mathfrak{s}$-monomorphisms** (or **$\mathfrak{s}$-injections**), **$\mathfrak{s}$-epimorphisms** (or **$\mathfrak{s}$-surjections**) and **$\mathfrak{s}$-bijections**, and the set of all such from $\mathbf{A}$ to $\mathbf{B}$ are denoted by $\mathbf{A} \rightarrowtail_{\mathfrak{s}} \mathbf{B}$, $\mathbf{A} \twoheadrightarrow_{\mathfrak{s}} \mathbf{B}$ and $\mathbf{A} \Rightarrow_{\mathfrak{s}} \mathbf{B}$, respectively.

We call $f : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$ a **$\mathfrak{s}$-isomorphism**, if f is bijective and $f^{-1} : \mathbf{B} \to_{\mathfrak{s}} \mathbf{A}$. The set of all isomorphisms from $\mathbf{A}$ onto $\mathbf{B}$ is denoted by $\mathbf{A} \cong_{\mathfrak{s}} \mathbf{B}$. We call $\mathbf{A}$ and $\mathbf{B}$ **$\mathfrak{s}$-isomorphic** if $\mathbf{A} \cong_{\mathfrak{s}} \mathbf{B}$ is non-empty.

We write $\mathsf{End}_{\mathfrak{s}}(\mathbf{A})$ for $\mathbf{A} \to_{\mathfrak{s}} \mathbf{A}$, the elements of which are called **$\mathfrak{s}$-endomorphisms of $\mathbf{A}$**. An **$\mathfrak{s}$-automorphism** of $\mathbf{A}$ is an $\mathfrak{s}$-endomorphism that is an $\mathfrak{s}$-isomorphism, and an **$\mathfrak{s}$-involution** of $\mathbf{A}$ is an $\mathfrak{s}$-automorphism f with $f^{-1} = f$. Let $\mathsf{Aut}_{\mathfrak{s}}(\mathbf{A})$ and $\mathsf{involutions}_{\mathfrak{s}}(\mathbf{A})$ denote the set of all $\mathfrak{s}$-automorphisms and $\mathfrak{s}$-involutions of $\mathbf{A}$, respectively. □

Definition 1.205 (Transportable Constructs) [Ada83, 18] We say that $\mathfrak{s}$ is a **transportable** construct, if, for each object $\mathbf{A}$, set B and bijection $f : \mathsf{uni}(\mathbf{A}) \Rightarrow B$, there exists precisely one object $\mathbf{B} \in \mathsf{Obj}_{\mathfrak{s}}(B)$ such that $f : \mathbf{A} \cong_{\mathfrak{s}} \mathbf{B}$, in which case we denote this unique object by $f_{\mathfrak{s}}[\mathbf{A}]$. □

Definition 1.206 (Granularity) [Ada83, 1D] We say that $\mathbf{A}$ is **$\mathfrak{s}$-finer** than $\mathbf{B}$, written $\mathbf{A} \preceq_{\mathfrak{s}} \mathbf{B}$, if $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbf{B})$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} \in \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$. We say that $\mathbf{A}$ is **$\mathfrak{s}$-coarser** than $\mathbf{B}$, written $\mathbf{A} \succeq_{\mathfrak{s}} \mathbf{B}$, if $\mathbf{B} \preceq_{\mathfrak{s}} \mathbf{A}$. □

Remark 1.207 $\preceq_{\mathfrak{s}}$ *quasi*-orders the *set* $\mathsf{Obj}_{\mathfrak{s}}$ (of all $\mathfrak{s}$-objects) and the *class* $\mathsf{Obj}_{\mathfrak{s}}(A)$ (of all $\mathfrak{s}$-objects with universe A).

Proof. $\boxed{\text{Reflexive}}$ Let $\mathbf{A}$ be an $\mathfrak{s}$-object. Then *certainly* $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbf{A})$, and by definition of a construct, $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathbf{A} \to_{\mathfrak{s}} \mathbf{A}$. $\boxed{\text{Transitive}}$ Suppose that $\mathbf{A} \preceq_{\mathfrak{s}} \mathbf{B} \preceq_{\mathfrak{s}} \mathbf{C}$. Then, $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbf{B})$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$, and, $\mathsf{uni}(\mathbf{B}) = \mathsf{uni}(\mathbf{C})$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{B})} : \mathbf{B} \to_{\mathfrak{s}} \mathbf{C}$. So $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbf{C})$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} = \mathsf{id}_{\mathsf{uni}(\mathbf{B})}\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathbf{A} \to_{\mathfrak{s}} \mathbf{C}$, the equality following by universe coincidence and the membership by compositional closure of morphisms in constructs. ◇

Convention 1.208 (Quasi-order Terminology) [Ada83, 22-23] We inherit the terminology of quasi-orders in a '$\preceq_{\mathfrak{s}}$'-prefixed form. For example, we shall call a construct **$\preceq_{\mathfrak{s}}$-ordered**, by which we mean that the quasi-order $\preceq_{\mathfrak{s}}$ is an order.

Remark 1.209 $\mathfrak{s}$ is $\preceq_{\mathfrak{s}}$-ordered iff $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{B})} : \mathbf{B} \to_{\mathfrak{s}} \mathbf{A}$ implies $\mathbf{A} = \mathbf{B}$.

Proof. $\boxed{\Rightarrow}$ Suppose that $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{B})} : \mathbf{B} \to_{\mathfrak{s}} \mathbf{A}$. Since $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathsf{uni}(\mathbf{A}) \to \mathsf{uni}(\mathbf{B})$, $\mathsf{uni}(\mathbf{A}) = \mathsf{do}(\mathsf{id}_{\mathsf{uni}(\mathbf{A})}) = \mathsf{rg}(\mathsf{id}_{\mathsf{uni}(\mathbf{A})}) \subseteq \mathsf{uni}(\mathbf{B})$, and symmetrically, $\mathsf{uni}(\mathbf{B}) \subseteq \mathsf{uni}(\mathbf{A})$, and so $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbf{B})$. Since $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbf{B})$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$, $\mathbf{A} \preceq_{\mathfrak{s}} \mathbf{B}$, and symmetrically, $\mathbf{B} \preceq_{\mathfrak{s}} \mathbf{A}$. By assumed anti-symmetry, $\mathbf{A} = \mathbf{B}$. $\boxed{\Leftarrow}$ Suppose that $\mathbf{A} \preceq_{\mathfrak{s}} \mathbf{B}$ and $\mathbf{B} \preceq_{\mathfrak{s}} \mathbf{A}$. Then, $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbf{B})$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{A})} : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$, $\mathsf{uni}(\mathbf{B}) = \mathsf{uni}(\mathbf{A})$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{B})} : \mathbf{B} \to_{\mathfrak{s}} \mathbf{A}$. So by assumption, $\mathbf{A} = \mathbf{B}$. ◇

Remark 1.210 Consequently, every transportable construct is $\preceq$ -ordered.

Remark 1.211 $\mathfrak{s}$ is $\preceq$ -discrete iff it is object-small.

Definition 1.212 (Fibre Terminology) [Ada83] Construct $\mathfrak{s}$ is called **fibre-small** if it is *transportable* and *object-small*. When we employ 'fibre' prefixed order terminology, we mean the analogous '$\preceq$ ' prefixed terminology under the assumption that the construct under discussion is fibre-small. For example, a construct is **fibre-discrete** (**fibre-complete**) iff fibre-small and $\preceq$ -discrete (resp. $\preceq$ -complete). □

Remark 1.213 Fibre-small constructs are $\preceq$ -ordered (by Remark 1.210 on page 45).

Remark 1.214 $\mathfrak{s}$ is fibre-discrete iff it is transportable and $\preceq$ -discrete (by Remark 1.211 on page 45).

Remark 1.215 [Ada83, 23] $\mathfrak{s}$ is fibre-discrete iff, for all $\mathbf{A}, \mathbf{B} \in \mathsf{Obj}_{\mathfrak{s}}(A)$, $\mathsf{id}_A : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$ implies $\mathbf{A} = \mathbf{B}$.

Example 1.216 (Unary Matrices)

The set of all unary matrices over sets together with unary-matrix-homomorphisms forms a construct, which we denote by $\mathsf{C(umx)}$.

Remark 1.217 $\mathsf{C(umx)}$-isomorphisms and unary-matrix-isomorphisms coincide.

Remark 1.218 $\mathsf{M} \preceq_{\mathsf{C(umx)}} \mathsf{N}$ iff $\mathsf{uni}(\mathsf{M}) = \mathsf{uni}(\mathsf{N})$ and $\mathsf{D_M} \subseteq \mathsf{D_N}$.

Remark 1.219 $\mathsf{C(umx)}$ is transportable.

□

1.4.1.2 Subconstructs and Isomorphic Constructs

Definition 1.220 (Subconstructs and Isomorphic Constructs) [Ada83, D6, 1E] A **subconstruct** of a construct $\mathfrak{s}$, is a construct $\mathfrak{t}$, such that every $\mathfrak{t}$-object is an $\mathfrak{s}$-object and, for all $\mathfrak{t}$-objects $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \rightarrow_{\mathfrak{t}} \mathbf{B} \subseteq \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$. A **full subconstruct** of a construct $\mathfrak{s}$, is a subconstruct $\mathfrak{t}$, such that, for all $\mathfrak{t}$-objects $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \rightarrow_{\mathfrak{t}} \mathbf{B} = \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$. Constructs $\mathfrak{s}$ and $\mathfrak{t}$ are called **concretely isomorphic** if, for each set A, there exists a bijection $\mathcal{I}_A : \mathsf{Obj}_{\mathfrak{s}}(A) \Rightarrow \mathsf{Obj}_{\mathfrak{t}}(A)$, such that, for all $\mathfrak{s}$-objects $\mathbf{A}$ and $\mathbf{B}$ and any function $f : \mathsf{uni}(\mathbf{A}) \rightarrow \mathsf{uni}(\mathbf{B})$, $f : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$ iff $f : \mathcal{I}_A(\mathbf{A}) \rightarrow_{\mathfrak{t}} \mathcal{I}_B(\mathbf{B})$. □

1.4.1.3 Subobjects, Images and Embeddings

Definition 1.221 (Subobjects) Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathfrak{s}$-objects. We call $\mathbf{B}$ an **$\mathfrak{s}$-subobject** of $\mathbf{A}$, denoted $\mathbf{B} \lhd_{\mathfrak{s}} \mathbf{A}$, if, (i), $\mathsf{uni}(\mathbf{B}) \subseteq \mathsf{uni}(\mathbf{A})$, (ii), the inclusion function ${}_{\mathsf{uni}(\mathbf{B})}\hookrightarrow_{\mathsf{uni}(\mathbf{A})}$ from $\mathsf{uni}(\mathbf{B})$ into $\mathsf{uni}(\mathbf{A})$ is an $\mathfrak{s}$-morphism of $\mathbf{B}$ into $\mathbf{A}$ (in which case we speak of the **inclusion morphism** denoted by ${}_{\mathbf{B}}\hookrightarrow^{\mathfrak{s}}_{\mathbf{A}}$), and (iii), for all $\mathfrak{s}$-objects $\mathbf{C}$ and functions $f : \mathsf{uni}(\mathbf{C}) \rightarrow \mathsf{uni}(\mathbf{B})$, ${}_{\mathsf{uni}(\mathbf{B})}\hookrightarrow_{\mathsf{uni}(\mathbf{A})} f : \mathbf{C} \rightarrow_{\mathfrak{s}} \mathbf{A}$ implies $f : \mathbf{C} \rightarrow_{\mathfrak{s}} \mathbf{B}$. (The converse implication is generally true.) Let $\mathsf{Sb}_{\mathfrak{s}}(\mathbf{A})$ denote the set of $\mathfrak{s}$-subobjects of $\mathbf{A}$ (which always contains $\mathbf{A}$). For $M \subseteq \mathsf{uni}(\mathbf{A})$, let $\mathsf{Sb}^{\mathbf{A}}_{\mathfrak{s}}(M)$ denote the set of all $\mathfrak{s}$-subobjects $\mathbf{B}$ of $\mathbf{A}$ with $M = \mathsf{uni}(\mathbf{B})$. Whenever we call a *set* M an $\mathfrak{s}$-object, or use the notation $[M]^{\mathbf{A}}_{\mathfrak{s}}$, we mean that $\mathsf{Sb}^{\mathbf{A}}_{\mathfrak{s}}(M) = \{[M]^{\mathbf{A}}_{\mathfrak{s}}\}$, i.e., there is precisely one subobject of $\mathbf{A}$ with universe M. □

Remark 1.222 $\mathbf{A} \lhd_{\mathfrak{s}} \mathbf{A}$.

Definition 1.223 (Subuniverses) We call a subset M of $\mathsf{uni}(\mathbf{A})$ an **$\mathfrak{s}$-subuniverse** of $\mathbf{A}$, denoted $M \lhd_{\mathfrak{s}} \mathbf{A}$, if $\mathsf{Sb}^{\mathbf{A}}_{\mathfrak{s}}(M)$ is non-empty. The *set* of all $\mathfrak{s}$-subuniverses of $\mathbf{A}$ is denoted by $\mathsf{Su}_{\mathfrak{s}}(\mathbf{A})$. □

Warning 1.224 This constructural notion of subuniverse is incompatible with the structural notion of subuniverse given in $\oint$1.5.5 (see Warning 1.305 on page 61).

Definition 1.225 (Hereditary Constructs) [Ada83, 37] A construct $\mathfrak{s}$ is called **hereditary** (resp. **non-empty hereditary**) if, for all $\mathfrak{s}$-objects $\mathbf{A}$, every subset (resp. every non-empty subset) of $\mathsf{uni}(\mathbf{A})$ is an $\mathfrak{s}$-subuniverse of $\mathbf{A}$. □

Remark 1.226 [Ada83, 36] If $\mathbf{B} \lhd_{\mathfrak{s}} \mathbf{A}$, $\mathsf{uni}(\mathbf{C}) = \mathsf{uni}(\mathbf{B})$ and $\mathsf{id}_{\mathsf{uni}(\mathbf{B})} \in \mathbf{C} \rightarrow_{\mathfrak{s}} \mathbf{A}$, then $\mathbf{B} \succeq_{\mathfrak{s}} \mathbf{C}$, i.e., $\mathbf{B}$ is the coarsest object with universe $\mathsf{uni}(\mathbf{B})$ such that $\mathsf{id}_{\mathsf{uni}(\mathbf{B})}$ is a morphism. Consequently, in a transportable construct, **subsets determine subobjects**; i.e., if $\mathbf{A}$ and $\mathbf{A}'$ are both subobjects of $\mathbf{B}$ and $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbf{A}')$, then $\mathbf{A} = \mathbf{A}'$. So a hereditary transportable construct admits *precisely one* subobject per subset.

Example 1.227 (Submatrices of Unary Matrices)

Unary submatrices and $\mathsf{C}(\mathsf{umx})$-subobjects coincide.

□

Definition 1.228 (Object Generation) [Ada83, R5] We call M a **set of generators** for $\mathbf{A}$ in construct $\mathfrak{s}$ if $\mathbf{A}$ is the only subobject of $\mathbf{A}$ that contains M, and say that **$\mathbf{A}$ has m-generators**, if $\mathbf{A}$ has a set of generators of cardinality $\mathbf{m}$. □

Many transportable constructs, while failing to be hereditary, have the property that the set of subuniverses of each object form a closed set system over the universe. In such constructs, the constructions of subobject and subuniverse generation is well-defined.

Definition 1.229 (Intersections and Subobject Generation) [Ada83, 39] We say that a construct $\mathfrak{s}$ has **subobject intersections**, if, for any $\mathfrak{s}$-object $\mathbf{A}$, $\mathsf{Su}_{\mathfrak{s}}(\mathbf{A})$ forms a closed system over $\mathsf{uni}(\mathbf{A})$. Let $\mathfrak{s}$ be a construct *with intersections.* We say that a subobject $\mathbf{B}$ of $\mathbf{A}$ is **generated by** M **(in $\mathbf{A}$)**, if $\mathbf{B}$ is the least subobject of $\mathbf{A}$ with $M \subseteq \mathsf{uni}(\mathbf{B})$. The existence of this unique object is guaranteed in a construct having intersections. □

Definition 1.230 (Constructs with Images and Inverse-images) [Ada83, 46] A construct $\mathfrak{s}$ is said to **have images** (resp. **have inverse-images**), if, for each morphism f of $\mathbf{A}$ in $\mathbf{B}$, $\mathsf{Sb}^{\mathbf{B}}_{\mathfrak{s}}(\mathsf{rg}(f)) \neq \emptyset$ (resp. $\mathsf{Sb}^{\mathbf{A}}_{\mathfrak{s}}(\mathsf{do}(f)) \neq \emptyset$). A *transportable* construct $\mathfrak{s}$ has images (resp. has preimages), iff, for each morphism f of $\mathbf{A}$ in $\mathbf{B}$, the image $\mathsf{rg}(f)$ determines a *unique* subobject of $\mathbf{B}$ (the inverse-image $\mathsf{do}(f)$ determines a *unique* subobject of $\mathbf{A}$), which we denote by $\mathrm{f}_{\mathfrak{s}}[\mathbf{A}]$ (resp. by $\mathrm{f}^{-1}_{\mathfrak{s}}[\mathbf{B}]$). □

Remark 1.231 [Ada83, 46] In a construct with images, every morphism f of $\mathbf{A}$ into $\mathbf{B}$ can be factored as $f = {}_{\mathbf{C}}{\hookrightarrow}^{\mathfrak{s}}_{\mathbf{B}}\, h$, where h is a surjective morphism of $\mathbf{A}$ onto $\mathbf{C} \lhd_{\mathfrak{s}} \mathbf{B}$ and ${}_{\mathbf{C}}{\hookrightarrow}^{\mathfrak{s}}_{\mathbf{B}}$ is the inclusion morphism.

Definition 1.232 (Embeddings) [Ada83, 56, Ex g] An object $\mathbf{B}$ **can be embedded** into object $\mathbf{A}$, iff $\mathbf{B}$ is isomorphic to some subobject of $\mathbf{A}$. □

For ‘algebraic constructs’, $\mathbf{B}$ can be embedded into object $\mathbf{A}$, iff there exists an injective morphism from $\mathbf{B}$ into $\mathbf{A}$. This accounts for the fact that algebraists see injective homomorphisms as *embeddings*, while order theorists do not (see Definition 1.173 on page 39 and Definition 1.289 on page 58).

1.4.1.4 Free Objects

Definition 1.233 (Free Objects) [Ada83, 1H] An object $\mathbf{F}$ is called **$\mathfrak{s}$-free over** V, if $V \subseteq \mathsf{uni}(\mathbf{F})$, $\mathbf{F}$ is an $\mathfrak{s}$-object and, for each $\mathfrak{s}$-object $\mathbf{A}$ and each function $f : V \to \mathsf{uni}(\mathbf{A})$, there exists a *unique* $\mathfrak{s}$-morphism $f_{\mathbf{F}}$ from $\mathbf{F}$ into $\mathbf{A}$ extending f, in which case we say that $\mathbf{F}$ is **$\mathfrak{s}$-freely generated by** V. $\square$

Remark 1.234 [Ada83, 49-50] If $\mathbf{F}$ is $\mathfrak{s}$-free over V and $f, g : \mathbf{F} \to_{\mathfrak{s}} \mathbf{A}$, then, $f_{|V} = g_{|V}$ iff $f = g$.

Remark 1.235 [Ada83, 50] If $\mathbf{F}$ is $\mathfrak{s}$-free over V then V is a set of generators for $\mathbf{F}$.

Remark 1.236 [Ada83, 50] If $\mathbf{F}_1$ and $\mathbf{F}_2$ are $\mathfrak{s}$-free over V_1 and V_2 respectively, then, $\mathsf{card}(V_1) = \mathsf{card}(V_2)$ implies $\mathbf{F}_1 \cong_{\mathfrak{s}} \mathbf{F}_2$. $\square$

These isomorphisms are realised by extending the cardinal bijections between V_1 and V_2.

Remark 1.237 If $\mathbf{F}$ is $\mathfrak{s}$-free over V and $\mathbf{B} \cong_{\mathfrak{s}} \mathbf{F}$, then $\mathbf{B}$ is $\mathfrak{s}$-free over some W with $\mathsf{card}(V) = \mathsf{card}(W)$.

Definition 1.238 (Free Object on n Free Generators) [Ada83, 50] We say that $\mathbf{F}$ is a **free object on n free generators** if it is a free object with generators V with $\mathsf{card}(V) = \mathbf{n}$. $\square$

The following observation will prove important in the sequel. Note that this result requires that $\mathfrak{s}$ has *images*.

Remark 1.239 If $\mathfrak{s}$ has *images*, $\mathbf{F}$ is $\mathfrak{s}$-freely generated by V, $f : V \to \mathsf{uni}(\mathbf{F})$ and $V \subseteq f[V]$, then the unique $\mathbf{F}$-endomorphism extending f is surjective. Consequently, if $\mathbf{F}$ is a free object on ω generators V, $X \subseteq V$ with $V - X$ infinite, and $f : X \to \mathsf{uni}(\mathbf{F})$, then there exists a surjective endomorphism of $\mathbf{F}$ extending f.

Proof. Let f' be the unique $\mathbf{F}$-endomorphism extending f. Since $\mathfrak{s}$ has images, there exists a subobject $\mathbf{A}$ of $\mathbf{F}$ with $\mathsf{uni}(\mathbf{A}) = \mathsf{rg}(f')$. Since $V \subseteq f[V]$, $V \subseteq \mathsf{uni}(\mathbf{A})$, and so $\mathbf{A} = \mathbf{F}$, by Remark 1.235. $\diamond$

1.4.1.5 Quotients

Definition 1.240 (Quotients and Congruences) [Ada83, 1G] Let $\mathbf{A}$ be any $\mathfrak{s}$-object, α an equivalence relation on $\mathsf{uni}(\mathbf{A})$ and $\mathbf{B} \in \mathsf{Obj}_{\mathfrak{s}}(\coprod \alpha)$. We call $\mathbf{B}$ an **$\mathfrak{s}$-quotient object of A by** α, if

1. the quotient surjection q_α is an $\mathfrak{s}$-morphism of $\mathbf{A}$ onto $\mathbf{B}$, and
2. for all $\mathfrak{s}$-objects $\mathbf{C}$ and each function f from $\coprod \alpha$ into $\mathsf{uni}(\mathbf{C})$, if $f\mathsf{q}_\alpha$ is an $\mathfrak{s}$-morphism of $\mathbf{A}$ in $\mathbf{C}$, then f is an $\mathfrak{s}$-morphism of $\mathbf{B}$ into $\mathbf{C}$.

Let $\mathsf{Quot}^{\mathfrak{s}}_{\mathbf{A}}(\alpha)$ denote the set of all $\mathfrak{s}$-quotient objects of $\mathbf{A}$ by α. An **$\mathfrak{s}$-congruence on $\mathbf{A}$** is any equivalence relation α on $\mathsf{uni}(\mathbf{A})$ with $\mathsf{Quot}^{\mathfrak{s}}_{\mathbf{A}}(\alpha) \neq \emptyset$. Let $\mathsf{Con}^{\mathfrak{s}}(\mathbf{A})$ denote the set of all $\mathfrak{s}$-congruences on $\mathbf{A}$. We say that construct $\mathfrak{s}$ **has kernels** if all kernels of $\mathfrak{s}$-morphisms are $\mathfrak{s}$-congruences. A construct is called **cohereditary** if every equivalence on every object induces a quotient of this object. $\square$

Remark 1.241 [Ada83, 44] If $\mathfrak{s}$ is transportable, then, if $\mathbf{B}$ and $\mathbf{C}$ are both quotients of $\mathbf{A}$ by α, then $\mathbf{B} = \mathbf{C}$.

Convention 1.242 ($\mathbf{A} /^{\mathfrak{s}} \alpha$) Let $\mathfrak{s}$ be a transportable construct and let $\alpha \in \mathsf{Con}^{\mathfrak{s}}(\mathbf{A})$. Let $\mathbf{A} /^{\mathfrak{s}} \alpha$ denote the unique $\mathfrak{s}$-quotient of $\mathbf{A}$ by α.

Example 1.243 (Quotients of Unary Matrices)

> **Remark 1.244** The construct of unary matrices is (transportable) and cohereditary.
>
> *Proof.* (*We must show that* q_α *is a matrix homomorphism of* M *onto* $\mathsf{M}/\ \alpha$.) But by definition, $\mathsf{q}_\alpha\left[\mathsf{D}_{\mathsf{M}}\right] = \mathsf{D}_{\mathsf{M}/\alpha}$, so *certainly*, $\mathsf{q}_\alpha\left[\mathsf{D}_{\mathsf{M}}\right] \subseteq \mathsf{D}_{\mathsf{M}/\alpha}$. $\diamond$
>
> **Remark 1.245** For each unary matrix M and an equivalence relation α on $\mathsf{uni}(\mathsf{M})$, $\mathsf{M}/\alpha = \langle \coprod \alpha, \mathsf{q}_\alpha\left[\mathsf{D}_{\mathsf{M}}\right] \rangle$.
>
> *Proof.* Assume that L is a unary matrix and that f a function from $\coprod \alpha$ into $\mathsf{uni}(\mathsf{L})$, such that $f\mathsf{q}_\alpha$ is a unary matrix homomorphism of M in L. (*We must show that* f *is a homomorphism of* $\mathsf{M}/\ \alpha$ *into* L.) $f\left[\mathsf{D}_{\mathsf{M}/\alpha}\right] = f\left[\mathsf{q}_\alpha\left[\mathsf{D}_{\mathsf{M}}\right]\right] = (f\mathsf{q}_\alpha)\left[\mathsf{D}_{\mathsf{M}}\right] \subseteq \mathsf{D}_{\mathsf{L}}$, the final inclusion following since $f\mathsf{q}_\alpha$ is a unary matrix homomorphism of M in L, by assumption. $\diamond$

$\square$

Remark 1.246 [Ada83, 44] For transportable constructs $\mathfrak{s}$, congruences determine quotients; that is, if $\alpha \in \mathsf{Con}^{\mathfrak{s}}(\mathbf{A})$, $\mathbf{B}, \mathbf{C} \in \mathsf{Obj}_{\mathfrak{s}}(\coprod \alpha)$ and $\mathbf{B}$ and $\mathbf{C}$ are both $\mathfrak{s}$-quotients of $\mathbf{A}$ by α, then $\mathbf{B}$ = $\mathbf{C}$ (by Remark 1.241 on page 49).

Remark 1.247 If $\alpha \in \mathsf{Con}^{\mathfrak{s}}(\mathbf{A})$ and $\mathbf{B} \in \mathsf{Obj}_{\mathfrak{s}}(\coprod \alpha)$, then, $\mathbf{B}$ is a quotient object of $\mathbf{A}$ by α iff $\mathbf{A}$ is the finest object on $\mathsf{part}(\alpha)$ for which the quotient map $\mathsf{q}_{\alpha} \cdot : \mathsf{uni}(\mathbf{A}) \to \mathsf{uni}(\mathbf{B})$ is an $\mathfrak{s}$-morphism of $\mathbf{A}$ in $\mathbf{B}$ [Ada83, 45]. $\square$

Remark 1.248 [Ada83, 45] In a construct with kernels, every morphism f of $\mathbf{A}$ into $\mathbf{B}$ can be factored as $f = g\left(\mathsf{q}_{\equiv_f}\right)$, where $\mathbf{C} \in \mathsf{Quot}^{\mathfrak{s}}_{\mathbf{A}}(\equiv_f)$ and g is an injective morphism of $\mathbf{C}$ into $\mathbf{B}$.

Remark 1.249 [Ada83, 48] In a construct with images and kernels, every morphism f of $\mathbf{A}$ in $\mathbf{B}$ can be factored as $f = \left({}_{\mathbf{C}}{\hookrightarrow}^{\mathfrak{s}}_{\mathbf{B}}\right) h\left(\mathsf{q}_{\equiv_f}\right)$, where $\mathbf{C}$ is a quotient of $\mathbf{A}$ by $\equiv_f$, h is a bijective morphism of $\mathbf{A}$ in $\mathbf{C}$ and ${}_{\mathbf{C}}{\hookrightarrow}^{\mathfrak{s}}_{\mathbf{B}}$ is the inclusion morphism.

1.4.1.6 Sources, Initial Objects and Products

Definition 1.250 (Sources) [Ada83, 58] A **source** $\{A \xrightarrow{f_i} \mathbf{B}_i\}_I$ (on set A) is a set $\{\langle f_i, \mathbf{B}_i, \rangle : i \in I\}$, such that, for each $i \in I$, $\mathbf{B}_i$ is an object and $f_i : A \to \mathsf{uni}(\mathbf{B})_i$. In this definition I may be a *class*. We call the source $\{A \xrightarrow{f_i} \mathbf{B}_i\}_I$ an $\mathfrak{s}$-source if all $\mathbf{B}_i$ are $\mathfrak{s}$-objects. We say that source $\{A \xrightarrow{f_i} \mathbf{B}_i\}_I$ **separates points** if, for all $a, b \in A$ with $a \neq b$, there exists $i \in I$ with $f_i(a) \neq f_i(b)$. $\square$

Definition 1.251 (Initial Objects) [Ada83, 58-65] An **initial object** of an $\mathfrak{s}$-source $\{A \xrightarrow{f_i} \mathbf{B}_i\}_I$, is an $\mathfrak{s}$-object $\mathbf{A}$ on A, such that, for all $i \in I$, f_i is an $\mathfrak{s}$-morphism from $\mathbf{A}$ into $\mathbf{B}_i$, and, for each $\mathfrak{s}$-object $\mathbf{C}$ and function $g : \mathsf{uni}(\mathbf{C}) \to A$, if $f_i g$ is an $\mathfrak{s}$-morphism from $\mathbf{C}$ into $\mathbf{B}_i$ for all $i \in I$, then g is an $\mathfrak{s}$-morphism from $\mathbf{C}$ into $\mathbf{A}$.

A construct $\mathfrak{s}$ is called **initially unique**, if each source sc that has *some* initial structure, has a *unique* initial structure, which we denote by $\prod \mathsf{sc}$. A construct $\mathfrak{s}$ is called **initially complete**, if each source has a unique initial structure, and is called **initially mono-complete**, if every $\mathfrak{s}$-source that separates points has a unique initial structure.

An **$\mathfrak{s}$-splitting** of an $\mathfrak{s}$-object $\mathbf{B}$ is an $\mathfrak{s}$-object $\mathbf{A}$ for which there exists a surjective $\mathfrak{s}$-morphism f from $\mathbf{A}$ onto $\mathbf{B}$ with $\mathbf{A}$ an initial object of the singleton source $\{\mathsf{uni}(\mathbf{A}) \xrightarrow{f} \mathbf{B}\}$. We say the construct $\mathfrak{s}$ **has splitting** if each singleton $\mathfrak{s}$-source has an initial object. $\square$

Remark 1.252 [Ada83, 58] $\mathbf{A}$ is an initial object of an $\mathfrak{s}$-source $\{\mathsf{uni}(\mathbf{A}) \xrightarrow{f_i} \mathbf{B}_i\}_I$ iff, for each $\mathfrak{s}$-object $\mathbf{C}$ and function $g : \mathsf{uni}(\mathbf{C}) \to \mathsf{uni}(\mathbf{A})$, g is an $\mathfrak{s}$-morphism from $\mathbf{C}$ into $\mathbf{A}$ iff $f_i g$ is an $\mathfrak{s}$-morphism from $\mathbf{C}$ into $\mathbf{B}_i$ for all $i \in I$.

Remark 1.253 [Ada83, 59] If $\{\mathsf{uni}(\mathbf{A}) \xrightarrow{f} \mathbf{B}\}$ is a singleton $\mathfrak{s}$-source and f a bijection, then $\mathbf{A}$ is an initial object iff f is an $\mathfrak{s}$-isomorphism. Consequently, initially complete constructs are transportable.

Remark 1.254 [Ada83, 59] If $\mathbf{A}$ is an initial object of an $\mathfrak{s}$-source $\{\mathsf{uni}(\mathbf{A}) \xrightarrow{f_i} \mathbf{B}_i\}_I$, then $\mathbf{A}$ is the *coarsest* object on $\mathsf{uni}(\mathbf{A})$ such that $\forall\, [i \in I]\ f_i : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}_{\mathrm{i}}$.

Remark 1.255 Consequently, $\preceq$-ordered constructs are initially-unique.

Remark 1.256 [Ada83, 63] A construct is initially complete iff it has splitting and is initially mono-complete.

Definition 1.257 (Products) [Ada83, 2B] A **product** of $\mathfrak{s}$-objects $\{\mathbf{B}_i : i \in I\}$ is an initial object of the source $\{\prod_{i \in I} \mathsf{uni}(\mathbf{B})_i \xrightarrow{\boldsymbol{\pi}_i} \mathbf{B}_i\}_I$, where $\boldsymbol{\pi}_i$ is the i-th projection map. Precisely when a product of $\{\mathbf{B}_i : i \in I\}$ exists *and* this product is unique, we shall denote this unique product by $\prod_{i \in I} \mathbf{B}_i$. We write $\mathbf{B}^I$ for $\prod_{i \in I} \mathbf{B}$. We say that $\mathfrak{s}$ **has products** (is **product complete**) if each set of $\mathfrak{s}$-objects has a (unique) Cartesian product. $\square$

Remark 1.258 [Ada83, 71] A fibre-small construct is initially mono-complete iff it has Cartesian products and is hereditary.

Remark 1.259 [Ada83, 73] If $\mathbf{A}$ is a Cartesian product of $\{\mathbf{B}_i : i \in I\}$, then, for any set $\{f_i : i \in I\}$ of $\mathfrak{s}$-morphisms from an $\mathfrak{s}$-object $\mathbf{C}$ into $\mathbf{B}_i$, there exists a unique $\mathfrak{s}$-morphism f from $\mathbf{C}$ into $\mathbf{A}$ with $f_i = \boldsymbol{\pi}_i f$.

1.4.2 Categories

Definition 1.260 (Categories) [Ada83] A **category** $\mathfrak{x}$ is determined by

1. a set $\mathsf{Obj}_{\mathfrak{x}}$, the elements of which are called **$\mathfrak{x}$-objects** or just **objects**,
2. for all $\mathfrak{x}$-objects $\mathbf{A}$ and $\mathbf{B}$, a pairwise disjoint set $\mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$, the elements of which are called **$\mathfrak{x}$-morphisms from A to B** or just **morphisms**, and
3. an 'operation', called **$\mathfrak{x}$-composition**, or just **composition**, which assigns, to arbitrary $\mathfrak{x}$-objects $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, and arbitrary $\mathfrak{x}$-morphisms $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$ and $g : \mathbf{B} \rightarrow_{\mathfrak{x}} \mathbf{C}$, a morphism $g \cdot^{\mathfrak{x}} f \in \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{C}$,

such that

1. $h \cdot^{\mathfrak{x}} (g \cdot^{\mathfrak{x}} f) = (h \cdot^{\mathfrak{x}} g) \cdot^{\mathfrak{x}} f$, for all $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$, $g : \mathbf{B} \rightarrow_{\mathfrak{x}} \mathbf{C}$ and $h : \mathbf{C} \rightarrow_{\mathfrak{x}} \mathbf{D}$.
2. for each $\mathfrak{x}$-object $\mathbf{A}$, there exists a morphism, called the **$\mathfrak{x}$-identity on A**, $\mathsf{id}^{\mathfrak{x}}_{\mathbf{A}} \in \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{A}$, such that, for all $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$ $f \cdot^{\mathfrak{x}} \mathsf{id}^{\mathfrak{x}}_{\mathbf{A}} = f$ and $\mathsf{id}^{\mathfrak{x}}_{\mathbf{B}} \cdot^{\mathfrak{x}} f = f$.

Where unambiguous, we shall write fg for $f \cdot^{\mathfrak{x}} g$. The class of all $\mathfrak{x}$-morphisms is denoted $\mathsf{Mph}_{\mathfrak{x}}$. If $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$, we call $\mathbf{A}$ the **domain (object)** of f and call $\mathbf{B}$ the **range (object)** of f. A category $\mathfrak{x}$ is called **small**, if $\mathsf{Obj}_{\mathfrak{x}} \in \mathsf{Set}$, otherwise it is called **large**. A **$\mathfrak{x}$-retraction** (resp. **$\mathfrak{x}$-co-retraction**), is a morphism $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$, such that there exists a morphism $g : \mathbf{B} \rightarrow_{\mathfrak{x}} \mathbf{A}$, with $gf = \mathsf{id}^{\mathfrak{x}}_{\mathbf{A}}$ (resp. $fg = \mathsf{id}^{\mathfrak{x}}_{\mathbf{B}}$). A **$\mathfrak{x}$-monomorphisms** (**$\mathfrak{x}$-epimorphism**), is a morphism $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$, such that, for each pair of morphisms $g_1, g_2 : \mathbf{C} \rightarrow_{\mathfrak{x}} \mathbf{A}$ ($g_1, g_2 : \mathbf{B} \rightarrow_{\mathfrak{x}} \mathbf{C}$), if $fg_1 = fg_2$ ($g_1 f = g_2 f$), then $g_1 = g_2$. A **$\mathfrak{x}$-isomorphism**, is a morphism $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$, such that there exists a morphism $g : \mathbf{B} \rightarrow_{\mathfrak{x}} \mathbf{A}$, called the **inverse morphism**, with $fg = \mathsf{id}^{\mathfrak{x}}_{\mathbf{B}}$ and $gf = \mathsf{id}^{\mathfrak{x}}_{\mathbf{A}}$. □

Remark 1.261 [Ada83, 112] Retractive monomorphisms and co-retractive epimorphisms are isomorphisms.

Definition 1.262 (Functors between Categories) Let $\mathfrak{x}$ and $\mathfrak{y}$ be two categories. A **$\langle \mathfrak{x}, \mathfrak{y} \rangle$-functor** F, is a mapping which,

1. to each $\mathfrak{x}$-object $\mathbf{A}$, assigns an $\mathfrak{y}$-object $\mathsf{F}(\mathbf{A})$, and
2. to each $\mathfrak{x}$-morphism $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$, assigns a $\mathfrak{y}$-morphism $\mathsf{F}(f) \in \mathsf{F}(\mathbf{A}) \rightarrow_{\mathfrak{y}} \mathsf{F}(\mathbf{B})$, in such a way that,
   (a) F preserves composition, i.e., $\mathsf{F}(g)\mathsf{F}(f) = \mathsf{F}(gf)$, for all morphisms $f : \mathbf{A} \rightarrow_{\mathfrak{x}} \mathbf{B}$ and $g : \mathbf{B} \rightarrow_{\mathfrak{x}} \mathbf{C}$, and
   (b) F preserves identities, i.e., $\mathsf{F}(\mathsf{id}^{\mathfrak{x}}_{\mathbf{A}}) = \mathsf{id}^{\mathfrak{y}}_{\mathsf{F}\mathbf{A}}$, for all $\mathfrak{x}$-objects $\mathbf{A}$.

We write $\mathsf{F} : \mathfrak{x} \to \mathfrak{y}$ iff F is a $\langle \mathfrak{x}, \mathfrak{y} \rangle$-functor. For category $\mathfrak{x}$, define the **$\mathfrak{x}$-identity functor** $\mathsf{ID}_{\mathfrak{x}} : \mathfrak{x} \to \mathfrak{x}$ by the rule $\mathsf{ID}_{\mathfrak{x}}(\mathbf{A}) = \mathbf{A}$ and $\mathsf{ID}_{\mathfrak{x}}(f) = f$. For $\mathsf{F} : \mathfrak{x} \to \mathfrak{y}$ and $\mathsf{G} : \mathfrak{y} \to \mathfrak{z}$, define the **composite functor** $\mathsf{GF} : \mathfrak{x} \to \mathfrak{z}$ by the rule $\mathsf{GF}(\mathbf{A}) = \mathsf{G}(\mathsf{F}(\mathbf{A}))$ and $\mathsf{GF}(f) = \mathsf{G}(\mathsf{F}(f))$. A functor $\mathsf{F} : \mathfrak{x} \to \mathfrak{y}$, is called an **isomorphism of categories** if, there exists a functor $\mathsf{G} : \mathfrak{y} \to \mathfrak{x}$, called the **inverse functor**, with $\mathsf{FG} = \mathsf{ID}_{\mathfrak{y}}$ and $\mathsf{GF} = \mathsf{ID}_{\mathfrak{x}}$, in which case we call $\mathfrak{x}$ and $\mathfrak{y}$ **isomorphic**. □

Definition 1.263 (Natural Transformations) Let $\mathfrak{x}$ and $\mathfrak{y}$ be categories and $\mathsf{F}, \mathsf{G} : \mathfrak{x} \to \mathfrak{y}$. A natural transformation $\boldsymbol{n} : \mathsf{F} \to \mathsf{G}$ is a map assigning to each $\mathfrak{x}$-object $\mathbf{A}$ a $\mathfrak{y}$-morphism $\boldsymbol{n}_{\mathbf{A}} : \mathsf{F}(\mathbf{A}) \to \mathsf{G}(\mathbf{A})$ such that, for each $\mathfrak{x}$-morphism $f : \mathbf{A} \to \mathbf{B}$, $\boldsymbol{n}_{\mathbf{B}}\mathsf{F}(f) = \mathsf{G}(f)\boldsymbol{n}_{\mathbf{A}}$; the latter condition is referred to as **naturality**. □

1.5 Algebras and Structures

In this section we present the theory of *structures* and *universal algebras*. Algebras play a pivotal role in the theory of algebraic logic, both as the domain over which sentential calculi are defined, and as the 'target' of the algebraization process. Structures are also important, particularly *matrices over algebras*, which form the standard models of sentential calculi. In this presentation, we shall pay particular attention to *congruence relations on structures*, and the related notion of the *Leibniz equivalence relation* on a structure, given the central role played by these notions in the theory of algebraic logics.

In ∮1.5.1 we consider *types* of algebras and structures, and define *algebras* and *structures* in ∮1.5.2. The notions of *classes* of structures and *class operators* are considered in ∮1.5.3. In ∮1.5.4 structure *homomorphisms* are introduced, and *substructures* and *subuniverses* are considered in ∮1.5.5. *Products* of structures are the topic of ∮1.5.6, although the theory of *reduced products* and *ultraproducts* is only considered in ∮1.5.9. *Terms*, *term algebras*, *term functions* and *polynomials* form the content of ∮1.5.7.

While congruences on *algebras* have been well-understood for some time, it is only recently that a suitable theory of congruences on *structures* has begun to emerge [Elg97],[Elg98]. In ∮1.5.8 we consider congruences on algebras and structures, paying careful attention to the fundamental differences that arise between algebra congruences and structure congruences. One of the most striking differences is that congruences on algebras form an (algebraic) closed system while congruences on structures do not. In particular, the square relation on the universe of a structure is *not* a structure congruence. There is, however, a *largest* congruence on a structure, and this largest congruence is known as the *Leibniz equivalence relation* of the structure [BP89a],[Elg97],[Elg98]. The Leibniz relation plays an important role in the theory of algebraizable logics, and it is clear that this relation will come to play an important role in universal algebra itself, since the Leibniz relation may be used to find the largest algebra congruence compatible with a particular subset of the universe of the algebra. As such, it is not surprising that the protoalgebraicity of certain logics, arising from universal algebraic considerations, may be characterized in *purely* universal algebraic terms as forms of *coherence* (defined in ∮10). The related notion of the *quotient* of a structure is also considered in this section.

Reduced products and *ultraproducts* of algebras and structures are considered in ∮1.5.9, and *free structures* are introduced in ∮1.5.10. The *free algebra* determined by a class of algebras proves important in this text. We shall show how closed systems on such free structures give rise to sentential calculi that reflect both the consequences of the closed system and the equational truths of the given class (see $\oint 8$).

The *model theory* of algebras and structures is introduced in $\oint 1.5.11$. We consider both the model theory of structures *with equality*, which is important, in the case of algebras, in the consideration of the algebraic semantics of a sentential calculus, and the model theory of structures *without equality*, which is important when considering matrices as models of sentential calculi. With model theory in hand, we present, in $\oint 1.5.11$, an important model theoretic characterization of the Leibniz relation, which essentially demonstrates that the Leibniz relation is an elementary approximation of the second-order notion that two entities are equal provided they satisfy precisely the same properties; this second order notion of equality being first introduced by Leibniz.

The equational and quasi-equational theory of varieties and quasivarieties of algebras, is presented in $\oint 1.5.13$ and $\oint 1.5.14$, respectively. The results of this section are used repeatedly throughout the rest of this text. Important characterizations of relative congruence generation and congruences class are given.

Finally, a number of examples of structures and algebras, used subsequently in the sequel, are presented in $\oint 1.5.15$.

1.5.1 Types of Algebras and Structures

Definition 1.264 (Elementary Types) A **type** $\mathfrak{e}$, is defined by specifying its **relation symbols** $\mathsf{Symb}_\mathsf{r}(\mathfrak{e})$, its **operation symbols** $\mathsf{Symb}_\mathsf{o}(\mathfrak{e})$ and its **constant symbols** $\mathsf{Symb}_\mathsf{c}(\mathfrak{e})$, and specifying an **arity function** $\mathsf{ar}^\mathfrak{e}(\cdot)$, that assigns to each **$\mathfrak{e}$-operation symbol** $\star$ an integral positive arity $\mathsf{ar}^\mathfrak{e}(\star)$, and assigns to each **$\mathfrak{e}$-relation symbol** $\bowtie$ an integral positive arity $\mathsf{ar}^\mathfrak{e}(\bowtie)$. A type $\mathfrak{e}$ is called **finite** if it has finitely many relation symbols, finitely many operation symbols and finitely many constant symbols. We call a type $\mathfrak{e}$ a **type of algebras** or an **algebraic type** (**type of relations** or a **relational type**) if it has no relation symbols (resp. no operation symbols). We call $\mathfrak{e}'$ a **subtype** of $\mathfrak{e}$ if every $\mathfrak{e}'$-operation symbol is an $\mathfrak{e}$-operation symbol with $\mathsf{ar}^{\mathfrak{e}'}(\star) = \mathsf{ar}^\mathfrak{e}(\star)$, and every $\mathfrak{e}'$-relation symbol is an $\mathfrak{e}$-relation symbol with $\mathsf{ar}^{\mathfrak{e}'}(\bowtie) = \mathsf{ar}^\mathfrak{e}(\bowtie)$, and every $\mathfrak{e}'$-constant symbol is an $\mathfrak{e}$-constant symbol, in which case we call $\mathfrak{e}$ a **supertype** of $\mathfrak{e}'$. With each elementary type $\mathfrak{e}$, we associate the **algebraic subtype** $\mathfrak{e}_{|\mathsf{o}}$ (resp. **relational subtype** $\mathfrak{e}_{|\mathsf{r}}$), obtained from $\mathfrak{e}$ by removing all relation symbols (resp. all operation symbols), and 'mending' the arity function appropriately. We often introduce types by merely describing their arities (see the next Example).

Unless otherwise specified, $\mathfrak{e}$, $\mathfrak{r}$ and $\mathfrak{a}$, shall denote a fixed but arbitrary elementary type, relational type and algebraic type, respectively. If $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are elementary types, then $\mathfrak{e}_1 \cup \mathfrak{e}_2$ denotes the elementary type obtained by 'unioning' $\mathfrak{e}_1$ and $\mathfrak{e}_2$, and $\mathfrak{e}_1 \oplus \mathfrak{e}_2$ denotes the elementary type obtained by 'unioning' $\mathfrak{e}_1$ and $\mathfrak{e}_2$ after renaming symbols so that no clashes occur. □

Example 1.265 (The Types of Modern-Groups and (Universal) Groups)

> A type of groups in *modern* algebra, is the algebraic type with a single binary operation; we often denote this operation symbol by $*$, although many other symbols will be used in different contexts (see the next convention). We shall call this type, the **type of groupoids**. In universal algebra, groups have an algebraic type (the **type of groups**) with a binary operation symbol, a unary operation symbol, and a constant symbol, which we often denote

by $*$, $^{-1}$ and $\mathbf{1}$ respectively, although many other symbolisms will be encountered in differing contexts in this text. We call the algebraic type with subtype groupoid and one constant symbol $\mathbf{1}$ the **type of monoids**.

□

Convention 1.266 It is common practice to change the symbols of a type. For example, in group theory, arbitrary group theory and abelian group theory typically invoke differing symbolisms. While it is possible to free the specification of a elementary type from a particular symbolism, as is typically the approach taken in model theory texts, such an approach is unnecessarily technical and cumbersome for the reader and user of the language. Instead, we shall freely, but unambiguously, shift between symbolisms for a particular elementary type, demanding that the reader is mathematically mature enough to handle such shifts. In keeping with this convention, we shall often introduce types by specifying arities only.

Example 1.267 (The Type of Sets)

The elementary type with no symbols whatsoever, is called the **type of sets**.

□

Example 1.268 (Type of Matrices over Algebras)

Let $\mathfrak{a}$ be a type of algebras. The **type of unary-matrices over** $\mathfrak{a}$ is the elementary type extending the type $\mathfrak{a}$ by a single unary relation symbol, typically denoted by the symbol D. The **type of unary-matrices** is the type of unary-matrices over the type of sets.

□

1.5.2 Algebras and Structures

Definition 1.269 (Structures and Algebras) A structure $\mathbf{A}$ is an object determined by the following datum:

1. $\mathsf{uni}(\mathbf{A})$ which *must be non-empty*;
2. $\mathsf{type}(\mathbf{A})$ which is an elementary type, called the **type**;
3. $\mathsf{relations}(\mathbf{A}) = \{\bowtie^{\mathbf{A}} : \bowtie\, \in \mathsf{Symb}_{\mathsf{r}}(\mathsf{type}(\mathbf{A}))\}$, where each $\bowtie^{\mathbf{A}}$ is an $\mathsf{ar}(\bowtie)$-relation on $\mathsf{uni}(\mathbf{A})$;
4. $\mathsf{operations}(\mathbf{A}) = \{\star^{\mathbf{A}} : \star \in \mathsf{Symb}_{\mathsf{o}}(\mathsf{type}(\mathbf{A}))\}$, where each $\star^{\mathbf{A}}$ is an $0 \neq \mathsf{ar}(\star)$-ary operation on $\mathsf{uni}(\mathbf{A})$;
5. $\mathsf{constants}(\mathbf{A}) = \{\mathbf{0}^{\mathbf{A}} : \mathbf{0} \in \mathsf{Symb}_{\mathsf{c}}(\mathsf{type}(\mathbf{A}))\}$, where each $\mathbf{0}^{\mathbf{A}}$ is an element of $\mathsf{uni}(\mathbf{A})$.

The members of $\mathsf{relations}(\mathbf{A})$, $\mathsf{operations}(\mathbf{A})$ and $\mathsf{constants}(\mathbf{A})$, are respectively called the **fundamental relations**, the **fundamental operations** and the **fundamental constants** of $\mathbf{A}$. By an **$\mathfrak{e}$-structure**, where $\mathfrak{e}$ is an elementary type, we mean a structure with type $\mathfrak{e}$. Structures of the same elementary type are called **similar**. Structures of relational types are called **relational structures** and structures of algebraic types are called **algebras**.

We say that a structure has **finite type** if its elementary type is finite. A structure is **finite** if its universe is finite, and is **trivial** if its universe contains precisely one element and, for each relation symbol $\bowtie$, $\bowtie^{\mathbf{A}} = \emptyset$. We call a structure **small** if its universe is a set.

We tend to drop the superscripts $\mathbf{A}$ wherever unambiguously possible. It is convenient to treat/view constants as fundamental operations with arity 0, and when we speak of a fundamental operation with arity 0, we shall mean a fundamental constant. $\square$

The following conventions, in keeping with Convention 1.392 on page 74, describe some common conventions for expressing fundamental relations of structures.

Convention 1.270 (Unary and Binary Relational Notations) For a binary relation symbol $\bowtie$, we may write $a \bowtie^{\mathbf{A}} b$ for $\langle a, b\rangle \in \bowtie^{\mathbf{A}}$, and for a unary relation symbol $\square$, we may write a is $\square^{\mathbf{A}}$, $a \in \square^{\mathbf{A}}$ or even $\square^{\mathbf{A}} a$, for $\langle a\rangle \in \square^{\mathbf{A}}$, as context appropriate.

Warning 1.271 (Empty Universes) We do not permit structures on empty universes. Model theorists do not consider empty structures [Men87, 46],[vD83, 59]. Structures with empty universes, i.e., no 'things', cannot be the models of any elementary theory, since $\exists[x]$ true is always a theorem. Algebraists follow model theorists [Hun74, D 1.1][RMT87, D 1.1]. Certain order texts consider the empty set as an order on the empty set [DP90, D 1.2]. Category theorists admit universal algebras with empty universes [Ada83, 23-24], and do so fundamentally, in that algebras are cited as examples of constructs having subobject intersection [Ada83, 38]; if empty universes are proscribed, however, this need not be the case (see Warning 1.305 on page 61 and the counter example that follows that warning). This dichotomy poses problems for algebraists, particularly with regard to the notion of a subuniverse (see Definition 1.302 on page 60).

Convention 1.272 When presenting *algebras*, it is convenient to confuse operations and constants, i.e., treating constants as operations of arity zero. When we say that $\langle \mathsf{uni}(\mathbf{A}); \star_1^{\mathbf{A}}, \ldots, \star_m^{\mathbf{A}}\rangle$ is an *algebra* of type $\langle n_1, \ldots, n_m\rangle$, we mean the algebra with operations $\{\star_i^{\mathbf{A}} : n_i > 0, \mathsf{ar}(\star_i^{\mathbf{A}}) = n_i, 1 \leq i \leq m\}$.

Example 1.273 (Set Structures)

> We conflate a structure of type sets, with its universe, and conflate a set with the structures of type sets whose universe is this set.

$\square$

Example 1.274 (Pointed Sets)

> The type of pointed sets has a single constant symbol, typically denoted by 0. Algebras of this type are called **pointed-sets** (or **pointed-sets** if A is a set).

$\square$

Definition 1.275 (Reducts and Expansions) Let $\mathbf{A}'$ be an $\mathfrak{e}'$-structure and $\mathbf{A}$ an $\mathfrak{e}$-structure. We call $\mathbf{A}'$ a **reduct** of $\mathbf{A}$ and call $\mathbf{A}$ an **expansion** of $\mathbf{A}'$, if both structures have the same universe, $\mathfrak{e}'$ is a subtype of $\mathfrak{e}$, and all fundamental operations and relations of $\mathbf{A}'$ coincide with their $\mathbf{A}$ counterparts, in which case we call $\mathbf{A}'$ *the* $\mathfrak{e}'$**-reduct of** $\mathbf{A}$. For $\mathfrak{e}$-structure $\mathbf{A}$, let $\mathsf{alg}(\mathbf{A})$ denote the $\mathfrak{e}_{|\mathsf{o}}$-reduct of $\mathbf{A}$, called the **algebraic reduct** of $\mathbf{A}$, and let $\mathsf{rel}(\mathbf{A})$ denote the $\mathfrak{e}_{|\mathsf{r}}$-reduct of $\mathbf{A}$, called the **relational reduct** of $\mathbf{A}$. $\square$

Example 1.276 (Unary Algebra-Matrices)

Let $\mathfrak{a}$ be a type of algebras. An **unary $\mathfrak{a}$-matrix**, is a structure of the type of unary-matrices over $\mathfrak{a}$, in which case we call $\mathsf{D}_{\mathbf{M}}$ the **designator**. The $\mathfrak{a}$-reduct of a unary $\mathfrak{a}$-matrix $\mathbf{M}$ is denoted by $\mathsf{alg}(\mathbf{M})$, and is called the **algebra of** the matrix. A **unary algebra-matrix** is a unary $\mathfrak{a}$-matrix for some type $\mathfrak{a}$ of algebras. For an arbitrary algebra-matrix $\mathbf{M}$, we denote the 'algebra'-subtype of $\mathsf{type}(\mathbf{M})$ by $\mathsf{type_s}(\mathbf{M})$, which we call the **algebra subtype** of the matrix. For an algebra $\mathbf{A}$, a unary $\mathbf{A}$-matrix is a unary $\mathsf{type}(\mathbf{A})$-matrix with algebra $\mathbf{A}$. We may present a unary $\mathbf{A}$-matrix $\mathbf{M}$ by $\langle \mathsf{alg}(\mathbf{M}), \mathsf{D}_{\mathbf{M}} \rangle$. In such a presentation, we may write $\langle \mathbf{A}, a \rangle$ for $\langle \mathbf{A}, \{a\} \rangle$, where context unambiguous. A unary **matrix** is a unary algebra-matrix whose algebra subtype is the type of sets.

□

1.5.3 Class Operators

We introduce the following **class operators** mapping classes of $\mathfrak{a}$-algebras to classes of $\mathfrak{a}$-algebras.

Definition 1.277 (Class Operators) Let $\mathfrak{e}$ be an elementary type. An **$\mathfrak{e}$-class operator**, or simply a **class operator** where unambiguous, is an operator on the set of all *classes* of $\mathfrak{e}$-structures. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two class operators, we write $\mathcal{O}_1\mathcal{O}_2$ for the composition of the operators. We write $\mathcal{O}_1 \leq \mathcal{O}_2$ to abbreviate '$\mathcal{O}_1(\mathcal{K}) \subseteq \mathcal{O}_2(\mathcal{K})$ for all classes of $\mathfrak{e}$-structures $\mathcal{K}$'. Clearly '$\leq$' orders the set of all classes of $\mathfrak{e}$-structures. A class $\mathcal{K}$ of $\mathfrak{e}$-structures is said to be **closed under an operator** $\mathcal{O}$ if $\mathcal{O}(\mathcal{K}) \subseteq \mathcal{K}$. □

1.5.4 Structure Homomorphisms

1.5.4.1 Structure Homomorphisms

Definition 1.278 (Structure Homomorphisms) Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathfrak{e}$-structures. A function $f : \mathsf{uni}(\mathbf{A}) \to \mathsf{uni}(\mathbf{B})$ is called an **$\mathfrak{e}$-homomorphism** from $\mathbf{A}$ into $\mathbf{B}$ if it **preserves relations**, i.e., for each $\bowtie \in \mathsf{Symb_r}(\mathfrak{e})$ and $\mathsf{a} \in \mathsf{uni}(\mathbf{A})^{\mathsf{ar}(\bowtie)}$,

$$\mathsf{a} \in \bowtie^{\mathbf{A}} \text{ implies } \underset{\rightarrow}{f}(\mathsf{a}) \in \bowtie^{\mathbf{B}},$$

it **commutes with operations** (or **preserves operations**), i.e., for each $\star \in \mathsf{Symb_o}(\mathfrak{e})$ and $\mathsf{a} \in \mathsf{uni}(\mathbf{A})^{\mathsf{ar}(\star)}$,

$$f(\star^{\mathbf{A}}(\mathsf{a})) = \star^{\mathbf{B}}(f(\mathsf{a}_{(0)}), \ldots, f(\mathsf{a}_{(\mathsf{ar}(\star)-1)})) \quad \text{and}$$

it **preserves constants**, i.e., for each $\mathbf{0} \in \mathsf{Symb_c}(\mathfrak{e})$,

$$f(\mathbf{0}^{\mathbf{A}}) = \mathbf{0}^{\mathbf{B}}.$$

We denote the set of all homomorphisms from structure $\mathbf{A}$ into $\mathbf{B}$ by $\mathbf{A} \to \mathbf{B}$, and write $f : \mathbf{A} \to \mathbf{B}$ for $f \in \mathbf{A} \to \mathbf{B}$. □

Remark 1.279 The three conditions determining a homomorphism may be unified, viewing operations and constants as relations. It is not hard to see that f is a homomorphism iff, for all constant symbols $\mathbf{0}$, operation symbols $\star$ and relation symbols $\bowtie$, we have

$$f\left[\{\mathbf{0}^{\mathbf{A}}\}\right] \subseteq \{\mathbf{0}^{\mathbf{B}}\}, \qquad \underset{\rightarrow}{f}\left[\star^{\mathbf{A}}\right] \subseteq {\star^{\mathbf{B}}}_{|\mathsf{rg}(f)} \subseteq \star^{\mathbf{B}} \qquad \text{and} \qquad \underset{\rightarrow}{f}\left[\bowtie^{\mathbf{A}}\right] \subseteq {\bowtie^{\mathbf{B}}}_{|\mathsf{rg}(f)} \subseteq \bowtie^{\mathbf{B}}.$$

Proof. $\boxed{\text{Homomorphism} \Rightarrow}$ $\boxed{\text{Operations}}$ Consider an operation symbol $\star$, with $R^{\mathbf{A}}$ and $R^{\mathbf{B}}$ defined by (say) $\langle a_0, \ldots a_{\mathsf{ar}(\star)}, a\rangle \in R^{\mathbf{A}}$ iff $\star^{\mathbf{A}}(a_0, \ldots, a_{\mathsf{ar}(\star)}) = a$, etc. Suppose that $\langle a_0, \ldots, a_{\mathsf{ar}(\star)}, a\rangle \in R^{\mathbf{A}}$, i.e., $\star^{\mathbf{A}}(a_0, \ldots, a_{\mathsf{ar}(\star)}) = a$. Since f is a homomorphism, $\star^{\mathbf{B}}(f(a_0), \ldots, f(a_{\mathsf{ar}(\star)})) = f(a)$, so $\langle f(a_0), \ldots, f(a_{\mathsf{ar}(\star)}), f(a)\rangle \in R^{\mathbf{B}}$ So $f(\langle a_0, \ldots, a_{\mathsf{ar}(\star)}, a\rangle) \in R^{\mathbf{B}}$. $\boxed{\text{Constants}}$ Similar. $\boxed{\text{Relations}}$ Trivial. $\boxed{\Rightarrow \text{Homomorphism}}$ $\boxed{\text{Operations}}$ Certainly, $\langle a_0, \ldots, a_{\mathsf{ar}(\star)}, \star^{\mathbf{A}}(a_0, \ldots, a_{\mathsf{ar}(\star)})\rangle \in R^{\mathbf{A}}$. So by assumption, $\langle f(a_0), \ldots, f(a_{\mathsf{ar}(\star)}), f(\star^{\mathbf{A}}(a_0, \ldots, a_{\mathsf{ar}(\star)}))\rangle \in R^{\mathbf{B}}$. In other words, $f(\star^{\mathbf{A}}(a_0, \ldots, a_{\mathsf{ar}(\star)})) = \star^{\mathbf{B}}(f(a_0), \ldots, f(a_{\mathsf{ar}(\star)}))$. $\boxed{\text{Constants}}$ Similar. $\boxed{\text{Relations}}$ Trivial. $\diamond$

Remark 1.280 The composition of structure homomorphisms is a homomorphism, and the identity function is a homomorphism.

Definition 1.281 (Constructs of Structures with Homomorphisms) It is convenient, for an elementary type $\mathfrak{e}$, to denote the construct of all (small) $\mathfrak{e}$-structures with homomorphisms by the same symbol $\mathfrak{e}$ that denotes the type. $\square$

Definition 1.282 (Special Homomorphism) Surjective and injective structure homomorphisms are called **epimorphisms** and **monomorphisms** respectively. We call f an **isomorphism** from $\mathbf{A}$ onto $\mathbf{B}$ if it is bijective, a homomorphism and the inverse function f^{-1} is a homomorphism from $\mathbf{B}$ onto $\mathbf{A}$. We call structures $\mathbf{A}$ and $\mathbf{B}$ **isomorphic** if there exists an isomorphism between them. Bijective homomorphisms are called **weak-isomorphisms**. We denote the set of all epimorphisms (resp. monomorphisms, and isomorphisms) from $\mathbf{A}$ to $\mathbf{B}$ by $\mathbf{A} \twoheadrightarrow \mathbf{B}$ (resp. $\mathbf{A} \rightarrowtail \mathbf{B}$, and $\mathbf{A} \cong \mathbf{B}$). A homomorphism (resp. isomorphism) from a structure to itself is called a **endomorphism** (resp. **automorphism**). An **involution** f is an automorphism with $f^{-1} = f$. $\square$

Remark 1.283 For an elementary type $\mathfrak{e}$, (small) structure epimorphisms, monomorphisms and isomorphisms coincide with $\mathfrak{e}$-epimorphisms, $\mathfrak{e}$-monomorphisms and $\mathfrak{e}$-isomorphisms, respectively.

Remark 1.284 [BS81] Between *algebras*, weak-isomorphisms are isomorphisms. $\square$

The previous remark not withstanding, weak isomorphisms need not be isomorphisms. For example, bijective order preserving functions need not be order-isomorphisms [DP90].

Remark 1.285 Let f be an injection of $\mathsf{uni}(\mathbf{A})$ into $\mathsf{uni}(\mathbf{B})$. The following conditions are equivalent.

1. f is a homomorphism.
2. f is a monomorphism.
3. $f\left[\{\mathbf{0}^{\mathbf{A}}\}\right] = \{\mathbf{0}^{\mathbf{B}}\}$, $\underset{\longrightarrow}{f}\left[\star^{\mathbf{A}}\right] = \star^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$ and $\underset{\longrightarrow}{f}\left[\bowtie^{\mathbf{A}}\right] \subseteq \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$, for all constant symbols $\mathbf{0}$, operation symbols $\star$ and relation symbols $\bowtie$.

Proof. The only non-triviality is (2) implies (3). Let $\mathsf{a} \in \mathsf{uni}(\mathbf{A})^{\mathsf{ar}(\star)+1}$ with $\underset{\longrightarrow}{f}(\mathsf{a}) \in \star^{\mathbf{B}}$. (We must show that $\mathsf{a} \in \star^{\mathbf{A}}$.) Re-functionalizing notation, we have $\star^{\mathbf{B}}(f(\mathsf{a}_{(0)}), \ldots, f(\mathsf{a}_{(\mathsf{ar}(\star))})) = f(\mathsf{a}_{(\mathsf{ar}(\star)+1)}))$. By operational-closure, there exists $a \in \mathsf{uni}(\mathbf{A})$ with $\star^{\mathbf{A}}(\mathsf{a}_{(0)}, \ldots, \mathsf{a}_{(\mathsf{ar}(\star))}) = a$. Since f is assumed to be a homomorphism, $\star^{\mathbf{B}}(f(\mathsf{a}_{(0)}), \ldots, f(\mathsf{a}_{(\mathsf{ar}(\star))})) = f(a)$. So $f(\mathsf{a}_{(\mathsf{ar}(\star)+1)})) = f(a)$, and so by injectivity of f, $\mathsf{a}_{(\mathsf{ar}(\star)+1)}) = a$. Hence $\star^{\mathbf{A}}(\mathsf{a}_{(0)}, \ldots, \mathsf{a}_{(\mathsf{ar}(\star))}) = \mathsf{a}_{(\mathsf{ar}(\star)+1)})$. De-functionalizing, $\mathsf{a} \in \star^{\mathbf{A}}$. $\diamond$

Remark 1.286 If f and g are isomorphisms (homomorphisms) then so is fg, when defined.

Definition 1.287 (The Class Operator $\mathcal{I}$) Let $\mathcal{K}$ be a class of $\mathfrak{e}$-structures. Let $\mathcal{I}(\mathcal{K})$ denote the class of all $\mathfrak{e}$-structures isomorphic to some member of $\mathcal{K}$. $\square$

Example 1.288 (Unary Algebra-Matrix Homomorphisms)

> Let $\mathbf{M}$ and $\mathbf{N}$ be unary $\mathfrak{a}$-matrices and $f : \mathsf{uni}(\mathbf{M}) \to \mathsf{uni}(\mathbf{N})$. Then f is a homomorphism from $\mathbf{M}$ into $\mathbf{N}$ iff f is a homomorphism from $\mathsf{alg}(\mathbf{M})$ into $\mathsf{alg}(\mathbf{N})$, and $f\,[\mathsf{D}_{\mathbf{M}}] \subseteq \mathsf{D}_{\mathbf{N}}$. Moreover, f is an isomorphism of $\mathbf{M}$ onto $\mathbf{N}$ iff f is an isomorphism of $\mathsf{alg}(\mathbf{M})$ onto $\mathsf{alg}(\mathbf{N})$ and $f\,[\mathsf{D}_{\mathbf{M}}] = \mathsf{D}_{\mathbf{N}}$.

$\square$

1.5.4.2 Algebra-Homomorphisms between Structures

It is useful to distinguish those functions between structures that are homomorphisms between the algebra reducts of these structures, which we call *algebra-homomorphisms*.

Definition 1.289 (Algebra-Homomorphisms between Structures)
An **algebra-homomorphism** (resp. **algebra-embedding**, **algebra-epimorphism** and **algebra-isomorphism**) from *structure* $\mathbf{A}$ to structure $\mathbf{B}$ is a homomorphism (resp. monomorphism, epimorphism and isomorphism) from $\mathsf{alg}(\mathbf{A})$ to $\mathsf{alg}(\mathbf{B})$. We denote the set of all algebra-homomorphisms (resp. algebra-embeddings, algebra-epimorphisms and algebra-isomorphisms) from $\mathbf{A}$ to $\mathbf{B}$ by $\mathbf{A} \to_{\mathsf{a}} \mathbf{B}$ (resp. $\mathbf{A} \rightarrowtail_{\mathsf{a}} \mathbf{B}$, $\mathbf{A} \twoheadrightarrow_{\mathsf{a}} \mathbf{B}$ and $\mathbf{A} \cong_{\mathsf{a}} \mathbf{B}$). $\square$

Remark 1.290 Between *algebras*, *algebra-homomorphisms* and *homomorphisms* coincide.

1.5.4.3 Strict Homomorphisms

We have already noted that, while bijective homomorphisms between *algebras* are isomorphisms, a bijective homomorphism between *structures* need not be an isomorphism. Consider a bijective homomorphism f from $\mathbf{A}$ onto $\mathbf{B}$. For f to be an isomorphism, we required that the inverse function f^{-1} be a homomorphism from $\mathbf{B}$ onto $\mathbf{A}$. Since the inverse function is already an algebra-isomorphism, the only additional requirements fall on the behaviour of f with respect to the fundamental relations. We require that, for all $\bowtie \in \mathsf{Symb}_{\mathsf{r}}(\mathfrak{e})$,

$$\underrightarrow{(f^{-1})}\left[\bowtie^{\mathbf{B}}\right] \subseteq \bowtie^{\mathbf{A}}.$$

The additional requirement can be rephrased so as to replace the image under the inverse-function from the definition (inverse functions only being definable for bijections) with a functional inverse-image (which is defined for all functions), i.e.,

$$(\underrightarrow{f})^{-1}\left[\bowtie^{\mathbf{B}}\right] \subseteq \bowtie^{\mathbf{A}}.$$

Definition 1.291 (Strict Homomorphisms) A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is called **strict**, if, for all $\bowtie \in \mathsf{Symb_r}(\mathfrak{e})$, $(\underset{\rightarrow}{f})^{-1}\left[\bowtie^{\mathbf{B}}\right] \subseteq \bowtie^{\mathbf{A}}$. □

Remark 1.292 For a function f from $\mathsf{uni}(\mathbf{A})$ into $\mathsf{uni}(\mathbf{B})$, the following conditions are equivalent.

1. f is a strict homomorphism from $\mathbf{A}$ into $\mathbf{B}$.
2. f is a homomorphism from $\mathbf{A}$ into $\mathbf{B}$ and $\underset{\rightarrow}{f}(\mathsf{a}) \in \bowtie^{\mathbf{B}} \to \mathsf{a} \in \bowtie^{\mathbf{A}}$, for all $\bowtie \in \mathsf{Symb_r}(\mathfrak{e})$.
3. f is an algebra-homomorphism and $\mathsf{a} \in \bowtie^{\mathbf{A}} \leftrightarrow \underset{\rightarrow}{f}(\mathsf{a}) \in \bowtie^{\mathbf{B}}$, for all $\bowtie \in \mathsf{Symb_r}(\mathfrak{e})$.
4. $f\left[\{\mathbf{0}^{\mathbf{A}}\}\right] = \{\mathbf{0}^{\mathbf{B}}\}$, $\underset{\rightarrow}{f}\left[\star^{\mathbf{A}}\right] \subseteq \star^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$ and $\underset{\rightarrow}{f}\left[\bowtie^{\mathbf{A}}\right] = \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Assume that $\underset{\rightarrow}{f}(\mathsf{a}) \in \bowtie^{\mathbf{B}}$. By assumption of strictness, $\underset{\rightarrow}{f}^{-1}[\![\underset{\rightarrow}{f}(\mathsf{a})]\!] \subseteq \bowtie^{\mathbf{A}}$, and *certainly* $\mathsf{a} \in \underset{\rightarrow}{f}^{-1}[\![\underset{\rightarrow}{f}(\mathsf{a})]\!]$. $\boxed{(2)\Rightarrow(3)}$ Follows immediately from assumption (2) and the definition of a (structure) homomorphism. $\boxed{(3)\Rightarrow(4)}$ That $f\left[\{\mathbf{0}^{\mathbf{A}}\}\right] = \{\mathbf{0}^{\mathbf{B}}\}$ and $\underset{\rightarrow}{f}\left[\star^{\mathbf{A}}\right] \subseteq \star^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$ follows by Remark 1.279 on page 56. $\boxed{\underset{\rightarrow}{f}\left[\bowtie^{\mathbf{A}}\right] \subseteq \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}}$ $\underset{\rightarrow}{f}\left[\bowtie^{\mathbf{A}}\right] \subseteq \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$, since $\mathsf{a} \in \bowtie^{\mathbf{A}} \to \underset{\rightarrow}{f}(\mathsf{a}) \in \bowtie^{\mathbf{B}}$, by assumption (3). $\boxed{\underset{\rightarrow}{f}\left[\bowtie^{\mathbf{A}}\right] \supseteq \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}}$ Let $\mathsf{b} \in \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$. (We must show that there exists $\mathsf{a} \in \bowtie^{\mathbf{A}}$ with $\underset{\rightarrow}{f}(\mathsf{a}) \in \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$.) Since $\mathsf{b} \in \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$, there exists $\mathsf{a} \in \underset{\longrightarrow}{\mathsf{uni}(\mathbf{A})}$ with $\underset{\rightarrow}{f}(\mathsf{a}) = \mathsf{b} \in \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$. (It suffices to show that $\mathsf{a} \in \bowtie^{\mathbf{A}}$.) Since $\underset{\rightarrow}{f}(\mathsf{a}) \in \bowtie^{\mathbf{B}}{}_{|\mathsf{rg}(f)}$, by assumption (3), $\mathsf{a} \in \bowtie^{\mathbf{A}}$. $\boxed{(4)\Rightarrow(1)}$ Follows trivially by Remark 1.279 on page 56. ⋄

The notion of strictness only adds additional restrictions on the behaviours of a homomorphism with regard to the relations, and does not restrict how the homomorphism behaves with respect to operations and constants. Homomorphisms are always 'strict' with respect to constants, but this is not generally the case for operations.

Counter Example 1.293 (Homomorphisms are not 'strict' for operations)

Consider the ring-homomorphism f from the additive group $\mathbb{Z}$ into itself mapping every integer to 0. While $f(1) + f(2) = 0 + 0 = 0 = f(0)$, it is not true that $1 + 2 = 0$.

□

Remark 1.294 The composition of strict structure homomorphisms is a strict homomorphism, and the identity function is a strict homomorphism.

Definition 1.295 (Constructs of Structures with Strict Homomorphisms) For an elementary type $\mathfrak{e}$, let $\mathfrak{c}^{\mathsf{s}}_{\mathfrak{e}}$ denote the construct of all (small) $\mathfrak{e}$-structures with strict homomorphisms. □

Remark 1.296 Bijective strict homomorphisms are isomorphisms.

Proof. Let f be a bijective strict homomorphism from $\mathbf{A}$ onto $\mathbf{B}$. (We must show that f^{-1} is a homomorphism from $\mathbf{B}$ onto $\mathbf{B}$.) We already know that f^{-1} is an algebra-homomorphism. Let $\bowtie \in \mathsf{Symb_r}(\mathfrak{e})$. (We must show that $f^{-1}\left[\bowtie^{\mathbf{B}}\right] \subseteq \bowtie^{\mathbf{A}}$.) But this is precisely the property that the strictness of f promises. ⋄

1.5.4.4 Reductive Homomorphisms

Definition 1.297 (Reductive Homomorphisms) A homomorphism f from $\mathbf{A}$ into $\mathbf{B}$ is called **reductive** (or a **contraction**) if f is a strict epimorphism, in which case $\mathbf{B}$ is called a **reduction** (or **contraction**) of $\mathbf{A}$ and $\mathbf{A}$ is called an **expansion** of $\mathbf{B}$. Structures $\mathbf{A}$ and $\mathbf{B}$ are called **relatives**, denoted $\mathbf{A} \backsim \mathbf{B}$, if they have isomorphic contractions. $\square$

1.5.4.5 Homomorphic Images

Definition 1.298 (Homomorphic Images) A structure $\mathbf{B}$ is called a **homomorphic image** of a structure $\mathbf{A}$ if there exists a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{B}$. $\square$

Definition 1.299 (The Class Operators $\mathcal{H}$, $\mathcal{C}$ and $\mathcal{E}$) Let $\mathcal{K}$ be a class of $\mathfrak{e}$-structures. Let $\mathcal{H}(\mathcal{K})$ ($\mathcal{C}(\mathcal{K})$, $\mathcal{E}(\mathcal{K})$) denote the class of all $\mathfrak{e}$-structures that are the homomorphic image (resp. contraction, expansion) of some member of $\mathcal{K}$. $\square$

1.5.5 Substructures and Subuniverses

Definition 1.300 (Substructures and Subalgebras) Let $\mathfrak{e}$ be an elementary type and let $\mathbf{A}$ and $\mathbf{B}$ be $\mathfrak{e}$-structures. We call $\mathbf{B}$ a **substructure** of $\mathbf{A}$, written $\mathbf{B} \lhd_{\mathfrak{e}} \mathbf{A}$ or $\mathbf{B} \lhd \mathbf{A}$, if $\mathsf{uni}(\mathbf{B}) \subseteq \mathsf{uni}(\mathbf{A})$, and

1. for each relation symbol $\bowtie$, $\bowtie^{\mathbf{B}} = \bowtie^{\mathbf{A}} \cap \mathsf{uni}(\mathbf{B})^{\mathsf{ar}(\bowtie)}$,
2. for each operation symbol $\star$, $\star^{\mathbf{B}}$ is the restriction of $\star^{\mathbf{A}}$ to $\mathsf{uni}(\mathbf{B})^{\mathsf{ar}(\star)}$, and
3. for each constant symbol $\mathbf{0}$, $\mathbf{0}^{\mathbf{B}} = \mathbf{0}^{\mathbf{A}}$.

If $\mathbf{B}$ is a substructure of $\mathbf{A}'$ and $\mathbf{A}'$ is the $\mathfrak{e}'$-reduct of $\mathbf{A}$, then $\mathbf{B}$ is called an **$\mathfrak{e}'$-subreduct** of $\mathbf{A}$. We say that $\mathbf{A}$ can be **embedded** into $\mathbf{B}$ if there exists an isomorphism of $\mathbf{A}$ onto a substructure of $\mathbf{B}$. $\square$

Definition 1.301 (The Class Operator $\mathcal{S}$) Let $\mathcal{K}$ be a class of $\mathfrak{e}$-structures. Let $\mathcal{S}(\mathcal{K})$ denote the class of all $\mathfrak{e}$-structures that are isomorphic to a substructure of some member of $\mathcal{K}$. $\square$

Definition 1.302 (Subuniverses) A subset $C \subseteq \mathsf{uni}(\mathbf{A})$ is called a **subuniverse** of $\mathbf{A}$, denoted $C \leq \mathbf{A}$, if C is closed under the fundamental operations and constants of $\mathbf{A}$, i.e., if $\mathbf{0} \in \mathsf{Symb}_{\mathsf{c}}(\mathbf{A})$ then $\mathbf{0}^{\mathbf{A}} \in C$, and if $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathbf{A})$ and $c_1, \ldots, c_{\mathsf{ar}(\star)} \in C$ then $\star^{\mathbf{A}}(c_1, \ldots, c_{\mathsf{ar}(\star)}) \in C$. Let $\mathsf{Su}(\mathbf{A})$ denote the set of all subuniverses of $\mathbf{A}$. The set $\mathsf{Su}(\mathbf{A})$ of all subuniverses of structure $\mathbf{A}$ determines an *algebraic* closed system. The associated algebraic lattice is denoted by $\mathbf{Su}(\mathbf{A})$. The associated algebraic closure operator is denoted by $\|\cdot\|^{\mathbf{A}}_{\mathsf{su}}$, for which we may drop the superscript $\mathbf{A}$ when unambiguous. We say that $X \subseteq \mathsf{uni}(\mathbf{A})$ **generates** $\mathbf{A}$ (or $\mathbf{A}$ is **generated by** X) if $\|X\|_{\mathsf{su}} = \mathsf{uni}(\mathbf{A})$. A substructure $\mathbf{B}$ of $\mathbf{A}$ is said to be **finitely generated** if $B = \|Y\|_{\mathsf{su}}$, for some finite $Y \subseteq B$. A structure $\mathbf{A}$ is called **finitely generated** if $\mathbf{A}$, considered as a substructure of $\mathbf{A}$, is finitely generated. $\square$

Remark 1.303 If $\mathbf{B}$ is a substructure of $\mathbf{A}$, then $\mathsf{uni}(\mathbf{B})$ is a subuniverse of $\mathbf{A}$.

Remark 1.304 The empty-set is a subuniverse iff $\mathbf{A}$ has no fundamental constants iff $\mathfrak{e}$ has no constant symbols, and this is the only case of a subuniverse that is not the universe of a substructure.

Warning 1.305 ($\mathsf{Su}_{\mathfrak{e}}(\mathbf{A}) \neq \mathsf{Su}(\mathbf{A})$) The notion of subuniverse given above does not coincide with the constructural notion of subuniverse given in Definition 1.223 on page 46, in that empty subuniverses are permitted while algebras with empty universes are proscribed (apparently for model-theoretic reasons, see Warning 1.271 on page 55). This is in keeping with both [BS81] and [RMT87]. Some universal algebraic texts define subuniverses of algebras in the manner of Definition 1.223, (i.e., subuniverses are the universes of subalgebras), while still positing a notion of subuniverse generation (for example [vA95] and this authors' [Bar95]). This is not entirely sound, since subuniverses defined in the latter manner need not form closed systems (see the following counter example). This 'mistake' is understandable. In category theory algebras with empty universes are permitted [Ada83, 23-24], thereby ensuring that constructs of algebras have subobject intersection [Ada83, 38].

Counter Example 1.306 (Non-Empty Subuniverse need not form Closed Systems)

Consider the unary-algebra on $\{a, b\}$ with $\boldsymbol{u}(a) = a$ and $\boldsymbol{u}(b) = b$. Both $\{a\}$ and $\{b\}$ are subuniverses, but their intersection is empty.

□

A useful characterization of subuniverse generation, via closure under term functions, is given in Theorem 1.344 on page 65.

Remark 1.307 If $\mathbf{B}$ and $\mathbf{C}$ are both substructures of $\mathbf{A}$ and $\mathsf{uni}(\mathbf{C}) \subseteq \mathsf{uni}(\mathbf{B})$, then $\mathbf{C}$ is a substructure of $\mathbf{B}$.

Remark 1.308 Since $\mathbf{Su}(\mathbf{A})$ is an algebraic lattice, the compact elements of $\mathbf{Su}(\mathbf{A})$ are precisely the universes of the finitely generated subuniverses of $\mathbf{A}$.

Theorem 1.309 [BS81] Let $f : \mathbf{A} \to \mathbf{B}$.

1. If $\mathbf{A}$ is generated by X and $f' : \mathbf{A} \to \mathbf{B}$ with $f_{|X} = f'_{|X}$, then $f = f'$.
2. $\forall\, [U \in \mathsf{Su}(\mathbf{A})]\ f[U] \in \mathsf{Su}(\mathbf{B})$.
3. $\forall\, [V \in \mathsf{Su}(\mathbf{B})]\ f^{-1}[V] \in \mathsf{Su}(\mathbf{A})$.

□

Notice how condition 3. of the previous theorem is 'like' the notion of continuity in the theory of topological closed system, viewing the subuniverses of an algebra as the closed sets of a 'closed system'. It is this property, and others like it, that motivates our study in Part II of 'continuous functions' between closed systems. In particular, see Example 5.44 on page 188, which follows from the previous theorem, and which is key in giving rise to the logic of subuniverses of Example 6.80 on page 242.

The following definition of the image and pre-image structures is well-defined by (2) and (3) of the previous theorem.

Definition 1.310 (Image Structures) For a homomorphism f from structure $\mathbf{A}$ into structure $\mathbf{B}$, the substructure of $\mathbf{B}$ with universe $f\left[\mathsf{uni}(\mathbf{A})\right]$ is denoted by $f\left[\mathbf{A}\right]$, the substructure of $\mathbf{A}$ with universe $f^{-1}\left[\mathsf{uni}(\mathbf{B})\right]$ is denoted by $f^{-1}\left[\mathbf{B}\right]$, which we call the **image structure** and **pre-image structure** respectively. □

Example 1.311 (Unary Submatrices)

Substructures of unary $\mathbf{A}$-matrices are called **submatrices**.

□

1.5.6 Products

1.5.6.1 Direct Products

Definition 1.312 (Direct Products of Algebra) [BS81] Let $\langle \mathbf{A}_i : i \in I \rangle$ be an indexed system of $\mathfrak{a}$-algebras. The **direct product** of $\langle \mathbf{A}_i : i \in I \rangle$, denoted $\prod_I \mathbf{A}_i$, is the algebra with universe $\prod_I \mathsf{uni}(\mathbf{A}_i)$ such that, for each $\star \in \mathsf{Symb_o}(\mathfrak{a})$ and $\mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{ar}(\star)} \in \prod_I \mathsf{uni}(\mathbf{A}_i)$, $\star^{\mathbf{A}}(\mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{ar}(\star)}) = \langle \star^{\mathbf{A}_i}(\mathsf{a}_{1(i)}, \ldots, \mathsf{a}_{\mathsf{ar}(\star)_{(i)}}) : i \in I \rangle$. If $I = \{1, 2, \ldots, n\}$ we also write $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ for $\prod_I \mathbf{A}_i$ and $\mathbf{A}^n$ for $\mathbf{A} \times \cdots \times \mathbf{A}$ (n 'factors'). □

Remark 1.313 Observe that the direct product $\prod \emptyset$ of the empty system of $\mathfrak{a}$-algebras is a trivial algebra of type $\mathfrak{a}$.

Remark 1.314 For an indexed system $\langle \mathbf{A}_i : i \in I \rangle$ of $\mathfrak{a}$-algebras and $j \in I$, the j-th projection map $\boldsymbol{\pi}_j$ is a homomorphism from $\prod_I \mathbf{A}_i$ onto $\mathbf{A}_j$.

Remark 1.315 The construct $\mathfrak{a}$ has products. These products are precisely those of Definition 1.312.

Remark 1.316 If f is a homomorphism from algebra $\mathbf{A}$ into $\mathbf{B}$, then $\underset{\rightarrow}{f}$ is a homomorphism from $\mathbf{A}^I$ into $\mathbf{B}^I$.

Remark 1.317 If α is a congruence on algebra $\mathbf{A}$, then $\underset{\rightarrow}{\alpha}$ is a congruence on $\mathbf{A}^I$.

Definition 1.318 (Direct Products of Structures) [BS81, 204] Let $\langle \mathbf{A}_i : i \in I \rangle$ be an indexed system of $\mathfrak{e}$-structures. The **direct product** of $\langle \mathbf{A}_i : i \in I \rangle$, denoted $\prod_I \mathbf{A}_i$, is the structure with universe $\prod_I \mathsf{uni}(\mathbf{A}_i)$, algebra reduct $\prod_I \mathbf{A}_i$, and whose fundamental relations are defined by $\bowtie^{\prod_I \mathbf{A}_i}(\mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{ar}(\bowtie)})$ iff $\forall\,[i \in I]\ \bowtie^{\mathbf{A}_i}(\mathsf{a}_{1(i)}, \ldots, \mathsf{a}_{\mathsf{ar}(\bowtie)_{(i)}})$, for all relation symbols $\bowtie \in \mathsf{Symb_r}(\mathfrak{e})$. □

Remark 1.319 [Elg98] The projection functions, which we already know to be algebra homomorphisms, are in fact structure homomorphisms, although not generally strict.

Definition 1.320 (The Class Operator $\mathcal{P}$) Let $\mathcal{K}$ be a class of $\mathfrak{e}$-structures. Let $\mathcal{P}(\mathcal{K})$ denote the class of all $\mathfrak{e}$-structures that are direct products of some members of $\mathcal{K}$. □

1.5.6.2 Subdirect Products

Definition 1.321 (Subdirect Products and Embeddings) [Elg98] A structure $\mathbf{A}$ is called a **subdirect product** of the indexed system $\langle \mathbf{A}_i \,:\, i \in I \rangle$ of structures if $\mathbf{A}$ is a substructure of $\prod_I \mathbf{A}_i$ and, for each $j \in I$, $\boldsymbol{\pi}_{j|A}$ is surjective. We say that a monomorphism f from $\mathbf{A}$ into $\prod_I \mathbf{A}_i$ is a **subdirect embedding**, if $f[\mathbf{A}]$ is a subdirect product of $\langle \mathbf{A}_i \,:\, i \in I \rangle$. □

Remark 1.322 [Elg98] Subdirect embeddings are always strict, and so the name embedding is justified.

Definition 1.323 (Subdirect Irreducibility) [Elg98] A structure $\mathbf{A}$ is called **completely subdirectly irreducible** if, for every subdirect embedding $f : \mathbf{A} \to \prod_I \mathbf{A}_i$, there exists $j \in I$ such that $\boldsymbol{\pi}_j f : \mathbf{A} \to \mathbf{A}_j$ is an isomorphism. Completely subdirectly irreducible *algebras* are also called **subdirectly irreducible**. □

Remark 1.324 [BS81] Birkhoff proved that every *algebra* $\mathbf{A}$ is isomorphic to a subdirect product of subdirectly irreducible homomorphic images of $\mathbf{A}$.

Definition 1.325 (The Class Operator $\mathcal{P}_S$) Let $\mathcal{K}$ be a class of $\mathfrak{e}$-structures. Let $\mathcal{P}_S(\mathcal{K})$ denote the class of all $\mathfrak{e}$-structures that are subdirect products of some members of $\mathcal{K}$. □

1.5.7 Terms, Term Algebras, Term Functions and Polynomials

An important algebra, known as the *term algebra*, is definable on the set of terms of any elementary language or type. In the case of an algebraic type $\mathfrak{a}$, the term algebra also has type $\mathfrak{a}$. Generally, the term algebra of an elementary type $\mathfrak{e}$ has algebraic type $\mathfrak{e}_{|\mathsf{o}}$.

Definition 1.326 (Terms) Let $\mathfrak{e}$ be an elementary type and let V be a set of **variables**. The **$\mathfrak{e}$-term over** V are given by the production-rule

$$p ::= x \,|\, \mathbf{0} \,|\, \star(p_1, \ldots, p_n),$$

where x ranges over V, $\mathbf{0}$ ranges over $\mathsf{Symb_c}(\mathfrak{e})$, $\star$ ranges over $\mathsf{Symb_o}(\mathfrak{e})$ of non-zero arity, and $n = \mathsf{ar}(\star)$. (See the subsequent convention for an explanation of this **production-rule** notation). Brackets are used, conventionally, to resolve precedence. We shall often write a term p as $p(x_1, \ldots, x_n)$ to indicate that the variables occurring in p are among $\{x_1, \ldots, x_n\}$. A term p is called n**-ary** if the number of variables appearing explicitly in the formal string p is at most n. An n-ary term p may be assumed to be an m-ary term, where integer $m \geq n$, provided the set of variables V is large enough. The number of distinct variables occurring in a term p is called the **arity** of the term, and is denoted by $\mathsf{ar}(p)$. When we say that $P(x_1, \ldots, x_n)$ is a set of terms, we shall mean that P is a set of terms, and that for each $p \in P$, the variables occurring in p are among $\{x_1, \ldots, x_n\}$. We denote all **$\mathfrak{e}$-terms over** V by $\mathsf{Tm}_V^{\mathfrak{e}}$, and the set of all $\mathfrak{e}$-terms over V of arity n is denoted by $\mathsf{Tm}_V^{\mathfrak{e}}(n)$, dropping the superscript $\mathfrak{e}$ when context unambiguous or unimportant.

It is convenient to assume that $\mathsf{V} = \{\mathsf{v}_0, \mathsf{v}_1, \ldots\}$ is an arbitrary, fixed and globally chosen *denumerably infinite* set of variables. When we drop a variable set parameter from notations, for example when we write $\mathsf{Tm}^{\mathfrak{e}}$ or Tm, we shall implicitly mean the variable set V. □

Convention 1.327 (Production Rules) We require that the reader be able to interpret **production-rule notation**, and that they do so *without* us giving a formal definition. Computer scientists and logicians are fluent with this convention [BdRV01].

Informally, we read the above definition as follows. When parsing a finite symbol sequence, it is an $\mathfrak{e}$-term over V if either, (i) it is an element of V, or, (ii) it is an $\mathfrak{e}$-constant symbol, or, (iii) the sequence starts with a symbol $\star$, where $\star$ is an $\mathfrak{e}$-operation symbol of arity $n > 0$, and is followed by a symbol sequence of the form $(p_1, \ldots, p_n)$, where for each $i = 1, \ldots, n$, the symbol sequence p_i can be recognized itself as an $\mathfrak{e}$-term by this algorithm.

The key to interpreting such notation, is to recognize which 'symbols' of the production-rule are 'variables', after the ranges of the other 'symbols' have been specified.

Remark 1.328 If every variable of V is a variable of W, then every $\mathfrak{e}$-term over V is an $\mathfrak{e}$-term over W.

Example 1.329 (Groupoid and Group Terms)

> x, y, $x * x$, $y * y$, $x * y$, $y * x$ and $x * (x * y)$, are all groupoid terms over variables $\{x, y\}$. All groupoid terms are group terms. $\mathbf{1}$, $y^{-\mathbf{1}}$, $x*\mathbf{1}$, $y*y^{-\mathbf{1}}$, $x * y^{-\mathbf{1}}$ and $((x * \mathbf{1})^{-\mathbf{1}})*((y*\mathbf{1})*y^{-\mathbf{1}})$, are all group terms over variables $\{x, y\}$, but not groupoid terms.

□

Remark 1.330 There are no $\mathfrak{e}$-terms over V iff V is empty *and* $\mathfrak{e}$ has no constant symbols.

Remark 1.331 Constants and variables are the only terms arising from relational types.

Definition 1.332 (Term Algebras) Given any elementary type $\mathfrak{e}$ and a set of variables V, if $\mathsf{Tm}_V^{\mathfrak{e}} \neq \emptyset$, then the **$\mathfrak{e}$-term algebra over** V (or the **absolutely free $\mathfrak{e}$-term algebra** over V), denoted $\mathbf{Tm}_V^{\mathfrak{e}}$, is the $\mathfrak{e}_{|\mathsf{o}}$-algebra, with universe $\mathsf{Tm}_V^{\mathfrak{e}}$, whose $\mathfrak{e}$-terms over V fundamental operations are defined by

$$\star^{\mathbf{Tm}_V}(p_1, \ldots, p_{\mathsf{ar}(\star)}) = \star(p_1, \ldots, p_{\mathsf{ar}(\star)}),$$

for each $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathfrak{e})$ and $p_1, \ldots, p_{\mathsf{ar}(\star)} \in \mathsf{Tm}_V^{\mathfrak{e}}$. We drop the subscript V in the case that V is our standard denumerably infinite variable set V. □

Remark 1.333 $\mathbf{Tm}_V$ exists if and only if $V \cup \mathsf{Symb}_{\mathsf{c}}(\mathfrak{e}) \neq \emptyset$.

Convention 1.334 (Existence of Term Algebras) Conventionally, when we speak about a term algebra without explicit reference to its existence, we mean implicitly that it exists.

Remark 1.335 If $t(x_1, \ldots, x_n) \in \mathsf{Tm}_V^{\mathfrak{e}}$, then $t^{\mathbf{Tm}_V}(x_1, \ldots, x_n) = t$.

Remark 1.336 $\mathbf{Tm}_V$ is generated by V.

Definition 1.337 (Substitutions, Transpositions and Invariance) Endomorphisms of $\mathbf{Tm}$ are called **substitutions**. The substitution that transposes two variables y and z and leaves all other variables fixed shall be called the **transposition of** y **and** z, denoted (y, z). A set of terms P shall be called **invariant** (resp. p**-invariant**, where p is a term), if $\sigma[P] \subseteq P$ for every substitution σ (resp, every substitution σ that fixes p, i.e., that has $\sigma(p) = p$). □

Definition 1.338 (Interpreting Terms) For a structure $\mathbf{A}$ and term $p(x_1, \ldots, x_{\mathsf{ar}(p)})$, we recursively define an $\mathsf{ar}(p)$-ary operation $p^{\mathbf{A}}$ on $\mathsf{uni}(\mathbf{A})$, called the **term function**, by

1. $p^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(p)}) = \mathbf{0}^{\mathbf{A}}$, if $p = \mathbf{0} \in \mathsf{Symb_c}(\mathfrak{a})$;
2. $p^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(p)}) = a_i$, if $p(x_1, \ldots, x_{\mathsf{ar}(p)}) = x_i$;
3. $p^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(p)}) = \star^{\mathbf{A}}(p_1^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(p)}), \ldots, p_{\mathsf{ar}(\star)}^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(p)}))$, if $p(x_1, \ldots, x_{\mathsf{ar}(p)}) = \star(p_1(x_1, \ldots, x_{\mathsf{ar}(p)}), \ldots, p_{\mathsf{ar}(\star)}(x_1, \ldots, x_{\mathsf{ar}(p)}))$ for some non-constant operation symbol $\star$ and terms $p_1, \ldots, p_{\mathsf{ar}(\star)}$.

For a set of terms $P(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in \mathsf{uni}(\mathbf{A})$, we write $P^{\mathbf{A}}(a_1, \ldots, a_n)$ for $\{p^{\mathbf{A}}(a_1, \ldots, a_n) : p \in P\}$. $\square$

Remark 1.339 Term functions are **compatible with homomorphisms**. That is, for homomorphism $f : \mathbf{A} \to \mathbf{B}$ and term p, $f(p^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(p)})) = p^{\mathbf{B}}(f(a_1), \ldots, f(a_{\mathsf{ar}(p)}))$.

Definition 1.340 (Polynomials) Let $\mathbf{A}$ be an $\mathfrak{c}$-structure, $n, m \in \omega$, p an $(n+m)$-ary term and $a_1, \ldots, a_m \in \mathsf{uni}(\mathbf{A})$. Define an n-ary operation U on $\mathsf{uni}(\mathbf{A})$ by $U(e_1, \ldots, e_n) = p^{\mathbf{A}}(e_1, \ldots, e_n, a_1, \ldots, a_m)$. Such operations are called **polynomials** on $\mathbf{A}$. Let $\mathsf{Pol}^{\mathrm{n}}_{\mathbf{A}}(\mathbf{A})$ denote the set of all n-ary polynomials on $\mathbf{A}$. $\square$

Remark 1.341 Clearly, $\mathsf{Pol}^1_{\mathfrak{c}}(\mathbf{A}) \subseteq \mathsf{Pol}^2_{\mathfrak{c}}(\mathbf{A}) \subseteq \ldots$, and the term function $t^{\mathbf{A}}$ is an n-ary polynomial on $\mathbf{A}$, for each n-ary term $t \in \mathsf{Tm}(n)$.

Remark 1.342 Every n-ary polynomial on $\mathbf{A}$ may be considered as an m-ary polynomial on $\mathbf{A}$, for integer $m \geq n$.

Remark 1.343 Polynomials, like term functions, are compatible with homomorphisms.

Theorem 1.344 ([RMT87]) Let $\mathbf{A}$ be an algebra of type $\mathfrak{a}$, $S \subseteq \mathsf{uni}(\mathbf{A})$ and $s_1, \ldots, s_n \in \mathsf{uni}(\mathbf{A})$. Then $\|S\|^{\mathbf{A}}_{\mathsf{su}} = \{t^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(t)}) : a_1, \ldots, a_{\mathsf{ar}(t)} \in \mathsf{uni}(\mathbf{A}), t \in \mathsf{Tm}\}$ and $\|s_1, \ldots, s_n\|^{\mathbf{A}}_{\mathsf{su}} = \{t^{\mathbf{A}}(s_1, \ldots, s_n) : t \in \mathsf{Tm}(n)\}$.

Proposition 1.345 [BP92, Pr 5.1][vA95, Pr 1.8.2 (ii)] Let f be a reduction from $\mathbf{A}$ onto $\mathbf{B}$. Then, for any *algebra*-homomorphism g from the (absolutely free) term algebra $\mathbf{Tm}$ into $\mathsf{alg}(\mathbf{A})$, any relation symbol $\bowtie$ and any $\boldsymbol{p} \in \mathsf{Tm}^{\mathsf{ar}(\bowtie)}$, $\underset{\rightarrow}{g}(\boldsymbol{p}) \in \bowtie^{\mathbf{A}} \leftrightarrow \underset{\rightarrow}{f}(\underset{\rightarrow}{g}(\boldsymbol{p})) \in \bowtie^{\mathbf{B}}$.

Example 1.346 (Unary Algebra-Matrix Homomorphisms) [BP92, Pr 5.1][vA95, Pr 1.8.2 (ii)]

> A matrix-homomorphism f from $\mathbf{M}$ into $\mathbf{N}$ is reductive iff f is surjective and $\mathsf{D_M} = f^{-1}[\mathsf{D_N}]$, in which case for any term p and *algebra*-homomorphism g from the (absolutely free) term algebra $\mathbf{Tm}$ into $\mathsf{alg}(\mathbf{M})$, $g(p) \in \mathsf{D_M}$ iff $f(g(p)) \in \mathsf{D_N}$.

$\square$

1.5.8 Quotients and Congruences

Recall that for an object $\mathbf{A}$ of a construct $\mathfrak{s}$ and an equivalence relation α on the universe of that object, the equivalence α is called a congruence if a (at least one) $\mathfrak{s}$-quotient object of $\mathbf{A}$ by α exists (see Definition 1.240 on page 48). For an elementary type $\mathfrak{e}$, the construct of all $\mathfrak{e}$-structures, being transportable, has *unique* quotient objects, when they exist (see Remark 1.241 on page 49).

1.5.8.1 Quotients of Relational Structures

Proposition 1.347 (Constructural Quotients of Relational Structures) Let $\mathfrak{r}$ be a type of relational structures. The construct $\mathfrak{r}$ is (transportable) and cohereditary. For each $\mathfrak{r}$-structure $\mathbf{A}$ and an equivalence relation α on $\mathsf{uni}(\mathbf{A})$, $\mathbf{A}/\alpha$ is the $\mathfrak{r}$-structure defined by $\bowtie^{\mathbf{A}/\alpha} = \mathsf{q}_\alpha\left[\bowtie^{\mathbf{A}}\right]$, for each relation symbol $\bowtie$.

Proof. $\boxed{(1)}$ (We must show that q_α is a homomorphism of $\mathbf{A}$ onto $\mathbf{A}/\ \alpha$.) Let $\bowtie$ be a relation symbol. (We must show that $\mathsf{q}_\alpha\left[\bowtie^{\mathbf{A}}\right] \subseteq \bowtie^{\mathbf{A}/\alpha}$.) But by definition, $\bowtie^{\mathbf{A}/\alpha} = \mathsf{q}_\alpha\left[\bowtie^{\mathbf{A}}\right]$, so *certainly*, $\mathsf{q}_\alpha\left[\bowtie^{\mathbf{A}}\right] \subseteq \bowtie^{\mathbf{A}/\alpha}$. $\boxed{(2)}$ Assume that $\mathbf{B}$ is a $\mathfrak{r}$-structure and that f a function from $\coprod \alpha$ into $\mathsf{uni}(\mathbf{B})$, such that $f\mathsf{q}_\alpha$ is a structure homomorphism of $\mathbf{A}$ in $\mathbf{B}$. (We must show that f is a homomorphism of $\mathbf{A}/\ \alpha$ into $\mathbf{B}$.) Let $\bowtie$ be a relation symbol. $f\left[\bowtie^{\mathbf{A}/\alpha}\right] = f\left[\mathsf{q}_\alpha\left[\bowtie^{\mathbf{A}}\right]\right] = (f\mathsf{q}_\alpha)\left[\bowtie^{\mathbf{A}}\right] \subseteq \bowtie^{\mathbf{B}}$, the final inclusion following since $f\mathsf{q}_\alpha$ is a homomorphism of $\mathbf{A}$ in $\mathbf{B}$, by assumption. ◇

So from a constructural perspective, if $\mathfrak{r}$ is a relational type, then every $\mathfrak{r}$-structure may be (uniquely) factored by *any* equivalence relation on its universe. In other words, equivalence relations and constructural $\mathfrak{r}$-congruences coincide. (Of course, for *special* subconstructs of the construct of *all* $\mathfrak{r}$-structures, not all equivalences need be congruences.)

Remark 1.348 For a relational structure $\mathbf{A}$, the quotient map is a *contraction* (i.e., strict epimorphism) of $\mathbf{A}$ onto $\mathbf{A}/\ \alpha$, and so $\mathbf{A}/\ \alpha$ is a *reduction* of $\mathbf{A}$.

1.5.8.2 Congruences

In contrast to relational structures, not all equivalence relations on algebras admit quotients. The operations of algebras, when viewed as relations, have special properties arising from their functional nature, which relies heavily on the properties of the equality relation. Viewing these operations as relations, when one factors by an arbitrary equivalence relation (in the relational sense), the resulting relations on the factor structure need not be *operations*, i.e., the operational result need not be unique. Consequently one needs to distinguish which equivalence relations are *congruences.*

Definition 1.349 (Congruences) [Elg98, 208] Let $\mathbf{A}$ be a structure. A binary relation α on $\mathsf{uni}(\mathbf{A})$ is called a **compatible relation** on/of $\mathbf{A}$ if α **preserves the fundamental operations** of $\mathbf{A}$ (we may also say that the fundamental operations are **compatible with** α), that is, for all $\star \in \mathsf{Symb_o}(\mathbf{A})$,

$$a_1\ \alpha\ b_1, \ldots, a_{\mathsf{ar}(\star)}\ \alpha\ b_{\mathsf{ar}(\star)} \text{ implies } \star^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(\star)})\ \alpha\ \star^{\mathbf{A}}(b_1, \ldots, b_{\mathsf{ar}(\star)}),$$

and α is **compatible with the fundamental relations** of $\mathbf{A}$, that is, for each $\bowtie \in \mathsf{Symb_r}(\mathbf{A})$,

$$\mathtt{a} \in \bowtie^{\mathbf{A}} \text{ and } \mathtt{a} \underset{\rightarrow}{\alpha} \mathtt{b} \text{ implies } \mathtt{b} \in \bowtie^{\mathbf{A}}.$$

A compatible equivalence relation on $\mathbf{A}$ is called a **congruence relation** (or just a **congruence**) on $\mathbf{A}$. The set of all congruence relations (resp. compatible relations) on $\mathbf{A}$ is denoted by $\mathsf{Con}(\mathbf{A})$ (resp. $\mathsf{Cpat}(\mathbf{A})$). The congruence classes of congruences of $\mathbf{A}$ are called **cosets of A**. □

Remark 1.350 The compatible relations on an *algebra* $\mathbf{A}$ are precisely the subuniverses of $\mathbf{A}^2$, and hence $\mathsf{Con}(\mathbf{A}) \subseteq \mathsf{Su}(\mathbf{A}^2)$.

Remark 1.351 Binary compatible relations **preserve term functions**. That is, if α is a compatible relation on $\mathbf{A}$ and p is a term, then, $a_1 \ \alpha \ b_1, \ldots, a_{\mathsf{ar}(p)} \ \alpha \ b_{\mathsf{ar}(p)}$ implies $p^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(p)}) \ \alpha \ p^{\mathbf{A}}(b_1, \ldots, b_{\mathsf{ar}(p)})$. They also **preserve polynomials**.

Theorem 1.352 ([RMT87]) Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathfrak{a}$-algebras, $t(x_1, \ldots, x_n)$ an n-ary $\mathfrak{a}$-term, α a binary compatible relation on $\mathbf{A}$ and $h : \mathbf{A} \to \mathbf{B}$ a homomorphism. If $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \alpha$, then $\langle t^{\mathbf{A}}(a_1, \ldots, a_n), t^{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \alpha$, and if $a_1, \ldots, a_n \in \mathsf{uni}(\mathbf{A})$, then $h(t^{\mathbf{A}}(a_1, \ldots, a_n)) = t^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$. □

Note that the following result pertains *only to algebras* and *not* structures more generally (see Theorem 1.360 on page 68).

Remark 1.353 For an *algebra* $\mathbf{A}$, $\mathsf{Con}(\mathbf{A})$ and $\mathsf{Cpat}(\mathbf{A})$ form algebraic closed systems over $\mathsf{uni}(\mathbf{A})^2$.

Definition 1.354 (The Closed System of Congruences on an Algebra) Let $\mathbf{A}$ be an *algebra*. We denote the algebraic closure operator associated with the algebraic closed system $\mathsf{Con}(\mathbf{A})$ by $\|\cdot\|_{\Theta_{\mathbf{A}}}$, and the associated inclusion-ordered algebraic lattice by $\mathbf{Con}(\mathbf{A})$. The least element of $\mathbf{Con}(\mathbf{A})$ is the diagonal relation $=_{\mathsf{uni}(\mathbf{A})}$ on $\mathsf{uni}(\mathbf{A})$, which we denote by $\perp_{\mathbf{A}}$, and the greatest element is the square relation $\blacksquare_{\mathsf{uni}(\mathbf{A})}$ on $\mathsf{uni}(\mathbf{A})$, which we denote by $\blacksquare_{\mathbf{A}}$. □

The generation of congruences is described by Mal'cev's Lemma, of which we require the following simple corollary.

Lemma 1.355 ([Dud83],[Mal54]) For an algebra $\mathbf{A}$, a non-empty binary relation α on $\mathsf{uni}(\mathbf{A})$ and $c, d \in \mathsf{uni}(\mathbf{A})$, the following conditions are equivalent.

1. $\langle c, d\rangle \in \|\alpha\|_{\Theta_{\mathbf{A}}}$.

2. There exist binary polynomials $\mathsf{B}_1, \ldots, \mathsf{B}_n \in \mathsf{Pol}_2(\mathbf{A})$ and pairs $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \alpha$, for some integer $n \geq 1$, such that $c = \mathsf{B}_1(a_1, b_1)$, $\mathsf{B}_i(b_i, a_i) = \mathsf{B}_{i+1}(a_{i+1}, b_{i+1})$, for $1 \leq i < n$, and $\mathsf{B}_n(b_n, a_n) = d$.

3. There exist unary polynomials $\mathsf{U}_1, \ldots, \mathsf{U}_n \in \mathsf{Pol}_1(\mathbf{A})$ and pairs $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \alpha \cup \overleftarrow{\alpha}$, for some integer $n \geq 1$, such that $c = \mathsf{U}_1(a_1)$, $\mathsf{U}_i(b_i) = \mathsf{U}_{i+1}(a_{i+1})$, for $1 \leq i < n$, and $\mathsf{U}_n(b_n) = d$.

Corollary 1.356 [Mal54] If $\mathbf{A}$ is an algebra and $\emptyset \neq Y \subseteq \mathsf{uni}(\mathbf{A})$, then Y is a congruence class of some congruence on $\mathbf{A}$ iff, for any $a, b \in Y$ and any unary polynomial U of $\mathbf{A}$, whenever $\mathsf{U}(a) \in Y$, we have $\mathsf{U}(b) \in Y$. □

Theorem 1.357 [BS81] Let $\mathbf{A}$ and $\mathbf{B}$ be *algebras* and $f : \mathbf{A} \to \mathbf{B}$. The following conditions are valid.

1. $\forall\,[\alpha \in \mathsf{Con}(\mathbf{B})]\ \underset{\rightarrow}{f}^{-1}[\alpha] \in \mathsf{Con}(\mathbf{A})$.
2. $\equiv_f \in \mathsf{Con}(\mathbf{A})$.

□

Observe how the first condition of the previous theorem is analogous to the condition of continuity in a topological closed system, when one views $\mathsf{Con}(\mathbf{A})$ and $\mathsf{Con}(\mathbf{B})$ as constituting the closed sets of a closed system. Of course, the set of congruences on $\mathbf{A}$, while forming a closed system on $\mathsf{uni}(\mathbf{A})^2$, generally do not constitute the closed sets of a *topological* closed system; the empty set is not a congruence, and, more importantly, the union of two congruences need not itself be a congruence.

Proposition 1.358 If $\mathbf{A}$ and $\mathbf{B}$ are *algebras*, $f : \mathbf{A} \to \mathbf{B}$ and $\alpha \in \mathsf{Con}(\mathbf{A})$, then $\underset{\rightarrow}{f}[\alpha] \in \mathsf{Con}(\mathbf{B})$.

Proof. Let $\star$ be an operation symbol, and suppose that $\langle c_i, d_i\rangle \in \underset{\rightarrow}{f}[\alpha]$, for $1 \leq i \leq \mathsf{ar}(\star)$. (*We must show that* $\langle \star^{\mathbf{B}}(c_1, \ldots, c_{\mathsf{ar}(\star)}), \star^{\mathbf{B}}(d_1, \ldots, d_{\mathsf{ar}(\star)})\rangle \in \underset{\rightarrow}{f}[\alpha]$.) By definition, for each $1 \leq i \leq \mathsf{ar}(\star)$, there exist $\langle a_i, b_i\rangle \in \alpha$ with $f(a_i) = c_i$ and $f(b_i) = d_i$. Hence $\star^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(\star)})\,\alpha\,\star^{\mathbf{A}}(b_1, \ldots, b_{\mathsf{ar}(\star)})$ and so $f(\star^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(\star)}))\ \underset{\rightarrow}{f}[\alpha]\ f(\star^{\mathbf{A}}(b_1, \ldots, b_{\mathsf{ar}(\star)}))$. Hence $\star^{\mathbf{B}}(c_1, \ldots, c_{\mathsf{ar}(\star)}) = \star^{\mathbf{B}}(f(a_1), \ldots, f(a_{\mathsf{ar}(\star)})) = f(\star^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(\star)}))\ \underset{\rightarrow}{f}[\alpha]\ f(\star^{\mathbf{A}}(b_1, \ldots, b_{\mathsf{ar}(\star)})) = \star^{\mathbf{B}}(f(b_1), \ldots, f(b_{\mathsf{ar}(\star)})) = \star^{\mathbf{B}}(d_1, \ldots, d_{\mathsf{ar}(\star)})$. ◇

Definition 1.359 (Congruential Properties) Let $n \geq 2$ be an integer. An algebra $\mathbf{A}$ is said to be **congruence n-permutable** if $(\alpha \circ^n \beta) = (\beta \circ^n \alpha)$, for all $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$. A class $\mathcal{K}$ of algebras is said to be *congruence n-permutable* if every algebra in $\mathcal{K}$ is congruence n-permutable. In the literature, congruence 2-permutable algebras are usually called **congruence permutable**. An algebra $\mathbf{A}$ is called **congruence distributive** (resp. **congruence modular**) if **Con**$(\mathbf{A})$ is a distributive (resp. modular) lattice. An algebra $\mathbf{A}$ is called **fully congruence regular** (or just **congruence regular**) if, for all $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$ and $a \in \mathsf{uni}(\mathbf{A})$, if $\alpha[\![a]\!] = \beta[\![a]\!]$ then $\alpha = \beta$. □

1.5.8.3 The Leibniz Relation

Except for trivial structures, the congruences on a structure do not form a closed system. In fact, except for trivial cases, the square relation $\blacksquare_{\mathsf{uni}(\mathbf{A})}$ on a structure $\mathbf{A}$ is not a congruence [Elg98, 208],[BP89a, T1.5]. There is always, however, a greatest congruence on a structure. This congruence is known as the *Leibniz relation*.

Theorem 1.360 [Elg98, 208],[BP89a, T1.5] For a structure $\mathbf{A}$, $\mathsf{Con}(\mathbf{A})$ is a principal ideal of $\mathsf{Con}(\mathsf{alg}(\mathbf{A}))$.

Definition 1.361 (The Leibniz Relation of a Structure) For a structure $\mathbf{A}$, let $\Omega_{\mathbf{A}}$ denote the largest element of $\mathsf{Con}(\mathbf{A})$. $\square$

Remark 1.362 By definition, $\Omega_{\mathbf{A}}$ is the largest $\mathsf{alg}(\mathbf{A})$ congruence compatible with the fundamental relations of $\mathbf{A}$. $\square$

The following result follows from Theorem 1.360 and Remark 4.80 on page 157.

Corollary 1.363 For a structure $\mathbf{A}$, $\mathsf{Con}(\mathbf{A})$ is an algebraic closed system on $\Omega_{\mathbf{A}}$.

Proposition 1.364 ([Elg97],[BP92]) Let $\mathbf{A}$ and $\mathbf{B}$ be two $\mathfrak{e}$-structures and f a homomorphism from $\mathbf{A}$ into $\mathbf{B}$.

1. If f is strict then, for all $\alpha \in \mathsf{Con}(\mathbf{A})$, $\underset{\rightarrow}{f}^{-1}[\alpha] \in \mathsf{Con}(\mathbf{A})$.
2. If f is a strict epimorphism, then, for all $\alpha \in [\equiv_f\rangle_{\mathbf{Con}(\mathbf{A})}$, $\underset{\rightarrow}{f}[\alpha] \in \mathsf{Con}(\mathbf{B})$, and consequently, $\equiv^{\Omega}_{\mathbf{A}} = \underset{\rightarrow}{f}^{-1}[\equiv^{\Omega}_{\mathbf{B}}]$ and $\underset{\rightarrow}{f}[\equiv^{\Omega}_{\mathbf{A}}] = \equiv^{\Omega}_{\mathbf{B}}$.

$\square$

1.5.8.4 Quotients

The compatibility property of a congruence allows one to introduce a structure on the quotient of the universe.

Definition 1.365 (Quotient Structures) Let $\mathbf{A}$ be an $\mathfrak{e}$-structure and $\alpha \in \mathsf{Con}(\mathbf{A})$. Define an $\mathfrak{a}$-structure $\mathbf{A}/\alpha$, on universe A/α, with

$$\begin{aligned} \mathbf{0}^{\mathbf{A}/\alpha} &= \alpha[\![\mathbf{0}^{\mathbf{A}}]\!], \\ \star^{\mathbf{A}/\alpha}(\mathfrak{u}_1, \ldots, \mathfrak{u}_{\mathsf{ar}(\star)}) &= \alpha\left[\star^{\mathbf{A}}\left[\mathfrak{u}_1 \times \ldots \times \mathfrak{u}_{\mathsf{ar}(\star)}\right]\right] \qquad \text{and} \\ \bowtie^{\mathbf{A}/\alpha} &= \bowtie^{\mathbf{A}}/\alpha, \end{aligned}$$

for each $\mathbf{0} \in \mathsf{Symb}_{\mathsf{c}}(\mathfrak{e})$, $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathfrak{e})$, $\bowtie \in \mathsf{Symb}_{\mathsf{r}}(\mathfrak{e})$. $\mathbf{A}/\alpha$ is indeed an $\mathfrak{a}$-structure, called the **quotient structure** of $\mathbf{A}$ by α. Let $\mathbf{A}$ be a structure and $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$ with $\alpha \subseteq \beta$. Define $\beta/\alpha = \{\langle a/\alpha, b/\alpha\rangle \in (A/\alpha)^2 \ : \ \langle a, b\rangle \in \beta\}$. $\square$

Definition 1.366 (Quotient Homomorphism) [BS81] Let $\mathbf{A}$ be a structure and let $\alpha \in \mathsf{Con}(\mathbf{A})$. The **quotient map** $\mathsf{q}_\alpha : \mathbf{A} \to \mathbf{A}/\alpha$ is a *strict* epimorphism, which is sometimes referred to as the **quotient homomorphism** or the **canonical homomorphism**. $\square$

Remark 1.367 $\equiv_{\mathsf{q}_\alpha} = \alpha$.

Theorem 1.368 (The Isomorphism Theorems of Algebras) Let $\mathbf{A}$ and $\mathbf{B}$ be algebras.

1. **[First Homomorphism Theorem]** Let $f : \mathbf{A} \to \mathbf{B}$ be a surjective homomorphism. Then there exists an isomorphism $g : \mathbf{A}/(\equiv_f) \to \mathbf{B}$, such that $f = g\mathsf{q}_{\equiv_f}$, where $\mathsf{q}_{\equiv_f} : \mathbf{A} \to \mathbf{A}/\equiv_f$ is the canonical homomorphism.

2. [**Second Homomorphism Theorem**] Let $\mathbf{A}$ be an algebra and $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$ with $\alpha \subseteq \beta$. Then the map $f : (A/\alpha)/(\beta/\alpha) \to A/\beta$ defined by $f\big((a/\alpha)/(\beta/\alpha)\big) = a/\beta$, is an isomorphism from $(\mathbf{A}/\alpha)/(\beta/\alpha)$ to $\mathbf{A}/\beta$.

3. [**Correspondence Theorem**] Let $\alpha \in \mathsf{Con}(\mathbf{A})$. The filter sublattice $[\alpha)_{\mathbf{Con}(\mathbf{A})}$ of algebraic lattice $\mathbf{Con}(\mathbf{A})$ is isomorphic to the algebraic lattice $\mathbf{Con}(\mathbf{A}/\alpha)$ under the map taking $\beta \mapsto \beta/\alpha$.

Proposition 1.369 [BS81] Let $\mathbf{A}$ be an *algebra* and $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$. If $\alpha \subseteq \beta$, then $\beta/\alpha \in \mathsf{Con}((\mathbf{A}/\alpha))$.

Remark 1.370 If $\mathbf{A}$ is an *algebra* and $f : \mathbf{A} \cong \mathbf{B}$ and $\alpha \in \mathsf{Con}(\mathbf{A})$, then $\underset{\to}{f}[\alpha] \in \mathsf{Con}(\mathbf{B})$ and $(\mathbf{A}/\,\alpha) \cong (\mathbf{B}/(\underset{\to}{f}[\alpha]))$.

Example 1.371 (Congruences and Quotients of Unary Algebra-Matrices)

Let $\mathbf{M}$ be an algebra-matrix. By an $\mathbf{M}$-congruence or a congruence on/of $\mathbf{M}$, we mean an $\mathsf{alg}(\mathbf{M})$-congruence. Let α be an $\mathbf{M}$-congruence. By the quotient of $\mathbf{M}$ by congruence α, denoted $\mathbf{M}/\alpha$, we mean the matrix $\langle \mathsf{alg}(\mathbf{M})/\alpha, \mathsf{q}_\alpha[\mathsf{D}_\mathbf{M}]\rangle$.

□

1.5.8.5 Relative Congruences

When studying classes of algebras that are *not* closed under homomorphic images, in particular *quasivarieties* of algebras (see Section 1.5.14), not all congruences on an algebra of the class are the kernels of 'valid' homomorphisms, i.e., homomorphisms that yield homomorphic images that lie in the class. Consequently, we need to be able to distinguish 'valid' congruences from 'invalid' congruences. To this end, we now consider the notion of a *relative congruence*.

Definition 1.372 (Relative Congruences) Given a class $\mathcal{K}$ of algebras and an algebra $\mathbf{A}$ (not necessarily in $\mathcal{K}$), the $\mathcal{K}$**-congruences** (or **relative congruences**, if $\mathcal{K}$ is understood) of $\mathbf{A}$ are the congruences α of $\mathbf{A}$ for which $\mathbf{A}/\alpha \in \mathcal{K}$. We use $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ to denote the set of all relative congruences of $\mathbf{A}$. The congruence classes of $\mathcal{K}$-congruences of $\mathbf{A}$ are called $\mathcal{K}$**-cosets of A** or just **relative cosets**. □

Note 1.373 (Relative Congruences of Algebras outside the Class) Relative congruences are defined on *all* algebras and not just those algebras in $\mathcal{K}$. Relative congruences permit one to step to an algebra $\mathbf{A}$ outside of the class; when $\mathbf{A}$ is factored by a relative congruence, the factor algebra lies in the class. This construction is similar in nature to the construction of a factor field; from the original *field* a *ring* is constructed which is then factored to obtain the factor *field* [Hun74]. □

For classes of algebras generally, relative congruences are less well-behaved than congruences. For example, $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ need not form a closed system over $\mathsf{uni}(\mathbf{A})^2$ (not even for $\mathbf{A} \in \mathcal{K}$), and so no notion of relative congruence generation is definable. In Section 1.5.14 we introduce the notion of a *quasivariety*, and for classes of algebras that form quasivarieties, the relative congruence form an algebraic closed system, in which case relative congruence generation is defined, although more difficult to characterize (see Lemma 1.452 on page 87) than is the case for congruence generation (see Lemma 1.355 on page 67). Of course, for any class *closed under homomorphic images*, for example *varieties* of algebras (see Section 1.5.13), for any algebra *in* such a class, relative congruences and congruences coincide. This, however, need not be true for algebras *outside* of the class.

Theorem 1.374 Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathfrak{a}$-algebras, $\mathcal{K}$ a class of $\mathfrak{a}$-algebras closed under $\mathcal{I}$ and $\mathcal{S}$ and $f : \mathbf{A} \to \mathbf{B}$. Then $\forall\, [\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{B})]\ \underset{\rightarrow}{f}^{-1}[\alpha] \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.

Proof. Suppose that $f : \mathbf{A} \to \mathbf{B}$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{B})$. By Theorem 1.357, $\equiv_f, \underset{\rightarrow}{f}^{-1}[\alpha] \in \mathsf{Con}(\mathbf{A})$. By Remark 1.76, $\equiv_f \subseteq \underset{\rightarrow}{f}^{-1}[\alpha]$. So by the second isomorphism theorem, $\mathbf{A}/\underset{\rightarrow}{f}^{-1}[\alpha]$ is isomorphic to $(\mathbf{A}/\equiv_f)/(\underset{\rightarrow}{f}^{-1}[\alpha]/\equiv_f)$. Since $f[\mathbf{A}] \lhd \mathbf{B}$, it follows from a result in [BP89a], that $f[\mathbf{A}]/\alpha_{|f[\mathsf{uni}(\mathbf{A})]}$ is isomorphic to a subalgebra of $\mathbf{B}/\alpha$, and hence by assumption $f[\mathbf{A}]/\alpha_{|f[\mathsf{uni}(\mathbf{A})]} \in \mathcal{K}$. (*It suffices, by assumption, to show that* $f[\mathbf{A}]/\alpha_{|f[\mathsf{uni}(\mathbf{A})]}$ *is isomorphic to* $(\mathbf{A}/\equiv_f)/(\underset{\rightarrow}{f}^{-1}[\alpha]/\equiv_f)$.) By the first homomorphism theorem, $\overleftarrow{f}_{[\![]\!]} : f[\mathbf{A}] \cong \mathbf{A}/\equiv_f$, where $\overleftarrow{f}_{[\![]\!]}$ is the pre-pole function taking $b \in f[\mathsf{uni}(\mathbf{A})]$ to $f^{-1}[\![b]\!]$. (*By Remark 1.370, it suffices to show that* $\underset{\longrightarrow}{\overleftarrow{f}_{[\![]\!]}}[\alpha_{|f[\mathsf{uni}(\mathbf{A})]}] = \underset{\rightarrow}{f}^{-1}[\alpha]/\equiv_f$.)

$$\begin{aligned}
\underset{\longrightarrow}{\overleftarrow{f}_{[\![]\!]}}[\alpha_{|f[\mathsf{uni}(\mathbf{A})]}] &= \{\langle \overleftarrow{f}_{[\![]\!]}(b_1), \overleftarrow{f}_{[\![]\!]}(b_2)\rangle \ :\ \langle b_1, b_2\rangle \in \alpha_{|f[\mathsf{uni}(\mathbf{A})]}\} \\
&= \{\langle f^{-1}[\![b_1]\!], f^{-1}[\![b_2]\!]\rangle \ :\ \langle b_1, b_2\rangle \in \alpha_{|f[\mathsf{uni}(\mathbf{A})]}\} \\
&= \{\langle f^{-1}[\![b_1]\!], f^{-1}[\![b_2]\!]\rangle \ :\ b_1, b_2 \in f[\mathsf{uni}(\mathbf{A})], \langle b_1, b_2\rangle \in \alpha\} \\
&= \{\langle a_1/\equiv_f, a_2/\equiv_f\rangle \ :\ \langle a_1, a_2\rangle \in f^{-1}[\alpha]\} \\
&= \underset{\rightarrow}{f}^{-1}[\alpha]/\equiv_f
\end{aligned}$$

$\diamond$

Definition 1.375 (Relative Congruential Properties) Let $\mathcal{K}$ be a class of algebras. An algebra $\mathbf{A}$ (not necessarily in $\mathcal{K}$) is called **$\mathcal{K}$-regular** (or **fully $\mathcal{K}$-regular** or **relatively congruence regular**) if, for all $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a \in \mathsf{uni}(\mathbf{A})$, if $\alpha[\![a]\!] = \beta[\![a]\!]$ then $\alpha = \beta$. For a term $\mathbf{0}$ constant over $\mathcal{K}$, $\mathbf{A}$ is called **relatively congruence point regular at 0** if, for all $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\alpha[\![\mathbf{0}^{\mathbf{A}}]\!] = \beta[\![\mathbf{0}^{\mathbf{A}}]\!]$ then $\alpha = \beta$. $\square$

1.5.9 Reduced Products and Ultraproducts

Reduced products and ultraproducts arise from model-theoretic considerations, in that quasi-equational classes of structures are closed under the formation of ultraproducts (see $\oint$1.5.14). While these structures are well-understood and relatively simple for classes of structures with equality, for classes of structures without equality, they have only recently been clearly understood, and then only in particular contexts [Elg97],[Elg98]. Central to this recent understanding, is the theory of algebraizable logics, and the primary results of [Elg98] are a generalization of Blok and Pigozzi's theory, which concerns a single n-ary relation over an algebra, to encompass arbitrarily many finite arity relations over an algebra, i.e., structures.

Recall the definition of a reduced product given in $\oint$1.1.7, and in particular the definition of the equivalence relation $\mho_{\mathcal{F}}^{\boldsymbol{F}}$ associated with a family $\boldsymbol{F}$ and a filter $\mathcal{F}$ on $\mathsf{idx}(\boldsymbol{F})$.

Remark 1.376 [BS81] If $\boldsymbol{F} = \langle \mathbf{A}_i \ : \ i \in I \rangle$ is an indexed system of algebras and $\mathcal{F} \in \mathsf{Filter}(I)$, then $\mho^{\boldsymbol{F}}_{\mathcal{F}}$ is a congruence on $\prod_I \mathbf{A}_i$. □

We purposely present the definition of reduced products of algebras and structures separately, since, as we shall see, the former is a real quotient, while the latter is generally not a quotient.

Definition 1.377 (Reduced Products and Ultraproducts of Algebras) [BS81] Let $\boldsymbol{F} = \langle \mathbf{A}_i \ : \ i \in I \rangle$ be an indexed system of $\mathfrak{a}$-algebras, and $\mathcal{F} \in \mathsf{Filter}(I)$. The quotient algebra $(\prod_I \mathbf{A}_i)/\mho^{\boldsymbol{F}}_{\mathcal{F}}$ is called a **reduced product** of $\langle \mathbf{A}_i \ : \ i \in I \rangle$, and is denoted $\prod_I \mathbf{A}_i/\mathcal{F}$. A reduced product $\prod_I \mathbf{A}_i/\mathcal{F}$ is called an **ultraproduct** of $\langle \mathbf{A}_i \ : \ i \in I \rangle$ if $\mathcal{F}$ is an ultrafilter over I. □

Reduced products and ultraproducts of algebras are, by definition, quotient algebras. We now consider the reduced products of structures. The reader is urged to notice how, in the following definition, while the algebraic reduct is a standard quotient (it is simply the reduced product as in the previous definition), the relational components are not defined as would be for a quotient (i.e., as in Definition 1.365 on page 69).

Definition 1.378 (Reduced Products and Ultraproducts of Structures) [BS81] Let $\langle \mathbf{A}_i \ : \ i \in I \rangle$ be an indexed system of $\mathfrak{e}$-structures, and $\mathcal{F} \in \mathsf{Filter}(I)$. The **reduced product** of $\langle \mathbf{A}_i \ : \ i \in I \rangle$ by $\mathcal{F}$ is the structure with algebra $\prod_I \mathbf{A}_i/\mathcal{F}$, and whose fundamental relations are defined by

$$\bowtie^{\prod_I \mathbf{A}_i/\mathcal{F}} (\mathfrak{u}_1, \ldots, \mathfrak{u}_{\mathsf{ar}(\bowtie)}) \text{ iff } \{ i \in I \ : \bowtie^{\mathbf{A}_i} (\mathfrak{u}_{1\,(i)}, \ldots, \mathfrak{u}_{\mathsf{ar}(\bowtie)\,(i)}) \} \in \mathcal{F}.$$

A reduced product is called an **ultraproduct** of $\langle \mathbf{A}_i \ : \ i \in I \rangle$ if $\mathcal{F}$ is an ultrafilter over I. □

Remark 1.379 [Elg98] The quotient map associated with a reduced or ultra product need not be strict. □

Consequently, reduced products and ultraproducts of structures, are not really *quotients*, since by Remark 1.348 on page 66, the quotient map onto a quotient structure must be strict.

Definition 1.380 (The Class Operators $\mathcal{P}_R$ and $\mathcal{P}_U$) Let $\mathcal{K}$ be a class of $\mathfrak{e}$-structures. Let $\mathcal{P}_R(\mathcal{K})$ denote the class of all $\mathfrak{e}$-structures that are reduced products of some members of $\mathcal{K}$, and let $\mathcal{P}_S(\mathcal{K})$ denote the class of all $\mathfrak{e}$-structures that are ultraproducts of some members of $\mathcal{K}$. □

1.5.10 Free Structures

Recall the notation of a free object for a construct given in ∮1.4.1.4. We shall now consider free structures, and in particular free algebras. Free algebras prove important in the sequel. In particular, in ∮8, we shall describe a general technique for inducing sentential logics from closed systems over free algebras.

We begin by reformulating the constructural definition of a free object for the case of structures, but with an essential difference: we demand that the free structure be *generated* (in the sense of Definition 1.302 on page 60) by its free generators. This is because we admit the possibility of a free structure for a class $\mathcal{K}$ of structures that does not necessarily belong to $\mathcal{K}$, and so Remark 1.235 on page 48 does not necessarily apply.

Definition 1.381 (Universal Mapping Property) Let $\mathcal{K}$ be a class of $\mathfrak{a}$-structures, $\mathbf{F}$ an $\mathfrak{e}$-structure and $X \subseteq \mathsf{uni}(\mathbf{F})$ with $\mathbf{F}$ *generated* by X. Algebra $\mathbf{F}$ is said to have the **universal mapping property** for $\mathcal{K}$ over X, if, for every structure $\mathbf{A} \in \mathcal{K}$ and every map $h : X \to \mathsf{uni}(\mathbf{A})$, there exists a homomorphism $g : \mathbf{F} \to \mathbf{A}$ extending h, i.e., $g_{|X} = h$. In this case, X is called a set of **free generators** of $\mathbf{F}$, and $\mathbf{F}$ is said to be **freely generated** by X. $\square$

Free algebras for a class $\mathcal{K}$ are easily described as certain quotients of term algebras. Unfactored term algebras provide us with the simplest examples of algebras with the universal mapping property (for any class of algebras).

Theorem 1.382 [BS81] For any algebraic type $\mathfrak{a}$ and a set of variables V, if the term algebra $\mathbf{Tm}_V$ exists, then it has the universal mapping property for the class of all $\mathfrak{a}$-algebras. $\square$

The term algebra, while free for any class of algebras, is never going to be a member of any interesting class of algebras, essentially because of its *absolute* freedom. The following construction yields a more interesting free algebra for a class of algebras, and is often a member of that class (see Theorem 1.385 on page 73). Note that the congruence described in the following definition is indeed a congruence, since the congruences on an algebra form a closed system.

Definition 1.383 ($\equiv_{\mathcal{K}}^{V}$, $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ and $\mathbf{F}_{\mathcal{K}}$) [BS81] Suppose $\mathbf{Tm}_V$ exists. For a class $\mathcal{K}$ of $\mathfrak{a}$-algebras, we define a *congruence relation* $\equiv_{\mathcal{K}}^{V}$ on $\mathbf{Tm}_V$ by

$$\equiv_{\mathcal{K}}^{V} = \bigcap \{\alpha \in \mathsf{Con}(\mathbf{Tm}_V) \ : \ \mathbf{Tm}_V/\alpha \in \mathcal{IS}(\mathcal{K})\},$$

The algebra $\mathbf{Tm}_V / \equiv_{\mathcal{K}}^{V}$ is denoted by $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$. We write $\mathbf{F}_{\mathcal{K}}$ for $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ when V is our global chosen denumerably infinite set of standard variables V. $\square$

Remark 1.384 [BS81] $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ exists iff $\mathbf{Tm}_V$ exists.

Theorem 1.385 [BS81] If $\mathbf{Tm}_V$ exists, then $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ has the universal mapping property for $\mathcal{K}$ over $\overline{[V]}$ and $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}} \in \mathcal{ISP}(\mathcal{K})$.

Definition 1.386 ($\mathcal{K}$-Free Algebras) The algebra $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ is called the **$\mathcal{K}$-free algebra over** $\bar{V}$. When V is finite, say $V = \{x_1, \ldots, x_n\}$, we often write $\mathbf{F}_{\mathcal{K}}^{\overline{x}_1,\ldots,\overline{x}_n}$ for $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$. Let $\bar{\cdot}^{\mathcal{K}_V}$ denote the canonical homomorphism from the term algebra $\mathbf{Tm}_V$ onto $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$. We denote the $\bar{\cdot}^{\mathcal{K}_V}$-image of terms P by $\overline{[P]}^{\mathcal{K}_V}$, the $\bar{\cdot}^{\mathcal{K}_V}$-pre-pole of $\boldsymbol{p} \in \mathsf{uni}(\mathbf{F}_{\mathcal{K}}^{\overline{[V]}})$ by $\underline{[\![\boldsymbol{p}]\!]}_{\mathcal{K}_V}$ and the $\bar{\cdot}^{\mathcal{K}_V}$-pre-image of $\boldsymbol{P} \subseteq \mathsf{uni}(\mathbf{F}_{\mathcal{K}}^{\overline{[V]}})$ by $\underline{[\boldsymbol{P}]}_{\mathcal{K}_V}$. We *sometimes* write $\bar{P}^{\mathcal{K}_V}$ (with a short bar) for $\overline{[P]}^{\mathcal{K}_V}$, although only in very simple expressions since this notation is misleading. We may drop the subscript/superscript V from this notation in the case that $V = \mathsf{V}$, or when context unambiguous or determinable from the argument, and may drop the subscript/superscript $\mathcal{K}$ when context unambiguous. $\square$

Remark 1.387 [BS81] If $\mathcal{K}$ has a non-trivial member and $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ exists, then, for each $x \in V$, $V \cap \overline{x} = \{x\}$, and hence $\mathsf{card}(\bar{V}) = \mathsf{card}(V)$. In this case, $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ is uniquely determined, up to isomorphism, by $\mathsf{card}(V)$.

Convention 1.388 For cardinal $\mathbf{m}$, let $\mathbf{F}_{\mathcal{K}}^{\mathbf{m}}$ denote the unique $\mathcal{K}$-free algebra (up to isomorphism) over a set of $\mathbf{m}$ free generators, provided it exists. When we speak of *a* free algebra, or when we use free algebra associated notation, we implicitly imply that such a free algebra exists.

1.5.11 The Model Theory of Structures

Any **elementary theory** attempts to abstract a 'universe' of **things**. Terms are the nouns of our language, abstracting things. Formulae allow us to make statements about things. In this text, we necessarily make use of elementary logics with equality and without equality. Logics with equality can be handled in many different ways. The distinction between elementary theories with and without equality lies at the core of the theory of algebraizable logics. Propositional logics can be modelled as elementary relational theories without equality, while the target algebraic theory is standardly modelled as an elementary algebraic theory with equality. In order for the 'linking' to potentially occur, the target theory, with equality, is first interpreted as an elementary theory without equality, treating equality as an extra binary relation symbol and 'porting' the equality axioms (see $\oint$1.5.11.2).

Warning 1.389 (Models in Practise) Computer scientists [OMG04, 7.2.8] and applied mathematicians appear to use the word 'model' in the opposite manner to which the word is used by model theorists. For the former, the 'model' is the abstract formal system, which is obtained by 'modelling' some system. For the latter, the system being abstracted is the 'model', the abstraction is the 'theory'.

1.5.11.1 Elementary Languages

Definition 1.390 (Elementary Languages) Any elementary language L is defined by specifying its elementary type, denoted $\mathsf{type}(\mathsf{L})$, and its **variable symbols** (or just **variables**), denoted $\mathsf{Var}(\mathsf{L})$. For reasons of resolving ambiguities, we assume that no variable symbol appears as a symbol of $\mathsf{type}(\mathsf{L})$. It is useful to syntactically confuse languages with their 'underlying' types, speaking, for example, of **L-operation symbols** and **L-terms**, and writing Tm^{L} for $\mathsf{Tm}^{\mathsf{type}(\mathsf{L})}_{\mathsf{Var}(\mathsf{L})}$, etc. □

Definition 1.391 (Atomic Formulae) [Men87] We define the set $\mathsf{Form}_{\mathsf{a}}(\mathsf{L})$, of **atomic L-formulae**, or just **atomic formulae**, by the production-rule

$$\mathsf{r} ::= \bowtie (p_1, \ldots, p_n),$$

where $\bowtie$ ranges over all relation symbols, $n = \mathsf{ar}(\bowtie)$ and $p_1, \ldots, p_n$ are terms. □

Convention 1.392 For an n-ary relation symbol $\bowtie$, we often denote the atomic formula $\bowtie (p_1, \ldots, p_n)$ by (the formal symbol sequence) $\langle p_1, \ldots, p_n \rangle$ are $\bowtie$ or by $p_1 \bowtie p_2 \bowtie \ldots \bowtie p_n$, as context appropriate. For a unary relation symbol $\Box$, we may also write p is $\Box$ for $\langle p \rangle$ are $\Box$.

Recall Convention 1.327 pertaining to *production rules.*

Definition 1.393 (Formulae) [Men87] The set $\mathsf{Form}(\mathsf{L})$, of **L-formulae** or just **formulae**, are defined by the production-rule

$$\eta ::= \mathsf{r} \mid \eta \longrightarrow \zeta \mid \neg\eta \mid \forall\, [x]\, \eta$$

where r ranges over atomic formulae and x ranges over the variables of the language. We write $\eta \,\mathsf{and}\, \zeta$ for $\neg(\eta \longrightarrow \neg\zeta)$, $\eta \vee \zeta$ for $(\neg\eta) \longrightarrow \zeta$, $\eta \leftrightarrow \zeta$ for $(\eta \longrightarrow \zeta)\,\mathsf{and}(\zeta \longrightarrow \eta)$, and write $\exists\,[x]\,\eta$ for $\neg(\forall[x]\,\neg\eta)$. We may also write $\eta\textbf{,}\ \zeta$ for $\eta\,\mathsf{and}\,\zeta$. For integer $n \geq 1$ and formulae $\eta_0, \ldots, \eta_n$, we may write $\bigwedge_{i\leq n}\eta_i$ for $\eta_0\,\mathsf{and}\ldots\mathsf{and}\,\eta_n$ and $\bigvee_{i\leq n}\eta_i$ for $\eta_0 \vee \ldots \vee \eta_n$. We write $\forall[x_0, \ldots, x_n]\,\eta$ for $\forall[x_0]\ \ldots\forall[x_n]\,\eta$ and $\exists[x_0, \ldots, x_n]\,\eta$ for $\exists[x_0]\ \ldots\exists[x_n]\,\eta$. We use brackets (informally) to resolve precedence. $\square$

Our logical connectives are emboldened to contrast them with our use of the same symbols of our set-theory and lattice theory. Compare the elementary connective $\longrightarrow$ with the set theoretic symbol $\rightarrow$, and the elementary and with the lattice $\wedge$.

Remark 1.394 There are no L-formulae if L has no relation symbols. $\square$

Consequently, purely *algebraic* theories, have to come *with equality.*

Example 1.395 (Group and Groupoid Formulae)

There are no group formulae *without equality.*

$\square$

Example 1.396 (Binary Matrices)

The **type of binary-matrices** is the elementary type with a single binary relation symbol D. Variables are the only terms. Examples of binary matrix formulae, include

$$\text{(non-trivial)} \quad \exists[x,y]\, x\mathsf{D}y, \tag{1.61}$$

$$\text{(trivial)} \quad \forall[x,y]\, \neg x\mathsf{D}y, \tag{1.62}$$

$$\text{(proper)} \quad \exists[x,y]\, \neg x\mathsf{D}y \tag{1.63}$$

$$\text{(improper)} \quad \forall[x,y]\, x\mathsf{D}y, \tag{1.64}$$

$$\text{(somewhere-reflexive)} \quad \exists[x]\, x\mathsf{D}x, \tag{1.65}$$

$$\text{(anti-reflexive)} \quad \forall[x]\, \neg x\mathsf{D}x, \tag{1.66}$$

$$\text{(non-reflexive)} \quad \exists[x]\, \neg x\mathsf{D}x, \tag{1.67}$$

$$\text{(reflexive)} \quad \forall[x]\, x\mathsf{D}x, \tag{1.68}$$

$$\text{(symmetric)} \quad \forall[x,y]\, x\mathsf{D}y \longrightarrow y\mathsf{D}x, \qquad \text{and} \tag{1.69}$$

$$\text{(transitive)} \quad \forall[x,y,z]\, x\mathsf{D}y\textbf{,}\ y\mathsf{D}z \longrightarrow x\mathsf{D}z. \tag{1.70}$$

$\square$

Definition 1.397 (Subformula, Free and Bound Variables) [vD83, 64-65] Informally, a formula η is called a **subformula** of ζ if η occurs in the construction of ζ as an application of the production rule of Definition 1.393 on page 74. We speak of the **occurrences** of terms and subformulae in a formulae. A particular occurrence of a variable x in a formula η is said to **belong to** an occurrence of a subformula ζ of η if this occurrence of x is also an occurrence of ζ.

An occurrence of a variable x in a formula η is called **free** if x does not belong to any occurrence of a subformula of the form $\forall x(\zeta)$ or $\exists x(\zeta)$, otherwise the occurrence is called **bound**. A variable x is called **free for** η if some occurrence of x in η is free.

When we write $\eta(x_1, \ldots, x_n)$ we mean that η is a formula all of whose free variables are among $x_1, \ldots, x_n$. Given a formula $\eta(x_1, \ldots, x_n)$, when we write $\eta(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ we mean the formula obtained by replacing *all* free occurrences of x_i in η by term t. $\square$

Definition 1.398 (Special Formulae) Let L be an elementary language. A **L-sentence** is a L-formula with no free variables. An **open formula** has no bound variables, i.e., has no occurrences of quantifiers. A formula η is said to be in **prenix form** if it is of the form

$$Q_1[x_1] \ldots Q_n[x_n]\zeta(x_1, \ldots, x_n, \vec{y}),$$

where $\zeta(x_1, \ldots, x_n, \vec{y})$ is an open formula and each Q_i is a quantifier, in which case we call $\zeta(x_1, \ldots, x_n, \vec{y})$ the **matrix** of η. A **universal formula** is a formula in prenix form, all of whose quantifiers are universal. A **basic Horn formula** is an *open* formula of the form $\mathsf{r}_1 \text{ and} \ldots \text{and } \mathsf{r}_n \longrightarrow \mathsf{r}$, where the $\mathsf{r}_1, \ldots, \mathsf{r}_n, \mathsf{r}$ are all atomic formulae. A **Horn formula** is a formula in prenix form with a basic Horn formula as matrix. A **universal Horn formula** is a Horn formula that is universal.

Let sets of all L-sentences, universal L-sentences, universal Horn L-sentences, L-sentences without equality, universal L-sentences without equality and universal Horn L-sentences without equality, are denoted by $\mathsf{sentences}(\mathsf{L})$, $\mathsf{sentences}_{\forall}(\mathsf{L})$, $\mathsf{sentences}_{\forall\mathsf{H}}(\mathsf{L})$, $\mathsf{sentences}_{\not\approx}(\mathsf{L})$, $\mathsf{sentences}_{\not\approx\forall}(\mathsf{L})$ and $\mathsf{sentences}_{\not\approx\forall\mathsf{H}}(\mathsf{L})$, respectively. $\square$

Remark 1.399 Our definition of a Horn formula is taken from [vA95]. An alternative common phrasing is the following (taken from [BS81]): a formula is said to be in **disjunctive form** (**conjunctive form**) if it is open and of the form $\bigvee_i \mathsf{and}_j\, \eta_{ij}$ (resp. $\mathsf{and}_i \bigvee_j \eta_{ij}$), where each formula η_{ij} is atomic or (once) negated atomic. A **basic Horn formula** is an open formula of disjunctive form $\bigvee_{i\leq n} \eta_i$, but such that *no more than one* η_i is atomic. A **Horn formula** is a formula in prenix form with a basic Horn formula as matrix. A **universal Horn formula** is a Horn formula that is universal.

Example 1.400 (Binary Matrix Horn Formulae)

Binary-matrix formulae (1.69) and (1.70) are Horn formulae.

$\square$

We briefly consider the notion of a substitution. Unlike the case for *sentential logics*, the variables in first-order languages are *object* variables. Consequently, a substitution must be a mechanism for substituting terms for variables in formulae, such that the resulting formula says for the substituted terms, what the original formula said for the substituting variables. The presence of bound variables makes substitution into formulae technical to define.

Definition 1.401 (Simultaneous Substitutions) [vD83, 66] A **simultaneous substitution** σ is an assignment of a term, denoted by $\sigma(x)$, to each variable x. Let $\left[\frac{x_1}{p_1}, \ldots, \frac{x_n}{p_n}\right]$ denote the substitution mapping variables x_i to terms p_i, for each $i < n$, and fixing all other variables. We deterministically extend a substitution to a mapping from terms to terms, by recursively defining $\sigma(\mathbf{0}) = \mathbf{0}$ if $\mathbf{0}$ is a constant symbol, and $\sigma(\star(p_1, \ldots, p_n)) = \star(\sigma(p_1), \ldots, \sigma(p_n))$ if $\star$ is an operation symbol and $p_1, \ldots, p_n$ are terms. $\square$

Example 1.402 (Simultaneous Substitutions of Group Terms)

> Consider the group formula $(x * y)$. Let us *simultaneously* substitute $x * y$ for x and $y * x^{\mathbf{-1}}$ for y, obtaining $\left[\frac{x}{x*y}, \frac{y}{y*x^{\mathbf{-1}}}\right](x * y) = (x * y) * (y * x^{\mathbf{-1}})$. Observe that $\left[\frac{x}{x*y}\right]\left[\frac{y}{y*x^{\mathbf{-1}}}\right](x * y) = \left[\frac{x}{x*y}\right](x * (y * x^{\mathbf{-1}})) = ((x * y) * (y * (x * y)^{\mathbf{-1}})) \neq (x * y) * (y * x^{\mathbf{-1}})$, hence the meaning of the term '*simultaneous* substitution'.

$\square$

Definition 1.403 (Extending Simultaneous Substitutions) [vD83, 66-67] We extend a substitution to a mapping from formulae to formulae, by recursively defining,

1. $\sigma(\bowtie (p_1, \ldots, p_n)) = \bowtie (\sigma(p_1), \ldots, \sigma(p_n))$ for atomic formula $\bowtie (p_1, \ldots, p_n)$,
2. $\sigma(\neg\eta) = \neg\sigma(\eta)$,
3. $\sigma(\eta \longrightarrow \zeta) = \sigma(\eta) \longrightarrow \sigma(\zeta)$,
4. For formula η, $\left[\frac{x}{p}\right](\forall\, [y]\, \eta) = \begin{cases} \forall\, [y]\, \left[\frac{x}{p}\right](\eta) & ; \quad \text{if } x \neq y, \\ \forall\, [y]\, \eta & ; \quad \text{otherwise.} \end{cases}$.

$\square$

Example 1.404 (Simultaneous Substitutions into Group Matrix Formulae)

> Consider the type of binary matrices over groups. Then $\left[\frac{x}{z}\right](\forall\, [x]\, x\mathsf{D}y) = \forall\, [x]\, x\mathsf{D}y$ and $\left[\frac{y}{z}\right](\forall\, [x]\, x\mathsf{D}y) = \forall\, [x]\, \left[\frac{y}{z}\right](x\mathsf{D}y) = \forall\, [x]\, x\mathsf{D}z$.

$\square$

1.5.11.2 Elementary Languages with Equality

Elementary theories *with equality*, abstract a universe of things in which one may sensibly ask whether or not *any* two things are **identical**. *Syntactically*, this identity relation is abstracted by introducing a binary relation symbol such as $\approx$ into our language, and adding **equality axioms** to the theory, that is, assuming that $x \approx x$, $x \approx y \longrightarrow y \approx x$ and $x \approx y \,\mathsf{and}\, y \approx z \longrightarrow x \approx z$.

There are commonly two approaches to handling equality in first-order theories. The first approach, common in model theory texts dealing only with theories with equality and in texts on universal algebra, treats the equality relation as a meta-symbol, not occurring in the elementary type. This approach is inherently incompatible with the theory of first-order structures (without equality). For example, the structures from which models are drawn, are simply the structures without equality, while the language, however, permits formulae with equality.

The second approach, which is the approach that we shall adopt, places the equality symbol at the same ontological level as the other relation symbols in the type. The advantage of this approach is that the theory may be naturally bootstrapped from the theory of structures without equality. The need to designate the equality symbol and treat it meta-logically cannot be avoided however, since, while the notion of 'equivalence' may be defined in first-order languages, first-order languages lack the richness to tie down the notion of 'equality'. With the second approach, it is necessary to place a meta-logical restriction on which structures may be considered as models of languages with equality.

While the machinery of this section is 'overly technical', this degree of technicality is only required in the setting up, and we shall adopt a simplifying convention in the next section.

Definition 1.405 (Elementary Languages with Equality) We say that an elementary *type* has **equality**, if it contains the binary relation symbol $\approx$, otherwise we say that it is **without equality**. With each elementary type $\mathfrak{e}$ without equality, we associate the elementary type $\mathfrak{e}_{\approx}$ with equality, specified by $\mathsf{Symb_c}(\mathfrak{e}_{\approx}) = \mathsf{Symb_c}(\mathfrak{e})$, $\mathsf{Symb_o}(\mathfrak{e}_{\approx}) = \mathsf{Symb_o}(\mathfrak{e})$, $\mathsf{Symb_r}(\mathfrak{e}_{\approx}) = \mathsf{Symb_r}(\mathfrak{e}) \cup \{\approx\}$, where all 'inherited' symbols have the same arity, and the extra relation symbol $\approx$ is binary. With each elementary type $\mathfrak{e}$ with equality, we associate the elementary type $\mathfrak{e}_{\not\approx}$ without equality, specified by $\mathsf{Symb_c}(\mathfrak{e}_{\not\approx}) = \mathsf{Symb_c}(\mathfrak{e})$, $\mathsf{Symb_o}(\mathfrak{e}_{\not\approx}) = \mathsf{Symb_o}(\mathfrak{e})$, $\mathsf{Symb_r}(\mathfrak{e}_{\not\approx}) = \mathsf{Symb_r}(\mathfrak{e}) - \{\approx\}$, where all 'inherited' symbols have the same arity.

By our convention of treating languages as types, we may sensibly speak of an elementary language with equality, etc. For elementary language without equality, $\mathsf{L}_{\approx}$ denotes the language with equality determined by type $\mathsf{type}(\mathsf{L})_{\approx}$ and variables $\mathsf{Var}(\mathsf{L})$, and for elementary language with equality, $\mathsf{L}_{\not\approx}$ denotes the language without equality determined by type $\mathsf{type}(\mathsf{L})_{\not\approx}$ and variables $\mathsf{Var}(\mathsf{L})$.

□

Example 1.406 (Groupoid and Group Formulae with Equality)

Important groupoid formulae with equality are

$$\text{(associativity)} \qquad (x * y) * z \approx x * (y * z), \tag{1.71}$$

$$\text{(commutativity)} \qquad x * y \approx y * x, \tag{1.72}$$

$$\text{(existential-identity)} \qquad \exists\,[e]\,\forall\,[y]\; e * y \approx y \;\mathsf{and}\; y * e \approx y, \tag{1.73}$$

$$\text{(existential-inverse)} \qquad \forall\,[x]\,\exists\,[x']\; x' * x \approx e \;\mathsf{and}\; x * x' \approx e. \tag{1.74}$$

With 'universal groups' existential-identity and existential-inverse are 'expressible' by the group formulae (with equality)

$$\text{(identity)} \qquad \mathbf{1} * y \approx y \;\mathsf{and}\; y * \mathbf{1} \approx y, \tag{1.75}$$

$$\text{(inverse)} \qquad x^{\mathbf{-1}} * x \approx \mathbf{1} \;\mathsf{and}\; x * x^{\mathbf{-1}} \approx \mathbf{1}. \tag{1.76}$$

□

Definition 1.407 (Identities and Quasi-identities) Let L be an elementary language with equality. An **L-identity**, or just **identity**, is an expression of the form $p_1 \approx p_2$, where p_1 and p_2 are both terms. A **L-quasi-identity**, or just **quasi-identity**, is a formula of the form $\left(\bigwedge_{i\le n} I_i\right) \to I$, where $I_0, \ldots, I_n, I$ are identities. Let $\mathsf{Identity}(\mathsf{L})$ denote the set of all L-identities and $\mathsf{QuasiIdentity}(\mathsf{L})$ the set of all L-quasi-identities.

For $P \cup \{p\} \subseteq \mathsf{Tm}$, we abbreviate $\{q \approx r : q, r \in P\}$ by $P \approx P$, and abbreviate $\{q \approx p : q \in P\}$ by $P \approx p$ and $\{p \approx q \;:\; q \in P\}$ by $p \approx P$. For a finite set $\mathcal{I} = \{I_1, \ldots, I_n\}$ of identities, $\bigwedge\mathcal{I}$ abbreviates $I_1 \;\mathsf{and} \ldots \mathsf{and}\; I_n$. For a set of identities $\mathcal{J}$, an expression of the form $\left(\bigwedge_{i\le n} I_i\right) \to \mathcal{J}$ abbreviates the set of quasi-identities $\{\left(\bigwedge_{i\le n} I_i\right) \to J \;:\; J \in \mathcal{J}\}$. □

Example 1.408 (Groupoid and Group Identities and Quasi-identities)

(1.71) and (1.72) are identities, while (1.73) (1.74) are not. The group formulae (1.75) and (1.76), while not identities, are logically equivalent, in the 'without equality' sense, to the identities

$$\text{(left-identity)} \qquad \mathbf{1} * y \approx y, \tag{1.77}$$

$$\text{(right-identity)} \qquad y * \mathbf{1} \approx y, \tag{1.78}$$

$$\text{(left-inverse)} \qquad x^{\mathbf{-1}} * x \approx \mathbf{1}, \text{and} \tag{1.79}$$

$$\text{(right-inverse)} \qquad x * x^{\mathbf{-1}} \approx \mathbf{1}. \tag{1.80}$$

Examples of group quasi-identities are

$$\text{(left-identity-uniqueness)} \qquad x * y \approx y \rightarrow x \approx \mathbf{1}, \tag{1.81}$$

$$\text{(right-identity-uniqueness)} \qquad x * y \approx x \rightarrow y \approx \mathbf{1}, \tag{1.82}$$

$$\text{(left-inverse-uniqueness)} \qquad y * x \approx \mathbf{1} \rightarrow y \approx x^{\mathbf{-1}}, \tag{1.83}$$

$$\text{(right-inverse-uniqueness)} \qquad x * y \approx \mathbf{1} \rightarrow y \approx x^{\mathbf{-1}}, \tag{1.84}$$

□

Remark 1.409 Quasi-identities (1.81) to (1.84), demonstrate how elementary language *with equality* can express *uniqueness*. This property of uniqueness, however, will require special *model theoretic interpretation*, over and above the normal interpretations given to (binary) relations in model theories of languages *without equality*, in order to distinguish it from the property of *equivalence*.

Definition 1.410 (Denoting Uniqueness) If η is a formula, and y is a variable free for η, then the formula $\forall\,[y]\,\eta(y) \leftrightarrow x \approx y$ is written $\eta(\mathbf{!}x)$, and read 'η **holds uniquely of** x, The formula $\exists\,[x]\,\eta(\mathbf{!}x)$ is written $\exists\mathbf{!}\,[x]\,\eta(x)$ and read '**there exists a unique** x **such that** $\eta(x)$'. □

While it is convenient to bootstrap languages and logics with equality via the machinery of languages and logics more generally (i.e., without necessarily having equality), practically, given our universal algebraic needs, we shall conventionally treat logics with equality as primary or common, and logics without equality as singular.

Convention 1.411 We introduce the following conventions.

1. Arbitrary elementary *types* (e.g., type meta-variables) are assumed to be *without equality*.
2. No elementary type *with equality* shall be 'atomically' introduced or specified.
3. The only machinery formally available to introduce types with equality is via the operator $\cdot_{\approx}$ given in Definition 1.405, and in this case only for technical reasons, in which case we redraw the readers attention to this convention.
4. When speaking of some arbitrary elementary *language*, without explicit specification of its equality perspective, we do so because the subsequent usage is independent of the language being with or without equality.

5. An elementary language L, may only be specified by an elementary type $\mathfrak{e}$ that is *without equality*. This language, however, is to be taken as the language *with equality* determined by $\mathfrak{e}_{\approx}$.

6. The function $\mathsf{type}(\mathsf{L})$ is redefined, in the case of languages with equality, to return $\mathfrak{e}$ instead of $\mathfrak{e}_{\approx}$.

7. For elementary type $\mathfrak{e}$ (without equality by convention (1)) and variables V, $\mathfrak{e}_V$ denotes the language (*with equality* by convention (4)) determined by type $\mathfrak{e}$ and variables V. By convention (6), $\mathsf{type}(\mathfrak{e}_V)$ is unambiguously $\mathfrak{e}$.

8. An elementary type $\mathfrak{e}$ (without equality by convention), may be treated as the elementary language $\mathfrak{e}_{\mathsf{V}}$ (with equality by convention), where V is the conventional fixed denumerable variable set. Unambiguously, $\mathsf{type}(\mathfrak{e}) = \mathfrak{e}$.

9. The only way to specify a language without equality is via the operator $\cdot_{\not\approx}$ given in Definition 1.405. The function $\mathsf{type}(*)$ remains unchanged for languages without equality.

1.5.11.3 Interpretations

By our convention of confusing elementary languages with their elementary types, we may speak of an L-structure, where L is an elementary language, by which we mean a $\mathsf{type}(\mathsf{L})$-structure, and conventionally, $\mathsf{type}(\mathsf{L})$ is *never* a type with equality.

Definition 1.412 (Interpretation) [BS81, 194-195] Let L be an elementary language. An **L-interpretation** i, is a triple $\langle \mathsf{lang}(\mathsf{i}), \mathsf{structure}(\mathsf{i}), \mathsf{i}(\cdot)\rangle$ where $\mathsf{lang}(\mathsf{i}) = \mathsf{L}$, $\mathsf{structure}(\mathsf{i})$ is an L-structure and $\mathsf{i}(\cdot) : \mathsf{Var}(\mathsf{L}) \to \mathsf{uni}(\mathsf{structure}(\mathsf{i}))$. We speak of an **interpretation of L in $\mathbf{A}$**, by which we mean an L-interpretation i with $\mathsf{structure}(\mathsf{i}) = \mathbf{A}$. It is convenient to syntactically confuse an interpretation with its elementary language, speaking, for example, of **i-variables**, by which we mean $\mathsf{lang}(\mathsf{i})$-variables, etc. The set of all interpretations of L in $\mathbf{A}$, is denoted by $\mathsf{L} \longrightarrow_{\mathsf{i}} \mathbf{A}$. For an interpretation i, i-variables $x_1, \ldots, x_n$ and elements $a_1, \ldots, a_n \in \mathsf{uni}(\mathsf{structure}(\mathsf{i}))$, let $\mathsf{i}\left[\frac{x_1,\ldots,x_n}{a_1,\ldots,a_n}\right]$ denote the interpretation of $\mathsf{lang}(\mathsf{i})$ in $\mathsf{structure}(\mathsf{i})$ that maps $x_i \mapsto a_i$ and agrees with i on all other i-variables. □

Definition 1.413 (Interpreting Terms) [BS81, 194-195] We extend our perspective of an interpretation i in $\mathbf{A}$, by *extending* $\mathsf{i}(\cdot)$ to a function from i-terms into $\mathbf{A}$, by the recursive definition

1. $\mathsf{i}(\mathbf{0}) = \mathbf{0}^{\mathbf{A}}$, for each constant symbol $\mathbf{0}$;

2. $\mathsf{i}(\star(p_1, \ldots, p_k)) = \star^{\mathbf{A}}(\mathsf{i}(p_1), \ldots, \mathsf{i}(p_k))$, for each non-constant k-ary operation symbol $\star$ and all terms $p_1, \ldots, p_k$.

□

1.5.11.4 Satisfaction

While the following definition applies to formulae of languages *with equality*, it works just as well for formulae of languages *without equality*, since these are just special cases of the former; in which case condition (1.85) of the following definition *never* applies.

Definition 1.414 (The Satisfaction Relation) [Men87, p.g., 48] Let i be an interpretation of L in $\mathbf{A}$. For an L-formula η (with or without equality), we define the notion that i **satisfies** η, written $\models_{\mathsf{i}} \eta$, recursively, for all terms p and q, $\bowtie \in \mathsf{Symb_r(L)}$, terms $p_1, \ldots, p_{\mathsf{ar}(\bowtie)}$, formulae η and ψ, and variables x, as follows,

$$\models_{\mathsf{i}} p \approx q \quad \text{iff} \quad \mathsf{i}(p) = \mathsf{i}(q), \tag{1.85}$$

$$\models_{\mathsf{i}} \bowtie(p_1, \ldots, p_{\mathsf{ar}(\bowtie)}) \quad \text{iff} \quad \langle \mathsf{i}(p_1), \ldots, \mathsf{i}(p_{\mathsf{ar}(\bowtie)})\rangle \in \bowtie^{\mathbf{A}}, \tag{1.86}$$

$$\models_{\mathsf{i}} \eta \longrightarrow \zeta \quad \text{iff} \quad \models_{\mathsf{i}} \eta \text{ implies } \models_{\mathsf{i}} \zeta, \tag{1.87}$$

$$\models_{\mathsf{i}} \neg\eta \quad \text{iff} \quad \text{not } \models_{\mathsf{i}} \eta, \qquad \text{and} \tag{1.88}$$

$$\models_{\mathsf{i}} \forall[x]\, \eta \quad \text{iff} \quad \forall\, [a \in \mathsf{uni}(\mathbf{A})]\ \models_{\mathsf{i}\left[\frac{x}{a}\right]} \eta. \tag{1.89}$$

For a set Γ of formulae, we write $\models_{\mathsf{i}} \Gamma$ if and only if $\models_{\mathsf{i}} \eta$, for all $\eta \in \Gamma$. $\square$

Remark 1.415 The satisfaction relation depends only on the symbols occurring in the formula and on the free variables of that formula.

Convention 1.416 Consequent to the previous remark, we may write $\models_{\mathbf{A}} \eta(a_1, \ldots, a_n)$ for $\models_{\mathsf{i}\left[\frac{x_1,\ldots,x_n}{a_1,\ldots,a_n}\right]} \eta$, where $\mathsf{ar}(\eta) = n$ and $\{x_1, \ldots, x_n\}$ are the free variables occurring in η, and i is *any* interpretation into $\mathbf{A}$. For *sentences* Ψ, we may write $\models_{\mathbf{A}} \Psi$, without reference to any assignment.

Definition 1.417 (Extended Satisfaction) [Men87, 48] We extend the satisfaction relation $\models$, to a relation between L-structures and L-formulae, defined by

$$\models_{\mathbf{A}} \eta \text{ iff } \models_{\mathbf{A}} \forall[x_1, \ldots, x_m]\, \eta,$$

where $x_1, \ldots, x_m$ are precisely the free variables of η. For a set of formulae Γ, we define

$$\models_{\mathbf{A}} \Gamma \text{ iff } \forall\, [\eta \in \Gamma]\ \models_{\mathbf{A}} \eta.$$

For a set of structures $\mathcal{K}$, we define

$$\models_{\mathcal{K}} \Gamma \text{ iff } \forall\, [\mathbf{A} \in \mathcal{K}]\ \models_{\mathbf{A}} \Gamma.$$

We write $\models_{\mathcal{K}} \eta$ for $\models_{\mathcal{K}} \{\eta\}$. $\square$

Remark 1.418 Notice how the distinction between *sentences* and other formulae is eliminated.

1.5.11.5 Models

Definition 1.419 (Models) [Men87, 48] Let L be an elementary language, $\mathcal{K}$ a set of L-structures and $\mathbf{A}$ an L-structure. We call $\mathcal{K}$ (resp. $\mathbf{A}$) a **model of** Γ if $\models_{\mathcal{K}} \Gamma$ (resp. $\models_{\mathbf{A}} \Gamma$). $\square$

Definition 1.420 (The Semantic Theorems of $\mathcal{K}$) For a set $\mathcal{K}$ of L-structures, we let $\mathsf{theorems}^{\mathsf{L}}(\mathcal{K})$ denote the set of all L-formulae for which $\mathcal{K}$ is a model, which we call the **semantic theorems of** $\mathcal{K}$. $\square$

Proposition 1.421 Let L be a language with equality. Any class $\mathcal{K}$ of L-structures satisfies the **equality axioms**,

$$\text{(reflexivity)} \qquad x \approx x, \tag{1.90}$$

$$\text{(symmetry)} \qquad x \approx y \longrightarrow y \approx x, \tag{1.91}$$

$$\text{(transitivity)} \qquad x \approx y \,\mathsf{and}\, y \approx z \longrightarrow x \approx z, \tag{1.92}$$

$$\text{(operation-compatibility)} \qquad \left(\mathsf{and}_{i=1}^{n}\, x_i \approx y_i\right) \longrightarrow \star(x_1, \ldots, x_n) \approx \star(y_1, \ldots, y_n), \quad \text{and} \tag{1.93}$$

$$\text{(relation-compatibility)} \qquad \left(\mathsf{and}_{j=1}^{m}\, {x'}_j \approx {y'}_j\right) \longrightarrow \left(\bowtie ({x'}_1, \ldots, {x'}_m) \longrightarrow \bowtie ({y'}_1, \ldots, {y'}_m)\right), \tag{1.94}$$

where x, y and z are any three distinct *fixed* variables, $\star$ ranges over operation symbols of *non-zero* arity n and the variables x_i and y_i are all distinct, $\bowtie$ ranges over relation symbols of arity m and the variables x'_i and y'_i are all distinct, etc.

Definition 1.422 (Special Theorems of $\mathcal{K}$) For an elementary language L, with or without equality, let $\mathsf{theorems}^{\mathsf{L}}_{\forall}(\mathcal{K}) = \mathsf{theorems}^{\mathsf{L}}(\mathcal{K}) \cap \mathsf{sentences}^{\mathsf{L}}_{\forall}(\mathsf{L})$, $\mathsf{theorems}^{\mathsf{L}}_{\forall\mathsf{H}}(\mathcal{K}) = \mathsf{theorems}^{\mathsf{L}}(\mathcal{K}) \cap \mathsf{sentences}^{\mathsf{L}}_{\forall\mathsf{H}}(\mathsf{L})$, $\mathsf{theorems}^{\mathsf{L}}_{\not\approx}(\mathcal{K}) = \mathsf{theorems}^{\mathsf{L}}(\mathcal{K}) \cap \mathsf{sentences}^{\mathsf{L}}_{\not\approx}(\mathsf{L})$, $\mathsf{theorems}^{\mathsf{L}}_{\not\approx\forall}(\mathcal{K}) = \mathsf{theorems}^{\mathsf{L}}(\mathcal{K}) \cap \mathsf{sentences}^{\mathsf{L}}_{\not\approx\forall}(\mathsf{L})$ and $\mathsf{theorems}^{\mathsf{L}}_{\not\approx\forall\mathsf{H}}(\mathcal{K}) = \mathsf{theorems}^{\mathsf{L}}(\mathcal{K}) \cap \mathsf{sentences}^{\mathsf{L}}_{\not\approx\forall\mathsf{H}}(\mathsf{L})$. □

In order to collect *the* models of some set of formulae, we need to cap the 'size' of the potential models. For our needs, *classes* of (*small*) structures suffice. Recall that sets of *small* structures must be *classes*, and if $\mathcal{K}$ is a *class* of structures, then each structure in $\mathcal{K}$ is *small* (see Definition 1.7 on page 15 and the subsequent remark).

Definition 1.423 (The Models of Γ) For a set Γ of L-formulae, let $\mathsf{Mod}^{\mathfrak{e}}(\Gamma)$ denote the *class* of all *small* $\mathfrak{e}$-structures $\mathbf{A}$, such that $\models_{\mathbf{A}} \Gamma$. □

1.5.11.6 Elementary Classes

Definition 1.424 (Elementary Classes) A class $\mathcal{K}$ of L-structures is called **elementary** if there exists a set of L-sentences Γ such that $\mathcal{K} = \mathsf{Mod}^{\mathsf{L}}(\Gamma)$, in which case Γ is called an **axiomatization of $\mathcal{K}$**. □

Remark 1.425 The requirement that the formulae in an axiomatization be sentences is simply one of convenience.

Remark 1.426 $\Gamma \subseteq \mathsf{theorems}^{\mathsf{L}}(\mathcal{K})$ iff $\mathcal{K} \subseteq \mathsf{Mod}^{\mathsf{L}}(\Gamma)$.

Example 1.427 (Elementary Groupoid Theories a.k.a. Modern-Group Theory)

> A groupoid is commutative iff it satisfies $x * y \approx y * x$ and is associative (i.e., a semigroup) iff it satisfies $(x * y) * z \approx x * (y * z)$. Modern-monoids are semigroups satisfying $\exists[e]\, \forall[x]\, x * e \approx x \approx x * e$ and modern-groups are semigroups satisfying
>
> $$\exists[e] \ \left((\forall[x]\, x * e \approx x \approx x * e) \,\mathsf{and}\, \left(\forall[x]\, \exists[x']\, x' * x \approx x * x' \approx e\right)\right)$$
>
> So the classes of all groupoids, commutative groupoids, semigroups, modern-monoids, modern-groups and abelian modern-groups, are all elementary classes.

□

In the next example, observe how much 'simpler' the characterizing formulae are, compared to the previous example. The complex existential formulae have been replaced by identities. This is the advantage of 'typing' the inverse operation and the identity.

Example 1.428 (Elementary (Universal) Monoid and Group Theories)

Monoids are precisely the algebras of type monoid, with a semigroup reduct, satisfying $x * \mathbf{1} \approx \mathbf{1} * x \approx x$. A **group G** is an algebra of type group with a monoid reduct satisfying $x * x^{-\mathbf{1}} = x^{-\mathbf{1}} * x = \mathbf{1}$. So the classes of monoids, groups and abelian groups are all elementary.

□

Example 1.429 (Rings and Fields)

The **type of rings** has algebraic type $\langle +, -, \mathbf{0}, \cdot \rangle$ with arities $\langle 2, 1, 0, 2 \rangle$, where $\cdot$ is usually denoted 'invisibly'. A **ring** is an algebra of type ring, whose $\langle +, -, \mathbf{0} \rangle$-reduct is an abelian group, whose $\langle \cdot \rangle$-reduct is a semigroup, and which further satisfies the identities $x \cdot (y+z) \approx (x \cdot y) + (x \cdot z)$ and $(x+y) \cdot z \approx (x \cdot z) + (y \cdot z)$. If the semigroup reduct of a ring commutes, then we say that the ring **commutes** and speak of a **commutative ring**.

The **type of ring with unit** extends the type of rings by a single constant symbol $\mathbf{1}$. A **ring with unit** is an algebra of type ring with unit, with a ring reduct (the natural one) and a monoid $\langle \cdot, \mathbf{1} \rangle$-reduct. A commutative ring $\mathbf{R}$ with unit is called an **integral domain**, if it satisfies the formulae $\mathbf{0} \not\approx \mathbf{1}$ and $x \not\approx \mathbf{0}\,\mathsf{and}(xy \approx xz \vee yx \approx zx) \longrightarrow y \approx z$, and is called a **division-ring**, if $\mathbf{0} \not\approx \mathbf{1}$ and $x \not\approx \mathbf{0} \longrightarrow \exists[y]\, xy \approx \mathbf{1} \approx yx$. A **field** is a commutative division ring.

□

Remark 1.430 Note further the simple nature of the identities defining semigroups, monoids, groups, abelian groups, rings, commutative rings and rings with unit, contrasting these with the more complex formulae defining integral domains, division rings and fields. The later formulae contain the 'not-equals' relation, implication and existential quantifiers. The former, on the other hand, are essentially defined by sets of universally quantified equations. The former classes of algebras, defined by such equations, are known as *equational classes* or *varieties*, and yield a relatively simple model theory (see $\oint$1.5.13.2).

1.5.12 Characterizing the Leibniz Relation

Recall that by Theorem 1.360, the congruences $\mathsf{Con}(\mathbf{A})$, of a structure $\mathbf{A}$, form a principal ideal of the 'algebra congruences' $\mathsf{Con}(\mathsf{alg}(\mathbf{A}))$[Elg98, 208],[BP89a, T1.5]. Since principal ideals, by definition, have a top element, we were able to define $\Omega_{\mathbf{A}}$ to be the largest element of the principal ideal $\mathsf{Con}(\mathbf{A})$. We now characterize the Leibniz relation $\Omega_{\mathbf{A}}$.

Theorem 1.431 [Elg98, 207],[BP89a] For a structure $\mathbf{A}$, $a\,\Omega_{\mathbf{A}}\,b$ iff, for every formula $\phi(x, \vec{y})$, without equality, and all $\vec{c} \in \mathsf{uni}(\mathbf{A})$, $\models_{\mathbf{A}} \phi(a, \vec{c})$ iff $\models_{\mathbf{A}} \phi(b, \vec{c})$.

Corollary 1.432 [Elg98, 207] If, for some $\bowtie \in \mathsf{Symb_r}(\mathbf{A})$, $\bowtie^{\mathbf{A}} \neq \mathsf{uni}(\mathbf{A})^{\mathsf{ar}(\bowtie)}$ and $\bowtie^{\mathbf{A}} \neq \emptyset$, then $\Omega_{\mathbf{A}} \neq \blacksquare_{\mathbf{A}}$.

Corollary 1.433 If, for every $\bowtie \in \mathsf{Symb_r}(\mathbf{A})$, $\bowtie^{\mathbf{A}} = \mathsf{uni}(\mathbf{A})^{\mathsf{ar}(\bowtie)}$ or $\bowtie^{\mathbf{A}} \neq \emptyset$, then $\Omega_{\mathbf{A}} = \blacksquare_{\mathbf{A}}$.

1.5.13 The Equational Theory of Algebras and Varieties

1.5.13.1 Equations over Algebras

The primary content of this document, presented in Part V, concerns an 'algebraization' technique more general than that of Blok and Pigozzi. The need for such a technique arose in an attempt to 'algebraize' logics arising naturally from algebras, but which are inherently unalgebraizable in the sense of Blok and Pigozzi's. In Chapter 8, we shall describe a general technique which we have been using to construct these logics. Central to this technique, is to take some closed system arising naturally from an algebra or class of algebras, such as subuniverses, ideals or cosets, and to consider this closed system over a (universe of a) free algebra. While typically such closed systems over the free algebra are finitary and 'structural' ('structural' in a sense to be described later), the points of such closed sets are not terms but sets of terms, and so cannot constitute the theory of a propositional calculus. By unioning such a closed set, however, one obtains a set of terms, and hence a potential theory.

Recall Definition 1.383 on page 73, describing the construction of a free algebra $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ for a given class $\mathcal{K}$ of algebras, essentially by factoring the term algebra $\mathbf{Tm}_V$ by the congruence

$$\equiv_{\mathcal{K}}^{V} = \bigcap \{\alpha \in \mathsf{Con}(\mathbf{Tm}_V) \ : \ \mathbf{Tm}_V/\alpha \in \mathcal{IS}(\mathcal{K})\}. \tag{1.95}$$

Our primary aim in this sub-section, is to give a model-theoretic description of this universal-algebraic construction, thereby characterizing each *point* of a free algebra by describing (in terms of equality modulo $\mathcal{K}$) the *set of terms* constituting this point.

Theorem 1.434 ([RMT87]) Let $\mathcal{K}$ be *any* class of $\mathfrak{a}$-algebras and $p, q \in \mathsf{Tm}_V^{\mathfrak{a}}$. If $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ exists, then the following conditions are equivalent.

1. $\mathcal{K} \models \ p \approx q$.
2. $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}} \models \ p \approx q$.
3. $\bar{p} = \bar{q}$ in $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$.
4. For any set W with $W \supseteq V$, $\mathbf{F}_{\mathcal{K}}^{\overline{[W]}} \models \ p \approx q$.

□

The following corollary to Theorem 1.434 provides a model-theoretic characterization of $\equiv_{\mathcal{K}}^{V}$ and the points of the associated $\mathcal{K}$-free algebra.

Corollary 1.435 If $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ exists, then $p \equiv_{\mathcal{K}}^{V} q$ iff $\models_{\mathcal{K}} p \approx q$.

Proof. By Theorem 1.434 on page 84, $\models_{\mathcal{K}} p \approx q$ iff $\bar{p} = \bar{q}$ in $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ iff $\equiv_{\mathcal{K}}^{V}[\![p]\!] = \equiv_{\mathcal{K}}^{V}[\![q]\!]$ iff $p \equiv_{\mathcal{K}}^{V} q$. ◇

Consequently, each point of the free algebra $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}}$ is a set of terms all of which are 'equal' over $\mathcal{K}$, and any two distinct points have no terms that are commonly 'equal' over $\mathcal{K}$. It is this observation that we shall exploit in Chapter 8 as a means of obtaining propositional calculi (whose theories are sets of *terms*) from closed set systems over the universes of free algebras (whose closed sets are sets of points from the free algebra, which are *not* sets of terms).

1.5.13.2 Varieties

Definition 1.436 (Varieties) A non-empty class $\mathcal{V}$ of $\mathfrak{a}$-algebras is called a **variety** if it is an elementary class axiomatizable by identities/equations only. The intersection of varieties of $\mathfrak{a}$-algebras is again a variety of $\mathfrak{a}$-algebras, and the class of all $\mathfrak{a}$-algebras is a variety. Consequently, there exists a smallest variety containing a given class $\mathcal{K}$, which we denote by $\mathcal{V}\langle\mathcal{K}\rangle$. We say that $\mathcal{V}\langle\mathcal{K}\rangle$ is the **variety generated by** $\mathcal{K}$. A variety $\mathcal{V}$ is called **finitely generated** if $\mathcal{V} = \mathcal{V}\langle\mathcal{K}\rangle$ for some finite set $\mathcal{K}$ of *finite* algebras. A variety $\mathcal{V}$ that is a subclass of a class $\mathcal{K}$ is called a **subvariety** of $\mathcal{K}$. We also regard $\mathcal{V}\langle\cdot\rangle$ as a class operator. □

Theorem 1.437 (Birkhoff) [BS81, Theorem 9.5, Pg. 61] $\mathcal{V}\langle\mathcal{K}\rangle = \mathcal{HSP}(\mathcal{K})$.

Remark 1.438 If $\mathcal{V}$ is a variety and $\mathbf{A} \in \mathcal{V}$, then $\mathsf{Con}(\mathbf{A}) = \mathsf{Con}^{\mathcal{V}}(\mathbf{A})$. This follows from the $\mathcal{H}$ in the previous theorem. For algebras *outside* of the variety $\mathcal{V}$, relative congruences and congruences need not coincide.

Theorem 1.439 (Birkhoff's Theorem) [RMT87] Every member of a variety $\mathcal{V}$ is isomorphic to a subdirect product of subdirectly irreducible members of $\mathcal{V}$

Theorem 1.440 ([BS81]) (i) If $\langle \mathbf{A}_i : i \in I\rangle$ is a system of finite algebras with $\{\mathbf{A}_i : i \in I\} = \{\mathbf{B}_1, \ldots, \mathbf{B}_n\}$, for some natural number n, and X is an ultrafilter over I, then $(\prod_I \mathbf{A}_i)/\Theta_X$ is isomorphic to one of the algebras $\mathbf{B}_1, \ldots, \mathbf{B}_n$. In particular, if $\mathcal{K}$ is a finite set of finite algebras, then $\mathcal{P}_U(\mathcal{K}) \subseteq \mathcal{I}(\mathcal{K})$. (ii) [Jónsson] Let $\mathcal{V}\langle\mathcal{K}\rangle$ be a congruence distributive variety. If $\mathbf{A}$ is a subdirectly irreducible algebra in $\mathcal{V}\langle\mathcal{K}\rangle$, then $\mathbf{A} \in \mathcal{HSP}_U(\mathcal{K})$, and hence $\mathcal{V}\langle\mathcal{K}\rangle = \mathcal{IP}_S\mathcal{HSP}_U(\mathcal{K})$. (iii) [Jónsson] If $\mathcal{K}$ is a finite set of finite algebras and $\mathcal{V}\langle\mathcal{K}\rangle$ is congruence distributive, then the subdirectly irreducible algebras of $\mathcal{V}\langle\mathcal{K}\rangle$ are in $\mathcal{HS}(\mathcal{K})$, and $\mathcal{V}\langle\mathcal{K}\rangle = \mathcal{IP}_S\mathcal{HSK}$.

Convention 1.441 (Treating Algebraic Type $\mathfrak{a}$ as a Variety and a Construct) It is convenient to conflate a *type* $\mathfrak{a}$ of algebras with the *variety* of all $\mathfrak{a}$-algebras and with the *construct* of all $\mathfrak{a}$-algebras with homomorphisms.

1.5.13.3 Mal'cev Conditions

A characterization of a formula of varieties (or more precisely, a class of varieties) by the existence of certain terms and the satisfaction of certain identities involving these terms shall be referred to as a **Mal'cev characterization** of the said formula. (For a more formally precise definition, see [Tay73].) For example, it is well known [Mal54] that a variety $\mathcal{V}$ is congruence permutable if and only if

[$\mathcal{M}$] There exists a ternary term $p(x, y, z)$ (over $V = \{x, y, z\}$) such that $\mathcal{V}$ satisfies the identities $p(x, x, y) \approx y$ and $p(x, y, y) \approx x$.

The condition [$\mathcal{M}$] is an example of a **Mal'cev condition**. The above characterization of congruence permutability (or more precisely, of the class of congruence permutable varieties) is thus a Mal'cev characterization. Congruence n-permutability is also characterized by a Mal'cev condition.

Theorem 1.442 [Hag73] A variety $\mathcal{V}$ is congruence n-permutable iff there exist ternary terms $\Delta_1, \ldots, \Delta_{n-1}$ such that $\mathcal{V}$ satisfies the identities

$$\begin{aligned} x &\approx \Delta_1(x,y,y), && (1.96)\\ \Delta_{i-1}(x,x,y) &\approx \Delta_i(x,y,y), \quad \text{for } i = 2, \ldots, n-1, \text{ and} && (1.97)\\ \Delta_{n-1}(x,x,y) &\approx y. && (1.98) \end{aligned}$$

□

The following Mal'cev characterization of congruence modular varieties was discovered by Day [Day69].

Theorem 1.443 [Day69] A variety $\mathcal{V}$ is congruence modular iff there exist integer $n \geq 2$ and quaternary terms $p_0, \ldots, p_n$, such that $\mathcal{V}$ satisfies the identities

$$p_i(x,y,y,x) \approx x, \quad \text{for } i = 1, \ldots, n \tag{1.99}$$

and the identities

$$\begin{aligned} x &\approx p_0(x,y,z,w), && (1.100)\\ p_i(x,x,w,w) &\approx p_{i+1}(x,x,w,w), \quad \text{for even } i,\ 0 \leq i < n, && (1.101)\\ p_i(x,y,y,w) &\approx p_{i+1}(x,y,y,w), \quad \text{for odd } i,\ 1 \leq i < n, \text{ and} && (1.102)\\ p_n(x,y,z,w) &\approx w. && (1.103) \end{aligned}$$

□

Congruence regularity and congruence point regularity, when satisfied by all algebras of a variety, also have Mal'cev characterizations.

Theorem 1.444 [Dud83] For a variety $\mathcal{V}$ the following conditions are equivalent.

1. Every algebra in $\mathcal{V}$ is congruence regular.
2. There exists integer $k \geq 1$, quaternary terms $p_1, \ldots, p_k$ and ternary terms $\Delta_1, \ldots, \Delta_k$ such that $\mathcal{V}$ satisfies the identities

$$\Delta_1(x,x,z) \approx z, \quad \text{for } i = 1, \ldots, k \tag{1.104}$$

and the identities

$$\begin{aligned} x &\approx p_1(\Delta_1(x,y,z),x,y,z), && (1.105)\\ p_i(z,x,y,z) &\approx p_{i+1}(\Delta_{i+1}(x,y,z),x,y,z), \quad \text{for } i = 1, \ldots, k-1, \text{ and} && (1.106)\\ p_i(z,x,y,z) &\approx y. && (1.107) \end{aligned}$$

□

It follows, from the previous three results, that congruence regular varieties must be congruence modular and congruence n-permutable for some $n > 2$, a result first observed by Hagemann [Hag73].

Corollary 1.445 [Hag73] Congruence regular varieties are congruence modular and congruence n-permutable for some $n > 2$. If k is as in (2) of the previous theorem, then the variety is congruence $k+1$-permutable.

1.5.14 The Quasi-Equational Theory of Algebras and Quasivarieties

Definition 1.446 (Quasivarieties) A non-empty class $\mathcal{K}$ of $\mathfrak{a}$-algebras is called a **quasivariety** if it is an elementary class axiomatizable by quasi-identities and identities only. The intersection of quasivarieties of $\mathfrak{a}$-algebras is again a quasivariety of $\mathfrak{a}$-algebras. There exists a smallest quasivariety containing a given class $\mathcal{K}$, which we denote by $\mathcal{Q}\langle\mathcal{K}\rangle$. We say that $\mathcal{Q}\langle\mathcal{K}\rangle$ is the **quasivariety generated by** $\mathcal{K}$. A quasivariety $\mathcal{K}$ is called **finitely generated** if $\mathcal{K} = \mathcal{Q}\langle\mathcal{K}'\rangle$ for some finite set $\mathcal{K}'$ of *finite* algebras. A quasivariety $\mathcal{K}'$ that is a subclass of a class $\mathcal{K}$ is called a **subquasivariety** of $\mathcal{K}$. We regard $\mathcal{Q}\langle\cdot\rangle$ as a class operator. □

Theorem 1.447 (Mal'cev) [BS81] $\mathcal{Q}\langle\mathcal{K}\rangle = \mathcal{ISPP}_U(\mathcal{K})$.

Remark 1.448 If $\mathcal{K}$ is a quasivariety then $\mathbf{F}_{\mathcal{K}}^{\overline{[V]}} \in \mathcal{K}$.

Remark 1.449 Unlike varieties (see Remark 1.438 on page 85), congruences on algebras of a quasivariety $\mathcal{K}$ need not be $\mathcal{K}$-congruences, since quasivarieties are not generally closed under homomorphic images (the lack of $\mathcal{H}$ in the previous theorem).

Proposition 1.450 [vA95, P 0.4.3] Let $\mathcal{K}$ be a class of $\mathfrak{a}$-algebras and $\mathbf{A}$ an $\mathfrak{a}$-algebra, not necessarily in $\mathcal{K}$. If $\mathcal{K}$ is closed under $\mathcal{I}$, $\mathcal{S}$ and $\mathcal{P}$, then $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ forms a closed set system over $\mathsf{uni}(\mathbf{A})^2$, and if $\mathcal{K}$ is a quasivariety, then $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ is an algebraic closed system.

Definition 1.451 (Relative Congruence Generation) If $\mathcal{K}$ is a *quasivariety* of $\mathfrak{a}$-algebras and $\mathbf{A}$ an $\mathfrak{a}$-algebra, the algebraic closure operator, associated with the algebraic closed set system $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, is denoted by $\|\cdot\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$, and the associated algebraic lattice is denoted by $\mathbf{Con}^{\mathcal{K}}(\mathbf{A})$. We denote the least $\mathcal{K}$-congruence on $\mathbf{A}$ by $\bot^{\mathcal{K}}_{\mathbf{A}}$. □

The following characterization of $\mathcal{K}$-congruence generation, taken from [BR99], is a variant of a result in [CD90].

Lemma 1.452 For a *quasivariety* $\mathcal{K}$, an algebra $\mathbf{A}$, a subset Y of A^2 and elements $a, b \in A$, we have $\langle a, b\rangle \in \|Y\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$ iff there exist $l, m \in \omega$, a quasi-identity $[\bigwedge_{i<l} p_i(\vec{x}) \approx p'_i(\vec{x})]$ and $[\bigwedge_{j<m} q_j(\vec{x}) \approx q'_j(\vec{x})] \rightarrow r(\vec{x}) \approx s(\vec{x})$ satisfied by $\mathcal{K}$ and elements $\vec{c} \in A$, such that for $i < l$ and $j < m$, we have $\langle p_i^{\mathbf{A}}(\vec{c}), {p'_i}^{\mathbf{A}}(\vec{c})\rangle \in Y$, $q_j^{\mathbf{A}}(\vec{c}) = {q'_j}^{\mathbf{A}}(\vec{c})$, $r^{\mathbf{A}}(\vec{c}) = a$ and $s^{\mathbf{A}}(\vec{c}) = b$.

Remark 1.453 $\underset{\rightarrow}{f}[\|\alpha\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}] \subseteq \left\| \underset{\rightarrow}{f}[\alpha] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$ for any endomorphism f of algebra $\mathbf{A}$ and $\alpha \subseteq \mathsf{uni}(\mathbf{A})^2$. □

Definition 1.454 (The Equational Consequence Relation) Let Σ be a set of identities. We write $\Sigma \models_{\mathcal{K}} p \approx q$ iff, for all $\mathbf{A} \in \mathcal{K}$ and every homomorphism f from the absolutely free term algebra into $\mathbf{A}$, if $f(p') = f(q')$, for all $p' \approx q' \in \Sigma$, then $f(p) = f(q)$. For identities Σ and Σ', we write $\Sigma \models_{\mathcal{K}} \Sigma'$ iff $\Sigma \models_{\mathcal{K}} p \approx q$ for all $p \approx q \in \Sigma'$, and write $\Sigma \mathrel{=\!\mid\!\models}_{\mathcal{K}} \Sigma'$ iff $\Sigma \models_{\mathcal{K}} \Sigma'$ and $\Sigma' \models_{\mathcal{K}} \Sigma$.

We may write $\mathbf{A}$ for $\mathcal{K} = \{\mathbf{A}\}$ in these notations. When a single formula appears where a set of formulae is expected, the formula abbreviates the singleton containing that formula. In such notions, we may separate sets of formulae (and formulae) by commas; these commas are to be taken as unions. $\square$

Theorem 1.455 (Finitariness) [BP89a] If $\Gamma \models_{\mathcal{K}} \eta$, then there exists a finite subset $\Gamma' \subseteq_{\mathsf{f}} \Gamma$, such that $\Gamma' \models_{\mathcal{K}} \eta$.

Theorem 1.456 (Structurality) [BP89a] If $\Gamma \models_{\mathcal{K}} \eta$, the formulae in $\Gamma \cup \{\eta\}$ are all *open*, and σ is a substitution to *open* formulae, then $\sigma[\Gamma] \models_{\mathcal{K}} \sigma(\eta)$.

For quasivarieties $\mathcal{K}$, an essentially well-known characterization of $\models_{\mathcal{K}}$ is stated below in the precise form needed here, so as to avoid repeated explanation.

Lemma 1.457 For a quasivariety $\mathcal{K}$ and a set $\Sigma \cup \{r \approx s\}$ of equations, the following conditions are equivalent.

1. $\langle r, s\rangle \in \|\{\langle g, h\rangle : g \approx h \in \Sigma\}\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}$.
2. $\Sigma \models_{\mathcal{K}} r \approx s$.
3. For all algebras $\mathbf{A}$ and all $\tilde{a} \in A^\omega$, $\langle r^{\mathbf{A}}(\tilde{a}), s^{\mathbf{A}}(\tilde{a})\rangle \in \left\|\{\langle g^{\mathbf{A}}(\tilde{a}), h^{\mathbf{A}}(\tilde{a})\rangle : g \approx h \in \Sigma\}\right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$.
4. For all algebras $\mathbf{A} \in \mathcal{K}$ and all $\tilde{a} \in A^\omega$, $\langle r^{\mathbf{A}}(\tilde{a}), s^{\mathbf{A}}(\tilde{a})\rangle \in \left\|\{\langle g^{\mathbf{A}}(\tilde{a}), h^{\mathbf{A}}(\tilde{a})\rangle : g \approx h \in \Sigma\}\right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$.
5. There exists a finite $\Sigma' \subseteq \Sigma$ (whose occurring variables are among $\vec{x} \in V$, say) such that $\mathcal{K}$ satisfies the quasi-identity $\left[\bigwedge_{g \approx h \in \Sigma'} g(\vec{x}) \approx h(\vec{x})\right] \to r(\vec{x}) \approx s(\vec{x})$.
6. $\langle \tilde{r}, \tilde{s}\rangle \in \|\{\langle g, h\rangle : g \approx h \in \Sigma\}\|_{\Theta^{\mathcal{K}}_{\mathbf{F}_{\mathcal{K}}}}$.

Definition 1.458 (Trivial Quasivarieties) A quasivariety $\mathcal{K}$ is called **trivial** if, for all terms p and q, $\models_{\mathcal{K}} p \approx q$, otherwise $\mathcal{K}$ is called **non-trivial**. $\square$

Remark 1.459 By structurality, $\mathcal{K}$ is trivial iff, there exist distinct variables x and y, such that $\models_{\mathcal{K}} x \approx y$.

Definition 1.460 (Equationally Definable Constants and Unary Terms) Let $\mathcal{K}$ be a quasivariety of algebras and $p(x_1, \ldots, x_n)$ a term. We say that $p(x_1, \ldots, x_n)$ is an **equationally definable constant** modulo $\mathcal{K}$ (or just a **$\mathcal{K}$-constant**) if, for all terms $q_1, \ldots, q_n$ and $r_1, \ldots, r_n$, $\models_{\mathcal{K}} p(q_1, \ldots, q_n) \approx p(r_1, \ldots, r_n)$; in this case we may just denote the term by p, sensibly write $p^{\mathbf{A}}$ for $\mathbf{A} \in \mathcal{K}$, and sensibly write $\alpha[\![p^{\mathbf{A}}]\!]$ for *any* algebra $\mathbf{A}$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.

We say that $p(x_1, \ldots, x_n)$ is an **equationally definable unary term** modulo $\mathcal{K}$ (or just is **$\mathcal{K}$-unary**) in variable x_i if, for all terms $q_1, \ldots, q_{i-1}, q_{i+1} \ldots, q_n$ and $r_1, \ldots, r_{i-1}, r_{i+1} \ldots, r_n$, $\models_{\mathcal{K}} p(q_1, \ldots, q_{i-1}, x_i, q_{i+1} \ldots, q_n) \approx p(r_1, \ldots, r_{i-1}, x_i, r_{i+1} \ldots, r_n)$; in this case we may just denote the term by $p(x_i)$, sensibly write $p^{\mathbf{A}}(a)$ for $\mathbf{A} \in \mathcal{K}$ and $a \in \mathsf{uni}(\mathbf{A})$, and sensibly write $\alpha[\![p^{\mathbf{A}}(a)]\!]$ for *any* algebra $\mathbf{A}$, $a \in \mathsf{uni}(\mathbf{A})$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. $\square$

Remark 1.461 By structurality, $p(x_1, \ldots, x_n)$ is a $\mathcal{K}$-constant iff, there exist distinct variables $x_1, \ldots, x_n, y_1, \ldots, y_n$, such that $\models_{\mathcal{K}} p(x_1, \ldots, x_n) \approx p(y_1, \ldots, y_n)$.

Remark 1.462 Constant symbols are $\mathcal{K}$-constants.

Note 1.463 (Definable Constants and Subuniverses) Equationally definable constants, unlike fundamental constants, are not forced into the least subuniverse when interpreted in an algebra of the class. There exist quasivarieties of algebras with no constant symbols in their type, but with equationally definable constants; for algebras of such quasivarieties, the empty-set *is* a subuniverse. □

Remark 1.464 By structurality, $p(x_1, \dots, x_n)$ is $\mathcal{K}$-unary in x_i iff, there exist distinct variables $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n, y_1, \dots y_{i-1}, y_{i+1}, \dots, \dots, y_n$, such that

$$\models_{\mathcal{K}} p(z_1, \dots, z_{i-1}, x_i, z_{i+1}, \dots, z_n) \approx p(y_1, \dots, z_{i-1}, x_i, y_{i+1}, \dots, y_n).$$

Remark 1.465 Unary terms are $\mathcal{K}$-unary.

1.5.14.1 Quasi-Mal'cev Conditions

A **quasi-Mal'cev** condition is a characterization of a property satisfied by a class of algebras in terms of the satisfaction by the class of certain quasi-identities and identities. The Mal'cev characterization of congruence regularity, given in Theorem 1.444, can be given a much simpler (and far more intuitive) quasi-Mal'cev characterization. The following quasi-Mal'cev characterization was discovered by Csákány [Csá70] over a decade before the first Mal'cev characterization was discovered by Duda [Dud83].

Theorem 1.466 [Csá70] For a variety $\mathcal{V}$ the following conditions are equivalent.

1. Every algebra in $\mathcal{V}$ is congruence regular.
2. There exists integer $k \geq 1$ and ternary terms $\Delta_1, \dots, \Delta_k$ such that $\mathcal{V}$ satisfies the quasi-identities
$$\bigwedge\nolimits_{1 \leq i \leq k} \Delta_i(x, y, z) \approx z \leftrightarrow x \approx y. \tag{1.108}$$

1.5.15 Further Examples

In the following example, we introduce lattices and semilattices as algebras, and note the one-to-one correspondence between lattice-orders and lattice-algebras.

Example 1.467 (Lattices and Semilattices as Algebras)

> **Definition 1.468 (Semilattices)** A $\boxdot$**-semilattice** is an algebra of type $\langle \overset{2}{\boxdot} \rangle$ that is a commutative semigroup satisfying the **identity of idempotence** $x \boxdot x \approx x$. A $\boxdot$**-semilattice expansion**, is any algebra with a $\boxdot$-reduct that is a $\boxdot$-semilattice. □
>
> If $\mathbf{P}$ is a join (resp. meet) semilattice-order, then $\langle \mathsf{uni}(\mathbf{P}); \vee^{\mathbf{P}} \rangle$ (resp. $\langle \mathsf{uni}(\mathbf{P}); \wedge^{\mathbf{P}} \rangle$) is a semilattice. Conversely, if $\mathbf{P}$ is a semilattice (in the algebraic sense), then $a \leq b$ if and only if $a \boxdot^{\mathbf{P}} b = b$ defines a join-semilattice-order $\mathbf{P}'$ on $\mathsf{uni}(\mathbf{P})$ with $\boxdot^{\mathbf{P}} = \vee^{\mathbf{P}'}$. These operations are mutually inverse.

Definition 1.469 (Lattices) A **lattice** is an algebra of type $\mathfrak{t}(\mathsf{lat}) \doteq \langle \overset{2}{\wedge}, \overset{2}{\vee} \rangle$, that is a $\wedge$-semilattice expansion and a $\vee$-semilattice expansion, satisfying the **absorption** identities, $x \vee (x \wedge y) \approx x$ and $x \wedge (x \vee y) \approx x$. A **lattice expansion** is an algebra with a $\langle \wedge, \vee \rangle$-reduct that is a lattice. A lattice expansion $\mathbf{P}$ is called **distributive** if it satisfies the identity $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$, or the equivalent identity $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$, and is called **modular** if it satisfies the identity $((x \wedge z) \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z)$. □

If $\mathbf{P}$ is a lattice-order, then $\langle \mathsf{uni}(\mathbf{P}); \vee^{\mathbf{P}}, \wedge^{\mathbf{P}} \rangle$ is a lattice. Conversely, if $\mathbf{P}$ is a lattice (in the algebraic sense), then $a \vee^{\mathbf{P}} b \approx b$ iff $a \wedge^{\mathbf{P}} b \approx a$, and $a \leq b$ if and only if $a \vee^{\mathbf{P}} b = b$ defines a lattice-order $\mathbf{P}'$ on $\mathsf{uni}(\mathbf{P})$ with $\vee^{\mathbf{P}} = \vee^{\mathbf{P}'}$ and $\wedge^{\mathbf{P}} = \wedge^{\mathbf{P}'}$. These operations are mutually inverse.

Convention 1.470 ($\leq$) When working with a lattice $\mathbf{P}$, we shall often invoke this partial order, which we shall denote by $\leq^{\mathbf{P}}$. For lattice terms p and q, $p \leq q$ abbreviates the identity $p \vee q \approx q$ and $p \geq q$ abbreviates the identity $p \wedge q \approx q$. For $\emptyset \neq P \subseteq \mathsf{Tm}$ and $q \in \mathsf{Tm}$, $P \leq q$ abbreviates $\{p \leq q \,:\, p \in P\}$ (which abbreviates $\{p \vee q \approx q \,:\, p \in P\}$) and $P \geq q$ abbreviates $\{p \geq q \,:\, p \in P\}$ (which abbreviates $\{p \wedge q \approx q \,:\, p \in P\}$), and $P \wedge q$ shall abbreviate $\{p \wedge q \,:\, p \in P\}$ and $P \vee q$ shall abbreviate $\{p \vee q \,:\, p \in P\}$.

Definition 1.471 (Bounded Lattices) A 0-**lattice** is a lattice expansion of type $\mathfrak{t}(\mathsf{lat}_0) \doteq \langle \overset{2}{\wedge}, \overset{2}{\vee}, \overset{0}{0} \rangle$ and satisfying $0 \wedge x \approx 0$. A 1-**lattice** is a lattice expansion of type $\mathfrak{t}(\mathsf{lat}_1) \doteq \langle \overset{2}{\wedge}, \overset{2}{\vee}, \overset{0}{1} \rangle$ with a lattice reduct and satisfying $1 \vee x \approx 1$. We define a 0-**lattice expansion** and a 1-**lattice expansion** in the obvious manner. A 01-**lattice** is an algebra of type $\mathfrak{t}(\mathsf{lat}_{01}) \doteq \langle \overset{2}{\wedge}, \overset{2}{\vee}, \overset{0}{0}, \overset{0}{1} \rangle$ that is both a 0-lattice expansion and a 1-lattice expansion. We define a 01-**lattice expansion** in the obvious manner. □

Definition 1.472 (Complemented Lattices) A 0-**complemented-lattice** is a 0-lattice expansion of type $\mathfrak{t}(\mathsf{lat}'_0) \doteq \langle \overset{2}{\wedge}, \overset{2}{\vee}, \overset{1}{'}, \overset{0}{0} \rangle$ satisfying

$$x \wedge x' \approx 0. \tag{1.109}$$

A 1-**lattice** is a 1-lattice expansion of type $\mathfrak{t}(\mathsf{lat}'_1) \doteq \langle \overset{2}{\wedge}, \overset{2}{\vee}, \overset{1}{'}, \overset{0}{1} \rangle$ satisfying

$$x \vee x' \approx 1. \tag{1.110}$$

A **complemented-lattice** is a 01-lattice expansion of type $\mathfrak{t}(\mathsf{lat}'_{01}) \doteq \langle \overset{2}{\wedge}, \overset{2}{\vee}, \overset{1}{'}, \overset{0}{0}, \overset{0}{1} \rangle$ satisfying (1.109) and (1.110). We define a 0-**complemented-lattice expansion**, a 1-**complemented-lattice expansion** and a **complemented-lattice expansion** in the obvious manner. □

Remark 1.473 A complemented lattice is 0-complemented and 1-complemented.

Convention 1.474 When we speak of a **quasivariety of lower-unbounded lattice expansions** (resp. **quasivariety of upper-unbounded lattice expansions**), we mean a quasivariety of lattice expansions, with at least one lower-unbounded (resp. upper-unbounded) member.

Warning 1.475 When we say that $\mathcal{K}$ is a quasivariety of lower-unbounded lattice expansions, we do *not* mean that *every* member of $\mathcal{K}$ is lower-unbounded.

Definition 1.476 (Boolean Algebras) A **boolean algebra** is a distributive complemented-lattice. □

□

Warning 1.477 Lattice-order homomorphisms and lattice homomorphisms do *not* coincide, hence Definition 1.188 on page 42.

Convention 1.478 (Lattices vs. Lattice-Orders) When we speak of a **lattice**, we shall mean a lattice algebra. Orders that are lattices are called *lattice-orders*. *Complete lattices* (and hence *algebraic lattices*) are special lattice-*orders*.

We now give a standard presentation of M-theory, i.e., structures with one relational symbol and an arbitrary algebra-reduct. These structures, known in the context of algebraic logic as matrices, form the standard models of deductive systems in the sense of [BP89a].

Example 1.479 (M-Theory)

Definition 1.480 (Type of Matrices over Algebras) Let $\mathfrak{a}$ be a type of algebras and n a positive integer. The **type of n-matrices over $\mathfrak{a}$**, denoted $\mathfrak{a}^n_{\mathsf{D}}$, is the elementary type 'extending type $\mathfrak{a}$ by a single n-ary relation symbol, symbolically chosen so as not to 'collide' with any n-ary relation symbol of $\mathfrak{a}$, and typically denoted by the symbol D. We call n the **dimension**. The **type of n-matrices** is the type of n-matrices over the type of sets. □

Convention 1.481 (Dimension) Unless specified to the contrary, matrices under consideration have the same algebraic-type and dimension.

Definition 1.482 (Algebra-Matrices) Let $\mathfrak{a}$ be a type of algebras. An **$\mathfrak{a}$-matrix of dimension** n, is a structure of the type $\mathfrak{a}^n_{\mathsf{D}}$ (of n-matrices over $\mathfrak{a}$), in which case we call n the **dimension**. We tend to write $\mathsf{D}_{\mathbf{M}}$ for $\mathsf{D}^{\mathbf{M}}$, which we call the **designator**. An **$\mathfrak{a}$-matrix** is an $\mathfrak{a}$-matrix of dimension n, for some n. The $\mathfrak{a}$-reduct of an $\mathfrak{a}$-matrix $\mathbf{M}$ is denoted by $\mathsf{alg}(\mathbf{M})$, and is called the **scalar algebra of** the matrix. For an $\mathfrak{a}$-matrix $\mathbf{M}$, we write $\underrightarrow{\mathsf{uni}}(\mathbf{M})$ for $\mathsf{uni}(\mathbf{M})^{\mathsf{dim}(\mathbf{M})}$.

An **algebra-matrix** is an $\mathfrak{a}$-matrix for some type $\mathfrak{a}$ of algebras. For an arbitrary algebra-matrix $\mathbf{M}$, we denote the 'algebra'-subtype of $\mathsf{type}(\mathbf{M})$ by $\mathsf{type}_{\mathsf{s}}(\mathbf{M})$, which we call the **scalar subtype** of the matrix, and denote the dimension by $\mathsf{dim}(\mathbf{M})$. For an algebra $\mathbf{A}$, an $\mathbf{A}$-matrix is a $\mathsf{type}(\mathbf{A})$-matrix with scalar algebra $\mathbf{A}$. When the particular dimension is clear from the context, we may present an $\mathbf{A}$-matrix $\mathbf{M}$ by $\langle \mathsf{alg}(\mathbf{M}), \mathsf{D}_{\mathbf{M}} \rangle$. In such a presentation, we may write $\langle \mathbf{A}, a \rangle$ for $\langle \mathbf{A}, \{a\} \rangle$, where context unambiguous. We denote the product algebra $\mathsf{alg}(\mathbf{M})^{\mathsf{dim}(\mathbf{M})}$ by $\underrightarrow{\mathsf{alg}(\mathbf{M})}$. A **matrix** is a algebra-matrix whose algebra subtype is the type of sets.

Substructures of $\mathfrak{a}$-matrices (of the same dimension) are called **submatrices**. Structure homomorphisms, isomorphisms, etc between matrices (of the same dimension) are called **matrix-homomorphisms**, **matrix-isomorphisms**, etc, although we commonly drop the prefix 'matrix-' where ever unambiguous. □

Remark 1.483 $\mathbf{M}$ is a submatrix of $\mathbf{N}$ iff they have the same dimension, $\mathsf{alg}(\mathbf{M})$ is a subalgebra of $\mathsf{alg}(\mathbf{N})$, and $\mathsf{D}_{\mathsf{M}} = \mathsf{D}_{\mathsf{N}} \cap \mathsf{uni}(\mathsf{M})^{\mathsf{dim}(\mathsf{M})}$.

Remark 1.484 Function f is a matrix-homomorphism from $\mathbf{M}$ into $\mathbf{N}$ iff f is a homomorphism from $\mathsf{alg}(\mathbf{M})$ into $\mathsf{alg}(\mathbf{N})$, and $f\left[\mathsf{D}_{\mathbf{M}}\right] \subseteq \mathsf{D}_{\mathbf{N}}$.

Remark 1.485 f is an isomorphism of $\mathbf{M}$ onto $\mathbf{N}$ iff f is an isomorphism of $\mathsf{alg}(\mathbf{M})$ onto $\mathsf{alg}(\mathbf{N})$, and $f\left[\mathsf{D}_{\mathbf{M}}\right] = \mathsf{D}_{\mathbf{N}}$.

Remark 1.486 A matrix-homomorphism f from $\mathbf{M}$ into $\mathbf{N}$ is **reductive** iff f is surjective and $\mathsf{D}_{\mathsf{M}} = f^{-1}[\mathsf{D}_{\mathsf{N}}]$.

Remark 1.487 Matrix reductions are precisely the strict epimorphisms, by Remark 1.292 on page 59.

Proposition 1.488 [BP92, Pr 5.1][vA95, Pr 1.8.2 (ii)] Let f be a reductive (matrix) homomorphism from *algebra*-matrix $\mathbf{M}$ onto *algebra*-matrix $\mathbf{N}$. Then, for any term p and (algebra) homomorphism g from the (absolutely free) term algebra $\mathbf{Tm}$ into $\mathsf{alg}(\mathbf{M})$, $g(p) \in \mathsf{D}_{\mathsf{M}}$ iff $f(g(p)) \in \mathsf{D}_{\mathsf{N}}$.

Definition 1.489 (Compatibility) We say that a binary relation α on $\mathsf{uni}(\mathbf{M})$ is **compatible** with $\mathtt{A} \subseteq \underrightarrow{\mathsf{uni}}(\mathbf{M})$ if $\mathsf{a} \in \mathtt{A}, \mathsf{a} \underrightarrow{\alpha} \mathsf{b} \rightarrow \mathsf{b} \in \mathtt{A}$. □

Remark 1.490 [vA95, 90] A binary relation α on $\mathsf{uni}(\mathbf{M})$ is compatible with $\mathtt{A} \subseteq \underrightarrow{\mathsf{uni}}(\mathbf{M})$ iff $\mathsf{a} \in \mathtt{A} \rightarrow \underrightarrow{\alpha}[\![\mathsf{a}]\!] \subseteq \mathtt{A}$.

Definition 1.491 (Congruences, Quotients and Subquotients) Let $\mathbf{M}$ be an algebra-matrix. By an **M-congruence** or a **congruence** on/of $\mathbf{M}$, we mean an $\mathsf{alg}(\mathbf{M})$-congruence. Let α be an $\mathbf{M}$-congruence. By the quotient of $\mathbf{M}$ by congruence α, denoted $\mathbf{M}/\alpha$, we mean the matrix $\langle \mathsf{alg}(\mathbf{M})/\alpha, \mathsf{q}_{\alpha}\left[\mathsf{D}_{\mathbf{M}}\right]\rangle$. We shall explicitly speak of a **structure congruence** when we mean a congruence in the sense of Definition 1.349. □

Lemma 1.492 [BP88, L 3.3][vA95, L 1.8.7] If $\mathbf{N}$ is a submatrix of algebra-matrix $\mathbf{M}$ and α is an $\mathsf{alg}(\mathbf{M})$-congruence that is compatible with $\mathsf{D}_{\mathbf{M}}$, then the matrix $\langle \mathsf{alg}(\mathbf{N})/\alpha \cap \mathsf{uni}(\mathbf{N})^2, \mathsf{D}_{\mathbf{N}}\rangle$ is embeddable in $\mathbf{M}/\alpha$. □

Recall that by Theorem 1.360 and Definition 1.361, the Leibniz relation $\Omega_{\mathbf{A}}$ is the largest congruence on structure $\mathbf{A}$. The Leibniz relation on a matrix has a particularly simple characterization.

Corollary 1.493 [BP89a, T1.5] $\Omega_{\mathbf{M}}$ is the largest $\mathsf{alg}(\mathbf{M})$ congruence compatible with $\mathsf{D}_{\mathbf{M}}$. □

Recall further, the characterization of the Leibniz relation given in Theorem 1.431. The following characterization follows from Theorem 1.431 together with Lemma 1.355.

Theorem 1.494 [BP89a, T1.5],[vA95, T1.7.3] For an $\mathfrak{a}$-matrix $\mathbf{M}$, $a\,\Omega_{\mathbf{M}}\,b$ iff, for all $\mathfrak{a}$-terms $p_1(x,\vec{y}), \ldots, p_{\mathsf{dim}(\mathbf{M})}(x,\vec{y})$ and all $\vec{c} \in \mathsf{uni}(\mathbf{M})$,

$$\langle p_1^{\mathbf{M}}(a,\vec{c}), \ldots, p_{\mathsf{dim}(\mathbf{M})}^{\mathbf{M}}(a,\vec{c})\rangle \in \mathsf{D}_{\mathbf{M}} \text{ iff } \langle p_1^{\mathbf{M}}(b,\vec{c}), \ldots, p_{\mathsf{dim}(\mathbf{M})}^{\mathbf{M}}(b,\vec{c})\rangle \in \mathsf{D}_{\mathbf{M}}. \tag{1.111}$$

□

□

Chapter 2

On the Algebraization of Sentential Calculi

In this chapter we present a terse summary of the standard theory of algebraizable logics. For a more extensive survey, the reader is urged to consider the excellent [Cze01] and [FRP03]. In order to distinguish these logics from more general notions of logics considered in Part III, we shall call these logics *sentential calculi*. In the literature these logics are generally referred to as n-deductive systems and the term 'sentential' is (generally) reserved for 1-deductive systems.

In ∮2.1 we define *signatures of sentential calculi*, where the signature encodes **both** the type of algebras under consideration and the dimension (i.e., the n in n-deductive system). The reason that we encode the dimension in the signature is to maintain compatibility with our theory of logics over constructs (see Part III); we note that most texts on algebraic logic equate the signature and the type of algebras, and encode the dimension in the deductive system. The auxiliary notions of *formulae*, *terms* and *substitutions* are also introduced. In ∮2.2 we define *sentential calculi* in terms of *axioms* and *rules*, as well as the auxiliary notions of *consequence*, *theories* and *theorems*, and we present some of the basic properties of sentential calculi. The matrix model theory of sentential calculi is developed in ∮2.3, and a motivation for such a model theory is given. The related notions of *semantic consequence* and *filters* are defined, and the standard results are summarized. We end this section by considering the *Leibniz relation* and *reduced* matrix models. A number of examples are provided in ∮2.4.

We then turn to the algebraization of sentential calculi. *Equivalent* sentential calculi are considered in ∮2.5, where the notion of equivalence pertains to sentential calculi of the same algebraic type but with possibly different *dimensions*. Note that we call a binary relationship between the formulae of a sentential n-calculus and a sentential m-calculus a *formal $\langle n, m\rangle$-translation*. In ∮2.6, we summarize the standard theory of *algebraization* and in ∮2.7 we summarize the theory of *protoalgebraic* sentential calculi.

While the reader familiar with the theory of algebraic logic may omit this chapter, such readers are urged to view the examples and in particular, to note the notions introduced to denote these logics, since we shall refer to them later in the text.

2.1 Languages of Sentential Calculi

Definition 2.1 (Signatures of Sentential Calculi) A **signature of sentential calculi** $\mathfrak{p}$ is determined by its **type** $\mathsf{type}(\mathfrak{p})$, which is a type of universal algebras, the operation symbols of which are called **connective symbols**, and its **dimension** $\mathsf{dim}(\mathfrak{p})$, which is a non-zero natural, such that $\mathsf{Symb_c}(\mathsf{type}(\mathfrak{p})) \cap \mathsf{V} = \emptyset$, where V is our globally chosen denumerably infinite set of variables, which we call **sentential variables** in this context. We tend to denote meta-variables of sentential calculi by the symbols x, y and z, together with the usual subscripts etc. Where ambiguity is avoidable, we may simply speak of *variables*. By a **signature of sentential n-calculi**, we mean a signature of sentential calculi with dimension n.

Throughout this chapter, unless specified to the contrary, $\mathfrak{p}$ shall denote an arbitrary signature of sentential calculi with type $\mathfrak{a}$, and arbitrary algebras are to be taken as $\mathfrak{a}$-algebras. □

Warning 2.2 (Signatures in the Literature) Our usage of the term 'signature' is non-standard. In the literature of sentential n-calculi, the term 'signature' corresponds to what we call the 'type', i.e., the standard usage of 'signature' does not encode the dimension. We have adopted the approach of encoding the dimension in the signature so as to maintain compatibility with our theory of *logics over constructs* developed in Part III.

Remark 2.3 The requirement that $\mathsf{Symb_c}(\mathsf{type}(\mathfrak{p})) \cap \mathsf{V} = \emptyset$ is to ensure that variable symbols do not clash with constant symbols in the definitions of formulae.

Definition 2.4 (Terms/Scalar-Formulae and Term/Scalar-Formulae Algebras) Let $\mathfrak{p}$ be a signature of sentential calculi and let $\emptyset \neq V \subseteq \mathsf{V}$. We denote the absolutely free $\mathsf{type}(\mathfrak{p})$-term algebra over scalar variables V by $\mathbf{Tm}_V(\mathfrak{p})$, writing $\mathsf{Tm}_V(\mathfrak{p})$ for $\mathsf{uni}(\mathbf{Tm}_V(\mathfrak{p}))$, the members of which are called **terms** or **scalar-formulae** (over scalar variables V). We drop all references to V in the case that $V = \mathsf{V}$. □

Definition 2.5 (Formulae and the Formulae Algebra) Let $\mathfrak{p}$ be a signature of sentential calculi. We write $\mathbf{Fm}(\mathfrak{p})$ for $(\mathbf{Tm}(\mathfrak{p}))^{\mathsf{dim}(\mathfrak{p})}$, and write $\mathsf{Fm}(\mathfrak{p})$ for $\mathsf{uni}(\mathbf{Fm}(\mathfrak{p}))$, the members of which are called **formulae**. We say that a scalar variable **occurs** in formula ϕ if it occurs in any of the scalar-formulae/terms in $\{\phi_{(0)}, \ldots, \phi_{(\mathsf{dim}(\mathfrak{p})-1)}\}$. Where unambiguous, we drop the '$(\mathfrak{p})$' from these notions, for example, writing $\mathbf{Fm}$ for $\mathbf{Fm}(\mathfrak{p})$. □

Convention 2.6 (1-Calculi) In keeping with our convention conflating unary relations and subsets (see Convention 1.93 on page 27), for signatures of sentential 1-calculi, we may syntactically conflate the scalar notions with their (dimensioned) counterparts. In particular, we may conflate a term/scalar-formula p and a formula $\langle p \rangle$. Typically this usage is confined to examples.

Definition 2.7 (Substitutions) A **$\mathfrak{p}$-substitution** is an endomorphism of $\mathbf{Tm}(\mathfrak{p})$. The set of all $\mathfrak{p}$-substitutions is denoted by $\mathsf{Sub}(\mathfrak{p})$. □

Remark 2.8 Substitutions are a 'scalar' notion.

Definition 2.9 (Substituting into Formulae) Let σ be a $\mathfrak{p}$-substitution. We shall also (conflating symbols) treat σ as an operator on $\mathsf{Fm}(\mathfrak{p})$, defined by $\sigma(\langle p_1, \ldots, p_1\rangle) = \langle \sigma(p_1), \ldots, \sigma(p_1)\rangle$. $\square$

Remark 2.10 If σ is a $\mathfrak{p}$-substitution, then (the extended operator) σ is an endomorphism of $\mathsf{Fm}(\mathfrak{p})$.

2.2 Sentential Calculus

2.2.1 Axioms, Rules and Derivations

Definition 2.11 (Axioms and Rules) A $\mathfrak{p}$-*axiom* ϖ is determined by its **conclusion** $\mathsf{conc}(\varpi)$, which is a $\mathfrak{p}$-formula. Let $\mathsf{Ax}(\mathfrak{p})$ denote the set of all $\mathfrak{p}$-axioms. A $\mathfrak{p}$-*rule* Λ is determined by its **premise** $\mathsf{prem}(\Lambda)$, which is a non-empty finite set of $\mathfrak{p}$-formulae of cardinality $\mathsf{ar}(\Lambda)$, and its **conclusion** $\mathsf{conc}(\Lambda)$, which is a $\mathfrak{p}$-formula. Let $\mathsf{Rl}(\mathfrak{p})$ denote the set of all $\mathfrak{p}$-rules. $\square$

Note 2.12 (Axioms are *not* Formulae) While an axiom is certainly *determined* by the formula that is its conclusion, axioms and formulae are *statically incomparable* entities. For example, for a formula ϕ, the expression '$\mathsf{conc}(\phi)$' is a *syntax error.* We have *purposely* avoided conflating axioms and formulae since this leads to *ambiguity.* $\square$

Convention 2.13 (Presenting Axioms and Rules) It is convenient to specify/present a rule Λ by (some) $\Lambda_1, \ldots, \Lambda_n \boldsymbol{\vdash} \mathsf{conc}(\Lambda)$, where $\mathsf{prem}(\Lambda) = \{\Lambda_1, \ldots, \Lambda_n\}$, and an axiom ϖ by $\boldsymbol{\vdash} \mathsf{conc}(\varpi)$, where '$\boldsymbol{\vdash}$ ' is an emboldened version of the symbol '$\vdash$', which we shall use to denote the *consequence relation* (see Definition 2.19 on page 96). At times we may conflate axioms with their conclusions, i.e., conflate axioms and formulae.

Remark 2.14 ([vA98]) We must stress that there is no order to formulae of the premise of a rule, despite the order of a presentation. This non-ordered nature is typical of **Hilbert systems**. Order dependent rules are typical of **Gentzen systems**. An important current area of research is the relationship between Hilbert and Gentzen systems.

Definition 2.15 (Direct Derivability) A formula ψ is **directly derivable** from formulae Φ by a rule Λ, if there exists a substitution σ with $\sigma\,[\mathsf{prem}(\Lambda)] \subseteq \Phi$ and $\sigma(\mathsf{conc}(\Lambda)) = \psi$. Let ir be a set of rules. We say that formulae Φ are **closed under direct derivability** by a set of rules ir, if it contains every formula directly derivable from itself by every rule of ir. $\square$

2.2.2 Sentential Calculi

Definition 2.16 (Sentential Calculi) [vA95, p.g. 74-76] A **sentential calculus** $\mathcal{S}$ is determined by its **signature** $\mathsf{sig}(\mathcal{S})$, which is a signature of sentential calculi, its **axioms** $\mathsf{Ax}(\mathcal{S})$, where $\mathsf{Ax}(\mathcal{S}) \subseteq \mathsf{Ax}(\mathsf{sig}(\mathcal{S}))$, and its rules $\mathsf{Rl}(\mathcal{S})$, where $\mathsf{Rl}(\mathcal{S}) \subseteq \mathsf{Rl}(\mathsf{sig}(\mathcal{S}))$. We speak of a **sentential n-calculus**, by which we mean a sentential calculus whose signature has dimension n. Where unambiguous, we may also call sentential n-calculi **n-deductive systems**, although we only do so when *logics over constructs* are *not* (currently) under consideration. Throughout this chapter, unless specified to the contrary, $\mathcal{S}$ shall denote an arbitrary sentential calculi of signature $\mathfrak{p}$ (of type $\mathfrak{a}$). $\square$

Convention 2.17 (Calculi-centric Notation) We write $\mathsf{type}(\mathcal{S})$, $\mathsf{Var_s}(\mathcal{S})$, $\mathsf{dim}(\mathcal{S})$, $\mathbf{Tm}(\mathcal{S})$, $\mathsf{Tm}(\mathcal{S})$, $\mathbf{Fm}(\mathcal{S})$, $\mathsf{Fm}(\mathcal{S})$ and $\mathsf{Sub}(\mathcal{S})$, for $\mathsf{type}(\mathsf{sig}(\mathcal{S}))$, $\mathsf{Var_s}(\mathsf{sig}(\mathcal{S}))$, $\mathsf{dim}(\mathsf{sig}(\mathcal{S}))$, $\mathbf{Tm}(\mathsf{sig}(\mathcal{S}))$, $\mathsf{Tm}(\mathsf{sig}(\mathcal{S}))$, $\mathbf{Fm}(\mathsf{sig}(\mathcal{S}))$, $\mathsf{Fm}(\mathsf{sig}(\mathcal{S}))$ and $\mathsf{Sub}(\mathsf{sig}(\mathcal{S}))$, respectively. To avoid ambiguity, analogous convention for rules and axioms are *not* introduced.

Definition 2.18 (Derivations) A **derivation** of formula ϕ from formulae Γ in $\mathcal{S}$, is a non-empty finite sequence $\psi_0, \ldots, \psi_{n-1}$ of formulae, such that $\psi_{n-1} = \phi$ and, for each $i \in n$,

1. $\psi_i \in \Gamma$, or
2. $\psi_i \in \sigma\left[\mathsf{Ax}(\mathcal{S})\right]$, for some substitution σ, or
3. there exists a rule $\Lambda \in \mathsf{Rl}(\mathcal{S})$ and a substitution σ with $\sigma\left[\mathsf{prem}(\Lambda)\right] \subseteq \{\psi_0, \ldots, \psi_{i-1}\}$ and $\psi_i = \sigma(\mathsf{conc}(\Lambda)$,

in which case we call $\psi_0, \ldots, \psi_{n-1}$ a **derivation** of ϕ from Γ (in $\mathcal{S}$). $\square$

Definition 2.19 (The Consequence Relation and Equal Sentential Calculi) With each sentential calculus $\mathcal{S}$, we associate the binary relationship $\vdash_{\mathcal{S}}$, called the **consequence relation**, from $\mathfrak{P}(\mathsf{Fm}(\mathcal{S}))$ to $\mathsf{Fm}(\mathcal{S})$, defined by $\Gamma \vdash_{\mathcal{S}} \phi$ iff ϕ is a member of the smallest set of formulae that includes Γ, includes $\sigma\left[\mathsf{Ax}(\mathcal{S})\right]$ for every substitution σ, and is closed under direct derivability by the rules $\mathsf{Rl}(\mathcal{S})$. We write $\vdash_{\mathcal{S}} \phi$ for $\emptyset \vdash_{\mathcal{S}} \phi$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two sentential calculi. We say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are **equivalent** if they have the same signature and $\vdash_{\mathcal{S}_1} = \vdash_{\mathcal{S}_2}$, which we denote by $\mathcal{S}_1 \equiv \mathcal{S}_2$. $\square$

We require the notion of *equivalence* since the same consequence relation can be described by different axiomatizations. Some texts take the consequence relation as the *starting point* for the definition of a logic, in which case there is no need for the notion of equivalence.

Convention 2.20 It is convenient, once a sentential calculus has been determined up to *equivalence*, for example by the conditions of Theorem 2.22, to name or symbolise this calculus. This simply means that the name or symbolism stands for *some* sentential calculus with the determining properties.

Remark 2.21 In particular, $\vdash_{\mathcal{S}} \psi$, for all $\mathcal{S}$-axioms ψ, and $\mathsf{prem}(\Lambda) \vdash_{\mathcal{S}} \mathsf{conc}(\Lambda)$, for all $\mathcal{S}$-rules Λ.

Theorem 2.22 [LS58] The consequence relations of sentential calculi are characterizable as precisely those binary relationships $\vdash$ from $\mathfrak{P}(\mathsf{Fm}(\mathfrak{p}))$ to $\mathsf{Fm}(\mathfrak{p})$ satisfying

1. If $\phi \in \Gamma$ then $\Gamma \vdash \phi$.
2. If $\Gamma \vdash \phi$ and $\Gamma \subseteq \Phi$ then $\Phi \vdash \phi$.
3. If $\Gamma \vdash \phi$ and, for each $\psi \in \Gamma$, $\Phi \vdash \psi$, then $\Phi \vdash \phi$.

4. (**Finitariness**) If $\Gamma \vdash \phi$ then $\Gamma' \vdash \phi$ for some *finite* subset Γ' of Γ.

5. (**Structurality**) If $\Gamma \vdash \phi$ then, for every substitution σ, $\sigma[\Gamma] \vdash \sigma(\phi)$.

$\square$

From conditions (1) through to (4) of the previous theorem, we see that the consequence relation of a sentential calculus $\mathcal{S}$ is an *algebraic* point-consequence relation on $\mathsf{Fm}(\mathcal{S})$, by Proposition 4.77 on page 157. Consequently, $\vdash_{\mathcal{S}}$ must be the consequence relation of a *formal system*. We now define that formal system. It is not immediately clear what this formal system is, due to the role played by substitution in the definition of derivation. This role is fundamental, and is tightly tied to *structurality*.

Definition 2.23 (Associating Sentential Calculi with Formal Systems) With each sentential calculus $\mathcal{S}$, we associate a formal system $F(\mathcal{S})$ with language $\mathsf{Fm}(\mathcal{S})$, and formally axiomatized by all axioms $\bigcup_{\sigma \in \mathsf{Sub}(\mathcal{S})} \{\vdash \sigma(\mathsf{conc}(\varpi)) \,:\, \varpi \in \mathsf{Ax}(\mathcal{S})\}$, and all rules $\bigcup_{\sigma \in \mathsf{Sub}(\mathcal{S})} \{\sigma[\mathsf{prem}(\Lambda)] \vdash \sigma(\mathsf{conc}(\Lambda)) \,:\, \Lambda \in \mathsf{Rl}(\mathcal{S})\}$. $\square$

For a proof of the following result, see Lemma 6.35 on page 231.

Proposition 2.24 $\Gamma \vdash_{\mathcal{S}} \phi$ iff $\Gamma \vdash_{F(\mathcal{S})} \phi$.

Convention 2.25 (Conflating Sentential Calculi with their Formal Systems) Consequent to the previous proposition, we shall conflate the sentential calculus $\mathcal{S}$ with its associated formal system $F(\mathcal{S})$, thereby inheriting the notations and results of formal systems, and hence of algebraic closed systems. In particular, we obtain the notions of the (finitary closed system of) **theories** of $\mathcal{S}$, denoted $\mathsf{Th}(\mathcal{S})$, the algebraic **theory lattice**, denoted $\mathbf{Th}(\mathcal{S})$, the **theorems** of $\mathcal{S}$, denoted $\mathsf{Thm}(\mathcal{S})$, and the consequence operator, denoted $\|\cdot\|_{\mathcal{S}} : \mathfrak{P}(\mathsf{Fm}(\mathcal{S})) \to \mathsf{Th}(\mathcal{S})$.

The following result characterizes the theories and consequence operators of sentential calculi. Note that in [vA95, L 1.5.3], condition (1) of this result is described as necessary. It is in fact sufficient (see Corollary 6.16 on page 226).

Theorem 2.26 [BP89a]

1. $\mathsf{Th}(\mathcal{S})$ is an algebraic closed system satisfying

$$\forall\, [T \in \mathsf{Th}(\mathcal{S})]\; \forall\, [\sigma \in \mathsf{Sub}(\mathcal{S})]\; \sigma^{-1}[T] \in \mathsf{Th}(\mathcal{S}). \tag{2.1}$$

2. $\|\cdot\|_{\mathcal{S}}$ is an algebraic closure operator on $\mathsf{Fm}(\mathfrak{p})$ satisfying

$$\forall\, [\Gamma \subseteq \mathsf{Fm}(\mathcal{S})]\; \forall\, [\sigma \in \mathsf{Sub}(\mathcal{S})]\; \sigma[\|\Gamma\|_{\mathcal{S}}] \subseteq \|\sigma[\Gamma]\|_{\mathcal{S}}. \tag{2.2}$$

Further, (1) characterizes the theories of a sentential calculus and (2) characterizes the consequence operator of a sentential calculus.

2.3 The Matrix Model Theory of Sentential Calculi

The reader, unfamiliar with algebraic logic, is warned that the models to be encountered here, when applied to the case of the classical sentential calculus, do *not* yield the models that one encounters in a traditional treatment of the classical sentential calculus. In such a treatment, models are phrased in terms of interpretations into the two element Boolean algebra; i.e., models are 'truth tables'. That such a notion of a model yields soundness and completeness theorems is precisely because the classical sentential calculus is **algebraizable** with the variety of Boolean algebras as the unique algebraic semantics, together with the very unusual fact that this variety has a single (two-element) algebra as a model for the entire variety.

The characterization of a universal sentential calculus as a first-order Horn theory, allows one to associate a semantics with each sentential calculus, such that the soundness and completeness theorems of first-order predicate calculus yield soundness and completeness theorems for all sentential calculi.

2.3.1 Suggesting Matrix Models

As first observed by Bloom [BP89a], a sentential calculus is characterizable in first-order terms. We now demonstrate how the sentential logics of dimension n and type $\mathfrak{a}$ may be put into one-to-one correspondence with elementary *Horn* $\mathfrak{a}^n_{\mathsf{D}}$-theories.

Definition 2.27 ($\mathfrak{p}$-Matrices) Let $\mathfrak{p}_{\mathsf{D}}$ denote the type of $\mathsf{type}(\mathfrak{p})$-matrices of dimension $\mathsf{dim}(\mathfrak{p})$, structures of which are called **$\mathfrak{p}$-matrices**. Throughout this chapter, unless specified to the contrary, arbitrary matrices are to be taken as $\mathfrak{p}$-matrices. □

Definition 2.28 ('Translating' from $\mathfrak{p}$ to $\mathfrak{p}_{\mathsf{D}}$) [Blo75][BP89a] Let $\boxtimes(\mathfrak{p})$ denote the elementary language *without equality* with type $\mathfrak{p}_{\mathsf{D}}$ and variables V. With each $\mathfrak{p}$-formula ϕ we associate the open $\boxtimes(\mathfrak{p})$-formula

$$\boxtimes(\phi) = \mathsf{D}(\phi_{(0)}, \dots, \phi_{(\mathsf{dim}(\mathfrak{p})-1)}),$$

with each $\mathfrak{p}$-axiom $\vdash \phi$ we associate the universal $\boxtimes(\mathfrak{p})$-sentence

$$\boxtimes(\vdash \phi) = \forall[\vec{z}]\ \boxtimes(\phi),$$

where $\vec{z}$ are the variables occurring in ϕ, and with each $\mathfrak{p}$-rule $\phi_1, \dots, \phi_n \vdash \phi$ we associate the universal Horn $\boxtimes(\mathfrak{p})$-sentence

$$\boxtimes(\phi_1, \dots, \phi_n \vdash \phi) = \forall[\vec{z}]\ \boxtimes(\phi_1)\,\mathsf{and}\dots\mathsf{and}\,\boxtimes(\phi_n) \to \boxtimes(\phi),$$

where $\vec{z}$ are the variables occurring in the formulae $\{\phi, \phi_1, \dots, \phi_n\}$. □

Definition 2.29 ($\mathsf{E}^{\mathsf{D}}_{\mathcal{S}}$) Let $\mathcal{S}$ be a sentential calculus. Let $\mathsf{E}^{\mathsf{D}}_{\mathcal{S}}$ denote the elementary universal $\boxtimes(\mathfrak{p})$-theory (without equality) axiomatized by $\boxtimes[\mathsf{Ax}(\mathcal{S})] \cup \boxtimes[\mathsf{RI}(\mathcal{S})]$. □

Theorem 2.30 [BP89a] Let $\mathcal{S}$ be a universal sentential calculus and let $\phi_1, \dots, \phi_n, \phi$ be a finite sequence of formulae. Then $\vdash_{\mathcal{S}} \phi$ iff $\vdash_{\mathsf{E}^{\mathsf{D}}_{\mathcal{S}}} \boxtimes(\vdash \phi)$ and $\phi_1, \dots, \phi_n \vdash_{\mathcal{S}} \phi$ iff $\vdash_{\mathsf{E}^{\mathsf{D}}_{\mathcal{S}}} \boxtimes(\phi_1, \dots, \phi_n \vdash \phi)$. □

Bloom demonstrated, in the one-dimensional case, how the models of $\mathsf{E}^{\mathsf{D}}_{\mathcal{S}}$, which are $\mathsf{type}(\mathcal{S})$-matrices of dimension $\mathsf{dim}(\mathcal{S})$, may serve as the models of $\mathcal{S}$, and Blok and Pigozzi have demonstrated that the same is true more generally. This technique, which we describe next, leads to the notion of a **matrix model** of a predicate calculus, as well as soundness and completeness theorems.

2.3.2 Matrix Models

2.3.2.1 Interpreting Formulae in Matrices

Definition 2.31 (Interpretations into Matrices) Let $\mathbf{M}$ be a $\mathfrak{p}$-matrix. An **interpretation** into $\mathbf{M}$ is a homomorphism from $\mathsf{Fm}(\mathfrak{p})$ into $\mathsf{alg}(\mathbf{M})$. We denote the set of all interpretations into $\mathbf{M}$ by $\mathsf{Int}(\mathfrak{p}, \mathbf{M})$. For a sentential calculus $\mathcal{S}$, we write $\mathsf{Int}(\mathcal{S}, \mathbf{M})$ for $\mathsf{Int}(\mathsf{sig}(\mathcal{S}), \mathbf{M})$. □

2.3.2.2 Semantic Consequence

As in any model theory, the *ideal* first step in establishing that the 'concrete' entities called matrices indeed provide a suitable realm for modelling sentential calculi, would be to demonstrate a technique for *inducing* a *sentential calculi* from a *matrix* (or set of matrices), in a manner 'independently of sentential calculi'. The consequence-relation of such an induced calculi would be called a *semantic-consequence relation* determined by the matrix (or set of matrices) and denoted with the symbol $\models$, read 'models'. If this can be achieved, then given a *particular* sentential calculus $\mathcal{S}$, its consequence relation $\vdash_{\mathcal{S}}$ may be compared to the induced semantic-consequence relation $\models$, thereby providing a means for evaluating the *inducing* matrix's (or set of matrices') 'model-worthiness' modulo $\mathcal{S}$. Since sentential calculi are determined by their consequence relations, satisfying the conditions of Theorem 2.22 on page 96, one may equivalently establish a method for associating with each matrix (or set of matrices) a relation $\models$, from sets of formulae to formulae, satisfying these conditions.

The *standard technique*, described now, proceeds in the latter manner. With each matrix $\mathbf{M}$, we shall associate a binary relationship $\models^{\mathbf{M}}$ from $\mathfrak{P}(\mathsf{Fm}(\mathfrak{p}))$ to $\mathsf{Fm}(\mathfrak{p})$. Unfortunately, while $\models^{\mathbf{M}}$ satisfies conditions (1), (2), (3) and (5) of Theorem 2.22, it does not satisfy condition (4); it is not necessarily *finitary* [vA95, 81]. Syntactically, however, $\models^{\mathbf{M}}$ may still be 'compared' to $\vdash_{\mathcal{S}}$, and so may still be used for a test of the model-worthiness of $\mathbf{M}$ for $\mathcal{S}$.

Definition 2.32 (The Semantic Consequence Relation) With each $\mathfrak{p}$-matrix $\mathbf{M}$, we associate the binary relationship $\models^{\mathbf{M}}$ from $\mathfrak{P}(\mathsf{Fm}(\mathfrak{p}))$ to $\mathsf{Fm}(\mathfrak{p})$ defined by

$$\Gamma \models^{\mathbf{M}} \phi \text{ iff } \forall\, [\mathsf{i} \in \mathsf{Int}(\mathfrak{p}, \mathbf{M})]\; \underset{\rightarrow}{\mathsf{i}}[\Gamma] \subseteq \mathsf{D}_{\mathbf{M}} \rightarrow \underset{\rightarrow}{\mathsf{i}}(\phi) \in \mathsf{D}_{\mathbf{M}}. \tag{2.3}$$

For a set $\mathcal{M}$ of $\mathfrak{p}$-matrices, we associate the binary relationship $\models^{\mathcal{M}}$ from $\mathfrak{P}(\mathsf{Fm}(\mathfrak{p}))$ to $\mathsf{Fm}(\mathfrak{p})$ defined by

$$\Gamma \models^{\mathcal{M}} \phi \text{ iff } \forall\, [\mathbf{M} \in \mathcal{M}]\; \Gamma \models^{\mathbf{M}} \phi. \tag{2.4}$$

□

Remark 2.33 [BP89a] We stress, that while the relationships $\models^{\mathbf{M}}$ and $\models^{\mathcal{M}}$ satisfy conditions (1), (2), (3) and (5) of Theorem 2.22 on page 96, condition (4) of *finitariness* is *not* satisfied. As such, $\models^{\mathbf{M}}$ and $\models^{\mathcal{M}}$ are certainly the consequence relations of closed-systems on $\mathsf{Fm}(\mathfrak{p})$, and these consequence relations satisfy the condition of structurality. *In cases where they are finitary*, these relations would be the consequence relations of sentential calculi.

Definition 2.34 (Finitary Matrices) A class $\mathcal{M}$ of matrices is called **finitary** if $\models^{\mathcal{M}}$ is *finitary*. A single matrix $\mathbf{M}$ is called finitary if $\{\mathbf{M}\}$ is finitary. $\square$

Lemma 2.35 [BP89a] Any matrix over a finite algebra is finitary. $\square$

The problem of characterizing finitary matrices more generally, and the problem of characterizing finitary classes of matrices, are both still open.

2.3.2.3 Matrix Models

Definition 2.36 (Matrix Models) Let $\mathbf{M}$ be a matrix. We say that $\mathbf{M}$ is a **(matrix) model** of/for $\mathcal{S}$ (or an **$\mathcal{S}$-matrix**) if $\vdash_{\mathcal{S}} \subseteq \models^{\mathbf{M}}$, i.e.

$$\Gamma \vdash_{\mathcal{S}} \phi \to \Gamma \models^{\mathbf{M}} \phi. \tag{2.5}$$

We denote the set of all matrix models of $\mathcal{S}$ by $\mathsf{MMod}(\mathcal{S})$. For a particular algebra $\mathbf{A}$, let $\mathsf{MMod}_{\mathbf{A}}(\mathcal{S}) = \{\mathbf{M} \in \mathsf{MMod}(\mathcal{S}) : \mathsf{alg}(\mathbf{M}) = \mathbf{A}\}$. We write $\mathsf{FMMod}(\mathcal{S})$ for $\mathsf{MMod}_{\mathsf{Fm}(\mathcal{S})}(\mathcal{S})$, the members of which are called **formula (matrix) models**. $\square$

Proposition 2.37 [BP92, Pr 5.1] Submatrices of $\mathcal{S}$-matrices are $\mathcal{S}$-matrices. $\square$

The following result follows from the previous proposition, together with Proposition 1.119 on page 31.

Proposition 2.38 [BP92, Pr 5.1] If $\mathbf{N}$ is a reduction of $\mathbf{M}$, then, $\mathbf{M}$ is an $\mathcal{S}$-matrix iff $\mathbf{N}$ is an $\mathcal{S}$-matrix. $\square$

For a proof of the following result, see Proposition 7.73 on page 269.

Proposition 2.39 For sentential $\mathfrak{p}$-calculi $\mathcal{S}$ and $\mathcal{S}'$, the following conditions are equivalent.

1. $\mathcal{S} \preceq \mathcal{S}'$.
2. $\mathsf{MMod}(\mathcal{S}') \subseteq \mathsf{MMod}(\mathcal{S})$.
3. $\mathsf{FMMod}(\mathcal{S}') \subseteq \mathsf{FMMod}(\mathcal{S})$.
4. For each algebra $\mathbf{A}$, $\mathsf{MMod}_{\mathbf{A}}(\mathcal{S}') \subseteq \mathsf{MMod}_{\mathbf{A}}(\mathcal{S})$.

Proposition 2.40 [BP89a] For a sentential calculus $\mathcal{S}$, the $\mathcal{S}$-matrices are precisely the $\mathsf{E}^{\mathsf{D}}_{\mathcal{S}}$-models.

2.3.2.4 Filters - Realizing Matrix Models

The matrix models of a sentential calculus are described abstractly. We now consider the notion of a filter of a logic on a matrix, which yields a more constructive technique (particularly in the case of protoalgebraic logics, see Theorem 2.139) for realizing matrix models.

Definition 2.41 ($\mathcal{S}$-Filters of Matrices and Algebras) [BP89b] Let $\mathbf{M}$ be a matrix. A set $F \subseteq \underrightarrow{\mathsf{uni}}(\mathbf{M})$ is called an $\mathcal{S}$**-filter** of $\mathbf{M}$ if $\mathsf{D}_{\mathbf{M}} \subseteq F$ and

$$\mathsf{i} \in \mathsf{Int}(\mathcal{S}, \mathbf{M}), \Gamma \vdash_{\mathcal{S}} \phi, \underrightarrow{\mathsf{i}}[\Gamma] \subseteq F \rightarrow \underrightarrow{\mathsf{i}}(\phi) \in F. \tag{2.6}$$

The set of all $\mathcal{S}$-filters on $\mathbf{M}$, denoted $\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$, forms an algebraic closed-system on $\underrightarrow{\mathsf{uni}}(\mathbf{M})$. The corresponding closure operator is denoted by $\|\cdot\|^{\mathbf{M}}_{\mathsf{fi}_{\mathcal{S}}}$ and the associated algebraic inclusion-ordered lattice is denoted $\mathbf{Fi}_{\mathcal{S}}(\mathbf{M})$. By an $\mathcal{S}$**-filter** of an algebra $\mathbf{A}$, we mean the $\mathcal{S}$-filters of the matrix $\langle \mathbf{A}, \emptyset \rangle$. We write $\mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$, $\|\cdot\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})$ for $\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \emptyset \rangle)$, $\|\cdot\|^{\langle \mathbf{A}, \emptyset \rangle}_{\mathsf{fi}_{\mathcal{S}}}$ and $\mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \emptyset \rangle)$, respectively.

With each $f : \mathbf{M} \rightarrow \mathbf{N}$, where $\mathbf{M}$ and $\mathbf{N}$ are matrices, we associate the function $f^{\mathcal{S}} : \mathfrak{P}(\underrightarrow{\mathsf{uni}}(\mathbf{M})) \rightarrow \mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$, defined by $f^{\mathcal{S}}(\mathtt{A}) = \|\underrightarrow{f}[\mathtt{A}]\|^{\mathbf{N}}_{\mathsf{fi}_{\mathcal{S}}}$, for all $\mathtt{A} \subseteq \underrightarrow{\mathsf{uni}}(\mathbf{M})$. For $f : \mathbf{A} \rightarrow \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are algebras, and $\mathtt{B} \subseteq \mathfrak{P}(\mathsf{uni}(\mathbf{B})^{\mathsf{dim}(\mathcal{S})})$, we define the function $f^{\mathcal{S}}_{\mathtt{B}} : \mathfrak{P}(\mathsf{uni}(\mathbf{A})^{\mathsf{dim}(\mathcal{S})}) \rightarrow \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{B}, \mathtt{B} \rangle)$, defined by $f^{\mathcal{S}}_{\mathtt{B}}(\mathtt{A}) = \|\underrightarrow{f}[\mathtt{A}]\|^{\langle \mathbf{B}, \mathtt{B} \rangle}_{\mathsf{fi}_{\mathcal{S}}}$, for all $\mathtt{A} \subseteq \mathsf{uni}(\mathbf{A})^{\mathsf{dim}(\mathcal{S})}$, writing $f^{\mathcal{S}}$ for $f^{\mathcal{S}}_{\emptyset}$. □

Proposition 2.42 [BP86],[BP88],[BP89a],[BP89b] Let $\mathbf{A}$ and $\mathbf{B}$ be algebras, $f : \mathbf{A} \rightarrow \mathbf{B}$, $\mathtt{A} \subseteq F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$ and $\mathtt{B} \subseteq G \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{B})$.

1. $f^{G} \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, f^{-1}[\mathtt{B}] \rangle) \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$.

2. If f is surjective and $\equiv_{f}$ is compatible with F then $f[F] \in \mathsf{Filter}(\mathcal{S}) f[\mathtt{A}]$.

3. $f^{\mathcal{S}}_{\mathtt{B}\,|\mathsf{Fi}_{\mathcal{S}}()\mathbf{A}} : \mathbf{Fi}_{\mathcal{S}}(\mathbf{A}) : \rightarrow_{\blacktriangledown} \mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{B}, \mathtt{B} \rangle)$.

Corollary 2.43 Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathsf{sig}(\mathcal{S})$-matrices and f a *surjective* (matrix) homomorphism from $\mathbf{M}$ onto $\mathbf{N}$. If $F \in \mathsf{Fi}_{\mathcal{S}}(\mathsf{N})$ then $f^{-1}[F] \in \mathsf{Fi}_{\mathcal{S}}(\mathsf{M})$.

Remark 2.44 Consequently, if σ is a *bijective* substitution then

$$\sigma[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X \rangle)] \subseteq \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, \sigma[X] \rangle) \text{ and } \sigma[T] = \|\sigma[X]\|^{\mathbf{Tm}}_{\mathsf{fi}_{\mathcal{S}}}.$$

Lemma 2.45 If X is σ-invariant then so is $T = \|X\|^{\mathbf{Tm}}_{\mathsf{fi}_{\mathcal{S}}}$. If $T = \|X\|^{\mathbf{Tm}}_{\mathsf{fi}_{\mathcal{S}}}$ is σ-invariant, then $X \vdash_{\mathcal{S}} \sigma[X]$. □

We highlight the instances of the previous results that will be most important in the sequel.

Proposition 2.46 Let $\mathcal{S}$ be a propositional calculus, σ a substitution and X a subset of Tm that is σ-invariant.

1. $\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X \rangle)$ is closed under σ^{-1}.

2. For any $G \subseteq \mathsf{Tm}$, $\sigma^{\mathcal{S}}_{X}(\|G\|^{\langle \mathbf{Tm}, X \rangle}_{\mathsf{fi}_{\mathcal{S}}}) = \|\sigma[G]\|^{\langle \mathbf{Tm}, X \rangle}_{\mathsf{fi}_{\mathcal{S}}}$.

3. $\sigma^{\mathcal{S}}_{X\,|\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X \rangle)}$ is a join-complete semilattice endomorphism of $\mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X \rangle)$.

4. If σ is an involution of $\mathbf{Tm}$, then $\sigma[\mathsf{Fi}_{\mathcal{S}}(\langle\mathbf{Tm}, X\rangle)] = \mathsf{Fi}_{\mathcal{S}}(\langle\mathbf{Tm}, X\rangle)$.

Theorem 2.47 $\mathbf{M} \in \mathsf{MMod}(\mathcal{S})$ iff $\mathsf{D}_{\mathbf{M}} \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$.

Proof. $\boxed{\Rightarrow}$ Assume that $\mathbf{M} \in \mathsf{MMod}(\mathcal{S})$. Certainly $\mathsf{D}_{\mathbf{M}} \subseteq \mathsf{D}_{\mathbf{M}}$. Suppose that $\mathsf{i} \in \mathsf{Int}(\mathcal{S}, \mathbf{M})$, $\Gamma \vdash_{\mathcal{S}} \phi$ and $\underset{\rightarrow}{\mathsf{i}}[\Gamma] \subseteq \mathsf{D}_{\mathbf{M}}$. (*We must show that* $\underset{\rightarrow}{\mathsf{i}}(\phi) \in \mathsf{D}_{\mathbf{M}}$.) Since $\mathbf{M} \in \mathsf{MMod}(\mathcal{S})$ and $\Gamma \vdash_{\mathcal{S}} \phi$, $\Gamma \models^{\mathbf{M}} \phi$. Since $\Gamma \models^{\mathbf{M}} \phi$, $\mathsf{i} \in \mathsf{Int}(\mathcal{S}, \mathbf{M})$ and $\underset{\rightarrow}{\mathsf{i}}[\Gamma] \subseteq \mathsf{D}_{\mathbf{M}}$, it follows, by (2.3), that $\underset{\rightarrow}{\mathsf{i}}(\phi) \in \mathsf{D}_{\mathbf{M}}$. $\boxed{\Leftarrow}$ Assume that $\mathsf{D}_{\mathbf{M}} \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$. Suppose that $\Gamma \vdash_{\mathcal{S}} \phi$. (*We must show that* $\Gamma \models^{\mathbf{M}} \phi$.) Let $\mathsf{i} \in \mathsf{Int}(\mathcal{S}, \mathbf{M})$ and suppose that $\underset{\rightarrow}{\mathsf{i}}[\Gamma] \subseteq \mathsf{D}_{\mathbf{M}}$. (*We must show that* $\underset{\rightarrow}{\mathsf{i}}(\phi) \in \mathsf{D}_{\mathbf{M}}$.) Since $\mathsf{D}_{\mathbf{M}} \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$, $\mathsf{i} \in \mathsf{Int}(\mathcal{S}, \mathbf{M})$ and $\underset{\rightarrow}{\mathsf{i}}[\Gamma] \subseteq \mathsf{D}_{\mathbf{M}}$, $\underset{\rightarrow}{\mathsf{i}}(\phi) \in \mathsf{D}_{\mathbf{M}}$. $\diamond$

Corollary 2.48 $\mathsf{MMod}_{\mathbf{A}}(\mathcal{S}) = \{\langle\mathbf{A}, F\rangle \,:\, F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})\}$.

Lemma 2.49 [BP89a] Let $\mathcal{S}$ be a sentential calculus, $\emptyset \neq V \subseteq \mathsf{Var}_{\mathsf{s}}(\mathcal{S})$ and $\Gamma \cup \{\phi, \psi\} \subseteq \mathsf{Fm}_V(\mathcal{S})$.

1. $\mathbf{M} \in \mathsf{MMod}_{\mathsf{Tm}_V(\mathcal{S})}(\mathcal{S})$ iff $\mathsf{D}_{\mathbf{M}} = \|\mathsf{D}_{\mathbf{M}}\|_{\mathcal{S}} \cap \mathsf{Fm}_V(\mathcal{S})$.

2. $\mathbf{M} \in \mathsf{MMod}_{\mathsf{Tm}_V(\mathcal{S})}(\mathcal{S})$ implies $\mathsf{Fi}_{\mathcal{S}}(\mathbf{M}) = \{T \cap \mathsf{Fm}_V(\mathcal{S}) \,:\, \mathsf{D}_{\mathbf{M}} \subseteq T \in \mathsf{Th}(\mathcal{S})\}$.

3. $T \in \mathsf{Th}(\mathcal{S})$ implies $\|\Gamma\|_{\mathsf{fi}_{\mathcal{S}}}^{\langle\mathsf{Fm}_V(\mathcal{S}), T \cap \mathsf{Fm}_V(\mathcal{S})\rangle} = \|T \cup \Gamma\|_{\mathcal{S}} \cap \mathsf{Fm}_V(\mathcal{S})$.

4. $\Gamma \cup \{\phi\} \vdash_{\mathcal{S}} \psi$ iff $\psi \in \|\{\phi\}\|_{\mathsf{fi}_{\mathcal{S}}}^{\langle\mathsf{Tm}_V(\mathcal{S}), \|\Gamma\|_{\mathcal{S}} \cap \mathsf{Fm}_V(\mathcal{S})\rangle}$.

Corollary 2.50 $\mathsf{MMod}_{\mathsf{Tm}_V(\mathcal{S})}(\mathcal{S}) = \{\langle\mathsf{Tm}_V(\mathcal{S}), T \cap \mathsf{Fm}_V(\mathcal{S})\rangle \,:\, T \in \mathsf{Th}(\mathcal{S})\}$.

Corollary 2.51 [BP89a] Let $\mathcal{S}$ be a sentential calculus and $\Gamma \cup \{\phi, \psi\} \subseteq \mathsf{Fm}(\mathcal{S})$.

1. $\mathbf{M} \in \mathsf{FMMod}(\mathcal{S})$ iff $\mathsf{D}_{\mathbf{M}} \in \mathsf{Th}(\mathcal{S})$.

2. $\mathbf{M} \in \mathsf{FMMod}(\mathcal{S})$ implies $\mathsf{Fi}_{\mathcal{S}}(\mathbf{M}) = \{T \,:\, \mathsf{D}_{\mathbf{M}} \subseteq T \in \mathsf{Th}(\mathcal{S})\}$.

3. $\mathbf{M} \in \mathsf{FMMod}(\mathcal{S})$ implies $\|\Gamma\|_{\mathsf{fi}_{\mathcal{S}}}^{\langle\mathsf{Fm}(\mathcal{S}), \mathbf{M}\rangle} = \|\mathsf{D}_{\mathbf{M}} \cup \Gamma\|_{\mathcal{S}}$.

4. $\Gamma \cup \{\phi\} \vdash_{\mathcal{S}} \psi$ iff $\psi \in \|\{\phi\}\|_{\mathsf{fi}_{\mathcal{S}}}^{\langle\mathsf{Tm}(\mathcal{S}), \|\Gamma\|_{\mathcal{S}}\rangle}$.

Corollary 2.52 $\mathsf{FMMod}(\mathcal{S}) = \{\langle\mathsf{Tm}(\mathcal{S}), T\rangle \,:\, T \in \mathsf{Th}(\mathcal{S})\}$.

Corollary 2.53 $\mathsf{Fi}_{\mathcal{S}}(\mathbf{Tm}) = \mathsf{Th}(\mathcal{S})$.

Lemma 2.54 [BP89a]

$$\begin{aligned} \|\mathtt{A}\|_{\mathsf{fi}_{\mathcal{S}}}^{\mathbf{M}} = \{&\mathsf{a} \in \underset{\longrightarrow}{\mathsf{uni}}(\mathbf{M}) \,:\, \exists\, [\mathsf{b}_0, \ldots, \mathsf{b}_{n-1} \in \underset{\longrightarrow}{\mathsf{uni}}(\mathbf{M})]\; \mathsf{a} = \mathsf{b}_{n-1} \text{ and } \forall\, [i \leq n-2] \\ &\mathsf{b}_i \in \mathsf{D}_{\mathbf{M}} \cup \mathtt{A} \text{ or} \\ &\exists\, [\Gamma \cup \{\phi\} \subseteq_{\mathsf{f}} \mathsf{Fm}(\mathcal{S}), \mathsf{i} \in \mathsf{Int}(\mathcal{S}, \mathbf{M})]\; \Gamma \vdash_{\mathcal{S}} \phi,\, \underset{\rightarrow}{\mathsf{i}}[\Gamma] \subseteq \{\mathsf{b}_0, \ldots, \mathsf{b}_{i-1}\}, \mathsf{b}_i = \underset{\rightarrow}{\mathsf{i}}(\phi)\}. \end{aligned} \tag{2.7}$$

Corollary 2.55 [BP89a] Let $\mathcal{S}$ be a sentential calculus, $\mathbf{M}$ a $\mathsf{sig}(\mathcal{S})$-matrix. If $\mathtt{b} \in \|\{\mathtt{a}\}\|_{\mathsf{fi}_{\mathcal{S}}}^{\mathbf{M}}$, then there exists a finitely generated subalgebra $\mathbf{A}$ of $\mathsf{alg}(\mathbf{M})$ with $\mathtt{a}, \mathtt{b} \in \mathsf{uni}(\mathbf{A})^{\mathsf{dim}(\mathcal{S})}$ and $\mathtt{b} \in \|\{\mathtt{a}\}\|_{\mathsf{fi}_{\mathcal{S}}}^{\langle\mathbf{A}, \mathsf{D}_{\mathbf{M}} \cap \mathsf{uni}(\mathbf{A})^{\mathsf{dim}(\mathcal{S})}\rangle}$.

2.3.3 Constituting Models and Semantics

Definition 2.56 (Constituting Models and Semantics) Let $\mathfrak{p}$ be a signature of sentential calculi and $\mathcal{M}$ a set of $\mathfrak{p}$-matrices. We say that $\mathcal{M}$ **constitutes a model** of/for $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \models^{\mathcal{M}}$, i.e.

$$\Gamma \vdash_{\mathcal{S}} \phi \to \Gamma \models^{\mathcal{M}} \phi, \tag{2.8}$$

and say that $\mathcal{M}$ *constitutes a semantics* of/for $\mathcal{S}$ if $\vdash_{\mathcal{S}} = \models^{\mathcal{M}}$, i.e.

$$\Gamma \vdash_{\mathcal{S}} \phi \leftrightarrow \Gamma \models^{\mathcal{M}} \phi. \tag{2.9}$$

$\square$

Remark 2.57 $\mathcal{M}$ constitutes a model of $\mathcal{S}$ iff every $\mathbf{M} \in \mathcal{M}$ is a model for $\mathcal{S}$.

Remark 2.58 $\mathsf{MMod}(\mathcal{S})$ constitutes a model for $\mathcal{S}$.

Remark 2.59 Any subset of $\mathsf{MMod}(\mathcal{S})$, *including* $\emptyset$, constitutes a model for $\mathcal{S}$.

Remark 2.60 While it is common that a single matrix be a model, it is rarely the case that a single matrix 'be a semantics', and so we introduce no special notation to this end. $\square$

The demonstration of a matrix model for $\mathcal{S}$ amounts to 'proving $\mathcal{S}$ sound', while the existence of a matrix semantics for $\mathcal{S}$ 'proves soundness and completeness'. Blok and Pigozzi demonstrate that a matrix-semantics exists for any sentential calculus, formalized in the next theorem.

Theorem 2.61 (Soundness and Completeness Theorem) [BP92, T4.2] $\mathsf{MMod}(\mathcal{S})$ and $\mathsf{FMMod}(\mathcal{S})$ both constitute a matrix semantics for $\mathcal{S}$.

Remark 2.62 The proof of this result depends necessarily on the property of **structurality**, namely, that for every substitution σ,

$$\Gamma \vdash \phi \to \sigma[\Gamma] \vdash \sigma(\phi). \tag{2.10}$$

2.3.4 Leibniz Analysis

Recall the definition of the Leibniz relation $\Omega_{\mathbf{A}}$, associated with a structure $\mathbf{A}$, given in Definition 1.361 on page 69. The Leibniz relation was first introduced in [Wój73]. Blok and Pigozzi coined the term 'Leibniz operator' and 'Leibniz congruence' [BP86], and were the first to see that deductive systems can be classified usefully by the properties of the Leibniz operator, in a manner analogous to the Mal'cev classification of varieties (for example, see $\oint$2.7).

Definition 2.63 (The $\mathcal{S}$-Leibniz Relation and Operator) [BP89a] With each matrix $\mathbf{M}$, we associate the function $\Omega^{\mathcal{S}}_{\mathbf{M}}(\cdot) : \mathsf{Fi}_{\mathcal{S}}(\mathbf{M}) \to \mathsf{Con}(\mathsf{alg}(\mathbf{M}))$, defined by $\Omega^{\mathcal{S}}_{\mathbf{M}}(F) = \Omega_{\langle \mathsf{alg}(\mathbf{M}), \mathrm{F}\rangle}$, which is called the **$\mathcal{S}$-Leibniz operator** or just the **Leibniz operator**. For an algebra $\mathbf{A}$, we write $\Omega^{\mathcal{S}}_{\mathbf{A}}$ for $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \emptyset\rangle}$, and write $\Omega^{\mathcal{S}}$ for $\Omega^{\mathcal{S}}_{\mathsf{Tm}(\mathcal{S})}$. $\square$

Remark 2.64 Note that in the definition of the *operator* $\Omega^{\mathcal{S}}_{\mathbf{M}}$ in terms of the Leibniz *relation*, $\mathcal{S}$ is serving *no role other* than to simply restrict the domain. $\square$

Since $\mathcal{S}$-filters are preserved under inverse images of *surjective* matrix homomorphisms, inverse images of filters may serve as input to the $\mathcal{S}$-Leibniz operator. The following important result demonstrates that the $\mathcal{S}$-Leibniz operator '*commutes*' with the taking of inverse images, under *surjective* matrix homomorphisms, of filters. Our reading of the literature of algebraizable logics, has lead us to feel that this result is a central result, consequently our 'upgrading' of Blok and Pigozzi's lemma to a theorem.

Definition 2.65 (1.8.8) An $\mathcal{S}$-matrix $\mathbf{M}$ is called **reduced**, if $\Omega^{\mathcal{S}}_{\mathbf{M}}(\mathsf{D}_{\mathbf{M}}) = =_{\mathsf{uni}(\mathbf{M})}$. The set of all reduced $\mathcal{S}$-matrices is denoted by $\mathsf{MMod}_*(\mathcal{S})$. □

Theorem 2.66 [BP89a] $\mathsf{MMod}_*(\mathcal{S})$ constitutes a matrix-semantics for $\mathcal{S}$. □

Most of the following lemma is contained or implicit in [BP86], [BP88],[BP89a],[BP92], or follows by standard arguments about closure operators. Note that the restriction of a congruence on an algebra to a subalgebra is itself a congruence on the subalgebra.

Theorem 2.67 [BP92, L5.4]

If f is a *surjective* (matrix) homomorphism from $\mathbf{M}$ onto $\mathbf{N}$ and $F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$, then $\Omega^{\mathcal{S}}_{\mathbf{M}}(f^{-1}[F]) = f^{-1}\left[\Omega^{\mathcal{S}}_{\mathbf{N}}(F)\right]$.

Lemma 2.68

1. Let $h : \mathsf{M} \to \mathsf{N}$ be a surjective matrix homomorphism. Then for all $F \in \mathsf{Fi}_{\mathcal{S}}(\mathsf{N})$, $\Omega_{\mathsf{M}}(h^{-1}[F]) = h^{-1}[\Omega_{\mathsf{N}}(F)]$.
2. For every $\mathcal{S}$-theory T and every surjective substitution σ, $\sigma^{-1}[\Omega(T)] = \Omega(\sigma^{-1}[T])$, and hence the set $\Omega[\mathsf{Th}(\mathcal{S})]$ is closed under inverse substitution.

Corollary 2.69 [BP89a] If σ is a *surjective* substitution and $T \in \mathsf{Th}(\mathcal{S})$, then $\Omega^{\mathcal{S}}(\sigma^{-1}[T]) = \sigma^{-1}\left[\Omega^{\mathcal{S}}(T)\right]$.

2.4 Examples

Example 2.70 (The Sentential Calculi of Equivalences)

Let $\mathfrak{a}$ be a type of algebras.

Definition 2.71 (The Sentential Calculi of Equivalences) [vA95, 76] Let $S^2(\mathfrak{a}, \equiv)$ be the sentential calculus with signature $\mathfrak{a}$, dimension two, the *single* axiom

$$\vdash \langle x, x\rangle \tag{2.11}$$

and the two rules

$$\langle x, y\rangle \vdash \langle y, x\rangle, \qquad \text{and} \tag{2.12}$$

$$\langle x, y\rangle, \langle y, z\rangle \vdash \langle x, z\rangle, \tag{2.13}$$

where x, y and z are any three distinct *fixed* variables. □

Remark 2.72 $\mathsf{Th}(S^2(\mathfrak{a}, \equiv)) = \mathsf{ER}(\mathsf{Tm}) = \mathsf{Th}(F^2(\mathsf{Tm}, \equiv))$.

Remark 2.73 [vA95, 87] For any $\mathfrak{a}$-algebra $\mathbf{A}$, $\mathsf{Fi}_{S^2(\mathfrak{a},\equiv)}(\mathbf{A}) = \mathsf{ER}(\mathsf{uni}(\mathbf{A}))$.

□

Example 2.74 (The Sentential Calculi of Congruences) [vA95, 76]

Let $\mathfrak{a}$ be a type of algebras.

Definition 2.75 (The Sentential Calculi of Congruences) Let $S^2(\mathfrak{a}, \Theta)$ be the sentential calculus with signature $\mathfrak{a}$, dimension two, the rules and axioms of $S^2(\mathfrak{a}, \equiv)$, the additional rules,

$$\langle x_1, y_1\rangle, \ldots, \langle x_{\mathsf{ar}(\star)}, y_{\mathsf{ar}(\star)}\rangle \vdash \langle \star(x_1, \ldots, x_{\mathsf{ar}(\star)}), \star(y_1, \ldots, y_{\mathsf{ar}(\star)})\rangle, \tag{2.14}$$

one for each $\mathfrak{a}$-operation symbol $\star$, where the variables are chosen to be distinct. □

Remark 2.76 $\mathsf{Th}(S^2(\mathfrak{a}, \Theta)) = \mathsf{Con}(\mathbf{Tm}) = \mathsf{Th}(F^2(\mathbf{Tm}, \Theta))$.

Remark 2.77 [vA95, 87-88] For any $\mathfrak{a}$-algebra $\mathbf{A}$, $\mathsf{Fi}_{S^2(\mathfrak{a},\Theta)}(\mathbf{A}) = \mathsf{Con}(\mathbf{A})$.

□

Example 2.78 (The Sentential Calculi of Relative Congruences) [vA95, 76-77]

Let $\mathcal{K}$ be an $\mathfrak{a}$-quasivariety and I and Q an axiomatization of $\mathcal{K}$ in terms of identities and quasi-identities respectively.

Definition 2.79 (The Sentential Calculi of Relative Congruences) Let $S^2(\Theta^{\mathcal{K}})$ be the sentential 2-calculus axiomatized by all axioms and rules of $S^2(\mathfrak{a}, \Theta)$, all axioms

$$\vdash \langle p, q\rangle \quad \text{where } p \approx q \in I \tag{2.15}$$

and all rules

$$\langle p_1, q_1\rangle, \ldots, \langle p_i, q_i\rangle \vdash \langle p, q\rangle \quad \text{where } p_1 \approx q_1, \ldots, p_i \approx q_i \longrightarrow p \approx q \in Q. \tag{2.16}$$

□

Remark 2.80 $S^2(\Theta^{\mathcal{K}})$ depends only on $\mathcal{K}$ and not on any particular axiomatization.

Remark 2.81 For any $\Gamma \cup \phi \subseteq \mathsf{Tm}$, $\Gamma \vdash_{S^2(\Theta^{\mathcal{K}})} \phi$ iff $\Gamma \models_{\mathcal{K}} \phi$. □

Consequently, the quasi-equational theory of $\mathcal{K}$ is perfectly captured as a sentential 2-deductive system.

Convention 2.82 (Conflating the quasi-equational theory of $\mathcal{K}$ and $\vdash_{S^2(\Theta^{\mathcal{K}})}$)
We shall often conflate $\models_{\mathcal{K}}$ and $\vdash_{S^2(\Theta^{\mathcal{K}})}$, for example, speaking of the structurality and finitariness of $\models_{\mathcal{K}}$.

Remark 2.83 $\mathsf{Th}(S^2(\Theta^{\mathcal{K}})) = \mathsf{Con}^{\mathcal{K}}(\mathbf{Tm}) = \mathsf{Th}(F^2(\mathcal{K}, \Theta^{\mathbf{Tm}}))$.

Remark 2.84 [vA95, 88-89] For any $\mathfrak{a}$-algebra $\mathbf{A}$, not necessarily in $\mathcal{K}$, $\mathsf{Fi}_{S^2(\Theta^{\mathcal{K}})}(\mathbf{A}) = \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.

□

In the next example we consider the logic $S(\mathcal{K},\tau)$, where τ is a finite set of pairs of unary terms and $\mathcal{K}$ is a quasivariety of algebras, introduced in [BR99]. This class of logics includes *all* algebraizable sentential 1-calculi up to equivalence (see Theorem 2.121). This logic is strongly correlated to questions of (generalizations of) point regularity and coherence in universal algebra, as well as to questions pertaining to the solutions to finite sets of equations in *one* variable. In this text, we generalize the logic $S(\mathcal{K},\tau)$ on two fronts.

At the simpler level, for the sake of completeness, we extend this logic from a sentential 1-calculus to a sentential n-calculus, by replacing τ with a finite set $\mathfrak{N}$ of pairs of terms in n-variables, obtaining the sentential n-calculus $S^n(\mathcal{K},\mathfrak{N})$; in the case that the terms of $\mathfrak{N}$ are unary, $S^n(\mathcal{K},\mathfrak{N}) = S(\mathcal{K},\mathfrak{N})$. We shall show that this class of logics encompasses, up to equivalence, *all* algebraizable sentential n-calculi; we develop a number of results concerning this logic from which those of [BR99] obtain as special cases. As mentioned in the previous paragraph, the logic $S(\mathcal{K},\tau)$ correlates with generalizations of the notion of *point* regularity in universal algebra; more formally, $S(\mathcal{K},\tau)$ is algebraizable iff $\mathcal{K}$ is $\langle\mathcal{K},\tau\rangle$-regular (see Definition 2.124 of Example 2.124 as well as Theorem 2.125 of the same example). We shall see that an analogous relationship holds for $S^n(\mathcal{K},\mathfrak{N})$ and a suitable generalization of this notion of regularity which we call $\langle\mathcal{K},\mathfrak{N}\rangle$-regularity (see Definition 11.1 on page 359 and Theorem 11.8 on page 362). While $\langle\mathcal{K},\tau\rangle$-regularity encompasses the well-known condition of point regularity in universal algebra (see Definition 1.375 on page 71), neither $\langle\mathcal{K},\tau\rangle$-regularity nor (the more general) $\langle\mathcal{K},\mathfrak{N}\rangle$-regularity, encompass full regularity (see Definition 1.375).

Our generalization of $S(\mathcal{K},\tau)$ on the second front is aimed to rectify this situation. In $\oint$12 we shall introduce a family $S(\mathcal{K},\mathfrak{B}_*)$ of sentential 1-calculi , where $\mathfrak{B}$ is a finite set of pairs of *binary* equations; this family encompasses the logics $S(\mathcal{K},\tau)$, as well as a logic we call the *membership logic*. A notion of regularity, called $\langle\mathcal{K},\mathfrak{B}_*\rangle$*-regularity*, is introduced in $\oint$11.2. While this notion of regularity encompasses $\langle\mathcal{K},\tau\rangle$-regularity, full regularity and some other existing regularity conditions in universal algebra, and $\langle\mathcal{K},\tau\rangle$-regularity corresponds with a logical condition on $S(\mathcal{K},\mathfrak{B}_*)$ much like algebraizability, this logical condition is *not* algebraizability; in fact, $S(\mathcal{K},\mathfrak{B}_*)$ is generally 'inherently unalgebraizable' in the sense of [BP89a]. It is to this end that we develop our theory of *parameterized algebraization*. We shall see that, under certain circumstances, $S(\mathcal{K},\mathfrak{B}_*)$ has $\mathcal{K}$ as its (unique) *parameterized* equivalent algebraic semantics iff $\mathcal{K}$ is $\langle\mathcal{K},\mathfrak{B}_*\rangle$-regular, and in particular, $\mathcal{K}$ is (fully) $\mathcal{K}$-regular iff $\mathcal{K}$ is the (unique) *parameterized* equivalent algebraic semantics for the membership logic determined by $\mathcal{K}$.

Example 2.85 (The Logic $S(\mathcal{K},\tau)$) [BR99]

Let $\mathfrak{a}$ be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and τ a finite set of pairs of *unary* terms. Such a set of terms is called a *unary system of equations* or just **unary systems**.

Warning 2.86 In [BR99], what we call a unary system of equations is called a *translation*. We reserve the word 'translation' for a more general notion of translation (see $\oint$5); unary system of equations are *special* translations in our sense.

In calling τ a system of equations, we are clearly implicitly identifying a pair of terms $\langle\delta,\epsilon\rangle$ with the identity $\delta\approx\epsilon$. We formalize the identification in the following definition.

Definition 2.87 ($\tau^{\approx}$) Let $\tau^{\approx}$ denote the binary relationship from $\mathsf{Tm}\mathfrak{a}$ to $\mathsf{Identity}(\mathfrak{a})$ defined by $\tau^{\approx}[\![p]\!] = \{\delta(p)\approx\epsilon(p) \,:\, \langle\delta,\epsilon\rangle\in\tau\}$. □

We now give a definition of the sentential 1-calculus $S(\mathcal{K}, \tau)$. Note that the approach that we are taking is slightly different to that taken in [BR99], in that we describe $S(\mathcal{K}, \tau)$ in terms of axioms and rules with Theorem 2.89 as a consequence, whereas in [BR99] they take (2.19) of Theorem 2.89 as the definition and prove that this formula well-defines a (structural and finitary) sentential 1-calculus; no formal axiomatization is given. That our approach coincides with theirs follows from our Proposition 9.8 on page 314 (in a more general context).

Definition 2.88 (The Logic $S(\mathcal{K}, \tau)$) Let $S(\mathcal{K}, \tau)$ denote the sentential 1-calculus determined by all axioms

$$\vdash p, \quad \text{where} \quad \models_{\mathcal{K}} \tau^{\approx}[\![p]\!], \tag{2.17}$$

and all rules

$$P \vdash p, \quad \text{where} \quad \models_{\mathcal{K}} \bigwedge \tau^{\approx}[P] \to \tau^{\approx}[\![p]\!]. \tag{2.18}$$

□

Theorem 2.89 For all $P \cup \{p\} \subseteq \mathsf{Tm}$,

$$P \vdash_{S(\mathcal{K},\tau)} p \quad \text{iff} \quad \tau^{\approx}[P] \models_{\mathcal{K}} \tau^{\approx}[\![p]\!]. \tag{2.19}$$

□

Some of the $S(\mathcal{K}, \tau)$-filters can be given a particularly simple description in terms of the *simultaneous solutions* to the system of equations $\tau^{\approx}$ over $\mathbf{A}$ modulo $\mathcal{K}$.

Definition 2.90 ($\langle\alpha, \tau\rangle$-Classes) For $\alpha \in \mathsf{Con}(\mathbf{A})$, let $\tau^{\mathbf{A}}/\alpha = \{a \in \mathsf{uni}(\mathbf{A}) : \forall\, [\langle\delta, \epsilon\rangle \in \tau]\ \delta^{\mathbf{A}}(a)\, \alpha\, \epsilon^{\mathbf{A}}(a)\}$, which we call the **$\tau$-class determined by** α or the **$\langle\alpha, \tau\rangle$-class**; such classes are called **$\langle\mathcal{K}, \tau\rangle$-classes**. We write τ/α for $\tau^{\mathbf{Tm}}/\alpha$. For an algebra $\mathbf{A}$, let $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A}) = \{\tau^{\mathbf{A}}/\alpha : \alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})\}$; this set forms an algebraic closed system over the universe of $\mathbf{A}$ (see Corollary 5.92 of Example 5.90 on page 200 for an alternative proof of this result). □

Note that the third assertion of the following result cannot generally be strengthened to an equality, even for algebras in the quasivariety [BR99].

Proposition 2.91 $\mathsf{Th}(S(\mathcal{K}, \tau)) = \mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{Tm})$, $\mathsf{Fi}_{S(\mathcal{K},\tau)}(\mathbf{F}) = \mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{F})$, where $\mathbf{F}$ is a $\mathcal{K}$-free algebra on ω-free generators, and $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A}) \subseteq \mathsf{Fi}_{S(\mathcal{K},\tau)}(\mathbf{A})$, where $\mathbf{A}$ is any $\mathfrak{a}$-algebra.

□

A special case of the logic $S(\mathcal{K}, \tau)$ obtains when $\tau = \{\langle x, 0\rangle\}$, where x is a variable and 0 is a $\mathcal{K}$-constant (see Definition 1.460 on page 88); these logics are called *assertional logics* in [BR99].

Example 2.92 (The Assertional Logics) [BR99]

Let $\mathfrak{a}$ be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and 0 a $\mathcal{K}$-constant.

Definition 2.93 (The Assertional Logic) Let $\mathbf{0}(x)$ denote the unary system $\{\langle x, 0\rangle\}$. By $S(\mathcal{K}, 0)$, we mean $S(\mathcal{K}, \mathbf{0})$; this logic is called the **assertional logic determined by** 0. Any use of the symbolism $S(\mathcal{K}, 0)$ implicitly implies that 0 is a $\mathcal{K}$-constant. □

Recall that $P \approx 0$ abbreviates $\{p \approx 0 : p \in P\}$ (see Definition 1.407 on page 78).

Corollary 2.94 For all $P \cup \{p\} \subseteq \mathsf{Tm}$,

$$P \vdash_{S(\mathcal{K},0)} p \quad \text{iff} \quad P \approx 0 \models_{\mathcal{K}} p \approx 0. \tag{2.20}$$

□ □

Clearly $\mathbf{0^A}/\alpha = \alpha[\![0^{\mathbf{A}}]\!]$. So by Proposition 2.91,

$$\mathsf{Th}(S(\mathcal{K},0)) = \{\alpha[\![0]\!] \,:\, \alpha \in \mathsf{Con}(\mathbf{Tm})\}, \tag{2.21}$$

$$\mathsf{Fi}_{S(\mathcal{K},0)}(\mathbf{F}) = \{\alpha[\![0^{\mathbf{F}}]\!] \,:\, \alpha \in \mathsf{Con}(\mathbf{F})\} \quad \text{and} \tag{2.22}$$

$$\{\alpha[\![0^{\mathbf{A}}]\!] \,:\, \alpha \in \mathsf{Con}(\mathbf{A})\} \subseteq \mathsf{Fi}_{S(\mathcal{K},0)}(\mathbf{A}), \tag{2.23}$$

where $\mathbf{F}$ is a $\mathcal{K}$-free algebra and $\mathbf{A}$ is any $\mathfrak{a}$-algebra. Consequently, it is not surprising that properties of $S(\mathcal{K},0)$ correlate with universal algebraic (relative) congruential properties at the point 0.

□

2.5 Equivalent Sentential Calculi

In order to phrase the notion that a sentential calculus be *algebraizable*, we need to be able to speak of *equivalent* sentential calculi, when the calculi under consideration may have different dimensions (although they must have the same type), this the problem of algebraization establish a form of equivalence between a sentential n-calculus and the sentential 2-calculus $S^2(\Theta^{\mathcal{K}})$, for some quasivariety $\mathcal{K}$. To this end, we introduce the notion of a *formal translation* between sentential calculi. Note that the prefixing of all the notions in section by the word 'formal' is not standard in the literature. In Part III, we shall be considering *logics* as models and semantics of other logics, and so as to distinguish that notion of semantics from the notion of semantics introduced in this section, we prefix all these notions with the word 'formal'.

Definition 2.95 (Formal Translations between Sentential Calculi) [BP89a] Let $\mathfrak{a}$ be a type of algebras and n and m non-zero naturals. A **formal** $\langle n, m\rangle$**-translation** $\boldsymbol{\tau}$ is a finite set of m-tuples of terms in n-variables. With each algebra $\mathbf{A}$, we associate the relationship $\boldsymbol{\tau}^{\mathbf{A}}$ from $\mathsf{uni}(\mathbf{A})^n$ to $\mathsf{uni}(\mathbf{A})^m$ defined by

$$\boldsymbol{\tau}^{\mathbf{A}}[\![\langle a_1, \ldots, a_n\rangle]\!] = \{\langle p_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, p_m^{\mathbf{A}}(a_1, \ldots, a_n)\rangle \,:\, \langle p_1, \ldots, p_m\rangle \in \boldsymbol{\tau}\}. \tag{2.24}$$

We drop the superscript $\mathbf{A}$ in the case that $\mathbf{A} = \mathbf{Tm}$.

Let $\mathcal{S}_1$ be a sentential n-calculus and $\mathcal{S}_2$ be a sentential m-calculus, both of type $\mathfrak{a}$. A **formal translation from** $\mathcal{S}_1$ **to** $\mathcal{S}_2$ is a $\langle n, m\rangle$-translation (see Definition 2.95 on page 108). For a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$, we define a function $\tau^\star(\cdot) : \mathfrak{P}(\mathsf{Fm}(\mathcal{S})) \to \mathsf{Th}(\mathcal{S})$ by $\tau^\star(\Gamma) = \|\tau\,[\Gamma]\|_{\mathcal{S}_2}$, and a function $\tau^\blacktriangleleft(\cdot) : \mathfrak{P}(\mathsf{Fm}(\mathcal{S}_2)) \to \mathfrak{P}(\mathsf{Fm}(\mathcal{S}_2))$ by $\tau^\blacktriangleleft(\Phi) = \overleftarrow{\tau}\,\lfloor\Phi\rfloor$.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sentential calculi of type $\mathfrak{a}$ with (possibly) different dimensions. We call $\mathcal{S}_2$ a **formal semantics** for $\mathcal{S}_1$, if there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that, for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathcal{S}_1)$,

$$\Gamma \vdash_{\mathcal{S}_1} \phi \text{ iff } \boldsymbol{\tau}\,[\Gamma] \vdash_{\mathcal{S}_2} \boldsymbol{\tau}[\![\phi]\!], \tag{2.25}$$

in which case we call τ a **formal semantic translation** from $\mathcal{S}_1$ to $\mathcal{S}_2$.

For a substitution σ, define a function $\sigma_{\mathcal{S}} : \mathfrak{P}(\mathsf{Fm}(\mathcal{S})) \to \mathsf{Th}(\mathcal{S})$ by $\sigma_{\mathcal{S}}(\Gamma) = \|\sigma[\Gamma]\|_{\mathcal{S}}$. We say that function $\mathsf{f} : \mathsf{Th}(\mathcal{S}_1) \to \mathsf{Th}(\mathcal{S}_2)$ **commutes with** σ if $\sigma_{\mathcal{S}_2}(\mathsf{f}(T)) = \mathsf{f}(\sigma_{\mathcal{S}_1}(T))$, for all $T \in \mathsf{Th}(\mathcal{S}_1)$, and say that f **commutes**, if it commutes with all substitutions. $\square$

The following theorem of Block and Pigozzi characterizes formal translations in terms of an *isomorphism* from the theory lattice of $\mathcal{S}_1$ onto a *join-complete subsemilattice* of the theory lattice of $\mathcal{S}_2$ that is *compact* in the theory lattice of $\mathcal{S}_2$ and which *commutes*. Theorems of such form, relating translations with theory lattice isomorphism are known generically in the literature as **Blok-Pigozzi theorems**.

Theorem 2.96 [BP89a] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sentential calculi.

1. If $\mathcal{S}_2$ is a formal semantics for $\mathcal{S}_1$ with formal semantic translation τ then
   (a) $\tau^{\star}{}_{|\mathsf{Th}(\mathcal{S}_1)}$ commutes,
   (b) $\tau^{\star}{}_{|\mathsf{Th}(\mathcal{S}_1)} : \mathbf{Th}(\mathcal{S}_1) \cong \tau^{\star}[\mathbf{Th}(\mathcal{S}_1)] \triangleleft_{\blacktriangledown} \mathbf{Th}(\mathcal{S}_2)$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{Th}(\mathcal{S}_1)]}$, and
   (c) $\tau^{\star}[\mathbf{Th}(\mathcal{S}_1)]$ is compact in $\mathbf{Th}(\mathcal{S}_2)$.
2. If $\mathsf{f} : \mathbf{Th}(\mathcal{S}_1) \cong \mathsf{f}[\mathbf{Th}(\mathcal{S}_1)] \triangleleft_{\blacktriangledown} \mathbf{Th}(\mathcal{S}_2)$, f commutes with all *surjective* substitutions and $\mathsf{f}[\mathbf{Th}(\mathcal{S}_1)]$ is *compact* in $\mathbf{Th}(\mathcal{S}_2)$, then there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that $\mathcal{S}_2$ is a formal semantics for $\mathcal{S}_1$ with formal semantic translation τ, $\tau^{\star}{}_{|\mathsf{Th}(\mathcal{S}_1)} = \mathsf{f}$ and $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{Th}(\mathcal{S}_1)]} = \mathsf{f}^{-1}$.

We now consider formal equivalent semantics. When a logic is a formal equivalent semantics for another logics, these two logics are in essence logically indistinguishable. The notion of 'untranslation' is non-standard; there is no name for this concept in the literature. We introduce it for reasons of clarity.

Definition 2.97 (Formal Equivalent Semantics) [BP89a] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sentential calculi of type $\mathfrak{a}$ with (possibly) different dimensions, $\boldsymbol{\tau}$ a formal translation from $\mathcal{S}_1$ to $\mathcal{S}_2$ and $\boldsymbol{\pi}$ a formal translation from $\mathcal{S}_2$ to $\mathcal{S}_1$. We say that $\boldsymbol{\pi}$ **untranslates** $\boldsymbol{\tau}$ if

$$\boldsymbol{\pi}\left[\boldsymbol{\tau}[\![\phi]\!]\right] \dashv\vdash_{\mathcal{S}_1} \phi. \tag{2.26}$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sentential calculi of type $\mathfrak{a}$ with (possibly) different dimensions. We call $\mathcal{S}_2$ a **formal equivalent semantics** for $\mathcal{S}_1$, if $\mathcal{S}_2$ is a formal semantics for $\mathcal{S}_1$ with some formal semantic translation $\boldsymbol{\tau}$, $\mathcal{S}_1$ is a formal semantics for $\mathcal{S}_2$ with some formal semantic translation $\boldsymbol{\pi}$, $\boldsymbol{\pi}$ untranslates $\boldsymbol{\tau}$ and $\boldsymbol{\tau}$ untranslates $\boldsymbol{\pi}$, in which case we call τ and π **formal equivalent semantic translations**. $\square$

Remark 2.98 By structurality, if $\boldsymbol{\tau}$ is a formal translation from $\mathcal{S}_1$ to $\mathcal{S}_2$ and $\boldsymbol{\pi}$ is a formal translation from $\mathcal{S}_2$ to $\mathcal{S}_1$, then $\boldsymbol{\pi}$ untranslates $\boldsymbol{\tau}$ iff for some distinct 1-variables $x_1, \ldots, x_{\mathsf{dim}(\mathcal{S}_1)}$

$$\boldsymbol{\pi}\left[\boldsymbol{\tau}[\![\langle x_1, \ldots, x_{\mathsf{dim}(\mathcal{S}_1)}\rangle]\!]\right] \dashv\vdash_{\mathcal{S}_1} \langle x_1, \ldots, x_{\mathsf{dim}(\mathcal{S}_1)}\rangle. \tag{2.27}$$

Lemma 2.99 [BP89a] Let τ be a formal translation from $\mathcal{S}_1$ to $\mathcal{S}_2$ and π a formal translation from $\mathcal{S}_2$ to $\mathcal{S}_1$. The following conditions are equivalent.

1. $\mathcal{S}_2$ is a formal equivalent semantics for $\mathcal{S}_1$ with formal equivalent semantic translations τ and π.

2. $\mathcal{S}_2$ is a formal semantics for $\mathcal{S}_1$ with formal semantic translations τ, and τ untranslates π.

3. $\mathcal{S}_1$ is a formal semantics for $\mathcal{S}_2$ with formal semantic translations π, and π untranslates τ.

We now give the *Blok-Pigozzi theorem* for formal equivalent semantics, which relates formal equivalent semantics with (full) lattice isomorphism between the theory lattices that commutes.

Theorem 2.100 [BP89a] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sentential calculi.

1. If $\mathcal{S}_2$ is a formal equivalent semantics for $\mathcal{S}_1$ with formal semantic translation τ then
   (a) $\tau^\star{}_{|\mathsf{Th}(\mathcal{S}_1)}$ commutes,
   (b) $\tau^\star{}_{|\mathsf{Th}(\mathcal{S}_1)} : \mathbf{Th}(\mathcal{S}_1) \cong \mathbf{Th}(\mathcal{S}_2)$ with inverse isomorphism $\tau^\blacktriangleleft{}_{|\mathbf{Th}(\mathcal{S}_2)}$.

2. If $\mathsf{f} : \mathbf{Th}(\mathcal{S}_1) \cong \mathbf{Th}(\mathcal{S}_2)$ and f commutes with all *surjective* substitutions, then there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that $\mathcal{S}_2$ is a formal equivalent semantics for $\mathcal{S}_1$ with formal equivalent semantic translation τ, $\tau^\star{}_{|\mathsf{Th}(\mathcal{S}_1)} = \mathsf{f}$ and $\tau^\blacktriangleleft{}_{|\tau^\star[\mathsf{Th}(\mathcal{S}_1)]} = \mathsf{f}^{-1}$.

Proposition 2.101 [BP89a] The notion of formal equivalent semantics is an equivalence relation of the class of all sentential calculi of type $\mathfrak{a}$. □

Consequent to the previous proposition, we may define a notion of *formal equivalence* between sentential calculi. Note that while *equivalent* sentential calculi are *formally equivalent*, the converse is not generally true, even for sentential calculi of the same dimension [vA95]. Had we defined a sentential calculi as being determined by a structural and finitary consequence relation rather than in terms of axioms and rules, as is done in some text, there would be no need for the notion of *equivalence*: it would just be *equality*. It is only because the same consequence relation can be described by different axiomatizations that we require the notion of equivalence.

Definition 2.102 (Formal Equivalence) We shall call two sentential calculi $\mathcal{S}_1$ and $\mathcal{S}_2$ **formally equivalent**, denoted by $\mathcal{S}_1 \cong \mathcal{S}_2$, if $\mathcal{S}_2$ is a formal equivalent semantics of $\mathcal{S}_1$. □

2.5.1 Examples

The following example, demonstrates that if the relative congruence logics of two quasivarieties are formally equivalent, then these logics are *equal* as are the quasivarieties. So in this case, formal equivalence and equality coincide. This relationship does not hold for sentential calculi more generally [vA95, 101-103]. It is consequent to this observation, together with Proposition 2.101, that whenever a sentential calculus is algebraizable, it has a *unique* equivalent algebraic semantics.

Example 2.103

Let $\mathcal{K}$ and $\mathcal{Q}$ be two quasivarieties of $\mathfrak{a}$-algebras.

Proposition 2.104 [BP89b, C 4.6] If $S^2(\Theta^{\mathcal{K}})$ is formally equivalent to $S^2(\Theta^{\mathcal{Q}})$ then $S^2(\Theta^{\mathcal{K}}) = S^2(\Theta^{\mathcal{Q}})$ and $\mathcal{K} = \mathcal{Q}$.

□

2.6 Algebraization

By definition, a formal translation from a sentential n-calculus to a sentential 2-calculus is a finite set of pairs of terms in n variables.

Definition 2.105 (Algebraic Semantics) [BP89a] Let $\mathcal{S}$ be a sentential n-calculus of type $\mathfrak{a}$ and $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras. We call $\mathcal{K}$ an **algebraic semantics** for $\mathcal{S}$ if $S^2(\Theta^{\mathcal{K}})$ is a formal semantics for $\mathcal{S}$ with formal semantic translation τ in which case we call τ **defining equations** for $\mathcal{S}$ in $\mathcal{K}$. □

Remark 2.106 $\mathcal{K}$ is an algebraic semantics for $\mathcal{S}$ with defining equations τ iff

$$\forall\,[\Gamma \cup \phi \subseteq \mathsf{Fm}(\mathcal{S})] \quad \Gamma \vdash_{\mathcal{S}} \phi \text{ iff } \tau^{\approx}[\Gamma] \models_{\mathcal{K}} \tau^{\approx}[\![\phi]\!], \tag{2.28}$$

□

By definition, a formal translation from a sentential 2-calculus to a sentential n-calculus is a finite set $\mathbf{\Delta}$ of n-formulae in 2 variables, i.e., $\mathbf{\Delta}$ is a finite set of n-tuples of terms in 2 variables. In this case of sentential *one*-calculi, defining equations are *unary systems of equations* (see Example 2.85).

Definition 2.107 ($\sigma_{\mathcal{K}}$) For a substitution σ, we write $\sigma_{\mathcal{K}}$ for $\sigma_{S^2(\Theta^{\mathcal{K}})}$. □

The following *Block-Pigozzi theorem* for algebraic semantics follows at once from definitions and Theorem 2.96. (See Theorem 13.13 on page 396 for a (more general) proof of the 1-calculus case.) Note that this result and Corollary 2.114 are the *canonical* Blok-Pigozzi theorems.

Corollary 2.108 [BP89a] Let $\mathcal{S}$ be a sentential calculus and $\mathcal{K}$ a quasivariety of algebras.

1. If $\mathcal{K}$ is an algebraic semantics for $\mathcal{S}$ with defining equations τ then
   (a) ${\tau^{\star}}_{|\mathsf{Th}(\mathcal{S})}$ commutes,
   (b) ${\tau^{\star}}_{|\mathsf{Th}(\mathcal{S})} : \mathbf{Th}(\mathcal{S}) \cong \tau^{\star}[\mathbf{Th}(\mathcal{S})] \lhd_{\blacktriangledown} \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ with inverse isomorphism ${\tau^{\blacktriangleleft}}_{|\tau^{\star}[\mathsf{Th}(\mathcal{S})]}$, and
   (c) $\tau^{\star}[\mathbf{Th}(\mathcal{S})]$ is compact in $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$.
2. If $\mathsf{f} : \mathbf{Th}(\mathcal{S}) \cong \mathsf{f}[\mathbf{Th}(\mathcal{S})] \lhd_{\blacktriangledown} \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$, f commutes with all *surjective* substitutions and $\mathsf{f}[\mathbf{Th}(\mathcal{S})]$ is *compact* in $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$, then, there exists a system of $\mathsf{dim}(\mathcal{S})$-ary equation τ, such that $\mathcal{K}$ is an algebraic semantics for $\mathcal{S}$ with defining equation τ, ${\tau^{\star}}_{|\mathsf{Th}(\mathcal{S})} = \mathsf{f}$ and ${\tau^{\blacktriangleleft}}_{|\tau^{\star}[\mathsf{Th}(\mathcal{S})]} = \mathsf{f}^{-1}$.

Remark 2.109 A formal translation from a sentential 2-calculus to a sentential n-calculus is a formal $\langle 2, n\rangle$-translation, i.e., is a finite set of n-formulae in two variables, i.e., a finite set of n-tuples of terms all in the same two variables.

Definition 2.110 (Equivalent Algebraic Semantics) [BP89a] We call a quasivariety $\mathcal{K}$ an equivalent algebraic semantics for $\mathcal{S}$ if $S^2(\Theta^{\mathcal{K}})$ is a formal equivalent semantics for $\mathcal{S}$ with formal equivalent semantic translations τ and $\mathbf{\Delta}$, in which case τ are called **defining equations** for $\mathcal{S}$ in $\mathcal{K}$ and $\mathbf{\Delta}$ are called **equivalence formulae** for $\mathcal{S}$ and $\mathcal{K}$. □

Remark 2.111 By definition, $\mathcal{K}$ is an equivalent algebraic semantics for $\mathcal{S}$ with defining equations τ and equivalence formulae $\boldsymbol{\Delta}$ iff (2.28) holds and

$$\begin{gathered}\forall\,[\{\langle p_j, q_j\rangle\ : j \in J\} \cup \{\langle p, q\rangle\} \subseteq \mathsf{Tm}^2] \\ \{p_j \approx q_j\ : j \in J\} \models_{\mathcal{K}} p \approx q \text{ iff } \boldsymbol{\Delta}\,[\{\langle p_j, q_j\rangle\ : j \in J\}] \vdash_{\mathcal{S}} \boldsymbol{\Delta}[\![\langle p, q\rangle]\!],\end{gathered} \tag{2.29}$$

$$\forall\,[\{\langle p, q\rangle\} \subseteq \mathsf{Tm}^2] \quad \tau^{\approx}\,[\boldsymbol{\Delta}[\![\langle p, q\rangle]\!]] \mathrel{=\!\mid\models}_{\mathcal{K}} p \approx q \quad \text{and} \tag{2.30}$$

$$\forall\,[\phi \in \mathsf{Fm}(\mathcal{S})] \quad \boldsymbol{\Delta}\,[\tau[\![\phi]\!]] \dashv\vdash_{\mathcal{S}} \phi. \tag{2.31}$$

Remark 2.112 [BP89a] By Remark 2.98, (2.30) is equivalent to

$$\tau^{\approx}\,[\boldsymbol{\Delta}[\![\langle x, y\rangle]\!]] \mathrel{=\!\mid\models}_{\mathcal{K}} x \approx y, \tag{2.32}$$

where x and y are any distinct 1-variables, and (2.31) is equivalent to

$$\boldsymbol{\Delta}\left[\tau[\![\langle x_1, \ldots, x_{\mathsf{dim}(\mathcal{S}_1)}\rangle]\!]\right] \dashv\vdash_{\mathcal{S}} \langle x_1, \ldots, x_{\mathsf{dim}(\mathcal{S}_1)}\rangle, \tag{2.33}$$

where $x_1, \ldots, x_{\mathsf{dim}(\mathcal{S}_1)}$ are any distinct 1-variables. □

The following characterization follows from Lemma 2.99.

Corollary 2.113 [BP89a] For a quasivariety $\mathcal{K}$, sentential n-calculus $\mathcal{S}$, n-ary system of equation τ and formal $\langle 2, n\rangle$-translation $\boldsymbol{\Delta}$, the following conditions are equivalent.

1. $\mathcal{K}$ is an equivalent algebraic semantics for $\mathcal{S}$ with defining equations τ and equivalence formulae $\boldsymbol{\Delta}$.
2. $\mathcal{S}$ and $\mathcal{K}$ satisfy (2.28) and (2.30).
3. $\mathcal{S}$ and $\mathcal{K}$ satisfy (2.28) and (2.32).
4. $\mathcal{S}$ and $\mathcal{K}$ satisfy (2.29) and (2.31).
5. $\mathcal{S}$ and $\mathcal{K}$ satisfy (2.29) and (2.33).

□

The following characterization follows immediately from Theorem 2.100.

Corollary 2.114 [BP89a] Let $\mathcal{S}$ be a sentential calculus and $\mathcal{K}$ a quasivariety of algebras.

1. If $\mathcal{K}$ is an equivalent algebraic semantics for $\mathcal{S}$ with defining equations τ then
   (a) ${\tau^{\star}}_{|\mathsf{Th}(\mathcal{S})}$ commutes,
   (b) ${\tau^{\star}}_{|\mathsf{Th}(\mathcal{S})} : \mathbf{Th}(\mathcal{S}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ with inverse isomorphism ${\tau^{\blacktriangleleft}}_{|\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})}$.
2. If $\mathsf{f} : \mathbf{Th}(\mathcal{S}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ and f commutes with all *surjective* substitutions, then there exists a system of $\mathsf{dim}(\mathcal{S})$-ary equation τ and a finite set $\boldsymbol{\Delta}$ of n-formulae in 2 variables, such that $\mathcal{K}$ is an equivalent algebraic semantics for $\mathcal{S}$ with defining equation τ and equivalence formulae $\boldsymbol{\Delta}$, ${\tau^{\star}}_{|\mathsf{Th}(\mathcal{S})} = \mathsf{f} = {\boldsymbol{\Delta}^{\blacktriangleleft}}_{|\mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})}$ and ${\tau^{\blacktriangleleft}}_{|\tau^{\star}[\mathsf{Th}(\mathcal{S})]} = \mathsf{f}^{-1} = {\boldsymbol{\Delta}^{\star}}_{|\mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})}$.

□

An algebraizable sentential calculus may have more than one set of defining equations and more than one set of equivalence formulae. As the next result shows, which is mostly an immediate corollary to Proposition 2.104 of Example 2.103, such distinct defining equations and equivalence formulae must be inter-derivable, and the equivalent algebraic semantics must be unique.

Corollary 2.115 [BP89a] Let $\mathcal{K}$ and $\mathcal{K}'$ be two equivalent algebraic semantics for $\mathcal{S}$ with respective defining equations τ and τ', and equivalence formulae $\mathbf{\Delta}$ and $\mathbf{\Delta}'$. Then $\mathbf{\Delta}(\langle x, y\rangle) \dashv\vdash_{\mathcal{S}} \mathbf{\Delta}'(\langle x, y\rangle)$, $\mathcal{K} = \mathcal{K}'$ and $\tau(\langle x_1, \ldots, x_{\mathsf{dim}(\mathcal{S})}\rangle) \mathrel{=\!|\!\models}_{\mathcal{K}} \tau'(\langle x_1, \ldots, x_{\mathsf{dim}(\mathcal{S})}\rangle)$.

Definition 2.116 (Algebraizable Sentential Calculi and Quasivarieties of Logic) [BP89a] A sentential calculus is called **algebraizable** if it has an equivalent algebraic semantics. A **quasivariety of logic** is a quasivariety that is the equivalent algebraic semantics of some sentential calculus. □

Algebraicity is characterizable in terms of the Leibniz operator constituting an isomorphism from the filter lattice onto the relative congruence lattice of each algebra. Notice that a necessary condition for such an isomorphism to exist is that the Leibniz operator must preserve the inclusion order of filters. This property is an equivalent condition of the property known as protoalgebraicity, which we consider in the next section. Consequently, protoalgebraicity is a necessary condition for algebraicity.

Theorem 2.117 [BP89a] The following conditions are equivalent.

1. $\mathcal{K}$ is the equivalent algebraic semantics for S.
2. For each algebra $\mathbf{A}$, $\Omega^{\mathcal{S}} : \mathsf{Fi}_{\mathcal{S}}(\mathbf{A}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{A})$.
3. $\Omega^{\mathcal{S}} : \mathbf{Th}(\mathcal{S}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$.

□

The following result characterizes algebraicity internally to a logic (i.e., without *apriori* reference to a quasivariety).

Theorem 2.118 [BP89a] The following conditions on a sentential calculus $\mathcal{S}$ are equivalent, where $\mathcal{K}$ is the quasivariety generated by $\{\mathbf{Tm}/\Omega_{\mathbf{Tm}}(T) \,:\, T \in \mathsf{Th}(\mathcal{S}_{:X})\}$.

1. $\mathcal{S}$ is algebraizable.
2. $\Omega^{\mathcal{S}}_{\mathbf{A}} : \mathbf{Fi}_{\mathcal{S}}(\mathbf{A}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{A})$, for every $\mathfrak{a}$-algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$.
3. $\Omega^{\mathcal{S}} : \mathbf{Th}(\mathcal{S}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$.

□

Algebraizability is characterizable in purely logical terms. In the following result we give one such characterization of algebraizable sentential *one*-calculi, noting that analogous characterizations of algebraizable sentential n-calculi exist [BP89a].

Theorem 2.119 For a sentential 1-calculi, the following conditions are equivalent.

1. $\mathcal{S}$ is algebraizable.

2. There exists a system of unary equations τ and a finite set of binary terms $\mathbf{\Delta}$ such that $\mathcal{S}$ satisfies (2.31),

$$\vdash_{\mathcal{S}} \mathbf{\Delta}(x,x), \tag{2.34}$$

$$\mathbf{\Delta}(x,y) \vdash_{\mathcal{S}} \mathbf{\Delta}(y,x) \quad \text{and} \tag{2.35}$$

$$\mathbf{\Delta}(x,y), \mathbf{\Delta}(y,z) \vdash_{\mathcal{S}} \mathbf{\Delta}(x,z), \tag{2.36}$$

and, for every $l \in \omega$, every l-ary fundamental operation symbol $\star$ and any variables $\vec{u}, \vec{v}$

$$\mathbf{\Delta}(u_0,v_0), \ldots, \mathbf{\Delta}(u_{l-1},v_{l-1}) \vdash_{\mathcal{S}} \mathbf{\Delta}(\star(\vec{u}), \star(\vec{v})). \tag{2.37}$$

□

Notice how the ternary terms in equivalent condition (2) are 'congruence like' or 'congruential'.

2.6.1 Examples

The sentential 1-calculi having *some* algebraic semantics are precisely, up to *equivalence* (and *not* just *formal equivalence*), the logics $S(\mathcal{K},\tau)$ of [BR99].

Example 2.120 (The Sentential 1-Calculi having some Algebraic Semantics) [BR99]

Theorem 2.121 The sentential 1-calculi having *some* algebraic semantics are precisely, up to *equivalence*, the logics $S(\mathcal{K},\tau)$.

Proof. That $\mathcal{K}$ is an algebraic semantics for $S(\mathcal{K},\tau)$ with defining equations τ, follows from (2.19) of Theorem 2.89 and (2.28) of Remark 2.106, where the latter is interpreted for the 1 dimensional case. Suppose that sentential 1-calculus $\mathcal{S}$ has an algebraic semantics $\mathcal{K}$ with defining equations τ. Then τ is a unary system; hence $S(\mathcal{K},\tau)$ is well-defined. The equivalence of $\mathcal{S}$ and $S(\mathcal{K},\tau)$ follows by (2.19) of Theorem 2.89 and (2.28) of Remark 2.106. ◇

□

So the analysis of algebraic sentential 1-calculi can be reduced to the analysis of the logics $S(\mathcal{K},\tau)$, where $\mathcal{K}$ is a quasivariety and τ a unary system of equations. Consequently, the question as to when is $S(\mathcal{K},\tau)$ algebraizable is of the utmost importance. Since $S(\mathcal{K},\tau)$ is determined entirely by $\mathcal{K}$ and τ, this question is phrasable entirely within the discourse of *universal algebra*. In [BR99], it was shown that $S(\mathcal{K},\tau)$ is algebraizable iff $\mathcal{K}$ is $\langle\mathcal{K},\tau\rangle$-regular. In the following example, we define the condition that an algebra and quasivariety be $\langle\mathcal{K},\tau\rangle$-regular and formalize this relationship for ease of subsequent reference.

Example 2.122 (The Algebraization of $S(\mathcal{K},\tau)$) [BR99]

Let $\mathfrak{a}$-be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and τ a finite set of pairs of unary terms. It follows from (2.28) of Remark 2.106, together with (2.19) of Theorem 2.89, that $\mathcal{K}$ is *always* an algebraic semantics for $S(\mathcal{K},\tau)$ with defining equations τ.

Proposition 2.123 $\mathcal{K}$ is *always* an algebraic semantics for $S(\mathcal{K}, \tau)$ with defining equations τ.

Definition 2.124 ($\langle \mathcal{K}, \tau \rangle$-Regularity) We say that $\mathbf{A}$ is **$\langle \mathcal{K}, \tau \rangle$-regular** if, for all $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\tau^{\mathbf{A}}/\alpha = \tau^{\mathbf{A}}/\beta$, for each $1 \leq i \leq n$, then $\alpha = \beta$. We say that $\mathcal{K}$ is **$\langle \mathcal{K}, \tau \rangle$-regular** if every algebra in $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-regular. □

The following characterizes the algebraizability of $S(\mathcal{K}, \tau)$. The equivalence of (1) and (3) follows from Proposition 2.123, (2.19) of Theorem 2.89 and Corollary 2.113. The equivalence of (2) and (3) follows by standard universal algebraic arguments. For more general proofs of this result, see Theorem 11.4 on page 360 and Theorem 11.8 on page 362, or Theorem 11.18 on page 366 and Theorem 15.17 on page 427; the later proof is obtained from our theory of parameterized algebraization, while the former from our generalization of the arguments of [BR99] from sentential 1-calculi to sentential n-calculi.

Theorem 2.125 The following conditions are equivalent.

1. $S(\mathcal{K}, \tau)$ is algebraizable.
2. $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-regular.
3. There exists a finite set $\mathbf{\Delta}$ of *binary* terms such that

$$\tau^{\approx}[\mathbf{\Delta}(\langle x, y \rangle)] \mathrel{=\!\!\mid\!\models}_{\mathcal{K}} x \approx y. \tag{2.38}$$

□

Note that in [BR99], the following notion of filter determination is called ideal determination; we use the term 'filter' rather than 'ideal' in order to maintain compatibility with terminology used later in our text. The term 'filter' is also more appropriate in the *logical* context, although in the *universal algebraic* context the term 'ideal' is (generally) more appropriate.

Definition 2.126 ($\langle \mathcal{K}, \tau \rangle$-Filter Determination) Let $\mathbf{A}$ be an algebra not necessarily in $\mathcal{K}$. We say that $\mathbf{A}$ is **$\langle \mathcal{K}, \tau \rangle$-filter determined** if, $\tau^{\mathbf{A}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Fi}_{S^n(\mathcal{K},\tau)}(\mathbf{A})$. We say that $\mathcal{K}$ is **τ-filter determined** if every algebra in $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-filter determined. □

Remark 2.127 If $\mathbf{A}$ is $\langle \mathcal{K}, \tau \rangle$-filter determined then $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A}) = \mathbf{Fi}_{S^n(\mathcal{K},\tau)}(\mathbf{A})$.

Theorem 2.128 The following conditions are equivalent.

1. $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-regular.
2. Every algebra is $\langle \mathcal{K}, \tau \rangle$-filter determined.
3. $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-filter determined.

□

We briefly consider that algebraization of the assertional logic and its relationship to the well-known universal algebraic condition of *point regularity* (see Definition 1.375 on page 71).

Example 2.129 (The Algebraization of the Assertional Logics) [BR99]

Let $\mathfrak{a}$ be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and 0 a $\mathcal{K}$-constant. Recall that with each $\mathcal{K}$-constant 0 we associate the unary system of equations $\mathbf{0}(x) = \{\langle x, 0\rangle\}$ (see Definition 2.93 of Example 2.92).

Remark 2.130 An algebra $\mathbf{A}$ is relatively congruence point regular at 0 iff $\mathbf{A}$ is $\langle \mathcal{K}, \mathbf{0}\rangle$-regular.

Corollary 2.131 The following conditions are equivalent.

1. $S(\mathcal{K}, \mathbf{0})$ is algebraizable.
2. $\mathcal{K}$ is relatively congruence point regular at 0.
3. There exists a finite set of binary terms $\mathbf{\Delta}(x, y)$, such that

$$\mathbf{\Delta}(x, y) \approx 0 \dashv\models_{\mathcal{K}} x \approx y. \tag{2.39}$$

□

The reader unfamiliar with the quasi-Mal'cev condition (3) characterizing relative point regularity should observe how an equation in a relatively 0-regular quasivariety is 'equivalent' to a finite set of equations all with 0 on the right. Groups (adopting $\langle +, -, 0\rangle$ notation) are 0-regular since

$$x - y \approx 0 \dashv\models_{\mathcal{K}} x \approx y. \tag{2.40}$$

This is precisely why terms may be 'brought to one side' of equations in modern algebra.

□

In Counter Example 3.1 on page 124, we shall demonstrate that the well-known variety of *quasigroups*, while being *fully congruence regular* (see Definition 1.359 on page 68), is the equivalent algebraic semantics of no sentential 1-calculus, and further, contains a non-trivial (fully congruence regular) subvariety that is not even the *algebraic semantics* of *any* non-trivial sentential 1-calculus. As such, the universal algebraic condition of full regularity, unlike point regularity, appears to fall outside the domain of algebraic logic. The primary aim of this text is to remedy this problem by one, associating with each quasivariety $\mathcal{K}$ a sentential 1-calculus called its *membership logic*, and two, developing a theory of *parameterized algebraization*, such that $\mathcal{K}$ is the (suitably parameterized) *parameterized equivalent algebraic* semantics for its membership logic precisely when $\mathcal{K}$ is *relatively congruence regular* (see Definition 1.375 on page 71). This brings varieties such as quasigroups into the algebraic logic fold, since in the case that $\mathcal{K}$ is a variety, relative congruence regularity corresponds to full regularity.

2.7 Protoalgebraicity

As mentioned earlier, a necessary condition for algebraicity is that the Leibniz operator must preserve the order of filters. This property is equivalent to the condition known as protoalgebraicity which we consider next.

Definition 2.132 (Protoalgebraicity and Filter Correspondence) [BP89a]
$\mathcal{S}$ is called **protoalgebraic** if, for all $T \in \mathsf{Th}(\mathcal{S})$,

$$\phi \ \underrightarrow{\Omega^{\mathcal{S}}(T)} \ \psi \text{ implies } T \cup \{\phi\} \vdash_{\mathcal{S}} \psi \text{ and } T \cup \{\psi\} \vdash_{\mathcal{S}} \phi. \tag{2.41}$$

We say that $\mathcal{S}$ has the **filter correspondence property** if, for all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, all $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$ and every $F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$,

$$\underset{\to}{f}^{-1}\left[\| \underset{\to}{f}[F] \|^{\mathbf{N}}_{\mathsf{fi}_{\mathcal{S}}}\right] = F. \tag{2.42}$$

□

Theorem 2.133 [BP89a] Every algebraizable sentential calculus is protoalgebraic. □

Recall the definition of $f^{\mathcal{S}}(\cdot)$, given in Definition 2.41.

Remark 2.134 Formula (2.42) is more succinctly phrased as $\underset{\to}{f}^{-1}\left[f^{\mathcal{S}}(F)\right] = F$. □

The following characterizations of protoalgebraicity have been summarized from [vA95].

Theorem 2.135 [BP89a] The following conditions are equivalent.

1. $\mathcal{S}$ is protoalgebraic.
2. $\mathcal{S}$ has the filter correspondence property.
3. For every algebra $\mathbf{A}$, $\alpha \in \mathsf{Con}(\mathbf{A})$ and $F, G \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$ with $F \subseteq G$, if α is compatible with F then α is compatible with G.
4. If $f : \mathbf{A} \twoheadrightarrow \mathbf{B}$, $F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$ and $G \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{B})$, then $F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} \underset{\to}{f}^{-1}[G] = \underset{\to}{f}^{-1}\left[\| \underset{\to}{f}[F] \cup G \|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}\right]$.
5. For all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$ and $F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$, $\underset{\to}{f}^{-1}\left[\| \underset{\to}{f}[F] \|^{\mathsf{alg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{S}}}\right] = F$.
6. For $\mathcal{S}$-theory T, $\mathcal{S}$-matrix $\mathbf{N}$ and $f : \langle \mathbf{Tm}, T \rangle \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, $\underset{\to}{f}^{-1}\left[f^{\mathcal{S}}(T)\right] = T$.
7. For $\mathcal{S}$-theory T, $\mathcal{S}$-matrix $\mathbf{N}$ and $f : \langle \mathbf{Tm}, T \rangle \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, $\underset{\to}{f}^{-1}\left[\| \underset{\to}{f}[T] \|^{\mathsf{alg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{S}}}\right] = T$.
8. For every $\alpha \in \mathsf{Con}(\mathbf{Tm})$ and $\mathcal{S}$-theories T and R with $T \subseteq R$, if α is compatible with T then α is compatible with R.
9. If $f : \mathbf{Tm} \twoheadrightarrow \mathbf{B}$, $T \in \mathsf{Th}(\mathcal{S})$ and $G \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{B})$, then $T \vee^{\mathbf{Th}(\mathcal{S})} \underset{\to}{f}^{-1}[G] = \underset{\to}{f}^{-1}\left[\| \underset{\to}{f}[T] \cup G \|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}\right]$.
10. For every algebra $\mathbf{A}$, $\Omega^{\mathcal{S}}_{\mathbf{A}}(\cdot)_{|\mathsf{Fi}_{\mathcal{S}}(\mathbf{A})}$ is $\subseteq$-preserving.
11. For every algebra $\mathbf{A}$ and $F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$, if $\mathsf{b} \; \underrightarrow{\Omega^{\mathcal{S}}_{\mathbf{A}}(F)} \; \mathsf{c}$ then $\mathsf{b} \in \|\{\mathsf{c}\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $\mathsf{c} \in \|\{\mathsf{b}\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$.
12. $\Omega^{\mathcal{S}}_{\mathbf{Tm}}(\cdot)_{|\mathsf{Th}(\mathcal{S})}$ is $\subseteq$-preserving.

□

We now consider a characterization of protoalgebraicity in terms of the existence of certain formulae satisfied by $\mathcal{S}$. This result, from [Pal03], is a correction to an erroneous result in [BP92]. We require the following definition.

Definition 2.136 (Inserting Terms into Formula) For $\phi \in \mathsf{Tm}^{n-1}$, $p \in \mathsf{Tm}$ and $i \leq n$, let $\phi[p/i]$ denote $\langle \phi_{(0)}, \phi_{(1)}, \ldots, \phi_{(i-1)}, p, \phi_{(i)}, \ldots, \phi_{(n-2)} \rangle \in \mathsf{Tm}^n$. □

Theorem 2.137 [Pal03] A sentential n-calculus $\mathcal{S}$ is protoalgebraic iff there exist $\mathcal{S}$-formulae $\mathbf{\Delta}_1(x,y,\vec{z}),\ldots,\mathbf{\Delta}_k(x,y,\vec{z})$, for some natural k, in $n+1$-variables, such that

$$\vdash_{\mathcal{S}} \mathbf{\Delta}_1(x,x,\vec{z}) \quad \text{and} \tag{2.43}$$

$$\vec{z}[y/j], \mathbf{\Delta}_1(x,y,\vec{z}),\ldots,\mathbf{\Delta}_k(x,y,\vec{z}) \vdash_{\mathcal{S}} \vec{z}[x/j], \quad \text{for all } j \leq n. \tag{2.44}$$

□

In the case of sentential 1-calculi, a far simpler characterization of protoalgebraicity obtains. In $\oint$16 we shall provide a new characterization of protoalgebraic sentential n-calculi in the spirit of this result (see Theorem 16.39 on page 453 and Corollary 16.39 on page 453).

Corollary 2.138 [BP89a] A sentential 1-calculus $\mathcal{S}$ is protoalgebraic iff there exist terms $\Delta_1(x,y), \Delta_k(x,y)$, for some $k > 0$, such that

$$\vdash_{\mathcal{S}} \Delta_i(x,x), \quad \text{for all } i \leq k, \text{ and} \tag{2.45}$$

$$y \cup \{\Delta_i(x,y) \,:\, i \leq k\} \vdash_{\mathcal{S}} x. \tag{2.46}$$

□

Generally, filter generation has no simple description. For protoalgebraic sentential calculi, a relatively simple characterization pertains.

Theorem 2.139 [BP89a] Let $\mathcal{S}$ be a protoalgebraic sentential calculus, $\mathbf{A}$ an algebra and $\mathtt{A} \subseteq \mathsf{uni}(\mathbf{A})^{\mathsf{dim}(\mathcal{S})}$, then

$$\|\mathtt{A}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}} = \{\underset{\rightarrow}{\mathsf{i}}(\phi) \,:\, \mathsf{i} \in \mathsf{Int}(\mathbf{G},\mathbf{A}), \Gamma \vdash_{\mathcal{S}} \phi, \underset{\rightarrow}{\mathsf{i}}[\Gamma] \subseteq \mathtt{A} \cup \|\emptyset\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}\}. \tag{2.47}$$

2.7.1 Examples

The ‘canonical’ example of a protoalgebraic sentential calculus is the logic $S^2(\mathfrak{a},\equiv)$ (see Example 2.70) whose theories are precisely the equivalence relations on the term algebra and whose filters are precisely the equivalence relations on the universes of $\mathfrak{a}$-algebras [vA95]. While the proof of the protoalgebraicity of $S^2(\mathfrak{a},\equiv)$ is not that difficult, we are able to give a much simpler proof in Example 16.42 on page 454, based on a characterization of protoalgebraic sentential n-calculi developed in $\oint$16.

In the following example we apply Corollary 2.138 to the logic $S(\mathcal{K},\tau)$, obtaining a characterization of the protoalgebraicity of $S(\mathcal{K},\tau)$ in terms of purely algebraic conditions on $\mathcal{K}$; in fact, characterized as a *quasi-Mal'cev* condition.

Example 2.140 (The Logic $S(\mathcal{K},\tau)$) [BR99]

Let $\mathfrak{a}$ be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and τ a finite set of pairs of unary terms. The following characterization of the protoalgebraicity of $S(\mathcal{K},\tau)$ follows immediately from Corollary 2.138 together with Theorem 2.89 of Example 2.85.

Corollary 2.141 $S(\mathcal{K},\tau)$ is protoalgebraic iff there exist a finite set $\mathbf{\Delta}$ of *binary* terms such that

$$\models_{\mathcal{K}} \tau^{\approx}\left[\mathbf{\Delta}(x,x)\right], \quad \text{and} \tag{2.48}$$

$$\models_{\mathcal{K}} \bigwedge \tau^{\approx}[\![y]\!] \mathsf{\ and\ } \bigwedge \tau^{\approx}\left[\mathbf{\Delta}(x,y)\right] \rightarrow \tau^{\approx}[\![\langle x\rangle]\!]. \tag{2.49}$$

□

In [BR99] a 'concrete' realization of this quasi-Mal'cev condition was obtained as a condition on algebras which they termed having $\mathcal{K}$-*coherent* $\langle\mathcal{K},\tau\rangle$-*classes*. Recall the definition of the $\langle\alpha,\tau\rangle$-class $\tau^{\mathbf{A}}/\alpha$ (see Definition 2.90), as well as the condition that a binary relation be compatible with a subset of its domain (see Definition 1.65 on page 24).

Definition 2.142 (Having $\mathcal{K}$-Coherent $\langle\mathcal{K},\tau\rangle$-Classes) We say that $A \subseteq \mathsf{uni}(\mathbf{A})$ is **$\langle\mathcal{K},\tau\rangle$-coherent** if, for all $\beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\tau^{\mathbf{A}}/\beta \subseteq A$ then β is compatible with A, i.e., $\beta\,[A] = A$ (equivalently $\beta\,[A] \subseteq A$). We say that $\mathbf{A}$ (resp. $\mathcal{K}$) has **$\mathcal{K}$-coherent $\langle\mathcal{K},\tau\rangle$-classes** if, (resp. for each $\mathbf{A} \in \mathcal{K}$ and) for all $\alpha,\beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\tau^{\mathbf{A}}/\beta \subseteq \tau^{\mathbf{A}}/\alpha$ then β is compatible with $\tau^{\mathbf{A}}/\alpha$, i.e., $\beta\left[\tau^{\mathbf{A}}/\alpha\right] = \tau^{\mathbf{A}}/\alpha$ (equivalently $\beta\left[\tau^{\mathbf{A}}/\alpha\right] \subseteq \tau^{\mathbf{A}}/\alpha$). □

Theorem 2.143 The following conditions are equivalent.

1. $S(\mathcal{K},\tau)$ is protoalgebraic.
2. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\tau\rangle$-classes.
3. Every $\mathfrak{a}$-algebra has $\mathcal{K}$-coherent $\langle\mathcal{K},\tau\rangle$-classes.
4. The $S(\mathcal{K},\tau)$-filters of every algebra are $\langle\mathcal{K},\tau\rangle$-coherent.

□

Having $\mathcal{K}$-coherent $\langle\mathcal{K},\tau\rangle$-classes is a necessary condition for $\mathcal{K}$ to be $\langle\mathcal{K},\tau\rangle$-regular.

Proposition 2.144 If $\mathcal{K}$ is $\langle\mathcal{K},\tau\rangle$-regular (equivalently $S(\mathcal{K},\tau)$ is algebraizable) then $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\tau\rangle$-classes.

In $\oint$10.2 we consider a generalization of this notion of coherence from finite sets of pairs of *unary* terms to finite sets of pairs of n-*ary* terms, and provide a quasi-Mal'cev characterization of this condition. In $\oint$9.1.2 we introduce a generalization of the sentential 1-calculus $S(\mathcal{K},\tau)$; this logic $S^n(\mathcal{K},\tau)$, is a sentential n-calculus and τ is a finite set of pairs of n-*ary* terms. We obtain a generalization of the previous corollary, characterizing the protoalgebraicity of $S^n(\mathcal{K},\tau)$ in terms of $\mathcal{K}$ satisfying this more general notion of coherence. The proof of this result is only given in $\oint$16, since it is based on the simpler characterization of protoalgebraic sentential n-calculi which we obtain in that chapter.

□

We highlight the case of 'point coherence' and the assertional logics.

Example 2.145 (Protoalgebraic Assertional Logics) [BR99]

Let 0 be a $\mathcal{K}$-constant.

Definition 2.146 (Point Coherent Congruence Classes) We say that $\mathbf{A}$ has **$\mathcal{K}$-coherent $\langle\mathcal{K},0\rangle$-classes** if, for all $a \in \mathsf{uni}(\mathbf{A})$ and $\alpha,\beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\beta[\![0^{\mathbf{A}}]\!] \subseteq \alpha[\![0^{\mathbf{A}}]\!]$ then β is compatible with $\alpha[\![0^{\mathbf{A}}]\!]$. We extend this definition to $\mathcal{K}$ in the natural manner. □

Recall the definition of the unary system $\mathbf{0}$ given in Definition 2.93.

Remark 2.147 $\mathbf{A}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, 0\rangle$-classes iff $\mathbf{A}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathbf{0}\rangle$-classes. $\square$

Note that the quasi-Mal'cev condition (3) in the following characterization is simpler than that which would be obtained by a direct interpretation of Corollary 2.141; this simplification follows by simple standard universal algebraic arguments (see the proof of Corollary 10.21 on page 350 for a similar argument).

Corollary 2.148 The following conditions are equivalent.

1. $S(\mathcal{K}, \mathbf{0})$ is protoalgebraic.
2. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, 0\rangle$-classes.
3. There exists a finite set $\mathbf{\Delta}$ of binary terms such that

$$\models_{\mathcal{K}} \mathbf{\Delta}(x, x) \approx 0, \quad \text{and} \tag{2.50}$$

$$\models_{\mathcal{K}} \bigwedge \mathbf{\Delta}(x, 0) \approx 0 \rightarrow x \approx 0. \tag{2.51}$$

$\square$

Chapter 3

The Problem with Full Regularity in Algebraic Logic

For the sake of the following discussion, let us call a quasivariety of algebras *logical* if it is the *equivalent algebraic semantics* of some sentential 1-calculus. The logical quasivarieties have been *completely characterized* universal algebraically in [BR99]: these are precisely the quasivarieties that are $\langle \mathcal{K}, \tau \rangle$-*regular* for some unary system τ (see Definition 2.124 on page 115 of our text; in [BR99], unary systems are called translations). The notion of $\langle \mathcal{K}, \tau \rangle$-*regular* arises naturally from the consideration of algebraizable sentential 1-calculi, where the unary system is just the set of *defining equations* from the sentential 1-calculus to the quasivariety (see Definition 2.105 on page 111). In [BR99], a sentential 1-calculus $S(\mathcal{K}, \tau)$ is defined (see Example 2.85 on page 106 of our text), determined by a quasivariety $\mathcal{K}$ and a unary system of equations τ, and it is shown that $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-regular iff $S(\mathcal{K}, \tau)$ is algebraizable, in which case $\mathcal{K}$ is its (unique) algebraic semantics and τ is a set of defining equations (see Theorem 2.125 on page 115 of our text). In fact, *all* algebraizable sentential 1-calculi are equivalent to a logic $S(\mathcal{K}, \tau)$. Of particular interest to the *algebraist*, is the fact that the notion of $\langle \mathcal{K}, \tau \rangle$-regularity encompasses the well-known (to algebraists) notion of *point regularity* [Slo59],[Fic68],[Fic70] (see Definition 1.375 on page 71 and Example 2.129 on page 116). Since the algebraizability of a sentential calculus is characterizable purely logically, that is, without any apriori reference to a quasivariety [BP89a] (see Theorem 2.119 on page 113 of our text), the *purely universal algebraic* condition that a quasivariety be *relatively point regular* is characterizable *purely logical* in the *assertional logic* of the quasivariety; the condition of relative point regularity has been *ported* from algebra to logic.

A natural question arises as to whether or not the condition of *full congruence regularity* (see Definition 1.359 on page 68) can also be subject to such *algebraic logical* treatment. The condition of *full regularity* becomes important in algebra when varieties have no *equationally definable constants* and hence cannot be *point regular*. An example of such a variety is the variety of *quasigroups* [BS81] which is a well-known generalization of *groups*. The simple answer is *no*, as demonstrated by the following counter example. We shall now show, that despite being *fully congruence regular*, the *variety of quasigroups* cannot be the *equivalent algebraic semantics* of *any* sentential 1-calculus, in other words, the *fully congruence regular variety of quasigroups* is not *logical*. Further, the variety of quasigroups has a non-trivial (fully congruence regular) subvariety

$\mathcal{Q}'$ that is not even the *algebraic semantics* of *any* non-trivial sentential 1-calculus; a fairly dramatic failure from the perspective of algebraic logic. Since every sentential 1-calculus with an algebraic semantics $\mathcal{K}$ and defining equations τ is equivalent to the logic $S(\mathcal{K}, \tau)$, it follows that for every system τ of unary equations, the sentential 1-calculus $S(\mathcal{Q}', \tau)$ is trivial, in other words, this logic has only one theory, namely the set of all terms.

Counter Example 3.1 (Quasigroups and Steiner Quasigroups)

Quasigroups arose originally from *non-euclidean geometry*, in particular from the analysis of *Steiner triple systems*.

Definition 3.2 (Steiner Triple System) [BS81] A **Steiner triple system** $\mathcal{S}$ is determined by its universe $\mathsf{uni}(\mathcal{S})$, which is a set, and a set of three element subsets of $\mathsf{uni}(\mathcal{S})$, denoted by $\mathsf{Line}(\mathcal{S})$, the members of which are called **lines**, such that, for all $\langle a, b\rangle \in \mathsf{uni}(\mathcal{S})$ with $a \neq b$, there exists a unique $L \in \mathsf{Line}(\mathcal{S})$ with $\{a, b\} \subseteq L$. The **order** of a Steiner triple system $\mathcal{S}$ is the cardinality of $\mathsf{uni}(\mathcal{S})$. $\square$

Lemma 3.3 [BS81] If $\mathcal{S}$ is a finite Steiner triple system then

$$\mathsf{card}(\mathsf{Line}(\mathcal{S})) = \mathsf{card}(\mathsf{uni}(\mathcal{S})) \times (\mathsf{card}(\mathsf{uni}(\mathcal{S})) - 1) \div 6 \quad \text{and} \tag{3.1}$$

$$\mathsf{card}(\mathsf{uni}(\mathcal{S})) \equiv 1 \text{ or } 3 \mod 6. \tag{3.2}$$

$\square$

There is one Steiner triple system of order 1 and it has no lines, there is no Steiner triple system of order 2 and there is one Steiner triple system of order 3; this system has one line which is the universe. The Steiner triple system of next higher order has order seven, and it is the unique (up to renaming of points of the universe) Steiner triple system on $\{1, 2, 3, 4, 5, 6, 7\}$, denoted by $\mathcal{S}_7$, with

$$\mathsf{Line}(\mathcal{S}_7) = \big\{\{1, 2, 3\}, \{3, 4, 5\}, \{1, 5, 6\}, \{1, 4, 7\}, \{2, 5, 7\}, \{3, 6, 7\}, \{2, 4, 6\}\big\}.$$

Note that $\mathcal{S}_7$ is the well-known non-euclidean geometry on seven points (see [BS81, 100] for a visual realization). The geometric notion of a Steiner triple system is realizable algebraically as a *Steiner quasigroup*, introduced next.

Definition 3.4 (Steiner Quasigroups) [BS81] A **Steiner quasigroups** (or **squag**) is a groupoid satisfying the identities

$$x * x \approx x, \tag{3.3}$$

$$x * y \approx y * y \quad \text{and} \tag{3.4}$$

$$x * (x * y) \approx y. \tag{3.5}$$

$\square$

Steiner quasigroups characterize Steiner triple systems algebraically, as demonstrated by the following result.

Theorem 3.5 [BS81] If $\mathcal{S}$ is a Steiner triple system, then

$$a *^{\mathcal{S}} b = c \text{ if } \{a, b, c\} \in \mathsf{Line}(\mathcal{S}) \quad \text{and} \tag{3.6}$$

$$a *^{\mathcal{S}} a = a, \tag{3.7}$$

defines a squag on $\mathsf{uni}(\mathcal{S})$. If $\mathbf{G}$ is a squag, then

$$\mathsf{Line}(\mathbf{G}) = \{\{a, b, c\} \,:\, \mathsf{card}(\{a, b, c\}) = 3,\ a *^{\mathbf{G}} b = c,\ a *^{\mathbf{G}} c = b,\ b *^{\mathbf{G}} c = a\} \tag{3.8}$$

defines a Steiner triple system on $\mathsf{uni}(\mathbf{G})$. Further, these operations are mutually inverse. □

We now introduce *quasigroups*. Quasigroups abstract both Steiner quasigroups and groups. Since quasigroups are defined by *equations*, the class of all quasigroups is a *variety* of algebras. Observe that there is one operation of multiplication, and this operation is generally not commutative. Note further, that there are two operations of division: right-division $\backslash$ and left-division $/$. The identities capture the fact that right-dividing a by b and then right multiplying the answer by b yields a, and multiplying a by b on the right and then right-dividing the answer by b yields a; symmetrically for left-division.

Definition 3.6 (Quasigroups) [BS81] A **quasigroup** is an algebra $\mathbf{Q} = \langle Q; \cdot, /, \backslash\rangle$ of type $\langle 2, 2, 2\rangle$, satisfying the identities

$$(x/y)y \approx x, \tag{3.9}$$

$$(xy)/y \approx x, \tag{3.10}$$

$$x(x\backslash y) \approx y \quad \text{and} \tag{3.11}$$

$$x\backslash(xy) \approx y. \tag{3.12}$$

□

Notice that quasigroups capture the notion of multiplication and division without providing a *multiplicative identity*. In fact, it is not hard to show that the variety of quasigroups has *no* equationally definable constant terms at *all*, let alone a multiplicative identity (the proof is similar to the proof of Theorem 3.10 below).

In the next result, we show that the variety of quasigroups is *fully congruence regular*; without equationally definable constants, this variety cannot be *point regular*, and as such, full congruence regularity is the best that we can hope for. It is insightful, from the point of view of this text, to view full congruence regularity as a *parameterized analogue* of point regularity; instead of a single constant term 0 determining the congruence classes that must coincide, the constant 0 is replaced with an *arbitrary point*; this arbitrary point is the *parameter* in the definition of full congruence regularity.

Theorem 3.7 [Bar95] The variety of quasigroups is fully congruence regular. Consequently, the variety of quasigroups is congruence modular and congruence permutable.

Proof. The terms $p_1(u, x, y, z) = (y(x\backslash z))/(x\backslash u)$ and $\Delta_1(x, y, z) = y(x\backslash z)$ realize the Mal'cev condition of Theorem 1.444 on page 86. The outstanding assertions follow by Corollary 1.445. ◇

The following result demonstrates that every group may be viewed as a quasigroup. Consequently, any Mal'cev condition that holds for the variety of quasigroups will hold for the variety of groups and any variety of algebras that have group reducts, e.g., in the varieties of rings and of Boolean algebras. So these varieties are all congruence regular, congruence modular and congruence permutable.

Remark 3.8 Let $\mathbf{G} = \langle G; \cdot, ^{-1}, 1\rangle$ be a group. Define two binary operations $/$ and $\backslash$ on G by $a/b = ab^{-1}$ and $a\backslash b = a^{-1}b$, for all $a, b \in G$. Then $\langle G; \cdot, /, \backslash\rangle$ is a quasigroup.

Proof. We verify the identities 3.9 to 3.12 below. $(x/y)y = (xy^{-1})y \approx x(y^{-1}y) \approx x1 \approx 1$, $(xy)/y = (xy)y^{-1} \approx x(yy^{-1}) \approx x1 \approx x$, $x(x\backslash y) = x(x^{-1}y) \approx (xx^{-1})y \approx y$, and, $x\backslash(xy) = x^{-1}(xy) \approx (x^{-1}x)y \approx y$. $\diamond$

It can be shown that Steiner quasigroups give rise to special quasigroups [BS81]. So Steiner triple systems describe a subquasivariety of quasigroups. It suffices for our needs merely to demonstrate how Steiner triple systems give rise to quasigroups.

Theorem 3.9 [BS81] If $\mathcal{S}$ is a Steiner triple system, then

$$a \cdot a = a, \tag{3.13}$$

$$a \cdot b = c \text{ iff } \{a, b, c\} \in \mathsf{Line}(\mathcal{S}), \tag{3.14}$$

$$a/b \text{ is the unique } c \in \mathsf{uni}(\mathcal{S}) \text{ such that } c \cdot b = a \quad \text{and} \tag{3.15}$$

$$a\backslash b \text{ is the unique } \ c \in \mathsf{uni}(\mathcal{S}) \text{ such that } a \cdot c = b, \tag{3.16}$$

defines a quasigroup on $\mathsf{uni}(\mathcal{S})$. $\square$

We now turn to our main result, which demonstrates that full congruence regularity falls outside of the domain of algebraizable logics.

Theorem 3.10 The (fully congruence regular) variety of quasigroups $\mathcal{Q}$ is not the equivalent algebraic semantics of any sentential 1-calculus, and has a nontrivial subvariety that is not an algebraic semantics for any nontrivial sentential 1-calculus.

Proof. To verify the first assertion above, it suffices, by Corollary 2.113 on page 112, to show that the variety $\mathcal{Q}$ of quasigroups does not satisfy any quasi-identity of the form

$$\bigwedge\nolimits_{i<n}\bigwedge\nolimits_{j<m}\delta_i(\Delta_j(x,y)) \approx \varepsilon_i(\Delta_j(x,y)) \to x \approx y, \tag{3.17}$$

where $\delta_0, \ldots, \delta_{n-1}, \varepsilon_0, \ldots, \varepsilon_{n-1}$ are unary and $\Delta_0, \ldots, \Delta_{m-1}$ binary terms.

Consider the Steiner triple system $\mathcal{S}_7$ (defined above) on $A = \{1, 2, 3, 4, 5, 6, 7\}$ and the associated quasigroup $\mathbf{G}$, as defined as in Theorem 3.9.

If p is a unary term of $\mathcal{Q}$'s type and $a \in A$, then $t^{\mathbf{G}}(a) = a$ (by induction on p's complexity). Now if $\mathcal{Q}$ satisfied (3.17), we could infer from $\delta_i^{\mathbf{G}}(\Delta_j^{\mathbf{G}}(1,2)) = \Delta_j^{\mathbf{G}}(1,2) = \varepsilon_i^{\mathbf{G}}(\Delta_j^{\mathbf{G}}(1,2))$ the contradiction that $1 = 2$ in $\mathbf{G}$.

Let $\mathcal{V}$ be the subvariety of $\mathcal{Q}$ generated by the algebra $\mathbf{G}$. Suppose $\mathcal{V}$ is an algebraic semantics for a nontrivial sentential 1-calculus $\mathcal{S}$, with defining equations $\delta_i \approx \varepsilon_i$, $i < n$, in the sense of [BP89a, Definition 2.2] (see Definition 2.105 on page 111 of our text). As observed above, $\mathbf{G}$ (and therefore $\mathcal{V}$) satisfies $\delta_i(x) \approx x \approx \varepsilon_j(x)$ for all $i, j < n$. Thus, for every term r and each $i < n$, $\mathcal{V}$ satisfies $\delta_i(r) \approx \varepsilon_i(r)$, whence $\vdash_{\mathcal{S}} r$, contradicting the nontriviality of $\mathcal{S}$. $\diamond$

$\square$

This is indeed unfortunate for the algebraist. The primary aim of this text is to remedy this short coming. Briefly, with each quasivariety $\mathcal{K}$, we shall associate a sentential 1-calculus called the *membership logic* of the quasivariety (see Definition 5.62 of Example 5.57 on page 191); the membership logic is determined by its theories which are the congruence classes of the relative congruences on the term algebra, called *relative cosets*, together with the empty set, called the *improper coset*. We shall show that despite its 1-deductive character and despite the previous counter example, the membership logic of a quasivariety $\mathcal{K}$ interprets much of the (2-deductive) equational consequence relation $\models_{\mathcal{K}}$, and *all* of $\models_{\mathcal{K}}$ when $\mathcal{K}$ is a variety; in fact, any Mal'cev condition applicable to a variety is discernible in its membership logic. Further, the condition that a *quasivariety* be *relatively regular* is discernible (purely logically) in the membership logic of that quasivariety, in which case the membership logic interprets *all* of its equational consequence relation. When a quasivariety is relatively regular, there exist strong *two-way* relationships between the consequence relation $\vdash$ of its membership logic and the equational consequence relation $\models_{\mathcal{K}}$ of the quasivariety that are *similar in spirit* to the relationships between an algebraizable logic and its equivalent algebraic semantics (although these relationships must be different in the light of the previous counter example); namely, there exists a finite set $\boldsymbol{\Delta}$ of *ternary* terms such that

$$z, P \vdash p \text{ iff } P \approx z \models_{\mathcal{K}} p \approx z, \tag{3.18}$$

$$\begin{aligned}&\{p_i \approx q_i \;:\; i \in I\} \models_{\mathcal{K}} p \approx q \text{ iff}\\ &\qquad z, \{\Delta(p_i, q_i, z) \;:\; \Delta \in \boldsymbol{\Delta}, i \in I\} \vdash \{\Delta(p_i, q_i, z) \;:\; \Delta \in \boldsymbol{\Delta}\},\end{aligned} \tag{3.19}$$

$$z, \boldsymbol{\Delta}(p, z, z) \dashv\vdash z, p, \tag{3.20}$$

$$\boldsymbol{\Delta}(x, y, z) \approx z \mathrel{=\!\mid\models_{\mathcal{K}}} x \approx y. \tag{3.21}$$

Observe that if one replaced all references to z with a constant 0 and performed obvious simplifications (the ternary terms would become binary by 'hiding the 0' and noting that 0 is a theorem of the assertional logic), these formulae would characterize the relationship between the consequence relation $\vdash$ of the *assertional logic* of a relatively 0-regular quasivariety $\mathcal{K}$ and the equational consequence relation $\models_{\mathcal{K}}$ of its *equivalent algebraic semantics* $\mathcal{K}$ [BR99];

$$P \vdash p \text{ iff } P \approx 0 \models_{\mathcal{K}} p \approx 0, \tag{3.22}$$

$$\{p_i \approx q_i \;:\; i \in I\} \models_{\mathcal{K}} p \approx q \text{ iff } \{\Delta(p_i, q_i) \;:\; \Delta \in \boldsymbol{\Delta}, i \in I\} \vdash \{\Delta(p_i, q_i) \;:\; \Delta \in \boldsymbol{\Delta}\}, \tag{3.23}$$

$$\boldsymbol{\Delta}(p, 0) \dashv\vdash p, \tag{3.24}$$

$$\boldsymbol{\Delta}(x, y) \approx 0 \mathrel{=\!\mid\models_{\mathcal{K}}} x \approx y. \tag{3.25}$$

We shall show that (3.21) characterizes the relative congruence regularity of a quasivariety just as (3.25) characterizes the relative point regularity of a quasivariety [BR99]. The increase by *one* of the arities of the *ternary* terms $\boldsymbol{\Delta}$ of (3.21) over the *binary* terms $\boldsymbol{\Delta}$ of (3.25), is typical of the relationship between terms in a quasi-Mal'cev characterization of a 'full' condition versus the terms in the quasi-Mal'cev condition of the analogous 'point' version of that condition; the 'point' of the latter is replaced by a variable in the former and can no longer be 'hidden in the terms'. In the discourse of this text, we would view (3.21) a *parameterized* version of (3.25); in fact, we may view (3.18) through to (3.21) as *parameterized* analogues of (3.22) through to (3.25). Extending this idea of *parameterization* to the notion of equivalent algebraic semantics more generally, we shall develop a theory of *parameterized algebraization*, that encompasses *both* the relationship between the membership logic of a relatively regular quasivariety and the determining quasivariety described above, *and* the standard relationship between an algebraizable sentential 1-calculus and its equivalent algebraic semantics as described in [BP89a].

Open Problem 3.11 Is the variety $\mathcal{Q}$ of quasigroups an equivalent algebraic semantics for some *interesting* sentential n-calculus for $n > 1$? It is, of course, an equivalent algebraic semantics for $S^2(\Theta^{\mathcal{Q}})$.

Part II

Elementary and Concrete Closure

While developing our theory of parameterized algebraization, we noticed that many distinct arguments in the standard theory of algebraic logics, as well as in our theory, could be unified by the notion of a *continuous translation* between closed systems, and that many of these arguments could be given an *elementary* footing; notions from the standard theory unified from this perspective include, *structurality*, *filters* and *models* of logics, *algebraic and equivalent algebraic semantics*, the *filter correspondence property*, and the family of logics $S(\mathcal{K}, \tau)$ [BR99] which encompass all algebraizable sentential 1-calculi.

The elementary setting is obtained by considering the standard well-known objects of closure, namely closed systems, closure operators and consequence relations, as elementary structures with order reducts. In the case of closure operators, this perspective is well-known [DP90]. We shall show that closed systems and consequence relations too may be formulated as elementary structures over orders, and that these elementary objects are in *one-to-one correspondence* with elementary closure operators via relationships that reflect the *standard* correspondences between (concrete) closed systems, closure operators and consequence relations. While the realization of closure operators as elementary structures is trivial, i.e., as an order with an order-preserving, increasing and idempotent unary operation $\|\cdot\|$, the realization of the other objects of closure is less trivial. In the case of closed systems, for example, we need to give elementary meaning to *closure under arbitrary non-empty intersections*. The key to such a realization is the observation that it is not the notion of meet that is inherently non-elementary, rather it is the description of the set over which the meet is to be taken that is generally non-elementary. In particular, the meet (or join) of an *elementarily definable set* is an elementary notion. In the case of elementary closed systems (with a unary relation symbol cl), we exploit the fact that *principal filters* are elementarily definable, as is the *intersection* of two elementarily definable sets. In the case of consequence relations (with a binary relation symbol $\cdot \vdash \cdot$), we use the fact that the *poles of fundamental binary relations* are elementarily definable. In addition to elementary closure operators, closed systems and consequence relations, we introduce two other elementary closure related notions, namely *elementary closed equivalence relations* (having a binary relation symbol $\cdot \dashv\vdash \cdot$) and *elementary proto-Leibniz relations* (with a ternary relation symbol $\cdot \simeq(\cdot) \cdot$). These structures too are in one-to-one correspondence with elementary closure operators, and the same is true for their concrete forms and concrete closure operators. The former is an characterization of an *abbreviation* commonly encountered in logic, i.e., writing $\Gamma \dashv\vdash \Phi$ as an abbreviation for

$$\Gamma \vdash \Phi \text{ and } \Phi \vdash \Gamma.$$

It is our intuition that the latter is an elementary realization of the second order relation of Leibniz, and *not* the Leibniz relation encountered in algebraic logic [BP89a]. In fact, we discovered the relation in an attempt to understand the relation defined by the right-hand-side of the implication in the formulae

$$\phi \, \Omega^{\mathcal{S}}(T) \, \psi \text{ implies } T \cup \{\phi\} \vdash_{\mathcal{S}} \psi \text{ and } T \cup \{\psi\} \vdash_{\mathcal{S}} \phi,$$

characterizing or, in some texts, defining protoalgebraicity.

While these elementary classes are in one-to-one correspondence, each class admits different structure homomorphisms (all of which are order-preserving functions, which we call *weak-translations*), so while we conflate them in the discourse, tending to speak only of elementary closed systems, we distinguish between the different types of homomorphism, speaking of $\|\cdot\|$-

homomorphisms, cl-homomorphisms, $\vdash$-homomorphisms, etc. We characterize these homomorphisms, as well as their strict and reflecting variants, where appropriate. Of more interest to us are *galois relations* between the underlying orders, these are *pairs* of weak translations that satisfy particular elementary properties; we call these galois pairs *translations*. By a (strict) continuous translation, we mean a translation whose 'forward' weak-translation is a (strict) $\|\cdot\|$-homomorphism. We characterize continuous and strictly continuous translations, and show, by means of example, how these notions, unify many of the arguments and constructions in algebraic logic. In particular, structurality is the requirement that all substitutions by continuous, homomorphisms between algebras constitute continuous translations between the closed sets of filters of some sentential calculus, and the filter correspondence property may be characterized in terms of the strict continuity of reductive matrix homomorphisms. One of the characterizations of continuity is that the 'backward' translation maps closed points to closed points, just as the pre-image by continuous functions of closed sets are closed in topology; this is our justification for the term *continuous*. We also develop the theory of the product closed system determined by a translation from an order to an elementary closed system, and show how this construction gives rise to the class of sentential 1-calculi $S(\mathcal{K}, \tau)$ [BR99], which include all algebraizable sentential 1-calculi; later in the text we extend this construction to n-calculi. The last of the elementary discourse concerns *iseomorphisms*. We are able to prove 'one direction' of the theory of equivalent logics in this elementary setting; that is we show that iseomorphic translations imply that the suborders of closed points must be isomorphic.

We also consider 'concrete' closed systems and 'concrete' translations between them, which are grounded binary relationships between their universes (the set over which the power-order is taken); 'concrete' translations may alternatively be viewed as multi-maps, i.e., functions from a set to a power-set. We show that 'concrete' translations are precisely the translations between 'concrete' closed systems. Stronger characterizations of continuous and strictly continuous 'concrete' translations are obtained. 'Concrete' *iseomorphisms* are also characterized. In particular, we are able to establish that isomorphisms between closed orders give rise to iseomorphic 'concrete' translations. The *product* of a *source* defined and characterized, where a source is determined *multiple* translations from one set to multiple closed systems. We show how the *semantic consequence* relation determined by a matrix may be realized as the product of a source. Products of sources are used often in our theory of *parameterized algebraization*. We also develop the dual theory of the *quotient* of a *sink*, where a source is determined *multiple* translations from multiple closed systems to one set, and we demonstrate how the *filters* of sentential calculi arise as the quotient of a sink.

In [BJ06], a theory of *transformers between closure operators*, was published, as part of a generalization of the theory of algebraic logic. Transformers are special functions between the *lattices* of closed sets of two closure operators. We show that transformers between closure operators and *strictly* continuous relationships between closed systems are in essence the same notion. We explicate, and duly reference, the relationships between our (independently obtained) notions and theirs.

A number of examples of *sentential calculi*, pertinent to the sequel, are obtained as examples in this part, including the *sentential calculus of subuniverse* (see Example 5.47 on page 188) and our most important logic in this text, the *membership logic* of a quasivariety (see Example 5.57

on page 191).

We must note that many of the ideas in this part have been derived from and extend notions and results that we have found in textbooks of *topology*, in particular, [Eng68], [AE88], and [Kel50], and has been inspired by Tarski's note that 'formalized deductive disciplines form the field of research of metamathematics roughly in the same sense in which *spatial* entities form the field of research in *geometry* [Tar56]'.

Chapter 4

Closure

In this chapter, we introduce various notions of *elementary* closure and consider the relationships between these elementary notions and the standard 'concrete' objects of closure, i.e., closed systems, closure operators and consequence relations. In $\oint$4.1 we define five elementary structures, all with order reducts, namely *elementary closure operators*, *elementary closed systems*, *elementary consequence relations*, *elementary closed equivalence relations* and *elementary proto-Leibniz relations*, and demonstrate that each of these elementary classes of structures are in one-to-one correspondence with any other. In $\oint$4.2 we consider the *concrete* instances of these structures, that is, we consider the case that the underlying order is the inclusion ordered power-set of some set, and show that in the case of closure operators, closed systems and consequence relations, the concrete versions of these elementary structures coincide with the well-known second-order objects with the same name. We have not seen concrete analogues of elementary closed equivalence relations and proto-Leibniz relations in the literature; although the latter object is implicit in the definition of protoalgebraicity. Finally, in $\oint$4.3 we consider *algebraic* (also known as *finitary*) closed systems, where we will characterize these objects in terms of a logic-like notion called a *formal system*. A number of examples of formal systems are presented; these will evolve into logics over constructs and ultimately into 'inherently unalgebraizable' sentential calculi to which we shall apply our theory of parameterized algebraization (see Part V). In the next chapter we shall consider the various *structure homomorphisms* between these structures, and unify many of the arguments and constructions from algebraic logic under the umbrella of *continuous translations*.

4.1 Elementary Closure

In this section we define and inter-relate our five elementary structures of closure, as well as define and characterize the notion of granularity.

4.1.1 Elementary Closure Operators

Of our five elementary structures of closure, the elementary closure operator is by far the simplest to define, since the well-known definition of a (concrete) closure operator, as an idempotent, $\subseteq$-increasing and $\subseteq$-preserving operator, is immediately expressible over an order, as an idempotent, $\geq$-increasing and $\geq$-preserving operation. The elementary closure operator is also well-known [DP90].

Definition 4.1 (Elementary Closure Operators) The **type of elementary closure operators**, denoted $\mathsf{type(eco)}$, has a binary relation symbol $\leq$ and a unary operation symbol $\|\cdot\|$. An **elementary closure operator** is a $\mathsf{type(eco)}$-structure $\mathfrak{c} = \langle \mathsf{uni_e}(\mathfrak{c}); \leq^{\mathfrak{c}}; \|\cdot\|_{\mathfrak{c}} \rangle$ whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{c}}$ and called the **underlying order**, and such that $\mathfrak{c}$ satisfies the (further) axioms

$$\text{(order-preserving)} \qquad x \leq y \longrightarrow \|x\| \leq \|y\| \,, \tag{4.1}$$

$$\text{(increasing)} \qquad x \leq \|x\| \qquad \text{and} \tag{4.2}$$

$$\text{(idempotent)} \qquad \|\|x\|\| \approx \|x\| \,, \tag{4.3}$$

in which case we call $\|\cdot\|_{\mathfrak{c}}$ the **closure operator** and write $\mathsf{uni_e}(\mathfrak{c})$ for $\mathsf{uni}(\mathbf{P}_{\mathfrak{c}})$ which we call the **elementary universe** (or just **universe** when unambiguous). When we call $\mathfrak{c}$ an **elementary closure operator on order $\mathbf{P}$**, we mean that $\mathbf{P}_{\mathfrak{c}} = \mathbf{P}$. For an order $\mathbf{P}$ and an operator $\|\cdot\|$ on $\mathsf{uni_e}(\mathbf{P})$, when we say that $\|\cdot\|$ determines a (elementary) closure operator on $\mathbf{P}$, or say that $\langle \mathbf{P}; \|\cdot\| \rangle$ is a (elementary) closure operator, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; \|\cdot\| \rangle$ is an elementary closure operator. Let $\mathsf{ECO}(\mathbf{P})$ denote the set of (elementary) closure operators on order $\mathbf{P}$. $\square$

Warning 4.2 Our use of the subscript 'e' in the term $\mathsf{uni_e}(\mathfrak{c})$ of these and subsequent notations, is to distinguish the *elementary universe* from the *'concrete' universe* when working with concrete closure operators (see $\oint$4.2).

Recall that order-preserving functions preserve upper bounds, but do not generally preserve *least* upper bounds, and that *least* upper bound preserving functions are order preserving.

Lemma 4.3 Let u be an idempotent increasing operator on the universe of order $\mathbf{P}$. Then u is order-preserving iff $\mathsf{u} : \mathbf{P} \to_{\blacktriangledown} \mathsf{u}[\mathbf{P}]$.

Proof. $\boxed{\Rightarrow}$ Assume that u is order-preserving and increasing on $\mathbf{P}$. Let $A \subseteq \mathsf{uni}(\mathbf{P})$ and suppose that $\blacktriangledown^{\mathbf{P}} A$ exists. Since order preserving functions preserve upper bounds, $\mathsf{u}(\blacktriangledown^{\mathbf{P}} A)$ is an upper bound of $\mathsf{u}[A]$ in $\mathsf{u}[\mathbf{P}]$. Let $\mathsf{u}(b)$ be any upper bound of $\mathsf{u}[A]$ in $\mathsf{u}[\mathbf{P}]$. Now, for any $a \in A$, $a \leq \mathsf{u}(a)$, since u is increasing, and $\mathsf{u}(a) \leq \mathsf{u}(b)$, since $\mathsf{u}(b)$ is an upper bound of $\mathsf{u}[A]$ in $\mathsf{u}[\mathbf{P}]$. Hence $a \leq \mathsf{u}(b)$ and so $\mathsf{u}(b)$ is an upper bound of A in $\mathbf{P}$. Hence $\blacktriangledown^{\mathbf{P}} A \leq \mathsf{u}(b)$; hence $\mathsf{u}(\blacktriangledown^{\mathbf{P}} A) \leq \mathsf{u}(\mathsf{u}(b)) = \mathsf{u}(b)$, since u is increasing and idempotent. So $\mathsf{u}(\blacktriangledown^{\mathbf{P}} A)$ is the least upper bound of $\mathsf{u}[A]$ in $\mathsf{u}[\mathbf{P}]$. $\boxed{\Leftarrow}$ Follows trivially, since $\blacktriangledown$-preserving functions are order preserving. $\diamond$

The following corollary to the previous lemma characterizes the elementary closure operators on a given order.

Corollary 4.4 Let $\|\cdot\|$ be an operator on the universe of order $\mathbf{P}$. Then $\|\cdot\|$ determines an elementary closure operator on $\mathbf{P}$ iff $\|\cdot\|$ is increasing, idempotent and $\|\cdot\| : \mathbf{P} \to_{\blacktriangledown} \mathsf{u}[\mathbf{P}]$.

4.1.2 Elementary Closed Systems

While elementary closure operators are naturally expressible, less clear is how to define an elementary closed system over an order, given the role of arbitrary intersection in the standard definition (see Definition 1.196 on page 43), and how to do so while still requiring elementary closure operators and elementary closed systems to be in one-to-one correspondence under the standard concrete correspondence. The key property that we need to formulate *elementarily* is the property that every set is contained in a least closed set.

Definition 4.5 (Elementary Closed Systems) The **type of elementary closed systems**, denoted $\mathsf{type(ecs)}$, has a binary relation symbol $\leq$ and a unary relation symbol cl. An **elementary closed system** is a $\mathsf{type(ecs)}$-structure whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{c}}$ and called the **underlying order**, and is such that $\mathfrak{c}$ satisfies the axiom

$$\forall[x]\,\exists[z]\,(z \text{ is } \mathsf{cl}\,\mathsf{and}\, x \leq z\,\mathsf{and}(\forall[y]\, y \text{ is } \mathsf{cl}\,\mathsf{and}\, x \leq y \rightarrow z \leq y)), \tag{4.4}$$

in which case we call $\mathsf{cl}_{\mathfrak{c}}\cdot$ the associated **closed relation** and write $\mathsf{uni_e}(\mathfrak{c})$ for $\mathsf{uni}(\mathbf{P}_{\mathfrak{c}})$ which we call the **elementary universe** (or just **universe** when unambiguous). We tend to conflate the unary relation $\mathsf{cl}_{\mathfrak{c}}$ with the set $\{c : c \text{ is } \mathsf{cl}_{\mathfrak{c}}\}$, hence writing either $a\,\text{is}\,\mathsf{cl}_{\mathfrak{c}}$ or $a \in \mathsf{cl}_{\mathfrak{c}}$, as appropriate. When we call $\mathfrak{c}$ an **elementary closed system on order P**, we mean that $\mathbf{P}_{\mathfrak{c}} = \mathbf{P}$. For an order $\mathbf{P}$ and a unary relation cl on $\mathsf{uni}(\mathbf{P})$, when we say that cl determines a (elementary) closed system on $\mathbf{P}$, or say that $\langle \mathbf{P}; \mathsf{cl}\rangle$ is a (elementary) closed system, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; \mathsf{cl}\rangle$ is an elementary closure operator. Let $\mathsf{ECS}(\mathbf{P})$ denote the set of (elementary) closed systems on order $\mathbf{P}$. The suborder of $\mathbf{P}_{\mathfrak{c}}$ induced by $\mathbf{P}_{\mathfrak{c}}$ on (the set) $\mathsf{cl}_{\mathfrak{c}}$ is denoted by (emboldened) $\mathbf{cl}_{\mathfrak{c}}$. □

The following, apparently second-order, characterization of the elementary closed systems on an order is an immediate rephrasing of (4.4). While (4.5) is second-order as phrased, the set $[a\rangle_{\mathbf{P}_{\mathfrak{c}}} \cap \mathsf{cl}$ is elementarily definable, and hence the assertion that it has a meet and that this meet is closed is expressible as a elementary formulae, viz. (4.4).

Proposition 4.6 Let $\mathbf{P}$ be an order and cl a unary relation on $\mathsf{uni}(\mathbf{P})$. Then cl determines an elementary closed system on $\mathbf{P}$ iff

$$\forall\,[a \in \mathsf{uni}(\mathbf{P})]\ \blacktriangle\big([a\rangle_{\mathbf{P}_{\mathfrak{c}}} \cap \mathsf{cl}\big) \text{ exists and is } \mathsf{cl}. \tag{4.5}$$

□

Consequently, if $\mathfrak{c}$ is an elementary closed system then, for each $a \in \mathsf{uni_e}(\mathfrak{c})$, there exists a (unique) $\leq^{\mathfrak{c}}$-least closed point above a. We highlight some useful consequences for ease of later reference, first introducing facilitating notation.

Definition 4.7 (Closed-Cover) Let $\mathfrak{c}$ be an elementary closed system. With each $a \in \mathsf{uni_e}(\mathfrak{c})$, we associate the set $\mathsf{cover}_{\mathfrak{c}}(a)$, which we call the **closed-cover** of a, defined by

$$\mathsf{cover}_{\mathfrak{c}}(a) = [a\rangle_{\mathbf{P}_{\mathfrak{c}}} \cap \mathsf{cl}_{\mathfrak{c}} \doteq \{b\,\text{is}\,\mathsf{cl}_{\mathfrak{c}} \,:\, a \leq b\}.$$

□

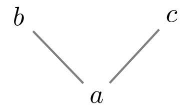


Figure 4.1: see Example 4.9

Corollary 4.8 For an elementary closed system $\mathfrak{c}$ the following formulae are valid.

$$\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a)\ \text{exists}, \tag{4.6}$$

$$\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a)\ \text{is}\ \mathsf{cl}_{\mathfrak{c}}, \tag{4.7}$$

$$a \le \blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a), \tag{4.8}$$

$$\mathsf{cover}_{\mathfrak{c}}(a) = \mathsf{cover}_{\mathfrak{c}}(\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a)), \tag{4.9}$$

$$\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a) \in \mathsf{cover}_{\mathfrak{c}}(a), \tag{4.10}$$

$$a\ \text{is}\ \mathsf{cl}_{\mathfrak{c}}\ \text{ iff }\ a = \blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a) \quad \text{and} \tag{4.11}$$

$$\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a) = \overset{\mathbf{cl}_{\mathfrak{c}}}{\blacktriangle}\,\mathsf{cover}_{\mathfrak{c}}(a). \tag{4.12}$$

Example 4.9 (The Discrete Closed System on P)

For any ordered set $\mathbf{P}$, $\mathsf{uni}(\mathbf{P})$ determines a closed system on $\mathbf{P}$. We call this closed system the **discrete closed system** on $\mathbf{P}$. The ordered set described in Figure 4.1, admits only the discrete closed system.

□

We now aim to show that elementary closed systems and closure operators are in one-to-one correspondence.

Definition 4.10 (Associating Elementary Closed Systems and Closure Operators) With each elementary closure operator $\mathfrak{c}$, we associate the elementary closed system $\mathsf{ecs}(\mathfrak{c})$ on $\mathbf{P}_{\mathfrak{c}}$, for which we tend to write $\mathsf{cl}_{\mathfrak{c}}$ for $\mathsf{cl}_{\mathsf{ecs}(\mathfrak{c})}$, determined by

$$a\ \text{is}\ \mathsf{cl}_{\mathfrak{c}}\ \text{ iff }\ \|a\|_{\mathfrak{c}} = a. \tag{4.13}$$

With each elementary closed system $\mathfrak{c}$, we associate the elementary closure operator $\mathsf{eco}(\mathfrak{c})$ on $\mathbf{P}_{\mathfrak{c}}$, for which we tend to write $\|\cdot\|_{\mathfrak{c}}$ for $\|\cdot\|_{\mathsf{eco}(\mathfrak{c})}$, determined by

$$\|a\|_{\mathfrak{c}} = \blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a), \tag{4.14}$$

this operator being well-defined by (4.6). □

Proof. $\boxed{\text{ecs}(\mathfrak{c})\text{ is an elementary closed system}}$ Let $a \in \mathsf{uni}(\mathsf{ecs}(\mathfrak{c}))$ and let $c = \|a\|_{\mathfrak{c}}$. Then by idempotence, $\|c\|_{\mathfrak{c}} = \big\|\,\|a\|_{\mathfrak{c}}\big\|_{\mathfrak{c}} = \|a\|_{\mathfrak{c}} = c$, and so by definition, $c\ \text{is}\ \mathsf{cl}_{\mathsf{ecs}(\mathfrak{c})}$. Suppose that $b\ \text{is}\ \mathsf{cl}_{\mathsf{ecs}(\mathfrak{c})}$ and $a \le b$. By definition, $b = \|b\|_{\mathfrak{c}}$, and by order preservation, $\|a\|_{\mathfrak{c}} \le \|b\|_{\mathfrak{c}}$. Then $c = \|a\|_{\mathfrak{c}} \le \|b\|_{\mathfrak{c}} = b$, as required. $\boxed{\text{eco}(\mathfrak{c})\text{ is an elementary closure operator}}$ $\boxed{\text{Increasing}}$ By definition and (4.8), $a \le \|a\|_{\mathsf{eco}(\mathfrak{c})}$. $\boxed{\text{Order preserving}}$ If $a \le b$, then $\mathsf{cover}_{\mathfrak{c}}(b) \subseteq \mathsf{cover}_{\mathfrak{c}}(a)$ and hence $\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a) \le \blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a)$; i.e., $\|a\|_{\mathsf{eco}(\mathfrak{c})} \le \|b\|_{\mathsf{eco}(\mathfrak{c})}$. $\boxed{\text{Idempotent}}$ By (4.9), $\mathsf{cover}_{\mathfrak{c}}(a) = \mathsf{cover}_{\mathfrak{c}}(\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a))$, and hence $\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(\blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a)) = \blacktriangle\,\mathsf{cover}_{\mathfrak{c}}(a)$; i.e., $\Big\|\,\|a\|_{\mathsf{eco}(\mathfrak{c})}\Big\|_{\mathsf{eco}(\mathfrak{c})} = \|a\|_{\mathsf{eco}(\mathfrak{c})}$. ⋄

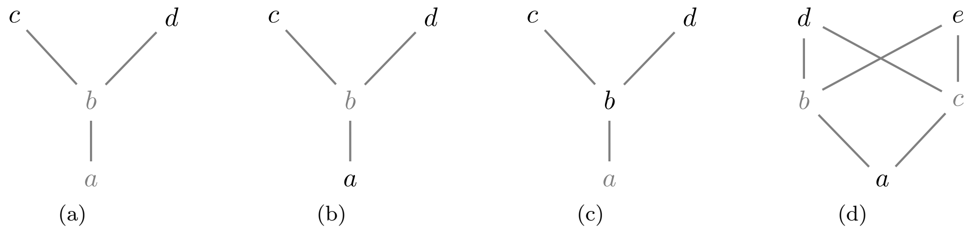


Figure 4.2: see Example 4.17

Proposition 4.11 For order $\mathbf{P}$, $\mathsf{ecs}(\cdot)$ and $\mathsf{eco}(\cdot)$ define mutually inverse bijections between $\mathsf{ECO}(\mathbf{P})$ and $\mathsf{ECS}(\mathbf{P})$.

Proof. (*It suffices to prove that* $\mathsf{ecs}(\cdot)$ *is injective and* $\mathsf{ecs}(\mathsf{eco}(\mathfrak{c})) = \mathfrak{c}$.)
$\boxed{\mathsf{ecs}(\cdot)\text{ is injective}}$ Let $\mathfrak{c}, \mathfrak{d} \in \mathsf{ECO}(\mathbf{P})$ and suppose that $\mathsf{cl}_{\mathsf{ecs}(\mathfrak{c})} = \mathsf{cl}_{\mathsf{ecs}(\mathfrak{d})}$. Let $a \in \mathsf{uni}(\mathbf{P})$. (*It suffices to show that* $\|a\|_{\mathfrak{c}} = \|a\|_{\mathfrak{d}}$.) By increasingness of $\|\cdot\|_{\mathfrak{d}}$, $a \le \|a\|_{\mathfrak{d}}$, and so by increasingness of $\|\cdot\|_{\mathfrak{c}}$, $\|a\|_{\mathfrak{c}} \le \big\| \|a\|_{\mathfrak{d}} \big\|_{\mathfrak{c}}$. Since $\big\| \|a\|_{\mathfrak{d}} \big\|_{\mathfrak{d}} = \|a\|_{\mathfrak{d}}$ (by idempotence), $\|a\|_{\mathfrak{d}} \in \mathsf{cl}_{\mathsf{ecs}(\mathfrak{d})}$ by definition, and so $\|a\|_{\mathfrak{d}} \in \mathsf{cl}_{\mathsf{ecs}(\mathfrak{c})}$, since $\mathsf{cl}_{\mathsf{ecs}(\mathfrak{c})} = \mathsf{cl}_{\mathsf{ecs}(\mathfrak{d})}$ by assumption. In other words, $\big\| \|a\|_{\mathfrak{d}} \big\|_{\mathfrak{c}} = \|a\|_{\mathfrak{d}}$. Hence $\|a\|_{\mathfrak{c}} \le \big\| \|a\|_{\mathfrak{d}} \big\|_{\mathfrak{c}} = \|a\|_{\mathfrak{d}}$. By a symmetric argument, $\|a\|_{\mathfrak{d}} \le \|a\|_{\mathfrak{c}}$, and so $\|a\|_{\mathfrak{c}} = \|a\|_{\mathfrak{d}}$ by the anti-symmetry of $\le$. $\boxed{\mathsf{ecs}(\mathsf{eco}(\mathfrak{c})) = \mathfrak{c}}$ Let $\mathfrak{c} \in \mathsf{ECS}(\mathbf{P})$. Now $a \,\mathsf{is}\, \mathsf{cl}_{\mathsf{ecs}(\mathsf{eco}(\mathfrak{c}))}$ [iff] $a = \|a\|_{\mathsf{eco}(\mathfrak{c})}$ [iff] $a = \blacktriangle\, \mathsf{cover}_{\mathfrak{c}}(a)$ [iff by (4.11)] $a \,\mathsf{is}\, \mathsf{cl}_{\mathfrak{c}}$. $\diamond$

Convention 4.12 (Conflating Elementary Closure Operators and Closed Systems) Consequent to the previous definition and proposition we shall (tend to) syntactically conflate elementary closure operators and closed systems, and, as such, treat (4.13) and (4.14) as properties of these conflated structures

Remark 4.13 Rephrasing (4.13) in the light of the previous results,

$$\mathsf{cl}_{\mathfrak{c}} = \{\|a\|_{\mathfrak{c}} \;:\; a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})\}. \tag{4.15}$$

Remark 4.14 $\|a\|_{\mathfrak{c}}$ is the least $\mathfrak{c}$-closed c above a (by (4.14) and (4.10)). This property is used so often that we shall refer to it as the **minimality property of elementary closure operators** or simply **minimality** where unambiguous. Further, by (4.13) and (4.2),

$$a \,\mathsf{is}\, \mathsf{cl}_{\mathfrak{c}} \text{ iff } \|a\|_{\mathfrak{c}} \le a. \tag{4.16}$$

Corollary 4.15 If $\mathfrak{c}$ is an elementary closed system and $A \subseteq \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$ such that $\blacktriangledown A$ exists, then $\blacktriangledown^{\mathbf{cl}_{\mathfrak{c}}}\{\|a\|_{\mathfrak{c}} \;:\; a \in A\}$ exists and

$$\|\blacktriangledown A\|_{\mathfrak{c}} = \overset{\mathbf{cl}_{\mathfrak{c}}}{\blacktriangledown}\{\|a\|_{\mathfrak{c}} \;:\; a \in A\}. \tag{4.17}$$

Proof. By (4.15), $\|\blacktriangledown A\|_{\mathfrak{c}} \in \mathsf{cl}_{\mathfrak{c}}$. For each $a \in A$, $a \le \blacktriangledown A$, and hence $\|a\|_{\mathfrak{c}} \le \|\blacktriangledown A\|_{\mathfrak{c}}$, by order-preservation. So $\|\blacktriangledown A\|_{\mathfrak{c}}$ is an upper bound of $\{\|a\|_{\mathfrak{c}} \;:\; a \in A\}$ in $\mathbf{cl}_{\mathfrak{c}}$. Suppose that $c \in \mathsf{cl}_{\mathfrak{c}}$ is an upper bound of $\{\|a\|_{\mathfrak{c}} \;:\; a \in A\}$ in $\mathbf{cl}_{\mathfrak{c}}$. Then, for all $a \in A$, $a \le \|a\|_{\mathfrak{c}} \le c$. Hence $\blacktriangledown A \le c$ and so by order-preservation $\|\blacktriangledown A\|_{\mathfrak{c}} \le \|c\|_{\mathfrak{c}} = c$, the final equality following from (4.13). $\diamond$

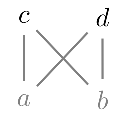


Figure 4.3: see Counter Example 4.18

Proposition 4.16 If $\mathfrak{c}$ is an elementary closed system then $\mathbf{cl}_\mathfrak{c} \lhd_\blacktriangle \mathbf{P}_\mathfrak{c}$.

Proof. Let $A \subseteq \mathsf{cl}_\mathfrak{c}$. Suppose that $\blacktriangle^{\mathbf{cl}_\mathfrak{c}} A$. Certainly $\blacktriangle^{\mathbf{cl}_\mathfrak{c}} A$ is a $\mathbf{P}_\mathfrak{c}$-lower bound of A. If a is a $\mathbf{P}_\mathfrak{c}$-lower bound of A, then $A \subseteq \mathsf{cover}_\mathfrak{c}(a)$, and so $a \leq \blacktriangle^{\mathbf{P}_\mathfrak{c}} \mathsf{cover}_\mathfrak{c}(a) \overset{(4.12)}{=} \blacktriangle^{\mathbf{cl}_\mathfrak{c}} \mathsf{cover}_\mathfrak{c}(a) \leq \blacktriangle^{\mathbf{cl}_\mathfrak{c}} A$. So $\blacktriangle^{\mathbf{cl}_\mathfrak{c}} A = \blacktriangle^{\mathbf{P}_\mathfrak{c}} A$. Conversely, suppose that $\blacktriangle^{\mathbf{P}_\mathfrak{c}} A$ exists. (*It suffices to show that* $\blacktriangle^{\mathbf{P}_\mathfrak{c}} A \in \mathsf{cl}_\mathfrak{c}$.) Since $A \subseteq \mathsf{cover}_\mathfrak{c}(\blacktriangle^{\mathbf{P}_\mathfrak{c}} A)$, $\left\| \blacktriangle^{\mathbf{P}_\mathfrak{c}} A \right\|_\mathfrak{c} = \blacktriangle^{\mathbf{P}_\mathfrak{c}} \mathsf{cover}_\mathfrak{c}(\blacktriangle^{\mathbf{P}_\mathfrak{c}} A)$ is a $\mathbf{P}_\mathfrak{c}$-lower bound of A, and hence $\left\| \blacktriangle^{\mathbf{P}_\mathfrak{c}} A \right\|_\mathfrak{c} \leq \blacktriangle^{\mathbf{P}_\mathfrak{c}} A$. Hence $\blacktriangle^{\mathbf{P}_\mathfrak{c}} A = \left\| \blacktriangle^{\mathbf{P}_\mathfrak{c}} A \right\|_\mathfrak{c} \in \mathsf{cl}_\mathfrak{c}$ by (4.16). ◇

Example 4.17

The black elements X of the ordered sets described in Figure 4.2 (for example, $X = \{c, d\}$ for (a)), determine closed systems only in the case of (c), while (a), (b) and (d) fail since the sets of black elements are not $\wedge$-consistent. In the case of (a), c and d have a meet in $\mathbf{P}$ but not in $\mathbf{P}_{|X}$, in (b), c and d have a meet in $\mathbf{P}$ and in $\mathbf{P}_{|X}$, but the meets differ, while in (d), d and e do not have a meet in $\mathbf{P}$ but do have a meet in $\mathbf{P}_{|X}$.

□

The necessary condition of Proposition 4.16 is not sufficient, as demonstrated by the following counter-example.

Counter Example 4.18

Consider the ordered set $\mathbf{P}$ defined by $a \dashv c, d$ and $b \dashv c, d$, and consider the set $X = \{c, d\}$ (see Figure 4.3). X is not a closed system, since it does not contain a least element above of a. On the other hand, all downsets meet X and X is a $\blacktriangle$-consistent subset of $\mathbf{P}$, as all subsets of X have neither $\mathbf{P}$-meets nor $\mathbf{P}_{|X}$-meets.

□

4.1.3 Elementary Consequence Relations

The standard formulation[1] of a (concrete) consequence relation (see Definition 4.47), as a set-point relationship, does not (to our knowledge) admit an elementary formulation, since points fall outside the domain of discourse. It is common practice, given a set-point consequence relation $\vdash$, to introduce a set-set consequence relation as an *abbreviation*, that is, to write $A \vdash B$ for $\forall [b \in B]\, A \vdash b$. The essence of (concrete) consequence can be captured by this set-set relation, and these relations admit an elementary abstraction. We shall show that such elementary consequence relations are in one-to-one correspondence with elementary closed systems, and that their concrete forms, as set-set relations, are in one-to-one correspondence with standardly formulated set-point consequence relations (see Definition 4.47 of ∮4.2).

[1] While we have not seen this 'standard' consequence relation defined outside of a logical setting, we are certain that such a *concrete* formulation must be well-known.

Definition 4.19 (Elementary Consequence Relations) The **type of elementary consequence relations**, denoted $\mathsf{type}(\mathsf{ecr})$, has a binary relation symbol $\leq$ and a binary relation symbol $\vdash$. An **elementary consequence relation** is a $\mathsf{type}(\mathsf{ecr})$-structure $\mathfrak{c}$ whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{c}}$ and called the **underlying order**, and is such that $\mathfrak{c}$ satisfies the axioms

$$\text{(inversion)} \qquad x \leq y \longrightarrow y \vdash x, \tag{4.18}$$

$$\text{(transitivity)} \qquad x \vdash y \,\mathsf{and}\, y \vdash z \longrightarrow x \vdash z \qquad \text{and} \tag{4.19}$$

$$\text{(limit)} \qquad \forall[x]\, \exists[y]\, x \vdash y \,\mathsf{and}(\forall[z]\, x \vdash z \longrightarrow z \leq y), \tag{4.20}$$

in which case we call $\vdash_{\mathfrak{c}}$ the associated **consequence relation**. When we call $\mathfrak{c}$ an **elementary consequence relation on order P**, we mean that $\mathbf{P}_{\mathfrak{c}} = \mathbf{P}$. For an order $\mathbf{P}$ and a binary relation $\vdash$ on $\mathsf{uni}(\mathbf{P})$, when we say that $\vdash$ determines a (elementary) consequence relation on $\mathbf{P}$, or say that $\langle \mathbf{P}; \vdash \rangle$ is a (elementary) consequence relation, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; \vdash \rangle$ is an elementary consequence relation. Let $\mathsf{ECR}(\mathbf{P})$ denote the set of (elementary) consequence relations on order $\mathbf{P}$. □

Analogously to the case for closed systems, (4.20) is an elementary expression of the second-order condition that there exists a least point '$\vdash$-reachable' from a given point. The reason that this second-order condition has an elementary characterization is because the poles of fundamental binary relations are elementarily definable sets. More precisely, we have the following characterization of the elementary consequence relations on a given order $\mathbf{P}$. The proof is immediate.

Proposition 4.20 Let $\mathbf{P}$ be an order and $\vdash$ a binary relation on the universe of $\mathbf{P}$. Then $\vdash$ determines a consequence relation on $\mathbf{P}$ iff (4.18) and (4.19) hold and

$$\blacktriangledown \vdash [\![a]\!] \text{ exists and } a \vdash (\blacktriangledown \vdash [\![a]\!]). \tag{4.21}$$

□

We enumerate some basic properties of elementary consequence relations.

Remark 4.21 Let $\mathfrak{c}$ be an elementary consequence relation. The following formulae are all valid.

$$\text{(pre-up-preserving)} \qquad a \vdash b \text{ and } a \leq c \text{ implies } c \vdash b, \tag{4.22}$$

$$\text{(post-down-preserving)} \qquad a \vdash b \text{ and } c \leq b \text{ implies } a \vdash c, \tag{4.23}$$

$$\text{(reflexive)} \qquad a \vdash a, \tag{4.24}$$

$$a \leq \blacktriangledown \vdash [\![a]\!], \tag{4.25}$$

$$(\blacktriangledown \vdash [\![a]\!]) \vdash a \quad \text{and} \tag{4.26}$$

$$a \vdash b \text{ iff } b \leq \blacktriangledown \vdash [\![a]\!]. \tag{4.27}$$

Proof. $\boxed{\text{(4.22) and (4.22)}}$ By inversion and transitivity. $\boxed{\text{(4.24)}}$ By inversion and transitivity. $\boxed{\text{(4.25)}}$ By (4.21), $\blacktriangledown \vdash [\![a]\!]$ exists, and by (already established) reflexivity, $a \in \vdash [\![a]\!]$; hence $a \leq \blacktriangledown \vdash [\![a]\!]$. $\boxed{\text{(4.26)}}$ By inversion and (4.25). $\boxed{\text{(4.27)}}$ $\boxed{\Rightarrow}$ $a \vdash b$ [implies] $b \in \vdash [\![a]\!]$ [implies] $b \leq \blacktriangledown \vdash [\![a]\!]$ $\boxed{\Leftarrow}$ $b \leq (\blacktriangledown \vdash [\![a]\!])$ [implies by inversion] $(\blacktriangledown \vdash [\![a]\!]) \vdash b$ [implies by (4.21) and transitivity] $a \vdash b$. ◇

Definition 4.22 (Associating Consequence Relations and Closure Operators) With each elementary consequence relation $\mathfrak{c}$, we associate the elementary closure operator $\mathsf{eco}(\mathfrak{c})$, for which we tend to abbreviate $\|a\|_{\mathsf{eco}(\mathfrak{c})}$ by $\|a\|_{\mathfrak{c}}$, defined by

$$\|a\|_{\mathfrak{c}} = \blacktriangledown \vdash_{\mathfrak{c}} [\![a]\!], \tag{4.28}$$

this operator being well-defined by (4.21). With each elementary closure operator $\mathfrak{c}$, we associate the elementary consequence relation $\mathsf{ecr}(\mathfrak{c})$ in $\mathbf{P}_{\mathfrak{c}}$, for which we tend to abbreviate $\vdash_{\mathsf{ecr}(\mathfrak{c})}$ by $\vdash_{\mathfrak{c}}$, where

$$a \vdash_{\mathfrak{c}} b \leftrightarrow b \leq \|a\|_{\mathfrak{c}} . \tag{4.29}$$

□

Proof. $\boxed{\text{eco}(\mathfrak{c}) \text{ is an elementary closure operator}}$ Let $\mathfrak{c}$ be an elementary consequence relation. $\boxed{\text{Order preserving}}$ $a \leq b$ [implies by inversion] $b \vdash_{\mathfrak{c}} a$ [implies by $\vdash_{\mathfrak{c}}$-transitivity] $\vdash_{\mathfrak{c}} [\![a]\!] \subseteq \vdash_{\mathfrak{c}} [\![b]\!]$ [implies] $\blacktriangledown \vdash_{\mathfrak{c}} [\![a]\!] \leq \blacktriangledown \vdash_{\mathfrak{c}} [\![b]\!]$ [implies] $\|a\|_{\mathsf{eco}(\mathfrak{c})} \leq \|b\|_{\mathsf{eco}(\mathfrak{c})}$. $\boxed{\text{Increasing}}$ (4.24) [implies] $a \in \vdash_{\mathfrak{c}} [\![a]\!]$ [implies] $a \leq \blacktriangledown \vdash_{\mathfrak{c}} [\![a]\!]$ [implies] $a \leq \|a\|_{\mathsf{eco}(\mathfrak{c})}$. $\boxed{\text{Idempotent}}$ (4.21) [implies] $a \vdash_{\mathfrak{c}} (\blacktriangledown \vdash_{\mathfrak{c}} [\![a]\!])$ [implies] $a \vdash_{\mathfrak{c}} \|a\|_{\mathsf{eco}(\mathfrak{c})}$ [implies by $\vdash_{\mathfrak{c}}$-transitivity] $\vdash_{\mathfrak{c}} [\![\|a\|_{\mathsf{eco}(\mathfrak{c})}]\!] \subseteq \vdash_{\mathfrak{c}} [\![a]\!]$ [implies] $\blacktriangledown \vdash_{\mathfrak{c}} [\![\|a\|_{\mathsf{eco}(\mathfrak{c})}]\!] \leq \blacktriangledown \vdash_{\mathfrak{c}} [\![a]\!]$ [implies] $\left\| \|a\|_{\mathsf{eco}(\mathfrak{c})} \right\|_{\mathsf{eco}(\mathfrak{c})} \leq \|a\|_{\mathsf{eco}(\mathfrak{c})}$, which suffices. $\boxed{\text{ecr}(\mathfrak{c}) \text{ is an elementary consequence relation}}$ We use Proposition 4.20. $\boxed{\text{Inversion (4.18)}}$ $a \leq b$ [implies by $\|\cdot\|_{\mathfrak{c}}$-increasingness] $a \leq b \leq \|b\|_{\mathfrak{c}}$ [implies] $b \vdash_{\mathsf{ecr}(\mathfrak{c})} a$. $\boxed{\text{Transitivity (4.19)}}$ $a \vdash_{\mathsf{ecr}(\mathfrak{c})} b$ and $c \vdash_{\mathsf{ecr}(\mathfrak{c})} a$ [implies] $b \leq \|a\|_{\mathfrak{c}}$ and $a \leq \|c\|_{\mathfrak{c}}$ [implies by order-preservation and idempotence] $b \leq \|a\|_{\mathfrak{c}}$ and $\|a\|_{\mathfrak{c}} \leq \|c\|_{\mathfrak{c}}$ [implies] $b \leq \|c\|_{\mathfrak{c}}$ [implies] $c \vdash_{\mathsf{ecr}(\mathfrak{c})} b$. $\boxed{\text{Claim: } \blacktriangledown \vdash_{\mathsf{ecr}(\mathfrak{c})} [\![a]\!] = \|a\|_{\mathfrak{c}}}$ Suppose that $b \in \vdash_{\mathsf{ecr}(\mathfrak{c})} [\![a]\!]$, i.e., $a \vdash_{\mathsf{ecr}(\mathfrak{c})} b$. By definition, $b \leq \|a\|_{\mathfrak{c}}$. So $\|a\|_{\mathfrak{c}}$ is an upper bound of $\vdash_{\mathsf{ecr}(\mathfrak{c})} [\![a]\!]$. Suppose that c is an upper bound of $\vdash_{\mathsf{ecr}(\mathfrak{c})} [\![a]\!]$, i.e., if $a \vdash_{\mathsf{ecr}(\mathfrak{c})} b$ then $b \leq c$, i.e., if $b \leq \|a\|_{\mathfrak{c}}$ then $b \leq c$. Certainly, $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{c}}$, hence $\|a\|_{\mathfrak{c}} \leq c$, which suffices. $\boxed{\text{(4.21)}}$ (*In the light of the previous claim, it suffices to show that* $a \vdash_{\mathsf{ecr}(\mathfrak{c})} \|a\|_{\mathfrak{c}}$.) Now, $a \vdash_{\mathsf{ecr}(\mathfrak{c})} \|a\|_{\mathfrak{c}}$ [iff] $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{c}}$ [iff] true. ◇

Proposition 4.23 For order $\mathbf{P}$, $\mathsf{eco}(\cdot)$ and $\mathsf{ecr}(\cdot)$ define mutually inverse bijections, between $\mathsf{ENR}(\mathbf{P})$ and $\mathsf{ECO}(\mathbf{P})$.

Proof. (*It suffices to prove that* $\mathsf{ecr}(\cdot)$ *is injective and* $\mathsf{ecr}(\mathsf{eco}(\mathfrak{c})) = \mathfrak{c}$.)
$\boxed{\text{ecr}(\cdot) \text{ is injective}}$ Suppose that $\mathfrak{c}, \mathfrak{d} \in \mathsf{ECO}(\mathbf{P})$ and $\mathsf{ecr}(\mathfrak{c}) = \mathsf{ecr}(\mathfrak{d})$, i.e., $\vdash_{\mathsf{ecr}(\mathfrak{c})} = \vdash_{\mathsf{ecr}(\mathfrak{d})}$. Now, $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{d}}$ [iff] $a \vdash_{\mathsf{ecr}(\mathfrak{d})} \|a\|_{\mathfrak{c}}$ [iff] $a \vdash_{\mathsf{ecr}(\mathfrak{c})} \|a\|_{\mathfrak{c}}$ [iff] $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{c}}$ [iff] true. So $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{d}}$. Symmetrically, $\|a\|_{\mathfrak{d}} \leq \|a\|_{\mathfrak{c}}$. Hence $\|\cdot\|_{\mathfrak{c}} = \|\cdot\|_{\mathfrak{d}}$, i.e., $\mathfrak{c} = \mathfrak{d}$. $\boxed{\text{ecr}(\text{eco}(\mathfrak{c})) = \mathfrak{c}}$ Let $\mathfrak{c} \in \mathsf{ENR}(\mathbf{P})$. $a \vdash_{\mathsf{ecr}(\mathsf{eco}(\mathfrak{c}))} b$ [iff] $b \leq \|a\|_{\mathsf{eco}(\mathfrak{c})}$ [iff] $b \leq \blacktriangledown \vdash_{\mathfrak{c}} [\![a]\!]$ [iff by (4.27)] $a \vdash_{\mathfrak{c}} b$. ◇

Convention 4.24 (Conflating Consequence Relations and Closure Operators) In the light of the previous definition and proposition, we shall tend to syntactically conflate elementary consequence relations and elementary closure operators and (hence) elementary closed systems, and hence treating (4.28) and (4.29) as properties of these conflated structures.

Corollary 4.25 For an elementary closure operator $\mathfrak{c}$,

$$\|a\|_{\mathfrak{c}} = \|b\|_{\mathfrak{c}} \text{ iff } a \vdash_{\mathfrak{c}} b \text{ and } b \vdash_{\mathfrak{c}} a, \tag{4.30}$$

$$a \vdash_{\mathfrak{c}} \|a\|_{\mathfrak{c}}\,, \tag{4.31}$$

$$\|a\|_{\mathfrak{c}} \vdash_{\mathfrak{c}} a, \tag{4.32}$$

$$a \,\text{is}\, \mathsf{cl}_{\mathfrak{c}} \text{ iff } a \vdash_{\mathfrak{c}} b \rightarrow b \leq a \quad \text{and} \tag{4.33}$$

$$a \vdash_{\mathfrak{c}} b \text{ iff } \forall\,[g \,\text{is}\, \mathsf{cl}_{\mathfrak{c}}]\; a \leq g \rightarrow b \leq g. \tag{4.34}$$

Proof.
$\boxed{(4.30)}$ $\boxed{\Rightarrow}$ Suppose that $\|a\|_{\mathfrak{c}} = \|b\|_{\mathfrak{c}}$. Since elementary closure operators are increasing, $b \leq \|b\|_{\mathfrak{c}} = \|a\|_{\mathfrak{c}}$ and $a \leq \|a\|_{\mathfrak{c}} = \|b\|_{\mathfrak{c}}$, i.e., $b \leq \|a\|_{\mathfrak{c}}$ and $a \leq \|b\|_{\mathfrak{c}}$. So by (4.29), $a \vdash_{\mathfrak{c}} b$ and $b \vdash_{\mathfrak{c}} a$. $\boxed{\Leftarrow}$ Suppose that $a \vdash_{\mathfrak{c}} b$ and $b \vdash_{\mathfrak{c}} a$. Then by (4.29), $b \leq \|a\|_{\mathfrak{c}}$ and $a \leq \|b\|_{\mathfrak{c}}$. Since elementary closure operators are order-preserving and idempotent, $\|b\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{c}}$ and $\|a\|_{\mathfrak{c}} \leq \|b\|_{\mathfrak{c}}$. Hence $\|a\|_{\mathfrak{c}} = \|b\|_{\mathfrak{c}}$. $\boxed{(4.31)}$ $a \vdash_{\mathfrak{c}} \|a\|_{\mathfrak{c}}$ [iff by (4.29)] $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{c}}$ [iff] true. $\boxed{(4.32)}$ $\|a\|_{\mathfrak{c}} \vdash_{\mathfrak{c}} a$ [iff by (4.29)] $a \leq \big\| \|a\|_{\mathfrak{c}} \big\|_{\mathfrak{c}}$ [iff by idempotence] $a \leq \|a\|_{\mathfrak{c}}$ [iff by increasingness] true. $\boxed{(4.33)}$ $\boxed{\Rightarrow}$ Assume that $a \,\text{is}\, \mathsf{cl}_{\mathfrak{c}}$ and $a \vdash_{\mathfrak{c}} b$. Then by (4.13), $\|a\|_{\mathfrak{c}} = a$, and by (4.29), $b \leq \|a\|_{\mathfrak{c}}$. Hence $b \leq a$. $\boxed{\Leftarrow}$ Assume that $a \vdash_{\mathfrak{c}} b \rightarrow b \leq a$. Since by (4.31), $a \vdash_{\mathfrak{c}} \|a\|_{\mathfrak{c}}$, by assumption $\|a\|_{\mathfrak{c}} \leq a$. So by (4.16), $a \,\text{is}\, \mathsf{cl}_{\mathfrak{c}}$. $\boxed{(4.34)}$ $\boxed{\Rightarrow}$ Suppose that $a \vdash_{\mathfrak{c}} b$. Then by (4.29), $b \leq \|a\|_{\mathfrak{c}}$. Let $g \,\text{is}\, \mathsf{cl}_{\mathfrak{c}}$ such that $a \leq g$. Then $\|a\|_{\mathfrak{c}} \overset{(4.1)}{\leq} \|g\|_{\mathfrak{c}} \overset{(4.13)}{=} g$. Hence $b \leq g$. $\boxed{\Leftarrow}$ Since $a \overset{(4.2)}{\leq} \|a\|_{\mathfrak{c}} \overset{(4.15)}{\in} \mathsf{cl}_{\mathfrak{c}}$, by assumption, $b \leq \|a\|_{\mathfrak{c}}$. So by (4.29), $a \vdash_{\mathfrak{c}} b$. ◇

4.1.4 Elementary Closed Equivalence Relations

In logic, it is common practice given a set-point consequence relation $\vdash$, to abbreviate '$A \vdash B$ and $B \vdash A$' by '$A \dashv\vdash B$', where $A \vdash B$ abbreviates $\forall\,[b \in B]\; A \vdash b$. This set-set relation $\dashv\vdash$ is an equivalence relation; it is this equivalence relation that we now aim to characterize in the elementary context. We note that we have not seen the *concrete* set-set relation $\dashv\vdash$ characterized in the literature.

Definition 4.26 (Elementary Closed Equivalence Relations) The **type of elementary closed equivalence relations**, denoted $\mathsf{type}(\mathsf{ece})$, has a binary relation symbol $\leq$ and a binary relation symbol $\dashv\vdash$. An **elementary closed equivalence relation** (or just an **elementary closed equivalence**) is a $\mathsf{type}(\mathsf{ece})$-structure $\mathfrak{c}$ whose $\leq$-reduct is an order, which we denote by $\mathbf{P}_{\mathfrak{c}}$ and call the **underlying order**, and is such that $\mathfrak{c}$ satisfies the axioms

$$\text{(reflexive)} \qquad x \dashv\vdash x, \tag{4.35}$$

$$\text{(symmetric)} \qquad x \dashv\vdash y \longrightarrow y \dashv\vdash x, \tag{4.36}$$

$$\text{(transitive)} \qquad x \dashv\vdash y \,\mathsf{and}\, y \dashv\vdash z \longrightarrow x \dashv\vdash z, \tag{4.37}$$

$$\text{(up-transference)} \qquad \forall[x, y, z]\; x \dashv\vdash z \,\mathsf{and}\, y \geq x \longrightarrow \exists[u]\; u \geq z \,\mathsf{and}\, y \dashv\vdash u \tag{4.38}$$

$$\text{(limit)} \qquad \forall[x]\; \exists[y]\; x \dashv\vdash y \,\mathsf{and} (\forall[z]\; x \dashv\vdash z \longrightarrow z \leq y), \tag{4.39}$$

in which case we call $\dashv\vdash_{\mathfrak{c}}$ the associated **closed equivalence relation**. When we call $\mathfrak{c}$ an **elementary closed equivalence relation on order $\mathbf{P}$**, we mean that $\mathbf{P}_{\mathfrak{c}} = \mathbf{P}$. For an order $\mathbf{P}$ and an operator $\dashv\vdash$ on $\mathsf{uni}(\mathbf{P})$, when we say that $\dashv\vdash$ determines a (elementary) closed equivalence relation on $\mathbf{P}$, or say that $\langle \mathbf{P}; \dashv\vdash \rangle$ is a (elementary) closed equivalence relation, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; \dashv\vdash \rangle$ is an elementary closed equivalence relation. Let $\mathsf{ECE}(\mathbf{P})$ denote the set of (elementary) closed equivalence relations on order $\mathbf{P}$. $\square$

Proposition 4.27 Let $\mathbf{P}$ be an order and $\dashv\vdash$ a binary relation on $\mathsf{uni}(\mathbf{P})$. Then $\dashv\vdash$ determines an elementary closed equivalence on $\mathbf{P}$ iff $\dashv\vdash$ is an equivalence relation satisfying (4.38) and such that for all $a \in \mathsf{uni}(\mathbf{P})$,

$$\blacktriangledown \dashv\vdash [\![a]\!] \text{ exists and } a \dashv\vdash \big(\blacktriangledown \dashv\vdash [\![a]\!]\big). \tag{4.40}$$

Lemma 4.28 If $\mathfrak{c}$ is an elementary closed equivalence, then

$$a \leq \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!], \tag{4.41}$$

$$a \dashv\vdash_{\mathfrak{c}} b \quad \text{implies} \quad b \leq \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!], \tag{4.42}$$

$$a \dashv\vdash_{\mathfrak{c}} b \quad \text{iff} \quad \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!] = \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!], \tag{4.43}$$

$$a \dashv\vdash_{\mathfrak{c}} b \quad \text{iff} \quad \exists\, [c \geq b]\, a \dashv\vdash_{\mathfrak{c}} c \text{ and } \exists\, [d \geq a]\, b \dashv\vdash_{\mathfrak{c}} d. \tag{4.44}$$

Proof. $\boxed{(4.41)}$ By reflexivity $a \in \dashv\vdash_{\mathfrak{c}} [\![a]\!]$, hence $a \leq \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!]$. $\boxed{(4.42)}$ $a \dashv\vdash_{\mathfrak{c}} b$ implies $b \in \dashv\vdash_{\mathfrak{c}} [\![a]\!]$ implies $b \leq \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!]$. $\boxed{(4.43)}$ $\boxed{\Rightarrow}$ If $a \dashv\vdash_{\mathfrak{c}} b$, then since $\dashv\vdash_{\mathfrak{c}}$ is an equivalence relation, $\dashv\vdash_{\mathfrak{c}} [\![a]\!] = \dashv\vdash_{\mathfrak{c}} [\![b]\!]$, and so $\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!] = \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!]$. $\boxed{\Leftarrow}$ If $\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!] = \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!]$, then by (4.40), $a \dashv\vdash_{\mathfrak{c}} \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!] = \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!] \dashv\vdash_{\mathfrak{c}} b$, and so by transitivity, $a \dashv\vdash_{\mathfrak{c}} b$. $\boxed{(4.44)}$ $\boxed{\Rightarrow}$ Suppose that $a \dashv\vdash_{\mathfrak{c}} b$. Let $c = \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!]$. Then by (4.40) and (4.43), $a \dashv\vdash_{\mathfrak{c}} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!]) = c$, and $b \leq c$ (by (4.41)). The outstanding implication follows symmetrically. $\boxed{\Leftarrow}$ Suppose that there exists $c \geq b$ with $a \dashv\vdash_{\mathfrak{c}} c$ and there exists $d \geq a$ with $b \dashv\vdash_{\mathfrak{c}} d$. Then $b \leq c \overset{(4.41)}{\leq} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![c]\!]) \overset{(4.43)}{=} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!])$, and $a \leq d \overset{(4.41)}{\leq} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![d]\!]) \overset{(4.43)}{=} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!])$. Hence

$$b \leq (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!]) \quad \text{and} \tag{i}$$

$$a \leq (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!]). \tag{ii}$$

Now by (4.40), $a \dashv\vdash_{\mathfrak{c}} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!])$, and so by (ii) and (4.38), there exists $e \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$ such that

$$e \geq (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!]) \quad \text{and} \tag{iii}$$

$$(\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!]) \dashv\vdash_{\mathfrak{c}} e. \tag{iv}$$

So, $e \overset{(iv)}{\dashv\vdash_{\mathfrak{c}}} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!]) \overset{(4.40)}{\dashv\vdash_{\mathfrak{c}}} b$, and hence by transitivity and (4.43),

$$(\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![e]\!]) = (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!]). \tag{v}$$

Consequently, $(\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!]) \overset{(iii)}{\leq} e \overset{(4.41)}{\leq} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![e]\!]) \overset{(v)}{=} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!])$. By a symmetric argument, $(\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!]) \leq (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!])$. So $(\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!]) = (\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![b]\!])$, and hence $a \dashv\vdash_{\mathfrak{c}} b$ by (4.43). $\diamond$

Definition 4.29 (Relating Closed-Equivalences and Power Consequence Relations) With each elementary consequence relation $\mathfrak{c}$ we associate the elementary closed equivalence $\mathsf{ece}(\mathfrak{c})$ on $\mathbf{P}_{\mathfrak{c}}$, for which we tend to write $\dashv\vdash_{\mathfrak{c}}$ for $\dashv\vdash_{\mathsf{ece}(\mathfrak{c})}$, determined by

$$a \dashv\vdash_{\mathfrak{c}} b \leftrightarrow a \vdash_{\mathfrak{c}} b \text{ and } b \vdash_{\mathfrak{c}} a. \tag{4.45}$$

With each elementary closed equivalence $\mathfrak{c}$ we associate the elementary consequence relation $\mathsf{ecr}(\mathfrak{c})$ on $\mathbf{P}_{\mathfrak{c}}$, for which we tend to write $\vdash_{\mathfrak{c}}$ for $\vdash_{\mathsf{ecr}(\mathfrak{c})}$, determined by

$$a \vdash_{\mathfrak{c}} b \leftrightarrow \exists\, [c \geq b]\, a \dashv\vdash_{\mathfrak{c}} c. \tag{4.46}$$

$\square$

Proof. $\boxed{\mathsf{ece}(\mathfrak{c}) \text{ is an elementary closed equivalence}}$ We shall use Proposition 4.27. Let $\mathfrak{c}$ be an elementary consequence relation. $\boxed{\text{Reflexive}}$ $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} a$ [iff] $a \vdash_{\mathfrak{c}} a$ and $a \vdash_{\mathfrak{c}} a$ [iff] true. $\boxed{\text{Symmetric}}$ $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} b$ [implies] $a \vdash_{\mathfrak{c}} b$ and $b \vdash_{\mathfrak{c}} a$ [implies] $b \vdash_{\mathfrak{c}} a$ and $a \vdash_{\mathfrak{c}} b$ [implies] $b \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} a$. $\boxed{\text{Transitive}}$ $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} b$ and $b \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} c$ [implies] $a \vdash_{\mathfrak{c}} b$ and $b \vdash_{\mathfrak{c}} a$ and $b \vdash_{\mathfrak{c}} c$ and $c \vdash_{\mathfrak{c}} b$ [implies] $a \vdash_{\mathfrak{c}} b$ and $b \vdash_{\mathfrak{c}} c$ and $c \vdash_{\mathfrak{c}} b$ and $b \vdash_{\mathfrak{c}} a$ [implies (by transitivity of $\vdash_{\mathfrak{c}}$)] $a \vdash_{\mathfrak{c}} c$ and $c \vdash_{\mathfrak{c}} a$ [implies] $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} c$. $\boxed{(4.38)}$ Suppose that (i), $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} c$, and (ii), $b \geq a$. Consider $d = \|b\|_{\mathfrak{c}}$. (*We must show that $d \geq c$ and that $b \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} d$.*) Now $\|d\|_{\mathfrak{c}} = \big\| \|b\|_{\mathfrak{c}} \big\|_{\mathfrak{c}} = \|b\|_{\mathfrak{c}}$ and so by (4.30), $b \vdash_{\mathfrak{c}} d$ and $d \vdash_{\mathfrak{c}} a$; hence $b \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} d$, as required. By (i) and definition, $a \vdash_{\mathfrak{c}} c$ and $c \vdash_{\mathfrak{c}} a$, and so by (4.30), we have (iii) $\|a\|_{\mathfrak{c}} = \|c\|_{\mathfrak{c}}$. By (ii) and order-preservation, we have (iv) $\|a\|_{\mathfrak{c}} \leq \|b\|_{\mathfrak{c}}$. Hence $c \leq \|c\|_{\mathfrak{c}} \overset{\text{(iii)}}{=} \|a\|_{\mathfrak{c}} \overset{\text{(iv)}}{\leq} \|b\|_{\mathfrak{c}} = d$, i.e., $c \leq d$, as required. $\boxed{(4.40)}$ $\boxed{\text{Claim: } a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \|a\|_{\mathfrak{c}}}$ By (4.31) and (4.32), $a \vdash_{\mathfrak{c}} \|a\|_{\mathfrak{c}}$ and $\|a\|_{\mathfrak{c}} \vdash_{\mathfrak{c}} a$, and so $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \|a\|_{\mathfrak{c}}$. $\boxed{\blacktriangledown \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \llbracket a \rrbracket \text{ exists}}$ (*We shall show that $\|a\|_{\mathfrak{c}} = \blacktriangledown \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \llbracket a \rrbracket$.*) Suppose that $b \in \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \llbracket a \rrbracket$, i.e., $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} b$, i.e., $a \vdash_{\mathfrak{c}} b$ and $b \vdash_{\mathfrak{c}} a$. So by (4.30), $\|a\|_{\mathfrak{c}} = \|b\|_{\mathfrak{c}}$. Hence by increasingness of elementary closure operators, $b \leq \|b\|_{\mathfrak{c}} = \|a\|_{\mathfrak{c}}$. So $\|a\|_{\mathfrak{c}}$ is an upper bound of $\dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \llbracket a \rrbracket$. Let c be any upper bound of $\dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \llbracket a \rrbracket$. By the previous claim, $\|a\|_{\mathfrak{c}} \in \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \llbracket a \rrbracket$ and so $\|a\|_{\mathfrak{c}} \leq c$. $\boxed{a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} (\blacktriangledown \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \llbracket a \rrbracket)}$ (*Since we have shown that $\|a\|_{\mathfrak{c}} = \blacktriangledown \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \llbracket a \rrbracket$, it suffices to prove that $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \|a\|_{\mathfrak{c}}$; this has already been established in the previous claim and so no further proof is required.*) $\boxed{\mathsf{ecr}(\mathfrak{c}) \text{ determines an elementary consequence relation}}$ We invoke Proposition 4.20. Let $\mathfrak{c}$ be an elementary equivalence. $\boxed{(4.18) \text{ Inversion}}$ Suppose that $a \leq b$. By reflexivity, $b \dashv\vdash_{\mathfrak{c}} b$, and since $a \leq b$, by definition $b \vdash_{\mathsf{ecr}(\mathfrak{c})} a$. $\boxed{(4.19) \text{ Transitivity}}$ Assume that $a \vdash_{\mathsf{ecr}(\mathfrak{c})} b$ and $b \vdash_{\mathsf{ecr}(\mathfrak{c})} c$. Then by definition, there exists d such that $a \dashv\vdash_{\mathfrak{c}} d$ and $b \leq d$ and, there exists e such that $b \dashv\vdash_{\mathfrak{c}} e$ and $c \leq e$. Let $u = \blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$; this join exists by (4.40). Then $a \dashv\vdash_{\mathfrak{c}} u$ again by (4.40). (*It suffices to show that $c \leq u$, since then $a \vdash_{\mathsf{ecr}(\mathfrak{c})} c$.*) Since $b \dashv\vdash_{\mathfrak{c}} e$ and $b \leq d$, there exists, by (4.38), $v \geq e$ with $d \dashv\vdash_{\mathfrak{c}} v$. Since $a \dashv\vdash_{\mathfrak{c}} d$ and $d \dashv\vdash_{\mathfrak{c}} v$, by transitivity, $a \dashv\vdash_{\mathfrak{c}} v$, and so $v \leq u$, by the definition of u, and hence $c \leq e \leq v \leq u$. Hence $c \leq u$. $\boxed{(4.21)}$ By (4.40), $\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$ exists. $\boxed{\text{Claim: } (\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket) \in \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket}$ Since $(\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket) \geq (\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket)$ and $a \dashv\vdash_{\mathfrak{c}} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket)$ (the latter assertion by (4.40)), by definition, $a \vdash_{\mathsf{ecr}(\mathfrak{c})} (\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket)$, i.e., $(\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket) \in \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket$. $\boxed{\blacktriangledown \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket \text{ exists}}$ (*We shall show that $\blacktriangledown \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket = \blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$. Since we have already established, in the previous claim, that $(\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket) \in \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket$, it suffices to show that $\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$ is an upper bound of $\vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket$.*) Let $b \in \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket$, i.e., $a \vdash_{\mathsf{ecr}(\mathfrak{c})} b$, i.e., there exists $c \geq b$ such that $a \dashv\vdash_{\mathfrak{c}} c$. Since $c \in \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$, $c \leq \blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$. Hence $b \leq \blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$. So $\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$ is an upper bound of $\vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket$. $\boxed{a \vdash_{\mathsf{ecr}(\mathfrak{c})} (\blacktriangledown \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket)}$ (*Since we have established that $\blacktriangledown \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket = \blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket$ and that $(\blacktriangledown \dashv\vdash_{\mathfrak{c}} \llbracket a \rrbracket) \in \vdash_{\mathsf{ecr}(\mathfrak{c})} \llbracket a \rrbracket$, there is nothing more to prove.*) $\diamond$

Proposition 4.30 For order $\mathbf{P}$, $\mathsf{ece}(\cdot)$ and $\mathsf{ecr}(\cdot)$ define mutually inverse bijections between $\mathsf{ECR}(\mathbf{P})$ and $\mathsf{ECE}(\mathbf{P})$.

Proof. It suffices to prove that $\mathsf{ecr}(\mathsf{ece}(\mathfrak{c})) = \mathfrak{c}$, for all $\mathfrak{c} \in \mathsf{ECR}(\mathbf{P})$, and that $\mathsf{ece}(\mathsf{ecr}(\mathfrak{c})) = \mathfrak{c}$, for all $\mathfrak{c} \in \mathsf{ECE}(\mathbf{P})$.

$\boxed{\mathsf{ecr}(\mathsf{ece}(\mathfrak{c})) = \mathfrak{c}}$ Let $\mathfrak{c} \in \mathsf{ECR}(\mathbf{P})$. $\boxed{\vdash_{\mathsf{ecr}(\mathsf{ece}(\mathfrak{c}))} \subseteq \vdash_{\mathfrak{c}}}$ Suppose that $a \vdash_{\mathsf{ecr}(\mathsf{ece}(\mathfrak{c}))} b$. Then there exists $c \geq b$ with $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} c$. So $a \vdash_{\mathfrak{c}} c$ and $c \vdash_{\mathfrak{c}} a$. Since $c \geq b$, $c \vdash_{\mathfrak{c}} b$, by (4.18), and so by (4.19), $a \vdash_{\mathfrak{c}} b$. $\boxed{\vdash_{\mathsf{ecr}(\mathsf{ece}(\mathfrak{c}))} \supseteq \vdash_{\mathfrak{c}}}$ Suppose that $a \vdash_{\mathfrak{c}} b$. Then by (4.29), $b \leq \|a\|_{\mathfrak{c}}$. Since $a \vdash_{\mathfrak{c}} \|a\|_{\mathfrak{c}}$, by (4.31), and $\|a\|_{\mathfrak{c}} \vdash_{\mathfrak{c}} a$, by (4.32), it follows, by (4.45), that $a \dashv\vdash_{\mathsf{ece}(\mathfrak{c})} \|a\|_{\mathfrak{c}}$. Letting $c = \|a\|_{\mathfrak{c}}$, we have $c \geq b$ and $a \dashv\vdash_{\mathfrak{c}} c$. Hence $a \vdash_{\mathsf{ecr}(\mathsf{ece}(\mathfrak{c}))} b$, by definition. $\boxed{\mathsf{ece}(\mathsf{ecr}(\mathfrak{c})) = \mathfrak{c}}$ Let $\mathfrak{c} \in \mathsf{ECE}(\mathbf{P})$. Then $a \dashv\vdash_{\mathsf{ece}(\mathsf{ecr}(\mathfrak{c}))} b$ [iff] ($a \vdash_{\mathsf{ecr}(\mathfrak{c})} b$ and $b \vdash_{\mathsf{ecr}(\mathfrak{c})} a$) [iff] there exists $c \geq b$ with $a \dashv\vdash_{\mathfrak{c}} c$ and there exists $d \geq a$ with $b \dashv\vdash_{\mathfrak{c}} d$ [iff by (4.44)] $a \dashv\vdash_{\mathfrak{c}} b$. $\diamond$

Convention 4.31 (Conflating Elementary Closed Equivalences and Consequences) Consequent to the previous definition and proposition, we tend to conflate elementary closed equivalences and elementary consequence relations (thereby further extending the earlier substitutions), and as such treating (4.45) and (4.46) as properties of these conflated structures.

Corollary 4.32 Let $\mathfrak{c}$ be a closed system. Then,

$$a \dashv\vdash_{\mathfrak{c}} \|a\|_{\mathfrak{c}}\,, \tag{4.47}$$

$$\|a\|_{\mathfrak{c}} \;=\; \blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!], \tag{4.48}$$

$$\blacktriangledown \dashv\vdash_{\mathfrak{c}} [\![a]\!] = \blacktriangledown \vdash_{\mathfrak{c}} [\![a]\!] \quad \text{and} \tag{4.49}$$

$$a \dashv\vdash_{\mathfrak{c}} b \text{ iff } \|a\|_{\mathfrak{c}} = \|b\|_{\mathfrak{c}}\,. \tag{4.50}$$

Proof. $\boxed{\text{(4.47), (4.48) and (4.49)}}$ The proof of these facts is implicit in the proof of Definition 4.29 together with Proposition 4.30. $\boxed{\text{(4.50)}}$ Follows immediately from (4.45) and (4.30). ⋄

4.1.5 Elementary Proto-Leibniz Relations

The Leibniz relation can be viewed as a parameterized binary relation, parameterized by the set argument, and hence as a ternary relationship between a set and two points. Recall the definition of protoalgebraicity given in Definition 2.132 on page 116. In the case of a sentential 1-calculus $\mathcal{S}$, $\mathcal{S}$ is called *protoalgebraic* if, for all $T \in \mathsf{Th}(\mathcal{S})$,

$$\phi\, \Omega^{\mathcal{S}}(T)\, \psi \text{ implies } T \cup \{\phi\} \vdash_{\mathcal{S}} \psi \text{ and } T \cup \{\psi\} \vdash_{\mathcal{S}} \phi. \tag{4.51}$$

Our intention now is to analyse the ternary relationship defined by the *right hand side* of this expression. In order to obtain an elementary abstraction of this ternary relationship on an order, we must first replace the points by sets (since sets correspond to points in the universe of the order, while points lie outside of the elementary domain of discourse), considering the ternary relationship defined by

$$A \cup C \vdash D \text{ and } A \cup D \vdash C. \tag{4.52}$$

While this ternary relation can be given sensible meaning on an order which is a $\vee$-semilattice ($\vee$-semilattices are still elementary), we have found an abstraction of this ternary relation over orders more generally; one that coincides with the elementary ternary relation defined by (4.52) in the case that the underlying order is a $\vee$-semilattice. More precisely, given an elementary closure operator $\mathfrak{c}$, the formula

$$\forall\, [a \leq a']\; c \leq \|a'\|_{\mathfrak{c}} \leftrightarrow d \leq \|a'\|_{\mathfrak{c}} \tag{4.53}$$

defines a ternary relationship. Note that (4.53) is equivalent to

$$\forall\, [a \leq a']\; a' \vdash_{\mathfrak{c}} c \leftrightarrow a' \vdash_{\mathfrak{c}} d. \tag{4.54}$$

In the case that $\mathbf{P}_{\mathfrak{c}}$ is a $\vee$-semilattice, the ternary relation defined by (4.53) is precisely the ternary relationship defined by

$$a \vee c \vdash d \text{ and } a \vee d \vdash c. \tag{4.55}$$

In this section, we shall show that the ternary relation defined by (4.53) with respect to an elementary closure operator $\mathfrak{c}$, can be characterized as an elementary structure on an order (i.e., an order with a ternary binary relation), and we shall show that these structures are in a one-to-one correspondence with elementary closure operators (and hence with elementary closed systems, etc.) We call these structures *elementary proto-Leibniz relations*. Note that the prefix 'proto' is with respect to the Blok/Pigozzi usage of 'Leibniz relation', which is an *elementary approximation* of the *second-order* relation of Leibniz (although elementary in a different context to our usage in this chapter). It is our *intuition* that the elementary proto-Leibniz relation is the *second-order* relation of Leibniz in an *elementary* setting. In $\oint$4.2.4, we shall consider the elementary proto-Leibniz relation in a concrete setting, relating it back to the ternary relationship between a set and two points, as defined by

$$A \cup \{c\} \vdash d \text{ and } A \cup \{d\} \vdash c.$$

Definition 4.33 (The Elementary Proto-Leibniz Relations) The **type of elementary consequence relations**, denoted $\mathsf{type}(\mathsf{epl})$, has a binary relation symbol $\leq$ and a ternary relation symbol $\cdot \bumpeq(\cdot)\, \cdot$. An **elementary proto-Leibniz relation** is a $\mathsf{type}(\mathsf{epl})$-structure $\mathfrak{c}$ whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{c}}$ and called the **underlying order**, and is such that $\mathfrak{c}$ satisfies the axioms

$$x \bumpeq(v)\; x, \tag{4.56}$$

$$x \bumpeq(v)\; y \longrightarrow y \bumpeq(v)\; x, \tag{4.57}$$

$$x \bumpeq(v)\; y \,\mathsf{and}\, y \bumpeq(v)\; z \longrightarrow x \bumpeq(v)\; z, \tag{4.58}$$

$$v \leq v' \,\mathsf{and}\, v \bumpeq(v)\; x \longrightarrow \exists[x']\; x \leq x' \,\mathsf{and}\, v' \bumpeq(v')\; x', \tag{4.59}$$

$$\forall[v]\; \exists[v'] \; \big(v \bumpeq(v)\; v' \,\mathsf{and} (\forall[z]\; v \bumpeq(v)\; z \longrightarrow z \leq v') \text{ and } (\forall[y]\; v' \bumpeq(v')\; y \longrightarrow y \leq v')\big), \tag{4.60}$$

$$\big(x \bumpeq(v)\; y \,\mathsf{and}\, v \leq v' \,\mathsf{and}\, v' \bumpeq(v')\; x' \,\mathsf{and}\, x \leq x'\big) \longrightarrow \exists[y \leq y']\; \big(v' \bumpeq(v')\; y'\big) \quad \text{and} \tag{4.61}$$

$$\forall[x, y, v]\; \Bigg(\bigg(\forall[v \leq v']\; \big(\exists[x \leq x']\; v' \bumpeq(v')\; x'\big) \leftrightarrow \big(\exists[y \leq y']\; v' \bumpeq(v')\; y'\big)\bigg) \longrightarrow x \bumpeq(v)\; y\Bigg), \tag{4.62}$$

in which case we call $\bumpeq^{\mathfrak{c}}(\cdot)$ the associated **proto-Leibniz relation**. When we call $\mathfrak{c}$ an **elementary proto-Leibniz relation on order P**, we mean that $\mathbf{P}_{\mathfrak{c}} = \mathbf{P}$. For an order $\mathbf{P}$ and a ternary relation $\bumpeq^{\mathfrak{c}}(\cdot)$ on $\mathsf{uni}(\mathbf{P})$, when we say that $\bumpeq^{\mathfrak{c}}(\cdot)$ determines a (elementary) proto-Leibniz relation on $\mathbf{P}$, or say that $\langle \mathbf{P}; \bumpeq^{\mathfrak{c}}(\cdot)\rangle$ is a (elementary) proto-Leibniz relation, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; \bumpeq^{\mathfrak{c}}(\cdot)\rangle$ is an elementary proto-Leibniz relation. Let $\mathsf{EPL}(\mathbf{P})$ denote the set of (elementary) proto-Leibniz relations on order $\mathbf{P}$. It is convenient to view an elementary proto-Leibniz relation as a parameterized *binary* relation, parameterized by the argument in the parenthesis; we call this argument the **parameter**. $\square$

Proposition 4.34 Let $\bumpeq(\cdot)$ be a ternary relation on the universe of order $\mathbf{P}$. Then $\bumpeq(\cdot)$ determines an elementary proto-Leibniz relation on $\mathbf{P}$ iff, for each $a \in \mathsf{uni}(\mathbf{P})$, $\bumpeq(a)$ is an equivalence relation on $\mathsf{uni}(\mathbf{P})$, and

$$\blacktriangledown \Bumpeq(a)\llbracket a\rrbracket \text{ exists,} \tag{4.63}$$

$$a \Bumpeq(a)\ (\blacktriangledown \Bumpeq(a)\llbracket a\rrbracket)\,, \tag{4.64}$$

$$\blacktriangledown\big(\Bumpeq(\blacktriangledown \Bumpeq(a)\llbracket a\rrbracket)\llbracket \blacktriangledown \Bumpeq(a)\llbracket a\rrbracket\rrbracket\big) = \blacktriangledown \Bumpeq(a)\llbracket a\rrbracket, \tag{4.65}$$

$$a \le a' \text{ implies } \blacktriangledown \Bumpeq(a)\llbracket a\rrbracket \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket, \tag{4.66}$$

$$c \Bumpeq(a)\ d \text{ iff } \forall\,[a \le a']\,(c \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket \text{ iff } d \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket) \tag{4.67}$$

Proof.

$\boxed{\Rightarrow}$ Clearly $\Bumpeq(a)$ is an equivalence relation and (4.63) and (4.64) follow from (4.60). $\boxed{\scriptstyle (4.66)}$ Suppose that $a \le a'$. Since $a \Bumpeq(a)\ (\blacktriangledown \Bumpeq(a)\llbracket a\rrbracket)$ by (4.63), by (4.59), there exists $c \in \Bumpeq(a')\llbracket a'\rrbracket$ with $\blacktriangledown \Bumpeq(a)\llbracket a\rrbracket \le c$. Since $c \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$, $\blacktriangledown \Bumpeq(a)\llbracket a\rrbracket \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$. $\boxed{\scriptstyle (4.65)}$ Let $b = \blacktriangledown(\Bumpeq^{\mathfrak{c}}(a)\llbracket a\rrbracket)$. (*We must show that* $\blacktriangledown\big(\Bumpeq(b)\llbracket b\rrbracket\big) = b$.) Since $b \in \Bumpeq(b)\llbracket b\rrbracket$ by reflexivity, $b \le \blacktriangledown\big(\Bumpeq(b)\llbracket b\rrbracket\big)$. Conversely, if $c \in \Bumpeq(b)\llbracket b\rrbracket$, i.e., $b \Bumpeq(b)\ c$, then by (4.60), (4.63) and (4.64), together with the uniqueness of least upper bounds, $c \le b$; hence $\blacktriangledown\big(\Bumpeq(b)\llbracket b\rrbracket\big) \le b$. $\boxed{\scriptstyle (4.67)}$ $\boxed{\Rightarrow}$ Assume that $c \Bumpeq^{\mathfrak{c}}(a')\ d$ and let $a \le a'$. Suppose that $c \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$. Since $a' \Bumpeq^{\mathfrak{c}}(a')\ \big(\blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket\big)$ by already established (4.64), by (4.61), there exists $d \le d'$ such that $a' \Bumpeq^{\mathfrak{c}}(a')\ d'$. Since $d' \in \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$ be already established reflexivity, $d' \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$. Hence $d \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$. Symmetrically, $d \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$ implies $c \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$. $\boxed{\Leftarrow}$ Assume that $\forall\,[a \le a']\,(c \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket \text{ iff } d \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket)$. (*We shall apply (4.62).*) Let $a \le a'$. Suppose that there exists $c \le c'$ such that $a' \Bumpeq(a')\ c'$. (*We shall show that there exists* $d \le d'$ *such that* $a' \Bumpeq(a')\ d'$.) Let $d' = \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$. Since $a' \Bumpeq(a')\ c'$, $c' \in \Bumpeq(a')\llbracket a'\rrbracket$, and so $c \le c' \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$. Since $a \le a'$, we have, by assumption, $d \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket = d'$. Further $a' \Bumpeq(a')\ \big(\blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket\big) = d'$, by already established (4.65). Symmetrically, if there exists $d \le d'$ such that $a' \Bumpeq(a')\ d'$, then there exist exists $c \le c'$ such that $a' \Bumpeq(a')\ c'$. So by (4.62), $c \Bumpeq^{\mathfrak{c}}(a)\ d$.

$\boxed{\Leftarrow}$ It is easily shown that (4.56), (4.57), (4.58) are satisfied. $\boxed{\scriptstyle (4.60)}$ Let $a' = \blacktriangledown \Bumpeq(a)\llbracket a\rrbracket$, which exists by (4.63). By (4.64), $a \Bumpeq(a)\ a'$. If $a \Bumpeq(a)\ c$, i.e., $c \in \Bumpeq(a)\llbracket a\rrbracket$, then certainly $c \le \blacktriangledown \Bumpeq(a)\llbracket a\rrbracket = a'$. Finally, if $a' \Bumpeq(a')\ d$, then $d \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$, which exists by (4.63), and since $\blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket = \blacktriangledown \Bumpeq(a)\llbracket a\rrbracket$ by (4.65), we have $d \le a'$. $\boxed{\scriptstyle (4.59)}$ Suppose that $a \le a'$ and $a \Bumpeq(a)\ c$. Let $c' \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$. By (4.63), $a' \Bumpeq(a')\ c'$. Since $c \in \Bumpeq(a)\llbracket a\rrbracket$, $c \le \blacktriangledown \Bumpeq(a)\llbracket a\rrbracket \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket = c'$, where the second inequality follows by (4.66). So (4.59) is satisfied. $\boxed{\scriptstyle (4.61)}$ Suppose that $c \Bumpeq(a)\ d$, $a \le a'$, $a' \Bumpeq(a')\ c'$ and $c \le c'$. Let $d' = \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$. (*It suffices to show that* $d \le d'$ *and* $a' \Bumpeq(a')\ d'$.) Since $a' \Bumpeq(a')\ c'$, $c \le c' \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$. By already established (4.64), $a' \Bumpeq(a')\ d'$. Since $c \Bumpeq(a)\ d$, $a \le a'$ and $c \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket$, we have, by the forward implication of (4.67), that $d \le \blacktriangledown \Bumpeq(a')\llbracket a'\rrbracket = d'$. $\boxed{\scriptstyle (4.62)}$ Let $c, d, a \in \mathsf{uni}(\mathbf{P})$. Assume that $\forall[a \le a']\,\big(\exists[c \le c']\ a' \Bumpeq(a')\ c'\big) \leftrightarrow \big(\exists[d \le d']\ a' \Bumpeq(a')\ d'\big)$. (*We shall apply the reverse implication of (4.67).*) Let $a \le a'$. Suppose that $c \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$. (*We shall show that* $d \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$.) Let $c' = \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$. Then $c \le c'$ and, by already established (4.65), $a' \Bumpeq(a')\ c'$. So by assumption, there exists $d \le d'$ such that $a' \Bumpeq(a')\ d'$. Since $d' \in \Bumpeq(a')\llbracket a'\rrbracket$, $d \le d' \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$. Symmetrically, if $c \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$ then $d \le \blacktriangledown \Bumpeq^{\mathfrak{c}}(a')\llbracket a'\rrbracket$. Hence by (4.67), $c \Bumpeq(a)\ d$. $\diamond$

Definition 4.35 (Relating Elementary Closure Operators and Proto-Leibniz Relations) With each elementary closure operator $\mathfrak{c}$, we associate the elementary proto-Leibniz relation $\mathsf{epl}(\mathfrak{c})$ on $\mathbf{P}_{\mathfrak{c}}$, for which we tend to write $\Bumpeq^{\mathfrak{c}}(\cdot)$ for $\Bumpeq^{\mathsf{epl}(\mathfrak{c})}(\cdot)$, determined by

$$c \Bumpeq^{\mathfrak{c}}(a)\ d \text{ iff } \forall\,[a \le a']\ c \le \|a'\|_{\mathfrak{c}} \leftrightarrow d \le \|a'\|_{\mathfrak{c}}\,. \tag{4.68}$$

With each elementary proto-Leibniz relation $\mathfrak{c}$, we associate the elementary closure operator $\mathsf{eco}(\mathfrak{c})$ on $\mathbf{P}_{\mathfrak{c}}$, for which we tend to write $\|\cdot\|_{\mathfrak{c}}$ for $\|\cdot\|_{\mathsf{eco}(\mathfrak{c})}$, determined by

$$\|a\| = \blacktriangledown \bumpeq(a)[\![a]\!], \tag{4.69}$$

this operator being well-defined by (4.63). □

Proof. $\boxed{\mathsf{epl}(\mathfrak{c}) \text{ is an elementary proto-Leibniz relation}}$ Let $\mathfrak{c}$ be an elementary closure operator. We invoke Proposition 4.34. $\boxed{\text{Reflexivity}}$ Let $c \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$. For all $a \leq a'$, $c \leq \|a'\|_{\mathfrak{c}}$ iff $c \leq \|a'\|_{\mathfrak{c}}$. Hence $c \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ c$. $\boxed{\text{Symmetry}}$ Suppose that $c \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ d$. For all $a \leq a'$, $(d \leq \|a'\|_{\mathfrak{c}}$ iff $c \leq \|a'\|_{\mathfrak{c}})$ [iff] $(c \leq \|a'\|_{\mathfrak{c}}$ iff $d \leq \|a'\|_{\mathfrak{c}})$ [iff] (true). So $d \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ c$. $\boxed{\text{Transitivity}}$ Suppose that $c \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ d$ and $d \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ e$. For all $a \leq a'$, $c \leq \|a'\|_{\mathfrak{c}}$ iff $d \leq \|a'\|_{\mathfrak{c}}$ iff $e \leq \|a'\|_{\mathfrak{c}}$, and so $c \leq \|a'\|_{\mathfrak{c}}$ iff $e \leq \|a'\|_{\mathfrak{c}}$. So $c \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ e$. $\boxed{\text{Claim 1: } \|a\|_{\mathfrak{c}} \in \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)[\![a]\!]}$ Let $a \leq a'$. If $a \leq \|a'\|_{\mathfrak{c}}$ then $\|a\|_{\mathfrak{c}} \leq \|a'\|_{\mathfrak{c}}$, and if $\|a\|_{\mathfrak{c}} \leq \|a'\|_{\mathfrak{c}}$ then $a \leq \|a'\|_{\mathfrak{c}}$. Hence $a \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ \|a\|_{\mathfrak{c}}$ $\boxed{\text{Claim 2: } \blacktriangledown \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)[\![a]\!] = \|a\|_{\mathfrak{c}}.}$ (*By Claim 1, it suffices to show that* $\|a\|_{\mathfrak{c}}$ *is an upper bound of* $\bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)[\![a]\!]$.) Let $c \in \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)[\![a]\!]$, i.e., $a \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ c$. Since $a \leq a$ and $a \leq \|a\|_{\mathfrak{c}}$ we must have $c \leq \|a\|_{\mathfrak{c}}$. $\boxed{(4.63)}$ By Claim 2. $\boxed{(4.64)}$ By Claim 1 and Claim 2. $\boxed{(4.66)}$ By Claim 2, $\blacktriangledown \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)[\![a]\!] = \|a\|_{\mathfrak{c}}$. So (4.66) follows from order preservation of elementary closure operators. $\boxed{(4.65)}$ Let $a' = \blacktriangledown \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)[\![a]\!]$, which exists by already established (4.63). (*We must show that* $\blacktriangledown\big(\bumpeq^{\mathsf{epl}(\mathfrak{c})}(a')[\![a']\!]\big) = a'$.) By Claim 2 and idempotence of closure operators, we have $\blacktriangledown\big(\bumpeq^{\mathsf{epl}(\mathfrak{c})}(a')[\![a']\!]\big) = \|a'\|_{\mathfrak{c}} = \big\| \|a\|_{\mathfrak{c}} \big\|_{\mathfrak{c}} = \|a\|_{\mathfrak{c}} = a'$. $\boxed{(4.67)}$ $c \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)\ d$ [iff] $\forall\, [a \leq a']\ c \leq \|a'\|_{\mathfrak{c}}$ iff $d \leq \|a'\|_{\mathfrak{c}}$ [iff, by Claim 2] $\forall\, [a \leq a']\ c \leq \blacktriangledown \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a')[\![a']\!]$ iff $d \leq \blacktriangledown \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a')[\![a']\!]$. $\boxed{\mathsf{eco}(\mathfrak{c}) \text{ is an elementary closure operator}}$ Let $\mathfrak{c}$ be an elementary proto-Leibniz relation. $\boxed{\text{Order preserving}}$ By (4.66). $\boxed{\text{Increasing}}$ By reflexivity, $a \in \bumpeq^{\mathfrak{c}}(a)[\![a]\!]$, and hence $a \leq \blacktriangledown \bumpeq^{\mathfrak{c}}(a)[\![a]\!] = \|a\|_{\mathsf{eco}(\mathfrak{c})}$. $\boxed{\text{Idempotent}}$ By (4.65). ◇

Proposition 4.36 For an order $\mathbf{P}$, $\mathsf{epl}(\cdot)$ and $\mathsf{eco}(\cdot)$ determine mutually inverting bijections between $\mathsf{ECO}(\mathbf{P})$ and $\mathsf{EPL}(\mathbf{P})$.

Proof. $\boxed{\mathsf{epl}(\mathsf{eco}(\mathfrak{c})) = \mathfrak{c}}$ Let $\mathfrak{c}$ be an elementary proto-Leibniz relation on $\mathbf{P}$. $c \bumpeq^{\mathsf{epl}(\mathsf{eco}(\mathfrak{c}))}(a)\ d$ [iff] $\forall[a \leq a']c \leq \|a'\|_{\mathsf{eco}(\mathfrak{c})}$ iff $d \leq \|a'\|_{\mathsf{eco}(\mathfrak{c})}$ [iff] $\forall\, [a \leq a']\ c \leq \blacktriangledown \bumpeq^{\mathfrak{c}}(a')[\![a']\!]$ iff $d \leq \blacktriangledown \bumpeq^{\mathfrak{c}}(a')[\![a']\!]$ [iff, by (4.67)] $c \bumpeq^{\mathfrak{c}}(a)\ d$. $\boxed{\mathsf{eco}(\mathsf{epl}(\mathfrak{c})) = \mathfrak{c}}$ Let $\mathfrak{c}$ be an elementary closure operator on $\mathbf{P}$. Observe that $\bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)[\![a]\!] = \{c\ :\ a \bumpeq^{\mathsf{epl}(\mathfrak{c})}(c)\} = \{c\ :\ \forall\, [a \leq a']\ a \leq \|a'\|_{\mathfrak{c}}$ iff $c \leq \|a'\|_{\mathfrak{c}}\} = \{c\ :\ \forall\, [a \leq a']$ true iff $c \leq \|a'\|_{\mathfrak{c}}\} = \{c\ :\ \forall\, [a \leq a']\ c \leq \|a'\|_{\mathfrak{c}}\} = \{c\ :\ c \leq \|a\|_{\mathfrak{c}}\}$, and hence $\|a\|_{\mathsf{eco}(\mathsf{epl}(\mathfrak{c}))} = \blacktriangledown \bumpeq^{\mathsf{epl}(\mathfrak{c})}(a)[\![a]\!] = \blacktriangledown\{c\ :\ c \leq \|a\|_{\mathfrak{c}}\} = \|a\|_{\mathfrak{c}}$. ◇

Convention 4.37 (Conflating Elementary Proto-Leibniz Relations and Closure Operators) Consequent to the previous definition and proposition, we tend to conflate elementary proto-Leibniz relations and elementary closure operators (thereby further extending the earlier substitutions), and as such treating (4.68) and (4.69) as properties of these conflated structures.

We now collect some basic properties of the elementary proto-Leibniz relation. Note in particular (4.72) which asserts that the elementary proto-Leibniz relation, unlike the Blok-Pigozzi Leibniz relation, is order preserving. More precisely,

$$\bumpeq(\cdot)\colon \mathbf{P}_{\mathfrak{c}} \to_{\leq} \langle \mathsf{ER}(\mathsf{uni}_{\mathsf{e}}(\mathfrak{c}); \subseteq\rangle. \tag{4.70}$$

It is the emulation of this property by the Leibniz relation that characterizes protoalgebraic logics.

Proposition 4.38 For an elementary closed system $\mathfrak{c}$, the following formulae are all valid.

$$\text{if } g \in \mathsf{cl}_{\mathfrak{c}}, a \leq g, c \leq g \text{ and } c \Bumpeq(a)\; d \text{ then } d \leq g, \tag{4.71}$$

$$a \leq a' \text{ implies } \Bumpeq(a) \subseteq \Bumpeq(a'), \tag{4.72}$$

$$c \Bumpeq(a)\; \|c\| \quad \text{and} \tag{4.73}$$

$$\Bumpeq(a) = \Bumpeq(\|a\|). \tag{4.74}$$

Proof. $\boxed{(4.71)}$ Suppose that $g \in \mathsf{cl}_{\mathfrak{c}}$, $a \leq g$, $c \leq g$, and $c \Bumpeq(a)\; d$. Since $a \leq g$ and $c \Bumpeq(a)\; d$, we have $c \leq \|g\| = g$ iff $d \leq \|g\| = g$, by (4.68). Since $c \leq g$, it must be true that $d \leq g$. $\boxed{(4.72)}$ Suppose that $c \Bumpeq(a)\; d$. Let $a' \leq b$. Then $a \leq \|b\|$, and since $c \Bumpeq(a)\; d$, $c \leq \|b\|$ iff $d \leq \|b\|$, by (4.68). So by (4.68), $c \Bumpeq(a')\; d$. $\boxed{(4.73)}$ By (4.64) and (4.69). $\boxed{(4.74)}$ Since $a \leq \|a\|$, it follows by (4.72) that, $\Bumpeq(a) \subseteq \Bumpeq(\|a\|)$. Suppose that $c \Bumpeq(\|a\|)\; d$. For any $a \leq b$, setting $b' = \|b\|$, we have $\|a\| \leq b'$, and since $c \Bumpeq(\|a\|)\; d$, $c \leq \|b'\| = \big\|\, \|b\|\, \big\| = \|b\|$ iff $d \leq \|b'\| = \big\|\, \|b\|\, \big\| = \|b\|$, by (4.68). So by (4.68), $c \Bumpeq(a)\; d$. $\diamond$

We now consider the special case where the underlying order of the elementary closed system is a $\vee$-semilattice. Note that such closed systems are still elementary.

Theorem 4.39 If $\mathbf{P}_{\mathfrak{c}}$ is a $\vee$-semilattice then

$$c \Bumpeq(a)\; d \text{ iff } (a \vee c \vdash d \text{ and } a \vee d \vdash c)\,. \tag{4.75}$$

Proof. $\boxed{\Rightarrow}$ Suppose that $c \Bumpeq(a)\; d$. Since $a \leq a \vee c$ and $c \leq \|a \vee c\|$, by (4.68) we have $d \leq \|a \vee c\|$, i.e., $a \vee c \vdash d$. Symmetrically $a \vee d \vdash c$. $\boxed{\Leftarrow}$ Assume that $a \vee c \vdash d$ and $a \vee d \vdash c$. Let $a \leq b$. Suppose that $c \leq \|b\|$. Since $a \leq \|b\|$, we have $a \vee c \leq \|b\|$, and since $a \vee c \vdash d$, we have $d \leq \|b\|$. Symmetrically, if $d \leq \|b\|$ then $c \leq \|b\|$. Hence $c \Bumpeq(a)\; d$, by (4.68). $\diamond$

4.1.6 Granularity

The following definition of the granularity relationship, allowing one to compare closed systems over the same order, is the natural generalization of the granularity relationship from logic and from topology.

Definition 4.40 (Strength and Weakness) Let $\mathfrak{c}$ and $\mathfrak{d}$ be elementary closed systems. We say that $\mathfrak{c}$ is **finer** than $\mathfrak{d}$, denoted $\mathfrak{c} \preceq \mathfrak{d}$, if $\mathbf{P}_{\mathfrak{c}} = \mathbf{P}_{\mathfrak{d}}$ and $\mathsf{cl}_{\mathfrak{c}} \supseteq \mathsf{cl}_{\mathfrak{d}}$, and say that $\mathfrak{c}$ is **coarser** than $\mathfrak{d}$, denoted $\mathfrak{c} \succeq \mathfrak{d}$, if $\mathfrak{d} \preceq \mathfrak{c}$. $\square$

Proposition 4.41 For closed systems $\mathfrak{c}$ and $\mathfrak{d}$ with the same universe, the following conditions are equivalent.

1. $\mathfrak{c} \preceq \mathfrak{d}$.
2. $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{d}}$.
3. $\vdash_{\mathfrak{c}} \subseteq \vdash_{\mathfrak{d}}$, i.e., $a \vdash_{\mathfrak{c}} b$ implies $a \vdash_{\mathfrak{d}} b$.

Proof. $\boxed{(1) \Rightarrow (2)}$ By assumption, $\|a\|_{\mathfrak{d}} \in \mathsf{cl}_{\mathfrak{c}}$, and since $a \leq \|a\|_{\mathfrak{d}}$, by minimality, $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{d}}$. $\boxed{(2) \Rightarrow (3)}$ Suppose that $a \vdash_{\mathfrak{c}} b$. Then $b \leq \|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{d}}$, so $a \vdash_{\mathfrak{d}} b$. $\boxed{(3) \Rightarrow (1)}$ Suppose that g is $\mathfrak{d}$-closed. Suppose that $g \vdash_{\mathfrak{c}} b$. Then, by assumption (3), $g \vdash_{\mathfrak{d}} b$. Since g is $\mathfrak{d}$-closed, $b \leq g$. So g is $\mathfrak{c}$-closed. $\diamond$

4.2 Concrete Closure

While it is clear that elementary closure operators coincide with the standard (concrete) definition of closure operators in the case that the underlying order is the inclusion-ordered power-set, it is less clear that this is indeed the case for closed systems and consequence relations; in the case of elementary closed equivalence relations and proto-Leibniz relations, we know of no standard concrete analogues in the literature. In this section, we shall show that elementary closed systems and consequence relations do indeed coincide with their standard concrete counterparts when interpreted over the complete power-order, and we shall characterize the concrete analogues of the elementary closed equivalence relation and proto-Leibniz relation. In this section, we shall also show that the closed systems over a given set themselves form a closed system, which we call the closed system of closed systems, which we use as a means to introduce the well-known notions of a basis for a closed system and the closed system generated by a system. We also define the filtration of a closed system by a given set, which is the closed system obtained by selecting only those closed sets containing this given set; the notion of a filtration is used extensively in Part VI, where we explain how our theory of parameterized algebraization may be explained from with a non-parameterized theory of equivalent logics over constructs. Finally, we show how matrices may be viewed as 'little' closed systems, an idea we exploit in unifying the matrix model theory of logics over constructs, and hence sentential calculi, (see $\oint$7.6) within our theory of continuous translations between closed systems (developed in the next chapter).

Definition 4.42 (Concrete Closure) An elementary closure operator (resp. closed system, consequence relation, closed equivalence etc.) $\mathfrak{c}$ is called **complete** if the underlying order is a complete lattice, and is called **concrete** if $\mathbf{P}_{\mathfrak{c}} = \mathfrak{P}(A)$ for some (unique) *non-empty* set A; in the latter case, we write $\mathsf{uni}(\mathfrak{c})$ for A, which we call the **universe** (and never use the term 'universe' synonymously for 'elementary universe'), and call $\mathfrak{c}$ a **closure operator over** A (resp. **closed system over** A, **consequence relation over** A, **closed equivalence over** A) or just a **closure operator** (resp. **closed system**, **consequence relation**, **closed equivalence**). Arbitrary (concrete) closed systems are denoted by $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$, with the usual adornments, and use of this symbolism shall imply that the closed systems are concrete. By our convention of conflating closed systems, closure operators, etc., we may speak of $\mathbb{C}$ being a closure operator, etc. The set of all closed systems over A is denoted by $\mathsf{ClSys}(A)$. □

Clearly concrete closure operators as defined above coincide with the standard definition of a closure operator (see, for example, [BS81]). Of course, concrete closure operators are complete. It is not immediately clear that concrete closed systems coincide with the standard notion of a closed system as given in Definition 1.196. We shall now show that the two notions indeed coincide.

In the case of *complete* closed systems, the necessary condition of Proposition 4.16 is sufficient, as demonstrated by the following characterization of *complete* closed systems. Condition (2) of this characterization shows that our notion of complete closed system coincides with the standard notion of a closed system on a complete lattice[DP90].

Proposition 4.43 Let $\mathbf{P}$ be a *complete lattice* and $\mathsf{cl} \subseteq \mathsf{uni}(\mathbf{P})$. The following conditions are equivalent.

1. cl determines an elementary closed system on $\mathbf{P}$.

2. $\mathbf{P}_{|_{\mathsf{cl}}} \lhd_{\blacktriangle} \mathbf{P}$.

3. $\forall\, [A \subseteq \mathsf{cl}]\ \blacktriangle^{\mathbf{P}} A \in \mathsf{cl}$.

4. $1^{\mathbf{P}} \in \mathsf{cl}$ and $\forall\, [\emptyset \neq A \subseteq \mathsf{cl}]\ \blacktriangle A \in \mathsf{cl}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ By Proposition 4.16. $\boxed{(2)\Rightarrow(3)}$ Trivial, since $\mathbf{P}$ is complete. $\boxed{(3)\Rightarrow(1)}$ Let $a \in \mathsf{uni}(\mathbf{P})$. Since $\mathbf{P}$ is complete, $\blacktriangle^{\mathbf{P}}([a\rangle_{\mathbf{P}} \cap \mathsf{cl})$ exists, and by assumption, $\blacktriangle^{\mathbf{P}}([a\rangle_{\mathbf{P}} \cap \mathsf{cl}) \in \mathsf{cl}$. So the result follows by Proposition 4.6. $\boxed{(3)\Leftrightarrow(4)}$ Trivial, since $\mathbf{P}$ is complete. $\diamond$

Remark 4.44 Consequently our notion of a (concrete) closed system, coincides with the standard notion of a closed system as given in Definition 1.196. $\square$

We enumerate a few well-known results about (concrete) closed systems that are used routinely in the sequel.

Proposition 4.45 ([BS81]) Let $\mathbb{C}$ be a closed system. Then $\mathbf{cl}_{\mathfrak{C}}$ is a complete lattice, with

$$\overset{\mathbf{cl}_{\mathfrak{C}}}{\blacktriangle} \mathcal{A} = \begin{cases} \mathsf{uni}(\mathbb{C}) & ; \quad \mathcal{A} = \emptyset, \\ \bigcap \mathcal{A} & ; \quad \text{otherwise} \end{cases} \quad \text{and} \tag{4.76}$$

$$\overset{\mathbf{cl}_{\mathfrak{C}}}{\blacktriangledown} \mathcal{A} = \bigcap \left\{ G \in \mathsf{cl}_{\mathbb{C}} \ :\ \bigcup \mathcal{A} \subseteq G \right\}, \tag{4.77}$$

for all $\mathcal{A} \subseteq \mathsf{cl}_{\mathbb{C}}$.

Remark 4.46 ([BS81]) For a closed system $\mathbb{C}$,

$$\overset{\mathbf{cl}_{\mathfrak{C}}}{\blacktriangledown} \mathcal{A} = \left\| \bigcup \mathcal{A} \right\|_{\mathfrak{C}} = \left\| \bigcup \|[\mathcal{A}]\|_{\mathbb{C}} \right\|_{\mathbb{C}} = \left\| \bigcup \{ \|A\|_{\mathbb{C}} \ :\ A \in \mathcal{A} \} \right\|_{\mathbb{C}}. \tag{4.78}$$

$\square$

We now show that the concrete set-set consequence relation is in natural one-to-one correspondence with the standard set-point consequence relation. Note that our set-set consequence is typically introduced as an abbreviating notion via (4.83). We have seen no characterization of the set-set consequence, and, in particular, we have seen no characterization that has an elementary abstraction.

Definition 4.47 (Point-Consequence Relations) A point consequence relation $\mathbb{C}$ is determined by its universe $\mathsf{uni}(\mathbb{C})$ and a binary relationship $\vdash^{\cdot}_{\mathbb{C}}$ from $\mathfrak{P}(\mathsf{uni}(\mathbb{C}))$ to $\mathsf{uni}(\mathbb{C})$ satisfying

$$a \in A \text{ implies } A \vdash^{\cdot}_{\mathbb{C}} a, \tag{4.79}$$

$$B \subseteq A \text{ and } B \vdash^{\cdot}_{\mathbb{C}} a \text{ implies } A \vdash^{\cdot}_{\mathbb{C}} a, \text{ and} \tag{4.80}$$

$$B \vdash^{\cdot}_{\mathbb{C}} a \text{ and } \forall\, [b \in B]\ A \vdash^{\cdot}_{\mathbb{C}} b \text{ implies } A \vdash^{\cdot}_{\mathbb{C}} a. \tag{4.81}$$

With each (concrete) consequence relation $\mathbb{C}$, we associate the point-consequence relation $\mathsf{pc}(\mathbb{C})$, for which we tend to write $\vdash^{\cdot}_{\mathbb{C}}$ for $\vdash^{\cdot}_{\mathsf{pc}(\mathbb{C})}$, with universe $\mathsf{uni}(\mathbb{C})$ and determined by

$$A \vdash^{\cdot}_{\mathbb{C}} a \text{ iff } A \vdash_{\mathbb{C}} \{a\}. \tag{4.82}$$

With each point consequence relation $\mathbb{C}$, we associate the (concrete) consequence relation $\mathsf{cc}(\mathbb{C})$, for which we tend to write $\vdash_{\mathbb{C}}$ for $\vdash_{\mathsf{cc}(\mathbb{C})}$, with universe $\mathsf{uni}(\mathbb{C})$ and determined by

$$A \vdash_{\mathbb{C}} B \text{ iff } \forall\,[b \in B]\; A \dot{\vdash}_{\mathbb{C}} b. \tag{4.83}$$

Further, $\mathsf{pc}(\cdot)$ and $\mathsf{cc}(\cdot)$ are mutually inverting bijections between the (concrete) consequence relations over X and the point-consequence relations over X. Consequently, we syntactically conflate point-consequence relations and (concrete) consequence relations (and hence conflated with concrete closure operators and closed systems), speaking only of **consequence relations**, and treating (4.82) and (4.83) as properties of these conflated structures. We further strengthen this convention by writing '$\vdash$' for '$\dot{\vdash}$' and allowing the 'type' of the right-hand parameter to distinguish the two types of relations. $\square$

Proof. $\boxed{\text{pc}(\mathbb{C}) \text{ is a point-consequence relation}}$ $\boxed{(4.79)}$ $a \in A$ [implies] $\{a\} \subseteq A$ [implies by (4.18)] $A \vdash_{\mathbb{C}} \{a\}$ [implies] $A \dot{\vdash}_{\mathsf{pc}(\mathbb{C})} a$. $\boxed{(4.80)}$ $B \subseteq A$ and $B \dot{\vdash}_{\mathsf{pc}(\mathbb{C})} a$ [implies] $B \subseteq A$ and $B \vdash_{\mathbb{C}} \{a\}$ [implies by (4.18)] $A \vdash_{\mathbb{C}} B$ and $B \vdash_{\mathbb{C}} \{a\}$ [implies by (4.19)] $A \vdash_{\mathbb{C}} \{a\}$ [implies] $A \dot{\vdash}_{\mathsf{pc}(\mathbb{C})} a$. $\boxed{(4.81)}$ $B \dot{\vdash}_{\mathsf{pc}(\mathbb{C})} a$ and $\forall\,[b \in B]\; A \dot{\vdash}_{\mathsf{pc}(\mathbb{C})} b$ [implies] $B \vdash_{\mathbb{C}} \{a\}$ and $\forall\,[b \in B]\; A \vdash_{\mathbb{C}} \{b\}$ [implies *] $B \vdash_{\mathbb{C}} \{a\}$ and $A \vdash_{\mathbb{C}} B$ [implies by (4.19)] $A \vdash_{\mathbb{C}} \{a\}$ [implies] $A \dot{\vdash}_{\mathsf{pc}(\mathbb{C})} a$, where the 'starred' implication is justified as follows: since $\forall\,[b \in B]\; A \vdash_{\mathbb{C}} \{b\}$, $\forall\,[b \in B]\; \{b\} \in \vdash_{\mathbb{C}}[\![A]\!]$, and so $B = \bigcup_{b \in B}\{b\} \subseteq \bigcup \vdash_{\mathbb{C}}[\![A]\!]$; since $A \vdash_{\mathbb{C}} (\bigcup \vdash_{\mathbb{C}}[\![A]\!])$ by (4.21) and $B \subseteq \bigcup \vdash_{\mathbb{C}}[\![A]\!]$, it follows by (4.23) that $A \vdash_{\mathbb{C}} B$. $\boxed{\text{cc}(\mathbb{C}) \text{ is a consequence relation}}$ We invoked Proposition 4.20. $\boxed{(4.18)}$ $A \subseteq B$ [implies] $\forall\,[a \in A]\; a \in B$ [implies by (4.79)] $\forall\,[a \in A]\; B \dot{\vdash}_{\mathbb{C}} a$ [implies] $B \vdash_{\mathsf{cc}(\mathbb{C})} A$. $\boxed{(4.19)}$ $A \vdash_{\mathsf{cc}(\mathbb{C})} B$ and $B \vdash_{\mathsf{cc}(\mathbb{C})} C$ [implies] $\forall\,[b \in B]\; A \dot{\vdash}_{\mathbb{C}} b$ and $\forall\,[c \in C]\; B \dot{\vdash}_{\mathbb{C}} c$ [implies] $\forall\,[c \in C]\; (B \dot{\vdash}_{\mathbb{C}} c$ and $\forall\,[b \in B]\; A \dot{\vdash}_{\mathbb{C}} b)$ [implies by (4.81)] $\forall\,[c \in C]\; A \dot{\vdash}_{\mathbb{C}} c$ [implies] $A \vdash_{\mathsf{cc}(\mathbb{C})} C$ $\boxed{(4.21)}$ (*It suffices to show that* $A \vdash_{\mathsf{cc}(\mathbb{C})} (\bigcup \vdash_{\mathsf{cc}(\mathbb{C})}[\![A]\!])$.) Let $b \in \bigcup \vdash_{\mathsf{cc}(\mathbb{C})}[\![A]\!]$. (*We must show that* $A \dot{\vdash}_{\mathbb{C}} b$.) By definition, there exists B such that $b \in B$ and $A \vdash_{\mathsf{cc}(\mathbb{C})} B$. So $\forall\,[b' \in B]\; A \dot{\vdash}_{\mathbb{C}} b'$. Since $b \in B$, $A \dot{\vdash}_{\mathbb{C}} b$. $\boxed{\text{Mutually inverting bijections}}$ (*It suffices to show that* $\mathsf{cc}(\mathsf{pc}(\mathbb{C})) = \mathbb{C}$ *and* $\mathsf{pc}(\mathsf{cc}(\mathbb{C})) = \mathbb{C}$.) $\boxed{\mathsf{cc}(\mathsf{pc}(\mathbb{C})) = \mathbb{C}}$ Let $\mathbb{C}$ be a (concrete) consequence relation. Now $A \vdash_{\mathsf{cc}(\mathsf{pc}(\mathbb{C}))} B$ [iff] $\forall\,[b \in B]\; A \dot{\vdash}_{\mathsf{pc}(\mathbb{C})} b$ [iff] $\forall\,[b \in B]\; A \vdash_{\mathbb{C}} \{b\}$ [iff] $A \vdash_{\mathbb{C}} B$, the final equivalence being justified as follows. If $A \vdash_{\mathbb{C}} B$ then, for all $b \in B$, since $\{b\} \subseteq B$. $A \vdash_{\mathbb{C}} \{b\}$, by (4.23). Conversely, if $\forall\,[b \in B]\; A \vdash_{\mathbb{C}} \{b\}$, then $\forall\,[b \in B]\; \{b\} \in \vdash_{\mathbb{C}}[\![A]\!]$, and so $B \subseteq \bigcup \vdash_{\mathbb{C}}[\![A]\!]$; since $A \vdash_{\mathbb{C}} (\bigcup \vdash_{\mathbb{C}}[\![A]\!])$ by (4.21), it follows, by (4.23), that $A \vdash_{\mathbb{C}} B$. $\boxed{\mathsf{pc}(\mathsf{cc}(\mathbb{C})) = \mathbb{C}}$ Let $\mathbb{C}$ be a point-consequence relation. Now $A \dot{\vdash}_{\mathsf{pc}(\mathsf{cc}(\mathbb{C}))} b$ [iff] $A \vdash_{\mathsf{cc}(\mathbb{C})} \{b\}$ [iff] $\forall\,[b' \in \{b\}]\; A \dot{\vdash}_{\mathbb{C}} b'$ [iff] $A \dot{\vdash}_{\mathbb{C}} b$. $\diamond$

We list some well-known properties for ease of subsequent reference.

Corollary 4.48 If $\mathbb{C}$ is a closed system then

$$\|A\|_{\mathbb{C}} = \{a \,:\, A \vdash_{\mathbb{C}} a\} \quad \text{and} \tag{4.84}$$

$$A \vdash_{\mathbb{C}} a \text{ iff } a \in \|A\|_{\mathbb{C}}. \tag{4.85}$$

Example 4.49 (Topological Spaces)

The closed sets of a topological closed system form a closed system, the closure operator of a topological closed system is a closure operator, and the **nearness relation** between a point a and a set A, defined by a is near A iff every open neighbourhood of a meets A, is a (reversed) point-consequence relation. Further, our conflating convention conflates these three objects, i.e., the *closure operator associated with the closed system* of closed sets of a topological closed system is the *closure operator of that closed system*, etc.

$\square$

Warning 4.50 We shall subsequently (tend to) omit explicit reference to those properties of elementary closed systems (etc.) that are immediate analogues of the familiar properties of 'concrete' closed systems (etc.).

Convention 4.51 It is convenient to conflate closure operators with their fundamental operation. For example, we may call an operator $\|\cdot\| : \mathfrak{P}(X) \to \mathfrak{P}(X)$ a closure operator over X. Analogous conventions pertain to closed systems, consequence relations and closed equivalences.

4.2.1 The Closed System of all Closed Systems

Consider all the closed systems that can be formed over a given set A. From these closed systems, it is possible to form a closed system $\mathbf{C}(\mathbf{C}, A)$, over $\mathfrak{P}(A)$, by considering *all* the closed sets of a closed system over A, as a *single* closed set of $\mathbf{C}(\mathbf{C}, A)$, thereby putting the closed systems over A into one-to-one correspondence with the closed sets of $\mathbf{C}(\mathbf{C}, A)$.

Definition 4.52 (The Closed System of all Closed Systems) With each A, we associate the $\mathfrak{P}(\mathcal{A})$-closed system $\mathbf{C}(\mathbf{C}, A)$, determined by closed system $\{\mathsf{cl}_{\mathbb{C}} : \mathbb{C} \in \mathsf{ClSys}(A)\}$, which we call the **closed system of all closed systems** over A. $\square$

Proof. $\boxed{\text{Universe}}$ The discrete closed system over A has closed sets $\mathfrak{P}(A)$, and so $\mathfrak{P}(A) \in \mathsf{cl}_{\mathbf{C}(\mathbf{C},\mathrm{A})}$. $\boxed{\text{Intersection}}$ Let $\emptyset \neq \mathfrak{A} \subseteq \mathsf{cl}_{\mathbf{C}(\mathbf{C},\mathrm{A})}$. (*We need to show that* $\bigcap \mathfrak{A}$ *determines a closed system on* A.) By definition, for each $\mathcal{A} \in \mathfrak{A}$, there exists a closed system $\mathbb{C}_{\mathcal{A}}$ on A with $\mathcal{A} = \mathsf{cl}_{\mathbb{C}_{\mathcal{A}}}$. Note that

$$B \in \bigcap \mathfrak{A} \text{ iff } \forall\, [\mathcal{A} \in \mathfrak{A}]\; B \in \mathsf{cl}_{\mathbb{C}_{\mathcal{A}}}. \tag{4.86}$$

$\boxed{\text{Universe}}$ For each $\mathcal{A} \in \mathfrak{A}$, $\mathcal{A}$ is a closed system over A, and so $A \in \mathcal{A}$. So $A \in \bigcap \mathfrak{A}$. $\boxed{\text{Intersection}}$ Let $\emptyset \neq \mathcal{A} \subseteq \bigcap \mathfrak{A} \subseteq \mathfrak{P}(A)$. Note that $\bigcap \mathcal{A} \in \mathfrak{P}(A)$. (*We must show that* $\bigcap \mathcal{A} \in \bigcap \mathfrak{A}$.) $\forall\, [B \in \mathcal{A}]\; B \in \bigcap \mathfrak{A}$. So by (4.86), $\forall\, [B \in \mathcal{A}]\; \forall\, [\mathcal{A} \in \mathfrak{A}]\; B \in \mathsf{cl}_{\mathbb{C}_{\mathcal{A}}}$. So $\forall\, [\mathcal{A} \in \mathfrak{A}]\; \mathcal{A} \subseteq \mathsf{cl}_{\mathbb{C}_{\mathcal{A}}}$. So $\forall\, [\mathcal{A} \in \mathfrak{A}]\; \bigcap \mathcal{A} \in \mathsf{cl}_{\mathbb{C}_{\mathcal{A}}}$. Hence by (4.86), $\bigcap \mathcal{A} \in \bigcap \mathfrak{A}$.

$\diamond$

Remark 4.53 The map $\mathbb{C} \mapsto \mathsf{cl}_{\mathbb{C}}$ is a bijection from $\mathsf{ClSys}(A)$ onto $\mathsf{cl}_{\mathbf{C}(\mathbf{C},\mathrm{A})}$, by definition.

Convention 4.54 (Conflating $\mathsf{cl}_{\mathbf{C}(\mathbf{C},\mathrm{A})}$ and $\mathsf{ClSys}(A)$) We shall treat each $\mathcal{G} \in \mathsf{cl}_{\mathbf{C}(\mathbf{C},\mathrm{A})}$ as the unique closed system $\mathbb{C}$ over A with $\mathsf{cl}_{\mathbb{C}} = \mathcal{G}$, and treat each closed system $\mathbb{C}$ over A as the unique $\mathcal{G} \in \mathsf{cl}_{\mathbf{C}(\mathbf{C},\mathrm{A})}$ with $\mathsf{cl}_{\mathbb{C}} = \mathcal{G}$. For example, we may write 'let $\mathbb{C}$ be a closed system in $\mathsf{cl}_{\mathbf{C}(\mathbf{C},\mathrm{A})}$'.

The above convention permits comparison of the granularity relation $\preceq$ (restricted to closed systems over A) and the inclusion-ordered complete lattice $\mathbf{cl}_{\mathbf{C}(\mathbf{C},A)}$. It is unfortunate that these two orders, as presented, are dual-isomorphic.

Remark 4.55 For $\mathbb{C}, \mathbb{D} \in \mathsf{ClSys}(A)$, $\mathbb{C} \preceq \mathbb{D}$ iff $\mathbb{D} \leq^{\mathbf{cl}_{\mathbf{C}(\mathbf{C},\mathrm{A})}} \mathbb{C}$.

Remark 4.56 Since $\mathbf{cl}_{\mathbf{C}(\mathbf{C},A)}$ is a complete lattice, the granularity relationship, restricted to closed systems over a particular universe, is a *complete* lattice-order. $\square$

As such, we have a mechanism for taking meets and joins of *closed systems* (over A), although we must be careful to distinguish $\preceq$-meets and $\preceq$-joins from $\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}$-meets and $\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}$-joins.

Remark 4.57 $\blacktriangledown^{\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}} = \blacktriangle^{\preceq}$ and $\blacktriangle^{\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}} = \blacktriangledown^{\preceq}$.

Remark 4.58 For $\mathcal{A} \subseteq \mathfrak{P}(A)$, $\|\mathcal{A}\|_{\mathbf{C}(\mathbb{C},A)} = \bigcap\{\mathsf{cl}_{\mathbb{C}} : \mathbb{C} \in \mathsf{ClSys}(A), \mathcal{A} \subseteq \mathsf{cl}_{\mathbb{C}}\}$.

4.2.2 Bases

Definition 4.59 (Basis) A system $\mathcal{G}$ is called a **basis** for a closed system $\mathbb{C}$ iff every $\mathbb{C}$-closed set other than the universe is a non-empty possibly infinite intersection of members of $\mathcal{G}$. In this case we say that $\mathbb{C}$ is **generated by basis** $\mathcal{G}$. With each system $\mathfrak{X}$ we associate the closed system $\mathbf{C}(\mathfrak{X}, \mathsf{basis})$ with universe $\mathsf{uni}(\mathfrak{X})$ and defined by $\mathsf{cl}_{\mathbf{C}(\mathfrak{X},\mathsf{basis})} = \{\mathsf{uni}(\mathfrak{X})\} \cup \{\bigcap_{\emptyset \neq \mathcal{B} \subseteq \mathfrak{X}} \mathcal{B}\}$. □

Proof. Trivially, $\mathsf{uni}(\mathbf{C}(\mathfrak{X}, \mathsf{basis})) = \mathsf{uni}(\mathfrak{X}) \in \{\mathsf{uni}(\mathfrak{X})\} \cup \{\bigcap_{\emptyset \neq \mathcal{B} \subseteq \mathfrak{X}} \mathcal{B}\}$. Let $\emptyset \neq \mathcal{C} \subseteq \{\mathsf{uni}(\mathfrak{X})\} \cup \{\bigcap_{\emptyset \neq \mathcal{B} \subseteq \mathfrak{X}} \mathcal{B}\}$. If $\mathcal{C} = \{\mathsf{uni}(\mathfrak{X})\}$, then $\bigcap \mathcal{C} = \mathsf{uni}(\mathfrak{X}) \in \{\mathsf{uni}(\mathfrak{X})\} \cup \{\bigcap_{\emptyset \neq \mathcal{B} \subseteq \mathfrak{X}} \mathcal{B}\}$. Suppose that $\mathcal{C} \neq \{\mathsf{uni}(\mathfrak{X})\}$. Let $\mathcal{C}' = \mathcal{C} - \{\mathsf{uni}(\mathfrak{X})\}$. Clearly $\mathcal{C}' \neq \emptyset$, $\mathcal{C}' \subseteq \{\bigcap_{\emptyset \neq \mathcal{B} \subseteq \mathfrak{X}} \mathcal{B}\}$ and $\bigcap \mathcal{C} = \bigcap \mathcal{C}'$. So, for each $A \in \mathcal{C}'$, there exists $\emptyset \neq \mathcal{B}_A \subseteq \mathfrak{X}$ with $A = \bigcap \mathcal{B}_A$. So $\bigcap \mathcal{C} = \bigcap \mathcal{C}' = \bigcap_{A \in \mathcal{C}'} A = \bigcap_{A \in \mathcal{C}'} \bigcap \mathcal{B}_A \in \{\mathsf{uni}(\mathfrak{X})\} \cup \{\bigcap_{\emptyset \neq \mathcal{B} \subseteq \mathfrak{X}} \mathcal{B}\}$. ◇

Remark 4.60 For $\mathfrak{X} \subseteq \mathfrak{P}(A)$, $\mathbf{C}(\mathfrak{X}, \mathsf{basis})$ is the coarsest closed system $\mathbb{C}$ over A with $\mathfrak{X} \subseteq \mathsf{cl}_{\mathbb{C}}$.

Remark 4.61 For $\mathfrak{X}_1 \subseteq \mathfrak{X}_2 \subseteq \mathfrak{P}(A)$, $\mathbf{C}(\mathfrak{X}_2, \mathsf{basis}) \preceq \mathbf{C}(\mathfrak{X}_1, \mathsf{basis})$.

Remark 4.62 $\mathcal{G}$ is a basis for a closed system $\mathbb{C}$ iff $\mathbb{C} = \mathbf{C}(\mathcal{G}, \mathsf{basis})$.

Remark 4.63 For $\mathcal{G} \subseteq \mathfrak{P}(A)$, $\mathcal{G}$ is a basis for closed system $\mathbb{C}$ over A iff $\mathbb{C} = \|\mathcal{G}\|_{\mathbf{C}(\mathbb{C},A)}$.

Proof. $\boxed{\Rightarrow}$ Suppose that $\mathcal{G}$ is a basis for $\mathbb{C}$. Let $\mathbb{C}' = \|\mathcal{G}\|_{\mathbf{C}(\mathbb{C},A)}$. (*It suffices to show that* $\mathsf{cl}_{\mathbb{C}'} = \mathsf{cl}_{\mathbb{C}}$.) Since $\mathcal{G} \subseteq \mathsf{cl}_{\mathbb{C}}$, $\mathsf{cl}_{\mathbb{C}'} \subseteq \mathsf{cl}_{\mathbb{C}}$, by Remark 4.58. Let $G \in \mathsf{cl}_{\mathbb{C}}$. If $G = A$, then certainly $G \in \mathsf{cl}_{\mathbb{C}'}$. Otherwise, $G = \bigcap \mathcal{B}$, for some $\emptyset \neq \mathcal{B} \subseteq \mathcal{G}$. Since $\mathcal{B} \subseteq \mathsf{cl}_{\mathbb{C}'}$, by Remark 4.58, and $\mathbb{C}'$ is a closed system, $G = \bigcap \mathcal{B} \in \mathsf{cl}_{\mathbb{C}'}$. $\boxed{\Leftarrow}$ Assume that $\mathbb{C} = \|\mathcal{G}\|_{\mathbf{C}(\mathbb{C},A)}$. Let $G \in \mathsf{cl}_{\mathbb{C}}$. Let $\mathbb{C}' = \mathbf{C}(\mathcal{G}, \mathsf{basis})$. By definition, $\mathcal{G} \subseteq \mathsf{cl}_{\mathbb{C}'}$, so by Remark 4.58 and assumption, $\mathsf{cl}_{\mathbb{C}} \subseteq \mathsf{cl}_{\mathbf{C}(\mathcal{G},\mathsf{basis})}$, and hence $G \in \mathsf{cl}_{\mathbf{C}(\mathcal{G},\mathsf{basis})}$. So by the definition of $\mathbf{C}(\mathcal{G}, \mathsf{basis})$, either G is the universe, or, there exists $\emptyset \neq \mathcal{B} \subseteq \mathcal{G}$ with $G = \bigcap B$. So $\mathcal{G}$ is a basis for $\mathbb{C}$. ◇

Remark 4.64 $\mathbf{C}(\mathfrak{X}, \mathsf{basis}) = \|\mathfrak{X}\|_{\mathbf{C}(\mathfrak{X},\mathsf{basis})}$.

Remark 4.65 If $\mathfrak{X} \subseteq \mathfrak{P}(A)$ and $\mathfrak{X} = \bigcup_{i \in I} \mathcal{A}_i$, where $\mathcal{A}_i \subseteq \mathfrak{P}(A)$ for each $i \in I$, then $\mathbf{C}(\mathfrak{X}, \mathsf{basis}) = \blacktriangledown_{i \in I}^{\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}} \mathbf{C}(\mathcal{A}_i, \mathsf{basis}) = \blacktriangle_{i \in I}^{\preceq} \mathbf{C}(\mathcal{A}_i, \mathsf{basis})$.

Proof. For each $i \in I$, $\mathcal{A}_i \subseteq \mathsf{cl}_{\mathbf{C}(\mathcal{A}_i,\mathsf{basis})} \subseteq \mathsf{cl}_{\blacktriangledown_{i \in I}^{\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}} \mathbf{C}(\mathcal{A}_i,\mathsf{basis})}$. Hence $\mathfrak{X} = \bigcup_{i \in I} \mathcal{A}_i \subseteq \mathsf{cl}_{\blacktriangledown_{i \in I}^{\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}} \mathbf{C}(\mathcal{A}_i,\mathsf{basis})}$, and so $\mathsf{cl}_{\mathbf{C}(\mathfrak{X},\mathsf{basis})} \subseteq \mathsf{cl}_{\blacktriangledown_{i \in I}^{\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}} \mathbf{C}(\mathcal{A}_i,\mathsf{basis})}$, by Remark 4.60. Conversely, for each $i \in I$, $\mathcal{A}_i \subseteq \mathfrak{X}$, hence $\mathbf{C}(\mathfrak{X}, \mathsf{basis}) \preceq \mathbf{C}(\mathcal{A}_i, \mathsf{basis})$, i.e., $\mathsf{cl}_{\mathbf{C}(\mathcal{A}_i,\mathsf{basis})} \subseteq \mathsf{cl}_{\mathbf{C}(\mathfrak{X},\mathsf{basis})}$. Hence $\mathsf{cl}_{\blacktriangledown_{i \in I}^{\mathbf{cl}_{\mathbf{C}(\mathbb{C},A)}} \mathbf{C}(\mathcal{A}_i,\mathsf{basis})} \subseteq \mathsf{cl}_{\mathbf{C}(\mathfrak{X},\mathsf{basis})}$. ◇

4.2.3 Filtrations

Next we define the notion of a *filtration* closed system, which is the closed system obtained from closed system $\mathbb{C}$ by selecting only those closed sets containing a given subset of the universe. It is easily seen that this definition well-defines a closed system.

Definition 4.66 (Closed-Filtrations) Let $\mathbb{C}$ be a closed system and $A \subseteq \mathsf{uni}(\mathbb{C})$. By the **closed-filtration of** $\mathbb{C}$ **by** A, denoted $\mathbb{C}_{:A}$, we mean the closed system with $\mathsf{uni}(\mathbb{C}_{:A}) = \mathsf{uni}(\mathbb{C})$ and $\mathsf{cl}_{\mathbb{C}_{:\mathrm{A}}} = \{G \in \mathsf{cl}_{\mathbb{C}} \ : \ A \subseteq G\}$. □

Remark 4.67 Let $\mathbb{C}$ be a closed system and $A \subseteq \mathsf{uni}(\mathbb{C})$.

1. If $\mathbb{C}$ is finitary then so is $\mathbb{C}_{:A}$.
2. $\|B\|_{\mathbb{C}_{:A}} = \|B \cup A\|_{\mathbb{C}}$.
3. $B \vdash_{\mathbb{C}_{:A}} a$ iff $A \cup B \vdash_{\mathbb{C}} a$.

Proof. $\boxed{(3)}$ $B \vdash_{\mathbb{C}_{:A}} a$ [iff] $a \in \|B\|_{\mathbb{C}_{:A}}$ [iff, by (2)] $a \in \|B \cup A\|_{\mathbb{C}}$ [iff] $A \cup B \vdash_{\mathbb{C}} a$. ⋄

4.2.4 The Point Proto-Leibniz Relations

The elementary proto-Leibniz relation, when realized in a concrete context, is a parameterized binary relation between *subsets* of the concrete universe. Recalling the motivating formula (4.51), we shall now consider the parameterized (by a set) binary relationship between points of the concrete universe defined by

$$A \cup \{c\} \vdash d \text{ and } A \cup \{d\} \vdash c.$$

Definition 4.68 (The Point Proto-Leibniz Relation) For a closed system $\mathbb{C}$, $A \subseteq \mathsf{uni}(\mathbb{C})$ and $c, d \in \mathsf{uni}(\mathbb{C})$, we write $c \boldsymbol{\Leftrightarrow}^{\mathbb{C}}(A)\ d$ for $\{c\} \Leftrightarrow^{\mathbb{C}}(A)\ \{d\}$; this convention implicitly defines a parameterized equivalence relation between points of the concrete universe which we shall refer to as the *point proto-Leibniz relation.* □

That the point proto-Leibniz relation is an equivalence relation follows by Proposition 4.34. The following characterization follows by definition and Theorem 4.39.

Proposition 4.69 The following conditions are equivalent.

1. $c \boldsymbol{\Leftrightarrow}^{\mathbb{C}}(A)\ d$.
2. $\forall\,[A \subseteq B]\ c \in \|B\| \leftrightarrow d \in \|B\|$.
3. $A \cup \{c\} \vdash d$ and $A \cup \{d\} \vdash c$.

□

The following result sheds light on the behaviour of the Leibniz relation when a logic is protoalgebraic (see the example following this theorem).

Theorem 4.70 $\asymp^{\mathbb{C}}(A)$ is the $\subseteq$-largest equivalence relation compatible with all $\mathbb{C}$-closed sets containing A.

Proof. $\boxed{\text{Compatibility}}$ Suppose that $G \in \mathsf{cl}_{\mathbb{C}}$ and $A \subseteq G$. Let $a \in G$ and suppose that $a \asymp^{\mathbb{C}}(A)\; b$. Now, $A \subseteq G$ so $a \in \|G\|_{\mathbb{C}} = G$ iff $b \in \|G\|_{\mathbb{C}} = G$. Since $a \in G$, $b \in G$. $\boxed{\text{Maximality}}$ Let α be any equivalence relation compatible with all $\mathbb{C}$-closed sets containing A. Suppose that $c\; \alpha\; d$. (*We must show that* $c \asymp^{\mathbb{C}}(A)\; d$.) Let $A \subseteq B \subseteq \mathsf{uni}(\mathbb{A})$. Suppose that $c \in \|B\|$. Since $A \subseteq \|B\|$, α is compatible with $\|B\|$. Hence, since $c \in \|B\|$ and $c\; \alpha\; d$, $d \in \|B\|$. Symmetrically, $d \in \|B\|$ implies $c \in \|B\|$. Hence $c \asymp^{\mathbb{C}}(A)\; d$. $\diamond$

Example 4.71 (Protoalgebraic Sentential Calculi)

Recall the definition of protoalgebraicity given in Definition 2.132 on page 116. In the case of a sentential 1-calculus $\mathcal{S}$, $\mathcal{S}$ is protoalgebraic if, for all $T \in \mathsf{Th}(\mathcal{S})$,

$$\phi\, \Omega^{\mathcal{S}}(T)\, \psi \text{ implies } T \cup \{\phi\} \vdash_{\mathcal{S}} \psi \text{ and } T \cup \{\psi\} \vdash_{\mathcal{S}} \phi. \tag{4.87}$$

Conflating $\mathcal{S}$ with the closed system determined by all $\mathcal{S}$-theories, we may rephrase the definition of protoalgebraicity in terms of an inclusion relationship between $\Omega^{\mathcal{S}}(T)$ and $\asymp^{\mathcal{S}}(T)$. More precisely, we have the following.

Remark 4.72 A sentential 1-calculus $\mathcal{S}$ is protoalgebraic iff, for all $T \in \mathsf{Th}(\mathcal{S})$, $\Omega^{\mathcal{S}}(T) \subseteq \asymp^{\mathcal{S}}(T)$. $\square$

Consequently, the following result obtains by the previous theorem.

Corollary 4.73 If $\mathcal{S}$ is protoalgebraic, then, for all $T \in \mathsf{Th}(\mathcal{S})$, $\Omega^{\mathcal{S}}(T)$ is compatible with all theories containing T.

$\square$

We enumerate a few properties of the point proto-Leibniz relation. Observe that we have to often be careful to eliminate cases involving the empty-set: since images of the empty-set are always empty. This problem is avoided by the elementary proto-Leibniz relation, since this relation, in the concrete setting, is a parameterized relation between sets and as such, may sensibly relate the empty-set to some other set.

Proposition 4.74 Let $\mathbb{C}$ be a closed system.

1. If $A \neq \emptyset$ then A lies within a single equivalence-class of $\asymp^{\mathbb{C}}(A)$.
2. For all $A \cup \{a\} \subseteq \mathsf{uni}(\mathbb{A})$. $\asymp^{\mathbb{C}}(A \cup \{a\})\, [A \cup \{a\}] = \asymp^{\mathbb{C}}(A \cup \{a\})[\![a]\!]$.
3. If $A \neq \emptyset$ then $\asymp^{\mathbb{C}}(A)\, [A] = \|A\|$.
4. $\asymp(A) = \asymp(\|A\|)$.

Proof. $\boxed{(1)}$ (*The proof essentially follows since* A *lies in any closed set that contains it.*) Let $a \in A \neq \emptyset$ and let $b \in \asymp^{\mathbb{C}}(A)\, [A]$. Then there exists $a' \in A$ with $a' \asymp^{\mathbb{C}}(A)\; b$. Since $\asymp^{\mathbb{C}}(A)$ is an equivalence relation, it suffices to show that $a \asymp^{\mathbb{C}}(A)\; a'$. For all $A \subseteq B$, $a \in A \subseteq \|B\|$ and $a' \in A \subseteq \|B\|$, so certainly $a \in \|B\|$ iff $a' \in \|B\|$; so $a' \asymp^{\mathbb{C}}(A)\; b$. $\boxed{(2)}$ Follows immediately from (1). $\boxed{(3)}$ Let $a' \in A \neq \emptyset$. By (2),

$\rightleftharpoons (A)\,[A] = \rightleftharpoons (A)[\![a']\!]$. $\boxed{\rightleftharpoons (A)[\![a']\!] \subseteq \|A\|}$ Let $a \in \rightleftharpoons (A)[\![a']\!]$. Since $A \subseteq A$, $a' \in A \subseteq \|A\|$ and $a' \rightleftharpoons (A)\ a$, $a \in \|A\|$. $\boxed{\|A\| \subseteq \rightleftharpoons (A)[\![a']\!]}$ Let $a \in \|A\|$. For all $A \subseteq B$, $a \in \|A\| \subseteq \|B\|$ and $a' \in A \subseteq \|B\|$, so certainly $a \in \|B\|$ iff $a' \in \|B\|$; so $a \rightleftharpoons (A)\ a'$. Hence $a \in \rightleftharpoons (A)[\![a']\!]$. $\boxed{(4)}$ By (4.74). $\diamond$

Non-empty closed sets may be characterized in terms of the point proto-Leibniz relation, as demonstrated by the following result. To see that this result cannot be strengthened to include the empty-set, consider any constrained closed system and observe that (3) would assert that the empty-set is closed, which would be a contradiction.

Corollary 4.75 For $A \neq \emptyset$ the following conditions are equivalent.

1. $A \in \mathsf{cl}_{\mathbb{C}}$.

2. $\rightleftharpoons^{\mathbb{C}}(A)\,[A] \subseteq A$.

3. $\rightleftharpoons^{\mathbb{C}}(A)\,[A] = A$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Suppose that $A \in \mathsf{cl}$. By (3) of the previous proposition, $\rightleftharpoons (A)\,[A] = \|A\| = A$, so *certainly*, $\rightleftharpoons (A)\,[A] \subseteq A$. $\boxed{(2)\Rightarrow(3)}$ The required converse inclusion is trivial since $\rightleftharpoons (A)$ is an equivalence relation, and hence reflexive. $\boxed{(3)\Rightarrow(1)}$ Suppose that $\rightleftharpoons (A)\,[A] = A$. Then by (3) of the previous proposition, $A = \|A\|$, and hence A is closed. $\diamond$

4.2.5 Unary Matrices as Closed Systems

Recall ∮2.3, where *matrices* where employed as models of sentential calculi. In ∮7, we shall be considering *logics* as models of (other) logics. In order to explicate the relationship between matrices as models of logics and logics as models of logics, it proves useful to view matrices as 'small' logics. In the discourse of closed systems, this amounts to viewing matrices as 'small' closed systems. To this end we introduce the following definitions.

Definition 4.76 (Unary Matrix-Closed Systems) With any given unary matrix M, we associate the two closed systems $\mathbb{C}(\mathsf{M})$ and $\mathbb{C}(\mathsf{M}, \emptyset)$, both with universe $\mathsf{uni}(\mathsf{M})$, determined by $\mathsf{cl}_{\mathbb{C}(\mathsf{M})} = \{\mathsf{D}_\mathsf{M}, \mathsf{uni}(\mathsf{M})\}$ and $\mathsf{cl}_{\mathbb{C}(\mathsf{M},\emptyset)} = \{\emptyset, \mathsf{D}_\mathsf{M}, \mathsf{uni}(\mathsf{M})\}$ respectively, which we call the **constrained** and **unconstrained (matrix-)closed system** determined by M. $\square$

4.3 Algebraic Closure and Formal Systems

In this section we briefly summarize the *well-known* results pertaining to (concrete) *algebraic* (or *finitary*) closed systems and show how these closed systems may be characterized in terms of a logic-like notion of a *formal-system*. Numerous examples of algebraic closed systems and formal-systems, pertinent to the sequel, are introduced. We make no claims of originality of the results obtained in this section, since it is our belief that they are all well-known, in some form, to practitioners in the field.

4.3.1 Algebraic Closure

Recall the definition of an algebraic closure operator given in Definition 1.198 on page 43. We prefer the alternative term 'finitary' rather than 'algebraic' given the particular use of the latter term in logic (as in 'algebraic logic'). Later in this text we shall tend to use the term 'finitary' rather than 'algebraic'. Note that the terms 'algebraic' and 'finitary' are *not* interchangeable in *lattice theory.*

Proposition 4.77 For a closed system $\mathbb{C}$ the following conditions are equivalent.

1. $\mathbb{C}$ is algebraic.
2. [BS81] $\|A\|_{\mathbb{C}} = \bigcup\{\|B\|_{\mathbb{C}} \,:\, B \subseteq A \text{ and } B \text{ is finite}\}$, for all $A \subseteq \mathsf{uni}(\mathbb{C})$.
3. For all $A \subseteq \mathsf{uni}(\mathbb{C})$ and $a \in \mathsf{uni}(\mathbb{C})$,

$$A \vdash_{\mathbb{C}} a \text{ implies } \exists\,[A' \subseteq_{\mathsf{f}} A]\; A' \vdash_{\mathbb{C}} a. \tag{4.88}$$

Proof. $\boxed{(2)\Rightarrow(3)}$ Suppose that $A \vdash_{\mathbb{C}} a$, i.e., $a \in \|A\| = \bigcup\{\|A'\| \,:\, A' \subseteq_{\mathsf{f}} A\}$ by assumption. So for some finite $A' \subseteq_{\mathsf{f}} A$, $a \in \|A'\|$, i.e., $A' \vdash_{\mathbb{C}} a$. $\boxed{(3)\Rightarrow(2)}$ $\|A\| = \{a \,:\, A \vdash_{\mathbb{C}} a\} = \{a \,:\, \exists\,[A' \subseteq_{\mathsf{f}} A]\; A' \vdash_{\mathbb{C}} a\} = \bigcup_{A' \subseteq_{\mathsf{f}} A}\{a \,:\, A' \vdash_{\mathbb{C}} a\} = \bigcup_{A' \subseteq_{\mathsf{f}} A} \|A'\| = \bigcup\{\|A'\| \,:\, A' \subseteq_{\mathsf{f}} A\}$. $\diamond$

Recall the notions of a *compact* element of a complete lattice and an algebraic lattice (see ∮1.2.4).

Theorem 4.78 [BS81, T5.1] If $\mathbb{C}$ is an algebraic closed system, then $\mathbf{cl}_{\mathbb{C}}$ is an algebraic lattice and the compact elements of $\mathbf{cl}_{\mathbb{C}}$ are precisely the finitely generated closed sets. □

Remark 4.79 Any closed system over a finite universe is algebraic. □

Remark 4.80 If $\mathbb{C}$ is an algebraic closed system and $G \in \mathsf{cl}_{\mathbb{C}}$, then the principal $\mathbf{cl}_{\mathbb{C}}$-ideal $\langle G]_{\mathbf{cl}_{\mathbb{C}}}$ forms a algebraic closed system on G. □

4.3.1.1 Examples

Counter Example 4.81 (Topological Closed Systems need not be Algebraic)

The closed system of closed sets of a topological closed system need not be algebraic, as demonstrated by the standard topology on $\mathbb{R}$. For example, the set $\{[0, 1 - \frac{1}{n+2}] \,:\, n \in \mathbb{N}\}$ is a directed set of closed intervals for which the union is not closed.

□

Example 4.82 (Upsets, Downsets and Convexities of Orders)

Let $\mathbf{P}$ be an order. It is easily seen that $\mathsf{Dn}(\mathbf{P})$, $\mathsf{Up}(\mathbf{P})$ and $\mathsf{Cx}(\mathbf{P})$ all form *unconstrained* closed systems on $\mathsf{uni}(\mathbf{P})$.

Definition 4.83 (The Consequence Relations of Upsets, Downsets and Convexities) Let $\vdash^{\mathbf{P}}_{\mathsf{dn}}$ (resp. $\vdash^{\mathbf{P}}_{\mathsf{up}}$, $\vdash^{\mathbf{P}}_{\mathsf{cx}}$) denote the point-consequence relation determined by the closed system $\mathsf{Dn}(\mathbf{P})$ (resp. $\mathsf{Up}(\mathbf{P})$, $\mathsf{Cx}(\mathbf{P})$). □

Remark 4.84 Let $A \cup \{a\} \subseteq \mathsf{uni}(\mathbf{P})$.

1. $A \vdash^{\mathbf{P}}_{\mathsf{dn}} a$ iff $\exists\, [b \in A]\; a \leq b$.
2. $A \vdash^{\mathbf{P}}_{\mathsf{up}} a$ iff $\exists\, [b \in A]\; b \leq a$.
3. $A \vdash^{\mathbf{P}}_{\mathsf{cx}} a$ iff $\exists\, [b, c \in A]\; b \leq a \leq c$.

Proposition 4.85 For an order $\mathbf{P}$, the closed systems $\mathsf{Dn}(\mathbf{P})$, $\mathsf{Up}(\mathbf{P})$ and $\mathsf{Cx}(\mathbf{P})$ all form algebraic closed systems. In fact, $\mathsf{Dn}(\mathbf{P})$ and $\mathsf{Up}(\mathbf{P})$ are closed under *arbitrary* unions.

Proof. As an example, we prove that $\mathsf{Cx}(\mathbf{P})$ forms an *algebraic* closed system. Let $\mathcal{A} \subseteq \mathsf{Cx}(\mathbf{P})$ that is directed. Let $a, c \in \bigcup \mathcal{A}$ and suppose that $a \leq b \leq c$. (*We must show that* $b \in \bigcup \mathcal{A}$.) There exists $A, C \in \mathcal{A}$ with $a \in A$ and $c \in C$. By directedness, there exists $B \in \mathcal{A}$ with $a, c \in B$. Since B is convex, $a, c \in B$ and $a \leq b \leq c$, $b \in B \subseteq \bigcup \mathcal{A}$. $\diamond$

Definition 4.86 (The Closure Operators of Upsets, Downsets and Convexities) The closure operators associated with the closed systems $\mathsf{Dn}(\mathbf{P})$, $\mathsf{Up}(\mathbf{P})$ and $\mathsf{Cx}(\mathbf{P})$ are denoted by $\|\cdot\|^{\mathbf{P}}_{\mathsf{dn}}$, $\|\cdot\|^{\mathbf{P}}_{\mathsf{up}}$ and $\|\cdot\|^{\mathbf{P}}_{\mathsf{cx}}$, respectively. $\square$

Remark 4.87 $\|A\|^{\mathbf{P}}_{\mathsf{dn}}$ (resp. $\|A\|^{\mathbf{P}}_{\mathsf{up}}$, $\|A\|^{\mathbf{P}}_{\mathsf{cx}}$) is the smallest downset (resp. upset, convexity) containing A.

$\square$

In the next example we consider closed systems formed from lattice ideals and filters. These closed systems, together with the closed systems of convexities on a lattice, provide important examples in this text; they shall lead us to logics that are generally unalgebraizable in the standard sense, but for which our more general notion of parametrized algebraizability is applicable. Lattice ideals and filters provide canonical examples of closed systems, but with a caveat as we shall see. Recall that non-empty lattice ideals and filters are not permitted. Note that all applications of this example will pertain to lattice *algebras* rather than lattice *orders* (see Example 1.467 on page 89).

Example 4.88 (Ideals and Filters of Lattices)

Remark 4.89 [RMT87, 48] A lattice is lower-bounded (upper-bounded) iff set of all lattice ideals (resp. filters) forms a *constrained* closed system. In the case that a lattice is not lower-bounded (upper-bounded), the set of all lattice ideals (resp. filters) together with the empty-set forms an *unconstrained* closed set. $\square$

While the following definitions are phrased in terms of lattices, we intend that they may be applied to lattice *expansions* in the obvious manner.

Definition 4.90 ($\mathsf{Id}_{\diamond}(\mathbf{P})$ and $\mathsf{Fl}_{\diamond}(\mathbf{P})$) Let $\mathbf{P}$ be a lattice. Let $\mathsf{Id}_{\diamond}(\mathbf{P})$ and (resp. $\mathsf{Fl}_{\diamond}(\mathbf{P})$) denote the set of all lattice ideals (resp. lattice filters) of $\mathbf{P}$ together with the empty-set precisely when $\mathbf{P}$ is not lower-bounded (resp. upper-bounded). By the previous remark, $\mathsf{Id}_{\diamond}(\mathbf{P})$ and $\mathsf{Fl}_{\diamond}(\mathbf{P})$ form closed systems over $\mathsf{uni}(\mathbf{P})$. The associated closure operators are denoted by $\|\cdot\|^{\mathbf{P}}_{\mathsf{id}_{\diamond}}$ and $\|\cdot\|^{\mathbf{P}}_{\mathsf{fi}_{\diamond}}$, respectively, and the associated complete lattices are denoted by

$\mathbf{Id}_\diamond(\mathbf{P})$ and $\mathbf{Fl}_\diamond(\mathbf{P})$, respectively. Let $\vdash^{\mathbf{P}}_{\mathsf{id}_\diamond}$ and $\vdash^{\mathbf{P}}_{\mathsf{fi}_\diamond}$ denote the point-consequence relations determined by the closed system $\mathsf{Id}_\diamond(\mathbf{P})$ and $\mathsf{Fl}_\diamond(\mathbf{P})$, respectively, and let $\dashv\vdash^{\mathbf{P}}_{\mathsf{id}_\diamond}$ and $\dashv\vdash^{\mathbf{P}}_{\mathsf{fi}_\diamond}$ denote the associated closed-equivalence relations, respectively. $\square$

Remark 4.91 [RMT87, 48] For a lattice $\mathbf{P}$, the closed systems $\mathsf{Id}_\diamond(\mathbf{P})$ and $\mathsf{Fl}_\diamond(\mathbf{P})$ are algebraic.

Remark 4.92 For a *finite* subset $\emptyset \neq A \subseteq_{\mathsf{f}} \mathsf{uni}(\mathbf{P})$, $\|A\|^{\mathbf{P}}_{\mathsf{id}_\diamond} = \langle \blacktriangledown^{\mathbf{P}} A]_{\mathbf{P}}$ and $\|A\|^{\mathbf{P}}_{\mathsf{fi}_\diamond} = [\blacktriangle^{\mathbf{P}} A\rangle_{\mathbf{P}}$.

Proof. (*We prove the first assertion, the second being dual.*) Certainly $\langle \blacktriangledown^{\mathbf{P}} A]_{\mathbf{P}}$ is an ideal (in fact a principal ideal), and one which contains A, since for any $a \in A$, $a \leq \blacktriangledown^{\mathbf{P}} A \in \langle \blacktriangledown^{\mathbf{P}} A]_{\mathbf{P}}$, and hence $a \in \langle \blacktriangledown^{\mathbf{P}} A]_{\mathbf{P}}$. Hence $\|A\|^{\mathbf{P}}_{\mathsf{id}_\diamond} \subseteq \langle \blacktriangledown^{\mathbf{P}} A]_{\mathbf{P}}$, by minimality of $\|A\|^{\mathbf{P}}_{\mathsf{id}_\diamond}$. Conversely, $\langle \blacktriangledown^{\mathbf{P}} A]_{\mathbf{P}} \subseteq \|A\|^{\mathbf{P}}_{\mathsf{id}_\diamond}$, since for any $a \in \langle \blacktriangledown^{\mathbf{P}} A]_{\mathbf{P}}$, $a \leq \blacktriangledown^{\mathbf{P}} A \in \|A\|^{\mathbf{P}}_{\mathsf{id}_\diamond}$, and hence $a \in \|A\|^{\mathbf{P}}_{\mathsf{id}_\diamond}$. $\diamond$

Remark 4.93 Let $A \cup \{a, b\} \subseteq \mathsf{uni}(\mathbf{P})$ with $A \neq \emptyset$.

1. $A \vdash^{\mathbf{P}}_{\mathsf{id}_\diamond} a$ iff $\exists\, [A' \subseteq_{\mathsf{f}} A]\; a \leq \blacktriangledown^{\mathbf{P}} A'$.
2. $\{a\} \vdash^{\mathbf{P}}_{\mathsf{id}_\diamond} b$ iff $b \leq a$.
3. $\{a\} \dashv\vdash^{\mathbf{P}}_{\mathsf{id}_\diamond} \{b\}$ iff $a = b$.
4. $\vdash^{\mathbf{P}}_{\mathsf{id}_\diamond} a$ iff $\mathbf{P}$ is lower-bounded with lower-bound a.

Proof. (*We prove (1).*)
$\boxed{\Rightarrow}$ Suppose that $A \vdash^{\mathbf{P}}_{\mathsf{id}_\diamond} a$. By algebraicity, there exists finite $A' \subseteq A$ with $A' \vdash^{\mathbf{P}}_{\mathsf{id}_\diamond} a$. So $a \in \|A'\|^{\mathbf{P}}_{\mathsf{id}_\diamond} = \langle \blacktriangledown^{\mathbf{P}} A']_{\mathbf{P}}$, by Remark 4.92. Hence $a \leq \blacktriangledown^{\mathbf{P}} A'$. $\boxed{\Leftarrow}$ Suppose $\exists\, [A' \subseteq_{\mathsf{f}} A]\; a \leq \blacktriangledown^{\mathbf{P}} A'$. Then $a \in \langle \blacktriangledown^{\mathbf{P}} A']_{\mathbf{P}} = \|A'\|^{\mathbf{P}}_{\mathsf{id}_\diamond}$, by Remark 4.92. Hence $A' \vdash^{\mathbf{P}}_{\mathsf{id}_\diamond} a$, and so by (4.80), $A \vdash^{\mathbf{P}}_{\mathsf{id}_\diamond} a$. $\diamond$

Remark 4.94 Let $A \cup \{a, b\} \subseteq \mathsf{uni}(\mathbf{P})$ with $A \neq \emptyset$.

1. $A \vdash^{\mathbf{P}}_{\mathsf{fi}_\diamond} a$ iff $\exists\, [A' \subseteq_{\mathsf{f}} A]\; a \geq \blacktriangle^{\mathbf{P}} A'$.
2. $\{a\} \vdash^{\mathbf{P}}_{\mathsf{fi}_\diamond} b$ iff $b \geq a$.
3. $\{a\} \dashv\vdash^{\mathbf{P}}_{\mathsf{fi}_\diamond} \{b\}$ iff $a = b$.
4. $\vdash^{\mathbf{P}}_{\mathsf{fi}_\diamond} a$ iff $\mathbf{P}$ is upper-bounded with upper-bound a.

$\square$

We shall have occasion, to require that the empty-set be an ideal or filter, independently of the bounded nature of a particular lattice.

Remark 4.95 Adding the empty-set to a closed system $\mathbb{C}$ yields a closed system $\mathbb{C}'$, which is algebraic iff $\mathbb{C}$ is algebraic.

Definition 4.96 ($\mathsf{Id}_{\diamond_\emptyset}(\mathbf{P})$ and $\mathsf{Fl}_{\diamond_\emptyset}(\mathbf{P})$) Let $\mathbf{P}$ be a lattice. Let $\mathsf{Id}_{\diamond_\emptyset}(\mathbf{P}) = \mathsf{Id}_\diamond(\mathbf{P}) \cup \emptyset$ and $\mathsf{Fl}_{\diamond_\emptyset}(\mathbf{P}) = \mathsf{Fl}_\diamond(\mathbf{P}) \cup \emptyset$. By the previous remark, $\mathsf{Id}_{\diamond_\emptyset}(\mathbf{P})$ and $\mathsf{Fl}_{\diamond_\emptyset}(\mathbf{P})$ form algebraic closed systems over $\mathsf{uni}(\mathbf{P})$. The associated algebraic closure operators are denoted by $\|\cdot\|^{\mathbf{P}}_{\mathsf{id}_{\diamond_\emptyset}}$ and $\|\cdot\|^{\mathbf{P}}_{\mathsf{fi}_{\diamond_\emptyset}}$, respectively, and the associated complete lattices are denoted by $\mathbf{Id}_{\diamond_\emptyset}(\mathbf{P})$ and $\mathbf{Fl}_{\diamond_\emptyset}(\mathbf{P})$, respectively. Let $\vdash^{\mathbf{P}}_{\mathsf{id}_{\diamond_\emptyset}}$ and $\vdash^{\mathbf{P}}_{\mathsf{fi}_{\diamond_\emptyset}}$ denote the point-consequence relations determined by the closed system $\mathsf{Id}_{\diamond_\emptyset}(\mathbf{P})$ and $\mathsf{Fl}_{\diamond_\emptyset}(\mathbf{P})$, respectively, and let $\dashv\vdash^{\mathbf{P}}_{\mathsf{id}_{\diamond_\emptyset}}$ and $\dashv\vdash^{\mathbf{P}}_{\mathsf{fi}_{\diamond_\emptyset}}$ denote the associated closed-equivalence relations, respectively. $\square$

Remark 4.97 Remarks 4.93 and 4.94 remain valid when $\vdash^{\mathbf{P}}_{\mathsf{id}_\diamond}$, $\vdash^{\mathbf{P}}_{\mathsf{fi}_\diamond}$, $\dashv\vdash^{\mathbf{P}}_{\mathsf{id}_\diamond}$ and $\dashv\vdash^{\mathbf{P}}_{\mathsf{fi}_\diamond}$, are replaced by $\vdash^{\mathbf{P}}_{\mathsf{id}_{\diamond_\emptyset}}$, $\vdash^{\mathbf{P}}_{\mathsf{fi}_{\diamond_\emptyset}}$, $\dashv\vdash^{\mathbf{P}}_{\mathsf{id}_{\diamond_\emptyset}}$ and $\dashv\vdash^{\mathbf{P}}_{\mathsf{fi}_{\diamond_\emptyset}}$, respectively, *except* that cases (4) *never* occur.

□

Example 4.98 (Subuniverses of Groups)

The set of subuniverses of a group form an algebraic closed system over the universe of that group. This closed system is not grounded, as every subuniverse of a group will contain the identity, i.e., $\emptyset$ is not a subuniverse of a group. Nor is this closed system generally context-free. For example, the **Klein 4-group** has three non-trivial subuniverses, and the union of any two of these is not a subuniverse [Fra94, Figure 1.4].

□

Example 4.99 (Equivalential Closed Systems)

Let X be a set.

Remark 4.100 The set $\mathsf{ER}(X)$ of all equivalence relations on X forms an algebraic closed system over X^2.

Remark 4.101 $\Bbbk_{\mathsf{ER}(X)} = =_X$.

Definition 4.102 (Equivalential Closed Systems) Any closed system with universe X^2 that is courser than $\mathsf{ER}(X)$ is called **equivalential** on/over X. □

□

The most important examples of closed systems in this text are the closed systems consisting of the equivalence classes of the equivalence relations of equivalential closed systems. Since the equivalence classes of an equivalence relation form a partition, the intersection of any two distinct equivalence classes of the same equivalence relation is always empty. As a consequence, we shall (generally) need to add the empty set as an additional 'equivalence class'.

Example 4.103 (Cosets of Equivalential Closed Systems)

Let $\mathbb{E}$ be an equivalential closed system on non-empty X.

Definition 4.104 (Proper Cosets) Let $\mathsf{PrpCos}(\mathbb{E}) = \{\alpha[\![a]\!] : a \in X, \alpha \in \mathsf{cl}_{\mathbb{E}}\}$, the members of which are called **proper cosets** of $\mathbb{E}$. □

The following result demonstrates that the *only* reason why the proper cosets of an equivalential closed system may fail to form a closed system is that the empty-set is not a proper-coset. Consequently, by adding the empty-set to the proper cosets a closed system is obtained.

Lemma 4.105 If $\emptyset \neq \{\alpha_i[\![a_i]\!] : i \in I\} \subseteq \mathsf{PrpCos}(\mathbb{E})$ and $a \in \bigcap_{i\in I} \alpha_i[\![a_i]\!]$, then

$$\bigcap_{i\in I}(\alpha_i[\![a]\!]) = (\bigcap_{i\in I} \alpha_i)[\![a]\!]. \tag{4.89}$$

Consequently, $\{\emptyset\} \cup \mathsf{PrpCos}(\mathbb{E})$ is a closed system.

Proof. Let $\mathcal{A} = \{\emptyset\} \cup \{\alpha \,/a \;:\; a \in X, \alpha \in \mathsf{cl}_{\mathbb{E}}\}$. (*We shall prove that $\mathcal{A}$ is a closed system; the outstanding assertions are implicit in the proof.*) $\boxed{\text{Top}}$ Since $\blacksquare_X \in \mathbb{E}$, $X = \blacksquare_X[\![a]\!] \in \mathcal{A}$, where a is any point in X. $\boxed{\bigcap}$ Let $\emptyset \neq \mathcal{B} \subseteq \mathcal{A}$. If $\bigcap \mathcal{B} = \emptyset$, then $\bigcap \mathcal{B} \in \mathcal{A}$. Suppose $\bigcap \mathcal{B} \neq \emptyset$. Then $\emptyset \notin \mathcal{B}$. Let $a \in \bigcap \mathcal{B}$. Then $\mathcal{B} = \{\alpha_i[\![a]\!] \;:\; i \in I\}$, for some I and $\alpha_i \in \mathsf{cl}_{\mathbb{E}}$. Define a binary relationship r from X^2 to X by $\langle c, d\rangle \; r \; e$ iff $c = a$ and $d = e$. Note that this relationship is expanding and that $r\,[\alpha] = \alpha[\![a]\!]$ for any equivalence relation α on X. Then $\bigcap\{\alpha_i[\![a]\!] \;:\; i \in I\} = \bigcap\{r\,[\alpha_i] \;:\; i \in I\} = r\left[\bigcap_{i\in I} \alpha_i\right] = (\bigcap_{i\in I} \alpha_i)[\![a]\!] \in \mathcal{A}$, where the second equality follows by (1.13) of Table 1.1 on page 18, and the (final) membership follows since $\bigcap_{i\in I} \alpha_i \in \mathsf{cl}_{\mathbb{E}}$ since $\mathbb{E}$ is a closed system. ◇

Note that the only case in which $\bigcap \mathsf{PrpCos}(\mathbb{E}) \neq \emptyset$ occurs when $\mathsf{cl}_{\mathbb{E}} = \{\blacksquare_X\}$, i.e., $\mathbb{E}$ is trivial; in this case $\mathsf{PrpCos}(\mathbb{E}) = \{X\}$ in which case $\mathsf{PrpCos}(\mathbb{E})$ is a trivial closed system.

Definition 4.106 (Closed Systems of Cosets) Let $\mathsf{Cos}(\mathbb{E})$ (resp. $\mathsf{Cos}_{\emptyset}(\mathbb{E})$) denote the closed system with universe X and whose closed sets are the proper cosets of $\mathbb{E}$ together with the empty-set (called the **improper coset**) precisely when $\mathbb{E}$ is non-trivial (resp. aways), which we call the **coset closed system** (resp. **forced coset closed system**) of $\mathbb{E}$. We conflate these closed systems with their closed sets and denote the complete lattices $\mathbf{cl}_{\mathsf{Cos}(\mathbb{E})}$ and $\mathbf{cl}_{\mathsf{Cos}_{\emptyset}(\mathbb{E})}$ by $\mathbf{Cos}(\mathbb{E})$ and $\mathbf{Cos}_{\emptyset}(\mathbb{E})$, respectively. □

Remark 4.107 $\mathsf{Cos}(\mathbb{E})$ is unconstrained iff $\mathbb{E}$ is non-trivial. $\mathsf{Cos}(\mathbb{E})$ is trivial iff $\mathbb{E}$ is trivial.

Remark 4.108 $\mathsf{Cos}_{\emptyset}(\mathbb{E})$ is unconstrained. $\mathsf{Cos}_{\emptyset}(\mathbb{E})$ is almost-trivial iff $\mathbb{E}$ is trivial. □

The following characterization of (non-empty) coset generation in terms of $\mathbb{E}$-closure follows easily from the fact that the $\mathbb{E}$-closed sets are equivalence relations.

Remark 4.109 For $A \neq \emptyset$, $\|A\|_{\mathsf{Cos}(\mathbb{E})} = \|A\|_{\mathsf{Cos}_{\emptyset}(\mathbb{E})} = (\left\| A^2 \right\|_{\mathbb{E}})\,[A] = (\left\| A^2 \right\|_{\mathbb{E}})[\![a]\!]$, for any $a \in A$. □

The following characterization of $\mathsf{Cos}(\mathbb{E})$-consequence proves important in the sequel.

Proposition 4.110 For non-trivial $\mathbb{E}$, $A \vdash_{\mathsf{Cos}(\mathbb{E})} a$ iff $A \neq \emptyset$ and $\forall[\alpha \in \mathsf{cl}_{\mathbb{E}}]\forall[b \in X]\, A \times \{b\} \subseteq \alpha \to a\,\alpha\,b$. For trivial $\mathbb{E}$, $A \vdash_{\mathsf{Cos}(\mathbb{E})} a$, for all $A \cup \{a\} \subseteq X$.

Proof. (*We prove the first assertion; the second is trivial.*) $\boxed{\Rightarrow}$ Suppose that $A \vdash_{\mathsf{Cos}(\mathbb{E})} a$. Then $A \neq \emptyset$ by Remark 4.107. Let $a' \in A$. Then by Remark 4.109, $\langle a, a'\rangle \in \left\| A^2 \right\|_{\mathbb{E}}$. Let $\alpha \in \mathsf{cl}_{\mathbb{E}}$ and $b \in X$ with $A \times \{b\} \subseteq \alpha$. Since α is an equivalence relation, $A^2 \subseteq \alpha$. Hence $\left\| A^2 \right\|_{\mathbb{E}} \subseteq \alpha$. So $a\,\alpha\,a'\,\alpha\,b$, and hence by transitivity, $a\,\alpha\,b$. $\boxed{\Leftarrow}$ Assume that $A \neq \emptyset$ and $\forall\,[\alpha \in \mathsf{cl}_{\mathbb{E}}]\;\forall\,[b \in X]\; A \times \{b\} \subseteq \alpha \to a\,\alpha\,b$. Consider $\alpha = \left\| A^2 \right\|_{\mathbb{E}}$ and $b \in A$. Certainly, $A \times \{b\} \subseteq \alpha$ and so by assumption, $a\,\alpha\,b$. Hence $a \in \alpha\,[A]$, and so $a \in \|A\|_{\mathsf{Cos}(\mathbb{E})}$ by Remark 4.109. ◇

Corollary 4.111 $A \vdash_{\mathsf{Cos}_{\emptyset}(\mathbb{E})} a$ iff $A \neq \emptyset$ and $\forall\,[\alpha \in \mathsf{cl}_{\mathbb{E}}]\;\forall\,[b \in X]\; A \times \{b\} \subseteq \alpha \to a\,\alpha\,b$. □

The algebraicity of an equivalential closed system is reflected in its coset closed systems.

Proposition 4.112 If $\mathbb{E}$ is algebraic then $\mathsf{Cos}(\mathbb{E})$ and $\mathsf{Cos}_{\emptyset}(\mathbb{E})$ are both algebraic.

Proof. (*It suffices to prove that* $\mathsf{Cos}(\mathbb{E})$ *is algebraic, since adding the empty-set to an algebraic closed system yields an algebraic closed system.*) If $\mathbb{E}$ is trivial then $\mathsf{Cos}(\mathbb{E})$ is trivial and hence algebraic. Assume that $\mathbb{E}$ is non-trivial and algebraic. Let $\mathcal{B}$ be a non-empty $\subseteq$-directed subset of $\mathsf{cl}_{\mathbb{E}}$. If $\mathcal{B} = \{\emptyset\}$ then $\bigcup \mathcal{B} = \emptyset \in \mathsf{Cos}(\mathbb{E})$. We may assume without loss of generality that $\emptyset \notin \mathcal{B}$. So $\mathcal{B} = \{\alpha_i[\![a]\!] \,:\, i \in I\}$, for some I and $\alpha_i \in \mathsf{cl}_{\mathbb{E}}$. For each $i \in I$, let $\alpha'_i = \left\| (\alpha_i[\![a]\!])^2 \right\|_{\mathbb{E}}$. Since $(\alpha_i[\![a]\!])^2 \subseteq \alpha_i$, $\alpha'_i = \left\| (\alpha_i[\![a]\!])^2 \right\|_{\mathbb{E}} \subseteq \|\alpha_i\|_{\mathbb{E}} = \alpha_i$, hence $\alpha_i[\![a]\!] \subseteq \alpha'_i[\![a]\!] \subseteq \alpha_i[\![a]\!]$ and so $\alpha'_i[\![a]\!] = \alpha_i[\![a]\!]$. Let r be the binary relationship defined in the proof of Lemma 4.105. Then $\bigcup\{\alpha_i[\![a]\!] \,:\, i \in I\} = \bigcup\{\alpha'_i[\![a]\!] \,:\, i \in I\} = \bigcup\{r\,[\alpha'_i] \,:\, i \in I\} = r\left[\bigcup_{i\in I} \alpha'_i\right] = (\bigcup_{i\in I} \alpha'_i)[\![a]\!]$, by (1.9) of Table 1.1. (*It suffices to show that* $\{\alpha'_i \,:\, i \in I\}$ *is* $\subseteq$*-directed.*) Let $i, j \in I$. (*We must show that there exists* $k \in I$ *with* $\alpha'_i \cup \alpha'_j \subseteq \alpha'_k$.) Since $\mathcal{B}$ is directed, there exists $k \in I$ with $\alpha_i[\![a]\!] \cup \alpha_j[\![a]\!] \subseteq \alpha_k[\![a]\!]$. So $\alpha'_i \cup \alpha'_j \subseteq \alpha'_k$. $\diamond$

$\square$

In this text, we are particularly interested in the closed system $\mathsf{Cos}(\mathsf{Con}(\mathbf{A}))$ of all cosets of congruences on an algebra $\mathbf{A}$, and the closed system $\mathsf{Cos}(\mathsf{Con}^{\mathcal{K}}(\mathbf{A}))$ of all cosets of (relative) $\mathcal{K}$-congruences on $\mathbf{A}$, where $\mathcal{K}$ is a quasivariety; since the congruences and relative congruences (with respect to a quasivariety) form finitary closed systems (see Remark 1.353 on page 67 and Proposition 1.450 on page 87), $\mathsf{Cos}(\mathsf{Con}(\mathbf{A}))$ and $\mathsf{Cos}(\mathsf{Con}^{\mathcal{K}}(\mathbf{A}))$ are finitary closed systems, by Proposition 4.112.

Example 4.113 (Closed Systems of Cosets of Congruences)

We begin by introducing the closed system of (non-relative) cosets of an algebra. In the case of (non-relative) cosets we shall always force the improper-coset to be a coset. The reason for this is that in the next chapter we shall see that the cosets on the term algebra (with ω variables V) constitute the theories of a sentential 1-calculus. Since the closed system of congruences on this term algebra is never trivial, this sentential calculus has no theories; consequently the empty-set is always a filter of this sentential calculus on any algebra. Since we aim to show that the cosets on an algebra coincide with these filters, we must ensure that the empty-set is always a coset.

Definition 4.114 (Closed Systems of Cosets of Congruences) Let $\mathbf{A}$ be an algebra of type $\mathfrak{a}$. We denote the finitary closed system $\mathsf{Cos}_{\emptyset}(\mathsf{Con}(\mathbf{A}))$ of all cosets of congruences on an algebra $\mathbf{A}$, by $\mathsf{Cos}(\mathbf{A})$. We identify $\mathsf{Cos}(\mathbf{A})$ with $\mathsf{cl}_{\mathsf{Cos}(\mathbf{A})}$, the members of which are called **cosets of A**, and we write $\|\cdot\|^{\mathbf{A}}_{\mathsf{cos}}$ for $\|\cdot\|_{\mathsf{Cos}(\mathbf{A})}$. The algebraic lattice $\mathbf{cl}_{\mathsf{Cos}(\mathbf{A})}$ is denoted by $\mathbf{Cos}(\mathbf{A})$. $\square$

Note that by definition $\mathsf{Cos}(\mathbf{A})$ is always unconstrained. Hence if $A \vdash_{\mathsf{Cos}(\mathbf{A})} a$ then $A \neq \emptyset$. The following characterization of *non-empty* closure follows from Remark 4.109 of Example 4.103. Of course $\|\emptyset\|^{\mathbf{A}}_{\mathsf{cos}} = \emptyset$.

Remark 4.115 $\|B\|^{\mathbf{A}}_{\mathsf{cos}} = (\left\| B^2 \right\|_{\Theta_{\mathbf{A}}})\,[B] = (\left\| B^2 \right\|_{\Theta_{\mathbf{A}}})[\![b]\!]$, for $\emptyset \neq B \subseteq \mathsf{uni}(\mathbf{A})$ and $b \in B$. $\square$

The following characterization of $\mathsf{Cos}(\mathbf{A})$-consequence follows at once from Proposition 4.110 of Example 4.103.

Corollary 4.116 $A \vdash_{\mathsf{Cos}(\mathbf{A})} a$ iff $A \neq \emptyset$ and $\forall[\alpha \in \mathsf{Con}(\mathbf{A})]\forall[b \in \mathsf{uni}(\mathbf{A})]\, A \times \{b\} \subseteq \alpha \rightarrow a\,\alpha\,b$. $\square$

We now turn to the cosets of *relative* congruences. In this case we need to be more subtle. Consider a quasivariety $\mathcal{K}$. In the next chapter we shall associate a sentential 1-calculus with $\mathcal{K}$, called the *membership logic*, which has as its theories all the cosets of (relative) $\mathcal{K}$-congruences on the term algebra (over ω-variables V). There is only one case in which the closed system of $\mathcal{K}$-congruences on this term algebra is trivial, namely when $\mathcal{K}$ is trivial, i.e., $\models_{\mathcal{K}} p \approx q$ for all terms p and q. In all other cases, there are at least two relative congruences on this term algebra. In the latter case, when $\mathcal{K}$ is non-trivial, the improper coset must be a coset of the closed system of relative congruences of the term algebra, and so the membership logic will have no theorems; hence the *empty-set will always be a filter* of the membership logic on any algebra. We aim to establish that when $\mathcal{K}$ is relatively congruence regular, the relative cosets on an algebra coincide with the filters of the membership logic on that algebra; consequently, we need to ensure that the improper-coset is a relative coset on all algebras, even if the closed system of relative congruences on an algebra is trivial. In the case where $\mathcal{K}$ is trivial, for any algebra $\mathbf{A}$, the closed system of relative congruences on $\mathbf{A}$ is *trivial*. We *could* take the approach of forcing the empty-set to always be a relative coset in this case, thereby ensuring a uniform definition of the closed systems of relative cosets. We do *not*, however, take this approach, since we wish the membership logic to reflect, as strongly as possible, the quasi-equational theory of $\mathcal{K}$, and so would like the triviality of $\mathcal{K}$ to be reflected in the membership logic. For this reason, when $\mathcal{K}$ is trivial, we define the relative coset closed systems to be trivial. The benefits of adopting this approach will become apparent when we obtain characterizations of the membership logic in terms of the quasi-equational theory of $\mathcal{K}$. This approach also ensure compatibility with generalizations of the membership logic that we shall also be introducing.

Definition 4.117 (Closed Systems of Relative Cosets of Congruences) Let $\mathbf{A}$ be an algebra of type $\mathfrak{a}$ and $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras. Let

$$\mathsf{Cos}^{\mathcal{K}}(\mathbf{A}) = \begin{cases} \mathsf{Cos}_{\emptyset}(\mathsf{Con}^{\mathcal{K}}(\mathbf{A})) & ; \quad \text{if } \mathcal{K} \text{ is non-trivial,} \\ \mathsf{Cos}(\mathsf{Con}^{\mathcal{K}}(\mathbf{A})) & ; \quad \text{otherwise.} \end{cases}$$

Note that by Proposition 4.112, the closed system $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$ is algebraic. We identify $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$ with $\mathsf{cl}_{\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})}$, the members of which are called **$\mathcal{K}$-cosets of $\mathbf{A}$**, and we write $\|\cdot\|^{\mathbf{A}}_{\mathsf{cos}^{\mathcal{K}}}$ for $\|\cdot\|_{\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})}$. The algebraic lattice $\mathbf{cl}_{\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})}$ is denoted by $\mathbf{Cos}^{\mathcal{K}}(\mathbf{A})$. $\square$

By definition, $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$ is unconstrained iff $\mathcal{K}$ is non-trivial, and $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$ is trivial iff $\mathcal{K}$ is trivial. The following characterization of relative coset generation follows from definition and Remark 4.109 of Example 4.103.

Remark 4.118 $\|B\|^{\mathbf{A}}_{\mathsf{cos}^{\mathcal{K}}} = (\left\| B^2 \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}})\,[B] = (\left\| B^2 \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}})[\![b]\!]$, for $\emptyset \neq B \subseteq \mathsf{uni}(\mathbf{A})$ and $b \in B$. If $\mathcal{K}$ is non-trivial, then $\|\emptyset\|^{\mathbf{A}}_{\mathsf{cos}^{\mathcal{K}}} = \emptyset$, otherwise $\|\emptyset\|^{\mathbf{A}}_{\mathsf{cos}^{\mathcal{K}}} = \mathsf{uni}(\mathbf{A})$. $\square$

Proposition 4.110 of Example 4.103 yields the following characterization of $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$-consequence.

Corollary 4.119 If $\mathcal{K}$ is non-trivial, then $A \vdash_{\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})} a$ iff $A \neq \emptyset$ and $\forall\,[\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})]\,\forall\,[b \in \mathsf{uni}(\mathbf{A})]\; A \times \{b\} \subseteq \alpha \rightarrow a\,\alpha\,b$. If $\mathcal{K}$ is trivial, then $A \vdash_{\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})} a$ for all $A \cup \{a\} \subseteq \mathsf{uni}(\mathbf{A})$.

$\square$

4.3.2 Formal Systems

Algebraic closed systems may be characterized as *formal-systems*, which are logic-like entities supporting *axioms*, *rules* and *derivations*. This notion of a formal system has been derived from the notion with the same name presented in [Cur76].

Definition 4.120 (Formal Systems) A **formal system** $\mathfrak{F}$, is determined by the following three parameters.

1. A set $\mathsf{lg}(\mathfrak{F})$ called the **language of** $\mathfrak{F}$, the elements of which are called **$\mathfrak{F}$-formula**.

2. A set $\mathsf{FAx}(\mathfrak{F})$ of $\mathfrak{F}$-formulae, called the **axioms**.

3. A set $\mathsf{FRl}(\mathfrak{F})$ of *rules*, where each *rule* Λ in $\mathsf{FRl}(\mathfrak{F})$ is determined by its **premise** $\mathsf{prem}(\Lambda)$, which is a finite set of formulae, and its conclusion $\mathsf{conc}(\Lambda)$, which is a formula.

With each formal system $\mathfrak{F}$, we associate a binary relationship $\vdash_{\mathfrak{F}}$ between *sets of* $\mathfrak{F}$-formulae and (*single*) $\mathfrak{F}$-formulae, called the **formal consequence relation**, defined by $\Gamma \vdash_{\mathfrak{F}} \eta$ iff there exists a derivation of η from Γ (modulo $\mathfrak{F}$), where a **derivation of η from Γ (modulo $\mathfrak{F}$)** is a non-empty finite sequence $\zeta_0, \ldots, \zeta_{n-1}$, such that $\zeta_{n-1} = \eta$ and, for each $i \in n$,

1. $\zeta_i \in \mathsf{FAx}(\mathfrak{F}) \cup \Gamma$, or

2. there exists a rule $\Lambda \in \mathsf{FRl}(\mathfrak{F})$ with $\mathsf{prem}(\Lambda) \subseteq \{\zeta_0, \ldots, \zeta_{i-1}\}$ and $\mathsf{conc}(\Lambda) = \zeta_i$.

We write $\vdash$ for $\vdash_{\mathfrak{F}}$ wherever context unambiguous. We call formal systems $\mathfrak{F}_1$ and $\mathfrak{F}_2$ *equivalent* if they have the same language and the same formal consequence relation. $\square$

Convention 4.121 (Specifying Rules and Axioms) When we call $\boldsymbol{\vdash} \phi$ an axiom, we mean the axiom with conclusion ϕ, and when we call $\phi_1, \ldots, \phi_n \boldsymbol{\vdash} \phi$ a rule, we mean the rule with premise $\{\phi_1, \ldots, \phi_n\}$ and conclusion ϕ. Note that the symbol '$\boldsymbol{\vdash}$ ' is an emboldened version of the symbol '$\vdash$'; this is so as to avoid potential ambiguity.

Unlike the sentential calculi, formal systems have no notion of 'variables' and hence no notion of 'closure under substitution' (also known as 'structurality'). For this reason, it is useful to introduce a mechanism for describing axioms and rules by means of schema. Note that the notion of a schema is merely 'syntactic sugar', and has no intrinsic meaning, and no attempt is made to formalize this notion. It is analogous to the axiom schema of set-theory.

Definition 4.122 (Schema) When we describe an axiom or rule by prefixing the notions of the previous conventions with a emboldened universal quantifier (with a list of emboldened 'variable' symbols) and mention these 'variable' symbols in the axioms, we are describing a set of axioms or rules, obtained by successively replacing the 'variable' symbols by all possible formulae of the language (for a use-case, see Example 4.153 on page 170). $\square$

The following technicalities are aimed at establishing a one-to-one correspondence between formal systems and algebraic (point) consequence relations, after which we shall conflate formal systems with their (unique) associated algebraic point-consequence relations, thereby inheriting the notations and theory of algebraic closure.

Proposition 4.123 If $\mathfrak{F}$ is a formal system then $\vdash_{\mathfrak{F}}$ is an algebraic (point) consequence relation over $\mathsf{Fm}(\mathfrak{F})$.

Proof. $\boxed{\text{(4.79)}}$ Follows immediately from (1) of Definition 4.120. $\boxed{\text{(4.80)}}$ Follows since, if $\Phi \subseteq \Gamma$ then any derivation of η from Φ is also a derivation of η from Γ. $\boxed{\text{(4.81)}}$ We must show that if $\Gamma \vdash \zeta$, for all $\zeta \in \Phi$, then $\Gamma \vdash \xi$. It suffices to prove that for any derivation $\langle \eta_1, \ldots, \eta_n \rangle$ of η_n from Φ, $\Gamma \vdash \eta_n$. We proceed by induction on the length of derivations from Φ. $\boxed{\text{Base Case}}$ Let $\langle \xi \rangle$ be a derivation of ξ from Φ of length one. Then either ξ is an axiom, in which case $\Gamma \vdash \xi$, otherwise, $\xi \in \Phi$, in which case $\Gamma \vdash \xi$ by assumption. $\boxed{\text{Induction Hypothesis}}$ Assume that for any derivation $\langle \eta_1, \ldots, \eta_n \rangle$ of η_n from Φ, of length $n < m$, $\Gamma \vdash \eta_n$. $\boxed{\text{Inductive Step}}$ Let $\langle \eta_1, \ldots, \eta_m \rangle$ be a derivation of η_m from Φ. By the induction hypothesis, $\Gamma \vdash \{\eta_1, \ldots, \eta_{m-1}\}$. If η_m is an axiom or in Φ, then $\langle \eta_m \rangle$ is a derivation of η_m from Φ of length one, and the result follows from the base case. Otherwise, there exists a rule $\xi_1, \ldots, \xi_{r-1} \Vdash \xi_r$, with $\{\xi_1, \ldots, \xi_{r-1}\} \subseteq \{\eta_1, \ldots, \eta_{m-1}\}$ and $\xi_r = \eta_m$. So, for each $i = 1 \ldots r-1$, $\Gamma \vdash \xi_i$; let $\langle \zeta^i_1, \ldots, \zeta^i_{j(i)} \rangle$ be a derivation of ξ_i from Γ. Then $\langle \zeta^1_1, \ldots, \zeta^1_{j(1)}, \zeta^2_1, \ldots, \zeta^2_{j(2)}, \ldots, \zeta^{r-1}_1, \ldots, \zeta^{r-1}_{j(r-1)}, \xi_r \rangle$ is a derivation of η_m from Γ. $\boxed{\text{(4.88)}}$ It suffices to prove that for any derivation $\langle \eta_1, \ldots, \eta_n \rangle$ of η_n from Γ, there exists a finite subset Γ' of Γ, such that $\Gamma' \vdash \eta_n$. We proceed by induction on the length of such derivations. $\boxed{\text{Base Case}}$ Let $\langle \eta_1 \rangle$ be a derivation of η_1 from Γ of length one. If η_1 is an axiom, then $\emptyset \vdash \eta_1$, and $\emptyset$ is a finite subset of Γ. If $\eta_1 \in \Gamma$, then $\{\eta_1\} \vdash \eta_1$, and $\{\eta_1\}$ is a finite subset of Γ. $\boxed{\text{Induction Hypothesis}}$ Assume that for any derivation $\langle \eta_1, \ldots, \eta_n \rangle$ of η_n from Γ, of length $n < m$, there exists a finite subset Γ' of Γ, such that $\Gamma' \vdash \eta_n$. $\boxed{\text{Inductive Proof}}$ Let $\langle \eta_1, \ldots, \eta_m \rangle$ be a derivation of η_m from Γ. If η_m is an axiom or in Γ, then $\langle \eta_m \rangle$ is a derivation of η_m from Γ of length one, and the result follows from the base case. Otherwise, there exists a rule $\xi_1, \ldots, \xi_{r-1} \Vdash \xi_r$ with $\{\xi_1, \ldots, \xi_{r-1}\} \subseteq \{\eta_1, \ldots, \eta_{m-1}\}$ and $\xi_r = \eta_m$. Now, for each $i = 1 \ldots r-1$, there exists a derivation of ξ_i from Γ of length strictly less than m, and so by the induction hypothesis, there exists a finite subset Γ'_i of Γ, such that $\Gamma'_i \vdash \xi_i$; let $\langle \zeta^i_1, \ldots, \zeta^i_{j(i)} \rangle$ be a derivation of ξ_i from Γ'_i. Then $\langle \zeta^1_1, \ldots, \zeta^1_{j(1)}, \zeta^2_1, \ldots, \zeta^2_{j(2)}, \ldots, \zeta^{r-1}_1, \ldots, \zeta^{r-1}_{j(r-1)}, \xi_r \rangle$ is a derivation of η_m from $\Gamma'_1 \cup \ldots \cup \Gamma'_{r-1}$, which is a finite subset of Γ. $\diamond$

Definition 4.124 (Approximating Power-Point Relations with Formal Systems) With each set A and binary relationship $\rhd$ from $\mathfrak{P}(A)$ to A, we associate the formal theory $F(\rhd, \mathsf{aprx})$, with language A, with all axioms $\Vdash \eta$ where $\emptyset \rhd \eta$, and all rules $\eta_1, \ldots, \eta_n \Vdash \eta$ where $\{\eta_1, \ldots, \eta_n\} \rhd \eta$. $\square$

Proposition 4.125

1. If $\vdash$ is an algebraic (point) consequence relation over A, then $\vdash_{F(\vdash, \mathsf{aprx})} = \vdash$.

2. If $\mathfrak{F}$ is a formal system, then $\vdash_{F(\vdash_{\mathfrak{F}}, \mathsf{aprx})} = \vdash_{\mathfrak{F}}$.

Proof. It suffices to prove only (1), since (2) follows immediately from Proposition 4.123 and (1). $\boxed{\vdash \subseteq \vdash_{F(\vdash, \mathsf{aprx})}}$ Suppose that $\Gamma \vdash \eta$. Then, by assumption (4.88), $\{\eta_1, \ldots, \eta_n\} \vdash \eta$, for some finite subset $\{\eta_1, \ldots, \eta_n\}$ of Γ. If $\{\eta_1, \ldots, \eta_n\} = \emptyset$, then, by definition, η is a $F(\vdash, \mathsf{aprx})$-axiom, in which case $\Gamma \vdash_{F(\vdash, \mathsf{aprx})} \eta$. Otherwise $\{\eta_1, \ldots, \eta_n\}$ is non-empty, hence $\langle \eta_1, \ldots, \eta_n, \eta \rangle$ is a $F(\vdash, \mathsf{aprx})$-rule, and hence $\{\eta_1, \ldots, \eta_n\} \vdash_{F(\vdash, \mathsf{aprx})} \eta$. Since we know that $\vdash_{F(\vdash, \mathsf{aprx})}$ satisfies (4.80), by Proposition 4.123, and $\{\eta_1, \ldots, \eta_n\} \subseteq \Gamma$, it follows that $\Gamma \vdash_{F(\vdash, \mathsf{aprx})} \eta$. $\boxed{\vdash_{F(\vdash, \mathsf{aprx})} \subseteq \vdash}$ It suffices to show that if $\langle \eta_1, \ldots, \eta_m \rangle$ is a derivation of η_m from Γ in $F(\vdash, \mathsf{aprx})$, then $\Gamma \vdash \eta_m$. We proceed by induction on m. $\boxed{\text{Base Case}}$ Let

$\langle\eta_1\rangle$ be a derivation of η_1 from Γ of length one. Then either η_1 is a $F(\vdash,\mathsf{aprx})$-axiom, in which case, by the definition of a $F(\vdash,\mathsf{aprx})$-axiom, $\emptyset\vdash\eta_1$, and since $\emptyset\subseteq\Gamma$, it follows, by assumption (4.80), that $\Gamma\vdash\eta_1$; otherwise, $\eta_1\in\Gamma$, in which case $\Gamma\vdash\eta_1$ by assumption (4.79). $\boxed{\text{Inductive Hypothesis}}$ Assume that if $n<m$ and $\langle\eta_1,\ldots,\eta_n\rangle$ is a derivation of η_n from Γ in $F(\vdash,\mathsf{aprx})$, then $\Gamma\vdash\eta_n$. $\boxed{\text{Inductive Step}}$ Let $\langle\eta_1,\ldots,\eta_m\rangle$ be a derivation of η_m from Γ in $F(\vdash,\mathsf{aprx})$. By the inductive hypothesis, $\Gamma\vdash\eta_i$, for $i=1$ to $m-1$. If η_m is an $F(\vdash,\mathsf{aprx})$-axiom or η_m is in Γ, then the base case may be invoked without loss of generality. Otherwise, there exists a $F(\vdash,\mathsf{aprx})$-rule $\xi_1,\ldots,\xi_{r-1}\boldsymbol{\vdash}\xi_r$, with $\{\xi_1,\ldots,\xi_{r-1}\}\subseteq\{\eta_1,\ldots,\eta_{m-1}\}$ and $\xi_r=\eta_m$. By the definition of a $F(\vdash,\mathsf{aprx})$-rule, $\{\xi_1,\ldots,\xi_{r-1}\}\vdash\xi_r=\eta_m$, Hence by (4.80) $\{\eta_1,\ldots,\eta_{m-1}\}\vdash\xi_r=\eta_m$. So, $\Gamma\vdash\eta_i$, for $i=1$ to $m-1$, and $\{\eta_1,\ldots,\eta_{m-1}\}\vdash\xi_r=\eta_m$, hence $\Gamma\vdash\eta_m$ by (4.81). $\diamond$

Propositions 4.123 and 4.125, together with Definition 4.47 and Proposition 4.77, characterize the formal consequence relations of formal systems as precisely the algebraic point-consequence relations over the language of the theory. More formally we have the following.

Corollary 4.126 If $\vdash$ is a binary relationship from $\mathfrak{P}(A)$ to A, then $\vdash$ is an algebraic (point) consequence relation over A iff $\vdash$ is the formal consequence relation of some formal system with language A.

Definition 4.127 (Theorems, Theories and Consequence Operators) We write $\mathsf{Thm}(\mathfrak{F})$ for the constraint $\Bbbk_{\vdash_\mathfrak{F}}$, the members of which are called **$\mathfrak{F}$-theorems**, $\mathsf{Th}(\mathfrak{F})$ for $\mathsf{cl}_{\vdash_\mathfrak{F}}$, the members of which are called **$\mathfrak{F}$-theories**, $\|\cdot\|_\mathfrak{F}$ for $\|\cdot\|_{\vdash_\mathfrak{F}}$, which we call the **formal consequence operator**, $\dashv\vdash_\mathfrak{F}$ for $\dashv\vdash_{\vdash_\mathfrak{F}}$ which we call the **formal equivalence relation**. $\square$

Corollary 4.128 A formal system is uniquely determined by its language and any one of the following datum (all over its language).

1. Its theories, as an algebraic closed system.
2. Its consequence relation, as an algebraic point-consequence relation.
3. Its consequence operator, as an algebraic closure operator.

$\square$

We formalize the previous corollary by means of the following definition.

Definition 4.129 (Formal Systems determined by Algebraic Closed Systems) With each algebraic point-consequence relation $\mathbb{C}$, let $F(\mathbb{C})$ denote the formal system with language $\mathsf{uni}(\mathbb{C})$ and formal consequence relation $\vdash_\mathbb{C}$. Since we conventionally conflate algebraic closed systems, algebraic point-consequence relations and algebraic closure operators, we may use any one of these objects in place of $\mathbb{C}$ in this definition. $\square$

Convention 4.130 (Formal-Axiomatization) When a formal system $\mathfrak{F}$ has been determined by some means other than by formal axioms and rules, we shall adopt the convention that $\mathsf{FAx}(\mathfrak{F})$ and $\mathsf{FRl}(\mathfrak{F})$ are some axioms and rules determining $\mathfrak{F}$. When we speak of a **formal-axiomatization** of $\mathfrak{F}$ when mean a specification of axioms and rules determining $\mathfrak{F}$.

Remark 4.131 $\mathsf{Thm}(\mathfrak{F}) \in \mathsf{Th}(\mathfrak{F})$.

Remark 4.132 $\mathsf{FAx}(\mathfrak{F}) \subseteq \mathsf{Thm}(\mathfrak{F})$.

4.3.2.1 Examples

The subuniverses $\mathsf{Su}(\mathbf{A})$ of an algebra $\mathbf{A}$ form an algebraic closed system (see Definition 1.302 on page 60). In the following example we describe a formal-axiomatization of these systems.

Example 4.133 (The Formal Systems of Subuniverses)

With each $\mathfrak{a}$-algebra $\mathbf{A}$, we associate the formal-system $F(\mathbf{A}, \mathsf{su})$, axiomatized by

1. all axioms $\vdash \mathbf{0}^{\mathbf{A}}$, where $\mathbf{0} \in \mathsf{Symb_c}(\mathfrak{a})$, and
2. all rules $a_1, \ldots, a_{\mathsf{ar}(\star)} \vdash \star^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(\star)})$, for all $\star \in \mathsf{Symb_o}(\mathfrak{a})$ and $a_1, \ldots, a_{\mathsf{ar}(\star)} \in \mathsf{uni}(\mathbf{A})$.

Proposition 4.134 $\mathsf{Th}(F(\mathbf{A}, \mathsf{su})) = \mathsf{Su}(\mathbf{A})$.

□

Recall that the compatible relations $\mathsf{Cpat}(\mathbf{A})$ on an algebra $\mathbf{A}$ are precisely the subuniverses of $\mathbf{A}^2$ (see Remark 1.350 on page 67) and that $\mathsf{Cpat}(\mathbf{A})$ forms an algebraic closed system.

Example 4.135 (Formal Systems of Compatible Relations)

Let $\mathbf{A}$ be a $\mathfrak{a}$-algebra.

Definition 4.136 (Formal Systems of Compatible Relations) We denote the formal system $F(\mathbf{A}^2, \mathsf{su})$ of $\mathbf{A}^2$ by $F^2(\mathbf{A}, \mathsf{cp})$, which we call the **formal system of compatible relations on A**. Any formal system on $\mathsf{uni}(\mathbf{A})^2$ that is courser than $F^2(\mathbf{A}, \mathsf{cp})$ is called **A-compatible**. □

Corollary 4.137 $F^2(\mathbf{A}, \mathsf{cp})$ is axiomatized by

1. all axioms $\vdash \langle \mathbf{0}^{\mathbf{A}}, \mathbf{0}^{\mathbf{A}} \rangle$, where $\mathbf{0} \in \mathsf{Symb_c}(\mathfrak{a})$, and
2. all rules $\langle a_1, b_1 \rangle, \ldots, \langle a_{\mathsf{ar}(\star)}, b_{\mathsf{ar}(\star)} \rangle \vdash \langle \star^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(\star)}), \star^{\mathbf{A}}(b_1, \ldots, b_{\mathsf{ar}(\star)}) \rangle$, for all $\star \in \mathsf{Symb_o}(\mathfrak{a})$ and $a_1, \ldots, a_{\mathsf{ar}(\star)}, b_1, \ldots, b_{\mathsf{ar}(\star)} \in \mathsf{uni}(\mathbf{A})$.

□

Recall Example 4.99 on page 160, where we asserted that the set of all equivalence relations $\mathsf{ER}(X)$ on a given set X forms an algebraic closed system. Consequently, this closed system must be formally-axiomatizable.

Example 4.138 (The Formal Systems of Equivalence Relations)

Definition 4.139 (The Formal Systems of Equivalence Relations) With each set X, associate the formal system $F^2(X, \equiv)$ with language X^2, formally axiomatized by all axioms and rules

$$\vdash \langle a, a\rangle, \tag{4.90}$$

$$\langle a, b\rangle \vdash \langle b, a\rangle \quad \text{and} \tag{4.91}$$

$$\langle a, b\rangle, \langle b, c\rangle \vdash \langle a, c\rangle, \tag{4.92}$$

where a, b and c range over X. A formal system coarser than $F^2(X, \equiv)$ is called **equivalential**. □

Remark 4.140 The theorems of $F^2(X, \equiv)$ are precisely the diagonal relation $=_X$.

Remark 4.141 $F^2(X, \equiv)$-theories are precisely the equivalence relations over X.

Remark 4.142 $\|\alpha\|_{F^2(X,\equiv)}$ is the smallest equivalence relation on X containing all pairs in α.

□

Recall that the set of all congruences $\mathsf{Con}(\mathbf{A})$, on an algebra $\mathbf{A}$, forms an *algebraic* closed system, as do the set of all relative congruences $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, where $\mathcal{K}$ is a quasivariety of algebras, not necessarily containing $\mathbf{A}$ (see Remark 1.353 on page 67 and Proposition 1.450 on page 87). Consequently, these closed systems must be formally axiomatizable.

Example 4.143 (The Formal Systems of Congruences and Relative Congruences)

Let $\mathbf{A}$ be an $\mathfrak{a}$-algebra and $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras.

Definition 4.144 (The Formal Systems of Congruences) Let $F^2(\mathbf{A}, \Theta)$ be the formal system with language $\mathsf{uni}(\mathbf{A})^2$, determined by all axioms and rules of $F^2(\mathsf{uni}(\mathbf{A}), \equiv)$ and in addition all rules

$$\langle a_1, b_1\rangle, \dots, \langle a_{\mathsf{ar}(\star)}, b_{\mathsf{ar}(\star)}\rangle \vdash \langle \star^{\mathbf{A}}(a_1, \dots, a_{\mathsf{ar}(\star)}), \star^{\mathbf{A}}(b_1, \dots, b_{\mathsf{ar}(\star)})\rangle, \tag{4.93}$$

for all $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathfrak{a})$ and $a_1, \dots, a_{\mathsf{ar}(\star)}, b_1, \dots, b_{\mathsf{ar}(\star)} \in \mathsf{uni}(\mathbf{A})$. □

The proofs of the following remarks follow routinely from the definition of a congruence and the definition of $F^2(\mathbf{A}, \Theta)$.

Remark 4.145 $\mathsf{Thm}(F^2(\mathbf{A}, \Theta)) = =_{\mathsf{uni}(\mathbf{A})}$.

Remark 4.146 $\mathsf{Th}(F^2(\mathbf{A}, \Theta)) = \mathsf{Con}(\mathbf{A})$. □

The following definition of the formal system of relative congruences, and the proof of the subsequent remark, follow from Lemma 1.452 on page 87.

Definition 4.147 (The Formal System of Relative Congruences) Let $F^2(\mathbf{A}, \Theta^{\mathcal{K}})$ denote the formal system with language $\mathsf{uni}(\mathbf{A})^2$, determined by all axioms

$$\vdash \langle r^{\mathbf{A}}(\vec{c}), r'^{\mathbf{A}}(\vec{c})\rangle, \tag{4.94}$$

where $\mathcal{K} \models \bigwedge_{j<m} q_j(\vec{x}) \approx q'_j(\vec{x}) \to r(\vec{x}) \approx r'(\vec{x})$ and ${q_j}^{\mathbf{A}}(\vec{c}) \approx {q'_j}^{\mathbf{A}}(\vec{c})$, and all rules

$$\langle {p_1}^{\mathbf{A}}(\vec{c}), {p'_1}^{\mathbf{A}}(\vec{c})\rangle, \ldots, \langle {p_{n-1}}^{\mathbf{A}}(\vec{c}), {p'_{n-1}}^{\mathbf{A}}(\vec{c})\rangle \vdash \langle r^{\mathbf{A}}(\vec{c}), r'^{\mathbf{A}}(\vec{c})\rangle, \tag{4.95}$$

where $\mathcal{K} \models \bigwedge_{i<n} p_i(\vec{x}) \approx p'_i(\vec{x})$ and $\bigwedge_{j<m} q_j(\vec{x}) \approx q'_j(\vec{x}) \to r(\vec{x}) \approx r'(\vec{x})$ and ${q_j}^{\mathbf{A}}(\vec{c}) = {q'_j}^{\mathbf{A}}(\vec{c})$.
□

Remark 4.148 $\mathsf{Thm}(F^2(\mathbf{A}, \Theta^{\mathcal{K}})) = \perp^{\mathbf{A}}_{\mathcal{K}}$, and $\mathsf{Th}(F^2(\mathbf{A}, \Theta^{\mathcal{K}})) = \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $\|\alpha\|_{F^2(\mathbf{A},\Theta^{\mathcal{K}})} = \|\alpha\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$.

□

Recall that by Example 4.113, the closed systems $\mathsf{Cos}(\mathbf{A})$ and $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$, of all cosets of congruences and relative congruences of an algebra, are algebraic, and hence formally-axiomatizable. In the following example we provide a formal-axiomatization of $F(\mathsf{Con}(\mathbf{A}), \mathsf{cos})$ based on Corollary 1.356 on page 68.

Example 4.149 (Formal Systems of Cosets of Congruences)

Let $\mathbf{A}$ be an algebra of type $\mathfrak{a}$.

Definition 4.150 (The Coset Formal Systems of an Algebra) Let $F(\mathbf{A}, \mathsf{cos})$ denote the formal system with language $\mathsf{uni}(\mathbf{A})$, no axioms and all rules

$$a, b, \mathsf{U}(a) \vdash \mathsf{U}(b), \tag{4.96}$$

where $a, b \in \mathsf{uni}(\mathbf{A})$ and $\mathsf{U} \in \mathsf{Pol}_1(\mathbf{A})$. □

We now demonstrate that $F(\mathbf{A}, \mathsf{cos})$ indeed formally axiomatizes $\mathsf{Cos}(\mathbf{A})$.

Proposition 4.151 $\mathsf{Th}(F(\mathbf{A}, \mathsf{cos})) = \mathsf{Cos}(\mathbf{A})$.

Proof. We show that $\|B\|_{\mathsf{Cos}(\mathbf{A})} = \|B\|_{F(\mathbf{A},\mathsf{cos})}$. By definition, $\|\emptyset\|_{\mathsf{Cos}(\mathbf{A})} = \emptyset = \|B\|_{F(\mathbf{A},\mathsf{cos})}$. Let $B \neq \emptyset$. By Corollary 1.356, $\|B\|_{F(\mathbf{A},\mathsf{cos})}$ is certainly a congruence class of some congruence on $\mathbf{A}$, and certainly contains B, so by minimality, $\|B\|_{\mathsf{Cos}(\mathbf{A})} \subseteq \|B\|_{F(\mathbf{A},\mathsf{cos})}$. We shall show that any point derivable from B in $F(\mathbf{A}, \mathsf{cos})$ must lie in $\|B\|_{\mathsf{Cos}(\mathbf{A})}$. We proceed inductively on the length of such derivations.
$\boxed{\text{Base Case}}$ Suppose that a is derivable from B be a derivation of length one. Since there are no $F(\mathbf{A}, \mathsf{cos})$-axioms, $a \in B$, and hence $a \in \|B\|_{\mathsf{Cos}(\mathbf{A})}$. $\boxed{\text{Inductive Hypothesis}}$ Assume that for any derivation of length n or less of a from B in $F(\mathbf{A}, \mathsf{cos})$, $a \in \|B\|_{\mathsf{Cos}(\mathbf{A})}$. $\boxed{\text{Inductive Step}}$ Suppose that $a_1, \ldots, a_{n+1}$ is a derivation from B in $F(\mathbf{A}, \mathsf{cos})$, for which no shorter derivation exists. By the inductive hypothesis, $a_1, \ldots, a_n \in \|B\|_{\mathsf{Cos}(\mathbf{A})}$. By the minimality of the derivation, there exists a $F(\mathbf{A}, \mathsf{cos})$-rule $a, b, \mathsf{U}(a) \vdash \mathsf{U}(b)$, with $a, b, \mathsf{U}(a) \in \{a_1, \ldots, a_n\} \subseteq \|B\|_{\mathsf{Cos}(\mathbf{A})}$ and $\mathsf{U}(b) = a_{n+1}$. Since $\|B\|_{\mathsf{Cos}(\mathbf{A})}$ is a congruence class, by Corollary 1.356, $a_{n+1} = \mathsf{U}(b) \in \|B\|_{\mathsf{Cos}(\mathbf{A})}$. ◇

We rephrase the previous result for ease of later reference.

Remark 4.152 $F(\mathbf{A}, \mathsf{cos})$ is formally-axiomatized by no axioms and all rules

$$a, b, p^{\mathbf{A}}(a, c_1, \ldots, c_{\mathsf{ar}(p)-1}) \vdash p^{\mathbf{A}}(b, c_1, \ldots, c_{\mathsf{ar}(p)-1}), \tag{4.97}$$

for all $a, b \in \mathsf{uni}(\mathbf{A})$, terms p and $a, c_1, \ldots, c_{\mathsf{ar}(p)-1} \in \mathsf{uni}(\mathbf{A})$. □

We know of no *simple* formal-axiomatization of $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$ generally. We shall show, however, that under certain circumstances, such as when the quasivariety $\mathcal{K}$ is relatively congruence regular, the relative cosets on an algebra coincide with the filters of the *membership logic* on that algebra (the membership logic is still to be defined, see Example 5.57 on page 191).

$\square$

Recall Example 4.82 on page 157 and Example 4.88 on page 158 where we introduced the closed systems $\mathsf{Cx}(\mathbf{P})$, $\mathsf{Id}_\Diamond(\mathbf{P})$ and $\mathsf{Fl}_\Diamond(\mathbf{P})$ and of convexities, ideals and filters of a lattice $\mathbf{P}$, as well as Example 4.85, where we noted that these closed systems are all *algebraic*. As such, these algebraic closed systems must be formally-axiomatizable.

Example 4.153 (Formal Systems of Lattice Ideals, Filters and Convexities)
Let $\mathbf{P}$ be a lattice expansion.

Definition 4.154 (The Formal System of Lattice Ideals) Let $F(\mathbf{P}, \mathsf{id})$ denote the formal system with language $\mathsf{uni}(\mathbf{P})$, with all rules described by the following *rule schema*,

$$\forall[\boldsymbol{x}, \boldsymbol{y}]\ \boldsymbol{x} \vdash \boldsymbol{x} \wedge^{\mathbf{P}} \boldsymbol{y} \quad \text{and} \tag{4.98}$$

$$\forall[\boldsymbol{x}, \boldsymbol{y}]\ \boldsymbol{x}, \boldsymbol{y} \vdash \boldsymbol{x} \vee^{\mathbf{P}} \boldsymbol{y}, \tag{4.99}$$

and the single axiom

$$\vdash 0, \tag{4.100}$$

precisely when $\mathbf{P}$ is lower-bounded and 0 is the lower-bound. $\square$

Proposition 4.155 $\mathsf{Th}(F(\mathbf{P}, \mathsf{id})) = \mathsf{Id}_\Diamond(\mathbf{P})$, and consequently, $\vdash_{F(\mathbf{P},\mathsf{id})} = \vdash_{\mathsf{Id}_\Diamond(\mathbf{P})}$ and $\|\cdot\|_{F(\mathbf{P},\mathsf{id})} = \|\cdot\|^{\mathbf{P}}_{\mathsf{Id}_\Diamond}$.

Proof. $\boxed{\mathsf{Th}(F(\mathbf{P},\mathsf{id})) \subseteq \mathsf{Id}_\Diamond(\mathbf{P})}$ Let $T \in \mathsf{Th}(F(\mathbf{P}, \mathsf{id}))$. If $T = \emptyset$ then $F(\mathbf{P}, \mathsf{id})$ has no axioms, which is only possible if $\mathbf{P}$ is not lower-bounded, in which case $T = \emptyset \in \mathsf{Id}_\Diamond(\mathbf{P})$. Suppose that $T \neq \emptyset$. Suppose that $a \in T$ and $b \leq a$. Since $a \vdash_{F(\mathbf{P},\mathsf{id})} b \wedge a$ and $a \in T$, $b \wedge a \in T$, since T is a theory. But $b \wedge a = b$ since $b \leq a$, so $b \in T$. Suppose that $a, b \in T$. Since $\{a, b\} \vdash_{F(\mathbf{P},\mathsf{id})} b \vee a$ and $a, b \in T$, $b \vee a \in T$, since T is a theory. So $T \in \mathsf{Id}_\Diamond(\mathbf{P})$. $\boxed{\mathsf{Id}_\Diamond(\mathbf{P}) \subseteq \mathsf{Th}(F(\mathbf{P},\mathsf{id}))}$ Let $\mathsf{I} \in \mathsf{Id}_\Diamond(\mathbf{P})$. If $\mathsf{I} = \emptyset$, then $\mathbf{P}$ is not lower-bounded and so $F(\mathbf{P}, \mathsf{id})$ has no axioms, in which case $\mathsf{I} = \emptyset \in \mathsf{Th}(F(\mathbf{P}, \mathsf{id}))$. Suppose that $\mathsf{I} \neq \emptyset$. We shall show that anything derivable from I is contained in I, by proceeding inductively on the length of derivations from I in $F(\mathbf{P}, \mathsf{id})$. $\boxed{\text{Base Case}}$ Suppose that a is derivable from I by a derivation of length one. So either $a \in \mathsf{I}$, which suffices, else $\vdash a$ is an axiom. In the latter case, $\mathbf{P}$ is lower-bounded with lower-bound a, and so $a \in \mathsf{I}$. $\boxed{\text{Induction Hypothesis}}$ Assume that any point derivable from I by a derivation of length n or less, is contained in I. $\boxed{\text{Inductive Step}}$ Suppose that $a_1, \ldots, a_{n+1}$ is a derivation from I. By the inductive hypothesis, $\{a_1, \ldots, a_n\} \subseteq \mathsf{I}$. If either $a_{n+1} \in \mathsf{I}$ or $\vdash a_{n+1}$ is an axiom, the result follows as in the base case. Otherwise, there exists a rule Λ with $\mathsf{prem}(\Lambda) \subseteq \{a_1, \ldots, a_n\} \subseteq \mathsf{I}$ and $\mathsf{conc}(\Lambda) = a_{n+1}$. If $\Lambda = a \vdash a \wedge b$, for some $a, b \in \mathsf{uni}(\mathbf{P})$, then $a \in \mathsf{I}$ and $a_{n+1} = a \wedge b \leq a$, and so $a_{n+1} \in \mathsf{I}$. Otherwise, $\Lambda = a, b \vdash a \vee b$, for some $a, b \in \mathsf{uni}(\mathbf{P})$, in which case, $a, b \in \mathsf{I}$ and $a_{n+1} = a \vee b$, and so $a_{n+1} \in \mathsf{I}$. $\diamond$

Proposition 4.156 An alternative axiomatization of $F(\mathbf{P}, \mathsf{id})$ is given by (4.99) and (4.100) above and the rule schema

$$\forall[\boldsymbol{x}, \boldsymbol{y}]\ \boldsymbol{x} \vee^{\mathbf{P}} \boldsymbol{y} \vdash \boldsymbol{y}. \tag{4.101}$$

Proof. $\boxed{\mathsf{Th}(F(\mathbf{P},\mathsf{id})) \subseteq \mathsf{Id}_\diamondsuit(\mathbf{P})}$ Let $T \in \mathsf{Th}(F(\mathbf{P},\mathsf{id}))$. If $T = \emptyset$ then $F(\mathbf{P},\mathsf{id})$ has no axioms, which is only possible if $\mathbf{P}$ is not lower-bounded, in which case $T = \emptyset \in \mathsf{Id}_\diamondsuit(\mathbf{P})$. Suppose that $T \neq \emptyset$. Suppose that $a \in T$ and $b \leq a$. Then $a \vee b = a \in T$. So by the rule $a \vee b \vdash b$, $b \in T$. If $a, b \in T$, then by the rule $a, b \vdash a \vee b$, $b \vee a \in T$. So $T \in \mathsf{Id}_\diamondsuit(\mathbf{P})$. $\boxed{\mathsf{Id}_\diamondsuit(\mathbf{P}) \subseteq \mathsf{Th}(F(\mathbf{P},\mathsf{id}))}$ Let $\mathsf{I} \in \mathsf{Id}_\diamondsuit(\mathbf{P})$. If $\mathsf{I} = \emptyset$, then $\mathbf{P}$ is not lower-bounded and so $F(\mathbf{P},\mathsf{id})$ has no axioms, in which case $\mathsf{I} = \emptyset \in \mathsf{Th}(F(\mathbf{P},\mathsf{id}))$. Suppose that $\mathsf{I} \neq \emptyset$. We shall show that anything derivable from I is contained in I, by proceeding inductively on the length of derivations from I in $F(\mathbf{P},\mathsf{id})$. $\boxed{\text{Base Case}}$ Suppose that a is derivable from I by a derivation of length one. So either $a \in \mathsf{I}$, which suffices, else $\vdash a$ is an axiom. In the latter case, $\mathbf{P}$ is lower-bounded with lower-bound a, and so $a \in \mathsf{I}$. $\boxed{\text{Induction Hypothesis}}$ Assume that any point derivable from I by a derivation of length n or less, is contained in I. $\boxed{\text{Inductive Step}}$ Suppose that $a_1, \ldots, a_{n+1}$ is a derivation from I. By the inductive hypothesis, $\{a_1, \ldots, a_n\} \subseteq \mathsf{I}$. If either $a_{n+1} \in \mathsf{I}$ or $\vdash a_{n+1}$ is an axiom, the result follows as in the base case. Otherwise, there exists a rule Λ with $\mathsf{prem}(\Lambda) \subseteq \{a_1, \ldots, a_n\} \subseteq \mathsf{I}$ and $\mathsf{conc}(\Lambda) = a_{n+1}$. If $\Lambda = a \vee b \vdash b$, for some $a, b \in \mathsf{uni}(\mathbf{P})$, then $a \vee b \in \mathsf{I}$ and $a_{n+1} = b \leq a \vee b$, and so $a_{n+1} \in \mathsf{I}$. Otherwise, $\Lambda = a, b \vdash a \vee b$, for some $a, b \in \mathsf{uni}(\mathbf{P})$, in which case, $a, b \in \mathsf{I}$ and $a_{n+1} = a \vee b$, and so $a_{n+1} \in \mathsf{I}$. $\diamond$

Definition 4.157 (The Formal System of Lattice Filters) Let $F(\mathbf{P},\mathsf{fi})$ denote the formal system with language $\mathsf{uni}(\mathbf{P})$, determined by all rules described by the following *rule schema*

$$\forall[\boldsymbol{x},\boldsymbol{y}]\ \boldsymbol{x} \vdash \boldsymbol{x} \vee^{\mathbf{P}} \boldsymbol{y} \quad \text{and} \tag{4.102}$$

$$\forall[\boldsymbol{x},\boldsymbol{y}]\ \boldsymbol{x},\boldsymbol{y} \vdash \boldsymbol{x} \wedge^{\mathbf{P}} \boldsymbol{y}, \tag{4.103}$$

and the single axiom

$$\vdash 1, \tag{4.104}$$

precisely when $\mathbf{P}$ is upper-bounded and 1 is the upper-bound. $\square$

Proposition 4.158 $\mathsf{Th}(F(\mathbf{P},\mathsf{fi})) = \mathsf{Fl}_\diamondsuit(\mathbf{P})$, and consequently, $\vdash_{F(\mathbf{P},\mathsf{fi})} = \vdash_{\mathsf{Fl}_\diamondsuit(\mathbf{P})}$ and $\|\cdot\|_{F(\mathbf{P},\mathsf{fi})} = \|\cdot\|^{\mathbf{P}}_{\mathsf{fi}_\diamondsuit}$.

Proposition 4.159 An alternative axiomatization of $F(\mathbf{P},\mathsf{fi})$ is given by (4.103) and (4.104) above and the rule schema

$$\forall[\boldsymbol{x},\boldsymbol{y}]\ \boldsymbol{x} \wedge^{\mathbf{P}} \boldsymbol{y} \vdash \boldsymbol{y}. \tag{4.105}$$

Definition 4.160 (The Formal System of Lattice Convexities) Let $F(\mathbf{P},\mathsf{cx})$ denote the formal system with language $\mathsf{uni}(\mathbf{P})$, with no axioms and all rules $a \wedge b, b \vee c \vdash b$, where $a, b, c \in \mathsf{uni}(\mathbf{P})$. $\square$

Proposition 4.161 $\mathsf{Th}(F(\mathbf{P},\mathsf{cx})) = \mathsf{Cx}(\mathbf{P})$, and consequently, $\vdash_{F(\mathbf{P},\mathsf{cx})} = \vdash_{\mathsf{Cx}(\mathbf{P})}$ and $\|\cdot\|_{F(\mathbf{P},\mathsf{cx})} = \|\cdot\|^{\mathbf{P}}_{\mathsf{cx}}$.

Proof. $\boxed{\mathsf{Th}(F(\mathbf{P},\mathsf{cx})) \subseteq \mathsf{Cx}(\mathbf{P})}$ Let $T \in \mathsf{Th}(F(\mathbf{P},\mathsf{cx}))$. Suppose that $a, c \in T$ and $a \leq b \leq c$. Since $a \leq b$, $a \wedge b = a \in T$, and since $b \leq c$, $b \vee c = c \in T$. So by the rule $a \wedge b, b \vee c \vdash b$, $b \in T$. So $T \in \mathsf{Cx}(\mathbf{P})$. $\boxed{\mathsf{Cx}(\mathbf{P}) \subseteq \mathsf{Th}(F(\mathbf{P},\mathsf{cx}))}$ Let $\mathsf{C} \in \mathsf{Cx}(\mathbf{P})$. We shall show that anything derivable from C is contained in C, by proceeding inductively on the length of derivations from C in $F(\mathbf{P},\mathsf{cx})$. $\boxed{\text{Base Case}}$ Suppose that a is derivable from C by a derivation of length one. Since there are no axioms, $a \in \mathsf{C}$, which suffices. $\boxed{\text{Induction Hypothesis}}$ Assume that any point derivable from C by a derivation of length n or less, is contained in C. $\boxed{\text{Inductive Step}}$ Suppose that $a_1, \ldots, a_{n+1}$ is a derivation from C. By the inductive hypothesis, $\{a_1, \ldots, a_n\} \subseteq \mathsf{C}$. If $a_{n+1} \in \mathsf{C}$ this suffices. Since there are no axioms, there exists a rule $a \wedge b, b \vee c \vdash b$ with $\{a \wedge b, b \vee c\} \subseteq \{a_1, \ldots, a_n\} \subseteq \mathsf{C}$ and $b = a_{n+1}$. Since $a \wedge b \leq b \leq b \vee c$ and $a \wedge b, b \vee c \in \mathsf{C}$, by definition of a convexity, $a_{n+1} = b \in \mathsf{C}$. $\diamond$

$\square$

At times, it will prove useful to 'force' the empty-set to be an ideal or filter, independently of the bounded nature of a particular lattice. Recall Definition 4.96 of Example 4.88 on page 158, where we defined the algebraic closed systems $\mathsf{Id}_{\diamond_\emptyset}(\mathbf{P})$ and $\mathsf{Fl}_{\diamond_\emptyset}(\mathbf{P})$, of all lattice ideals and filters, respectively, together with the empty-set.

Example 4.162 (Formal Systems of Lattice Ideals and Filters with Empty-set)

Let $\mathbf{P}$ be a lattice expansion.

Definition 4.163 (Formal Systems of Lattice Ideals and Filters with Empty-set) Let $F(\mathbf{P}, \mathsf{id}_\emptyset)$ denote the formal system with language $\mathsf{uni}(\mathbf{P})$, formally-axiomatized by (4.98) and (4.99) (only), and let $F(\mathbf{P}, \mathsf{fi}_\emptyset)$ denote the formal system with language $\mathsf{uni}(\mathbf{P})$, formally-axiomatized by (4.102) and (4.103) (only) of Definition 4.157 of the same example. $\square$

The proofs of the following remarks and results are similar to those of the previous example, and as such are omitted.

Proposition 4.164 $\mathsf{Th}(F(\mathbf{P}, \mathsf{id}_\emptyset)) = \mathsf{Id}_{\diamond_\emptyset}(\mathbf{P})$ and $\mathsf{Th}(F(\mathbf{P}, \mathsf{fi}_\emptyset)) = \mathsf{Fl}_{\diamond_\emptyset}(\mathbf{P})$.

Proposition 4.165 $F(\mathbf{P}, \mathsf{id}_\emptyset)$ is alternatively formally-axiomatized by (4.99) and (4.101).

Proposition 4.166 $F(\mathbf{P}, \mathsf{fi}_\emptyset)$ is alternatively formally-axiomatized by (4.103) and (4.105).

$\square$

Chapter 5

Translations

As mentioned in the introduction, one of the aims of this text is to unify apparently disparate arguments and constructions from the field of algebraic logic, both from the standard theory and our parameterized theory, that appeared to us to have a distinctly similar character; in fact a character that we recognised from topology. Let us now consider one such similarity. Recall Theorem 2.22 on page 96 characterizing the consequence relations of sentential calculi as precisely those binary relationships $\vdash$ from $\mathfrak{P}(\mathsf{Fm}(\mathcal{S}))$ to $\mathsf{Fm}(\mathcal{S})$ satisfying five conditions, and note in particular, condition (5) known as *structurality*, which states that for every substitution σ and for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathcal{S})$,

$$\Gamma \vdash \phi \text{ implies } \sigma[\Gamma] \vdash \sigma(\phi). \tag{5.1}$$

Now recall that a sentential calculus $\mathcal{S}_2$ is called a *formal semantics* for sentential calculus $\mathcal{S}_1$, if there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that, for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathcal{S}_1)$,

$$\Gamma \vdash_{\mathcal{S}_1} \phi \text{ iff } \boldsymbol{\tau}[\Gamma] \vdash_{\mathcal{S}_2} \boldsymbol{\tau}[\![\phi]\!] \tag{5.2}$$

(see Definition 2.95 on page 108). Let us now unify these two notions, beginning by weakening the latter and calling $\mathcal{S}_2$ a *formal model* of $\mathcal{S}_1$ if there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that, for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathcal{S}_1)$,

$$\Gamma \vdash_{\mathcal{S}_1} \phi \text{ implies } \boldsymbol{\tau}[\Gamma] \vdash_{\mathcal{S}_2} \boldsymbol{\tau}[\![\phi]\!] \tag{5.3}$$

(we shall analyse this condition in $\oint$17.3 of Part VI, characterizing such τ as precisely the formal translations that *commute with substitutions*, the latter condition playing a pivotal role in algebraic logic; see in particular Corollary 17.27 on page 475). There are two fundamental differences between (5.1) and (5.3). Firstly, (5.1) is phrased with respect to a *single logic* while (5.3) involves *two logics*; in this respect, the former is an 'endo' variant of the latter. Secondly, and more importantly, the (5.1) involves a *function* while the latter involves, what we have found to be best viewed as, a *grounded binary relationship* (see $\oint$1.1.2); which we call concrete *translations* (concrete since the consequence relations involved are concrete). Since functions are special concrete translations, (5.1) may be viewed as a special case of (5.3).

Returning to *concrete closed systems* more generally, in the chapter we shall consider concrete translations τ, from the universe of a closed system $\mathbb{C}$ to the universe of a closed system $\mathbb{D}$, that satisfy the property that for all $A \cup \{a\} \subseteq \mathsf{uni}(\mathbb{C})$,

$$A \vdash_{\mathbb{C}} a \text{ implies } \boldsymbol{\tau}[A] \vdash_{\mathbb{D}} \boldsymbol{\tau}[\![a]\!]; \tag{5.4}$$

we call this property *continuity*, a term that we shall justify in the course of this chapter. We shall show that τ is continuous iff for all $A \cup B \subseteq \mathsf{uni}(\mathbb{C})$,

$$A \vdash_{\mathbb{C}} B \text{ implies } \boldsymbol{\tau}[A] \vdash_{\mathbb{D}} \boldsymbol{\tau}[B]\,; \tag{5.5}$$

since images of binary relationships are inclusion-order preserving, continuous translations are (concrete instances of) *elementary consequence relation homomorphisms* (see Definition 1.278 on page 56); we call the latter $\vdash$*-homomorphisms*. Observe that from this perspective, (5.2) may be viewed as demanding that τ be a $\vdash$*-strict* $\vdash$-homomorphism (see Definition 1.291 on page 59); that is, a $\vdash$-homomorphism that is strict with respect to $\vdash$, although *not* necessarily a strict $\vdash$-homomorphism.

Not all $\vdash$-homomorphisms between *concrete* closed systems need be *translations*, although they are certainly inclusion-order preserving functions between the power-sets of the universes. It is possible to give an *elementary* characterization of translations, that is, to distinguish those order preserving functions between orders that, in the concrete case (i.e., between inclusion-ordered power-sets) are translations. To this end, order-preserving functions between orders shall be called *weak-translations*, while the term *translation* is reserved, in the elementary case, for those weak-translations that coincide with translations in the concrete setting (actually *non-singular* translations and concrete translations coincide); informally, elementary translations are pairs of weak-translations $\langle \tau, \tau^{\blacktriangleleft} \rangle$, one 'forward' and one 'back', that constitute *galois relations* in the sense of [DP90]. In the case of a concrete translation τ, the 'back' weak-translation is the *reduced pre-image* function determined by τ (defined shortly). We shall show that an elementary translation is continuous iff $\tau^{\blacktriangleleft}$ maps closed points to closed points. In the concrete case, this latter condition formulates as requiring that the *reduced pre-image of closed sets be closed*, and since reduced preimages of functions coincide with preimages, for functions, requires that the *pre-image of closed sets be closed*; this justifies are usage of the term 'continuous', since this last property characterizes *continuous functions between topological closed systems.*

In $\oint$5.1 we define the notions of a *weak-translation* and *translation* between elementary closed systems and consider their *concrete* formulations. Weak-translations are simply the order-preserving functions between the order-reducts (see Definition 1.173 on page 39). In particular, we characterize concrete translations, in the elementary setting, as *non-singular* (elementary) translations. We name the various *homomorphisms* between elementary objects of closure: $\vdash$-homomorphisms, cl-homomorphisms, $\|\cdot\|$-homomorphisms and $\dashv\vdash$-homomorphisms. In $\oint$5.2 we characterize these homomorphisms, and give examples showing how well-known results pertaining to *structurality* and *filters* may be obtained even at this very weak setting (i.e., the theory of elementary *weak*-translations). While $\vdash$-homomorphisms (and continuous translations) are our primary concern in this text, the importance of considering cl-homomorphisms becomes clear in the next section.

Continuous, $\vdash$-reflecting and strictly continuous elementary *translations* are the primary focus of $\oint$5.3. Additional characterizations of continuity are obtained over and above those obtained for weak-translations in the previous section (these of course apply to translations). Key characterizations include the condition that the ‘back’ weak-translation be $\vdash$-*homomorphism*. Characterizations of $\vdash$-reflecting and strictly continuous elementary translations are also provided. In particular, strictly continuous elementary translations are characterized in terms of the *product* closed system determined by a translation from an order to an elementary closed system. The notion of an *iseomorphism* is introduced and characterized; iseomorphisms are generalizations of the notion of one logic being a formal equivalent semantics of another (see Definition 2.97 on page 109). We show that if two elementary closed systems are iseomorphic then their closed point suborders are *isomorphic*, although we are unable to establish a converse in the elementary setting.

The rest of the chapter is concerned with *concrete closed systems* and *concrete translations*. In ∮5.4 we characterize continuous translations, strictly continuous translations and iseomorphisms in terms of $\blacktriangledown$-preserving functions, $\blacktriangledown$-embeddings and isomorphisms (see Definition 1.180 on page 40), respectively, between the closed set lattices, thereby obtaining converses of the elementary results obtained in the previous section. In the concrete setting we are also able to define the *product* of a *source* of multiple translations from a set to multiple closed systems; this construction depends on the existence of a *basis* of a closed system, a notion we are unable to define in the elementary context. As an application of products of sources, we demonstrate how the *semantic consequence* relation determined by a matrix may be realized as the product of a source. Products of sources are also used often in our theory of *parameterized algebraization*. The dual notion of a *quotient* of a *sink* of multiple translations from multiple closed systems to a single set is also considered; as an example, we show how the *filters* of a logic arise as the quotient of all interpretations from the closed system of all theories into the universe of an algebra.

A number of *sentential calculi*, pertinent to the sequel, are developed in this chapter, including, the sentential calculi $S(\mathfrak{a}, \mathsf{su})$ of *subuniverses* (see Example 5.47 on page 188), the sentential calculus $S(\mathfrak{a}, \mathsf{cos})$ of *cosets* and the *membership logic* $S(\mathcal{K}, \mathsf{mem})$ or logic of *relative cosets* (see Example 5.57 on page 191).

5.1 Translations

In this section we introduce the notions of a *weak-translation* and a *translation* between elementary closed systems, and define *concrete translations* between concrete closed systems. While it is useful to conflate the various elementary objects of closure, each elementary class admits *different* structure homomorphisms. Since all our elementary structures of closure have order reducts, the homomorphisms between these structures must all be *order-preserving functions*. We call such functions *weak-translations*. The reader is urged to recall ∮1.2.6 on the various functions between orders.

Definition 5.1 (Weak-Translations between Elementary Closed Systems) In the context of elementary closed systems, we shall call order-preserving functions between orders **weak-translations**. Let $\mathfrak{c}$ and $\mathfrak{d}$ be elementary closed systems. A **weak-translation from $\mathfrak{c}$ to $\mathfrak{d}$** is a weak-translation from $\mathbf{P}_{\mathfrak{c}}$ into $\mathbf{P}_{\mathfrak{d}}$. With each weak-translation τ from $\mathfrak{c}$ into $\mathfrak{d}$ we associate the function $\tau^\star : \mathsf{uni}(\mathfrak{c}) \to \mathsf{cl}_{\mathfrak{d}}$ defined by $\tau^\star(a) = \|\tau(a)\|_{\mathfrak{d}}$. We call $\tau^\star$ the **closure of** τ. □

Warning 5.2 We shall (tend to) invoke the order-preserving properties of weak-translations without explicit mention.

Remark 5.3 If τ is a weak-translation from $\mathfrak{c}$ into $\mathfrak{d}$ then so is $\tau^\star$; further

$$\tau^\star(a) \dashv\vdash_{\mathfrak{d}} \tau(a) \quad \text{and} \tag{5.6}$$

$$\tau^\star(a) = \|\tau(a)\|_{\mathfrak{d}} \leq \|\tau(\|a\|_{\mathfrak{c}})\|_{\mathfrak{d}} = \tau^\star(\|a\|_{\mathfrak{c}}). \tag{5.7}$$

□

As noted earlier, while we tend to conflate elementary closure operators, closed systems, etc., not all these (non-conflated) types of structures admit the same homomorphisms. We now consider these homomorphisms, all of which are *weak-translations*. For convenience we (tend to) work with a single type of structure, namely elementary closed systems, and name the different types of homomorphisms so as to reflect these different structures.

Definition 5.4 (Homomorphisms between Elementary Closed Systems) Let τ be a weak-translation from $\mathfrak{c}$ into $\mathfrak{d}$. We call τ a $\|\cdot\|$**-homomorphism** if, for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$\tau(\|a\|_{\mathfrak{c}}) = \|\tau(a)\|_{\mathfrak{d}}\,, \tag{5.8}$$

a cl**-homomorphism** (or **closed**) if, for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$a \,\text{is}\, \mathsf{cl}_{\mathfrak{c}} \rightarrow \tau(a) \,\text{is}\, \mathsf{cl}_{\mathfrak{d}}, \tag{5.9}$$

a $\vdash$**-homomorphism** if, for all $a, b \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$a \vdash_{\mathfrak{c}} b \rightarrow \tau(a) \vdash_{\mathfrak{d}} \tau(b) \tag{5.10}$$

and a $\dashv\vdash$**-homomorphism** if, for all $a, b \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$a \dashv\vdash_{\mathfrak{c}} b \rightarrow \tau(a) \dashv\vdash_{\mathfrak{d}} \tau(b). \tag{5.11}$$

We call τ cl**-reflecting** if, for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$\tau(a) \,\text{is}\, \mathsf{cl}_{\mathfrak{d}} \rightarrow a \,\text{is}\, \mathsf{cl}_{\mathfrak{c}}, \tag{5.12}$$

$\vdash$**-reflecting** if, for all $a, b \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$\tau(a) \vdash_{\mathfrak{d}} \tau(b) \rightarrow a \vdash_{\mathfrak{c}} b \tag{5.13}$$

and $\dashv\vdash$**-reflecting** if, for all $a, b \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$\tau(a) \dashv\vdash_{\mathfrak{d}} \tau(b) \rightarrow a \dashv\vdash_{\mathfrak{c}} b. \tag{5.14}$$

We call τ cl**-strict** (resp. , $\vdash$**-strict** and $\dashv\vdash$**-strict**) if it is a $\vdash$-reflecting cl-homomorphism (resp. $\vdash$-reflecting $\vdash$-homomorphism, $\dashv\vdash$-reflecting $\dashv\vdash$-homomorphism). □

While it is interesting to know which results from algebraic logic can be captured at the level of weak-translations, many results cannot be obtained in this context. For example, the pre-image of a theory by a substitution is a theory [vA95, L 1.5.3] (see Theorem 2.26 on page 97 of our text) and filters are preserved under homomorphic pre-images [BP89a] (see Proposition 2.42 on page 101 of our text). Similar results obtain for, what we have termed, the *reduced pre-image* (introduced shortly) of formal translations in the context of a formal equivalent semantics [BP89a] (see Theorem 2.96 on page 109 of our text). Weak-translations between (elementary) orders admit no (useful) notion of pre-image. We have found that the most useful elementary abstraction of the *reduced* pre-image is what is known as a *galois connection* [DP90], which is a pair of weak translations between orders, one forward and one backwards, satisfying special elementary conditions; these galois pairs we call *translations*. We shall justify the term 'translation' shortly (in the concrete context).

Definition 5.5 (Elementary Translations) Let $\mathbf{P}$ and $\mathbf{Q}$ be orders. A **translation from $\mathbf{P}$ to $\mathbf{Q}$** is a pair $\tau = \langle \tau^{\blacktriangleright}, \tau^{\blacktriangleleft} \rangle$ where $\tau^{\blacktriangleright}$ is a weak-translation from $\mathbf{P}$ to $\mathbf{Q}$ and $\tau^{\blacktriangleleft}$ is a weak-translation from $\mathbf{Q}$ into $\mathbf{P}$, such that, for all $a \in \mathsf{uni}(\mathbf{P})$ and $b \in \mathsf{uni}(\mathbf{Q})$,

$$a \leq \tau^{\blacktriangleleft}(\tau^{\blacktriangleright}(a)) \quad \text{and} \tag{5.15}$$

$$\tau^{\blacktriangleright}(\tau^{\blacktriangleleft}(b)) \leq b. \tag{5.16}$$

Wherever unambiguous, we denote $\tau^{\blacktriangleright}$ by τ. Let $\mathbf{P} \rightleftarrows \mathbf{Q}$ denote the set of all translations from $\mathbf{P}$ to $\mathbf{Q}$. We write $\tau : \mathbf{P} \rightleftarrows \mathbf{Q}$ for $\tau \in \mathbf{P} \rightleftarrows \mathbf{Q}$.

Let $\mathfrak{c}$ and $\mathfrak{d}$ be elementary closed systems. A **translation τ from $\mathfrak{c}$ to $\mathfrak{d}$** is a translation from $\mathbf{P}_{\mathfrak{c}}$ to $\mathbf{P}_{\mathfrak{d}}$. Let $\mathfrak{c} \rightleftarrows \mathfrak{d}$ denote the set of all translations from $\mathfrak{c}$ to $\mathfrak{d}$; we write $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ for $\tau \in \mathfrak{c} \rightleftarrows \mathfrak{d}$. We call a *translation* $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ **continuous** (resp. **strictly continuous**) if it is a $\vdash$-homomorphism (resp. $\vdash$-strict). □

Warning 5.6 (Continuous) Our use of the term 'continuous', while compatible with the usage in topology, is non-standard with the usage of this term in algebraic lattice theory; we use the term '$\sqcup$-preserving' in the latter case (see Definition 1.180 on page 40 and Warning 1.181).

Warning 5.7 We shall (tend to) invoke properties (5.15) and (5.15) without explicit mention.

Recall that we call a function between orders $\blacktriangledown$-preserving (resp. $\blacktriangle$-preserving) if it preserves those joins (resp. meets) that *exist* (see Definition 1.180 on page 40); no assertion is made about the actual *existence* of such joins and meets.

Proposition 5.8 If τ is a translation from order $\mathbf{P}$ to $\mathbf{Q}$, then

$$\tau^{\blacktriangleleft} : \mathbf{Q} \rightarrow_{\blacktriangle} \mathbf{P}, \tag{5.17}$$

$$\tau : \mathbf{P} \rightarrow_{\blacktriangledown} \mathbf{Q} \quad \text{and} \tag{5.18}$$

$$a \leq \tau^{\blacktriangleleft}(b) \text{ iff } \tau(a) \leq b. \tag{5.19}$$

Proof. Suppose that $B \subseteq \mathsf{uni}(\mathbf{Q})$ such that $\blacktriangle B$ exists. Since order-preserving functions preserve bounds, $\tau^{\blacktriangleleft}(\blacktriangle B)$ is a lower-bound of $\tau^{\blacktriangleleft}[B]$. Let a be a lower-bound of $\tau^{\blacktriangleleft}[B]$. Then $\tau(a)$ is a lower-bound of $\tau[\tau^{\blacktriangleleft}[B]]$. (*We must show that* $a \leq \tau^{\blacktriangleleft}(\blacktriangle B)$.) In fact, $\tau(a)$ is a lower-bound of B, since if $b \in B$, then by assumption, $a \leq \tau^{\blacktriangleleft}(b)$, and so $\tau(a) \leq \tau(\tau^{\blacktriangleleft}(b)) \leq b$. Hence $\tau(a) \leq \blacktriangle B$, and so $a \leq \tau^{\blacktriangleleft}(\tau(a)) \leq \tau^{\blacktriangleleft}(\blacktriangle B)$. The proof of the outstanding assertion follows symmetrically. $\boxed{(5.19)}$ If $a \leq \tau^{\blacktriangleleft}(b)$ then $\tau(a) \leq \tau(\tau^{\blacktriangleleft}(b)) \leq b$. If $\tau(a) \leq b$ then $a \leq \tau^{\blacktriangleleft}(\tau(a)) \leq \tau^{\blacktriangleleft}(b)$. ◇

Warning 5.9 Conditions (5.17) and (5.18) are *not* elementary conditions.

We turn now to *concrete translations* which will be a grounded binary relationship paired with its reduced pre-image (defined shortly). Our aim is to characterize concrete translations in the elementary setting. To this end we shall distinguish certain elementary translations as *singular*. The *non-singular* elementary translation will characterize the concrete translations in the elementary setting. We begin with some observations. Recall that a function f from (the universe of) order $\mathbf{P}$ to (the universe of) $\mathbf{Q}$ is called 1-*preserving* if, *in the case* that $\mathbf{P}$ has a greatest element then $\mathbf{Q}$ too has a greatest element and f maps the greatest element of $\mathbf{P}$ to the greatest element of $\mathbf{Q}$ (see Definition 1.180 on page 40). Note that if $\mathbf{P}$ has *no* greatest element then f *is* automatically 1-preserving. Similarly for 0-preserving functions.

Remark 5.10 If $\tau : \mathbf{P} \rightleftarrows \mathbf{Q}$,

$$\tau^{\blacktriangleleft} : \mathbf{Q} \to_1 \mathbf{P} \quad \text{and} \tag{5.20}$$

$$\tau : \mathbf{P} \to_0 \mathbf{Q}; \tag{5.21}$$

in particular, if $\mathbf{P}$ has 0 then $\mathbf{Q}$ has 0, and if $\mathbf{Q}$ has 1 then $\mathbf{P}$ has 1. □

It need not be the case, however, that $\tau^{\blacktriangleleft} : \mathbf{Q} \to_0 \mathbf{P}$ nor that $\tau : \mathbf{P} \to_1 \mathbf{Q}$, even for complete orders. Translations for which the *former* property fail will be isolated as *singular*.

Counter Example 5.11 (Singular Translations)

> Let $\mathbf{P}$ and $\mathbf{Q}$ both be two element topped and bottomed orders, and consider the translation defined by $\tau(0^{\mathbf{P}}) = 0^{\mathbf{Q}} = \tau(1^{\mathbf{P}})$ and $\tau^{\blacktriangleleft}(0^{\mathbf{Q}}) = 1^{\mathbf{P}} = \tau^{\blacktriangleleft}(1^{\mathbf{Q}})$.

□

The following notation denoting non-singular translations has been chosen to reflect the standard symbolism for *multi-maps* (which we shall call *concrete translations*), since we shall see that non-singular translations characterize multi-maps in the elementary context.

Definition 5.12 (Singular and Non-Singular Translation) We call a translation $\tau : \mathbf{P} \rightleftarrows \mathbf{Q}$ **non-singular** if $\tau^{\blacktriangleleft} : \mathbf{Q} \to_0 \mathbf{P}$, otherwise we call it **singular**. Let $\mathbf{P} \multimap \mathbf{Q}$ denote the set of all non-singular translations from $\mathbf{P}$ to $\mathbf{Q}$. We write $\tau : \mathbf{P} \multimap \mathbf{Q}$ for $\tau \in \mathbf{P} \multimap \mathbf{Q}$. This notation has been chosen to coincide with the notation for a *concrete translation* (or *multi-map*) given in Definition 5.17. We call $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ non-singular is $\tau : \mathbf{P}_{\mathfrak{c}} \multimap \mathbf{P}_{\mathfrak{d}}$. The set of all non-singular translations $\mathfrak{c} \multimap \mathfrak{d}$; we write $\tau : \mathfrak{c} \multimap \mathfrak{d}$ for $\tau \in \mathfrak{c} \multimap \mathfrak{d}$. □

Remark 5.13 If $\tau : \mathbf{P} \multimap \mathbf{Q}$ and $\mathbf{P}$ has 0, then $\tau(a) = 0^{\mathbf{Q}}$ implies $a = 0^{\mathbf{P}}$.

Proof. If $\tau(a) = 0^{\mathbf{Q}}$, then $a \leq \tau^{\blacktriangleleft}(\tau(a)) = \tau^{\blacktriangleleft}(0^{\mathbf{Q}}) = 0^{\mathbf{P}}$, the final equality following by non-singularity. ◇

To the end of defining *concrete translations*, we now define the *reduced pre-image* of a *binary relationship*. The reduced pre-image plays an important role in algebraic logic. We have already implicitly used this notion (see Definition 2.95 on page 108). We shall encounter reduced pre-images extensively in this text. For example, we shall show that the logic $S(\mathcal{K}, \tau)$ of [BR99] (see

Example 2.85 on page 106) arises as the reduced pre-image of τ viewed as a binary relationship from Tm to Tm^2 (see Proposition 9.10 on page 315).

Definition 5.14 (Reduced Images) Let r be a binary relationship. For $A \subseteq \mathsf{do}(r)$, we define

$$r\lfloor A\rfloor = \{b \in \mathsf{rg}(r) \ : \ \overleftarrow{r}[\![b]\!] \subseteq A\},$$

which we call the **reduced-image** of A under r. The *function* $r\lfloor\cdot\rfloor$, from $\mathfrak{P}(\mathsf{do}(r))$ into $\mathfrak{P}(\mathsf{co}(r))$, is denoted by $r_{\lfloor\rfloor}(\cdot)$, which we call the **reduced-image function**. The set $r_{\lfloor\rfloor}[\mathfrak{P}(\mathsf{do}(r))]$, of all reduced-images under r, is denoted by $\mathsf{RedIm}(r)$. The reduced-images of $\overleftarrow{r}$, are called the **reduced-pre-images** of r. We write $\mathsf{PreRedIm}(r)$, for $\mathsf{RedIm}(\overleftarrow{r})$. $\square$

In the definition of the reduced-image, we insist that $b \in \mathsf{rg}(r)$, so as to avoid admitting points b with $\overleftarrow{r}[\![b]\!] = \emptyset \subseteq A$. We shall mostly encounter reduced *pre*-images. The reduced pre-image of B selects all points of the ground whose pole lies *entirely* within B. In other words,

$$\overleftarrow{r}\lfloor B\rfloor = \{a \in \mathsf{gr}(r) \ : \ r[\![a]\!] \subseteq B\}.$$

We enumerate some important properties of reduced images and pre-images. Some of these are only given in reduced image form and the reader is expected to formulate the reduced pre-image analogue.

Lemma 5.15 Let r be a binary relationship. The formulae of Table 5.1 are all valid. Further,

1. $r\lfloor A\rfloor \subseteq r[A]$.
2. $b \in r[A] - r\lfloor A\rfloor$ iff $\exists\,[a \in A, a' \notin A]\; a\ r\ b$ and $a'\ r\ b$
3. $r[\overleftarrow{r}\lfloor B\rfloor] \subseteq B$.
4. $A \cap \mathsf{gr}(r) \subseteq \overleftarrow{r}\lfloor r[A]\rfloor$.
5. $\overleftarrow{r}[\neg B] = \overset{\mathsf{gr}(r)}{\neg}\overleftarrow{r}\lfloor B\rfloor$.
6. $r[A] \subseteq B$ iff $A \subseteq \overleftarrow{r}\lfloor B\rfloor$.

Proof. $\boxed{\text{(5.22)}}$ Suppose that $A \subseteq A'$, and let $b \in r\lfloor A\rfloor$. So $b \in \mathsf{rg}(\alpha)$ and $\overleftarrow{r}[\![b]\!] \subseteq A \subseteq A'$, so $b \in r\lfloor A'\rfloor$. $\boxed{\text{(5.24)}}$ There exists no $b \in \mathsf{rg}(\alpha)$ such that $\overleftarrow{r}[\![b]\!] \subseteq \emptyset$, since $\emptyset \neq \overleftarrow{r}[\![b]\!]$ as $b \in \mathsf{rg}(\alpha)$. $\boxed{\text{(5.26)}}$ Let $b \in \mathsf{rg}(r)$. Certainly $\overleftarrow{r}[\![b]\!] \subseteq \mathsf{gr}(r)$, and so $b \in r\lfloor\mathsf{gr}(r)\rfloor$. So $\mathsf{rg}(r) \subseteq r\lfloor\mathsf{gr}(r)\rfloor$. The converse inclusion is trivial (by definition), so $\mathsf{rg}(r) = r\lfloor\mathsf{gr}(r)\rfloor$. The remaining equality follows by (5.22). $\boxed{\text{(5.28)}}$ By (5.22). $\boxed{\text{(5.30)}}$ By (5.22), for each $i \in I$, $r\lfloor\bigcap_{i\in I} A_i\rfloor \subseteq r\lfloor A_i\rfloor$. So $\bigcap_{i\in I} r\lfloor A_i\rfloor \subseteq r\lfloor\bigcap_{i\in I} A_i\rfloor$. Let $b \in r\lfloor\bigcap_{i\in I} A_i\rfloor$. So $b \in \mathsf{rg}(r)$ and $\overleftarrow{r}[\![b]\!] \subseteq \bigcap_{i\in I} A_i$. So, for each $i \in I$, $\overleftarrow{r}[\![b]\!] \subseteq A_i$, so $b \in r\lfloor A_i\rfloor$. So $b \in \bigcap_{i\in I} r\lfloor A_i\rfloor$. $\boxed{\text{(1)}}$ If $r\lfloor A\rfloor = \emptyset$, then trivial. Assume that $r\lfloor A\rfloor \neq \emptyset$. Let $b \in r\lfloor A\rfloor$, i.e., $\emptyset \neq \overleftarrow{r}[\![b]\!] \subseteq A$. So there *certainly* exists at least one point $a \in A$ with $a\ r\ b$. So $b \in r[A]$. $\boxed{\text{(2)}}$ $\boxed{\Rightarrow}$ Suppose that $b \in r[A] - r\lfloor A\rfloor$. Since $b \in r[A]$, there exists $a \in A$ with $a\ r\ b$. So $\emptyset \neq \overleftarrow{r}[\![b]\!]$. Since $b \notin r\lfloor A\rfloor$, $\overleftarrow{r}[\![b]\!] \not\subseteq A$. So there must exist $a' \notin A$ with $a'\ r\ b$. $\boxed{\Leftarrow}$ Suppose that $\exists\,[a \in A, a' \notin A]\; a\ r\ b$ and $a'\ r\ b$. Since $a\ r\ b$ and $a \in A$, $b \in r[A]$. But since $a'\ r\ b$ and $a' \notin A$, $\overleftarrow{r}[\![b]\!] \not\subseteq A$, and so $b \notin r\lfloor A\rfloor$. $\boxed{\text{(3)}}$ Let $b \in r[\overleftarrow{r}\lfloor B\rfloor]$. So there exists $a \in \overleftarrow{r}\lfloor B\rfloor$ with $a\ r\ b$. Since $a \in \overleftarrow{r}\lfloor B\rfloor$, $\emptyset \neq \overleftarrow{\overleftarrow{r}}[\![a]\!] \subseteq B$, i.e., $\emptyset \neq r[\![a]\!] \subseteq B$. So, since $a\ r\ b$, $b \in B$. $\boxed{\text{(4)}}$ Let $a \in A \cap \mathsf{gr}(r)$. So $a \in \mathsf{gr}(r)$ and $r[\![a]\!] \subseteq r[A]$. So $a \in \overleftarrow{r}\lfloor r[A]\rfloor$. $\boxed{\text{(5)}}$ $\boxed{\overleftarrow{r}[\neg B] \subseteq \overset{\mathsf{gr}(r)}{\neg}\overleftarrow{r}\lfloor B\rfloor}$ Let

$r\lfloor\cdot\rfloor$ is $\subseteq$-preserving. (5.22)	$\overleftarrow{r}\lfloor\cdot\rfloor$ is $\subseteq$-preserving. (5.23)
$r\lfloor\emptyset\rfloor = \emptyset.$ (5.24)	$\overleftarrow{r}\lfloor\emptyset\rfloor = \emptyset.$ (5.25)
$r\lfloor\mathsf{gr}(r)\rfloor = r\lfloor\mathsf{do}(r)\rfloor = \mathsf{rg}(r).$ (5.26)	$\overleftarrow{r}\lfloor\mathsf{rg}(r)\rfloor = \overleftarrow{r}\lfloor\mathsf{co}(r)\rfloor = \mathsf{gr}(r).$ (5.27)
$\bigcup_{i\in I} r\lfloor A_i\rfloor \subseteq r\left\lfloor \bigcup_{i\in I} A_i \right\rfloor.$ (5.28)	$\bigcup_{i\in I} \overleftarrow{r}\lfloor B_i\rfloor \subseteq \overleftarrow{r}\left\lfloor \bigcup_{i\in I} B_i \right\rfloor.$ (5.29)
$\bigcap_{i\in I} r\lfloor A_i\rfloor = r\left\lfloor \bigcap_{i\in I} A_i \right\rfloor.$ (5.30)	$\bigcap_{i\in I} \overleftarrow{r}\lfloor B_i\rfloor = \overleftarrow{r}\left\lfloor \bigcap_{i\in I} B_i \right\rfloor.$ (5.31)

Table 5.1: Fundamental properties of reduced-images and reduced-pre-images. (See Lemma 5.15 on page 179.)

$a \in \overleftarrow{r}[\neg B]$. So $a \notin \overleftarrow{r}\lfloor B\rfloor$ and $a \in \mathsf{gr}(r)$. So $a \in \overset{\mathsf{gr}(r)}{\neg}\overleftarrow{r}\lfloor B\rfloor$. $\boxed{\overleftarrow{r}[\neg B] \supseteq \overset{\mathsf{gr}(r)}{\neg}\overleftarrow{r}\lfloor B\rfloor}$ Let $a \in \overset{\mathsf{gr}(r)}{\neg}\overleftarrow{r}\lfloor B\rfloor$. So $a \in \mathsf{gr}(r)$ and $a \notin \overleftarrow{r}\lfloor B\rfloor$. So there exists some $a\ r\ b$ with $b \in \neg B$. So $a \in \overleftarrow{r}[\neg B]$. $\boxed{(6)}$ $\boxed{\Rightarrow}$ Suppose that $r[A] \subseteq B$. Let $a \in A$. Then $r[\![a]\!] \subseteq B$. So $\overset{\leftleftarrows}{r}[\![a]\!] = r[\![a]\!] \subseteq B$. Hence by definition, $a \in \overleftarrow{r}\lfloor B\rfloor$. So $A \subseteq \overleftarrow{r}\lfloor B\rfloor$. $\boxed{\Leftarrow}$ Suppose that $A \subseteq \overleftarrow{r}\lfloor B\rfloor$. Let $b \in r[A]$. So there exists $a \in A$ with $b \in r[\![a]\!]$. Since $a \in \overleftarrow{r}\lfloor B\rfloor$, $\overset{\leftleftarrows}{r}[\![a]\!] \subseteq B$. So $b \in r[\![a]\!] = \overset{\leftleftarrows}{r}[\![a]\!] \subseteq B$. Hence $r[A] \subseteq B$. $\diamond$

In the case of functions, viewed as binary relationships, the reduced pre-image and the functional inverse image coincide.

Remark 5.16 If f is a function, then $\overleftarrow{f}\lfloor\cdot\rfloor = f^{-1}[\cdot]$.

Proof. It suffices to show that $f^{-1}[B] - \overleftarrow{f}\lfloor B\rfloor = \emptyset$. Suppose to the contrary, that $\exists\, [b \in B, b' \notin B]\ b\overleftarrow{f}a$ and $b'\overleftarrow{f}a$. Then by (2) of Lemma 5.15, f cannot be a function, which is a contradiction. $\diamond$

We now introduce *concrete translations* between sets; these are just the *grounded* binary relationships. We shall show that concrete translations, when paired with their reduced pre-images, constitute elementary translations between the associated inclusion-ordered power-sets.

Definition 5.17 (Concrete Translation) A **(concrete) translation** (or **multi-map**) from a *non-empty* set A to a *non-empty* set C is a *grounded* binary relationship from A to C. The set of all translations from A to C is denoted by $A \multimap C$. We write $\tau : A \multimap C$ for $\tau \in A \multimap C$. For $\tau : A \multimap C$, define a function $\tau^{\blacktriangleleft}(\cdot) : \mathfrak{P}(C) \to \mathfrak{P}(A)$ by $\tau^{\blacktriangleleft}(D) = \overleftarrow{\tau}\lfloor D\rfloor$, and a function $\tau(\cdot) : \mathfrak{P}(A) \to \mathfrak{P}(C)$ by $\tau(C) = \tau[C]$, i.e., $\tau^{\blacktriangleleft}(\cdot) = (\overleftarrow{\tau})_{\lfloor\rfloor}(\cdot)$ and $\tau(\cdot) = \tau_{[]}(\cdot)$; we shall also write $\tau(a)$ for $\tau(\{a\})$ and $\tau^{\blacktriangleleft}(c)$ for $\tau^{\blacktriangleleft}(\{c\})$, for $a \in A$ and $c \in C$. We say that $\tau : A \multimap C$ is **finitary** if, for all $a \in A$, $\tau(a)$ is finite. $\square$

There is no need to continually speak of 'concrete' translations, since the distinction between concrete translations and elementary translation is clear from the parameters; elementary translations are between orders while concrete translations are between sets. In the later case, these give rise to elementary translations between the associated power-ordered sets, as we shall soon see.

Remark 5.18 If $\tau : A \multimap C$, then

$$a \in \tau^{\blacktriangleleft}(D) \text{ iff } \tau[\![a]\!] \subseteq D \quad \text{and} \tag{5.32}$$

$$B \subseteq \tau^{\blacktriangleleft}(D) \text{ iff } \tau[B] \subseteq D. \tag{5.33}$$

□

The following result demonstrates that (concrete) translations from A to C are in a natural one-to-one correspondence with the *non-singular* translations from $\mathfrak{P}(A)$ to $\mathfrak{P}(B)$; consequently no ambiguity arises from the overloaded symbolisms.

Proposition 5.19 If $\tau : A \multimap C$ then $\langle \tau(\cdot), \tau^{\blacktriangleleft}(\cdot)\rangle : \mathfrak{P}(A) \multimap \mathfrak{P}(C)$. If $\tau : \mathfrak{P}(A) \multimap \mathfrak{P}(C)$ then $\tau' : A \multimap C$ and $\tau'^{\blacktriangleleft} = \tau^{\blacktriangleleft}$, where τ' is the binary relationship from A to B defined by $\tau'[\![a]\!] = \tau(\{a\})$. Further, these two operations are mutually inverting.

Proof. $\boxed{\langle \tau(\cdot), \tau^{\blacktriangleleft}(\cdot)\rangle : \mathfrak{P}(A) \multimap \mathfrak{P}(C)}$ $\tau(\cdot)$ and $\tau^{\blacktriangleleft}(\cdot)$ are $\subseteq$-preserving by (1.1) of Table 1.1 on page 18 and (5.23) of Table 5.1, respectively. (5.15) follows by groundedness and (4) of Lemma 5.15, while (5.16) follows by (3). of Lemma 5.15. Non-singularity follows from (5.27) of Table 5.1 together with groundedness. $\boxed{\tau' : A \multimap C}$ Since τ is non-singular and B is non-empty, $\tau(\{a\}) \neq \emptyset$ by Remark 5.13, and so τ' is a well-defined and grounded binary relationship from A to B. $\boxed{\tau'^{\blacktriangleleft} = \tau^{\blacktriangleleft}}$ For $\emptyset \neq D$, $a \in \tau^{\blacktriangleleft}(D)$ [iff] $\{a\} \subseteq \tau^{\blacktriangleleft}(D)$ [iff by (5.19)] $\tau(\{a\}) \subseteq D$ [iff] $\tau'[\![\{a\}]\!] \subseteq D$ [iff by (5.32)] $a \in \tau'^{\blacktriangleleft}(D)$. Further, $\tau^{\blacktriangleleft}(\emptyset) = \emptyset$ by non-singularity, while $\tau'^{\blacktriangleleft}(\emptyset) = \emptyset$ by (5.25) of Table 5.1. $\boxed{\text{Mutually inverting}}$ Let $\tau : A \multimap C$, let $\pi = \langle \tau(\cdot), \tau^{\blacktriangleleft}(\cdot)\rangle$ and let π' be the binary relationship from A to B defined by $\pi'[\![a]\!] = \pi(\{a\})$. Then $\pi'[\![a]\!] = \pi(\{a\}) = \tau(\{a\}) = \tau[\{a\}] = \tau[\![a]\!]$. So $\tau = \pi'$. Conversely, let $\tau : \mathfrak{P}(A) \multimap \mathfrak{P}(C)$, let τ' be the binary relationship from A to B defined by $\tau'[\![a]\!] = \tau(\{a\})$ and let $\pi = \langle \tau'(\cdot), \tau'^{\blacktriangleleft}(\cdot)\rangle$. For $\emptyset \neq B$, $\pi(B) = \tau'(B) = \tau'[B] = \bigcup_{b\in B} \tau'[\![b]\!] = \bigcup_{b\in B} \tau(\{b\}) \overset{(5.18)}{=} \tau(\bigcup_{b\in B}\{b\}) = \tau(B)$, $\pi(\emptyset) = \emptyset = \tau(\emptyset)$ by (5.21), and $\pi^{\blacktriangleleft}(D) = \tau'^{\blacktriangleleft}(D) = \tau^{\blacktriangleleft}(D)$, the final equality being established earlier. ◇

Remark 5.20 If A and C are both non-empty and $f : A \to C$, then $f : A \multimap C$, in which case $f^{\blacktriangleleft}(D) = f^{-1}[D]$. □

While we have chosen the term 'translation' since translations between closed systems generalize formal translation between logics, we should perhaps motivate the use of this term more generally. The only possible objection could come from logicians, since it is this use of translation that we are generalizing. The simplest motivation is that there are logicians who consider orders and lattices as logics, for example frame theorist [BdRV01] and quantum logicians [Coh89]. To such logicians, our elementary closed systems are logics with languages and so the term translation is entirely justified.

5.2 Weak-Translations

In this section, we characterize $\vdash$-homomorphisms, cl-homomorphisms and $\|\cdot\|$-homomorphisms. While we are primarily interested in $\vdash$-homomorphism and, in particular, continuous translations, it shall transpire that the theory of cl-homomorphisms and $\|\cdot\|$-homomorphisms informs the theory of continuous translations.

5.2.1 $\vdash$-Homomorphisms

In the following result we characterize $\vdash$-homomorphisms. Note that since translations are weak-translations, this result yields characterizations of *continuous translations*. While more characterizations of continuous translations will be obtained in the next section, one of the most important characterizations, used extensively in the sequel and familiar to readers acquainted with algebraic logic (both as a property of substitutions and formal semantic translations between logics), namely equivalent condition (5), obtains at this weak elementary level.

Theorem 5.21 Let τ be a weak-translation from $\mathfrak{c}$ into $\mathfrak{d}$. The following conditions are equivalent.

1. τ is a $\vdash$-homomorphism.
2. τ is a $\dashv\vdash$-homomorphism.
3. $\tau(\|a\|_{\mathfrak{c}}) \leq \tau^{\star}(a)$ (equiv. $\tau(\|a\|_{\mathfrak{c}}) \leq \|\tau(a)\|_{\mathfrak{d}}$), for all $a \in \mathsf{uni}(\mathfrak{c})$.
4. $\tau^{\star}(\|a\|_{\mathfrak{c}}) \leq \tau^{\star}(a)$ (equiv. $\|\tau(\|a\|_{\mathfrak{c}})\|_{\mathfrak{d}} \leq \|\tau(a)\|_{\mathfrak{d}}$) , for all $a \in \mathsf{uni}(\mathfrak{c})$.
5. $\tau^{\star}(\|a\|_{\mathfrak{c}}) = \tau^{\star}(a)$ (equiv. $\|\tau(\|a\|_{\mathfrak{c}})\|_{\mathfrak{d}} = \|\tau(a)\|_{\mathfrak{d}}$), for all $a \in \mathsf{uni}(\mathfrak{c})$.
6. $\tau^{\star}$ is a $\vdash$-homomorphism.
7. $\tau(a) \vdash_{\mathfrak{d}} \tau(\|a\|)$, for all $a \in \mathsf{uni}(\mathfrak{c})$.
8. $\tau(a) \dashv\vdash_{\mathfrak{d}} \tau(\|a\|)$, for all $a \in \mathsf{uni}(\mathfrak{c})$.
9. $\tau(a) \dashv\vdash_{\mathfrak{d}} \tau^{\star}(\|a\|)$, for all $a \in \mathsf{uni}(\mathfrak{c})$.
10. If $a \vdash_{\mathfrak{c}} b$, c is $\mathsf{cl}_{\mathfrak{c}}$ and $\tau(a) \leq c$, then $\tau(b) \leq c$.

Proof. $\boxed{(1)\Rightarrow(2)}$ By (4.45). $\boxed{(2)\Rightarrow(1)}$ Suppose that τ is a $\dashv\vdash$-homomorphism and $a \vdash_{\mathfrak{c}} b$. By (4.46), there exists $c \geq b$ with $a \dashv\vdash_{\mathfrak{c}} c$. By order-preservation and assumption, $\tau(c) \geq \tau(b)$ and $\tau(a) \dashv\vdash_{\mathfrak{d}} \tau(c)$, and so by (4.46), $\tau(a) \vdash_{\mathfrak{d}} \tau(b)$. $\boxed{(1)\Rightarrow(3)}$ By (4.31), $a \vdash_{\mathfrak{c}} \|a\|_{\mathfrak{c}}$, and so by assumption, $\tau(a) \vdash_{\mathfrak{d}} \tau(\|a\|_{\mathfrak{c}})$. Hence by (4.29), $\tau(\|a\|_{\mathfrak{c}}) \leq \|\tau(a)\|_{\mathfrak{d}}$. $\boxed{(3)\Rightarrow(4)}$ Follows by order-preservation and idempotence of elementary closure operators. $\boxed{(4)\Rightarrow(5)}$ By (5.7). $\boxed{(5)\Rightarrow(1)}$ Suppose that $a \vdash_{\mathfrak{c}} b$. Then by (4.29), $b \leq \|a\|_{\mathfrak{c}}$. Since τ preserves order, $\tau(b) \leq \tau(\|a\|_{\mathfrak{c}}) \overset{(4.2)}{\leq} \big\|\tau(\|a\|_{\mathfrak{c}})\big\|_{\mathfrak{d}} \overset{(5)}{=} \|\tau(a)\|_{\mathfrak{d}}$. Hence by (4.29), $\tau(a) \vdash_{\mathfrak{d}} \tau(d)$. $\boxed{(4)\Leftrightarrow(6)}$ By the *already established* equivalence of (1) and (3), $\tau^{\star}$ is a $\vdash$-homomorphism iff $\tau^{\star}(\|a\|_{\mathfrak{c}}) \leq \|\tau^{\star}(a)\|_{\mathfrak{d}}$ iff $\big\|\tau(\|a\|_{\mathfrak{c}})\big\|_{\mathfrak{d}} \leq \big\|\|\tau(a)\|_{\mathfrak{d}}\big\|_{\mathfrak{d}}$ iff $\big\|\tau(\|a\|_{\mathfrak{c}})\big\|_{\mathfrak{d}} \leq \|\tau(a)\|_{\mathfrak{d}}$ iff (4). $\boxed{(1)\Rightarrow(7)}$ $a \vdash_{\mathfrak{c}} \|a\|_{\mathfrak{c}}$, so by assumption (1), $\tau(a) \vdash_{\mathfrak{d}} \tau(\|a\|_{\mathfrak{c}})$. $\boxed{(7)\Rightarrow(3)}$ Since, by assumption (7), $\tau(a) \vdash_{\mathfrak{d}} \tau(\|a\|_{\mathfrak{c}})$, we have, by (4.29), $\tau(\|a\|_{\mathfrak{c}}) \leq \|\tau(a)\|_{\mathfrak{d}}$. $\boxed{(7)\Rightarrow(8)}$ By assumption (7), $\tau(a) \vdash_{\mathfrak{d}} \tau(\|a\|)$. Further, by order-increasingness of elementary closure operators, order-preservation of weak-translation and (4.29), it is (generally) true that $\tau(\|a\|_{\mathfrak{c}}) \vdash_{\mathfrak{d}} \tau(a)$, and so the result follows by (4.45). $\boxed{(8)\Rightarrow(7)}$ By (4.45). $\boxed{(8)\Leftrightarrow(9)}$ It is (generally) true that $\tau(\|a\|_{\mathfrak{c}}) \dashv\vdash_{\mathfrak{d}} \big\|\tau(\|a\|_{\mathfrak{c}})\big\|_{\mathfrak{d}}$, by (4.47), the result follows by the transitivity and symmetry of $\dashv\vdash_{\mathfrak{d}}$. $\boxed{(3)\Rightarrow(10)}$ Suppose that $a \vdash_{\mathfrak{c}} b$, c is $\mathsf{cl}_{\mathfrak{c}}$ and $\tau(a) \leq c$. Since $a \vdash_{\mathfrak{c}} b$, it follows from (4.29) that $b \leq \|a\|_{\mathfrak{c}}$. Hence $\tau(b) \leq \tau(\|a\|_{\mathfrak{c}})$. So by assumption (3) and minimality, $\tau(b) \leq \|\tau(a)\|_{\mathfrak{d}} \leq c$, since $\tau(a) \leq c$ and c is $\mathsf{cl}_{\mathfrak{c}}$. $\boxed{(10)\Rightarrow(1)}$ Suppose that $a \vdash_{\mathfrak{c}} b$. Then $\|\tau(a)\|_{\mathfrak{d}} \in \mathsf{cl}_{\mathfrak{d}}$ and $\tau(a) \leq \|\tau(a)\|_{\mathfrak{d}}$, so by assumption (10), $\tau(b) \leq \|\tau(a)\|_{\mathfrak{d}}$. Hence $\tau(a) \vdash_{\mathfrak{d}} \tau(b)$, by (4.29). $\diamond$

Corollary 5.22 If τ is a $\vdash$-homomorphism from $\mathfrak{c}$ into $\mathfrak{d}$ and $\mathfrak{c}' \preceq \mathfrak{c}$, then τ is a $\vdash$-homomorphism from $\mathfrak{c}'$ into $\mathfrak{d}$. $\square$

The following property of $\vdash$-*reflecting* homomorphisms proves useful to the sequel.

Proposition 5.23 (Pullback) Suppose that weak-translation τ is $\vdash$-reflecting from $\mathfrak{d}$ into $\mathfrak{e}$. Then, for all elementary closed systems $\mathfrak{c}$ and weak-translations π from $\mathfrak{c}$ to $\mathfrak{d}$, if $\tau\pi$ is a $\vdash$-translation from $\mathfrak{c}$ to $\mathfrak{e}$, then π is a $\vdash$-translation from $\mathfrak{c}$ to $\mathfrak{d}$.

Proof. If $a \vdash_{\mathfrak{c}} b$ then $(\tau\pi)(a) \vdash_{\mathfrak{e}} (\tau\pi)(b)$. So by $\vdash$-reflection, $\pi(a) \vdash_{\mathfrak{d}} \tau(b)$, as required. $\diamond$

Remark 5.24 The composition of $\vdash$-homomorphisms is a $\vdash$-homomorphism.

5.2.1.1 Examples

The following examples demonstrate that even our *weakest* notion of a $\vdash$-homomorphism between elementary closed systems, unifies the notions of *structurality* and *filter* in the theory of sentential calculi.

Example 5.25 (Structurality in Sentential Calculi)

Theorem 2.22, together with the definition of a continuous function, characterizes the consequence relations of sentential calculi of signature $\mathfrak{p}$, as precisely those finitary consequence relations $\vdash$ over $\mathsf{Fm}(\mathfrak{p})$ such that, for every $\mathcal{S}$-substitution σ, $\underset{\rightarrow}{\sigma}$ is continuous from $\vdash$ into itself.

$\square$

Example 5.26 (Filter Closed Systems)

The proofs of following results follow immediately from the definition of an $\mathcal{S}$-filter (see Definition 2.41 on page 101) and (10) of Theorem 5.21.

Proposition 5.27 Let $\mathcal{S}$ be a sentential n-calculus of type $\mathfrak{a}$, $\mathbf{M}$ an $\mathfrak{a}$-matrix and $F \subseteq \mathsf{uni}(\mathbf{M})$. Then F is an $\mathcal{S}$-filter of $\mathbf{M}$ iff $\mathsf{D}_{\mathbf{M}} \subseteq F$ and, for every interpretation i of $\mathcal{S}$ into $\mathbf{M}$, $\underset{\rightarrow}{\mathsf{i}}$ is continuous from the finitary closed system of $\mathcal{S}$-theories into the closed system $\mathbb{C}(\langle \mathsf{uni}(\mathbf{M}), F\rangle)$.

Proposition 5.28 Let $\mathcal{S}$ be a sentential n-calculus of type $\mathfrak{a}$ and $\mathbf{M}$ an $\mathfrak{a}$-matrix. Every interpretation of $\mathcal{S}$ into $\mathsf{alg}(\mathbf{M})$ is continuous from the finitary closed system of $\mathcal{S}$-theories into the finitary closed system of $\mathcal{S}$-filters of $\mathbf{M}$.

$\square$

5.2.2 cl-Homomorphisms

We turn now to cl-homomorphisms. The importance of cl-homomorphisms will become evident in the next section, where we will show that a translation τ is continuous iff $\tau^{\blacktriangleleft}$ is closed, i.e., a cl-homomorphism. Consequently, the following characterization of cl-homomorphisms yields a characterization of continuous translations when interpreted with $\tau^{\blacktriangleleft}$ playing the role τ.

Proposition 5.29 Let τ be a weak-translation from $\mathfrak{c}$ into $\mathfrak{d}$. The following conditions are equivalent.

1. τ is a cl-homomorphism.
2. $\tau(\|a\|_{\mathfrak{c}}) \geq \|\tau(\|a\|_{\mathfrak{c}})\|_{\mathfrak{d}}$ (i.e., $\tau(\|a\|_{\mathfrak{c}}) \geq \tau^\star(\|a\|_{\mathfrak{c}})$) for all $a \in \mathsf{uni}(\mathfrak{c})$.
3. $\tau(\|a\|_{\mathfrak{c}}) = \|\tau(\|a\|_{\mathfrak{c}})\|_{\mathfrak{d}}$ (i.e., $\tau(\|a\|_{\mathfrak{c}}) = \tau^\star(\|a\|_{\mathfrak{c}})$) for all $a \in \mathsf{uni}(\mathfrak{c})$.
4. $\tau(g) = \|\tau(g)\|_{\mathfrak{d}}$ (i.e., $\tau(g) = \tau^\star(g)$) for all $g \in \mathsf{cl}_{\mathfrak{c}}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ By assumption, $\tau(\|a\|_{\mathfrak{c}})$ is $\mathfrak{d}$-closed, and so by minimality, $\tau(\|a\|_{\mathfrak{c}}) \geq \left\|\tau(\|a\|_{\mathfrak{c}})\right\|_{\mathfrak{d}}$. $\boxed{(2)\Rightarrow(3)}$ The converse inequality is always true, since elementary closure operators are increasing. $\boxed{(3)\Rightarrow(4)}$ Let $g \in \mathsf{cl}_{\mathfrak{c}}$. Then $\|\tau(g)\|_{\mathfrak{d}} \overset{(4.13)}{=} \left\|\tau(\|g\|_{\mathfrak{c}})\right\|_{\mathfrak{d}} \overset{(3)}{=} \tau(\|g\|_{\mathfrak{c}}) \overset{(4.13)}{=} \tau(g)$. $\boxed{(4)\Rightarrow(1)}$ By (4.13). ⋄

The following necessary condition proves useful to the sequel.

Remark 5.30 If τ is a cl-homomorphism then $\tau(\|a\|_{\mathfrak{c}}) \geq \|\tau(a)\|_{\mathfrak{d}}$.

Proof. Since $a \leq \|a\|_{\mathfrak{c}}$ and τ preserves order, $\tau(a) \leq \tau(\|a\|_{\mathfrak{c}})$. Since $\tau(\|a\|_{\mathfrak{c}})$ is closed by assumption, $\|\tau(a)\|_{\mathfrak{d}} \leq \tau(\|a\|_{\mathfrak{c}})$ by minimality. ⋄

Open Problem 5.31 Comparing the necessary condition of the previous remark to equivalent condition (2) of the previous proposition, show that this necessary condition is not sufficient.

Note that cl-homomorphisms are abundant in this text, since the closure of any weak-translation must trivially be a cl-homomorphism. More precisely, we have the following.

Remark 5.32 If τ is a weak-translation from $\mathfrak{c}$ into $\mathfrak{d}$ then $\tau^\star$ is a a cl-homomorphism.

5.2.2.1 Examples

Example 5.33 (Algebra Homomorphisms between Subuniverse Closed Systems)

Let $\mathbf{A}$ and $\mathbf{B}$ be two $\mathfrak{a}$-algebras. The proof of the following result follows from condition (3) of Theorem 1.309 on page 61.

Proposition 5.34 Every homomorphism f from algebra $\mathbf{A}$ into algebra $\mathbf{B}$ is closed (and continuous) from $F(\mathbf{A}, \mathsf{su})$ into $F(\mathbf{B}, \mathsf{su})$.

□

Recall the definition of the *promotion* $\underset{\rightarrow[2]}{f}$ of a function f given in Definition 1.28 on page 19.

Example 5.35 (Promoted Homomorphisms between Congruence Closed Systems)

Let $\mathbf{A}$ and $\mathbf{B}$ be two $\mathfrak{a}$-algebras. The following result follows from Proposition 1.358 on page 68.

Proposition 5.36 If $f : \mathbf{A} \to \mathbf{B}$ then $\underset{\rightarrow[2]}{f}$ is closed from $\mathsf{Con}(\mathbf{A})$ into $\mathsf{Con}(\mathbf{B})$.

□

5.2.3 $\|\cdot\|$-Homomorphisms

The importance of $\|\cdot\|$-homomorphisms is the fact that a weak-translation is a $\vdash$-homomorphism iff its closure is a $\|\cdot\|$-homomorphism (see Corollary 5.38). We begin by showing that the $\|\cdot\|$-homomorphisms are precisely the closed $\vdash$-homomorphisms.

Proposition 5.37 Let τ be a weak-translation from $\mathfrak{c}$ into $\mathfrak{d}$. The following conditions are equivalent.

1. τ is a $\|\cdot\|$-homomorphism.

2. τ is a $\vdash$-homomorphism and a cl-homomorphism.

Proof. $\boxed{(1)\Rightarrow(2)}$ Suppose that τ is a $\|\cdot\|$-homomorphism. $\boxed{\text{cl-homomorphism}}$ If c is $\mathsf{cl}_{\mathfrak{c}}$, then $f(c) \overset{(4.13)}{=} f(\|c\|_{\mathfrak{c}}) = \|f(c)\|_{\mathfrak{d}} \overset{(4.15)}{\text{is}}$ $\mathsf{cl}_{\mathfrak{d}}$. $\boxed{\vdash\text{-homomorphism}}$ $a \vdash_{\mathfrak{c}} b$ [implies by (4.29)] $b \leq \|a\|_{\mathfrak{c}}$ [implies by order-preservation and assumption] $f(b) \leq f(\|a\|_{\mathfrak{c}}) = \|f(a)\|_{\mathfrak{d}}$ [implies by (4.29)] $f(a) \vdash_{\mathfrak{d}} f(b)$. $\boxed{(2)\Rightarrow(1)}$ By Remark 5.30, $f(\|a\|_{\mathfrak{c}}) \geq \|f(a)\|_{\mathfrak{d}}$, and by (3) of Theorem 5.21, $\tau(\|a\|_{\mathfrak{c}}) \leq \|\tau(a)\|_{\mathfrak{d}}$. Hence $f(\|a\|_{\mathfrak{c}}) = \|f(a)\|_{\mathfrak{d}}$, as required. $\diamond$

The following further characterization of $\vdash$-homomorphisms follows from equivalent condition (6) of Theorem 5.21, Remark 5.32 and equivalent condition (2) of Proposition 5.37.

Corollary 5.38 Let τ be a weak-translation from $\mathfrak{c}$ into $\mathfrak{d}$. The following conditions are equivalent.

1. τ is a $\vdash$-homomorphism.

2. $\tau^{\star}$ is $\|\cdot\|$-homomorphism.

5.3 Translations

In this section we focus on continuous translations. In the context of translations, we are able to obtain many more characterizations of continuous translations over and above those characterizations of $\vdash$-homomorphisms obtained in the previous section. In particular, we shall show that a translation τ is continuous iff $\tau^{\blacktriangleleft}$ is closed. An important necessary condition is also derived, namely, if a translation τ is continuous then its closure $\tau^{\star}$ defined a $\blacktriangledown$-preserving function between the orders of closed points. We have not been able to establish a converse to this result in the elementary setting, that is, to show that all $\blacktriangledown$-preserving functions between the orders of closed points arise in this manner; in the concrete case this is indeed the case (see Theorem 5.108). We also characterize $\vdash$-reflecting translations and strictly continuous translations. Strictly continuous translations generalize the notion that one logic be a formal semantics for another (see Definition 2.95 on page 108). An important characterization of strictly continuous translations is obtained in terms of the notion of the product of a translation from an order to an elementary closed system; this notion of product will play an important role in the sequel, since we shall see that the logic $S(\mathcal{K}, \tau)$ of [BR99] and our generalization of this logic to sentential n-deductive systems all arise as such products. In the elementary context we have been unable to define the notion of the product of multiple translation, since this notion requires that the closed system have a basis and a basis is only definable in non-elementary settings (see ∮4.2.2 and ∮5.4.3.1). The final topic of this section is iseomorphisms between elementary closed systems; these are pairs of continuous translations (in opposite direction) that are *mutually untranslating*; we shall show that in this case these translations are in fact strict. Iseomorphisms generalize formal equivalent semantics of sentential calculi (see Definition 2.97 on page 109).

We first note two simple properties satisfied by translations generally.

Remark 5.39 Let τ be a translation from $\mathfrak{c}$ to $\mathfrak{d}$. For all $h \in \mathsf{cl}_{\mathfrak{d}}$,

$$\tau^{\star}(\tau^{\blacktriangleleft}(h)) \leq h \tag{5.34}$$

and for all $g \in \mathsf{cl}_{\mathfrak{c}}$,

$$\tau^{\star}(g) \geq \tau^{\star}(\tau^{\blacktriangleleft}(\tau^{\star}(g))). \tag{5.35}$$

Proof. $\boxed{\text{(5.34)}}$ $\tau^{\star}(\tau^{\blacktriangleleft}(h)) = \|\tau(\tau^{\blacktriangleleft}(h))\|_{\mathfrak{d}} \leq \|h\|_{\mathfrak{d}} = h$. $\boxed{\text{(5.35)}}$ $\tau^{\star}(g) \geq \tau(\tau^{\blacktriangleleft}(\tau^{\star}(g)))$, hence $\tau^{\star}(g) = \|\tau^{\star}(g)\|_{\mathfrak{d}} \geq \|\tau(\tau^{\blacktriangleleft}(\tau^{\star}(g)))\|_{\mathfrak{d}} = \tau^{\star}(\tau^{\blacktriangleleft}(\tau^{\star}(g)))$. ⋄

5.3.1 Continuous Translations

In the following result, we characterize continuous translations. Note that continuous translations are $\vdash$-homomorphisms, so the characterizations of $\vdash$-homomorphisms of Theorem 5.21 and Corollary 5.38 also pertain. Note that in the proof of this result, we establish the equivalence of the elementary conditions (1) through to (11) independently of the non-elementary conditions (12) and (13), and do so by elementary arguments.

Theorem 5.40 For a translation τ from $\mathfrak{c}$ to $\mathfrak{d}$, the following conditions are equivalent.

1. τ is continuous from $\mathfrak{c}$ into $\mathfrak{d}$.
2. The *weak-translation* τ satisfies any of the equivalent conditions of Theorem 5.21.
3. The *weak-translation* $\tau^{\blacktriangleleft}$ is a cl-homomorphism from $\mathfrak{d}$ into $\mathfrak{c}$, i.e., $\tau^{\blacktriangleleft}(h) \in \mathsf{cl}_{\mathfrak{c}}$, for all $h \in \mathsf{cl}_{\mathfrak{d}}$.
4. $\tau^{\blacktriangleleft}(\|c\|) \geq \tau^{\blacktriangleleft\star}(\|c\|)$ (i.e., $\tau^{\blacktriangleleft}(\|c\|) \geq \|\tau^{\blacktriangleleft}(\|c\|)\|$), for all $c \in \mathsf{uni}(\mathfrak{d})$.
5. $\tau^{\blacktriangleleft}(\|c\|) = \tau^{\blacktriangleleft\star}(\|c\|)$ (i.e., $\tau^{\blacktriangleleft}(\|c\|) = \|\tau^{\blacktriangleleft}(\|c\|)\|$), for all $c \in \mathsf{uni}(\mathfrak{d})$.
6. $\tau^{\blacktriangleleft}(h) = \tau^{\blacktriangleleft\star}(h)$ (i.e., $\tau^{\blacktriangleleft}(h) = \|\tau^{\blacktriangleleft}(h)\|$), for all $h \in \mathsf{cl}_{\mathfrak{d}}$.
7. $\|a\| \leq \tau^{\blacktriangleleft}(\tau^{\star}(a))$ (i.e., $\|a\| \leq \tau^{\blacktriangleleft}(\|\tau(a)\|)$), for all $a \in \mathsf{uni}(\mathfrak{c})$.
8. $\tau^{\blacktriangleleft}(\|c\|_{\mathfrak{d}}) \geq \tau^{\blacktriangleleft\star}(c)$ (i.e., $\tau^{\blacktriangleleft}(\|c\|_{\mathfrak{d}}) \geq \|\tau^{\blacktriangleleft}(c)\|$), for all $c \in \mathsf{uni}(\mathfrak{d})$.
9. $\langle \tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}, \tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} \rangle$ is a translation from $\mathbf{cl}_{\mathfrak{c}}$ into $\mathbf{cl}_{\mathfrak{d}}$.
10. $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \to \mathbf{cl}_{\mathfrak{d}}$ and $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} : \mathbf{cl}_{\mathfrak{d}} \to \mathbf{cl}_{\mathfrak{c}}$.

11. $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} : \mathbf{cl}_{\mathfrak{d}} \to \mathbf{cl}_{\mathfrak{c}}$.

12. $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \to_{\blacktriangledown} \mathbf{cl}_{\mathfrak{d}}$ and $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} : \mathbf{cl}_{\mathfrak{d}} \to_{\blacktriangle} \mathbf{cl}_{\mathfrak{c}}$.

13. $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} : \mathbf{cl}_{\mathfrak{d}} \to_{\blacktriangle} \mathbf{cl}_{\mathfrak{c}}$.

Proof.

$\boxed{(1)\Leftrightarrow(2)}$ Trivial. $\boxed{(1)\Rightarrow(3)}$ Let $h \in \mathsf{cl}_{\mathfrak{d}}$. Suppose that $\tau^{\blacktriangleleft}(h) \vdash_{\mathfrak{c}} b$. (*It suffices to show that* $b \leq \tau^{\blacktriangleleft}(h)$, *since then* $\tau^{\blacktriangleleft}(h) \in \mathsf{cl}_{\mathfrak{c}}$ *by (4.33).*) Since $\tau^{\blacktriangleleft}(h) \vdash_{\mathfrak{c}} b$, by assumption (1), $\tau(\tau^{\blacktriangleleft}(h)) \vdash_{\mathfrak{d}} \tau(b)$. By (5.16), $\tau(\tau^{\blacktriangleleft}(h)) \leq h$, and so $h \vdash_{\mathfrak{d}} \tau(b)$. Hence $\tau(b) \leq h$, since $h \in \mathsf{cl}_{\mathfrak{d}}$, and since weak-translations preserve order, $\tau^{\blacktriangleleft}(\tau(b)) \leq \tau^{\blacktriangleleft}(h)$. So by (5.15), $b \leq \tau^{\blacktriangleleft}(\tau(b)) \leq \tau^{\blacktriangleleft}(h)$. $\boxed{(3)\Leftrightarrow(4)\Leftrightarrow(5)\Leftrightarrow(6)}$ By Proposition 5.29. $\boxed{(6)\Rightarrow(7)}$ Since $a \leq \tau^{\blacktriangleleft}(\tau(a)) \leq \tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})$, $\|a\|_{\mathfrak{c}} \leq \left\|\tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})\right\|_{\mathfrak{c}} = \tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})$ by assumption (6). $\boxed{(7)\Rightarrow(8)}$ By assumption (7), $\|\tau^{\blacktriangleleft}(c)\|_{\mathfrak{c}} \leq \tau^{\blacktriangleleft}(\|\tau(\tau^{\blacktriangleleft}(c))\|) \leq \tau^{\blacktriangleleft}(\|c\|)$. $\boxed{(8)\Rightarrow(1)}$ Suppose that $a \vdash b$. Then $b \leq \|a\|$, and hence $\tau(b) \leq \tau(\|a\|)$. Since $a \leq \tau^{\blacktriangleleft}(\tau(a))$, it follows that $\|a\|_{\mathfrak{c}} \leq \|\tau^{\blacktriangleleft}(\tau(a))\|_{\mathfrak{c}} \leq \tau^{\blacktriangleleft}(\|\tau(a)\|)$, the final inequality following assumption (8). Hence $\tau(b) \leq \tau(\|a\|) \leq \tau(\tau^{\blacktriangleleft}(\|\tau(a)\|)) \leq \|\tau(a)\|$. Hence, $\tau(a) \vdash \tau(b)$. $\boxed{(3) \text{ and } (7) \Rightarrow(9)}$ By assumption (3), $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} : \mathsf{cl}_{\mathfrak{d}} \to \mathsf{cl}_{\mathfrak{c}}$, and certainly $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathsf{cl}_{\mathfrak{c}} \to \mathsf{cl}_{\mathfrak{d}}$. Further, $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}}$ is order-preserving from $\mathbf{cl}_{\mathfrak{d}}$ into $\mathbf{cl}_{\mathfrak{c}}$, and $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}$ is order-preserving from $\mathbf{cl}_{\mathfrak{c}}$ into $\mathbf{cl}_{\mathfrak{d}}$. Now by (7), for each $g \in \mathsf{cl}_{\mathfrak{c}}$, $\tau^{\blacktriangleleft}(\tau^{\star}(g)) \geq \|g\|_{\mathfrak{c}} = g$. Further, it is (generally) true that for each $h \in \mathsf{cl}_{\mathfrak{d}}$, $\tau^{\star}(\tau^{\blacktriangleleft}(h)) \leq h$, by (5.34). $\boxed{(9)\Rightarrow(10)\Rightarrow(11)\Rightarrow(3)}$ Trivial $\boxed{(9)\Rightarrow(12)}$ By Proposition 5.8. $\boxed{(12)\Rightarrow(13)\Rightarrow(3)}$ Trivial. ◇

Note that while condition (12) of the previous theorem cannot be weakened to $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \to_{\blacktriangledown} \mathbf{cl}_{\mathfrak{d}}$ only (see Counter-Example 5.107 on page 204), we shall show that, in the *concrete* case, *all* ▼-preserving functions from $\mathbf{cl}_{\mathbb{C}}$ to $\mathbf{cl}_{\mathbb{D}}$ arise in this manner (see Theorem 5.108).

The following result enumerates useful *necessary* conditions for a translation to by a $\vdash$-translation.

Corollary 5.41 If τ is continuous from $\mathfrak{c}$ into $\mathfrak{d}$ then, for all $g \in \mathsf{cl}_{\mathfrak{c}}$,

$$g \leq \tau^{\blacktriangleleft}(\tau^{\star}(g)) \quad \text{and} \tag{5.36}$$

$$\tau^{\star}(g) = \tau^{\star}(\tau^{\blacktriangleleft}(\tau^{\star}(g))). \tag{5.37}$$

Proof. $\boxed{(5.36)}$ By (7) of the previous theorem, $g = \|g\| \leq \tau^{\blacktriangleleft}(\tau^{\star}(g))$. $\boxed{(5.37)}$ By (5.36), $g \leq \tau^{\blacktriangleleft}(\tau^{\star}(g))$, hence $\tau^{\star}(g) \leq \tau^{\star}(\tau^{\blacktriangleleft}(\tau^{\star}(g)))$. The converse inequality is generally valid by (5.35). ◇

5.3.1.1 Examples

We shall now consider a number of examples, many of which shall prove useful in the sequel. We begin by extending Proposition 5.28 of Example 5.26.

Example 5.42 (Filter Closed Systems)

The following result follows from Corollary 2.43 on page 101 and (3) of Theorem 5.40.

Proposition 5.43 Let $\mathcal{S}$ be a sentential calculus, $\mathbf{M}$ and $\mathbf{N}$ be $\mathsf{sig}(\mathcal{S})$-matrices and f be a *surjective* (matrix) homomorphism from $\mathbf{M}$ onto $\mathbf{N}$. Then f is continuous from the finitary closed system of $\mathcal{S}$-filters of $\mathbf{M}$ onto the finitary closed system of $\mathcal{S}$-filters of $\mathbf{N}$.

□

The examples that follow identify important continuous functions between some of the closed systems introduced earlier in this text. In cases where these closed systems are defined over algebras and are finitary, and where the identified functions are homomorphisms, the endomorphic case, when applied to the term algebra, implies that these closed systems are precisely the theories of some sentential calculus; this follows in the light of Example 5.25. Consequently, a number of important sentential calculi will be introduced amongst these examples. In ∮6, where we introduce the notion of a *logic over a construct*, the endomorphic case of these same results will permit us to define *structural* logics on each algebra itself, as opposed to merely the term algebra.

Recall Definition 1.302 on page 60 where we asserted that the set $\mathsf{Su}(\mathbf{A})$ of all subuniverses of a structure $\mathbf{A}$ form an algebraic closed system, and recall that the compatible relations $\mathsf{Cpat}(\mathbf{A})$ on an algebra $\mathbf{A}$ are precisely the subuniverses of $\mathbf{A}^2$ (see Remark 1.350 on page 67) and that $\mathsf{Cpat}(\mathbf{A})$ forms an algebraic closed system.

Example 5.44 (Homomorphisms between Subuniverse Closed Systems)

Let $\mathbf{A}$ and $\mathbf{B}$ be two $\mathfrak{a}$-algebras. The proof of the following result follows from condition (3) of Theorem 1.309 on page 61, together with Theorem 5.40.

Proposition 5.45 Every homomorphism f from algebra $\mathbf{A}$ into algebra $\mathbf{B}$ is a continuous function from $\mathsf{Su}(\mathbf{A})$ into $\mathsf{Su}(\mathbf{B})$. □

The following corollary follows from the previous proposition and Remark 1.316 on page 62.

Corollary 5.46 For every homomorphism f from $\mathbf{A}$ into algebra $\mathbf{B}$, $\underset{\rightarrow}{f}$ is continuous from $\mathsf{Cpat}(\mathbf{A})$ into $\mathsf{Cpat}(\mathbf{B})$.

□

It follows, from Proposition 5.45, that every substitution of $\mathbf{Tm}$ is continuous from $\mathsf{Su}(\mathbf{Tm})$ into itself, and consequently the $\mathsf{Su}(\mathbf{Tm})$-closed sets (which are precisely the subuniverses of $\mathbf{Tm}$) constitute the theories of a sentential 1-calculus. We now identify this logic. The reader is urged to recall the definition of the formal system $F(\mathbf{A}, \mathsf{su})$ of subuniverses of an algebra $\mathbf{A}$, given in Example 4.133 on page 167.

Example 5.47 (The Sentential Calculi of Subuniverses)

Let $\mathfrak{a}$ be a type of algebras.

Definition 5.48 (The Sentential Calculi of Subuniverses) Let $S(\mathfrak{a}, \mathsf{su})$ denote the sentential 1-calculus determined by

1. all axioms $\vdash \mathbf{0}$ where $\mathbf{0} \in \mathsf{Symb_c}(\mathfrak{a})$, and
2. all rules $x_1, \ldots, x_{\mathsf{ar}(\star)} \vdash \star(x_1, \ldots, x_{\mathsf{ar}(\star)})$, where $\star \in \mathsf{Symb_o}(\mathfrak{a})$ and $x_1, \ldots, x_{\mathsf{ar}(\star)}$ is some choice of distinct variables.

We call this logic the **sentential calculus of $\mathfrak{a}$-subuniverses**. □

In the following result we demonstrate that the $S(\mathfrak{a}, \mathsf{su})$-filters on an algebra $\mathbf{A}$ are precisely the subuniverses of $\mathbf{A}$.

Proposition 5.49 For any $\mathfrak{a}$-algebra $\mathbf{A}$, $\mathsf{Fi}_{S(\mathfrak{a},\mathsf{su})}(\mathbf{A}) = \mathsf{Su}(\mathbf{A}) = \mathsf{Th}(F(\mathbf{A},\mathsf{su}))$.

Proof. $\boxed{\mathsf{Fi}_{S(\mathfrak{a},\mathsf{su})}(\mathbf{A}) \subseteq \mathsf{Su}(\mathbf{A})}$ Let $F \in \mathsf{Fi}_{S(\mathfrak{a},\mathsf{su})}(\mathbf{A})$. Let $\mathbf{0} \in \mathsf{Symb_c}(\mathfrak{a})$. By definition, $\vdash_{S(\mathfrak{a},\mathsf{su})} \mathbf{0}$. Let i be any interpretation of $S(\mathfrak{a},\mathsf{su})$ into $\mathbf{A}$. Since $F \in \mathsf{Fi}_{S(\mathfrak{a},\mathsf{su})}(\mathbf{A})$, $\mathbf{0}^{\mathbf{A}} = \mathsf{i}(\mathbf{0}) \in F$. Let $\star \in \mathsf{Symb_o}(\mathfrak{a})$ and $a_1, \ldots, a_{\mathsf{ar}(\star)} \in F$. By definition, $\{x_1, \ldots, x_{\mathsf{ar}(\star)}\} \vdash_{S(\mathfrak{a},\mathsf{su})} \star(x_1, \ldots, x_{\mathsf{ar}(\star)})$, for some distinct variables $x_1, \ldots, x_{\mathsf{ar}(\star)}$. Let i be any interpretation of $S(\mathfrak{a},\mathsf{su})$ into $\mathbf{A}$ mapping each $x_i \mapsto a_i$. Since $F \in \mathsf{Fi}_{S(\mathfrak{a},\mathsf{su})}(\mathbf{A})$, $\{x_1, \ldots, x_{\mathsf{ar}(\star)}\} \vdash_{S(\mathfrak{a},\mathsf{su})} \star(x_1, \ldots, x_{\mathsf{ar}(\star)})$ and $\mathsf{i}\left[\{x_1, \ldots, x_{\mathsf{ar}(\star)}\}\right] = \{a_1, \ldots, a_{\mathsf{ar}(\star)}\} \subseteq F$, $\star^{\mathbf{A}}(a_1, \ldots, a_{\mathsf{ar}(\star)}) = \mathsf{i}(\star(x_1, \ldots, x_{\mathsf{ar}(\star)})) \in F$. Consequently, F is closed under fundamental constant and fundamental operations, and hence is a subuniverse. $\boxed{\mathsf{Su}(\mathbf{A}) \subseteq \mathsf{Fi}_{S(\mathfrak{a},\mathsf{su})}(\mathbf{A})}$ Let $F \in \mathsf{Su}(\mathbf{A})$. Suppose that $P \vdash_{S(\mathfrak{a},\mathsf{su})} p$, $\mathsf{i} : \mathbf{Tm} \to \mathbf{A}$ with $\mathsf{i}[P] \subseteq F$. (*We show that $p \in F$ by induction on the length of derivations from F.*) $\boxed{\text{Base Case}}$ Suppose that p is derivable from P by a derivation of length 1. If $p \in P$, then $\mathsf{i}(p) \in F$. Otherwise, there exists an axiom $\vdash \mathbf{0}$ and a substitution σ, with $\sigma(\mathbf{0}) = p$. Since $\sigma(\mathbf{0}) = \mathbf{0}$, $p = \mathbf{0}$. Hence $\mathsf{i}(p) = \mathsf{i}(\mathbf{0}) = \mathbf{0}^{\mathbf{A}} \in F$, since subuniverses are closed under fundamental constants. $\boxed{\text{Inductive Hypothesis}}$ Suppose that for any term p derivable from P by a derivation of length less than m, $\mathsf{i}(p) \in F$. $\boxed{\text{Inductive Step}}$ Let $p_1, \ldots, p_m$ be a derivation of $p = p_m$ from P with no shorter derivation of p from P. Then there exists a rule $x_1, \ldots, x_{\mathsf{ar}(\star)} \vdash \star(x_1, \ldots, x_{\mathsf{ar}(\star)})$, where $\star \in \mathsf{Symb_o}(\mathfrak{a})$ and $x_1, \ldots, x_{\mathsf{ar}(\star)}$ are distinct variables and a substitution σ with $\sigma\left[\{x_1, \ldots, x_{\mathsf{ar}(\star)}\}\right] \subseteq \{p_1, \ldots, p_{m-1}\}$ and $\sigma(\star(x_1, \ldots, x_{\mathsf{ar}(\star)})) = p$. By the induction hypothesis, $\{\mathsf{i}(\sigma(x_1)), \ldots, \mathsf{i}(\sigma(x_{\mathsf{ar}(\star)}))\} = \mathsf{i}\left[\sigma\left[\{x_1, \ldots, x_{\mathsf{ar}(\star)}\}\right]\right] \subseteq \mathsf{i}\left[\{p_1, \ldots, p_{m-1}\}\right] \subseteq F$, and since F is a subuniverse and i and σ homomorphisms, $\mathsf{i}(p) = \mathsf{i}(\sigma(\star(x_1, \ldots, x_{\mathsf{ar}(\star)}))) = \star^{\mathbf{A}}(\mathsf{i}(\sigma(x_1)), \ldots, \mathsf{i}(\sigma(x_{\mathsf{ar}(\star)}))) \in F$. ◇

Since the theories of a sentential calculus coincide with its filters on the term algebra, it follows immediately that the $S(\mathfrak{a},\mathsf{su})$-theories are precisely the subuniverses of the term algebra.

Corollary 5.50 $\mathsf{Th}(S(\mathfrak{a},\mathsf{su})) = \mathsf{Su}(\mathbf{Tm}) = F(\mathbf{Tm},\mathsf{su})$.

□

In the following example we show that promoted homomorphisms between algebras are continuous between the closed systems of congruences (and relative congruences) on those algebras. Given our interest in cosets in this text, the importance of this result will become evident in the subsequent example, where we show that if a promoted function is continuous between two equivalential closed systems, then the function is continuous between the corresponding coset closed systems. Combining these two examples we obtain the fact that homomorphisms between algebras are continuous between the corresponding closed systems of cosets of congruences (and relative congruences) on those algebras. Consequently, the cosets of congruences and relative congruences on the term algebra must determine sentential 1-calculi.

Example 5.51 (Promoted Homomorphisms between Congruence Closed Systems)

Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathfrak{a}$-algebras, $\mathcal{K}$ a quasivariety, not necessarily containing $\mathbf{A}$ or $\mathbf{B}$, and $f : \mathbf{A} \to \mathbf{B}$. The following obtains from condition (1) of Theorem 1.357 on page 68, Theorem 1.374 on page 71 and Theorem 5.40.

Proposition 5.52 $\underset{\to}{f}$ is continuous from $\mathsf{Con}(\mathbf{A})$ into $\mathsf{Con}(\mathbf{B})$ and from $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ into $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$.

$\square$

We now show that if a promoted function is continuous between two equivalential closed systems, then that function is continuous between the corresponding coset closed systems.

Example 5.53 (Equivalential Closed Systems and their Cosets)

Let $\mathbb{E}$ and $\mathbb{F}$ be equivalential closed systems over A^2 and B^2 respectively, and let f be a function from A into B.

Proposition 5.54 Suppose that $\underset{\rightarrow}{f}$ is continuous from $\mathbb{E}$ into $\mathbb{F}$.

1. f is continuous from $\mathsf{Cos}_\emptyset(\mathbb{E})$ into $\mathsf{Cos}_\emptyset(\mathbb{F})$.
2. If $\mathbb{E}$ is non-trivial, then f is continuous from $\mathsf{Cos}(\mathbb{E})$ into $\mathsf{Cos}(\mathbb{F})$.
3. If $\mathbb{E}$ and $\mathbb{F}$ are both trivial, then f is continuous from $\mathsf{Cos}(\mathbb{E})$ into $\mathsf{Cos}(\mathbb{F})$.

Proof. $\boxed{(1)}$ Let $\mathfrak{u} \in \mathsf{Cos}_\emptyset(\mathbb{F})$. (*By Theorem 5.40, it suffices to show that* $f^{-1}[\mathfrak{u}] \in \mathsf{Cos}_\emptyset(\mathbb{E})$.) $\boxed{\mathfrak{u} = \emptyset}$ If $\mathfrak{u} = \emptyset$ then $f^{-1}[\mathfrak{u}] = \emptyset \in \mathsf{Cos}_\emptyset(\mathbb{E})$. $\boxed{\mathfrak{u} \neq \emptyset}$ In this case $\mathfrak{u} = \alpha[\![b]\!]$, for some $\alpha \in \mathsf{cl}_\mathbb{F}$ and $b \in B$. (*We must show that* $f^{-1}[\alpha[\![b]\!]] \in \mathsf{Cos}_\emptyset(\mathbb{E})$.) $\boxed{\alpha[\![b]\!] \cap \mathsf{rg}(f) = \emptyset}$ Suppose that $\alpha[\![b]\!] \cap \mathsf{rg}(f) = \emptyset$. In this case, $f^{-1}[\alpha[\![b]\!]] = f^{-1}[\emptyset] = \emptyset \in \mathsf{Cos}_\emptyset(\mathbb{E})$. $\boxed{\alpha[\![b]\!] \cap \mathsf{rg}(f) \neq \emptyset}$ In this case, there exists $a \in A$ with $f(a) \in \alpha[\![b]\!] \cap \mathsf{rg}(f)$. Then $f^{-1}[\alpha[\![b]\!]] = f^{-1}[\alpha[\![f(c)]\!]] = (\underset{\rightarrow}{f}^{-1}[\alpha])[\![a]\!]$, by Remark 1.60 on page 24. Since $\underset{\rightarrow}{f}$ is continuous from $\mathbb{E}$ into $\mathbb{F}$, $\underset{\rightarrow}{f}^{-1}[\alpha] \in \mathsf{cl}_\mathbb{E}$, and so $f^{-1}[\alpha[\![b]\!]] = (\underset{\rightarrow}{f}^{-1}[\alpha])[\![a]\!] \in \mathsf{Cos}_\emptyset(\mathbb{E})$, as required. $\boxed{(2)}$ The proof is similar to the proof of (1), using the fact that $\emptyset \in \mathsf{Cos}(\mathbb{E})$. $\boxed{(3)}$ Follows trivially, since $\mathsf{Cos}(\mathbb{E})$ and $\mathsf{Cos}(\mathbb{F})$ are both trivial closed systems. $\diamond$

$\square$

As a consequence of the previous two examples, homomorphisms between algebras are continuous between the corresponding closed systems of cosets of congruences (and relative congruences) on those algebras.

Example 5.55 (Homomorphisms between Coset Closed Systems)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and let $\mathbf{A}$ and $\mathbf{B}$ be two $\mathfrak{a}$-algebras. The following obtain immediately from Proposition 5.52 of Example 5.51 and Proposition 5.54 of Example 5.53.

Corollary 5.56 Every homomorphism from $\mathbf{A}$ into $\mathbf{B}$ is continuous from $\mathsf{Cos}(\mathbf{A})$ into $\mathsf{Cos}(\mathbf{B})$ and from $\mathsf{Cos}^\mathcal{K}(\mathbf{A})$ into $\mathsf{Cos}^\mathcal{K}(\mathbf{B})$.

$\square$

Since $\mathsf{Cos}(\mathbf{Tm})$ and $\mathsf{Cos}^\mathcal{K}(\mathbf{Tm})$ are both finitary closed systems (see Example 4.114 on page 162) over Tm, and every substitution is continuous from these closed systems to themselves (by the previous example), they must both be characterizable by sentential 1-calculi. In the following example we describe the first of these logics, i.e., the *non-relative* version. The reader is urged to recall the definition of the formal system $F(\mathbf{A}, \mathsf{cos})$ of cosets of $\mathbf{A}$, given in Example 4.149 on page 169.

Example 5.57 (The Sentential Calculi of Cosets and Membership)

Let $\mathfrak{a}$ be a type of algebras.

Definition 5.58 (The Sentential Calculi of Cosets) Let $S(\mathfrak{a}, \mathsf{cos})$ denote the sentential 1-calculus determined by no axioms and all rules

$$x, y, p(x, z_1, \ldots, z_{\mathsf{ar}(p)-1}) \vdash p(y, z_1, \ldots, z_{\mathsf{ar}(p)-1}), \tag{5.38}$$

for all terms p and some choice of distinct variables $x, y, z_1, \ldots, z_{\mathsf{ar}(p)-1}$. □

Remark 5.59 $S(\mathfrak{a}, \mathsf{cos})$ has no theorems. □

The following result demonstrates that the $S(\mathfrak{a}, \mathsf{cos})$-filters on an algebra are *precisely* the cosets of that algebra.

Proposition 5.60 $\mathsf{Fi}_{S(\mathfrak{a},\mathsf{cos})}(\mathbf{A}) = \mathsf{Cos}(\mathbf{A}) = \mathsf{Th}(F(\mathbf{A}, \mathsf{cos}))$.

Proof. $\boxed{\mathsf{Fi}_{S(\mathfrak{a},\mathsf{cos})}(\mathbf{A}) \subseteq \mathsf{Cos}(\mathbf{A})}$ Let $F \in \mathsf{Fi}_{S(\mathfrak{a},\mathsf{cos})}(\mathbf{A})$. If $F = \emptyset$, then $F \in \mathsf{Cos}(\mathbf{A})$. If $F \neq \emptyset$, the result follows by Corollary 1.356 on page 68 and (5.38). $\boxed{\mathsf{Cos}(\mathbf{A}) \subseteq \mathsf{Fi}_{S(\mathfrak{a},\mathsf{cos})}(\mathbf{A})}$ Let $F \in \mathsf{Cos}(\mathbf{A})$. If $F = \emptyset$, then $F \in \mathsf{Fi}_{S(\mathfrak{a},\mathsf{cos})}(\mathbf{A})$ by Remark 5.59. Suppose that $F \neq \emptyset$, $P \vdash_{S(\mathfrak{a},\mathsf{cos})} p$, $\mathsf{i} : \mathbf{Tm} \to \mathbf{A}$ with $\mathsf{i}[P] \subseteq F$. (*We show that* $\mathsf{i}(p) \in F$ *by induction on the length of derivations from* F.) $\boxed{\text{Base Case}}$ Suppose that p is derivable from P by a derivation of length 1. Since $S(\mathfrak{a}, \mathsf{cos})$ has no axioms, $p \in P$, and hence $\mathsf{i}(p) \in F$. $\boxed{\text{Inductive Hypothesis}}$ Suppose that for any term p derivable from P by a derivation of length less than m, $\mathsf{i}(p) \in F$. $\boxed{\text{Inductive Step}}$ Let $p_1, \ldots, p_m$ be a derivation of $p = p_m$ from P with no shorter derivation of p from P. Then there exists a rule $x, y, q(x, z_1, \ldots, z_{\mathsf{ar}(p)-1}) \vdash q(y, z_1, \ldots, z_{\mathsf{ar}(p)-1})$ and a substitution σ with $\sigma\left[\{x, y, q(x, z_1, \ldots, z_{\mathsf{ar}(p)-1})\}\right] \subseteq \{p_1, \ldots, p_{m-1}\}$ and $\sigma(q(y, z_1, \ldots, z_{\mathsf{ar}(p)-1})) = p$. By the induction hypothesis, $\mathsf{i}\left[\sigma\left[\{x, y, q(x, z_1, \ldots, z_{\mathsf{ar}(p)-1})\right]\}\right] \subseteq F$, and hence $\mathsf{i}(\sigma(x)), \mathsf{i}(\sigma(y)), q^{\mathbf{A}}(\mathsf{i}(\sigma(x)), \mathsf{i}(\sigma(z_1)), \ldots, \mathsf{i}(\sigma(z_{\mathsf{ar}(p)-1}))) \subseteq F$. Since F is a non-empty coset, by Corollary 1.356, $\mathsf{i}(p) = \mathsf{i}(\sigma(q(y, z_1, \ldots, z_{\mathsf{ar}(p)-1}))) = q^{\mathbf{A}}(\mathsf{i}(\sigma(y)), \mathsf{i}(\sigma(z_1)), \ldots, \mathsf{i}(\sigma(z_{\mathsf{ar}(p)-1}))) \in F$. ◇

Since filters of a sentential calculus on the term algebra coincide with the theories of that calculus, we immediately obtain that the $S(\mathfrak{a}, \mathsf{cos})$-theories are precisely the cosets of the term algebra.

Corollary 5.61 $\mathsf{Th}(S(\mathfrak{a}, \mathsf{cos})) = \mathsf{Cos}(\mathbf{Tm}) = \mathsf{Th}(F(\mathbf{Tm}, \mathsf{cos}))$. □

As mentioned above, $\mathsf{Cos}^{\mathcal{K}}(\mathbf{Tm})$ *must* be characterizable as a *sentential* 1-*calculus*. We now identify this logic (recall Convention 2.20 on page 96), which we call the *membership logic* of $\mathcal{K}$.

Definition 5.62 (The Relative Coset or Membership Logic) For a quasivariety $\mathcal{K}$ of $\mathfrak{a}$-algebras, let $S(\mathcal{K}, \mathsf{mem})$ denote the sentential 1-calculus with $\mathsf{Th}(S(\mathcal{K}, \mathsf{mem})) = \mathsf{Cos}^{\mathcal{K}}(\mathbf{Tm})$, which we call the **membership logic determined by** $\mathcal{K}$ (or the $\mathcal{K}$**-membership logic**, the **relative coset logic determined by** $\mathcal{K}$ or even the $\mathcal{K}$**-coset logic**). □

The following characterization of consequence in the membership logic $S(\mathcal{K}, \mathsf{mem})$ follows immediately from Corollary 4.119 on page 163, together with Lemma 1.457 on page 88. Note that since we insist that the *improper-coset* be a member of $\mathsf{Cos}^{\mathcal{K}}(\mathbf{Tm})$, $S(\mathcal{K}, \mathsf{mem})$ has no theorems.

Proposition 5.63 For $P \cup \{p\} \subseteq \mathsf{Tm}$ with $P \neq \emptyset$, the following conditions are equivalent.

1. $P \vdash_{S(\mathcal{K},\mathsf{mem})} p$.
2. $P \approx P \models_{\mathcal{K}} P \approx p$.
3. $P \approx q \models_{\mathcal{K}} q \approx p$, for some $q \in P$.
4. $P' \approx P' \models_{\mathcal{K}} P' \approx p$, for some finite $P' \subseteq_{\mathsf{f}} P$.

□

Characterizing the $\mathcal{K}$-membership logic is a central focus of this text. It is our most important logic in this text, and arguably, the most important sentential 1-calculi arising from algebra from the *algebraists perspective.* This logic, however, is 'inherently unalgebraizable' in the sense of [BP89a]. Our primary reason for developing our theory of *parameterized algebraization* (see Part V) was in an attempt to remedy this problem and bring the membership logic into the fold of *algebraic logic.*

Recall that the *downsets* $\mathsf{Dn}(\mathbf{P})$, the *upsets* $\mathsf{Up}(\mathbf{P})$ and the *convexities* $\mathsf{Cx}(\mathbf{P})$, on an order $\mathbf{P}$, all form algebraic closed systems (see Example 4.82 on page 157). Be aware that these closed systems are *concrete*; the underlying order is the inclusion-order complete power-set of the universe of $\mathbf{P}$ and *not* $\mathbf{P}$. In the following example, we show that the continuous *functions* from $\mathsf{Dn}(\mathbf{P})$ into $\mathsf{Dn}(\mathbf{Q})$ and from $\mathsf{Up}(\mathbf{P})$ into $\mathsf{Up}(\mathbf{Q})$, are precisely the order-preserving functions from $\mathbf{P}$ into $\mathbf{Q}$. While we are not really interested in these closed systems in this text, the results obtained in this example are precursors to analogous results obtained in the subsequent example concerning continuous functions between the closed systems of *ideals* (and *filters*) of *lattices.*

Example 5.64 (Closed Systems of Order Upsets, Downsets and Convexities)

Let $\mathbf{P}$ and $\mathbf{Q}$ be orders. The following characterization of continuous functions from $\mathsf{Dn}(\mathbf{P})$ into $\mathsf{Dn}(\mathbf{Q})$ (resp. from $\mathsf{Up}(\mathbf{P})$ into $\mathsf{Up}(\mathbf{Q})$), follows by Remark 1.187 on page 41 and Theorem 5.40.

Proposition 5.65 The continuous functions from $\mathsf{Dn}(\mathbf{P})$ into $\mathsf{Dn}(\mathbf{Q})$ (resp. from $\mathsf{Up}(\mathbf{P})$ into $\mathsf{Up}(\mathbf{Q})$) are *precisely* the order-preserving functions from $\mathbf{P}$ into $\mathbf{Q}$.

Proposition 5.66 If $f : \mathbf{P} \to_{\leq} \mathbf{Q}$ then f is continuous from $\mathsf{Cx}(\mathbf{P})$ into $\mathsf{Cx}(\mathbf{Q})$.

Proof. Let B be a convexity of $\mathbf{Q}$. (*It suffices, by Theorem 5.40, to show that* $f^{-1}[B]$ *is convex.*) Suppose that $a, c \in f^{-1}[B]$ and $a \leq b \leq c$. By order-preservation, $f(a) \leq f(b) \leq f(c)$. Since $f(a), f(c) \in B$ and B is a convexity, $f(b) \in B$. Hence $b \in f^{-1}[B]$. ◇

□

The following example, demonstrating that lattice-homomorphisms are continuous between the closed systems of lattice-ideals (resp. lattice-filters), will ultimately lead to a sentential 1-calculus of lattice-ideals (resp. lattice-filters) modulo a quasivariety of lattice expansions (see Example 12.61 on page 389).

Example 5.67 (Closed Systems of Lattice Ideals and Filters)

Let $\mathbf{P}$ and $\mathbf{Q}$ be lattices.

Proposition 5.68 Let $f : \mathsf{uni}(\mathbf{P}) \to \mathsf{uni}(\mathbf{Q})$.

1. If $\mathbf{P}$ is lower-unbounded and f is a lattice homomorphism then f is continuous from $\mathsf{Id}_\Diamond(\mathbf{P})$ into $\mathsf{Id}_\Diamond(\mathbf{Q})$.
2. If $\mathbf{P}$ and $\mathbf{Q}$ are both lower-bounded and f is a lattice 0-homomorphism then f is continuous from $\mathsf{Id}_\Diamond(\mathbf{P})$ into $\mathsf{Id}_\Diamond(\mathbf{Q})$.

Proof. $\boxed{(1)}$ Let B be an ideal of $\mathbf{Q}$. (*It suffices, by Theorem 5.40, to show that $f^{-1}[B]$ is an ideal.*) If $f^{-1}[B] = \emptyset$, then $f^{-1}[B] = \emptyset \in \mathsf{Id}_\Diamond(\mathbf{P})$, since $\mathbf{P}$ is lower-unbounded by assumption. Otherwise, $f^{-1}[B] \neq \emptyset$. By Proposition 5.65 of Example 5.64, we know that $f^{-1}[B]$ is a downset. Let $a, b \in f^{-1}[B]$. So $f(a), f(b) \in B$. (*We must show that $a \vee b \in f^{-1}[B]$.*) Since f is a homomorphism, $f(a \vee b) = f(a) \vee f(b) \in B$, since B is an ideal. Hence $a \vee b \in f^{-1}[B]$.
$\boxed{(2)}$ Let B be an ideal of $\mathbf{Q}$. (*It suffices, by Theorem 5.40, to show that $f^{-1}[B]$ is an ideal.*) By assumption $B \neq \emptyset$. Further, $f^{-1}[B] \neq \emptyset$, since $f(0) \in B$. The proof proceeds as in (1).
$\diamond$

The following result follows dually and as such the proof is omitted.

Proposition 5.69 Let $\mathbf{P}$ and $\mathbf{Q}$ be lattices and $f : \mathsf{uni}(\mathbf{P}) \to \mathsf{uni}(\mathbf{Q})$.

1. If $\mathbf{P}$ is upper-unbounded and f is a lattice homomorphism then f is continuous from $\mathsf{Fl}_\Diamond(\mathbf{P})$ into $\mathsf{Fl}_\Diamond(\mathbf{Q})$.
2. If $\mathbf{P}$ and $\mathbf{Q}$ are both upper-bounded and f is a lattice 1-homomorphism then f is continuous from $\mathsf{Fl}_\Diamond(\mathbf{P})$ into $\mathsf{Fl}_\Diamond(\mathbf{Q})$.

$\square$

5.3.2 $\vdash$-Reflecting Translations

We characterize $\vdash$-reflecting translations as an intermediate step to characterizing strictly continuous translations. We begin by noting a property that is true of translations generally.

Remark 5.70 If τ is a translation from $\mathfrak{c}$ into $\mathfrak{d}$ then, for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$\|a\|_{\mathfrak{c}} \leq \tau^{\blacktriangleleft}(\tau^{\star}(\|a\|_{\mathfrak{c}})). \tag{5.39}$$

Proof. $\|a\|_{\mathfrak{c}} \leq \tau^{\blacktriangleleft}(\tau(\|a\|_{\mathfrak{c}})) \leq \tau^{\blacktriangleleft}(\|\tau(\|a\|_{\mathfrak{c}})\|_{\mathfrak{d}})$. $\diamond$

The following result characterizes $\vdash$-reflecting translations.

Proposition 5.71 Let τ be a translation from $\mathfrak{c}$ to $\mathfrak{d}$. The following conditions are equivalent.

1. τ is $\vdash$-reflecting.
2. $a \vdash_{\mathfrak{c}} \tau^{\blacktriangleleft}(\tau^{\star}(a))$, for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$.
3. $\tau^{\blacktriangleleft}(\tau^{\star}(a)) \leq \|a\|_{\mathfrak{c}}$ (equiv. $\tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}}) \leq \|a\|_{\mathfrak{c}}$), for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$.
4. $\tau^{\blacktriangleleft}(\tau^{\star}(\|a\|_{\mathfrak{c}})) \leq \|a\|_{\mathfrak{c}}$ (equiv. $\tau^{\blacktriangleleft}(\|\tau(\|a\|_{\mathfrak{c}})\|_{\mathfrak{d}}) \leq \|a\|_{\mathfrak{c}}$), for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$.
5. $\tau^{\blacktriangleleft}(\tau^{\star}(\|a\|_{\mathfrak{c}})) = \|a\|_{\mathfrak{c}}$ (equiv. $\tau^{\blacktriangleleft}(\|\tau(\|a\|_{\mathfrak{c}})\|_{\mathfrak{d}}) = \|a\|_{\mathfrak{c}}$), for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$.

6. $\tau^{\blacktriangleleft}(\tau^{\star}(g)) \leq g$ (equiv. $\tau^{\blacktriangleleft}(\|\tau(g)\|_{\mathfrak{d}}) \leq g$), for all $g \in \mathsf{cl}_{\mathfrak{c}}$.

7. $\tau^{\blacktriangleleft}(\tau^{\star}(g)) = g$ (equiv. $\tau^{\blacktriangleleft}(\|\tau(g)\|_{\mathfrak{d}}) = g$), for all $g \in \mathsf{cl}_{\mathfrak{c}}$.

8. $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathsf{cl}_{\mathfrak{c}} \Rightarrow \tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]$ with (unique) inverse $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$.

9. $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \cong \tau^{\star}[\mathbf{cl}_{\mathfrak{c}}] \lhd \mathbf{cl}_{\mathfrak{d}}$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Since $\tau(\tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})) \leq \|\tau(a)\|_{\mathfrak{d}}$, $\tau(a) \vdash_{\mathfrak{d}} \tau(\tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}}))$. Consequently, by assumption (1), $a \vdash_{\mathfrak{c}} \tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})$. $\boxed{(2)\Rightarrow(3)}$ Trivial. $\boxed{(3)\Rightarrow(4)}$ By idempotence of elementary closure operators. $\boxed{(4)\Rightarrow(1)}$ Suppose that $\tau(a) \vdash_{\mathfrak{d}} \tau(b)$, i.e., $\tau(b) \leq \|\tau(a)\|$. By the increasing nature of closure operators, $\tau(b) \leq \left\|\tau(\|a\|_{\mathfrak{c}})\right\|_{\mathfrak{d}}$; hence $b \leq \tau^{\blacktriangleleft}(\tau(b)) \leq \tau^{\blacktriangleleft}(\left\|\tau(\|a\|_{\mathfrak{c}})\right\|_{\mathfrak{d}}) \overset{(4)}{\leq} \|a\|_{\mathfrak{c}}$. So $a \vdash_{\mathfrak{c}} b$. $\boxed{(4)\Rightarrow(5)}$ By (5.39). $\boxed{(5)\Rightarrow(4)}$ Trivial. $\boxed{(4)\Leftrightarrow(6)}$ Trivial. $\boxed{(5)\Leftrightarrow(7)}$ Trivial. $\boxed{(7)\Rightarrow(8)}$ Trivially, $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}$ is a surjective function onto $\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}] \subseteq \mathsf{cl}_{\mathfrak{d}}$. Suppose that $g, h \in \mathsf{cl}_{\mathfrak{c}}$ with $\tau^{\star}(g) = \tau^{\star}(h)$. Then by assumption (7), $g = \tau^{\blacktriangleleft}(\tau^{\star}(g)) = \tau^{\blacktriangleleft}(\tau^{\star}(h)) = h$, and so $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}$ is injective. That $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$ is the inverse follows by assumption (7) and (already established) injectivity of $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}$. $\boxed{(8)\Rightarrow(9)}$ Trivial, since $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}$ and $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$ are both order-preserving. $\boxed{(9)\Rightarrow(7)}$ Follows by assumed inverse. ◇

Counter Example 5.72 (Continuity and $\vdash$-Reflection are Independent Conditions)

Let $\mathbf{P}$ be the three element linear order determined by $0^{\mathbf{P}} < a < 1^{\mathbf{P}}$ and $\mathbf{Q}$ be the three element linear order determined by $0^{\mathbf{Q}} < b < 1^{\mathbf{Q}}$. Let $\mathfrak{c}$ be the discrete closed system on $\mathbf{P}$ and $\mathfrak{d}$ the closed system on $\mathbf{Q}$ determined by $\left\|0^{\mathbf{Q}}\right\|_{\mathfrak{d}} = 0^{\mathbf{Q}}$ and $\|b\|_{\mathfrak{d}} = \left\|1^{\mathbf{Q}}\right\|_{\mathfrak{d}} = 1^{\mathbf{Q}}$. Let $\tau^{\blacktriangleright} : \{0^{\mathbf{P}}, a, 1^{\mathbf{P}}\} \to \{0^{\mathbf{Q}}, b, 1^{\mathbf{Q}}\}$ be defined by $\tau^{\blacktriangleright}(0^{\mathbf{P}}) = 0^{\mathbf{Q}}$, $\tau^{\blacktriangleright}(a) = b$ and $\tau^{\blacktriangleright}(1^{\mathbf{P}}) = 1^{\mathbf{Q}}$, and let $\tau^{\blacktriangleleft} : \{0^{\mathbf{Q}}, b, 1^{\mathbf{Q}}\} \to \{0^{\mathbf{P}}, a, 1^{\mathbf{P}}\}$ be defined by $\tau^{\blacktriangleleft}(0^{\mathbf{Q}}) = 0^{\mathbf{P}}$, $\tau^{\blacktriangleleft}(b) = a$ and $\tau^{\blacktriangleleft}(1^{\mathbf{Q}}) = 1^{\mathbf{P}}$.

It is easily seen that $\tau = \langle \tau^{\blacktriangleright}, \tau^{\blacktriangleleft} \rangle$ is a non-singular translation from $\mathfrak{c}$ to $\mathfrak{d}$. By (7) of Theorem 5.40, τ is continuous, since $\tau^{\blacktriangleleft}(\left\|\tau^{\blacktriangleright}(0^{\mathbf{P}})\right\|_{\mathfrak{d}}) = \tau^{\blacktriangleleft}(\left\|0^{\mathbf{Q}}\right\|_{\mathfrak{d}}) = \tau^{\blacktriangleleft}(0^{\mathbf{Q}}) = 0^{\mathbf{P}} = \left\|0^{\mathbf{P}}\right\|_{\mathfrak{c}}$, $\tau^{\blacktriangleleft}(\left\|\tau^{\blacktriangleright}(1^{\mathbf{P}})\right\|_{\mathfrak{d}}) = \tau^{\blacktriangleleft}(\left\|1^{\mathbf{Q}}\right\|_{\mathfrak{d}}) = \tau^{\blacktriangleleft}(1^{\mathbf{Q}}) = 1^{\mathbf{P}} = \left\|1^{\mathbf{P}}\right\|_{\mathfrak{c}}$ and $\tau^{\blacktriangleleft}(\|\tau^{\blacktriangleright}(a)\|_{\mathfrak{d}}) = \tau^{\blacktriangleleft}(\left\|1^{\mathbf{Q}}\right\|_{\mathfrak{d}}) = \tau^{\blacktriangleleft}(1^{\mathbf{Q}}) = 1^{\mathbf{P}} \geq a = \|a\|_{\mathfrak{c}}$, Since $\tau^{\blacktriangleleft}(\|\tau^{\blacktriangleright}(a)\|_{\mathfrak{d}}) = 1^{\mathbf{P}} \not\leq a = \|a\|_{\mathfrak{c}}$, τ cannot be $\vdash$-reflecting by (3) of Proposition 5.71.

By a similar argument, $\tau' = \langle \tau^{\blacktriangleleft}, \tau^{\blacktriangleright} \rangle$ is a non-singular translation from $\mathfrak{d}$ to $\mathfrak{c}$ that is $\vdash$-reflecting but not continuous.

□

Observe that (9) of the previous proposition does not (erroneously) imply (13) of Theorem 5.40, due to the restriction placed on $\tau^{\blacktriangleleft}$ in (9). Note further, that (12) of Theorem 5.40 does not (erroneously) imply (8) of the previous proposition, since while certainly $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \to_{\blacktriangledown} \mathbf{cl}_{\mathfrak{d}}$ implies $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathsf{cl}_{\mathfrak{c}} \Rightarrow \tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]$, it does not necessarily imply that $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$ is the inverse of $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}$. Consequently, the explication of the inverse in equivalent condition (8) of the previous proposition cannot be dropped.

5.3.3 Strictly Continuous Translations

We now turn to *strictly continuous translations.* Strictly continuous translations play an important role in the theory of *algebraizable logics*, since they abstract the notion that one logic be a formal semantics for another (see Definition 2.95 on page 108), which generalizes the notion of an algebraic semantics. We shall further show how the filter correspondence property (and hence protoalgebraicity) can be characterized in terms of every reductive matrix homomorphism being strictly continuous between the closed system of matrix filters (see Example 5.76).

Note that in the following characterization of strictly continuous translations, we establish the equivalence of the elementary conditions (1) through to (7) independently of the non-elementary conditions (8) and (9), and do so by elementary arguments.

Theorem 5.73 Let τ be a translation from $\mathfrak{c}$ to $\mathfrak{d}$. The following conditions are equivalent.

1. τ is strictly continuous.
2. $\tau^{\blacktriangleleft}(\tau^{\star}(a)) = \|a\|_{\mathfrak{c}}$, for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$.
3. τ is continuous and $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}$ is injective.
4. τ is continuous, $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}}$ is injective and $\tau^{\star}(g) = \tau^{\star}(\tau^{\blacktriangleleft}(\tau^{\star}(g)))$, for all $g \in \mathsf{cl}_{\mathfrak{c}}$.
5. τ is continuous and $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \rightarrowtail_{\blacktriangledown} \mathbf{cl}_{\mathfrak{d}}$.
6. τ is continuous and $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \cong \tau^{\star}[\mathbf{cl}_{\mathfrak{c}}] \lhd \mathbf{cl}_{\mathfrak{d}}$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$.
7. $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} : \mathbf{cl}_{\mathfrak{d}} \to \mathbf{cl}_{\mathfrak{c}}$ and $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \cong \tau^{\star}[\mathbf{cl}_{\mathfrak{c}}] \lhd \mathbf{cl}_{\mathfrak{d}}$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$.
8. τ is continuous and $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \cong \tau^{\star}[\mathbf{cl}_{\mathfrak{c}}] \lhd_{\blacktriangledown} \mathbf{cl}_{\mathfrak{d}}$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$.
9. $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} : \mathbf{cl}_{\mathfrak{d}} \to_{\blacktriangle} \mathbf{cl}_{\mathfrak{c}}$ and $\tau^{\star}{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \cong \tau^{\star}[\mathbf{cl}_{\mathfrak{c}}] \lhd_{\blacktriangledown} \mathbf{cl}_{\mathfrak{d}}$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]}$.

Proof. $\boxed{(1)\Leftrightarrow(2)}$ By (7) of Theorem 5.40 and (3) of Proposition 5.71. $\boxed{(1)\Rightarrow(3)}$ By (8) of Proposition 5.71. $\boxed{(3)\Rightarrow(4)}$ By (5.37) of Corollary 5.41. $\boxed{(4)\Rightarrow(1)}$ By (7) of Proposition 5.71, it suffices to show that $\tau^{\blacktriangleleft}(\tau^{\star}(g)) = g$ for all $g \in \mathsf{cl}_{\mathfrak{c}}$. Let $g \in \mathsf{cl}_{\mathfrak{c}}$. By assumption, $\tau^{\star}(g) = \tau^{\star}(\tau^{\blacktriangleleft}(\tau^{\star}(g)))$. Further, by (3) of Theorem 5.40, $\tau^{\blacktriangleleft}(\tau^{\star}(g)) \in \mathsf{cl}_{\mathfrak{c}}$, since $\tau^{\star}(g) \in \mathsf{cl}_{\mathfrak{d}}$. Hence, by assumed injectivity of $\tau^{\star}$, $g = \tau^{\blacktriangleleft}(\tau^{\star}(g))$. $\boxed{(3)\Leftrightarrow(5)}$ By (12) of Theorem 5.40. $\boxed{(1)\Rightarrow(6)}$ By (9) of Proposition 5.71. $\boxed{(6)\Rightarrow(5)}$ Trivial $\boxed{(6)\Leftrightarrow(7)}$ By equivalent condition (11) of Theorem 5.40.

$\boxed{(1)\Rightarrow(8)}$ (*By (9) of Proposition 5.71, it suffices to prove that* $\tau^{\star}[\mathbf{cl}_{\mathfrak{c}}] \lhd_{\blacktriangledown} \mathbf{cl}_{\mathfrak{d}}$.) Let $B \subseteq \tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]$.

Suppose that $\blacktriangledown^{\tau^{\star}[\mathbf{cl}_{\mathfrak{c}}]} B$ exists. Certainly $\blacktriangledown^{\tau^{\star}[\mathbf{cl}_{\mathfrak{c}}]} B$ is an upper-bound of B in $\mathbf{cl}_{\mathfrak{d}}$. Let h be an upper-bound of B in $\mathbf{cl}_{\mathfrak{d}}$. Since $\tau^{\blacktriangleleft}(h)$ is an upper-bound of $\tau^{\blacktriangleleft}[B]$ in $\mathbf{cl}_{\mathfrak{c}}$ (since $\tau^{\blacktriangleleft}$ is order-preserving and $\tau^{\blacktriangleleft}(h)$ is closed by (3) of Theorem 5.40), $\tau^{\star}(\tau^{\blacktriangleleft}(h))$ is an upper-bound of $\tau^{\star}[\tau^{\blacktriangleleft}[B]]$ in $\tau^{\star}[\mathbf{cl}_{\mathfrak{c}}]$. By injectivity and the fact that $B \subseteq \tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]$, $\tau^{\star}[\tau^{\blacktriangleleft}[B]] = B$, and so $\tau^{\star}(\tau^{\blacktriangleleft}(h))$ is an upper-bound of B in $\tau^{\star}[\mathbf{cl}_{\mathfrak{c}}]$. Hence $\blacktriangledown^{\tau^{\star}[\mathbf{cl}_{\mathfrak{c}}]} B \leq \tau^{\star}(\tau^{\blacktriangleleft}(h))$. By (5.34), $\tau^{\star}(\tau^{\blacktriangleleft}(h)) \leq h$, and so $\blacktriangledown^{\tau^{\star}[\mathbf{cl}_{\mathfrak{c}}]} B \leq h$. Hence $\blacktriangledown^{\tau^{\star}[\mathbf{cl}_{\mathfrak{c}}]} B = \blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B$.

Suppose that $\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B$ exists. (*It suffices, by Remark 1.151, to show that* $\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B \in \tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]$.) Certainly $\tau^{\blacktriangleleft}(\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B)$ is an upper-bound of $\tau^{\blacktriangleleft}[B]$ in $\mathbf{P}_{\mathfrak{c}}$ (*in fact, by (3) of Theorem 5.40,* $\tau^{\blacktriangleleft}(\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B) \in \mathsf{cl}_{\mathfrak{d}}$*, and so* $\tau^{\blacktriangleleft}(\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B)$ *is an upper-bound of* $\tau^{\blacktriangleleft}[B]$ *in* $\mathbf{cl}_{\mathfrak{c}}$*, but this fact is not required*). Hence $\tau^{\star}(\tau^{\blacktriangleleft}(\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B))$ is an upper-bound of $\tau^{\star}[\tau^{\blacktriangleleft}[B]] = B$ in $\mathbf{cl}_{\mathfrak{d}}$, since $\tau^{\star}(\tau^{\blacktriangleleft}(\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B))$ is closed. So $\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B \leq \tau^{\star}(\tau^{\blacktriangleleft}(\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B))$. Conversely, since $\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B \in \mathsf{cl}_{\mathfrak{d}}$, we have $\tau^{\star}(\tau^{\blacktriangleleft}(\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B)) \leq \blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B$, by (5.34). Hence $\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B = \tau^{\star}(\tau^{\blacktriangleleft}(\blacktriangledown^{\mathbf{cl}_{\mathfrak{d}}} B)) \in \tau^{\star}[\mathsf{cl}_{\mathfrak{c}}]$. $\boxed{(8)\Rightarrow(5)}$ Trivial $\boxed{(8)\Leftrightarrow(9)}$ By equivalent condition (13) of Theorem 5.40. ◇

The condition of continuity cannot be dropped from conditions (3) through to (8) of the previous theorem, as demonstrated by the following counter-example.

Counter Example 5.74 (Continuity Cannot be Dropped)

Let $\mathbb{C}$ be the (concrete) closed system over $\{a, b\}$ with $\mathsf{cl}_{\mathbb{C}} = \{\emptyset, \{a, b\}\}$, let $\mathbb{D}$ be the discrete (concrete) closed system over $\{c, d\}$ and let $\tau : \mathbb{C} \multimap \mathbb{D}$ be the (concrete) translation defined by $\tau[\![a]\!] = \{c\}$ and $\tau[\![b]\!] = \{d\}$. By (7) of Theorem 5.40, τ is not continuous, since $\tau^{\blacktriangleleft}(\|\tau(\{a\})\|_{\mathbb{D}}) = \tau^{\blacktriangleleft}(\|\{c\}\|_{\mathbb{D}}) = \tau^{\blacktriangleleft}(\{c\}) = \{a\} \not\supseteq \{a, b\} = \|\{a\}\|_{\mathbb{D}}$. On the other hand, $\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbf{cl}_{\mathbb{C}} \cong \tau^{\star}\left[\mathbf{cl}_{\mathbb{C}}\right] \lhd_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{cl}_{\mathbb{C}}]}$. So continuity cannot be dropped from (8), and hence cannot be dropped from either (3) or (5). Further, $\tau^{\star}(\tau^{\blacktriangleleft}(\tau^{\star}(\emptyset))) = \tau^{\star}(\tau^{\blacktriangleleft}(\|\tau(\emptyset)\|_{\mathbb{C}})) = \tau^{\star}(\tau^{\blacktriangleleft}(\|\emptyset\|_{\mathbb{C}})) = \tau^{\star}(\tau^{\blacktriangleleft}(\emptyset)) = \tau^{\star}(\emptyset)$ and $\tau^{\star}(\tau^{\blacktriangleleft}(\tau^{\star}(\{a, b\}))) = \tau^{\star}(\tau^{\blacktriangleleft}(\|\tau(\{a, b\})\|_{\mathbb{C}})) = \tau^{\star}(\tau^{\blacktriangleleft}(\|\{c, d\}\|_{\mathbb{C}})) = \tau^{\star}(\tau^{\blacktriangleleft}(\{c, d\})) = \tau^{\star}(\{a, b\})$, and hence continuity cannot be dropped from (4).

$\square$

Open Problem 5.75 Show that if τ is $\vdash$-reflecting, it need not be the case that $\tau^{\star}\left[\mathbf{cl}_{\mathfrak{c}}\right] \lhd_{\blacktriangledown} \mathbf{cl}_{\mathfrak{d}}$ (compare (8) of Theorem 5.73 with (9) of Proposition 5.71).

5.3.3.1 Examples

In the following example, we characterise the condition that a sentential calculus be protoalgebraic, in terms of consequence reflection and strict continuity. It is the (equivalent) condition that the logic have the *filter correspondence property* that we characterize in terms of consequence reflection and strict continuity of reductive matrix homomorphisms. Consequently, even the filter correspondence property is unified within the framework of continuous translations between *elementary* closed systems.

Example 5.76 (Protoalgebraicity and Filter Correspondence in Sentential Calculi)

Recall that a sentential calculus $\mathcal{S}$ is said to have the *filter correspondence property* if, for all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, all $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$ and every $F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$,

$$\underset{\rightarrow}{f}^{-1}\left[\|\underset{\rightarrow}{f}[F]\|^{\mathbf{N}}_{\mathsf{fi}_{\mathcal{S}}}\right] = F \tag{5.40}$$

(see Definition 2.132 on page 116), and recall that by Theorem 2.135 on page 117, $\mathcal{S}$ has the filter correspondence property iff it is protoalgebraic.

Comparing (5.40) with (8) of Proposition 5.71, we see that the condition that $\mathcal{S}$ have the filter correspondence property may be characterized in terms of the consequence reflection of reductive matrix homomorphisms.

Corollary 5.77 $\mathcal{S}$ has the filter correspondence property iff, for all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, all $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, f is consequence reflecting from $\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$ onto $\mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$. $\square$

Since reductions are surjective, if $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, then f is continuous from $\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$ onto $\mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$, by Proposition 5.43 of Example 5.26. So $\mathcal{S}$ has the filter correspondence property iff, for all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, all $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, f is strictly continuous from $\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$ onto $\mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$. We collect these and other consequences in the following theorem.

Theorem 5.78 For a sentential calculus $\mathcal{S}$, the following conditions are equivalent.

1. $\mathcal{S}$ is protoalgebraic.
2. For all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, all $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, f is consequence reflecting from $\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$ onto $\mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$.

3. For all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, all $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, f is strictly continuous from $\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})$ onto $\mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$.
4. $f^{\mathcal{S}}{}_{|\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})} : \mathsf{Fi}_{\mathcal{S}}(\mathbf{M}) \Rightarrow f^{\mathcal{S}}[\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})] \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$ (is a bijection) with (unique) inverse $f^{\blacktriangleleft}{}_{|f^{\mathcal{S}}[\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})]}$.
5. $f^{\mathcal{S}}{}_{|\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})} : \mathbf{Fi}_{\mathcal{S}}(\mathbf{M}) \cong f^{\mathcal{S}}[\mathbf{Fi}_{\mathcal{S}}(\mathbf{M})] \lhd_{\blacktriangledown} \mathbf{Fi}_{\mathcal{S}}(\mathbf{N})$ with inverse isomorphism $f^{\blacktriangleleft}{}_{|f^{\mathcal{S}}[\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})]}$.

□

In the next example we locate the notion of *formal semantics* (see Definition 2.95 on page 108) from the theory of sentential calculi within the framework of *strictly continuous translations* between *elementary* closed systems.

Example 5.79 (Formal Semantics in Sentential Calculi)

Let τ be a *formal translation* (see Definition 2.95 on page 108) from sentential n-calculi $\mathcal{S}_1$ to sentential m-calculi $\mathcal{S}_2$, both of type $\mathfrak{a}$. Recall the binary relationship $\boldsymbol{\tau}^{\mathbf{A}}$ from $\mathsf{uni}(\mathbf{A})^n$ to $\mathsf{uni}(\mathbf{A})^m$ defined by (2.24) of Definition 2.95, and recall that we write $\boldsymbol{\tau}$ for $\boldsymbol{\tau}^{\mathbf{Tm}}$. The binary relationship $\boldsymbol{\tau}^{\mathbf{A}}$ is in fact a translation from $\mathsf{uni}(\mathbf{A})^n$ to $\mathsf{uni}(\mathbf{A})^m$. The following relationship between formal semantics and strict continuity follows immediately from definitions.

Remark 5.80 $\mathcal{S}_2$ is a formal semantics for $\mathcal{S}_1$ with formal semantic translation τ iff τ is strictly continuous from $\mathsf{Th}(\mathcal{S}_1)$ to $\mathsf{Th}(\mathcal{S}_2)$. □

We enumerate some of the characterizations of formal semantics that follow from Theorem 5.73. Note that these characterizations follow by *elementary* arguments.

Corollary 5.81 The following conditions are equivalent.

1. $\mathcal{S}_2$ is a formal semantics for $\mathcal{S}_1$ with formal semantic translation τ.
2. $\tau^{\blacktriangleleft}(\|\tau[\Gamma]\|_{\mathcal{S}_2}) = \|\Gamma\|_{\mathcal{S}_1}$, for all $\Gamma \subseteq \mathsf{Fm}(\mathcal{S}_1)$.
3. $\tau^{\blacktriangleleft}{}_{|\mathsf{Th}(\mathcal{S}_2)} : \mathbf{Th}(\mathcal{S}_2) \to \mathbf{Th}(\mathcal{S}_2)$ and $\tau^{\star}{}_{|\mathsf{Th}(\mathcal{S}_1)} : \mathbf{Th}(\mathcal{S}_1) \cong \tau^{\star}[\mathbf{Th}(\mathcal{S}_1)] \lhd \mathbf{Th}(\mathcal{S}_2)$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{Th}(\mathcal{S}_1)]}$.
4. $\tau^{\blacktriangleleft}{}_{|\mathsf{Th}(\mathcal{S}_2)} : \mathbf{Th}(\mathcal{S}_2) \to_{\blacktriangle} \mathbf{Th}(\mathcal{S}_2)$ and $\tau^{\star}{}_{|\mathsf{Th}(\mathcal{S}_1)} : \mathbf{Th}(\mathcal{S}_1) \cong \tau^{\star}[\mathbf{Th}(\mathcal{S}_1)] \lhd_{\blacktriangledown} \mathbf{Th}(\mathcal{S}_2)$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{Th}(\mathcal{S}_1)]}$.

□

5.3.4 Product by a Single Translation

In *topology*, the notion of the *coarsest space* determined by a function from a set into a space plays a central role in the theory of continuous functions. We generalize this construction to obtain the notion of the *product of a translation* from an order to an elementary closed system. This closed system is the *coarsest* closed system on that order for which the translation is continuous (it is in fact *strictly* continuous), and any *finer* closed system on the order will have the property that the translation is *continuous*. Without a notion of a basis, we are unable to extend this construction to *multiple* translations as we are able to do in the *concrete* context (see ∮5.4.3.1).

Definition 5.82 (Product by a Single Translation) Let $\mathbf{P}$ be an order, $\mathfrak{d}$ an elementary closed system and τ a translation from $\mathbf{P}$ to $\mathbf{P}_{\mathfrak{d}}$. Let $\tau^{\blacktriangleleft}_{\mathbf{P}}[\mathfrak{d}]$ denote the elementary closed system on $\mathbf{P}$ determined by $\mathsf{cl}_{\tau^{\blacktriangleleft}_{\mathbf{P}}[\mathfrak{d}]} = \tau^{\blacktriangleleft}[\mathsf{cl}_{\mathfrak{d}}]$, which is called the **product of $\mathfrak{d}$ by τ (on $\mathbf{P}$)**. The subscript $\mathbf{P}$ is unnecessary, since the translation encodes the order; we shall tend to omit it, except for situations in which it provides additional clarity. $\square$

Proof. Let $a \in \mathsf{uni}(\mathbf{P})$ and let $A = \{g \in \tau^{\blacktriangleleft}[\mathsf{cl}_{\mathfrak{d}}] \ : \ a \leq g\}$. (*We must show that* $\blacktriangle^{\mathbf{P}} A$ *exists and that* $\blacktriangle^{\mathbf{P}} A \in A$.) Observe that $a \leq \tau^{\blacktriangleleft}(\tau(a)) \leq \tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}}) \in \tau^{\blacktriangleleft}[\mathsf{cl}_{\mathfrak{d}}]$, and so $\tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}}) \in A$. (*It suffices to show that* $\tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})$ *is a lower-bound of* A.) Let $g \in A$, i.e., $a \leq g$ and $g = \tau^{\blacktriangleleft}(h)$ for some $h \in \mathsf{cl}_{\mathfrak{d}}$. Then $\tau(a) \leq \tau(g) = \tau(\tau^{\blacktriangleleft}(h)) \leq h$. So $\|\tau(a)\|_{\mathfrak{d}} \leq h$; hence $\tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}}) \leq \tau^{\blacktriangleleft}(h) = g$. $\diamond$

Remark 5.83 Implicit in the proof of the previous definition is the fact that

$$\|a\|_{\tau^{\blacktriangleleft}[\mathfrak{d}]} = \tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}}). \tag{5.41}$$

Remark 5.84 If $\mathfrak{d} \preceq \mathfrak{d}'$ then $\tau^{\blacktriangleleft}[\mathfrak{d}] \preceq \tau^{\blacktriangleleft}[\mathfrak{d}']$.

Proposition 5.85 τ is strictly continuous from $\tau^{\blacktriangleleft}[\mathfrak{d}]$ to $\mathfrak{d}$. Consequently, the following conditions are equivalent.

1. $A \vdash_{\tau^{\blacktriangleleft}[\mathbb{D}]} b$.
2. $\tau[A] \vdash_{\mathbb{D}} \tau[\![b]\!]$.
3. $\tau[A] \subseteq H \rightarrow \tau[\![b]\!] \subseteq H$, for all $H \in \mathsf{cl}_{\mathbb{D}}$.

Proof. If $a \vdash_{\tau^{\blacktriangleleft}[\mathfrak{d}]} b$, then by (5.41), $b \leq \|a\|_{\tau^{\blacktriangleleft}[\mathfrak{d}]} = \tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})$; hence $\tau(b) \leq \tau(\tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})) \leq \|\tau(a)\|_{\mathfrak{d}}$; hence $\tau(a) \vdash_{\mathfrak{d}} \tau(b)$. Conversely, if $\tau(a) \vdash_{\mathfrak{d}} \tau(b)$, then $\tau(b) \leq \|\tau(a)\|_{\mathfrak{d}}$; hence $\|\tau(b)\|_{\mathfrak{d}} \leq \|\tau(a)\|_{\mathfrak{d}}$; hence $\tau^{\blacktriangleleft}(\|\tau(b)\|_{\mathfrak{d}}) \leq \tau^{\blacktriangleleft}(\|\tau(a)\|_{\mathfrak{d}})$; hence by (5.41), $b \leq \|b\|_{\tau^{\blacktriangleleft}[\mathfrak{d}]} \leq \|a\|_{\tau^{\blacktriangleleft}[\mathfrak{d}]}$; hence $a \vdash_{\tau^{\blacktriangleleft}[\mathfrak{d}]} b$. $\diamond$

A useful result that we invoke often in this text is the fact that $\tau^{\blacktriangleleft}_{\mathbf{P}}[\mathfrak{d}]$ is the coarsest of all elementary closed systems $\mathfrak{c}$ on $\mathbf{P}$ such that τ is continuous from $\mathfrak{c}$ to $\mathfrak{d}$. More precisely, we have the following.

Proposition 5.86 τ is continuous from $\mathfrak{c}$ to $\mathfrak{d}$ iff $\mathfrak{c} \preceq \tau^{\blacktriangleleft}[\mathfrak{d}]$.

Proof. $\boxed{\Rightarrow}$ $a \vdash_{\mathfrak{c}} b$ [implies by assumption] $\tau(a) \vdash_{\mathfrak{d}} \tau(b)$ [implies by previous proposition] $a \vdash_{\tau^{\blacktriangleleft}[\mathfrak{d}]} b$. $\boxed{\Leftarrow}$ $a \vdash_{\mathfrak{c}} b$ [implies by assumption] $a \vdash_{\tau^{\blacktriangleleft}[\mathfrak{d}]} b$ [implies by previous proposition] $\tau(a) \vdash_{\mathfrak{d}} \tau(b)$. $\diamond$

Recall Proposition 5.85 characterizing the consequence relation of a product in terms of the consequence relation of the 'target' closed systems. In the case of the product by a single *surjective function*, it is possible to characterize the consequence relation of the single '*target*' closed system in terms of the consequence relation of the product. More precisely, we have the following result which proves key to our theory of *canons and archologies* (see $\oint$8).

Proposition 5.87 Let f be a *surjective* function onto $\mathsf{uni}(\mathbb{D})$, let $C \cup \{d\} \subseteq \mathsf{uni}(\mathbb{D})$, let $a_c \in f^{-1}[\![c]\!]$, for each $c \in C$, $b_d \in f^{-1}[\![d]\!]$, and let $A_C \subseteq \mathsf{do}(f)$ such that $f[A_C] = C$. The following conditions are equivalent.

1. $C \vdash_{\mathbb{D}} d$.

2. $f^{\blacktriangleleft}(C) \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} f^{-1}[\![d]\!]$.

3. $\{a_c : c \in C\} \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} b_d$.

4. $A_C \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} b_d$.

Consequently, $\|C\|_{\mathbb{D}} = f\left[\left\| f^{-1}[C] \right\|_{f^{\blacktriangleleft}[\mathbb{D}]}\right] = f\left[\|\{a_c : c \in C\}\|_{f^{\blacktriangleleft}[\mathbb{D}]}\right] = f\left[\|A_C\|_{f^{\blacktriangleleft}[\mathbb{D}]}\right]$.

Proof. $\boxed{(1)\Leftrightarrow(2)}$ By Proposition 5.85 on page 198, $f^{\blacktriangleleft}(C) \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} f^{-1}[\![d]\!]$ iff $f[f^{\blacktriangleleft}(C)] \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} f\left[f^{-1}[\![d]\!]\right]$ iff $C \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} \{d\}$ iff $C \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} d$, the penultimate equivalence following by surjectivity and (1.49) of Table 1.2 on page 21. $\boxed{(3)\Leftrightarrow(1)}$ By the equivalence of conditions (1) and (2) of Proposition 5.85, $\{b_d : c \in C\} \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} b_d$ iff $f[\{b_d : c \in C\}] \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} f(b_d)$. Since $f[\{b_d : c \in C\}] = C$ and $f(b_d) = d$, the result follows. $\boxed{(4)\Leftrightarrow(1)}$ Similarly, $A_C \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} b_d$ iff $f[A_C] \vdash_{f^{\blacktriangleleft}[\mathbb{D}]} f(b_d)$. Since $f[A_C] = C$ and $f(b_d) = d$, the result follows. ⋄

Recall the definition of a *finitary* (concrete) translation, given in Definition 5.17. The next result demonstrates that in the case of a *finitary* (concrete) translation into an *algebraic* (concrete) closed system, the *product* closed system must also be *algebraic*.

Theorem 5.88 If $\mathbb{D}$ is an algebraic (concrete) closed system and τ is *finitary* (concrete) translation to $\mathsf{uni}(\mathbb{D})$, then $\tau^{\blacktriangleleft}[\mathbb{D}]$ is algebraic.

Proof. Suppose that $\mathbb{D}$ is finitary. Let $\mathcal{C} \subseteq \mathsf{cl}_{\tau^{\blacktriangleleft}[\mathbb{D}]}$ with $\mathcal{C}$ being $\subseteq$-directed. For each $G \in \mathcal{C}$, there exists $H_G \in \mathsf{cl}_{\mathbb{D}}$ with $G = \tau^{\blacktriangleleft}(H_G)$. Let $\mathcal{D} = \{\tau^{\star}(G) : G \in \mathcal{C}\}$. (*We first show that $\mathcal{D}$ is directed.*) Let $\tau^{\star}(G_1), \tau^{\star}(G_2) \in \mathcal{D}$. By assumed directedness of $\mathcal{C}$, there exists $G_3 \in \mathsf{cl}_{\mathbb{D}}$ with $G_1 \cup G_2 \subseteq G_3$. Since $\tau^{\star}$ is $\subseteq$-preserving, $\tau^{\star}(G_1) \cup \tau^{\star}(G_2) \subseteq \tau^{\star}(G_3) \in \mathcal{D}$. Hence $\mathcal{D}$ is directed.

So by the assumed finitariness of $\mathbb{D}$, $\bigcup\mathcal{D} \in \mathsf{cl}_{\mathbb{D}}$, and hence $\tau^{\blacktriangleleft}(\bigcup\mathcal{D}) \in \mathsf{cl}_{\tau^{\blacktriangleleft}[\mathbb{D}]}$. (*It suffices to show that* $\tau^{\blacktriangleleft}(\bigcup\mathcal{D}) = \bigcup\mathcal{C}$.) $\boxed{\supseteq}$ $\tau^{\blacktriangleleft}(\bigcup\mathcal{D}) = \tau^{\blacktriangleleft}(\|\bigcup\mathcal{D}\|_{\mathbb{D}}) = \tau^{\blacktriangleleft}(\left\|\bigcup_{G\in\mathcal{C}} \tau^{\star}(\tau^{\blacktriangleleft}(H_G))\right\|_{\mathbb{D}}) = \tau^{\blacktriangleleft}(\left\|\bigcup_{G\in\mathcal{C}} \|\tau(\tau^{\blacktriangleleft}(H_G))\|_{\mathbb{D}}\right\|_{\mathbb{D}}) = \tau^{\blacktriangleleft}(\left\|\left\|\bigcup_{G\in\mathcal{C}} \tau(\tau^{\blacktriangleleft}(H_G))\right\|_{\mathbb{D}}\right\|_{\mathbb{D}}) = \tau^{\blacktriangleleft}(\left\|\bigcup_{G\in\mathcal{C}} \tau(\tau^{\blacktriangleleft}(H_G))\right\|_{\mathbb{D}}) \overset{(i)}{=} \tau^{\blacktriangleleft}(\left\|\tau(\bigcup_{G\in\mathcal{C}} \tau^{\blacktriangleleft}(H_G))\right\|_{\mathbb{D}}) = \tau^{\blacktriangleleft}(\tau^{\star}(\bigcup_{G\in\mathcal{C}} \tau^{\blacktriangleleft}(H_G))) = \tau^{\blacktriangleleft}(\tau^{\star}(\bigcup\mathcal{C})) = \overleftarrow{\tau}\lfloor\|\tau(\bigcup\mathcal{C})\|_{\mathbb{D}}\rfloor \overset{(ii)}{\supseteq} \overleftarrow{\tau}\lfloor\tau(\bigcup\mathcal{C})\rfloor \overset{(iii)}{\supseteq} \bigcup\mathcal{C} \cap \mathsf{gr}(\tau) \overset{(iv)}{=} \bigcup\mathcal{C}$, where (i) follows since images of unions are unions of images, by (1.9) of Table 1.1 on page 18, (ii) follows since reduced-pre-images are $\subseteq$-preserving, by (5.23) of Table 5.1 on page 180, (iii) follows by (4) of Lemma 5.15, and (iv) follows the groundedness of τ. $\boxed{\subseteq}$ Let $a \in \tau^{\blacktriangleleft}(\bigcup\mathcal{D})$. So $\tau(a) \subseteq \bigcup\mathcal{D} = \bigcup\{\tau^{\star}(G) : G \in \mathcal{C}\}$. Since τ is finitary, there exists $G_1, \ldots, G_n \in \mathcal{C}$, with $\tau(a) \subseteq \tau^{\star}(G_1) \cup \ldots \cup \tau^{\star}(G_n)$. By directedness of $\mathcal{D}$, there exists $G \in \mathcal{C}$ with $\tau^{\star}(G_1) \cup \ldots \cup \tau^{\star}(G_n) \subseteq \tau^{\star}(G)$. So $\tau(a) \subseteq \tau^{\star}(G)$ and hence $a \in \tau^{\blacktriangleleft}(\tau^{\star}(G))$. Since τ is strictly continuous from $\tau^{\blacktriangleleft}[\mathbb{D}]$ into $\mathbb{D}$, by Proposition 5.85, and hence $\vdash$-reflecting, it follows, by (7) of Proposition 5.71, that $\tau^{\blacktriangleleft}(\tau^{\star}(G)) = G$. So $a \in G \subseteq \bigcup\mathcal{C}$. ⋄

For *surjective functions* the converse obtains. Note that functions are special finitary translations.

Proposition 5.89 If f is surjective, then $\mathbb{D}$ is algebraic iff $f^{\blacktriangleleft}[\mathbb{D}]$ is algebraic.

Proof. $\boxed{\Rightarrow}$ By Theorem 5.88. $\boxed{\Leftarrow}$ Suppose that $f^{\blacktriangleleft}[\mathbb{D}]$ is finitary. Let $\mathcal{C} \subseteq \mathsf{cl}_{\mathbb{D}}$ with $\mathcal{C}$ being $\subseteq$-directed. (*We must show that* $\bigcup\mathcal{C} \in \mathsf{cl}_{\mathbb{D}}$.) Let $\mathcal{C}' = \{f^{-1}[H] : H \in \mathcal{C}\} \subseteq \mathsf{cl}_{\mathsf{f}^{\blacktriangleleft}[\mathbb{D}]}$. $\boxed{\text{Claim: } \mathcal{C}' \text{ is } \subseteq\text{-directed}}$ Suppose that $f^{-1}[H_1], f^{-1}[H_2] \in \mathcal{C}'$, where $H_1, H_1 \in \mathcal{C}$. Since $\mathcal{C}$ is $\subseteq$-directed, there exists $H \in \mathcal{C}$ with

$H_1 \cup H_1 \subseteq H$. Then $f^{-1}[H_1] \cup f^{-1}[H_2] \subseteq f^{-1}[H] \in \mathcal{C}'$. So, by the previous claim and assumed finitariness of $f^{\blacktriangleleft}[\mathbb{D}]$, $f^{-1}[\bigcup \mathcal{C}] = f^{-1}\left[\bigcup_{H\in\mathcal{C}} H\right] = \bigcup_{H\in\mathcal{C}} f^{-1}[H] = \bigcup \mathcal{C}' \in \mathsf{cl}_{f^{\blacktriangleleft}[\mathbb{D}]}$. So there exists $H \in \mathsf{cl}_{\mathbb{C}}$ with $f^{-1}[\bigcup \mathcal{C}] = f^{-1}[H]$. So by assumed surjectivity and Remark 1.40 on page 20, $\bigcup \mathcal{C} = H \in \mathsf{cl}_{\mathbb{C}}$. $\diamond$

5.3.4.1 Examples

Recall the example of the sentential 1-calculus $S(\mathcal{K}, \tau)$ of [BR99], determined by a unary system of equations τ and a quasivariety $\mathcal{K}$ of algebras (see Example 2.85 on page 106). In the following example we show how $S(\mathcal{K}, \tau)$ arises as the *product of a translation*.

Example 5.90 ($S(\mathcal{K}, \tau)$)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and τ a unary system. For each $\mathfrak{a}$-algebra $\mathbf{A}$, let $\tau^{\mathbf{A}}$ denote the (concrete) translation from $\mathsf{uni}(\mathbf{A})$ to $\mathsf{uni}(\mathbf{A})^2$, defined by

$$\tau^{\mathbf{A}}[\![a]\!] = \{\langle \delta^{\mathbf{A}}(a), \epsilon^{\mathbf{A}}(a)\rangle \,:\, \langle \delta, \epsilon\rangle \in \tau\}.$$

Observe that for all $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$,

$$\tau^{\mathbf{A}^{\blacktriangleleft}} = \tau^{\mathbf{A}}/\alpha,$$

where $\tau^{\mathbf{A}}/\alpha$ is the $\langle \alpha, \tau\rangle$-class (see Definition 2.90 on page 107). So $\langle \mathcal{K}, \tau\rangle$-classes coincide with the *reduced* pre-images (under $\tau^{\mathbf{A}}$) of relative congruences.

Remark 5.91 $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A}) = \tau^{\mathbf{A}^{\blacktriangleleft}}[\mathsf{Con}^{\mathcal{K}}(\mathbf{A})]$. □

Since the translation $\tau^{\mathbf{A}}$ is *finitary* (because τ is finite) and $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ is an *algebraic* closed system, it follows by Theorem 5.88 that $\tau^{\mathbf{A}^{\blacktriangleleft}}[\mathsf{Con}^{\mathcal{K}}(\mathbf{A})]$ is algebraic.

Corollary 5.92 [BR99] The closed system $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A})$ is algebraic. □

Since $\tau^{\mathbf{A}}$ is strictly continuous from $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, by Proposition 5.85, the following result obtains immediately.

Corollary 5.93 $A \vdash_{\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A})} a$ iff $\tau^{\mathbf{A}}[\![a]\!] \subseteq \left\| \tau^{\mathbf{A}}[A] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$. □

Consequently, it follows immediately from Lemma 1.457 on page 88, that

$$P \vdash_{\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{Tm})} p \quad \text{iff} \quad \tau^{\approx}[P] \models_{\mathcal{K}} \tau^{\approx}[\![p]\!]. \tag{5.42}$$

Comparing (5.42) with (2.19) of Theorem 2.89 on page 107, the following obtains immediately.

Corollary 5.94 [BR99] $\mathsf{Th}(S(\mathcal{K}, \tau)) = \mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{Tm})$.

Proposition 5.95 If $f : \mathbf{A} \to \mathbf{B}$, then f is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A})$ into $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{B})$.

Proof. It is not hard to show that $\tau^{\mathbf{B}} f = f\tau^{\mathbf{A}}$ (see Lemma 9.17 on page 317 for a proof in a more general context). Now f is continuous from $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ into $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, by Example 5.51, and $\tau^{\mathbf{A}}$ is (strictly) continuous from $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. So by Remark 5.24, (viewing f as a translation) $f\tau^{\mathbf{A}}$ is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, and hence so is $\tau^{\mathbf{B}} f$. Since $\tau^{\mathbf{B}}$ is strictly continuous from $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{B})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, and hence consequence reflecting, f is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{A})$ into $\mathsf{Sol}^{\mathcal{K}}_{\tau}(\mathbf{B})$, by Proposition 5.23. $\diamond$

□

5.3.5 Iseomorphisms

We now consider iseomorphisms between elementary closed systems, which essentially are pairs of continuous translations (in opposite direction) that are *mutually untranslating*, in which case these translations are strict. The reader is urged to recall the notion of a formal equivalent semantics of a sentential calculi (see Definition 2.97 on page 109). We begin by defining the property that one translation *untranslates* another.

Definition 5.96 (Untranslation) Let $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ and $\pi : \mathfrak{d} \rightleftarrows \mathfrak{c}$. We say that π **untranslates** τ if, for all $a \in \mathsf{uni_e}(\mathfrak{c})$, $(\pi\tau)(a) \dashv\vdash_{\mathfrak{c}} a$. We call τ and π **mutually untranslating** if τ untranslates π and π untranslates τ. □

Remark 5.97 If $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ and $\pi : \mathfrak{d} \rightleftarrows \mathfrak{c}$, then

$$b \leq \tau^{\blacktriangleleft}(\pi^{\blacktriangleleft}(\|a\|_{\mathfrak{c}})) \text{ iff } \pi^{\star}(\tau(b)) \leq \|a\|_{\mathfrak{c}} . \tag{5.43}$$

Proof. $b \leq \tau^{\blacktriangleleft}(\pi^{\blacktriangleleft}(\|a\|_{\mathfrak{c}}))$ [iff by (5.19)] $\tau(b) \leq \pi^{\blacktriangleleft}(\|a\|_{\mathfrak{c}})$ [iff by (5.19)] $\pi(\tau(b)) \leq \|a\|_{\mathfrak{c}}$ [iff] $\|\pi(\tau(b))\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{c}}$ [iff] $\pi^{\star}(\tau(b)) \leq \|a\|_{\mathfrak{c}}$. ◇

Proposition 5.98 For $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ and $\pi : \mathfrak{d} \rightleftarrows \mathfrak{c}$, the following conditions are equivalent.

1. π untranslates τ.
2. $\|a\|_{\mathfrak{c}} = \pi^{*}(\tau(a))$, for all $a \in \mathsf{uni_e}(\mathfrak{c})$.
3. $\|a\|_{\mathfrak{c}} \leq \pi^{*}(\tau(a))$ and $\|a\|_{\mathfrak{c}} = \tau^{\blacktriangleleft}(\pi^{\blacktriangleleft}(\|a\|_{\mathfrak{c}}))$, for all $a \in \mathsf{uni_e}(\mathfrak{c})$.

Proof. $\boxed{(1)\Leftrightarrow(2)}$ Trivial. $\boxed{(2)\Rightarrow(3)}$ Let $a \in \mathsf{uni_e}(\mathfrak{c})$. Trivially, $\|a\|_{\mathfrak{c}} \leq \pi^{*}(\tau(a))$. Observe that *for all* $b \in \mathsf{uni_e}(\mathfrak{c})$, $b \leq \tau^{\blacktriangleleft}(\pi^{\blacktriangleleft}(\|a\|_{\mathfrak{c}}))$ [iff by (5.43)] $\pi^{\star}(\tau(b)) \leq \|a\|_{\mathfrak{c}}$ [iff (by assumption (2))] $\|b\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{c}}$ [iff] $b \leq \|a\|_{\mathfrak{c}}$. So by (1.58), $\|a\|_{\mathfrak{c}} = \tau^{\blacktriangleleft}(\pi^{\blacktriangleleft}(\|a\|_{\mathfrak{c}}))$. $\boxed{(3)\Rightarrow(2)}$ Let $a \in \mathsf{uni_e}(\mathfrak{c})$. Then, $\pi^{\star}(\tau(\|a\|_{\mathfrak{c}})) \leq \|a\|_{\mathfrak{c}}$ [iff by (5.43)] $\|a\|_{\mathfrak{c}} \leq \tau^{\blacktriangleleft}(\pi^{\blacktriangleleft}(\|a\|_{\mathfrak{c}}))$ [iff by assumption] $\|a\|_{\mathfrak{c}} \leq \|a\|_{\mathfrak{c}}$ [iff] true. So $\pi^{\star}(\tau(\|a\|_{\mathfrak{c}})) \leq \|a\|_{\mathfrak{c}}$. The converse inequality is assumed. ◇

We now define the notion of iseomorphic translations between closed systems and the notion of iseomorphic elementary closed systems, and show that if two closed systems are iseomorphic then their orders of closed points are isomorphic. The reader familiar with the theory of algebraizable logics will notice that our condition of iseomorphic is apparently weaker than expected, in that we require only continuity where strictly continuous translations are expected. We shall show that our condition is equivalent to the stronger one (see (2) of Lemma 5.100 and (5) to (7) of Theorem 5.101).

Definition 5.99 (Iseomorphisms) We call $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ an **iseomorphism** from $\mathfrak{c}$ to $\mathfrak{d}$ if τ is continuous, and there exists continuous $\pi : \mathfrak{d} \rightleftarrows \mathfrak{c}$ such that τ and π are mutually untranslating, in which case we call elementary closed systems $\mathfrak{c}$ and $\mathfrak{d}$ **iseomorphic** and call π ***an* inverse iseomorphism of** τ (or just an **inverse** where unambiguous). □

Lemma 5.100 Suppose that τ is an iseomorphism from $\mathfrak{c}$ to $\mathfrak{d}$ with inverse π.

1. $a \dashv\vdash_{\mathfrak{c}} \pi^*(\tau^*(a))$ and $c \dashv\vdash_{\mathfrak{d}} \tau^*(\pi^*(c))$, for all $a \in \mathsf{uni_e}(\mathfrak{c})$ and $c \in \mathsf{uni}(\mathfrak{d})$.

2. τ and π are strictly continuous.

3. $\tau^*(a) = \pi^{\blacktriangleleft}(\|a\|_{\mathfrak{c}})$ and $\pi^*(c) = \tau^{\blacktriangleleft}(\|c\|_{\mathfrak{d}})$, for all $a \in \mathsf{uni_e}(\mathfrak{c})$ and $c \in \mathsf{uni}(\mathfrak{d})$.

4. ${\tau^*}_{|\mathsf{cl}_{\mathfrak{c}}} = {\pi^{\blacktriangleleft}}_{|\mathsf{cl}_{\mathfrak{c}}}$ and ${\pi^*}_{|\mathsf{cl}_{\mathfrak{d}}} = {\tau^{\blacktriangleleft}}_{|\mathsf{cl}_{\mathfrak{d}}}$.

5. If $\pi' : \mathfrak{d} \rightleftarrows \mathfrak{c}$ is continuous and τ and π' are mutually untranslating, then $\pi(c) \dashv\vdash_{\mathfrak{c}} \pi'(c)$, for all $c \in \mathsf{uni}(\mathfrak{d})$, and consequently $\pi^\star = (\pi')^\star$.

6. $\tau^*(\tau^{\blacktriangleleft}(h)) = h$ and $\pi^*(\pi^{\blacktriangleleft}(g)) = g$, for all $h \in \mathsf{cl}_{\mathfrak{d}}$ and $g \in \mathsf{cl}_{\mathfrak{c}}$.

7. ${\tau^*}_{|\mathsf{cl}_{\mathfrak{c}}}$ is surjective onto $\mathsf{cl}_{\mathfrak{d}}$ and ${\pi^*}_{|\mathsf{cl}_{\mathfrak{d}}}$ is surjective onto $\mathsf{cl}_{\mathfrak{c}}$.

Proof. $\boxed{(1)}$ By the continuity of τ and (8) of Theorem 5.21, $\tau(a) \dashv\vdash_{\mathfrak{d}} \tau^\star(\|a\|)$, so substituting $\pi(c)$ for a, we obtain $\tau(\pi(c)) \dashv\vdash_{\mathfrak{d}} \tau^\star(\|\pi(c)\|)$, i.e., $\tau(\pi(c)) \dashv\vdash_{\mathfrak{d}} \tau^\star(\pi^\star(c))$. Since by assumption, $\tau(\pi(c)) \dashv\vdash_{\mathfrak{d}} c$, by the transitivity of $\dashv\vdash_{\mathfrak{d}}$, $c \dashv\vdash_{\mathfrak{c}} \pi^*(\tau^*(c))$. The remaining formula follows symmetrically. $\boxed{(2)}$ Suppose that $\tau(a) \vdash_{\mathfrak{d}} \tau(b)$. By the continuity of π, $\pi(\tau(a)) \vdash_{\mathfrak{c}} \pi(\tau(b))$. Since by assumption, $a \dashv\vdash_{\mathfrak{c}} \pi(\tau(a))$ and $b \dashv\vdash_{\mathfrak{c}} \pi(\tau(b))$, it follows that $a \vdash_{\mathfrak{c}} b$. The remaining assertion follows symmetrically. $\boxed{(3)}$ Observe that for all $b \in \mathsf{uni_e}(\mathfrak{c})$, $b \leq \pi^*(c)$ [iff] $b \leq \|\pi(c)\|_{\mathfrak{c}}$ [iff] $\pi(c) \vdash_{\mathfrak{c}} b$ [iff (since τ is strictly continuous by (2))] $\tau(\pi(c)) \vdash_{\mathfrak{d}} \tau(b)$ [iff (since $\tau(\pi(c)) \dashv\vdash_{\mathfrak{d}} c$)] $c \vdash_{\mathfrak{d}} \tau(b)$ [iff] $\tau(b) \leq \|c\|_{\mathfrak{d}}$ [iff (by (5.19))] $b \leq \tau^{\blacktriangleleft}(\|c\|_{\mathfrak{d}})$. So $\pi^*(c) = \tau^{\blacktriangleleft}(\|c\|_{\mathfrak{d}})$, by (1.58). The remaining assertion follows symmetrically. $\boxed{(4)}$ Follows immediately from (3) and the idempotence of closure operators. $\boxed{(5)}$ Follows easily from (3) and (5.6). $\boxed{(6)}$ Since h is closed, $\tau^{\blacktriangleleft}(h) = \tau^{\blacktriangleleft}(\|h\|_{\mathfrak{c}}) = \pi^*(h)$ by (3). So $\tau^*(\tau^{\blacktriangleleft}(h)) = \tau^*(\pi^*(h)) \dashv\vdash_{\mathfrak{d}} h$ by (1). Since $\tau^*(\tau^{\blacktriangleleft}(h))$ and h are both closed, $\tau^*(\tau^{\blacktriangleleft}(h)) = h$. The remaining assertion follows symmetrically. $\boxed{(7)}$ Follows at once from (6) and (3) of Theorem 5.40. $\diamond$

So while the inverse of an iseomorphism is not necessarily unique, the inverse is unique up to closure. That is, if π and π' are both inverse translations of iseomorphism $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$, then $\pi^\star(c) = (\pi')^\star(c) = \tau^{\blacktriangleleft}(\|c\|_{\mathfrak{d}})$.

Theorem 5.101 For $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ and $\pi : \mathfrak{d} \rightleftarrows \mathfrak{c}$, the following conditions are equivalent.

1. τ is an iseomorphism from $\mathfrak{c}$ to $\mathfrak{d}$ with inverse π.

2. τ and π are continuous and, for all $c \in \mathsf{uni}(\mathfrak{d})$ and $a \in \mathsf{uni_e}(\mathfrak{c})$,
$$\tau^*(\pi(c)) = \|c\|_{\mathfrak{d}} \quad \text{and} \quad \pi^*(\tau(a)) = \|a\|_{\mathfrak{c}}\,.$$

3. τ and π are continuous and, for all $c \in \mathsf{uni}(\mathfrak{d})$ and $a \in \mathsf{uni_e}(\mathfrak{c})$,
$$\tau^*(\pi^\star(c)) = \|c\|_{\mathfrak{d}} \quad \text{and} \quad \pi^*(\tau^\star(a)) = \|a\|_{\mathfrak{c}}\,.$$

4. τ and π are continuous and, for all $h \in \mathsf{cl}_{\mathfrak{d}}$ and $g \in \mathsf{cl}_{\mathfrak{c}}$,
$$\tau^*(\pi^\star(h)) = h \quad \text{and} \quad \pi^*(\tau^\star(g)) = g.$$

5. τ and π are strictly continuous and mutually untranslating.

6. τ is strictly continuous and τ untranslates π.

7. π is strictly continuous and π untranslates τ.

Proof. $\boxed{1\Leftrightarrow 2}$ By Proposition 5.98. $\boxed{2\Rightarrow 3}$ $\tau^*(\pi^\star(c)) = \tau^*(\|\pi(c)\|_{\mathfrak{c}}) = \tau^*(\pi(c)) = \|c\|_{\mathfrak{d}}$, the second equality following by the continuity of π and (5) of Theorem 5.21. Similarly, $\pi^*(\tau^\star(a)) = \|a\|_{\mathfrak{c}}$. $\boxed{3\Rightarrow 4}$ By idempotence of closure operators. $\boxed{4\Rightarrow 2}$ $\tau^*(\pi(c)) = \tau^*(\|\pi(c)\|_{\mathfrak{c}}) = \tau^*(\pi^*(c)) = \tau^*(\pi^*(\|c\|_{\mathfrak{d}})) = \|c\|_{\mathfrak{d}}$, where the first equality follows by continuity of π and (5) of Theorem 5.21, the third by continuity of τ and (5) of Theorem 5.21, and the final equality follows by assumption. Similarly, $\pi^*(\tau(a)) = \|a\|_{\mathfrak{c}}$. $\boxed{1\Rightarrow 5}$ By 2 of Lemma 5.100. $\boxed{5\Rightarrow 1}$ Trivial. $\boxed{5\Rightarrow 6,7}$ Trivial. $\boxed{6\Rightarrow 5}$ Note that by the strict continuity of τ and (2) of Theorem 5.73, for all $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$,

$$\tau^{\blacktriangleleft}(\tau^\star(a)) = \|a\|_{\mathfrak{c}}. \tag{a}$$

Note further, that by Proposition 5.98, for all $c \in \mathsf{uni}(\mathfrak{d})$,

$$\tau^*(\pi(c)) = \|c\|_{\mathfrak{d}} \quad \text{and} \tag{b}$$

$$\pi^{\blacktriangleleft}(\tau^{\blacktriangleleft}(\|c\|_{\mathfrak{d}})) = \|c\|_{\mathfrak{d}}\,. \tag{c}$$

Since $\pi^*(\tau(a)) = \|\pi(\tau(a))\|_{\mathfrak{c}} \overset{(a)}{=} \tau^{\blacktriangleleft}(\tau^\star(\pi(\tau(a)))) \overset{(b)}{=} \tau^{\blacktriangleleft}(\|\tau(a)\|) = \tau^{\blacktriangleleft}(\tau^\star(a)) \overset{(a)}{=} \|a\|_{\mathfrak{c}}$, π untranslates τ, by Proposition 5.98. Further, $\pi^{\blacktriangleleft}(\pi^\star(c)) = \pi^{\blacktriangleleft}(\|\pi(c)\|_{\mathfrak{c}}) \overset{(a)}{=} \pi^{\blacktriangleleft}(\tau^{\blacktriangleleft}(\tau^\star(\pi(c)))) \overset{(b)}{=} \pi^{\blacktriangleleft}(\tau^{\blacktriangleleft}(\|c\|_{\mathfrak{d}})) \overset{(c)}{=} \|c\|_{\mathfrak{d}}$, so π is strictly continuous by (2) of Theorem 5.73. $\boxed{7\Rightarrow 5}$ Symmetric to proof of 6⇒5. ⋄

Consequently, if $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ is an iseomorphism, then $\pi^{\blacktriangleleft}$ determines an order-isomorphism from $\mathbf{cl}_{\mathfrak{c}}$ onto $\mathbf{cl}_{\mathfrak{d}}$. In more detail, we have the following result. Note that this theorem is essentially [BJ06, T3.8] (see Theorem 5.145 on page 213 of our text) in an elementary context.

Theorem 5.102 Let $\mathfrak{c}$ and $\mathfrak{d}$ be two elementary closed systems. If $\tau : \mathfrak{c} \rightleftarrows \mathfrak{d}$ is an iseomorphism with inverse iseomorphism $\pi : \mathfrak{d} \rightleftarrows \mathfrak{c}$, then $\pi^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{c}}} = \tau^*{}_{|\mathsf{cl}_{\mathfrak{c}}} : \mathbf{cl}_{\mathfrak{c}} \cong \mathbf{cl}_{\mathfrak{d}}$ with inverse isomorphism $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathfrak{d}}} = \pi^*{}_{|\mathsf{cl}_{\mathfrak{d}}}$.

Proof. Follows at once from (2), (4) and (7) of Lemma 5.100, together with (8) of Theorem 5.73. ⋄

As noted earlier, we are unable to provide a converse to this result in the elementary setting.

Open Problem 5.103 Show that there exist non-iseomorphic elementary closed systems $\mathfrak{c}$ and $\mathfrak{d}$ with $\mathbf{cl}_{\mathfrak{c}} \cong \mathbf{cl}_{\mathfrak{d}}$.

5.4 Concrete Translations

We now consider (concrete) translations between concrete closed systems. Recall Theorem 2.96 on page 109 concerning formal semantics. If one eliminates commutivity and compactness from this result, the reader will notice that in developing our theory of strictly continuous elementary translations, we have establish (1) of this theorem but not (2) (see Theorem 5.73); similarly for iseomorphisms and Theorem 2.100 on page 110 concerning equivalent semantics. We now show how the second statement of these results obtain in the concrete case, and relate these results to Blok and Jónsson theory of *similar consequence operators* [BJ06]. We shall begin this task starting with continuity, which is a weaker notion than that required for the aforementioned results. In this section we shall also define the product of *multiple* concrete translations, and show how the *semantic consequence* relation $\models^{\mathbf{M}}$ (see Definition 2.32 on page 99) determined by a matrix $\mathbf{M}$ may be realized as such a product.

Convention 5.104 (Concrete Closed Systems) For the remainder of this chapter, all closed systems and all translations are concrete, unless specified to the contrary.

5.4.1 Continuity

In the following characterization of continuous concrete translations, the reader is urged to distinguish equivalent condition (2) from the definition of continuity.

Proposition 5.105 Let $\mathbb{C}$ and $\mathbb{D}$ be (concrete) closed systems and $\tau : \mathbb{C} \multimap \mathbb{D}$. The following conditions are equivalent.

1. τ is continuous.
2. $A \vdash_{\mathbb{C}} a$ implies $\tau(A) \vdash_{\mathbb{D}} \tau(\{a\})$, for all $A \cup \{a\} \subseteq \mathsf{uni}_{\mathsf{e}}(\mathfrak{c})$.
3. $\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbf{cl}_{\mathbb{C}} \rightarrow_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$ and $\tau^{\star}(\|\{a\}\|) = \|\tau(\{a\})\| \qquad (\forall\, [a \in \mathsf{uni}(\mathbb{C})]\,)$.

Proof. $\boxed{(1)\Leftrightarrow(2)}$ Trivial. $\boxed{(1)\Rightarrow(3)}$ By (5) of Theorem 5.21 and (12) of Theorem 5.40. $\boxed{(3)\Rightarrow(1)}$ $\|\tau(\|A\|_{\mathbb{C}})\|_{\mathbb{D}} = \left\|\tau(\|\bigcup_{a\in A}\{a\}\|_{\mathbb{C}})\right\|_{\mathbb{D}} \overset{(4.78)}{=} \left\|\tau(\|\bigcup_{a\in A}\|\{a\}\|_{\mathbb{C}}\|_{\mathbb{C}})\right\|_{\mathbb{D}} \overset{(4.78)}{=} \left\|\tau(\blacktriangledown^{\mathbf{cl}_{\mathbb{C}}}_{a\in A}\|\{a\}\|_{\mathbb{C}})\right\|_{\mathbb{D}} = \tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}}(\blacktriangledown^{\mathbf{cl}_{\mathbb{C}}}_{a\in A}\|\{a\}\|_{\mathbb{C}}) = \blacktriangledown^{\mathbf{cl}_{\mathbb{D}}}_{a\in A}\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}}(\|\{a\}\|_{\mathbb{C}}) \overset{(4.78)}{=} \|\bigcup_{a\in A}\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}}(\|\{a\}\|_{\mathbb{C}})\|_{\mathbb{D}} = \|\bigcup_{a\in A}\|\tau(\|\{a\}\|_{\mathbb{C}})\|_{\mathbb{D}}\|_{\mathbb{D}} = \|\bigcup_{a\in A}\|\tau(\{a\})\|_{\mathbb{D}}\|_{\mathbb{D}} \overset{(4.78)}{=} \|\bigcup_{a\in A}\tau(\{a\})\|_{\mathbb{D}} = \|\tau(A)\|_{\mathbb{D}}$. ◇

Warning 5.106 We may invoke equivalent condition (2) of the previous result without explicit reference.

Condition (3) cannot be weakened to $\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbf{cl}_{\mathbb{C}} \rightarrow_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$ only.

Counter Example 5.107 ($\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbf{cl}_{\mathbb{C}} \rightarrow_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}} \not\rightarrow$ Continuity)

> Let $\mathbb{C}$, $\mathbb{D}$ and $\tau : \mathbb{C} \multimap \mathbb{D}$ be defined as in Counter-Example 5.74 on page 196. In that counter-example, we noted that τ is not continuous. It is easily seen, however, that $\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbf{cl}_{\mathbb{C}} \rightarrow_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$.

□

If τ is continuous from $\mathbb{C}$ into $\mathbb{D}$, then (3) of the previous proposition (or by (12) of Theorem 5.40), $\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbf{cl}_{\mathbb{C}} \rightarrow_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$. In the following result, we show that all $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \rightarrow_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$, arise in this manner.

Theorem 5.108 If $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \rightarrow_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$ then any translation $\tau : \mathbb{C} \multimap \mathbb{D}$ satisfying $\|\tau(a)\|_{\mathbb{D}} = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$, is continuous and satisfies $\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$; one such translation is defined by $\tau[\![a]\!] = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$.

Proof. Let G be any $\mathbb{C}$-closed set. If $G = \emptyset$ (and hence $\|\emptyset\|_{\mathbb{C}} = \emptyset$), then, since join complete semilattice homomorphisms preserve bottoms, $\tau^\star(\emptyset) = \|\tau(\emptyset)\|_{\mathbb{D}} = \|\emptyset\|_{\mathbb{D}} = \mathsf{f}(\|\emptyset\|_{\mathbb{C}}) = \mathsf{f}(\emptyset)$. Otherwise, $\tau^\star(G) = \|\tau(G)\|_{\mathbb{D}} = \left\|\bigcup_{a\in G}\tau[\![a]\!]\right\|_{\mathbb{D}} \overset{(4.78)}{=} \blacktriangledown^{\mathbf{cl}_{\mathbb{D}}}_{a\in G}\|\tau[\![a]\!]\|_{\mathbb{D}} = \blacktriangledown^{\mathbf{cl}_{\mathbb{D}}}_{a\in G}\mathsf{f}(\|a\|_{\mathbb{C}}) = \mathsf{f}\left(\blacktriangledown^{\mathbf{cl}_{\mathbb{C}}}_{a\in G}\|a\|_{\mathbb{C}}\right) = \mathsf{f}(G)$. Consequently, ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$. Finally, $\tau^\star(\|\{a\}\|_{\mathbb{C}}) = \mathsf{f}(\|\{a\}\|_{\mathbb{C}}) = \|\tau(\{a\})\|_{\mathbb{C}}$ and, since ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$, ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbf{cl}_{\mathbb{C}} \to_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$; hence τ is continuous from $\mathbb{C}$ to $\mathbb{D}$ by equivalent condition (3) of Proposition 5.105. Note that it is trivially true that the translation defined by $\tau[\![a]\!] = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$, satisfies $\|\tau(a)\|_{\mathbb{D}} = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$. $\diamond$

The following characterization of granularity in terms of the continuity of the identity function, follows immediately from Proposition 4.41 on page 148 and Theorem 5.40 on page 186.

Proposition 5.109 $\mathbb{C} \preceq \mathbb{C}'$ iff $\mathsf{id}_{\mathsf{uni}(\mathbb{C})}$ is a (well-defined) continuous bijection from $\mathbb{C}'$ onto $\mathbb{C}$.

Proof. $\mathbb{C} \preceq \mathbb{C}'$ [iff] $\mathsf{uni}(\mathbb{C}) = \mathsf{uni}(\mathbb{C}')$ and $\mathsf{cl}_{\mathbb{C}} \supseteq \mathsf{cl}_{\mathbb{C}'}$ [iff] $\mathsf{uni}(\mathbb{C}) = \mathsf{uni}(\mathbb{C}')$ and $\forall\,[G \in \mathsf{cl}_{\mathbb{C}'}]\; G \in \mathsf{cl}_{\mathbb{C}}$ [iff] $\mathsf{id}_{\mathsf{uni}(\mathbb{C})}$ is a well-defined bijection from $\mathbb{C}'$ onto $\mathbb{C}$ and $\forall\,[G \in \mathsf{cl}_{\mathbb{C}'}]\; G \in \mathsf{cl}_{\mathbb{C}}$ [iff] $\mathsf{id}_{\mathsf{uni}(\mathbb{C})}$ is a well-defined bijection from $\mathbb{C}'$ onto $\mathbb{C}$ and $\forall\,[G \in \mathsf{cl}_{\mathbb{C}'}]\; {\mathsf{id}_{\mathsf{uni}(\mathbb{C})}}^{-1}[G] \in \mathsf{cl}_{\mathbb{C}}$ [iff] $\mathsf{id}_{\mathsf{uni}(\mathbb{C})}$ is a well-defined bijection from $\mathbb{C}'$ onto $\mathbb{C}$ and is continuous from $\mathbb{C}'$ into $\mathbb{C}$. $\diamond$

5.4.2 Strict Continuity

If τ is strictly continuous from $\mathbb{C}$ into $\mathbb{D}$, then by equivalent condition (8) of Theorem 5.73 on page 195, ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbf{cl}_{\mathbb{C}} \cong {\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}}(\mathbf{cl}_{\mathbb{C}}) \lhd_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$. We now characterize such $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \cong \mathsf{f}\,[\mathbf{cl}_{\mathbb{C}}] \lhd_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$. Note that, in the *concrete* case, such f are precisely those $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \rightarrowtail_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$, by Remark 1.190.

Theorem 5.110 If $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \rightarrowtail_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$ then any translation $\tau : \mathbb{C} \multimap \mathbb{D}$ satisfying $\|\tau(a)\|_{\mathbb{D}} = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$, is strictly continuous and satisfies ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$ and ${\tau^\blacktriangleleft}_{|\mathsf{f}[\mathsf{cl}_{\mathbb{C}}]} = \mathsf{f}^{-1}$; one such translation is defined by $\tau[\![a]\!] = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$.

Proof. By Theorem 5.108, τ is continuous and satisfies ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$. (*We first show that* ${\tau^\blacktriangleleft}_{|\mathsf{f}(\mathsf{cl}_{\mathbb{C}})} = \mathsf{f}^{-1}$.) Let $H \in \mathsf{f}[\mathsf{cl}_{\mathbb{C}}]$. Note that H is $\mathbb{D}$-closed by assumption. Then $a \in \tau^\blacktriangleleft(H)$ iff $\|\{a\}\|_{\mathbb{C}} \subseteq \tau^\blacktriangleleft(H)$ iff $\tau^*(\|\{a\}\|_{\mathbb{C}}) \subseteq H$ iff $\left\|\tau^*(\|\{a\}\|_{\mathbb{C}})\right\|_{\mathbb{D}} \subseteq H$ iff $\mathsf{f}(\|a\|_{\mathbb{C}}) \subseteq H$ iff $\|a\|_{\mathbb{C}} \subseteq \mathsf{f}^{-1}(H)$ iff $a \in \mathsf{f}^{-1}(H)$, where the first equivalence follows since τ is continuous and so by (3) of Theorem 5.40, $\tau^\blacktriangleleft(H)$ is $\mathbb{C}$-closed, the second follows by (6) of Lemma 5.15 and the fact that H is closed, the third follows since H is $\mathbb{D}$-closed, the fourth follows since ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$, and the fifth follows since $\mathsf{f}^{-1}(H)$ is $\mathbb{C}$-closed. (*It remains to show that τ is $\vdash$-reflecting and hence strictly continuous.*) For all $H \in \mathsf{cl}_{\mathbb{C}}$, $\tau^\blacktriangleleft(\tau^\star(H)) = \tau^\blacktriangleleft(\mathsf{f}(H)) = \mathsf{f}^{-1}(\mathsf{f}(H)) = H$, and so τ is consequence reflecting by (7) of Proposition 5.71. $\diamond$

Theorem 5.111 Let $\mathbb{C}$ and $\mathbb{D}$ be *algebraic* closed systems.

1. If $\tau : \mathbb{C} \multimap \mathbb{D}$ is *finitary* and strictly continuous, then ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}}[\mathbf{cl}_{\mathbb{C}}]$ is *compact* in $\mathbf{cl}_{\mathbb{D}}$.
2. If $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \rightarrowtail_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$ and $\mathsf{f}(\mathbf{cl}_{\mathbb{C}})$ is *compact* in $\mathbf{cl}_{\mathbb{D}}$, then there exists a *finitary* strictly continuous translation $\tau : \mathbb{C} \multimap \mathbb{D}$ with ${\tau^\star}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$ and ${\tau^\blacktriangleleft}_{|\mathsf{f}(\mathsf{cl}_{\mathbb{C}})} = \mathsf{f}^{-1}$; one such translation is defined by $\tau[\![a]\!] = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$.

Proof. $\boxed{(1)}$ By (8) of Theorem 5.73, ${\tau^\star}_{|\mathsf{cl}_\mathbb{C}} : \mathbf{cl}_\mathbb{C} \cong {\tau^\star}_{|\mathsf{cl}_\mathbb{C}}(\mathbf{cl}_\mathbb{C}) \lhd_\blacktriangledown \mathbf{cl}_\mathbb{D}$ with inverse isomorphism ${\tau^\blacktriangleleft}_{|\tau^\star(\mathsf{cl}_\mathbb{C})}$. By Remark 1.190 on page 42, $\mathsf{Cmp}_\blacktriangledown(\mathbf{cl}_\mathbb{D}) \cap {\tau^\star}_{|\mathsf{cl}_\mathbb{C}}(\mathsf{cl}_\mathbb{C}) \subseteq \mathsf{Cmp}_\blacktriangledown({\tau^\star}_{|\mathsf{cl}_\mathbb{C}}(\mathbf{cl}_\mathbb{C}))$. Let $H \in \mathsf{Cmp}_\blacktriangledown({\tau^\star}_{|\mathsf{cl}_\mathbb{C}}(\mathbf{cl}_\mathbb{C}))$. Since isomorphisms preserve compact elements, by Remark 1.191 on page 42, $\tau^\blacktriangleleft(H) \in \mathsf{Cmp}_\blacktriangledown(\mathbf{cl}_\mathbb{C})$, and hence is finitely generated since $\mathbb{C}$ is assumed to be algebraic. So $\tau^\blacktriangleleft(G) = \|B\|_\mathbb{C}$ for some finite $B \subseteq_\mathsf{f} \mathsf{uni}(\mathbb{C})$. Then $G = \tau^\star(\tau^\blacktriangleleft(G)) = \tau^\star(\|B\|_\mathbb{C}) = \tau^\star(B)$, by (5) of Theorem 5.21 and the assumed continuity of τ. So $G = \|\tau(B)\|_\mathbb{D}$, and hence is finitely generated, since $\tau(B)$ is finite as τ is assumed to be finitary and B is finite. Since $\mathbb{D}$ is finitary, $G \in \mathsf{Cmp}_\blacktriangledown(\mathbf{cl}_\mathbb{D}) \cap {\tau^\star}_{|\mathsf{cl}_\mathbb{C}}(\mathsf{cl}_\mathbb{C})$. $\boxed{(2)}$ Let $a \in \mathsf{uni}(\mathbb{C})$. Since $\|\{a\}\|_\mathbb{C}$ is finitely generated and $\mathbb{C}$ is finitary, $\|\{a\}\|_\mathbb{C}$ is compact in $\mathbb{C}$. Since isomorphisms preserve compact elements, $\mathsf{f}(\|\{a\}\|_\mathbb{C})$ is a compact element of $\mathsf{f}(\mathbf{cl}_\mathbb{C})$. Since $\mathsf{f}(\mathbf{cl}_\mathbb{C})$ is assumed to be compact in $\mathbf{cl}_\mathbb{D}$, $\mathsf{f}(\|\{a\}\|_\mathbb{C})$ is a compact element of $\mathbf{cl}_\mathbb{D}$, and hence is finitely generated, since $\mathbb{D}$ is assumed to be algebraic. So there exists some finite subset $B_a \subseteq_\mathsf{f} \mathsf{f}(\|\{a\}\|_\mathbb{C})$ with $\|B_a\|_\mathbb{D} = \mathsf{f}(\|\{a\}\|_\mathbb{C})$. The remaining assertions follow by Theorem 5.110. $\diamond$

5.4.3 Products and Quotients

5.4.3.1 Products

Recall the definition of the product of a *single* translation from an order to an elementary closed system, given in ∮5.3.4. We shall require the notion of a product of *multiple* translations. The definition that we shall use makes crucial use of the notion of the closed system generated by a basis, and as such is non-elementary. We can see no natural elementary form of this construction.

Definition 5.112 (Products of Sources) A **source of closed systems** sc is determined by a *class* of pairs $\mathsf{Arrow}(\mathsf{sc})$, the members of which are called **source-arrows**, and a set $\mathsf{uni}(\mathsf{sc})$, called the **universe**, such that, for each source-arrow $\langle\tau, \mathbb{D}\rangle \in \mathsf{Arrow}(\mathsf{sc})$, $\tau : \mathsf{uni}(\mathsf{sc}) \multimap \mathsf{uni}(\mathbb{D})$. For a closed system $\mathbb{D}$, a set A and $\tau : A \multimap \mathsf{uni}(\mathbb{D})$, let $\langle\tau, \mathbb{D}\rangle$ denote the source of closed systems determined by the single source-arrow $\langle\tau, \mathbb{D}\rangle$ and universe A, which we call a **singleton source**. A source of closed systems is called **functional** if the translation component of each source-arrow is a function.

We shall call a source sc **continuous from** closed system $\mathbb{C}$ if $\mathsf{uni}(\mathbb{C}) = \mathsf{uni}(\mathsf{sc})$ and, for each $\langle\tau, \mathbb{D}\rangle \in \mathsf{sc}$, τ is continuous from $\mathbb{C}$ into $\mathbb{D}$. We denote the class of all closed systems $\mathbb{C}$ with source sc continuous from closed system $\mathbb{C}$ by $\mathsf{CContFrom}(\mathsf{sc})$.

With each source sc we associate the closed system $\mathsf{sc}^\blacktriangleleft$ generated by basis $\{\tau^\blacktriangleleft(H) : \langle\tau, \mathbb{D}\rangle \in \mathsf{sc}$ and $H \in \mathsf{cl}_\mathbb{D}\}$, which we call the **product closed system** induced by source sc. $\square$

The following result characterizes the consequence relation of $\mathsf{sc}^\blacktriangleleft$. Note that while the third equivalent condition is essentially trivially equivalent to the second, we have included it to highlight the relationship between products and the *semantic consequence relation* determined by a matrix (see of (2.3) of Definition 2.32 on page 99 and Example 5.121 to follow shortly).

Theorem 5.113 The following conditions are equivalent.

1. $A \vdash_{\mathsf{sc}^\blacktriangleleft} b$.

2. $\tau(A) \vdash_\mathbb{D} \tau(b)$, for all $\langle\tau, \mathbb{D}\rangle \in \mathsf{sc}$.

3. $\tau(A) \subseteq H \rightarrow \tau(b) \subseteq H$, for all $\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}$ and $H \in \mathsf{cl}_{\mathbb{D}}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Let $\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}$. For any $H \in \mathsf{cl}_{\mathbb{D}}$, $\tau^{\blacktriangleleft}(H) \in \mathsf{cl}_{\mathsf{sc}^{\blacktriangleleft}}$ by construction. The result follows by (3) of Theorem 5.40. $\boxed{(2)\Rightarrow(1)}$ Assume that $\forall\, [\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}]\ \tau(A) \vdash_{\mathbb{D}} \tau(b)$. Suppose that $G \in \mathsf{cl}_{\mathsf{sc}^{\blacktriangleleft}}$ and that $A \subseteq G$. (*We must show that $b \in G$.*) If G is the universe, then *certainly* $b \in G$. Otherwise, there exists some J-indexed set $\mathcal{C} = \{\langle \langle \tau_j, \mathbb{D}_j \rangle, H_j \rangle\}$, for some J, such that $\langle \tau_j, \mathbb{D}_j \rangle \in \mathsf{sc}$, $H_j \in \mathsf{cl}_{\mathbb{D}_j}$ and $G = \bigcap_{j \in J} \tau_j{}^{\blacktriangleleft}(H_j)$. For each $j \in J$: $A \subseteq G \subseteq \tau_j{}^{\blacktriangleleft}(H_j)$, so $\tau_j(A) \subseteq \tau_j(\tau_j{}^{\blacktriangleleft}(H_i)) = \tau_j(\overleftarrow{\tau_j}\, \lfloor H_i \rfloor) \subseteq H_j$, by (3) of Lemma 5.15. Hence by assumption, $\tau_j[\![b]\!] \subseteq H_j$; so $b \in \overleftarrow{\tau_j}\, \lfloor H_j \rfloor = \tau_j{}^{\blacktriangleleft}(H_j)$. So $b \in \bigcap_{j \in J} \tau_j{}^{\blacktriangleleft}(H_j) = G$. $\boxed{(2)\Leftrightarrow(3)}$ By (4.34). $\diamond$

Corollary 5.114 $\forall\, [\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}]$ τ is continuous from $\mathsf{sc}^{\blacktriangleleft}$ into $\mathbb{D}$; i.e., $\mathsf{sc}^{\blacktriangleleft} \in \mathsf{CContFrom}(\mathsf{sc})$.

Corollary 5.115 $\mathsf{CContFrom}(\mathsf{sc}) = \langle \mathsf{sc}^{\blacktriangleleft}]_{\preceq}$, i.e., source sc is continuous from $\mathbb{C}$ iff $\mathbb{C} \preceq \mathsf{sc}^{\blacktriangleleft}$.

Proof. $\boxed{\text{Satisfies}}$ By Corollary 5.114, $\mathsf{sc}^{\blacktriangleleft}$ is *a* closed system on $\mathsf{uni}(\mathsf{sc})$ such that, for each $\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}$, τ is continuous from $\mathsf{sc}^{\blacktriangleleft}$ into $\mathbb{D}$. $\boxed{\text{Coarsest}}$ Suppose that $\mathbb{C}$ is finer than $\mathsf{sc}^{\blacktriangleleft}$. Let $\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}$. (*We must show that τ is continuous from $\mathbb{C}$ into $\mathbb{D}$.*) Suppose that $A \vdash_{\mathbb{C}} b$. Since, by assumption, $\mathbb{C} \preceq \mathsf{sc}^{\blacktriangleleft}$, it follows, by Proposition 4.41 on page 148, that $A \vdash_{\mathsf{sc}^{\blacktriangleleft}} b$. Then by Proposition 5.113 on page 206, $\tau(A) \vdash_{\mathbb{D}} \tau(b)$. $\diamond$

The product of a singleton source coincides with the product of a single translation, the latter (elementary) notion being defined in $\oint$5.3.4.. More precisely, we have the following result, which follows at once from Theorem 5.113 and Proposition 5.85.

Corollary 5.116 For $\tau : A \multimap B$ and $\mathbb{D} \in \mathsf{ClSys}(B)$, $\tau^{\blacktriangleleft}_{\mathfrak{P}(A)}[\mathbb{D}] = \langle \tau, \mathbb{D} \rangle^{\blacktriangleleft}$.

Definition 5.117 ($\tau^{\blacktriangleleft}[\mathbb{C}]$) For $\tau : A \multimap B$ and $\mathbb{D} \in \mathsf{ClSys}(B)$, we shall write $\tau^{\blacktriangleleft}[\mathbb{D}]$ for $\tau^{\blacktriangleleft}_{\mathfrak{P}(A)}[\mathbb{D}]$. $\Box$

Corollary 5.118 $\mathsf{sc}^{\blacktriangleleft} = \blacktriangledown^{\mathbf{cl}_{\mathbb{C}^{\mathbb{C}}_{\mathsf{uni}(\mathsf{sc})}}}_{\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}} \tau^{\blacktriangleleft}[\mathbb{D}] = \blacktriangle^{\preceq}_{\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}} \tau^{\blacktriangleleft}[\mathbb{D}]$.

Proof. By definition, $\mathsf{sc}^{\blacktriangleleft}$ has a basis $\bigcup_{\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}} \mathsf{cl}_{\tau^{\blacktriangleleft}[\mathbb{D}]}$. The result follows by Remark 4.65 on page 153. $\diamond$

Remark 5.119 In particular, for each $\langle \tau, \mathbb{D} \rangle \in \mathsf{sc}$, $\mathsf{sc}^{\blacktriangleleft} \preceq \tau^{\blacktriangleleft}[\mathbb{D}]$. $\Box$

Open Problem 5.120 Show that Theorem 5.88 cannot be extended to the product of two finitary translations, nor even two functions (see Open Problems 9.27 on page 320 and 9.96 on page 334).

5.4.3.2 Examples

In the following example we demonstrate how the *semantic consequence* relation $\models^{\mathbf{M}}$ determined by a $\mathfrak{p}$-matrix $\mathbf{M}$ may be realized as the *product of a source.* This example motivates the approach that we shall be taking with regard to *abstractions* of logics in $\oint$7.

Example 5.121 (Models of Sentential Calculi)

Let $\mathfrak{p}$ be a signature of sentential calculi and $\mathbf{M}$ a $\mathfrak{p}$-matrix. Consider the constrained matrix-closed system $\mathbb{C}(\mathbf{M})$ and the source $\mathbf{Tm} \overset{\mathfrak{p}}{\ltimes} \mathbf{M} \doteq \{\langle \underset{\rightarrow}{\mathsf{i}}, \mathbb{C}(\mathbf{M})\rangle \ : \ \mathsf{i} \in \mathsf{Int}(\mathfrak{p}, \mathbf{M})\}$ with universe $\mathsf{Fm}(\mathfrak{p})$. By Theorem 5.113,

$$\Gamma \vdash_{\mathbf{Tm} \overset{\mathfrak{p}}{\ltimes} \mathbf{M}^{\blacktriangleleft}} \phi \text{ iff } \forall\, [\mathsf{i} \in \mathsf{Int}(\mathfrak{p}, \mathbf{M})] \ \underset{\rightarrow}{\mathsf{i}}[\Gamma] \subseteq H \rightarrow \underset{\rightarrow}{\mathsf{i}}(\phi) \in \mathsf{D}_{\mathbf{M}}. \tag{5.44}$$

Comparing (5.44) with (2.3) of Definition 2.32 on page 99, we see that $\vdash_{\mathbf{Tm} \overset{\mathfrak{p}}{\ltimes} \mathbf{M}^{\blacktriangleleft}} = \models^{\mathbf{M}}$.

□

5.4.3.3 Quotients

We turn now to the dual notion of the quotient of a sink. Just as products of sources pertain to the *abstraction* process in logic, quotients of sinks are key to the *modelling* process.

Definition 5.122 (Sink Closed Systems) A **sink of closed systems** sk is determined by a *class* of pairs $\mathsf{Arrow}(\mathsf{sk})$, the members of which are called **sink-arrows**, and an associated set $\mathsf{uni}(\mathsf{sk})$, called the **universe**, such that, for each sink-arrow $\langle \mathbb{C}, \tau\rangle \in \mathsf{Arrow}(\mathsf{sk})$, $\tau : \mathsf{uni}(\mathbb{C}) \multimap \mathsf{uni}(\mathsf{sk})$. For a closed system $\mathbb{C}$, set B and $\tau : \mathsf{uni}(\mathbb{C}) \multimap B$, let $\langle \mathbb{C}, \tau\rangle$ denote the sink of closed systems determined by the single sink-arrow $\langle \mathbb{C}, \tau\rangle$ and universe B, which we call a **singleton sink**. A sink of closed systems is called **functional** if the translation component of each sink arrow is a function.

We shall call a sink of closed systems sk **continuous into** closed system $\mathbb{D}$ if $\mathsf{uni}(\mathbb{D}) = \mathsf{uni}(\mathsf{sk})$ and, for each $\langle \mathbb{C}, \tau\rangle \in \mathsf{sk}$, τ is continuous from $\mathbb{C}$ into $\mathbb{D}$, and we denote the set of all closed systems $\mathbb{D}$ with sink sk continuous into closed system $\mathbb{D}$ by $\mathsf{CContInto}(\mathsf{sk})$.

With each sink of closed systems sk we associate the closed system $\coprod \mathsf{sk}$ with universe $\mathsf{uni}(\mathsf{sk})$ and $\mathsf{cl}_{\coprod \mathsf{sk}} = \{H \subseteq \mathsf{uni}(\mathsf{sk}) : \forall\, [\langle \mathbb{A}, \tau\rangle \in \mathsf{sk}]\ \tau^{\blacktriangleleft}(H) \in \mathsf{cl}_{\mathbb{A}}\}$, which we call the **sink closed system** induced by sink sk. □

Proof. (*We must show that* $\{H \subseteq \mathsf{uni}(\mathsf{sk}) : \forall\, [\langle \mathbb{A}, \tau\rangle \in \mathsf{sk}]\ \tau^{\blacktriangleleft}(H) \in \mathsf{cl}_{\mathbb{A}}\}$ *constitutes a closed system.*) $\boxed{\text{Universe}}$ For all $\langle \tau, \mathbb{C}\rangle \in \mathsf{sk}$, $\tau^{\blacktriangleleft}(\mathsf{uni}(\mathsf{sk})) = \mathsf{uni}(\mathbb{C}) \in \mathsf{cl}_{\mathbb{C}}$. $\boxed{\text{Intersection}}$ Let $\emptyset \neq \mathcal{A} \subseteq \{H \subseteq \mathsf{uni}(\mathsf{sk}) : \forall\, [\langle \mathbb{A}, \tau\rangle \in \mathsf{sk}]\ \tau^{\blacktriangleleft}(H) \in \mathsf{cl}_{\mathbb{A}}\}$. For all $\langle \tau, \mathbb{C}\rangle \in \mathsf{sk}$, by (5.31) of Table 5.1, $\tau^{\blacktriangleleft}(\bigcap \mathcal{A}) = \bigcap_{G \in \mathcal{A}} \tau^{\blacktriangleleft}(G) \in \mathsf{cl}_{\mathbb{C}}$, since, for all $G \in \mathcal{A}$, $\tau^{\blacktriangleleft}(G) \in \mathsf{cl}_{\mathbb{C}}$. ◇

Remark 5.123 $\forall\, [\langle \mathbb{C}, \tau\rangle \in \mathsf{sk}]$ τ is continuous from $\mathbb{C}$ into $\coprod \mathsf{sk}$; i.e., $\coprod \mathsf{sk} \in \mathsf{CContInto}(\mathsf{sk})$.

Proof. The reduced preimages of $\coprod \mathsf{sk}$-closed sets under τ are $\mathbb{C}$-closed by construction. So the result follows by equivalent condition (3) of Theorem 5.40. ◇

As in the case of products, not only is $\coprod \mathsf{sk} \in \mathsf{CContInto}(\mathsf{sk})$, but further, the closed systems in $\mathsf{CContInto}(\mathsf{sk})$ are characterizable in terms of their granularity with respect to $\coprod \mathsf{sk}$.

Theorem 5.124 $\mathsf{CContInto}(\mathsf{sk}) = [\coprod \mathsf{sk}\rangle_{\preceq}$, i.e., sink sk is continuous into $\mathbb{D}$ iff $\coprod \mathsf{sk} \preceq \mathbb{D}$.

Proof. $\boxed{\mathsf{CContInto}(\mathsf{sk}) \subseteq [\coprod \mathsf{sk}\rangle_{\preceq}}$ Let $\mathbb{D} \in \mathsf{CContInto}(\mathsf{sk})$. (*We must show that* $\coprod \mathsf{sk} \preceq \mathbb{D}$.) Let $H \in \mathsf{cl}_{\mathbb{D}}$. (*It suffices to show, by Proposition 4.41 on page 148, that* $H \in \mathsf{cl}_{\coprod \mathsf{sk}}$.) Let $\langle \tau, \mathbb{C}\rangle \in \mathsf{sk}$. (*We must show,* *by the definition of the sink closed system, that* $\tau^{\blacktriangleleft}(H) \in \mathsf{cl}_{\mathbb{C}}$.) Since $\mathbb{D} \in \mathsf{CContInto}(\mathsf{sk})$, τ is continuous from $\mathbb{C}$ into $\mathbb{D}$, and so by equivalent condition (3) of Theorem 5.40, $\tau^{\blacktriangleleft}(H) \in \mathsf{cl}_{\mathbb{C}}$, since $H \in \mathsf{cl}_{\mathbb{D}}$. $\boxed{\mathsf{CContInto}(\mathsf{sk}) \supseteq [\coprod \mathsf{sk})_{\preceq}}$ Let $\coprod \mathsf{sk} \preceq \mathbb{D}$. Let $\langle \tau, \mathbb{C} \rangle \in \mathsf{sk}$. (*We must show that* τ *is continuous from* $\mathbb{C}$ *to* $\mathbb{D}$.) Suppose that $A \vdash_{\mathbb{C}} b$. Then by the already established continuity of τ from $\mathbb{C}$ into $\coprod \mathsf{sk}$, $\tau[A] \vdash_{\coprod \mathsf{sk}} \tau[\![b]\!]$. Since $\coprod \mathsf{sk} \preceq \mathbb{D}$, $\tau[A] \vdash_{\mathbb{D}} \tau[\![b]\!]$, by Proposition 4.41 on page 148. ⋄

The reader should compare the following characterization of the closed sets of $\coprod \mathsf{sk}$, with the definition of the filter of a sentential calculus, given in Definition 2.41 on page 101.

Theorem 5.125 H is $\coprod \mathsf{sk}$-closed iff $\forall\, [\langle \tau, \mathbb{C} \rangle \in \mathsf{sk}]\ A \vdash_{\mathbb{C}} b \rightarrow \big(\tau[A] \subseteq H \rightarrow \tau[\![b]\!] \subseteq H\big)$.

Proof. We shall say that a subset B of $\mathsf{uni}(\mathsf{sk})$ is sk-*closed* if $\forall\, [\langle \tau, \mathbb{C} \rangle \in \mathsf{sk}]\ A \vdash_{\mathbb{C}} b \rightarrow \big(\tau[A] \subseteq H \rightarrow \tau[\![b]\!] \subseteq H\big)$. Let $\mathcal{B}$ denote the set of all sk-closed subsets of $\mathsf{uni}(\mathsf{sk})$.
$\boxed{\mathcal{B}\text{ form a closed system}}$ $\boxed{\mathsf{uni}(\mathsf{sk}) \in \mathcal{B}}$ Trivial. $\boxed{\text{Closed under non-empty } \bigcap}$ Let $\emptyset \neq \mathcal{C} \subseteq \mathcal{B}$. Let $\langle \tau, \mathbb{C} \rangle \in \mathsf{sk}$ and suppose that $A \vdash_{\mathbb{C}} b$ and $\tau[A] \subseteq \bigcap \mathcal{C}$. For each $B \in \mathcal{C}$, $\tau[A] \subseteq B$, and so by definition, $\tau[\![b]\!] \subseteq B$. Hence $\tau[\![b]\!] \subseteq \bigcap \mathcal{C}$. Let $\mathbb{D}$ denote the closed system on $\mathsf{uni}(\mathsf{sk})$ determined by closed sets $\mathcal{B}$.
$\boxed{\coprod \mathsf{sk} \preceq \mathbb{D}}$ Let $\langle \mathbb{C}, \tau \rangle \subseteq \mathsf{sk}$. (*By Theorem 5.124 on page 208, it suffices to show that* τ *is continuous from* $\mathbb{C}$ *into* $\mathbb{D}$.) Suppose that $A \vdash_{\mathbb{C}} b$. Consider $\|\tau[A]\|_{\mathbb{D}}$, which must by sk-closed by definition. Since $\tau[A]$ is contained in $\|\tau[A]\|_{\mathbb{D}}$, it follows, from sk-closure, that $\tau[b] \subseteq \|\tau[A]\|_{\mathbb{D}}$. Consequently $\tau[A] \vdash_{\mathbb{D}} \tau(b)$, as required. $\boxed{\mathbb{D} \preceq \coprod \mathsf{sk}}$ Let $G \in \mathsf{cl}_{\coprod \mathsf{sk}}$. (*We must show that* G *is* sk-*closed.*) Let $\langle \mathbb{C}, \tau \rangle \in \mathsf{sk}$ and suppose that $A \vdash_{\mathbb{C}} b$ and $\tau[A] \subseteq G$. By Remark 5.123, τ is continuous from $\mathbb{C}$ into $\coprod \mathsf{sk}$, and hence $\tau[A] \vdash_{\coprod \mathsf{sk}} \tau[\![b]\!]$. Since G is $\coprod \mathsf{sk}$-closed and contains $\tau[A]$, it must also contain $\tau[\![b]\!]$. ⋄

Corollary 5.126 $C \vdash_{\coprod \mathsf{sk}} c$ iff c lies in every set D that contains C and satisfies $\forall\, [\langle \tau, \mathbb{C} \rangle \in \mathsf{sk}]\ A \vdash_{\mathbb{C}} b \rightarrow \big(\tau[A] \subseteq C \rightarrow \tau[\![b]\!] \subseteq C\big)$. □

The following important result concerning the finitariness of the quotient closed system applies only to *functional* sinks.

Proposition 5.127 Let sk be a *functional* sink of closed systems such that, for each $\langle \mathbb{C}, f \rangle \in \mathsf{sk}$, $\mathbb{C}$ is finitary. Then $\coprod \mathsf{sk}$ is finitary.

Proof. Let $\mathcal{C}$ be a non-empty $\subseteq$-directed set of $\coprod \mathsf{sk}$-closed sets. (*We must show that* $\bigcup \mathcal{C}$ *is* $\coprod \mathsf{sk}$-*closed.*) Let $\langle \mathbb{C}, f \rangle \in \mathsf{sk}$. By definition, $\overleftarrow{f}_{[]}[\mathcal{C}] \subseteq \mathsf{cl}_{\mathbb{C}}$; where $\overleftarrow{f}_{[]}$ is the pre-image function (see Definition 1.31 on page 19). Let $f^{-1}[H_1], f^{-1}[H_2] \in \overleftarrow{f}_{[]}[\mathcal{C}]$ where $H_1, H_2 \in \mathcal{C}$. Since $\mathcal{C}$ is $\subseteq$-directed, there exists $H_3 \in \mathcal{C}$, with $H_1 \cup H_2 \subseteq H_3$. So $f^{-1}[H_1] \cup f^{-1}[H_2] \subseteq f^{-1}[H_3]$. Hence $\overleftarrow{f}_{[]}[\mathcal{C}]$ is $\subseteq$-directed. By assumed finitariness of $\mathbb{C}$, $\bigcup \overleftarrow{f}_{[]}[\mathcal{C}] \in \mathsf{cl}_{\mathbb{C}}$. Since $\bigcup \overleftarrow{f}_{[]}[\mathcal{C}] = \overleftarrow{f}[\bigcup \mathcal{C}]$ (by (1.42) of Table 1.2 on page 21), $f^{-1}[\bigcup \mathcal{C}] \in \mathsf{cl}_{\mathbb{C}}$. Since for all $\langle \mathbb{C}, f \rangle \in \mathsf{sk}$, $f^{-1}[\bigcup \mathcal{C}] \in \mathsf{cl}_{\mathbb{C}}$, by definition $\bigcup \mathcal{C}$ is $\coprod \mathsf{sk}$-closed. ⋄

Open Problem 5.128 Can quotients be interpreted in the elementary setting? What about the quotient by one elementary translation?

5.4.3.4 Products of Quotients and Quotients of Products

Now some combined results. Beginning with a source sc, we obtain the product $\mathsf{sc}^{\blacktriangleleft}$. For each $\langle\tau,\mathbb{D}\rangle \in \mathsf{sc}$, consider the singleton sink $\langle \mathsf{sc}^{\blacktriangleleft}, \tau\rangle$, and the associated quotient closed system $\tau[\mathsf{sc}^{\blacktriangleleft}]$. In the next result, which is an immediate corollary of Theorem 5.115 and Theorem 5.124, we note that $\tau[(\mathsf{sc}^{\blacktriangleleft})] \preceq \mathbb{D}$. A symmetric relationship obtains if we begin with a sink instead.

Proposition 5.129

1. $\forall\,[\langle\tau,\mathbb{D}\rangle \in \mathsf{sc}]\;\tau[\mathsf{sc}^{\blacktriangleleft}] \preceq \mathbb{D}$.

2. $\forall\,[\langle\mathbb{C},\tau\rangle \in \mathsf{sk}]\;\mathbb{C} \preceq \tau^{\blacktriangleleft}[\coprod \mathsf{sk}]$.

Proof. $\boxed{(1)}$ By Theorem 5.124, it suffices to show that τ is continuous from $\mathsf{sc}^{\blacktriangleleft}$ to $\mathbb{D}$. But this is true by Proposition 5.113. $\boxed{(2)}$ By Proposition 5.113, it suffices to show that τ is continuous from $\mathbb{C}$ to $\coprod \mathsf{sk}$. But this is true by Theorem 5.124. $\diamond$

Remark 5.130 In particular, $\tau[\tau^{\blacktriangleleft}[\mathbb{D}]] \preceq \mathbb{D}$ and $\mathbb{C} \preceq \tau^{\blacktriangleleft}[\tau[\mathbb{C}]]$.

5.4.3.5 Examples

In the next example we consider how *filters* of sentential calculi arise as the *quotient of a sink*. This example motivates the approach that we shall be taking to filters of logics in $\oint 7$.

Example 5.131 (Filters of Sentential Calculi)

Let $\mathcal{S}$ be a sentential $\mathfrak{p}$-calculi and $\mathbf{A}$ an algebra. Let $\mathcal{S} \overset{\mathfrak{p}}{\rtimes} \mathbf{A} = \{\langle\mathcal{S},\mathsf{i}\rangle \,:\, \mathsf{i} \in \mathsf{Int}(\mathcal{S},\mathbf{A})\}$, denote the sink with universe $\mathsf{uni}(\mathbf{A})^{\mathsf{dim}(\mathfrak{p})}$. By Theorem 5.125,

$$F \in \mathsf{cl}_{\coprod \mathcal{S} \overset{\mathfrak{p}}{\rtimes} \mathbf{A}} \text{ iff } \forall\,[\mathsf{i} \in \mathsf{Int}(\mathcal{S},\mathbf{A})]\;\Gamma \vdash_{\mathcal{S}} \phi \rightarrow \big(\underset{\rightarrow}{\mathsf{i}}\,[\Gamma] \subseteq F \rightarrow \underset{\rightarrow}{\mathsf{i}}(\phi) \in F\big). \tag{5.45}$$

Comparing (5.45) with (2.6) of Definition 2.41 on page 101, demonstrates that $\mathsf{cl}_{\coprod \mathcal{S} \overset{\mathfrak{p}}{\rtimes} \mathbf{A}} = \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$.

$\square$

5.4.4 Iseomorphisms

We now demonstrate that, in the concrete case, iseomorphisms characterize isomorphisms between lattices of closed sets. This result is a generalization of the analogous argument given in [BP89a].

Theorem 5.132 Let $\mathbb{C}$ and $\mathbb{D}$ be two closed systems and suppose that $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \cong \mathbf{cl}_{\mathbb{D}}$. Let $\tau : \mathbb{C} \multimap \mathbb{D}$ be a translation such that $\|\tau(a)\|_{\mathbb{D}} = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$, and let $\pi : \mathbb{D} \multimap \mathbb{C}$ be a translation such that $\|\pi[\![b]\!]\|_{\mathbb{C}} = \mathsf{f}^{-1}(\|\{b\}\|_{\mathbb{D}})$, for each $b \in \mathsf{uni}(\mathbb{D})$. Then

1. τ is an iseomorphism from $\mathbb{C}$ to $\mathbb{D}$ with inverse iseomorphism π,

2. $\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} = \pi^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$ and

3. $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathbb{D}}} = \pi^{\star}{}_{|\mathsf{cl}_{\mathbb{D}}} = \mathsf{f}^{-1}$.

One such realization of τ and π is given by $\tau(a) = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$, for each $a \in \mathsf{uni}(\mathbb{C})$, $\pi[\![b]\!] = \mathsf{f}^{-1}(\|\{b\}\|_{\mathbb{D}})$, for each $b \in \mathsf{uni}(\mathbb{D})$. Further, if $\mathbb{C}$ and $\mathbb{D}$ are both finitary, then these translations may both be chosen to be finitary.

Proof. Certainly $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \rightarrowtail_{\blacktriangledown} \mathbf{cl}_{\mathbb{D}}$, and so by Theorem 5.110 and the surjectivity of f, any translation $\tau : \mathbb{C} \multimap \mathbb{D}$, satisfying $\|\tau(a)\|_{\mathbb{D}} = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$ for each $a \in \mathsf{uni}(\mathbb{C})$, is a strictly continuous translation and satisfies $\tau^{\star}{}_{|\mathsf{cl}_{\mathbb{C}}} = \mathsf{f}$ and $\tau^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathbb{D}}} = \mathsf{f}^{-1}$. Similarly, since $\mathsf{f}^{-1} : \mathbf{cl}_{\mathbb{D}} \rightarrowtail_{\blacktriangledown} \mathbf{cl}_{\mathbb{C}}$, any translation $\pi : \mathbb{D} \multimap \mathbb{C}$, satisfying $\|\pi[\![b]\!]\|_{\mathbb{C}} = \mathsf{f}^{-1}(\|\{b\}\|_{\mathbb{D}})$ for each $b \in \mathsf{uni}(\mathbb{D})$, is a strictly continuous translation and satisfies $\pi^{\star}{}_{|\mathsf{cl}_{\mathbb{D}}} = \mathsf{f}^{-1}$ and $\pi^{\blacktriangleleft}{}_{|\mathsf{cl}_{\mathbb{C}}} = (\mathsf{f}^{-1})^{-1} = \mathsf{f}$. Finally, for any $A \subseteq \mathsf{uni}(\mathbb{C})$, by (5) of Theorem 5.21 and the continuity of τ and π, $\|(\pi\tau)[A]\|_{\mathbb{C}} = \|\pi(\tau(A))\|_{\mathbb{C}} = \big\|\pi(\big\|\tau(\|A\|_{\mathbb{C}})\big\|_{\mathbb{D}})\big\|_{\mathbb{C}} = \pi^{\star}(\tau^{\star}(\|A\|_{\mathbb{C}})) = \mathsf{f}^{-1}(\mathsf{f}(\|A\|_{\mathbb{C}})) = \|A\|_{\mathbb{C}}$. Hence $(\pi\tau)[A] \dashv\vdash_{\mathbb{C}} A$, and symmetrically, $(\tau\pi)[C] \dashv\vdash_{\mathbb{D}} C$, for all $C \subseteq \mathsf{uni}(\mathbb{D})$. $\diamond$

Note that it is possible for the lattice of closed sets of an algebraic closed system $\mathbb{C}$ to be *isomorphic* to the lattice of closed sets of $\mathbb{D}$, but with $\mathbb{D}$ failing to be algebraic [HW94][1].

5.4.5 Transformers between Closure Operators

Blok and Jónsson developed a theory of *similar consequence operators* based on the notion of a transformer [BJ06]. For completeness, we present the notions of transformers, and show that transformers and translations are essentially the same notion. While we present our theory in terms of translations, throughout this chapter translations may be replaced with transformers. Those results that coincide with results in [BJ06] when translations are replaced with transformers are duly referenced. Note that for ease of compatibility with our theory, we have reversed the *emphasised* direction of transformers from that given in [BJ06].

Definition 5.133 (Transformers and Untransformers) [BJ06] A **transformer $\boldsymbol{\tau}$** from set A to set C is $\blacktriangledown$-preserving function from $\mathfrak{P}(A)$ into $\mathfrak{P}(C)$. An **untransformer $\boldsymbol{\upsilon}$** from set A to set C is $\blacktriangle$-preserving function from $\mathfrak{P}(A)$ into $\mathfrak{P}(C)$. For a transformer $\boldsymbol{\tau}$ from A to C, we define $\boldsymbol{\tau}^{\blacktriangleleft} : \mathfrak{P}(C) \to \mathfrak{P}(A)$ by $\boldsymbol{\tau}^{\blacktriangleleft}(D) = \{a \in A \ : \ \boldsymbol{\tau}(\{a\}) \subseteq D\}$. $\square$

Example 5.134 (Translations as Transformers)

> Let $\tau : A \multimap C$ be a translation. Since images of binary relations preserve arbitrary unions, the image function $\tau_{[]}(\cdot)$ is a transformer of A to C. Since reduced-images preserve arbitrary intersections, $\tau^{\blacktriangleleft}(\cdot)$ is an untransformer from C to A. Notice that $\tau^{\blacktriangleleft} = (\tau_{[]})^{\blacktriangleleft}$, and so potential ambiguity is avoided.

$\square$

Proposition 5.135 If $\boldsymbol{\tau}$ is a transformer from A to C, then the translation $\tau : A \multimap C$, defined by $\tau[\![a]\!] = \boldsymbol{\tau}(\{a\})$ for all $a \in A$, satisfies $\tau_{[]} = \boldsymbol{\tau}$.

Proof. $\tau_{[]}(B) = \bigcup_{b \in B} \tau[\![b]\!] = \bigcup_{b \in B} \boldsymbol{\tau}(\{b\}) = \boldsymbol{\tau}(\bigcup_{b \in B}\{b\}) = \boldsymbol{\tau}(B)$, the penultimate equality following by the $\blacktriangledown$-preservation of transformers. $\diamond$

Consequent to the previous example and the previous proposition, translations and transformers are in one-to-one correspondence.

[1] We would like to thank James Raftery for pointing this fact out to us.

Proposition 5.136 [BJ06] Let $\boldsymbol{\tau}$ be a transformer from A to C.

1. $\boldsymbol{\tau}^{\blacktriangleleft}$ is an untransformer from C to A.
2. For all $B \subseteq A$ and $D \subseteq C$, $B \subseteq \boldsymbol{\tau}^{\blacktriangleleft}(\boldsymbol{\tau}(B))$ and $\boldsymbol{\tau}(\boldsymbol{\tau}^{\blacktriangleleft}(D)) \subseteq D$.
3. $\boldsymbol{\tau}\boldsymbol{\tau}^{\blacktriangleleft}\boldsymbol{\tau} = \boldsymbol{\tau}$ and $\boldsymbol{\tau}^{\blacktriangleleft}\boldsymbol{\tau} \cdot \boldsymbol{\tau}^{\blacktriangleleft} = \boldsymbol{\tau}^{\blacktriangleleft}$.

□

Our notion of a product of a single transformation coincides with Blok and Jónssons' notion of a $\boldsymbol{\tau}$-transform of a closure operator. We explicate the relationship between the two notions.

Definition 5.137 ($\boldsymbol{\tau}$-Transforms of a Closure Operator) [BJ06, D3.3, T3.4] Let $\mathfrak{c}$ and $\mathfrak{d}$ be closure operators, A any set and $\boldsymbol{\tau}$ a transformer from A to $\mathsf{uni}(\mathfrak{d})$. Define a power-operator $\mathfrak{d}^{\boldsymbol{\tau}}$ on A by $\mathfrak{d}^{\boldsymbol{\tau}} = \boldsymbol{\tau}^{\blacktriangleleft}\mathfrak{d}\boldsymbol{\tau}$. □

Conflating transformers with transformations, by (5.41), $\mathfrak{d}^{\boldsymbol{\tau}}$ is the closure operator of $\boldsymbol{\tau}^{\blacktriangleleft}[\mathfrak{d}]$, by Proposition 5.85 and (3) of Theorem 5.40, $\tau^{\blacktriangleleft}$ maps $\mathsf{cl}_{\mathfrak{d}}$ into $\mathsf{cl}_{\mathfrak{d}^{\tau}}$, and trivially, $\mathfrak{d}\boldsymbol{\tau}$ maps $\mathsf{cl}_{\mathfrak{d}^{\tau}}$ into $\mathsf{cl}_{\mathfrak{d}}$.

We now compare our notion of iseomorphism with Blok and Jónssons' notion of similarity and similarities induced by transformers. Given the one-to-one correspondences between closed systems and closure operators and between transformers and translations, we shall present the theory developed in [BJ06] in terms of translations and closed systems.

Definition 5.138 (Similarities and Similar Closed Systems) [BJ06, D3.2] Let $\mathbb{C}$ and $\mathbb{D}$ be two closed systems. An isomorphism $\mathsf{f} : \mathbf{cl}_{\mathbb{C}} \cong \mathbf{cl}_{\mathbb{D}}$ is called a **similarity** between $\mathbb{C}$ and $\mathbb{D}$, in which case we write $\mathsf{f} : \mathbb{C} \sim \mathbb{D}$. If a similarity exists between $\mathbb{C}$ and $\mathbb{D}$ then $\mathbb{C}$ and $\mathbb{D}$ are called **similar**, denoted $\mathbb{C} \sim \mathbb{D}$. □

Remark 5.139 If $\mathsf{f} : \mathbb{C} \sim \mathbb{D}$ then $\mathsf{f}^{-1} : \mathbb{D} \sim \mathbb{C}$. □

By theorems 5.102 and 5.132, $\mathbb{C}$ is similar to $\mathbb{D}$ iff $\mathbb{C}$ is iseomorphic to $\mathbb{D}$.

Corollary 5.140 The following conditions are equivalent.

1. $\mathbb{C}$ and $\mathbb{D}$ are iseomorphic.
2. $\mathbb{C} \sim \mathbb{D}$.

Definition 5.141 (Similarities Induced by Translations) [BJ06, D3.5] Let $\mathbb{C}$ and $\mathbb{D}$ be closure operators. A similarity f from $\mathbb{C}$ to $\mathbb{D}$ is said to be **induced by translations** τ and π, from $\mathsf{uni}(\mathbb{C})$ to $\mathsf{uni}(\mathbb{D})$ and from $\mathsf{uni}(\mathbb{D})$ to $\mathsf{uni}(\mathbb{C})$ respectively, if, for all $A \subseteq \mathsf{uni}(\mathbb{C})$, $\mathsf{f}(\|A\|_{\mathbb{C}}) = \tau^{\star}(A)$, and $\mathsf{f}^{-1} = \pi^{\star}$, in which case we write $\mathsf{f} : \mathbb{C} \overset{\tau;\pi}{\sim} \mathbb{D}$. By $\mathbb{C} \overset{\tau;\pi}{\sim} \mathbb{D}$ we mean that some similarity from $\mathbb{C}$ to $\mathbb{D}$ is induced by τ and π. □

Corollary 5.142 Let $\mathbb{C}$ and $\mathbb{D}$ be closed systems.

1. If τ is an iseomorphism from $\mathbb{C}$ to $\mathbb{D}$ with inverse iseomorphism π, then ${\tau^{*}}_{|\mathsf{cl}_{\mathbb{C}}} : \mathbb{C} \overset{\tau;\pi}{\sim} \mathbb{D}$.

2. If $\mathsf{f} : \mathbb{C} \sim \mathbb{D}$ then $\mathsf{f} : \mathbb{C} \overset{\tau;\pi}{\sim} \mathbb{D}$, where τ and π are any translations satisfying $\|\tau[\![a]\!]\|_{\mathbb{D}} = \mathsf{f}(\|\{a\}\|_{\mathbb{C}})$ for each $a \in \mathsf{uni}(\mathbb{C})$, and $\|\pi[\![b]\!]\|_{\mathbb{C}} = \mathsf{f}^{-1}(\|\{b\}\|_{\mathbb{D}})$ for each $b \in \mathsf{uni}(\mathbb{D})$.

Proof. $\boxed{(1)}$ By Theorem 5.102, ${\tau^*}_{|\mathsf{cl}_{\mathbb{C}}}$ is a similarity from $\mathbb{C}$ to $\mathbb{D}$. Further, ${\tau^*}_{|\mathsf{cl}_{\mathbb{C}}}(\|A\|_{\mathbb{C}}) = \tau^*(\|A\|_{\mathbb{C}}) = \tau^\star(A)$, by assumed continuity of τ and (5) of Theorem 5.21. Similarly, $({\tau^*}_{|\mathsf{cl}_{\mathbb{C}}})^{-1} = \pi^\star$, $\boxed{(2)}$ By Theorem 5.132, $\mathsf{f}(\|A\|_{\mathbb{C}}) = \tau^\star(\|A\|_{\mathbb{C}})$ and τ is an iseomorphism. Since iseomorphisms are continuous by definition, $\tau^\star(\|A\|_{\mathbb{C}}) = \tau^\star(A)$, by (5) of Theorem 5.21. So $\mathsf{f}(\|A\|_{\mathbb{C}}) = \tau^\star(A)$. Symmetrically, $\mathsf{f}^{-1} = \pi^\star$. $\diamond$

The following theorem, which is an amalgam of Corollary 3.6 and Theorem 3.7 of [BJ06], characterizes $\mathbb{C} \overset{\tau;\pi}{\sim} \mathbb{D}$.

Theorem 5.143 [BJ06, C3.6, T3.7] The following conditions are equivalent.

1. $\mathbb{C} \overset{\tau;\pi}{\sim} \mathbb{D}$.
2. For all $C \subseteq \mathsf{uni}(\mathbb{D})$ and $A \subseteq \mathsf{uni}(\mathbb{C})$,

$$\begin{aligned} &\tau^\star(\|A\|_{\mathbb{C}}) = \tau^\star(A), && \pi^\star(\|C\|_{\mathbb{D}}) = \pi^\star(C), \\ &\tau^*(\pi^\star(\|C\|_{\mathbb{D}})) = \|C\|_{\mathbb{D}} \quad \text{and} && \pi^*(\tau^\star(\|A\|_{\mathbb{C}})) = \|A\|_{\mathbb{C}}\,. \end{aligned}$$

3. For all $C \subseteq \mathsf{uni}(\mathbb{D})$ and $A \subseteq \mathsf{uni}(\mathbb{C})$,

$$\tau^{\blacktriangleleft}(\tau^\star(A)) = \|A\|_{\mathbb{C}} \quad \text{and} \quad \tau^\star(\pi\,[C]) = \|C\|_{\mathbb{D}}\,.$$

4. For all $C \subseteq \mathsf{uni}(\mathbb{D})$ and $A \subseteq \mathsf{uni}(\mathbb{C})$,

$$\pi^{\blacktriangleleft}(\pi^\star(C)) = \|C\|_{\mathbb{D}} \quad \text{and} \quad \pi^\star(\tau\,[A]) = \|A\|_{\mathbb{C}}\,.$$

□

Notice that condition (2) is equivalent to condition (4) of Theorem 5.101, by (5) of Theorem 5.21, and that condition (3) is equivalent to condition (6) of Theorem 5.101, by (2) Proposition 5.98 and (2) of Theorem 5.73. Similarly condition (4) coincides with (7) of Theorem 5.101. So iseomorphisms and induced similarities coincide.

Corollary 5.144 $\mathbb{C} \overset{\tau;\pi}{\sim} \mathbb{D}$ iff τ is an iseomorphism with inverse π.

Theorem 5.145 [BJ06, T3.8] If $\mathsf{f} : \mathbb{C} \overset{\tau;\pi}{\sim} \mathbb{D}$, then $\mathsf{f} = \pi^{\blacktriangleleft}$ and $\mathsf{f}^{-1} = \tau^{\blacktriangleleft}$. □

Part III

Constructural Abstract Logic

In Part III, we consider the theory of *logics over objects of a construct*, where by a construct we mean a concrete category (see $\oint$1.4.1). Our *primary motivation* stems from the fact that we have been encountering 'logics' in universal algebra for which *no* sentential analogue is immediately clear. For example, the set of all subuniverses on the free algebra of a quasivariety is in essence a structural and finitary logic, when one takes the construct to be the quasivariety. We explore in detail the relationship between such logics over a quasivariety and the sentential calculi that they induce. These techniques and relationships have been the primary means by which we have been discovering sentential calculi that arise naturally from universal algebra but which are 'inherently unalgebraizable', and for which our more general theory of 'parametrized algebraization' applies. Our *secondary motivation* arises from an attempt to analyse the standard theory of logics; in particular, to better understand the notions of *structurality*, since the most natural formulation of one of our primary logics $D_\forall(\mathcal{K}, \mathfrak{B}_*)$ is *not* structural. This is only possible if we have enough freedom for logics to *not* be structural; sentential calculi in the sense of [BP89a] being structural by *definition*. We shall show that a logic is structural iff it has a semantics, i.e., the structural logics are precisely the sound and complete logics. It is for this reason that we have to develop a structural 'approximation' of the non-structural $D_\forall(\mathcal{K}, \mathfrak{B}_*)$. As a *further motivation*, we shall show, in Part VI, how much of our theory of *parameterized* algebraization and protoalgebraicity can be obtained from a non-parameterized theory of equivalence and protoalgebraicity for logics over constructs (with a suitable choice of morphisms).

In $\oint$6 we introduce logics over objects of constructs. We do so in such a manner that *finitariness* and *structurality* are 'optional' conditions. Numerous examples, pertinent to the sequel, are developed in this chapter.

In $\oint$7 we consider the role of *structurality*, and introduce the notions of *models*, *abstracts*, *filters* and *ideals*, which we shall define in terms of *continuity* of interpretations, *quotients of sinks* and *products of sources*. We treat *logics* as the models of other logics. In so doing, the role of *continuity* and the position of 'geometric' arguments in the theory becomes clearer. More importantly, given the primary aim of this text, this enables us to treat a logic simultaneously both as the model of another logic and as a primary logic with its own models; this property is essential for the development of our theory of *canons* (and their induced *ideals*) and *archologies*. This perspective still encompasses *matrix models*, which can be seen as 'little' logics, with only their designated set and universe as theories. We also consider the relationship between *logics as models* and *matrices as models*. As examples, we consider the models of the example logics introduced in $\oint$6.

As noted above, we have been finding natural logics in universal algebra that are defined on the algebras of a quasivariety. In $\oint$8 we demonstrate how such logics induce *sentential* calculi, and explore the relationship between the induced calculi and the originating logics. In particular, to show how structural logics over free algebras, which we call *canons*, induce sentential 1-calculi, which we call the *ideals* induced by canons. We show how a given sentential calculus can be 'smoothed' with respect to a given quasivariety, so as to better reflect the equational truths of that quasivariety. As examples, we consider the sentential calculi induced by the logics introduced in $\oint$6. Our theory of canons and their induced ideals is developed at the *constructural* level, in terms of free objects and subconstructs, and as such, has applications beyond just logics over free *algebras*.

We must note that our theory of *logics over constructs* is not the only generalization of sentential n-calculi. We shall now briefly discuss these other generalizations and motivate some advantages of our theory of logics over constructs over these theories.

In [FJ96], *closed systems over universes of algebras* are treated as logics; such logics are special cases of logics over constructs. The primary focus of [FJ96] is to view these logics over algebras as *models of sentential calculi* (in the spirit that we view the models of a logic over a construct as other logics over the same construct) rather than treating these logics as 'primary' logics and then considering their models. We require that a logic over an object of a construct, in particular, a logic over the free algebra of a quasivariety, to be viewed as 'primary' and hence model inducing.

In [BJ06], a similar abstraction to ours is developed, where *monoid actions* serve an analogous role to that played by *morphisms* in our theory. The advantage of a construct over monoid actions, is that the full theory of *models* of sentential calculi, *Leibniz equivalence*, *reduced models*, *protoalgebraicity* and *equivalent logics*, can be abstracted to this level of discourse.

The notion of a π-*institution* was introduced in [FS88] and has recently been well studied (see, amongst other, [Vou03],[Vou05],[Vou07b]); π-institutions are a (partially) categorical abstraction of *structural* logics and encompass *structural* logics over constructs. While the property of structurality may easily be removed from the definition of a π-institution and made a condition, such a theory does not currently exist in the literature; as such, some of our logics over constructs *cannot* be realized as π-institutions. A key advantage of logics over constructs versus π-institutions is the fact that logics over constructs, unlike π-institutions, admit a *simple* and *natural* model theory entirely in the spirit of the model theory of sentential calculi: the construct provides an arena with objects over which models can be considered in terms of (the continuity of) morphisms from the logic's object into the model object. While recently, a model theories for π-institutions have been developed [Vou05],[Vou07b], these model theories are *either*, more in the spirit of a weak form of formal semantics (see Definition 2.95 on page 108 of our text), being based on the notion of a (single) translation [Vou05], than in the spirit of a matrix model, which is based on multiple interpretations into a matrix (see Definition 2.32 on page 99 of our text), *or* fairly technical (and as yet unpublished) [Vou07b]. The major distinction between π-institutions and logics over constructs is that π-institutions cater for *multi-signature* logics while logics over constructs are inherently *single-signature* logics. Since our theory of logics over constructs has been developed to satisfy certain needs in this text, key among which is our theory of canons and their induced ideals, and none of our needs entail multi-signature logics, we feel that attempting to satisfy these needs by extending the theory of π-institutions would introduce unnecessary technical overhead to this text. We are not sure that the theory of canons and induced ideals can naturally be developed within the framework of π-institutions since it depends heavily on a well-defined model theory and free objects, the latter being far more abstractly defined in category theory than in construct theory. This not withstanding, we shall define π-institutions and demonstrate some basic relationships between π-institutions and logics over constructs. Further, in Part VI, we shall obtain some *new* results pertaining to equivalence between π-institutions that sharpen the results of [Vou03]; these new results are suggested by a theory of *equivalent logics from different constructs* that we developed as a means of providing an *alternative* explanation of our theory of parameterized algebraization from a non-parameterized perspective. In attempting to locate our theory of equivalence within the theory of deductively equivalent π-institutions [Vou03], we noticed that our theory worked in situations where the latter theory failed; this suggested improvements that could be made to the theory of equivalent π-institutions. We have found that by sitting between sentential calculi and π-institutions, the theory of logics over constructs informs *both* theories and mediates the 'resolution gap' between sentential calculi and π-institutions.

Chapter 6

Logics over Constructs

In this chapter we present our notion of *logics over constructs.* Essentially a logic in this sense is a closed system over the universe of an object, which in essence follows [Tar56], and generalizes [FJ96], where finitary closed systems over *algebras* are considered to be logics. Our primary aim in developing this theory is to provide a vehicle for explaining how the 'inherently unalgebraizable' logics that we have discovered (and to which our theory of *parameterized algebraization* pertains) arise naturally from well-known closed systems over the universes of algebras, and in particular, to show how 'structural' closed systems over free algebras, which we call *canons*, induce sentential 1-calculi, which we call the *ideals* induced by canons (this theory is developed in $\oint 8$).

In $\oint 6.1$ we present the basic theory of logics over constructs; this section is mostly definitional. We introduce the notion of a *language of logics*, which is simply an object. The members of the universe of such a language are called *formulae.* A logic is determined by a language and a *point-consequence relation* on the formulae of this language (see Definition 4.47 on page 150). As such, the logics over a language are in one-to-one correspondence with the *closed systems* over the formulae of that language (and hence in one-to-one correspondence with the *closed systems* and *closure operators* over its formulae). The *theories*, *theorems* and *consequence operator* of a logic are defined, and a *granularity* relationship between logics over the same language is inherited from the theory of closed systems. Finitary logics are simply logics whose theories form a *finitary* closed system. The theory developed up till this point makes no use of the construct other than as a 'source' of objects. By a *signature of logics* we simply mean a *construct*, and a *typed* logic is a logic whose language belongs to a particular signature. Interpretations between logics of the same signature $\mathfrak{s}$ are defined to be the $\mathfrak{s}$-morphisms between these objects. Endomorphic interpretations are called *substitutions.* A typed logic is called *structural* if all substitutions are *continuous* from the logic as a closed system into itself. The free objects on denumerably countable free generators are called *global languages* and the free generators are called *variables.* Logics over global languages are called *deductive systems*, and structural deductive systems are called *calculi.*

In $\oint 6.2$ we consider logics on global languages that arise by means of *axioms* and *rules*, together with a notion of *derivation.* These deductive systems are finitary calculi, which we term *propositional calculi.* Sentential 1-calculi of type $\mathfrak{a}$ are precisely the propositional calculi over the signature of all $\mathfrak{a}$-algebras with algebra homomorphisms.

In terms of level of abstraction, logics over constructs lie between sentential calculi [BP89a] and

π-institutions [FS88],[Vou03], and as such, we have found that the theory of logics over constructs informs both the theory of sentential calculi and the theory of π-institutions, and is a useful tool in better matching the theory of π-institutions to the theory of sentential calculi. A prime example of the latter is our 'more appropriate' theory of equivalent π-institutions developed in Part VI, since the 'improvements' that we provide are informed by our development of equivalent logics from different constructs; by being 'closer' to sentential calculi than workers in the field of π-institutions, it was easier to get this theory of equivalence to work, and then being closer to π-institutions than workers in the field of sentential calculi, it was easier to see how to improve the theory of equivalent π-institutions so as to make it work more generally (the original theory only fully works for π-institutions that are *term*). In $\oint$6.3 we briefly explore the relationship between logics over constructs and π-institutions.

A number of examples are considered in $\oint$6.4. We show how a quasivariety $\mathcal{K}$ may be considered as a signature of logics; in this case the global language is the $\mathcal{K}$-*free* algebra on ω free generators; unless $\mathcal{K}$ is trivial, $\mathcal{K}$-calculi are *never* sentential calculi, since the language of sentential calculi, when viewed as logics over constructs, is the (absolutely free) term algebra and not the $\mathcal{K}$-free algebra. The relationship between propositional calculi of signature $\mathcal{K}$ and sentential calculi is the topic of $\oint$8, where the former are examples of what we term *canons*. Of particular importance to the sequel is the propositional logic $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$, of signature $\mathcal{K}$, whose theories are precisely the subuniverses of the $\mathcal{K}$-free algebra.

6.1 Logics over Constructs

The notion of *language*, to be introduced at this point, while purposely sparse (a language is an object), is rich enough to permit us to define formulae (the elements of the universe of the language), and, in conjunction with a specified construct, substitutions (endomorphisms), interpretations into other languages (morphisms) and even variables (free generators of free objects). In the subsequent sections we shall define *logics over languages* effectively as closed systems over the universes of languages, and then establish the notion of a *structural* logic, essentially as a closed system over the universe of a language where all endomorphisms are continuous; thereby providing an abstraction of propositional calculi in the discourse of closed systems over objects of a construct. This abstraction permits a notion of a model, which at this level of abstraction, is just (another) logic. The reader is urged to recall the summary of construct theory given in $\oint$1.4.1.

Definition 6.1 (Languages and Formulae) By a **language $\mathbf{A}$**, we mean any *object* $\mathbf{A}$. We write $\mathsf{Fm}(\mathbf{A})$ for $\mathsf{uni}(\mathbf{A})$. The elements of $\mathsf{Fm}(\mathbf{A})$ are called **$\mathbf{A}$-formulae** or just **formulae**. $\square$

We now introduce a logic as a (concrete) consequence relation over the universe of an object. Note that since we have opted *not* to include *structurality* in the definition of a logic, no use of *constructs* is made in the definition of a logic, only *objects*. This permits us to locate the same logic in different constructs and then ask questions regarding structurality. In addition, *finitariness* is also not a *requirement*.

Definition 6.2 (Logics) A **logic** L, is determined by a language $\mathbf{lg}(\mathsf{L})$ (we write $\mathsf{Fm}(\mathsf{L})$ for $\mathsf{Fm}(\mathbf{lg}(\mathsf{L})))$ and a point-consequence relation $\vdash_{\mathsf{L}}$ over $\mathsf{Fm}(\mathsf{L})$, which we call the **consequence relation determined by** L, i.e., $\vdash_{\mathsf{L}}$ is a binary relationship from $\mathfrak{P}(\mathsf{Fm}(\mathsf{L}))$ to $\mathsf{Fm}(\mathsf{L})$ satisfying

$$\phi \in \Gamma \text{ implies } \Gamma \vdash \phi, \quad \text{for all } \Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L}) \tag{6.1}$$

$$\text{if } \Phi \subseteq \Gamma \text{ and } \Phi \vdash \phi \text{ then } \Gamma \vdash \phi, \quad \text{for all } \Gamma \cup \Phi \cup \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L}) \quad \text{and} \tag{6.2}$$

$$\text{if } \Phi \vdash \phi \text{ and } \forall\, [\psi \in \Phi]\, \Gamma \vdash \psi, \text{then } \Gamma \vdash \phi, \quad \text{for all } \Gamma \cup \Phi \cup \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L}). \tag{6.3}$$

We write $\vdash$ for $\vdash_{\mathsf{L}}$ wherever context unambiguous. We say that L **satisfies** formula ϕ if $\vdash_{\mathsf{L}} \phi$. The closed system associated with $\vdash_{\mathsf{L}}$ is denoted by $\mathsf{Th}(\mathsf{L})$, the members of which are called L**-theories** (or just **theories**) and whose constraint is called the set of L**-theorems** (or just **theorems**); the associated complete inclusion-ordered lattice is denoted by $\mathbf{Th}(\mathsf{L})$, which we call the **theory-lattice**, abbreviating $\blacktriangledown^{\mathbf{Th}(\mathsf{L})}$ and $\blacktriangle^{\mathbf{Th}(\mathsf{L})}$ by $\blacktriangledown^{\mathsf{L}}$ and $\blacktriangle^{\mathsf{L}}$ respectively. The closure operator determined by $\vdash_{\mathsf{L}}$ is denoted $\|\cdot\|_{\mathsf{L}}$, which we call the **consequence operator**.

By a **theory basis** for a logic L, we mean a basis of the closed system $\mathsf{Th}(\mathsf{L})$, i.e., a set $\mathcal{T} \subseteq \mathsf{Th}(\mathsf{L})$, such that, for all $T \in \mathsf{Th}(\mathsf{L})$ there exists $\emptyset \neq \mathcal{T}' \subseteq \mathcal{T}$ with $T = \bigcap \mathcal{T}'$. We say that logic L is **finer** than logic M, or that M is **coarser** than logic L, denoted by $\mathsf{L} \preceq \mathsf{M}$, iff $\mathbf{lg}(\mathsf{L}) = \mathbf{lg}(\mathsf{M})$ and $\mathsf{Th}(\mathsf{L}) \preceq \mathsf{Th}(\mathsf{M})$.

A logic L is called **discrete**, **trivial**, **indiscrete** (or **almost-trivial**), **constrained** and **unconstrained**, if the closed system $\mathsf{Th}(\mathsf{L})$ is discrete, trivial, indiscrete, constrained and unconstrained, respectively. The discrete $\mathbf{A}$-logic, trivial $\mathbf{A}$-logic and indiscrete $\mathbf{A}$-logic, are denoted by $L(\mathbf{A}, \bot)$, $L(\mathbf{A}, \top)$ and $L(\mathbf{A}, \top_{\emptyset})$, respectively.

With each language $\mathbf{A}$ and each closed system $\mathbb{C}$ with $\mathsf{uni}(\mathbf{A}) = \mathsf{uni}(\mathbb{C})$, we associate the $\mathbf{A}$-logic $L(\mathbf{A}, \mathbb{C})$ determined by $\vdash_{\mathbb{C}}$.

□

Remark 6.3 T is a theory iff, for all formulae ϕ, $T \vdash_{\mathsf{L}} \phi$ implies $\phi \in T$.

Remark 6.4 $\|\Gamma\|_{\mathsf{L}} = \{\phi \in \mathsf{uni}(\mathsf{L}) \,:\, \Gamma \vdash_{\mathsf{L}} \phi\}$.

The following characterizations of a logic follow immediately from the one-to-one correspondences between (point) consequence relations, closed systems, closure operators and complete $\mathfrak{P}$-concrete lattices.

Remark 6.5 Given a language $\mathbf{A}$, an $\mathbf{A}$-logic is determined by any one of the following:

1. Its theories $\mathsf{Th}(\mathcal{S})$, characterized as a closed system on $\mathsf{Fm}(\mathbf{A})$.

2. Its consequence operator $\|\cdot\|_{\mathcal{S}}$, characterized as a closure operator on $\mathsf{Fm}(\mathbf{A})$.

Remark 6.6 $\Gamma \vdash_{L(\mathbf{A},\bot)} \phi$ iff $\phi \in \Gamma$.

Finitary logics are simply logics whose closed system of theories is algebraic; this notion depends only on the theories of the logic and not on any construct that that object may be a member of.

Definition 6.7 (Finitary Logics and Formal-Axiomatizations) A logic L is called **finitary** if $\vdash_{\mathsf{L}}$ is finitary, i.e., for all $\Gamma, \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L})$, $\Gamma \vdash_{\mathsf{L}} \phi \to \exists\, [\Gamma' \subseteq_{\mathsf{f}} \Gamma]\, \Gamma' \vdash_{\mathsf{L}} \phi$. With each logic L, let

$\mathsf{L}_{|\mathsf{f}}$ denote the logic with language $\mathbf{lg}(\mathsf{L})$ determined by the formal system $F(\vdash_{\mathsf{L}}, \mathsf{aprx})$, as given in Definition 4.124 on page 165. We call $\mathsf{L}_{|\mathsf{f}}$ the **finitary approximation** of L.

When we speak of a **formal-axiomatization** of a *finitary* logic L, we mean an axiomatization of a formal system $\mathfrak{F}$ with $\mathsf{Fm}(\mathfrak{F}) = \mathsf{Fm}(\mathsf{L})$ and $\vdash_{\mathfrak{F}} = \vdash_{\mathsf{L}}$. The prefix 'formal' is to distinguish this notion from the notion of an axiomatization, which only applies to *structural* finitary logics (structural in a sense still to be defined). $\square$

By Proposition 4.125, the consequence relations of finitary logics on language $\mathbf{A}$ are precisely the consequence relations of formal systems on $\mathsf{Fm}(\mathbf{A})$.

Corollary 6.8 For a logic L, the following conditions are equivalent.

1. L is finitary.
2. $\vdash_{\mathsf{L}_{|\mathsf{f}}} = \vdash_{\mathsf{L}}$
3. There exists a formal system $\mathfrak{F}$ on $\mathsf{Fm}(\mathbf{A})$ with $\vdash_{\mathfrak{F}} = \vdash_{\mathsf{L}}$.

$\square$

The following characterizations of a logic follow immediately from the one-to-one correspondences between algebraic (point) consequence relations, algebraic closed systems, algebraic closure operators and algebraic $\mathfrak{P}$-concrete lattices.

Remark 6.9 Given an language $\mathbf{A}$, a finitary $\mathbf{A}$-logic is determined by any one of the following:

1. Its theories $\mathsf{Th}(\mathcal{S})$, characterized as an algebraic closed system on $\mathsf{Fm}(\mathbf{A})$.
2. Its consequence operator $\|\cdot\|_{\mathcal{S}}$, characterized as an algebraic closure operator on $\mathsf{Fm}(\mathbf{A})$.
3. Its consequence relation $\vdash_{\mathcal{S}}$, characterized as an algebraic point-consequence relation on $\mathsf{Fm}(\mathbf{A})$.

$\square$

All sentential calculi are finitary by (4) of Theorem 2.22 on page 96. The discrete, trivial and almost-trivial logics are all finitary.

Eliminating structurality from the definition of a logic permits us to define the filtration of a logic, which is the logic determined by selecting only those theories of a given logic that contain certain formulae. While this operation plays an important role in our theory of parameterized algebraization, generally the filtration of a sentential calculi is not a sentential calculi, since the filtration logic may fail to be structural. Recall the definition of the *filtration* $\mathbb{C}_{:A}$ of a closed system $\mathbb{C}$ by a set $A \subseteq \mathsf{uni}(\mathbb{C})$, given in Definition 4.66 on page 154.

Definition 6.10 (Filtration of a Logic) Let L be a logic and $\Gamma \subseteq \mathsf{Fm}(\mathsf{L})$. The logic $L(\mathbf{lg}(\mathsf{L}), \mathsf{Th}(\mathsf{L})_{:\Gamma})$, i.e., the logic on $\mathbf{lg}(\mathsf{L})$ whose theories are precisely those L-theories containing Γ, is called the **filtration logic** of L by Γ, and is denote $\mathsf{L}_{:\Gamma}$. $\square$

The following remark follows at once from Remark 4.67 on page 154.

Remark 6.11 Let L be a logic and $\Gamma \subseteq \mathsf{Fm}(\mathsf{L})$.

1. If L is finitary then so is $\mathsf{L}_{:\Gamma}$.
2. $\|\Phi\|_{\mathsf{L}:\Gamma} = \|\Phi \cup \Gamma\|_{\mathsf{L}}$.
3. $\Phi \vdash_{\mathsf{L};\Gamma} \Gamma$ iff $\Gamma \cup \Phi \vdash_{\mathsf{L}} \Gamma$.

$\square$

There is nothing in our definition of a logic linking the 'deductive apparatus' with its signature. Any closed system over the formulae of a language determines a logic over that language. As such, no notion of substitution is available, and hence a notion of structurality is lacking. To remedy this situation, we now introduce the notions of a *signature of languages* and a *typed logic*.

Definition 6.12 (Signatures and Typed Logics) By a **signature of languages** $\mathfrak{s}$ (or just a **signature** when unambiguous) we mean any *construct* $\mathfrak{s}$. Let $\mathfrak{s}$ be a signature. By an **$\mathfrak{s}$-language $\mathbf{A}$**, we mean an $\mathfrak{s}$-object. Let $\mathsf{languages}(\mathfrak{s})$ denote the set of all $\mathfrak{s}$-languages. For a set A, let $\mathsf{languages}_{\mathfrak{s}}(A) = \mathsf{Obj}_{\mathfrak{s}}(A)$. This class may be empty, in which case we say that **no $\mathfrak{s}$-languages are definable on** A. An $\mathfrak{s}$-morphism between $\mathfrak{s}$-languages is called an **$\mathfrak{s}$-interpretation** (or just an **interpretation**) and an $\mathfrak{s}$-endomorphism is called an **$\mathfrak{s}$-substitution** (or just a **substitution**). We also write $\mathsf{Int}_{\mathfrak{s}}(\mathbf{A}, \mathbf{B})$ for $\mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$ and $\mathsf{Sub}_{\mathfrak{s}}(\mathbf{A})$ for $\mathsf{End}_{\mathfrak{s}}(\mathbf{A})$. We call a logic L an **$\mathfrak{s}$-logic**, if $\mathbf{lg}(\mathsf{L})$ is an $\mathfrak{s}$-language. Let $\mathsf{logics}(\mathfrak{s})$ denote the class of all logics with $\mathfrak{s}$-languages. Let L and M be $\mathfrak{s}$-logics. By an **$\mathfrak{s}$-interpretation** from L into M and an **$\mathfrak{s}$-substitution** of L, we mean, respectively, an $\mathfrak{s}$-interpretation from $\mathbf{lg}(\mathsf{L})$ into $\mathbf{lg}(\mathsf{M})$ and an $\mathfrak{s}$-substitution of $\mathbf{lg}(\mathsf{L})$. We write $\mathsf{Int}_{\mathfrak{s}}(\mathsf{L}, \mathsf{M})$ for $\mathsf{Int}_{\mathfrak{s}}(\mathbf{lg}(\mathsf{L}), \mathbf{lg}(\mathsf{M}))$ and $\mathsf{Sub}_{\mathfrak{s}}(\mathsf{L})$ for $\mathsf{Sub}_{\mathfrak{s}}(\mathbf{lg}(\mathsf{L}))$. $\square$

While we are hesitant to use the term 'substitution' in the context of languages that are not necessarily free, this usage is entirely consistent with the usage of this term in the discourse of π-institutions [Vou03].

Given a signature of logics $\mathfrak{s}$ and an $\mathfrak{s}$-logic L, we introduce the notion that L be $\mathfrak{s}$-*structural*. This notion is the natural generalization of structurality of deductive systems given in [BP89a] to our more abstract setting. Structurality in the standard sense essentially requires that deduction, or consequence, be closed under substitution. More generally, we shall ask that consequence be closed under $\mathfrak{s}$-substitutions.

Convention 6.13 (Inheriting the Discourse of Continuity) While we do *not* conflate logics with their associated closed systems, it is convenient to 'inherit' the terminology of continuous and related functions defined in $\oint$5. For example, when we speak of a **continuous function from logic** L **into** M, we mean a continuous function from $\mathsf{Th}(\mathsf{L})$ into $\mathsf{Th}(\mathsf{M})$.

Definition 6.14 (Structurality) Let L be any $\mathfrak{s}$-logic. We call L **$\mathfrak{s}$-structural** (resp. **finitely $\mathfrak{s}$-structural**), or just **structural** (resp. **finitely structural**) where unambiguous, if, for all substitutions σ of L and formulae $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L})$ (resp. $\Gamma \cup \{\phi\} \subseteq_{\mathsf{f}} \mathsf{Fm}(\mathsf{L})$),

$$\Gamma \vdash_{\mathsf{L}} \phi \text{ implies } \sigma[\Gamma] \vdash_{\mathsf{L}} \sigma(\phi). \tag{6.4}$$

$\square$

Remark 6.15 If L is finitary, then it is structural iff it is finitely structural. $\square$

The alert reader will notice that $\mathfrak{s}$-structurality of L amounts to the requirement that all $\mathfrak{s}$-substitutions of L be *continuous* functions from L into itself. We formalize this observation.

Corollary 6.16 An $\mathfrak{s}$-logic L is $\mathfrak{s}$-structural iff every $\mathfrak{s}$-substitution of L is continuous from L into itself. Table 7.2 on page 274 enumerates numerous characterizations of $\mathfrak{s}$-structurality, obtained from the earlier results of $\oint 5$ as well as results obtained in the next chapter. Of these characterizations, (7.28) is so well-known and used so often in this text, that we may invoke it without explicit reference.

Remark 6.17 If $\mathbf{A}$ is an $\mathfrak{s}$-language then the discrete logic $L(\mathbf{A}, \perp)$ is $\mathfrak{s}$-structural by the previous corollary. $\square$

Given a signature $\mathfrak{s}$ of languages, we distinguish certain $\mathfrak{s}$-logics as *deductive systems* and *calculi*; the former being logics over free-languages, and consequently admitting a notion of a *variable*; the latter being structural deductive systems. The term *propositional calculi* is (effectively) reserved for *finitary* calculi, a notion introduced in $\oint 6.2$.

Definition 6.18 (Global Languages, Deductive Systems and Calculi) Let $\mathfrak{s}$ be a signature of languages. We say that an $\mathfrak{s}$-language $\mathbf{G}$ is **$\mathfrak{s}$-global**, or just **global**, if $\mathbf{G}$ is an $\mathfrak{s}$-free object over ω free generators. Languages that are not global are called **local**. Conventionally, when we mention a global language we are assuming that it exists. By a **global language** we mean a global $\mathfrak{s}$-language $\mathbf{G}$, for some signature $\mathfrak{s}$. Global $\mathfrak{s}$-languages are also called **languages of $\mathfrak{s}$-deductive systems**. Formulae of global languages are called **global formulae**. Let $\mathfrak{s}$ be a signature of languages and $\mathbf{G}$ a global language. We adopt the convention that $\mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$ is *some* set of ω $\mathfrak{s}$-free generators of $\mathbf{G}$. The elements of $\mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$ are called **$\mathfrak{s}$-variables** or just **variables**. When dealing with an arbitrary signature, we shall assume that the variables of a global language are $\mathsf{V} = \{\mathsf{v}_0, \mathsf{v}_1, \ldots\}$.

An $\mathfrak{s}$-logic $\mathcal{D}$ is called an **$\mathfrak{s}$-deductive-system**, if $\mathbf{lg}(\mathcal{D})$ is a global $\mathfrak{s}$-language. Let $\mathcal{D}$ be an $\mathfrak{s}$-deductive-system. Define $\mathsf{Var}_{\mathfrak{s}}(\mathcal{D}) = \mathsf{Var}_{\mathfrak{s}}(\mathsf{obj}(\mathcal{D}))$, the elements of which are called the **$\mathfrak{s}$-variables of $\mathcal{D}$**. Structural $\mathfrak{s}$-deductive-systems are called **$\mathfrak{s}$-calculi**. $\square$

6.2 Propositional Calculi

Definition 6.19 (Propositional Calculi) Let $\mathbf{G}$ be a global $\mathfrak{s}$-language. A $\mathbf{G}$-*axiom* ϖ is determined by its **conclusion** $\mathsf{conc}(\varpi)$, which is a $\mathbf{G}$-formula. Let $\mathsf{Ax}(\mathbf{G})$ denote the set of all $\mathbf{G}$-axioms. A $\mathbf{G}$-*rule* Λ is determined by its **premise** $\mathsf{prem}(\Lambda)$, which is a non-empty finite set of $\mathbf{G}$-formulae of cardinality $\mathsf{ar}(\Lambda)$, and its **conclusion** $\mathsf{conc}(\Lambda)$, which is a $\mathbf{G}$-formula. Let $\mathsf{Rl}(\mathbf{G})$ denote the set of all $\mathbf{G}$-rules. It is convenient to specify/present a rule Λ by (some) $\phi_1, \ldots, \phi_n \vdash \mathsf{conc}(\Lambda)$, where $\mathsf{prem}(\Lambda) = \{\phi_1, \ldots, \phi_n\}$, and an axiom ϖ by $\vdash \mathsf{conc}(\varpi)$.

Let L be a logic with global language $\mathbf{G}$, ϖ a $\mathbf{G}$-axiom and Λ a $\mathbf{G}$-rule. We say that L **satisfies the axiom** ϖ if L satisfies that axiom's conclusion $\mathsf{conc}(\varpi)$, and say that L **satisfies the rule** Λ if $\mathsf{prem}(\Lambda) \vdash_{\mathsf{L}} \mathsf{conc}(\Lambda)$.

Let $\mathbf{G}$ be a global $\mathfrak{s}$-language. A formula ϕ is **directly derivable** from formulae Γ by a rule Λ in $\mathbf{G}$, if there exists a substitution σ with $\sigma\,[\mathsf{prem}(\Lambda)] \subseteq \Gamma$ and $\sigma(\mathsf{conc}(\Lambda)) = \phi$. Let ir be a set of rules. We say that formulae Φ are **closed under direct derivability** by ir, if it contains every formula directly derivable from itself by the rules of $\Lambda \in \mathsf{ir}$.

A **propositional calculus** $\mathcal{P}$ is determined by its **signature** $\mathsf{sig}(\mathcal{P})$, its **language** $\mathbf{lg}(\mathcal{P})$, which is a global $\mathsf{sig}(\mathcal{P})$-language, its **axioms** $\mathsf{Ax}(\mathcal{P})$ (the members of which are call **$\mathcal{P}$-axioms**), where $\mathsf{Ax}(\mathcal{P}) \subseteq \mathsf{Ax}(\mathbf{lg}(\mathcal{P}))$, and its **rules** $\mathsf{Rl}(\mathcal{P})$ (the members of which are call **$\mathcal{P}$-rules**), where $\mathsf{Rl}(\mathcal{P}) \subseteq \mathsf{Rl}(\mathbf{lg}(\mathcal{P}))$. For a propositional calculus $\mathcal{P}$, we write $\mathsf{Var}(\mathcal{P})$, $\mathsf{Fm}(\mathcal{P})$ and $\mathsf{Sub}(\mathcal{P})$, for $\mathsf{Var}(\mathbf{lg}(\mathcal{P}))$, $\mathsf{Fm}(\mathbf{lg}(\mathcal{P}))$ and $\mathsf{Sub}(\mathbf{lg}(\mathcal{P}))$, respectively.

With each propositional calculus $\mathcal{P}$, we associate the binary relationship $\vdash_{\mathcal{P}}$, called the **consequence relation**, from $\mathfrak{P}(\mathsf{Fm}(\mathcal{P}))$ to $\mathsf{Fm}(\mathcal{P})$, defined by $\Gamma \vdash_{\mathcal{P}} \phi$ iff ϕ is a member of the smallest set of formulae that includes Γ, includes $\sigma\,[\mathsf{conc}[\mathsf{Ax}(\mathcal{P})]]$ for every substitution σ, and is closed under direct derivability by the rules $\mathsf{Rl}(\mathcal{P})$. We write $\vdash_{\mathcal{P}} \phi$ for $\emptyset \vdash_{\mathcal{P}} \phi$. For formulae Φ, we write $\Gamma \vdash_{\mathcal{P}} \Phi$ for $\forall\,[\psi \in \Phi]\; \Gamma \vdash_{\mathcal{P}} \psi$ and $\vdash_{\mathcal{P}} \Phi$ for $\emptyset \vdash_{\mathcal{P}} \Phi$. Two propositional $\mathfrak{s}$-calculi are called **equivalent** if they have the same consequence relation. $\square$

Note 6.20 (Axioms are *not* Formulae) As with sentential calculi, we have *purposely* avoided conflating axioms and formulae since this leads to *ambiguity*. Note that extending the notion of satisfaction from formulae to axioms does not lead to ambiguity, even in the case of propositional calculi. An arbitrary logic L may *satisfy an axiom* of the correct language, but it is syntactically incorrect to ask whether an axiom *is an axiom* of L; logics generally, unlike propositional calculi, do not *have axioms*. A propositional calculi $\mathcal{P}$ may *have an axiom*, that is, the axiom is one of the axioms in the *axiomatization* of $\mathcal{P}$; it may also *satisfy an axiom* that it does not *have*. By introducing the concept that an arbitrary logic may satisfy an axiom or rule, permits succinct phrasing of results such as Remark 6.23 and Proposition 6.31. $\square$

Remark 6.21 The previous definition can be reformulate to require that the pole $\vdash_{\mathcal{P}}[\![\Gamma]\!]$ be the smallest set of formulae that includes Γ, includes $\sigma\,[\mathsf{conc}[\mathsf{Ax}(\mathcal{P})]]$ for every substitution σ, and is closed under direct derivability by the rules $\mathsf{Rl}(\mathcal{P})$. $\square$

The following Lemma describes a recursive process for 'calculating' consequence. From this result, we shall obtain a simpler means of 'calculating' consequence, namely by means of a *derivation* (see Definition 6.24 and Lemma 6.28). The notion of a *derivation* is distinct from the notion of *direct derivability.*

Lemma 6.22 Let $\mathcal{P}$ be a propositional $\mathfrak{s}$-calculus. Define $\Gamma \vdash^0_{\mathcal{P}} \phi$ iff $\phi \in \Gamma$ or there exists a substitution σ and $\mathcal{P}$-axiom ϖ with $\sigma(\mathsf{conc}(\varpi)) = \phi$. For natural n, define $\Gamma \vdash^{n+1}_{\mathcal{P}} \phi$ iff $\Gamma \vdash^n_{\mathcal{P}} \phi$ or ϕ is directly derivable from $\vdash^n_{\mathcal{P}}[\![\Gamma]\!]$ in $\mathcal{P}$. The following statements are valid.

1. $\vdash_{\mathcal{P}} = \bigcup_{n\in\mathbb{N}} \vdash^n_{\mathcal{P}}$.

2. $\Gamma \vdash_{\mathcal{P}} \phi$ iff $\Gamma \vdash^n_{\mathcal{P}} \phi$ for some natural n.

Proof. $\boxed{(1)}$ $\boxed{\vdash_{\mathcal{P}} \subseteq \bigcup_{n\in\mathbb{N}} \vdash^n_{\mathcal{P}}}$ Consider the pole $(\bigcup_{n\in\mathbb{N}} \vdash^n_{\mathcal{P}})[\![\Gamma]\!]$. (*By minimality, it suffices to show that* $(\bigcup_{n\in\mathbb{N}} \vdash^n_{\mathcal{P}})[\![\Gamma]\!]$ *contains* Γ, *contains* $\sigma\,[\mathsf{conc}[\mathsf{Ax}(\mathcal{P})]]$ *for every substitution* σ, *and is closed under direct derivability by the rules* $\mathsf{Rl}(\mathcal{P})$.) Clearly, $\Gamma \cup \bigcup_{\sigma\in\mathsf{Sub}(\mathcal{P})} \sigma\,[\mathsf{conc}[\mathsf{Ax}(\mathcal{P})]] \subseteq \vdash^0_{\mathcal{P}}[\![\Gamma]\!] \subseteq (\bigcup_{n\in\mathbb{N}} \vdash^n_{\mathcal{P}})[\![\Gamma]\!]$. (*It* *remains to show that* $(\bigcup_{n\in\mathbb{N}} \vdash^n_{\mathcal{P}})[\![\Gamma]\!]$ *is closed under direct derivability by the rules* $\mathsf{Rl}(\mathcal{P})$.) Let σ be a substitution and Λ a $\mathcal{P}$-rule with $\sigma\,[\mathsf{prem}(\Lambda)] \subseteq (\bigcup_{n\in\mathbb{N}} \vdash^n_{\mathcal{P}})[\![\Gamma]\!]$. Since rules have finite premises, there exists natural m with $\sigma\,[\mathsf{prem}(\Lambda)] \subseteq (\bigcup_{n\in m} \vdash^n_{\mathcal{P}})[\![\Gamma]\!]$, and so by definition, $\sigma(\mathsf{conc}(\Lambda)) \in \vdash^{m+1}_{\mathcal{P}}[\![\Gamma]\!]$. $\boxed{\bigcup_{n\in\mathbb{N}} \vdash^n_{\mathcal{P}} \subseteq \vdash_{\mathcal{P}}}$ $\boxed{\text{Base Case}}$ Trivially $\vdash^0_{\mathcal{P}}[\![\Gamma]\!] \subseteq \vdash_{\mathcal{P}}[\![\Gamma]\!]$. $\boxed{\text{Inductive Hypothesis}}$ Assume that for all natural $m \leq n$, $\vdash^m_{\mathcal{P}}[\![\Gamma]\!] \subseteq \vdash_{\mathcal{P}}[\![\Gamma]\!]$. $\boxed{\text{Inductive Proof}}$ Suppose that $\Gamma \vdash^{n+1}_{\mathcal{P}} \phi$. If $\Gamma \vdash^n_{\mathcal{P}} \phi$, then $\Gamma \vdash_{\mathcal{P}} \phi$, by the inductive hypothesis. Otherwise, ϕ is directly derivable from $\vdash^n_{\mathcal{P}}[\![\Gamma]\!]$ in $\mathcal{P}$. Since, by the inductive hypothesis, $\vdash^m_{\mathcal{P}}[\![\Gamma]\!] \subseteq \vdash_{\mathcal{P}}[\![\Gamma]\!]$, and $\vdash_{\mathcal{P}}[\![\Gamma]\!]$ is closed under direct derivability, $\phi \in \vdash_{\mathcal{P}}[\![\Gamma]\!]$, i.e., $\Gamma \vdash_{\mathcal{P}} \phi$. $\boxed{\text{(2)}}$ Follows trivially from (1). ◇

Remark 6.23 A propositional calculus satisfies all its rules and axioms.

Definition 6.24 (Derivations) Let $\mathcal{P}$ be a propositional calculus. A **derivation** of formula ϕ from formulae Γ in $\mathcal{P}$, is a non-empty finite sequence $\psi_0, \ldots, \psi_{n-1}$ of formulae, such that $\psi_{n-1} = \phi$ and, for each $i \in n$,

1. $\psi_i \in \Gamma$, or
2. $\psi_i \in \sigma\,[\mathsf{conc}[\mathsf{Ax}(\mathcal{P})]]$, for some substitution σ, or
3. there exists a rule $\Lambda \in \mathsf{Rl}(\mathcal{P})$ and a substitution σ with $\sigma\,[\mathsf{prem}(\Lambda)] \subseteq \{\psi_0, \ldots, \psi_{i-1}\}$ and $\psi_i = \sigma(\mathsf{conc}(\Lambda)$,

in which case we call $\psi_0, \ldots, \psi_{n-1}$ a **derivation** of ϕ from Γ (in $\mathcal{P}$). We say that ϕ is **derivable** from Γ in $\mathcal{P}$ if there exists some derivation of ϕ from Γ (in $\mathcal{P}$). □

Remark 6.25 Derivability and direct derivability are very different.

Remark 6.26 If $\psi_0, \ldots, \psi_n$ is a derivation of ψ_n from Γ in $\mathcal{P}$ and $n > 1$, then $\psi_0, \ldots, \psi_{n-1}$ is a derivation of ψ_{n-1} from Γ in $\mathcal{P}$.

Remark 6.27 If, for each $1 \leq i \leq n$, there exists a derivation of ϕ_i from Γ_i, and there exists a derivation of ϕ from $\{\phi_1, \ldots, \phi_n\}$, then there exists a derivation of ϕ from $\bigcup_{1\leq i\leq n} \Gamma_i$.

Proof. Suppose that, for each $1 \leq i \leq n$, $\psi^i_1, \ldots, \psi^i_{m_i}$ is a derivation of $\psi^i_{m_i} = \phi_i$ from Γ_i, and that $\psi_1, \ldots, \psi_m$ is a derivation of $\psi_m = \phi$ from $\{\phi_1, \ldots, \phi_n\}$. We shall show how to construct a derivation of ϕ from $\bigcup_{1\leq i\leq n} \Gamma_i$. Begin with the concatenation $\psi^1_1, \ldots, \psi^1_{m_1}, \ldots, \psi^n_1, \ldots, \psi^n_{m_n}, \psi_1, \ldots, \psi_m$. The only reason why this sequence may fail to be a derivation of ϕ from $\bigcup_{1\leq i\leq n} \Gamma_i$, is that, for some of the $1 \leq j \leq m$, ψ_j may be occurring in the derivation $\psi_1, \ldots, \psi_m$ not by (2) or (3) of the definition of a derivation, but rather by (1); that is $\psi_j = \psi^k_{m_k} = \phi_k$ for some $1 \leq k \leq n$. In such a case, however, we may simply delete the 'offending' ψ_j, as it already occurs earlier (as $\psi^k_{m_k}$). After performing all such deletions, the resulting sequence is a valid derivation of ϕ from $\bigcup_{1\leq i\leq n} \Gamma_i$. ◇

Lemma 6.28 If $\mathcal{P}$ is a propositional $\mathfrak{s}$-calculus and $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathcal{P})$, then $\Gamma \vdash_{\mathcal{P}} \phi$ iff there exists a derivation of ϕ from Γ in $\mathcal{P}$.

Proof. $\boxed{\Rightarrow}$ We proceed inductively using Lemma 6.22. $\boxed{\text{Base Case}}$ Suppose that $\Gamma \vdash^0_{\mathcal{P}} \phi$. Then either $\phi \in \Gamma$ or there exists a substitution σ and a $\mathcal{P}$-axiom ϖ with $\sigma(\mathsf{conc}(\varpi)) = \phi$; in either case, ϕ is a length 1 derivation of ϕ from Γ. $\boxed{\text{Inductive Hypothesis}}$ Assume that for all $m \leq n$, if $\Gamma \vdash^m_{\mathcal{P}} \phi$ then there exists a derivation of ϕ from Γ. $\boxed{\text{Inductive Step}}$ Suppose that $\Gamma \vdash^{n+1}_{\mathcal{P}} \phi$. If $\Gamma \vdash^n_{\mathcal{P}} \phi$, then the result follows from the inductive hypothesis. Otherwise, ϕ is directly derivable from $\vdash^n_{\mathcal{P}}[\![\Gamma]\!]$ in $\mathcal{P}$. So there exists a rule Λ and a substitution σ with $\sigma\,[\mathsf{prem}(\Lambda)] \subseteq \vdash^n_{\mathcal{P}}[\![\Gamma]\!]$ and $\sigma(\mathsf{conc}(\Lambda)) = \phi$. Suppose that $\mathsf{prem}(\Lambda) = \{\zeta_1, \ldots, \zeta_m\}$. By the inductive hypothesis, for each $1 \leq i \leq m$, there exists a derivation $\psi_{i_1}, \ldots, \sigma(\zeta_i)$ of $\sigma(\zeta_i)$ from Γ. Then $\psi_{1_1}, \ldots, \sigma(\zeta_1), \ldots, \psi_{m_1}, \ldots, \sigma(\zeta_m), \sigma(\mathsf{conc}(\Lambda))$ is a derivation of $\sigma(\mathsf{conc}(\Lambda)) = \phi$ from Γ. $\boxed{\Leftarrow}$ We shall proceed inductively on the length of derivations. $\boxed{\text{Base Case}}$ Suppose that ϕ_1 is a derivation of ϕ_1 from Γ of length 1. Either $\phi_1 \in \Gamma$ or $\psi_1 \in \sigma\,[\mathsf{conc}[\mathsf{Ax}(\mathcal{P})]]$, for some substitution σ; in either case, trivially, $\Gamma \vdash_{\mathcal{P}} \phi_1$. $\boxed{\text{Induction Hypothesis}}$ Assume that for any derivation of ϕ_n from Γ of length less than some fixed n, $\Gamma \vdash_{\mathcal{P}} \phi_n$. $\boxed{\text{Inductive Step}}$ Let $\phi_1, \ldots, \phi_{n+1}$ be a derivation of ϕ_{n+1} from Γ. If $\phi_{n+1} \in \Gamma$ or $\psi_{n+1} \in \sigma\,[\mathsf{conc}[\mathsf{Ax}(\mathcal{P})]]$, for some substitution σ, then in either case, trivially, $\Gamma \vdash_{\mathcal{P}} \phi_{n+1}$. Otherwise, there exists a rule $\Lambda \in \mathsf{Rl}(\mathcal{P})$ and a substitution σ with $\sigma\,[\mathsf{prem}(\Lambda)] \subseteq \{\phi_0, \ldots, \phi_n\}$ and $\phi_{n+1} = \sigma(\mathsf{conc}(\Lambda))$. By the inductive hypothesis and Remark 6.26 on page 228, $\{\phi_1, \ldots, \phi_n\} \subseteq \vdash_{\mathcal{P}}[\![\Gamma]\!]$. So $\sigma\,[\mathsf{prem}(\Lambda)] \subseteq \{\phi_0, \ldots, \phi_n\} \subseteq \vdash_{\mathcal{P}}[\![\Gamma]\!]$, and since $\vdash_{\mathcal{P}}[\![\Gamma]\!]$ is closed under direct derivability, $\phi_{n+1} = \sigma(\mathsf{conc}(\Lambda)) \in \vdash_{\mathcal{P}}[\![\Gamma]\!]$, i.e., $\Gamma \vdash_{\mathcal{P}} \phi_{n+1}$. $\diamond$

Warning 6.29 We shall use the previous lemma so commonly, that we shall do so without explicit reference.

The following observation is one of the standard tools for 'calculating' consequence. While we never use this result, preferring the mechanism of a derivation, we present it because of its ubiquity in standard texts on sentential calculi [vA95, p.g. 52].

Remark 6.30 $\vdash_{\mathcal{P}}$ is recursively definable as follows.

1. If $\phi \in \Gamma$ then $\Gamma \vdash_{\mathcal{P}} \phi$.

2. If σ is a substitution and ϖ an axiom of $\mathcal{P}$ then $\Gamma \vdash_{\mathcal{P}} \sigma(\mathsf{conc}(\varpi))$.

3. If ϕ is directly derivable from Φ and, for each $\psi \in \Phi$ $\Gamma \vdash_{\mathcal{P}} \psi$, then $\Gamma \vdash_{\mathcal{P}} \phi$.

$\square$

Propositional calculi are 'concrete entities', determined by axioms and rules, together with a process that admits a notion of consequence. Logics on the other hand, are defined far more abstractly. In the next series of results and definitions, we establish the relationship between propositional calculi and logics, ultimately establishing that propositional calculi are precisely the finitary deductive systems. Note, however, that propositional calculi may have distinct axiomatizations yet the same consequence relation.

Proposition 6.31 If $\mathcal{P}$ is a propositional $\mathfrak{s}$-calculus and $\mathcal{D}$ is an $\mathfrak{s}$-deductive system that is finitely structural, then $\mathcal{P} \preceq \mathcal{D}$ iff $\mathcal{D}$ satisfies every $\mathcal{P}$-axiom and every $\mathcal{P}$-rule.

Proof. $\boxed{\Rightarrow}$ Trivial. $\boxed{\Leftarrow}$ Let Γ be a set of formulae. We proceed inductively on the length of derivations from Γ in $\mathcal{P}$. $\boxed{\text{Base Case}}$ Suppose that ϕ is derivable from Γ by a derivation of length one. If $\phi \in \Gamma$, then certainly $\Gamma \vdash_{\mathcal{D}} \phi$. Otherwise, there exists a $\mathcal{P}$-axiom ϖ and a substitution σ with $\sigma(\mathsf{conc}(\varpi)) = \phi$. By assumption, $\vdash_{\mathcal{D}} \mathsf{conc}(\varpi)$, and by assumed finite structurality of $\mathcal{D}$, $\vdash_{\mathcal{D}} \sigma(\mathsf{conc}(\varpi))$. The case follows since $\sigma(\mathsf{conc}(\varpi)) = \phi$. $\boxed{\text{Inductive Hypothesis}}$ Suppose that for any formula ϕ derivable from Γ in $\mathcal{P}$ by a derivation of length $m \leq n$, $\Gamma \vdash_{\mathcal{D}} \phi$. $\boxed{\text{Inductive Hypothesis}}$ Suppose that ϕ is derivable from Γ in $\mathcal{P}$ by a derivation of length $n+1$ and by no shorter derivation. Then there exists a $\mathcal{P}$-rule Λ and substitution σ with $\sigma[\mathsf{prem}(\Lambda)] \subseteq \Gamma$ and $\sigma(\mathsf{conc}(\Lambda)) = \phi$. By assumption, $\mathsf{prem}(\Lambda) \vdash_{\mathcal{D}} \mathsf{conc}(\Lambda)$, and by assumed finite structurality, $\sigma[\mathsf{prem}(\Lambda)] \vdash_{\mathcal{D}} \sigma(\mathsf{conc}(\Lambda))$. Since $\sigma(\mathsf{conc}(\Lambda)) = \phi$ and $\sigma[\mathsf{prem}(\Lambda)] \subseteq \Gamma$, it follows that $\Gamma \vdash_{\mathcal{D}} \phi$. ◇

Lemma 6.32 If $\mathcal{P}$ is a propositional $\mathfrak{s}$-calculus, then $\vdash_{\mathcal{P}}$ is the consequence relation of a finitary $\mathfrak{s}$-calculus.

Proof. $\boxed{(6.1)}$ Suppose that $\phi \in \Gamma$. Then ϕ is a length 1 derivation of ϕ from Γ, hence $\Gamma \vdash_{\mathcal{P}} \phi$. $\boxed{(6.2)}$ Suppose that $\Phi \subseteq \Gamma$ and $\Phi \vdash_{\mathcal{P}} \phi$. Since $\Phi \vdash_{\mathcal{P}} \phi$, there exists a derivation of ϕ from Φ, and since $\Phi \subseteq \Gamma$, by definition, this derivation is also a derivation of ϕ from Γ. Hence $\Gamma \vdash_{\mathcal{P}} \phi$. $\boxed{\text{Finitariness}}$ Suppose that $\Gamma \vdash_{\mathcal{P}} \phi$. Then there exists a derivation of ϕ from Γ. Since derivations are of finite length, this derivation is a derivation of ϕ from Γ' for some finite $\Gamma' \subseteq_{\mathsf{f}} \Gamma$. Hence $\Gamma' \vdash_{\mathcal{P}} \phi$. $\boxed{(6.3)}$ Suppose that $\Phi \vdash_{\mathcal{P}} \phi$, and $\forall\, [\psi \in \Phi]\; \Gamma \vdash_{\mathcal{P}} \psi$. Since $\Phi \vdash_{\mathcal{P}} \phi$, by (already established) finitariness, there exists $\{\psi_1, \ldots, \psi_n\} \subseteq \Phi$ with $\{\psi_1, \ldots, \psi_n\} \vdash_{\mathcal{P}} \phi$, and hence a derivation of ϕ from $\{\psi_1, \ldots, \psi_n\}$. For each $1 \leq i \leq n$, since $\Gamma \vdash_{\mathcal{P}} \psi_i$, there exists a derivation of ψ_i from Γ. So by Remark 6.27 on page 228, there exists a derivation of ϕ from $\bigcup_{1 \leq i \leq n} \Gamma = \Gamma$. Hence $\Gamma \vdash_{\mathcal{P}} \phi$. $\boxed{\text{Structurality}}$ Suppose that $\Gamma \vdash_{\mathcal{P}} \phi$ and let σ be a substitution. There exists a derivation $\psi_1, \ldots, \psi_n$ of $\psi_n = \phi$ from Γ. (*It suffices to show that* $\sigma(\psi_1), \ldots, \sigma(\psi_n)$ *is a derivation of* $\sigma(\psi_n) = \sigma(\phi)$ *from* $\sigma[\Gamma]$.) Let $1 \leq i \leq n$. (*We consider the three cases of the definition of a derivation.*) $\boxed{(1)}$ Suppose $\phi_i \in \Gamma$. Then $\sigma(\phi_i) \in \sigma[\Gamma]$. $\boxed{(2)}$ Suppose there exists a $\mathcal{P}$-axiom ϖ and substitution ρ with $\rho(\mathsf{conc}(\varpi)) = \phi_i$. Then $\sigma(\rho(\mathsf{conc}(\varpi))) = \sigma(\phi_i)$. So there exists a $\mathcal{P}$-axiom, namely ϖ, and a substitution, namely $\sigma\rho$ (the composition of morphisms being a morphism in a construct), with $(\sigma\rho)(\mathsf{conc}(\varpi)) = \sigma(\phi_i)$. $\boxed{(3)}$ The proof follows by substitution composition as in the previous case. ◇

We shall now show that propositional $\mathfrak{s}$-calculi and finitary $\mathfrak{s}$-calculi 'coincide'. The usefulness of the following definition and subsequent results, is that, given any finitary $\mathfrak{s}$-calculus, we can always find an axiomatization of a propositional $\mathfrak{s}$-calculus with the same consequence relation.

Definition 6.33 (Propositional Approximations of Deductive Systems) With each $\mathfrak{s}$-deductive system $\mathcal{D}$, we associate the propositional $\mathfrak{s}$-calculus $\mathcal{D}_{|_{\mathsf{p}}}$, determined by all axioms $\vdash \phi$ such that $\vdash_{\mathcal{D}} \phi$ and all rules $\Gamma \vdash \phi$ such that $\Gamma \vdash_{\mathcal{D}} \phi$, which we call the **propositional approximation** of $\mathcal{D}$. □

The importance of *finitely structural* deductive systems lies in the fact that, while not necessarily finitary nor structural, they can nevertheless be *soundly* approximated by a coarsest propositional calculus.

Proposition 6.34 If $\mathcal{D}$ is a finitely structural $\mathfrak{s}$-deductive system, then $\mathcal{D}_{|_{\mathsf{p}}}$ is a coarsest propositional $\mathfrak{s}$-calculus finer than $\mathcal{D}$.

Proof. By definition, every $\mathcal{D}_{|\mathsf{p}}$ axiom and rule is satisfied by $\mathcal{D}$, and by assumption $\mathcal{D}$ is finitely structural, and so by Proposition 6.31, $\mathcal{D}_{|\mathsf{p}} \preceq \mathcal{D}$.
Let $\mathcal{P}$ be any propositional $\mathfrak{s}$-calculus finer than $\mathcal{D}$. Suppose that $\Gamma \vdash_{\mathcal{P}} \phi$. Then by finitariness of $\mathcal{P}$, there exists $\Gamma' \subseteq_{\mathsf{f}} \Gamma$ with $\Gamma' \vdash_{\mathcal{P}} \phi$. Since $\mathcal{P}$ is finer than $\mathcal{D}$, $\Gamma' \vdash_{\mathcal{D}} \phi$, and so by definition, $\Gamma' \vdash_{\mathcal{D}_{|\mathsf{p}}} \phi$. Hence $\Gamma \vdash_{\mathcal{D}_{|\mathsf{p}}} \phi$. $\diamond$

Lemma 6.35 Let $\mathcal{D}$ be an $\mathfrak{s}$-calculus. Then $\vdash_{\mathcal{D}}$ is the consequence relation of some propositional $\mathfrak{s}$-calculus iff $\mathcal{D} = \mathcal{D}_{|\mathsf{p}}$.

Proof. (*We prove the forward implication, the converse implication being trivial.*)
$\boxed{\mathcal{D} \preceq \mathcal{D}_{|\mathsf{p}}}$ Suppose that $\Gamma \vdash_{\mathcal{D}} \phi$. By assumed finitariness of $\mathcal{D}$, there exists $\Gamma' \subseteq_{\mathsf{f}} \Gamma$ with $\Gamma' \vdash_{\mathcal{D}} \phi$. So by construction (and Remark 6.23), $\Gamma' \vdash_{\mathcal{D}_{|\mathsf{p}}} \phi$. By Lemma 6.32, $\Gamma \vdash_{\mathcal{D}_{|\mathsf{p}}} \phi$. $\boxed{\mathcal{D}_{|\mathsf{p}} \preceq \mathcal{D}}$ Since $\mathcal{D}$ is structural it is certainly finitely structural, and so by Proposition 6.34, $\mathcal{D}_{|\mathsf{p}} \preceq \mathcal{D}$. $\diamond$

Consequently propositional $\mathfrak{s}$-calculi and finitary $\mathfrak{s}$-calculi 'coincide', a statement formalized in the result which is an immediate corollary to the previous two lemmas. That the consequence relation of any $\mathfrak{a}$-structural finitary $\mathfrak{a}$-logic (where $\mathfrak{a}$ is a type of algebras) is the consequence relation of a propositional $\mathfrak{a}$-calculus, was originally proven by Los and Suszko [LS58].

Corollary 6.36 The consequence relations of propositional $\mathfrak{s}$-calculi systems are precisely the consequence relations of finitary $\mathfrak{s}$-calculi.

Convention 6.37 Consequent to the previous corollary, we shall conflate propositional calculi and finitary calculi. The problem with this conflation is that two *distinct but equivalent* propositional calculi are *equal* as logics, and a particular finitary calculi may have multiple axiomatizations. We could have avoided this problem entirely, by defining a propositional calculus to be a finitary calculus, and characterized propositional calculi as those logics that can be axiomatized. In essence, this is precisely what this convention is achieving. In practise, however, the approach we have adopted presents no real problems. In this case $\mathsf{Ax}(\mathcal{P})$ and $\mathsf{Rl}(\mathcal{P})$ refer to *some* axiomatization of $\mathcal{P}$, unless a particular axiomatization has been given.

Remark 6.38 Let $\mathfrak{s}$ be a signature of logics.

1. The discrete $\mathfrak{s}$-deductive system is a propositional $\mathfrak{s}$-calculus, axiomatized with no axioms and no rules.

2. The indiscrete $\mathfrak{s}$-deductive system is a propositional $\mathfrak{s}$-calculus, axiomatized by the single axiom $\vdash x$, where x is any variable.

3. The trivial $\mathfrak{s}$-deductive system is a propositional $\mathfrak{s}$-calculus, axiomatized by the single rule $\{x\} \vdash y$, where x and y are any two distinct variables.

$\square$

Recall the definition of the finitary approximation $\mathsf{L}_{|\mathsf{f}}$ of logic L given in Definition 6.7. This formal system was determined by *all* possible finite consequences of L. In the following definition we associate another formal system with a propositional calculus, determined only by (all substitution instances of) its *axiomatization*. The role of the *substitutions* in the definition is to 'de-structuralize' the propositional calculus.

Definition 6.39 (Associating Propositional Calculi with Formal Systems) With each propositional $\mathfrak{s}$-calculus $\mathcal{P}$, we associate a formal system $F(\mathcal{P})$ defined with language $\mathbf{lg}(F(\mathcal{P})) = \mathsf{Fm}(\mathcal{P})$, $\mathsf{FAx}(F(\mathcal{P})) = \{\vdash \sigma(\mathsf{conc}(\varpi)) \ : \ \varpi \in \mathsf{Ax}(\mathcal{P}), \sigma \in \mathsf{Sub}(\mathcal{P})\}$, and $\mathsf{FRl}(F(\mathcal{P})) = \{\sigma[\mathsf{prem}(\Lambda)] \vdash \sigma(\mathsf{conc}(\Lambda)) \ : \ \Lambda \in \mathsf{Rl}(\mathcal{P}), \sigma \in \mathsf{Sub}(\mathcal{P})\}$. □

Remark 6.40 For a propositional $\mathfrak{s}$-calculus $\mathcal{P}$, $\vdash_{F(\mathcal{P})} = \vdash_{\mathcal{P}}$ and $F(\mathcal{P}) = \mathcal{P}_{|_\mathsf{f}}$. □

Recall the closed system granularity relation $\preceq$ is a complete lattice order, when restricted to a particular universe (see Remark 4.56 on page 152). Since the logics over $\mathbf{A}$ are precisely the closed systems over $\mathsf{Fm}(\mathbf{A})$, the granularity relation of logics is a *complete* lattice order. We shall now establish that the join of an arbitrary number of *propositional calculi*, in the complete lattice of *all* logics over $\mathbf{A}$, is itself a *propositional calculi*. We found this result surprising.

Definition 6.41 (Joining Propositional Calculi) Let $\mathfrak{L}$ be a set of propositional $\mathfrak{s}$-calculi. Let $\blacktriangledown_{\mathsf{p}} \mathfrak{L}$ denote the propositional $\mathfrak{s}$-calculus, determined by rules $\bigcup_{\mathcal{P}\in\mathfrak{L}} \mathsf{Rl}(\mathcal{P})$ and axioms $\bigcup_{\mathcal{P}\in\mathfrak{L}} \mathsf{Ax}(\mathcal{P})$. For propositional $\mathfrak{s}$-calculi $\mathcal{P}_1$ and $\mathcal{P}_2$, we write $\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$ for $\blacktriangledown_{\mathsf{p}} \{\mathcal{P}_1, \mathcal{P}_2\}$. Conventionally, this subscripted 'p' join notation only applies to propositional calculi, and any use of this notion implicitly implies that the logics under consideration are propositional calculi of the same signature. □

In the following result we show that the join $\blacktriangledown^{\preceq} \mathfrak{L}$ of propositional calculi $\mathfrak{L}$ is equal to $\blacktriangledown_{\mathsf{p}} \mathfrak{L}$, and hence the $\preceq$-join of propositional calculi is a propositional calculus.

Theorem 6.42 $\blacktriangledown_{\mathsf{p}} \mathfrak{L} = \blacktriangledown^{\preceq} \mathfrak{L}$.

Proof. $\boxed{\text{Upper bound}}$ $\blacktriangledown_{\mathsf{p}} \mathfrak{L}$ is an $\preceq$-upper-bound of $\mathfrak{L}$ by Proposition 6.31. □
$\boxed{\text{Least Upper Bound}}$ Suppose that $\mathcal{E}$ is an upper $\preceq$-bound of $\mathfrak{L}$. Note that by Proposition 4.41 on page 148, if $\mathcal{P} \in \mathfrak{L}$ and $\Gamma \vdash_{\mathcal{P}} \phi$, then $\Gamma \vdash_{\mathcal{E}} \phi$; we shall repeatedly use this fact without explicit further reference. (*It suffices to show that for any derivation of ψ from Γ in $\blacktriangledown_{\mathsf{p}} \mathfrak{L}$, $\Gamma \vdash_{\mathcal{E}} \psi$. We proceed by induction on the length of such derivations.*)
$\boxed{\text{Base Case}}$ Let ϕ_0 be a derivation from Γ in $\blacktriangledown_{\mathsf{p}} \mathfrak{L}$ of length 1. $\boxed{\text{Member}}$ If $\phi_0 \in \Gamma$ then trivially $\mathcal{E}$ satisfies ϕ_0.
$\boxed{\text{Axiom}}$ Suppose that there exists a ($\blacktriangledown_{\mathsf{p}} \mathfrak{L}$)-axiom ϖ and substitution σ with $\sigma(\mathsf{conc}(\varpi)) = \phi_0$. This axiom must be a $\mathcal{P}$-axiom for some $\mathcal{P} \in \mathfrak{L}$. By structurality of $\mathcal{P}$, $\mathcal{P}$ also satisfies $\sigma(\mathsf{conc}(\varpi)) = \phi_0$. Hence $\mathcal{E}$ satisfies ϕ_0. □
$\boxed{\text{Induction Hypothesis}}$ Assume that for any derivation of ψ from Γ in $\blacktriangledown_{\mathsf{p}} \mathfrak{L}$ of length $m < n$, $\Gamma \vdash_{\mathcal{E}} \psi$. □
$\boxed{\text{Inductive Step}}$ Let $\phi_0, \ldots, \phi_n$ be a derivation from Γ in $\blacktriangledown_{\mathsf{p}} \mathfrak{L}$, for which no shorter derivation exists. (*We must show that* $\Gamma \vdash_{\mathcal{E}} \phi_n$.) So there exists a ($\blacktriangledown_{\mathsf{p}} \mathfrak{L}$)-rule Λ and substitution σ with $\sigma(\mathsf{conc}(\Lambda)) = \phi_n$, and $\sigma[\mathsf{prem}(\Lambda)] \subseteq \{\phi_0, \ldots, \phi_{n-1}\}$. By the inductive hypothesis, $\Gamma \vdash_{\mathcal{E}} \sigma[\mathsf{prem}(\Lambda)]$. By definition, Λ is a $\mathcal{P}$-rule for some $\mathcal{P} \in \mathfrak{L}$. By structurality, $\sigma[\mathsf{prem}(\Lambda)] \vdash_{\mathcal{P}} \sigma(\mathsf{conc}(\Lambda))$. Hence $\sigma[\mathsf{prem}(\Lambda)] \vdash_{\mathcal{E}} \sigma(\mathsf{conc}(\Lambda))$. Since $\Gamma \vdash_{\mathcal{E}} \sigma[\mathsf{prem}(\Lambda)]$ and $\sigma[\mathsf{prem}(\Lambda)] \vdash_{\mathcal{E}} \sigma(\mathsf{conc}(\Lambda))$, $\Gamma \vdash_{\mathcal{E}} \sigma(\mathsf{conc}(\Lambda))$. The result follows since $\sigma(\mathsf{conc}(\Lambda)) = \phi_n$. ◇

The following result describes theory generation in $\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$, recursively in terms of theory generation in $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. The result extends immediately to the join of finitely many propositional calculi in the light of the previous theorem and the associativity of lattice joins.

Lemma 6.43 $\|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2} = \bigcup_{i\in\omega} \Upsilon^i_\Gamma$, where $\Upsilon^0_\Gamma = \Gamma \cup \mathsf{Thm}(\mathcal{P}_1) \cup \mathsf{Thm}(\mathcal{P}_2)$, and $\Upsilon^{i+1}_\Gamma = \left\|\left\|\Upsilon^i_\Gamma\right\|_{\mathcal{P}_1}\right\|_{\mathcal{P}_2}$.

Proof. Note that $\Gamma \subseteq \Upsilon^0_\Gamma \subseteq \ldots \subseteq \Upsilon^i_\Gamma \subseteq \Upsilon^{i+1}_\Gamma \subseteq \ldots$.

$\boxed{\|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2} \subseteq \bigcup_{i\in\omega} \Upsilon^i_\Gamma}$ (*It suffices to show that* $\bigcup_{i\in\omega} \Upsilon^i_\Gamma$ *is a* $\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$*-theory. We shall show that any formula derivable from* $\bigcup_{i\in\omega} \Upsilon^i_\Gamma$ *in* $\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$ *is contained in* $\bigcup_{i\in\omega} \Upsilon^i_\Gamma$*, proceeding inductively on the length of such derivations.*) $\boxed{\text{Base Case}}$ Suppose that ϕ is derivable from $\bigcup_{i\in\omega} \Upsilon^i_\Gamma$ by a derivation of length 1. If $\phi \in \bigcup_{i\in\omega} \Upsilon^i_\Gamma$, then the required result is trivial. Otherwise, there exists a $\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$-axiom ϖ and a substitution σ with $\sigma(\mathsf{conc}(\varpi)) = \phi$. By definition, $\mathsf{conc}(\varpi) \in \mathsf{Thm}(\mathcal{P}_1) \cup \mathsf{Thm}(\mathcal{P}_2)$, and by structurality, $\sigma(\mathsf{conc}(\varpi)) \in \mathsf{Thm}(\mathcal{P}_1) \cup \mathsf{Thm}(\mathcal{P}_2)$. So $\phi = \sigma(\mathsf{conc}(\varpi)) \in \Upsilon^0_\Gamma \subseteq \bigcup_{i\in\omega} \Upsilon^i_\Gamma$. $\boxed{\text{Inductive Hypothesis}}$ Assume that if ϕ is derivable from $\bigcup_{i\in\omega} \Upsilon^i_\Gamma$ in $\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$ by a derivation of length m, then $\phi \in \bigcup_{i\in\omega} \Upsilon^i_\Gamma$. $\boxed{\text{Inductive Step}}$ Suppose that $\phi_1, \ldots \phi_{m+1}$ is a derivation from $\bigcup_{i\in\omega} \Upsilon^i_\Gamma$ in $\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$, for which no shorter derivation of ϕ_{m+1} from $\bigcup_{i\in\omega} \Upsilon^i_\Gamma$ exists. By the induction hypothesis, $\{\phi_1, \ldots \phi_m\} \subseteq \bigcup_{i\in\omega} \Upsilon^i_\Gamma$. By the assumed minimality of the derivation $\phi_1, \ldots \phi_{m+1}$, there exists a $\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$-rule Λ and a substitution σ with $\sigma[\mathsf{prem}(\Lambda)] \subseteq \{\phi_1, \ldots \phi_m\} \subseteq \bigcup_{i\in\omega} \Upsilon^i_\Gamma$ and $\sigma(\mathsf{conc}(\Lambda)) = \phi_{m+1}$. Since $\sigma[\mathsf{prem}(\Lambda)]$ is finite, there exists i with $\sigma[\mathsf{prem}(\Lambda)] \subseteq \Upsilon^i_\Gamma$. By definition, $\mathsf{prem}(\Lambda) \vdash_{\mathcal{P}_1} \mathsf{conc}(\Lambda)$ or $\mathsf{prem}(\Lambda) \vdash_{\mathcal{P}_2} \mathsf{conc}(\Lambda)$, and so by structurality, $\sigma[\mathsf{prem}(\Lambda)] \vdash_{\mathcal{P}_1} \sigma(\mathsf{conc}(\Lambda))$ or $\sigma[\mathsf{prem}(\Lambda)] \vdash_{\mathcal{P}_2} \sigma(\mathsf{conc}(\Lambda))$. So, since $\sigma[\mathsf{prem}(\Lambda)] \subseteq \Upsilon^i_\Gamma$, it follows by definition that $\phi_{m+1} = \sigma(\mathsf{conc}(\Lambda)) \in \Upsilon^{i+1}_\Gamma \subseteq \bigcup_{i\in\omega} \Upsilon^i_\Gamma$. $\Box$

$\boxed{\bigcup_{i\in\omega} \Upsilon^i_\Gamma \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}}$ (*It suffices to show that* $\Upsilon^i_\Gamma \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$*, for each* $i \in \omega$*. We proceed inductively.*) $\boxed{\text{Base Case}}$ Certainly $\Gamma \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$. Let $\phi \in \mathsf{Thm}(\mathcal{P}_1)$. Since $\mathcal{P}_1 \preceq \mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$, $\phi \in \mathsf{Thm}(\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2)$. So $\mathsf{Thm}(\mathcal{P}_1) \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$. Similarly, $\mathsf{Thm}(\mathcal{P}_2) \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$. So $\Upsilon^0_\Gamma \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$. $\boxed{\text{Inductive Hypothesis}}$ Assume that $\Upsilon^i_\Gamma \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$. $\boxed{\text{Inductive Step}}$ Suppose that $\Upsilon^i_\Gamma \vdash_{\mathcal{P}_1} \phi$. Since $\mathcal{P}_1 \preceq \mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2$, $\Upsilon^i_\Gamma \vdash_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2} \phi$. Since $\Upsilon^i_\Gamma \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$ and $\Upsilon^i_\Gamma \vdash_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2} \phi$, $\phi \in \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$. Hence $\left\|\Upsilon^i_\Gamma\right\|_{\mathcal{P}_1} \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$. Consequently, by a similar argument, $\Upsilon^{i+1}_\Gamma = \left\|\left\|\Upsilon^i_\Gamma\right\|_{\mathcal{P}_1}\right\|_{\mathcal{P}_2} \subseteq \|\Gamma\|_{\mathcal{P}_1 \vee_{\mathsf{p}} \mathcal{P}_2}$. $\diamond$

6.3 Locating Logics over Constructs as π-Institutions

We now show how $\mathfrak{s}$-*structural* logics over constructs can be locate a π-institutions [FS88]. Note that in the following definition we have introduced some non-standard conventions so as to simplify the notional burden and highlight parallels with the theory of logics over constructs (and abstract algebraic logics). The reader unfamiliar with category theory is urged to read $\oint$1.4.2.

Definition 6.44 (π-Institutions) [FS88],[Vou03] A **π-institution** $\mathcal{I}$ is determined by a category $\mathsf{Sign}_\mathcal{I}$ (we write $\mathsf{Sign}_\mathcal{I}$ for $\mathsf{Obj}_{\mathsf{Sign}_\mathcal{I}}$ the objects of which are called **signatures**), a functor $\mathsf{SEN}_\mathcal{I} : \mathsf{Sign}_\mathcal{I} \to \mathsf{Set}$, (we write $\mathsf{Sent}_\mathcal{I}(\mathfrak{S})$ for $\mathsf{SEN}_\mathcal{I}(\mathfrak{S})$ and $\overline{\sigma}^\mathcal{I}$ for $\mathsf{SEN}_\mathcal{I}(\sigma)$), and a $\mathsf{Sign}_\mathcal{I}$-indexed family $\{\|\cdot\|^\mathcal{I}_\mathfrak{S} : \mathfrak{S} \in \mathsf{Sign}_\mathcal{I}\}$, such that, for each $\mathfrak{S} \in \mathsf{Sign}_\mathcal{I}$, $\|\cdot\|^\mathcal{I}_\mathfrak{S}$ is a closure operator on the set $\mathsf{SEN}_\mathcal{I}(\mathfrak{S})$ and, for all $\mathfrak{S}, \mathfrak{T} \in \mathsf{Sign}$, $\sigma : \mathfrak{S} \to \mathfrak{T}$ and $\Gamma \subseteq \mathsf{Sent}(\mathfrak{S})$, $\overline{\sigma}^\mathcal{I}\left[\|\Gamma\|^\mathcal{I}_\mathfrak{S}\right] \subseteq \left\|\overline{\sigma}^\mathcal{I}[\Gamma]\right\|^\mathcal{I}_\mathfrak{T}$; the last condition being referred to as **structurality**. The members of $\mathsf{Sent}_\mathcal{I}(\mathfrak{S})$ are called **sentences over** $\mathfrak{S}$ or **$\mathfrak{S}$-sentences**. We tend to drop the institution symbol '$\mathcal{I}$' from these (and subsequently defined) notations whenever unambiguous. We call $\mathcal{I}$ **finitary** if, for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, $\|\cdot\|_{\mathfrak{S}}$ is a finitary closure operator.

Let $\mathcal{I}$ be a π-institution. The **category of $\mathcal{I}$-theories** $\mathsf{TH}(\mathcal{I})$ is determined by the class of objects $\{\langle \mathfrak{S}, T\rangle \ : \ \mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}, \|T\|_{\mathfrak{S}} = T\} \doteq \mathsf{Th}(\mathcal{I})$ and morphisms $\sigma : \langle \mathfrak{S}, T\rangle \to \langle \mathfrak{T}, R\rangle$ where $\sigma : \mathfrak{S} \to \mathfrak{T}$ such that $\overline{\sigma}\,[T] \subseteq R$. We define a functor $\mathsf{SIG}_{\mathcal{I}} : \mathsf{TH}(\mathcal{I}) \to \mathbf{Sign}_{\mathcal{I}}$ by letting $\mathsf{SIG}_{\mathcal{I}}(\langle \mathfrak{S}, T\rangle) = \mathfrak{S}$ and by letting $\mathsf{SIG}_{\mathcal{I}}(\sigma) : \mathfrak{S} \to \mathfrak{T}$ denote the underlying $\mathbf{Sign}_{\mathcal{I}}$-morphism; when ambiguity is avoided, we may write σ for $\mathsf{SIG}_{\mathcal{I}}(\sigma)$. Define a functor $\mathsf{THY}_{\mathcal{I}} : \mathbf{Sign}_{\mathcal{I}} \to \mathsf{TH}(\mathcal{I})$ by $\mathsf{THY}_{\mathcal{I}}(\mathfrak{S}) = \langle \mathfrak{S}, \|\emptyset\|^{\mathcal{I}}_{\mathfrak{S}}\rangle$ and $\mathsf{THY}_{\mathcal{I}}(\sigma) : \langle \mathfrak{S}, \|\emptyset\|^{\mathcal{I}}_{\mathfrak{S}}\rangle \to \langle \mathfrak{T}, \|\emptyset\|^{\mathcal{I}}_{\mathfrak{T}}\rangle$, with $\mathsf{SIG}_{\mathcal{I}}(\mathsf{THY}_{\mathcal{I}}(\sigma)) = \sigma$, which is well-defined by structurality, and a functor $\mathsf{THS}_{\mathcal{I}} : \mathbf{Sign}_{\mathcal{I}} \to \mathbf{Set}$ by $\mathsf{THS}_{\mathcal{I}}(\mathfrak{S}) = \{\langle \mathfrak{S}, T\rangle \ : \ T \in \mathsf{Th}_{\mathcal{I}}(\mathfrak{S})\}$ and $\mathsf{THS}_{\mathcal{I}}(\sigma) : \mathsf{THS}_{\mathcal{I}}(\mathfrak{S}) \to \mathsf{THS}_{\mathcal{I}}(\mathfrak{T})$ defined by $\mathsf{THS}_{\mathcal{I}}(\sigma)(\langle \mathfrak{S}, T\rangle) = \langle \mathfrak{T}, \|\mathsf{SEN}_{\sigma}(T)\|^{\mathcal{I}}_{\mathfrak{T}}\rangle$. The last functor is called the **theory functor**. We tend to write $\sigma^{\star}$ for $\mathsf{THS}_{\mathcal{I}}(\sigma)$ wherever unambiguous.

It is convenient to denote the point-consequence relation, closed system and lattice of closed sets, associated with $\|\cdot\|^{\mathcal{I}}_{\mathfrak{S}}$, by $\vdash^{\mathcal{I}}_{\mathfrak{S}}$, $\mathsf{Th}_{\mathcal{I}}(\mathfrak{S})$ and $\mathbf{Th}_{\mathcal{I}}(\mathfrak{S})$, respectively. For $\langle \mathfrak{S}, T\rangle \in \mathsf{Th}(\mathcal{I})$, let $\mathsf{th}(\langle \mathfrak{S}, T\rangle) = T$.

Let $\mathcal{I}$ be a π-institution. A **source signature - variable pair** is a pair $\langle \mathfrak{A}, x\rangle$, where $\mathfrak{A} \in \mathsf{sig}(\mathcal{I})$ and $x \in \mathsf{Sent}(\mathfrak{A})$, such that, for all $\mathfrak{S} \in \mathsf{sig}(\mathcal{I})$ and $\phi \in \mathsf{Sent}(\mathfrak{S})$, there exists $\sigma_{\langle \mathfrak{S}, \phi\rangle} : \mathfrak{A} \to \mathfrak{S}$ such that

$$\overline{\sigma_{\langle \mathfrak{S}, \phi\rangle}}(x) = \phi \quad \text{and} \tag{6.5}$$

$$\forall\,[\mathfrak{S}' \in \mathsf{sig}(\mathcal{I})]\ \forall\,[\rho : \mathfrak{S} \to \mathfrak{T}]\ \rho\sigma_{\langle \mathfrak{S}, \phi\rangle} = \sigma_{\langle \mathfrak{T}, \overline{\rho}(\phi)\rangle}; \tag{6.6}$$

in this case, $\mathfrak{A}$ is called a **source signature** and x is called a **source variable** or just a **variable**. A π-institution is called **term** if it has a source signature - variable pair. □

Warning 6.45 $\mathsf{Th}(\mathcal{I}) = \{\langle \mathfrak{S}, T\rangle \ : \ \mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}, T \in \mathsf{Th}_{\mathcal{I}}(\mathfrak{S})\}$.

In the following example we locate logics over constructs as π-institutions.

Example 6.46 (Logics over Constructs as π-Institutions)

With each signature $\mathfrak{s}$ of logics and each $\mathfrak{s}$-logic L, we associate the π-institution $\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}$, where $\mathbf{Sign}_{\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}}$ is the one object category consisting of object $\mathbf{lg}(\mathsf{L})$ together with all $\mathfrak{s}$-endomorphisms of $\mathbf{lg}(\mathsf{L})$ (the category composition is just functional composition and the category identity associated with $\mathbf{lg}(\mathsf{L})$ is the identity map), $\mathsf{SEN}_{\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}}$ maps $\mathbf{lg}(\mathsf{L})$ to $\mathsf{Fm}(\mathsf{L})$ and maps each $\mathfrak{s}$-endomorphism of $\mathbf{lg}(\mathsf{L})$ to itself, and $\|\cdot\|^{\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}}_{\mathbf{lg}(\mathsf{L})} = \|\cdot\|_{\mathsf{L}}$ (it is easily shown that $\mathsf{SEN}_{\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}}$ is a functor, and hence $\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}$ is an π-institution).

Warning 6.47 $\mathsf{sig}(\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}) = \{\mathbf{lg}(\mathsf{L})\} \neq \{\mathfrak{s}\}$.

Remark 6.48 $\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}$ is structural iff L is $\mathfrak{s}$-structural. □

Note that the π-institution $\mathcal{I}^{\xrightarrow{\mathfrak{a}}[n]}_{\mathcal{S}}$ associated with a sentential n-calculus $\mathcal{S}$ viewed as a $\xrightarrow{\mathfrak{a}}_{[n]}$-logic, differs from the π-institution $\mathcal{I}_{\mathcal{S}}$ associated with $\mathcal{S}$ as given in [Vou03]. In the latter definition, $\mathbf{Sign}_{\mathcal{I}_{\mathcal{S}}}$ is the one object category consisting of the single object V, where V is the set of *scalar* variables, and the morphisms are all functions from V into Tm (the category composition is just functional composition and the category identity associated with V is the *inclusion* map), $\mathsf{SEN}_{\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}}$ maps V to $\mathsf{Fm}(\mathcal{S}) = \mathsf{Tm}^{n}$ and maps each $f : \mathsf{V} \to P$ to $\underset{\to[n]}{f'}$, where f'

is the unique $\mathbf{Tm}$-endomorphism extending f, and the closure operator is the same for $\mathcal{I}_{\mathcal{S}}^{\overset{\mathfrak{a}}{\rightarrow}[n]}$. We cannot adopt this approach, because we permit non-global $\mathfrak{s}$-logics and hence generally have no variable set. In the case of sentential calculi, the two approaches are essentially the same.

Let $\mathbf{G}$ be a global $\mathfrak{s}$-language, $\mathcal{D}$ a logic on $\mathbf{G}$ and x an $\mathfrak{s}$-variable of $\mathbf{G}$.

Remark 6.49 $\langle \mathbf{G}, x\rangle$ is a source signature - variable pair for $\mathcal{I}_{\mathcal{D}}^{\mathfrak{s}}$. Consequently, $\mathcal{I}_{\mathcal{D}}^{\mathfrak{s}}$ is term.

Proof. For each $\phi \in \mathsf{Sent}_{\mathcal{I}_{\mathcal{D}}^{\mathfrak{s}}}(\mathbf{G}) = \mathsf{Fm}(\mathbf{G})$, let $\sigma_{\langle \mathbf{G},\phi\rangle}$ be the unique $\mathfrak{s}$-endomorphism of $\mathbf{G}$ mapping all $\mathfrak{s}$-variables of $\mathbf{G}$ to ϕ. By definition, $\sigma_{\langle \mathbf{G},\phi\rangle} : \mathbf{G} \rightarrow_{\mathsf{Sign}_{\mathcal{I}_{\mathcal{D}}^{\mathfrak{s}}}} \mathbf{G}$, and $\overline{\sigma_{\langle \mathbf{G},\phi\rangle}}(x) = \sigma_{\langle \mathbf{G},\phi\rangle}(x) = \phi$. Let $\rho : \mathbf{G} \rightarrow_{\mathsf{Sign}_{\mathcal{I}_{\mathcal{D}}^{\mathfrak{s}}}} \mathbf{G}$, i.e., ρ is an $\mathfrak{s}$-endomorphism of $\mathbf{G}$. (It suffices to show that $\rho\sigma_{\langle \mathbf{G},\phi\rangle} = \sigma_{\langle \mathbf{G},\rho(\phi)\rangle}$.) For any $\mathfrak{s}$-variable y of $\mathbf{G}$, $\rho\sigma_{\langle \mathbf{G},\phi\rangle}(y) = \rho(\phi) = \sigma_{\langle \mathbf{G},\rho(\phi)\rangle}(y)$. The result follows by $\mathfrak{s}$-freedom of $\mathbf{G}$. $\diamond$

$\square$

6.4 Examples

6.4.1 Languages and Signatures

We now consider examples of signatures that are important to the sequel, beginning with signatures of *universal logics*. Recall that we may conflate the type of algebras $\mathfrak{a}$ with the variety of all $\mathfrak{a}$-algebras.

Example 6.50 (Languages and Signatures of Universal Logics)

A **language of universal logics** is a universal algebra. This notion coincides with Font and Jansana's (not explicitly defined) language of a **universal logic** [FJ96]. When we call a type of algebra $\mathfrak{a}$ a **signature of universal logics**, we are conflating the type $\mathfrak{a}$ with the construct of all $\mathfrak{a}$-algebras with homomorphisms. Signatures of sentential calculi of dimension 1 (essentially) coincide with signatures of universal logics. We prefer to use the word 'universal' rather than 'sentential' in this context, since we permit logics on arbitrary $\mathfrak{a}$-algebras and not just the term algebra.

If $\mathfrak{a}$ is a signature of universal logics, the global language is the term algebra on ω-free generators. Unless specified to the contrary, the variables are taken to be V.

$\square$

This usefulness of logics over constructs given for our needs, is that we may treat a class of algebras, closed under subalgebras, as a signature of logics; the languages are the algebras of the class and the interpretations are the homomorphisms between these algebras. The requirement of closure under subalgebras is required to ensure that the class is closed under homomorphic images of homomorphisms *between algebras of the class*; which ensures that this construct is well-defined. Our reason for choosing all homomorphisms between algebras of the class $\mathcal{K}$ and which then *forces* us to require that $\mathcal{K}$ be closed under subalgebras, *rather* than simply selecting only the *surjective* homomorphisms between algebras of the class, is that we want the construct $\mathcal{K}$ to be a *full* subconstruct of the variety of all algebras of the type; this is necessary for the development of our theory of *canons* and their induced *ideals* (see $\oint 8$). In the sequel, the only classes of interest are quasivarieties; these are closed under subalgebras (see Theorem 1.447 on page 87).

Example 6.51 (Classes of Algebras as Signatures)

Let $\mathcal{K}$ be a class of algebras of the same type that is *closed under subalgebras*, i.e., $\mathcal{S}(\mathcal{K}) \subseteq \mathcal{K}$.

Definition 6.52 (The Signature of Logics $\mathcal{K}$) When we speak of the **signature of logics** $\mathcal{K}$, we mean the construct of all algebras in $\mathcal{K}$ with homomorphisms (between algebras of $\mathcal{K}$). This terminology is *only* defined for classes closed under subalgebras and any usage of this terminology shall *implicitly* imply that $\mathcal{K}$ is closed under subalgebras. $\square$

Remark 6.53 The signature of logics $\mathcal{K}$ is a full subconstruct of the construct of all $\mathfrak{a}$-algebras. $\square$

Typically, $\mathcal{K}$ will be a quasivariety. That $\mathcal{K}$ may not be *closed under homomorphic images* (for example when $\mathcal{K}$ is a quasivariety) is of no consequence, since this simply means that there exist some $\mathfrak{a}$-algebras *outside* of $\mathcal{K}$ that are the surjective images of $\mathfrak{a}$-homomorphisms from algebras *in* $\mathcal{K}$ (these being precisely the algebras isomorphic to the factorization of some algebra in $\mathcal{K}$ by a congruence that is not a $\mathcal{K}$-congruence). The requirement of closure under subalgebras is sufficient to ensure that $\mathcal{K}$ is closed under homomorphic images of homomorphisms between algebras *of* $\mathcal{K}$.

When viewing a class $\mathcal{K}$ of $\mathfrak{a}$-algebras as a signature of logics, the global language, if it exists, is the $\mathcal{K}$-free algebra on ω-free generators, which we assume to be $\overline{[\mathsf{V}]}$. Substitutions are endomorphisms of this free algebra. Quasivarieties always contain a global language.

$\square$

Locating sentential n-calculi in our discourse is less straight forward, given the partial 'vector' nature of these systems. The approach that we shall take adopts the perspective that each n-formula denotes a 'propositional object', rather than taking the standard perspective, where each (one) formula denotes a 'propositional object', and an n-formula relates n 'propositional objects'. Instead of parameterizing deductive systems, we have a single notion of deductive system (over a construct), and distinguish sentential n-calculi by their underlying constructs. We informally speak of '*flattening*' the theory of sentential n-calculi. This approach is entirely consistent with the modelling of sentential n-calculi as π-institutions [Vou03].

Example 6.54 (Signatures of n-Logics over $\mathfrak{s}$)

Before we demonstrate how to locate sentential n-calculi as logics over constructs, we require some definitions and results in construct theory.

Definition 6.55 (Wellfounded Roots) Let $\mathfrak{s}$ be a construct and $\mathbf{m}$ a non-zero cardinal. We say that construct $\mathfrak{s}$ has **wellfounded m-roots**, if, whenever $\mathbf{B}^{\mathbf{m}}$ and $\mathbf{C}^{\mathbf{m}}$ both exist and $\mathbf{B}^{\mathbf{m}} = \mathbf{C}^{\mathbf{m}}$, then $\mathbf{B} = \mathbf{C}$. We say that $\mathfrak{s}$ **yields m-vectors** if $\mathfrak{s}$ has unique $\mathbf{m}$-powers (i.e., they exist and are unique) and wellfounded $\mathbf{m}$-roots. An $\mathfrak{s}$-object $\mathbf{A}$ is said to **have root m**, if there exists a *unique* object, denoted $\sqrt[\mathbf{m}]{\mathbf{A}}$, with $\mathbf{A} = (\sqrt[\mathbf{m}]{\mathbf{A}})^{\mathbf{m}}$. $\square$

Remark 6.56 If $\mathfrak{s}$ yields $\mathbf{m}$-vectors, then for every $\mathfrak{s}$-object $\mathbf{A}$, $\mathbf{A}^{\mathbf{m}}$ exists, $\sqrt[\mathbf{m}]{\mathbf{A}^{\mathbf{m}}}$ is well-defined and equal to $\mathbf{A}$, $(\sqrt[\mathbf{m}]{\mathbf{A}^{\mathbf{m}}})^{\mathbf{m}} = \mathbf{A}^{\mathbf{m}}$, and $\sqrt[\mathbf{m}]{\mathbf{A}^{\mathbf{m}}}$ is the only $\mathfrak{s}$-object $\mathbf{B}$ such that $\mathbf{B}^{\mathbf{m}} = \mathbf{A}^{\mathbf{m}}$.

Definition 6.57 ('Vector' Constructs) For a construct $\mathfrak{s}$ and non-zero cardinal $\mathbf{m}$ such that $\mathfrak{s}$ yields $\mathbf{m}$-vectors, let $\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}$ denote the construct consisting of all objects $\mathbf{A}^{\mathbf{m}}$, where $\mathbf{A}$ is an $\mathfrak{s}$-object, and with $\mathbf{A}^{\mathbf{m}} \rightarrow_{\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}} \mathbf{B}^{\mathbf{m}} = \{ \underrightarrow{f}_{[\mathbf{m}]} : f : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}\}$, which we call the **construct of m-vectors over** $\mathfrak{s}$. When the particular cardinal is clear from the context, we write $\underrightarrow{\mathbf{A}}$ for $\mathbf{A}^{\mathbf{m}}$ and $\underrightarrow{f}$ for $\underrightarrow{f}_{[\mathbf{m}]}$. Arbitrary $\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}$-objects are denoted by $\underrightarrow{\mathbf{A}}$, $\underrightarrow{\mathbf{B}}$ and $\underrightarrow{\mathbf{C}}$, etc, from which one can assume that $\sqrt[\mathbf{m}]{\underrightarrow{\mathbf{A}}} = \mathbf{A}$, etc. Arbitrary $\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}$-morphisms from $\underrightarrow{\mathbf{A}}$ into $\underrightarrow{\mathbf{B}}$ are *either* denoted by $\underrightarrow{f}$, $\underrightarrow{g}$, etc., from which one may assume that $f : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$ and $\underrightarrow{f}_{[\mathbf{m}]} = \underrightarrow{f}$, etc, *or* by emboldened $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, etc. The latter notation is used when a promotable 'scalar' morphism is not directly apparent. For a $\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}$-morphism $\boldsymbol{f} : \underrightarrow{\mathbf{A}} \rightarrow_{\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}} \underrightarrow{\mathbf{B}}$, let $\sqrt[\mathbf{m}]{\boldsymbol{f}}$ denote the unique $\mathfrak{s}$-morphism from $\sqrt[\mathbf{m}]{\underrightarrow{\mathbf{A}}}$ into $\sqrt[\mathbf{m}]{\underrightarrow{\mathbf{B}}}$ with $\underrightarrow{\sqrt[\mathbf{m}]{\boldsymbol{f}}} = \boldsymbol{f}$. Conventionally, when we speak of the **construct** $\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}$, we are *implicitly* assuming that $\mathfrak{s}$ yields $\mathbf{m}$-vectors. □

Proof. (*We need to establish that* $\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}$ *is a well-defined construct.*) $\boxed{\text{Identity}}$ Let $\mathbf{A}^{\mathbf{m}}$ be a $\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}$-object. Then by definition, $\mathbf{A}$ is an $\mathfrak{s}$-object. So $\mathsf{id}_{\mathsf{uni}(\mathbf{A})}$ is an $\mathfrak{s}$-endomorphism of $\mathbf{A}$. So $\underrightarrow{\mathsf{id}_{\mathsf{uni}(\mathbf{A})}}_{[\mathbf{m}]}$ is an $\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}$-endomorphism of $\mathbf{A}^{\mathbf{m}}$. Finally, $\underrightarrow{\mathsf{id}_{\mathsf{uni}(\mathbf{A})}}_{[\mathbf{m}]} = \mathsf{id}_{\mathsf{uni}(\mathbf{A}^{\mathbf{m}})}$, by Remark 1.92 on page 27. $\boxed{\text{Composition}}$ Suppose that $\boldsymbol{f} : \mathbf{A}^{\mathbf{m}} \rightarrow_{\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}} \mathbf{B}^{\mathbf{m}}$ and $\boldsymbol{g} : \mathbf{B}^{\mathbf{m}} \rightarrow_{\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}} \mathbf{C}^{\mathbf{m}}$. Then $\boldsymbol{f} = \underrightarrow{f}_{[\mathbf{m}]}$ and $\boldsymbol{g} = \underrightarrow{g}_{[\mathbf{m}]}$ for some $f : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$ and $g : \mathbf{B} \rightarrow_{\mathfrak{s}} \mathbf{C}$. Now, $\boldsymbol{g}\boldsymbol{f} = \underrightarrow{g}_{[\mathbf{m}]}\underrightarrow{f}_{[\mathbf{m}]} = \underrightarrow{gf}_{[\mathbf{m}]} \in \mathbf{A}^{\mathbf{m}} \rightarrow_{\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}} \mathbf{C}^{\mathbf{m}}$, the final equality following by Remark 1.92 on page 27. ◇

Remark 6.58 If $\underrightarrow{f} : \underrightarrow{\mathbf{A}} : \rightarrow_{\underrightarrow{\mathfrak{s}}_{[\mathbf{m}]}} \underrightarrow{\mathbf{B}}$ and $\mathsf{a} \in \mathsf{uni}(\underrightarrow{\mathbf{A}})$, then for all $i, j \in \mathbf{m}$, if $\mathsf{a}_{(i)} = \mathsf{a}_{(j)}$, then $\underrightarrow{f}(\mathsf{a})_{(i)} = \underrightarrow{f}(\mathsf{a})_{(j)}$. □

We are now in a position to introduce the *signature of n-logics over* $\mathfrak{s}$, where $\mathfrak{s}$ is a signature and n any positive integer such that $\mathfrak{s}$ yields n-vectors. Let $\mathfrak{s}$ be a signature of (languages of) logics and n any positive integer such that $\mathfrak{s}$ yields n-vectors.

Definition 6.59 (Signatures of n-Logics) We call $\underrightarrow{\mathfrak{s}}_{[\mathrm{n}]}$ the **signature of n-logics over** $\mathfrak{s}$. We call an arbitrary signature $\mathfrak{t}$ a **signature of n-logics over** $\mathfrak{s}$, if $\mathfrak{t} = \underrightarrow{\mathfrak{s}}_{[\mathrm{n}]}$. Conventionally, any use of the $\underrightarrow{\mathfrak{s}}_{[\mathrm{n}]}$ shall *implicitly* imply that $\mathfrak{s}$ yields n-vectors. □

Discussion 6.60 (Variables of Sentential n-Calculi) In the definition of a *sentential n-calculus*, Blok and Pigozzi essentially begin with the $\mathfrak{a}$-language of a (one) deductive system $\mathbf{Tm}$ on ω-free generators, the elements of which are called **formulae**, from which a new 'language' is formed, whose elements are n-tuples of (one) formulae, and which are called **n-formulae** [BP92]. Given this approach, what is one to make of 'variables', and consequently, what is one to take as the *language* of such a logic?

The most common approach, consistent with [BP92] and, importantly, consistent with [Elg98] (which is essentially a 'complete' generalization of Blok and Pigozzi's theory, from structures with one relation symbol, to structures generally), is to view (one) formulae as denoting 'propositional things', and to see n-formulae as describing 'n-ary relations' between 'propositional things'. Consequently, n-substitutions, that is substitutions of n-formulae yielding n-formulae, are determined by maps from (one) variables to (one) formulae. When adopting this approach, 'n-variables' are not mentioned [BP92].

Some have attempted to introduce **n-variables**. For example, in [vA95], an n-variable is defined as an n-formula, *all of whose co-ordinates are* $\mathbf{Tm}$*-variables*. This notion is problematic. In particular, the set of n-variables do not form a set of free generators for the set of n-formulae (with respect to its n-substitutions). Consider, for example, the set of all 2-formulae (of type $\mathfrak{a}$) and consider the 'variable' $\langle x, x\rangle$. There is (generally) no 2-substitution extending the map $\langle x, x\rangle \mapsto \langle p, q\rangle$, for distinct one formulae p and q. Of course, n-substitutions are determined by maps from one variables to one formulae. $\square$

In the following series of results, we aim to show that if $\mathbf{F}$ is an $\mathfrak{s}$-free object, then $\underset{\rightarrow}{\mathbf{F}}$ is a $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$-free object, and aim to describe the free-generators of $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$. We begin with a technical observation.

Lemma 6.61 If $\underset{\rightarrow}{\mathbf{A}}$ is a $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$-free object on $\mathbf{l}$ free generators $\mathtt{V}$, then $\forall\,[\mathtt{x}, \mathtt{y} \in \mathtt{V}]\;\forall\,[i, j \in \mathbf{m}]\;\mathtt{x}_{(i)} = \mathtt{y}_{(j)} \leftrightarrow \mathtt{x} = \mathtt{y}$.

Proof. $\boxed{\rightarrow}$ Let $\mathtt{x}, \mathtt{y} \in \mathtt{V}$ and $i, j \in \mathbf{m}$, such that $\mathtt{x}_{(i)} = \mathtt{y}_{(j)}$. Suppose, to the contrary, that $\mathtt{x} \neq \mathtt{y}$. Let $\mathtt{a}, \mathtt{b} \in \mathsf{uni}(\mathbf{A})^{\mathbf{m}}$ with $\mathtt{a}_{(i)} \neq \mathtt{b}_{(j)}$. Let $\underset{\rightarrow}{\sigma}$ be any endomorphism of $\underset{\rightarrow}{\mathbf{A}}$ mapping $\mathtt{x} \mapsto \mathtt{a}$ and $\mathtt{y} \mapsto \mathtt{b}$. Then $\underset{\rightarrow}{\sigma}(\mathtt{x})_{(i)} = \mathtt{a}_{(i)} \neq \mathtt{b}_{(j)} = \underset{\rightarrow}{\sigma}(\mathtt{y})_{(j)}$. Hence $\sigma(\mathtt{x}_{(i)}) \neq \sigma(\mathtt{y}_{(j)})$, while $\mathtt{x}_{(i)} = \mathtt{y}_{(j)}$, which is (set-theoretically) impossible. So $\mathtt{x} = \mathtt{y}$, and hence $\mathtt{x}_{(i)} = \mathtt{x}_{(j)}$. $\boxed{\leftarrow}$ Trivial $\diamond$

We now characterize the $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$-free objects on $\mathbf{l}$ free generators in terms of the $\mathfrak{s}$-free objects on $\mathbf{lm}$ free generators, where $\mathbf{lm}$ is the cardinal product of $\mathbf{l}$ and $\mathbf{m}$.

Theorem 6.62 Let $\mathfrak{s}$ be a construct that yields $\mathbf{m}$-vectors.

1. Suppose that $\mathbf{F}$ is an $\mathfrak{s}$-free object on $\mathbf{lm}$ free generators V, for some cardinal $\mathbf{l}$. For any bijection $x. : \mathbf{l} \times \mathbf{m} \Rightarrow V$, $\underset{\rightarrow}{\mathbf{F}}$ is a $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$-free object on $\mathbf{l}$ free generators $\{\mathtt{x}_i \;:\; i \in \mathbf{l}\}$, where $\mathtt{x}_i$ is the unique element in $V^{\mathbf{m}}$ with $(\mathtt{x}_i)_{(j)} = x_{\langle i,j\rangle}$, for each $j \in \mathbf{m}$.
2. If $\underset{\rightarrow}{\mathbf{A}}$ is a $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$-free object on $\mathbf{l}$ free generators $\mathtt{V}$, then $\mathbf{A}$ is an $\mathfrak{s}$-free object on $\mathbf{lm}$ free generators $\{\mathtt{x}_{(i)} \;:\; \mathtt{x} \in \mathtt{V}, i \in \mathbf{m}\}$.
3. Construct $\mathfrak{s}$ contains a free-object on $\mathbf{lm}$ free generators iff $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$ contains a free-object on $\mathbf{l}$ free generators.

Proof. $\boxed{(1)}$ Let $\mathbf{F}$ be an $\mathfrak{s}$-object with $\mathbf{lm}$-variables V. Since $\mathbf{n} = \mathbf{lm}$, there exists a bijection $x. : \mathbf{l} \times \mathbf{m} \Rightarrow V$. For each $i \in \mathbf{l}$, let $\mathtt{x}_i$ denote the unique element in $V^{\mathbf{m}}$ with $\mathtt{x}_{i\,(j)} = x_{\langle i,j\rangle}$, for each $j \in \mathbf{m}$. Let $\mathtt{W} = \{\mathtt{x}_i \;:\; i \in \mathbf{l}\}$. Note the following.

- $\mathsf{card}(\mathtt{W}) = \mathbf{l}$.
- For each $\mathbf{F}$-variable $x_{\langle i,j\rangle}$, there exists precisely one $\mathtt{x} \in \mathtt{W}$ with $x_{\langle i,j\rangle}$ appearing as a co-ordinate of $\mathtt{x}$, namely $\mathtt{x}_i$.
- $x_{\langle i,j\rangle}$ appears in only one of the co-ordinates of $\mathtt{x}_i$, namely co-ordinate j.
- For all $\mathtt{x} \in \mathtt{W}$ and $k \in \mathbf{m}$, $\mathtt{x}_{(k)} = x_{\langle i,j\rangle}$ iff $\mathtt{x} = \mathtt{x}_i$ and $k = j$.

Let $\underset{\rightarrow}{\mathbf{A}}$ be any $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$-object and let $f : \mathtt{W} \rightarrow \mathsf{uni}(\underset{\rightarrow}{\mathbf{A}})$. Let g be the unique $\mathfrak{s}$-morphism of $\mathbf{F}$ into $\mathbf{A}$ extending the (well-defined) function from V to $\mathsf{uni}(\mathbf{A})$ mapping $x_{\langle i,j\rangle} \mapsto f(\mathtt{x}_i)_{(j)}$. Consider the $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$-morphism $\underset{\rightarrow}{g}$ from $\underset{\rightarrow}{\mathbf{F}}$ into $\underset{\rightarrow}{\mathbf{A}}$.
(*It suffices to show that* $\underset{\rightarrow}{g}$ *is the unique morphism from* $\underset{\rightarrow}{\mathbf{F}}$ *into* $\underset{\rightarrow}{\mathbf{A}}$ *such that* $\underset{\rightarrow}{g}_{|\mathtt{W}} = f$.)
$\boxed{\underset{\rightarrow}{g}_{|\mathtt{W}} = f}$ Let $\mathtt{v}_i \in \mathtt{W}$. For each $j \in \mathbf{m}$, $\underset{\rightarrow}{g}(\mathtt{x}_i)_{(j)} = g(\mathtt{x}_{i\,(j)}) = g(x_{\langle i,j\rangle}) = f(\mathtt{x}_i)_{(j)}$, and so $\underset{\rightarrow}{g}(\mathtt{x}_i) = f(\mathtt{x}_i)$, as required. $\boxed{\text{Uniqueness}}$ Suppose that $\underset{\rightarrow}{h} : \underset{\rightarrow}{\mathbf{F}} \rightarrow_{\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}} \underset{\rightarrow}{\mathbf{A}}$, with $\underset{\rightarrow}{h}(\mathtt{x}_i) = f(\mathtt{x}_i)$, for all $i \in \mathbf{l}$. Then, for all $x_{\langle i,j\rangle} \in V$, $h(x_{\langle i,j\rangle}) = \underset{\rightarrow}{h}(\mathtt{x}_i)_{(j)} = f(\mathtt{x}_i)_{(j)} = g(x_{\langle i,j\rangle})$. Since h and g agree on the variables of $\mathbf{F}$, $h = g$, and so $\underset{\rightarrow}{h} = \underset{\rightarrow}{g}$. $\boxed{(2)}$ Suppose that $\underset{\rightarrow}{\mathbf{A}}$ is a $\underset{\rightarrow}{\mathfrak{s}}_{[\mathbf{m}]}$-free object

on $\mathbf{l}$-free generators $\mathtt{W}$. By Lemma 6.61, the set $V \doteq \{\mathtt{x}_{(i)} : \mathtt{x} \in \mathtt{W}, i \in \mathbf{m}\}$ has cardinality $\mathbf{lm}$. Let $\mathbf{B}$ be any $\mathfrak{s}$-object and let $f : V \to \mathsf{uni}(\mathbf{B})$. Let $\underset{\to}{f'}$ denote the unique $\underset{\to}{\mathfrak{s}}_{[\mathbf{m}]}$-morphism from $\underset{\to}{\mathbf{A}}$ into $\underset{\to}{\mathbf{B}}$ mapping $\mathtt{x} \mapsto \langle f(\mathtt{x}_{(i)}) : i \in \mathbf{m}\rangle$, for each $\mathtt{x} \in \mathtt{W}$. By construction, for all $\mathtt{x} \in \mathtt{W}$ and $i \in \mathbf{m}$, $\underset{\to}{f'}(\mathtt{x})_{(i)} = f(\mathtt{x}_{(i)})$. By definition, f' is an $\mathfrak{s}$-morphism from $\mathbf{A}$ into $\mathbf{B}$.
(*It suffices to show that* f' *is the unique* $\mathfrak{s}$*-morphism from* $\mathbf{A}$ *into* $\mathbf{B}$ *with* $f'_{|V} = f$.) $\boxed{f'_{|V} = f}$ For all $\mathtt{x} \in \mathtt{W}$ and $i \in \mathbf{m}$, $f'(\mathtt{x}_{(i)}) = \underset{\to}{f'}(\mathtt{x})_{(i)} = f(\mathtt{x}_{(i)})$. So $f'_{|V} = f$. $\boxed{\text{Uniqueness}}$ Let $g : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$ with $g_{|V} = f$. Then, for all $\mathtt{x} \in \mathtt{W}$ and $i \in \mathbf{m}$, $\underset{\to}{g}(\mathtt{x})_{(i)} = g(\mathtt{x}_{(i)}) = f(\mathtt{x}_{(i)}) = f'(\mathtt{x}_{(i)}) = \underset{\to}{f'}(\mathtt{x})_{(i)}$. Hence, $\underset{\to}{g}(\mathtt{x}) = \underset{\to}{f'}(\mathtt{x})$, for all $\mathtt{x} \in \mathtt{W}$, and so by the assumed freedom of $\underset{\to}{\mathbf{A}}$, $\underset{\to}{g} = \underset{\to}{f'}$. So by definition, $g = f'$. $\boxed{(3)}$ Follows immediately from (1) and (2). ◇

We are most interested in describing the $\underset{\to}{\mathfrak{s}}_{[n]}$-free objects on ω free generators in terms of the $\mathfrak{s}$-free objects on ω free generators, where n is a non-zero natural number; the following characterization obtains immediately, since $\omega = n\omega$.

Corollary 6.63 Let $\mathfrak{s}$ be a construct that yields n-vectors.

1. If $\mathbf{F}$ is an $\mathfrak{s}$-free object on ω free generators $\{x_1, x_2, \ldots\}$ where $x_1, x_2, \ldots$ are all distinct, then $\underset{\to}{\mathbf{F}}$ is a $\underset{\to}{\mathfrak{s}}_{[n]}$-free object on ω free generators $\{\langle x_{i+1}, x_{i+2}, \ldots, x_{i+n}\rangle : i = 0, 1, \ldots\}$.
2. If $\underset{\to}{\mathbf{A}}$ is a $\underset{\to}{\mathfrak{s}}_{[n]}$-free object on ω free generators $\mathtt{V}$, then $\mathbf{A}$ is an $\mathfrak{s}$-free object on ω free generators $\{\mathtt{x}_{(i)} : \mathtt{x} \in \mathtt{V}, i \in n\}$.
3. There exists an $\mathfrak{s}$-free object on ω free generators iff there exists a $\underset{\to}{\mathfrak{s}}_{[n]}$-free object on ω free generators.

□

If $\mathfrak{s}$ has a global language $\mathbf{G}$ and if $\underset{\to}{\mathfrak{s}}_{[\mathrm{n}]}$ is well-defined, then $\underset{\to}{\mathbf{G}}$ is a global $\underset{\to}{\mathfrak{s}}_{[\mathrm{n}]}$-language with variables $\{\langle \mathsf{v}_{kn}, \ldots, \mathsf{v}_{kn+n-1}\rangle : k \geq 0\}$, by Theorem 6.62 on page 238.

Convention 6.64 (Variables of $\underset{\to}{\mathfrak{s}}_{[\mathrm{n}]}$-Global Languages) We shall denote the variable $\langle \mathsf{v}_{kn}, \ldots, \mathsf{v}_{kn+n-1}\rangle$ by $\mathtt{v}_k$, for each $k = 0, 1, \ldots$, and denote the set of all these variables by $\mathtt{V}$.

Note 6.65 (Determining Interpretations) Interpretations from $\underset{\to}{\mathbf{G}}$ into $\underset{\to}{\mathbf{A}}$ are determinable by two means, either *directly* by uniquely extending a mapping from $\mathtt{V}$ into $(\mathsf{uni}(\mathbf{A}))^n$, or *indirectly* by 'promoting' the extension of a mapping from V into $\mathsf{uni}(\mathbf{A})$, that is, finding the unique $\mathfrak{s}$-morphism f extending a mapping from V into $\mathsf{uni}(\mathbf{A})$, and then obtaining $\underset{\to}{f}$. □

□

In the next example we consider signatures of *universal n-logics*. These (in essence) coincide with signatures of sentential n-calculi. Recall that when we defined the signatures of n-sentential calculi, we included the dimension in the signature and distinguished the signature from the type, noting that this was non-standard practice (see Definition 2.1 on page 94 and the subsequent warning),

Example 6.66 (Signatures of Universal n-Logics)

Remark 6.67 For a type of algebras $\mathfrak{a}$, the construct of all $\mathfrak{a}$-algebras yields $\mathbf{m}$-vectors.

Definition 6.68 (Signatures of Universal n-Logics) Let $\mathfrak{a}$ be a type of algebras. The signature $\underset{\rightarrow}{\mathfrak{a}}_{[\mathrm{n}]}$ is called the **signature of universal n-logics** over $\mathfrak{a}$. □

The global $\underset{\rightarrow}{\mathfrak{a}}_{[\mathrm{n}]}$-language is $\mathbf{Tm}^n$.

Convention 6.69 (Variables and Formulae of $\mathbf{Tm}^n$) When dealing with signatures of universal n-logics, when we speak of a **1-variable** we mean an $\mathfrak{a}$-variable, i.e., a variable in V, whereas a variable shall mean an element of $\mathtt{V}$ (see the previous example), which we tend to call a **n-variable** as an aid to the reader. In this context, **terms** means elements of Tm and global formulae mean elements of Tm^n. A global $\underset{\rightarrow}{\mathfrak{a}}_{[\mathrm{n}]}$-formula $\boldsymbol{\phi} \in \mathsf{Tm}^n$ is of the form $\langle p_1, \ldots, p_n \rangle$, where $p_1, \ldots, p_n \in \mathsf{Tm}$; in this case we call $p_1, \ldots, p_n$ the **terms of $\boldsymbol{\phi}$**. Since the number of terms in global formula is finite, we may assume that the same 1-variables occur in all the terms. When we write $\boldsymbol{\phi}(x_1, \ldots, x_n)$, where $\boldsymbol{\phi} \in \mathsf{Tm}^n$, we mean that $x_1, \ldots, x_n$ are the 1-variables occurring in the terms of $\boldsymbol{\phi}$.

□

The following example serves to further justify our inclusion of *local* languages, rather than purely formal languages (which we shall call *global*), as suitable languages of logics.

Example 6.70 (The Language of Boolean Logics) [HG98]

Halmos and Givant define a **pre-Boolean algebra** to be any algebra of *type* BA, where BA is the type of boolean algebras. In [HG98], pre-Boolean algebras are viewed as logics (see Example 6.111 of our text).

□

6.4.2 Logics

Example 6.71 (Sentential Calculi)

Let $\mathfrak{a}$ be a type of algebras and n a non-zero natural. The notion of a propositional $\mathfrak{a}$-calculus (resp. propositional $\underset{\rightarrow}{\mathfrak{a}}_{[n]}$-calculus) coincides with the notion of a sentential 1-calculus (sentential n-calculus) given in ∮2, i.e., a 1-deductive system (resp. sentential n-calculus) in the sense of [BP89a] and [vA95, 51]. We shall tend to use the word 'sentential' rather than 'propositional' in the case of such calculi, so as to distinguish these from our notion of a propositional calculus over a construct.

□

Example 6.72 (Propositional $\mathcal{K}$-Calculi)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras. Considering $\mathcal{K}$ as the construct of all algebras in $\mathcal{K}$ together with $\mathfrak{a}$-homomorphisms, we obtain the notion of a *propositional $\mathcal{K}$-calculus*. We call such calculi **canonical** if its variables are the globally chosen denumerably infinite set of variables V.

□

Note that canonical propositional $\mathcal{K}$-calculi are *not* sentential 1-calculi, since these are not logics over the term algebra. Each formula of a $\mathcal{K}$-calculus is an element of a free $\mathcal{K}$-algebra, and hence is a set of terms, rather than a single term. Propositional $\mathcal{K}$-calculi occur commonly in universal algebra. In ∮8 we present machinery for inducing sentential 1-calculi from propositional

$\mathcal{K}$-calculi, a technique that underlies the logics that we have been discovering, such as the logic of membership, that tend to be inherently unalgebraizable in the sense of [BP89a], but for which our more general notion of parametrized algebraizability applies.

Recall that for a quasivariety $\mathcal{K}$, the $\mathcal{K}$-relative congruences $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ on any algebra $\mathbf{A}$ (of the same type of $\mathcal{K}$) form a *finitary* closed system (see Proposition 1.450 on page 87). Recall further the definitions of the sentential 2-calculi $S^2(\mathfrak{a}, \equiv)$, $S^2(\mathfrak{a}, \Theta)$ and $S^2(\Theta^{\mathcal{K}})$, given in Definition 2.74 of Example 2.74 on page 105. In the following example we introduce the logics of congruences and relative congruences on an algebra. As an application of our theory of *canons and their ideals* developed in $\oint 8$, we shall show how the logic $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$ of relative congruences on the free algebra, which is a *canon*, has the sentential 2-calculus $S^2(\Theta^{\mathcal{K}})$ as its *ideal* (see Example 8.57 on page 294).

Example 6.73 (The Relative Congruence Logics)

Let $\mathfrak{a}$ be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra on ω-free generators.

Definition 6.74 (The Relative Congruence Logics) For an $\mathfrak{a}$-algebra $\mathbf{A}$, not necessarily in $\mathcal{K}$, let $U^2(\mathbf{A}, \Theta^{\mathcal{K}})$ denote the *finitary* $\mathbf{A}^2$-logic $L\left(\mathbf{A}^2, \mathsf{Con}^{\mathcal{K}}(\mathbf{A})\right)$. We write $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$ for $U^2(\mathbf{F}_{\mathcal{K}}, \Theta^{\mathcal{K}})$. Viewing $\mathfrak{a}$ as the variety of all $\mathfrak{a}$-algebras, we may write $\mathfrak{a}$ for $\mathcal{K}$ in these definitions. $\square$

Remark 6.75 $S^2(\Theta^{\mathcal{K}}) \equiv U^2(\mathbf{Tm}, \Theta^{\mathcal{K}})$ and $S^2(\mathfrak{a}, \Theta) \equiv U^2(\mathbf{Tm}, \Theta^{\mathfrak{a}})$.

Proof. By Example 2.77 on page 105, $\mathsf{Fi}_{S^2(\Theta^{\mathcal{K}})}(\mathbf{A}) = \mathsf{Con}^{\mathcal{K}}(\mathsf{uni}(\mathbf{A}))$, and that, by Corollary 2.53 on page 102, $\mathsf{Fi}_{S^2(\Theta^{\mathcal{K}})}(\mathbf{Tm}) = \mathsf{Th}(S^2(\Theta^{\mathcal{K}}))$. Since theories determine a sentential calculus up to equivalence, $S^2(\Theta^{\mathcal{K}}) \equiv U^2(\mathbf{Tm}, \Theta^{\mathcal{K}})$. By a similar argument, $S^2(\mathfrak{a}, \Theta) \equiv U^2(\mathbf{Tm}, \Theta^{\mathfrak{a}})$. $\diamond$

The following characterization of consequence in $S^2(\Theta^{\mathcal{K}})$ and $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$, in terms of $\models_{\mathcal{K}}$, follows from the definitions and Lemma 1.457 on page 88.

Corollary 6.76 For $\{\langle p_i, q_i\rangle \,:\, i \in I\} \subseteq \mathsf{Tm}^2$, the following conditions are equivalent.

1. $\{\langle p_i, q_i\rangle \,:\, i \in I\} \vdash_{S^2(\Theta^{\mathcal{K}})} \langle p, q\rangle$.
2. $\{\langle \overline{p_i}, \overline{q_i}\rangle \,:\, i \in I\} \vdash_{\boldsymbol{S}^2(\Theta^{\mathcal{K}})} \langle \overline{p}, \overline{q}\rangle$.
3. $\{p_i \approx q_i \,:\, i \in I\} \models_{\mathcal{K}} p \approx q$.

Remark 6.77 $S^2(\mathfrak{a}, \equiv) \preceq S^2(\mathfrak{a}, \Theta) \preceq S^2(\Theta^{\mathcal{K}})$. $\square$

We now consider the logics $U^2(\mathbf{A}, \Theta^{\mathcal{K}})$, for $\mathbf{A}$ *in* $\mathcal{K}$, as $\underset{\rightarrow[2]}{\mathcal{K}}$-logics.

Proposition 6.78 For $\mathbf{A} \in \mathcal{K}$, $U^2(\mathbf{A}, \Theta^{\mathcal{K}})$ is $\underset{\rightarrow[2]}{\mathcal{K}}$-structural. Consequently, $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$ is a propositional $\underset{\rightarrow[2]}{\mathcal{K}}$-calculus.

Proof. Let $\mathbf{A} \in \mathcal{K}$ and let σ be a $\underset{\rightarrow[2]}{\mathcal{K}}$-substitution of $\mathbf{A}^2$. By definition of the signatures $\mathcal{K}$ and $\underset{\rightarrow[2]}{\mathcal{K}}$, $\sigma = \underset{\rightarrow[2]}{f}$, for some endomorphism f of $\mathbf{A}$. So by Proposition 5.52 of Example 5.51 on page 189, σ is continuous from the closed system $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ into itself, which suffices. The outstanding consequence follows since $\mathbf{F}_{\mathcal{K}}{}^2$ is the global $\underset{\rightarrow[2]}{\mathcal{K}}$-language by Corollary 6.63. $\diamond$

Consequently, $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$ must be axiomatizable.

$\square$

Recall the sentential calculus $S(\mathfrak{a}, \mathsf{su})$ of $\mathfrak{a}$-subuniverse introduced in Example 5.47 on page 188. We shall now consider the logics over algebras whose theories are the subuniverses. The most interesting of these, from the perspective of *canons* and their induced *ideals*, is the logic of subuniverses over the *free algebra* of a *quasivariety*, since this logic is generally not sentential.

Example 6.79 (The Subuniverse Logic on an Algebra)

Let $\mathbf{A}$ be an $\mathfrak{a}$-algebra, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra on ω-free generators $\overline{[\mathsf{V}]}$.

Definition 6.80 (The Subuniverse Logics) For an $\mathfrak{a}$-algebra $\mathbf{A}$, we shall denote the *finitary* $\mathbf{A}$-logic $L(\mathbf{A}, F(\mathbf{A}, \mathsf{su}))$, determined by the subuniverse closed system $F(\mathbf{A}, \mathsf{su})$, by $U(\mathbf{A}, \mathsf{su})$, which we call the **subuniverse logic *on* A**. By an **$\mathfrak{a}$-subuniverse logic** we mean a subuniverse logic on an algebra of type $\mathfrak{a}$. We write $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ for $U(\mathbf{F}_{\mathcal{K}}, \mathsf{su})$. $\square$

Note that by Corollary 5.50 (of Example 5.47), $S(\mathfrak{a}, \mathsf{su}) \equiv U(\mathbf{Tm}, \mathsf{su})$. The following characterization of $U(\mathbf{A}, \mathsf{su})$ follows by definition and Theorem 1.344 on page 65.

Remark 6.81 $A \vdash_{U(\mathbf{A},\mathsf{su})} a$ iff $a \in \|A\|^{\mathbf{A}}_{\mathsf{su}}$ iff, there exists a term $p(\vec{x})$ and $\vec{b} \in A$ with $p^{\mathbf{A}}(\vec{b}) = a$. $\square$

The $\mathfrak{a}$-subuniverse logics are all $\mathfrak{a}$-structural, by Proposition 5.45 of Example 5.44 on page 188, and hence, for any $\mathbf{A} \in \mathcal{K}$, $U(\mathbf{A}, \mathsf{su})$ is $\mathcal{K}$-structural. In particular, the subuniverse logic $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ is finitary and $\mathcal{K}$-structural; consequently $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ is a *propositional* $\mathcal{K}$-calculus, and as such must be axiomatizable.

Proposition 6.82 $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ is axiomatized by all axioms $\vdash \mathbf{0}^{\mathbf{F}_{\mathcal{K}}}$, where $\mathbf{0} \in \mathsf{Symb}_{\mathsf{c}}(\mathfrak{a})$, and all inference-rules $\{\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}}\} \vdash \star^{\mathbf{F}_{\mathcal{K}}}(\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}})$, one for each $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathfrak{a})$ and some distinct choice of $\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}} \in \overline{[\mathsf{V}]}$.

Proof. Let $\mathcal{P}$ be the propositional $\mathcal{K}$-calculus axiomatized by the axioms and rules described in the statement of this proposition.

$\boxed{\mathcal{P}\text{-theories are subuniverses}}$ Let T be a $\mathcal{P}$-theory. Clearly T contains all fundamental constants. Let $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathfrak{a})$ and $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{\mathsf{ar}(\star)} \in T$. Then $\star^{\mathbf{F}_{\mathcal{K}}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{\mathsf{ar}(\star)})$ is directly derivable from T via the rule $\{\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}}\} \vdash \star^{\mathbf{F}_{\mathcal{K}}}(\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}})$ and the unique $\mathbf{F}_{\mathcal{K}}$-substitution extending the mapping $\overline{x_i} \mapsto \boldsymbol{p}_i$ for each $1 \leq i \leq \mathsf{ar}(\star)$. $\boxed{\text{Subuniverses are } \mathcal{P}\text{-theories}}$ Let $\boldsymbol{P}$ be a subuniverse of $\mathbf{F}_{\mathcal{K}}$. (*It suffices to show that $\boldsymbol{P}$ contains all substitution instances of $\mathcal{P}$-axioms and is closed under direct derivability with respect to $\mathcal{P}$.*) $\boxed{\text{Substituted Axioms}}$ For each $\mathcal{P}$-axiom $\vdash \mathbf{0}^{\mathbf{F}_{\mathcal{K}}}$ and $\mathbf{F}_{\mathcal{K}}$-substitution $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}(\mathbf{0}^{\mathbf{F}_{\mathcal{K}}}) = \mathbf{0}^{\mathbf{F}_{\mathcal{K}}} \in \boldsymbol{P}$, the equality following since $\boldsymbol{\sigma}$ is a homomorphism and $\mathbf{0}$ a constant symbol, and the membership following since subuniverses contain all fundamental constants. $\boxed{\text{Direct Derivability}}$ Let $\boldsymbol{\sigma}$ be a $\mathbf{F}_{\mathcal{K}}$-substitution and $\{\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}}\} \vdash \star^{\mathbf{F}_{\mathcal{K}}}(\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}})$ a $\mathcal{P}$-inference rule with $\boldsymbol{\sigma}\left[\{\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}}\}\right] = \{\boldsymbol{\sigma}(\overline{x_1}), \ldots, \boldsymbol{\sigma}(\overline{x_{\mathsf{ar}(\star)}})\} \subseteq \boldsymbol{P}$. (*We must show that $\boldsymbol{\sigma}(\star^{\mathbf{F}_{\mathcal{K}}}(\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}})) \in \boldsymbol{P}$.*) But, $\boldsymbol{\sigma}(\star^{\mathbf{F}_{\mathcal{K}}}(\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}})) = \star^{\mathbf{F}_{\mathcal{K}}}(\boldsymbol{\sigma}(\overline{x_1}), \ldots, \boldsymbol{\sigma}(\overline{x_{\mathsf{ar}(\star)}})) \in \boldsymbol{P}$, the equality following since $\boldsymbol{\sigma}$ is a $\mathbf{F}_{\mathcal{K}}$-endomorphism and the membership following by operational-closure of subuniverses.

$\diamond$

While $\boldsymbol{S}(\mathcal{K},\mathsf{su})$ is a (universal) propositional $\mathcal{K}$-calculus, it is *not* a *sentential calculus*, since it is not a logic over the *term algebra*. Each formula of $\boldsymbol{S}(\mathcal{K},\mathsf{su})$ is a set of terms, rather than a single term. In Example 8.51 on page 293, we shall associate a sentential 1-calculus with $\boldsymbol{S}(\mathcal{K},\mathsf{su})$, denoted $S(\mathcal{K},\mathsf{su})$, using the machinery of *canons and their induced ideals* developed in $\oint 8$.

□

Recall Example 5.57 on page 191, where we introduced the sentential 1-calculus $S(\mathfrak{a},\mathsf{cos})$ of $\mathfrak{a}$-*cosets*, whose theories are precisely the *cosets* $\mathsf{Cos}(\mathbf{Tm})$ on the term algebra (i.e., the congruence classes together with the *improper-coset* which is the empty-set, see Example 4.113 on page 162), and the *membership logic* $S(\mathcal{K},\mathsf{mem})$, whose theories are precisely the (relative) $\mathcal{K}$-*cosets* $\mathsf{Cos}^{\mathcal{K}}(\mathbf{Tm})$ on the term algebra (i.e., the congruence classes of $\mathcal{K}$-relative congruences together with the *improper-coset* precisely when $\mathcal{K}$ is non-trivial; see the aforementioned example).

Example 6.83 (Axiomatizing the Membership Logic)

Since the membership logic of a quasivariety $\mathcal{K}$ is sentential, the membership logic must be axiomatized by all rules $P \vdash p$ such that $P \vdash p$, by Corollary 6.90 (note that P must be finite in such a rule). Since for non-empty P,

$$P \vdash_{S(\mathcal{K},\mathsf{mem})} p \text{ iff } P \approx P \models_{\mathcal{K}} P \approx p, \tag{6.7}$$

by Proposition 5.63 on page 192, the following axiomatization of $S(\mathcal{K},\mathsf{mem})$ obtains immediately.

Proposition 6.84 If $\mathcal{K}$ is trivial, then $S(\mathcal{K},\mathsf{mem})$ is axiomatized by the single axiom $\vdash x$, for some arbitrary $x \in \mathsf{V}$. If $\mathcal{K}$ is non-trivial, then $S(\mathcal{K},\mathsf{mem})$ is axiomatized with no axioms and all rules $P \vdash p$ for which $P \approx P \models_{\mathcal{K}} P \approx p$.

□

In the following example, we shall consider the cosets and relative cosets on arbitrary algebras as *universal logics*. Recall that the closed systems $\mathsf{Cos}(\mathbf{A})$ and $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$ of cosets and (relative) $\mathcal{K}$-cosets on $\mathcal{K}$, respectively, are both *finitary*, by Example 4.113.

Example 6.85 (The Universal Logics of Cosets)

Let $\mathbf{A}$ be an $\mathfrak{a}$-algebra, $\mathcal{K}$ a quasivariety of $\mathcal{K}$-algebras and $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra on $\overline{[\mathsf{V}]}$.

Definition 6.86 (The Logics of Cosets and Relative Cosets over Algebras)
The *finitary* $\mathbf{A}$-logics $L(\mathbf{A},\mathsf{Cos}(\mathbf{A}))$ and $L\left(\mathbf{A},\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})\right)$ are denoted by $U(\mathbf{A},\mathsf{cos})$ and $U(\mathbf{A},\mathsf{cos}^{\mathcal{K}})$ respectively, which we call the **coset logic** on $\mathbf{A}$ and the $\mathcal{K}$-**coset logic** (or **relative coset logic**) on $\mathbf{A}$ respectively. We write $\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})$ and $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ for $U(\mathbf{F}_{\mathcal{K}},\mathsf{cos})$ and $U(\mathbf{F}_{\mathcal{K}},\mathsf{cos}^{\mathcal{K}})$, respectively. □

By Proposition 5.60 of Example 5.57 on page 191, then sentential calculus $S(\mathfrak{a},\mathsf{cos})$ is equivalent to the logic $U(\mathbf{Tm},\mathsf{cos})$, and by definition, the membership logic $S(\mathcal{K},\mathsf{mem})$ is equivalent to $U(\mathbf{Tm},\mathsf{cos}^{\mathcal{K}})$. We record these observation for ease of future reference.

Remark 6.87 $S(\mathfrak{a},\mathsf{cos}) \equiv U(\mathbf{Tm},\mathsf{cos})$ and $S(\mathcal{K},\mathsf{mem}) \equiv U(\mathbf{Tm},\mathsf{cos}^{\mathcal{K}})$.

Warning 6.88 The theories of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ are the cosets of *all* congruences of the free algebra of $\mathcal{K}$ (*including* non-relative ones), while the theories of $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ are the cosets of *relative* congruences of the free algebra of $\mathcal{K}$.

Note that the logic $U(\mathbf{A}, \mathsf{cos})$ is unconstrained; hence $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ is unconstrained. Further, $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$ (and hence $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$) are unconstrained iff $\mathcal{K}$ is non-trivial, and these logics are trivial iff $\mathcal{K}$ is trivial. By Corollary 5.56 of Example 5.55 on page 190, the logics $U(\mathbf{A}, \mathsf{cos})$ and $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$ are both $\mathfrak{a}$-structural, and hence $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$ is $\mathcal{K}$-structural. Consequently, $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ and $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ are both finitary and $\mathcal{K}$-structural, and hence are *propositional $\mathcal{K}$-calculi* and consequently axiomatizable. Of course, neither of these logics are sentential calculi.

Recall the *formal*-axiomatization of $F(\mathbf{A}, \mathsf{cos})$ given in Remark 4.152 on page 169. This formal-axiomatization yields a simpler *axiomatization* of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$.

Proposition 6.89 $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ is axiomatized by no axioms and all rules

$$\overline{x}, \overline{y}, p^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{z_1}, \ldots, \overline{z_{\mathsf{ar}(p)-1}}) \vdash p^{\mathbf{F}_{\mathcal{K}}}(\overline{y}, \overline{z_1}, \ldots, \overline{z_{\mathsf{ar}(p)-1}}),$$

one for each term p and some distinct choice of variables $x, y, z_1, \ldots, z_{\mathsf{ar}(p)-1} \in \mathsf{V}$.

Proof. Let $\boldsymbol{\mathcal{P}}$ be the propositional $\mathcal{K}$-calculus described in the statement of the proposition. By Remark 4.152 on page 169, $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ is *formally*-axiomatized by all rules

$$\overline{q}, \overline{r}, p^{\mathbf{F}_{\mathcal{K}}}(\overline{q}, \overline{s_1}, \ldots, \overline{s_{\mathsf{ar}(p)-1}}) \vdash p^{\mathbf{F}_{\mathcal{K}}}(\overline{r}, \overline{s_1}, \ldots, \overline{s_{\mathsf{ar}(p)-1}}), \tag{6.8}$$

for all terms p, q, r and terms $s_1, \ldots, s_{\mathsf{ar}(p)-1}$. This formal-axiomatization must also be an axiomatization of the propositional $\mathcal{K}$-calculi $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$.
$\boxed{\boldsymbol{\mathcal{P}} \preceq \boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})}$ Certainly, every rule of $\boldsymbol{\mathcal{P}}$ is a rule of the form (6.8) and so is satisfied by $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$, and since there are no axioms of $\boldsymbol{\mathcal{P}}$, and $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ is structural, by Proposition 6.31, $\boldsymbol{\mathcal{P}} \preceq \boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$. $\boxed{\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos}) \preceq \boldsymbol{\mathcal{P}}}$ Let $\overline{q}, \overline{r}, p^{\mathbf{F}_{\mathcal{K}}}(\overline{q}, \overline{s_1}, \ldots, \overline{s_{\mathsf{ar}(p)-1}}) \vdash p^{\mathbf{F}_{\mathcal{K}}}(\overline{r}, \overline{s_1}, \ldots, \overline{s_{\mathsf{ar}(p)-1}})$ be a rule of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ of the form (6.8). Then $\overline{x}, \overline{y}, p^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{z_1}, \ldots, \overline{z_{\mathsf{ar}(p)-1}}) \vdash p^{\mathbf{F}_{\mathcal{K}}}(\overline{y}, \overline{z_1}, \ldots, \overline{z_{\mathsf{ar}(p)-1}})$, some distinct choice of variables $x, y, z_1, \ldots, z_{\mathsf{ar}(p)-1} \in \mathsf{V}$, is a rule of $\boldsymbol{\mathcal{P}}$. Let $\boldsymbol{\sigma}$ be a $\mathbf{F}_{\mathcal{K}}$-substitution extending $\overline{x} \mapsto \overline{q}$, $\overline{y} \mapsto \overline{r}$, $\overline{z_i} \mapsto \overline{s_i}$ for $i \in \{1, \ldots, \mathsf{ar}(p) - 1\}$. By structurality of $\boldsymbol{\mathcal{P}}$, $\boldsymbol{\sigma}(\overline{x}), \boldsymbol{\sigma}(\overline{y}), \boldsymbol{\sigma}(p^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{z_1}, \ldots, \overline{z_{\mathsf{ar}(p)-1}})) \vdash_{\boldsymbol{\mathcal{P}}} \boldsymbol{\sigma}(p^{\mathbf{F}_{\mathcal{K}}}(\overline{y}, \overline{z_1}, \ldots, \overline{z_{\mathsf{ar}(p)-1}}))$. Since $\boldsymbol{\sigma}$ is a $\mathbf{F}_{\mathcal{K}}$-endomorphism, $\boldsymbol{\sigma}(p^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{z_1}, \ldots, \overline{z_{\mathsf{ar}(p)-1}})) = p^{\mathbf{F}_{\mathcal{K}}}(\boldsymbol{\sigma}(\overline{x}), \boldsymbol{\sigma}(\overline{z_1}), \ldots, \boldsymbol{\sigma}(\overline{z_{\mathsf{ar}(p)-1}})) = p^{\mathbf{F}_{\mathcal{K}}}(\overline{q}, \overline{s_1}, \ldots, \overline{s_{\mathsf{ar}(p)-1}})$ and $\boldsymbol{\sigma}(p^{\mathbf{F}_{\mathcal{K}}}(\overline{y}, \overline{z_1}, \ldots, \overline{z_{\mathsf{ar}(p)-1}})) = p^{\mathbf{F}_{\mathcal{K}}}(\boldsymbol{\sigma}(\overline{y}), \boldsymbol{\sigma}(\overline{z_1}), \ldots, \boldsymbol{\sigma}(\overline{z_{\mathsf{ar}(p)-1}})) = p^{\mathbf{F}_{\mathcal{K}}}(\overline{r}, \overline{s_1}, \ldots, \overline{s_{\mathsf{ar}(p)-1}})$, and so $\overline{q}, \overline{r}, p^{\mathbf{F}_{\mathcal{K}}}(\overline{q}, \overline{s_1}, \ldots, \overline{s_{\mathsf{ar}(p)-1}}) \vdash_{\boldsymbol{\mathcal{P}}} p^{\mathbf{F}_{\mathcal{K}}}(\overline{r}, \overline{s_1}, \ldots, \overline{s_{\mathsf{ar}(p)-1}})$. So, with respect to the aforementioned axiomatization of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$, every rule of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ is satisfied by $\boldsymbol{\mathcal{P}}$. Since there are no axioms and $\boldsymbol{\mathcal{P}}$ is structural, by Proposition 6.31, $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos}) \preceq \boldsymbol{\mathcal{P}}$. $\diamond$

The identification of the logic $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ of non-relative cosets of $\mathbf{F}_{\mathcal{K}}$ and the explication of the previous axiomatization is not done frivolously. In the case that $\mathcal{K}$ is a non-trivial *variety*, $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ and $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ coincide. In $\oint 8$, we shall apply our theory of canons and their induced ideals to $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ and $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$, and obtain a simpler characterization of the *membership logic* of a non-trivial *variety* that is derived from the axiomatization of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ given in the previous result (see Example 8.60 on page 294).

The following characterization of $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$-consequence follows immediately from the characterization of $S(\mathcal{K}, \mathsf{mem})$-consequence given in Proposition 5.63 of Example 5.57 on page 191, together with Lemma 1.457 on page 88. Note that this result pertains generally, and not just for non-empty P.

Corollary 6.90 For $P \cup \{p\} \subseteq \mathsf{Tm}$, $P \vdash_{S(\mathcal{K},\mathsf{mem})} p$ iff $\overline{[P]} \vdash_{\boldsymbol{S}(\cos^{\mathcal{K}})} \overline{p}$. $\square$

This result, together with Lemma 6.35, yields an axiomatization of $\boldsymbol{S}(\cos^{\mathcal{K}})$.

Proposition 6.91 If $\mathcal{K}$ is trivial, then $\boldsymbol{S}(\cos^{\mathcal{K}})$ is axiomatized by the single axiom $\vdash \overline{x}$, for some arbitrary $x \in \mathsf{V}$. If $\mathcal{K}$ is non-trivial, then $\boldsymbol{S}(\cos^{\mathcal{K}})$ is axiomatized with no axioms and all rules $\overline{[P]} \vdash \overline{p}$ for which $P \approx P \models_{\mathcal{K}} P \approx p$. $\square$

The relationship between the propositional $\mathcal{K}$-calculus $\boldsymbol{S}(\cos^{\mathcal{K}})$ and the sentential calculus $S(\mathcal{K}, \mathsf{mem})$ is explored more fully in $\oint 8$. In the discourse of that chapter, the language $\mathbf{F}_{\mathcal{K}}$ is called $\mathfrak{a}$-*canonical* and the logic $\boldsymbol{S}(\cos^{\mathcal{K}})$ is called an $\mathfrak{a}$-*canon*. We show how such a canon induces a sentential calculus called the *ideal* (in this case the ideal is $S(\mathcal{K}, \mathsf{mem})$) such that the filters of the ideal on the canonical language are precisely the theories of the canon; in this particular case, demonstrating that the $S(\mathcal{K}, \mathsf{mem})$-filters on $\mathbf{F}_{\mathcal{K}}$ are precisely the relative congruences on $\mathbf{F}_{\mathcal{K}}$.

$\square$

Recall the definition of the sentential 1-calculus $S(\mathcal{K}, \tau)$ of [BR99], determined by a unary system of equations and a quasivariety $\mathcal{K}$.

Example 6.92 (The Logics of Solutions to Unary Equations)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras, τ a unary system of equations and $\mathbf{A}$ an $\mathfrak{a}$-algebra. Recall that by Corollary 5.92 of Example 5.90 on page 200, $\mathsf{Sol}_{\tau}^{\mathcal{K}}(\mathbf{A})$ is a finitary closed system.

Definition 6.93 (The Logics of Solutions of Unary Equations) Let $U_{\mathbf{A}}(\mathcal{K}, \tau)$ denote the *finitary* $\mathbf{A}$-logic $L\left(\mathbf{A}, \mathsf{Sol}_{\tau}^{\mathcal{K}}(\mathbf{A})\right)$. We write $\boldsymbol{S}(\mathcal{K}, \tau)$ for $U_{\mathbf{F}_{\mathcal{K}}}(\mathcal{K}, \tau)$. For a $\mathcal{K}$-constant 0, we write $U_{\mathbf{A}}(\mathcal{K}, 0)$ for $U_{\mathbf{A}}(\mathcal{K}, \mathbf{0})$ and write $\boldsymbol{S}(\mathcal{K}, 0)$ for $\boldsymbol{S}(\mathcal{K}, \mathbf{0})$, where $\mathbf{0}(x)$ is the unary system $\{\langle x, 0\rangle\}$. $\square$

Remark 6.94 $S(\mathcal{K}, \tau) \equiv U_{\mathbf{Tm}}(\mathcal{K}, \tau)$, by Proposition 2.91 on page 107. $\square$

The following characterization of the logic $U_{\mathbf{A}}(\mathcal{K}, \tau)$ follows immediately from Corollary 5.93 on page 200.

Proposition 6.95 For any $\mathbf{A}$, not necessarily in $\mathcal{K}$, the following conditions are equivalent.

1. $\Gamma \vdash_{U_{\mathbf{A}}(\mathcal{K}, \tau)} \phi$.
2. $\tau^{\mathbf{A}}[\Gamma] \vdash_{\mathsf{Con}^{\mathcal{K}}(\mathbf{A})} \tau^{\mathbf{A}}[\![\phi]\!]$.
3. $\tau^{\mathbf{A}}[\![\phi]\!] \subseteq \left\| \tau^{\mathbf{A}}[\Gamma] \right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}$
4. $\tau[\Gamma] \subseteq \alpha \rightarrow \tau[\![\phi]\!] \subseteq \alpha$, for all $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.

$\square$

The following characterization of consequence in $\boldsymbol{S}(\mathcal{K}, \tau)$ follows from the previous proposition together with Lemma 1.457 on page 88.

Corollary 6.96 The following conditions are equivalent.

1. $\overline{[P]} \vdash_{\boldsymbol{S}(\mathcal{K},\tau)} \overline{p}$.
2. $\tau^{\approx}[P] \models_{\mathcal{K}} \tau^{\approx}[\![p]\!]$.
3. $P \vdash_{S(\mathcal{K},\tau)} p$.

□

By Proposition 5.95 of Example 5.92 on page 200, for $\mathbf{A}$ *in* $\mathcal{K}$, $U_{\mathbf{A}}(\mathcal{K},\tau)$ is a finitary and $\mathcal{K}$-structural logic, and hence $\boldsymbol{S}(\mathcal{K},\tau)$ is a propositional $\mathcal{K}$-calculus. We record this fact for ease of later reference.

Proposition 6.97 $\boldsymbol{S}(\mathcal{K},\tau)$ is a propositional $\mathcal{K}$-calculus.

□

Recall Convention 1.474 of Example 1.467, where we introduced the notion of a quasivariety of *lower-unbounded* lattice expansions and a quasivariety of *upper-unbounded* lattice expansions, and in particular, note Warning 1.475 of that example. We shall now consider logics that arise from lattice ideals and filters of lattice extensions.

Example 6.98 (The Logics of Lattice Ideals and Filters)

Let $\mathbf{P}$ be a *lattice expansion*, $\mathbf{F}$ the free *lattice* on ω-free generators, $\mathcal{K}$ a quasivariety of *lattice expansions* and $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra on ω-free generators.

Definition 6.99 (Logics of Ideals, Filters and Convexities of Lattices) For a lattice expansion $\mathbf{P}$, let $U(\mathbf{P},\mathsf{id})$, $U(\mathbf{P},\mathsf{fi})$ and $U(\mathbf{P},\mathsf{cx})$ denote the *finitary* $\mathbf{P}$-logics $L(\mathbf{P},\mathsf{Id}_{\Diamond}(\mathbf{P}))$, $L(\mathbf{P},\mathsf{Fl}_{\Diamond}(\mathbf{P}))$ and $L(\mathbf{P},\mathsf{Cx}(\mathbf{P}))$, respectively. We write $\boldsymbol{S}(\textsf{lat-id})$, $\boldsymbol{S}(\textsf{lat-fi})$ and $\boldsymbol{S}(\textsf{lat-cx})$ for $U(\mathbf{F},\mathsf{id})$, $U(\mathbf{F},\mathsf{fi})$ and $U(\mathbf{F},\mathsf{cx})$, respectively, and write $\boldsymbol{S}(\mathcal{K},\mathsf{id})$, $\boldsymbol{S}(\mathcal{K},\mathsf{fi})$ and $\boldsymbol{S}(\mathcal{K},\mathsf{cx})$ for $U(\mathbf{F}_{\mathcal{K}},\mathsf{id})$, $U(\mathbf{F}_{\mathcal{K}},\mathsf{fi})$ and $U(\mathbf{F}_{\mathcal{K}},\mathsf{cx})$, respectively.

Let $U(\mathbf{P},\mathsf{id}_{\emptyset})$ and $U(\mathbf{P},\mathsf{fi}_{\emptyset})$ denote the *finitary* $\mathbf{P}$-logics $L\left(\mathbf{P},\mathsf{Id}_{\Diamond_{\emptyset}}(\mathbf{P})\right)$ and $L\left(\mathbf{P},\mathsf{Fl}_{\Diamond_{\emptyset}}(\mathbf{P})\right)$, respectively. We write $\boldsymbol{S}(\mathcal{K},\mathsf{id}_{\emptyset})$ and $\boldsymbol{S}(\mathcal{K},\mathsf{fi}_{\emptyset})$ for $U(\mathbf{F}_{\mathcal{K}},\mathsf{id}_{\emptyset})$ and $U(\mathbf{F}_{\mathcal{K}},\mathsf{fi}_{\emptyset})$, respectively. □

Notice that there are no 'non-emboldened' S versions of these logics, since we are implicitly reserving that symbol for sentential calculi, and *none* of these logics are sentential as none are defined on a term algebra; the term algebra is never a lattice nor a lattice expansion.

So as to simplify the discourse and remove any possible confusion we introduce the following definitions and conventions.

Convention 6.100 For a quasivariety $\mathcal{K}$ of 0-lattice (resp. 1-lattice) expansions, we shall write $\boldsymbol{S}_0(\mathcal{K},\mathsf{id})$ (resp. $\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})$) for $\boldsymbol{S}(\mathcal{K},\mathsf{id})$ (resp. $\boldsymbol{S}(\mathcal{K},\mathsf{fi})$). For a quasivariety $\mathcal{K}$ of lower-unbounded lattice (resp. upper-unbounded lattice) expansions, we shall write $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$ (resp. $\boldsymbol{S}_*(\mathcal{K},\mathsf{fi})$) for $\boldsymbol{S}(\mathcal{K},\mathsf{id})$ (resp. $\boldsymbol{S}(\mathcal{K},\mathsf{fi})$). Use of this symbolism shall implicitly imply the 'type' of the quasivariety. Conventionally, $\mathsf{type}(\mathcal{K})$ shall denote the type of the quasivariety $\mathcal{K}$.

The following result follows from Example 5.67 on page 192.

Corollary 6.101 With respect to their conventionally restricted quasivarieties $\mathcal{K}$, $\boldsymbol{S}_0(\mathcal{K},\mathsf{id})$, $\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})$, $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$ and $\boldsymbol{S}_*(\mathcal{K},\mathsf{fi})$, are all $\mathcal{K}$-structural, and hence *propositional* $\mathcal{K}$-calculi. □

The proof of the following characterizations, following from Remarks 4.93, 4.94 and 4.97 of Example 4.88 on page 158, together with Theorem 1.434 on page 84.

Remark 6.102 Let $P \cup \{p, q\} \subseteq \mathsf{Tm}$ with $P \neq \emptyset$. For $\boldsymbol{S} \in \{\boldsymbol{S}_0(\mathcal{K}, \mathsf{id}), \boldsymbol{S}_*(\mathcal{K}, \mathsf{id})\}$ the following are valid.

1. $\overline{[P]} \vdash_{\boldsymbol{S}} \overline{p}$ iff $\exists\,[0 \neq n \in \mathbb{N}, \{p_1, \ldots, p_n\} \subseteq_{\mathsf{f}} P]\ \models_{\mathcal{K}} p \leq p_1 \vee \ldots \vee p_n$.
2. $\{\overline{p}\} \vdash_{\boldsymbol{S}} \overline{q}$ iff $\models_{\mathcal{K}} q \leq p$.
3. $\{\overline{p}\} \dashv\vdash_{\boldsymbol{S}} \{\overline{q}\}$ iff $\models_{\mathcal{K}} p \approx q$.

Remark 6.103 $\vdash_{\boldsymbol{S}_0(\mathcal{K},\mathsf{id})} \overline{p}$ iff $\models_{\mathcal{K}} p \approx 0$.

Remark 6.104 Let $P \cup \{p, q\} \subseteq \mathsf{Tm}$ with $P \neq \emptyset$. For $\boldsymbol{S} \in \{\boldsymbol{S}_1(\mathcal{K}, \mathsf{fi}), \boldsymbol{S}_*(\mathcal{K}, \mathsf{fi})\}$ the following are valid.

1. $\overline{[P]} \vdash_{\boldsymbol{S}} \overline{p}$ iff $\exists\,[0 \neq n \in \mathbb{N}, \{p_1, \ldots, p_n\} \subseteq_{\mathsf{f}} P]\ \models_{\mathcal{K}} p \geq p_1 \wedge \ldots \vee p_n$.
2. $\{\overline{p}\} \vdash_{\boldsymbol{S}} \overline{q}$ iff $\models_{\mathcal{K}} q \geq p$.
3. $\{\overline{p}\} \dashv\vdash_{\boldsymbol{S}} \{\overline{q}\}$ iff $\models_{\mathcal{K}} p \approx q$.

Remark 6.105 $\vdash_{\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})} \overline{p}$ iff $\models_{\mathcal{K}} p \approx 1$.

Proposition 6.106 For distinct variables x and y in V, $\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})$ is axiomatized by the two rules

$$\overline{x} \vdash \overline{x} \wedge^{\mathbf{F}_{\mathcal{K}}} \overline{y} \quad \text{and} \tag{6.9}$$

$$\overline{x}, \overline{y} \vdash \overline{x} \vee^{\mathbf{F}_{\mathcal{K}}} \overline{y}. \tag{6.10}$$

An alternative axiomatization is given by (6.10) and

$$\overline{x} \vee^{\mathbf{F}_{\mathcal{K}}} \overline{y} \vdash \overline{y}. \tag{6.11}$$

Proof. Note that since $\mathbf{F}_{\mathcal{K}}$ is lower-unbounded, $\emptyset \in \mathsf{Id}_{\Diamond}(\mathbf{F}_{\mathcal{K}})$.
$\boxed{\text{Axiomatization 1}}$ Let $\boldsymbol{S}$ denote the propositional $\mathcal{K}$-calculus determined by the first axiomatization. Note that $\boldsymbol{S}$ has no theorems, being defined only in terms of rules. $\boxed{\mathsf{Th}(\boldsymbol{S}) \subseteq \mathsf{Id}_{\Diamond}(\mathbf{F}_{\mathcal{K}})}$ Let $\boldsymbol{T} \in \mathsf{Th}(\boldsymbol{S})$. If $\boldsymbol{T} = \emptyset$ then by the earlier note, $\boldsymbol{T} = \emptyset \in \mathsf{Id}_{\Diamond}(\mathbf{F}_{\mathcal{K}})$. Suppose that $\boldsymbol{T} \neq \emptyset$. Suppose that $\overline{p} \in \boldsymbol{T}$ and $\overline{q} \leq^{\mathbf{F}_{\mathcal{K}}} \overline{p}$. Since $\overline{x} \vdash \overline{x} \wedge^{\mathbf{F}_{\mathcal{K}}} \overline{y}$ is an $\boldsymbol{S}$-rule, by structurality, $\overline{p} \vdash_{\boldsymbol{S}} \overline{q} \wedge^{\mathbf{F}_{\mathcal{K}}} \overline{p}$, and since $\overline{p} \in \boldsymbol{T}$, we have $\overline{q} \wedge^{\mathbf{F}_{\mathcal{K}}} \overline{p} \in \boldsymbol{T}$, since $\boldsymbol{T}$ is a theory. But $\overline{q} \wedge^{\mathbf{F}_{\mathcal{K}}} \overline{p} = \overline{q}$ since $\overline{q} \leq^{\mathbf{F}_{\mathcal{K}}} \overline{p}$, so $\overline{q} \in \boldsymbol{T}$. Suppose that $\overline{p}, \overline{q} \in \boldsymbol{T}$. Since $\overline{x}, \overline{y} \vdash \overline{x} \vee^{\mathbf{F}_{\mathcal{K}}} \overline{y}$ is an $\boldsymbol{S}$-rule, by structurality, $\{\overline{p}, \overline{q}\} \vdash_{\boldsymbol{S}} \overline{q} \vee^{\mathbf{F}_{\mathcal{K}}} \overline{p}$, and since $\overline{p}, \overline{q} \in \boldsymbol{T}$, $\overline{q} \vee^{\mathbf{F}_{\mathcal{K}}} \overline{p} \in \boldsymbol{T}$, since $\boldsymbol{T}$ is a theory. So $\boldsymbol{T} \in \mathsf{Id}_{\Diamond}(\mathbf{F}_{\mathcal{K}})$. $\boxed{\mathsf{Id}_{\Diamond}(\mathbf{F}_{\mathcal{K}}) \subseteq \mathsf{Th}(\boldsymbol{S})}$ Let $\mathsf{I} \in \mathsf{Id}_{\Diamond}(\mathbf{F}_{\mathcal{K}})$. If $\mathsf{I} = \emptyset$, then $\mathsf{I} = \emptyset \in \mathsf{Th}(\boldsymbol{S})$. Suppose that $\mathsf{I} \neq \emptyset$. We shall show that anything derivable from I is contained in I, by proceeding inductively on the length of derivations from I in $\boldsymbol{S}$. $\boxed{\text{Base Case}}$ Suppose that $\overline{p}$ is derivable from I by a derivation of length one. Since $\boldsymbol{S}$ has no axioms, $\overline{p} \in \mathsf{I}$. $\boxed{\text{Induction Hypothesis}}$ Assume that any $\overline{p}$ derivable from I by a derivation of length n or less, is a member of I. $\boxed{\text{Inductive Step}}$ Suppose that $\overline{p}_1, \ldots, \overline{p}_{n+1}$ is a derivation from I and that $\overline{p}_{n+1}$ is derivable by no shorter derivation. By the inductive hypothesis, $\{\overline{p}_1, \ldots, \overline{p}_n\} \subseteq \mathsf{I}$. There exists a rule Λ and a $\mathbf{F}_{\mathcal{K}}$-substitution $\boldsymbol{\sigma}$ with $\boldsymbol{\sigma}\,[\mathsf{prem}(\Lambda)] \subseteq \{\overline{p}_1, \ldots, \overline{p}_n\} \subseteq \mathsf{I}$ and $\boldsymbol{\sigma}(\mathsf{conc}(\Lambda)) = \overline{p}_{n+1}$. If $\Lambda = \overline{x} \vdash \overline{x} \wedge^{\mathbf{F}_{\mathcal{K}}} \overline{y}$, then $\boldsymbol{\sigma}(\overline{x}) \in \mathsf{I}$ and $\overline{p}_{n+1} = \boldsymbol{\sigma}(\overline{x} \wedge^{\mathbf{F}_{\mathcal{K}}} \overline{y}) = \boldsymbol{\sigma}(\overline{x}) \wedge^{\mathbf{F}_{\mathcal{K}}} \boldsymbol{\sigma}(\overline{y}) \leq^{\mathbf{F}_{\mathcal{K}}} \boldsymbol{\sigma}(\overline{x}) \in \mathsf{I}$, and so $\overline{p}_{n+1} \in \mathsf{I}$ since I is an ideal. Otherwise, $\Lambda = \overline{x}, \overline{y} \vdash \overline{x} \vee^{\mathbf{F}_{\mathcal{K}}} \overline{y}$, in which case, $\boldsymbol{\sigma}(\overline{x}), \boldsymbol{\sigma}(\overline{y}) \in \mathsf{I}$ and $\overline{p}_{n+1} = \boldsymbol{\sigma}(\overline{x} \vee^{\mathbf{F}_{\mathcal{K}}} \overline{y}) = \boldsymbol{\sigma}(\overline{x}) \vee^{\mathbf{F}_{\mathcal{K}}} \boldsymbol{\sigma}(\overline{y})$, and so $\overline{p}_{n+1} \in \mathsf{I}$. $\boxed{\text{Axiomatization 2}}$ The proof is similar to the proof of the first axiomatization and the proof of Proposition 4.156 of Example 4.153 on page 170. $\Diamond$

The proofs of the following results are similar or dual to the proof of the previous result, and as such are omitted.

Proposition 6.107 $\boldsymbol{S}_0(\mathcal{K}, \mathsf{id})$ is axiomatized by the axiom $\vdash 0^{\mathbf{F}_\mathcal{K}}$ and either (6.9) and (6.10) or (6.10) and (6.11).

Proposition 6.108 For distinct variables x and y in V, $\boldsymbol{S}_*(\mathcal{K}, \mathsf{fi})$ is axiomatized by the two rules

$$\overline{x} \vdash \overline{x} \vee^{\mathbf{F}_\mathcal{K}} \overline{y} \quad \text{and} \tag{6.12}$$

$$\overline{x}, \overline{y} \vdash \overline{x} \wedge^{\mathbf{F}_\mathcal{K}} \overline{y}. \tag{6.13}$$

An alternative axiomatization is given by (6.13) and

$$\overline{x} \wedge^{\mathbf{F}_\mathcal{K}} \overline{y} \vdash \overline{y}. \tag{6.14}$$

Proposition 6.109 $\boldsymbol{S}_1(\mathcal{K}, \mathsf{fi})$ is axiomatized by the axiom $\vdash 1^{\mathbf{F}_\mathcal{K}}$ and either (6.12) and (6.13) or (6.13) and (6.14). □

Remarks 6.102 to 6.105 suggest the definitions of *sentential* calculi, which we introduce next. In ∮8 we shall explore the relationship between the propositional $\mathcal{K}$-calculi of the previous example and these sentential calculi.

Definition 6.110 (The Sentential Calculi $S_*(\mathcal{K}, \mathsf{id})$, $S_*(\mathcal{K}, \mathsf{fi})$, $S_0(\mathcal{K}, \mathsf{id})$ and $S_1(\mathcal{K}, \mathsf{fi})$) For a quasivariety $\mathcal{K}$ of lower-unbounded lattice expansions, let $S_*(\mathcal{K}, \mathsf{id})$ be the sentential 1-calculus of type $\mathsf{type}(\mathcal{K})$, axiomatized by no axioms and all rules

$$p_1, \ldots, p_n \vdash p, \quad \text{where} \quad \models_\mathcal{K} p \leq p_1 \vee \ldots \vee p_n. \tag{6.15}$$

For a quasivariety $\mathcal{K}$ of upper-unbounded lattice expansions, let $S_*(\mathcal{K}, \mathsf{fi})$ be the sentential 1-calculus of type $\mathsf{type}(\mathcal{K})$, axiomatized by no axioms and all rules

$$p_1, \ldots, p_n \vdash p, \quad \text{where} \quad \models_\mathcal{K} p \geq p_1 \wedge \ldots \vee p_n. \tag{6.16}$$

For a quasivariety $\mathcal{K}$ of 0-lattice expansions, let $S_0(\mathcal{K}, \mathsf{id})$ be the sentential 1-calculus of type $\mathsf{type}(\mathcal{K})$, axiomatized by all axioms

$$\vdash p, \quad \text{where} \quad \models_\mathcal{K} p \approx 0, \tag{6.17}$$

and all rules (6.15). For a quasivariety $\mathcal{K}$ of 1-lattice expansions, let $S_1(\mathcal{K}, \mathsf{fi})$ be the sentential 1-calculus of type $\mathsf{type}(\mathcal{K})$, axiomatized by all axioms

$$\vdash p, \quad \text{where} \quad \models_\mathcal{K} p \approx 1, \tag{6.18}$$

and all rules (6.16). Conventionally, the notations $S_*(\mathcal{K}, \mathsf{id})$, $S_*(\mathcal{K}, \mathsf{fi})$, $S_0(\mathcal{K}, \mathsf{id})$ and $S_1(\mathcal{K}, \mathsf{fi})$ shall imply the type of quasivariety. □

□

We conclude these examples by locating *Boolean logics* in the sense of [HG98, 37-] within our framework of logics over constructs.

Example 6.111 (Boolean Logics) [HG98]

Halmos and Givant define a **Boolean logic** (also called a **propositional logic** in [HG98]) to be a pair $\langle \mathbf{B}, \alpha \rangle$, where $\mathbf{B}$ is a pre-Boolean algebra and α is a **fully invariant** BA-congruence relation on $\mathbf{B}$ (i.e., for every endomorphism f of $\mathbf{B}$, $\underset{\rightarrow}{f}[\alpha] \subseteq \alpha$). Although Halmos and Givant do not explicitly mention a **language** of Boolean logics, the language implicit in the definition of Boolean-logics would be our notion of a BA-language.

Remark 6.112 Boolean algebras are pre-Boolean algebras, but the converse is not generally true. □

There are two ways of locating Boolean logics as examples of logics over constructs. The first approach is to view each Boolean logic $\langle \mathbf{B}, \alpha \rangle$ as a 'small' logic with two theories, namely, α and $\blacksquare_{\mathbf{B}}$, the latter congruence certainly being fully invariant. The other approach is to associate with each pre-Boolean algebra $\mathbf{B}$ the logic whose theories consist of all fully-invariant BA-congruences on $\mathbf{B}$; it is easily seen that the arbitrary intersection of fully-invariant congruences is a fully-invariant congruence.

□

Chapter 7

On Structurality and Models in Logic

In Example 5.121 on page 208, we demonstrated that the *semantic consequence* relation $\models^{\mathbf{M}}$ determined by a $\mathfrak{p}$-matrix $\mathbf{M}$ may be realized as the *product of the source* of all interpretations into the matrix, and in Example 5.131 we showed how the *filters* of sentential calculi may be characterized in terms of the *quotient of the sink* of all interpretations from the logic into an algebra of a matrix. In this chapter we shall exploit these observations as a means of defining the notions of *semantic consequence* (*abstraction*) and *filters* (*realization*). We have found it useful to view logics as the models and abstractions of logics, as opposed to (just) matrices.

In $\oint$7.1, we begin by defining what it means for a logic M to model another logic L, essentially by requiring that all *interpretations* from L into M be *continuous* (treating both logics as closed systems), in which case we call M a *model* of logic L and call L an *abstraction* of logic M. We then show how models and abstracts may be realized as the quotient of the sink of all interpretations from a logic into the model language, and as the product of all interpretations from the abstract language into a logic, respectively. The former notion gives rise to the notion of a *filter*, while the latter gives rise to the notions of *semantic consequence* and *ideal*. In this section we develop the theory of filters and ideals, and demonstrate inter-relationships between the two notions. We show that logic M models L iff every filter of L on the language of M is a theory of M. Filters and ideals in turn give rise to the *logics of filters* and the *logics of ideals* determined by a given logic. These logics prove useful in the characterization of *structurality*, which is the topic of $\oint$7.2. We show that a logic is structural precisely when *it is its own model* and, equivalently, when *it is its own abstraction*. It is then shown that the ideal logic and the filter logic of a logic are always structural, from which we deduce that a logic is structural precisely when it is *equal to its filter logic* on its language and, equivalently, when it is *equal to its ideal logic* on its language. Further characterizations of structurality are obtained in $\oint$7.4.

The notion of a *maximal model* is introduced in $\oint$7.3. In essence, logic M is a maximal model of logic L if M is the filter logic of L on the language of M (equivalently the theories of M and the filters of L on the language of M coincide). We give a sufficient condition for maximal modellability, which we show, in $\oint$16, to be necessary in the case that L is structural and *protoalgebraic*, in the sense of protoalgebraicity to be defined in $\oint$16.

In $\oint 7.4$ we consider logics as semantics of logics, and prove that a logic has a semantics precisely when it is structural. Since the property of having a semantics is essentially the property that the logic be sound and complete, structurality is equivalent to soundness-and-completeness.

The rest of the theory of this chapter is devoted to exploring the relationship between *logics as models* and *matrices as models*. As a precursor to this discussion, in $\oint 7.5$ we introduce the notion of a *language indexed model*, which is a *family* of logics, indexed by the languages of the signature, each of which is a model (of some logic under consideration). The key issue is that there is *one* model logic for each language. We show how an arbitrary set of models (with possibly repeated languages) may be *normalized* to obtain an equivalent language indexed model. With this machinery in hand, we turn to the topic of matrices as models. In $\oint 7.6$ we show that a matrix over a language of logics may be viewed as a 'small' logic, whose theories are the designated set and the set of all formulae. Consequently, we define a matrix to be a model of a logic if this associated small logic is a model. We show that this notion of matrix model coincides, in the case of sentential calculi and algebra-matrices, with the standard notion of a matrix model [BP89a] (see Definition 2.36 on page 100). In the converse direction, given a set of matrices, viewed as a set of small logics over possibly duplicate languages, we employ the normalization process developed in $\oint 7.5$ to induce an equivalent language indexed model. It is worth noting at this point that, given the 'flat' nature of logics over constructs, all matrices under consideration are *unary* matrices.

In $\oint 7.7$, we consider a number of examples. We show that the filters of the propositional $\mathcal{K}$-calculus $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ on an algebra $\mathbf{A} \in \mathcal{K}$ are precisely the subuniverses of that algebra, and so the logic of subuniverses of $\mathbf{A}$ constitutes a maximal model of $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ on $\mathbf{A}$. We also demonstrate that the filters of the membership logic on an algebra encompass all $\mathcal{K}$-cosets on that algebra.

Finally, we summarize many of the results obtained in this chapter in tabular form for ease of future reference.

7.1 Logics as Models of Logics

7.1.1 Constituting Models and Abstracts

Definition 7.1 (Constituting Models and Abstracts) Let $\mathfrak{s}$ be a signature of languages and let $\mathfrak{L}$ and $\mathfrak{M}$ be *non-empty* classes of $\mathfrak{s}$-logics. We say that $\mathfrak{M}$ **constitutes an $\mathfrak{s}$-model** of $\mathfrak{L}$, or that $\mathfrak{L}$ **constitutes an $\mathfrak{s}$-abstraction** of $\mathfrak{M}$, if, for all $\mathsf{L} \in \mathfrak{L}$, $\mathsf{M} \in \mathfrak{M}$ and $\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathsf{L}, \mathsf{M})$, i is continuous from L into M, i.e.,

$$\forall\,[\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L})] \quad \Gamma \vdash_{\mathsf{L}} \phi \rightarrow \mathsf{i}\,[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi). \tag{7.1}$$

We tend to drop the references to $\mathfrak{s}$ in these notions wherever unambiguous. □

Informally, we picture abstracts to the left and models to the right, and consequently view interpretations running from left to right. We tend to denote arbitrary languages, logics, sets of logics, formulae, sets of formulae and theories, by $\mathbf{A}$, L, $\mathfrak{L}$, ϕ, Γ and T, to the left respectively, and by $\mathbf{B}$, M, $\mathfrak{M}$, ψ, Φ and R, to the right respectively.

Remark 7.2 We leave it to the reader to formulate the characterizations of the property that $\mathfrak{M}$ constitutes a model of $\mathfrak{L}$ from the results of $\oint 5$ (see Theorem 5.21 on page 182, Theorem 5.40 on page 186 and Proposition 5.105 on page 204). □

7.1.2 Models and Abstracts

Definition 7.3 (Models and Abstracts) We say that a single *logic* M **is an $\mathfrak{s}$-model** of logics $\mathfrak{L}$ (or a **$\langle\mathfrak{L},\mathfrak{s}\rangle$-model** or simply a **$\mathfrak{L}$-model** when $\mathfrak{s}$ is understood), if $\{\mathsf{M}\}$ constitutes an $\mathfrak{s}$-model of $\mathfrak{L}$. We say that a single *logic* L **is an $\mathfrak{s}$-abstraction** of logics $\mathfrak{M}$ (or a **$\langle\mathfrak{M},\mathfrak{s}\rangle$-abstraction** or simply an **$\mathfrak{M}$-abstraction** when $\mathfrak{s}$ is understood), if $\{\mathsf{L}\}$ constitutes an $\mathfrak{s}$-abstraction of $\mathfrak{M}$. For sets $\mathfrak{L}$ and $\mathfrak{M}$ of $\mathfrak{s}$-logics and $\mathfrak{s}$-*languages* $\mathbf{A}$ and $\mathbf{B}$, we define

$$\begin{aligned}
\mathsf{Mod}^{\mathfrak{s}}_{\mathfrak{L}} &= \{\mathsf{M} \in \mathsf{logics}(\mathfrak{s}) \ : \ \mathsf{M} \text{ is a } \mathfrak{s}\text{-model of } \mathfrak{L}\},\\
\mathsf{Abs}^{\mathfrak{s}}_{\mathfrak{M}} &= \{\mathsf{L} \in \mathsf{logics}(\mathfrak{s}) \ : \ \mathsf{L} \text{ is an } \mathfrak{s}\text{-abstraction of } \mathfrak{M}\},\\
\mathsf{Mod}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B}) &= \{\mathsf{M} \in \mathsf{Mod}^{\mathfrak{s}}_{\mathfrak{L}} \ : \ \mathbf{lg}(\mathsf{M}) = \mathbf{B}\} \qquad \text{and}\\
\mathsf{Abs}^{\mathfrak{s}}_{\mathfrak{M}}(\mathbf{A}) &= \{\mathsf{L} \in \mathsf{Abs}^{\mathfrak{s}}_{\mathfrak{L}} \ : \ \mathbf{lg}(\mathsf{L}) = \mathbf{A}\}.
\end{aligned}$$

□

Corollary 7.4 Table 7.1 enumerates various characterizations of the property that M models L. Most of these obtain from Theorem 5.21 on page 182, Theorem 5.40 on page 186 and Proposition 5.105 on page 204; the others are derived later in this chapter.

Remark 7.5 The following conditions are equivalent.

1. $\mathfrak{M}$ constitutes a model of $\mathfrak{L}$.
2. $\mathfrak{M}$ constitutes a model of L, for all $\mathsf{L} \in \mathfrak{L}$.
3. M is a model of $\mathfrak{L}$, for all $\mathsf{M} \in \mathfrak{M}$.
4. M is a model of L, for all $\mathsf{L} \in \mathfrak{L}$ and $\mathsf{M} \in \mathfrak{M}$.

Remark 7.6

1. $\mathfrak{M} \subseteq \mathsf{Mod}_{\mathfrak{L}}$ iff $\mathfrak{M}$ constitutes a model of $\mathfrak{L}$. In particular, $\mathsf{Mod}_{\mathfrak{L}}$ constitutes a model of $\mathfrak{L}$.
2. $\mathfrak{M} \subseteq \mathsf{Abs}_{\mathfrak{L}}$ iff $\mathfrak{M}$ constitutes an abstraction of $\mathfrak{L}$. In particular, $\mathsf{Abs}_{\mathfrak{L}}$ constitutes an abstraction of $\mathfrak{M}$.
3. $\mathsf{Mod}_{\mathfrak{L}}(\mathbf{A})$ and $\mathsf{Mod}_{\mathfrak{L}}$ are the largest sets, of $\mathbf{A}$-logics and logics respectively, constituting models of $\mathfrak{M}$.
4. $\mathsf{Abs}_{\mathfrak{L}}(\mathbf{A})$ and $\mathsf{Abs}_{\mathfrak{L}}$ are the largest sets of $\mathbf{A}$-logics and logics respectively, constituting abstractions of $\mathfrak{L}$.

Proposition 7.7 $\mathfrak{M} \subseteq \mathsf{Mod}_{\mathfrak{L}}$ iff $\mathfrak{L} \subseteq \mathsf{Abs}_{\mathfrak{M}}$.

Remark 7.8 $\mathfrak{L} \subseteq \mathfrak{L}'$ implies $\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B}) \supseteq \mathsf{Mod}_{\mathfrak{L}'}(\mathbf{B})$.

Proof. Let $\mathsf{M} \in \mathsf{Mod}_{\mathfrak{L}'}(\mathbf{B})$. (*We must show that* $\mathsf{M} \in \mathsf{Mod}_{\mathfrak{L}}(\mathbf{B})$.) Let $\mathsf{L} \in \mathfrak{L}$ and $\mathsf{i} \in \mathsf{Int}(\mathsf{L},\mathsf{M})$. (*We must show that* i *is continuous from* L *into* M.) Since $\mathfrak{L} \subseteq \mathfrak{L}'$, by assumption, $\mathsf{L} \in \mathfrak{L}'$, and since $\mathsf{i} \in \mathsf{Int}(\mathsf{L},\mathsf{M})$ and $\mathsf{M} \in \mathsf{Mod}_{\mathfrak{L}'}(\mathbf{B})$, i is continuous from L into M. ⋄

Remark 7.9 $\mathfrak{M} \subseteq \mathfrak{M}'$ implies $\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A}) \supseteq \mathsf{Abs}_{\mathfrak{M}'}(\mathbf{A})$.

Proof. Let $\mathsf{L} \in \mathsf{Abs}_{\mathfrak{M}'}(\mathbf{A})$. (*We must show that* $\mathsf{L} \in \mathsf{Abs}_{\mathfrak{M}}(\mathbf{A})$.) Let $\mathsf{M} \in \mathfrak{M}$ and $\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathsf{M})$. (*We must show that* i *is continuous from* L *into* M.) Since $\mathfrak{M} \subseteq \mathfrak{M}'$, by assumption, $\mathsf{M} \in \mathfrak{M}'$, and since $\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathsf{M})$ and $\mathsf{L} \in \mathsf{Abs}_{\mathfrak{M}'}(\mathbf{A})$, i is continuous from L into M. $\diamond$

7.1.3 Realizing Abstracts and Models

We shall now demonstrate how to construct a logic $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ on any given language $\mathbf{B}$ that models a given class of logics $\mathfrak{L}$, and how to construct a logic $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ on a language $\mathbf{A}$ that abstracts a given class of logics $\mathfrak{M}$. While in the modeling case, the theories of $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ are the natural generalization to this level of discourse of the standard notion of *filters* (see Definition 2.41 on page 101), the approach taken here is slightly different.

Consider the case that $\mathfrak{L} = \{\mathsf{L}\}$. By definition, for a $\mathbf{B}$-logic M to be a model of L, every interpretation of L into M must be continuous. We already have the machinery to construct such a model. Recall the notion of the *quotient of a sink* given in Definition 5.122 on page 208. If we consider all the interpretations from $\mathbf{lg}(\mathsf{L})$ into $\mathbf{B}$ as forming a sink sk from the closed system $\mathsf{Th}(\mathsf{L})$ into $\mathsf{Fm}(\mathbf{B})$, then the quotient closed system $\coprod \mathsf{sk}$ induced on $\mathsf{Fm}(\mathbf{B})$ will be such that all the interpretations of $\mathbf{lg}(\mathsf{L})$ into $\mathbf{B}$ are continuous from $\mathsf{Th}(\mathsf{L})$ in itself; consequently the logic $L(\mathbf{B}, \coprod \mathsf{sk})$ models L. Comparing the characterization of the closed sets of $\coprod \mathsf{sk}$ given in Theorem 5.125 on page 209 with the standard definition of a filter of a propositional calculus given in Definition 2.41 on page 101, reveals that, in the case that L is a deductive system in the sense of [BP89a] and $\mathbf{B}$ is an algebra, the theories of $L(\mathbf{B}, \coprod \mathsf{sk})$ are precisely the L-filters of $\mathbf{B}$.

The reason we have adopted such an approach, is that, in a symmetrical manner, we can employ a *source* and *product* to induce a logic $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ on $\mathbf{A}$ which *abstracts* $\mathfrak{M}$. While the notion of abstracts are not *explicitly* part of the standard presentation of algebraic logics, we employ abstracts extensively in $\oint$8, where we induce standard deductive systems in the sense of [BP89a] (i.e., propositional $\mathfrak{a}$-calculi) from closed systems over the algebras of quasivarieties. We have chosen the word *ideal* for the theories of $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$, as a natural analogue to *filter*.

7.1.3.1 Products and Quotients of Logics

We begin by considering *sources* and *sinks* of logics, and their associated *products* and *quotients*. We shall employ these notions to define the notions of *models* and *abstractions* of logics. The reader is urged to recall the definitions and results of $\oint$5.4.3.1, and $\oint$5.4.3.3.

Definition 7.10 (Products and Quotients of Logics) By a **language-translation** $\tau : \mathbf{A} \multimap \mathbf{B}$ from language $\mathbf{A}$ to $\mathbf{B}$, we mean a translation $\tau : \mathsf{Fm}(\mathbf{A}) \multimap \mathsf{Fm}(\mathbf{B})$. For logics L and M and languages $\mathbf{A}$ and $\mathbf{B}$, We write $\tau : \mathsf{L} \multimap \mathbf{B}$ for $\tau : \mathbf{lg}(\mathsf{L}) \multimap \mathbf{B}$, $\tau : \mathbf{A} \multimap \mathsf{M}$ for $\tau : \mathbf{A} \multimap \mathbf{lg}(\mathsf{M})$ and $\tau : \mathsf{L} \multimap \mathsf{M}$ for $\tau : \mathbf{lg}(\mathsf{L}) \multimap \mathbf{lg}(\mathsf{M})$.

A **source of logics** sc is determined by a *class* of pairs $\mathsf{Arrow}(\mathsf{sc})$, the members of which are called **source-arrows**, and a language $\mathbf{lg}(\mathsf{sc})$ called the **source language** for which we write $\mathsf{Fm}(\mathsf{sc})$ for $\mathsf{Fm}(\mathbf{lg}(\mathsf{sc}))$, such that, for each source-arrow $\langle \tau, \mathsf{M} \rangle \in \mathsf{Arrow}(\mathsf{sc})$, $\tau : \mathbf{lg}(\mathsf{sc}) \multimap \mathsf{M}$. For a language $\mathbf{A}$, a logic M and a translation $\tau : \mathbf{A} \multimap \mathsf{M}$, let $\langle \tau, \mathsf{M} \rangle$ denote the source of logics determined by the single source-arrow $\langle \tau, \mathsf{M}\rangle$ and language $\mathbf{A}$, which we call a **singleton logic source**. A source of logics is called **functional** if all translations are functions.

A **sink of logics** sk is determined by a *class* of pairs $\mathsf{Arrow}(\mathsf{sk})$, the members of which are called **sink-arrows**, and an associated language $\mathbf{lg}(\mathsf{sk})$ called the **language of the sink** for which we write $\mathsf{Fm}(\mathsf{sk})$ for $\mathsf{Fm}(\mathbf{lg}(\mathsf{sk}))$, such that, for each sink-arrow $\langle \mathsf{L}, \tau\rangle \in \mathsf{Arrow}(\mathsf{sk})$, $\tau : \mathsf{L} \multimap \mathbf{lg}(\mathsf{sk})$. For a logic L, a language $\mathbf{B}$ and a translation $\tau : \mathsf{L} \multimap \mathbf{B}$, let $\langle \mathsf{L}, \tau\rangle$ denote the sink of logics determined by the single pair $\langle \mathsf{L}, \tau\rangle$ and language $\mathbf{B}$, which we call a **singleton logic sink**. A sink of logics is called **functional** if all translations are functions.

Conventionally, we shall conflate the source of logics sc with the *source of closed systems* determined by universe $\mathsf{Fm}(\mathsf{sc})$ and family $\{\langle \tau, \mathsf{Fm}(\mathsf{M})\rangle \in \mathsf{sc} \,:\, \langle \tau, \mathsf{M}\rangle \in \mathsf{sc}\}$, and dually for sinks of logic.

For source of logics sc and sink of logics sk, we write $\mathsf{sc}^{\blacktriangleleft}$ for $L\,(\mathbf{lg}(\mathsf{sc}), \mathsf{sc}^{\blacktriangleleft})$, which we call the **product logic by source** sc, and we write $\coprod \mathsf{sk}$ for $L\,(\mathbf{lg}(\mathsf{sk}), \coprod \mathsf{sk})$, which we call the **quotient logic by sink** sc. For $\tau : \mathbf{A} \multimap \mathsf{M}$, we write $\tau^{\blacktriangleleft}[\mathsf{M}]$ for $\langle \tau, \mathsf{M}\rangle^{\blacktriangleleft}$, and for $\tau : \mathsf{L} \multimap \mathbf{B}$, we write $\tau[\mathsf{L}]$ for $\coprod\langle \mathsf{L}, \tau\rangle$, which we call the **product of** M **by** τ and the **quotient of** L **by** τ, respectively. □

7.1.3.2 Realizing Models

Recall Example 5.131 on page 210, where we showed how the notion of the *filters* of a sentential calculus $\mathcal{S}$ on an algebra $\mathbf{A}$ (see Definition 2.41 on page 101) may be characterized in terms of the *quotient* of the *sink* of all interpretations from $\mathcal{S}$ as a closed system into the universe of $\mathbf{A}$. We shall take this characterization as the starting point for our definition of the filters of a logic.

Definition 7.11 (The Modelling Sink) With each class $\mathfrak{L}$ of $\mathfrak{s}$-logics and each $\mathfrak{s}$-language $\mathbf{B}$, we associate the sink of logics $\mathfrak{L} \overset{\mathfrak{s}}{\rtimes} \mathbf{B}$ determined by language $\mathbf{B}$ and arrows $\{\langle \mathsf{L}, \mathsf{i}\rangle \,:\, \mathsf{L} \in \mathfrak{L}, \mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathsf{L}, \mathbf{B})\}$, which we call the **modelling $\mathfrak{s}$-sink of $\mathfrak{L}$ on $\mathbf{B}$**. □

Recall that we say that a sink of closed systems sk is *continuous into* a closed system $\mathbb{D}$ if $\mathsf{uni}(\mathbb{D}) = \mathsf{uni}(\mathsf{sk})$ and, for each $\langle \mathbb{C}, f\rangle \in \mathsf{sk}$, f is continuous from $\mathbb{C}$ into $\mathbb{D}$, and that the set of all such closed systems is denoted by $\mathsf{CContInto}(\mathsf{sk})$ (see Definition 5.122 on page 208).

Remark 7.12 $\mathsf{Mod}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B}) = \{L\,(\mathbf{B}, \mathbb{D}) \,:\, \mathbb{D} \in \mathsf{CContInto}(\mathfrak{L} \overset{\mathfrak{s}}{\rtimes} \mathbf{B})\}$.

Proof.
$\mathsf{M} \in \mathsf{Mod}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})$ [iff] $\mathbf{lg}(\mathsf{M}) = \mathbf{B}$ and $\forall\,[\mathsf{L} \in \mathfrak{L}]\,\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathsf{L}, \mathsf{M})]$ i is continuous from $\mathsf{Th}(\mathsf{L})$ into $\mathsf{Th}(\mathsf{M})$ [iff] $\mathbf{lg}(\mathsf{M}) = \mathbf{B}$ and $\forall\,[\langle \mathsf{L}, \mathsf{i}\rangle \in \mathfrak{L} \overset{\mathfrak{s}}{\rtimes} \mathbf{B}]$ i is continuous from $\mathsf{Th}(\mathsf{L})$ into $\mathsf{Th}(\mathsf{M})$ [iff] $\mathbf{lg}(\mathsf{M}) = \mathbf{B}$ and $\mathsf{Th}(\mathsf{M}) \in \mathsf{CContInto}(\mathfrak{L} \overset{\mathfrak{s}}{\rtimes} \mathbf{B})$ [iff] $\exists\,[\mathbb{D} \in \mathsf{CContInto}(\mathfrak{L} \overset{\mathfrak{s}}{\rtimes} \mathbf{B})]\,\mathsf{M} = L\,(\mathbf{B}, \mathbb{D})$ [iff] $\mathsf{M} \in \{L\,(\mathbf{B}, \mathbb{D}) : \mathbb{D} \in \mathsf{CContInto}(\mathfrak{L} \overset{\mathfrak{s}}{\rtimes} \mathbf{B})\}$.
◇

Definition 7.13 (Model Logics and Filters) We write $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})$ for $\coprod \mathfrak{L} \overset{\mathfrak{s}}{\rtimes} \mathbf{B}$, which we call the **model logic** or **filter logic** of $\mathfrak{L}$ on $\mathbf{B}$, and write $\mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})$ for $\mathsf{Th}(\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B}))$, the members of which are called **$\mathfrak{L}$-filters of B with respect to $\mathfrak{s}$** (or **$\langle \mathfrak{L}, \mathfrak{s}\rangle$-filters of B** or just **$\mathfrak{L}$-filters of B** when $\mathfrak{s}$ is understood), and write $\mathbf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})$ for $\mathbf{Th}(\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B}))$, $\Vdash^{\mathbf{B},\mathfrak{s}}_{\mathfrak{L}}$ for $\vdash_{\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})}$ $\dashv\Vdash^{\mathbf{B},\mathfrak{s}}_{\mathfrak{L}}$ for $\dashv\vdash_{\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})}$, and $\|\cdot\|^{\mathbf{B}}_{\mathsf{fi}_{\mathfrak{L}}}$ for $\|\cdot\|_{\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})}$. (Our usage of the term 'filter', is justified by Proposition 7.20.) For $f : \mathbf{B} \to_{\mathfrak{s}} \mathbf{C}$, we define a function $f^{\mathfrak{L}}_{\mathfrak{s}} : \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B}) \to \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C})$ by $f^{\mathfrak{L}}_{\mathfrak{s}}(F) = \| f\,[A] \|^{\mathbf{C}}_{\mathsf{fi}_{\mathfrak{L}}}$. In the case that $\mathfrak{L} = \{\mathsf{L}\}$, we write L for $\mathfrak{L}$ in all these definitions. □

The following corollary to Theorem 5.124 on page 208 (with Remark 7.12 on page 255) demonstrates that the filter logic $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ is indeed a model of $\mathfrak{L}$, and characterizes such models in terms of their granularity with respect to $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$.

Corollary 7.14 $\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B}) = [\mathsf{F}_{\mathfrak{L}}(\mathbf{B})\rangle_{\preceq}$. □

So $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ is the finest model of $\mathfrak{L}$ on $\mathbf{B}$. $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ models 'as much of' $\mathfrak{L}$ as is possible over language $\mathbf{B}$. Any other model will have less 'deducability'. Further, $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ serves as a 'canon' for all other models, in the sense that any logic coarser than $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ is itself a model, and these are the only models. We formalize these remarks for ease of later reference.

Remark 7.15 $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ is a model of $\mathfrak{L}$.

Remark 7.16 M is a model of $\mathfrak{L}$ iff $\mathsf{F}_{\mathfrak{L}}(\mathbf{lg}(\mathsf{M})) \preceq \mathsf{M}$ iff $\mathsf{Th}(\mathsf{M}) \subseteq \mathsf{Fi}_{\mathfrak{L}}(\mathbf{lg}(\mathsf{M}))$.

Corollary 7.17 The following conditions are equivalent.

1. $\mathsf{F}_{\mathfrak{L}}(\mathbf{B}) \preceq \mathsf{F}_{\mathfrak{L}'}(\mathbf{B})$.
2. $\mathsf{Fi}_{\mathfrak{L}}(\mathbf{B}) \supseteq \mathsf{Fi}_{\mathfrak{L}'}(\mathbf{B})$.
3. $[\mathsf{F}_{\mathfrak{L}}(\mathbf{B})\rangle_{\preceq} \supseteq [\mathsf{F}_{\mathfrak{L}'}(\mathbf{B})\rangle_{\preceq}$.
4. $\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B}) \supseteq \mathsf{Mod}_{\mathfrak{L}'}(\mathbf{B})$.

Proof. $\boxed{(1)\Leftrightarrow(2)}$ By definition. $\boxed{(1)\Leftrightarrow(3)}$ Trivial. $\boxed{(3)\Leftrightarrow(4)}$ By Corollary 7.14 on page 256. ⋄

We now consider the effect of granularity of a *single* logic L, on the models induced by this logic. The following result confirms the heuristics that if $\mathsf{L} \preceq \mathsf{L}'$, i.e., $\vdash_{\mathsf{L}} \subseteq \vdash_{\mathsf{L}'}$, then it must be *easier to model* L than model L' (less left consequences to be modelled as right consequences), and so L must have *more models* than L'.

Proposition 7.18 If $\mathsf{L} \preceq \mathsf{L}'$ then $\mathsf{Mod}_{\mathsf{L}}(\mathbf{B}) \supseteq \mathsf{Mod}_{\mathsf{L}'}(\mathbf{B})$, $\mathsf{F}_{\mathsf{L}}(\mathbf{B}) \preceq \mathsf{F}_{\mathsf{L}'}(\mathbf{B})$ and $\mathsf{Fi}_{\mathsf{L}}(\mathbf{B}) \supseteq \mathsf{Fi}_{\mathsf{L}'}(\mathbf{B})$.

Proof. Assume that $\mathsf{L} \preceq \mathsf{L}'$, i.e., $\mathbf{lg}(\mathsf{L}) = \mathbf{lg}(\mathsf{L}')$ and (i) $\mathsf{Th}(\mathsf{L}) \supseteq \mathsf{Th}(\mathsf{L}')$. It suffices, by Corollary 7.17 on page 256, to show that $\mathsf{Mod}_{\mathsf{L}}(\mathbf{B}) \supseteq \mathsf{Mod}_{\mathsf{L}'}(\mathbf{B})$. Let $\mathsf{M} \in \mathsf{Mod}_{\mathsf{L}'}(\mathbf{B})$, i.e., $\mathbf{lg}(\mathsf{M}) = \mathbf{B}$ and (ii) for all $\mathsf{i} \in \mathsf{Int}(\mathsf{L}', \mathbf{B})$ and all $R \in \mathsf{Th}(\mathsf{M})$, $\mathsf{i}^{-1}[R] \in \mathsf{Th}(\mathsf{L}')$. (*We must show that* $\mathsf{M} \in \mathsf{Mod}_{\mathsf{L}}(\mathbf{B})$.) Certainly $\mathbf{lg}(\mathsf{M}) = \mathbf{B}$. Let $\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathbf{B})$ and let $R \in \mathsf{Th}(\mathsf{M})$. Since $\mathbf{lg}(\mathsf{L}) = \mathbf{lg}(\mathsf{L}')$, $\mathsf{i} \in \mathsf{Int}(\mathsf{L}', \mathbf{B})$, and since $R \in \mathsf{Th}(\mathsf{M})$, it follows from (ii), that $\mathsf{i}^{-1}[R] \in \mathsf{Th}(\mathsf{L}')$. But $\mathsf{Th}(\mathsf{L}) \supseteq \mathsf{Th}(\mathsf{L}')$ by (i), so $\mathsf{i}^{-1}[R] \in \mathsf{Th}(\mathsf{L})$. ⋄

Corollary 7.19 $\mathsf{Mod}_{\mathsf{L}}(\mathbf{B}) = \mathsf{Mod}_{\langle \mathsf{L}]_{\preceq}}(\mathbf{B})$

Proof. By Remark 7.8 on page 253, $\mathsf{Mod}_{\mathsf{L}}(\mathbf{B}) \supseteq \mathsf{Mod}_{(\mathsf{L}]_{\preceq}}(\mathbf{B})$. Suppose that $\mathsf{M} \in \mathsf{Mod}_{\mathsf{L}}(\mathbf{B})$ and $\mathsf{L}' \preceq \mathsf{L}$. (*We must show that* $\mathsf{M} \in \mathsf{Mod}_{\mathsf{L}'}(\mathbf{B})$.) Since $\mathsf{L}' \preceq \mathsf{L}$, by Proposition 7.18 on page 256, $\mathsf{Mod}_{\mathsf{L}}(\mathbf{B}) \subseteq \mathsf{Mod}_{\mathsf{L}'}(\mathbf{B})$. $\diamond$

We are able to characterize the model logic $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$, and in so doing, justify our usage of the term 'filter'. This result follows immediately from Definition 5.122 on page 208 together with Theorem 5.125 on page 209. The reader should compare this result with Definition 2.41 on page 101.

Corollary 7.20 The following conditions are equivalent.

1. $F \in \mathsf{Fi}_{\mathfrak{L}}(\mathbf{B})$.
2. $\forall\,[\mathsf{L} \in \mathfrak{L}]\,\forall\,[\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathbf{B})]\;\mathsf{i}^{-1}[F] \in \mathsf{Th}(\mathsf{L})$.
3. $\forall\,[\mathsf{L} \in \mathfrak{L}]\,\forall\,[\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathbf{B})]\;\Gamma \vdash_{\mathsf{L}} \phi$ and $\mathsf{i}[\Gamma] \subseteq F$ implies $\mathsf{i}[\phi] \in F$.

In particular, the following conditions are equivalent.

1. $F \in \mathsf{Fi}_{\mathsf{L}}(\mathbf{B})$.
2. $\forall\,[\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathbf{B})]\;\mathsf{i}^{-1}[F] \in \mathsf{Th}(\mathsf{L})$.
3. $\forall\,[\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathbf{B})]\;\Gamma \vdash_{\mathsf{L}} \phi$ and $\mathsf{i}[\Gamma] \subseteq F$ implies $\mathsf{i}[\phi] \in F$.

Remark 7.21 $\emptyset \in \mathsf{Fi}_{\mathsf{L}}(\mathbf{B})$ iff $\mathsf{Thm}(\mathsf{L}) = \emptyset$. $\square$

The model logic reflects finitariness, as formalized in the next result which is an immediate corollary to Proposition 5.127 on page 209.

Corollary 7.22 If all logics in $\mathfrak{L}$ are finitary then $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ is finitary.

Corollary 7.23 If $\mathfrak{L} \subseteq \mathfrak{L}'$ then $\mathsf{F}_{\mathfrak{L}}(\mathbf{B}) \preceq \mathsf{F}_{\mathfrak{L}'}(\mathbf{B})$.

Proof. $\mathsf{Fi}_{\mathfrak{L}}(\mathbf{B}) = \{F \subseteq \mathsf{Fm}(\mathbf{B}) : \forall\,[\mathsf{L} \in \mathfrak{L}]\,\forall\,[\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathbf{B})]\;\mathsf{i}^{-1}[F] \in \mathsf{Th}(\mathsf{L})\} \supseteq \{F \subseteq \mathsf{Fm}(\mathbf{B}) : \forall\,[\mathsf{L} \in \mathfrak{L}']\,\forall\,[\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathbf{B})]\;\mathsf{i}^{-1}[F] \in \mathsf{Th}(\mathsf{L})\} = \mathsf{Fi}_{\mathfrak{L}'}(\mathbf{B})$. $\diamond$

In the next result, we demonstrate that all $\mathfrak{s}$-morphisms between $\mathfrak{s}$-languages $\mathbf{B}$ and $\mathbf{C}$ are continuous from $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})$ into $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C})$.

Proposition 7.24 Let $\mathfrak{L}$ be a set of $\mathfrak{s}$-logics and let $\mathbf{B}$ and $\mathbf{C}$ be $\mathfrak{s}$-languages. If $f : \mathbf{B} \to_{\mathfrak{s}} \mathbf{C}$, then f is continuous from $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})$ into $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C})$. Consequently,

1. $f^{-1}[G] \in \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B}) \quad (\forall\,[G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C})]\,)$,
2. $A \Vdash^{\mathbf{B}}_{\mathfrak{L}} b$ implies $f[A] \Vdash^{\mathbf{C}}_{\mathfrak{L}} f(b) \quad (\forall\,[A \cup \{b\} \subseteq \mathsf{Fm}(\mathbf{B})]\,)$,
3. $f[A] \dashv\Vdash^{\mathbf{C}}_{\mathfrak{L}} f\left[\|A\|^{\mathbf{B}}_{\mathsf{fi}_{\mathfrak{L}}}\right] \quad (\forall\,[A \subseteq \mathsf{Fm}(\mathbf{B})]\,)$,
4. $f[A] \dashv\Vdash^{\mathbf{C}}_{\mathfrak{L}} f^{\mathfrak{L}}_{\mathfrak{s}}(\|A\|^{\mathbf{B}}_{\mathsf{fi}_{\mathfrak{L}}}) \quad (\forall\,[A \subseteq \mathsf{Fm}(\mathbf{B})]\,)$,

5. $f\left[\|A\|^{\mathbf{B}}_{\mathsf{fi}_{\mathfrak{L}}}\right] \subseteq f^{\mathfrak{L}}_{\mathfrak{s}}(A) \qquad (\forall\,[A \subseteq \mathsf{Fm}(\mathbf{B})]\,),$

6. $f^{\mathfrak{L}}_{\mathfrak{s}}(\|A\|^{\mathbf{B}}_{\mathsf{fi}_{\mathfrak{L}}}) = f^{\mathfrak{L}}_{\mathfrak{s}}(A) \qquad (\forall\,[A \subseteq \mathsf{Fm}(\mathbf{B})]\,),$

7. $\|A\|^{\mathbf{B}}_{\mathsf{fi}_{\mathfrak{L}}} \subseteq f^{-1}\left[f^{\mathfrak{L}}_{\mathfrak{s}}(A)\right] \qquad (\forall\,[A \subseteq \mathsf{Fm}(\mathbf{B})]\,),$

8. $f^{-1}\left[\|C\|^{\mathbf{C}}_{\mathsf{fi}_{\mathfrak{L}}}\right] \supseteq \|f^{-1}[C]\|^{\mathbf{B}}_{\mathsf{fi}_{\mathfrak{L}}} \qquad (\forall\,[C \subseteq \mathsf{Fm}(\mathbf{C})]\,),$

9. $f^{\mathfrak{L}}_{\mathfrak{s}} : \mathbf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B}) \rightarrow_{\blacktriangledown} \mathbf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C})$ and

10. $f^{-1}[\cdot]_{|\mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C})} : \mathbf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C}) \rightarrow_{\blacktriangle} \mathbf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B}).$

Proof. Let $G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C})$. Let $\mathsf{L} \in \mathfrak{L}$ and $\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathsf{L}, \mathbf{B})$. (*By Corollary 7.20, it suffices to show that* $\mathsf{i}^{-1}\left[f^{-1}[G]\right] \in \mathsf{Th}(\mathsf{L})$.) Since $f\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathsf{L}, \mathbf{C})$, $(f\mathsf{i})^{-1}[G] \in \mathsf{Th}(\mathsf{L})$. Since $(f\mathsf{i})^{-1}[G] = \mathsf{i}^{-1}\left[f^{-1}[G]\right]$, $\mathsf{i}^{-1}\left[f^{-1}[G]\right] \in \mathsf{Th}(\mathsf{L})$. ◇

The proof of the following result requires the axiom of choice.

Proposition 7.25 Let $\mathcal{D}$ be an $\mathfrak{s}$-*deductive system* with $\mathfrak{s}$-*global* language $\mathbf{G}$ and let $\mathbf{B}$ and $\mathbf{C}$ be $\mathfrak{s}$-languages. If $f : \mathbf{B} \twoheadrightarrow_{\mathfrak{s}} \mathbf{C}$ and $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{B})$ with $\equiv_f$ compatible with F, then $f[F] \in \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{C})$.

Proof. Suppose that $\Gamma \vdash_{\mathcal{D}} \phi$, $\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{G}, \mathbf{C})$ and $\mathsf{i}[\Gamma] \subseteq f[F]$. (*By Corollary 7.20, it suffices to show that* $\mathsf{i}(\phi) \in f[F]$.) For each variable $x \in \mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$, pick $b_x \in \overleftarrow{f}[\![\mathsf{i}(x)]\!]$, the pre-pole being non-empty since f is assumed to be surjective. Then $f(b_x) = \mathsf{i}(x)$. Let j be the unique interpretation of $\mathbf{G}$ into $\mathbf{B}$ mapping $x \mapsto b_x$, for all $x \in \mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$.

$\boxed{\text{Claim: } f\mathsf{j} = \mathsf{i}}$ $f\mathsf{j}$ and i are both are morphisms from $\mathbf{G}$ into $\mathbf{C}$. For each variable $x \in \mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$, $f\mathsf{j}(x) = f(\mathsf{j}(x)) = f(b_x) = \mathsf{i}(x)$. So equality follows by the $\mathfrak{s}$-freedom of $\mathbf{G}$.

So $f\mathsf{j}[\Gamma] = \mathsf{i}[\Gamma] \subseteq f[F]$, and so $\mathsf{j}[\Gamma] \subseteq f^{-1}[f[F]] = F$, by Remark 1.72 on page 25 and the assumed compatibility of $\equiv_f$ with F. Since F is a filter, $\Gamma \vdash_{\mathcal{D}} \phi$, $\mathsf{j} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{G}, \mathbf{B})$ and $\mathsf{j}[\Gamma] \subseteq F$, we have $\mathsf{j}(\phi) \in F$, by Corollary 7.20. Hence $\mathsf{i}(\phi) = f(\mathsf{j}(\phi)) \in f[F]$. ◇

7.1.3.3 Realizing Abstractions

We now consider the natural generalization, of the semantic consequence relation determined by a matrix (see Definition 2.32 on page 99), to our context. Recall Example 5.121 on page 208, where we showed that the semantic consequence relation $\models^{\mathbf{M}}$ determined by a $\mathfrak{p}$-matrix $\mathbf{M}$ may be realized as the *product of a source*. That example motivates the approach that we shall take to semantic consequence.

Definition 7.26 (The Abstraction Source) With each class $\mathfrak{M}$ of $\mathfrak{s}$-languages and each $\mathfrak{s}$-logic M, we associate the source of logics $\mathbf{A} \overset{\mathfrak{s}}{\ltimes} \mathfrak{M}$, determined by language $\mathbf{A}$ and arrows $\{\langle \mathsf{i}, \mathsf{M}\rangle : \mathsf{M} \in \mathfrak{M}, \mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{A}, \mathsf{M})\}$, which we call the **abstraction $\mathfrak{s}$-source of $\mathfrak{M}$ on $\mathbf{A}$**. □

Recall that we say that a source of closed systems sc is *continuous from* a closed system $\mathbb{C}$ if $\mathsf{uni}(\mathbb{C}) = \mathsf{uni}(\mathsf{sc})$ and, for each $\langle f, \mathbb{D}\rangle \in \mathsf{sc}$, f is continuous from $\mathbb{C}$ into $\mathbb{D}$, and that we denote the set of all such closed systems by $\mathsf{CContFrom}(\mathsf{sc})$ (see Definition 5.122 on page 208).

Remark 7.27 $\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A}) = \{L(\mathbf{A}, \mathbb{C}) : \mathbb{C} \in \mathsf{CContFrom}(\mathbf{A} \ltimes \mathfrak{M})\}$.

Proof. $\mathsf{L} \in \mathsf{Abs}_{\mathfrak{M}}(\mathbf{A})$ [iff] $\mathbf{lg}(\mathsf{L}) = \mathbf{A}$ and L is an abstraction of $\mathfrak{M}$ [iff] $\mathbf{lg}(\mathsf{L}) = \mathbf{A}$ and $\forall\, [\mathsf{M} \in \mathfrak{M}]\ \forall\, [\mathsf{i} \in \mathsf{Int}(\mathsf{L},\mathsf{M})]$ i is continuous from $\mathsf{Th}(\mathsf{L})$ into $\mathsf{Th}(\mathsf{M})$ [iff] $\mathbf{lg}(\mathsf{L}) = \mathbf{A}$ and $\forall\, [\langle \mathsf{i},\mathsf{M}\rangle \in \mathbf{A} \ltimes \mathfrak{M}]$ i is continuous from $\mathsf{Th}(\mathsf{L})$ into $\mathsf{Th}(\mathsf{M})$ [iff] $\mathbf{lg}(\mathsf{L}) = \mathbf{A}$ and $\mathsf{Th}(\mathsf{L}) \in \mathsf{CContFrom}(\mathbf{A} \ltimes \mathfrak{M})$ [iff] $\exists\, [\mathbb{C} \in \mathsf{CContFrom}(\mathbf{A} \ltimes \mathfrak{M})]\ \mathsf{L} = L\,(\mathbf{A},\mathbb{C})$ [iff] $\mathsf{L} \in \{L\,(\mathbf{A},\mathbb{C}) : \mathbb{C} \in \mathsf{CContFrom}(\mathbf{A} \ltimes \mathfrak{M})\}$ $\diamond$

Definition 7.28 (Abstract Logics and Ideals) We write $\mathsf{I}^{\mathfrak{s}}_{\mathfrak{M}}(\mathbf{A})$ for $\mathbf{A} \overset{\mathfrak{s}}{\ltimes} \mathfrak{M}^{\blacktriangleleft}$, which we call the **abstract logic** or **ideal logic** of $\mathfrak{M}$ on $\mathbf{A}$. We write $\mathsf{Id}^{\mathfrak{s}}_{\mathfrak{M}}(\mathbf{A})$ for $\mathsf{Th}(\mathsf{I}^{\mathfrak{s}}_{\mathfrak{M}}(\mathbf{A}))$, the members of which are called **$\mathfrak{M}$-ideals** on $\mathbf{A}$ (with respect to $\mathfrak{s}$), and write $\models^{\mathfrak{M},\mathfrak{s}}_{\mathbf{A}}$ for $\vdash_{\mathsf{I}^{\mathfrak{s}}_{\mathfrak{M}}(\mathbf{A})}$. Let $\mathsf{PId}^{\mathfrak{s}}_{\mathfrak{M}}(\mathbf{A}) = \{\mathsf{i}^{-1}[T] : \mathsf{M} \in \mathfrak{M}, \mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{A}, \mathbf{lg}(\mathsf{M})), T \in \mathsf{Th}(\mathsf{M})\}$, the members of which are called **principal $\mathfrak{M}$-ideals**. In the case that $\mathfrak{M} = \{\mathsf{M}\}$, we write M for $\mathfrak{M}$ in these definitions. □

The following result is an immediate corollary to Theorem 5.115 on page 207.

Corollary 7.29 $\mathsf{I}^{\mathfrak{s}}_{\mathfrak{M}}(\mathbf{A})$ is the coarsest $\mathbf{A}$-logic L such that every interpretation of $\mathbf{A}$ into the language of every logic $\mathsf{M} \in \mathfrak{M}$ is continuous from L into M.

Remark 7.30 The principal ideals $\mathsf{PId}_{\mathfrak{M}}(\mathbf{A})$ form a theory-base for the abstract logic $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$, by definition. □

Symmetrically to the fact that $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ is the finest model of $\mathfrak{L}$ on $\mathbf{B}$, we have the following result which follows immediately from Theorem 5.115 on page 207 and Remark 7.27 on page 258.

Corollary 7.31 $\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A}) = \langle \mathsf{I}_{\mathfrak{M}}(\mathbf{A})]_{\preceq}$. □

So $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ is the coarsest abstraction of $\mathfrak{M}$ on $\mathbf{A}$. Any other abstraction will have more 'deducability'. The logic $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ serves as a canon for abstracts, in the sense that any logic finer than $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ is itself an abstract, and these are the only abstracts. Notice how this relationship is the inverse of the analogues for models and $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$. We formalize these remarks for ease of later reference.

Remark 7.32 $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ is an abstraction of $\mathfrak{M}$.

Remark 7.33 L is an abstraction of $\mathfrak{M}$ iff $\mathsf{L} \preceq \mathsf{I}_{\mathfrak{M}}(\mathbf{A})$.

Corollary 7.34 The following conditions are equivalent.

1. $\mathsf{I}_{\mathfrak{M}}(\mathbf{A}) \preceq \mathsf{I}_{\mathfrak{M}'}(\mathbf{A})$.

2. $\mathsf{Id}_{\mathfrak{M}'}(\mathbf{A}) \subseteq \mathsf{Id}_{\mathfrak{M}}(\mathbf{A})$.

3. $\mathsf{PId}_{\mathfrak{M}'}(\mathbf{A}) \subseteq \mathsf{PId}_{\mathfrak{M}}(\mathbf{A})$.

4. $\langle \mathsf{I}_{\mathfrak{M}}(\mathbf{A})]_{\preceq} \subseteq \langle \mathsf{I}_{\mathfrak{M}'}(\mathbf{A})]_{\preceq}$.

5. $\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A}) \subseteq \mathsf{Abs}_{\mathfrak{M}'}(\mathbf{A})$.

Proof. $\boxed{(1)\Leftrightarrow(2)\Leftrightarrow(3)}$ By definition. $\boxed{(1)\Leftrightarrow(4)}$ Trivial. $\boxed{(4)\Leftrightarrow(5)}$ By Corollary 7.31 on page 259, $\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A}) = \langle \mathsf{I}_{\mathfrak{M}}(\mathbf{A})]_{\preceq}$ and $\mathsf{Abs}_{\mathfrak{M}'}(\mathbf{A}) = \langle \mathsf{I}_{\mathfrak{M}'}(\mathbf{A})]_{\preceq}$, so the result follows easily. ◇

Recall that $\mathfrak{L} \subseteq \mathfrak{L}'$ implies $\mathsf{F}_{\mathfrak{L}}(\mathbf{B}) \preceq \mathsf{F}_{\mathfrak{L}'}(\mathbf{B})$ (see Corollary 7.23 on page 257).

Remark 7.35 If $\mathfrak{M} \subseteq \mathfrak{M}'$ then $\mathsf{I}_{\mathfrak{M}'}(\mathbf{A}) \preceq \mathsf{I}_{\mathfrak{M}}(\mathbf{A})$.

Proof. $\mathsf{PId}_{\mathfrak{M}}(\mathbf{A}) = \{\mathsf{i}^{-1}[T] : \mathsf{M} \in \mathfrak{M}, \mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M}), T \in \mathsf{Th}(\mathsf{M})\} \subseteq \{\mathsf{i}^{-1}[T] : \mathsf{M} \in \mathfrak{M}', \mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M}), T \in \mathsf{Th}(\mathsf{M})\} = \mathsf{PId}_{\mathfrak{M}'}(\mathbf{A})$. The result follows by Corollary 7.34 on page 259. ◇

If $\mathsf{M} \preceq \mathsf{M}'$, i.e., $\vdash_{\mathsf{M}} \subseteq \vdash_{\mathsf{M}'}$, then it must be *easier to abstract* M' than abstract M (more right consequences permit more possible left consequences), and so M' must have *more abstracts* than M'.

Proposition 7.36 If $\mathsf{M} \preceq \mathsf{M}'$ then $\mathsf{Abs}_{\mathbf{A}}(\mathsf{M}) \subseteq \mathsf{Abs}_{\mathbf{A}}(\mathsf{M}')$, $\mathsf{I}_{\mathsf{M}}(\mathbf{A}) \preceq \mathsf{I}_{\mathsf{M}'}(\mathbf{A})$, $\mathsf{Id}_{\mathsf{M}'}(\mathbf{A}) \subseteq \mathsf{Id}_{\mathsf{M}}(\mathbf{A})$ and $\mathsf{PId}_{\mathsf{M}'}(\mathbf{A}) \subseteq \mathsf{PId}_{\mathsf{M}}(\mathbf{A})$.

Proof. Assume that $\mathsf{M} \preceq \mathsf{M}'$, i.e., $\mathbf{lg}(\mathsf{M}) = \mathbf{lg}(\mathsf{M}')$ and (i) $\mathsf{Th}(\mathsf{M}) \supseteq \mathsf{Th}(\mathsf{M}')$. (*It suffices, by Corollary 7.34 on page 259, to show that* $\mathsf{Abs}_{\mathbf{A}}(\mathsf{M}) \subseteq \mathsf{Abs}_{\mathbf{A}}(\mathsf{M}')$.) Let $\mathsf{L} \in \mathsf{Abs}_{\mathbf{A}}(\mathsf{M})$, i.e., $\mathbf{lg}(\mathsf{L}) = \mathbf{A}$ and (ii) for all $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$ and all $R \in \mathsf{Th}(\mathsf{M})$, $\mathsf{i}^{-1}[R] \in \mathsf{Th}(\mathsf{L})$. (*We must show that* $\mathsf{L} \in \mathsf{Abs}_{\mathbf{A}}(\mathsf{M}')$.) Certainly $\mathbf{lg}(\mathsf{L}) = \mathbf{A}$. Let $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M}')$ and let $R \in \mathsf{Th}(\mathsf{M}')$. (*We must show that* $\mathsf{i}^{-1}[R] \in \mathsf{Th}(\mathsf{L})$.) By (i) $R \in \mathsf{Th}(\mathsf{M})$, and since $\mathbf{lg}(\mathsf{M}) = \mathbf{lg}(\mathsf{M}')$, $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$, it follows, by (ii), that $\mathsf{i}^{-1}[R] \in \mathsf{Th}(\mathsf{L})$, as required. ◇

Corollary 7.37 $\mathsf{Abs}_{\mathsf{M}}(\mathbf{A}) = \mathsf{Abs}_{[\mathsf{M}\rangle_{\preceq}}(\mathbf{A})$.

Proof. By Remark 7.9 on page 254, $\mathsf{Abs}_{\mathsf{M}}(\mathbf{A}) \supseteq \mathsf{Abs}_{[\mathsf{M}\rangle_{\preceq}}(\mathbf{A})$. Suppose that $\mathsf{L} \in \mathsf{Abs}_{\mathsf{M}}(\mathbf{A})$ and $\mathsf{M} \preceq \mathsf{M}'$. (*We must show that* $\mathsf{L} \in \mathsf{Abs}_{\mathsf{M}'}(\mathbf{A})$.) Since $\mathsf{M} \preceq \mathsf{M}'$, by Proposition 7.36 on page 260, $\mathsf{Abs}_{\mathbf{A}}(\mathsf{M}) \subseteq \mathsf{Abs}_{\mathbf{A}}(\mathsf{M}')$. ◇

In the following corollary to Theorem 5.113 on page 206, we characterize the consequence relation $\models^{\mathfrak{M}}_{\mathbf{A}}$ of the ideal logic. The second equivalent condition in the characterization can be seen as a natural generalization of Blok and Pigozzi's technique by which a matrix or set of matrices (over algebras) induce consequence relations on the terms/formulae, which are best viewed as *semantic* consequence relations, which they employ in obtaining the notion that a matrix or set of matrices be a *model* of a deductive system [vA95, p.g. 80].

Corollary 7.38 (Characterizing Consequence in the Ideal Logic) The following conditions are equivalent.

1. $\Gamma \models^{\mathfrak{M}}_{\mathbf{A}} \phi$.
2. $\mathsf{i}[\Gamma] \in T$ implies $\mathsf{i}(\phi) \in T$, for all $\mathsf{M} \in \mathfrak{M}$, $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$ and $T \in \mathsf{Th}(\mathsf{M})$.
3. $\mathsf{i}[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi)$, for all $\mathsf{M} \in \mathfrak{M}$ and $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$.

In particular, the following are equivalent.

1. $\Gamma \models^{\mathsf{M}}_{\mathbf{A}} \phi$.

2. $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$, $T \in \mathsf{Th}(\mathsf{M})$ and $\mathsf{i}[\Gamma] \in T$ implies $\mathsf{i}(\phi) \in T$.

3. $\mathsf{i}[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi)$, for all $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$.

$\square$

The following result follows from the previous corollary, together with the fact that the identity function on a language is an interpretation of that language onto itself.

Lemma 7.39 If $\mathsf{M} \in \mathfrak{M}$ then $\mathsf{I}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{M})) \preceq \mathsf{M}$.

Proof. Suppose that $\Gamma \models^{\mathfrak{M}}_{\mathbf{lg}(\mathsf{M})} \phi$. Since $\mathsf{M} \in \mathfrak{M}$ and $\mathsf{id} : \mathbf{lg}(\mathsf{M}) \to \mathbf{lg}(\mathsf{M})$, by Corollary 7.38 on page 260, $\mathsf{id}\,[\Gamma] \vdash_{\mathsf{M}} \mathsf{id}(\phi)$, i.e., $\Gamma \vdash_{\mathsf{M}} \phi$. $\diamond$

7.1.4 Inter-relating Models and Abstracts

We now show that our notion of model coincides with (the natural generalization of) that of Blok and Pigozzi (see Definition 2.56 on page 103).

Theorem 7.40 Let L be a logic with language $\mathbf{A}$ and $\mathfrak{M}$ a set of logics. The following conditions are equivalent.

1. $\mathfrak{M}$ constitutes a model of L.

2. $\mathsf{L} \preceq \mathsf{I}_{\mathfrak{M}}(\mathbf{A})$.

3. $\Gamma \vdash_{\mathsf{L}} \phi$ implies $\Gamma \models^{\mathfrak{M}}_{\mathbf{A}} \phi$, for all $\Gamma \cup \phi \subseteq \mathsf{Fm}(\mathbf{A})$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Suppose that $\Gamma \vdash_{\mathsf{L}} \phi$. (*We must show that* $\Gamma \models^{\mathfrak{M}}_{\mathbf{A}} \phi$.) Let $\mathsf{M} \in \mathfrak{M}$ and i an interpretation of $\mathbf{A}$ into $\mathbf{lg}(\mathsf{M})$. (*By Corollary 7.38 on page 260, it suffices to show that* $\mathsf{i}[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi)$.) Since $\mathfrak{M}$ constitutes a model of L and $\mathsf{M} \in \mathfrak{M}$, i is continuous from L into M. So since $\Gamma \vdash_{\mathsf{L}} \phi$, $\mathsf{i}[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi)$. $\boxed{(2)\Leftarrow(1)}$ Suppose that $\mathsf{L} \preceq \mathsf{I}_{\mathfrak{M}}(\mathbf{A})$. Suppose that $\Gamma \vdash_{\mathsf{L}} \phi$, $\mathsf{M} \in \mathfrak{M}$ and $\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathsf{M})$. (*We must show that* $\mathsf{i}\,[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi)$.) Since $\Gamma \vdash_{\mathsf{L}} \phi$ and $\mathsf{L} \preceq \mathsf{I}_{\mathfrak{M}}(\mathbf{A})$, $\Gamma \models^{\mathfrak{M}}_{\mathbf{A}} \phi$, and so by Corollary 7.38, $\mathsf{i}\,[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi)$. $\boxed{(2)\Leftrightarrow(3)}$ Definitional. $\diamond$

Next, we aim to now show that $\mathfrak{M} \subseteq \mathsf{Mod}_{\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A})} = \mathsf{Mod}_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{B})$ and $\mathfrak{L} \subseteq \mathsf{Abs}_{\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B})} = \mathsf{Abs}_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{A})$. This result proves useful to the analysis of structurality of the next section.

Theorem 7.41 Let $\mathfrak{L}$ and $\mathfrak{M}$ be sets of logics and $\mathbf{A}$ and $\mathbf{B}$ languages.

1. $\mathsf{Mod}_{\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{B}) = \mathsf{Mod}_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{B}) = \langle \mathsf{F}_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{B})]_{\preceq}$.

2. $\mathsf{Abs}_{\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{A}) = \mathsf{Abs}_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{A}) = [\mathsf{I}_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{A})\rangle_{\preceq}$.

3. $\mathsf{F}_{\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{B}) = \mathsf{F}_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{B})$.

4. $\mathsf{I}_{\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{A}) = \mathsf{I}_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{A})$.

5. $\forall\,[\mathsf{M} \in \mathfrak{M}]\ \mathsf{M} \in \mathsf{Mod}_{\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{lg}(\mathsf{M})) = \mathsf{Mod}_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{lg}(\mathsf{M})) = \langle \mathsf{F}_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{lg}(\mathsf{M}))]_{\preceq}$.

6. $\forall\,[\mathsf{L} \in \mathfrak{L}]\ \mathsf{L} \in \mathsf{Abs}_{\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{lg}(\mathsf{L})) = \mathsf{Abs}_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{lg}(\mathsf{L})) = [\mathsf{I}_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{lg}(\mathsf{L}))\rangle_{\preceq}$.

7. $\mathfrak{M} \subseteq \mathsf{Mod}_{\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A})} = \mathsf{Mod}_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})}$.

8. $\mathfrak{L} \subseteq \mathsf{Abs}_{\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B})} = \mathsf{Abs}_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}$.

Proof. $\boxed{1}$ By Corollary 7.19 on page 256, $\mathsf{Mod}_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{B}) = \mathsf{Mod}_{\langle \mathsf{I}_{\mathfrak{M}}(\mathbf{A})]_{\preceq}}(\mathbf{B}) = \mathsf{Mod}_{\mathsf{Abs}_{\mathfrak{M}}(\mathbf{A})}(\mathbf{B})$, by Corollary 7.31 on page 259. The second equality holds by Corollary 7.14 on page 256. $\boxed{2}$ By Corollary 7.37 on page 260, $\mathsf{Abs}_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{A}) = \mathsf{Abs}_{[\mathsf{F}_{\mathfrak{L}}(\mathbf{B}))\rangle_{\preceq}}(\mathbf{A}) = \mathsf{Abs}_{\mathsf{Mod}_{\mathfrak{L}}(\mathbf{B})}(\mathbf{A})$, by Corollary 7.14 on page 256. The second equality holds by Corollary 7.31 on page 259. $\boxed{\text{3 and 4}}$ Follow from 1 and 2 by Corollary 7.17 on page 256 and Corollary 7.34 on page 259 respectively. $\boxed{5}$ Let $\mathsf{M} \in \mathfrak{M}$. (*We must show that* M *is a model of* $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ *on* $\mathbf{lg}(\mathsf{M})$.) Let $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$. (*We must show that* i *is continuous from* $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ *into* M.) But this is so by Remark 5.114 on page 207. The remaining equalities follow by (1). $\boxed{6}$ Let $\mathsf{L} \in \mathfrak{L}$. (*We must show that* L *is an abstract of* $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ *on* $\mathbf{lg}(\mathsf{L})$.) Let $\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathbf{B})$. (*We must show that* i *is continuous from* L *into* $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$.) But this is so by Remark 5.123 on page 208. The remaining equalities follow by (2). $\boxed{\text{7 and 8}}$ Follow by definitions from (5) and (6). $\diamond$

Of particular importance to us is the case that $\mathfrak{L} = \{\mathsf{L}\}$, which we explore in the next section on structurality.

7.2 On Structurality

We return to structurality. Recall the notion that a logic be structural, given in Definition 6.14 on page 225. The most important case of the previous section, occurs when $\mathfrak{L} = \{\mathsf{L}\} = \mathfrak{M}$. Beginning with a logic L, we consider the models and abstracts of L on its own language. These in turn give rise to their own abstracts and models on the language of L. Structurality is the key to understanding (and simplifying) these and other similar relationships.

Convention 7.42 (For this Section Only) We shall assume, for the duration of this section, that L is a logic with language $\mathbf{L}$.

We begin by noting that a logic is structural precisely when it models itself (equivalently abstracts itself), the proof of which follows directly from definitions.

Theorem 7.43 For a logic L, the following conditions are equivalent.

1. L is structural.

2. $\mathsf{L} \in \mathsf{Mod}_{\mathsf{L}}(\mathbf{L})$.

3. $\mathsf{L} \in \mathsf{Abs}_{\mathsf{L}}(\mathbf{L})$.

$\square$

Recall Theorem 7.41 on page 261. In the case that $\mathfrak{L} = \{\mathsf{L}\} = \mathfrak{M}$, we obtain the following relationship (*without* any assumption of structurality of L).

$$\mathsf{L} \in \mathsf{Mod}_{\mathsf{Abs}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) \cap \mathsf{Abs}_{\mathsf{Mod}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \left[\mathsf{I}_{\mathsf{F}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}), \mathsf{F}_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L})\right]_{\preceq}. \tag{7.2}$$

The key to the sequel is the structurality of both the model logic $\mathsf{F}_{\mathsf{L}}(\mathbf{L})$ and the abstract logic $\mathsf{I}_{\mathsf{L}}(\mathbf{L})$ (*independently of the structurality of* L), and the simplifications that follow.

Although our definitions have been phrased in terms of a signature, i.e., a construct of objects, this signature has played no role in the form thus far, other than serving as a convenient carrier of functions. We now consider results that take advantage of the underlying signature, in particular, the existence of an identity substitution and the closure of interpretations under composition.

Proposition 7.44 $\mathsf{I}_{\mathsf{L}}(\mathbf{L}) \preceq \mathsf{L} \preceq \mathsf{F}_{\mathsf{L}}(\mathbf{L})$.

Proof. $\boxed{\mathsf{I}_{\mathsf{L}}(\mathbf{L}) \preceq \mathsf{L}}$ Suppose that $\Gamma \vdash_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})} \phi$, i.e., $\Gamma \models^{\mathsf{L}}_{\mathbf{L}} \phi$. (*We must show that* $\Gamma \vdash_{\mathsf{L}} \phi$.) Since the identity function $\mathsf{id}_{\mathbf{L}}$ is a substitution of $\mathbf{L}$, by Corollary 7.38 on page 260, $\mathsf{id}_{\mathbf{L}}[\Gamma] \vdash_{\mathsf{L}} \mathsf{id}_{\mathbf{L}}(\phi)$. In other words, $\Gamma \vdash_{\mathsf{L}} \phi$. $\boxed{\mathsf{L} \preceq \mathsf{F}_{\mathsf{L}}(\mathbf{L})}$ Let $T \in \mathsf{Th}(\mathsf{F}_{\mathsf{L}}(\mathbf{L})) = \mathsf{Fi}_{\mathsf{L}}(\mathbf{L})$. Since the identity function $\mathsf{id}_{\mathbf{L}}$ is a substitution of $\mathbf{L}$, $T = \mathsf{id}_{\mathbf{L}}{}^{-1}[T] \in \mathsf{Th}(\mathsf{L})$, by Corollary 7.20 on page 257. $\diamond$

Notice the 'inverted' nature of the relationships of the previous proposition to those of Remark 7.16 and Remark 7.33.

Theorem 7.45 Let $\mathfrak{L}$ and $\mathfrak{M}$ be sets of logics and $\mathbf{A}$ and $\mathbf{B}$ languages.

1. $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ is structural.

2. $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ is structural.

Proof. $\boxed{\mathsf{I}_{\mathfrak{M}}(\mathbf{A}) \text{ is structural.}}$ Suppose that $\Gamma \vdash_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})} \phi$ and that $\sigma \in \mathsf{Sub}(\mathbf{A})$. (*We must show that* $\sigma[\Gamma] \vdash_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})} \sigma(\phi)$.) Let $\mathsf{M} \in \mathfrak{M}$ and $\mathsf{j} \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$. (*It suffices to show, by Corollary 7.38 on page 260, that* $\mathsf{j}[\sigma[\Gamma]] \vdash_{\mathsf{M}} \mathsf{j}(\sigma(\phi))$.) But, $\mathsf{j}\sigma \in \mathsf{Int}(\mathbf{A}, \mathsf{M})$, and since $\Gamma \vdash_{\mathsf{I}_{\mathfrak{M}}(\mathbf{A})} \phi$ (and $\mathsf{I}_{\mathfrak{M}}(\mathbf{A})$ is an abstraction of M), it follows by definition that $(\mathsf{j}\sigma)[\Gamma] \vdash_{\mathsf{M}} (\mathsf{j}\sigma)[\phi]$. $\boxed{\mathsf{F}_{\mathfrak{L}}(\mathbf{B}) \text{ is structural.}}$ Let $\sigma \in \mathsf{Sub}(\mathbf{B})$ and $\Gamma \vdash_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})} \phi$. (*We must show that* $\sigma[\Gamma] \vdash_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})} \sigma(\phi)$.) Now $\sigma[\Gamma] \subseteq \|\sigma[\Gamma]\|_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})} \in \mathsf{Th}(\mathsf{F}_{\mathfrak{L}}(\mathbf{B}))$ and $\Gamma \vdash_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})} \phi$; so by Proposition 7.20 *(ii)*, $\sigma(\phi) \in \|\sigma[\Gamma]\|_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})}$; hence $\sigma[\Gamma] \vdash_{\mathsf{F}_{\mathfrak{L}}(\mathbf{B})} \sigma(\phi)$. $\diamond$

Remark 7.46 The proof that $\mathsf{F}_{\mathfrak{L}}(\mathbf{B})$ is structural does not require 'constructural' arguments.

Remark 7.47 In particular, $\mathsf{I}_{\mathsf{L}}(\mathbf{L})$ and $\mathsf{F}_{\mathsf{L}}(\mathbf{L})$ are both structural.

Theorem 7.48 The following conditions are equivalent.

1. L is structural.

2. $\mathsf{L} = \mathsf{F}_{\mathsf{L}}(\mathbf{L})$.

3. $\mathsf{L} = \mathsf{I}_{\mathsf{L}}(\mathbf{L})$.

4. $\mathsf{F}_{\mathsf{L}}(\mathbf{L}) = \mathsf{I}_{\mathsf{L}}(\mathbf{L})$.

5. $\mathsf{I}_{\mathsf{F}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \mathsf{L}$.

6. $\mathsf{F}_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \mathsf{L}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Assume that L is structural. Then by Theorem 7.43, L is a model of $\mathbf{L}$. So by Corollary 7.14, $\mathsf{L} \succeq \mathsf{F}_{\mathsf{L}}(\mathbf{L})$. Conversely, by Proposition 7.44, it is generally true (independently of the structurality of L) that $\mathsf{L} \preceq \mathsf{F}_{\mathsf{L}}(\mathbf{L})$. The result follows since $\preceq$ is an order. $\boxed{(2)\Rightarrow(1)}$ Assume that $\mathsf{L} = \mathsf{F}_{\mathsf{L}}(\mathbf{L})$. By Theorem 7.45, $\mathsf{F}_{\mathsf{L}}(\mathbf{L})$ is structural, so L must be structural. $\boxed{(1)\Rightarrow(3)}$ Assume that L is structural. Then by Theorem 7.43, L is an abstract of $\mathbf{L}$. So by Corollary 7.31, $\mathsf{L} \preceq \mathsf{I}_{\mathsf{L}}(\mathbf{L})$. Conversely, by Proposition 7.44, it is generally true (independently of the structurality of L) that $\mathsf{L} \succeq \mathsf{I}_{\mathsf{L}}(\mathbf{L})$. $\boxed{(3)\Rightarrow(1)}$ Assume that $\mathsf{L} = \mathsf{I}_{\mathsf{L}}(\mathbf{L})$. By Theorem 7.45, $\mathsf{I}_{\mathsf{L}}(\mathbf{L})$ is structural, so L must be structural. $\boxed{(2) \text{ and } (3)\Rightarrow(4)}$ Trivial. $\boxed{(4) \Rightarrow (2)}$ By Proposition 7.44. $\boxed{(2) \text{ and } (3)\Rightarrow(5)}$ $\mathsf{I}_{\mathsf{F}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \mathsf{I}_{\mathsf{L}}(\mathbf{L}) = \mathsf{L}$, the first equality by (2) and the second by (3). $\boxed{(5)\Rightarrow(1)}$ $\mathsf{I}_{\mathsf{F}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L})$ is structural by Theorem 7.45, and hence so is L by assumption. $\boxed{(2) \text{ and } (3)\Rightarrow(6)}$ $\mathsf{F}_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \mathsf{F}_{\mathsf{L}}(\mathbf{L}) = \mathsf{L}$. $\boxed{(6)\Rightarrow(1)}$ $\mathsf{F}_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L})$ is structural by Theorem 7.45, and hence so is L by assumption. $\diamond$

Since by Remark 7.47 on page 263, $\mathsf{I}_{\mathsf{L}}(\mathbf{L})$ and $\mathsf{F}_{\mathsf{L}}(\mathbf{L})$ are both structural, we may 'substitute' both for L in the equations of the previous theorem, thereby obtaining the following useful simplifications.

Corollary 7.49 $\mathsf{I}_{\mathsf{L}}(\mathbf{L}) = \mathsf{F}_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \mathsf{I}_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \mathsf{I}_{\mathsf{F}_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L})}(\mathbf{L}) = \mathsf{F}_{\mathsf{I}_{\mathsf{I}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L})}(\mathbf{L})$.

Corollary 7.50 $\mathsf{F}_{\mathsf{L}}(\mathbf{L}) = \mathsf{F}_{\mathsf{F}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \mathsf{I}_{\mathsf{F}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L}) = \mathsf{I}_{\mathsf{F}_{\mathsf{F}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L})}(\mathbf{L}) = \mathsf{F}_{\mathsf{I}_{\mathsf{F}_{\mathsf{L}}(\mathbf{L})}(\mathbf{L})}(\mathbf{L})$.

While these simplifications are dependent on *constructural* arguments, they are independent of the *structurality* of L itself. Recall that by Proposition 7.44 on page 263, $\mathsf{L} \in [\mathsf{F}_{\mathsf{L}}(\mathbf{L}), \mathsf{I}_{\mathsf{L}}(\mathbf{L})]_{\preceq}$. If L is *structural*, then this interval collapses to a singleton, by Theorem 7.48 on page 263, in which case L is the coarsest model of itself and the finest abstract of itself. If L is not structural, then this interval contains at least three distinct logics, $\mathsf{F}_{\mathsf{L}}(\mathbf{L})$, L and $\mathsf{I}_{\mathsf{L}}(\mathbf{L})$. Consequently, this interval never has cardinality two.

7.3 Maximal Models

Definition 7.51 (Maximal Models) Let L be a logic and $\mathfrak{M}$ a set of logics. We say that $\mathfrak{M}$ **constitutes a maximal model of** L if, for each $\mathsf{M} \in \mathfrak{M}$, $\mathsf{Th}(\mathsf{M}) = \mathsf{Fi}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))$. $\square$

We note that a maximal model is indeed a model; this follows by Remark 7.16.

Remark 7.52 $\mathfrak{M}$ constitutes a maximal model of L iff $\mathfrak{M}$ constitutes a model of L and, for each $\mathsf{M} \in \mathfrak{M}$, $\mathsf{Fi}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M})) \subseteq \mathsf{Th}(\mathsf{M})$. $\square$

The following result describes a sufficient condition for maximal modellability. We do not believe that the converse of this result holds generally, although we have not yet tried to find a counter example. In $\oint 16$, we shall develop a theory of protoalgebraicity at this level of discourse. There we shall show that if L is a structural deductive system (not necessarily finitary) that is also protoalgebraic, then the condition of the next result is indeed necessary, and hence characterizes maximal modellability.

Lemma 7.53 Suppose that for each $F \in \mathsf{Fi}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))$, if $F \vdash_{\mathsf{M}} \mathsf{g}$, then there exists $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L})$ and $\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathsf{M})$, with $\Gamma \vdash_{\mathsf{L}} \phi$, $\mathsf{i}[\Gamma] \subseteq F$ and $\mathsf{i}(\phi) = \mathsf{g}$. Then $\mathsf{Fi}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M})) \subseteq \mathsf{Th}(\mathsf{M})$.

Proof. Let $F \in \mathsf{Fi}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))$ and suppose that $F \vdash_{\mathsf{M}} \mathsf{g}$. (*We must show that* $\mathsf{g} \in F$.) By assumption, there exists $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L})$ and $\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathsf{M})$, with $\Gamma \vdash_{\mathsf{L}} \phi$, $\mathsf{i}[\Gamma] \subseteq F$ and $\mathsf{i}(\phi) = \mathsf{g}$, and since $F \in \mathsf{Fi}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))$, $\mathsf{g} \in F$ by Corollary 7.20. ⋄

Corollary 7.54 Suppose that $\mathfrak{M}$ constitutes a model of L and, for each $\mathsf{M} \in \mathfrak{M}$ and $F \in \mathsf{Fi}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))$, if $F \vdash_{\mathsf{M}} \mathsf{g}$, then there exists $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathsf{L})$ and $\mathsf{i} \in \mathsf{Int}(\mathsf{L}, \mathsf{M})$, with $\Gamma \vdash_{\mathsf{L}} \phi$, $\mathsf{i}[\Gamma] \subseteq F$ and $\mathsf{i}(\phi) = \mathsf{g}$. Then $\mathfrak{M}$ constitutes a maximal model of L.

7.4 Logics as Semantics of Logics

Definition 7.55 (Semantics) Let L be an $\mathfrak{s}$-logic and $\mathfrak{M}$ a *non-empty* class of $\mathfrak{s}$-logics. We call $\mathfrak{M}$ an **$\mathfrak{s}$-semantics for** L (or a **$\langle \mathsf{L}, \mathfrak{s} \rangle$-semantics**) if, $\vdash_{\mathsf{L}} = \models^{\mathfrak{M},\mathfrak{s}}_{\mathbf{lg}(\mathsf{L})}$. In the case that $\mathfrak{M} = \{\mathsf{M}\}$, we write M for $\mathfrak{M}$ in these definitions, obtaining the notion that a logic M be a $\langle \mathsf{L}, \mathfrak{s} \rangle$-semantics. For $\mathfrak{s}$-language $\mathbf{B}$, let $\mathsf{Sem}^{\mathfrak{s}}_{\mathsf{L}}(\mathbf{B})$ denote the set of all $\mathbf{B}$-logic models of L. Let $\mathsf{Sem}^{\mathfrak{s}}_{\mathsf{L}}$ denote the class of all full logic models of L. □

Remark 7.56 The demonstration of a model of L amounts to 'proving L sound', while the existence of a semantics for L 'proves soundness and completeness'.

Remark 7.57 $\mathfrak{M}$ is a semantics for L iff $\mathsf{L} = \mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L}))$.

Remark 7.58 Trivially, for any language $\mathbf{A}$, $\mathfrak{M}$ is a semantics for $\mathsf{l}_{\mathfrak{M}}(\mathbf{A})$.

Proposition 7.59 $\mathfrak{M}$ is a semantics for L iff $\mathfrak{M}$ constitutes a model of L and $\mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L})) \preceq \mathsf{L}$.

Proof. $\boxed{\Rightarrow}$ Suppose that $\mathfrak{M}$ is a semantics for L. By Remark 7.57 on page 265, $\mathsf{L} = \mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L}))$ and so certainly $\mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L})) \preceq \mathsf{L}$. By (1) of Theorem 7.41 on page 261, it is generally true that $\mathfrak{M} \subseteq \mathsf{Mod}_{\mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L}))}$. In the case that $\mathfrak{M}$ is a semantics for L, since $\mathsf{L} = \mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L}))$, $\mathsf{Mod}_{\mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L}))} = \mathsf{Mod}_{\mathsf{L}}$. $\boxed{\Leftarrow}$ Since $\mathfrak{M}$ constitutes a model of L by assumption, $\mathsf{L} \preceq \mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L}))$, by Theorem 7.40 on page 261, and since $\mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L})) \preceq \mathsf{L}$ by assumption, $\mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L})) = \mathsf{L}$. ⋄

Lemma 7.60 If $\mathfrak{M}$ is a semantics for L and $\mathfrak{M}' \subseteq \mathsf{Mod}_{\mathsf{L}}$ then $\mathfrak{M} \cup \mathfrak{M}'$ is a semantics for L.

Proof. (*It suffices to show that* $\mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L})) = \mathsf{l}_{\mathfrak{M}\cup\mathfrak{M}'}(\mathbf{lg}(\mathsf{L}))$.)
$\boxed{\mathsf{l}_{\mathfrak{M}\cup\mathfrak{M}'}(\mathbf{lg}(\mathsf{L})) \preceq \mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L}))}$ Follows by Remark 7.35. $\boxed{\mathsf{l}_{\mathfrak{M}}(\mathbf{lg}(\mathsf{L})) \preceq \mathsf{l}_{\mathfrak{M}\cup\mathfrak{M}'}(\mathbf{lg}(\mathsf{L}))}$ Assume that $\Gamma \models^{\mathfrak{M}}_{\mathbf{lg}(\mathsf{L})} \phi$. Suppose that $\mathsf{M} \in \mathfrak{M} \cup \mathfrak{M}'$ and $\mathsf{i} : \mathsf{Int}(\mathsf{L}, \mathbf{B})$. (*By Proposition 7.59 on page 265, it suffices to show that* $\mathsf{i}[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi)$.) Since $\Gamma \models^{\mathfrak{M}}_{\mathbf{lg}(\mathsf{L})} \phi$, $\Gamma \vdash_{\mathsf{L}} \phi$, since $\mathfrak{M}$ is a semantics for L. By Proposition 7.59 on page 265, $\mathfrak{M}$ constitutes a model for L and by assumption $\mathfrak{M}'$ constitutes a model for L; consequently $\mathfrak{M} \cup \mathfrak{M}'$ constitutes a model of L. So $\mathsf{i}[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi)$ by definition. ⋄

Note that given the abstract nature of our notion of logic, in particular, without an implicit assumption of structurality, we are able to add structurality as an equivalent condition to the standard completeness theorem for logics.

Theorem 7.61 (Completeness) Let L be a logic on language $\mathbf{L}$. The following conditions are equivalent.

1. L is structural.
2. L is a semantics for itself.
3. $\mathsf{F}_{\mathsf{L}}(\mathbf{L})$ is a semantics for L.
4. $\mathsf{Mod}_{\mathsf{L}}(\mathbf{L})$ is a semantics for L
5. $\mathsf{Mod}_{\mathsf{L}}$ is a semantics for L.
6. L has some semantics.

Proof. $\boxed{(1)\Leftrightarrow(2)}$ L is structural iff (by Theorem 7.48) $\mathsf{L} = \mathsf{I}_{\mathbf{L}}(\mathsf{L})$ iff L is a semantics for itself, by Remark 7.57. $\boxed{(1) \text{ and } (2)\Rightarrow(3)}$ Follows by Theorem 7.48. $\boxed{(3)\Rightarrow(4)}$ By Remark 7.15 on page 256, $\mathsf{F}_{\mathsf{L}}(\mathbf{L}) \in \mathsf{Mod}_{\mathsf{L}}(\mathbf{L})$. The result follows by Lemma 7.60 on page 265. $\boxed{(4)\Rightarrow(5)}$ By Lemma 7.60. $\boxed{(5)\Rightarrow(6)}$ Trivial. $\boxed{(6)\Rightarrow(1)}$ Follows by Remark 7.57 on page 265 and Theorem 7.45 on page 263. $\diamond$

Consequently, *only* a structural logic can *possibly* have a semantics, in which case it does have a semantics.

7.5 Language-Indexed Models

Of particular importance to us are classes of logics where no two logics have the same underlying language, in which case we may index such classes on their languages. Such classes will play an analogous role to classes of algebras such as varieties and quasivarieties. In fact, most of our later examples arise in this manner. In $\oint$8.2, we study a special instance of such classes, which we call *archologies.*

Definition 7.62 (Language-Indexed Logics and Normalization) Let $\mathcal{Y}$ be a class of languages. By a **$\mathcal{Y}$-system of logics** we mean a $\mathcal{Y}$-indexed family of logics $\mathfrak{M}_{\mathcal{Y}} = \{\mathsf{M}_{\mathbf{B}} : \mathbf{B} \in \mathcal{Y}\}$, where M is a $\mathbf{B}$-logic. Let $\mathfrak{M}$ be a class of logics. The class of all languages $\mathbf{B}$ with $\mathsf{M} \in \mathfrak{M}$ for some $\mathbf{B}$-logic M, is denoted $\mathsf{Lang}(\mathfrak{M})$. For each language $\mathbf{B}$, let

$$L(\mathbf{B}, \langle\mathfrak{M}\rangle) = L\left(\mathbf{B}, \mathbb{C}\left(\bigcup\{\mathsf{Th}(\mathsf{M}) \;:\; \mathsf{M} \in \mathfrak{M}, \mathbf{lg}(\mathsf{M}) = \mathbf{B}\}, \mathsf{clbase}\right)\right),$$

and let $\mathfrak{M}^{\mathsf{N}}$ denote the $\mathsf{Lang}(\mathfrak{M})$-indexed system $\{L(\mathbf{B}, \langle\mathfrak{M}\rangle) \;:\; \mathbf{B} \in \mathsf{Lang}(\mathfrak{M})\}$, which we call the **normalization of $\mathfrak{M}$**. $\square$

Lemma 7.63 $\mathsf{I}_{\mathfrak{M}^{\mathsf{N}}}(\mathbf{A}) = \mathsf{I}_{\mathfrak{M}}(\mathbf{A})$.

Proof. $\boxed{\mathsf{I}_{\mathfrak{M}^{\mathsf{N}}}(\mathbf{A}) \preceq \mathsf{I}_{\mathfrak{M}}(\mathbf{A})}$ Suppose than $\Gamma \models^{\mathfrak{M}}_{\mathbf{A}} \phi$, $\mathsf{M} \in \mathfrak{M}^{\mathsf{N}}$, $\mathbf{lg}(\mathsf{M}) = \mathbf{B}$, $\mathsf{i} \in \mathsf{Int}(\mathbf{A}, \mathbf{B})$, $R \in \mathsf{Th}(\mathsf{M})$ and $\mathsf{i}[\Gamma] \subseteq R$. (*It suffices, by Corollary 7.38, to show that* $\mathsf{i}(\phi) \in R$.) If $R = \mathsf{Fm}(\mathbf{B})$ then certainly $\mathsf{i}(\phi) \in R$. Otherwise, $R = \bigcap_{i\in I} R_i$, for some $\emptyset \neq I$ and $R_i \in \mathsf{Th}(\mathsf{M}_i)$, where $\mathsf{M}_i \in \mathfrak{M}$ and $\mathbf{lg}(\mathsf{M}_i) = \mathbf{B}$. Let $i \in I$. Clearly, $\mathsf{i}[\Gamma] \subseteq R_i$. Since $\Gamma \models^{\mathfrak{M}}_{\mathbf{A}} \phi$, $\mathsf{i} : \mathsf{Int}(\mathbf{A}, \mathbf{B})$, $\mathsf{M}_i \in \mathfrak{M}$, it follows, by Corollary 7.38, that $\mathsf{i}(\phi) \in R_i$. Consequently, $\mathsf{i}(\phi) \in \bigcap_{i\in I} R_i = R$, as required. $\boxed{\mathsf{I}_{\mathfrak{M}}(\mathbf{A}) \preceq \mathsf{I}_{\mathfrak{M}^{\mathsf{N}}}(\mathbf{A})}$ Suppose than $\Gamma \models^{\mathfrak{M}^{\mathsf{N}}}_{\mathbf{A}} \phi$, $\mathsf{M} \in \mathfrak{M}$, $\mathbf{lg}(\mathsf{M}) = \mathbf{B}$,

$\mathsf{i} : \mathsf{Int}(\mathbf{A}, \mathbf{B})$, $R \in \mathsf{Th}(\mathsf{M})$ and $\mathsf{i}[\Gamma] \subseteq R$. (*It suffices, by Corollary 7.38, to show that* $\mathsf{i}(\phi) \in R$.) By construction, there exists a unique $\mathsf{M}' \in \mathfrak{M}^{\mathsf{N}}$ with $\mathbf{lg}(\mathsf{M}') = \mathbf{B}$, and again by construction $R \in \mathsf{Th}(\mathsf{M}')$. Since $\Gamma \models^{\mathfrak{M}^{\mathsf{N}}}_{\mathbf{A}} \phi$, $\mathsf{M}' \in \mathfrak{M}^{\mathsf{N}}$, $\mathbf{lg}(\mathsf{M}) = \mathbf{B}$, $\mathsf{i} : \mathsf{Int}(\mathbf{A}, \mathbf{B})$, and $\mathsf{i}[\Gamma] \subseteq R \in \mathsf{Th}(\mathsf{M}')$, by Corollary 7.38, $\mathsf{i}(\phi) \in R$. $\diamond$

7.6 Matrix Models

We now explore the relationship between logics as models and matrices as models. The key observation is that every matrix may be thought of as a logic, but not conversely. Consequently, we will be able to syntactically treat matrices as logics and isolate certain logics as 'matrix-logics'. It turns out that the subclass of all 'matrix-logics' of $\mathsf{Mod}^{\mathfrak{s}}_{\mathsf{L}}$ also serves as an $\mathfrak{s}$-semantics for L. It is these logics that logicians, or at least first-order logicians, focus on, and hence Blok and Pigozzi's notion of a 'matrix-semantics' of a logic. We shall see that there are important first-order reasons for adopting such a stance, in that, for finitary logics, these matrices are first-order definable languages. There are, however, reasons for examining models other than just the matrix models. We shall see that by adopting this stance, finitary structural logics are much easier to recognize, particularly those arising from algebras. Another reason is that L will seldom have a single language based matrix-semantics (i.e., one matrix) but may have a single logic on a single language that serves as a semantics.

At this level of discourse, the dimensional aspects of n-propositional calculi have been 'factored away'; being 'hidden' by choice of morphism. Consequently, the matrix theory appropriate for $\mathfrak{s}$-logics, is that of *unary* matrices over objects.

7.6.1 Language Matrices

We begin by briefly introducing the notion of a (unary) matrix over a language.

Definition 7.64 (Language Matrices) A language-matrix $\mathbf{M}$ is determined by its *language* $\mathbf{lg}(\mathbf{M})$, which is a language of logics, and its **designated formulae** $\mathsf{D}_{\mathbf{M}} \subseteq \mathsf{Fm}(\mathbf{lg}(\mathbf{M}))$. We write $\mathsf{Fm}(\mathbf{M})$ for $\mathsf{Fm}(\mathbf{lg}(\mathbf{M}))$, which we call the **formulae** of the language-matrix. For a language of logics $\mathbf{A}$, an $\mathbf{A}$-matrix is a language-matrix with language $\mathbf{A}$. For a signature $\mathfrak{s}$ of languages of logics, an $\mathfrak{s}$-matrix is a matrix whose language is an $\mathfrak{s}$-language. We may present a language-matrix $\mathbf{M}$ by $\langle \mathbf{lg}(\mathbf{M}), \mathsf{D}_{\mathbf{M}} \rangle$. In such a presentation, we may write $\langle \mathbf{lg}(\mathbf{M}), \phi \rangle$ for $\langle \mathbf{lg}(\mathbf{M}), \{\phi\} \rangle$.

Let $\mathfrak{s}$ be a signature and $\mathbf{M}$ and $\mathbf{N}$ $\mathfrak{s}$-matrices. An **$\mathfrak{s}$-matrix-morphism** f from $\mathbf{M}$ into $\mathbf{N}$ is an $\mathfrak{s}$-morphism from $\mathbf{lg}(\mathbf{M})$ into $\mathbf{lg}(\mathbf{N})$ such that $f[\mathsf{D}_{\mathbf{M}}] \subseteq \mathsf{D}_{\mathbf{N}}$. We inherit the language of unary-matrices over algebras in the natural manner, speaking of **$\mathfrak{s}$-matrix-epimorphisms**, **$\mathfrak{s}$-matrix-monomorphisms**, **$\mathfrak{s}$-matrix-isomorphisms**, **strict $\mathfrak{s}$-matrix-morphisms**, **$\mathfrak{s}$-matrix-reductions**, **$\mathfrak{s}$-submatrices**, etc, where the signature $\mathfrak{s}$ takes the role of the algebraic type $\mathfrak{a}$, in the natural manner. The set of $\mathfrak{s}$-matrix morphisms ($\mathfrak{s}$-matrix-epimorphisms, $\mathfrak{s}$-matrix-monomorphisms, $\mathfrak{s}$-matrix-isomorphisms, strict $\mathfrak{s}$-matrix-morphisms and $\mathfrak{s}$-matrix-reductions) from $\mathbf{M}$ into/onto $\mathbf{N}$ is denoted by $\mathbf{M} \to_{\mathfrak{s}} \mathbf{N}$ (resp. $\mathbf{M} \twoheadrightarrow_{\mathfrak{s}} \mathbf{N}$, $\mathbf{M} \rightarrowtail_{\mathfrak{s}} \mathbf{N}$, $\mathbf{M} \cong_{\mathfrak{s}} \mathbf{N}$, $\mathbf{M} \to^{\mathsf{s}}_{\mathfrak{s}} \mathbf{N}$, $\mathbf{M} \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$). $\square$

In our context, language-matrices are always *unary* matrices. In this chapter, except where possible confusion arises, as when language-matrices and algebra n-matrices of n-sentential calculi are compared, we tend just to speak of matrices.

Definition 7.65 (The Construct of $\mathfrak{s}$-Matrices) We associate with any signature $\mathfrak{s}$, the construct $\mathsf{Mx}_{\mathfrak{s}}$, called the construct of **$\mathfrak{s}$-matrices** with matrix-homomorphisms. □

7.6.2 Matrices as Models of Logics

Recall ∮4.2.5, in which, with any given unary-matrix M, we associate two closed systems, both with universe $\mathsf{uni}(\mathsf{M})$, namely the *(generally) constrained closed system* $\mathbb{C}(\mathsf{M})$ with $\mathsf{cl}_{\mathbb{C}(\mathsf{M})} = \{\mathsf{D}_{\mathsf{M}}, \mathsf{uni}(\mathsf{M})\}$, and the *unconstrained closed system* $\mathbb{C}(\mathsf{M}, \emptyset)$, with $\mathsf{cl}_{\mathbb{C}(\mathsf{M},\emptyset)} = \{\emptyset, \mathsf{D}_{\mathsf{M}}, \mathsf{uni}(\mathsf{M})\}$. We are most interested in the *constrained closed system* associated with a matrix; topologists may be more interested in the other.

Definition 7.66 (Language-Matrices to Logics) With each language $\mathbf{A}$-matrix $\mathbf{M}$, we denote the $\mathbf{A}$-logic $L(\mathbf{lg}(\mathbf{M}), \mathbb{C}(\mathbf{M}))$ by $L(\mathbf{M})$. □

Convention 7.67 (Conflating M with $L(\mathbf{M})$) It is convenient to confuse a language-matrix $\mathbf{M}$ with its induced $\mathbf{lg}(\mathbf{M})$-logic $L(\mathbf{M})$, thereby inheriting the notations and appropriate notions from the previous sections of this chapter. For example, for a logic L, we may call matrix $\mathbf{M}$ an **L-model**, and speak of a set $\mathcal{M}$, of language-matrices, as constituting a **model**, an **abstract** or a **semantics** for L. Further, for language matrices $\mathbf{M}$ and $\mathbf{N}$, and sets of language-matrices $\mathcal{M}$ and $\mathcal{N}$, the following notions are all well-defined: $\mathsf{F}^{\mathfrak{s}}_{\mathcal{M}}(\mathbf{B})$, $\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{M}}(\mathbf{B})$, $\Vdash^{\mathbf{B},\mathfrak{s}}_{\mathcal{M}}$, $\mathsf{I}^{\mathfrak{s}}_{\mathcal{N}}(\mathbf{A})$, $\mathsf{Id}^{\mathfrak{s}}_{\mathcal{N}}(\mathbf{A})$, $\models^{\mathcal{N},\mathfrak{s}}_{\mathbf{A}}$ and $\mathsf{PId}^{\mathfrak{s}}_{\mathcal{N}}(\mathbf{A})$, as well as $\mathsf{F}^{\mathfrak{s}}_{\mathbf{M}}(\mathbf{B})$, $\mathsf{Fi}^{\mathfrak{s}}_{\mathbf{M}}(\mathbf{B})$, $\Vdash^{\mathbf{B},\mathfrak{s}}_{\mathbf{M}}$, $\mathsf{I}^{\mathfrak{s}}_{\mathbf{N}}(\mathbf{A})$, $\mathsf{Id}^{\mathfrak{s}}_{\mathbf{N}}(\mathbf{A})$, $\models^{\mathbf{N},\mathfrak{s}}_{\mathbf{A}}$ and $\mathsf{PId}^{\mathfrak{s}}_{\mathbf{N}}(\mathbf{A})$.

Definition 7.68 (Matrix Models and Abstracts) Let $\mathsf{MMod}^{\mathfrak{s}}_{\mathbf{B}}(\mathfrak{L})$ denote the set of all L matrix-models on language $\mathbf{B}$ and $\mathsf{MMod}^{\mathfrak{s}}(\mathfrak{L})$ the set of all L matrix-models on *some* $\mathsf{sig}(\mathsf{L})$-language. Let $\mathsf{MAbs}^{\mathfrak{s}}_{\mathbf{A}}(\mathfrak{M})$ denote the set of all L matrix-abstracts on language $\mathbf{A}$ and $\mathsf{MAbs}^{\mathfrak{s}}(\mathfrak{M})$ the set of all M matrix-abstracts on *some* $\mathsf{sig}(\mathsf{M})$-language. By a **$\langle\mathfrak{L},\mathfrak{s}\rangle$-matrix** (or simply a **$\mathfrak{L}$-matrix** when $\mathfrak{s}$ is understood) we mean a matrix that is an $\mathfrak{s}$-model of $\mathfrak{L}$.

In keeping with earlier conventions, we may write L for $\mathfrak{L} = \{\mathsf{L}\}$ and M for $\mathfrak{M} = \{\mathsf{M}\}$ in all of these notations. □

Remark 7.69 $\mathbf{M}$ is a $\mathfrak{L}$-matrix iff $\mathsf{D}_{\mathbf{M}}$ is a $\mathfrak{L}$-filter on $\mathbf{lg}(\mathbf{M})$. □

The following result, which follows trivially from (2) of Corollary 7.38 on page 260, highlights the relationship between our notion of a logic as model on the one hand and Blok and Pigozzi's notion of a matrix as model on the other (see Definition 2.36 on page 100).

Proposition 7.70 $\Gamma \models^{\mathcal{M},\mathfrak{s}}_{\mathbf{A}} \phi$ iff $\forall\,[\mathbf{N} \in \mathcal{M}, \mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{A}, \mathbf{lg}(\mathbf{N}))]\ \mathsf{i}[\Gamma] \subseteq \mathsf{D}_{\mathbf{N}} \to \mathsf{i}(\phi) \in \mathsf{D}_{\mathbf{N}}$.

Remark 7.71 [vA95, 60] Generally, the logic $\mathsf{I}^{\mathfrak{s}}_{\mathcal{M}}(\mathbf{A})$ is not finitary. □

Recall Theorem 2.61 on page 103, namely the soundness and completeness theorem for propositional calculi, which states that the set $\mathsf{MMod}(\mathcal{P})$ (of all matrix-models of propositional calculi $\mathcal{P}$)

constitutes a matrix semantics for $\mathcal{P}$. The formulae matrix-models $\mathsf{FMMod}(\mathcal{P})$ also constitutes a semantics. In Remark 2.62 on page 103, we noted that the proof of this result depends *necessarily* on structurality. We are now in a position to justify that remark.

The next theorem is the natural analogue of Theorem 2.61 on page 103. Notice that the theorem is formulated 'locally', that is not just for deductive systems but for logics more generally, and that structurality is positioned as an equivalent equivalent condition for completeness (all logics, in our sense, being sound). The result follows immediately from Theorem 7.61, Proposition 7.70 and Lemma 7.63.

Theorem 7.72 For a logic L with language $\mathbf{A}$, the following conditions are equivalent.

1. L is $\mathfrak{s}$-structural.
2. $\mathsf{MMod}^{\mathfrak{s}}_{\mathbf{A}}(\mathsf{L})$ is a matrix semantics for L.
3. $\mathsf{MMod}^{\mathfrak{s}}(\mathsf{L})$ is a matrix semantics for L.

Proposition 7.73 If $\mathsf{L} \preceq \mathsf{L}'$ then $\mathsf{MMod}^{\mathfrak{s}}(\mathsf{L}') \subseteq \mathsf{MMod}^{\mathfrak{s}}(\mathsf{L})$ and, for any $\mathfrak{s}$-language $\mathbf{B}$ $\mathsf{MMod}^{\mathfrak{s}}_{\mathbf{B}}(\mathsf{L}') \subseteq \mathsf{MMod}^{\mathfrak{s}}_{\mathbf{B}}(\mathsf{L})$. If L and M are both structural and $\mathsf{MMod}^{\mathfrak{s}}(\mathsf{L}') \subseteq \mathsf{MMod}^{\mathfrak{s}}(\mathsf{L})$, then $\mathsf{L} \preceq \mathsf{L}'$. □

Recall the definition of the *filtration logic* $\mathsf{L}_{:\Gamma}$ of a logic L by $\Gamma \subseteq \mathsf{Fm}(\mathsf{L})$, given in Definition 6.10 on page 224.

Definition 7.74 (Filters of Matrices) For a set $\mathfrak{L}$ of $\mathfrak{s}$-logics and an $\mathfrak{s}$-matrix $\mathbf{M}$, $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{M})$ denotes the filtration logic $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{lg}(\mathbf{M}))_{:\mathsf{D}_{\mathbf{M}}}$, i.e., the logic on $\mathbf{lg}(\mathbf{M})$ whose theories are precisely those $\mathfrak{L}$-filters of $\mathbf{lg}(\mathbf{M})$ containing the designator $\mathsf{D}_{\mathbf{M}}$, which we call the **model logic** or **filter logic** of $\mathfrak{L}$ on $\mathbf{M}$. We inherit all the related filter notation of Definition 7.13 in the obvious fashion, for example writing $\mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{M})$ for $\mathsf{Th}(\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{M}))$, the members of which are called **$\mathfrak{L}$-filters of $\mathbf{M}$ with respect to $\mathfrak{s}$** (or **$\langle\mathfrak{L},\mathfrak{s}\rangle$-filters of $\mathbf{M}$** or just **$\mathfrak{L}$-filters of $\mathbf{M}$** when $\mathfrak{s}$ is understood). In particular, for $f : \mathbf{M} \to_{\mathfrak{s}} \mathbf{N}$, we define a function $f^{\mathfrak{L}}_{\mathfrak{s}} : \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{M}) \to \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{N})$ by $f^{\mathfrak{L}}_{\mathfrak{s}}(F) = \| f\,[F]\,\|^{\mathsf{N}}_{\mathsf{fi}_{\mathfrak{L}}}$, which is distinguished from the identically notated version of Definition 7.13, by being an operation on $\mathfrak{s}$-matrix morphisms rather than an operation on $\mathfrak{s}$-morphisms. For languages $\mathbf{A}$ and $\mathbf{B}$, $f : \mathbf{A} \to_{\mathfrak{s}} \mathbf{B}$ and $B \subseteq \mathsf{Fm}(\mathbf{B})$, we define a function $f^{\mathfrak{L}}_{B} : \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{A}) \to \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\langle \mathbf{B}, B\rangle)$, by $f^{\mathfrak{L}}_{B}(F) = \| f\,[F]\,\|^{\langle \mathbf{B},B\rangle}_{\mathsf{fi}_{\mathfrak{L}}}$. As usual, we drop the signature $\mathfrak{s}$ from these notations whenever the signature is unambiguously determinable or unimportant. □

The following remark follows at once from Remark 6.11 on page 225 and Corollary 7.23.

Remark 7.75 Let $\mathfrak{L}$ be a set of $\mathfrak{s}$-logics and $\mathbf{M}$ an $\mathfrak{s}$-matrix.

1. If $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{lg}(\mathbf{M}))$ is finitary then so is $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{M})$.
2. If every logic of $\mathfrak{L}$ is finitary then so is $\mathsf{F}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{M})$.
3. $\|X\|^{\mathbf{M}}_{\mathsf{fi}_{\mathfrak{L}}} = \|X \cup \mathsf{D}_{\mathbf{M}}\|^{\mathbf{lg}(\mathbf{M})}_{\mathsf{fi}_{\mathfrak{L}}}$.

□

The following important result follows immediately from Corollary 1.80 on page 25 and Proposition 7.24.

Proposition 7.76 Let $\mathfrak{L}$ be a set of $\mathfrak{s}$-logics, let $\mathbf{M}$ and $\mathbf{N}$ be $\mathfrak{s}$-matrices and let $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$. Then, for every $G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{N})$, $f^{-1}[G] \in \mathsf{Fi}^{\mathfrak{s}}_{\mathfrak{L}}(\mathbf{M})$.

7.7 Examples

Example 7.77 (The Subuniverse Logics)

Recall Example 6.79 on page 242, where we introduced the subuniverse logic $U(\mathbf{A}, \mathsf{su})$ on an algebra $\mathbf{A}$, and recall Example 6.80 on page 242. By Example 5.44 on page 188, every homomorphism from an algebra $\mathbf{A}$ into an algebra $\mathbf{B}$ is a continuous function from the closed system $F(\mathbf{A}, \mathsf{su})$ into closed system $F(\mathbf{A}, \mathsf{su})$. Consequently we have the following.

Remark 7.78 For any two classes $\mathfrak{L}$ and $\mathfrak{M}$ of $\mathfrak{a}$-subuniverse logics, $\mathfrak{L}$ constitutes a model and an abstraction of $\mathfrak{M}$. □

Let $\mathbf{A}$ and $\mathbf{B}$ be two $\mathfrak{a}$-algebras. Since $U(\mathbf{B}, \mathsf{su})$ is a model of $U(\mathbf{A}, \mathsf{su})$, it follows, by Remark 7.16, that $\mathsf{F}^{\mathfrak{a}}_{U(\mathbf{A},\mathsf{su})}(\mathbf{B}) \preceq U(\mathbf{B}, \mathsf{su})$. In other words we have the following.

Remark 7.79 $\mathsf{Su}(\mathbf{B}) \subseteq \mathsf{Fi}^{\mathfrak{a}}_{\mathbf{B}}(U(\mathbf{A}, \mathsf{su}))$. □

Generally, however, there can be other filters.

Counter Example 7.80 ($\mathsf{Su}(\mathbf{B}) \subsetneq \mathsf{Fi}^{\mathfrak{a}}_{\mathbf{B}}(U(\mathbf{A}, \mathsf{su}))$)

Consider the unary algebra $\mathbf{A}$ with universe $\{a\}$ and $\boldsymbol{u}^{\mathbf{A}}(a) = a$, and the unary algebra $\mathbf{B}$ with universe $\{a, b\}$ and $\boldsymbol{u}^{\mathbf{B}}(a) = b$ and $\boldsymbol{u}^{\mathbf{B}}(b) = b$. Clearly, $\mathsf{Su}(\mathbf{A}) = \{\emptyset, \{a\}\}$ and $\mathsf{Su}(\mathbf{B}) = \{\emptyset, \{b\}, \{a, b\}\}$. There is a filter on $\mathbf{B}$ that is not a subuniverse, namely $\{a\}$. To see this, note that there are two homomorphisms from $\mathbf{A}$ to $\mathbf{B}$, namely f mapping $a \mapsto a$ and g mapping $a \mapsto b$, and observe that $\overleftarrow{f}_{[]}[\mathsf{Su}(\mathbf{B})] = \{\emptyset, \{a\}\} = \mathsf{Su}(\mathbf{A})$ and $\overleftarrow{g}_{[]}[\mathsf{Su}(\mathbf{B})] = \{\emptyset, \{a\}\} = \mathsf{Su}(\mathbf{A})$.

□

Since $U(\mathbf{A}, \mathsf{su})$ is $\mathfrak{a}$-structural (by Example 6.79 on page 242), by Theorem 7.48, $\mathsf{F}^{\mathfrak{a}}_{U(\mathbf{A},\mathsf{su})}(\mathbf{A}) = U(\mathbf{A}, \mathsf{su})$.

Remark 7.81 $\mathsf{Su}(\mathbf{A}) = \mathsf{Fi}^{\mathfrak{a}}_{\mathbf{A}}(U(\mathbf{A}, \mathsf{su}))$. □

Recall the axiomatization of $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ given in Proposition 6.82 of Example 6.80 on page 242.

Proposition 7.82 For a quasivariety $\mathcal{K}$ of algebras and $\mathbf{A} \in \mathcal{K}$, $\mathsf{Fi}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{A}) = \mathsf{Su}(\mathbf{A}) = \mathsf{Th}(F(\mathbf{A}, \mathsf{su}))$; consequently $F(\mathbf{A}, \mathsf{su})$ is a *maximal* model of $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$.

Proof. $\boxed{\mathsf{Fi}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{A}) \subseteq \mathsf{Su}(\mathbf{A})}$ Let $F \in \mathsf{Fi}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{A})$. Let $\mathbf{0} \in \mathsf{Symb}_{\mathsf{c}}(\mathfrak{a})$. By Proposition 6.82 on page 242, $\vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \mathbf{0}^{\mathbf{F}}$. Let i be any interpretation of $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ into $\mathbf{A}$. Since $F \in \mathsf{Fi}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{A})$, $\mathbf{0}^{\mathbf{A}} = \mathsf{i}(\mathbf{0}^{\mathbf{F}}) \in F$. Let $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathfrak{a})$ and $a_1, \ldots, a_{\mathsf{ar}(\star)} \in F$. By Proposition 6.82, $\{\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}}\} \vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \star^{\mathbf{F}}(\overline{x_1}, \ldots, \overline{x_{\mathsf{ar}(\star)}})$, for some distinct variables $x_1, \ldots, x_{\mathsf{ar}(\star)}$. Let i be any interpretation of $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ into $\mathbf{A}$ mapping each $\overline{x_i} \mapsto a_i$. Since $F \in \mathsf{Fi}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{A})$,

$\{\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}}\} \vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \star^{\mathbf{F}}(\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}})$ and $\mathsf{i}\left[\{\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}}\}\right] = \{a_1,\ldots,a_{\mathsf{ar}(\star)}\} \subseteq F$, $\star^{\mathbf{A}}(a_1,\ldots,a_{\mathsf{ar}(\star)}) = \mathsf{i}(\star^{\mathbf{F}}(\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}})) \in F$. Consequently, F is closed under fundamental constant and fundamental operations, and hence is a subuniverse. $\boxed{\mathsf{Su}(\mathbf{A}) \subseteq \mathsf{Fi}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{A})}$

Let $F \in \mathsf{Su}(\mathbf{A})$. Suppose that $\overline{[P]} \vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \overline{p}$, $\mathsf{i} : \mathbf{Tm} \to \mathbf{A}$ with $\mathsf{i}\left[\overline{[P]}\right] \subseteq F$. (*We show that* $\overline{p} \in F$ *by induction on the length of derivations from* F.) $\boxed{\text{Base Case}}$ Suppose that $\overline{p}$ is derivable from $\overline{[P]}$ by a derivation of length 1. If $\overline{p} \in \overline{[P]}$, then $\mathsf{i}(\overline{p}) \in F$. Otherwise, there exists an axiom $\vdash \mathbf{0}^{\mathbf{F}}$ and a substitution σ, with $\sigma(\mathbf{0}^{\mathbf{F}}) = \overline{p}$. Since homomorphisms preserve fundamental constants, $\overline{p} = \mathbf{0}^{\mathbf{F}}$. Hence $\mathsf{i}(\overline{p}) = \mathsf{i}(\mathbf{0}^{\mathbf{F}}) = \mathbf{0}^{\mathbf{A}} \in F$, since subuniverses are closed under fundamental constants. $\boxed{\text{Inductive Hypothesis}}$ Suppose that for any term $\overline{p}$ derivable from $\overline{[P]}$ by a derivation of length less than m, $\mathsf{i}(\overline{p}) \in F$. $\boxed{\text{Inductive Step}}$ Let $\overline{p_1},\ldots,\overline{p_m}$ be a derivation of $\overline{p} = \overline{p_m}$ from $\overline{[P]}$ with no shorter derivation of $\overline{p}$ from $\overline{[P]}$. Then there exists a rule $\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}} \vdash \star^{\mathbf{F}}(\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}})$, where $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathfrak{a})$ and $x_1,\ldots,x_{\mathsf{ar}(\star)}$ are distinct variables and a substitution σ with $\sigma\left[\{\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}}\}\right] \subseteq \{\overline{p_1},\ldots,\overline{p_{m-1}}\}$ and $\sigma(\star^{\mathbf{F}}(\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}})) = \overline{p}$. By the induction hypothesis, $\{\mathsf{i}(\sigma(\overline{x_1})),\ldots,\mathsf{i}(\sigma(\overline{x_{\mathsf{ar}(\star)}}))\} = \mathsf{i}\left[\sigma\left[\{\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}}\}\right]\right] \subseteq \mathsf{i}\left[\{\overline{p_1},\ldots,\overline{p_{m-1}}\}\right] \subseteq F$, and since F is a subuniverse and i and σ homomorphisms, $\mathsf{i}(\overline{p}) = \mathsf{i}(\sigma(\star^{\mathbf{F}}(\overline{x_1},\ldots,\overline{x_{\mathsf{ar}(\star)}}))) = \star^{\mathbf{A}}(\mathsf{i}(\sigma(\overline{x_1})),\ldots,\mathsf{i}(\sigma(\overline{x_{\mathsf{ar}(\star)}}))) \in F$. $\diamond$

$\square$

Recall that by Proposition 5.60 of Example 5.57 on page 191, the filters of the sentential calculus $S(\mathsf{cos})$ on an algebra $\mathbf{A}$ are precisely the cosets on $\mathbf{A}$. We have not yet established an analogous relationship between the filters of the membership logic $S(\mathcal{K},\mathsf{mem})$ and relative cosets. In the following example, we employ the machinery of this chapter to show that every relative coset is a filter of the membership logic. Generally, however, there are filters that are not cosets.

Example 7.83 (The Logics of Cosets and Relative Cosets)

Before considering the case of the membership logic and relative cosets, we note that the universal logic $U(\mathbf{A},\mathsf{cos})$ of cosets on $\mathbf{A}$ is a *maximal* $\mathfrak{a}$-model of the sentential calculus $S(\mathsf{cos})$ of cosets. Recall that by Corollary 5.56 of Example 5.55 on page 190, every homomorphism from $\mathbf{A}$ into $\mathbf{B}$ is continuous from $\mathsf{Cos}(\mathbf{A})$ into $\mathsf{Cos}(\mathbf{B})$; consequently, $U(\mathbf{A},\mathsf{cos})$ is an $\mathfrak{a}$-model of $U(\mathbf{B},\mathsf{cos})$.

Remark 7.84 For all $\mathfrak{a}$-algebras $\mathbf{A}$ and $\mathbf{B}$, $U(\mathbf{A},\mathsf{cos})$ is an $\mathfrak{a}$-model of $U(\mathbf{B},\mathsf{cos})$. $\square$

Proposition 7.85 $U(\mathbf{A},\mathsf{cos})$ is a *maximal* $\mathfrak{a}$-model of $S(\mathsf{cos})$.

Proof. By the previous Remark, $U(\mathbf{A},\mathsf{cos})$ is an $\mathfrak{a}$-model of $U(\mathbf{B},\mathsf{cos})$. By Remark 6.87 of Example 6.85 on page 243, $S(\mathfrak{a},\mathsf{cos}) \equiv U(\mathbf{Tm},\mathsf{cos})$; hence $U(\mathbf{A},\mathsf{cos})$ is a maximal $\mathfrak{a}$-model of $U(\mathbf{B},\mathsf{cos})$, since $\mathsf{Th}(U(\mathbf{A},\mathsf{cos})) \doteq \mathsf{Cos}(\mathbf{A}) = \mathsf{Fi}^{\mathfrak{a}}_{\mathbf{A}}(S(\mathsf{cos}))$, by Proposition 5.60 on page 191. $\diamond$

We now turn to the membership logic and relative cosets.

Proposition 7.86 For any $\mathfrak{a}$-algebra $\mathbf{A}$, $U(\mathbf{A},\mathsf{cos}^{\mathcal{K}})$ is an $\mathfrak{a}$-model of both $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ and $S(\mathcal{K},\mathsf{mem})$. Consequently, for $\mathbf{A} \in \mathcal{K}$, $U(\mathbf{A},\mathsf{cos}^{\mathcal{K}})$ is a $\mathcal{K}$-model of both $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ and $S(\mathcal{K},\mathsf{mem})$.

Proof. By Corollary 5.56 of Example 5.55 on page 190, every homomorphism from $\mathbf{A}$ into $\mathbf{B}$ is continuous from $\mathsf{Cos}(\mathbf{A})$ into $\mathsf{Cos}(\mathbf{B})$; consequently, $U(\mathbf{A},\mathsf{cos})$ is an $\mathfrak{a}$-model of $U(\mathbf{B},\mathsf{cos})$.

So by definition, $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$ is an $\mathfrak{a}$-model of both $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ and $S(\mathcal{K}, \mathsf{mem})$. $\diamond$

It follows immediately, by Remark 7.16, that relative cosets are all filters of the membership logic.

Corollary 7.87 For any $\mathfrak{a}$-algebra $\mathbf{A}$, $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A}) \subseteq \mathsf{Fi}^{\mathfrak{a}}_{\mathbf{A}}(S(\mathcal{K}, \mathsf{mem})) = \mathsf{Fi}^{\mathcal{K}}_{\mathbf{A}}(S(\mathcal{K}, \mathsf{mem}))$. For $\mathbf{A} \in \mathcal{K}$, $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A}) \subseteq \mathsf{Fi}^{\mathfrak{a}}_{\mathbf{A}}(\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})) = \mathsf{Fi}^{\mathcal{K}}_{\mathbf{A}}(\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}}))$. $\square$

Recall that in Example 4.149 on page 169 we noted that we knew of no simple formal-axiomatization of the algebraic closed system $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$. The previous result demonstrates that the filters of the membership logic on an algebra encompass all $\mathcal{K}$-cosets on that algebra; generally, however, there are filters that are not $\mathcal{K}$-cosets, in other words, $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$ is not a maximal model of the membership logic. We shall see, however, that in the case that $\mathcal{K}$ is relatively congruence regular (see Definition 1.375 on page 71), $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$ is in fact a maximal model, and hence the filters of the membership logic perfectly describe the relative cosets on an algebra.

Recall the definition of the propositional $\mathcal{K}$-calculus $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ of *non-relative* cosets on the $\mathcal{K}$-free algebra $\mathbf{F}_{\mathcal{K}}$, given in Example 6.86 on page 243, and the axiomatization given in Proposition 6.89 of that example.

Theorem 7.88 For each $\mathbf{A} \in \mathcal{K}$, $U(\mathbf{A}, \mathsf{cos})$ is a maximal $\mathfrak{a}$-model of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$, i.e., $\mathsf{Fi}^{\mathfrak{a}}_{\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})}(\mathbf{A}) = \mathsf{Cos}(\mathbf{A})$.

Proof. By Remark 7.84, $U(\mathbf{A}, \mathsf{cos})$ is an $\mathfrak{a}$-model of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$, and so it suffices to show that every $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$-filter on $\mathbf{A}$ is a coset. Let F be a $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$-filter on $\mathbf{A}$. By Proposition 5.60 of Example 5.57 on page 191, it suffices to show that F is a $S(\mathfrak{a}, \mathsf{cos})$-filter. Suppose that $P \vdash_{S(\mathfrak{a}, \mathsf{cos})} p$, $\mathsf{i} : \mathbf{Tm} \to \mathbf{A}$ with $\mathsf{i}[P] \subseteq F$. Since $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ is an $\mathfrak{a}$-model of $S(\mathfrak{a}, \mathsf{cos})$ (by Remark 7.84), the canonical homomorphism is continuous from $S(\mathfrak{a}, \mathsf{cos})$ onto $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$. Hence $\overline{[P]} \vdash_{\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})} \overline{p}$. Let $\bar{\mathsf{i}}$ be the unique interpretation of $\mathbf{F}_{\mathcal{K}}$ into $\mathbf{A}$ mapping $\overline{x}$ to $\mathsf{i}(x)$, for each $x \in \mathsf{V}$. Note that for each $x \in \mathsf{V}$, $\bar{\mathsf{i}}(\overline{x}) = \mathsf{i}(x)$, and so by the absolute freedom of the term algebra, $\bar{\mathsf{i}}(\bar{\cdot}) = \mathsf{i}(\cdot)$. Hence $\bar{\mathsf{i}}\left[\overline{[P]}\right] = \mathsf{i}[P] \subseteq F$. Since $\overline{[P]} \vdash_{\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})} \overline{p}$ and F is a $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$-filter, $\mathsf{i}(p) = \bar{\mathsf{i}}(\overline{p}) \in F$. $\diamond$

$\square$

We shall now consider the filters of the *logics of lattice ideals and filters*, introduced in Example 6.98 on page 246.

Example 7.89 (Logics of Lattice Ideals and Filters)

Recall that conventionally, the usage of the notations $\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})$, $\boldsymbol{S}_*(\mathcal{K}, \mathsf{fi})$, $\boldsymbol{S}_0(\mathcal{K}, \mathsf{id})$ and $\boldsymbol{S}_1(\mathcal{K}, \mathsf{fi})$, implicitly imply the 'type' of quasivariety $\mathcal{K}$ of lattice expansions under consideration (see Convention 6.100 of Example 12.61 on page 389).

Proposition 7.90 For $\mathbf{P} \in \mathcal{K}$, $\mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})}(\mathbf{P}) = \mathsf{Id}_{\diamond\emptyset}(\mathbf{P})$, i.e., $\mathsf{F}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})}(\mathbf{P}) = U(\mathbf{P}, \mathsf{id}_{\emptyset})$.

Proof. Recall the axiomatization of $\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})$ given in Proposition 6.106 on page 247. Let $\mathbf{F} = \mathbf{lg}(\boldsymbol{S}_*(\mathcal{K}, \mathsf{id}))$. Note that since $\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})$ has no theorems, $\emptyset \in \mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})}(\mathbf{P})$, by Remark 7.21.

$\boxed{\mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}(\mathbf{P}) \subseteq \mathsf{Id}_{\Diamond_\emptyset}(\mathbf{P})}$ Let $F \in \mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}(\mathbf{P})$. If $F = \emptyset$ then $F \in \mathsf{Id}_{\Diamond_\emptyset}(\mathbf{P})$. Otherwise $F \neq \emptyset$. $\boxed{\text{Downset}}$ Suppose that $a \leq^{\mathbf{P}} b \in F$. Let i be any interpretation mapping $\overline{x} \mapsto b$ and $\overline{y} \mapsto a$. Since $\overline{x} \vdash_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})} \overline{x} \wedge^{\mathbf{F}} \overline{y}$ and $\mathsf{i}(\overline{x}) = b \in F$, we have $a = b \wedge^{\mathbf{P}} a = \mathsf{i}(\overline{x}) \wedge^{\mathbf{P}} \mathsf{i}(\overline{y}) = \mathsf{i}(\overline{x} \wedge^{\mathbf{F}} \overline{y}) \in F$, since F is a $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$-filter. $\boxed{\vee\text{-Closed}}$ Suppose that $a, b \in F$. Let i be any interpretation mapping $\overline{x} \mapsto a$ and $\overline{y} \mapsto b$. Since $\{\overline{x}, \overline{y}\} \vdash_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})} \overline{x} \vee^{\mathbf{F}} \overline{y}$ and $\mathsf{i}[\{\overline{x}, \overline{y}\}] = \{b, a\} \subseteq F$, $a \vee^{\mathbf{P}} b = \mathsf{i}(\overline{x}) \vee^{\mathbf{P}} \mathsf{i}(\overline{y}) = \mathsf{i}(\overline{x} \vee^{\mathbf{F}} \overline{y}) \in F$, since F is a $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$-filter. $\boxed{\mathsf{Id}_{\Diamond_\emptyset}(\mathbf{P}) \subseteq \mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}(\mathbf{P})}$ Since $\mathbf{P} \in \mathcal{K}$, every homomorphism from $\mathbf{F}$ into $\mathbf{P}$ is continuous from $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$ into $U(\mathbf{P}, \mathsf{id}_\emptyset)$, and hence $U(\mathbf{P}, \mathsf{id}_\emptyset)$ is a $\mathcal{K}$-model of $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$. So by Remark 7.16, $\mathsf{Id}_{\Diamond_\emptyset}(\mathbf{P}) \subseteq \mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}(\mathbf{P})$. $\Diamond$

The proofs of the following are similar or dual to the proof of the previous proposition, and as such are omitted.

Proposition 7.91 For $\mathbf{P} \in \mathcal{K}$, $\mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = \mathsf{Fl}_{\Diamond_\emptyset}(\mathbf{P})$, i.e., $\mathsf{F}^{\mathcal{K}}_{\boldsymbol{S}_*(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = U(\mathbf{P}, \mathsf{fi}_\emptyset)$.

Proposition 7.92 For $\mathbf{P} \in \mathcal{K}$, $\mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_0(\mathcal{K},\mathsf{id})}(\mathbf{P}) = \mathsf{Id}_{\Diamond}(\mathbf{P})$, i.e., $\mathsf{F}^{\mathcal{K}}_{\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = U(\mathbf{P}, \mathsf{fi})$.

Proposition 7.93 For $\mathbf{P} \in \mathcal{K}$, $\mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = \mathsf{Fl}_{\Diamond}(\mathbf{P})$, i.e., $\mathsf{F}^{\mathcal{K}}_{\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = U(\mathbf{P}, \mathsf{id})$.

Corollary 7.94 For $\mathbf{P} \in \mathcal{K}$, $U(\mathbf{P}, \mathsf{id}_\emptyset)$ (resp. $U(\mathbf{P}, \mathsf{fi}_\emptyset)$, $U(\mathbf{P}, \mathsf{fi})$ and $U(\mathbf{P}, \mathsf{id})$) is a maximal $\mathcal{K}$-model of $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$ (resp. $\boldsymbol{S}_*(\mathcal{K},\mathsf{fi})$, $\boldsymbol{S}_0(\mathcal{K},\mathsf{id})$ and $\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})$).

$\square$

M models L	
$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]$ i is continuous from L into M	
$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]$ i is continuous from $\mathsf{Th}(\mathsf{L})$ into $\mathsf{Th}(\mathsf{M})$ (7.3)	$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]\ R \in \mathsf{Th}(\mathsf{M}) \to \mathsf{i}^{-1}[R] \in \mathsf{Th}(\mathsf{L})$ (7.4)
$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]\ \mathsf{i}[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}\left[\|\Gamma\|_{\mathsf{L}}\right]$ (7.5)	$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]\ \mathsf{i}^{-1}\left[\|\Phi\|_{\mathsf{M}}\right] \supseteq \left\| \mathsf{i}^{-1}\left[\|\Phi\|_{\mathsf{M}}\right] \right\|_{\mathsf{L}}$ (7.6)
$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]\ \mathsf{i}\left[\|\Gamma\|_{\mathsf{L}}\right] \subseteq \|\mathsf{i}[\Gamma]\|_{\mathsf{M}}$ (7.7)	$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]\ \mathsf{i}^{-1}\left[\|\Phi\|_{\mathsf{M}}\right] = \left\| \mathsf{i}^{-1}\left[\|\Phi\|_{\mathsf{M}}\right] \right\|_{\mathsf{L}}$ (7.8)
$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]\ \left\| \mathsf{i}[\|\Gamma\|_{\mathsf{L}}] \right\|_{\mathsf{M}} \subseteq \|\mathsf{i}[\Gamma]\|_{\mathsf{M}}$ (7.9)	$\forall[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})] R \in \mathsf{Th}(\mathsf{M}) \to \left\| \mathsf{i}^{-1}[R] \right\|_{\mathsf{L}} = \mathsf{i}^{-1}[R]$ (7.10)
$\forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]\ \left\| \mathsf{i}[\|\Gamma\|_{\mathsf{L}}] \right\|_{\mathsf{M}} = \|\mathsf{i}[\Gamma]\|_{\mathsf{M}}$ (7.11)	
$\mathsf{M} \in \mathsf{Mod}^{\mathsf{s}}_{\mathsf{L}}$ (7.12)	$\mathsf{L} \in \mathsf{Abs}^{\mathsf{s}}_{\mathsf{M}}(\mathbf{lg}(\mathsf{L}))$ (7.13)
$\mathsf{M} \succeq \mathsf{F}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))$ (7.14)	$\mathsf{L} \preceq \mathsf{I}^{\mathsf{s}}_{\mathsf{M}}(\mathbf{lg}(\mathsf{L}))$ (7.15)
$\Phi \vdash_{\mathsf{F}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))} \psi \to \Phi \vdash_{\mathsf{M}} \psi$ (7.16)	$\Gamma \vdash_{\mathsf{L}} \phi \to \Gamma \vdash_{\mathsf{I}^{\mathsf{s}}_{\mathsf{M}}(\mathbf{lg}(\mathsf{L}))} \phi$ (7.17)
$\mathsf{id}^{\mathbf{lg}(\mathsf{M})}_{\mathsf{s}}$ is continuous from $\mathsf{F}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))$ onto M (7.18)	$\mathsf{id}^{\mathbf{lg}(\mathsf{L})}_{\mathsf{s}}$ is a continuous from L onto $\mathsf{I}^{\mathsf{s}}_{\mathsf{M}}(\mathbf{lg}(\mathsf{L}))$ (7.19)
$\Gamma \vdash_{\mathsf{L}} \phi \to \mathsf{i}[\Gamma] \vdash_{\mathsf{M}} \mathsf{i}(\phi), \forall\,[\mathsf{i} \in \mathsf{Int}_{\mathsf{s}}(\mathsf{L},\mathsf{M})]$ (7.20)	$\mathsf{cl}_{\mathsf{M}} \subseteq \mathsf{cl}_{\mathsf{F}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{lg}(\mathsf{M}))}$ (7.21)

Table 7.1: The statements of this table are equivalent (see Corollary 7.4, Remark 7.16, Remark 7.33 and Theorem 7.40).

L is structural	
$\forall\,[\sigma \in \mathsf{Sub}_{\mathsf{s}}(\mathsf{L}), \Gamma \subseteq \mathsf{Fm}(\mathsf{L})]\quad \sigma[\Gamma] \vdash \sigma\left[\|\Gamma\|\right]$ (7.22)	$\forall\,[\sigma \in \mathsf{Sub}_{\mathsf{s}}(\mathsf{L}), \phi \in \mathsf{Fm}(\mathsf{L})]\quad \sigma^{\star}(\|\{\phi\}\|_{\mathsf{L}}) = \sigma^{\star}(\{\phi\})$ and $\sigma^{\star}{}_{\restriction \mathsf{Th}(\mathsf{L})} : \mathbf{Th}(\mathsf{L}) \to_{\blacktriangledown} \mathbf{Th}(\mathsf{M})$ (7.23)
$\forall\,[\sigma \in \mathsf{Sub}_{\mathsf{s}}(\mathsf{L}), \Gamma \subseteq \mathsf{Fm}(\mathsf{L})]\quad \sigma\left[\|\Gamma\|\right] \subseteq \sigma^{\star}(\Gamma)$ (7.24)	$\mathsf{L} \in \mathsf{Mod}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{lg}(\mathsf{L}))$ (7.25)
$\forall\,[\sigma \in \mathsf{Sub}_{\mathsf{s}}(\mathsf{L}), \Gamma \subseteq \mathsf{Fm}(\mathsf{L})]\quad \sigma^{\star}(\|\Gamma\|) \subseteq \sigma^{\star}(\Gamma)$ (7.26)	$\mathsf{L} \in \mathsf{Abs}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{lg}(\mathsf{L}))$ (7.27)
$\forall\,[\sigma \in \mathsf{Sub}_{\mathsf{s}}(\mathsf{L}), \Gamma \subseteq \mathsf{Fm}(\mathsf{L})]\quad \sigma^{\star}(\|\Gamma\|) = \sigma^{\star}(\Gamma)$ (7.28)	$\mathsf{L} = \mathsf{F}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{lg}(\mathsf{L}))$ (7.29)
$\forall\,[\sigma \in \mathsf{Sub}_{\mathsf{s}}(\mathsf{L}), \Gamma \subseteq \mathsf{Fm}(\mathsf{L})]\quad \sigma[\Gamma] \dashv\vdash \sigma\left[\|\Gamma\|\right]$ (7.30)	$\mathsf{L} = \mathsf{I}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{lg}(\mathsf{L}))$ (7.31)
$\forall\,[\sigma \in \mathsf{Sub}_{\mathsf{s}}(\mathsf{L}), \Gamma \subseteq \mathsf{Fm}(\mathsf{L})]\quad \sigma[\Gamma] \dashv\vdash \sigma^{\star}(\|\Gamma\|)$ (7.32)	L is a semantics for itself (7.33)
L has some semantics (7.34)	$\mathsf{F}^{\mathsf{s}}_{\mathsf{L}}(\mathbf{L})$ is a semantics for L (7.35)

Table 7.2: Characterizing structurality. (See Corollary 6.16, Theorem 7.43, Theorem 7.48 and Theorem 7.61)

Chapter 8

Canons and Archologies

In this chapter we shall make precise a technique that we have been using to induce sentential calculi from familiar closed systems arising in universal algebra. One of the most striking features of algebraic logic, when encountered by ourselves for the first time coming from a universal algebraic background, is the fact that algebraic logicians make primary use of the *term algebra*; this is striking since the term algebra is typically of little interest to the algebraist. Far more familiar to the universal algebraist is the *free algebra* determined by some class of algebras. The term algebra, being purely a syntactic object, encodes no semantic information, while the free algebra on the other hand, which obtains from the term algebra by factorization by a congruence encoding the equational theory of the class of algebras (see Definition 1.383 on page 73), contains much semantic information, and in the case of varieties, is a semantically rich algebra indeed.

One of the motivations for considering logics over constructs, lies in the fact that we can consider a logic directly on the free algebra $\mathbf{F}_{\mathcal{K}}$ of a quasivariety $\mathcal{K}$ as a (primary) propositional $\mathcal{K}$-calculus, and then consider the algebras of the quasivariety as the languages over which filters obtain, without having to find some sentential calculus achieving the same result. Consider the example of the subuniverse logics. Recall that, for a given a quasivariety $\mathcal{K}$, the logic of subuniverses of the $\mathcal{K}$-free algebra on ω free generators is denoted by $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ (see Example 6.79 on page 242). Considering the quasivariety $\mathcal{K}$ as a signature of logics (i.e., the construct of all $\mathcal{K}$ algebras with homomorphisms), we showed that $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ is a propositional $\mathcal{K}$-calculus and we provided an axiomatization of this logic (see Example 6.80 on page 242). We showed further that the filters of this logic on an algebra $\mathbf{A}$ in $\mathcal{K}$ are precisely the subuniverses of $\mathbf{A}$ (see Proposition 7.82 of Example 7.77 on page 270), consequently demonstrating that the logic $F(\mathbf{A}, \mathsf{su})$ is a maximal model of $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ (with respect to the signature $\mathcal{K}$). *What sentential calculus best reflects the propositional $\mathcal{K}$-calculus $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$?* One possibility is the sentential calculus $S(\mathfrak{a}, \mathsf{su})$ (see Example 5.47 on page 188). This logic, however, reflects none of the semantics of $\mathcal{K}$, being defined purely in terms of the type $\mathfrak{a}$ of algebras. As such, no $\mathcal{K}$ dependent properties may be characterized in terms of properties satisfied by $S(\mathfrak{a}, \mathsf{su})$.

The primary aim of this chapter is to explicate a technique for inducing sentential 1-calculi from such propositional $\mathcal{K}$-calculi. In the discourse of this chapter, we shall be calling a language such as $\mathbf{F}_{\mathcal{K}}$ an $\mathfrak{a}$-*canonical language* and a logic on an $\mathfrak{a}$-canonical language is called an $\mathfrak{a}$-*canon.* The quasivariety $\mathcal{K}$ can be viewed as a full subconstruct of the construct $\mathfrak{a}$ (of all $\mathfrak{a}$-algebras with homomorphisms), and in the discourse of this chapter, such a signature of logics is called an $\mathfrak{a}$-*archetype*. A class of logics such as $\{F(\mathbf{A}, \mathsf{su}) : \mathbf{A} \in \mathcal{K}\}$ is called an $\mathfrak{a}$-*archology* (certain restrictions apply in the definition of an archology). By definition, an archology contains a canon, and all the logics of the archology model this canon. We shall show how an $\mathfrak{a}$-canon induces a sentential calculus (called the *ideal*) that reflects both the consequences of the logic and the equational semantics encoded in the $\mathfrak{a}$-canonical language $\mathbf{F}_{\mathcal{K}}$. Note, however, that the discourse of this chapter is phrased in terms of constructs and subconstructs, and the case of types of algebras and quasivarieties is a special case.

While our notion of an archology, because of the restrictions imposed, can be viewed as a π-institution (the restrictions ensure that the π-institution is structural), it is not its *multi-signature* nature of an archology that is of interest to us. Rather, it is its *proto-typical* nature that is important.

In $\oint$8.1 we consider canons and their induced calculi, and in $\oint$8.2 we consider archologies and their relationships to their canon and the ideal induced by the canon.

8.1 Canons

In $\oint$8.1.1, we define the notion of an $\mathfrak{s}$-canonical language, which is itself a global language (with respect to the full subconstruct consisting only of itself) and introduce machinery for converting between substitutions of the global $\mathfrak{s}$-language $\mathbf{G}$ and substitutions of a canonical language $\mathbf{F}$. Canons are defined in $\oint$8.1.2, and two techniques are considered for inducing logics on $\mathbf{G}$ from the canon logic on $\mathbf{F}$. The simpler of the techniques involves the product of the singleton source consisting of the canonical morphism from $\mathbf{G}$ onto $\mathbf{F}$, which we call the *form* of the canon. This technique induces a finitary $\mathfrak{s}$-deductive system, in the case that the canon is finitary, but not necessarily a structural $\mathfrak{s}$-deductive system. The second technique is to take the induced $\mathfrak{s}$-deductive system to be the ideal logic $\mathsf{I}^{\mathfrak{s}}_{\boldsymbol{\mathcal{D}}}(\mathbf{G})$ (see Definition 7.28 on page 259) which we call the *ideal* of the canon. The ideal is always $\mathfrak{s}$-structural, but not necessarily finitary; not even when the canon is finitary. We show that, in the case that the canon is structural, the ideal and form coincide, a property that, in fact, characterizes the *structurality* of the canon. Consequently, in the case that the canon is structural and *finitary*, this induced logic is a *propositional* $\mathfrak{s}$-calculi, and hence, in the case that $\mathfrak{s}$ is a type of algebras, this induced logic is a *sentential* 1-*calculus*. We show that if a canon is *structural* then the *theory lattice of the ideal* (which is equal to the form) is *isomorphic* to the *theory lattice of the canon*.

Of particular importance is the *discrete canon*, introduced in $\oint$8.1.3, which is simply the discrete logic on a canonical language. This logic is both finitary and structural, and so its ideal and form coincide, yielding a minimal propositional $\mathfrak{s}$-calculus reflecting the semantic truths encoded in the canonical language. In the case of a quasivariety $\mathcal{K}$ and the canon $\mathbf{F}_{\mathcal{K}}$, the induced sentential calculus is a logic that reflects the equational truths of $\mathcal{K}$; we call this sentential calculus the *equational logic of* $\mathcal{K}$. We present an axiomatization of the ideal logic induced by the discrete canon, and hence an axiomatization of the equational logic of $\mathcal{K}$. In $\oint$8.1.4 we demonstrate how to *axiomatize* the ideal of a structural and finitary canon in terms of an *axiomatization of the canon* and an *axiomatization of the ideal induced by the discrete canon*.

In $\oint$8.1.5 we turn the problem around. Beginning with an $\mathfrak{s}$-deductive system $\mathcal{D}$ and an $\mathfrak{s}$-

canonical language $\mathbf{F}$, we consider the *filter logic* (also called the *model logic*) $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ induced by $\mathcal{D}$ on $\mathbf{F}$ (see Definition 7.13 on page 255), which is always a structural $\mathfrak{s}$-canon on $\mathbf{F}$, and hence it in turn induces a structural $\mathfrak{s}$-calculus (the form and ideal coincide since $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ is structural), which we call the logic of $\mathcal{D}$-filters on $\mathcal{K}$. The theory lattice of this logic is isomorphic to the lattice of $\mathcal{D}$ filters on $\mathbf{F}$. In the case that $\mathcal{D}$ is a sentential calculus and $\mathcal{K}$ a quasivariety of algebras, this induced sentential calculus is essentially $\mathcal{D}$ enriched so as to encode the equational truths of $\mathcal{K}$. Informally we speak of 'smoothing' or 'fitting' $\mathcal{D}$ to $\mathcal{K}$.

Finally, in ∮8.1.6 we present a number of examples. Some of the examples demonstrate how familiar sentential calculi such as $S(\mathcal{K}, \tau)$ arise from canons, while others give rise to 'inherently unalgebraizable' logics to which our *parametrized* theory of algebraization successfully applies (see Part V).

Convention 8.1 Throughout this chapter, let $\mathfrak{s}$ be a signature with a global language on ω-free generators. This global language will be denoted by $\mathbf{G}$ and we shall assume that $\mathsf{Var}_{\mathfrak{s}}(\mathbf{G}) = \mathsf{V}$, where V is a global chosen denumerably infinite set of variables. All $\mathfrak{s}$-deductive systems are over $\mathbf{G}$. Arbitrary $\mathfrak{s}$-deductive systems are denoted by $\mathcal{D}$, and arbitrary propositional $\mathfrak{s}$-calculi (i.e., $\mathfrak{s}$-structural and finitary $\mathfrak{s}$-deductive systems) are denoted by $\mathcal{P}$.

8.1.1 Canonical Languages

We begin by identifying certain languages of a signature $\mathfrak{s}$ as $\mathfrak{s}$-*canonical*. Given any $\mathfrak{s}$-language $\mathbf{A}$, we consider the construct consisting of *only* this language and all $\mathfrak{s}$-substitutions of $\mathbf{A}$. This construct is a *full* subconstruct of $\mathfrak{s}$. If in this subconstruct the single language $\mathbf{A}$ is *global*, i.e., it is freely generated (in the subconstruct) with ω-free generators, we call the language $\mathfrak{s}$-*canonical*. In the next section we shall consider logics on an $\mathfrak{s}$-canonical language, which we call $\mathfrak{s}$-*canons*, and shall demonstrate how such canons *induce* logics on the global $\mathfrak{s}$-language $\mathbf{G}$.

Definition 8.2 (Canonical Languages) Let $\mathfrak{s}$ be a signature of logics. An $\mathfrak{s}$-language $\mathbf{F}$ is called **$\mathfrak{s}$-canonical** if, in the *full* subconstruct of $\mathfrak{s}$ consisting of $\mathbf{F}$ only, $\mathbf{F}$ is freely generated by ω free generators. In this situation we refer to $\mathbf{G}$ as the **root language**. We shall denote the full subconstruct consisting of $\mathbf{F}$ by $\mathfrak{s}_{\mathbf{F}}$ and denote the $\mathfrak{s}_{\mathbf{F}}$-free generators of $\mathbf{F}$ by $\overline{[\mathsf{V}]} = \{\overline{\mathsf{v}} : \mathsf{v} \in \mathsf{V}\}$. The unique $\mathfrak{s}$-interpretation of $\mathbf{G}$ onto $\mathbf{F}$ extending $\mathsf{v} \mapsto \overline{\mathsf{v}}$ is denoted by $\overline{\cdot}^{\mathfrak{s}_{\mathbf{F}}}$, and we denote the prepole and pre-image of this interpretation by $\underline{[\![\cdot]\!]}_{\mathfrak{s}_{\mathbf{F}}}$ and $\underline{[\cdot]}_{\mathfrak{s}_{\mathbf{F}}}$, respectively, although we tend to drop the subscript $\mathfrak{s}_{\mathbf{F}}$ where ever unambiguous. Arbitrary $\mathbf{F}$-formulae are denoted by (emboldened) $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, etc, and arbitrary sets of $\mathbf{F}$-formulae are denoted by (emboldened) $\boldsymbol{\Gamma}$, etc., in contrast to $\mathbf{G}$-formulae, denoted by ϕ, ψ, etc, and sets of $\mathbf{G}$-formulae, denoted by Γ, etc. We shall *implicitly* extend this emboldening convention wherever possible. For example, an arbitrary $\mathfrak{s}_{\mathbf{F}}$-deductive system is denoted by $\boldsymbol{\mathcal{D}}$, in contrast to an arbitrary $\mathfrak{s}$-deductive system which is denoted by $\mathcal{D}$. □

The following example highlights our most important case. Given a quasivariety $\mathcal{K}$ of $\mathfrak{a}$-algebras, we consider the $\mathcal{K}$-*free* algebra $\mathbf{F}_{\mathcal{K}}$ on ω-free generators as an $\mathfrak{a}$-*canonical* language, where $\mathfrak{a}$ is the signature of all $\mathfrak{a}$-algebras with homomorphisms. In this case the *root* language is the *term algebra*. In the next section, we shall show how certain $\mathbf{F}_{\mathcal{K}}$-logics (i.e., logics whose theories are closed systems over the universe of the $\mathcal{K}$-free algebra) induce *sentential calculi*.

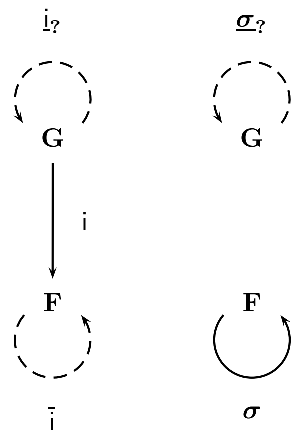


Figure 8.1: Converting Interpretations and Substitutions (see Definition 8.4)

Example 8.3 (Free Algebras of Quasivarieties)

> Let $\mathfrak{a}$ be a type of algebras, which we conflate with the construct of all $\mathfrak{a}$-algebras with homomorphisms. Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras. Then the $\mathcal{K}$-free algebra $\mathbf{F}_{\mathcal{K}}$ over ω-free generators $\overline{[\mathsf{V}]}$ is an $\mathfrak{a}$-*canonical* language; the *root* language is the term algebra on V, which we denote by $\mathbf{Tm}$.

□

In the following definition, we introduce mechanisms for converting *interpretations* of $\mathbf{G}$ into $\mathbf{F}$ to *substitutions* of $\mathbf{G}$ and $\mathbf{F}$ respectively, and for converting $\mathbf{F}$-substitutions to $\mathbf{G}$-substitutions. See Figure 8.1 for a visual representation of these conversions.

Definition 8.4 (Converting Interpretations and Substitutions) For any $\mathfrak{s}$-interpretation $\mathsf{i} : \mathbf{G} \to \mathbf{F}$, let $\bar{\mathsf{i}}$ denote the unique $\mathbf{F}$-substitution with $\bar{\mathsf{i}}(\overline{x}) = \mathsf{i}(x)$ for all variables $x \in V$, and let $\underline{\mathsf{i}}_?$ denote the unique $\mathbf{G}$-substitution with $\underline{\mathsf{i}}_?(\mathsf{v}) = ?(\mathsf{i}(\mathsf{v}))$, for all variables $\mathsf{v} \in \mathsf{V}$, where $? : V \to \mathsf{Fm}(\mathbf{G})$ is some function such that $?(\overline{\mathsf{v}}) \in \underline{[\![\mathsf{i}(\mathsf{v})]\!]}$, for each $\mathsf{v} \in \mathsf{V}$. For any $\mathbf{F}$-substitution $\boldsymbol{\sigma}$, pick a function $? : \overline{[V]} \to \mathsf{Fm}(\mathbf{G})$ such that $?(\overline{\mathsf{v}}) \in \underline{[\![\boldsymbol{\sigma}(\overline{\mathsf{v}})]\!]}$, for each $\mathsf{v} \in \mathsf{V}$, and let $\underline{\boldsymbol{\sigma}}_?$ denote the unique $\mathbf{G}$-substitution with $\underline{\boldsymbol{\sigma}}_?(\mathsf{v}) = ?(\overline{\mathsf{v}})$, for all $\mathsf{v} \in \mathsf{V}$. We drop the subscripts $?$ from these notations when the particular choice function is unimportant. □

We briefly remark on some important relationships between these operations of conversion and the *canonical image* function.

Remark 8.5 $\bar{\mathsf{i}}(\overline{\phi}) = \mathsf{i}(\phi)$.

Proof. Since $\bar{\mathsf{i}} : \mathbf{F} \to_{\mathfrak{s}_{\mathbf{F}}} \mathbf{F}$, and $\mathfrak{s}_{\mathbf{F}}$ is a subconstruct of $\mathfrak{s}$, $\bar{\mathsf{i}} : \mathbf{F} \to_{\mathfrak{s}} \mathbf{F}$, and since $\bar{\cdot} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{F}$, the composition $\bar{\mathsf{i}}(\bar{\cdot}) : \mathbf{G} \to_{\mathfrak{s}} \mathbf{F}$. But then, $\mathsf{i} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{F}$ and $\bar{\mathsf{i}}(\bar{\cdot})$ coincide on the $\mathfrak{s}$-free generators of $\mathbf{G}$ by construction, and hence must be equal generally. ◇

Remark 8.6 $\overline{\underline{\boldsymbol{\sigma}}_?(\phi)} = \boldsymbol{\sigma}(\overline{\phi})$.

Proof. $\overline{\underline{\boldsymbol{\sigma}}_{?}(\cdot)}$ and $\boldsymbol{\sigma}(\bar{\cdot})$ are both compositions of $\mathfrak{s}$-morphisms, and are hence both $\mathfrak{s}$-morphisms from $\mathbf{G}$ into $\mathbf{F}$, and since for each $\mathsf{v} \in \mathsf{V}$, $\overline{\underline{\boldsymbol{\sigma}}_{?}(\mathsf{v})} = \overline{?(\bar{\mathsf{v}})} = \boldsymbol{\sigma}(\bar{\mathsf{v}})$ (as $?(\bar{\mathsf{v}}) \in \underline{[\![\boldsymbol{\sigma}(\bar{\mathsf{v}})]\!]}$), they must be equal by the freedom of $\mathbf{G}$. ◇

Remark 8.7 $\overline{\underline{\mathsf{i}}_{?}(\phi)} = \mathsf{i}(\phi)$.

Proof. By morphism composition $\overline{\underline{\mathsf{i}}_{?}(\cdot)} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{F}$. For each $\mathsf{v} \in \mathsf{V}$, $\overline{\underline{\mathsf{i}}_{?}(\mathsf{v})} = \mathsf{i}(\mathsf{v})$. So by $\mathbf{G}$-freedom over V, $\overline{\underline{\mathsf{i}}_{?}(\cdot)} = \mathsf{i}$. ◇

8.1.2 Canons

We now consider $\mathfrak{s}$-*canons*, which are logics over an $\mathfrak{s}$-canonical language. With each $\mathfrak{s}$-canon we associate two logics on the *root* language, one called the *form* of the canon and the other the *ideal* of the canon. While the ideal is far more complex to characterize than the form, the canon is always a *semantics* for the ideal. On the other hand, while the canon need not be a semantics for its form, the form is always finitary whenever the canon if finitary; this is not generally true for the ideal. A key result is that the form and ideal coincide whenever the canon is $\mathfrak{s}$-structural; this coincidence in fact characterizes the $\mathfrak{s}$-structurality of the canon.

Definition 8.8 (Canons, Forms and Ideals) An **$\mathfrak{s}$-canonical logic** or **$\mathfrak{s}$-canon** is a logic on an $\mathfrak{s}$-canonical language. All arbitrary $\mathfrak{s}$-canons are to be taken over $\mathbf{F}$. Unless specified to the contrary, $\boldsymbol{\mathcal{D}}$ shall denote a fixed but arbitrary $\mathfrak{s}$-canon. Since an $\mathfrak{s}$-canonical language $\mathbf{F}$ forms a *full* subconstruct of $\mathfrak{s}$, $\mathfrak{s}$-substitutions of $\mathbf{F}$ and $\mathfrak{s}_{\mathbf{F}}$-substitutions of $\mathbf{F}$ coincide, and consequently we tend to drop explicit references to signatures. In particular, an $\mathfrak{s}$-canonical logic $\boldsymbol{\mathcal{D}}$ is $\mathfrak{s}$-structural iff it is $\mathfrak{s}_{\mathbf{F}}$-structural, and so we just speak of structurality. A **propositional $\mathfrak{s}$-canon** is an $\mathfrak{s}$-canon that is both finitary and structural. Arbitrary propositional $\mathfrak{s}$-canons are denoted by emboldened $\boldsymbol{\mathcal{P}}$.

We denote the product logic $\bar{\cdot}^{\blacktriangleleft}[\boldsymbol{\mathcal{D}}]$ by $\underline{\boldsymbol{\mathcal{D}}}$ (see Definition 7.10 on page 254), which is an $\mathfrak{s}$-deductive system on $\mathbf{G}$, which we call the **form** of $\boldsymbol{\mathcal{D}}$, and we denoted the logic $\Vdash^{\mathfrak{s}}_{\boldsymbol{\mathcal{D}}}(\mathbf{G})$ by $\boldsymbol{\mathcal{D}}^{\imath}$ (see Definition 7.28 on page 259), which we call the **ideal** of $\boldsymbol{\mathcal{D}}$. We call a canon **ideal** if its form and ideal coincide, in which case we speak of the **ideal/form**. □

While $\boldsymbol{\mathcal{D}}$ is always a semantics for its ideal $\boldsymbol{\mathcal{D}}^{\imath}$, by Remark 7.57 on page 265, it is generally difficult to characterize the ideal $\boldsymbol{\mathcal{D}}^{\imath}$, as it is defined as the product of a non-singleton source. On the other hand, while the form $\underline{\boldsymbol{\mathcal{D}}}$ is far more easily characterized, as it is the product of a *single* and *surjective* function, it need not be *structural*, in which case it can have *no* semantics (see Theorem 7.61 on page 266). We shall see that in the case that $\boldsymbol{\mathcal{D}}$ is *structural*, these two logics coincide, in which case the simpler characterizations of $\underline{\boldsymbol{\mathcal{D}}}$ pertain.

We begin by analysing the simpler $\underline{\boldsymbol{\mathcal{D}}}$. The following result is a special case of Theorem 5.115 on page 207, Proposition 5.85 on page 198, (5.41) of Remark 5.83 on page 198 and Proposition 5.87 on page 198.

Theorem 8.9 Let $\boldsymbol{\mathcal{D}}$ be an $\mathfrak{s}$-canon.

1. $\underline{\boldsymbol{\mathcal{D}}}$ is the *coarsest* $\mathfrak{s}$-deductive system $\mathcal{D}$ on $\mathbf{G}$ for which $\bar{\cdot}$ is continuous from $\mathcal{D}$ into $\boldsymbol{\mathcal{D}}$.

2. $\bar{\cdot}$ is strictly continuous from $\underline{\boldsymbol{\mathcal{D}}}$ onto $\boldsymbol{\mathcal{D}}$.

3. $\mathsf{Th}(\underline{\boldsymbol{\mathcal{D}}}) = \{\underline{[\boldsymbol{T}]} \,:\, \boldsymbol{T} \in \mathsf{Th}(\boldsymbol{\mathcal{D}})\}$.

4. The following conditions are equivalent.

   (a) $\Gamma \vdash_{\underline{\boldsymbol{\mathcal{D}}}} \phi$.

   (b) $\overline{[\Gamma]} \vdash_{\boldsymbol{\mathcal{D}}} \overline{\phi}$.

   (c) $\forall\, [\boldsymbol{T} \in \mathsf{Th}(\boldsymbol{\mathcal{D}})]\ \overline{[\Gamma]} \subseteq \boldsymbol{T} \to \overline{\phi} \in \boldsymbol{T}$.

5. $\|\Gamma\|_{\underline{\boldsymbol{\mathcal{D}}}} = \underline{\left[\left\|\overline{[\Gamma]}\right\|_{\boldsymbol{\mathcal{D}}}\right]}$.

6. $\|\boldsymbol{\Gamma}\|_{\boldsymbol{\mathcal{D}}} = \overline{\left[\left\|\underline{[\boldsymbol{\Gamma}]}\right\|_{\underline{\boldsymbol{\mathcal{D}}}}\right]}$.

7. $\underline{\boldsymbol{\mathcal{D}}}$ is finitary iff $\boldsymbol{\mathcal{D}}$ is finitary.

8. Let $\boldsymbol{\Gamma} \cup \{\boldsymbol{\phi}\} \subseteq \mathsf{Fm}(\boldsymbol{\mathcal{D}})$, let $\phi_{\boldsymbol{\psi}} \in \underline{[\![\boldsymbol{\psi}]\!]}$, for each $\boldsymbol{\psi} \in \boldsymbol{\Gamma}$, $\phi_{\boldsymbol{\phi}} \in \underline{[\![\boldsymbol{\phi}]\!]}$, and let $\Gamma_{\boldsymbol{\Gamma}} \subseteq \mathsf{Fm}(\mathbf{G})$ such that $\overline{[\Gamma_{\boldsymbol{\Gamma}}]} = \boldsymbol{\Gamma}$. The following conditions are equivalent.

   (a) $\boldsymbol{\Gamma} \vdash_{\boldsymbol{\mathcal{D}}} \boldsymbol{\phi}$.

   (b) $\underline{[\boldsymbol{\Gamma}]} \vdash_{\underline{\boldsymbol{\mathcal{D}}}} \underline{[\![\boldsymbol{\phi}]\!]}$.

   (c) $\{\phi_{\boldsymbol{\psi}} \,:\, \boldsymbol{\psi} \in \boldsymbol{\Gamma}\} \vdash_{\underline{\boldsymbol{\mathcal{D}}}} \phi_{\boldsymbol{\phi}}$.

   (d) $\Gamma_{\boldsymbol{\Gamma}} \vdash_{\underline{\boldsymbol{\mathcal{D}}}} \phi_{\boldsymbol{\phi}}$.

   Further, $\|\boldsymbol{\Gamma}\|_{\boldsymbol{\mathcal{D}}} = \overline{\left[\|\{\phi_{\boldsymbol{\psi}} \,:\, \boldsymbol{\psi} \in \boldsymbol{\Gamma}\}\|_{\underline{\boldsymbol{\mathcal{D}}}}\right]} = \overline{\left[\|\Gamma_{\boldsymbol{\Gamma}}\|_{\underline{\boldsymbol{\mathcal{D}}}}\right]}$.

□

Observe that (4) characterizes arbitrary form consequences in terms of certain canon consequences, while (8) characterizes *arbitrary* canon consequences in terms of certain form consequences. Thus there is a direct and simple relationship between the canon and its form. Note further, that (5) characterizes form theory generation in terms of canon theory generation, while (6) characterizes canon theory generation in terms of form theory generation. While the ideal is always structural (where as the form is generally non-structural), such direct and simple correlations do not obtain generally between the canon and its ideal. Since the form and ideal coincide in the case that the canon is structural (still to be shown), in this case the well-behaved ideal inherits these direct and simple correlations from the generally badly-behaved form.

We now show that the kernel of the canonical homomorphism is compatible with the form theories. Note that this kernel identifies formulae of the root language that map to the same formula of the canonical language, i.e., it identifies those root formulae that are 'equal modulo the canonical language'. Consequently, this result shows that for any root formulae ϕ in a form theory, any other root formula 'equal to ϕ modulo the canon' must also lie in that theory. This is an important property that will play a major role in the theory developed in this chapter. In our most important case, where the free algebra of a quasivariety $\mathcal{K}$ is viewed as an $\mathfrak{a}$-canonical language, this result implies that the form theories, which are subsets of terms, are closed modulo the equational theory of $\mathcal{K}$, i.e., if p lies in a form theory T and $\models_{\mathcal{K}} p \approx q$, then $q \in T$. While the following result does not hold generally for *ideal* theories, when the ideal and form coincide (which occurs, as we shall show, precisely when the cannon is *structural*), the ideal of course inherits this property of compatibility.

Corollary 8.10 The theories of $\underline{\mathcal{D}}$ are compatible with the kernel $\equiv_{\bar{\cdot}^{\mathbf{F}}}$ of the canonical homomorphism $\bar{\cdot}^{\mathbf{F}} : \mathbf{G} \twoheadrightarrow_{\mathfrak{s}} \mathbf{F}$.

Proof. Let $T \in \mathsf{Th}(\underline{\mathcal{D}})$, $\phi \in T$ with $\overline{\phi} = \overline{\psi}$. Certainly $\{\overline{\phi}\} \vdash_{\mathcal{D}} \overline{\psi}$, and so $\{\phi\} \vdash_{\underline{\mathcal{D}}} \psi$, by (4) of Theorem 8.9. Since T is a $\underline{\mathcal{D}}$-theory and $\phi \in T$, $\psi \in T$. ⋄

We now turn to the interplay between the form $\underline{\mathcal{D}}$ and the ideal $\mathcal{D}^{\imath}$, with the aim of establishing their equality *precisely* in the case that the canon $\mathcal{D}$ is *structural*. We begin by recalling some properties of $\mathcal{D}^{\imath}$ in the case at hand. The following result follows immediately from Theorem 7.45 on page 263, Corollary 7.31 on page 259 and Corollary 7.29 on page 259.

Theorem 8.11 Let $\mathcal{D}$ be an $\mathfrak{s}$-canonical logic.

1. $\mathcal{D}^{\imath}$ is $\mathfrak{s}$-structural.

2. $\mathcal{D}^{\imath}$ is the *coarsest* $\mathfrak{s}$-abstraction of $\mathcal{D}$ on $\mathbf{G}$. Consequently, $\mathcal{D}$ is an $\mathfrak{s}$-model of $\mathcal{D}^{\imath}$.

3. Every $\mathfrak{s}$-interpretation of $\mathbf{G}$ into $\mathbf{F}$ is continuous from $\mathcal{D}^{\imath}$ into $\mathcal{D}$.

4. In particular, $\bar{\cdot}$ is continuous from $\mathcal{D}^{\imath}$ onto $\mathcal{D}$.

□

By (4) of the previous theorem, $\bar{\cdot}$ is continuous from the ideal onto the canon, and by (1) of Theorem 8.9, the form is the *coarsest* $\mathfrak{s}$-deductive system with this property; consequently the ideal is finer than the form.

Corollary 8.12 $\mathcal{D}^{\imath} \preceq \underline{\mathcal{D}}$ □

Since all $\mathfrak{s}$-abstractions of $\mathcal{D}$ on $\mathbf{G}$ are finer than $\mathcal{D}^{\imath}$, by (2) of Theorem 8.11, unless the form and ideal coincide, the form is *not* an $\mathfrak{s}$-abstraction of $\mathcal{D}$ and the canon is *not a model* of the form. In the following result we show that the *form and ideal coincide* precisely when the *canon is structural*; so generally the form is not an $\mathfrak{s}$-abstraction of $\mathcal{D}$. Note that further characterizations of the structurality of $\mathcal{D}$ are given in Theorem 8.37 on page 291.

Theorem 8.13 For an $\mathbf{F}$-canon $\mathcal{D}$, the following are equivalent.

1. $\mathcal{D}$ is structural.

2. $\mathcal{D}$ is ideal, i.e., $\mathcal{D}^{\imath} = \underline{\mathcal{D}}$.

3. $\mathcal{D}$ is a semantics for $\underline{\mathcal{D}}$.

4. $\underline{\mathcal{D}}$ is structural.

Proof. $\boxed{(1)\Rightarrow(2)}$ $\boxed{\boldsymbol{\mathcal{D}}^{\imath} \preceq \underline{\boldsymbol{\mathcal{D}}}}$ By Corollary 8.12. $\boxed{\boldsymbol{\mathcal{D}}^{\imath} \succeq \underline{\boldsymbol{\mathcal{D}}}}$ Suppose that $\Gamma \vdash_{\underline{\boldsymbol{\mathcal{D}}}} \phi$. (*We must show that* $\Gamma \models^{\boldsymbol{\mathcal{D}},\mathfrak{s}}_{\mathbf{G}} \phi$.) Let $\mathsf{i} : \mathbf{G} \to \mathbf{F}$. (*We must show that* $\mathsf{i}[\Gamma] \vdash_{\boldsymbol{\mathcal{D}}} \mathsf{i}(\phi)$.) Since $\Gamma \vdash_{\underline{\boldsymbol{\mathcal{D}}}} \phi$, $\overline{[\Gamma]} \vdash_{\boldsymbol{\mathcal{D}}} \overline{\phi}$. Consider the $\mathbf{F}$-endomorphism $\bar{\mathsf{i}}$ given in Definition 8.4 on page 278. By the assumed structurality of $\boldsymbol{\mathcal{D}}$, $\bar{\mathsf{i}}\left[\overline{[\Gamma]}\right] \vdash_{\boldsymbol{\mathcal{D}}} \bar{\mathsf{i}}(\overline{\phi})$. By Remark 8.5 on page 278, $\bar{\mathsf{i}}\left[\overline{[\Gamma]}\right] = \mathsf{i}[\Gamma]$ and $\bar{\mathsf{i}}(\overline{\phi}) = \mathsf{i}(\phi)$. So $\mathsf{i}[\Gamma] \vdash_{\boldsymbol{\mathcal{D}}} \mathsf{i}(\phi)$, as required. $\boxed{(2)\Rightarrow(3)}$ Trivial since $\boldsymbol{\mathcal{D}}$ is generally a semantics for $\boldsymbol{\mathcal{D}}^{\imath}$. $\boxed{(3)\Rightarrow(4)}$ By Theorem 7.61 on page 266, only structural logics can have semantics, so $\underline{\boldsymbol{\mathcal{D}}}$ must be structural. $\boxed{(4)\Rightarrow(1)}$ Suppose that $\underline{\boldsymbol{\mathcal{D}}}$ is structural. Suppose that $\boldsymbol{\Gamma} \vdash_{\boldsymbol{\mathcal{D}}} \boldsymbol{\phi}$ and that $\boldsymbol{\sigma}$ is an $\mathbf{F}$-substitution. (We must show that $\boldsymbol{\sigma}[\boldsymbol{\Gamma}] \vdash_{\boldsymbol{\mathcal{D}}} \boldsymbol{\sigma}(\boldsymbol{\phi})$.) Since $\boldsymbol{\Gamma} \vdash_{\boldsymbol{\mathcal{D}}} \boldsymbol{\phi}$, we have $\underline{[\boldsymbol{\Gamma}]} \vdash_{\underline{\boldsymbol{\mathcal{D}}}} \underline{[\![\boldsymbol{\phi}]\!]}$, by (8) of Theorem 8.9. Pick a $\boldsymbol{\mathcal{D}}$-substitution $\underline{\boldsymbol{\sigma}_?}$ as in Definition 8.4 on page 278. By the assumed $\mathfrak{s}$-structurality of $\underline{\boldsymbol{\mathcal{D}}}$, $\underline{\boldsymbol{\sigma}_?}\left[\underline{[\boldsymbol{\Gamma}]}\right] \vdash_{\underline{\boldsymbol{\mathcal{D}}}} \underline{\boldsymbol{\sigma}_?}\left[\underline{[\![\boldsymbol{\phi}]\!]}\right]$. So by (4) of Theorem 8.9, $\overline{\left[\underline{\boldsymbol{\sigma}_?}\left[\underline{[\boldsymbol{\Gamma}]}\right]\right]} \vdash_{\boldsymbol{\mathcal{D}}} \overline{\left[\underline{\boldsymbol{\sigma}_?}\left[\underline{[\![\boldsymbol{\phi}]\!]}\right]\right]}$. But, by Remark 8.6 on page 278, $\overline{\left[\underline{\boldsymbol{\sigma}_?}\left[\underline{[\boldsymbol{\Gamma}]}\right]\right]} = \boldsymbol{\sigma}[\boldsymbol{\Gamma}]$ and $\overline{\left[\underline{\boldsymbol{\sigma}_?}\left[\underline{[\![\boldsymbol{\phi}]\!]}\right]\right]} = \{\boldsymbol{\sigma}(\boldsymbol{\phi})\}$ as required. ◇

The following corollary summarises our most import case.

Theorem 8.14 $\underline{\boldsymbol{\mathcal{D}}}$ is a propositional $\mathfrak{s}$-calculus iff $\boldsymbol{\mathcal{D}}$ is a propositional $\mathfrak{s}_{\mathbf{F}}$-calculus, in which case $\underline{\boldsymbol{\mathcal{D}}} = \boldsymbol{\mathcal{D}}^{\imath}$ and $\boldsymbol{\mathcal{D}}$ is a semantics for $\underline{\boldsymbol{\mathcal{D}}}$. □

We now demonstrate that in the case that $\boldsymbol{\mathcal{D}}$ is an *ideal* canon (equivalently $\boldsymbol{\mathcal{D}}$ is *structural*), the form theory lattice (which equals the ideal theory lattice) is *isomorphic* to the $\boldsymbol{\mathcal{D}}$-theory lattice.

Theorem 8.15 If $\boldsymbol{\mathcal{D}}$ is a structural $\mathfrak{s}$-canon then $\left(\overline{[\cdot]}\right)_{|\mathsf{Th}(\underline{\boldsymbol{\mathcal{D}}})} : \mathbf{Th}(\underline{\boldsymbol{\mathcal{D}}}) \cong \mathbf{Th}(\boldsymbol{\mathcal{D}})$, with inverse isomorphism $\left(\underline{[\cdot]}\right)_{|\mathsf{Th}(\boldsymbol{\mathcal{D}})}$.

Proof. $\boxed{\text{Isomorphism}}$ (*It suffices to show that this function is well-defined onto and an $\subseteq$-embedding.*) $\boxed{\text{Well defined}}$ (*We need to show that the described function is well-defined.*) Let T be a $\underline{\boldsymbol{\mathcal{D}}}$-theory. (*We must show that $\overline{[T]}$ is a $\boldsymbol{\mathcal{D}}$-theory. It suffices to show that* $\left\|\overline{[T]}\right\|_{\boldsymbol{\mathcal{D}}} = \overline{[T]}$.) By Theorem 8.9, $\left\|\overline{[T]}\right\|_{\boldsymbol{\mathcal{D}}} = \overline{\left[\underline{\left\|\overline{[T]}\right\|_{\boldsymbol{\mathcal{D}}}}\right]} = \overline{\left[\|T\|_{\underline{\boldsymbol{\mathcal{D}}}}\right]} = \overline{[T]}$, the second equality following since T is a $\underline{\boldsymbol{\mathcal{D}}}$-theory and hence is $\mathbf{F}$-compatible by Corollary 8.24, and the third equality following since T is a $\underline{\boldsymbol{\mathcal{D}}}$-theory. $\boxed{\text{Onto}}$ Let $\boldsymbol{T}$ be a $\boldsymbol{\mathcal{D}}$-theory. Consider $\underline{[\boldsymbol{T}]}$. Trivially, $\overline{\left[\underline{[\boldsymbol{T}]}\right]} = \boldsymbol{T}$. (*We must show that* $\underline{[\boldsymbol{T}]} \in \mathsf{Th}(\underline{\boldsymbol{\mathcal{D}}})$. *It suffices to show that* $\left\|\underline{[\boldsymbol{T}]}\right\|_{\underline{\boldsymbol{\mathcal{D}}}} = \underline{[\boldsymbol{T}]}$.) By Theorem 8.9, $\left\|\underline{[\boldsymbol{T}]}\right\|_{\underline{\boldsymbol{\mathcal{D}}}} = \underline{\left[\left\|\overline{\underline{[\boldsymbol{T}]}}\right\|_{\boldsymbol{\mathcal{D}}}\right]} = \underline{\left[\|\boldsymbol{T}\|_{\boldsymbol{\mathcal{D}}}\right]} = \underline{[\boldsymbol{T}]}$, the final equality following since $\boldsymbol{T}$ is a $\boldsymbol{\mathcal{D}}$-theory. $\boxed{\text{Order Embedding}}$ Let T and R be two $\underline{\boldsymbol{\mathcal{D}}}$-theories. If $T \subseteq R$, then trivially, $\overline{[T]} \subseteq \overline{[R]}$. Suppose that $\overline{[T]} \subseteq \overline{[R]}$. Then since T and R are $\underline{\boldsymbol{\mathcal{D}}}$-theories, by Theorem 8.9, $T = \|T\|_{\underline{\boldsymbol{\mathcal{D}}}} = \underline{\left[\left\|\overline{[T]}\right\|_{\boldsymbol{\mathcal{D}}}\right]} \subseteq \underline{\left[\left\|\overline{[R]}\right\|_{\boldsymbol{\mathcal{D}}}\right]} = \|R\|_{\underline{\boldsymbol{\mathcal{D}}}} = R$, the inclusion following since $\overline{[T]} \subseteq \overline{[R]}$. $\boxed{\text{Inverse Isomorphism}}$ (*It suffices to show that* $\underline{[\cdot]}_{|\mathsf{Th}(\boldsymbol{\mathcal{D}})} = \left(\overline{[\cdot]}\right)_{|\mathsf{Th}(\underline{\boldsymbol{\mathcal{D}}})}^{-1}$.) Let $T \in \mathsf{Th}(\underline{\boldsymbol{\mathcal{D}}})$. (*Since the bijectivity of* $\left(\overline{[\cdot]}\right)_{|\mathsf{Th}(\underline{\boldsymbol{\mathcal{D}}})}$ *has already been established, it suffices to show that* $\underline{\left[\overline{[T]}\right]} = T$.) Since all $\underline{\boldsymbol{\mathcal{D}}}$-theories are $\mathbf{F}$-compatible, by Corollary 8.24, $\underline{\left[\overline{[T]}\right]} = T$. ◇

Open Problem 8.16 The following theorem demonstrates a situation where all existing notions of equivalent logics, including our notion of equivalence developed in ∮17 and the notion of deductively equivalent π-institutions [Vou03] appear to 'miss the mark' (unless we have misunderstood the categorical notion of an *adjoint equivalence* [Vou03], in which case we apologise and retract this open problem). While $\left(\overline{[\cdot]}\right)_{|\mathsf{Th}(\underline{\boldsymbol{\mathcal{D}}})} : \mathbf{Th}(\underline{\boldsymbol{\mathcal{D}}}) \cong \mathbf{Th}(\boldsymbol{\mathcal{D}})$, in the case that $\boldsymbol{\mathcal{D}}$ is a structural $\mathfrak{s}$-canon, these logics can never be equivalent with respect to the existing notions of equivalence, since there is no *useful* one-to-one correspondence between **G**-substitutions and **F**-substitutions. Note however, the set of **G**-substitutions can be partitioned by the kernel of the map taking σ to the unique **F**-substitution mapping $\overline{[x]}$ to $\overline{[\sigma(x)]}$, for each **G**-variable x, and this partition is in natural one-to-one correspondence with the set of **F**-substitutions.

A similar situation occurs when one begins with a π-institution $\mathcal{I}$ and constructs a new ('concrete') π-institution $\mathcal{I}'$, whose signatures are the sets of $\mathcal{I}'$-sentences, whose substitutions are the $\mathsf{SEN}_{\mathcal{I}}(\sigma)$, for each $\mathcal{I}$-substitution, and the functor $\mathsf{SEN}_{\mathcal{I}'}$ is the identity functor. In many senses, $\mathcal{I}$ and $\mathcal{I}'$ are equivalent, yet in the case that there exist signatures $\mathfrak{S}$ and $\mathfrak{T}$ two *distinct* $\mathcal{I}$-substitutions $\sigma : \mathfrak{S} \to \mathfrak{T}$ and $\rho : \mathfrak{S} \to \mathfrak{T}$ with $\mathsf{SEN}_{\mathcal{I}}(\sigma) = \mathsf{SEN}_{\mathcal{I}}(\rho)$, $\mathcal{I}$ and $\mathcal{I}'$ are (generally) not deductively equivalent.

Can a theory of deductive equivalence be developed that would include the equivalence of $\underline{\boldsymbol{\mathcal{D}}}$ and $\boldsymbol{\mathcal{D}}$ as above, and the equivalence of $\mathcal{I}$ and $\mathcal{I}'$? $\square$

8.1.3 The Discrete Canon and Canon Compatibility

Recall that by Corollary 8.10, the theories of the form are closed (or compatible) with 'equality modulo the canonical language'. Since we are only interested in the case where the ideal and form coincide, i.e., when the canon is structural, the ideal theories are always closed under 'canon equality'. It is this observation that will lead us to an *axiomatization of the ideal* in terms of an *axiomatization of the canon*; informally, we shall show that the ideal may be axiomatized by taking each rule and axiom from the *canon axiomatization* and converting it to a rule or axiom over the *root language* by picking a representative from each canon formula in the original rule or axiom, and then, adding *additional rules* that reflect 'equality modulo the canonical language'. In our most important case, where $\mathcal{K}$ is a quasivariety of algebras and the $\mathfrak{a}$-canonical language is the $\mathcal{K}$-free algebra, we will need to add additional rules $p \vdash q$ whenever $\models_{\mathcal{K}} p \approx q$.

We shall now focus on the *simplest possible case*, namely the discrete logic on a canonical language. Note that the discrete logic may be axiomatized by the empty-set of rules and the empty-set of axioms, and note further that the discrete logic on a canonical language is *structural*; hence it is *ideal*.

Definition 8.17 (The Discrete Canon) We shall denote the discrete canon $L(\mathbf{F}, \perp)$ by $\mathbf{F}_\perp$ and $\underline{\mathbf{F}_\perp}$ by $\underline{\mathbf{F}}_{\mathfrak{s}}$, dropping the subscript $\mathfrak{s}$ wherever unambiguous. We call $\mathbf{F}_\perp$ the **discrete canon determined by F** and call $\underline{\mathbf{F}}_{\mathfrak{s}}$ the **discrete ideal determined by F**. $\square$

The discrete logic $\mathbf{F}_\perp$ is finitary, and structural, by Remark 6.17 on page 226, i.e., $\mathbf{F}_\perp$ is a propositional $\mathfrak{s}_{\mathbf{F}}$-calculus. Consequently, by Theorem 8.14, $\underline{\mathbf{F}}_{\mathfrak{s}}$ is ideal, and so the 'full machinery' of $\oint$8.1.2 pertains. Note that by definition, the discrete-canon has no theorems.

We now show that the form of the *discrete-canon* is the *finest* of all the *forms of canons*. Consequently, the discrete-ideal is finer than the ideal of any *ideal* canon.

Proposition 8.18 For any $\mathfrak{s}$-canon $\boldsymbol{\mathcal{D}}$, $\underline{\mathbf{F}} \preceq \underline{\boldsymbol{\mathcal{D}}}$. Consequently, $\underline{\mathbf{F}}$ is finer than the ideal of any *ideal* canon.

Proof. Since $\mathbf{F}_\perp \preceq \boldsymbol{\mathcal{D}}$, $\underline{\mathbf{F}} \preceq \underline{\boldsymbol{\mathcal{D}}}$ by Remark 5.84 on page 198. $\diamond$

For ease of future reference, we interpret the results of the previous sub-section in the case of the discrete-canon, which, as we have noted, is ideal. Notice the extra *existential* equivalent conditions of (7), which follow since $\boldsymbol{\Gamma} \vdash_{\mathbf{F}_\perp} \boldsymbol{\phi}$ iff $\boldsymbol{\phi} \in \boldsymbol{\Gamma}$.

Corollary 8.19 Let $\mathbf{F}$ be an $\mathfrak{s}$-canonical language.

1. $\underline{\mathbf{F}}$ is a propositional $\mathfrak{s}$-calculus, $(\mathbf{F}_\perp)^\imath = \underline{\mathbf{F}}$ and $\mathbf{F}_\perp$ is a semantics for $\underline{\mathbf{F}}$.
2. $\underline{\mathbf{F}}$ is the *coarsest* $\mathfrak{s}$-deductive system $\mathcal{D}$ on $\mathbf{G}$ for which $\bar{\cdot}$ is continuous from $\mathcal{D}$ into $\mathbf{F}_\perp$.
3. $\bar{\cdot}$ is strictly continuous from $\underline{\mathbf{F}}$ onto $\mathbf{F}_\perp$.
4. $\underline{\mathbf{F}}$ is the *coarsest* $\mathfrak{s}$-abstraction of $\mathbf{F}_\perp$ on $\mathbf{G}$.
5. Every $\mathfrak{s}$-interpretation of $\mathbf{G}$ into $\mathbf{F}_\mathcal{K}$ is continuous from $\underline{\mathbf{F}}$ into $\mathbf{F}_\perp$.
6. $\mathsf{Th}(\underline{\mathbf{F}}) = \{\underline{[\boldsymbol{\Gamma}]} \ : \ \boldsymbol{\Gamma} \subseteq \mathsf{Fm}(\mathbf{F})\}$.
7. $\Gamma \vdash_{\underline{\mathbf{F}}} \phi$ iff $\overline{\phi} \in \overline{[\Gamma]}$ iff $\forall\,[\boldsymbol{\Gamma} \subseteq \mathsf{Fm}(\mathbf{F})]\ \overline{[\Gamma]} \subseteq \boldsymbol{\Gamma} \to \overline{\phi} \in \boldsymbol{\Gamma}$ iff $\exists\,[\psi \in \Gamma]\ \overline{\phi} = \overline{\psi}$ iff $\exists\,[\psi \in \Gamma]\ \psi \vdash_{\underline{\mathbf{F}}} \phi$.
8. $\|\Gamma\|_{\underline{\mathbf{F}}} = \underline{\left[\overline{[\Gamma]}\right]}$.
9. $\overline{\left[\left\|\underline{[\boldsymbol{\Gamma}]}\right\|_{\underline{\mathbf{F}}}\right]} = \boldsymbol{\Gamma}$.
10. Let $\boldsymbol{\Gamma} \cup \{\boldsymbol{\phi}\} \subseteq \mathsf{Fm}(\mathbf{F}_\perp)$, let $\phi_{\boldsymbol{\psi}} \in \underline{[\![\boldsymbol{\psi}]\!]}$, for each $\boldsymbol{\psi} \in \boldsymbol{\Gamma}$, $\phi_{\boldsymbol{\phi}} \in \underline{[\![\boldsymbol{\phi}]\!]}$, and let $\Gamma_{\boldsymbol{\Gamma}} \subseteq \mathsf{Fm}(\mathbf{G})$ such that $\overline{[\Gamma_{\boldsymbol{\Gamma}}]} = \boldsymbol{\Gamma}$. Then $\boldsymbol{\phi} \in \boldsymbol{\Gamma}$ iff $\underline{[\boldsymbol{\Gamma}]} \vdash_{\underline{\mathbf{F}}} \underline{[\![\boldsymbol{\phi}]\!]}$ iff $\{\phi_{\boldsymbol{\psi}} \ : \ \boldsymbol{\psi} \in \boldsymbol{\Gamma}\} \vdash_{\underline{\mathbf{F}}} \phi_{\boldsymbol{\phi}}$ iff $\Gamma_{\boldsymbol{\Gamma}} \vdash_{\underline{\mathbf{F}}} \phi_{\boldsymbol{\phi}}$. Further, $\boldsymbol{\Gamma} = \overline{\left[\|\{\phi_{\boldsymbol{\psi}} \ : \ \boldsymbol{\psi} \in \boldsymbol{\Gamma}\}\|_{\underline{\mathbf{F}}}\right]} = \overline{\left[\|\Gamma_{\boldsymbol{\Gamma}}\|_{\underline{\mathbf{F}}}\right]}$.

$\square$

Observe that the kernel of the canonical homomorphism is interpreted in the deductive machinery of the discrete-ideal. More precisely we have the following.

Remark 8.20 The binary relation on $\mathsf{Fm}(\mathbf{G})$ determined by $\{\phi\} \vdash_{\underline{\mathbf{F}}} \psi$ is an equivalence relation equal to the kernel of $\bar{\cdot}$. $\square$

Since the discrete-canon is structural and finitary, the discrete-ideal is a propositional $\mathfrak{s}$-calculus (by Theorem 8.14); consequently the discrete-ideal must be axiomatizable. We now present an axiomatization of the discrete-ideal. Note that this axiomatization will play a central role in the axiomatization of the *ideal* of any *finitary ideal canon* (see $\oint$8.1.4).

Proposition 8.21 $\underline{\mathbf{F}}$ is axiomatized with no axioms and all rules $\{\phi\} \vdash \psi$ where $\overline{\phi} = \overline{\psi}$.

Proof. Let $\mathcal{P}$ be the propositional $\mathfrak{s}$-calculus on $\mathbf{G}$ determined by all rules $\{\phi\} \vdash \psi$ where $\overline{\phi} = \overline{\psi}$. By (7) of Corollary 8.19 on page 284, $\underline{\mathbf{F}}$ satisfies all these rules, and since $\underline{\mathbf{F}}$ is structural, $\mathcal{P} \preceq \underline{\mathbf{F}}$, by Proposition 6.31 on page 229. Conversely, if $\Gamma \vdash_{\underline{\mathbf{F}}} \phi$, then by (7) of the aforementioned corollary, there exists $\psi \in \Gamma$ with $\overline{\phi} = \overline{\psi}$. So $\{\psi\} \vdash \phi$ is a $\mathcal{P}$-rule and so $\psi \vdash_\mathcal{P} \phi$. Since $\psi \in \Gamma$, $\Gamma \vdash_\mathcal{P} \phi$. $\diamond$

We have been speaking informally about theories on the root language being compatible with 'equality modulo the canonical language'. We now make this notion precise.

Definition 8.22 (Canonical Compatibility) Let $\mathbf{F}$ be a canonical language. We call a set Γ of $\mathbf{G}$-formulae **compatible with F** or **F-compatible**, if $\underline{\left[\overline{[\Gamma]}\right]} = \Gamma$, and we say that a $\mathbf{G}$-deductive system $\mathcal{D}$ is **compatible with F** or **F-compatible**, if every $\mathcal{D}$-theory is $\mathbf{F}$-compatible. $\square$

Remark 8.23 Γ is $\mathbf{F}$-compatible iff $\underline{\left[\overline{[\Gamma]}\right]} \subseteq \Gamma$. $\square$

Observe that the $\mathbf{F}$-compatible $\mathfrak{s}$-deductive systems are *precisely* those $\mathfrak{s}$-deductive systems coarser than the discrete-ideal (by (7) of Corollary 8.19 on page 284); hence the discrete-ideal must be $\mathbf{F}$-compatible and, by Proposition 8.18, the form of *any* canon is $\mathbf{F}$-compatible. Consequently, the ideal of any *ideal* canon must also be $\mathbf{F}$-compatible.

Corollary 8.24 An $\mathfrak{s}$-deductive system $\mathcal{D}$ is $\mathbf{F}$-compatible iff $\underline{\mathbf{F}} \preceq \mathcal{D}$. In particular, $\underline{\mathbf{F}}$ is $\mathbf{F}$-compatible, as is $\underline{\boldsymbol{\mathcal{D}}}$, for any $\mathfrak{s}$-canon $\boldsymbol{\mathcal{D}}$, as is the ideal of any ideal canon.

8.1.4 Axiomatization

We turn now to the problem of *axiomatizing* the ideal of a *finitary ideal* cannon, i.e., a *finitary and structural* cannon. If $\boldsymbol{\mathcal{P}}$ is a *propositional* $\mathfrak{s}$-canon, i.e., both structural and finitary, then $\boldsymbol{\mathcal{P}}^{\imath} = \underline{\boldsymbol{\mathcal{P}}}$ is a propositional $\mathfrak{s}$-calculus, and hence $\underline{\boldsymbol{\mathcal{P}}}$ must be axiomatizable. The following theorem describes one such axiomatization derived from a given axiomatization of $\boldsymbol{\mathcal{P}}$. Informally, the axiomatization is obtained by selecting a root language representative of each canon rule and axiom, and in addition, adding an axiomatization of the *discrete-ideal*.

Theorem 8.25 If $\boldsymbol{\mathcal{P}}$ is a *propositional* $\mathfrak{s}$-canon axiomatized by axioms ax and rules ir, then the propositional $\mathfrak{s}$-calculus $\underline{\boldsymbol{\mathcal{P}}}$ is axiomatized by the axioms and rules described by the following procedure.

1. For each $\boldsymbol{\varpi} \in \mathsf{ax}$, choose an axiom $\varpi_{\boldsymbol{\varpi}}$ such that $\overline{\varpi_{\boldsymbol{\varpi}}} = \boldsymbol{\varpi}$.

2. For each $\boldsymbol{\Lambda} \in \mathsf{ir}$, choose a rule $\Lambda_{\boldsymbol{\Lambda}}$ such that $\overline{[\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}})]} = \boldsymbol{\Lambda}$ and $\overline{\mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}})} = \mathsf{conc}(\boldsymbol{\Lambda})$.

3. In addition, choose any set of rules axiomatizing $\underline{\mathbf{F}}$.

Proof. Let $\mathcal{P}$ be the propositional $\mathfrak{s}$-calculus axiomatized by the axioms and rules described.
$\boxed{\mathcal{P} \preceq \underline{\boldsymbol{\mathcal{P}}}}$ (*Since $\underline{\boldsymbol{\mathcal{P}}}$ is structural, it suffices, by Proposition 6.31 on page 229, to show that $\underline{\boldsymbol{\mathcal{P}}}$ satisfies each $\mathcal{P}$-axiom and each $\mathcal{P}$-rule.*) Since $\underline{\boldsymbol{\mathcal{P}}}$ is $\mathbf{F}$-compatible, by Corollary 8.24 on page 285, and is a propositional $\mathbf{F}$-calculus, $\underline{\mathbf{F}} \preceq \underline{\boldsymbol{\mathcal{P}}}$, by Corollary 8.24, so certainly $\underline{\boldsymbol{\mathcal{P}}}$ satisfies all rules of $\underline{\mathbf{F}}$ ($\underline{\mathbf{F}}$ has no axioms). We consider the remaining axioms and rules. $\boxed{\text{Axiom}}$ Let $\boldsymbol{\varpi} \in \mathsf{ax}$. (*We must show that* $\vdash_{\underline{\boldsymbol{\mathcal{P}}}} \varpi_{\boldsymbol{\varpi}}$.) Certainly $\vdash_{\boldsymbol{\mathcal{P}}} \boldsymbol{\varpi}$, and so by (8) of Theorem 8.9, $\vdash_{\underline{\boldsymbol{\mathcal{P}}}} \varpi_{\boldsymbol{\varpi}}$. $\boxed{\text{Rule}}$ Let $\boldsymbol{\Lambda} \in \mathsf{ir}$. (*We must show that* $\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}}) \vdash_{\underline{\boldsymbol{\mathcal{P}}}} \mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}})$.) Certainly $\mathsf{prem}(\boldsymbol{\Lambda}) \vdash_{\boldsymbol{\mathcal{P}}} \mathsf{conc}(\boldsymbol{\Lambda})$, and so by (8) of Theorem 8.9, $\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}}) \vdash_{\underline{\boldsymbol{\mathcal{P}}}} \mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}})$. $\boxed{\underline{\boldsymbol{\mathcal{P}}} \preceq \mathcal{P}}$ (*By (4) of Theorem 8.9, it suffices to show that, for all* $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathbf{G})$, *if* $\overline{[\Gamma]} \vdash_{\boldsymbol{\mathcal{P}}} \overline{\phi}$ *then* $\Gamma \vdash_{\mathcal{P}} \phi$.) We proceed by induction of the length of derivations in $\boldsymbol{\mathcal{P}}$. $\boxed{\text{Note}}$ For any $\mathbf{G}$-formulae ϕ and ψ, if $\overline{\phi} = \overline{\psi}$, then $\psi \vdash_{\underline{\mathbf{F}}} \phi$, and so $\psi \vdash_{\mathcal{P}} \phi$. $\boxed{\text{Base Case}}$ Suppose that $\overline{\phi}$ is derivable from $\overline{[\Gamma]}$ by a derivation of length one. Two cases arise. $\boxed{\overline{\phi} \in \overline{[\Gamma]}}$ If $\overline{\phi} \in \overline{[\Gamma]}$, then there exists $\psi \in \Gamma$ with $\overline{\phi} = \overline{\psi}$, and so $\psi \vdash_{\mathcal{P}} \phi$ as noted. So, since $\psi \in \Gamma$, $\Gamma \vdash_{\mathcal{P}} \phi$. $\boxed{\text{Substitution instance of axiom}}$ Otherwise, there exists a $\boldsymbol{\mathcal{P}}$-substitution $\boldsymbol{\sigma}$ and a

$\boldsymbol{\mathcal{P}}$-axiom $\boldsymbol{\varpi}$ with $\boldsymbol{\sigma}(\boldsymbol{\varpi}) = \overline{\phi}$. Then $\varpi_{\boldsymbol{\varpi}}$ is a $\mathcal{P}$-axiom and so $\vdash_{\mathcal{P}} \varpi_{\boldsymbol{\varpi}}$. Choose a $\mathbf{G}$-substitution $\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}$ as described in Definition 8.4 on page 278. Then by structurality, $\vdash_{\mathcal{P}} \underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}(\varpi_{\boldsymbol{\varpi}})$. By Remark 8.6 on page 278, $\overline{\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}(\varpi_{\boldsymbol{\varpi}})} = \boldsymbol{\sigma}(\overline{\varpi_{\boldsymbol{\varpi}}}) = \boldsymbol{\sigma}(\boldsymbol{\varpi}) = \overline{\phi}$, and so by the earlier note, $\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}(\varpi_{\boldsymbol{\varpi}}) \vdash_{\mathcal{P}} \phi$. Since $\vdash_{\mathcal{P}} \underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}(\varpi_{\boldsymbol{\varpi}})$ and $\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}(\varpi_{\boldsymbol{\varpi}}) \vdash_{\mathcal{P}} \phi$, we have $\vdash_{\mathcal{P}} \phi$. So certainly, $\Gamma \vdash_{\mathcal{P}} \phi$. $\boxed{\text{Induction Hypothesis}}$ Assume that for for any $\mathbf{G}$-formula ϕ, if $\overline{\phi}$ is derivable from $\overline{[\Gamma]}$ by a derivation of length $m \leq n$, then $\Gamma \vdash_{\mathcal{P}} \phi$. $\boxed{\text{Inductive Proof}}$ Suppose that ϕ is a $\mathbf{G}$-formula such that $\overline{\phi}$ is derivable from $\overline{[\Gamma]}$ by a derivation $\overline{\phi_1}, \ldots, \overline{\phi_{n+1}}$ of length $n+1$. So $\overline{\phi} = \overline{\phi_{n+1}}$ and we may assume without loss of generality that $\phi = \phi_{n+1}$. By the inductive hypothesis,

$$\Gamma \vdash_{\mathcal{P}} \{\phi_1, \ldots, \phi_n\}. \tag{i}$$

In the cases that $\overline{\phi_{n+1}} \in \overline{[\Gamma]}$ or $\overline{\phi_{n+1}}$ is a $\boldsymbol{\mathcal{P}}$-substitution image of a $\boldsymbol{\mathcal{P}}$-axiom, the result follows by the base case. Otherwise, there exists a $\boldsymbol{\mathcal{P}}$-substitution $\boldsymbol{\sigma}$ and a $\boldsymbol{\mathcal{P}}$-rule $\boldsymbol{\Lambda}$ with $\boldsymbol{\sigma}[\mathsf{prem}(\boldsymbol{\Lambda})] \subseteq \{\overline{\phi_1}, \ldots, \overline{\phi_n}\}$ and $\boldsymbol{\sigma}(\mathsf{conc}(\boldsymbol{\Lambda})) = \overline{\phi}$. Choose a $\mathbf{G}$-substitution $\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}$ as described in Definition 8.4 on page 278. Since $\boldsymbol{\sigma}[\mathsf{prem}(\boldsymbol{\Lambda})] \subseteq \{\overline{\phi_1}, \ldots, \overline{\phi_n}\}$, by Remark 8.6, $\overline{[\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}[\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}})]]} = \boldsymbol{\sigma}\left[\overline{[\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}})]}\right] = \boldsymbol{\sigma}[\mathsf{prem}(\boldsymbol{\Lambda})] \subseteq \{\overline{\phi_1}, \ldots, \overline{\phi_n}\}$, so by the earlier note,

$$\{\phi_1, \ldots, \phi_n\} \vdash_{\mathcal{P}} \underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}[\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}})]. \tag{ii}$$

Since $\boldsymbol{\Lambda}$ is a $\boldsymbol{\mathcal{P}}$-rule, by definition, $\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}}) \vdash \mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}})$ is a $\mathcal{P}$-rule; so $\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}}) \vdash_{\mathcal{P}} \mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}})$, and hence by structurality,

$$\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}[\mathsf{prem}(\Lambda_{\boldsymbol{\Lambda}})] \vdash_{\mathcal{P}} \underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}(\mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}})). \tag{iii}$$

Again by Remark 8.6, $\overline{\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}(\mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}}))} = \boldsymbol{\sigma}(\overline{\mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}})}) = \boldsymbol{\sigma}(\mathsf{conc}(\boldsymbol{\Lambda})) = \overline{\phi}$, and so by the earlier note,

$$\underline{\boldsymbol{\sigma}}_{\boldsymbol{?}}(\mathsf{conc}(\Lambda_{\boldsymbol{\Lambda}})) \vdash_{\mathcal{P}} \phi. \tag{iv}$$

So by (i), (ii), (iii) and (iv) and '$\vdash_{\mathcal{P}}$-transitivity' (i.e., (6.3)), $\Gamma \vdash_{\mathcal{P}} \phi$. $\diamond$

Observe that we have solved the problem of axiomatizing the ideal (which is a sentential calculus) of a finitary $\mathcal{K}$-structural $\mathfrak{a}$-canon on the $\mathcal{K}$-free algebra in terms of an axiomatization of the canon. This axiomatization consists of picking a representative of each rule and axiom of the cannon axiomatization, and, in addition, adding a rule $p \vdash q$ whenever $\models_{\mathcal{K}} p \approx q$; these latter rules encode the equational theory of the quasivariety $\mathcal{K}$. Of course, if this equational theory can be characterized more *succinctly*, then we do not need to encode *all* identities such that $\models_{\mathcal{K}} p \approx q$.

8.1.5 The Filter Canon

With the aim of showing that the theories of an *ideal* canon on $\mathbf{F}$ coincide with the filters of *its ideal* on $\mathbf{F}$, we turn the focus (of the discourse so far in this section) around. Instead of beginning with a canon, we now start with a logic $\mathcal{D}$ on the *root* language and consider its filters on the canonical language; these filters form a canon, which we call the *filter canon* (in the discourse of $\oint 7$ this logic is called the *filter logic* or the *model logic*). This filter canon in turn induces a form and ideal back on the *root* language. By Theorem 7.45 on page 263, the filter canon is *always structural*, so by Theorem 8.13, the *filter canon is always ideal* (independently of the structurality of $\mathcal{D}$), and so the form and ideal of the filter cannon *always* coincide; we call this logic the *ideal of $\mathcal{D}$ modulo* $\mathbf{F}$. Note that $\mathcal{D}$ and its ideal modulo $\mathbf{F}$ are both logics on the *root* language. We shall show that $\mathcal{D}$ is *structural* iff $\mathcal{D}$ *coincides with its ideal modulo* $\mathbf{F}$. We can then turn the focus back, and show that a canon $\boldsymbol{\mathcal{D}}$ on $\mathbf{F}$ is *structural* precisely when its *theories* coincide with the *filters of its ideal* on $\mathbf{F}$.

Definition 8.26 (Filter Canons and Ideals modulo F) Let $\mathcal{D}$ be an $\mathfrak{s}$-deductive system (i.e., a logic on $\mathbf{G}$) and $\mathbf{F}$ an $\mathfrak{s}$-canonical language. The filter/model logic $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ on $\mathbf{F}$ is called the **filter canon** of $\mathcal{D}$ on $\mathbf{F}$ and $\underline{\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})}$ is denoted by $\mathcal{D}_{|\mathbf{F}}$, which we call the **ideal of $\mathcal{D}$ modulo F**. □

Warning 8.27 (Two Ideal Logics) We have two notions of ideal logic at play in this chapter: the *ideal of a canon* and the *ideal of an $\mathfrak{s}$-deductive system modulo* $\mathbf{F}$. The *ideal of a cannon* is a logic on the *root* language. The *ideal of an $\mathfrak{s}$-deductive system $\mathcal{D}$ modulo* $\mathbf{F}$ and $\mathcal{D}$ are *both* logics on the *root* language.

As we have already noted, since the filter canon of $\mathcal{D}$ is *always structural*, it must be ideal, and consequently its form and ideal must coincide. Consequently, if $\mathcal{D}$ is *finitary* then, by Corollary 7.22 on page 257, the filter canon $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ of $\mathcal{D}$ is *finitary*, and so, by Theorem 8.9, the ideal $\mathcal{D}_{|\mathbf{F}}$ of $\mathcal{D}$ modulo $\mathbf{F}$ must also be *finitary* and hence a *propositional* $\mathfrak{s}$-calculus. Notice that the propositional nature of the ideal $\mathcal{D}_{|\mathbf{F}}$ of $\mathcal{D}$ modulo $\mathbf{F}$ depends *only* on the finitariness of $\mathcal{D}$ and is *independent* of the structurality of $\mathcal{D}$. Note further, that by Corollary 8.24, the ideal $\mathcal{D}_{|\mathbf{F}}$ of $\mathcal{D}$ modulo $\mathbf{F}$ is $\mathbf{F}$-*compatible*. We record these observation for ease of future reference.

Theorem 8.28 Let $\mathcal{D}$ be an $\mathfrak{s}$-deductive system and $\mathbf{F}$ an $\mathfrak{s}$-canonical language.

1. $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ is structural.
2. $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ is ideal, i.e., $\mathcal{D}_{|\mathbf{F}} = (\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F}))^{\imath}$.
3. $\mathcal{D}_{|\mathbf{F}}$ is structural and $\mathbf{F}$-compatible.
4. $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ is a semantics for $\mathcal{D}_{|\mathbf{F}}$.
5. If $\mathcal{D}$ if finitary then $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ and $\mathcal{D}_{|\mathbf{F}}$ are finitary and $\mathcal{D}_{|\mathbf{F}}$ is a *propositional* $\mathfrak{s}$-calculus.

□

$\mathcal{D}_{|\mathbf{F}}$ is a structural $\mathfrak{s}$-calculus that may be viewed as the logic of $\mathcal{D}$-filters on $\mathbf{F}$. Recall that $\mathsf{F}^{\mathfrak{s}}_{\mathcal{P}}(\mathbf{F})$-consequence is also denoted by $\Vdash^{\mathbf{F},\mathfrak{s}}_{\mathcal{P}}$ and that $\|\cdot\|_{\mathsf{F}^{\mathfrak{s}}_{\mathcal{P}}(\mathbf{F})}$ is denoted by $\|\cdot\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}$. The following result, explicated for ease of future reference, follows immediately from the previous theorem together with Theorem 8.9 and Theorem 8.11. Note that (8) characterizes *$\mathcal{D}$-filter generation on* $\mathbf{F}$ in terms of $\mathcal{D}_{|\mathbf{F}}$*-theory generation* (recall that generally, *arbitrary* filter generation is complex to characterize (see Lemma 2.54 on page 102)). This result will shortly lead us to a characterization of $\mathcal{D}$-filter generation on $\mathbf{F}$ in terms of *$\mathcal{D}$-theory generation*, in the case that $\mathcal{D}$ is *structural*, via which we shall be able to achieve our aim of showing that the theories of an *ideal* canon on $\mathbf{F}$ coincide with the filters of *its ideal* on $\mathbf{F}$. We ask the reader to be patient with regard to a few more repetitions (in slightly different and simpler forms) of the following (now familiar) result; each simpler repetition will lead us closer to our aim.

Corollary 8.29 Let $\mathcal{D}$ be an $\mathfrak{s}$-deductive system.

1. $\mathcal{D}_{|\mathbf{F}}$ is the *coarsest* $\mathfrak{s}$-deductive system $\mathcal{D}$ on $\mathbf{G}$ for which $\bar{\cdot}$ is continuous from $\mathcal{D}$ into the filter logic $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$.

2. $\overline{\cdot}$ is strictly continuous from $\mathcal{D}_{|\mathbf{F}}$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$.

3. $\mathcal{D}_{|\mathbf{F}}$ is the *coarsest* $\mathfrak{s}$-abstraction of $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ on $\mathbf{G}$.

4. Every $\mathfrak{s}$-interpretation of $\mathbf{G}$ into $\mathbf{F}_{\mathcal{K}}$ is continuous from $\mathcal{D}_{|\mathbf{F}}$ into $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$.

5. $\mathsf{Th}(\mathcal{D}_{|\mathbf{F}}) = \{\underline{F} \ : \ F \in \mathsf{Fi}_{\mathcal{D}}(\mathbf{F})\}$.

6. $\Gamma \vdash_{\mathcal{D}_{|\mathbf{F}}} \phi$ iff $\overline{\Gamma} \vdash_{\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})} \overline{\phi}$ iff $\overline{\phi} \in \|\overline{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}$ iff $\forall\, [F \in \mathsf{Fi}_{\mathcal{D}}(\mathbf{F})]\ \overline{\Gamma} \subseteq F \to \overline{\phi} \in F$.

7. $\|\Gamma\|_{\mathcal{D}_{|\mathbf{F}}} = \underline{\left[\|\overline{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}\right]}$.

8. $\|\boldsymbol{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}} = \overline{\left[\left\|\underline{\boldsymbol{\Gamma}}\right\|_{\mathcal{D}_{|\mathbf{F}}}\right]}$.

9. $\mathcal{D}_{|\mathbf{F}}$ is finitary iff $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ is finitary.

10. Let $\boldsymbol{\Gamma} \cup \{\boldsymbol{\phi}\} \subseteq \mathsf{Fm}(\mathbf{F})$, let $\phi_{\boldsymbol{\psi}} \in \underline{[\![\boldsymbol{\psi}]\!]}$, for each $\boldsymbol{\psi} \in \boldsymbol{\Gamma}$, $\phi_{\boldsymbol{\phi}} \in \underline{[\![\boldsymbol{\phi}]\!]}$, and let $\Gamma_{\boldsymbol{\Gamma}} \subseteq \mathsf{Fm}(\mathbf{G})$ such that $\overline{\Gamma_{\boldsymbol{\Gamma}}} = \boldsymbol{\Gamma}$. Then $\boldsymbol{\phi} \in \|\boldsymbol{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}$ iff $\underline{\boldsymbol{\Gamma}} \vdash_{\mathcal{D}_{|\mathbf{F}}} \underline{[\![\boldsymbol{\phi}]\!]}$ iff $\{\phi_{\boldsymbol{\psi}} \ : \ \boldsymbol{\psi} \in \boldsymbol{\Gamma}\} \vdash_{\mathcal{D}_{|\mathbf{F}}} \phi_{\boldsymbol{\phi}}$ iff $\Gamma_{\boldsymbol{\Gamma}} \vdash_{\mathcal{D}_{|\mathbf{F}}} \phi_{\boldsymbol{\phi}}$, and $\|\boldsymbol{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}} = \overline{\left[\|\{\phi_{\boldsymbol{\psi}} \ : \ \boldsymbol{\psi} \in \boldsymbol{\Gamma}\}\|_{\mathcal{D}_{|\mathbf{F}}}\right]} = \overline{\left[\|\Gamma_{\boldsymbol{\Gamma}}\|_{\mathcal{D}_{|\mathbf{F}}}\right]}$.

11. $\left(\overline{\cdot}\right)_{|\mathsf{Th}(\mathcal{D}_{|\mathbf{F}})} : \mathbf{Th}(\mathcal{D}_{|\mathbf{F}}) \cong \mathbf{Fi}_{\mathcal{D}}(\mathbf{F})$, with inverse isomorphism $\left(\underline{\cdot}\right)_{|\mathsf{Fi}_{\mathcal{D}}(\mathbf{F})}$.

□

The next result demonstrates that the ideal $\mathcal{D}_{|\mathbf{F}}$ of $\mathcal{D}$ modulo $\mathbf{F}$ is always coarser than $\mathcal{D}$. Informally, this is because $\mathcal{D}_{|\mathbf{F}}$ has only those $\mathcal{D}$ theories that are $\mathbf{F}$-compatible; alternatively $\mathcal{D}_{|\mathbf{F}}$ has gained additional consequences reflecting 'equality modulo $\mathbf{F}$'. We shall shortly make these statements precise.

Proposition 8.30 If $\mathcal{D}$ is an $\mathfrak{s}$-deductive system then $\mathcal{D} \preceq \mathcal{D}_{|\mathbf{F}}$.

Proof. Suppose that $\Gamma \vdash_{\mathcal{D}} \phi$. Since $\overline{\cdot}$ is an interpretation of $\mathbf{G}$ in $\mathbf{F}$ and $\overline{\Gamma} \subseteq \|\overline{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}$, $\overline{\phi} \in \|\overline{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}$. In other words, $\overline{\Gamma} \vdash_{\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})} \overline{\phi}$. Hence $\Gamma \vdash_{\mathcal{D}_{|\mathbf{F}}} \phi$ by (6) of Corollary 8.29. ◇

We aim to show that the $\mathcal{D}_{|\mathbf{F}}$-theories are precisely those $\mathcal{D}$-theories that are $\mathbf{F}$-compatible. We begin with a technical observation.

Lemma 8.31 If $\mathcal{D}$ is structural and T is an $\mathbf{F}$-compatible $\mathcal{D}$-theory, then $\|\overline{T}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}} = \overline{T}$.

Proof. (*It suffices to show that $\overline{T}$ is a $\mathcal{D}$-filter on* $\mathbf{F}$.) Suppose that i is an interpretation of $\mathbf{G}$ into $\mathbf{F}$, $\Gamma \vdash_{\mathcal{D}} \phi$ and that $\mathsf{i}\,[\Gamma] \subseteq \overline{T}$. (*By Corollary 7.20 on page 257, it suffices to show that* $\mathsf{i}(\phi) \in \overline{T}$.) Choose a substitution $\underline{\mathsf{i}}_?$ as in Definition 8.4. By Remark 8.7, $\overline{\underline{\mathsf{i}}_?(\cdot)} = \mathsf{i}(\cdot)$. So $\overline{\underline{\mathsf{i}}_?\,[\Gamma]} \subseteq \overline{T}$. So $\underline{\mathsf{i}}_?\,[\Gamma] \subseteq \underline{\left[\overline{\underline{\mathsf{i}}_?\,[\Gamma]}\right]} \subseteq \underline{\left[\overline{T}\right]} = T$, since T is $\mathbf{F}$-compatible. Since $\Gamma \vdash_{\mathcal{D}} \phi$, by structurality, $\underline{\mathsf{i}}_?\,[\Gamma] \vdash_{\mathcal{D}} \underline{\mathsf{i}}_?(\phi)$, and since T is a theory containing $\underline{\mathsf{i}}_?\,[\Gamma]$, $\underline{\mathsf{i}}_?(\phi) \in T$. Hence $\mathsf{i}(\phi) = \overline{\underline{\mathsf{i}}_?(\phi)} \in \overline{T}$. ◇

We now characterize the $\mathcal{D}_{|\mathbf{F}}$-theories as precisely those $\mathcal{D}$-theories that are $\mathbf{F}$-compatible. As a consequence, we characterize the $\mathbf{F}$-compatible $\mathfrak{s}$-deductive systems as precisely those logics that coincide with their ideal modulo $\mathbf{F}$.

Proposition 8.32 Let $\mathcal{D}$ be a *structural* $\mathfrak{s}$-deductive system.

1. The theories of $\mathcal{D}_{|\mathbf{F}}$ are precisely the $\mathbf{F}$-compatible $\mathcal{D}$-theories.

2. The following conditions are equivalent.

   (a) $\mathcal{D}$ is $\mathbf{F}$-compatible.

   (b) $\mathcal{D}_{|\mathbf{F}} \preceq \mathcal{D}$.

   (c) $\mathcal{D} = \mathcal{D}_{|\mathbf{F}}$.

Proof. $\boxed{(1)}$ (*It suffices, by Proposition 8.30 to show that every $\mathcal{D}$-theory that is $\mathbf{F}$-compatible is a $\mathcal{D}_{|\mathbf{F}}$-theory.*) Let T be an $\mathbf{F}$-compatible $\mathcal{D}$-theory. Suppose that $T \vdash_{\mathcal{D}_{|\mathbf{F}}} \phi$. (*We must show that $\phi \in T$. Since T is $\mathbf{F}$-compatible, it suffices to show that $\overline{\phi} \in \overline{[T]}$.*) Since $T \vdash_{\mathcal{D}_{|\mathbf{F}}} \phi$, by (6) of Corollary 8.29, $\overline{\phi} \in \|\overline{[T]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}} = \overline{[T]}$ by Lemma 8.31. $\boxed{(2)}$ [(a)$\Rightarrow$(b)] By (1). [(b)$\Rightarrow$(c)] Follows since generally $\mathcal{D} \preceq \mathcal{D}_{|\mathbf{F}}$, by Proposition 8.30. [(c)$\Rightarrow$(a)] Follows since all theories of $\mathcal{D}_{|\mathbf{F}}$ are $\mathbf{F}$-compatible by (3) of Theorem 8.28.

$\diamond$

So in the case that $\mathcal{D}$ is structural and $\mathbf{F}$-compatible, we may substitute $\mathcal{D}$ for $\mathcal{D}_{|\mathbf{F}}$ in Corollary 8.29. We highlight this case for ease of future reference. The result in this form is important, in that it characterizes $\mathcal{D}$-filter generation on $\mathbf{F}$ in a simple manner in terms of $\mathcal{D}$-theory generation, and establishes an isomorphism between the theory lattice of $\mathcal{D}$ and the $\mathcal{D}$-filter lattice on $\mathbf{F}$. It also leads as directly to our aim of showing that the theories of an *ideal* canon on $\mathbf{F}$ coincide with the filters of *its ideal* on $\mathbf{F}$.

Corollary 8.33 Let $\mathcal{D}$ be a *structural* $\mathfrak{s}$-deductive system that is $\mathbf{F}$-*compatible.*

1. $\mathcal{D}$ is the *coarsest* $\mathfrak{s}$-deductive system $\mathcal{D}$ on $\mathbf{G}$ for which $\bar{\cdot}$ is continuous from $\mathcal{D}$ into the filter logic $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$.

2. $\bar{\cdot}$ is strictly continuous from $\mathcal{D}$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$.

3. $\mathsf{Th}(\mathcal{D}) = \{\underline{[F]} \;:\; F \in \mathsf{Fi}_{\mathcal{D}}(\mathbf{F})\}$.

4. $\Gamma \vdash_{\mathcal{D}} \phi$ iff $\overline{\phi} \in \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}$ iff $\forall\,[F \in \mathsf{Fi}_{\mathcal{D}}(\mathbf{F})]\; \overline{[\Gamma]} \subseteq F \rightarrow \overline{\phi} \in F$.

5. $\|\Gamma\|_{\mathcal{D}} = \underline{\left[\|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}\right]}$.

6. $\|\boldsymbol{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}} = \overline{\left[\left\|\underline{[\boldsymbol{\Gamma}]}\right\|_{\mathcal{D}}\right]}$.

7. $\mathcal{D}$ is a propositional $\mathfrak{s}$-calculus iff $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ is finitary.

8. Let $\boldsymbol{\Gamma} \cup \{\boldsymbol{\phi}\} \subseteq \mathsf{Fm}(\mathbf{F})$, let $\phi_{\boldsymbol{\psi}} \in \underline{[\![\boldsymbol{\psi}]\!]}$, for each $\boldsymbol{\psi} \in \boldsymbol{\Gamma}$, $\phi_{\boldsymbol{\phi}} \in \underline{[\![\boldsymbol{\phi}]\!]}$, and let $\Gamma_{\boldsymbol{\Gamma}} \subseteq \mathsf{Fm}(\mathbf{G})$ such that $\overline{[\Gamma_{\boldsymbol{\Gamma}}]} = \boldsymbol{\Gamma}$. Then $\boldsymbol{\phi} \in \|\boldsymbol{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}}$ iff $\underline{[\boldsymbol{\Gamma}]} \vdash_{\mathcal{D}} \underline{[\![\boldsymbol{\phi}]\!]}$ iff $\{\phi_{\boldsymbol{\psi}} \;:\; \boldsymbol{\psi} \in \boldsymbol{\Gamma}\} \vdash_{\mathcal{D}} \phi_{\boldsymbol{\phi}}$ iff $\Gamma_{\boldsymbol{\Gamma}} \vdash_{\mathcal{D}} \phi_{\boldsymbol{\phi}}$, and $\|\boldsymbol{\Gamma}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{D}}} = \overline{[\|\{\phi_{\boldsymbol{\psi}} \;:\; \boldsymbol{\psi} \in \boldsymbol{\Gamma}\}\|_{\mathcal{D}}]} = \overline{[\|\Gamma_{\boldsymbol{\Gamma}}\|_{\mathcal{D}}]}$.

9. $\left(\overline{[\cdot]}\right)_{|\mathsf{Th}(\mathcal{D})} : \mathbf{Th}(\mathcal{D}) \cong \mathbf{Fi}_{\mathcal{D}}(\mathbf{F})$, with inverse isomorphism $\left(\underline{[\cdot]}\right)_{|\mathsf{Fi}_{\mathcal{D}}(\mathbf{F})}$.

$\square$

We now consider the case of a *propositional* $\mathfrak{s}$-calculus $\mathcal{P}$. Since $\mathcal{P}$ is finitary, its associated ideal $\mathcal{P}_{|\mathbf{F}}$ modulo $\mathbf{F}$ is also a propositional $\mathfrak{s}$-calculus, by (5) of Theorem 8.28, and hence must be axiomatizable. We now aim to provide such an axiomatization in terms of an axiomatization of $\mathcal{P}$ and an axiomatization of the discrete canon $\underline{\mathbf{F}}$. Recall Definition 6.41 on page 232, where we defined the join of propositional calculi. By Theorem 6.42 on page 232, $\mathcal{P}$ and $\underline{\mathbf{F}}$ are both finer than $\mathcal{P} \vee_{\mathsf{p}} \underline{\mathbf{F}}$. So by Corollary 8.24, $\mathcal{P} \vee_{\mathsf{p}} \underline{\mathbf{F}}$ is $\mathbf{F}$-compatible.

Remark 8.34 If $\mathcal{P}$ is a propositional $\mathfrak{s}$-calculus then $\mathcal{P} \vee_{\mathsf{p}} \underline{\mathbf{F}}$ is $\mathbf{F}$-compatible. $\square$

The following useful characterization of $\mathcal{P} \vee_{\mathsf{p}} \underline{\mathbf{F}}$-theory generation follows directly from Lemma 6.43 on page 233 and the fact that $\underline{\mathbf{F}}$ has no theorems.

Lemma 8.35 For a propositional $\mathfrak{s}$-calculus $\mathcal{P}$, $\|\Gamma\|_{\mathcal{P} \vee_{\mathsf{p}} \underline{\mathbf{F}}} = \bigcup_{i\in\omega} \Upsilon^i_\Gamma$, where $\Upsilon^0_\Gamma = \Gamma \cup \mathsf{Thm}(\mathcal{P})$ else $\Upsilon^{i+1}_\Gamma = \big\|\big\|\Upsilon^i_\Gamma\big\|_{\mathcal{P}}\big\|_{\underline{\mathbf{F}}}$. $\square$

We now show that in the case that $\mathcal{P}$ is a *propositional* $\mathfrak{s}$-calculus, then the *ideal* $\mathcal{P}_{|\mathbf{F}}$ *of* $\mathcal{P}$ *modulo* $\mathbf{F}$ is simply the join of $\mathcal{P}$ and the *discrete ideal* $\underline{\mathbf{F}}$ *determined by* $\mathbf{F}$; so by the definition of the join of propositional calculi, $\mathcal{P}_{|\mathbf{F}}$ is axiomatized by the union of any axiomatization of $\mathcal{P}$ and any axiomatization of $\underline{\mathbf{F}}$. Recall that an axiomatization of the latter is described in Proposition 8.21.

Theorem 8.36 If $\mathcal{P}$ is a *propositional* $\mathfrak{s}$-calculus then $\mathcal{P}_{|\mathbf{F}} = \mathcal{P} \vee_{\mathsf{p}} \underline{\mathbf{F}}$, which is a propositional $\mathfrak{s}$-calculus.

Proof. (*We shall show that* $\|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}} = \overline{\left[\|\Gamma\|_{\mathcal{P} \vee_{\mathsf{p}} \underline{\mathbf{F}}}\right]}$, *for all* $\Gamma \subseteq \mathsf{Fm}(\mathbf{G})$, *the result following by (8) of Corollary 8.29 and Remark 8.34.*) By Lemma 8.35, $\overline{\|\Gamma\|_{\mathcal{P} \vee_{\mathsf{p}} \underline{\mathbf{F}}}} = \overline{[\bigcup_{i\in\omega} \Upsilon^i_\Gamma]} = \bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]}$, the last equality by (1.41) of Table 1.2 on page 21. It suffices to show that $\bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]}$ is a $\mathcal{P}$-filter of $\mathbf{F}$ and that $\bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]} \subseteq \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$, since, by definition, $\Gamma \subseteq \Upsilon^0_\Gamma$ and hence $\overline{[\Gamma]} \subseteq \overline{[\Upsilon^0_\Gamma]} \subseteq \bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]}$, hence $\bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]} = \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$ by the minimality of filter generation.

$\boxed{\bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]} \text{ is a } \mathcal{P}\text{-filter of } \mathbf{F}}$ Suppose that $\Phi \vdash_{\mathcal{P}} \phi$ and that $\mathsf{i} : \mathbf{G} \to \mathbf{F}$ with $\mathsf{i}[\Phi] \subseteq \bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]}$. (*We must show that* $\mathsf{i}(\phi) \in \bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]}$.) Choose any substitution $\underline{\mathsf{i}}$, as in Definition 8.4 on page 278. Since $\mathcal{P}$ is finitary, there exists finite $\Phi' \subseteq \Phi$ with $\Phi' \vdash_{\mathcal{P}} \phi$. Hence there exists $i \in \omega$ with $\mathsf{i}[\Phi'] \subseteq \overline{[\Upsilon^i_\Gamma]}$. By Remark 8.7 on page 279, for each $\psi \in \Phi'$, $\overline{\underline{\mathsf{i}}(\psi)} = \mathsf{i}(\psi) \in \overline{[\Upsilon^i_\Gamma]} \subseteq \overline{[\Upsilon^{i+1}_\Gamma]}$, and since $\Upsilon^{i+1}_\Gamma = \big\|\big\|\Upsilon^i_\Gamma\big\|_{\mathcal{P}}\big\|_{\underline{\mathbf{F}}}$ which is $\mathbf{F}$-compatible, $\underline{\mathsf{i}}(\psi) \in \Upsilon^{i+1}_\Gamma$. So $\underline{\mathsf{i}}[\Phi'] \subseteq \Upsilon^{i+1}_\Gamma$. Since $\Phi' \vdash_{\mathcal{P}} \phi$, by structurality of $\mathcal{P}$, $\underline{\mathsf{i}}[\Phi'] \vdash_{\mathcal{P}} \underline{\mathsf{i}}(\phi)$. So $\underline{\mathsf{i}}(\phi) \in \Upsilon^{i+2}_\Gamma$. Hence $\mathsf{i}(\phi) = \overline{\underline{\mathsf{i}}(\phi)} \in \overline{[\Upsilon^{i+2}_\Gamma]} \subseteq \bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]}$, the equality of $\mathsf{i}(\phi)$ and $\overline{\underline{\mathsf{i}}(\phi)}$ following by Remark 8.7.

$\boxed{\bigcup_{i\in\omega} \overline{[\Upsilon^i_\Gamma]} \subseteq \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}.}$ We proceed inductively. $\boxed{\text{Base Case}}$ Let $\phi \in \Upsilon^0_\Gamma = \Gamma \cup \mathsf{Thm}(\mathcal{P})$. (*We must show that* $\overline{\phi} \in \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$.) If $\phi \in \Gamma$, then $\overline{\phi} \in \overline{[\Gamma]} \subseteq \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$. Otherwise $\phi \in \mathsf{Thm}(\mathcal{P})$, i.e., $\emptyset \vdash_{\mathcal{P}} \phi$. Since $\overline{\cdot} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{F}$, $\overline{[\emptyset]} = \emptyset \subseteq \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$, $\emptyset \vdash_{\mathcal{P}} \phi$ and $\|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$ is a $\mathcal{P}$-filter, $\overline{\phi} \in \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$. $\boxed{\text{Inductive Hypothesis}}$ Assume that $\overline{[\Upsilon^i_\Gamma]} \subseteq \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$. $\boxed{\text{Inductive Proof}}$ Let $\overline{\phi} \in \overline{[\Upsilon^{i+1}_\Gamma]} = \overline{\left[\big\|\|\Upsilon^i_\Gamma\|_{\mathcal{P}}\big\|_{\mathbf{F}_\perp}\right]} = \overline{\left[\underline{\overline{[\|\Upsilon^i_\Gamma\|_{\mathcal{P}}]}}\right]} = \overline{[\|\Upsilon^i_\Gamma\|_{\mathcal{P}}]}$. (*We must show that* $\overline{\phi} \in \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$.) There exists $\psi \in \big\|\Upsilon^i_\Gamma\big\|_{\mathcal{P}}$ with $\overline{\phi} = \overline{\psi}$. So there exists $\Phi \subseteq \Upsilon^i_\Gamma$ with $\Phi \vdash_{\mathcal{P}} \psi$. Since the canonical homomorphism $\overline{\cdot} : \mathbf{G} \to \mathbf{F}$ is an interpretation, $\overline{[\Phi]} \subseteq \overline{[\Upsilon^i_\Gamma]} \subseteq \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$ (by the inductive hypothesis), $\Phi \vdash_{\mathcal{P}} \psi$ and $\|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$ is a $\mathcal{P}$-filter, $\overline{\phi} = \overline{\psi} \in \|\overline{[\Gamma]}\|^{\mathbf{F}}_{\mathsf{fi}_{\mathcal{P}}}$. $\diamond$

We now shift the focus back to beginning with a canon and inducing its form and ideal, in order to satisfy our aim of showing that the theories of an *ideal* canon (on $\mathbf{F}$) coincide with the filters of *its ideal* on $\mathbf{F}$, in other words, $\mathsf{Th}(\boldsymbol{\mathcal{D}}) = \mathsf{Fi}_{\underline{\boldsymbol{\mathcal{D}}}}(\mathbf{F})$. Consider a *structural* cannon $\boldsymbol{\mathcal{D}}$. By

Theorem 8.13, $\mathcal{D}$ is *ideal*, i.e., its form $\underline{\mathcal{D}}$ and ideal $\mathcal{D}^{\imath}$ coincide. Consequently its ideal/form is *structural* and $\mathbf{F}$-*compatible*, by Corollary 8.24, and hence its ideal/form satisfies the assumptions of the previous corollary. So it follows by (9) of the previous corollary, that the canonical image-function $\overline{[\cdot]}$ induces an isomorphism from the ideal/form theory lattice $\mathbf{Th}(\underline{\mathcal{D}})$ onto the ideal/form filter lattice $\mathbf{Fi}_{\underline{\mathcal{D}}}(\mathbf{F})$ on $\mathbf{F}$. Now recall that by Theorem 8.15, since $\mathcal{D}$ is assumed to be structural, the *same* canonical image-function also induces an isomorphism from the ideal/form theory lattice $\mathbf{Th}(\underline{\mathcal{D}})$ onto the $\mathcal{D}$-theory lattice $\mathbf{Th}(\mathcal{D})$. Consequently, in the case that the canon $\mathcal{D}$ is structural, the $\mathcal{D}$-theories must coincide with the ideal/form filters on $\mathbf{F}$, i.e., $\mathsf{Th}(\mathcal{D}) = \mathsf{Fi}_{\underline{\mathcal{D}}}(\mathbf{F})$. We now show that the coincidence of $\mathsf{Th}(\mathcal{D})$ and $\mathsf{Fi}_{\underline{\mathcal{D}}}(\mathbf{F})$ in fact *characterizes* the structurality of $\mathcal{D}$.

Theorem 8.37 For an $\mathfrak{s}$-canon $\mathcal{D}$, the following conditions are equivalent.

1. $\mathcal{D}$ is structural.
2. $\underline{\mathcal{D}} = \underline{\mathcal{D}}_{|\mathbf{F}}$.
3. $\mathcal{D} = \mathsf{F}^{\mathfrak{s}}_{\underline{\mathcal{D}}}(\mathbf{F})$.
4. $\mathsf{Th}(\mathcal{D}) = \mathsf{Fi}_{\underline{\mathcal{D}}}(\mathbf{F})$.
5. $\underline{\mathcal{D}}_{|\mathbf{F}} \preceq \underline{\mathcal{D}}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Since $\mathcal{D}$ is structural, $\underline{\mathcal{D}}$ is structural, by Theorem 8.13. Further, $\underline{\mathcal{D}}$ is $\mathbf{F}$-compatible, by Corollary 8.24. So by Proposition 8.32, $\underline{\mathcal{D}} = \underline{\mathcal{D}}_{|\mathbf{F}}$. $\boxed{(2)\Rightarrow(3)}$ $\mathbf{\Gamma} \vdash_{\mathcal{D}} \phi$ iff $\underline{[\mathbf{\Gamma}]} \vdash_{\underline{\mathcal{D}}} \underline{[\![\phi]\!]}$ iff $\underline{[\mathbf{\Gamma}]} \vdash_{\underline{\mathcal{D}}_{|\mathbf{F}}} \underline{[\![\phi]\!]}$ iff $\overline{\left[\underline{[\mathbf{\Gamma}]}\right]} \vdash_{\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})} \overline{\left[\underline{[\![\phi]\!]}\right]}$ iff $\mathbf{\Gamma} \vdash_{\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})} \phi$, the first equivalence following by Theorem 8.9, the second by assumption, the third by Corollary 8.29, and the fourth is trivial. $\boxed{(3)\Rightarrow(1)}$ $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{F})$ is always structural, by Theorem 8.28, and so by assumption, $\mathcal{D}$ is structural. $\boxed{(3)\Leftrightarrow(4)}$ Trivial. $\boxed{(2)\Rightarrow(5)}$ Trivial. $\boxed{(5)\Rightarrow(2)}$ Follows since generally $\mathcal{D} \preceq \mathcal{D}_{|\mathbf{F}}$, by Proposition 8.30. $\diamond$

8.1.6 Examples

In order to axiomatize the ideal of any ideal cannon, it is essential to axiomatize the discrete ideal. In the following example, we consider the $\mathcal{K}$-free algebra $\mathbf{F}_{\mathcal{K}}$, of a quasivariety $\mathcal{K}$ of $\mathfrak{a}$-algebras, as an $\mathfrak{a}$-canonical language (see Example 8.3 on page 278), and characterize its discrete ideal which is a sentential 1-calculus; we call this sentential calculus the $\mathcal{K}$-*equational logic*. This logic will play a central role in the characterization of the induced ideal of any ideal logic on $\mathbf{F}_{\mathcal{K}}$.

Example 8.38 (The Sentential Calculus of Equality over a Quasivariety)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra over ω-free generators $\overline{[\mathsf{V}]}$, which is $\mathfrak{a}$-canonical.

Definition 8.39 (The $\mathcal{K}$-Equational Logic) We denote the sentential 1-calculus $\underline{\mathbf{F}_{\mathcal{K}}}$ by $S(\mathcal{K}, \approx)$, which we call the **$\mathcal{K}$-equational logic**. We write $\vdash^{\mathcal{K}}_{\approx}$ for $\vdash_{S(\mathcal{K},\approx)}$ and $\|\cdot\|^{\mathcal{K}}_{\approx}$ for $\|\cdot\|_{\mathcal{K}}$. $\square$

Recall that by Corollary 1.435 on page 84, $\overline{p} = \overline{q}$ iff $\models_{\mathcal{K}} p \approx q$. The following result follows immediately from this fact together with Corollary 8.19 and Proposition 8.21.

Corollary 8.40 Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras.

1. $P \vdash_{\approx}^{\mathcal{K}} p$ iff $\exists\, [q \in P]\ \models_{\mathcal{K}} p \approx q$.
2. $\|P\|_{\approx}^{\mathcal{K}} = \{p \,:\, \exists\, [q \in P]\ \models_{\mathcal{K}} p \approx q\}$.
3. $\mathsf{Th}(S(\mathcal{K}, \approx)) = \{P \,:\, \forall\, [p \in P]\ \models_{\mathcal{K}} p \approx q$ implies $q \in P\}$.
4. $S(\mathcal{K}, \approx)$ is axiomatized with *no axioms* and all inference-rules $\{p\} \vdash q$ such that $\models_{\mathcal{K}} p \approx q$.

Definition 8.41 ($\mathcal{K}$-Equational Closure) We shall say that terms P are $\mathcal{K}$-**equationally closed** if they are $\mathbf{F}_{\mathcal{K}}$-compatible, and call an $\mathfrak{a}$-deductive system $\mathcal{K}$**-equationally compatible** if it is $\mathbf{F}_{\mathcal{K}}$-compatible. □

Remark 8.42 P is $\mathcal{K}$-equationally closed iff $\forall\, [p \in P]\ \models_{\mathcal{K}} p \approx q$ implies $q \in P$ (by Corollary 1.435).

Remark 8.43 The theories of $\vdash_{\approx}^{\mathcal{K}}$ are precisely the $\mathcal{K}$-equationally closed sets of terms.

Remark 8.44 $S(\mathcal{K}, \mathsf{su})$ is $\mathcal{K}$-equationally compatible (by Corollary 8.24).

□

In the next example, we begin with a sentential 1-calculus $\mathcal{S}$ of type $\mathfrak{a}$ and a quasivariety $\mathcal{K}$ of $\mathfrak{a}$-algebras, and we consider the *ideal* $\mathcal{S}_{|\mathbf{F}_{\mathcal{K}}}$ *of* $\mathcal{S}$ *modulo* $\mathbf{F}_{\mathcal{K}}$, where $\mathbf{F}_{\mathcal{K}}$ is the $\mathcal{K}$-free algebra on ω-free generators considered as an $\mathfrak{a}$-canonical language. The ideal $\mathcal{S}_{|\mathbf{F}_{\mathcal{K}}}$ is the coarsest 'refinement' of $\mathcal{S}$ such that all theories are $\mathcal{K}$-equationally closed.

Example 8.45 (Refining Sentential 1-Calculus to Quasivariety)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-calculi and $\mathcal{S}$ any sentential 1-calculus of type $\mathfrak{a}$.

Definition 8.46 (The Sentential Calculus $\mathcal{S}_{|\mathcal{K}}$) The $\mathfrak{a}$-calculus $\mathcal{S}_{|\mathbf{F}_{\mathcal{K}}}$ is denoted by $\mathcal{S}_{|\mathcal{K}}$, which we call the **refinement of $\mathcal{S}$ to $\mathcal{K}$**. □

The following axiomatization of $\mathcal{S}_{|\mathcal{K}}$ follows from Theorem 8.36.

Corollary 8.47 An axiomatization of $\mathcal{S}_{|\mathcal{K}}$ consists of any axiomatization of $\mathcal{S}$ together with all inference-rules $\{p\} \vdash q$ such that $\{p\} \vdash q$ iff $\models_{\mathcal{K}} p \approx q$.

Open Problem 8.48 Make precise the manner in which '$\mathcal{S}_{|\mathcal{K}}$ refines $\mathcal{S}$ modulo $\mathcal{K}$' by proving the following.

Conjecture 8.49 For any $\mathfrak{a}$-algebra $\mathbf{A}$, $\mathsf{q}_{\perp_{\mathcal{K}}^{\mathbf{A}}}[\cdot] : \mathbf{Fi}_{\mathcal{S}_{|\mathcal{K}}}(\mathbf{A}) \cong \mathbf{Fi}_{\mathcal{S}}(\mathbf{A}/\perp_{\mathcal{K}}^{\mathbf{A}})$.

Conjecture 8.50 For any $\mathbf{A} \in \mathcal{K}$, $\mathsf{Fi}_{\mathcal{S}_{|\mathcal{K}}}(\mathbf{A}) = \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$.

□

Recall Example 6.80 on page 242, where we defined and axiomatized the *propositional* $\mathcal{K}$-calculus $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ of all subuniverses on $\mathbf{F}_{\mathcal{K}}$. We shall now apply the theory of canons and their ideals to associate a sentential 1-calculus with $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$. The reader is urged to recall the definition of the sentential 1-calculus $S(\mathfrak{a}, \mathsf{su})$ of $\mathfrak{a}$-*subuniverse* introduced in Example 5.47 on page 188.

Example 8.51 (The Sentential Calculi of Subuniverses)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{F}_{\mathcal{K}}$ the $\mathfrak{a}$-canonical language as in Example 8.80. The root language is the term algebra. We shall consider $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ as an $\mathfrak{a}$-canon.

Definition 8.52 (The Sentential Calculi of $\mathcal{K}$-Subuniverses) Viewing $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ as an $\mathfrak{a}$-canon, let $S(\mathcal{K}, \mathsf{su})$ denote $\underline{\boldsymbol{S}(\mathcal{K}, \mathsf{su})}$. ☐

Since, by construction, the signature $\mathcal{K}$ is a full subconstruct of the signature $\mathfrak{a}$, and since $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ is a *propositional* $\mathcal{K}$-calculus, it is a structural and finitary $\mathfrak{a}$-canon. So by Theorem 8.11, $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ is ideal and hence $S(\mathcal{K}, \mathsf{su}) = \boldsymbol{S}(\mathcal{K}, \mathsf{su})^{\imath}$ and $S(\mathcal{K}, \mathsf{su})$ is $\mathfrak{a}$-structural, and by (7) of Theorem 8.9, $S(\mathcal{K}, \mathsf{su})$ is finitary. Hence $S(\mathcal{K}, \mathsf{su})$ is a *sentential* 1-*calculus*.

Corollary 8.53 $S(\mathcal{K}, \mathsf{su})$ is a sentential 1-calculus and $S(\mathcal{K}, \mathsf{su}) = \boldsymbol{S}(\mathcal{K}, \mathsf{su})^{\imath}$. ☐

By Proposition 7.82 of Example 7.77 on page 270, $\mathsf{Fi}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{F}_{\mathcal{K}}) = \mathsf{Su}(\mathbf{A}) = F(\mathbf{F}_{\mathcal{K}}, \mathsf{su})$, so by (3) of Theorem 8.9,

$$\mathsf{Th}(S(\mathcal{K}, \mathsf{su})) = \{\underline{[\boldsymbol{P}]} \,:\, \boldsymbol{P} \in \mathsf{Su}(\mathbf{F}_{\mathcal{K}})\}, \tag{8.1}$$

by (7) of Theorem 8.9,

$$\mathsf{Su}(\mathbf{F}_{\mathcal{K}}) = \{\overline{[T]} \,:\, T \in \mathsf{Th}(S(\mathcal{K}, \mathsf{su}))\} \tag{8.2}$$

and by (3) of Theorem 8.9,

$$P \vdash_{S(\mathcal{K},\mathsf{su})} p \text{ iff } \overline{[P]} \vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \overline{p}. \tag{8.3}$$

In fact, by Theorem 8.15, we have the following.

Corollary 8.54 $\overline{[\cdot]} : \mathbf{Th}(S(\mathcal{K}, \mathsf{su})) \cong \mathbf{Su}(\mathbf{F}_{\mathcal{K}})$ with inverse isomorphism $\underline{[\cdot]}$. ☐

Recall the axiomatization of $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$ given in Example 6.80. The following axiomatization of the sentential calculus $S(\mathcal{K}, \mathsf{su})$, obtains from that axiomatization and Theorem 8.25.

Theorem 8.55 $S(\mathcal{K}, \mathsf{su})$ is axiomatized by all axioms $\vdash \mathbf{0}$, where $\mathbf{0} \in \mathsf{Symb}_{\mathsf{c}}(\mathfrak{a})$, and all inference-rules $\{x_1, \ldots, x_{\mathsf{ar}(\star)}\} \vdash \star(x_1, \ldots, x_{\mathsf{ar}(\star)})$, one for each $\star \in \mathsf{Symb}_{\mathsf{o}}(\mathfrak{a})$ and some distinct choice of variables $x_1, \ldots, x_{\mathsf{ar}(\star)}$, together with any set of rules axiomatizing $S(\mathcal{K}, \approx)$, for example all inference-rules $\{p\} \vdash q$ such that $\{p\} \vdash q$ iff $\models_{\mathcal{K}} p \approx q$. ☐

The following result follows from the axiomatization given in the previous theorem, the axiomatization of $S(\mathcal{K}, \mathsf{su})$ given in Example 5.47 on page 188 and Corollary 8.47.

Corollary 8.56 $S(\mathcal{K}, \mathsf{su}) = S(\mathfrak{a}, \mathsf{su})_{|\mathcal{K}}$.

☐

Recall Example 6.74 on page 241, where we defined the $\mathbf{F}_{\mathcal{K}}{}^2$-logic $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$, and showed that this logic is a propositional $\underset{\to[2]}{\mathcal{K}}$-calculus. In the next example, we shall obtain an axiomatization of the propositional calculus using the theory developed in this section.

Example 8.57 (The Relative Congruence Canon of a Quasivariety)

Let $\mathfrak{a}$ be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra on ω-free generators $\overline{\texttt{V}}$.

Since the signature $\mathcal{K}$ is a full subconstruct of the signature $\mathfrak{a}$, the signature $\underset{\rightarrow[2]}{\mathcal{K}}$ is a full subconstruct of the signature $\underset{\rightarrow[2]}{\mathfrak{a}}$. Consequently, the global $\underset{\rightarrow[2]}{\mathcal{K}}$-language $\mathbf{F}_{\mathcal{K}}{}^2$ is $\underset{\rightarrow[2]}{\mathfrak{a}}$-canonical with root language $\mathbf{Tm}^2$. Since $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$ is a propositional $\underset{\rightarrow[2]}{\mathcal{K}}$-calculus (by Proposition 6.78 on page 241), $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$ is an $\underset{\rightarrow}{\mathfrak{a}}{}_{[2]}$-structural and finitary $\underset{\rightarrow}{\mathfrak{a}}{}_{[2]}$-canon. We note this observation for ease of future reference.

Remark 8.58 $\boldsymbol{S}^2(\Theta^{\mathcal{K}})$ is an $\underset{\rightarrow}{\mathfrak{a}}{}_{[2]}$-structural and finitary $\underset{\rightarrow}{\mathfrak{a}}{}_{[2]}$-canon. $\square$

Corollary 8.59 $\boldsymbol{S}^2(\Theta^{\mathcal{K}})^{\imath} = \underline{\boldsymbol{S}^2(\Theta^{\mathcal{K}})} = S^2(\Theta^{\mathcal{K}})$.

Proof. By Remark 8.58 and Theorem 8.11, $\boldsymbol{S}^2(\Theta^{\mathcal{K}})^{\imath} = \underline{\boldsymbol{S}^2(\Theta^{\mathcal{K}})}$. By Corollary 6.76 on page 241, $\{\langle p_i, q_i\rangle : i \in I\} \vdash_{S^2(\Theta^{\mathcal{K}})} \langle p, q\rangle$ iff $\{\langle \overline{p_i}, \overline{q_i}\rangle : i \in I\} \vdash_{\boldsymbol{S}^2(\Theta^{\mathcal{K}})} \langle \overline{p}, \overline{q}\rangle$ iff $\{\langle p_i, q_i\rangle : i \in I\} \vdash_{\underline{\boldsymbol{S}^2(\Theta^{\mathcal{K}})}} \langle p, q\rangle$, where the final equivalence follows by (4) of Theorem 8.9. $\diamond$

$\square$

Recall the definition of the universal logic $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$, determined by its theories which are all relative $\mathcal{K}$-cosets on the $\mathcal{K}$-free algebra $\mathbf{F}_{\mathcal{K}}$, defined in Example 6.86 on page 243. In that example, we noted that while $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ is certainly not a sentential calculus, it is a propositional $\mathcal{K}$-calculus, and hence an *ideal* $\mathfrak{a}$-canon with $\mathfrak{a}$-canonical language $\mathbf{F}_{\mathcal{K}}$. In the next example, we shall show that the *ideal/form* of this canon is equivalent to the *membership logic* $S(\mathcal{K}, \mathsf{mem})$. Note that we have already established that $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ is an $\mathfrak{a}$-model of the membership logic (see Proposition 7.86 of Example 7.83 on page 271). We will now be able to conclude that $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ is a *maximal* $\mathfrak{a}$-model of the membership logic, in other words, the filters of the membership logic on $\mathbf{F}_{\mathcal{K}}$ are precisely the $\mathcal{K}$-cosets on $\mathbf{F}_{\mathcal{K}}$.

In this example, we shall also consider the logic $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ of *non-relative* cosets of $\mathbf{F}_{\mathcal{K}}$, which is also an *ideal* $\mathfrak{a}$-canon on $\mathbf{F}_{\mathcal{K}}$; we shall provide an axiomatization of the ideal/form of the canon $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$. Since $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ and $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ coincide in the case that $\mathcal{K}$ is a non-trivial *variety*, in this case the ideal/form of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ is equivalent to the membership logic, and so the axiomatization of the ideal/form of $\boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$ provides an axiomatization of the membership logic of a non-trivial *variety*; the axiomatization is simpler than the axiomatization of the membership logic generally.

Example 8.60 (The Membership Canon)

Let $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra on $\overline{\mathsf{V}}$. As mentioned before, $\mathbf{F}_{\mathcal{K}}$ is an $\mathfrak{a}$-canonical language with root language $\mathbf{Tm}$. We shall call the $\mathfrak{a}$-canon $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ the **canonical membership logic**. Note that by Example 6.85, the canonical membership logic is *ideal*, and so its form and ideal coincide, by Theorem 8.13.

We shall now show that the *membership logic* is equivalent to the *ideal/form* of the *canonical membership logic.*

Theorem 8.61 $S(\mathcal{K}, \mathsf{mem}) \equiv \boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})^{\imath} = \underline{\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})}$.

Proof. Since $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ is a propositional $\mathfrak{a}$-canon, $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})^{\imath} = \underline{\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})}$, by Theorem 8.13. By Corollary 6.90 of Example 6.85 on page 243, $P \vdash_{S(\mathcal{K},\mathsf{mem})} p$ iff $\overline{[P]} \vdash_{\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})} \overline{p}$, and so by (4) of Theorem 8.9, $P \vdash_{S(\mathcal{K},\mathsf{mem})} p$ iff $P \vdash_{\underline{\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})}} p$. Hence $S(\mathcal{K},\mathsf{mem}) \equiv \underline{\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})}$. $\diamond$

The following result follows immediately from the previous theorem, together with Theorem 8.15 and Theorem 8.37. Note that it follows immediately from this result, that the lattice of $\mathcal{K}$-cosets on the term algebra $\mathbf{Tm}$ is isomorphic to the lattice of $\mathcal{K}$-cosets on the $\mathcal{K}$-free algebra $\mathbf{F}_{\mathcal{K}}$.

Corollary 8.62 The following are all valid.

1. $\overline{[\cdot]} : \mathbf{Th}(S(\mathcal{K},\mathsf{mem})) \cong \mathbf{Th}(\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}}))$.
2. $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ is a maximal $\mathfrak{a}$-model of $S(\mathcal{K},\mathsf{mem})$.

$\square$

Recall the definition of the logic $\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})$ of *non-relative* cosets on the $\mathcal{K}$-free algebra $\mathbf{F}_{\mathcal{K}}$, introduced in Example 6.85 on page 243. In that example, we noted that $\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})$ is a propositional $\mathcal{K}$-calculus and an axiomatization was provided in Proposition 6.89. Consequently, $\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})$ is an *ideal* $\mathfrak{a}$-canon on $\mathfrak{a}$-canonical language $\mathbf{F}_{\mathcal{K}}$; as such, its form and ideal coincide, and an axiomatization of its ideal/form, which is a *sentential* 1-*calculus*, obtains from the aforementioned axiomatization of $\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})$, together with Theorem 8.25.

Theorem 8.63 $\underline{\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})} = \boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})^{\imath}$. This sentential 1-calculus is axiomatized by all rules

$$x, y, p(x, z_1, \dots, z_{\mathsf{ar}(p)-1}) \vdash p(y, z_1, \dots, z_{\mathsf{ar}(p)-1}),$$

one for each term p and some distinct choice of variables $x, y, z_1, \dots, z_{\mathsf{ar}(p)-1} \in \mathsf{V}$, and in addition, any axiomatization of $S(\mathcal{K},\approx)$. $\square$

Observe that in the case that $\mathcal{K}$ is a non-trivial *variety*, the *non-relative coset logic* $\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})$ coincides with the *canonical membership logic* $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$, and hence the ideal/form of $\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})$ must be equivalent to the *membership logic* of $\mathcal{K}$; in this case, the previous theorem provides an alternative (and simpler) characterization of the *membership logic of non-trivial variety* $\mathcal{K}$. Note that the reason why the variety must be non-trivial, is because $\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})$ always includes the improper-coset while $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ excludes the improper-coset precisely when $\mathcal{K}$ is trivial.

$\square$

Recall the definition of the sentential 1-calculus $S(\mathcal{K},\tau)$ of [BR99], determined by a unary system of equations and a quasivariety $\mathcal{K}$ (see Example 2.88 on page 107), and the definition of the propositional $\mathcal{K}$-calculus $\boldsymbol{S}(\mathcal{K},\tau)$ (see Example 6.93 on page 245). In the following example we show how the sentential calculus $S(\mathcal{K},\tau)$ arises as the ideal of the canon $\boldsymbol{S}(\mathcal{K},\tau)$.

Example 8.64 (The Logic $S(\mathcal{K},\tau)$)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and τ a unary system of equations.

Corollary 8.65 Let $\mathfrak{N}$ be a system of n-ary $\mathfrak{a}$-equations, $\mathcal{K}$ be a quasi-variety of $\mathfrak{a}$-algebras.

1. $S^n(\mathcal{K},\mathfrak{N}) \equiv \underline{\boldsymbol{S}^n(\mathcal{K},\mathfrak{N})} = \boldsymbol{S}^n(\mathcal{K},\mathfrak{N})^{\imath}$.
2. $\overline{[\cdot]} : \mathbf{Th}(S^n(\mathcal{K},\mathfrak{N})) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{F}_{\mathcal{K}}) = \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{F}_{\mathcal{K}})$ with inverse isomorphism $\underline{[\cdot]}$.

Proof. $\boxed{(1)}$ By Corollary 6.96 on page 245 and Theorem 8.9, $P \vdash_{S(\mathcal{K},\tau)} p$ iff $\overline{[P]} \vdash_{\boldsymbol{S}(\mathcal{K},\tau)} \overline{p}$ iff $\boldsymbol{\Gamma} \vdash_{\underline{\boldsymbol{S}(\mathcal{K},\tau)}} \boldsymbol{\phi}$. Hence $S(\mathcal{K},\tau) \equiv \underline{\boldsymbol{S}(\mathcal{K},\tau)}$. Since $S(\mathcal{K},\tau)$ is structural, $\underline{\boldsymbol{S}(\mathcal{K},\tau)} = \boldsymbol{S}(\mathcal{K},\tau)^{\imath}$, by Theorem 8.13. $\boxed{(2)}$ By (1), Proposition 2.91 on page 107 and Theorem 8.37. $\diamond$

$\square$

Recall the definition of the propositional $\mathcal{K}$-calculus $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$ given in Example 6.98 on page 246, which is an *ideal* canon, and hence its ideal/form is a sentential calculus. In the following example, we shall demonstrate that this sentential calculus is equivalent to the sentential calculus $S_*(\mathcal{K},\mathsf{id})$, also introduced in the aforementioned example.

Example 8.66 (Logics of Lattice Ideals and Filters)

Remark 8.67 For a quasivariety $\mathcal{K}$ of lower-unbounded lattices, $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$ is a structural and finitary $\mathsf{type}(\mathcal{K})$-canon, by Example 12.61 on page 389. $\square$

Since by definition, $\mathsf{Th}(\boldsymbol{S}_*(\mathcal{K},\mathsf{id})) = \mathbf{Id}_{\diamond}(\mathbf{F}_{\mathcal{K}})$, the following result follows at once from the previous remark, together with Theorem 8.11, Theorem 8.9, Theorem 8.15 and Theorem 8.25, and the axiomatization of $\boldsymbol{S}_*(\mathcal{K},\mathsf{id})$ given in Example 6.98.

Corollary 8.68 Let $\mathcal{K}$ be a quasivariety of lower-unbounded lattices.

1. $\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})} = \boldsymbol{S}_*(\mathcal{K},\mathsf{id})^{\imath}$.
2. $\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}$ is a sentential 1-calculus.
3. $\overline{[\cdot]} : \mathbf{Th}(\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}) \cong \mathbf{Id}_{\diamond}(\mathbf{F}_{\mathcal{K}})$ with inverse isomorphism $\underline{[\cdot]}$.
4. An axiomatization of $\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}$ is given by no axioms, the two rules

$$x \vdash x \wedge y \quad \text{and} \tag{8.4}$$

$$x, y \vdash x \vee y, \tag{8.5}$$

where x and y are two distinct variables, and all rules

$$p \vdash q, \quad \text{where} \quad \models_{\mathcal{K}} p \approx q. \tag{8.6}$$

$\square$

We have not given a name nor special notion to $\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}$, since, as shown in the next result, $\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}$ is equivalent to the sentential calculus $S_*(\mathcal{K},\mathsf{id})$.

Proposition 8.69 $S_*(\mathcal{K},\mathsf{id}) \equiv \underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}$.

Proof. $\boxed{S_*(\mathcal{K},\mathsf{id}) \preceq \underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}}$ *(It suffices to show that every $S_*(\mathcal{K},\mathsf{id})$-rule, as in Example 6.98 on page 248 (there are no axioms), is satisfied by $\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}$, in which case, since $S_*(\mathcal{K},\mathsf{id})$ and $\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}$ are both sentential, $S_*(\mathcal{K},\mathsf{id}) \preceq \underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}$ by Proposition 6.31 on page 229.)*
Let $p_1,\ldots,p_n \vdash p$ be a $S_*(\mathcal{K},\mathsf{id})$-rule. Hence $\models_{\mathcal{K}} p \le p_1 \vee \ldots \vee p_n$, and hence $\overline{p} \in \langle \overline{p_1} \vee^{\mathbf{F}_{\mathcal{K}}} \ldots \vee^{\mathbf{F}_{\mathcal{K}}} \overline{p_n}]_{\mathbf{F}_{\mathcal{K}}} = \|\overline{p_1},\ldots,\overline{p_n}\|^{\mathbf{F}_{\mathcal{K}}}_{\mathsf{id}_{\diamond}}$. So $\{\overline{p_1},\ldots,\overline{p_n}\} \vdash_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})} \overline{p}$, and hence $\{p_1,\ldots,p_n\} \vdash_{\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}} p$, by Theorem 8.9. $\boxed{\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})} \preceq S_*(\mathcal{K},\mathsf{id})}$ Suppose that $P \vdash_{\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}} p$. By finitariness, there exist $\emptyset \neq \{p_1,\ldots,p_n\} \subseteq P$ with $\{p_1,\ldots,p_n\} \vdash_{\underline{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})}} p$. Hence

$\{\overline{p_1}, \ldots, \overline{p_n}\} \vdash_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})} \overline{p}$ and so $\overline{p} \in \|\overline{p_1}, \ldots, \overline{p_n}\|^{\mathbf{F}_\mathcal{K}}_{\mathsf{id}_\Diamond} = \langle \overline{p_1} \vee^{\mathbf{F}_\mathcal{K}} \ldots \vee^{\mathbf{F}_\mathcal{K}} \overline{p_n}]_{\mathbf{F}_\mathcal{K}}$. So $\overline{p} \leq^{\mathbf{F}_\mathcal{K}} \overline{p_1} \vee^{\mathbf{F}_\mathcal{K}} \ldots \vee^{\mathbf{F}_\mathcal{K}} \overline{p_n}$, and hence $\models_\mathcal{K} p \leq p_1 \vee \ldots \vee p_n$. So $p_1, \ldots, p_n \vdash p$ is a $S_*(\mathcal{K}, \mathsf{id})$-rule, as in Example 6.98 on page 248, and so $\{p_1, \ldots, p_n\} \vdash_{S_*(\mathcal{K},\mathsf{id})} p$, and hence $P \vdash_{S_*(\mathcal{K},\mathsf{id})} p$. $\diamond$

The following result follows from the preceding corollary and proposition, together with Theorem 8.37.

Corollary 8.70 $\mathsf{Fi}_{S_*(\mathcal{K},\mathsf{id})}(\mathbf{F}_\mathcal{K}) = \mathsf{Id}_\Diamond(\mathbf{F}_\mathcal{K})$. $\square$

So $\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})$ is a semantics for $S_*(\mathcal{K}, \mathsf{id})$, by Corollary 8.68, Proposition 8.69 and Theorem 8.13.

Corollary 8.71 $\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})$ is a semantics for $S_*(\mathcal{K}, \mathsf{id})$. $\square$

We leave it to the reader to formulate the analogous results for $\boldsymbol{S}_0(\mathcal{K}, \mathsf{id})$, $\boldsymbol{S}_1(\mathcal{K}, \mathsf{fi})$ and $\boldsymbol{S}_*(\mathcal{K}, \mathsf{fi})$. We briefly mention a few important results for ease of future reference.

Corollary 8.72 $S_*(\mathcal{K}, \mathsf{fi}) \equiv \underline{\boldsymbol{S}_*(\mathcal{K}, \mathsf{fi})} = \boldsymbol{S}_*(\mathcal{K}, \mathsf{fi})^\imath$, $S_0(\mathcal{K}, \mathsf{id}) \equiv \underline{\boldsymbol{S}_0(\mathcal{K}, \mathsf{id})} = \boldsymbol{S}_0(\mathcal{K}, \mathsf{id})^\imath$ and $S_1(\mathcal{K}, \mathsf{fi}) \equiv \underline{\boldsymbol{S}_1(\mathcal{K}, \mathsf{fi})} = \boldsymbol{S}_1(\mathcal{K}, \mathsf{fi})^\imath$.

Corollary 8.73 $\boldsymbol{S}_*(\mathcal{K}, \mathsf{fi})$ is a semantics for $S_*(\mathcal{K}, \mathsf{fi})$. $\boldsymbol{S}_0(\mathcal{K}, \mathsf{id})$ and $\boldsymbol{S}_1(\mathcal{K}, \mathsf{fi})$ are semantics for $S_0(\mathcal{K}, \mathsf{id})$ and $S_1(\mathcal{K}, \mathsf{fi})$, respectively.

Corollary 8.74 $\mathsf{Fi}_{S_*(\mathcal{K},\mathsf{fi})}(\mathbf{F}_\mathcal{K}) = \mathsf{Fl}_\Diamond(\mathbf{F}_\mathcal{K})$, $\mathsf{Fi}_{S_0(\mathcal{K},\mathsf{id})}(\mathbf{F}_\mathcal{K}) = \mathsf{Id}_\Diamond(\mathbf{F}_\mathcal{K})$ and $\mathsf{Fi}_{S_1(\mathcal{K},\mathsf{fi})}(\mathbf{F}_\mathcal{K}) = \mathsf{Fl}_\Diamond(\mathbf{F}_\mathcal{K})$. $\square$

The following important result now follows immediately, together with (4) of Theorem 8.9 and remarks 6.102 through to 6.105 of Example 6.98.

Theorem 8.75 For appropriate quasivarieties and $P \cup \{p\}$ with $P \neq \emptyset$, the following are all valid.

1. $P \vdash_{S_*(\mathcal{K},\mathsf{id})} p$ iff $\overline{[P]} \vdash_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})} \overline{p}$.
2. $P \vdash_{S_*(\mathcal{K},\mathsf{fi})} p$ iff $\overline{[P]} \vdash_{\boldsymbol{S}_*(\mathcal{K},\mathsf{fi})} \overline{p}$.
3. $P \vdash_{S_0(\mathcal{K},\mathsf{id})} p$ iff $\overline{[P]} \vdash_{\boldsymbol{S}_0(\mathcal{K},\mathsf{id})} \overline{p}$, and $\vdash_{S_0(\mathcal{K},\mathsf{id})} p$ iff $\vdash_{\boldsymbol{S}_0(\mathcal{K},\mathsf{id})} \overline{p}$.
4. $P \vdash_{S_1(\mathcal{K},\mathsf{fi})} p$ iff $\overline{[P]} \vdash_{\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})} \overline{p}$, and $\vdash_{S_1(\mathcal{K},\mathsf{fi})} p$ iff $\vdash_{\boldsymbol{S}_1(\mathcal{K},\mathsf{fi})} \overline{p}$.

$\square$

Since $S_0(\mathcal{K}, \mathsf{id})$ and $S_1(\mathcal{K}, \mathsf{fi})$ both have theorems, they are candidates for the standard theory of algebraization. On the other hand, neither $S_*(\mathcal{K}, \mathsf{id})$ nor $S_*(\mathcal{K}, \mathsf{fi})$ have theorems, and except in the case that $\mathcal{K}$ is trivial, these logics cannot be protoalgebraic and hence cannot be algebraized. Consequently, they are potential candidates for our theory of *parametrized* algebraization, which we develop in Part V.

$\square$

8.2 Archologies

In the previous section we only considered a single logic, namely the canon. We now consider how a family of logics (such as the class of all subuniverse logics over the algebras of a quasivariety) containing a canon and such that each logic of the class models the canon, interacts with the ideal induced by the canon. We call such a class of logics an *archology*. Since the canon is a member of the archology, the canon is a model of itself. We shall see that it is, in fact, a *semantics* for itself, and consequently *structural*. So the ideal and form of the canon of an archology coincide, and this logic is always structural, and is finitary whenever the canon is finitary.

In $\oint$8.2.1, we introduce the notion of an archetypal subsignature of a signature $\mathfrak{s}$, which is essentially a full subconstruct of $\mathfrak{s}$ containing an $\mathfrak{s}$-canonical language. We demonstrate techniques for converting $\mathfrak{s}$-interpretations to interpretations in an archetypal subsignature and vice versa. Archologies are defined in $\oint$8.2.2 and a number of examples are considered in $\oint$8.2.3.

As we have noted before, in *some* cases, archologies can be viewed as π-institutions. It must be noted that the primary focus of an archology is its proto-typical nature; the archology is a family of non-sentential logics that serve as the defining model of a sentential logic we wish to induce. The multi-signature nature of an archology is of no concern to us. Further, there is *no* theory in *categorical abstract algebraic logic* (CAAL) that generalizes the definitions and results that we obtain in this chapter. Our theory of canons and archologies is entirely novel, both in *abstract algebraic logic* (AAL) and CAAL.

8.2.1 Archetypal Subsignatures

An *archetype* of a *signature* of logics is simply a full subconstruct of that signature that contains a global language. Our most important example of an archetype is the signature $\mathcal{K}$ (see Example 6.51 on page 236), where $\mathcal{K}$ is a quasivariety of $\mathfrak{a}$-algebras; the signature $\mathcal{K}$ is a full subconstruct of the signature $\mathfrak{a}$, and so, in the discourse of this section, $\mathcal{K}$ *is an* $\mathfrak{a}$-*archetype*.

Definition 8.76 (Archetypal Subsignatures) A subsignature $\mathfrak{t}$ of signature $\mathfrak{s}$ is called **archetypal** (or an **archetype of** $\mathfrak{s}$ or an $\mathfrak{s}$**-archetype**) if it is a full subconstruct of $\mathfrak{s}$ that contains an $\mathfrak{s}$-canonical language, denoted $\mathbf{F}_{\mathfrak{t}}$ and called the **canonical language**, which is $\mathfrak{t}$-free over $\{\overline{\mathsf{v}} \ : \ \mathsf{v} \in \mathsf{V}\}$. The global $\mathfrak{s}$-language is called the **root language**. Conventionally, unless specified to the contrary, we shall denote the global $\mathfrak{s}$-language by $\mathbf{G}$. □

Because we insist that an $\mathfrak{s}$-archetype $\mathfrak{t}$ be a *full* subconstruct of $\mathfrak{s}$, $\mathfrak{s}$-interpretations and $\mathfrak{t}$-interpretations, between languages of $\mathfrak{t}$, coincide, and so we may unambiguously speak of an interpretation when no other signatures are under consideration.

In the following definition we introduce mechanisms for converting interpretations from *the root language* $\mathbf{G}$ *into* $\mathfrak{t}$*-languages* to interpretations from the canonical language $\mathbf{F}_{\mathfrak{t}}$ into the same $\mathfrak{t}$-language, and vise versa.

Definition 8.77 (Converting Interpretations) Let $\mathfrak{t}$ be an archetype of $\mathfrak{s}$. For a language $\mathbf{A} \in \mathfrak{s}$ and an interpretation i of $\mathbf{G}$ into $\mathbf{A}$, let $\bar{\mathsf{i}}^{\mathbf{F}_{\mathfrak{t}}}$ denote the unique $\mathfrak{t}$ interpretation of $\mathbf{F}_{\mathfrak{t}}$ into $\mathbf{A}$ with $\bar{\mathsf{i}}^{\mathbf{F}_{\mathfrak{t}}}(\overline{\mathsf{v}}) = \mathsf{i}(\mathsf{v})$ for all variables $\mathsf{v} \in \mathsf{V}$. For any $\mathbf{A} \in \mathfrak{t}$ and $\mathfrak{t}$-interpretation $\mathsf{i} : \mathbf{F}_{\mathfrak{t}} \to \mathbf{A}$, let

$\underline{\mathsf{i}}_{\mathbf{G}}$ denote the $\mathfrak{s}$-interpretation of $\mathbf{G}$ into $\mathbf{A}$ mapping each $\mathsf{v} \in \mathbb{V}$ to $\mathsf{i}(\overline{\mathsf{v}})$. We tend to omit the superscripts $\mathbf{F}_{\mathfrak{t}}$ and $\mathbf{G}$ from these notions wherever unambiguous. $\square$

We remark on some important relationships between these conversion mechanisms and the canonical map from the root language to the canonical language.

Remark 8.78 $\bar{\mathsf{i}}(\overline{\phi}) = \mathsf{i}(\phi)$.

Proof. $\bar{\mathsf{i}}(\bar{\cdot})$ is a composition of $\mathfrak{s}$-morphisms and hence $\bar{\mathsf{i}}(\bar{\cdot}) : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$. Further, for all $\mathsf{v} \in \mathsf{V}$, $\bar{\mathsf{i}}(\overline{\mathsf{v}}) = \mathsf{i}(\mathsf{v})$. So $\bar{\mathsf{i}}(\bar{\cdot}) = \mathsf{i}$, by the $\mathfrak{s}$-freedom of $\mathbf{G}$. $\diamond$

Remark 8.79 $\underline{\mathsf{i}}(p) = \mathsf{i}(\overline{p})$.

Proof. $\underline{\mathsf{i}}(\cdot)$ and $\mathsf{i}\bar{\cdot}$ are both $\mathfrak{s}$-morphisms from $\mathbf{G}$ into $\mathbf{A}$. For each $\mathsf{v} \in \mathsf{V}$, $\underline{\mathsf{i}}(\mathsf{v}) = \mathsf{i}(\overline{\mathsf{v}}) = \mathsf{i}\overline{\mathsf{v}}$, and so $\underline{\mathsf{i}}(\cdot) = \mathsf{i}\bar{\cdot}$ by the $\mathfrak{s}$-freedom of $\mathbf{G}$. $\diamond$

In the following example we formalize our most important archetype.

Example 8.80 (Quasivarieties of Algebras)

> Let $\mathfrak{a}$ be a type of algebras and $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras. Recall that we view $\mathfrak{a}$ as the signature of all $\mathfrak{a}$-algebras with homomorphisms (see Example 6.50 on page 235) and we view $\mathcal{K}$ as the signature of all algebras in $\mathcal{K}$ and *all* homomorphisms between these algebras (see Example 6.51 on page 236). Consequently the signature $\mathcal{K}$ is a full subconstruct of the signature $\mathfrak{a}$; in fact $\mathcal{K}$ is an $\mathfrak{a}$-archetype with canonical language $\mathbf{F}_{\mathcal{K}}$, where $\mathbf{F}_{\mathcal{K}}$ is the $\mathcal{K}$ algebra over ω-free generators $\overline{[\mathsf{V}]}$. The root language is the term algebra.

$\square$

8.2.2 Archologies

Definition 8.81 (Archologies) An $\mathfrak{s}$-**archology** $\mathfrak{A}$ is determined by an $\mathfrak{s}$-archetype $\mathsf{sig}(\mathfrak{A})$ and a $\mathsf{sig}(\mathfrak{A})$-indexed set of $\mathsf{sig}(\mathfrak{A})$-logics $\{L_{\mathfrak{A}}(\mathbf{A}) \ : \ \mathbf{A} \in \mathsf{sig}(\mathfrak{A})\}$, such that, $\mathbf{lg}(L_{\mathfrak{A}}(\mathbf{A})) = \mathbf{A}$, for each $\mathbf{A} \in \mathsf{sig}(\mathfrak{A})$, and such that $\{L_{\mathfrak{A}}(\mathbf{A}) \ : \ \mathbf{A} \in \mathsf{sig}(\mathfrak{A})\}$ constitutes a $\mathsf{sig}(\mathfrak{A})$-model for $\mathsf{L}_{\mathbf{F}_{\mathsf{sig}(\mathfrak{A})}}$. We write $\mathbf{F}_{\mathfrak{A}}$ for $\mathbf{F}_{\mathsf{sig}(\mathfrak{A})}$, and conflate $\mathfrak{A}$ with $\{L_{\mathfrak{A}}(\mathbf{A}) \ : \ \mathbf{A} \in \mathsf{sig}(\mathfrak{A})\}$. We denote $\mathsf{L}_{\mathbf{F}_{\mathfrak{A}}}$ by $\overline{\mathfrak{A}}$, which we call the **canon**. Note that the canon $\overline{\mathfrak{A}}$ of an $\mathfrak{s}$-archology $\mathfrak{A}$ is an $\mathfrak{s}$-canon in the sense of $\oint$8.1 and so the earlier definition and results pertain. We denote the ideal $\overline{\mathfrak{A}}^{\imath}$ of the $\mathfrak{s}$-canon $\overline{\mathfrak{A}}$ by $\underline{\mathfrak{A}}$, which we call the **ideal**. An archology is called **canon-finitary** if $\overline{\mathfrak{A}}$ is finitary, and is called **finitary** if every logic in $\mathfrak{A}$ is finitary.

Since the signature $\mathsf{sig}(\mathfrak{A})$, of an archology $\mathfrak{A}$, is $\mathfrak{s}$-archetypal, we may unambiguously ignore the distinction between $\mathsf{sig}(\mathfrak{A})$ based notions and $\mathfrak{s}$ based notions when dealing with logics *in* $\mathfrak{A}$ and their languages. For example, a logic in $\mathfrak{A}$ is $\mathfrak{s}$-structural iff it is $\mathsf{sig}(\mathfrak{A})$-structural. So we tend to drop reference to the signatures wherever unambiguous. For logics and languages outside of the archology, such as $\underline{\mathfrak{A}}$, dropped signatures can only refer to $\mathfrak{s}$. $\square$

The reader is urged to take caution in distinguishing between $\overline{\mathfrak{A}}^{\imath}$ and $\mathfrak{A}^{\imath}$; the former is the abstraction of the single logic $\overline{\mathfrak{A}}$ (and by definition is denoted $\underline{\mathfrak{A}}$), while the latter is the abstraction of the entire class of logics making up the archology (by our convention of identifying this class with the symbol denoting the archology). As we shall now see, $\overline{\mathfrak{A}} \doteq \overline{\mathfrak{A}}^{\imath}$ and $\mathfrak{A}^{\imath}$ coincide.

Observe that while we only demand that an archology $\mathfrak{A}$ constitute a *model* of its canon $\overline{\mathfrak{A}}$, it is in fact a *semantics* for its canon: since $\overline{\mathfrak{A}} \in \mathfrak{A}$, it follows (by Lemma 7.39 on page 261) that $\mathfrak{A}^{\imath} \preceq \overline{\mathfrak{A}}$; conversely, since $\mathfrak{A}$ constitutes a model of $\overline{\mathfrak{A}}$, it follows (by Theorem 7.40 on page 261) that $\overline{\mathfrak{A}} \preceq \mathfrak{A}^{\imath}$. Consequently $\overline{\mathfrak{A}} = \mathfrak{A}^{\imath}$. We formalize this observation for ease of future reference.

Corollary 8.82 $\overline{\mathfrak{A}} = \mathfrak{A}^{\imath}$ and $\mathfrak{A}$ is a semantics for $\overline{\mathfrak{A}}$. □

Observe further that since $\mathbf{F}_{\mathfrak{A}} \in \mathsf{sig}(\mathfrak{A})$, the canon $\overline{\mathfrak{A}}$ is a model of itself, and so $\overline{\mathfrak{A}}$ is structural (by Theorem 7.43 on page 262). We record this observation and some consequents that follow from Theorem 8.13, Theorem 8.9 and Theorem 8.37.

Theorem 8.83 Let $\mathfrak{A}$ be an archology.

1. The canon $\overline{\mathfrak{A}}$ is ideal, i.e., $\underline{\mathfrak{A}} = \underline{(\overline{\mathfrak{A}})}$.

2. $\underline{\mathfrak{A}}$ is structural.

3. $\underline{\mathfrak{A}}$ is finitary iff $\mathfrak{A}$ is canon-finitary, in which case $\underline{\mathfrak{A}}$ is a *propositional* $\mathfrak{s}$-calculus.

4. $\overline{\mathfrak{A}}$ is a semantics for $\underline{\mathfrak{A}}$.

5. $\Gamma \vdash_{\underline{\mathfrak{A}}} \phi$ iff $\overline{[\Gamma]} \vdash_{\overline{\mathfrak{A}}} \overline{\phi}$.

6. $\mathsf{Th}(\overline{\mathfrak{A}}) = \mathsf{Fi}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{F}_{\mathfrak{A}})$, i.e., $\overline{\mathfrak{A}} = \mathsf{F}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{F}_{\mathfrak{A}})$.

□

We now show that the archology $\mathfrak{A}$ is also a semantics for its ideal $\underline{\mathfrak{A}}$. Since the canon $\overline{\mathfrak{A}}$ is a semantics for the ideal $\underline{\mathfrak{A}}$ and $\overline{\mathfrak{A}} \in \mathfrak{A}$, it suffices, by Lemma 7.60 on page 265, to show that the archology $\mathfrak{A}$ constitutes a model of the ideal $\underline{\mathfrak{A}}$.

Theorem 8.84 The archology $\mathfrak{A}$ is a semantics for its ideal $\underline{\mathfrak{A}}$.

Proof. Since $\overline{\mathfrak{A}}$ is a semantics for $\underline{\mathfrak{A}}$, by Theorem 8.83 on page 300, and $\overline{\mathfrak{A}} \in \mathfrak{A}$, it suffices by Lemma 7.60 on page 265, to show that $\mathfrak{A}$ constitutes a model of $\underline{\mathfrak{A}}$. Let $\mathsf{L} \in \mathfrak{A}$. Suppose that $\Gamma \vdash_{\underline{\mathfrak{A}}} \phi$ and that i is an $\mathfrak{s}$-interpretation of $\underline{\mathfrak{A}}$ in L. Let $T \in \mathsf{Th}(\mathsf{L})$ with $\mathsf{i}[\Gamma] \subseteq T$. (*We must show that* $\mathsf{i}(p) \in T$.) Let $\bar{\mathsf{i}}$ be the $\mathfrak{s}$-interpretation of $\overline{\mathfrak{A}}$ in L given in Definition 8.77 on page 298. By Remark 8.78 on page 299, $\bar{\mathsf{i}}[\overline{\Gamma}] = \mathsf{i}[\Gamma]$ and $\bar{\mathsf{i}}(\overline{\phi}) = \mathsf{i}(\phi)$. Since $\Gamma \vdash_{\underline{\mathfrak{A}}} \phi$, $\overline{\Gamma} \vdash_{\overline{\mathfrak{A}}} \overline{[\phi]}$, and since T is a L-theory, L is an $\mathfrak{s}$-model of $\overline{\mathfrak{A}}$ and $\bar{\mathsf{i}}[\overline{\Gamma}] = \mathsf{i}[\Gamma] \subseteq T$, $\mathsf{i}(\phi) = \bar{\mathsf{i}}(\overline{\phi}) \in T$. ◇

By a similar argument, we show that $\underline{\mathfrak{A}}$-filters and $\overline{\mathfrak{A}}$-filters coincide for languages in $\mathsf{sig}(\mathfrak{A})$.

Proposition 8.85 For all $\mathbf{A} \in \mathsf{sig}(\mathfrak{A})$, $\mathsf{Fi}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{A}) = \mathsf{Fi}^{\mathsf{sig}(\mathfrak{A})}_{\overline{\mathfrak{A}}}(\mathbf{A})$.

Proof. $\boxed{\mathbf{Fi}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{A}) \subseteq \mathbf{Fi}^{\mathsf{sig}(\mathfrak{A})}_{\overline{\mathfrak{A}}}(\mathbf{A})}$ Let $F \in \mathbf{Fi}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{A})$. Suppose that $\overline{[\Gamma]} \vdash_{\overline{\mathfrak{A}}} \overline{\phi}$ and $\mathsf{i} : \mathbf{F}_{\mathfrak{A}} :\to_{\mathsf{sig}(\mathfrak{A})} \mathbf{A}$ with $\mathsf{i}\left[\overline{[\Gamma]}\right] \subseteq F$. By (5) of Theorem 8.83, $\Gamma \vdash_{\underline{\mathfrak{A}}} \phi$. Let $\underline{\mathsf{i}}$ be the $\mathfrak{s}$-interpretation of $\mathbf{G}$ into $\mathbf{A}$, as defined in Definition 8.77. By Remark 8.79, $\underline{\mathsf{i}}\,[\Gamma] = \mathsf{i}\left[\overline{[\Gamma]}\right] \subseteq F$, and hence $\mathsf{i}(\overline{\phi}) = \underline{\mathsf{i}}(\phi) \in F$. $\boxed{\mathbf{Fi}^{\mathsf{sig}(\mathfrak{A})}_{\overline{\mathfrak{A}}}(\mathbf{A}) \subseteq \mathbf{Fi}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{A})}$

Let $F \in \mathbf{Fi}^{\mathsf{sig}(\mathfrak{A})}_{\overline{\mathfrak{A}}}(\mathbf{A})$. Suppose that $\Gamma \vdash_{\underline{\mathfrak{A}}} \phi$ and $\mathsf{i} : \mathbf{G} : \to_{\mathfrak{s}} \mathbf{A}$ with $\mathsf{i}\,[\Gamma] \subseteq F$. By (5) of Theorem 8.83, $\overline{[\Gamma]} \vdash_{\underline{\mathfrak{A}}} \overline{\phi}$. Let $\bar{\mathsf{i}}$ be the $\mathfrak{s}$-interpretation of $\mathbf{F}_{\mathfrak{A}}$ into $\mathbf{A}$, as defined in Definition 8.77. By Remark 8.78, $\bar{\mathsf{i}}\left[\overline{[\Gamma]}\right] = \mathsf{i}\,[\Gamma] \subseteq F$, and hence $\mathsf{i}(\phi) = \bar{\mathsf{i}}(\overline{\phi}) \in F$. $\diamond$

For each $\mathsf{sig}(\mathfrak{A})$-language $\mathbf{A}$, since $L_{\mathfrak{A}}(\mathbf{A})$ is a model of $\overline{\mathfrak{A}}$ (by the definition of an archology), $\mathsf{Th}(L_{\mathfrak{A}}(\mathbf{A})) \subseteq \mathsf{Fi}^{\mathsf{sig}(\mathfrak{A})}_{\overline{\mathfrak{A}}}(\mathbf{A}) \doteq \mathsf{Fi}^{\mathsf{sig}(\mathfrak{A})}_{\overline{\mathfrak{A}}}(\mathbf{A})$; so by the previous result, $\mathsf{Th}(L_{\mathfrak{A}}(\mathbf{A})) \subseteq \mathsf{Fi}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{A})$.

Corollary 8.86 For each $\mathsf{sig}(\mathfrak{A})$-language $\mathbf{A}$, $\mathsf{Th}(L_{\mathfrak{A}}(\mathbf{A})) \subseteq \mathsf{Fi}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{A})$, i.e., $\mathsf{F}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{A}) \preceq L_{\mathfrak{A}}(\mathbf{A})$. $\square$

As we noted above, for each $\mathsf{sig}(\mathfrak{A})$-language $\mathbf{A}$, $\mathsf{Th}(L_{\mathfrak{A}}(\mathbf{A})) \subseteq \mathsf{Fi}^{\mathsf{sig}(\mathfrak{A})}_{\overline{\mathfrak{A}}}(\mathbf{A})$. We shall call an archology maximal if the inclusion is always an equality.

Definition 8.87 (Maximal Archologies) An archology $\mathfrak{A}$ is called **maximal** if, for each $\mathsf{sig}(\mathfrak{A})$-language $\mathbf{A}$, $\mathsf{Fi}^{\mathsf{sig}(\mathfrak{A})}_{\overline{\mathfrak{A}}}(\mathbf{A}) = L_{\mathfrak{A}}(\mathbf{A})$. $\square$

While the notion of maximality is expressed in terms of the archology only, by Corollary 8.86 we may express this condition 'globally'.

Corollary 8.88 $\mathfrak{A}$ is maximal iff $\mathsf{Fi}^{\mathfrak{s}}_{\underline{\mathfrak{A}}}(\mathbf{A}) = \mathsf{Th}(L_{\mathfrak{A}}(\mathbf{A}))$, for each $\mathsf{sig}(\mathfrak{A})$-language $\mathbf{A}$.

8.2.3 Examples

In the following example we consider the archology obtained by considering the subuniverse logics of a quasivariety.

Example 8.89 (Archologies of Subuniverses of a Quasivariety)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras. In the following definition we define the archology of all subuniverse logics on algebras of a quasivariety $\mathcal{K}$. That this definition indeed well-defines an archology follows from Remark 7.78 of Example 7.77 on page 270.

Definition 8.90 (The Archology $\mathfrak{A}(\mathcal{K}, \mathsf{su})$) For a quasivariety of $\mathfrak{a}$-algebras let $\mathfrak{A}(\mathcal{K}, \mathsf{su})$ denote the archology $\{U(\mathbf{A}, \mathsf{su}) : \mathbf{A} \in \mathcal{K}\}$. The canon is $\boldsymbol{S}(\mathcal{K}, \mathsf{su})$. $\square$

Remark 8.91 The ideal $\underline{\mathfrak{A}(\mathcal{K}, \mathsf{su})} = S(\mathcal{K}, \mathsf{su})$, by Example 8.51. $\square$

For each each $\mathbf{A} \in \mathcal{K}$, $\mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{A}) = \mathsf{Su}(\mathbf{A}) = \mathsf{Th}(F(\mathbf{A}, \mathsf{su}))$, by Proposition 7.82 of Example 7.77, and by Proposition 8.85, $\mathsf{Fi}^{\mathfrak{a}}_{S(\mathcal{K},\mathsf{su})}(\mathbf{A}) = \mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}(\mathbf{A})$; consequently $\mathsf{Fi}^{\mathfrak{a}}_{S(\mathcal{K},\mathsf{su})}(\mathbf{A}) = \mathsf{Th}(F(\mathbf{A}, \mathsf{su}))$. So $\mathfrak{A}(\mathcal{K}, \mathsf{su})$ is maximal.

Proposition 8.92 The archology $\mathfrak{A}(\mathcal{K}, \mathsf{su})$ is maximal.

$\square$

Next we consider the archology determined by the universal logics of $\mathcal{K}$-cosets on algebras of $\mathcal{K}$. We employ the machinery developed in this section, to show that if $\mathcal{K}$ is a *variety*, then this archology is *maximal*. As a consequence, we shall see that for a *variety* $\mathcal{K}$, the filters of the *membership logic* on algebras in $\mathcal{K}$ are *precisely* the $\mathcal{K}$-cosets on those algebras.

Example 8.93 (The Archology of Relative Cosets)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras. By Proposition 7.86 of Example 7.83 on page 271, the universal logics $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$, for $\mathbf{A} \in \mathcal{K}$, determine an $\mathfrak{a}$-archology with archetype $\mathcal{K}$.

Definition 8.94 (The Archology of Relative Cosets) Let $\mathfrak{A}(\mathsf{cos}^{\mathcal{K}})$ denote the $\mathfrak{a}$-archology with archetype $\mathcal{K}$ determined by universal logics $\{U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}}) : \mathbf{A} \in \mathcal{K}\}$, which we call the **archology of $\mathcal{K}$-cosets**. □

The canon of this archology is $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})$ and, by Theorem 8.61 on page 294, the ideal is the membership logic $S(\mathcal{K}, \mathsf{mem})$. We shall now show that if $\mathcal{K}$ is a *variety*, then this archology is maximal, in other words, the *filters of the membership logic* and the *$\mathcal{K}$-cosets* coincide for algebras in $\mathcal{K}$.

Corollary 8.95 If $\mathcal{K}$ is a *variety*, then the archology $\mathfrak{A}(\mathsf{cos}^{\mathcal{K}})$ is maximal.

Proof. If $\mathcal{K}$ is trivial, then the result follows trivially. Suppose that $\mathcal{K}$ is a non-trivial variety. Then $\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}}) = \boldsymbol{S}(\mathcal{K}, \mathsf{nr\text{-}cos})$. Let $\mathbf{A} \in \mathcal{K}$. By Proposition 8.85, $\mathsf{Fi}^{\mathfrak{a}}_{S(\mathcal{K},\mathsf{mem})}(\mathbf{A}) = \mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}(\mathsf{cos}^{\mathcal{K}})}(\mathbf{A}) = \mathsf{Fi}^{\mathcal{K}}_{\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})}(\mathbf{A}) \overset{\text{(i)}}{=} \mathsf{Fi}^{\mathfrak{a}}_{\boldsymbol{S}(\mathcal{K},\mathsf{nr\text{-}cos})}(\mathbf{A}) \overset{\text{(ii)}}{=} \mathsf{Cos}(\mathbf{A}) \overset{\text{(iii)}}{=} \mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$, where (i) follows since the signature $\mathcal{K}$ is a full subconstruct of the signature $\mathfrak{a}$, (ii) follows by Theorem 7.88 of Example 7.83 on page 271 and (iii) follows since $\mathcal{K}$ is a non-trivial variety. ◇

Consequently, if $\mathcal{K}$ is a variety, then the filters of the membership logic on algebras in $\mathcal{K}$ are precisely the $\mathcal{K}$-cosets on those algebras.

Corollary 8.96 If $\mathcal{K}$ is a variety, then for each $\mathbf{A} \in \mathcal{K}$, $\mathsf{Fi}^{\mathfrak{a}}_{S(\mathcal{K},\mathsf{mem})}(\mathbf{A}) = \mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$.

Open Problem 8.97 Is the previous result still valid for arbitrary $\mathfrak{a}$-algebras?

□

We briefly consider the archologies of logics of lattice ideal and filter logics of a quasivariety of lattice expansions.

Example 8.98 (The Archologies of Logics of Lattice Ideals and Filters)

That the following well-define archologies, follows from Corollary 7.94 of Example 7.89 on page 272.

Definition 8.99 (The Archologies of Lattice Ideals and Filters) For a quasivariety $\mathcal{K}$ of lower-unbounded lattices (resp. upper-unbounded lattices, 0-lattices and 1-lattices), let $\mathfrak{A}_*(\mathcal{K}, \mathsf{id}) = \{U(\mathbf{P}, \mathsf{id}_\emptyset) : \mathbf{P} \in \mathcal{K}\}$ (resp. $\mathfrak{A}_*(\mathcal{K}, \mathsf{fi}) = \{U(\mathbf{P}, \mathsf{fi}_\emptyset) : \mathbf{P} \in \mathcal{K}\}$, $\mathfrak{A}_0(\mathcal{K}, \mathsf{id}) = \{U(\mathbf{P}, \mathsf{id}) : \mathbf{P} \in \mathcal{K}\}$ and $\mathfrak{A}_1(\mathcal{K}, \mathsf{fi}) = \{U(\mathbf{P}, \mathsf{fi}) : \mathbf{P} \in \mathcal{K}\}$), which constitutes a $\mathsf{type}(\mathcal{K})$-archology with canon $\boldsymbol{S}_*(\mathcal{K}, \mathsf{id})$ (resp. $\boldsymbol{S}_*(\mathcal{K}, \mathsf{fi})$, $\boldsymbol{S}_0(\mathcal{K}, \mathsf{id})$ and $\boldsymbol{S}_1(\mathcal{K}, \mathsf{fi})$). Conventionally, use of these notions shall imply the type of the quasivariety $\mathcal{K}$. □

Corollary 7.94 in fact says more.

Remark 8.100 $\mathfrak{A}_*(\mathcal{K}, \mathsf{id})$, $\mathfrak{A}_*(\mathcal{K}, \mathsf{fi})$, $\mathfrak{A}_0(\mathcal{K}, \mathsf{id})$ and $\mathfrak{A}_1(\mathcal{K}, \mathsf{fi})$ are all maximal archologies. □

The associated ideals of $\mathfrak{A}_*(\mathcal{K}, \mathsf{id})$, $\mathfrak{A}_*(\mathcal{K}, \mathsf{fi})$, $\mathfrak{A}_0(\mathcal{K}, \mathsf{id})$ and $\mathfrak{A}_1(\mathcal{K}, \mathsf{fi})$ are $S_*(\mathcal{K}, \mathsf{id})$, $S_*(\mathcal{K}, \mathsf{fi})$, $S_0(\mathcal{K}, \mathsf{id})$ and $S_1(\mathcal{K}, \mathsf{fi})$, respectively.

Corollary 8.101 For any $\mathbf{P} \in \mathcal{K}$, $\mathsf{Fi}_{S_*(\mathcal{K},\mathsf{id})}(\mathbf{P}) = \mathsf{Id}_\Diamond(\mathbf{P})$, $\mathsf{Fi}_{S_*(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = \mathsf{Fl}_\Diamond(\mathbf{P})$, $\mathsf{Fi}_{S_0(\mathcal{K},\mathsf{id})}(\mathbf{P}) = \mathsf{Id}_\Diamond(\mathbf{P})$ and $\mathsf{Fi}_{S_1(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = \mathsf{Fl}_\Diamond(\mathbf{P})$.

Open Problem 8.102 Characterize the $S_*(\mathcal{K}, \mathsf{id})$-filters on arbitrary $\mathsf{type}(\mathcal{K})$-algebras, etc. For a $\mathsf{type}(\mathcal{K})$-algebra $\mathbf{A}$, when is $\perp_\mathcal{K}$ compatible with all $S_*(\mathcal{K}, \mathsf{id})$-filters on $\mathbf{A}$, etc. How does this relate to protoalgebraicity, in the cases of $\mathsf{Fi}_{S_0(\mathcal{K},\mathsf{id})}(\mathbf{P}) = \mathsf{Id}_\Diamond(\mathbf{P})$ and $\mathsf{Fi}_{S_1(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = \mathsf{Fl}_\Diamond(\mathbf{P})$, and to parametrized protoalgebraicity in the cases of $\mathsf{Fi}_{S_*(\mathcal{K},\mathsf{id})}(\mathbf{P}) = \mathsf{Id}_\Diamond(\mathbf{P})$ and $\mathsf{Fi}_{S_*(\mathcal{K},\mathsf{fi})}(\mathbf{P}) = \mathsf{Fl}_\Diamond(\mathbf{P})$?

$\square$

Part IV

Regularity, Coherence and the Logics of Solutions to Equations

In part IV, we consider *four* interrelated subjects, the problem of *solving systems of equations*, the *sentential calculi arising from solutions to systems of equations*, *regularity* and *coherence*, and their relationship with *algebraizable logics*.

In [BR99] it was shown that, if a sentential 1-calculus has an equivalent algebraic semantics $\mathcal{K}$ with defining equations τ (see Definition 2.110 on page 111 of our text), where $\mathcal{K}$ is a quasivariety of algebras, then this logic is equivalent to the logic $S(\mathcal{K}, \tau)$ (see Example 2.85 on page 106 of our text, noting that defining equations τ in the case of sentential *one*-calculi are *unary systems of equations*), in which case $\mathcal{K}$ satisfies the condition that for any algebra $\mathbf{A}$ and any two $\mathcal{K}$-congruences α and β on $\mathbf{A}$, if the solutions to τ (as a unary system of equations) over $\mathbf{A}$ modulo α and the solutions to τ over $\mathbf{A}$ modulo β coincide, then α and β are equal. This condition was termed $\langle \mathcal{K}, \tau \rangle$*-regularity* in [BR99], since this condition encompasses a previously existing notion in universal algebra, known as *relative point regularity* (see Definition 1.375 on page 71), that is, if the unary system $\tau(x) = \{\langle x, 0 \rangle\}$, where 0 is an equationally definable constant term, then the condition of $\langle \mathcal{K}, \tau \rangle$-regularity coincides with the condition of relative point regularity at 0. They also showed that conversely, if $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-regular, then $S(\mathcal{K}, \tau)$ is algebraizable with (unique) equivalent algebraic semantics $\mathcal{K}$. Since the algebraizability of a sentential calculus is characterizable in *purely logical* terms (see Theorem 2.119 on page 113 of our text), the purely universal algebraic condition of $\langle \mathcal{K}, \tau \rangle$-regularity is characterizable in terms of purely logical conditions on $S(\mathcal{K}, \tau)$. Since point regularity is encompassed by $\langle \mathcal{K}, \tau \rangle$-regularity, Blok and Raftery have succeeded in bringing the (already existing) universal algebraic notion of point regularity into the domain of logic.

The same cannot be said for *full congruence regularity* nor *relative congruence regularity* (see Definition 1.359 on page 68 and Definition 1.375 on page 71). As we demonstrated in Counter Example 3.1 on page 124, the *fully congruence regular* variety of quasigroups can never be the equivalent algebraic semantics of *any* sentential 1-calculi. This not withstanding, in this text we shall present a logic called the *membership logic* of a quasivariety $\mathcal{K}$, and show that the condition that $\mathcal{K}$ be relatively regular (and hence that a *variety* $\mathcal{K}$ be *fully regular*) may be characterized in terms of purely logical conditions on its membership logic; these conditions, however, in the light of the aforementioned example, cannot be conditions characterizing the algebraizability of the membership logic. The primary aim of the theory of *parametrized algebraization*, developed in Part V of this text, is to unify the theory of relative regularity (and its relation to the membership logic) with the theory of $\langle \mathcal{K}, \tau \rangle$-regularity (and its relation to algebraizable logics).

Given that all algebraizable sentential 1-calculi are equivalent to a logic $S(\mathcal{K}, \tau)$ and that all algebraizable logics must be protoalgebraic [BP89a], the problem of characterizing the protoalgebraicity of $S(\mathcal{K}, \tau)$ is an important question in algebraic logic. Since the protoalgebraicity of a logic is characterizable in terms of the existence of formulae and the satisfaction of deductions involving those formulae (see Theorem 2.119 on page 113), and since $\mathcal{K}$ is always an algebraic semantics for $S(\mathcal{K}, \tau)$ (see Theorem 2.121 on page 114), the protoalgebraicity of $S(\mathcal{K}, \tau)$ is characterizable, in purely universal algebraic terms, as a *quasi-Mal'cev* condition satisfied by $\mathcal{K}$. In [BR99], 'concrete meaning' is given to this quasi-Mal'cev condition. The condition that an algebra *has coherent* $\langle \mathcal{K}, \tau \rangle$*-classes* was introduced in [BR99], and it was shown that $S(\mathcal{K}, \tau)$ is protoalgebraic iff the algebras of $\mathcal{K}$ (equivalently all algebras) have coherent $\langle \mathcal{K}, \tau \rangle$-classes. The use of the term 'coherence' was justified, in that, while the condition of having coherent $\langle \mathcal{K}, \tau \rangle$-classes does not encompass the already existing condition of having *coherent congruences classes* [Dud89], it is similar in spirit.

In this text, we shall generalize the theory developed in [BR99] on two fronts. In the *simpler* of the two generalizations, which is developed for the sake of *completeness* and its presentation is restricted to Part IV, we extend the arguments of [BR99] from 1-sentential calculi to n-sentential calculi more generally. The second of these generalizations involves extending the theory in a manner that unifies *both* the relationship between $\langle\mathcal{K},\tau\rangle$*-regularity* and the *algebraizability of* $S(\mathcal{K},\tau)$, and the relationship between *relative regularity* and the characterization of this condition as *logical properties satisfied by the membership logic*; we shall refer to this program as the **unification program**. This program concerns only sentential *one*-calculi. While the development of this program is begun in Part IV, it is only concluded in Part V. There are four key steps in this program. In Step 1, which is undertaken in Part IV, we find a notion of regularity that encompasses *both* $\langle\mathcal{K},\tau\rangle$-regularity and (full) $\mathcal{K}$-regularity. To this end, we introduce the notion of $\langle\mathcal{K},\mathfrak{B}_*\rangle$*-regularity*, where $\mathfrak{B}$ is a system of equations in *two* variables, called a *binary system of equations*, modelled as a finite set of pairs of terms in two variables. We may view a *unary* system of equations τ as a special *binary* system in which the second variable does not occur; in this case our condition of $\langle\mathcal{K},\tau_*\rangle$-regularity coincides with the condition of $\langle\mathcal{K},\tau\rangle$-regularity of [BR99]. Further, if $\mathfrak{B}=\{\langle x,y\rangle\}$, then $\langle\mathcal{K},\mathfrak{B}_*\rangle$-regularity and $\mathcal{K}$-regularity coincide. In Step 2, we introduce the sentential 1-calculus $S(\mathcal{K},\mathfrak{B}_*)$, where $\mathfrak{B}$ is a binary system of equations. When a unary system τ is viewed as a binary system with only one variable, $S(\mathcal{K},\mathfrak{B}_*)$ and $S(\mathcal{K},\tau)$ coincide, and in the case that $\mathfrak{B}=\{\langle x,y\rangle\}$, $S(\mathcal{K},\mathfrak{B}_*)$ is the *membership logic* mentioned above. Step 3 involves the development of a theory of parameterized algebraization. In one of the equivalent formulations of this notion, the parameter is a binary system of equations $\mathfrak{B}$; we speak of a sentential 1-calculus having a $\mathfrak{B}_*$-equivalent semantics $\mathcal{K}$ and of being $\mathfrak{B}_*$-algebraizable. When this binary system is taken to be a unary system, our parameterized theory coincides with the standard theory of [BP89a]. The final step involves establishing relationships between $\mathfrak{B}_*$-algebraizable logics, the $\mathfrak{B}_*$-algebraizability of the logic $S(\mathcal{K},\mathfrak{B}_*)$ and the $\langle\mathcal{K},\mathfrak{B}_*\rangle$-regularity of $\mathcal{K}$, and to do so in a manner that specializes to the theory of [BR99] and characterizes $\mathcal{K}$-regularity in terms of the $\mathfrak{B}_*$-algebraizability of the membership logic. We shall have need to also develop a theory of parameterized protoalgebraicity, which in turn raises the question of characterizing the parameterized protoalgebraicity of the logics $S(\mathcal{K},\mathfrak{B}_*)$ in terms of some concrete property satisfied by the algebras of $\mathcal{K}$; this shall turn out to be a generalization of the condition of having coherent $\langle\mathcal{K},\tau\rangle$-classes analogous to the generalization of $\langle\mathcal{K},\tau\rangle$-regularity by $\langle\mathcal{K},\mathfrak{B}_*\rangle$-regularity, which we term *having coherent* $\langle\mathcal{K},\mathfrak{B}_*\rangle$*-classes*; this condition is introduced and characterized in Part IV.

We shall now briefly outline the contents of Part IV.

In $\oint 9$ we consider *systems of equations* and their *solutions* over an algebra modulo a quasivariety of algebras. We first consider the general problem of solving a system of equations in n-variables. The solutions to such a system are sets of n-tuples of elements from the given algebra. Such solutions naturally give rise to sentential n-calculi; we introduce the sentential n-calculus $S^n(\mathcal{K},\mathfrak{N})$, determined by a quasivariety $\mathcal{K}$ and an n-ary system of equations $\mathfrak{N}$, which specializes to the logic $S(\mathcal{K},\tau)$ [BR99] (in the case that $n=1$). In this chapter, we also consider a framework for solving systems of equations in 2 variables by '*fixing*' one of the variables. The solutions obtained via this framework are sets of *points* from the algebra, as opposed to sets of *pairs of points*, as would be obtained by the general technique. Solving *binary* systems in this manner gives rise to sentential 1-calculi. We discuss these calculi in Part V. It should be noted, however, that the sentential 1-calculi that arise in this manner are generally 'inherently unalgebraizable' in the sense of [BP89a], primarily because they generally have *no theorems*. As such they cannot, except in the trivial case, be *protoalgebraic*, with protoalgebraicity being a precondition for algebraization. In Part V of this text, we shall introduce a more general notion of protoalgebraicity and algebraization, that encompasses the standard theory as a special case, and which can be usefully applied to logics without theorems.

In $\oint 10$ we consider two generalizations of $\langle \mathcal{K}, \tau \rangle$-regularity. The first generalization, developed for the sake of completeness, extends the unary system τ with a finite number of n-ary systems of equations. This notion is called $\langle \mathcal{K}, \langle \mathfrak{N}_1, \ldots \mathfrak{N}_m \rangle \rangle$-*regularity*; this condition coincides with $\langle \mathcal{K}, \tau \rangle$-regularity in the case that $n = 1$ and $m = 1$. In the case that $m = 1$, we speak of $\langle \mathcal{K}, \mathfrak{N} \rangle$-regularity. We prove that the sentential n-calculus $S^n(\mathcal{K}, \mathfrak{N})$ is *algebraizable* iff $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N} \rangle$-regular. The second generalization, developed as part of the first step of our unification program, is called $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity (mentioned earlier), where $\mathfrak{B}$ is a binary system of equations; $\langle \mathcal{K}, \mathfrak{B} \rangle$-regularity and $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity are distinct conditions. In Part V, we shall explore the relationship between a quasivariety $\mathcal{K}$ being $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regular and the (parameterized) $\mathfrak{B}_*$-algebraizability of the sentential 1-calculus $S(\mathcal{K}, \mathfrak{B}_*)$, which is introduced in that part.

In $\oint 11$ we consider the analogous generalizations of the condition that an algebra have *coherent* $\langle \mathcal{K}, \tau \rangle$-*classes*. For completeness we consider algebras *having coherent* $\langle \mathcal{K}, \mathfrak{N} \rangle$-*classes*, where $\mathfrak{N}$ is an n-ary system of equations. While it is indeed true that $S^n(\mathcal{K}, \mathfrak{N})$ is *protoalgebraic* iff all algebras in $\mathcal{K}$ have coherent $\langle \mathcal{K}, \mathfrak{N} \rangle$-classes, the proof of this result is delayed to Part VI, where it obtains easily from our new (and simpler) characterization of protoalgebraic sentential n-calculi obtained in that part as a consequence of our theory of protoalgebraic logics over constructs (see Corollary 16.41 of Example 16.40 on page 454).

Chapter 9

Solving Systems of Equations

A primary task in algebra is solving systems of equations. In this chapter we consider *finite systems of equations* over algebras, and the *solutions* of such systems over an algebra modulo a quasivariety of algebras. Systems of equations and their solutions modulo a quasivariety play an important role in the theory of algebraic logic. Not only do such systems provide examples of logics, but in fact, this family of logics includes *all* algebraizable logics. Generally, a system of equations in n-variables gives rise to an n-deductive systems in the sense of Blok and Pigozzi. We consider such systems in $\oint$9.1 and their associated logics in $\oint$9.1.2.

In $\oint$9.2, we consider a framework for solving *binary systems* of equations (i.e., systems in two variables) *parametrically* by '*fixing one of the variables*', essentially converting the binary system into a *unary system*. Key to solving binary systems in this manner, is the notion of a *pivot*, and the associated notion that the system under consideration *pivots*. This framework shall allow us to associate sentential 1-calculi with binary systems of equations (see $\oint$12). Generally these logics have no theorems and hence are generally 'inherently unalgebraizable'. Part V of this text is devoted to a *parametrized* algebraization technique for algebraizing such logics.

9.1 Solving Systems of Equations

We begin by considering finite systems of equations in n-variables, and the solutions to such systems modulo a quasivariety of algebras. In the next section, we shall consider solving systems of equations in two variables *parametrically*.

9.1.1 Systems of Equations, Instantiation and Solution

Recall the notion of a (concrete) translation between sets, given in Definition 5.17 on page 180.

Definition 9.1 (Systems of Equations) Let $\mathfrak{a}$ be a type of algebras and $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras. By a **system of $\mathfrak{a}$-equations**, or just a **system of equations** or even just a **system** where unambiguous, we mean a non-empty finite set $\mathfrak{N}$ of pairs of $\mathfrak{a}$-terms. Informally, we are identifying formal equations with pairs of terms. Formally, we define a translation $\mathfrak{N}^{\approx}$ from Tm^n to $\mathsf{Identity}(\mathfrak{a})$, determined by

$$\mathfrak{N}^{\approx}[\![\langle p_1,\ldots,p_n\rangle]\!] = \{\delta(p_1,\ldots,p_n) \approx \epsilon(p_1,\ldots,p_n) \;:\; \langle\delta,\epsilon\rangle \in \mathfrak{N}\}.$$

By the **terms of a system** we mean the terms in any of the pairs of the system. By the **variables of a system**, we mean the variables occurring in any of the terms of the system. We may assume that the variables of a system occur in all the terms of the system. Let $\mathsf{dim}(\mathfrak{N})$ denote the cardinality of the variables of $\mathfrak{N}$, which is always finite. We call $\mathsf{dim}(\mathfrak{N})$ the **dimension** of the system. When we speak of an n**-ary system of equations**, we mean a system of equations with dimension n. By a **unary system of equations** (resp. **binary system of equations**) we mean a system of equations with dimension 1 (resp. 2). By the **cardinality of a system**, we mean the number of pairs in that system. When we say that $\mathfrak{N}(x_1, \ldots, x_n)$ is a system of equations, we mean that the variables of the system are $\{x_1, \ldots, x_n\}$, and we are implying an ordering of the variables so as to facilitate instantiation of the variables of the system. Given n-ary systems $\mathfrak{N}$ and $\mathfrak{N}'$, we shall say that $\mathfrak{N}$ and $\mathfrak{N}'$ are $\mathcal{K}$**-equivalent**, if, for distinct variables $x_1, \ldots, x_n$, $\mathfrak{N}^{\approx}[\![\langle x_1, \ldots, x_n\rangle]\!] \dashv\models_{\mathcal{K}} \mathfrak{N}'^{\approx}[\![\langle x_1, \ldots, x_n\rangle]\!]$.

Let $\mathbf{A}$ be an $\mathfrak{a}$-algebra not necessarily in $\mathcal{K}$ and $\mathfrak{N}(x_1, \ldots, x_n)$ an n-system of equations. Let $\mathfrak{N}^{\mathbf{A}}$ denote the translation from $\mathsf{uni}(\mathbf{A})^n$ to $\mathsf{uni}(\mathbf{A})^2$, defined by

$$\mathfrak{N}^{\mathbf{A}}[\![\langle a_1, \ldots, a_n\rangle]\!] = \{\langle \delta^{\mathbf{A}}(a_1, \ldots, a_n), \epsilon^{\mathbf{A}}(a_1, \ldots, a_n)\rangle \ : \ \langle \delta, \epsilon\rangle \in \mathfrak{N}\}, \tag{9.1}$$

which we call the **instantiation relationship**. For $\alpha \subseteq \mathsf{uni}(\mathbf{A})^2$, let

$$\mathfrak{N}^{\mathbf{A}}/\alpha \doteq \overleftarrow{\mathfrak{N}^{\mathbf{A}}}\,\lfloor\alpha\rfloor = \mathfrak{N}^{\mathbf{A}\blacktriangleleft}(\alpha) = \{\langle a_1, \ldots, a_n\rangle \in \mathsf{uni}(\mathbf{A})^n \ : \ \mathfrak{N}^{\mathbf{A}}[\![\langle a_1, \ldots, a_n\rangle]\!] \subseteq \alpha\}, \tag{9.2}$$

which defines a function from $\mathsf{BRel}(\mathsf{uni}(\mathbf{A}))$ into $\mathfrak{P}((\mathsf{uni}(\mathbf{A}))^n)$, which we call the **solution function**. We call $\mathfrak{N}^{\mathbf{A}}/\alpha$ the **set of solutions to $\mathfrak{N}$ in A modulo** α or the $\mathfrak{N}$**-class of** α. We drop the superscript '$\mathbf{A}$' from these notions in the case that $\mathbf{A} = \mathbf{Tm}$. Let

$$\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A}) = \{\mathfrak{N}^{\mathbf{A}}/\alpha \ : \ \alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})\}.$$

Recall the definition of the product closed system determined by a single translation from a set into a closed system (see ∮5.3.4) and the definition of the formal-system $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ (see Example 4.143 on page 168). By definition (and conflating closed systems with their closed sets)

$$\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A}) = \tau^{\mathbf{A}\blacktriangleleft}[\mathsf{Con}^{\mathcal{K}}(\mathbf{A})];$$

hence $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ constitutes an *algebraic* closed system (by Theorem 5.88 on page 199 and the fact that $\tau^{\mathbf{A}}$ is a finitary translation); the associated algebraic lattice is denoted by $\mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$. □

The following remark follows by (1.1) of Table 1.1 on page 18 and (5.22) of Table 5.1 on page 180.

Remark 9.2 $\mathfrak{N}^{\mathbf{A}}\,[\cdot]$ and $\mathfrak{N}^{\mathbf{A}}/\cdot$ are $\subseteq$-preserving. □

The following remark follows since $\mathfrak{N}^{\mathbf{A}}$ is a translation and $\mathfrak{N}^{\mathbf{A}}/\cdot$ is its associated reduced pre-image (see Proposition 5.19 on page 181 and Proposition 5.8 on page 177).

Remark 9.3 Let $\alpha \subseteq \mathsf{uni}(\mathbf{A})^2$ and $\mathtt{A} \subseteq (\mathsf{uni}(\mathbf{A}))^n$.

1. $\mathfrak{N}^{\mathbf{A}}\left[\mathfrak{N}^{\mathbf{A}}/\alpha\right] \subseteq \alpha$.

2. $\mathtt{A} \subseteq \mathfrak{N}^{\mathbf{A}}/\mathfrak{N}^{\mathbf{A}}\,[\mathtt{A}]$.

3. $\mathfrak{N}^{\mathbf{A}}[\mathtt{A}] \subseteq \alpha$ iff $\mathtt{A} \subseteq \mathfrak{N}^{\mathbf{A}}/\alpha$.

$\square$

The following characterization of consequence in $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ follows immediately from Proposition 5.85 on page 198 and definitions.

Corollary 9.4 For any $\mathbf{A}$, not necessarily in $\mathcal{K}$, the following conditions are equivalent.

1. $\mathtt{A} \vdash_{\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})} \mathsf{a}$.

2. $\mathfrak{N}^{\mathbf{A}}[\mathtt{A}] \vdash_{\mathsf{Con}^{\mathcal{K}}(\mathbf{A})} \mathfrak{N}^{\mathbf{A}}[\![\mathsf{a}]\!]$.

3. $\mathfrak{N}^{\mathbf{A}}[\![\mathsf{a}]\!] \subseteq \left\| \mathfrak{N}^{\mathbf{A}}[\mathtt{A}] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$.

4. $\tau[\mathtt{A}] \subseteq \alpha \to \tau[\![\mathsf{a}]\!] \subseteq \alpha$, for all $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.

$\square$

We shall now demonstrate that the instantiation relationship is strictly continuous from the closed systems of solutions on a algebra (modulo $\mathcal{K}$) to the relative congruence lattice on that algebra. Consequently, the lattice of solutions is isomorphic to a join-complete subsemilattice of the relative congruence lattice that is compact in the relative congruence lattice.

Corollary 9.5 Let $\mathcal{K}$ be a quasi-variety of $\mathfrak{a}$-algebras and $\mathbf{A}$ an $\mathfrak{a}$-algebra not necessarily in $\mathcal{K}$.

1. $\mathfrak{N}^{\mathbf{A}}$ is strictly continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.

2. $\mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ is isomorphic to a $\blacktriangledown$-complete subsemilattice $\mathbf{P}$ of $\mathbf{Con}^{\mathcal{K}}(\mathbf{A})$ that is compact in $\mathbf{Con}^{\mathcal{K}}(\mathbf{A})$, under the function mapping $\mathtt{A} \mapsto \left\| \mathfrak{N}^{\mathbf{A}}[\mathtt{A}] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$ with inverse isomorphism $(\mathfrak{N}^{\mathbf{A}}/\cdot)_{|\mathsf{uni}(\mathbf{P})}$.

Proof. $\boxed{(1)}$ By Proposition 5.85 on page 198. $\boxed{(4)}$ By (2), equivalent condition (8) of Theorem 5.73 on page 195, and (1) of Theorem 5.111 on page 205 (together with the fact that $\mathfrak{N}^{\mathbf{A}}$ is concrete and finitary and $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ and $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ are both finitary, the former by (2).) $\diamond$

Recall the definition of the *promotion* $\underset{\to[n]}{\alpha}$ of a binary relation α (see Definition 1.89 on page 27). Note that for a congruence α on $\mathbf{A}$, the promotion $\underset{\to[n]}{\alpha}$ is an equivalence relation of $\mathsf{uni}(\mathbf{A})^n$. In the next result, we note that for a relative congruence α, the equivalence relation $\underset{\to[n]}{\alpha}$ is compatible with the solution set $(\mathfrak{N})^{\mathbf{A}}/\alpha$; the proof follows easily from definitions and the preservation of term functions by congruences (see Remark 1.351 on page 67).

Proposition 9.6 Let $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. Then $\underset{\to[n]}{\alpha}$ is compatible with $(\mathfrak{N})^{\mathbf{A}}/\alpha$, i.e., $\underset{\to[n]}{\alpha}\left[\mathfrak{N}^{\mathbf{A}}/\alpha\right] = \mathfrak{N}^{\mathbf{A}}/\alpha$. $\square$

9.1.2 The Logic $S^n(\mathcal{K}, \mathfrak{N})$ of Solutions to Equations

In this text, we shall consider two generalizations of the sentential 1-calculus $S(\mathcal{K}, \tau)$ of [BR99] (see Example 2.85 on page 106 of our text). In the first generalization, which we develop now, we extend the logic $S(\mathcal{K}, \tau)$ from a sentential 1-calculus determined by a unary system τ and a quasivariety $\mathcal{K}$, to a sentential n-calculus $S^n(\mathcal{K}, \mathfrak{N})$ determined by an n-ary system $\mathfrak{N}$ and a quasivariety $\mathcal{K}$: in the case that $\mathfrak{N}$ is a unary system, our logic $S^1(\mathcal{K}, \mathfrak{N})$ coincides with their logic $S(\mathcal{K}, \mathfrak{N})$. We show that $S^n(\mathcal{K}, \mathfrak{N})$ is *algebraizable* iff $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N} \rangle$*-regular*, thereby obtaining, as a special case, the result in [BR99] that $S(\mathcal{K}, \tau)$ is algebraizable iff $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-regular (see Theorem 2.125 on page 115 of our text). We shall also show that *all* algebraizable sentential n-calculi are formally equivalent to a logic $S^n(\mathcal{K}, \mathfrak{N})$. In [BR99] it was also shown that $S(\mathcal{K}, \tau)$ is *protoalgebraic* precisely when $\mathcal{K}$ has $\mathcal{K}$*-coherent* $\langle \mathcal{K}, \tau \rangle$*-classes* (see Example 2.140 on page 118 of our text). While it is indeed the case that $S^n(\mathcal{K}, \mathfrak{N})$ is *protoalgebraic* precisely when $\mathcal{K}$ has $\mathcal{K}$*-coherent* $\langle \mathcal{K}, \mathfrak{N} \rangle$*-classes*, we do not establish this result in this chapter, since we require a (new) characterization of protoalgebraic sentential n-calculi (see Corollary 16.39 on page 453) that we obtain from our theory of protoalgebraic logics over constructs which we develop in $\oint$16. In that chapter we will prove the aforementioned characterization of the protoalgebraicity of $S^n(\mathcal{K}, \mathfrak{N})$ (see Corollary 16.41 on page 454). The second generalization, introduced in $\oint$12, stems from our desire to bring *full congruence regularity* into the arena of *algebraic logic*. Unlike this second generalization, the first generalization is fairly routine. Except for the characterization of the protoalgebraicity of this logic, which we delay to $\oint$16, the results obtained are fairly simple generalizations of arguments from [BR99].

Before proceeding, the reader is urged to recall Example 2.85 on page 106 and Example 2.122 on page 114.

Definition 9.7 (The Sentential n-Calculus $S^n(\mathcal{K}, \mathfrak{N})$) Let $\mathfrak{a}$ be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and $\mathfrak{N}(x_1, \ldots, x_n)$ an n-ary system of equations. Let $S^n(\mathcal{K}, \mathfrak{N})$ denote the sentential n-calculus determined by all axioms

$$\vdash \phi, \quad \text{where} \quad \models_{\mathcal{K}} \mathfrak{N}^{\approx}[\![\phi]\!] \ , \tag{9.3}$$

and all rules

$$\Gamma \vdash \phi, \quad \text{where} \quad \mathfrak{N}^{\approx}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}^{\approx}[\![\phi]\!] \ . \tag{9.4}$$

In the case that $\mathfrak{N}$ is unary, we shall write $S(\mathcal{K}, \mathfrak{N})$ for $S^1(\mathcal{K}, \mathfrak{N})$. □

Proposition 9.8 For $\Gamma \cup \phi \subseteq \mathsf{Tm}^n$,

$$\Gamma \vdash_{S^n(\mathcal{K}, \mathfrak{N})} \phi \text{ iff } \mathfrak{N}^{\approx}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}^{\approx}[\![\phi]\!]. \tag{9.5}$$

Proof. $\boxed{\Rightarrow}$ We proceed by induction on the length of derivations in $S^n(\mathcal{K}, \mathfrak{N})$. $\boxed{\text{Base Case}}$ Suppose that ϕ is derivable from Γ by a derivation of length 1. If $\phi \in \Gamma$ then $\mathfrak{N}[\![\phi]\!] \subseteq \mathfrak{N}[\Gamma]$ and so $\mathfrak{N}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}[\![\phi]\!]$. Otherwise, there exists an axiom $\vdash \psi$ and a substitution σ with $\sigma(\psi) = \phi$. By definition, $\models_{\mathcal{K}} \mathfrak{N}[\![\psi]\!]$, and so by structurality of $\models_{\mathcal{K}}$, $\models_{\mathcal{K}} \sigma[\mathfrak{N}[\![\psi]\!]]$. So by Lemma 9.17, $\models_{\mathcal{K}} \mathfrak{N}[\![\sigma(\psi)]\!]$, i.e., $\models_{\mathcal{K}} \mathfrak{N}[\![\phi]\!]$, hence $\mathfrak{N}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}[\![\phi]\!]$. $\boxed{\text{Induction Hypothesis}}$ Assume that if any ϕ is derivable from Γ by a derivation of length k, then $\mathfrak{N}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}[\![\phi]\!]$. $\boxed{\text{Inductive Step}}$ Suppose that ϕ is derivable from Γ by a derivation $\phi_1, \ldots, \phi_{k+1}$ of length $k+1$, and by no shorter derivation. Then there exists a rule $\Phi \vdash \psi$ and a substitution σ such that $\sigma[\Phi] \subseteq \{\phi_1, \ldots, \phi_k\}$ and $\sigma(\psi) = \phi$. By definition, $\mathfrak{N}[\Phi] \models_{\mathcal{K}} \mathfrak{N}[\![\psi]\!]$, and so by structurality and Lemma 9.17, $\mathfrak{N}[\sigma[\Phi]] \models_{\mathcal{K}} \mathfrak{N}[\![\sigma(\psi)]\!]$.

By the inductive hypothesis, $\mathfrak{N}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}[\sigma[\Phi]]$, and so $\mathfrak{N}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}[\![\sigma(\psi)]\!]$, the result following since $\sigma(\psi) = \phi$. $\boxed{\Leftarrow}$ By finitariness of $\models_{\mathcal{K}}$ and definition. $\diamond$

A comparison of (9.5) with (2.28) of Remark 2.106, demonstrates that $\mathcal{K}$ is *always* an algebraic semantics for $S^n(\mathcal{K}, \mathfrak{N})$ with defining equations $\mathfrak{N}$, and that if $\mathcal{S}$ is a sentential n-calculus with algebraic semantics $\mathcal{K}$ and defining equations τ, then $\mathcal{S}$ is equivalent (and *not* just formally equivalent) to $S^n(\mathcal{K}, \mathfrak{N})$.

Corollary 9.9 The sentential n-calculi having algebraic semantics are precisely the logics $S^n(\mathcal{K}, \mathfrak{N})$, up to equivalence.

In the following result, which is a generalization of [BR99, L 5.1] from unary systems to n-ary systems (see Proposition 2.91 on page 107 of our text), we demonstrate that the *solution sets of* $\mathfrak{N}$ *modulo* $\mathcal{K}$ are all $S^n(\mathcal{K}, \mathfrak{N})$*-filters*. In the case of the term algebra or the $\mathcal{K}$-free algebra on ω-free generators, the solution sets and $S^n(\mathcal{K}, \mathfrak{N})$-filters *coincide*.

Proposition 9.10 For any algebra $\mathbf{A}$, not necessarily in $\mathcal{K}$, $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A}) \subseteq \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A})$, with equality whenever $\mathbf{A} = \mathbf{Tm}$ or $\mathbf{A}$ is the $\mathcal{K}$-free algebra on V.

Proof. $\boxed{\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A}) \subseteq \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A})}$ Suppose that $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, $\mathsf{i} \in \mathsf{Int}(\mathbf{Tm}, \mathbf{A})$, $\Gamma \vdash_{S^n(\mathcal{K},\mathfrak{N})} \phi$ and $\underset{\rightarrow_{[n]}}{\mathsf{i}}[\Gamma] \subseteq \mathfrak{N}^{\mathbf{A}}/\alpha$. (*We must show that* $\underset{\rightarrow_{[n]}}{\mathsf{i}}[\phi] \subseteq \mathfrak{N}^{\mathbf{A}}/\alpha$.) Since $\underset{\rightarrow_{[n]}}{\mathsf{i}}[\Gamma] \subseteq \mathfrak{N}^{\mathbf{A}}/\alpha$, $\mathfrak{N}^{\mathbf{A}}\left[\underset{\rightarrow_{[n]}}{\mathsf{i}}[\Gamma]\right] \subseteq \mathfrak{N}^{\mathbf{A}}\left[\mathfrak{N}^{\mathbf{A}}/\alpha\right] \subseteq \alpha$, by Remark 9.3, and so by Lemma 9.17, $\underset{\rightarrow_{[2]}}{\mathsf{i}}\left[\mathfrak{N}^{\mathbf{A}}[\Gamma]\right] \subseteq \alpha$. Further, since $\Gamma \vdash_{S^n(\mathcal{K},\mathfrak{N})} \phi$, $\mathfrak{N}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}[\![\phi]\!]$. Since $\alpha \in \mathsf{Fi}_{S^2(\Theta^{\mathcal{K}})}(\mathbf{A})$ (by Example 2.77), $\mathfrak{N}[\Gamma] \models_{\mathcal{K}} \mathfrak{N}[\![\phi]\!]$, and $\underset{\rightarrow_{[2]}}{\mathsf{i}}\left[\mathfrak{N}^{\mathbf{A}}[\Gamma]\right] \subseteq \alpha$, $\underset{\rightarrow_{[2]}}{\mathsf{i}}\left[\mathfrak{N}^{\mathbf{A}}[\![\phi]\!]\right] \subseteq \alpha$. So by Lemma 9.17, $\mathfrak{N}^{\mathbf{A}}\left[\underset{\rightarrow_{[n]}}{\mathsf{i}}[\phi]\right] \subseteq \alpha$. Hence by Remark 9.3, $\underset{\rightarrow_{[n]}}{\mathsf{i}}[\phi] \subseteq \mathfrak{N}^{\mathbf{A}}/\alpha$. $\boxed{\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{Tm}) \supseteq \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{Tm})}$ Let $T \in \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{Tm}) = \mathsf{Th}(S^n(\mathcal{K}, \mathfrak{N}))$. (*It suffices to show that* $\mathfrak{N}^{\mathbf{Tm}}/\left\|\mathfrak{N}^{\mathbf{Tm}}[T]\right\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} = T$.) By Remark 9.3 and inclusion preservation, $T \subseteq \mathfrak{N}^{\mathbf{Tm}}/\mathfrak{N}^{\mathbf{Tm}}[T] \subseteq \mathfrak{N}^{\mathbf{Tm}}/\left\|\mathfrak{N}^{\mathbf{Tm}}[T]\right\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}$. Conversely, let $\langle p_1, \ldots, p_n\rangle \in \mathfrak{N}^{\mathbf{Tm}}/\left\|\mathfrak{N}^{\mathbf{Tm}}[T]\right\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}$. So $\mathfrak{N}^{\mathbf{Tm}}[\![\langle p_1, \ldots, p_n\rangle]\!] \in \left\|\mathfrak{N}^{\mathbf{Tm}}[T]\right\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}$. Hence $\mathfrak{N}^{\mathbf{Tm}}[T] \models_{\mathcal{K}} \mathfrak{N}^{\mathbf{Tm}}[\![\langle p_1, \ldots, p_n\rangle]\!]$. So by Proposition 9.8, $T \vdash_{S^n(\mathcal{K},\mathfrak{N})} \langle p_1, \ldots, p_n\rangle$, and since T is a theory, $\langle p_1, \ldots, p_n\rangle \in T$. $\boxed{\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{F}_{\mathcal{K}}) \supseteq \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{F}_{\mathcal{K}})}$ Similar. $\diamond$

Since the filters of a sentential calculus on the term algebra coincide with the theories, it follows at once that the theories of $S^n(\mathcal{K}, \mathfrak{N})$ coincide with the filters of $S^n(\mathcal{K}, \mathfrak{N})$ on the term algebra; so $S^n(\mathcal{K}, \mathfrak{N})$ may be viewed as the *logic of (simultaneous) solutions* to the system of equations $\mathfrak{N}$ modulo the quasivariety $\mathcal{K}$.

Corollary 9.11 $\mathsf{Th}(S^n(\mathcal{K}, \mathfrak{N})) = \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{Tm})$. $\square$

Note that the inclusion $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A}) \subseteq \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A})$ is generally strict, even for $\mathbf{A} \in \mathcal{K}$ [BR99], so while all sets of solutions to $\mathfrak{N}$ on $\mathbf{A}$ modulo $\mathcal{K}$ are $S^n(\mathcal{K}, \mathfrak{N})$-filters, there are $S^n(\mathcal{K}, \mathfrak{N})$-filters that do not arise as a set of solutions to $\mathfrak{N}$ on $\mathbf{A}$ modulo $\mathcal{K}$. In the next chapter we shall show that $S^n(\mathcal{K}, \mathfrak{N})$ is algebraizable precisely when $\mathcal{K}$ is $\langle\mathcal{K}, \mathfrak{N}\rangle$-regular (defined in the next chapter), in which case it is indeed true that $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A}) = \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A})$ for all $\mathbf{A}$.

Note that since the solution function $\mathfrak{N}^{\mathbf{A}}/\cdot$ is order-preserving (by Remark 9.2), by Proposition 9.10, the solution function determines an order preserving function from the relative congruence lattice on an algebra $\mathbf{A}$ into the lattice of $S^n(\mathcal{K}, \mathfrak{N})$-filters on $\mathbf{A}$.

Corollary 9.12 $\mathfrak{N}^{\mathbf{A}}/\cdot$ defines an order-preserving function from $\mathbf{Con}^{\mathcal{K}}(\mathbf{A})$ into $\mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A})$. Whenever $\mathbf{A} = \mathbf{Tm}$ or $\mathbf{A}$ is the $\mathcal{K}$-free algebra on V, this function is *surjective*. □

We shall now consider an alternative route to the logic $S^n(\mathcal{K}, \mathfrak{N})$ employing our theory of *logics over constructs* developed in ∮6, and the theory of *canons and archologies* developed in ∮8.

Definition 9.13 (The Universal Logics of Solutions to n-ary Equations) Let $\mathfrak{N}$ be a system of n-ary $\mathfrak{a}$-equations, $\mathcal{K}$ be a quasi-variety of $\mathfrak{a}$-algebras, $\mathbf{A}$ an $\mathfrak{a}$-algebra not necessarily in $\mathcal{K}$ and $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra on ω-free generators. Let $U^n_{\mathbf{A}}(\mathcal{K}, \mathfrak{N})$ denote the *finitary* $\mathbf{A}^n$-logic $L\left(\mathbf{A}^n, \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})\right)$. We write $\boldsymbol{S}^n(\mathcal{K}, \mathfrak{N})$ for $U^n_{\mathbf{F}_{\mathcal{K}}}(\mathcal{K}, \mathfrak{N})$. We drop the superscript n in the case that $n = 1$. □

Note that $\boldsymbol{S}^n(\mathcal{K}, \mathfrak{N})$ is *not* a sentential n-calculus since its language is $\mathbf{F}_{\mathcal{K}}{}^n$ and not Tm^n, hence the emboldened symbol '$\boldsymbol{S}$'. It follows at once from Corollary 9.11 that the sentential n-calculus $S^n(\mathcal{K}, \mathfrak{N})$ is equivalent to $U^n_{\mathbf{Tm}}(\mathcal{K}, \mathfrak{N})$.

Remark 9.14 $S^n(\mathcal{K}, \mathfrak{N}) \equiv U^n_{\mathbf{Tm}}(\mathcal{K}, \mathfrak{N})$.

Proof. by Proposition 9.12 on page 316, the function $\mathfrak{N}/\cdot$ defines a *surjective* function from $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ into $\mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{Tm})$, and that, by Corollary 2.53 on page 102, $\mathsf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{Tm}) = \mathsf{Th}(S^n(\mathcal{K}, \mathfrak{N}))$. ⋄

The following characterization of the logic $U^n_{\mathbf{A}}(\mathcal{K}, \mathfrak{N})$ follows immediately from Corollary 9.4.

Proposition 9.15 For any $\mathbf{A}$, not necessarily in $\mathcal{K}$, the following conditions are equivalent.

1. $\Gamma \vdash_{U^n_{\mathbf{A}}(\mathcal{K},\mathfrak{N})} \phi$.
2. $\mathfrak{N}^{\mathbf{A}}[\Gamma] \vdash_{\mathsf{Con}^{\mathcal{K}}(\mathbf{A})} \mathfrak{N}^{\mathbf{A}}[\![\phi]\!]$.
3. $\mathfrak{N}^{\mathbf{A}}[\![\phi]\!] \subseteq \left\| \mathfrak{N}^{\mathbf{A}}[\Gamma] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$.
4. $\tau[\Gamma] \subseteq \alpha \to \tau[\![\phi]\!] \subseteq \alpha$, for all $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.

□

The following result, highlighting the relationship between $\boldsymbol{S}^n(\mathcal{K}, \mathfrak{N})$-consequence and $S^n(\mathcal{K}, \mathfrak{N})$-consequence, follows by Proposition 9.15, Proposition 9.8 and Lemma 1.457 on page 88.

Proposition 9.16 The following conditions are equivalent.

1. $\{\langle \overline{p^i_1}, \dots, \overline{p^i_n} \rangle \,:\, i \in I\} \vdash_{\boldsymbol{S}^n(\mathcal{K},\mathfrak{N})} \langle \overline{p_1}, \dots, \overline{p_n} \rangle$.
2. $\mathfrak{N}^{\approx}\left[\{\langle p^i_1, \dots, p^i_n \rangle \,:\, i \in I\}\right] \models_{\mathcal{K}} \mathfrak{N}^{\approx}[\![\langle p_1, \dots, p_n \rangle]\!]$.
3. $\{\langle p^i_1, \dots, p^i_n \rangle \,:\, i \in I\} \vdash_{S^n(\mathcal{K},\mathfrak{N})} \langle p_1, \dots, p_n \rangle$.

□

We shall now show that each logic $U^n_{\mathbf{A}}(\mathcal{K},\mathfrak{N})$ is $\underset{\rightarrow[n]}{\mathfrak{a}}$-*structural.* In fact, we shall establish a stronger result, namely that for any algebras $\mathbf{A}$ and $\mathbf{B}$, $U^n_{\mathbf{B}}(\mathcal{K},\mathfrak{N})$ is an $\underset{\rightarrow[n]}{\mathfrak{a}}$-*model* of $U^n_{\mathbf{A}}(\mathcal{K},\mathfrak{N})$; by definition, this amounts to showing that for each homomorphism f from $\mathbf{A}$ into $\mathbf{B}$, $\underset{\rightarrow}{f}$ is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ into $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{B})$, in other words, promoted algebra homomorphisms are continuous between the solution systems. This result depends intrinsically on the fact that the *instantiation relationship* 'commutes with homomorphisms'; we establish this fact first. Note that this 'commutivity' play a key role in the theory of *algebraizable logics* (see Corollary 2.108 on page 111).

Lemma 9.17 $\mathfrak{N}^{\mathbf{B}}\underset{\rightarrow[n]}{f} = \underset{\rightarrow[2]}{f}\mathfrak{N}^{\mathbf{A}}$, for all $\mathfrak{a}$-algebras $\mathbf{A}$ and $\mathbf{B}$ and all $f : \mathbf{A} \rightarrow \mathbf{B}$. In particular, $\mathfrak{N}^{\mathbf{A}}\left[\underset{\rightarrow[n]}{\mathsf{i}}[\Gamma]\right] = \underset{\rightarrow[2]}{\mathsf{i}}[\mathfrak{N}[\Gamma]]$, for any homomorphism $\mathsf{i} : \mathbf{Tm} \rightarrow \mathbf{A}$.

Proof.

$$\begin{aligned}
\mathfrak{N}^{\mathbf{B}}[\![\underset{\rightarrow[n]}{f}(\langle a_1,\ldots,a_n\rangle)]\!] &= \mathfrak{N}^{\mathbf{B}}[\![\langle f(a_1),\ldots,f(a_n)\rangle]\!] \\
&= \{\langle \delta^{\mathbf{B}}(f(a_1),\ldots,f(a_n)), \epsilon^{\mathbf{B}}(f(a_1),\ldots,f(a_n))\rangle \;:\; \langle\delta,\epsilon\rangle \in \mathfrak{N}\} \\
&= \{\langle f(\delta^{\mathbf{A}}(a_1,\ldots,a_n)), f(\epsilon^{\mathbf{A}}(a_1,\ldots,a_n))\rangle \;:\; \langle\delta,\epsilon\rangle \in \mathfrak{N}\} \\
&= \underset{\rightarrow[2]}{f}\left[\{\langle \delta^{\mathbf{A}}(a_1,\ldots,a_n), \epsilon^{\mathbf{A}}(a_1,\ldots,a_n)\rangle \;:\; \langle\delta,\epsilon\rangle \in \mathfrak{N}\}\right] \\
&= \underset{\rightarrow[2]}{f}\left[\mathfrak{N}^{\mathbf{A}}[\![\langle a_1,\ldots,a_n\rangle]\!]\right].
\end{aligned}$$

◇

We now show that the logics of solutions to n-ary equations $\mathfrak{a}$-model each other.

Proposition 9.18 For any $\mathfrak{a}$-algebras $\mathbf{A}$ and $\mathbf{B}$, $U^n_{\mathbf{B}}(\mathcal{K},\mathfrak{N})$ is an $\underset{\rightarrow[n]}{\mathfrak{a}}$-*model* of $U^n_{\mathbf{A}}(\mathcal{K},\mathfrak{N})$.

Proof. Let $f : \mathbf{A} \rightarrow \mathbf{B}$. (*By definition, we must show that* $\underset{\rightarrow}{f}$ *is continuous from* $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ *into* $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{B})$.) By Lemma 9.17, $\mathfrak{N}^{\mathbf{B}}\underset{\rightarrow[n]}{f} = \underset{\rightarrow[2]}{f}\mathfrak{N}^{\mathbf{A}}$. Now $\underset{\rightarrow[2]}{f}$ is continuous from $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ into $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, by Example 5.51 on page 189, and $\mathfrak{N}^{\mathbf{A}}$ is (strictly) continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. So by Remark 5.24 on page 183, (viewing $\underset{\rightarrow[2]}{f}$ as a translation) $\underset{\rightarrow[2]}{f}\mathfrak{N}^{\mathbf{A}}$ is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, and hence so is $\mathfrak{N}^{\mathbf{B}}\underset{\rightarrow[n]}{f}$. Since $\mathfrak{N}^{\mathbf{B}}$ is strictly continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{B})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, and hence consequence reflecting, $\underset{\rightarrow[n]}{f}$ is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$ into $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{B})$, by Proposition 5.23 on page 183. ◇

In particular, for $\mathbf{A} \in \mathcal{K}$, $U^n_{\mathbf{B}}(\mathcal{K},\mathfrak{N})$ is an $\underset{\rightarrow[n]}{\mathfrak{a}}$-*model* of $\boldsymbol{S}^n(\mathcal{K},\mathfrak{N})$. Consequently, the logics of solutions to n-ary equations over algebras *in* $\mathcal{K}$ determine an $\underset{\rightarrow}{\mathfrak{a}}_{[n]}$-*archology* with archetype $\underset{\rightarrow}{\mathcal{K}}_{[n]}$; noting that since the signature $\mathcal{K}$ is an $\mathfrak{a}$-archetype with canonical language $\mathbf{F}_{\mathcal{K}}$, where $\mathbf{F}_{\mathcal{K}}$ is the $\mathcal{K}$-free algebra on ω-free generators $\overline{[\mathsf{V}]}$, the signature $\underset{\rightarrow}{\mathcal{K}}_{[n]}$ is an $\underset{\rightarrow}{\mathfrak{a}}_{[n]}$-archetype with canonical language ${\mathbf{F}_{\mathcal{K}}}^n$.

Definition 9.19 (The Archology $\mathfrak{A}^n(\mathcal{K},\mathfrak{N})$) For a quasivariety $\mathcal{K}$ of algebras and a system $\mathfrak{N}$ of n-ary equations, $\{U^n_{\mathbf{A}}(\mathcal{K},\mathfrak{N}) \;:\; \mathbf{A} \in \mathcal{K}\}$ well-defines an $\underset{\rightarrow}{\mathfrak{a}}_{[n]}$-archology $\mathfrak{A}^n(\mathcal{K},\mathfrak{N})$ with *canonical language* $\mathbf{F}_{\mathfrak{A}^n(\mathcal{K},\mathfrak{N})} = {\mathbf{F}_{\mathcal{K}}}^n$, where $\mathbf{F}_{\mathcal{K}}$ is the $\mathcal{K}$-free algebra on ω-free generators $\overline{[\mathsf{V}]}$, *root language* $\mathbf{Tm}^n$ and *canon* $\overline{\mathfrak{A}^n(\mathcal{K},\mathfrak{N})} = \boldsymbol{S}^n(\mathcal{K},\mathfrak{N})$. □

Remark 9.20 Generally this archology is not maximal [BR99]. □

Since the archology $\mathfrak{A}^n(\mathcal{K},\mathfrak{N})$ is finitary and hence canon-finitary, its ideal $\underline{\mathfrak{A}^n(\mathcal{K},\mathfrak{N})}$ is a *sentential n-calculus*, by (3) of Theorem 8.83 on page 300. By (5) of the same theorem,

$$\{\langle p_1^i,\ldots,p_n^i\rangle \;:\; i\in I\} \vdash_{\underline{\mathfrak{A}^n(\mathcal{K},\mathfrak{N})}} \langle p_1,\ldots,p_n\rangle \text{ iff } \{\langle \overline{p_1^i},\ldots,\overline{p_n^i}\rangle \;:\; i\in I\} \vdash_{\boldsymbol{S}^n(\mathcal{K},\mathfrak{N})} \langle \overline{p_1},\ldots,\overline{p_n}\rangle,$$

and so by the equivalence of (1) and (3) of Proposition 9.16, $S^n(\mathcal{K},\mathfrak{N}) \equiv \underline{\boldsymbol{S}^n(\mathcal{K},\mathfrak{N})}$. Consequently, by Theorem 8.15 on page 282, the promotion of the canonical map from $\mathbf{Tm}$ to $\mathbf{F}_{\mathcal{K}}$ induces an isomorphism from $\mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{Tm})$ onto $\mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{F}_{\mathcal{K}})$.

Corollary 9.21 Let $\mathfrak{N}$ be a system of n-ary $\mathfrak{a}$-equations, $\mathcal{K}$ be a quasi-variety of $\mathfrak{a}$-algebras.

1. $S^n(\mathcal{K},\mathfrak{N}) \equiv \underline{\boldsymbol{S}^n(\mathcal{K},\mathfrak{N})} = \mathsf{I}^{\mathfrak{a}}_{\boldsymbol{S}^n(\mathcal{K},\mathfrak{N})}(\mathbf{Tm})$.
2. $\underset{\to[n]}{\overline{[\cdot]}} : \mathbf{Th}(S^n(\mathcal{K},\mathfrak{N})) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{F}_{\mathcal{K}}) = \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{F}_{\mathcal{K}})$ with inverse isomorphism $\underset{\to[n]}{\underline{[\cdot]}}$.

□

9.2 Solving Binary Systems Parametrically

We now turn to *binary systems* and a framework of solving such systems *parametrically.* This perspective on solving binary systems of equations arose from our attempt to unify the relationship between the condition of $\langle\mathcal{K},\tau\rangle$*-regularity and the algebraizability of the logic* $S(\mathcal{K},\tau)$ (as described in [BR99]; see Example 2.122 on page 114 of our text) with the relationship between *full $\mathcal{K}$-regularity and some suitable notion of parameterized algebraizability of the membership logic* (see Definition 5.62 of Example 5.57 on page 191). Since the variety of quasigroups contains a non-trivial subvariety $\mathcal{Q}'$ that is not the algebraic semantics of any non-trivial sentential 1-calculus (see ∮3), its membership logic cannot be equivalent to a logic $S(\mathcal{Q}',\tau)$, for any unary system τ, since $\mathcal{Q}'$ is always an algebraic semantics for $S(\mathcal{Q}',\tau)$. We shall see that the theories of the membership logic can be described in terms of the solutions to a *binary* system of equations, namely the binary system $\mathfrak{B}(x,y) = \{\langle x,y\rangle\}$, but not in the sense of the previous section, since the solutions to *binary* equations, leads to solutions which are sets of *pairs* and to sentential *two*-calculi. Since the membership logic is a sentential *one*-calculus, we require a framework for solving binary equations but which leads to solutions that are sets of *points* and not sets of *pairs* of points. This is achieved by obtaining *parameterized* sets of solutions; we fix one of the variables at a given point (the parameter), and find all the solutions for the other variable (on an algebra modulo a relative congruence).

To see how this pertains to the membership logic, recall that the membership logic is determined by its theories which are the cosets of the $\mathcal{K}$-relative congruences on the term algebra together with the empty-set when $\mathcal{K}$ is non-trivial (see Example 5.57). Consider the binary system of equations $\mathfrak{B}(x,y) = \{\langle x,y\rangle\}$ and a $\mathcal{K}$-relative congruence α on the term algebra. If we pick a term q and substitute q for y in $\mathfrak{B}$, we obtain the *unary* system $\mathfrak{B}_q(x) = \{\langle x,q\rangle\}$; unary if we consider q as fixed (more precisely, we can change the variable x so that it does not occur in q). Solving this unary system for x (with q fixed) modulo α (as described in the previous section), we obtain

$$\mathfrak{B}_q/\,\alpha = \{p \;:\; \langle p,q\rangle\in\alpha\} = \alpha[\![q]\!].$$

Consequently, if we iterate over all terms q and all relative congruences α, we obtain the set of all *proper* $\mathcal{K}$-cosets of the term algebra; if, in addition, we add the empty-set when $\mathcal{K}$ is non-trivial, we obtain precisely the theories of the *membership logic*.

Convention 9.22 (Systems of Binary Equations) When we say that $\mathfrak{B}(x, y)$ is a **binary system of equations** (or just a **binary system** where unambiguous), we shall call x the **left variable** and y the **right variable**. Unless specified to the contrary, $\mathfrak{B}(x, y)$ shall denote a fixed but arbitrary binary system.

9.2.1 Instantiation, Solution and Normals

We begin by considering the *solution set* $\mathfrak{B}_b^{\mathbf{A}}/\alpha$ of all the values for x that make the system true modulo a relative congruence α with y fixed to some point b in an algebra $\mathbf{A}$. Collecting all such solution sets for all possible points b and all relative congruences α on an algebra $\mathbf{A}$, we obtain the set of solution sets $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$. Unfortunately, the set $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ is *not* a closed system, since the intersection of solution sets need not be a solution set; while such an intersection certainly only contains solutions, it need not be a 'full' set of solutions, that is, the intersection need not be equal to $\mathfrak{B}_b^{\mathbf{A}}/\alpha$ for any b and relative congruence α. Treating the set $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ of solution sets as the basis of a closed system, we obtain the closed system $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$, the members of which are called *normals*.

Definition 9.23 (Instantiation, Solution and Normals) For each term p, define a translation, denoted by $\mathfrak{B}_p^{\approx}$, from Tm to $\mathsf{Identity}(\mathfrak{a})$ by $\mathfrak{B}_p^{\approx}[\![q]\!] = \{\delta(q, p) \approx \epsilon(q, p) \,:\, \langle \delta, \epsilon \rangle \in \mathfrak{B}\}$. Let $\mathbf{A}$ be an $\mathfrak{a}$-algebra and $a, b \in \mathsf{uni}(\mathbf{A})$. Define a translation $\mathfrak{B}_b^{\mathbf{A}}$ from $\mathsf{uni}(\mathbf{A})$ to $(\mathsf{uni}(\mathbf{A}))^2$, by

$$\mathfrak{B}_b^{\mathbf{A}}[\![c]\!] = \mathfrak{B}^{\mathbf{A}}[\![\langle c, b \rangle]\!]. \tag{9.6}$$

For a binary relation α on $\mathsf{uni}(\mathbf{A})$, let

$$\mathfrak{B}_b^{\mathbf{A}}/\alpha \doteq \overleftarrow{\mathfrak{B}_b^{\mathbf{A}}}\,\lfloor \alpha \rfloor = \mathfrak{B}_b^{\mathbf{A}\blacktriangleleft}(\alpha) = \{c \,:\, \mathfrak{B}_b^{\mathbf{A}}[\![c]\!] \subseteq \alpha\}, \tag{9.7}$$

which we call the (set of) **solutions to $\mathfrak{B}_*$ at b modulo α**. We drop the superscript $\mathbf{A}$ in the case that $\mathbf{A}$ is the term algebra over the standard denumerably infinite variable set V.

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras not necessarily containing $\mathbf{A}$, and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. We also call $\mathfrak{B}_b^{\mathbf{A}}/\alpha$ the **$\mathfrak{B}_b$-class of** α. Let

$$\begin{aligned}
\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) &= \{\mathfrak{B}_b^{\mathbf{A}}/\alpha \,:\, \alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})\} \quad \text{and} \\
\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) &= \bigcup_{b \in \mathsf{uni}(\mathbf{A})} \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}).
\end{aligned}$$

Arbitrary members of $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ are called **$\langle \mathcal{K}, \mathfrak{B}_* \rangle$-solutions on $\mathbf{A}$ determined by** b or **principal $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-normals of $\mathbf{A}$ determined by** b, and arbitrary members of $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ are called **$\langle \mathcal{K}, \mathfrak{B}_* \rangle$-solutions on $\mathbf{A}$** or **principal $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-normals of $\mathbf{A}$** (or briefly **$\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes**). Recall the definition of the product of a translation into a closed system (see Definition 5.82 on page 198). By definition (and conflating closed systems with their closed sets),

$$\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) = \mathfrak{B}_b^{\mathbf{A}\blacktriangleleft}[\mathsf{Con}^{\mathcal{K}}(\mathbf{A})];$$

hence $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ is an *algebraic* closed system (by Theorem 5.88 on page 199 and the fact that $\mathfrak{B}^{\mathbf{A}}_b$ is a finitary translation); the associated *algebraic* lattice is denoted by $\mathbf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$.

Let

$$\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) = \{\bigcap \mathcal{A} \,:\, \emptyset \neq \mathcal{A} \subseteq \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})\}, \tag{9.8}$$

the members of which are called **$\langle \mathfrak{B}, \mathcal{K} \rangle$-normals of $\mathbf{A}$**. Recall the definition of the product closed system determined by a source of closed systems (see Definition 5.112 on page 206). Let $\mathsf{sc}_{\mathbf{A}}(\mathcal{K}, \mathfrak{B}_*)$ denote the source of closed systems $\{\langle \mathfrak{B}^{\mathbf{A}}_b, \mathsf{Con}^{\mathcal{K}}(\mathbf{A})\rangle \,:\, b \in \mathsf{uni}(\mathbf{A})\}$. By definition,

$$\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) = \mathsf{sc}_{\mathbf{A}}(\mathcal{K}, \mathfrak{B}_*)^{\blacktriangleleft};$$

hence $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ forms a closed system; the associated complete lattice is denoted by $\mathbf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$.

We drop the '$\mathcal{K}$' from these notions in the case that $\mathcal{K}$ is the variety of all $\mathfrak{a}$-algebras, in which case $\mathsf{Con}^{\mathcal{K}}(\mathbf{A}) = \mathsf{Con}(\mathbf{A})$. The position of the subscript 'b', and the use of the subscript '$*$' in that position, in these (and future related) notions, is an aid to the eye, indicating that the right variable is the variable that is being *parameterized*. $\square$

Remark 9.24 $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ is a *basis* for $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$, by definition, and hence $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ is the coarsest closed system containing $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$. $\square$

The following result follows since $\mathfrak{B}^{\mathbf{A}}_b$ is a translation and $\mathfrak{B}^{\mathbf{A}}_b/\alpha$ is the associated reduced image (see Proposition 5.19 on page 181).

Remark 9.25 $\mathfrak{B}^{\mathbf{A}}_b \left[\mathfrak{B}^{\mathbf{A}}_b/\alpha\right] \subseteq \alpha$ and $\mathfrak{B}^{\mathbf{A}}_b/\alpha = \mathfrak{B}^{\mathbf{A}}_b/(\mathfrak{B}^{\mathbf{A}}_b \left[\mathfrak{B}^{\mathbf{A}}_b/\alpha\right])$. $\square$

The following characterization of $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$-consequence follows immediately from definitions and Theorem 5.113 on page 206.

Corollary 9.26 $A \vdash_{\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})} a$ iff $\forall\, [\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})]\, \forall\, [b \in \mathsf{uni}(\mathbf{A})]\, \mathfrak{B}^{\mathbf{A}}_b\, [A] \subseteq \alpha \rightarrow \mathfrak{B}^{\mathbf{A}}_b [\![a]\!] \subseteq \alpha$. $\square$

Since $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ is a product of *multiple* translations, this closed system need *not* be finitary.

Open Problem 9.27 Demonstrate that $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ need not be finitary. Under what conditions is it finitary? (See Open Problems 5.120 on page 207 and 9.96.)

The following result follows by Proposition 5.85 on page 198, Corollary 5.114 on page 207 and Remark 5.119 on page 207.

Corollary 9.28 Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras, $\mathbf{A}$ an $\mathfrak{a}$-algebra, not necessarily in $\mathcal{K}$, and $b \in \mathsf{uni}(\mathbf{A})$.

1. $\mathfrak{B}^{\mathbf{A}}_b$ is strictly continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.
2. For all $b \in \mathsf{uni}(\mathbf{A})$, $\mathfrak{B}^{\mathbf{A}}_b$ is continuous from $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$.
3. $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \preceq \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$.

Remark 9.29 If $f : \mathbf{A} \rightarrow \mathbf{B}$ then $\mathfrak{B}^{\mathbf{B}}_{f(b)} f = \underset{\rightarrow [2]}{f}\, \mathfrak{B}^{\mathbf{A}}_b$.

Proof. $\mathfrak{B}^{\mathbf{B}}_{f(b)}[\![f(a)]\!] = \{\langle \delta^{\mathbf{B}}(f(a), f(b)), \epsilon^{\mathbf{B}}(f(a), f(b))\rangle : \langle \delta, \epsilon\rangle \in \mathfrak{B}\} = \{\langle f(\delta^{\mathbf{A}}(a, b)), f(\epsilon^{\mathbf{A}}(a, b))\rangle : \langle \delta, \epsilon\rangle \in \mathfrak{B}\} = \underset{\rightarrow[2]}{f}\left[\{\langle \delta^{\mathbf{A}}(a, b), \epsilon^{\mathbf{A}}(a, b)\rangle : \langle \delta, \epsilon\rangle \in \mathfrak{B}\}\right] = \underset{\rightarrow[2]}{f}\left[\mathfrak{B}^{\mathbf{A}}_{b}[\![a]\!]\right]$. ⋄

Proposition 9.30 If $f : \mathbf{A} \to \mathbf{B}$, then f is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ into $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_{f(b)}}(\mathbf{B})$.

Proof. By Remark 9.29, $\mathfrak{B}^{\mathbf{B}}_{f(b)} f = \underset{\rightarrow[2]}{f} \mathfrak{B}^{\mathbf{A}}_{b}$. Now f is continuous from $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ into $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, by Example 5.51 on page 189, and $\mathfrak{B}^{\mathbf{A}}_{b}$ is (strictly) continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. So by Remark 5.24, (viewing f as a translation) $f\mathfrak{B}^{\mathbf{A}}_{b}$ is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, and hence so is $\mathfrak{B}^{\mathbf{B}}_{f(b)} f$. Since $\mathfrak{B}^{\mathbf{B}}_{f(b)}$ is strictly continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{B})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{B})$, and hence consequence reflecting, f is continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ into $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_{f(b)}}(\mathbf{B})$, by Proposition 5.23. ⋄

The proof of the following important result follows easily from definitions and the preservation of term functions by congruences (see Remark 1.351 on page 67).

Proposition 9.31 Let $\mathcal{K}$ be a quasivariety, $\mathbf{A}$ an algebra, $b \in \mathsf{uni}(\mathbf{A})$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. Then α is compatible with $\mathfrak{B}^{\mathbf{A}}_{b}/\alpha$, i.e., $\alpha[\![\mathfrak{B}^{\mathbf{A}}_{b}/\alpha]\!] = \mathfrak{B}^{\mathbf{A}}_{b}/\alpha$. □

There is no need to introduce a special *parameterized* notion of equivalent binary systems, since the notion of equivalence given in Definition 9.1 suffices, as demonstrated by the following (simple) result. Note that equivalent condition (1) is what we would ask of the parameterized version equivalence and (6) is the standard non-parameterized notion of equivalence.

Remark 9.32 The following conditions are equivalent.

1. For any two distinct variables z and x, $\mathfrak{B}^{\approx}_{z}[\![x]\!] \mathrel{=\!\!|\!\models_{\mathcal{K}}} \mathfrak{B}'^{\approx}_{z}[\![x]\!]$.
2. For any terms $P \cup \{p\}$, $\mathfrak{B}^{\approx}_{p}[P] \mathrel{=\!\!|\!\models_{\mathcal{K}}} \mathfrak{B}'^{\approx}_{p}[P]$.
3. For any algebra $\mathbf{A}$ (not necessarily in $\mathcal{K}$), $b \in \mathsf{uni}(\mathbf{A})$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, $\mathfrak{B}^{\mathbf{A}}_{b}/\alpha = \mathfrak{B}'^{\mathbf{A}}_{b}/\alpha$.
4. For any algebra $\mathbf{A} \in \mathcal{K}$, $b \in \mathsf{uni}(\mathbf{A})$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, $\mathfrak{B}^{\mathbf{A}}_{b}/\alpha = \mathfrak{B}'^{\mathbf{A}}_{b}/\alpha$.
5. For any algebra $\mathbf{A} \in \mathcal{K}$, $A \cup \{b\} \subseteq \mathsf{uni}(\mathbf{A})$, $\mathfrak{B}^{\mathbf{A}}_{b}[A] = \mathfrak{B}'^{\mathbf{A}}_{b}[A]$.
6. $\mathfrak{B}$ and $\mathfrak{B}'$ are $\mathcal{K}$-equivalent (in the sense of Definition 9.1).

Proof. $\boxed{(1)\Leftrightarrow(6)}$ Follows trivially since $\mathfrak{B}^{\approx}[\![\langle x, z\rangle]\!] = \mathfrak{B}^{\approx}_{z}[\![x]\!]$. $\boxed{(1)\Rightarrow(2)}$ Since x and z are distinct, it follows by the structurality of $\models_{\mathcal{K}}$, that for all $q \in P$, $\mathfrak{B}^{\approx}_{p}[\![q]\!] \mathrel{=\!\!|\!\models_{\mathcal{K}}} \mathfrak{B}'^{\approx}_{p}[\![q]\!]$. Hence $\mathfrak{B}^{\approx}_{p}[P] \mathrel{=\!\!|\!\models_{\mathcal{K}}} \mathfrak{B}'^{\approx}_{p}[P]$. $\boxed{(2)\Rightarrow(5)}$ Follows since $\mathbf{A} \in \mathcal{K}$. $\boxed{(5)\Rightarrow(1)}$ Trivial. $\boxed{(2)\Leftrightarrow(3)\Leftrightarrow(4)}$ By Lemma 1.457 on page 88. ⋄

9.2.2 Bases

The $\mathfrak{B}_{*}$-classes determined by the least relative congruence on an algebra, which we shall call *bases*, play an important role in this text. We shall see that, under certain circumstances, understanding the structure of a *variable base* sheds important light on solving the system more generally.

Definition 9.33 (Bases) For a binary system $\mathfrak{B}$, quasivariety $\mathcal{K}$ and an algebra $\mathbf{A}$, not necessarily in $\mathcal{K}$, we call $\mathfrak{B}_b^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}}$ the $\langle \mathcal{K}, \mathfrak{B}_* \rangle$**-base at** b (in $\mathbf{A}$). A p**-base** is a base $\mathfrak{B}_p/\bot_{\mathcal{K}}$, where p is a term; such a base is called a **term base**. A **variable base** is a z-base where z is a variable in V. A base is called **trivial** if it is the empty-set, otherwise it is called **non-trivial**. $\square$

The base $\mathfrak{B}_b^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}}$ consists of the solutions to the system with the right variable fixed to b modulo the least relative congruence, i.e., the values that when substituted into the system for the left variable (with b substituted for the right variable) make the system simultaneously true in the algebra $\mathbf{A}/\bot_{\mathcal{K}}^{\mathbf{A}}$ (recall that $\bot_{\mathcal{K}}^{\mathbf{A}}$ is the least congruence that factors $\mathbf{A}$ to an algebra in $\mathcal{K}$). In other words,

$$\mathfrak{B}_b^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}} = \{a \in \mathsf{uni}(\mathbf{A}) \ : \ \forall\, [\langle \delta, \epsilon \rangle \in \mathfrak{B}]\ \delta^{\mathbf{A}/\bot_{\mathcal{K}}^{\mathbf{A}}}(\bot_{\mathcal{K}}^{\mathbf{A}}[\![a]\!], \bot_{\mathcal{K}}^{\mathbf{A}}[\![b]\!]) = \epsilon^{\mathbf{A}/\bot_{\mathcal{K}}^{\mathbf{A}}}(\bot_{\mathcal{K}}^{\mathbf{A}}[\![a]\!], \bot_{\mathcal{K}}^{\mathbf{A}}[\![b]\!])\}.$$

If $\mathbf{A} \in \mathcal{K}$, then the base $\mathfrak{B}_b^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}}$ consists of the solutions to the system with the right variable fixed to b in $\mathbf{A}$, in other words,

$$\mathfrak{B}_b^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}} = \{a \in \mathsf{uni}(\mathbf{A}) \ : \ \forall\, [\langle \delta, \epsilon \rangle \in \mathfrak{B}]\ \delta^{\mathbf{A}}(a, b) = \epsilon^{\mathbf{A}}(a, b)\}.$$

Term bases and *variable bases* are *formal* solutions of *terms*, yielding information about solutions to the system in *all algebras of the quasivariety* and yielding $\mathcal{K}$-relative information about solutions to the system in *all* algebras. By Lemma 1.457 on page 88, we have that

$$\mathfrak{B}_p/\bot_{\mathcal{K}} = \{q \ : \models_{\mathcal{K}} \mathfrak{B}_p^{\approx}[\![q]\!]\} = \{q \ : \models_{\mathcal{K}} \bigwedge\nolimits_{\langle \delta, \epsilon \rangle \in \mathfrak{B}} \delta(q, p) \approx \epsilon(q, p)\} \tag{9.9}$$

and in particular,

$$\mathfrak{B}_z/\bot_{\mathcal{K}} = \{q \ : \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![q]\!]\} = \{q \ : \models_{\mathcal{K}} \bigwedge\nolimits_{\langle \delta, \epsilon \rangle \in \mathfrak{B}} \delta(q, z) \approx \epsilon(q, z)\}.$$

The following results demonstrate how knowledge of a *term base* provides partial knowledge of other bases on arbitrary algebras.

Proposition 9.34 Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{A}$ an $\mathfrak{a}$-algebra.

1. If $p \in \mathfrak{B}_p/\bot_{\mathcal{K}}$ and $\mathsf{i} : \mathbf{Tm} \to \mathbf{A}$ then $\mathsf{i}(q) \in \mathfrak{B}_{\mathsf{i}(p)}^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}}$. Consequently, $\mathsf{i}[\mathfrak{B}_p/\bot_{\mathcal{K}}] \subseteq \mathfrak{B}_{\mathsf{i}(p)}^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}}$.
2. For any substitution σ and term p, $\sigma[\mathfrak{B}_p/\bot_{\mathcal{K}}] \subseteq \mathfrak{B}_{\sigma(p)}/\bot_{\mathcal{K}}$, with equality whenever σ is an involution of $\mathbf{Tm}$.

Proof. $\boxed{(1)}$ By assumption and (9.9), $\models_{\mathcal{K}} \bigwedge_{\langle \delta, \epsilon \rangle \in \mathfrak{B}} \delta(q, p) \approx \epsilon(q, p)$. Hence $\langle \delta^{\mathbf{A}}(\mathsf{i}(q), \mathsf{i}(p)), \epsilon^{\mathbf{A}}(\mathsf{i}(q), \mathsf{i}(p)) \rangle \in \bot_{\mathcal{K}}^{\mathbf{A}}$, by Lemma 1.457 on page 88; i.e., $\mathsf{i}(q) \in \mathfrak{B}_{\mathsf{i}(p)}^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}}$. $\boxed{(2)}$ The first part is a special case of (1), while the equality assertion follows by a double application of the first part. $\diamond$

Open Problem 9.35 Is it true that if $f : \mathbf{A} \to \mathbf{B}$ then $f[\mathfrak{B}_a^{\mathbf{A}}/\bot_{\mathcal{K}}^{\mathbf{A}}] \subseteq \mathfrak{B}_{f(a)}^{\mathbf{B}}/\bot_{\mathcal{K}}^{\mathbf{B}}$?

Recall the definition of p-invariance given in Definition 1.337 on page 64. The following observation follows at once from (2) of Proposition 9.34.

Remark 9.36 The z-base $\mathfrak{B}_z/\bot_{\mathcal{K}}$ is z-invariant.

Remark 9.37 If some variable base is trivial (resp. non-trivial) then all variable bases are trivial (resp. non-trivial).

Remark 9.38 It is possible that variable bases are trivial, while some p-base is non-trivial. If variable bases are non-trivial then all p-bases are non-trivial.

9.2.3 Pivots

Closely related to the notion of the *variable bases* of a binary system, is the notion which we have termed the *pivots* of the system. A pivot is the instantiation of a variable base by another variable.

Definition 9.39 (Pivots) We denote $\mathfrak{B}_y^{\approx}[\mathfrak{B}_z/\bot_{\mathcal{K}}]$ by $[z \Rsh y]$, which we call the $\langle \mathcal{K}, \mathfrak{B}_* \rangle$**-pivot from** z **to** y or just the **pivot from** z **to** y where unambiguous; we may write $[z \Rsh_{\mathfrak{B}_*} y]$ or $[z \Rsh y]$ for $[z \Rsh_{\mathfrak{B}_*}^{\mathcal{K}} y]$ where unambiguous. □

Remark 9.40 For any term p, $\models_{\mathcal{K}} \mathfrak{B}_p^{\approx}[\mathfrak{B}_p/\bot_{\mathcal{K}}]$, consequently, $\models_{\mathcal{K}} [z \Rsh z]$. □

An import condition, introduced next, is the condition that a binary system *pivots.* We shall encounter this condition in the sequel (see Proposition 13.19 on page 399). It is a necessary condition for our theory of parametrized algebraization to succeed.

Definition 9.41 (Pivoting) We say that $\mathfrak{B}_*$ **pivots for** $\mathcal{K}$ if, for any variables y and z and any $P \cup \{p\} \subseteq \mathsf{Tm}$,

$$\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!] \ \text{ implies } \ [z \Rsh y] \cup \mathfrak{B}_y^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[\![p]\!], \tag{9.10}$$

and say that $\mathfrak{B}_*$ **pivots finitarily for** $\mathcal{K}$ if, for any variable z and any $P \cup \{p\} \subseteq \mathsf{Tm}$, whenever

$$\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!], \tag{9.11}$$

there exists a finite subset $Z \subseteq_{\mathsf{f}} \mathfrak{B}_z/\bot_{\mathcal{K}}$ and a finite subset $P' \subseteq_{\mathsf{f}} P$ such that, *for all* variables y,

$$\models_{\mathcal{K}} \bigwedge \mathfrak{B}_y^{\approx}[Z] \mathbin{\mathsf{and}} \bigwedge \mathfrak{B}_y^{\approx}[P'] \to \bigwedge \mathfrak{B}_y^{\approx}[\![p]\!]. \tag{9.12}$$

□

Discussion 9.42 (On Pivoting) In order to get a deeper understanding of the condition that $\mathfrak{B}_*$ pivots for $\mathcal{K}$, note the following. Suppose that

$$\mathfrak{B}_z^{\approx}[P(z, y, x_1, x_2, \ldots)] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p(z, y, x_1, x_2, \ldots)]\!]. \tag{9.13}$$

We could attempt to replace the *subscript* variable z in this expression by applying the transposition $\sigma = \langle z, y \rangle$ to obtain (by structurality)

$$\mathfrak{B}_y^{\approx}[P(y, z, x_1, x_2, \ldots)] \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[\![p(y, z, x_1, x_2, \ldots)]\!],$$

but in doing so we have swapped the positions of z and y in the terms $P \cup \{p\}$. When $\mathfrak{B}_*$ *pivots for* $\mathcal{K}$, we may replace the *subscript* variable z in (9.13) with y, without mutating P or p, provided we prepend or 'assume' the pivot from z to y, i.e., given (9.13), the following is valid,

$$[z \Rsh y] \cup \mathfrak{B}_y^{\approx}[P(z, y, x_1, x_2, \ldots)] \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[\![p(z, y, x_1, x_2, \ldots)]\!].$$

□

In fact, as will be seen in the examples to follow shortly, if the system pivots for $\mathcal{K}$, the pivot from z to y, in these examples, is an *equational* mechanism for *equating* z and y, *with respect to the system*, from *within the system* itself.

Proposition 9.43 The following conditions are equivalent.

1. $\mathfrak{B}_*$ pivots for $\mathcal{K}$.
2. For any variables y and z and any $P \cup \{p\} \subseteq \mathsf{Tm}$,

$$[y \Rsh z] \cup \mathfrak{B}^{\approx}_z\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!] \text{ implies } [z \Rsh y] \cup \mathfrak{B}^{\approx}_y\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_y[\![p]\!]. \tag{9.14}$$

3. For any variables y and z and any $P \cup \{p\} \subseteq \mathsf{Tm}$,

$$[y \Rsh z] \cup \mathfrak{B}^{\approx}_z\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!] \text{ iff } [z \Rsh y] \cup \mathfrak{B}^{\approx}_y\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_y[\![p]\!]. \tag{9.15}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ Suppose that $[y \Rsh z] \cup \mathfrak{B}^{\approx}_z\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]$. By assumption (1),

$$[z \Rsh y] \cup [y \Rsh y] \cup \mathfrak{B}^{\approx}_y\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_y[\![p]\!].$$

Since $\models_{\mathcal{K}} [y \Rsh y]$, we have

$$[z \Rsh y] \cup \mathfrak{B}^{\approx}_y\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_y[\![p]\!].$$

$\boxed{(2)\Rightarrow(3)}$ By symmetry. $\boxed{(3)\Rightarrow(1)}$ Suppose that $\mathfrak{B}^{\approx}_z\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]$. Then certainly, $[y \Rsh z] \cup \mathfrak{B}^{\approx}_z\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]$, and so by (3), $[z \Rsh y] \cup \mathfrak{B}^{\approx}_y\,[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_y[\![p]\!]$. $\diamond$

Remark 9.44 If $\mathfrak{B}_*$ *pivots finitarily for* $\mathcal{K}$ then $\mathfrak{B}_*$ *pivots for* $\mathcal{K}$.

Remark 9.45 The following conditions are equivalent.

1. $\mathfrak{B}_*$ pivots finitarily for $\mathcal{K}$.
2. For any variable z and any $P \cup \{p\} \subseteq \mathsf{Tm}$, (9.11) holds *iff* there exists a finite subset $Z \subseteq_{\mathsf{f}} \mathfrak{B}_z/\bot_{\mathcal{K}}$ and a finite subset $P' \subseteq_{\mathsf{f}} P$ such that, for all variables y, (9.12) holds.

Proof. $\boxed{(1)\Rightarrow(2)}$ Suppose that (1) is valid, and suppose that there exists a finite subset $Z \subseteq_{\mathsf{f}} \mathfrak{B}_z/\bot_{\mathcal{K}}$ and a finite subset $P' \subseteq_{\mathsf{f}} P$ such that, for all variables y, (9.12) holds. Taking $y = z$, in (9.12) yields

$$\models_{\mathcal{K}} \bigwedge \mathfrak{B}^{\approx}_z[Z] \textsf{ and} \bigwedge \mathfrak{B}^{\approx}_z[P'] \rightarrow \bigwedge \mathfrak{B}^{\approx}_z[\![p]\!],$$

and since $\models_{\mathcal{K}} \bigwedge \mathfrak{B}^{\approx}_z[Z]$, by Remark 9.40,

$$\models_{\mathcal{K}} \bigwedge \mathfrak{B}^{\approx}_z[P'] \rightarrow \bigwedge \mathfrak{B}^{\approx}_z[\![p]\!],$$

and consequently, (9.11) holds, which suffices. $\boxed{(2)\Rightarrow(1)}$ Trivial. $\diamond$

The property that $\mathfrak{B}_*$ *pivots finitarily* for $\mathcal{K}$ will play an important role in our theory of *parametrized algebraization* developed in Part V. We shall characterize this property in terms of $\mathcal{K}$ being a $\mathfrak{B}_*$-*algebraic semantics* for some sentential calculus (see Theorem 13.22 on page 399).

Definition 9.46 (Having Finite Pivots) A **realization of the pivots from** z is a subset $Z \subseteq \mathfrak{B}_z/\bot^{\mathcal{K}}$ such that for all variables y (including $y = z$), $\mathfrak{B}_y^{\approx}[Z] \models_{\mathcal{K}} [z \Rsh y]$. We say that $\mathfrak{B}_*$ has **finite pivots** for $\mathcal{K}$, if for *some* variable z, there exists a *finite* realization of the pivots from z. $\square$

The condition of *having finite pivots* is independent of the variable z. More precisely we have the following.

Remark 9.47 $\mathfrak{B}_*$ has finite pivots for $\mathcal{K}$ iff for *all* variables z, there exists a *finite* subset $Z \subseteq \mathfrak{B}_z/\bot^{\mathcal{K}}$ such that for all variables y (including $y = z$), $\mathfrak{B}_y^{\approx}[Z] \models_{\mathcal{K}} [z \Rsh y]$.

Proof. We repeatedly implicitly invoke structurality of $\models_{\mathcal{K}}$ and the involution case of (2) of Proposition 9.34. Suppose that for some variable z, there exists a finite subset $Z \subseteq \mathfrak{B}_z/\bot^{\mathcal{K}}$ such that for *all* variables y, $\mathfrak{B}_y^{\approx}[Z] \models_{\mathcal{K}} [z \Rsh y]$. Let w be any variable distinct from z and consider the transposition $\sigma = \langle z, w\rangle$ (which is an involution). Since $\models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[Z]$, $\models_{\mathcal{K}} \mathfrak{B}_w^{\approx}[\sigma[Z]]$, and so $\sigma[Z] \subseteq_{\mathsf{f}} \mathfrak{B}_w/\bot^{\mathcal{K}}$. (*It suffices to show that for all variables y,* $\mathfrak{B}_y^{\approx}[\sigma[Z]] \models_{\mathcal{K}} [w \Rsh y]$.) Let y be a variable distinct from z and w. Since $\mathfrak{B}_y^{\approx}[Z] \models_{\mathcal{K}} [z \Rsh y]$, $\mathfrak{B}_y^{\approx}[\sigma[Z]] \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[\sigma[\mathfrak{B}_z/\bot_{\mathcal{K}}]] = [w \Rsh y]$. We now consider the cases for $y = w$ and $y = z$. Since $\mathfrak{B}_z^{\approx}[Z] \models_{\mathcal{K}} [z \Rsh z]$, $\mathfrak{B}_w^{\approx}[\sigma[Z]] \models_{\mathcal{K}} \mathfrak{B}_w^{\approx}[\sigma[\mathfrak{B}_z/\bot_{\mathcal{K}}]] = [w \Rsh w]$, and since $\mathfrak{B}_w^{\approx}[Z] \models_{\mathcal{K}} \mathfrak{B}_w^{\approx}[\mathfrak{B}_z/\bot_{\mathcal{K}}]$, $\mathfrak{B}_z^{\approx}[\sigma[Z]] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\sigma[\mathfrak{B}_z/\bot_{\mathcal{K}}]] = [w \Rsh z]$. $\diamond$

Remark 9.48 Notice that the choice of finite Z in the definition of *pivoting finitarily* is made after the choice of $P \cup \{p\}$, while *having finite pivots* amounts to a *global choice* of a Z (and does not necessarily imply pivoting).

Remark 9.49 If $\mathfrak{B}_*$ *has finite pivots for* $\mathcal{K}$ and $\mathfrak{B}_*$ *pivots for* $\mathcal{K}$ then $\mathfrak{B}_*$ *pivots finitarily for* $\mathcal{K}$.

Definition 9.50 (Symmetric Pivots) We say that $\mathfrak{B}_*$ has **symmetric pivots** in $\mathcal{K}$ (or that $\mathfrak{B}_*$ is **pivot symmetric** in $\mathcal{K}$) if for all distinct variables z and y, $[z \Rsh y] \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathcal{K}} [y \Rsh z]$. $\square$

Remark 9.51 If $[z \Rsh y] \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathcal{K}} [y \Rsh z]$ for *some* distinct variables z and y, then $\mathfrak{B}_*$ has symmetric pivots in $\mathcal{K}$.

Proof. Let w be a variable distinct from y and z and consider the transposition $\sigma = \langle y, w\rangle$. Since $[z \Rsh y] \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathcal{K}} [y \Rsh z]$, by structurality, $\sigma[[z \Rsh y]] \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathcal{K}} \sigma[[y \Rsh z]]$, hence $\mathfrak{B}_w^{\approx}[\sigma[\mathfrak{B}_z/\bot_{\mathcal{K}}]] \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\sigma[\mathfrak{B}_y/\bot_{\mathcal{K}}]]$, and since σ is an involution, by the involution case of (2) of Proposition 9.34, $[z \Rsh w] \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathcal{K}} [w \Rsh z]$. Repeated application of this argument may be used to replace z with any other variable. $\diamond$

9.2.4 Examples

We now consider a series of examples that will play an important role in this text.

An important class of systems of binary equations, given the number of other important systems that they encompass, are those that we shall term *separable*; separable in the sense that all the terms on the left hand side of the equations are in one of the two variables, and all the terms on the right hand side are terms in the other variable. Of course, terms in one variable are *unary* terms. Such systems give rise to a family of logics, which we have termed the *logics of separable binary systems* (see Example 12.42 on page 384), a family of logics amenable to the theory of parametrized algebraization.

Example 9.52 (Separable Binary Systems)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras.

Definition 9.53 (Separable Binary Systems) We shall call a system $\mathfrak{B}$ of binary equations separable over $\mathcal{K}$ (or simply $\mathcal{K}$-separable) if there exists a binary system $\mathfrak{B}'(x,y) = \{\boldsymbol{u}_i(x) \approx \boldsymbol{u}'_i(y) : i \in n\}$, such that $\mathfrak{B}$ and $\mathfrak{B}'$ are $\mathcal{K}$-equivalent. □

For ease of discourse, when dealing with separable binary systems, we shall work directly with a system of the form $\{\boldsymbol{u}_i(x) \approx \boldsymbol{u}'_i(y) : i \in n\}$, rather than continually invoking $\mathcal{K}$-equivalence. Let $\mathfrak{B}(x,y) = \{\boldsymbol{u}_i(x) \approx \boldsymbol{u}'_i(y) : i \in n\}$ be a ($\mathcal{K}$-separable) binary system with *non-trivial* variable bases.

Remark 9.54 $\mathfrak{B}^{\approx}_r[\![q]\!] = \{\boldsymbol{u}_i(q) \approx \boldsymbol{u}'_i(r) : i \in n\}$.

Remark 9.55 $\mathfrak{B}_z/\bot_{\mathcal{K}} = \{p \ : \models_{\mathcal{K}} \bigwedge_{i\in n}\boldsymbol{u}_i(p) \approx \boldsymbol{u}'_i(z)\}$.

Lemma 9.56 Let $p \in \mathfrak{B}_z/\bot^{\mathcal{K}}$. Then

$$[z \Rsh y] \mathrel{=\joinrel|\joinrel\models_{\mathcal{K}}} \mathfrak{B}^{\approx}_y[\![p]\!] \mathrel{=\joinrel|\joinrel\models_{\mathcal{K}}} \{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z) \ : \ i \in n\} \mathrel{=\joinrel|\joinrel\models_{\mathcal{K}}} [y \Rsh z]. \tag{9.16}$$

Proof. $\boxed{[z \Rsh y] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_y[\![p]\!]}$ Trivial. $\boxed{\mathfrak{B}^{\approx}_y[\![p]\!] \models_{\mathcal{K}} \{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z) \ : \ i \in n\}}$ By definition, (i) $\models_{\mathcal{K}} \{\boldsymbol{u}_i(p) \approx \boldsymbol{u}'_i(z) \ : \ i \in n\}$ and (ii) $\mathfrak{B}^{\approx}_y[\![p]\!] = \{\boldsymbol{u}_i(p) \approx \boldsymbol{u}'_i(y) \ : \ i \in n\}$. Combining (i) and (ii), $\mathfrak{B}^{\approx}_y[\![p]\!] \models_{\mathcal{K}} \{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z) : i \in n\}$. $\boxed{\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z) \ : \ i \in n\} \models_{\mathcal{K}} [z \Rsh y]}$ Let $q \in \mathfrak{B}_z/\bot_{\mathcal{K}}$. By definition, $\models_{\mathcal{K}} \{\boldsymbol{u}_i(q) \approx \boldsymbol{u}'_i(z) \ : \ i \in n\}$. Hence $\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z) \ : \ i \in n\} \models_{\mathcal{K}} \{\boldsymbol{u}_i(q) \approx \boldsymbol{u}'_i(y) \ : \ i \in n\} = \mathfrak{B}^{\approx}_y[\![q]\!]$. Hence $\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z) \ : \ i \in n\} \models_{\mathcal{K}} [z \Rsh y]$. The remaining relationships follow from the symmetry of $\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z) \ : \ i \in n\}$. ⋄

Corollary 9.57 $\mathfrak{B}_*$ has finite pivots, has symmetric pivots and pivots in $\mathcal{K}$; consequently, $\mathfrak{B}_*$ pivots finitarily in $\mathcal{K}$. The singleton $\{p\}$, where $\models_{\mathcal{K}} \bigwedge_{i\in n}\boldsymbol{u}_i(p) \approx \boldsymbol{u}'_i(z)$, is a finite realization of the pivot from z.

Proof. $\boxed{\text{Finite Pivots}}$ Follows at once from Lemma 9.56. $\boxed{\text{Pivots}}$ Suppose that $\mathfrak{B}^{\approx}_z[P] \models_{\mathcal{K}}$

$\mathfrak{B}^{\approx}_{z}[\![p]\!]$. Then by Lemma 9.56,

$$\begin{aligned}[z \upharpoonright y] \cup \mathfrak{B}^{\approx}_{y}[P] &\models_{\mathcal{K}} \bigcup_{i \in n}\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z)\} \cup \bigcup_{p \in P}\bigcup_{i \in n}\{\boldsymbol{u}_i(p) \approx \boldsymbol{u}'_i(y)\} \\ &\models_{\mathcal{K}} \bigcup_{i \in n}\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z)\} \cup \bigcup_{p \in P}\bigcup_{i \in n}\{\boldsymbol{u}_i(p) \approx \boldsymbol{u}'_i(z)\} \\ &= \bigcup_{i \in n}\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z)\} \cup \mathfrak{B}^{\approx}_{z}[P] \\ &\models_{\mathcal{K}} \bigcup_{i \in n}\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z)\} \cup \mathfrak{B}^{\approx}_{z}[\![p]\!] \\ &= \bigcup_{i \in n}\{\boldsymbol{u}'_i(y) \approx \boldsymbol{u}'_i(z)\} \cup \bigcup_{i \in n}\{\boldsymbol{u}_i(p) \approx \boldsymbol{u}'_i(z)\} \\ &\models_{\mathcal{K}} \bigcup_{i \in n}\{\boldsymbol{u}_i(p) \approx \boldsymbol{u}'_i(y)\} \\ &= \mathfrak{B}^{\approx}_{y}[\![p]\!].\end{aligned}$$

$\diamond$

$\square$

The next example is a special case of separable binary systems, namely when $\mathfrak{B}(x, y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$, where $\boldsymbol{u}$ is a $\mathcal{K}$-unary term. Such systems give rise to logics that we have termed the logics of *identified membership* (see Example 12.46 on page 385).

Example 9.58 ($\mathfrak{B}(x, y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$)

Let $\boldsymbol{u}$ be a unary term over $\mathcal{K}$. We shall denote the binary translation $\mathfrak{B}(x, y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$ by (emboldened) $\boldsymbol{u}(x, y)$. We remark on some (obvious) properties.

Remark 9.59 The following are all valid.

1. $\boldsymbol{u}^{\approx}_{r}[\![q]\!] = \{q \approx \boldsymbol{u}(r)\}$.
2. $\boldsymbol{u}_z/\bot_{\mathcal{K}} = \{p : \models_{\mathcal{K}} p \approx \boldsymbol{u}(z)\}$, so $\boldsymbol{u}(z) \in \boldsymbol{u}_z/\bot_{\mathcal{K}}$.
3. $\{\boldsymbol{u}(y) \approx \boldsymbol{u}(z)\} \subseteq [y \upharpoonright z] \cap [z \upharpoonright y]$.
4. $\mathsf{PN}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A}) = \{\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!] \ : \ \alpha \in \mathsf{Con}_{\mathcal{K}}(\mathbf{A})\}$.

$\square$

The following results follow at once from Lemma 9.56 and Corollary 9.57 of the previous example.

Remark 9.60 $[y \upharpoonright z] \dashv\models_{\mathcal{K}} \{\boldsymbol{u}(y) \approx \boldsymbol{u}(z)\} \dashv\models_{\mathcal{K}} [z \upharpoonright y]$.

Proposition 9.61 $\boldsymbol{u}_*$ has finite pivots, has symmetric pivots and $\boldsymbol{u}_*$ pivots finitarily in $\mathcal{K}$. $\square$

Remark 9.62 For algebra $\mathbf{A}$, $b \in \mathsf{uni}(\mathbf{A})$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, $\boldsymbol{u}^{\mathbf{A}}_{b}/\,\alpha = \alpha[\![\boldsymbol{u}^{\mathbf{A}}(b)]\!]$.

$\square$

A special case, of the previous example, occurs when the unary term $\boldsymbol{u}$ is *idempotent* over $\mathcal{K}$. The specialness of such systems lies in the fact that the *normals*, with the empty-set removed if the empty-set is normal, coincide with the *principal* normals, and as a consequence, the normals form an *algebraic* closed system, as opposed to just a closed system.

Recall the definition of a $\mathcal{K}$-coset, given in Definition 1.372 on page 70. We take the opportunity to define the notion of a $\langle\mathcal{K},\boldsymbol{u}\rangle$*-coset on* $\mathbf{A}$, which are the $\mathcal{K}$-cosets on $\mathbf{A}$ containing $\boldsymbol{u}^{\mathbf{A}}(a)$ for some point a. A special case occurs when $\boldsymbol{u}$ is constant over $\mathcal{K}$. While the intersection of all $\langle\mathcal{K},\boldsymbol{u}\rangle$-cosets is non-empty in the case that $\boldsymbol{u}$ is a $\mathcal{K}$-constant (since this intersection contains $\perp^{\mathcal{K}}_{\mathbf{A}}[\![\boldsymbol{u}^{\mathbf{A}}]\!]$), when $\boldsymbol{u}$ is not constant over $\mathcal{K}$, the intersection of $\langle\mathcal{K},\boldsymbol{u}\rangle$-cosets is generally empty. In order to maintain compatibility with a more general notion of $\boldsymbol{u}$-cosets to be encountered later, while still encompassing *relative cosets* as a special case of $\langle\mathcal{K},\boldsymbol{u}\rangle$-cosets ($\langle\mathcal{K},\boldsymbol{u}\rangle$-cosets with $\boldsymbol{u}(y)=y$), we shall need to defined the $\langle\mathcal{K},\boldsymbol{u}\rangle$-cosets to include the empty set precisely when $\boldsymbol{u}$ is non-constant over $\mathcal{K}$. Note that results that pertain to idempotent $\boldsymbol{u}$ generally, that is, without explicit mention that $\boldsymbol{u}$ be non-constant over $\mathcal{K}$, also pertain to 0, but not conversely.

Example 9.63 (Idempotent $\langle\mathcal{K},\boldsymbol{u}\rangle$-Cosets)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and $\boldsymbol{u}$ a $\mathcal{K}$-unary term.

Definition 9.64 ($\mathcal{K}$-Idempotence) We say that $\boldsymbol{u}$ is **$\mathcal{K}$-idempotent** (or **idempotent over $\mathcal{K}$**) if, for some variable y, $\models_{\mathcal{K}} \boldsymbol{u}(\boldsymbol{u}(y)) \approx \boldsymbol{u}(y)$. □

Remark 9.65 The choice of variable, in the definition of idempotence, is immaterial.

Convention 9.66 For the remainder of this example, unless specified to the contrary, let $\boldsymbol{u}$ denote an arbitrary unary term *idempotent* over $\mathcal{K}$ and let $\mathbf{u}(x,y)=\{\langle x,\boldsymbol{u}(y)\rangle\}$. When we make reference to a term 0, we mean that this term is constant over $\mathcal{K}$ and that we are taking $\boldsymbol{u}=0$, in which case $\mathbf{u}(x,y)=\{\langle x,0\rangle\}$, which we denote by $\mathbf{0}(x)$.

We now introduce the notions of *proper idempotent $\boldsymbol{u}$-cosets* and *idempotent $\boldsymbol{u}$-cosets*, which, as shall transpire, are just special names for the principal normals (by definition) and non-empty normals (see Corollary 9.71) respectively.

Definition 9.67 (Proper Idempotent $\langle\mathcal{K},\boldsymbol{u}\rangle$-Cosets) Let $\boldsymbol{u}$ be a unary term idempotent over quasivariety $\mathcal{K}$. We define

$$\mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A},\boldsymbol{u}) \doteq \mathsf{Sol}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A}) = \{\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!] \,:\, \alpha\in\mathsf{Con}^{\mathcal{K}}(\mathbf{A}), a\in\mathsf{uni}(\mathbf{A})\},$$

the members of which are called **proper idempotent $\langle\mathcal{K},\boldsymbol{u}\rangle$-cosets** of $\mathbf{A}$. The subscript i is to draw attention to the fact that $\mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A},\boldsymbol{u})$ is *only* defined for $\boldsymbol{u}$ idempotent over $\mathcal{K}$. Let $\mathsf{Cos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A},\boldsymbol{u})$ denote the set of proper idempotent $\boldsymbol{u}$-cosets $\mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A},\boldsymbol{u})$ together with the empty-set (called the **improper $\langle\mathcal{K},\boldsymbol{u}\rangle$-coset**) *precisely* when $\boldsymbol{u}$ is *non-constant* over $\mathcal{K}$. We shall show that $\mathsf{Cos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A},\boldsymbol{u})$ constitutes a *finitary closed system* (see Theorem 9.73). Since $\mathcal{K}$-constants are $\mathcal{K}$-idempotent, we may simply speak of a 0**-coset**, where 0 is $\mathcal{K}$-constant, in which case we tend to write $\mathsf{Cos}^{\mathcal{K}}_{\mathsf{c}}(\mathbf{A},0)$ for $\mathsf{Cos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A},\boldsymbol{u})$; this notation is only defined for $\mathcal{K}$-constant 0 and any use of this notion implicitly implies that 0 is a $\mathcal{K}$-constant. □

Warning 9.68 Later we shall be introducing *another* notion of *coset* named a $\langle\mathcal{K},\mathfrak{B}_*\rangle$-coset; such cosets are special filters of a logic (see Definition 12.40), where $\mathfrak{B}$ is a binary system of equations. A special case of these, occurring when $\mathfrak{B}(x,y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$, are called $\langle\mathcal{K}, \boldsymbol{u}\rangle$-*cosets* (see Definition 12.47 on page 385). The reader is urged to distinguish between $\langle\mathcal{K}, \boldsymbol{u}\rangle$-cosets and *idempotent* $\langle\mathcal{K}, \boldsymbol{u}\rangle$-cosets, the former being defined for arbitrary unary $\boldsymbol{u}$, while the latter is only defined for $\boldsymbol{u}$ idempotent over $\mathcal{K}$. In the case that $\boldsymbol{u}$ is $\mathcal{K}$-idempotent, however, the two notions coincide (see Example 12.53 on page 386).

While we have not found a counter-example (see Open Problem 9.27), it is our intuition that generally (i.e., for arbitrary $\mathfrak{B}$), $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ need not be finitary, even for $\mathfrak{B}(x,y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$ for arbitrary (i.e., not necessarily $\mathcal{K}$-idempotent) $\boldsymbol{u}$. We shall now demonstrate that for $\mathcal{K}$-*idempotent* $\boldsymbol{u}$, $\mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A})$ is indeed finitary, essentially because the *normals* are precisely the *principal normals* together with the empty-set. We begin with some technical observations.

Remark 9.69 For any $\mathfrak{a}$-algebra $\mathbf{A}$ and any $\mathcal{K}$-congruence α on $\mathbf{A}$, if $b' \,\alpha\, \boldsymbol{u}^{\mathbf{A}}(b)$ then $\boldsymbol{u}^{\mathbf{A}}(b') \,\alpha\, b'$.

Proof. If $b' \,\alpha\, \boldsymbol{u}^{\mathbf{A}}(b)$, then $\boldsymbol{u}^{\mathbf{A}}(b') \,\alpha\, \boldsymbol{u}^{\mathbf{A}}(\boldsymbol{u}^{\mathbf{A}}(b)) \,\alpha\, \boldsymbol{u}^{\mathbf{A}}(b) \,\alpha\, b'$. ◇

The following result shows that when the intersection of idempotent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-cosets is non-empty, this intersection is itself an idempotent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-coset.

Lemma 9.70 Let $\emptyset \neq \{\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(b_i)]\!] \;:\; i \in I\} \subseteq \mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u})$. If there exists $b \in \bigcap_{i\in I}(\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(b_i)]\!])$ then $\bigcap_{i\in I}(\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(b_i)]\!]) = \bigcap_{i\in I}\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(b)]\!] \in \mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u})$.

Proof. Let $b \in \bigcap_{i\in I}(\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(b_i)]\!])$. By Lemma 4.105 of Example 4.103 on page 160, $\bigcap_{i\in I}(\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(b_i)]\!]) = (\bigcap_{i\in I}\alpha_i)[\![b]\!]$. (*It suffices to show that* $\langle b, \boldsymbol{u}^{\mathbf{A}}(b)\rangle \in (\bigcap_{i\in I}\alpha_i)$.) For each $i \in I$, $b \,\alpha_i\, \boldsymbol{u}^{\mathbf{A}}(b_i)$, hence $\boldsymbol{u}^{\mathbf{A}}(b) \,\alpha_i\, b$, by Remark 9.69. ◇

It follows immediately, from the preceding lemma and the definition of normals, that the proper $\boldsymbol{u}$-cosets are precisely the non-empty normals of $\boldsymbol{u}$. Note that while the normals are *never empty* in the case that $\boldsymbol{u}$ is $\mathcal{K}$-constant, the following result, as phrased, is still valid in this case.

Corollary 9.71 $\mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}) = \mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A}) - \{\emptyset\}$, for each $\mathfrak{a}$-algebra $\mathbf{A}$. If $\boldsymbol{u}$ is *not* a $\mathcal{K}$-constant, then $\mathsf{Cos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}) = \mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A}) \cup \{\emptyset\}$, for each $\mathfrak{a}$-algebra $\mathbf{A}$. □

The $\mathcal{K}$-constantness of a $\mathcal{K}$-unary term $\boldsymbol{u}$ is characterizable as the coincidence of proper idempotent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-cosets and normals.

Corollary 9.72 For a $\mathcal{K}$-idempotent unary term $\boldsymbol{u}$, the following conditions are equivalent.

1. $\boldsymbol{u}$ is constant over $\mathcal{K}$.
2. $\mathsf{Sol}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A}) = \mathsf{Sol}^{\mathcal{K}}_{\boldsymbol{u}_b}(\mathbf{A})$, for each $\mathfrak{a}$-algebra $\mathbf{A}$ and $a, b \in \mathsf{uni}(\mathbf{A})$.
3. $\mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}) = \mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A})$, for each $\mathfrak{a}$-algebra $\mathbf{A}$.
4. $\mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}) = \mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A})$, for all $\mathbf{A} \in \mathcal{K}$.

In this case, $\mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}) = \{\alpha[\![\boldsymbol{u}^{\mathbf{A}}]\!] \;:\; \alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})\} = \mathsf{Sol}^{\mathcal{K}}_{\boldsymbol{u}_b}(\mathbf{A})$, for any $b \in \mathsf{uni}(\mathbf{A})$.

Proof. We prove the only not entirely trivial case.
$\boxed{(4)\Rightarrow(1)}$ Consider $\mathbf{F}_{\mathcal{K}}$ and $\perp_{\mathcal{K}}$. Since $\perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{x})]\!], \perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{y})]\!] \in \mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A})$ and $\mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A})$ is closed under intersection, $\perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{y})]\!] \in \mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A}) \in \mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A}) = \mathsf{PrpCos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u})$. Hence

$\perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{x})]\!] \cap \perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{y})]\!] = \perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{p})]\!]$, for some term p. Hence $\perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{x})]\!] \cap \perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{y})]\!] \neq \emptyset$, hence $\perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{x})]\!] = \perp_{\mathcal{K}}[\![\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{y})]\!]$. So $\boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{y}) \perp_{\mathcal{K}} \boldsymbol{u}^{\mathbf{F}_{\mathcal{K}}}(\overline{x})$. So by Lemma 1.457 on page 88, $\models_{\mathcal{K}} \boldsymbol{u}(x) \approx \boldsymbol{u}(y)$, and hence $\boldsymbol{u}$ is constant over $\mathcal{K}$. $\diamond$

For non-constant $\boldsymbol{u}$, there is a subtle distinction between $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$ and $\mathsf{N}_{\boldsymbol{u}_*}^{\mathcal{K}}(\mathbf{A})$ that must be noted. It is certainly true, by Corollary 9.71, that $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u}) = \mathsf{N}_{\boldsymbol{u}_*}^{\mathcal{K}}(\mathbf{A}) \cup \{\emptyset\}$ and 'usually' (or at least for one algebra in $\mathcal{K}$) $\emptyset \in \mathsf{N}_{\boldsymbol{u}_*}^{\mathcal{K}}(\mathbf{A})$ by Corollary 9.72. It may be possible, however, that $\emptyset \notin \mathsf{N}_{\boldsymbol{u}_*}^{\mathcal{K}}(\mathbf{A})$ for some algebra $\mathbf{A}$, since $\boldsymbol{u}$ may be constant with respect to $\mathbf{A}$ while still failing to be $\mathcal{K}$-constant.

We now show that the idempotent $\boldsymbol{u}$-cosets form an algebraic closed system. In the constant case the result is essentially trivial, in the light of Corollary 9.72 and the fact that $\mathsf{Sol}_{\boldsymbol{u}_b}^{\mathcal{K}}(\mathbf{A})$ is a finitary closed system. In the non-constant case, we have effectively already established that $\boldsymbol{u}$-cosets form a closed system (by Lemma 9.70 or by Corollary 9.71), and so only finitariness requires proof. Note that while the fact that the $\boldsymbol{u}$-cosets form a closed system depends on idempotence, the proof of finitariness does not.

Theorem 9.73 For each algebra $\mathbf{A}$, $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$ is an *algebraic* closed system.

Proof. $\boxed{\boldsymbol{u} \text{ is } \mathcal{K}\text{-constant}}$ If $\boldsymbol{u}$ is a $\mathcal{K}$-constant, the result follows by Corollary 9.72 and the fact that $\mathsf{Sol}_{\boldsymbol{u}_b}^{\mathcal{K}}(\mathbf{A})$ is a finitary closed system. $\boxed{\boldsymbol{u} \text{ is not } \mathcal{K}\text{-constant}}$ Assume that $\boldsymbol{u}$ is not $\mathcal{K}$-constant. By Corollary 9.71, $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u}) = \mathsf{N}_{\boldsymbol{u}_*}^{\mathcal{K}}(\mathbf{A}) \cup \{\emptyset\}$. Since $\mathsf{N}_{\boldsymbol{u}_*}^{\mathcal{K}}(\mathbf{A})$ is a closed system, so is $\mathsf{N}_{\boldsymbol{u}_*}^{\mathcal{K}}(\mathbf{A}) \cup \{\emptyset\}$; adding the empty-set to a closed system yields a closed system. (*We prove finitariness.*) Let $\emptyset \neq G \subseteq \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$. Since the empty-set does not contribute to unions, we may assume, without loss of generality, that $\emptyset \neq G \subseteq \mathsf{PrpCos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$. So $G = \{\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] : i \in I\}$, for some I and $\alpha_i \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. For each $i \in I$, let $\alpha_i' = \big\|(\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!])^2\big\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}$. Since $(\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!])^2 \subseteq \alpha_i$, $\alpha_i' \subseteq \alpha_i$, hence $\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \subseteq \alpha_i'[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \subseteq \alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!]$ and so $\alpha_i'[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] = \alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!]$. $\boxed{\text{Claim: } \{\alpha_i' : i \in I\} \text{ is } \subseteq\text{-directed}}$ Let $i, j \in I$. Since G is directed, there exists $k \in I$ with $\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \cup \alpha_j[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \subseteq \alpha_k[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!]$. So $\alpha_i' \cup \alpha_j' \subseteq \alpha_k'$. $\Box$ Consequently, $\alpha \doteq \bigcup_{i \in I} \alpha_i' \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. Let $i \in I$. (*It suffices to show that* $\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] = \bigcup \mathcal{C}$.) $\boxed{\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \supseteq \bigcup \mathcal{C}}$ Let $b \in \bigcup \mathcal{C}$. Then there exists $j \in I$ with $b \in \alpha_j[\![\boldsymbol{u}^{\mathbf{A}}(a_j)]\!]$. By directedness of $\mathcal{C}$, there exists $k \in I$ with $\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \cup \alpha_j[\![\boldsymbol{u}^{\mathbf{A}}(a_j)]\!] \subseteq \alpha_k[\![\boldsymbol{u}^{\mathbf{A}}(a_k)]\!]$. So $b \,\alpha_k\, \boldsymbol{u}^{\mathbf{A}}(a_i)$ and $\boldsymbol{u}^{\mathbf{A}}(a_i) \,\alpha_k\, \boldsymbol{u}^{\mathbf{A}}(a_k)$. Hence $b \,\alpha_k'\, \boldsymbol{u}^{\mathbf{A}}(a_i)$. Since $\alpha_k' \subseteq \alpha$, $b \,\alpha\, \boldsymbol{u}^{\mathbf{A}}(a_i)$, i.e., $b \in \alpha[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!]$. $\boxed{\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \subseteq \bigcup \mathcal{C}}$ Let $b \in \alpha[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!]$. So there exists $j \in I$ with $b \,\alpha_j'\, \boldsymbol{u}^{\mathbf{A}}(a_i)$. By directedness of $\mathcal{C}$, there exists $k \in I$ with $\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \cup \alpha_j[\![\boldsymbol{u}^{\mathbf{A}}(a_j)]\!] \subseteq \alpha_k[\![\boldsymbol{u}^{\mathbf{A}}(a_k)]\!]$. Hence $\alpha_i' \cup \alpha_j' \subseteq \alpha_k'$. Since $b \,\alpha_j'\, \boldsymbol{u}^{\mathbf{A}}(a_i)$, $b \,\alpha_k'\, \boldsymbol{u}^{\mathbf{A}}(a_i)$, and since $\boldsymbol{u}^{\mathbf{A}}(a_i) \,\alpha_k'\, \boldsymbol{u}^{\mathbf{A}}(a_k)$ (since $\alpha_i[\![\boldsymbol{u}^{\mathbf{A}}(a_i)]\!] \subseteq \alpha_k[\![\boldsymbol{u}^{\mathbf{A}}(a_k)]\!]$), $b \,\alpha_k'\, \boldsymbol{u}^{\mathbf{A}}(a_k)$. We have already established that $\alpha_k' \subseteq \alpha_k$, hence $b \,\alpha_k\, \boldsymbol{u}^{\mathbf{A}}(a_k)$. Consequently, $b \in \bigcup \mathcal{C}$. $\diamond$

Corollary 9.74 The closed system $\mathsf{N}_{\boldsymbol{u}_*}^{\mathcal{K}}(\mathbf{A})$ is finitary. $\Box$

Convention 9.75 The algebraic lattice associated with the finitary closed system $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$ is denoted by $\mathbf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$. For $\mathcal{K}$-constant 0, we write $\mathbf{Cos}_{\mathsf{c}}^{\mathcal{K}}(\mathbf{A}, 0)$ for $\mathbf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, 0)$; a notation only defined for $\mathcal{K}$-constant terms 0.

The following characterization of closure in $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$ follows from definitions, Remark 5.83 on page 198 and Remark 9.69.

Remark 9.76 For $\emptyset \neq A \subseteq \mathsf{uni}(\mathbf{A})$ and $a \in A$, $\|A\|_{\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A},\boldsymbol{u})} = \left\| A \times \boldsymbol{u}^{\mathbf{A}}[A] \right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}[A] = \left\| A \times \{\boldsymbol{u}^{\mathbf{A}}(a)\} \right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}[\![a]\!] = \left\| A \times \{\boldsymbol{u}^{\mathbf{A}}(a)\} \right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!]$. If $\boldsymbol{u}$ is *not* a $\mathcal{K}$-constant, then $\|\emptyset\|_{\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A},\boldsymbol{u})} = \emptyset$.

Remark 9.77 For $\mathcal{K}$-constant 0, $\|A\|_{\mathsf{Cos}_{\mathsf{c}}^{\mathcal{K}}(\mathbf{A},0)} = \left\| A \times \{0^{\mathbf{A}}\} \right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}[\![0^{\mathbf{A}}]\!]$. In particular, $\|\emptyset\|_{\mathsf{Cos}_{\mathsf{c}}^{\mathcal{K}}(\mathbf{A},0)} = \bot_{\mathbf{A}}^{\mathcal{K}}[\![0^{\mathbf{A}}]\!]$. □

The following characterization of closure in $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A},\boldsymbol{u})$ follows by the previous remarks, Corollary 9.26 and Remark 9.69.

Proposition 9.78 For $\emptyset \neq A \subseteq \mathsf{uni}(\mathbf{A})$, $A \vdash_{\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A},\boldsymbol{u})} a$ iff $\forall [\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})] \forall [b \in \mathsf{uni}(\mathbf{A})]\, A \times \{\boldsymbol{u}^{\mathbf{A}}(b)\} \subseteq \alpha \rightarrow a\, \alpha\, \boldsymbol{u}^{\mathbf{A}}(b)$. If $\boldsymbol{u}$ is *not* a $\mathcal{K}$-constant then there are no consequences from the empty-set. □

Rephrasing the previous result for the case of constant 0, yields a slightly simpler formulation.

Corollary 9.79 For constant 0, $A \vdash_{\mathsf{Cos}_{\mathsf{c}}^{\mathcal{K}}(\mathbf{A},0)} a$ iff $\forall\, [\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})]\; A \times \{0^{\mathbf{A}}\} \subseteq \alpha \rightarrow a\, \alpha\, 0^{\mathbf{A}}$. □

The usefulness of the notion of idempotent $\langle \mathcal{K}, \boldsymbol{u} \rangle$-cosets lies in the fact that they unify both constant $\langle \mathcal{K}, 0 \rangle$-cosets (by definition) and $\mathcal{K}$-cosets, while still forming *finitary* closed systems. In the latter case, note that for a *variable* y, the idempotent $\langle \mathcal{K}, y \rangle$-cosets and the $\mathcal{K}$-cosets on an algebra coincide; the only potential mismatch occurs when $\mathcal{K}$ is trivial, but in this case every variable is a $\mathcal{K}$-constant, and so the problem is avoided.

Remark 9.80 $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, y) = \mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$, where y is a variable.

Proof. Certainly, the proper $\langle \mathcal{K}, y \rangle$-cosets coincide with the proper $\mathcal{K}$-cosets. In the case that $\mathcal{K}$ is non-trivial, y is not a $\mathcal{K}$-constant, in which case the improper $\langle \mathcal{K}, y \rangle$-coset is a member of $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A},\boldsymbol{u})$ and the improper $\mathcal{K}$-coset is a member of $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$, hence $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A},\boldsymbol{u}) = \mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$. In the case that $\mathcal{K}$ is trivial, y is a $\mathcal{K}$-constant, in which case $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A},\boldsymbol{u}) = \mathsf{Cos}_{\mathsf{c}}^{\mathcal{K}}(\mathbf{A}, y) = \{\mathsf{uni}(\mathbf{A})\} = \mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$, since the improper coset is not a member of $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$ when $\mathcal{K}$ is trivial. ◇

Since the idempotent $\langle \mathcal{K}, \boldsymbol{u} \rangle$-cosets of an algebra $\mathbf{A}$ form a finitary closed system over the universe of $\mathbf{A}$, they determine the theories of a finitary universal logic on $\mathbf{A}$. We now identify these logics.

Definition 9.81 (The Idempotent $\langle \mathcal{K}, \boldsymbol{u} \rangle$-Coset Logics)
The *finitary* $\mathbf{A}$-logic $L\left(\mathbf{A}, \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A},\boldsymbol{u})\right)$ is denoted by $U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$. We refer to these logics generically as the **idempotent $\langle \mathcal{K}, \boldsymbol{u} \rangle$-coset logics**. For a term 0 constant over $\mathcal{K}$, we may write $U_{\mathsf{c}}(\mathbf{A}, 0\text{-}\mathsf{cos}^{\mathcal{K}})$ for $U_{\mathsf{i}}(\mathbf{A}, 0\text{-}\mathsf{cos}^{\mathcal{K}})$. We refer to these logics generically as the **constant $\langle \mathcal{K}, 0 \rangle$-coset logics**. □

Note that by definition, $U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ has theorems iff $\boldsymbol{u}$ is constant over $\mathcal{K}$. Recall the definition of the universal logic $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$ of $\mathcal{K}$-cosets on $\mathbf{A}$ (see Example 6.85 on page 243) and the definition of the universal logic $U_{\mathbf{A}}(\mathcal{K}, 0)$, where 0 is a $\mathcal{K}$-constant (see Example 6.92 on page 245). By Remark 9.80 $U_{\mathsf{i}}(\mathbf{A}, y\text{-}\mathsf{cos}^{\mathcal{K}}) = U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$, and for $\mathcal{K}$-constant 0, $U_{\mathsf{c}}(\mathbf{A}, 0\text{-}\mathsf{cos}^{\mathcal{K}}) = U_{\mathbf{A}}(\mathcal{K}, 0)$.

We shall now show that for any algebras $\mathbf{A}$ and $\mathbf{B}$, $U_{\mathsf{i}}(\mathbf{B}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is an $\mathfrak{a}$-model of $U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$; this amounts to demonstrating that every homomorphism from $\mathbf{A}$ to $\mathbf{B}$ is continuous from $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$ to $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{B}, \boldsymbol{u})$. As a consequence, the idempotent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-logics are $\mathfrak{a}$-*structural*. Note that this result is valid even if $\boldsymbol{u}$ is constant over $\mathcal{K}$.

Proposition 9.82 $U_{\mathsf{i}}(\mathbf{B}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is an $\mathfrak{a}$-model of $U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$, for any $\mathfrak{a}$-algebras $\mathbf{A}$ and $\mathbf{B}$. Consequently, $U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is $\mathfrak{a}$-structural.

Proof. Let $f : \mathbf{A} \to \mathbf{B}$. (*We must show that f is continuous from $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$ to $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{B}, \boldsymbol{u})$.*) Let $\mathfrak{u} \in \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{B}, \boldsymbol{u})$. (*It suffices, by equivalent condition (3) of Theorem 5.40 on page 186, to show that $f^{-1}[\mathfrak{u}] \in \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$.*) $\boxed{\mathfrak{u} = \emptyset}$ Suppose that $\mathfrak{u} = \emptyset$. Then $\boldsymbol{u}$ is not a $\mathcal{K}$-constant, and hence $f^{-1}[\mathfrak{u}] = \emptyset \in \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$. $\boxed{\mathfrak{u} \neq \emptyset}$ Suppose that $\mathfrak{u} \neq \emptyset$. In this case $\mathfrak{u} = \alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!]$, for some $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{B})$ and $b \in \mathsf{uni}(\mathbf{A})$. (*We must show that $f^{-1}\left[\alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!]\right] \in \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{A}, \boldsymbol{u})$.*) $\boxed{\alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!] \cap \mathsf{rg}(f) = \emptyset}$ Suppose that $\alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!] \cap \mathsf{rg}(f) = \emptyset$. Then $\boldsymbol{u}$ cannot be a $\mathcal{K}$-constant, otherwise, for any $a \in \mathsf{uni}(\mathbf{A})$, $f(\boldsymbol{u}^{\mathbf{A}}(a)) = \boldsymbol{u}^{\mathbf{B}}(f(a))\,\alpha\,\boldsymbol{u}^{\mathbf{B}}(b)$, which would be a contradiction. Hence $f^{-1}\left[\alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!]\right] = f^{-1}[\emptyset] = \emptyset \in \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{B}, \boldsymbol{u})$. $\boxed{\alpha[\![b]\!] \cap \mathsf{rg}(f) \neq \emptyset}$ Assume that $\alpha[\![b]\!] \cap \mathsf{rg}(f) \neq \emptyset$. By Corollary 5.56 on page 190, f is continuous from $\mathsf{Cos}^{\mathcal{K}}(\mathbf{A})$ into $\mathsf{Cos}^{\mathcal{K}}(\mathbf{B})$, and hence, by equivalent condition (3) of Theorem 5.40, $f^{-1}\left[\alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!]\right]$ is a $\mathcal{K}$-coset of $\mathbf{A}$. (*It suffices to show that $\boldsymbol{u}^{\mathbf{A}}(a) \in f^{-1}\left[\alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!]\right]$.*) Since $\alpha[\![b]\!] \cap \mathsf{rg}(f) \neq \emptyset$, there exists $a \in \mathsf{uni}(\mathbf{A})$ with $f(a) \in \alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!]$. So by Remark 1.60 on page 24, $f^{-1}\left[\alpha[\![\boldsymbol{u}^{\mathbf{B}}(b)]\!]\right] = f^{-1}\left[\alpha[\![f(a)]\!]\right] = (\underset{\to}{f}^{-1}[\alpha])[\![a]\!]$. (*It suffices to show that $\boldsymbol{u}^{\mathbf{A}}(a) \in (\underset{\to}{f}^{-1}[\alpha])[\![a]\!]$.*) Now $f(\boldsymbol{u}^{\mathbf{A}}(a)) = \boldsymbol{u}^{\mathbf{B}}(f(a))\ \alpha\ \boldsymbol{u}^{\mathbf{B}}(\boldsymbol{u}^{\mathbf{B}}(b))\ \alpha\ \boldsymbol{u}^{\mathbf{B}}(b)\,\alpha\,f(a)$, so $\langle \boldsymbol{u}^{\mathbf{A}}(a), a\rangle \in \underset{\to}{f}^{-1}[\alpha]$. $\diamond$

Consequently, $U_{\mathsf{i}}(\mathbf{Tm}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ must be equivalent to a *sentential* 1-calculus, and if $\mathbf{F}_{\mathcal{K}}$ is the $\mathcal{K}$-free algebra on $\overline{[\mathsf{V}]}$, $U_{\mathsf{i}}(\mathbf{F}_{\mathcal{K}}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ must be equivalent to a *propositional* $\mathcal{K}$-calculus, which, by Theorem 8.14 on page 282, is an *ideal* $\mathfrak{a}$-canon; this logic is the canon for the $\mathfrak{a}$-archology of all idempotent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-logics on algebras of $\mathcal{K}$. We now identify these logics and this archology.

Definition 9.83 (The Sentential Calculi of Idempotent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-Cosets)
We write $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ for $U_{\mathsf{i}}(\mathbf{Tm}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ and $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ for $U_{\mathsf{i}}(\mathbf{F}_{\mathcal{K}}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$, which we call the **sentential calculus of idempotent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-cosets** and the **idempotent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-coset canon**, respectively. Let $\mathfrak{A}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ denote the $\mathfrak{a}$-archology determined by $\mathfrak{a}$-archetype $\mathcal{K}$ and logics $\{U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}}) \ : \ \mathbf{A} \in \mathcal{K}\}$; the canon of this archology is $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$. For a term 0 constant over $\mathcal{K}$, we may write $S_{\mathsf{c}}(0\text{-}\mathsf{cos}^{\mathcal{K}})$, for $S_{\mathsf{i}}(0\text{-}\mathsf{cos}^{\mathcal{K}})$ and $\boldsymbol{S}_{\mathsf{c}}(0\text{-}\mathsf{cos}^{\mathcal{K}})$ for $\boldsymbol{S}_{\mathsf{i}}(0\text{-}\mathsf{cos}^{\mathcal{K}})$. $\Box$

The *sentential calculi of idempotent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-cosets* encompass the *membership logics* $S(\mathcal{K}, \mathsf{mem})$ and the *assertional logics* $S(\mathcal{K}, 0)$. To see this, note that by Remark 9.80, $\mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{Tm}, y) = \mathsf{Cos}^{\mathcal{K}}(\mathbf{Tm})$, for variable y, and hence $S_{\mathsf{i}}(y\text{-}\mathsf{cos}^{\mathcal{K}})$ is equivalent to $S(\mathcal{K}, \mathsf{mem})$. Further for $\mathcal{K}$-constant 0, by (2.21) of Example 2.92 on page 107, $\mathsf{Th}(S_{\mathsf{c}}(0\text{-}\mathsf{cos}^{\mathcal{K}})) = \mathsf{Th}(S(\mathcal{K}, 0))$.

Note that for any $\mathfrak{a}$-algebra $\mathbf{A}$, since every $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is an $\mathfrak{a}$-model of $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ by Proposition 9.82, the idempotent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-cosets on $\mathbf{A}$ are all $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$-filters on $\mathbf{A}$.

The following characterization of consequence in $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ and $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$, in terms of $\models_{\mathcal{K}}$, follows from Proposition 9.78 and Corollary 9.79, together with Lemma 1.457 on page 88.

Corollary 9.84 For $P \cup \{p\} \subseteq \mathsf{Tm}$ with $P \neq \emptyset$, the following conditions are equivalent.

1. $P \vdash_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} p$.
2. $P \approx \boldsymbol{u}[P] \models_{\mathcal{K}} p \approx \boldsymbol{u}[P]$.
3. $P \approx \boldsymbol{u}(q) \models_{\mathcal{K}} p \approx \boldsymbol{u}(q)$, for some $q \in P$.
4. $P' \approx \boldsymbol{u}[P'] \models_{\mathcal{K}} p \approx \boldsymbol{u}[P']$, for some finite $P' \subseteq_{\mathsf{f}} P$.
5. $\overline{[P]} \vdash_{\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} \overline{p}$.

Further, $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ has a theorem iff $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ has a theorem iff $\boldsymbol{u}$ is a $\mathcal{K}$-constant. $\square$

Remark 9.85 For any term p, $\{p\} \vdash_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} \boldsymbol{u}(p)$.

Proof. Since $\models_{\mathcal{K}} \boldsymbol{u}(p) \approx \boldsymbol{u}(\boldsymbol{u}(p))$, certainly $p \approx \boldsymbol{u}(p) \models_{\mathcal{K}} \boldsymbol{u}(p) \approx \boldsymbol{u}(\boldsymbol{u}(p))$, the result following by Corollary 9.84. $\diamond$

Consequently, $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$-filters on $\mathbf{A}$ are $\boldsymbol{u}^{\mathbf{A}}$-closed. More precisely, we have the following.

Proposition 9.86 For any algebra $\mathbf{A}$, not necessarily in $\mathcal{K}$, and any non-empty $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$-filter F of $\mathbf{A}$, $\boldsymbol{u}^{\mathbf{A}}[F] \subseteq F$.

Proof. Let x be a variable and $a \in F$. By Remark 9.85, $\{x\} \vdash_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} \boldsymbol{u}(x)$. Since $a \in F$, $\boldsymbol{u}^{\mathbf{A}}(a) \in F$. $\diamond$

The following axiomatizations of $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ and $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ follow from Corollary 9.84 together with Lemma 6.35 on page 231.

Proposition 9.87 The propositional $\mathcal{K}$-calculus $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is axiomatized with all (proper) rules $\overline{[P]} \vdash \overline{p}$ for which $P \approx \boldsymbol{u}[P] \models_{\mathcal{K}} p \approx \boldsymbol{u}[P]$, and, if $\boldsymbol{u}$ is constant over $\mathcal{K}$, all axioms $\vdash p$ where $\models_{\mathcal{K}} p \approx \boldsymbol{u}$. $\square$

Proposition 9.88 The sentential 1-calculus $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is axiomatized with all (proper) rules $P \vdash p$ for which $P \approx \boldsymbol{u}[P] \models_{\mathcal{K}} p \approx \boldsymbol{u}[P]$, and, if $\boldsymbol{u}$ is constant over $\mathcal{K}$, all axioms $\vdash p$ where $\models_{\mathcal{K}} p \approx \boldsymbol{u}$. $\square$ $\square$

We shall now show that the ideal/form of the ideal canon $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is equivalent to $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$.

Theorem 9.89 $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}}) \equiv \underline{\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} = \boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})^{\imath} = \underline{\mathfrak{A}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})}$.

Proof. Since $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is $\mathfrak{a}$-structural, it is ideal, by Theorem 8.13 on page 281; consequently, $\underline{\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} = \boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})^{\imath}$. By Corollary 9.84, $P \vdash_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} p$ iff $\overline{[P]} \vdash_{\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} \overline{p}$, and hence $\vdash_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} = \vdash_{\underline{\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})}}$, by (4) of Theorem 8.9 on page 279. Note that $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})^{\imath} = \underline{\mathfrak{A}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})}$ is definitional since the archology $\mathfrak{A}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is well-defined with canon $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$. $\diamond$

Since by definition, $\mathsf{Th}(\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})) = \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}, \boldsymbol{u})$, the following result follows at once from the previous theorem, together with Theorem 8.11 and Theorem 8.15.

Corollary 9.90 $\overline{[\cdot]} : \mathbf{Th}(S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})) \cong \mathbf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}, \boldsymbol{u})$ with inverse isomorphism $\underline{[\cdot]}$. Further, $\mathsf{Fi}_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})}(\mathbf{F}_{\mathcal{K}}) = \mathsf{Cos}_{\mathsf{i}}^{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}, \boldsymbol{u})$. $\square$

Open Problem 9.91 In the light of Corollary 8.96 on page 302 and Open Problem 8.97, prove the following conjectures.

> **Conjecture 9.92** If $\mathcal{K}$ is a *variety*, then for each $\mathbf{A} \in \mathcal{K}$, $\mathsf{Fi}^{\mathfrak{a}}_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})}(\mathbf{A}) = \mathsf{Cos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u})$.
>
> **Conjecture 9.93** If $\mathcal{K}$ is a *variety*, then for every $\mathfrak{a}$-algebra $\mathbf{A}$, $\mathsf{Fi}^{\mathfrak{a}}_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})}(\mathbf{A}) = \mathsf{Cos}^{\mathcal{K}}_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u})$.

(In this regard, See Corollary 13.36 and Corollary 13.37 of Example 13.34 on page 402.)

□

Open Problem 9.94 Idempotent $\boldsymbol{u}$ may not be the only unary terms for which $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) = \mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) - \{\emptyset\}$. Prove or disprove the following conjecture.

> **Conjecture 9.95** $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) = \mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) - \{\emptyset\}$, for each $\mathfrak{a}$-algebra $\mathbf{A}$, iff, there exists a term $p(x)$ such that $\models_{\mathcal{K}} \boldsymbol{u}(\boldsymbol{u}(x)) \approx \boldsymbol{u}(p(x))$.

Also see Open Problem 12.29 on page 382.

Open Problem 9.96 Find a more general reason why $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ is *finitary* in the previous example. Notice that even though the binary system $\boldsymbol{u}$ has cardinality *one*, $\mathsf{N}^{\mathcal{K}}_{\boldsymbol{u}_*}(\mathbf{A})$ is still defined in terms of a product of multiple *functions*. It is possibly because we can view these multiple functions as defining a single finitary translation from $\mathsf{uni}(\mathbf{A})^2$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$; the product by this translation will induce a finitary closed system over $\mathsf{uni}(\mathbf{A})^2$ and the solutions $\boldsymbol{u}^{\mathbf{A}}_b/\alpha$ may be like the cosets of a (finitary) equivalential closed system (see Proposition 4.112 on page 161). (See Open Problem 5.120 on page 207 and Open Problem 9.27.)

In the following example, we demonstrate how solving binary systems parametrically, encompasses solving unary systems. The importance of this, lies in the fact that the logics that we shall be introducing later, determined by parametric solutions to binary systems, encompass the logics determined by solving unary systems. The logics determined by solving unary systems encompass all algebraizable 1-deductive systems in the sense of Blok and Pigozzi, and so the logics determined by solutions to binary systems parametrically, include *all* algebraizable 1-deductive systems (see ∮9.1.2). We shall also use the observations of the next example, to show that our theory of parametrized algebraization generalizes the standard theory of algebraization, i.e., our theory specializes to the standard theory.

Example 9.97 (Essentially Unary Systems)

Let $\mathfrak{B} = \{\langle \delta_i, \varepsilon_i \rangle : i < n\}$ be a binary system of equations.

> **Definition 9.98 (Essentially Unary Systems)** We call $\mathfrak{B}$ **essentially $\mathcal{K}$-unary** if each $\delta_i(x, y)$ and $\varepsilon_i(x, y)$ depend only on x; i.e., for each $i < n$, there exist unary $\delta'_i, \varepsilon'_i \in \mathsf{Tm}$ such that $\mathcal{K}$ satisfies $\delta_i(x, y) \approx \delta'_i(x)$ and $\varepsilon_i(x, y) \approx \varepsilon'_i(x)$. When working with an essentially $\mathcal{K}$-unary (binary) system $\mathfrak{B}$, it is convenient to assume that a unary system $\tau = \{\langle \delta'_i, \varepsilon'_i \rangle : i < n\}$, where the δ'_i and ε'_i are as above, is readily available, and we shall denote such a unary system (in this context) by $\mathfrak{B}'$, which we shall call a **unary realization of $\mathfrak{B}$ in $\mathcal{K}$**. □

Remark 9.99 If $\mathfrak{B}$ is $\mathcal{K}$-unary, then $\mathfrak{B}' = \{\langle \delta_i', \varepsilon_i' \rangle \,:\, i < n\}$ (as in the previous definition) is a translation in the sense of [BP89a] (they denote such systems by τ which they call a **translation**).

Remark 9.100 If $\mathfrak{B}$ is $\mathcal{K}$-unary, then, for any algebra $\mathbf{A}$, any $b \in A$ and any $\mathcal{K}$-congruence α of $\mathbf{A}$, we have $\mathfrak{B}_b^{\mathbf{A}}/\alpha = \mathfrak{B}'^{\mathbf{A}}/\alpha$, where $\mathfrak{B}$ is as in the previous definition.

Remark 9.101 If $\mathfrak{B}$ is $\mathcal{K}$-unary, then, $\mathfrak{B}$ has finite pivots, is pivot symmetric and pivots finitarily in $\mathcal{K}$. □

Note that essentially $\mathcal{K}$-unary binary systems include the systems $\mathbf{0}(x)$ where 0 is a $\mathcal{K}$-constant.

□

We now consider an example of (two) binary systems that pivot and have finite pivots (and hence pivot finitarily), but not pivot symmetric. The example is drawn from lattice theory, and the binary systems of (single) equations that we consider are closely tied to lattice ideals and filters.

Example 9.102 (Lattices Ideals and Filters)

Let $\mathcal{K}$ be an $\mathfrak{a}$-quasivariety of lattice expansions.

Definition 9.103 (The Ideal and Filter Systems $\vartriangle(x,y)$ and $\triangledown(x,y)$) We define

$$\vartriangle(x,y) = \{\langle x \vee y, y \rangle\} \quad \text{and} \tag{9.17}$$

$$\triangledown(x,y) = \{\langle x \wedge y, y \rangle\}. \tag{9.18}$$

□

The symbolisms have been chosen to reflect the shape of *principal ideals*, in the case of $\vartriangle$, and *principal filters*, in the case of $\triangledown$, rather than meets and joins (they would be upside down if that were the case).

Remark 9.104

1. $\vartriangle_q^{\approx}[\![p]\!] = \{p \vee q \approx q\} = \{p \leq q\}$.
2. $\vartriangle_z/\bot_{\mathcal{K}} = \{p \,:\models_{\mathcal{K}} p \leq z\}$, and so $z \in \vartriangle_z/\bot_{\mathcal{K}}$.
3. $[z \upharpoonright_{\vartriangle_*} y] \dashv\models_{\mathcal{K}} z \leq y$.
4. For an $\mathfrak{a}$-algebra $\mathbf{A}$, $b \in \mathsf{uni}(\mathbf{A})$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, $\vartriangle_b^{\mathbf{A}}/\,\alpha = \{a \,:\, (a \vee^{\mathbf{A}} b)\, \alpha\, b\}$.

Proof. (*We prove (3), the rest being trivial. By (1) and (2), it suffices to show that* $z \leq y \models_{\mathcal{K}} \vartriangle_z/\bot_{\mathcal{K}} \leq y$.) Let $p \in \vartriangle_z/\bot_{\mathcal{K}}$. (*It suffices to show that* $z \leq y \models_{\mathcal{K}} p \leq y$.) By (2), $\models_{\mathcal{K}} p \leq z$. So $z \leq y \models_{\mathcal{K}} p \leq y$. ◇

Remark 9.105

1. $\triangledown_q^{\approx}[\![p]\!] = \{p \wedge q \approx q\} = \{p \geq q\}$.
2. $\triangledown_z/\bot_{\mathcal{K}} = \{p \,:\models_{\mathcal{K}} p \geq z\}$, and so $z \in \triangledown_z/\bot_{\mathcal{K}}$.
3. $[z \upharpoonright_{\triangledown_*} y] \dashv\models_{\mathcal{K}} z \geq y$.

4. For an $\mathfrak{a}$-algebra $\mathbf{A}$, $b \in \mathsf{uni}(\mathbf{A})$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, $\triangledown_b^{\mathbf{A}} / \alpha = \{a : (a \wedge^{\mathbf{A}} b)\, \alpha\, b\}$.

Remark 9.106 $\vartriangle_*$ and $\triangledown_*$ both have finite pivots for $\mathcal{K}$.

Remark 9.107 If either $\vartriangle_*$ or $\triangledown_*$ are pivot symmetric for $\mathcal{K}$, then $\mathcal{K}$ is trivial.

We now show that the normals of $\vartriangle$ (resp. $\triangledown$) on a lattice expansion *in* $\mathcal{K}$, are lattice ideals (resp. filters). Note that since lattice ideals (resp. filters) are closed under arbitrary non-empty intersection, it suffices to prove that $\mathsf{Sol}^{\mathcal{K}}_{\vartriangle_*}(\mathbf{P}) \subseteq \mathsf{Id}_\Diamond(\mathbf{P})$ (resp. $\mathsf{Sol}^{\mathcal{K}}_{\triangledown_*}(\mathbf{P}) \subseteq \mathsf{Fl}_\Diamond(\mathbf{P})$).

Proposition 9.108 For $\mathbf{P} \in \mathcal{K}$, $\mathsf{N}^{\mathcal{K}}_{\vartriangle_*}(\mathbf{P}) \subseteq \mathsf{Id}_\Diamond(\mathbf{P})$ and $\mathsf{N}^{\mathcal{K}}_{\triangledown_*}(\mathbf{P}) \subseteq \mathsf{Fl}_\Diamond(\mathbf{P})$.

Proof. (*We shall prove that* $\mathsf{Sol}^{\mathcal{K}}_{\vartriangle_*}(\mathbf{P}) \subseteq \mathsf{Id}_\Diamond(\mathbf{P})$*, the proof that* $\mathsf{Sol}^{\mathcal{K}}_{\triangledown_*}(\mathbf{P}) \subseteq \mathsf{Fl}_\Diamond(\mathbf{P})$ *being dual. This will suffice, since lattice ideals and filters are closed under non-empty intersection. We shall repeatedly, implicitly make use of (4) of Remark 9.104.*)

$\boxed{\text{Downset}}$ Suppose that $c \le^{\mathbf{P}} b \in \vartriangle_a^{\mathbf{P}} / \alpha$. So $(b \vee^{\mathbf{A}} a)\, \alpha\, a$ and $c \vee^{\mathbf{P}} b = b$. Then $(c \vee^{\mathbf{A}} a)\, \alpha (c \vee^{\mathbf{A}} (b \vee^{\mathbf{A}} a)) = (c \vee^{\mathbf{A}} b) \vee^{\mathbf{A}} a = (b \vee^{\mathbf{A}} a)\, \alpha\, a$. Since $(c \vee^{\mathbf{A}} a)\, \alpha\, a$, $c \in \vartriangle_a^{\mathbf{P}} / \alpha$.

$\boxed{\vee\text{-Closed}}$ Suppose that $b, c \in \vartriangle_a^{\mathbf{P}} / \alpha$. Then $(b \vee^{\mathbf{A}} a)\, \alpha\, a$ and $(c \vee^{\mathbf{A}} a)\, \alpha\, a$. So $(b \vee^{\mathbf{P}} c) \vee^{\mathbf{A}} a = (b \vee^{\mathbf{P}} a) \vee^{\mathbf{A}} (c \vee^{\mathbf{P}} a)\, \alpha (a \vee^{\mathbf{A}} a) = a$, and hence $b \vee^{\mathbf{P}} c \in \vartriangle_a^{\mathbf{P}} / \alpha$. $\Diamond$

Finally, we shall show that if $\mathcal{K}$ is the[1] variety of distributive lattices, then $\vartriangle_*$ and $\triangledown_*$ pivot in $\mathcal{K}$ (and consequently pivot finitarily in $\mathcal{K}$). We shall require the following lemma.

Lemma 9.109 If $\mathcal{K}$ is the quasivariety of distributive lattices, then for any lattice term $p(z, y, \vec{x})$,

$$z \le y \models_{\mathcal{K}} p(z, y, \vec{x}) \vee y \approx p(y, z, \vec{x}) \vee y \quad \text{and} \tag{9.19}$$

$$z \ge y \models_{\mathcal{K}} p(z, y, \vec{x}) \wedge y \approx p(y, z, \vec{x}) \wedge y. \tag{9.20}$$

Proof. (*We shall prove (9.19), the proof of (9.20) being dual. We proceed by induction on the complexity of such terms p.*)

$\boxed{\text{Base Case}}$ If $p(z, y, \vec{x}) = z$, then 9.19 is simply $z \vee y \approx y \models_{\mathcal{K}} z \vee y \approx y \vee y$, which is certainly true, and the case where $p(z, y, \vec{x}) = y$ is similar. The case that p is some variable other that y or z holds trivially. So the result holds for all terms of complexity one. $\boxed{\text{Inductive Hypothesis}}$ Assume that 9.19 is true for all terms p of complexity less than $n > 1$. $\boxed{\text{Inductive Proof}}$ Suppose that p has complexity n. Then $p = q \star r$, for some $\star \in \{\vee, \wedge\}$, and by the induction hypothesis, we may assume that

$$z \vee y \approx y \models_{\mathcal{K}} q(z, y, \vec{x}) \vee y \approx q(y, z, \vec{x}) \vee y \quad \text{and}$$
$$z \vee y \approx y \models_{\mathcal{K}} r(z, y, \vec{x}) \vee y \approx r(y, z, \vec{x}) \vee y.$$

So

$$z \vee y \approx y \models_{\mathcal{K}} (q(z, y, \vec{x}) \vee y) \star (r(z, y, \vec{x}) \vee y) \approx (q(y, z, \vec{x}) \vee y) \star (r(y, z, \vec{x}) \vee y).$$

Suppose that $\star = \wedge$. In this case we have

$$z \vee y \approx y \models_{\mathcal{K}} (q(z, y, \vec{x}) \vee y) \wedge (r(z, y, \vec{x}) \vee y) \approx (q(y, z, \vec{x}) \vee y) \wedge (r(y, z, \vec{x}) \vee y). \tag{i}$$

By distributivity,

$$\models_{\mathcal{K}} (q(z, y, \vec{x}) \vee y) \wedge (r(z, y, \vec{x}) \vee y) \approx (q(z, y, \vec{x}) \wedge r(z, y, \vec{x})) \vee y \quad \text{and} \tag{ii}$$

$$\models_{\mathcal{K}} (q(y, z, \vec{x}) \vee y) \wedge (r(y, z, \vec{x}) \vee y) \approx (q(y, z, \vec{x}) \wedge r(y, z, \vec{x})) \vee y. \tag{iii}$$

[1] We thank an examiner of this thesis for noting that there is only *one quasi*variety of distributive lattices.

So by (i), (ii) and (iii),

$$z \vee y \approx y \models_{\mathcal{K}} (q(z, y, \vec{x}) \wedge r(z, y, \vec{x})) \vee y \approx (q(y, z, \vec{x}) \wedge r(y, z, \vec{x})) \vee y,$$

as required. Suppose that $\star = \vee$. In this case we have

$$z \vee y \approx y \models_{\mathcal{K}} (q(z, y, \vec{x}) \vee y) \vee (r(z, y, \vec{x}) \vee y) \approx (q(y, z, \vec{x}) \vee y) \vee (r(y, z, \vec{x}) \vee y). \tag{iv}$$

Then by associativity and idempotence,

$$\models_{\mathcal{K}} (q(z, y, \vec{x}) \vee y) \vee (r(z, y, \vec{x}) \vee y) \approx (q(z, y, \vec{x}) \vee r(z, y, \vec{x})) \vee y \quad \text{and} \tag{v}$$

$$\models_{\mathcal{K}} (q(y, z, \vec{x}) \vee y) \vee (r(y, z, \vec{x}) \vee y) \approx (q(y, z, \vec{x}) \vee r(y, z, \vec{x})) \vee y, \tag{vi}$$

and hence by (iv), (v) and (vi),

$$z \vee y \approx y \models_{\mathcal{K}} (q(z, y, \vec{x}) \vee r(z, y, \vec{x})) \vee y \approx (q(y, z, \vec{x}) \vee r(y, z, \vec{x})) \vee y,$$

as required. Note that distributivity is not required in the second case. $\diamond$

Proposition 9.110 If $\mathcal{K}$ is the variety of distributive lattices, then $\triangle_*$ and $\triangledown_*$ both pivot in $\mathcal{K}$; consequently $\triangle_*$ and $\triangledown_*$ both pivot finitarily in $\mathcal{K}$.

Proof. We shall prove that $\triangle_*$ pivots in $\mathcal{K}$, the proof that $\triangledown_*$ pivots in $\mathcal{K}$ being dual. Suppose that

$$P \vee z \approx z \models_{\mathcal{K}} p \vee z \approx z. \tag{i}$$

Let σ be the transposition $\langle z, y \rangle$. Then by (i) and structurality,

$$\sigma[P] \vee y \approx y \models_{\mathcal{K}} \sigma(p) \vee y \approx y. \tag{ii}$$

Generally (i.e., independently from (i) or (ii)), by Lemma 9.109,

$$z \leq y, P \vee y \approx y \models_{\mathcal{K}} \sigma[P] \vee y \approx y. \tag{iii}$$

By (ii) and (iii)

$$z \leq y, P \vee y \approx y \models_{\mathcal{K}} \sigma(p) \vee y \approx y. \tag{iv}$$

Applying Lemma 9.109 to (iv), we obtain

$$z \leq y, P \vee y \approx y \models_{\mathcal{K}} \sigma(\sigma(p)) \vee y \approx y, \tag{v}$$

and since σ is an involution, we have

$$z \leq y, P \vee y \approx y \models_{\mathcal{K}} p \vee y \approx y. \tag{vi}$$

The result follows by (3) of Remark 9.104. $\diamond$

Open Problem 9.111 Characterize those quasivarieties of lattices for which $\triangle_*$ and $\triangledown_*$ both pivot finitarily.

Open Problem 9.112 Develop an analogous theory for convex sets. Is distributivity a requirement in such a theory?

$\square$

We end with a counter-example[2] demonstrating that $\mathfrak{B}_*$ need not pivot in $\mathcal{K}$.

Counter Example 9.113 ($\mathfrak{B}_*$ need not pivot in $\mathcal{K}$)

A **polrim** is an algebra $\mathbf{A} = \langle A; \oplus, \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that for any $a, b, c \in A$,

1. $\langle A; \oplus, 0\rangle$ is a monoid (not necessarily commutative),
2. the binary relation $\leq$ on A, defined by $a \leq b$ iff $a \dot{-} b = 0$, is a partial order with least element 0,
3. $a \leq b$ implies that $c \oplus a \leq c \oplus b$ and $a \oplus c \leq b \oplus c$ and
4. $a \dot{-} b \leq c$ iff $a \leq c \oplus b$.

(See [RvA97] and its bibliography.) The class of all polrims is denoted LM. The followin observation follows easily.

Remark 9.114 $\models_{\mathsf{LM}} x, y \leq (y \dot{-} x) \oplus x$.

Remark 9.115 LM is a quasivariety (and is *not* a variety [Hig84]).

Remark 9.116 $\models_{\mathsf{LM}} [[(y \dot{-} x) \oplus x] \dot{-} z] \oplus z \approx z \to [(x \oplus y) \dot{-} (z \oplus z)] \oplus z \approx z$.

Proof. If $[[(b \dot{-} a) \oplus a] \dot{-} c] \oplus c = c$ then, by Remark 9.114, $c \geq (b \dot{-} a) \oplus a \geq b, a$, hence $a \oplus b \leq c \oplus c$, by (2) and (3); then $(a \oplus b) \dot{-} (c \oplus c) = 0$, by (2), whence $[(a \oplus b) \dot{-} (c \oplus c)] \oplus c = 0 \oplus c = c$, by (1). ◇

Lemma 9.117 There is *no* $\langle \oplus, \dot{-}, 0\rangle$-term $p(\vec{x}, y)$ for which $\mathsf{LM} \models p(\vec{x}, y) \oplus y \approx y$ and $\mathsf{LM} \not\models p(\vec{x}, y) \approx 0$.

Proof. LM is the equivalent semantics of an algebraizable fragment (with implication and a weak conjunction &) of intuitionistic propositional logic without the rules of 'exchange' and 'contraction', as defined in [OK85] (see [RvA97] for a proof). This fragment is 'equivalent', in a suitable sense, to a Gentzen system L in whose formalism the above claim amounts just to the underivability of the sequent $y \Rightarrow (p(\vec{x}, y) \& y)$, whenever $\mathsf{LM} \not\models p(\vec{x}, y) \approx 0$. (Here, $\Rightarrow$ is the Gentzen style derivability relation.) Establishing the claim is a straightforward proof-theoretic exercise, since L has a cut elimination theorem [OK85]. ◇

Let $\mathfrak{B} = \{\langle \delta(x, y), \epsilon(x, y)\rangle\}$, where $\delta(x, y) = x \oplus y$ and $\epsilon(x, y) = y$.

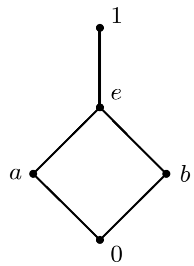
Corollary 9.118 For any variables y and z,

$$\mathfrak{B}_y/\bot^{\mathsf{LM}} = \{p(\vec{x}, y, z) \in \mathsf{Tm}^{\mathfrak{a}} : \mathsf{LM} \models p(\vec{x}, y, z) \approx 0\}. \tag{9.21}$$

□

Let $\mathbf{A} = \langle A; \oplus, \dot{-}, 0\rangle$ be the polrim whose operations and partial order $\leq$ are defined by the tables and figure below:

[2] We would like to thank James Raftery for this example.



$\oplus$	0	a	b	e	1
0	0	a	b	e	1
a	a	a	1	1	1
b	b	e	b	e	1
e	e	e	1	1	1
1	1	1	1	1	1

$\dot{-}$	0	a	b	e	1
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
e	e	b	a	0	0
1	1	1	a	a	0

Let $g(x,y,z) = [(y \dot{-} x) \oplus x] \dot{-} z$ and $t(x,y,z) = (x \oplus y) \dot{-} (z \oplus z)$.

Remark 9.119 $[y \Rsh z] \cup \mathfrak{B}_z(g(x,y,z)) \models_{\mathsf{LM}} \mathfrak{B}_z(t(x,y,z))$.

Proof. By Lemma 9.116,

$$\mathsf{LM} \models \delta(g(x,y,z),z) \approx \epsilon(g(x,y,z),z) \to \delta(t(x,y,z),z) \approx \epsilon(t(x,y,z),z),$$

from which the result follows immediately. $\diamond$

Lemma 9.120 $[y \Rsh z] \cup \mathfrak{B}_z(\sigma(g)) \not\models_{\mathsf{LM}} \mathfrak{B}_z(\sigma(t))$, where σ is the transposition (yz).

Proof. It suffices to show that $[y \Rsh z] \cup \mathfrak{B}_z(\sigma(g)) \not\models_{\mathbf{A}} \mathfrak{B}_z(\sigma(t))$, since $\mathbf{A} \in \mathsf{LM}$. By (9.21), if $p(\vec{x},y,z) \in \mathfrak{B}_y/\bot^{\mathsf{LM}}$ and $\vec{u},v,w \in \mathsf{uni}(\mathbf{A})$ then $p^{\mathbf{A}}(\vec{u},v,w) = 0$, so $\delta^{\mathbf{A}}(p^{\mathbf{A}}(\vec{u},v,w)) = 0 \oplus w = w = \epsilon^{\mathbf{A}}(p^{\mathbf{A}}(\vec{u},v,w))$. Thus, $[y \Rsh z]$ is a set of identities of $\mathbf{A}$. It is therefore enough to show that $\mathfrak{B}_z(\sigma(g)) \not\models_{\mathbf{A}} \mathfrak{B}_z(\sigma(t))$, i.e., that $\mathbf{A} \not\models g(x,z,y) \oplus z \approx z \to t(x,z,y) \oplus z \approx z$, i.e., that

$$\mathbf{A} \not\models [[(z \dot{-} x) \oplus x] \dot{-} y] \oplus z \approx z \to [(x \oplus z) \dot{-} (y \oplus y)] \oplus z \approx z.$$

If we interpret $\langle x,y,z \rangle$ as $\langle a,a,e \rangle$ in $\mathbf{A}$ then $[[(z \dot{-} x) \oplus x] \dot{-} y] \oplus z$ and z take the value e, but $[(x \oplus z) \dot{-} (y \oplus y)] \oplus z$ takes the value 1, completing the argument. $\diamond$

Considering Remark 9.119 together with the previous lemma, demonstrates that $\mathfrak{B}_*$ does not pivot in $\mathcal{K}$.

$\square$

Open Problem 9.121 Show that there exists $\mathfrak{B}$ and $\mathcal{K}$ such that $\mathfrak{B}_*$ *pivots finitarily for* $\mathcal{K}$ but that $\mathfrak{B}_*$ does not have finite pivots for $\mathcal{K}$.

Open Problem 9.122 Demonstrate that the condition that $\mathfrak{B}_*$ *pivots finitarily for* $\mathcal{K}$ and the condition that $\mathfrak{B}_*$ *has finite pivots for* $\mathcal{K}$, are independent.

Open Problem 9.123 Demonstrate that the conditions that $\mathfrak{B}_*$ *pivots for* $\mathcal{K}$, $\mathfrak{B}_*$ *has finite pivots for* $\mathcal{K}$, and $\mathfrak{B}_*$ *has symmetric pivots in* $\mathcal{K}$, are independent.

Chapter 10

Coherence

A group $\mathbf{G}$ has the property that if any subalgebra of $\mathbf{G}$ contains a congruence class of some congruence α on $\mathbf{G}$, then α is *compatible* with the universe of this subalgebra, i.e., the universe of this subalgebra is a *union of congruence classes* of α. This property was introduced as an object of study by Geiger in [Gei74], where such algebras are termed *coherent.* Geiger gave Mal'cev conditions characterizing varieties of coherent algebras, and deduced from these conditions that such varieties must be congruence permutable and congruence regular. Taylor has shown that a congruence permutable and congruence regular variety need not be coherent [Tay74]. In [Cha83], Chajda described a condition, involving the polynomials and principal congruences on an algebra, that together with congruence regularity and congruence permutability, characterizes coherent varieties. Clark and Fleischer established a 'local' link between coherence and congruence permutability [CF87], and Duda established a similar link between coherence and congruence regularity [Dud91].

Ursini considered the following weaker variant of coherence for algebras with a typed constant 0 [Urs94]: if any subalgebra of an algebra $\mathbf{A}$ contains the 0-class of a congruence α on $\mathbf{A}$, then this subalgebra is a union of congruence classes of α. We call such algebras *point coherent at* 0. Ursini showed that a variety of algebras with this property is congruence permutable, and characterized such varieties via a Mal'cev condition. Ursini also described a form of point regularity on the classical relations of an algebra, which we will call *classical point regularity* at 0, and showed that, at a varietal level, the conditions of point coherence at 0 and classical point regularity at 0 coincide.

In [Dud89] the notion of an algebra having **coherent congruence classes** was introduced, that is, any *congruence class* that contains a congruence class of any other congruence α, must be a union of α congruence classes, i.e., α must be compatible with the congruence class. Duda gave a Mal'cev condition characterizing varieties of such algebras.

In [BR99], a notion of coherence was introduced, called having $\mathcal{K}$*-coherent* $\langle\mathcal{K},\tau\rangle$*-classes* (see Definition 2.142 on page 119 of our text). This coherence condition encompasses *none* of the aforementioned condition, although it would encompass a *point* analogue of the condition of having *coherent congruence classes*, had such a condition existed in the literature at that time (see Definition 2.146 on page 119). This condition characterizes the protoalgebraicity of the 1-deductive systems $S(\mathcal{K},\tau)$ in terms of $\mathcal{K}$ and τ only, that is, purely universal algebraically and independently of $S(\mathcal{K},\tau)$ (see Theorem 2.143 on page 119). In [BR03], we introduced a generalization of this notion, which in the discourse of this text, is called '*having coherent* $\langle \mathcal{K}, \mathfrak{B}_* \rangle$*-classes*, a condition that characterizes the *parametrized protoalgebraicity* of 1-deductive systems that we shall be introducing in ∮12, where the notion of parametrized protoalgebraicity is introduced in Part V.

In this chapter we shall consider these conditions, as well as generalizations appropriate for the n-deductive analogues of $S(\mathcal{K}, \tau)$ (to be introduced in the next chapter). We shall also consider notions of *subuniverse* (or subalgebra) coherence, since in the sequel we shall be considering *logics of subuniverses*, and the protoalgebraicity of these logics relates closely to subuniverse coherence.

In ∮10.1 we consider notions of subuniverse coherence. In ∮10.3 we introduce a general condition of a *set* being $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-coherent, from which the condition of having $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes is derived in ∮10.3.1. A *weakened* version of this condition is considered in ∮10.3.2. We show that the weakened version coincides with the full version whenever $\mathfrak{B}_*$ *pivots* in $\mathcal{K}$. Since unary systems always pivot, the two versions coincide for *inherently unary* binary systems (and hence for unary systems), and this condition is precisely the condition characterized in [BR99]. A generalization appropriate for n-deductive systems is considered in ∮10.2 (in the next chapter we shall introduce a sentential n-calculus $S^n(\mathcal{K}, \mathfrak{N})$, where $\mathfrak{N}$ is a system of n-ary equations, and there we shall characterize the protoalgebraicity of $S^n(\mathcal{K}, \mathfrak{N})$ in terms of this final notion of coherence). Finally, a number of examples, that pertain to the sequel, are considered in ∮10.3.3.

10.1 Subuniverse Coherence

The first notion of coherence to be studied in universal algebra (originally called *coherence* [Gei74]) was the condition that if any subuniverse of an algebra contains a congruence class of some congruence on the algebra, then this subuniverse is a union of congruence classes of that congruence, i.e., that congruence is *compatible* with the subuniverse. In this text we shall call this condition *subuniverse coherence*. In this section we shall introduce various generalizations of subuniverse coherence and provide Mal'cev conditions characterizing these notions of coherence. Later in this text, we shall introduce the *sentential calculus of subuniverse modulo a quasivariety* (see Example 5.47 on page 188). This logic is generally 'inherently unalgebraizable' since it generally has no theorems; and as a consequence is a prime candidate for our theory of parameterized algebraization. In Example 14.21 on page 413 we shall explicate various relationships between the *parameterized protoalgebraicity* of this logic and the conditions of subuniverse conditions introduced in this section.

Definition 10.1 (Subuniverse Coherence) Let $\mathcal{K}$ be a quasivariety of algebras and let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ be $\mathcal{K}$-unary terms. We say that $\mathcal{K}$ is **subuniverse $\langle \mathcal{K}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent** if, for each $\mathbf{A} \in \mathcal{K}$ and $B \in \mathsf{Su}(\mathbf{A})$, $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a \in \mathsf{uni}(\mathbf{A})$, if $\alpha\left[\{\boldsymbol{u}_1^{\mathbf{A}}(a), \ldots, \boldsymbol{u}_n^{\mathbf{A}}(a)\}\right] \subseteq B$ then α is compatible with B, i.e., $\alpha[B] \subseteq B$ (equivalently $\alpha[B] = B$). For a variable y, subuniverse $\langle \mathcal{K}, y \rangle$-coherence is referred to as **subuniverse $\mathcal{K}$-coherence**, since the particular value of the variable is immaterial in this case. When we drop the $\mathcal{K}$ from these notions, for example speaking of $\mathcal{K}$ being **subuniverse $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent** or of $\mathcal{K}$ being **subuniverse coherent**, we mean that the quantification 'for any $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$' is to be replaced by 'for any $\alpha \in \mathsf{Con}(\mathbf{A})$'. We say that $\mathcal{K}$ is **subuniverse point $\langle \mathcal{K}, \mathbf{0}_1, \ldots, \mathbf{0}_n \rangle$-coherent**, if $\mathbf{0}_1, \ldots, \mathbf{0}_n$ are $\mathcal{K}$-constant terms and $\mathcal{K}$ is subuniverse $\langle \mathcal{K}, \mathbf{0}_1, \ldots, \mathbf{0}_n \rangle$-coherent. Use of the word 'point' in this context implies that the terms are equationally definable constants. □

In the following result we provide a *Mal'cev* characterization of the property that a quasivariety be subuniverse $\langle \mathcal{K}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent. Note that despite the fact that we are dealing with a quasivariety and relative congruences, this characterization is a Mal'cev condition and not a *quasi-Mal'cev* condition. Consequently, $\mathcal{K}$ is subuniverse $\langle \mathcal{K}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent iff $\mathcal{K}$ is subuniverse $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent iff the variety generated by $\mathcal{K}$ is $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent. So subuniverse coherence is in some sense an inherently a *non-relative* notion.

Theorem 10.2 Let $\mathcal{K}$ be a quasivariety of algebras, let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ be $\mathcal{K}$-unary terms, and let $\mathcal{V}$ be the variety generated by $\mathcal{K}$. The following conditions are equivalent.

1. $\mathcal{K}$ is subuniverse $\langle \mathcal{K}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent.
2. There exist ternary terms $\Delta_1, \ldots, \Delta_m$, an $m+1$-ary term p and a (selection) function $j. : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ such that

$$\models_{\mathcal{K}} \boldsymbol{u}_{j_i}(z) \approx \Delta_i(x, x, z), \quad \text{for each } 1 \leq i \leq m \text{ and} \tag{10.1}$$

$$\models_{\mathcal{K}} x \approx p(y, \Delta_1(x, y, z), \ldots, \Delta_m(x, y, z)). \tag{10.2}$$

3. $\mathcal{V}$ is subuniverse $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent.
4. $\mathcal{K}$ is subuniverse $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent.

Proof. We prove the equivalence of (1) and (2), the outstanding conditions following trivially in the light of the equational character of (10.1) and (10.2). $\boxed{(1) \Rightarrow (2)}$ Consider the 3-generated $\mathcal{K}$-free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\overline{x}, \overline{y}, \overline{z}) \in \mathcal{K}$. Let $\alpha = \|\langle \overline{x}, \overline{y} \rangle\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$. By assumption, $\overline{x} \in \left\| \{y\} \cup \alpha \left[\{\boldsymbol{u}_1^{\mathbf{F}}(\overline{z}), \ldots, \boldsymbol{u}_n^{\mathbf{F}}(\overline{z})\}\right] \right\|^{\mathbf{F}}_{\mathsf{su}}$. So by Theorem 1.344 on page 65, there exists a term p and $\overline{\Delta}_1, \ldots, \overline{\Delta}_m \in \alpha\left[\{\boldsymbol{u}_1^{\mathbf{F}}(\overline{z}), \ldots, \boldsymbol{u}_n^{\mathbf{F}}(\overline{z})\}\right]$, such that $\overline{x} = p^{\mathbf{F}}(\overline{y}, \overline{\Delta}_1, \ldots, \overline{\Delta}_m)$. The result follows by Lemma 1.457 on page 88. $\boxed{(2) \Rightarrow (1)}$ Let $\mathbf{A} \in \mathcal{K}$, $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, $a, b, c \in \mathsf{uni}(\mathbf{A})$, $B \in \mathsf{Su}(\mathbf{A})$, such that $\alpha\left[\{\boldsymbol{u}_1^{\mathbf{A}}(a), \ldots, \boldsymbol{u}_n^{\mathbf{A}}(a)\}\right] \subseteq B$, $b \in B$ and $b \,\alpha\, c$. By (10.1), for each i, $\Delta_i^{\mathbf{A}}(c, b, a) \,\alpha\, \Delta_i^{\mathbf{A}}(b, b, a) = \boldsymbol{u}_{j_i}^{\mathbf{A}}(a)$; hence $\Delta_i^{\mathbf{A}}(c, b, a) \in \alpha[\![\boldsymbol{u}_{j_i}^{\mathbf{A}}(a)]\!] \subseteq B$. So by (10.2) and Theorem 1.344, $c = q^{\mathbf{A}}(b, \Delta_0^{\mathbf{A}}(c, b, a), \ldots, \Delta_{n-1}^{\mathbf{A}}(c, b, a)) \in B$. ◇

For ease of future reference, we highlight the special case of subuniverse $\mathcal{K}$-coherence.

Corollary 10.3 Let $\mathcal{K}$ be a quasivariety and $\mathcal{V}$ the variety generated by $\mathcal{K}$. The following conditions are equivalent.

1. $\mathcal{K}$ is subalgebra $\mathcal{K}$-coherent.
2. There exist ternary terms $\Delta_1, \ldots, \Delta_m$ and an $m+1$-ary term p such that

$$\models_{\mathcal{K}} z \approx \Delta_i(x, x, z), \quad \text{for each } 1 \leq i \leq m \text{ and} \tag{10.3}$$

$$\models_{\mathcal{K}} x \approx p(y, \Delta_1(x, y, z), \ldots, \Delta_m(x, y, z)). \tag{10.4}$$

3. $\mathcal{V}$ is subalgebra coherent.
4. $\mathcal{K}$ is subalgebra coherent.

□

We turn briefly to subuniverse *point* coherence. Note that the quantification over all $a \in \mathsf{uni}(\mathbf{A})$ in the definition of subuniverse coherence is immaterial in the case of subuniverse *point* coherence. It is essentially this observation that leads to the simpler terms in the following characterization of subuniverse point coherence. This result follows easily from Theorem 10.2. The one direction, from binary terms to ternary terms is trivial, since z serves no purpose due to the constants. In the other direction, the binary terms obtain from the ternary, by substituting x for z.

Corollary 10.4 Let $\mathcal{K}$ be a quasivariety of algebras, let $\mathbf{0}_1, \ldots, \mathbf{0}_n$ be $\mathcal{K}$-constant terms, and let $\mathcal{V}$ be the variety generated by $\mathcal{K}$. The following conditions are equivalent.

1. $\mathcal{K}$ is subuniverse $\langle \mathcal{K}, \mathbf{0}_1, \ldots, \mathbf{0}_n \rangle$-coherent.

2. There exist binary terms $\Delta_1, \ldots, \Delta_m$, an $m+1$-ary term p and a $j. : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that

$$\models_{\mathcal{K}} \mathbf{0}_{j_i} \approx \Delta_i(x, x), \quad \text{for each } 1 \leq i \leq m \text{ and} \tag{10.5}$$

$$\models_{\mathcal{K}} x \approx p(y, \Delta_1(x, y), \ldots, \Delta_m(x, y)). \tag{10.6}$$

3. $\mathcal{V}$ is subuniverse $\langle \mathcal{K}, \mathbf{0}_1, \ldots, \mathbf{0}_n \rangle$-coherent.

□

We once again draw the readers attention to difference in arity between the terms of the Mal'cev characterization of subuniverse *point* coherence versus the terms of the Mal'cev characterization of subuniverse coherence. It is the *extra* variable 'z' in the latter terms that will account for the key role played by the variable z in our theory of parameterized algebraization.

10.2 Coherent $\mathfrak{N}$-Classes

In [BR99], a notion of coherence was introduced that was different in spirit to the existing conditions of coherence in the literature of universal algebra; the latter were variants of *subuniverse* coherence, and simply referred to as *coherence*, and the condition of having *coherent congruence classes*. Unlike the condition of $\langle \mathcal{K}, \tau \rangle$-*regularity* introduced in [BR99], the notion of *coherence* introduced in [BR99] encompassed no existing coherence conditions form the literature of universal algebra. We shall give a brief motivation of its introduction.

Recall our discussion (on page 307) concerning the relationship between algebraizable sentential 1-calculi, the condition that a quasivariety $\mathcal{K}$ be $\langle \mathcal{K}, \tau \rangle$-regular and the algebraizability of the logic $S(\mathcal{K}, \tau)$ introduced in [BR99]. In particular, recall that all algebraizable sentential 1-calculi are formally equivalent to a logic $S(\mathcal{K}, \tau)$, and as a consequence the study of algebraizable 1-calculi can be restricted to the analysis of the logics $S(\mathcal{K}, \tau)$. The notion of coherence introduced in [BR99] arises from considering the protoalgebraicity of $S(\mathcal{K}, \tau)$; $S(\mathcal{K}, \tau)$ is protoalgebraic iff $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-coherent [BR99] (see Theorem 2.143 on page 119 of our text). Further, this condition of coherence is a *necessary* condition for $\mathcal{K}$ to be $\langle \mathcal{K}, \tau \rangle$-regular (equivalently, for $S(\mathcal{K}, \tau)$ to be algebraizable); *coherence is to regularity* as *protoalgebraicity is to algebraizability.* We shall now give an informal description of the condition that an algebra be $\langle \mathcal{K}, \tau \rangle$-coherent.

The logic $S(\mathcal{K},\tau)$ is usefully viewed as the logic of solutions to τ modulo $\mathcal{K}$; in fact, the theories of $S(\mathcal{K},\tau)$ are precisely the solutions $\mathfrak{N}/\alpha$ for $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})$, the 'best behaved' $S(\mathcal{K},\tau)$-filters on $\mathbf{A}$ are the solutions $\mathfrak{N}^{\mathbf{A}}/\alpha$ for $\alpha \in \mathsf{Con}^{\mathbf{A}}(\mathbf{Tm})$, and if $\mathcal{K}$ is $\langle\mathcal{K},\tau\rangle$-regular (equivalently $S(\mathcal{K},\tau)$ is algebraizable) then the solutions $\mathfrak{N}^{\mathbf{A}}/\alpha$ are precisely the $S(\mathcal{K},\tau)$-filters of $\mathcal{K}$ [BR99] (see $\oint$9.1.2 for a generalization of these arguments to sentential n-calculi). The condition that an algebra $\mathbf{A}$ *has $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes* demands that if any solution $\mathfrak{N}^{\mathbf{A}}/\alpha$ contains *another* solution $\mathfrak{N}^{\mathbf{A}}/\beta$, then the relative congruence β must be *compatible* with the solution $\mathfrak{N}^{\mathbf{A}}/\alpha$, i.e., $\beta\left[\tau^{\mathbf{A}}/\alpha\right] = \tau^{\mathbf{A}}/\alpha$.

In this section, we shall consider the natural generalization of this condition of coherence from unary systems to n-ary systems. In $\oint$16 we shall show how this notion permits us to extend the results of [BR99] from sentential 1-calculi to sentential n-calculi more generally.

Definition 10.5 (Coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-Classes) Let $\mathfrak{N}$ be an n-ary system of equations. We say that $\mathbf{A}$ (resp. $\mathcal{K}$) has **$\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes** if, (resp. for each $\mathbf{A} \in \mathcal{K}$ and) for all $\alpha,\beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\mathfrak{N}^{\mathbf{A}}/\beta \subseteq \mathfrak{N}^{\mathbf{A}}/\alpha$ then $\underset{\rightarrow[n]}{\beta}$ is compatible with $\mathfrak{N}^{\mathbf{A}}/\alpha$, i.e., $\underset{\rightarrow[n]}{\beta}\left[\mathfrak{N}^{\mathbf{A}}/\alpha\right] = \mathfrak{N}^{\mathbf{A}}/\alpha$ (equivalently $\underset{\rightarrow[n]}{\beta}\left[\mathfrak{N}^{\mathbf{A}}/\alpha\right] \subseteq \mathfrak{N}^{\mathbf{A}}/\alpha$). □

Before turning to the characterizations of coherence, we note that having $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes is a *necessary* condition for an algebra to be $\langle\mathcal{K},\mathfrak{N}\rangle$-regular; i.e., regularity implies coherence. Note that this relationship between *regularity* and *coherence* reflects, in the light of the earlier discussion, the relationship between *algebraizability* and *protoalgebraicity*.

Proposition 10.6 If $\mathbf{A}$ is $\langle\mathcal{K},\mathfrak{N}\rangle$-regular then $\mathbf{A}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes.

Proof. Suppose that $\mathfrak{N}^{\mathbf{A}}/\beta \subseteq \mathfrak{N}^{\mathbf{A}}/\alpha$. By assumed regularity and Proposition 11.2 on page 359, $\beta \subseteq \alpha$; hence $\underset{\rightarrow[n]}{\beta} \subseteq \underset{\rightarrow[n]}{\alpha}$. Since $\underset{\rightarrow[n]}{\alpha}$ is compatible with $\mathfrak{N}^{\mathbf{A}}/\alpha$, by Proposition 9.6 on page 313, and $\underset{\rightarrow[n]}{\beta} \subseteq \underset{\rightarrow[n]}{\alpha}$, we have that $\underset{\rightarrow[n]}{\beta}$ is compatible with $\mathfrak{N}^{\mathbf{A}}/\alpha$ as required. ◇

The following result characterizes the property that $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes. For a further characterization in terms of the *protoalgebraicity* of the *n-deductive system* $S^n(\mathcal{K},\mathfrak{N})$ (introduced in $\oint$9.1.2) see Corollary 16.41 on page 454.

Theorem 10.7 For a quasivariety $\mathcal{K}$ of $\mathfrak{a}$-algebras and an n-ary system $\mathfrak{N}$ of equations, the following are equivalent.

1. All $\mathfrak{a}$-algebras have $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes.

2. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes.

3. There exists a finite set $\mathbf{\Delta}$ of $2n$-ary terms such that

$$\models_{\mathcal{K}} \mathfrak{N}^{\approx}\left[\mathbf{\Delta}(x_1,\ldots,x_n,x_1,\ldots,x_n)\right], \quad \text{and} \tag{10.7}$$

$$\models_{\mathcal{K}} \bigwedge\mathfrak{N}^{\approx}[\![\langle y_1,\ldots,y_n\rangle]\!] \text{ and } \bigwedge\mathfrak{N}^{\approx}\left[\mathbf{\Delta}(x_1,\ldots,x_n,y_1,\ldots,y_n)\right] \rightarrow \mathfrak{N}^{\approx}[\![\langle x_1,\ldots,x_n\rangle]\!]. \tag{10.8}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ Trivial. $\boxed{(2)\Rightarrow(3)}$ Consider the $2n$-generated $\mathcal{K}$-free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\overline{x_1},\ldots,\overline{x_n},\overline{y_1},\ldots,\overline{y_n}) \in \mathcal{K}$. Let $\alpha = \Theta^{\mathbf{F}}_{\mathcal{K}}(\langle\overline{x_1},\overline{y_1}\rangle,\ldots,\langle\overline{x_n},\overline{y_n}\rangle)$ and $\beta = \Theta^{\mathbf{F}}_{\mathcal{K}}(\mathfrak{N}^{\mathbf{F}}[(\mathfrak{N}^{\mathbf{F}}/\alpha) \cup \{\langle\overline{y_1},\ldots,\overline{y_n}\rangle\}])$. Note that $\langle\overline{y_1},\ldots,\overline{y_n}\rangle \in \mathfrak{N}^{\mathbf{F}}/\beta$. Now $\mathfrak{N}^{\mathbf{F}}/\alpha \subseteq \mathfrak{N}^{\mathbf{F}}/\beta$, so by assumption (2),

$\underset{\to[n]}{\alpha}[\mathfrak{N}^{\mathbf{F}}/\beta] = \mathfrak{N}^{\mathbf{F}}/\beta$. From $\langle\overline{y_1},\ldots,\overline{y_n}\rangle \in \mathfrak{N}^{\mathbf{F}}/\beta$ and $\langle\overline{x_1},\overline{y_1}\rangle,\ldots,\langle\overline{x_n},\overline{y_n}\rangle \in \alpha$, we infer $\langle\overline{x_1},\ldots,\overline{x_n}\rangle \in \mathfrak{N}^{\mathbf{F}}/\beta$. Hence $\mathfrak{N}^{\mathbf{F}}[\![\langle\overline{x_1},\ldots,\overline{x_n}\rangle]\!] \subseteq \beta = \Theta^{\mathbf{F}}_{\mathcal{K}}(\mathfrak{N}^{\mathbf{F}}_{\vec{z}}[(\mathfrak{N}^{\mathbf{F}}/\,\alpha) \cup \{\langle\overline{y_1},\ldots,\overline{y_n}\rangle\}])$. Since relative congruences form an algebraic closed system and $\mathfrak{N}^{\mathbf{F}}[\![\langle\overline{x_1},\ldots,\overline{x_n}\rangle]\!]$ is finite, there exist $\overline{\Delta_1},\ldots,\overline{\Delta_m} \in \mathfrak{N}^{\mathbf{F}}/\,\alpha$ and $\overline{\Delta'_1},\ldots,\overline{\Delta'_n} \in \{\langle\overline{y_1},\ldots,\overline{y_n}\rangle\}$, with $\mathfrak{N}^{\mathbf{F}}[\![\langle\overline{x_1},\ldots,\overline{x_n}\rangle]\!] \subseteq \Theta^{\mathbf{F}}_{\mathcal{K}}(\mathfrak{N}^{\mathbf{F}}[\overline{\Delta'_1},\ldots,\overline{\Delta'_n},\overline{\Delta_1},\ldots,\overline{\Delta_m}])$. The result follows from Lemma 1.457. $\boxed{(3)\Rightarrow(1)}$ Let $\mathbf{A}$ be an algebra and α and β two $\mathcal{K}$-congruences of $\mathbf{A}$ with $\mathfrak{N}^{\mathbf{A}}/\,\alpha \subseteq \mathfrak{N}^{\mathbf{A}}/\,\beta$. (*We must show that* $\underset{\to[n]}{\alpha}$ *is compatible with* $\mathfrak{N}^{\mathbf{A}}/\,\beta$.) Let $\langle a_1,b_1\rangle,\ldots,\langle a_n,b_n\rangle \in \alpha$ with $\langle a_1,\ldots a_n\rangle \in \mathfrak{N}^{\mathbf{A}}/\,\beta$. By (10.7) and Lemma 1.457, $\mathfrak{N}^{\mathbf{A}}[\mathbf{\Delta}^{\mathbf{A}}(a_1,\ldots a_n,a_1,\ldots a_n)] \subseteq \alpha$, and since $\langle a_1,b_1\rangle,\ldots,\langle a_n,b_n\rangle \in \alpha$, $\mathbf{\Delta}^{\mathbf{A}}(b_1,\ldots b_n,a_1,\ldots a_n) \in \mathfrak{N}^{\mathbf{A}}/\alpha \subseteq \mathfrak{N}^{\mathbf{A}}/\beta$, by assumption. So $\mathfrak{N}^{\mathbf{A}}[\mathbf{\Delta}^{\mathbf{A}}(b_1,\ldots b_n,a_1,\ldots a_n)] \subseteq \beta$, and since $\langle a_1,\ldots a_n\rangle \in \mathfrak{N}^{\mathbf{A}}/\,\beta$, $\mathfrak{N}^{\mathbf{A}}[\![\langle a_1,\ldots a_n\rangle]\!] \subseteq \beta$. Hence by (10.8) and Lemma 1.457, $\mathfrak{N}^{\mathbf{A}}[\![\langle b_1,\ldots b_n\rangle]\!] \subseteq \beta$, i.e., $\langle b_1,\ldots b_n\rangle \in \mathfrak{N}^{\mathbf{A}}/\,\beta$. ◇

10.2.1 Examples

Example 10.8 (Having Coherent $\langle\mathcal{K},\tau\rangle$-Classes) [BR99]

> Let $\mathcal{K}$ be a quasivariety and τ a unary system. The results of [BR99] concerning $\mathcal{K}$ having coherent $\langle\mathcal{K},\tau\rangle$-classes, where τ is a unary system, obtain immediately from the results of this section. In particular, the equivalence of the quasi-Mal'cev condition of Corollary 2.141 on page 119 and the condition that $\mathcal{K}$ has coherent $\langle\mathcal{K},\tau\rangle$-classes, follows immediately from Theorem 10.7.

□

10.3 $\langle\mathcal{K},\mathfrak{B}_*\rangle$-Coherence

None of the previous notions of coherence encompass the property that an algebra have *coherent congruence classes* in the sense of [Dud89]. We now develop a theory of coherence that encompasses *both* the condition of having coherent congruence classes and the condition of having coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes in the sense of [BR99]. In Example 14.28 on page 415, we shall relate this condition of coherence to the protoalgebraicity of the logics $S(\mathcal{K},\mathfrak{B}_*)$ introduced in ∮12. We call this condition *having coherent* $\langle\mathcal{K},\mathfrak{B}_*\rangle$*-classes*, where $\mathfrak{B}$ is a binary system. We shall introduce two such conditions, one weaker than the other, and shall show that the two conditions coincide when $\mathfrak{B}_*$ *pivots* for $\mathcal{K}$. This coincidence, when $\mathfrak{B}_*$ *pivots* for $\mathcal{K}$, sheds some light on why the pivoting of $\mathfrak{B}_*$ for $\mathcal{K}$ is a necessary condition for our theory of parameterized algebraization to 'work well'.

10.3.1 Coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-Classes

We begin by considering the *stronger* of the two conditions.

Definition 10.9 (Coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-Classes) Let $\mathcal{K}$ be a quasivariety of algebras, let $\mathfrak{B}$ be a binary system of equations and let $\mathbf{A}$ be an algebra, not necessarily in $\mathcal{K}$. A subset $A \subseteq \mathsf{uni}(\mathbf{A})$ is called $\langle\mathcal{K},\mathfrak{B}_*\rangle$**-coherent** if, for any $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a \in \mathsf{uni}(\mathbf{A})$, if $\mathfrak{B}^{\mathbf{A}}_a/\alpha \subseteq A$ then α is compatible with A, i.e., $\alpha\,[A] \subseteq A$ (equivalently $\alpha\,[A] = A$). We say that $\mathbf{A}$ (resp. $\mathcal{K}$) has $\mathcal{K}$**-coherent** $\langle\mathcal{K},\mathfrak{B}_*\rangle$**-classes** if, (resp. for each $\mathbf{A} \in \mathcal{K}$ and) for each $b \in \mathsf{uni}(\mathbf{A})$ and $\beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, the set $\mathfrak{B}^{\mathbf{A}}_b/\beta$ is $\langle\mathcal{K},\mathfrak{B}\rangle$-coherent. □

In the following result we provide a quasi-Mal'cev characterization of the condition that a quasivariety $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.

Theorem 10.10 For a quasivariety $\mathcal{K}$ of $\mathfrak{a}$-algebras and a binary system $\mathfrak{B}$ of equations, the following are equivalent.

1. All $\mathfrak{a}$-algebras have $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.
2. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.
3. There exists a finite set $\mathbf{\Delta}$ of quaternary terms such that

$$\models_{\mathcal{K}} \mathfrak{B}^{\approx}_{w}\left[\mathbf{\Delta}(x, x, w, z)\right], \quad \text{and} \tag{10.9}$$

$$\models_{\mathcal{K}} \bigwedge \mathfrak{B}^{\approx}_{z}[\![y]\!] \text{ and } \bigwedge \mathfrak{B}^{\approx}_{z}\left[\mathbf{\Delta}(x, y, w, z)\right] \longrightarrow \mathfrak{B}^{\approx}_{z}[\![x]\!]. \tag{10.10}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ Trivial. $\boxed{(2)\Rightarrow(3)}$ Consider the 4-generated $\mathcal{K}$-free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\overline{x}, \overline{y}, \overline{w}, \overline{z}) \in \mathcal{K}$. Let $\alpha = \Theta^{\mathbf{F}}_{\mathcal{K}}(\langle \overline{x}, \overline{y} \rangle)$, let $\beta = \Theta^{\mathbf{F}}_{\mathcal{K}}\big(\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[(\mathfrak{B}^{\mathbf{F}}_{\overline{w}}/\,\alpha) \cup \{\overline{y}\}]\big)$. Note that $\overline{y} \in \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$. Now $\mathfrak{B}^{\mathbf{F}}_{\overline{w}}/\,\alpha \subseteq \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$, so $\alpha\left[\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta\right] = \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$, by assumption (2). From, $\overline{y} \in \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$ and $\langle \overline{x}, \overline{y} \rangle \in \alpha$, we infer $\overline{x} \in \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$. Hence $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[\![\overline{x}]\!] \subseteq \beta = \Theta^{\mathbf{F}}_{\mathcal{K}}\big(\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[(\mathfrak{B}^{\mathbf{F}}_{\overline{w}}/\,\alpha) \cup \{\overline{y}\}]\big)$. Since relative congruences form an algebraic closed system and $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[\![\overline{x}]\!]$ is finite, there exist $\overline{\Delta_1}, \dots, \overline{\Delta_m} \in \mathfrak{B}^{\mathbf{F}}_{\overline{w}}/\,\alpha$ and $\overline{\Delta'_1}, \dots, \overline{\Delta'_n} \in \{\overline{y}\}$, with $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[\![\overline{x}]\!] \subseteq \Theta^{\mathbf{F}}_{\mathcal{K}}(\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[\overline{\Delta'_1}, \dots, \overline{\Delta'_n}, \overline{\Delta_1}, \dots, \overline{\Delta_m}])$. The result follows by Lemma 1.457. $\boxed{(3)\Rightarrow(1)}$ Let $\mathbf{A}$ be an algebra, $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a, b \in \mathsf{uni}(\mathbf{A})$, such that $\mathfrak{B}^{\mathbf{A}}_{a}/\,\alpha \subseteq \mathfrak{B}^{\mathbf{A}}_{b}/\,\beta$. (*We must show that* α *is compatible with* $\mathfrak{B}^{\mathbf{A}}_{b}/\,\beta$.) Let $\langle c, d \rangle \in \alpha$ with $c \in \mathfrak{B}^{\mathbf{A}}_{b}/\,\beta$. By (10.9) and Lemma 1.457, $\mathfrak{B}^{\mathbf{A}}_{a}(\mathbf{\Delta}^{\mathbf{A}}(c, c, a, b)) \subseteq \alpha$, and since $\langle c, d \rangle \in \alpha$, $\mathbf{\Delta}^{\mathbf{A}}(d, c, a, b) \in \mathfrak{B}^{\mathbf{A}}_{a}/\alpha \subseteq \mathfrak{B}^{\mathbf{A}}_{b}/\beta$, by assumption. Hence $\mathfrak{B}^{\mathbf{A}}_{b}[\mathbf{\Delta}^{\mathbf{A}}(d, c, a, b)] \subseteq \beta$, and since $c \in \mathfrak{B}^{\mathbf{A}}_{b}/\,\beta$, $\mathfrak{B}^{\mathbf{A}}_{b}[\![c]\!] \subseteq \beta$. So by (10.10) and Lemma 1.457, $\mathfrak{B}^{\mathbf{A}}_{b}[\![d]\!] \subseteq \beta$, i.e., $d \in \mathfrak{B}^{\mathbf{A}}_{b}/\,\beta$. $\diamond$

A comparison of this quasi-Mal'cev condition with the quasi-Mal'cev condition characterizing the property that $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \tau \rangle$-classes (see Corollary 2.141 on page 119), where τ is a unary system, reveals that while the two quasi-Mal'cev conditions are similar in form, the difference in arities of the corresponding Δs (four versus two) is *atypical* of the relationship in arities between terms in quasi-Mal'cev conditions for *full* versus *point* variants of the same notion: the difference is usually one and *not* two (compare, for example, the quasi-Mal'cev conditions characterizing $\mathcal{K}$-regularity versus relative point regularity). The essential reason that this occurs stems from the *two* points involved in the definition of having $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes, one point determining the 'original' $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-class and the other determining the 'containing' $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-class, versus *no points* in the definition of having $\mathcal{K}$-coherent $\langle \mathcal{K}, \tau \rangle$-classes. Note that while our definition of having $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes is not arbitrary, in that it has evolved from making our theory of parameterized protoalgebraicity 'work', the existence of two 'free variables' in the quasi-Mal'cev characterization does not 'work well' within our theory; we require that $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes in *this* sense, but somehow with a quasi-Mal'cev characterization involving only *one* 'free variable'. To this end, we now consider the notion of $\mathcal{K}$ having *weakly* $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes. While this condition has a quasi-Mal'cev characterization of the appropriate form, it is too *weak* for our logical requirements. Fortunately, if $\mathfrak{B}_*$ *pivots* for $\mathcal{K}$, then the two notions coincide. This is the primary reason why pivoting is a *necessary* condition for our theory of parameterized algebraization to succeed.

10.3.2 Weakly Coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-Classes

We now introduce and characterize the weaker of our two notions of coherence. Notice that the only difference between the two notions lies in the points determining the $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes. In this weaker form, we require coherence only when the two $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes involved are determined by the *same* point, where as in the stronger form we demand coherence even when the two $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes are determined by different points. As a consequence, the quasi-Mal'cev characterization that we obtain only has *one* 'free variable' as opposed to *two* in the strong case.

Definition 10.11 (Weakly Coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-Classes) We say that $\mathbf{A}$ (resp. $\mathcal{K}$) has **weakly $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes** if, (resp. for each $\mathbf{A}\in\mathcal{K}$) for each $a\in\mathsf{uni}(\mathbf{A})$ and $\alpha,\beta\in\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\mathfrak{B}^{\mathbf{A}}_a/\alpha\subseteq\mathfrak{B}^{\mathbf{A}}_a/\beta$ then α is compatible with $\mathfrak{B}^{\mathbf{A}}_a/\beta$, i.e., $\alpha\left[\mathfrak{B}^{\mathbf{A}}_a/\beta\right]\subseteq\mathfrak{B}^{\mathbf{A}}_a/\beta$ (equivalently $\alpha\left[\mathfrak{B}^{\mathbf{A}}_a/\beta\right]=\mathfrak{B}^{\mathbf{A}}_a/\beta$). $\square$

Remark 10.12 Clearly every algebra with $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes has weakly $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes.

Theorem 10.13 For a quasivariety $\mathcal{K}$ of $\mathfrak{a}$-algebras and a binary system $\mathfrak{B}$ of equations, the following are equivalent.

1. All $\mathfrak{a}$-algebras have weakly $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes.

2. $\mathcal{K}$ has weakly $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes.

3. There exists a finite set $\mathbf{\Delta}$ of ternary terms such that

$$\models_{\mathcal{K}}\mathfrak{B}^{\approx}_z\left[\mathbf{\Delta}(x,x,z)\right],\quad\text{and}\tag{10.11}$$

$$\models_{\mathcal{K}}\bigwedge\mathfrak{B}^{\approx}_z[\![y]\!]\ \mathsf{and}\ \bigwedge\mathfrak{B}^{\approx}_z\left[\mathbf{\Delta}(x,y,z)\right]\to\mathfrak{B}^{\approx}_z[\![x]\!].\tag{10.12}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ Trivial. $\boxed{(2)\Rightarrow(3)}$ Consider the 3-generated $\mathcal{K}$-free algebra $\mathbf{F}=\mathbf{F}_{\mathcal{K}}(\overline{x},\overline{y},\overline{z})\in\mathcal{K}$. Let $\alpha=\Theta^{\mathbf{F}}_{\mathcal{K}}(\langle\overline{x},\overline{y}\rangle)$ and $\beta=\Theta^{\mathbf{F}}_{\mathcal{K}}\big(\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[(\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\,\alpha)\cup\{\overline{y}\}]\big)$. Note that $\overline{y}\in\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$. Now $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\,\alpha\subseteq\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$, so by assumption (2), $\alpha\left[\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta\right]=\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$. From $\overline{y}\in\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$ and $\langle\overline{x},\overline{y}\rangle\in\alpha$, we infer $\overline{x}\in\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$. Hence $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[\![\overline{x}]\!]\subseteq\beta=\Theta^{\mathbf{F}}_{\mathcal{K}}\big(\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[(\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\,\alpha)\cup\{\overline{y}\}]\big)$. Since relative congruences form an algebraic closed system and $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[\![\overline{x}]\!]$ is finite, there exist $\overline{\Delta_1},\dots,\overline{\Delta_m}\in\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\,\alpha$ and $\overline{\Delta'_1},\dots,\overline{\Delta'_n}\in\{\overline{y}\}$, with $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[\![\overline{x}]\!]\subseteq\Theta^{\mathbf{F}}_{\mathcal{K}}(\mathfrak{B}^{\mathbf{F}}_{\overline{z}}[\overline{\Delta'_1},\dots,\overline{\Delta'_n},\overline{\Delta_1},\dots,\overline{\Delta_m}])$. The result follows from Lemma 1.457. $\boxed{(3)\Rightarrow(1)}$ Let $\mathbf{A}$ be an algebra, α and β two $\mathcal{K}$-congruences of $\mathbf{A}$ and $a\in\mathsf{uni}(\mathbf{A})$, with $\mathfrak{B}^{\mathbf{A}}_a/\,\alpha\subseteq\mathfrak{B}^{\mathbf{A}}_a/\,\beta$. We must show that α is compatible with $\mathfrak{B}^{\mathbf{A}}_a/\,\beta$. Let $\langle c,d\rangle\in\alpha$ with $c\in\mathfrak{B}^{\mathbf{A}}_a/\,\beta$. By (10.11) and Lemma 1.457, $\mathfrak{B}^{\mathbf{A}}_a[\mathbf{\Delta}^{\mathbf{A}}(c,c,a)]\subseteq\alpha$, and since $\langle c,d\rangle\in\alpha$, $\mathbf{\Delta}^{\mathbf{A}}(d,c,a)\in\mathfrak{B}^{\mathbf{A}}_a/\alpha\subseteq\mathfrak{B}^{\mathbf{A}}_a/\beta$, by assumption. So $\mathfrak{B}^{\mathbf{A}}_a[\mathbf{\Delta}^{\mathbf{A}}(d,c,a)]\subseteq\beta$, and since $c\in\mathfrak{B}^{\mathbf{A}}_a/\,\beta$, $\mathfrak{B}^{\mathbf{A}}_a[\![c]\!]\subseteq\beta$. Hence by (10.12) and Lemma 1.457, $\mathfrak{B}^{\mathbf{A}}_a[\![d]\!]\subseteq\beta$, i.e., $d\in\mathfrak{B}^{\mathbf{A}}_a/\,\beta$. $\diamond$

In the following result, we demonstrate that, in the case that $\mathfrak{B}_*$ *pivots* for $\mathcal{K}$, having *weakly* $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes implies having $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes. Consequently, when $\mathfrak{B}_*$ *pivots* for $\mathcal{K}$ then the two conditions of coherence coincide.

Theorem 10.14 If $\mathcal{K}$ has weakly $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes and $\mathfrak{B}_*$ *pivots* for $\mathcal{K}$, then $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{B}_*\rangle$-classes.

Proof. Let $\mathbf{A}$ be an algebra, $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a, b \in \mathsf{uni}(\mathbf{A})$, such that $\mathfrak{B}_a^{\mathbf{A}} / \alpha \subseteq \mathfrak{B}_b^{\mathbf{A}} / \beta$. (*We must show that α is compatible with $\mathfrak{B}_b^{\mathbf{A}} / \beta$.*) Let $\langle c, d\rangle \in \alpha$ with $c \in \mathfrak{B}_b^{\mathbf{A}} / \beta$. By (10.11) and Lemma 1.457, $\mathfrak{B}_a^{\mathbf{A}}(\mathbf{\Delta}^{\mathbf{A}}(c, c, a)) \subseteq \alpha$, and since $\langle c, d\rangle \in \alpha$, $\mathbf{\Delta}^{\mathbf{A}}(d, c, a) \in \mathfrak{B}_a^{\mathbf{A}}/\alpha \subseteq \mathfrak{B}_b^{\mathbf{A}}/\beta$, by assumption. Hence $\mathfrak{B}_b^{\mathbf{A}}[\mathbf{\Delta}^{\mathbf{A}}(d, c, a)] \subseteq \beta$, and since $c \in \mathfrak{B}_b^{\mathbf{A}} / \beta$, $\mathfrak{B}_b^{\mathbf{A}}[\![c]\!] \subseteq \beta$. By (10.12),

$$\models_{\mathcal{K}} \bigwedge \mathfrak{B}_z^{\approx}[\![y]\!] \text{ and } \bigwedge \mathfrak{B}_z^{\approx}[\mathbf{\Delta}(x, y, z)] \rightarrow \mathfrak{B}_z^{\approx}[\![x]\!],$$

hence

$$\mathfrak{B}_z^{\approx}[\![y]\!] \cup \mathfrak{B}_z^{\approx}[\mathbf{\Delta}(x, y, z)] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![x]\!].$$

Let w be a variable not in $\{x, y, z\}$. Since $\mathfrak{B}_*$ pivots for $\mathcal{K}$,

$$[z \Rsh w] \cup \mathfrak{B}_w^{\approx}[\![y]\!] \cup \mathfrak{B}_w^{\approx}[\mathbf{\Delta}(x, y, z)] \models_{\mathcal{K}} \mathfrak{B}_w^{\approx}[\![x]\!].$$

By finitariness of $\models_{\mathcal{K}}$, there exists a finite subset $P(x, y, z, w, \vec{u}) \subseteq \mathfrak{B}_z / \perp_{\mathcal{K}}$ such that

$$\mathfrak{B}_w^{\approx}[P(x, y, z, w, \vec{u})] \cup \mathfrak{B}_w^{\approx}[\![y]\!] \cup \mathfrak{B}_w^{\approx}[\mathbf{\Delta}(x, y, z)] \models_{\mathcal{K}} \mathfrak{B}_w^{\approx}[\![x]\!]. \tag{i}$$

Let $\vec{e} \in \mathsf{uni}(\mathbf{A})$. Let $p(x, y, z, w, \vec{u}) \in P(x, y, z, w, \vec{u})$. By definition, $\models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p(x, y, z, w, \vec{u})]\!]$, and so by Lemma 1.457, $\mathfrak{B}_a^{\mathbf{A}}[\![p^{\mathbf{A}}(d, c, a, c, \vec{e})]\!] \subseteq \alpha$. Hence, $p^{\mathbf{A}}(d, c, a, c, \vec{e}) \in \mathfrak{B}_a^{\mathbf{A}} / \alpha$. Since $\mathfrak{B}_a^{\mathbf{A}} / \alpha \subseteq \mathfrak{B}_b^{\mathbf{A}} / \beta$, we have $p^{\mathbf{A}}(d, c, a, c, \vec{e}) \in \mathfrak{B}_b^{\mathbf{A}} / \beta$, i.e., $\mathfrak{B}_b^{\mathbf{A}}[\![p^{\mathbf{A}}(d, c, a, c, \vec{e})]\!] \subseteq \beta$. So $\mathfrak{B}_b^{\mathbf{A}}\left[P^{\mathbf{A}}(d, c, a, c, \vec{e})\right] \subseteq \beta$. Hence $\mathfrak{B}_b^{\mathbf{A}}\left[P^{\mathbf{A}}(d, c, a, c, \vec{e})\right] \cup \mathfrak{B}_b^{\mathbf{A}}[\![c]\!] \cup \mathfrak{B}_b^{\mathbf{A}}[\mathbf{\Delta}^{\mathbf{A}}(d, c, a)] \subseteq \beta$, so by (i) and Lemma 1.457, $\mathfrak{B}_b^{\mathbf{A}}[\![d]\!] \subseteq \beta$, i.e., $d \in \mathfrak{B}_b^{\mathbf{A}} / \beta$. ⋄

Open Problem 10.15 Does $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity imply that $\mathcal{K}$ has coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes?

10.3.3 Examples

Recall the definition of a $\mathcal{K}$-separable binary system and the fact that such systems pivot for $\mathcal{K}$ (when they have non-trivial variable-bases).

Example 10.16 (Separable Binary Systems)

Let $\mathfrak{B}(x, y) = \{\boldsymbol{u}_i(x) \approx \boldsymbol{u}'_i(y) : i \in n\}$ be a ($\mathcal{K}$-separable) binary system with *non-trivial* variable bases.

Corollary 10.17 The following conditions are equivalent.

$\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.

$\mathcal{K}$ has weakly $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.

There exists a finite set $\mathbf{\Delta}$ of ternary terms such that

$$\models_{\mathcal{K}} \boldsymbol{u}_i(\Delta(x, x, z)) \approx \boldsymbol{u}'_i(z), \quad \forall\, [i = 1..n, \Delta \in \mathbf{\Delta}] \quad \text{and} \tag{10.13}$$

$$\begin{aligned}\models_{\mathcal{K}} \bigwedge\nolimits_{i=1..n} \Big(\boldsymbol{u}_i(y) \approx \boldsymbol{u}'_i(z) \text{ and} \bigwedge\nolimits_{\Delta \in \mathbf{\Delta}} \boldsymbol{u}_i(\Delta(x, y, z)) \approx \boldsymbol{u}'_i(z) \Big) \\ \rightarrow \bigwedge\nolimits_{i=1..n} \boldsymbol{u}_i(x) \approx \boldsymbol{u}'_i(z).\end{aligned} \tag{10.14}$$

□

We now consider the special case where $\mathfrak{B}(x, y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$, where $\boldsymbol{u}$ is a $\mathcal{K}$-unary term, since in this case, $\mathfrak{B}(x, y)$ always pivots and we are able to obtain a *simpler* quasi-Mal'cev characterization than that obtained directly from (3) of Corollary 10.17. This quasi-Mal'cev condition will be more familiar to those readers acquainted with the *Mal'cev* characterization of *varieties* having *coherent congruence classes* given in [Dud89].

Example 10.18 (Having $\mathcal{K}$-Coherent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-Classes)

Let $\mathcal{K}$ be a quasivariety and $\boldsymbol{u}$ a $\mathcal{K}$-unary term.

Definition 10.19 (Having $\mathcal{K}$-Coherent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-Classes) We say that an algebra $\mathbf{A}$ (resp. $\mathcal{K}$) has **$\mathcal{K}$-coherent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-classes** if, (resp. for all $\mathbf{A} \in \mathcal{K}$ and) for all $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a, b \in \mathsf{uni}(\mathbf{A})$, if $\beta[\![\boldsymbol{u}^{\mathbf{A}}(b)]\!] \subseteq \alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!]$ then β is compatible with $\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!]$. $\square$

Recall the definition of the binary system $\boldsymbol{u}$ determined by $\boldsymbol{u}$ (see Example 9.58 on page 327).

Remark 10.20 $\mathbf{A}$ has $\mathcal{K}$-coherent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-classes iff $\mathbf{A}$ has $\mathcal{K}$-coherent $\langle\mathcal{K}, \boldsymbol{u}_*\rangle$-classes. $\square$

The following result obtains since $\boldsymbol{u}$ pivots for $\mathcal{K}$ by Proposition 9.61 on page 327. Observe how the quasi-Mal'cev condition (3), which is a direct interpretation of the quasi-Mal'cev condition (3) of Corollary 10.17, may be replaced with the simpler quasi-Mal'cev condition (4). The ternary terms $\boldsymbol{\Delta}$ may be taken to be the *same* in both these conditions.

Corollary 10.21 The following conditions are equivalent.

1. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-classes.
2. For all $\mathbf{A} \in \mathcal{K}$, $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a \in \mathsf{uni}(\mathbf{A})$, if $\beta[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!] \subseteq \alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!]$ then β is compatible with $\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!]$.
3. There exists a finite set $\boldsymbol{\Delta}$ of ternary terms such that
$$\models_{\mathcal{K}} \boldsymbol{\Delta}(x, x, z) \approx \boldsymbol{u}(z), \quad \text{and} \tag{10.15}$$
$$\models_{\mathcal{K}} y \approx \boldsymbol{u}(z) \,\mathsf{and} \bigwedge \boldsymbol{\Delta}(x, y, z) \approx \boldsymbol{u}(z) \rightarrow x \approx \boldsymbol{u}(z). \tag{10.16}$$
4. There exists a finite set $\boldsymbol{\Delta}$ of ternary terms such that
$$\models_{\mathcal{K}} \boldsymbol{\Delta}(x, x, z) \approx \boldsymbol{u}(z), \quad \text{and} \tag{10.17}$$
$$\models_{\mathcal{K}} \bigwedge \boldsymbol{\Delta}(x, \boldsymbol{u}(z), z) \approx \boldsymbol{u}(z) \rightarrow x \approx \boldsymbol{u}(z). \tag{10.18}$$

Proof. $\boxed{(3)\Rightarrow(4)}$ Suppose that $\mathbf{A} \in \mathcal{K}$ and $a, c \in \mathsf{uni}(\mathbf{A})$ such that, for all $\Delta \in \boldsymbol{\Delta}$, $\Delta^{\mathbf{A}}(a, \boldsymbol{u}^{\mathbf{A}}(c), c) = \boldsymbol{u}^{\mathbf{A}}(c)$. (*We must show that* $a = \boldsymbol{u}^{\mathbf{A}}(c)$.) Let $b = \boldsymbol{u}^{\mathbf{A}}(c)$. Then $b = \boldsymbol{u}^{\mathbf{A}}(c)$ and $\Delta^{\mathbf{A}}(a, b, c) = \boldsymbol{u}^{\mathbf{A}}(c)$; hence by (10.16), $a = \boldsymbol{u}^{\mathbf{A}}(c)$. $\boxed{(4)\Rightarrow(3)}$ Suppose that $\mathbf{A} \in \mathcal{K}$ and $a, b, c \in \mathsf{uni}(\mathbf{A})$ such that $b = \boldsymbol{u}^{\mathbf{A}}(c)$ and, for all $\Delta \in \boldsymbol{\Delta}$, $\Delta^{\mathbf{A}}(a, b, c) = \boldsymbol{u}^{\mathbf{A}}(c)$. (*We must show that* $a = \boldsymbol{u}^{\mathbf{A}}(c)$.) For all $\Delta \in \boldsymbol{\Delta}$, since $b = \boldsymbol{u}^{\mathbf{A}}(c)$ and $\Delta^{\mathbf{A}}(a, b, c) = \boldsymbol{u}^{\mathbf{A}}(c)$, $\Delta^{\mathbf{A}}(a, \boldsymbol{u}^{\mathbf{A}}(c), c) = \boldsymbol{u}^{\mathbf{A}}(c)$. So by (10.18), $a = \boldsymbol{u}^{\mathbf{A}}(c)$. $\diamond$

$\square$

In [Dud89] the notion of an algebra having **coherent congruence classes** was introduced and a Mal'cev condition characterizing varieties of such algebras obtained. In the next example, we consider the relativization of this notion.

Example 10.22 ($\mathcal{K}$-Coherent $\mathcal{K}$-Classes)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras.

Definition 10.23 ($\mathcal{K}$-Coherent $\mathcal{K}$-Classes) We say that $\mathbf{A}$ has **$\mathcal{K}$-coherent $\mathcal{K}$-classes** if, for all $a, b \in \mathsf{uni}(\mathbf{A})$ and $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\beta[\![b]\!] \subseteq \alpha[\![a]\!]$ then β is compatible with $\alpha[\![a]\!]$. We extend this definition to $\mathcal{K}$ in the natural manner. □

Remark 10.24 $\mathbf{A}$ has $\mathcal{K}$-coherent $\mathcal{K}$-classes iff $\mathbf{A}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, x \rangle$-classes. □

We explicate the characterization that $\mathcal{K}$ has $\mathcal{K}$-coherent $\mathcal{K}$-classes for ease of subsequent reference, and note that [Dud89, Theorem 2] (for varieties) follows from this corollary, and that equivalent condition (2), which is apparently weaker than (1) is a new result even in the non-relative case.

Corollary 10.25 The following conditions are equivalent.

1. $\mathcal{K}$ has $\mathcal{K}$-coherent $\mathcal{K}$-classes.
2. For all $\mathbf{A} \in \mathcal{K}$, $a \in \mathsf{uni}(\mathbf{A})$ and $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $\beta[\![a]\!] \subseteq \alpha[\![a]\!]$ then β is compatible with $\alpha[\![a]\!]$.
3. There exists a finite set $\mathbf{\Delta}$ of ternary terms such that

$$\models_{\mathcal{K}} \mathbf{\Delta}(x, x, z) \approx z, \quad \text{and} \tag{10.19}$$

$$\models_{\mathcal{K}} \bigwedge \mathbf{\Delta}(x, z, z) \approx z \rightarrow x \approx z. \tag{10.20}$$

□

Recall the definition of an *essentially $\mathcal{K}$-unary system* given in Example 9.97 on page 334, and recall further that such systems always pivot for $\mathcal{K}$. For ease of highlighting the manner in which the theory of coherence developed in this section encompasses the condition of having $\mathcal{K}$-coherent $\langle \mathcal{K}, \tau \rangle$-classes of [BR99] (see Definition 2.142 on page 119), the following example is phrased in terms of *unary systems* rather than essentially $\mathcal{K}$-unary systems.

Example 10.26 (Unary Systems)

Let $\mathcal{K}$ be a quasivariety and τ a *unary system*. In the case of such systems, the quantification over points in both the notions of coherence of this section is superfluous, and so trivially the two notions coincide. This superfluous quantification is reflected in a simpler quasi-Mal'cev condition with no 'free variables'. The proof that this quasi-Mal'cev condition is equivalent to the one obtained by a direct interpretation of the quasi-Mal'cev condition of Theorem 10.13 is simple: in the one direction the ternary terms can be converted to binary terms by setting $z = x$, in the other direction the binary terms are converted to ternary terms in which z does not occur. We note the quasi-Mal'cev characterization of $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \tau \rangle$-classes given in [BR99] is precisely equivalent condition (3) of the following result (see Example 2.140 on page 118 of our text).

Corollary 10.27 The following conditions are equivalent.

1. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \tau_* \rangle$-classes.
2. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \tau \rangle$-classes.

3. There exists a finite set $\mathbf{\Delta}$ of *binary* terms such that

$$\models_{\mathcal{K}} \tau^{\approx}\left[\mathbf{\Delta}(x,x)\right], \quad \text{and} \tag{10.21}$$

$$\models_{\mathcal{K}} \bigwedge \tau^{\approx}[\![y]\!] \text{ and } \bigwedge \tau^{\approx}\left[\mathbf{\Delta}(x,y)\right] \to \tau^{\approx}[\![\langle x\rangle]\!]. \tag{10.22}$$

□

We now characterize the various notions of coherence with respect to lattice ideals and filters. The reader is urged to recall the definitions and results of Example 9.102 on page 335. We develop the theory for ideals, leaving the dual filter results to the reader.

Example 10.28 (Quasivarieties of Lattices)

We begin by characterizing the condition that $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\vartriangle_*\rangle$-classes. Recall that the $\langle\mathcal{K},\vartriangle_*\rangle$-classes of lattice expansions are all ideals (see Proposition 9.108 on page 336). Note that the $\langle\mathcal{K},\vartriangle_*\rangle$-classes do not encompass all ideals, since by the aforementioned proposition, $\mathsf{N}^{\mathcal{K}}_{\vartriangle_*}(\mathbf{P}) \subseteq \mathsf{Id}_{\Diamond}(\mathbf{P})$, while the $\langle\mathcal{K},\vartriangle_*\rangle$-classes are the members of $\mathsf{Sol}^{\mathcal{K}}_{\vartriangle_*}(\mathbf{P}) \subseteq \mathsf{N}^{\mathcal{K}}_{\vartriangle_*}(\mathbf{P})$. The following result is an immediate corollary to Theorem 10.10.

Corollary 10.29 For a quasivariety of lattice expansions, the following are equivalent.

1. All $\mathfrak{a}$-algebras have $\mathcal{K}$-coherent $\langle\mathcal{K},\vartriangle_*\rangle$-classes.
2. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\vartriangle_*\rangle$-classes.
3. There exists a finite set $\mathbf{\Delta}$ of quaternary terms such that

$$\models_{\mathcal{K}} \mathbf{\Delta}(x,x,w,z) \leq w, \quad \text{and} \tag{10.23}$$

$$\models_{\mathcal{K}} y \leq z, \mathbf{\Delta}(x,y,w,z) \leq z \to x \leq z. \tag{10.24}$$

□

We now characterize the property that $\mathcal{K}$ has *weakly* $\mathcal{K}$-coherent $\langle\mathcal{K},\vartriangle_*\rangle$-classes. This result follows by Theorem 10.13.

Corollary 10.30 For a quasivariety of lattice expansions, the following are equivalent.

1. All $\mathfrak{a}$-algebras have weakly $\mathcal{K}$-coherent $\langle\mathcal{K},\vartriangle_*\rangle$-classes.
2. $\mathcal{K}$ has weakly $\mathcal{K}$-coherent $\langle\mathcal{K},\vartriangle_*\rangle$-classes.
3. There exists a finite set $\mathbf{\Delta}$ of ternary terms such that

$$\models_{\mathcal{K}} \mathbf{\Delta}(x,x,z) \leq z, \quad \text{and} \tag{10.25}$$

$$\models_{\mathcal{K}} y \leq z, \mathbf{\Delta}(x,y,z) \leq z \to x \leq z. \tag{10.26}$$

□

In the case that $\vartriangle_*$ pivots in $\mathcal{K}$, having *weak* $\mathcal{K}$-coherent $\langle\mathcal{K},\vartriangle_*\rangle$-classes is equivalent to having $\mathcal{K}$-coherent $\langle\mathcal{K},\vartriangle_*\rangle$-classes, by Theorem 10.14.

Warning 10.31 In the light of Proposition 9.110 on page 337, which states that if $\mathcal{K}$ is distributive then $\vartriangle_*$ pivots in $\mathcal{K}$, it is tempting to erroneously conclude that

'A *distributive* quasivariety of lattice expansions has $\mathcal{K}$-coherent $\langle \mathcal{K}, \triangle_* \rangle$-classes iff it has weakly $\mathcal{K}$-coherent $\langle \mathcal{K}, \triangle_* \rangle$-classes.'

While we have not sought a counter-example, we can see no reason to suspect the validity of this statement. $\square$

We now consider the condition that every ideal be $\langle \mathcal{K}, \triangle_* \rangle$-coherent. Since every $\langle \mathcal{K}, \triangle_* \rangle$-class is an ideal, this condition is *stronger* than those considered above.

Theorem 10.32 For a quasivariety $\mathcal{K}$ of lattice expansions, the following conditions are equivalent, where $\mathcal{V}$ is the variety generated by $\mathcal{K}$.

1. For all $\mathbf{P} \in \mathcal{K}$, every ideal of $\mathbf{P}$ is $\langle \mathcal{K}, \triangle_* \rangle$-coherent.
2. There exist ternary terms $\Delta_1, \ldots, \Delta_m$, for some m, such that

$$\models_{\mathcal{K}} \Delta_i(x, x, z) \leq z, \quad \text{for all } 1 \leq i \leq m, \text{ and} \tag{10.27}$$

$$\models_{\mathcal{K}} x \leq \Delta_1(x, y, z) \vee \ldots \vee \Delta_m(x, y, z) \vee y. \tag{10.28}$$

3. For all $\mathbf{P} \in \mathcal{V}$, every ideal of $\mathbf{P}$ is $\langle \mathcal{V}, \triangle_* \rangle$-coherent.

Proof. $\boxed{(1)\Rightarrow(2)}$ Let $\mathbf{F}_{\mathcal{K}}$ denote the $\mathcal{K}$ free algebra on three generators $\{\overline{x}, \overline{y}, \overline{z}\}$. Let $\alpha = \|\langle \overline{x}, \overline{y} \rangle\|_{\Theta^{\mathcal{K}}_{\mathbf{F}_{\mathcal{K}}}}$ and $\mathsf{I} = \left\| \left(\triangle^{\mathbf{F}_{\mathcal{K}}}_{\overline{z}} / \alpha \right) \cup \{\overline{y}\} \right\|^{\mathbf{F}_{\mathcal{K}}}_{\mathsf{id}_\Diamond}$. Since $\triangle^{\mathbf{F}_{\mathcal{K}}}_{\overline{z}} / \alpha \subseteq \mathsf{I}$, $\langle \overline{x}, \overline{y} \rangle \in \alpha$ and $\overline{y} \in \mathsf{I}$, by assumption (1), $\overline{x} \in \mathsf{I}$. So by Remark 4.93 on page 159, there exist $\overline{\Delta}_1, \ldots, \overline{\Delta}_m \in \triangle^{\mathbf{F}_{\mathcal{K}}}_{\overline{z}} / \alpha$, such that $\overline{x} = \overline{\Delta}_1 \vee^{\mathbf{F}_{\mathcal{K}}} \ldots \vee^{\mathbf{F}_{\mathcal{K}}} \overline{\Delta}_m \vee^{\mathbf{F}_{\mathcal{K}}} \overline{y}$. The result follows by Lemma 1.457. $\boxed{(2)\Rightarrow(1)}$ Let $\mathbf{P} \in \mathcal{K}$, $\mathsf{I} \in \mathsf{Id}_\Diamond(\mathbf{P})$, $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{P})$ and $c \in \mathsf{uni}(\mathbf{P})$ with $\triangle^{\mathbf{P}}_c / \alpha \subseteq \mathsf{I}$. Suppose that $a \in \mathsf{I}$ and $a \, \alpha \, b$. Now $c \in \triangle^{\mathbf{P}}_c / \alpha \subseteq \mathsf{I}$, by (4) of Remark 9.104 on page 335, and $\Delta^{\mathbf{P}}_i(a, a, c) \leq^{\mathbf{P}} c$, by (10.27); hence $\Delta^{\mathbf{P}}_i(a, a, c) \in \mathsf{I}$. Now $\Delta^{\mathbf{P}}_i(b, a, c) \vee^{\mathbf{P}} c \;\alpha\; \Delta^{\mathbf{P}}_i(a, a, c) \vee^{\mathbf{P}} c = c$, and so by (4) of Remark 9.104, $\Delta^{\mathbf{P}}_i(b, a, c) \in \triangle^{\mathbf{P}}_c / \alpha \subseteq \mathsf{I}$. Since $a \in \mathsf{I}$, $\Delta^{\mathbf{P}}_1(b, a, c) \vee^{\mathbf{P}} \Delta^{\mathbf{P}}_m(b, a, c) \vee^{\mathbf{P}} a \in \mathsf{I}$, and since $b \leq^{\mathbf{P}} \Delta^{\mathbf{P}}_1(b, a, c) \vee^{\mathbf{P}} \Delta^{\mathbf{P}}_m(b, a, c) \vee^{\mathbf{P}} a$, by (10.27), $b \in \mathsf{I}$. The equivalence of (3) follows from the equational nature of (10.27) and (10.28). $\diamond$

The following corollary follows from the previous theorem and Proposition 9.108 on page 336.

Corollary 10.33 If $\mathcal{K}$ satisfies the equivalent conditions of Theorem 10.32, then $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \triangle_* \rangle$-classes.

Theorem 10.34 For a quasivariety $\mathcal{K}$ of lattice expansions, the following conditions are equivalent, where $\mathcal{V}$ is the variety generated by $\mathcal{K}$.

1. For all $\mathbf{P} \in \mathcal{K}$, every ideal of $\mathbf{P}$ is $\mathcal{K}$-coherent.
2. There exist ternary terms $\Delta_1, \ldots, \Delta_m$, for some m, such that

$$\models_{\mathcal{K}} \Delta_i(x, x, z) \approx z, \quad \text{for all } 1 \leq i \leq m, \text{ and} \tag{10.29}$$

$$\models_{\mathcal{K}} x \leq \Delta_1(x, y, z) \vee \ldots \vee \Delta_m(x, y, z) \vee y. \tag{10.30}$$

3. For all $\mathbf{P} \in \mathcal{V}$, every ideal of $\mathbf{P}$ is $\mathcal{K}$-coherent.

Proof. $\boxed{(1)\Rightarrow(2)}$ Let $\mathbf{F}_{\mathcal{K}}$ denote the $\mathcal{K}$ free algebra on three generators $\{\overline{x}, \overline{y}, \overline{z}\}$. Let $\alpha = \|\langle \overline{x}, \overline{y} \rangle\|_{\Theta^{\mathcal{K}}_{\mathbf{F}_{\mathcal{K}}}}$ and $\mathsf{I} = \|(\alpha[\![\overline{z}]\!]) \cup \{\overline{y}\}\|^{\mathbf{F}_{\mathcal{K}}}_{\mathsf{id}_\Diamond}$. Since $\alpha[\![\overline{z}]\!] \subseteq \mathsf{I}$, $\langle \overline{x}, \overline{y} \rangle \in \alpha$ and $\overline{y} \in \mathsf{I}$, by assumption (1), $\overline{x} \in \mathsf{I}$. So by Remark 4.93 on page 159, there exist $\overline{\Delta}_1, \ldots, \overline{\Delta}_m \in \alpha[\![\overline{z}]\!]$, such that $\overline{x} = \overline{\Delta}_1 \vee^{\mathbf{F}_\mathcal{K}} \ldots \vee^{\mathbf{F}_\mathcal{K}} \overline{\Delta}_m \vee^{\mathbf{F}_\mathcal{K}} \overline{y}$. The result follows by Lemma 1.457. $\boxed{(2)\Rightarrow(1)}$ Let $\mathbf{P} \in \mathcal{K}$, $\mathsf{I} \in \mathsf{Id}_\Diamond(\mathbf{P})$, $\alpha \in \mathsf{Con}^\mathcal{K}(\mathbf{P})$ and $c \in \mathsf{uni}(\mathbf{P})$ with $\alpha[\![c]\!] \subseteq \mathsf{I}$. Suppose that $a \in \mathsf{I}$ and $a\,\alpha\,b$. Now $c \in \alpha[\![c]\!] \subseteq \mathsf{I}$ and $\Delta_i^\mathbf{P}(a,a,c) = c$, by (10.29); hence $\Delta_i^\mathbf{P}(a,a,c) \in \mathsf{I}$. Now $\Delta_i^\mathbf{P}(b,a,c) \vee^\mathbf{P} c \;\alpha\; \Delta_i^\mathbf{P}(a,a,c) \vee^\mathbf{P} c = c$, so $\Delta_i^\mathbf{P}(b,a,c) \in \alpha[\![c]\!] \subseteq \mathsf{I}$. Since $a \in \mathsf{I}$, $\Delta_1^\mathbf{P}(b,a,c) \vee^\mathbf{P} \Delta_m^\mathbf{P}(b,a,c) \vee^\mathbf{P} a \in \mathsf{I}$, and since $b \leq^\mathbf{P} \Delta_1^\mathbf{P}(b,a,c) \vee^\mathbf{P} \Delta_m^\mathbf{P}(b,a,c) \vee^\mathbf{P} a$, by (10.29), $b \in \mathsf{I}$. The equivalence of (3) follows from the equational nature of (10.29) and (10.30). $\diamond$

Corollary 10.35 The equivalent condition of Theorem 10.34 implies the equivalent conditions of Theorem 10.32.

Open Problem 10.36 Find examples of lattice expansions satisfying these properties. One will probably have to 'type' a relative complementation operation of some sort (see Proposition 10.39).

Theorem 10.37 For a quasivariety of 0-lattice expansions, the following conditions are equivalent, where $\mathcal{V}$ is the variety generated by $\mathcal{K}$.

1. Every ideal of a member of $\mathcal{K}$ is $\langle \mathcal{K}, 0\rangle$-coherent.
2. The exists binary terms $\Delta_1(x,y), \Delta_m(x,y)$, for some $m > 0$, such that

$$\models_\mathcal{K} \Delta_i(x,x) \approx 0, \quad \text{for all } i \leq m, \text{ and} \tag{10.31}$$

$$\models_\mathcal{K} x \leq \Delta_1(x,y) \vee \Delta_m(x,y) \vee y. \tag{10.32}$$

3. Every ideal of a member of $\mathcal{V}$ is $\langle \mathcal{K}, 0\rangle$-coherent.

Proof. $\boxed{(1)\Rightarrow(2)}$ Let $\mathbf{F}$ be the $\mathcal{K}$-free algebra generated by $\overline{y}$ and $\overline{x}$, let $\alpha = \|\langle \overline{y}, \overline{x}\rangle\|_{\Theta^\mathcal{K}_\mathbf{F}}$, let $\mathsf{I} = \left\|\alpha[\![\overline{0}]\!] \cup \{\overline{y}\}\right\|^\mathbf{F}_{\mathsf{id}_\Diamond}$. Since $\alpha[\![\overline{0}]\!] \subseteq \mathsf{I}$, $\overline{y} \in \mathsf{I}$ and $\overline{y}\,\alpha\,\overline{x}$, by assumption, $\overline{x} \in \mathsf{I} = \left\|\alpha[\![\overline{0}]\!] \cup \{\overline{y}\}\right\|^\mathbf{F}_{\mathsf{id}_\Diamond}$. So $\alpha[\![\overline{0}]\!] \cup \{\overline{y}\} \vdash^\mathbf{F}_{\mathsf{id}_\Diamond} \overline{x}$. By Remark 4.93 of Example 4.88 on page 158, there exists $\overline{\Delta_1}, \ldots, \overline{\Delta_m} \in \alpha[\![\overline{0}]\!]$ such that $\overline{x} \leq^\mathbf{F} \overline{y} \vee^\mathbf{F} \overline{\Delta_1} \vee^\mathbf{F} \overline{\Delta_m}$. The result follows by Lemma 1.457 on page 88. $\boxed{(2)\Rightarrow(1)}$ Let $\mathbf{P} \in \mathcal{K}$, $\mathsf{I} \in \mathsf{Id}_\Diamond(\mathbf{P})$, and $\alpha \in \mathsf{Con}^\mathcal{K}(\mathbf{P})$ with $\alpha[\![0^\mathbf{P}]\!] \subseteq \mathsf{I}$. Suppose that $a \in \mathsf{I}$ and $a\,\alpha\,b$. For each $1 \leq i \leq m$, $\Delta_i^\mathbf{P}(b,b)\,\alpha\,0^\mathbf{P}$, by (10.31), hence $\Delta_i^\mathbf{P}(a,b)\,\alpha\,0^\mathbf{P}$, since $a\,\alpha\,b$, and hence $\Delta_i^\mathbf{P}(a,b) \in \mathsf{I}$. So $a, \Delta_1^\mathbf{P}(a,b), \ldots, \Delta_m^\mathbf{P}(a,b) \in \mathsf{I}$, hence $a \vee^\mathbf{P} \Delta_1^\mathbf{P}(a,b) \ldots \vee^\mathbf{P} \Delta_m^\mathbf{P}(a,b) \in \mathsf{I}$. By (10.32), $b \leq^\mathbf{P} a \vee^\mathbf{P} \Delta_1^\mathbf{P}(a,b) \ldots \vee^\mathbf{P} \Delta_m^\mathbf{P}(a,b)$, hence $b \in \mathsf{I}$. $\diamond$

Corollary 10.38 If $\mathcal{K}$ is a quasivariety of 0-lattice expansions satisfying the equivalent conditions of Theorem 10.32, then the equivalent conditions of Theorem 10.37 are also satisfied.

Proof. Let $\Delta'_1, \ldots, \Delta'_m$ be ternary terms satisfying (10.27) and (10.28). Define $\Delta_i(x,y) = \Delta'_i(x,y,0)$, for $i = 1, \ldots, m$. Then the binary terms $\Delta_1, \ldots, \Delta_m$ satisfy (10.31) and (10.32). $\diamond$

The equivalent conditions of Theorem 10.37 are too strong to be satisfied by any non-trivial *quasivariety* of 0-lattices. In order to find an example satisfying this condition, we need to have a 'typed' notion of *complementation*.

Proposition 10.39 If $\mathcal{K}$ is a quasivariety of *distributive* 0-*complemented*-lattice expansions, then $\mathcal{K}$ satisfies (10.31) and (10.32) for the single binary term $\Delta(x,y) = x \wedge y'$.

Proof. By (1.109), $\models_{\mathcal{K}} x \wedge x' \approx 0$, and hence $\models_{\mathcal{K}} \Delta(x,x) \approx 0$. Let $\mathbf{P} \in \mathcal{K}$ and $a, b \in \mathsf{uni}(\mathbf{P})$. Then $b \vee \Delta^{\mathbf{P}}(a,b) = b \vee (a \wedge b') \stackrel{\text{dst}}{=} (b \vee a) \wedge (b \vee b') \geq (b \vee a) \wedge (b) \stackrel{\text{dst}}{=} (b \wedge b) \vee (a \wedge b) \stackrel{\text{idp}}{=} b \vee (a \wedge b) \stackrel{\text{abs}}{=} b$. Since $\mathbf{P}$, a and b arbitrary, $\models_{\mathcal{K}} y \leq y \vee \Delta(x,y)$. $\diamond$

Open Problem 10.40 Can equivalent condition (2) of Theorem 10.32 be 'sharpened' for quasivarieties of 0-lattice expansions?

Open Problem 10.41 Develop an analogous theory for convex sets. Is distributivity a requirement in such a theory? (See Open Problem 9.112 on page 337).

$\square$

Chapter 11

Regularity

It is well known that if any two congruences of a group (ring, or Boolean algebra) have a congruence class in common, then these two congruences coincide, a property known as *congruence regularity* (see Definition 1.359 on page 68). Congruence regularity was first defined by A. I. Mal'cev in [Mal54]. The first Mal'cev characterization of congruence regular varieties was given by Grätzer in [Gra70], based on an unpublished sufficient condition discovered by A. Tarski. Grätzer's characterization is complicated and largely impractical for the purpose of establishing the regularity of any particular variety, but is important historically because of the research it generated. More useful characterizations were given by Csákány in [Csá70], Wille in [Wil70] and Duda in [Dud83] (see Theorem 1.444 on page 86). J. Hagemann proved in [Hag73] that congruence regular varieties are *congruence n-permutable* (for some integer $n \geq 2$) and *congruence modular*, by generating the appropriate Mal'cev terms from Wille's characterization (see Corollary 1.445 on page 87).

Many of the varieties of algebras arising from logic, for example the subvarieties of BCK-algebras, are not congruence regular, but do satisfy a weakened form of regularity. These varieties have a constant term 0, and satisfy the condition that if any two congruences on an algebra of the variety have common 0-classes, then these two congruences coincide. Such algebras are called *congruence point regular* at 0 (see Definition 1.375 on page 71). Congruence point regularity at a single constant symbol 0 was first considered by Slomiński [Slo59]. The first Mal'cev condition for point regular varieties was given by Fichtner in [Fic68]. In [Fic70], another Mal'cev condition was given, as well as a characterization involving a quasi-identity. In a parallel attempt to characterize point regular varieties, Grätzer studied 'congruence weakly 1-regular' algebras [Gra70]; algebras with at least one (not necessarily typed) congruence 'distinguishing' element. In [Hag73], Hagemann characterized those varieties point regular at n constant symbols, by examining a (more general) variant of this condition, called *congruence local regularity.* Although sharper characterizations have now been found, from this characterization, Hagemann was able to show that congruence point regular varieties must be congruence modular and congruence m-permutable, for some integer $m \geq 2$.

Given that congruence regular and congruence point regular varieties must be congruence n-permutable (for some n) and congruence modular, two questions arose. Firstly, are there weaker forms of congruence regularity that still imply congruence n-permutability (for some n) and congruence modularity at a varietal level? This question gave rise to the conditions of congruence *subregularity* [Tim75], congruence *weak n-regularity* [Gra70],[DMS87], and congruence *n-subregularity* [DMS87]. The idea of studying 1-subregularity is attributed to Tarski in [Gra70]. Each of these conditions, of which congruence subregularity is the weakest at a varietal level, force a variety to be congruence n-permutable and congruence modular [DMS87],[Dud87]. Secondly, is there a link between regularity and congruence modularity for algebras in general? This question was answered in [SBFT74], where it was shown that if every subalgebra of $\mathbf{A}^2$ is congruence regular, then $\mathbf{A}$ must be congruence modular. The result was sharpened in [DMS87], where it was shown that $\mathbf{A}$ is modular under the weaker assumption that every subalgebra of $\mathbf{A}^2$ be congruence subregular.

In [BR97] and [Bar95], we introduced a notion of **congruence term regularity at** $u_0, \ldots, u_{n-1}$, where $u_0, \ldots, u_{n-1}$ are unary terms, as a tool to abstract all the above notions, and Mal'cev characterizations of such varieties were presented.

A weakened form of congruence regularity, known as congruence *quasiregularity*, was introduced in [Thu58], where Thurston proved that, at a varietal level, this condition coincides with congruence regularity. Analogous weak variants of the other forms of congruence regularity were introduced in [DMS87] and [Dud87].

A quasivarietal notion of *τ-regularity* (with respect to a *unary system* τ, called a *translation* in that text), investigated in [BR99] (see Definition 2.124 on page 115 of our text), turns out to be incomparable in its generality with most of the above conditions (in particular with relative 1-subregularity) but it encompasses relative point regularity and characterizes the equivalent semantics of logics that are algebraizable in the sense of [BP89a] (see Example 2.129 on page 116 of our text). In [BR03] we introduced a quasivarietal notion of *$\mathfrak{B}_*$-regularity*, where $\mathfrak{B}_*$ is a *binary system of equations* (called a *parametrized translation*), which abstracted term regularity at (a single) u, and lifted congruence term regularity at u to a quasivarietal level, thereby unifying the notion of '$\mathfrak{B}_*$-regularity' (with respect to a *translation* $\mathfrak{B}_*$) and relative 1-subregularity (thus encompassing full regularity).

We shall now consider generalizations of both τ-regularity and $\mathfrak{B}_*$-regularity.

In $\oint$11.2 we consider $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$*-regularity*, where $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ are each binary systems of equations. This notion encompasses $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$-regularity, and as such encompasses all the universal algebraic conditions of regularity described above, as well as $\langle \mathcal{K}, \tau \rangle$-regularity. It shall turn out that $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity is closely linked to our parametrized theory of algebraizable logics, developed in Part V, in a manner analogous to the relationship between $\langle \mathcal{K}, \tau \rangle$-regularity and the theory of algebraizable 1-deductive systems.

The notion of $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regularity is *not* wide enough to encompass the regularity conditions satisfied by those quasivarieties that are the equivalent algebraic semantics of n-deductive systems. To this end, in $\oint$11.1, we consider the condition of $\langle \mathcal{K}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n \rangle$-regularity, where $\mathfrak{N}_1, \ldots, \mathfrak{N}_n$ are each systems of n-ary equations. We shall see, in $\oint$9.1.2, that the quasivarieties that are the equivalent algebraic semantics of n-deductive systems are precisely those quasivarieties that are $\langle \mathcal{K}, \mathfrak{N} \rangle$-regular, where $\mathfrak{N}$ is a system of n-ary equations. Note that while $\langle \mathcal{K}, \mathfrak{N} \rangle$-regularity encompasses $\langle \mathcal{K}, \tau \rangle$-regularity, where τ is a unary system of equations, it does not encompass $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity, where $\mathfrak{B}$ is a binary system of equations. Note that $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity and $\langle \mathcal{K}, \mathfrak{B} \rangle$-regularity are different concepts, hence our rigorous use of the sub-scripted asterisk in these notations.

11.1 $\langle \mathcal{K}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n \rangle$-Regularity

In [BR99], a notion of regularity was introduced that, for the first time, related regularity type conditions to the simultaneous solutions of equations: an algebra $\mathbf{A}$ was called $\langle \mathcal{K}, \tau \rangle$-*regular*, where τ is a unary system of equations and $\mathcal{K}$ a quasivariety, if, for any two $\mathcal{K}$-congruences α and β on $\mathbf{A}$, if the solution sets $\tau^{\mathbf{A}}/\alpha$ and $\tau^{\mathbf{A}}/\beta$ coincide, then $\alpha = \beta$ (see Definition 2.124 on page 115 of our text). This notion of regularity arises naturally from the consideration of algebraizable sentential 1-calculi, where the unary system is just the set of *defining equations* from the sentential 1-calculus to the quasivariety (see Definition 2.105 on page 111). In [BR99], a sentential 1-calculus $S(\mathcal{K}, \tau)$ is defined (see Example 2.85 on page 106 of our text), determined by a quasivariety $\mathcal{K}$ and a unary system of equations τ, and it is shown that $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-regular iff $S(\mathcal{K}, \tau)$ is algebraizable, in which case $\mathcal{K}$ is its (unique) algebraic semantics and τ is a set of defining equations (see Theorem 2.125 on page 115 of our text). Of particular interest to the algebraist, is the fact that the notion of $\langle \mathcal{K}, \tau \rangle$-regularity encompasses the well-known (to algebraists) notion of (point) regularity at 0 (see Definition 1.375 on page 71). Of importance to us is the fact that (full) congruence regularity (see Definition 1.359 on page 68) is *not* encompassed by the notion of $\langle \mathcal{K}, \tau \rangle$-regularity, and the primary aim of this text is to bring (full) congruence regularity into the fold of algebraic logic.

In this section, we consider a generalization of $\langle \mathcal{K}, \tau \rangle$-regularity, generalized along two 'axes'. Along the first 'axis', we consider n-ary systems of equations rather than just unary. The importance of this generalization, is that we are able to develop a theory in the spirit of [BR99], but for sentential n-calculi instead of just sentential 1-calculi. In $\oint$9.1.2, we shall associate a logic $S^n(\mathcal{K}, \mathfrak{N})$ with each quasivariety and n-ary system $\mathfrak{N}$, and we shall show that $\mathcal{K}$ is $\langle \mathcal{K}, \tau \rangle$-regular precisely when $S^n(\mathcal{K}, \mathfrak{N})$ is algebraizable, in which case $\mathcal{K}$ is its (unique) algebraic semantics and $\mathfrak{N}$ is a set of defining equations. We shall further show that, up to equivalence, all algebraizable sentential n-calculi arise in this manner. The generalization along the second 'axis' is to consider regularity with respect to *multiple* n-ary systems. This second generalization is more of interest to algebraists, in that it encompasses congruences point regularity at *multiple* points. We note that neither of these generalizations encompass (full) congruence regularity. In the next section we shall consider a notion of regularity that indeed encompasses (full) congruence regularity.

11.1.1 Definitions and Local Results

We begin by defining $\langle \mathcal{K}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n \rangle$-regularity, and considering a few *local* results, by which we mean results that pertain to a single particular algebra rather than the quasivariety as a whole. In $\oint$11.1.2 we shall consider such global results, and in particular, we shall provide *quasi-Mal'cev* characterizations of $\langle \mathcal{K}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n \rangle$-regularity.

Definition 11.1 ($\langle \mathcal{K}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n \rangle$-Regularity) Let $\mathcal{K}$ be a quasivariety of algebras, let $\mathfrak{N}_1, \ldots, \mathfrak{N}_n$ be systems of equations of the same dimension. and let $\mathbf{A}$ be an algebra, not necessarily in $\mathcal{K}$. We say that $\mathbf{A}$ is **$\mathcal{K}$-regular at $\mathfrak{N}_1, \ldots, \mathfrak{N}_n$** (or **$\langle \mathcal{K}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n \rangle$-regular**), if, for all $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $(\mathfrak{N}_i)^{\mathbf{A}}/\alpha = (\mathfrak{N}_i)^{\mathbf{A}}/\beta$, for each $1 \leq i \leq n$, then $\alpha = \beta$. We say that $\mathcal{K}$ is **$\langle \mathfrak{N}_1, \ldots, \mathfrak{N}_n \rangle$-regular**, if every algebra in $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n \rangle$-regular. □

The condition of regularity may be rephrased in terms of subset inclusion.

Proposition 11.2 **A** is $\langle\mathcal{K}, \mathfrak{N}_1, \dots, \mathfrak{N}_n\rangle$-regular iff, for all $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, if $(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \subseteq (\mathfrak{N}_i)^{\mathbf{A}}/\beta$, for each $1 \le i \le n$, then $\alpha \subseteq \beta$.

Proof. $\boxed{\Rightarrow}$ Assume that **A** is $\langle\mathcal{K}, \mathfrak{N}_1, \dots, \mathfrak{N}_n\rangle$-regular. Let $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, such that $(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \subseteq (\mathfrak{N}_i)^{\mathbf{A}}/\beta$, for each $1 \le i \le n$. Then, for each $1 \le i \le n$, $(\mathfrak{N}_i)^{\mathbf{A}}/\alpha = (\mathfrak{N}_i)^{\mathbf{A}}/\alpha \cap \beta$. Since $\alpha \cap \beta$ is a $\mathcal{K}$-congruence, by assumption, $\alpha = \alpha \cap \beta$, and hence $\alpha \subseteq \beta$. $\boxed{\Leftarrow}$ Trivial. $\diamond$

In the following result we show that an algebra **A** is $\langle\mathcal{K}, \mathfrak{N}_1, \dots, \mathfrak{N}_n\rangle$-regular iff, for each $\mathcal{K}$-congruence α, the $\mathcal{K}$-congruence generated by the instantiation of the solutions modulo α is precisely α.

Proposition 11.3 **A** is $\langle\mathcal{K}, \mathfrak{N}_1, \dots, \mathfrak{N}_n\rangle$-regular iff, for all $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$,

$$\left\| \bigcup_{1 \le i \le n} (\mathfrak{N}_i)^{\mathbf{A}} \left[(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}} = \alpha. \tag{11.1}$$

Proof.
$\boxed{\Rightarrow}$ By (1) of Remark 9.3, $\bigcup_{1 \le i \le n} (\mathfrak{N}_i)^{\mathbf{A}} \left[(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \right] \subseteq \alpha$; hence $\left\| \bigcup_{1 \le i \le n} (\mathfrak{N}_i)^{\mathbf{A}} \left[(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}} \subseteq \alpha$. Conversely, since $(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \subseteq (\mathfrak{N}_i)^{\mathbf{A}}/ \left\| \bigcup_{1 \le i \le n} (\mathfrak{N}_i)^{\mathbf{A}} \left[(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$, for each $1 \le i \le n$, $\alpha \subseteq \left\| \bigcup_{1 \le i \le n} (\mathfrak{N}_i)^{\mathbf{A}} \left[(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$ by Proposition 11.2. $\boxed{\Leftarrow}$ Let $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ with $(\mathfrak{N}_i)^{\mathbf{A}}/\alpha = (\mathfrak{N}_i)^{\mathbf{A}}/\beta$, for each $1 \le i \le n$. So $\bigcup_{1 \le i \le n} (\mathfrak{N}_i)^{\mathbf{A}} \left[(\mathfrak{N}_i)^{\mathbf{A}}/\alpha \right] = \bigcup_{1 \le i \le n} (\mathfrak{N}_i)^{\mathbf{A}} \left[(\mathfrak{N}_i)^{\mathbf{A}}/\beta \right]$. So by assumption, $\alpha = \beta$. $\diamond$

11.1.2 Global Characterizations

We now consider *global* characterizations, and in particular, obtain a *quasi-Mal'cev* characterization of the property that a quasivariety $\mathcal{K}$ be $\langle\mathcal{K}, \mathfrak{N}_1, \dots, \mathfrak{N}_m\rangle$-regular. The reader is urged to recall the definition of a *formal* $\langle n, m\rangle$*-translation* given in Definition 2.95 on page 108.

Theorem 11.4 Let $\mathfrak{N}_1, \dots, \mathfrak{N}_m$ be n-ary systems of equations. The following conditions on $\mathcal{K}$ and $\mathfrak{N}_1, \dots, \mathfrak{N}_m$ are equivalent (where k is any integer with $k \ge 2$).

1. Every algebra **A** is $\langle\mathcal{K}, \mathfrak{N}_1, \dots, \mathfrak{N}_m\rangle$-regular.
2. $\mathcal{K}$ is $\langle\mathfrak{N}_1, \dots, \mathfrak{N}_m\rangle$-regular.
3. The $\mathcal{K}$-free algebra on 2-free generators is $\langle\mathfrak{N}_1, \dots, \mathfrak{N}_m\rangle$-regular.
4. The $\mathcal{K}$-free algebra on k-free generators is $\langle\mathfrak{N}_1, \dots, \mathfrak{N}_m\rangle$-regular.
5. $\mathbf{F}_{\mathcal{K}}$ is $\langle\mathfrak{N}_1, \dots, \mathfrak{N}_m\rangle$-regular.
6. **Tm** is $\langle\mathfrak{N}_1, \dots, \mathfrak{N}_m\rangle$-regular.
7. There exist $\langle 2, n\rangle$-translations $\mathbf{\Delta}_1, \dots, \mathbf{\Delta}_k$ such that

$$\bigcup_{1 \le i \le k} (\mathfrak{N}_i)^{\approx} \left[\mathbf{\Delta}_i(\langle x, y\rangle) \right] \mathrel{=\!\mid\models_{\mathcal{K}}} x \approx y. \tag{11.2}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ Trivial. $\boxed{(2)\Rightarrow(7)}$ Consider the 2-generated $\mathcal{K}$-free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\overline{x}, \overline{y}) \in \mathcal{K}$. Let $\mathfrak{X} = \{\mathfrak{N}_1, \ldots, \mathfrak{N}_m\}$. Let $\alpha = \|\langle\overline{x}, \overline{y}\rangle\|_{\Theta_{\mathbf{F}_{\mathcal{K}}}}$ and $\beta = \left\|\bigcup_{\mathfrak{N}\in\mathfrak{X}} \mathfrak{N}^{\mathbf{F}}\left[\mathfrak{N}^{\mathbf{F}}/\alpha\right]\right\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$. Since $\bigcup_{\mathfrak{N}\in\mathfrak{X}} \mathfrak{N}^{\mathbf{F}}\left[\mathfrak{N}^{\mathbf{F}}/\alpha\right] \subseteq \alpha$, by Remark 9.3, $\beta \subseteq \alpha$, and hence $\mathfrak{N}^{\mathbf{F}}/\beta \subseteq \mathfrak{N}^{\mathbf{F}}/\alpha$, for each $\mathfrak{N} \in \mathfrak{X}$. But, by construction and Remark 9.3, for each $\mathfrak{N} \in \mathfrak{X}$, $\mathfrak{N}^{\mathbf{F}}/\alpha = \mathfrak{N}^{\mathbf{F}}/(\mathfrak{N}^{\mathbf{F}}\left[\mathfrak{N}^{\mathbf{F}}/\alpha\right]) \subseteq \mathfrak{N}^{\mathbf{F}}/\beta$. Hence $\mathfrak{N}^{\mathbf{F}}/\alpha = \mathfrak{N}^{\mathbf{F}}/\beta$, for each $\mathfrak{N} \in \mathfrak{X}$. So by assumption, $\alpha = \beta$. So $\langle\overline{x}, \overline{y}\rangle \in \left\|\bigcup_{\mathfrak{N}\in\mathfrak{X}} \mathfrak{N}^{\mathbf{F}}\left[\mathfrak{N}^{\mathbf{F}}/\alpha\right]\right\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$. Since relative congruences form an *algebraic* closed system, there exists $\beta' \subseteq_{\mathsf{f}} \bigcup_{\mathfrak{N}\in\mathfrak{X}} \mathfrak{N}^{\mathbf{F}}\left[\mathfrak{N}^{\mathbf{F}}/\alpha\right]$ with $\langle\overline{x}, \overline{y}\rangle \in \|\beta'\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$ by equivalent condition (2) of Proposition 4.77 on page 157. So, for each $\mathfrak{N} \in \mathfrak{X}$, there exist $\overline{\Delta^{\mathfrak{N}}_1}, \ldots, \overline{\Delta^{\mathfrak{N}}_{j_{\mathfrak{N}}}} \in \mathfrak{N}^{\mathbf{F}}/\alpha$, with $\langle\overline{x}, \overline{y}\rangle \in \left\|\bigcup_{\mathfrak{N}\in\mathfrak{X}} \mathfrak{N}^{\mathbf{F}}\left[\overline{\Delta^{\mathfrak{N}}_1}, \ldots, \overline{\Delta^{\mathfrak{N}}_{j_{\mathfrak{N}}}}\right]\right\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$. The result follows from Lemma 1.457 on page 88. $\boxed{(7)\Rightarrow(1)}$ Let $\mathbf{A}$ be any algebra and $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ with $(\mathfrak{N}_i)^{\mathbf{A}}/\alpha = (\mathfrak{N}_i)^{\mathbf{A}}/\beta$, for each $1 \le i \le n$. Let $\langle b, c\rangle \in \alpha$. For each $1 \le i \le n$, by Lemma 1.457 and (11.2), $(\mathfrak{N}_i)^{\mathbf{A}}\left[\mathbf{\Delta}^{\mathbf{A}}_i(\langle b, c\rangle)\right] \subseteq \alpha$, hence $\mathbf{\Delta}^{\mathbf{A}}_i(\langle b, c\rangle) \subseteq (\mathfrak{N}_i)^{\mathbf{A}}/\alpha = (\mathfrak{N}_i)^{\mathbf{A}}/\beta$, i.e., $(\mathfrak{N}_i)^{\mathbf{A}}\left[\mathbf{\Delta}^{\mathbf{A}}_i(\langle b, c\rangle)\right] \subseteq \beta$. By Lemma 1.457 and (11.2) again, $\langle b, c\rangle \in \beta$. By symmetry, $\alpha = \beta$. The proof of the remaining implications follow as in the proof of Theorem 11.18. ◇

11.1.3 $\langle\mathcal{K}, \mathfrak{N}\rangle$-Regularity

We now consider the case of a single n-ary system $\mathfrak{N}$. In this case we can characterize the property that a quasivariety $\mathcal{K}$ be $\langle\mathcal{K}, \mathfrak{N}\rangle$-regular in terms of a lattice isomorphism between the $\mathcal{K}$-congruence lattice on an algebra and the lattice of solutions to $\mathfrak{N}$.

Remark 11.5 $\mathbf{A}$ is $\langle\mathcal{K}, \mathfrak{N}\rangle$-regular iff, for each $b \in \mathsf{uni}(\mathbf{A})$, $\mathfrak{N}^{\mathbf{A}}/\cdot : \mathsf{Con}^{\mathcal{K}}(\mathbf{A}) \rightarrowtail \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$.

Corollary 11.6 The following conditions on $\mathcal{K}$ and $\mathfrak{N}$ are equivalent (where k is any integer with $k \ge 2$).

1. $\mathcal{K}$ is $\langle\mathfrak{N}\rangle$-regular.
2. For each algebra $\mathbf{A}$, $\mathfrak{N}^{\mathbf{A}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$.
3. For each $\mathbf{A} \in \mathcal{K}$, $\mathfrak{N}^{\mathbf{A}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$.
4. $\mathfrak{N}^{\mathbf{F}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{F}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{F})$, where $\mathbf{F}$ is the $\mathcal{K}$-free algebra on 2-free generators.
5. $\mathfrak{N}^{\mathbf{F}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{F}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{F})$, where $\mathbf{F}$ is the $\mathcal{K}$-free algebra on k-free generators.
6. $\mathfrak{N}^{\mathbf{F}_{\mathcal{K}}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{F}_{\mathcal{K}})$.
7. $\mathfrak{N}^{\mathbf{Tm}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{Tm})$.
8. There exists a finite set of binary terms $\mathbf{\Delta}(x, y)$, such that
$$\mathfrak{N}^{\approx}\left[\mathbf{\Delta}_i(x, y)\right] \dashv\models_{\mathcal{K}} x \approx y. \tag{11.3}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ By definition this map is surjective and by assumption injective; hence $\mathfrak{N}^{\mathbf{A}}/\cdot$ is a bijection from $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ onto $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}_b}(\mathbf{A})$. Trivially, this map is inclusion preserving. It suffices to prove that this map is order-reflecting. Suppose that $\mathfrak{N}^{\mathbf{A}}/\alpha \subseteq \mathfrak{N}^{\mathbf{A}}/\beta$. Then by assumption and Proposition 11.2, $\alpha \subseteq \beta$. $\boxed{(2)\Rightarrow(3),\ (4),\ (5),\ (6),\ (7)}$ Trivial. $\boxed{(3)\Rightarrow(1)}$ By Remark 11.5.
The proofs of (4)⇒(1), (5)⇒(1), (6)⇒(1) and (7)⇒(1), follow by Theorem 11.4 and Remark 11.5. ◇

Open Problem 11.7 Find a characterization of $\langle \mathcal{K}, \mathfrak{N}_1, \ldots, \mathfrak{N}_m\rangle$-regularity in the spirit of the previous theorem.

By Corollary 9.9, $\mathcal{K}$ is *always* an algebraic semantics for $S^n(\mathcal{K}, \mathfrak{N})$. We shall now show that $\mathcal{K}$ is an *equivalent* algebraic semantics for $S^n(\mathcal{K}, \mathfrak{N})$, precisely when $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-regular. Notice that the latter condition is purely universal algebraic.

Theorem 11.8 Let $\mathfrak{N}$ be a system of n-ary equations and $\mathcal{K}$ a quasivariety of algebras. The following conditions are equivalent.

1. $\mathcal{K}$ is the equivalent algebraic semantics for *some* sentential n-calculus with defining equations $\mathfrak{N}$.

2. $\mathcal{K}$ is an equivalent algebraic semantics for $S^n(\mathcal{K}, \mathfrak{N})$.

3. $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-regular.

Proof. $\boxed{(1)\Rightarrow(2)}$ Suppose that $\mathcal{K}$ is the equivalent algebraic semantics for some sentential calculus $\mathcal{S}$ with defining equations $\mathfrak{N}$ and some equivalence formulae $\mathbf{\Delta}$. By Corollary 9.9, $\mathcal{K}$ is an algebraic semantics for $S^n(\mathcal{K}, \mathfrak{N})$ with defining equations $\mathfrak{N}$. So $S^n(\mathcal{K}, \mathfrak{N})$ and $\mathcal{K}$ satisfy (2.28) and (2.32), and so the result follows by Corollary 2.113. $\boxed{(2)\Rightarrow(3)}$ Follows from (2.32) and (7) of Theorem 11.4 on page 360. $\boxed{(3)\Rightarrow(1)}$ By (7) of Theorem 11.4 on page 360, there exists a formal $\langle 2, n\rangle$-translation $\mathbf{\Delta}$ such that $\mathcal{K}$ satisfies (2.32). By Corollary 9.9, $\mathcal{K}$ is an algebraic semantics for $S^n(\mathcal{K}, \mathfrak{N})$ with defining equations $\mathfrak{N}$. So $S^n(\mathcal{K}, \mathfrak{N})$ and $\mathcal{K}$ satisfy (2.28) and (2.32), and so $\mathcal{K}$ is an equivalent algebraic semantics for $S^n(\mathcal{K}, \mathfrak{N})$ with defining equations $\mathfrak{N}$. ◇

Recall the definition of $\langle \mathcal{K}, \tau\rangle$*-filter determination* [BR99] (see Definition 2.126 on page 115 of our text). We generalize this notion to our context. Note that in [BR99] the term 'ideal' is used instead of 'filter'.

Definition 11.9 ($\langle \mathcal{K}, \mathfrak{N}\rangle$-Filter Determination) Let $\mathbf{A}$ be an algebra not necessarily in $\mathcal{K}$. We say that $\mathbf{A}$ is **$\langle \mathcal{K}, \mathfrak{N}\rangle$-filter determined** if, $\mathfrak{N}^{\mathbf{A}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A})$. We say that $\mathcal{K}$ is **$\mathfrak{N}$-filter determined** if every algebra in $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-filter determined. □

Remark 11.10 If $\mathbf{A}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-filter determined then $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A}) = \mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A})$. □

The following result is a generalization of [BR99, T 5.2] from unary systems to n-ary systems (see Theorem 2.128 on page 115 of our text).

Theorem 11.11 The following conditions are equivalent.

1. $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-regular.

2. Every algebra is $\langle \mathcal{K}, \mathfrak{N}\rangle$-filter determined.

3. $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-filter determined.

4. $\mathbf{Tm}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-filter determined.

5. $\mathbf{F}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-filter determined, where $\mathbf{F}$ is the $\mathcal{K}$-free algebra on 2 free generators.

6. $\mathbf{F}_{\mathcal{K}}$ is $\langle \mathcal{K}, \mathfrak{N}\rangle$-filter determined.

Proof. $\boxed{(1)\Rightarrow(2)}$ Let $\mathbf{A}$ be any algebra. By equivalent condition (2) of Corollary 11.6 on page 361 and Proposition 9.10, it suffices to prove that $\mathbf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A}) \subseteq \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{N}}(\mathbf{A})$. By equivalent condition (8) of Corollary 11.6, there exists a finite set of binary terms $\mathbf{\Delta}(x,y)$, such that

$$\mathfrak{N}^{\approx}\left[\mathbf{\Delta}(x,y)\right] \mathrel{=\!\!|\!\!\models}_{\mathcal{K}} x \approx y. \tag{i}$$

Suppose that $F \in \mathsf{Fi}_{S^n(\mathcal{K},\mathfrak{N})}(\mathbf{A})$. Let $\alpha = \left\| \mathfrak{N}^{\mathbf{A}}\left[F\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$. (*It suffices to show that* $F = \mathfrak{N}^{\mathbf{A}}/\left\| \mathfrak{N}^{\mathbf{A}}\left[F\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$.) By Remark 9.3 on page 312, $F \subseteq \mathfrak{N}^{\mathbf{A}}/\mathfrak{N}^{\mathbf{A}}\left[F\right] \subseteq \mathfrak{N}^{\mathbf{A}}/\left\| \mathfrak{N}^{\mathbf{A}}\left[F\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$. (*We prove the converse inclusion.*) Let $\mathsf{a} \in \mathfrak{N}^{\mathbf{A}}/\left\| \mathfrak{N}^{\mathbf{A}}\left[F\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$. Consider any vector $\vec{x} = \langle x_1, \ldots, x_n\rangle$ of n-distinct variables. By (i),

$$\mathfrak{N}^{\approx}\left[\mathbf{\Delta}\left[\mathfrak{N}[\![\vec{x}]\!]\right]\right] \mathrel{=\!\!|\!\!\models}_{\mathcal{K}} \mathfrak{N}^{\approx}[\![\vec{x}]\!] \tag{ii}$$

i.e.,

$$\mathbf{\Delta}\left[\mathfrak{N}[\![\vec{x}]\!]\right] \dashv\vdash_{S^n(\mathcal{K},\mathfrak{N})} \vec{x} \tag{iii}$$

(*It suffices to show that* $\mathbf{\Delta}\left[\mathfrak{N}[\![\mathsf{a}]\!]\right] \subseteq F$, *since then by (iii) and the fact that* F *is a* $S^n(\mathcal{K},\mathfrak{N})$*-filter, we have* $\mathsf{a} \in F$.)
Since $\mathsf{a} \in \mathfrak{N}^{\mathbf{A}}/\left\| \mathfrak{N}^{\mathbf{A}}\left[F\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$, $\mathfrak{N}^{\mathbf{A}}[\![\mathsf{a}]\!] \subseteq \mathfrak{N}^{\mathbf{A}}\left[\mathfrak{N}^{\mathbf{A}}/\left\| \mathfrak{N}^{\mathbf{A}}\left[F\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}\right] \subseteq \left\| \mathfrak{N}^{\mathbf{A}}\left[F\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$, by Remark 9.3 on page 312. Let $\langle \delta, \epsilon\rangle \in \mathfrak{N}$.
(*It suffices to show that* $\mathbf{\Delta}^{\mathbf{A}}[\![\langle \delta^{\mathbf{A}}(\mathsf{a}), \epsilon^{\mathbf{A}}(\mathsf{a})\rangle]\!] \subseteq F$.)
Since $\langle \delta^{\mathbf{A}}(\mathsf{a}), \epsilon^{\mathbf{A}}(\mathsf{a})\rangle \in \left\| \mathfrak{N}^{\mathbf{A}}\left[F\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$, by Lemma 1.452 on page 87, there exists a quasi-identity

$$[\textstyle\bigwedge_{i<l} p_i(\vec{x}) \approx p'_i(\vec{x})] \text{ and } [\textstyle\bigwedge_{j<k} q_j(\vec{x}) \approx q'_j(\vec{x})] \to r(\vec{x}) \approx s(\vec{x}) \tag{iv}$$

satisfied by $\mathcal{K}$ and elements $\vec{c} \in A$, such that for $i < l$ and $j < k$, we have

$$\langle p_i^{\mathbf{A}}(\vec{c}), {p'_i}^{\mathbf{A}}(\vec{c})\rangle \in \mathfrak{N}^{\mathbf{A}}\left[F\right], \tag{v}$$

$$q_j^{\mathbf{A}}(\vec{c}) = {q'_j}^{\mathbf{A}}(\vec{c}), \tag{vi}$$

$$r^{\mathbf{A}}(\vec{c}) = \delta^{\mathbf{A}}(\mathsf{a}) \quad \text{and} \quad s^{\mathbf{A}}(\vec{c}) = \epsilon^{\mathbf{A}}(\mathsf{a}). \tag{vii}$$

By (i) and (iv),

$$\begin{aligned}
&\mathfrak{N}^{\approx}\left[\mathbf{\Delta}\left[\{\langle p_i(\vec{x}), p'_i(\vec{x})\rangle \;:\; i < l\} \cup \{\langle q_i(\vec{x}), q'_i(\vec{x})\rangle \;:\; j < k\}\right]\right] \\
&\quad \models_{\mathcal{K}} [\textstyle\bigwedge_{i<l} p_i(\vec{x}) \approx p'_i(\vec{x})] \text{ and } [\textstyle\bigwedge_{j<k} q_j(\vec{x}) \approx q'_j(\vec{x})] \\
&\quad \models_{\mathcal{K}} r(\vec{x}) \approx s(\vec{x}) \\
&\quad \models_{\mathcal{K}} \mathfrak{N}^{\approx}\left[\mathbf{\Delta}[\![\langle r(\vec{x}), s(\vec{x})\rangle]\!]\right],
\end{aligned}$$

and so

$$\mathbf{\Delta}\left[\{\langle p_i(\vec{x}), p'_i(\vec{x})\rangle \;:\; i < l\} \cup \{\langle q_i(\vec{x}), q'_i(\vec{x})\rangle \;:\; j < k\}\right] \vdash_{S^n(\mathcal{K},\mathfrak{N})} \mathbf{\Delta}[\![\langle r(\vec{x}), s(\vec{x})\rangle]\!].$$

(*So showing that* $\mathbf{\Delta}^{\mathbf{A}}\left[\{\langle p_i(\vec{c}), p'_i(\vec{c})\rangle \;:\; i < l\} \cup \{\langle q_i(\vec{c}), q'_i(\vec{c})\rangle \;:\; j < k\}\right] \subseteq F$, *suffices, by (vii).*)
By (v), for each $i < l$, $\langle p_i^{\mathbf{A}}(\vec{c}), {p'_i}^{\mathbf{A}}(\vec{c})\rangle = \langle \delta'^{\mathbf{A}}(\mathsf{b}), \epsilon'^{\mathbf{A}}(\mathsf{b})\rangle$, for some $\langle \delta', \epsilon'\rangle \in \mathfrak{N}$ and $\mathsf{b} \in F$; so by (iii), $\mathbf{\Delta}^{\mathbf{A}}[\![\langle p_i^{\mathbf{A}}(\vec{c}), {p'_i}^{\mathbf{A}}(\vec{c})\rangle]\!] = \mathbf{\Delta}^{\mathbf{A}}[\![\langle \delta'^{\mathbf{A}}(\mathsf{b}), \epsilon'^{\mathbf{A}}(\mathsf{b})\rangle]\!] \subseteq \mathbf{\Delta}^{\mathbf{A}}\left[\mathfrak{N}^{\mathbf{A}}[\![\mathsf{b}]\!]\right] \subseteq F$. By (i), we have $\vdash_{S^n(\mathcal{K},\mathfrak{N})} \mathbf{\Delta}(x,x)$, which together with (vi), yields $\mathbf{\Delta}^{\mathbf{A}}[\![\langle q_j^{\mathbf{A}}(\vec{c}), {q'_j}^{\mathbf{A}}(\vec{c})\rangle]\!] \in F$, for each $j < k$. $\boxed{(2)\Rightarrow(3),\ (4),\ (5)\text{ and }(6)}$ Trivial. $\boxed{(3)\text{ or }(4)\text{ or }(5)\text{ or }(6)\Rightarrow(1)}$ By Corollary 11.6 and Proposition 9.10. $\diamond$

While not all protoalgebraic logics need be equivalent to such a logic $S^n(\mathcal{K},\mathfrak{N})$, since protoalgebraicity is a necessary condition for algebraization, it is important to characterize the protoalgebraicity of the logics $S^n(\mathcal{K},\mathfrak{N})$. In [BR99], the case for the sentential 1-calculus $S(\mathcal{K},\tau)$ is considered (where τ is a unary systems of equations), and it is shown that $S(\mathcal{K},\tau)$ is protoalgebraic precisely when $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\tau\rangle$-classes (see Example 2.140 on page 118 of our text).

In Example 16.40 on page 454 we characterize the protoalgebraicity of $S^n(\mathcal{K},\mathfrak{N})$ in terms of $\mathcal{K}$ having $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes, where the result follows more directly from a new characterization of protoalgebraic n-deductive systems obtained from the machinery of Chapter 16.

11.1.4 Examples

We explicate the quasi-Mal'cev characterization of $\langle\tau_1,\ldots,\tau_m\rangle$-regularity, where each τ_i is a *unary* system, for purposes of a later comparison with the quasi-Mal'cev characterization of $\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{m*}\rangle$-regularity (defined in the next section), where each $\mathfrak{B}_i$ is a *binary* system.

Example 11.12 ($\langle\tau_1,\ldots,\tau_m\rangle$-regular)

Let $\tau_1,\ldots,\tau_m$ be unary systems of equations and $\mathcal{K}$ a quasivariety.

Corollary 11.13 $\mathcal{K}$ is $\langle\tau_1,\ldots,\tau_m\rangle$-regular iff there exist finite sets $\boldsymbol{\Delta}_1,\ldots,\boldsymbol{\Delta}_k$ of *binary* terms such that

$$\bigcup_{1\le i\le k}(\tau_i)^{\approx}\left[\boldsymbol{\Delta}_i(\langle x,y\rangle)\right] \mathrel{=\!\mid\!\models}_{\mathcal{K}} x\approx y. \tag{11.4}$$

□

Example 11.14 ($\langle\mathcal{K},\tau\rangle$-Regularity) [BR99]

The characterizations of $\langle\mathcal{K},\tau\rangle$-regularity, where τ is a unary system, given in [BR99] follow immediately from Corollary 11.13. In particular, we obtain the equivalence of conditions (1) and (3) of Theorem 2.125 on page 115 of our text.

□

We end this section by showing that Polrims (see Counter Example 9.113 on page 338) are relatively point regular. This result is well-known.

Example 11.15 (Polrims are Relatively Point Regular)

Polrims are $\{\langle x,0\rangle\}$-regular, as witnessed by the terms $\boldsymbol{\Delta}=\{\Delta_0(x,y,z),\Delta_1(x,y,z)\}=\{x\dot{-}y,y\dot{-}x\}$ (or $\boldsymbol{\Delta}=\{\Delta(x,y,z)\}=\{(x\dot{-}y)\oplus(y\dot{-}x)\}$).

□

11.2 $\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-Regularity

While $\langle\mathcal{K},\mathfrak{N}_1,\ldots,\mathfrak{N}_n\rangle$-regularity encompasses both point regularity at a single 0 and point regularity at multiple $0_1,\ldots,0_n$, it does not encompass (full) regularity. Full regularity is of particular importance in quasivarieties without definable constant terms. One such variety is the variety of quasigroups, which is fully congruence regular and has no equationally definable constant terms (see Example 3.1). We now develop a theory of regularity that encompasses full regularity, n-subregularity, point regularity at multiple $0_1,\ldots,0_n$ and $\langle\mathcal{K},\tau\rangle$-regularity for unary system τ. It does *not*, however, encompass $\langle\mathcal{K},\mathfrak{N}\rangle$-regularity, where $\mathfrak{N}$ is an n-ary system with $n\ge 2$.

11.2.1 Definitions and Local Results

We begin by defining the notion of $\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regularity, where $\mathfrak{B}_i$ are *binary* systems of equations, and establishing some local results. In $\oint$11.2.2 we consider global characterizations and obtain a *quasi-Mal'cev* characterization of $\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regularity.

Definition 11.16 ($\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-Regularity) Let $\mathcal{K}$ be a quasivariety of algebras, let $\mathfrak{B}_1,\ldots,\mathfrak{B}_n$ be binary systems of equations and let $\mathbf{A}$ be an algebra, not necessarily in $\mathcal{K}$. We say that $\mathbf{A}$ is **$\mathcal{K}$-regular at $\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}$** or **$\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regular** (resp. **regular at $\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}$** or **$\langle\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regular**), if, for all $\alpha,\beta\in\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ (resp. for all $\alpha,\beta\in\mathsf{Con}(\mathbf{A})$) and $b\in\mathsf{uni}(\mathbf{A})$, if $(\mathfrak{B}_i)_b^{\mathbf{A}}/\alpha=(\mathfrak{B}_i)_b^{\mathbf{A}}/\beta$, for each $1\le i\le n$, then $\alpha=\beta$. We say that $\mathcal{K}$ is **$\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regular** (resp. **$\langle\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regular**), if every algebra in $\mathcal{K}$ is $\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n}\rangle$-regular (resp. $\langle\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regular). □

As with $\langle\mathcal{K},\mathfrak{N}_1,\ldots,\mathfrak{N}_n\rangle$-regularity, $\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regularity may be characterized in terms of subset inclusion as well as in terms of the $\mathcal{K}$-congruence generated by the instantiation of solutions.

Proposition 11.17 The following conditions are equivalent.

1. $\mathbf{A}$ is $\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regular.
2. For all $\alpha,\beta\in\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $b\in\mathsf{uni}(\mathbf{A})$, if $(\mathfrak{B}_i)_b^{\mathbf{A}}/\alpha\subseteq(\mathfrak{B}_i)_b^{\mathbf{A}}/\beta$, for each $1\le i\le n$, then $\alpha\subseteq\beta$.
3. For all $a\in\mathsf{uni}(\mathbf{A})$ and $\alpha\in\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$,

$$\left\|\bigcup_{1\le i\le n}(\mathfrak{B}_i)_a^{\mathbf{A}}\left[(\mathfrak{B}_i)_a^{\mathbf{A}}/\alpha\right]\right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}=\alpha. \tag{11.5}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ Assume that $\mathbf{A}$ is $\langle\mathcal{K},\mathfrak{B}_1,\ldots,\mathfrak{B}_n\rangle$-regular. Let $\alpha,\beta\in\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $b\in\mathsf{uni}(\mathbf{A})$, such that $(\mathfrak{B}_i)_b^{\mathbf{A}}/\alpha\subseteq(\mathfrak{B}_i)_b^{\mathbf{A}}/\beta$, for each $1\le i\le n$. Then, for each $1\le i\le n$, $(\mathfrak{B}_i)_b^{\mathbf{A}}/\alpha=(\mathfrak{B}_i)_b^{\mathbf{A}}/\alpha\cap\beta$. Since $\alpha\cap\beta$ is a $\mathcal{K}$-congruence, by assumption, $\alpha=\alpha\cap\beta$, and hence $\alpha\subseteq\beta$. $\boxed{(2)\Rightarrow(3)}$ By Remark 9.25, $\bigcup_{1\le i\le n}(\mathfrak{B}_i)_a^{\mathbf{A}}\left[(\mathfrak{B}_i)_a^{\mathbf{A}}/\alpha\right]\subseteq\alpha$, and so $\left\|\bigcup_{1\le i\le n}(\mathfrak{B}_i)_a^{\mathbf{A}}\left[(\mathfrak{B}_i)_a^{\mathbf{A}}/\alpha\right]\right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}\subseteq\alpha$. Conversely, since $(\mathfrak{B}_i)_a^{\mathbf{A}}/\alpha\subseteq(\mathfrak{B}_i)_a^{\mathbf{A}}/\left\|\bigcup_{1\le i\le n}(\mathfrak{B}_i)_a^{\mathbf{A}}\left[(\mathfrak{B}_i)_a^{\mathbf{A}}/\alpha\right]\right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}$, for each $1\le i\le n$, $\alpha\subseteq\left\|\bigcup_{1\le i\le n}(\mathfrak{B}_i)_a^{\mathbf{A}}\left[(\mathfrak{B}_i)_a^{\mathbf{A}}/\alpha\right]\right\|_{\Theta_{\mathbf{A}}^{\mathcal{K}}}$, by assumption (2) $\boxed{(3)\Leftarrow(1)}$ Let $\alpha,\beta\in\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a\in\mathsf{uni}(\mathbf{A})$, with $(\mathfrak{B}_i)_a^{\mathbf{A}}/\alpha=(\mathfrak{B}_i)_a^{\mathbf{A}}/\beta$, for each $1\le i\le n$. So $\bigcup_{1\le i\le n}(\mathfrak{B}_i)_a^{\mathbf{A}}\left[(\mathfrak{B}_i)_a^{\mathbf{A}}/\alpha\right]=\bigcup_{1\le i\le n}(\mathfrak{B}_i)_a^{\mathbf{A}}\left[(\mathfrak{B}_i)_a^{\mathbf{A}}/\beta\right]$. So by assumption, $\alpha=\beta$. ◇

11.2.2 Global Characterizations

We now establish a quasi-Mal'cev characterization of the property that a quasivariety be $\langle\mathcal{K},\mathfrak{B}_{1*},\ldots,\mathfrak{B}_{n*}\rangle$-regular. The reader is urged to compare this quasi-Mal'cev condition (7) to the analogous quasi-Mal'cev condition characterizing $\langle\tau_1,\ldots,\tau_m\rangle$-regularity, where $\tau_1,\ldots,\tau_m$ are unary systems of equations, given in Corollary 11.13 of Example 11.12. In particular, note how the binary terms of the later quasi-Mal'cev condition 'have become' ternary terms in the former. This increase (by one) in arity is typical of the relationship between the quasi-Mal'cev condition for a notion at a point and the quasi-Mal'cev condition for the analogous full notion; this relationship is reflected in our theory of parameterized algebraization versus the standard theory for sentential 1-calculi. For example, *parameterized defining equations* are equations in *two* variables while defining equations (for sentential 1-calculi) are equations in one variable (see Definition 2.105 on page 111 and Definition 13.1 on page 392).

Theorem 11.18 The following conditions on $\mathcal{K}$ and $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ are equivalent (where k is any integer with $k \geq 3$).

1. Every algebra $\mathbf{A}$ is $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regular.

2. $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regular.

3. The $\mathcal{K}$-free algebra on 3-free generators is $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regular.

4. The $\mathcal{K}$-free algebra on k-free generators is $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regular.

5. $\mathbf{F}_{\mathcal{K}}$ is $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regular.

6. $\mathbf{Tm}$ is $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regular.

7. There exist finite sets of ternary terms $\mathbf{\Delta}_1(x, y, z), \ldots, \mathbf{\Delta}_n(x, y, z)$, such that

$$\bigcup_{1 \leq i \leq n} (\mathfrak{B}_i)^{\approx}_{z} \left[\mathbf{\Delta}_i(x, y, z)\right] \mathrel{=\!\!|\!\!\models_{\mathcal{K}}} x \approx y. \tag{11.6}$$

Proof. $\boxed{\text{(1)}\Rightarrow\text{(2), (3), (4), (5), (6)}}$ Trivial. $\boxed{\text{(2)}\Rightarrow\text{(7)}}$ Consider the 3-generated $\mathcal{K}$-free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\overline{x}, \overline{y}, \overline{z}) \in \mathcal{K}$. Let $\mathfrak{X} = \{\mathfrak{B}_1, \ldots, \mathfrak{B}_n\}$. Let $\alpha = \|\langle \overline{x}, \overline{y} \rangle\|_{\Theta_{\mathbf{F}_{\mathcal{K}}}}$ and $\beta = \left\| \bigcup_{\mathfrak{B} \in \mathfrak{X}} \mathfrak{B}^{\mathbf{F}}_{\overline{z}} \left[\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$. Since $\bigcup_{\mathfrak{B} \in \mathfrak{X}} \mathfrak{B}^{\mathbf{F}}_{\overline{z}} \left[\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha\right] \subseteq \alpha$, by Remark 9.25, $\beta \subseteq \alpha$, and hence $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta \subseteq \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha$, for each $\mathfrak{B} \in \mathfrak{X}$. But, by construction and Remark 9.25, for each $\mathfrak{B} \in \mathfrak{X}$, $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha = \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/(\mathfrak{B}^{\mathbf{F}}_{\overline{z}} \left[\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha\right]) \subseteq \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$. Hence $\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha = \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\beta$, for each $\mathfrak{B} \in \mathfrak{X}$. So by assumption, $\alpha = \beta$. So $\langle \overline{x}, \overline{y} \rangle \in \left\| \bigcup_{\mathfrak{B} \in \mathfrak{X}} \mathfrak{B}^{\mathbf{F}}_{\overline{z}} \left[\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$. Since relative congruences form an *algebraic* closed system, there exist $\beta' \subseteq_{\mathsf{f}} \bigcup_{\mathfrak{B} \in \mathfrak{X}} \mathfrak{B}^{\mathbf{F}}_{\overline{z}} \left[\mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha\right]$ with $\langle \overline{x}, \overline{y} \rangle \in \|\beta'\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$ by equivalent condition (2) of Proposition 4.77 on page 157. So, for each $\mathfrak{B} \in \mathfrak{X}$, there exists $\overline{\Delta^{\mathfrak{B}}_1}, \ldots, \overline{\Delta^{\mathfrak{B}}_{j_{\mathfrak{B}}}} \in \mathfrak{B}^{\mathbf{F}}_{\overline{z}}/\alpha$, with $\langle \overline{x}, \overline{y} \rangle \in \left\| \bigcup_{\mathfrak{B} \in \mathfrak{X}} \mathfrak{B}^{\mathbf{F}}_{\overline{z}} \left[\overline{\Delta^{\mathfrak{B}}_1}, \ldots, \overline{\Delta^{\mathfrak{B}}_{j_{\mathfrak{B}}}}\right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$. The result follows from Lemma 1.457 on page 88. The previous argument can be easily adapted, using the structurality of $\models_{\mathcal{K}}$, to prove that (3)⇒(7), (4)⇒(7), (5)⇒(7) and (6)⇒(7).

$\boxed{\text{(7)}\Rightarrow\text{(1)}}$ Let $\mathbf{A}$ be any algebra, $a \in \mathsf{uni}(\mathbf{A})$ and $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ with $(\mathfrak{B}_i)^{\mathbf{A}}_a/\alpha = (\mathfrak{B}_i)^{\mathbf{A}}_a/\beta$, for each $1 \leq i \leq n$. Let $\langle b, c \rangle \in \alpha$. For each $1 \leq i \leq n$, by Lemma 1.457 and (11.6), $(\mathfrak{B}_i)^{\mathbf{A}}_a \left[\mathbf{\Delta}^{\mathbf{A}}_i(b, c, a)\right] \subseteq \alpha$, hence $\mathbf{\Delta}^{\mathbf{A}}_i(b, c, a) \subseteq (\mathfrak{B}_i)^{\mathbf{A}}_a/\alpha = (\mathfrak{B}_i)^{\mathbf{A}}_a/\beta$, i.e., $(\mathfrak{B}_i)^{\mathbf{A}}_a \left[\mathbf{\Delta}^{\mathbf{A}}_i(b, c, a)\right] \subseteq \beta$. By Lemma 1.457 and (11.6) again, $\langle b, c \rangle \in \beta$. By symmetry, $\alpha = \beta$. ⋄

We now consider the case of a single binary system $\mathfrak{B}$. It is this case that will prove most useful in the sequel. We now aim to characterize $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity in terms of an isomorphism between the lattice of principal normals and the congruence lattice. We begin by noting that $\mathbf{A}$ is $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regular iff $\mathfrak{B}^{\mathbf{A}}_b/\cdot$ is injective, which follows immediately from the definition of $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity.

Remark 11.19 $\mathbf{A}$ is $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regular iff, for each $b \in \mathsf{uni}(\mathbf{A})$, $\mathfrak{B}_b^{\mathbf{A}}/\cdot : \mathsf{Con}^{\mathcal{K}}(\mathbf{A}) \rightarrowtail \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$.

Corollary 11.20 The following conditions on $\mathcal{K}$ and $\mathfrak{B}$ are equivalent (where k is any integer with $k \geq 3$).

1. $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regular.
2. For each algebra $\mathbf{A}$ and $b \in \mathsf{uni}(\mathbf{A})$, $\mathfrak{B}_b^{\mathbf{A}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$.
3. For each $\mathbf{A} \in \mathcal{K}$ and $b \in \mathsf{uni}(\mathbf{A})$, $\mathfrak{B}_b^{\mathbf{A}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$.
4. For each term p, $\mathfrak{B}_{\overline{p}}^{\mathbf{F}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{F}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{B}_{\overline{p}}}(\mathbf{F})$, where $\mathbf{F}$ is the $\mathcal{K}$-free algebra on 3-free generators.
5. For each term p, $\mathfrak{B}_{\overline{p}}^{\mathbf{F}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{F}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{B}_{\overline{p}}}(\mathbf{F})$, where $\mathbf{F}$ is the $\mathcal{K}$-free algebra on k-free generators.
6. For each term p, $\mathfrak{B}_{\overline{p}}^{\mathbf{F}_{\mathcal{K}}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{F}_{\mathcal{K}}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{B}_{\overline{p}}}(\mathbf{F}_{\mathcal{K}})$.
7. For each term p, $\mathfrak{B}_{p}^{\mathbf{Tm}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm}) \cong \mathbf{Sol}^{\mathcal{K}}_{\mathfrak{B}_{p}}(\mathbf{Tm})$.

Proof. $\boxed{(1)\Rightarrow(2)}$ By definition this map is surjective and by assumption injective; hence $\mathfrak{B}_b^{\mathbf{A}}/\cdot : \mathsf{Con}^{\mathcal{K}}(\mathbf{A}) \Rightarrow \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ (i.e., $\mathfrak{B}_b^{\mathbf{A}}/\cdot$ is a bijection from $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ onto $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$). Trivially, this map is inclusion preserving. It suffices to prove that this map is order-reflecting. Suppose that $\mathfrak{B}_b^{\mathbf{A}}/\alpha \subseteq \mathfrak{B}_b^{\mathbf{A}}/\beta$. Then by assumption and Proposition 11.17, $\alpha \subseteq \beta$. $\boxed{(2)\Rightarrow(3),\ (4),\ (5),\ (6),\ (7)}$ Trivial. $\boxed{(3)\Rightarrow(1)}$ By Remark 11.19.
The proofs of (4)⇒(1), (5)⇒(1), (6)⇒(1) and (7)⇒(1), follow by Theorem 11.18 and Remark 11.19. ◇

11.2.3 Examples

In the following example we show how the well-known characterizations of *relative (full) congruence regularity* obtain from the results of this section. We introduce the more general notion of *term-regularity* with respect to a finite set of unary terms; the non-relative version of this notion was introduced and characterized in [Bar95].

Example 11.21 (Term Regularity and Full Regularity)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{A}$ an $\mathfrak{a}$-algebra.

Definition 11.22 (Term Regularity and Full Regularity)
For $\mathcal{K}$-unary terms $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$, $\langle \mathcal{K}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$**-regularity** (resp. $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$**-regularity**) means $\langle \mathcal{K}, \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regularity (resp. $\langle \mathfrak{B}_{1*}, \ldots, \mathfrak{B}_{n*} \rangle$-regularity), where $\mathfrak{B}_i(x, y) = \{x, \boldsymbol{u}_i(y)\}$ for each $i = 1, \ldots, n$.

By $\mathcal{K}$**-regularity** (resp. **regularity**), also called **relative regularity**, we mean $\langle \mathcal{K}, x \rangle$-regularity (resp. $\langle x \rangle$-regularity), where x is any variable. The term **full** $\mathcal{K}$**-regularity** (resp. **full regularity**) is a further synonym for $\mathcal{K}$-regularity (resp. regularity).

□

We note that $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-regularity has been called **term regularity at $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$** in the literature [Bar95], although we avoid this nomenclature in this this text. Clearly regularity and $\mathcal{K}$-regularity as defined above coincides with the notions of regularity and $\mathcal{K}$-regularity given in Definition 1.359 on page 68 and Definition 1.375 on page 71.

Corollary 11.23 $\mathcal{K}$ is $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-regular iff there exist finite sets of ternary terms $\boldsymbol{\Delta}_1(x,y,z), \ldots, \boldsymbol{\Delta}_n(x,y,z)$, such that

$$\bigwedge\nolimits_{1 \leq i \leq n} \bigwedge\nolimits_{\Delta \in \boldsymbol{\Delta}_i} \Delta(x,y,z) \approx \boldsymbol{u}_i(z) \mathrel{=\!\mid\!\models}_{\mathcal{K}} x \approx y. \tag{11.7}$$

Corollary 11.24 $\mathcal{K}$ is relatively regular iff there exists a finite set $\boldsymbol{\Delta}$ of ternary terms such that

$$\bigwedge\nolimits_{\Delta \in \boldsymbol{\Delta}_i} \Delta(x,y,z) \approx z \mathrel{=\!\mid\!\models}_{\mathcal{K}} x \approx y. \tag{11.8}$$

□

We shall now show that the characterization of n-subregularity given in [DMS87], and hence the characterizations of subregularity given in [Tim75], obtain from the results of the previous example. Note that the results of the following example are *relative* generalizations of the aforementioned non-relative characterizations; the latter obtain from the former by taking $\mathcal{K}$ to be a variety rather than a quasivariety.

Example 11.25 (Relative Subregularity)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras and $\mathbf{A}$ an $\mathfrak{a}$-algebra.

Definition 11.26 (Relative Subregularity) We say that $\mathbf{A}$ is **$\mathcal{K}$-regular at** A (resp. **regular at** A), where $\emptyset \neq A \subseteq \mathsf{uni}(\mathbf{A})$, if, for all $\alpha, \beta \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ (resp. $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$), if $\alpha[\![a]\!] = \beta[\![a]\!]$ for all $a \in A$, then $\alpha = \beta$. For non-zero natural l, we say that $\mathbf{A}$ is **$\langle \mathcal{K}, l \rangle$-regular** (resp. **$l$-regular**) if there exist $a_1, \ldots, a_n \in \mathsf{uni}(\mathbf{A})$, such that $\mathbf{A}$ is $\mathcal{K}$-regular (resp. regular) at $\{a_1, \ldots, a_n\}$. Algebra $\mathbf{A}$ is called **$\mathcal{K}$-subregular** (resp. **subregular**) if, for each $\mathbf{B} \lhd \mathbf{A}$, $\mathbf{A}$ is $\mathcal{K}$-regular (resp. regular) at $\mathsf{uni}(\mathbf{B})$. For non-zero natural l, we say that $\mathbf{A}$ is **$\langle \mathcal{K}, l \rangle$-subregular** (resp. **$l$-subregular**) if, for each $\mathbf{B} \lhd \mathbf{A}$, there exist $b_1, \ldots, b_n \in \mathsf{uni}(\mathbf{B})$, such that $\mathbf{A}$ is $\mathcal{K}$-regular (resp. regular) at $\{b_1, \ldots, b_n\}$. We extend these notions to quasivarieties in the obvious manner; for example, we say that $\mathcal{K}$ is $\langle \mathcal{K}, l \rangle$-subregular if every algebra in $\mathcal{K}$ is $\langle \mathcal{K}, l \rangle$-subregular. □

The proof of the following result follows by standard universal algebraic arguments and as such is omitted.

Theorem 11.27 The following conditions are equivalent.

1. $\mathcal{K}$ is $\langle \mathcal{K}, l \rangle$-subregular.
2. $\mathcal{K}$ is $\langle \mathcal{K}, l \rangle$-regular.
3. The $\mathcal{K}$-free algebra on ω free generators is $\langle \mathcal{K}, l \rangle$-regular.
4. The $\mathcal{K}$-free algebra on 3 free generators is $\langle \mathcal{K}, l \rangle$-subregular.
5. $\mathcal{K}$ is $\langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_l \rangle$-regular, for some $\mathcal{K}$-unary terms $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_l$. Further, $\mathcal{K}$ is $\mathcal{K}$-subregular iff it is $\langle \mathcal{K}, l \rangle$-subregular for some non-zero natural l.

□

We now give an example of a quasivariety $\mathcal{K}$ that is relatively 1-subregular, but which is neither relatively regular nor relatively point regular.

Example 11.28

Let A denote the set of integers and $\mathbf{A} = \langle A; d^{\mathbf{A}} \rangle$ the algebra of type $\langle 3 \rangle$, where $d^{\mathbf{A}}(a, b, c) = |a - b| + |c|$ for $a, b, c \in A$. The quasivariety $\mathcal{K}$ generated by $\mathbf{A}$ is relatively 1-subregular, since it is relatively regular at $\boldsymbol{u}(x) = d(x, x, x)$. It is therefore $\langle \mathcal{K}, \boldsymbol{u} \rangle$-coset determined, but is not fully $\mathcal{K}$-regular and has no equationally definable constant.

Observe that $\mathcal{K}$ satisfies $d(x, y, z) \approx \boldsymbol{u}(z) \leftrightarrow x \approx y$ (since $\mathbf{A}$ does). By Corollary 11.23, $\mathcal{K}$ is relatively regular at $\boldsymbol{u}$. Inasmuch as the range of every term function on $\mathbf{A}$ (other than a projection) consists of non-negative integers, there is no finite set $\{\Delta_j : j < m\}$ of ternary terms such that $\mathcal{K}$ satisfies $[\bigwedge_{j<m} \Delta_j(x, y, z) \approx z] \leftrightarrow x \approx y$; so by Corollary 11.24, $\mathcal{K}$ is not fully $\mathcal{K}$-regular. Further, for each positive integer n, $\{n\}$ is a subuniverse of $\mathbf{A}$. Thus, every element of A lies outside some subuniverse of $\mathbf{A}$, whence $\mathcal{K}$ has no equationally definable constant.

Open Problem 11.29 Does $\boldsymbol{u}_*$ pivot for $\mathcal{K}$?

□

Recall that the quasivariety LM of *polrims* is relatively 0-regular (see Example 11.15). We shall now demonstrate a binary system $\mathfrak{B}$ for which this quasivariety is not $\langle \mathsf{LM}, \mathfrak{B}_* \rangle$-regular.

Counter Example 11.30 (Polrims)

Let $\mathfrak{B} = \{\langle x \oplus y, y \rangle\}$ and consider that polrim $\mathbf{A}$ defined in Counter Example 9.113 on page 338. LM is *not* $\langle \mathsf{LM}, \mathfrak{B}_* \rangle$-regular, since $\mathbf{A}$ itself fails to be $\langle \mathsf{LM}, \mathfrak{B}_* \rangle$-regular. Indeed, the partition $\{\{0, a\}, \{b, e, 1\}\}$ of A corresponds to an LM-congruence α of $\mathbf{A}$ with $\mathfrak{B}_e^{\mathbf{A}} / \alpha = \{c \in A : \langle c \oplus e, e \rangle \in \alpha\} = A = \mathfrak{B}_e^{\mathbf{A}} / A^2$, but $\alpha \neq A^2$.

□

Part V

Parameterized Algebraization

In Part IV, we concluded Step 1 of our *unification program*, by developing a theory of $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-*regularity*, that encompassed both $\langle \mathcal{K}, \tau \rangle$-*regularity* [BR99] and (full) $\mathcal{K}$-*regularity*. In Part V, we shall tackle the remaining three steps.

Step 2 of our program is undertaken in $\oint 12$. We introduce a family of sentential *one*-calculi $S(\mathcal{K}, \mathfrak{B}_*)$, determined by a binary system $\mathfrak{B}$ and a quasivariety $\mathcal{K}$. Recall that the set $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ of all parameterized solution sets $\mathfrak{B}^{\mathbf{A}}_b/\alpha$ does not generally form a closed system, and so we took $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ as the basis of a closed system $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$. This closed system is generally non-finitary and so $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{Tm})$ cannot be the theories of a sentential calculus. We shall show that the sentential calculus $S(\mathcal{K}, \mathfrak{B}_*)$ is the *best sentential approximation* of the universal logic on the term algebra with theories $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{Tm})$. This family of sentential calculi encompass the logics $S(\mathcal{K}, \tau)$ of [BR99], and hence encompass *all* sentential 1-calculi having an *algebraic semantics* and consequently all *algebraizable* sentential 1-calculi. This family also encompasses the *sentential calculi of idempotent* $\langle \mathcal{K}, \boldsymbol{u} \rangle$-*cosets*, and hence includes the *membership logics*.

Step 3, undertaken in $\oint 13$, $\oint 14$ and $\oint 15$, involves the development of a theory of *parameterized* algebraization that specializes to the standard Blok-Pigozzi theory of algebraizable logics for a certain choice of parameter. The task of analyzing logics such as the *membership logic* would be eased if we could firmly locate them, relative to more familiar systems, within a theoretical framework that includes the Blok-Pigozzi theory of algebraization as developed in [BP89a] and subsequent papers. We shall offer and work through an extension (encompassing membership logics in particular) of Blok and Pigozzi's theory. This will have the dual effect of widening the class of quasivarieties susceptible (in a broad sense) to logical investigation, particularly by methods originating in *algebraic* logic. Indeed, a (suitably parametrized) general notion corresponding to 'quasivariety of logic' will apply even to the *congruence regular* variety of quasigroups, despite the fact that this variety is not the *equivalent algebraic semantics* of *any* sentential 1-calculus and contains a non-trivial (congruence regular) subvariety that is not even the *algebraic semantics* of any non-trivial sentential 1-calculus (see Counter Example 3.1 on page 124). In one of the equivalent formulations of our parameterized theory, the parameter is a binary system of equations $\mathfrak{B}$; we speak of a sentential 1-calculus having a $\mathfrak{B}_*$-*equivalent semantics* $\mathcal{K}$ and of being $\mathfrak{B}_*$-*algebraizable*. When this binary system is taken to be a unary system, our parameterized theory coincides with the standard theory of [BP89a].

The final step, undertaken in a series of examples developed throughout Part V, involves establishing relationships between $\mathfrak{B}_*$-algebraizable logics, the $\mathfrak{B}_*$-algebraizability of the logic $S(\mathcal{K}, \mathfrak{B}_*)$ and the $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity of $\mathcal{K}$, and to do so in a manner that specializes to the theory of [BR99] and characterizes $\mathcal{K}$-regularity in terms of the suitably parameterized algebraizability of the membership logic. The relationship between the parameterized protoalgebraicity of the logic $S(\mathcal{K}, \mathfrak{B}_*)$ and the condition that $\mathcal{K}$ have $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes is also considered.

We shall now briefly outline the contents of Part V.

In $\oint 12$ we define the sentential calculi $S(\mathcal{K}, \mathfrak{B}_*)$ and develop some results concerning its model theory. We show how this family of logics includes the logic $S(\mathcal{K}, \tau)$ of [BR99] (this coincidence occurs when the binary system is essentially $\mathcal{K}$-unary), includes the *logics of idempotent* $\langle \mathcal{K}, \boldsymbol{u} \rangle$-*cosets* (in the case that $\mathfrak{B}(x, y) = \{\langle x, \boldsymbol{u}(y) \rangle\}$) and hence encompasses the *membership logic*. A number of other instances of this logic are highlighted, including the *logics of separable binary equations*, the *logics of identified membership* $S(\mathcal{K}, \boldsymbol{u})$, where $\boldsymbol{u}$ is a *not necessarily idempotent* unary term, and the *logics of lattice ideals and filters*. The logics of identified membership encompass the sentential calculi of idempotent $\langle \mathcal{K}, \boldsymbol{u} \rangle$-cosets and hence the membership logics.

In $\oint 13$ we consider the notion that a logic have a *parametrized algebraic semantics*. There are two equivalent forms of expressing the parameter of the theory, either as a pair $\langle X, z \rangle$, where X is a set of terms (i.e., formulae) and z is a variable, or as a binary system $\mathfrak{B}$. We shall show that a quasivariety $\mathcal{K}$ is always a $\langle \{\boldsymbol{u}(z)\}, z \rangle$-*algebraic semantics* for the logic of identified membership $S(\mathcal{K}, \boldsymbol{u})$, and consequently, $\mathcal{K}$ is always a $\langle \{z\}, z \rangle$-algebraic semantics for its membership logic.

In $\oint 14$ we consider the various conditions of *parametrized protoalgebraicity*, including a condition we term *almost protoalgebraic*, the latter condition requiring that the Leibniz operator be inclusion-preserving when restricted to non-empty theories. These conditions are applied to the logics introduced earlier. For example, we shall show that the *membership logic* of a quasivariety is 'almost-protoalgebraic' iff the quasi-variety has $\mathcal{K}$-coherent $\mathcal{K}$-congruence classes.

Parametrized equivalent algebraic semantics are considered in $\oint 15$. We shall show that the regularity of a quasivariety may be characterized in terms of that quasivariety being a $\langle \{z\}, z \rangle$-equivalent algebraic semantics for its membership logic. Finally, in $\oint 15.6$, we consider various 'zones of applicability' of the theory of parametrized algebraization.

In Part VI we shall provide an alternative perspective on our parameterized theory from a *non*-parameterized theory of protoalgebraicity and equivalence for logics over constructs.

Chapter 12

The Logics of Parametric Solutions to Binary Equations

In this chapter we consider a generalization of the family of sentential 1-calculi $S(\mathcal{K}, \tau)$ of [BR99]. This generalization is analogous to the manner in which the conditions of $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-regularity and having $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes generalize the conditions of $\langle \mathcal{K}, \tau \rangle$-regularity and having $\mathcal{K}$-coherent $\langle \mathcal{K}, \tau \rangle$-classes introduced in [BR99]. With each quasivariety $\mathcal{K}$ and each *binary* system $\mathfrak{B}$, we shall associate a sentential 1-calculus $S(\mathcal{K}, \mathfrak{B}_*)$, which we view as the *sentential calculus of parametric solutions to* $\mathfrak{B}$ (in the sense of $\oint$9.2). Note that while the logic $S(\mathcal{K}, \mathfrak{B}_*)$ is different to the logic $S^2(\mathcal{K}, \mathfrak{B})$ (the former is a sentential 1-calculus while the latter is a sentential 2-calculus) it is not inappropriate to view the former as an attempt to approximate the latter as a sentential 1-calculus. Generally the logic $S(\mathcal{K}, \mathfrak{B}_*)$ is 'inherently unalgebraizable' in the sense of [BP89a], since typically it has no theorems in which case it can never be protoalgebraic (recall that protoalgebraicity is a necessary condition for algebraizability).

12.1 The Logics of Parametric Solutions to Binary Equations

Recall that $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ is the closed system on the universe of $\mathbf{A}$ determined by the basis $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ of all parameterized solution sets $\mathfrak{B}^{\mathbf{A}}_b/\alpha$, where the parameter b ranges over the universe of $\mathbf{A}$ and α ranges over all relative congruences on $\mathbf{A}$. As a first attempt at identifying the logics of parametric solutions of $\mathfrak{B}_*$, we simply consider the *universal logics* on algebras $\mathbf{A}$ determined by theories $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$. Note that while it would be most desirable to take $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ as the theories, generally $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ is not a closed system.

Convention 12.1 Throughout this section, unless specified to the contrary, let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras, $\mathbf{A}$ an $\mathfrak{a}$-algebra, $\mathbf{F}_{\mathcal{K}}$ the $\mathcal{K}$-free algebra on ω-free generators and $\mathfrak{B}$ a *binary* system of $\mathfrak{a}$-equations, all fixed but arbitrary. All logics under consideration are universal $\mathfrak{a}$-logics; as such, any unprefixed references to structurality refers to $\mathfrak{a}$-structurality.

Definition 12.2 (The Universal Logics of Parametric Solutions to Binary Equations) Let $U_\forall(\mathbf{A},\mathcal{K},\mathfrak{B}_*)$ denote the $\mathbf{A}$-logic $L\left(\mathbf{A},\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})\right)$, which we call the **universal logics of parametric solutions to $\mathfrak{B}_*$ modulo $\mathcal{K}$**. We write $\boldsymbol{D}_\forall(\mathcal{K},\mathfrak{B}_*)$ for $U_\forall(\mathbf{F}_{\mathcal{K}},\mathcal{K},\mathfrak{B}_*)$, where $\mathbf{F}_{\mathcal{K}}$ is the $\mathcal{K}$-free algebra on ω-free generators $\overline{[\mathsf{V}]}$, and write $D_\forall(\mathcal{K},\mathfrak{B}_*)$ for $U_\forall(\mathbf{Tm},\mathcal{K},\mathfrak{B}_*)$. □

The reason for the subscript '$\forall$' will become apparent. The reason for the 'D' rather than the 'S' in these notions, is that, as we shall see later, these logics are (generally) neither *finitary* nor *structural*; in particular, $D_\forall(\mathcal{K},\mathfrak{B}_*)$ is not (generally) a *sentential calculus*. By definition, the parametric solutions $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ form a theory basis for $U_\forall(\mathbf{A},\mathcal{K},\mathfrak{B}_*)$. Consequently, $U_\forall(\mathbf{A},\mathcal{K},\mathfrak{B}_*)$ is the *coarsest* $\mathbf{A}$-logic having all the parametric solutions $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ as theories (by Remark 9.24 on page 320). Consequently, while $D_\forall(\mathcal{K},\mathfrak{B}_*)$ is not (generally) a *sentential calculus*, it is the 'best' $\mathbf{Tm}$-logic of parametric solutions to $\mathfrak{B}_*$ modulo $\mathcal{K}$. We shall soon introduce a sentential calculus $S(\mathcal{K},\mathfrak{B}_*)$ which will serve as a *sentential approximation* of $D_\forall(\mathcal{K},\mathfrak{B}_*)$.

The following characterization of $U_\forall(\mathbf{A},\mathcal{K},\mathfrak{B}_*)$-consequence follows immediately from Corollary 9.26 on page 320.

Corollary 12.3 $A \vdash_{U_\forall(\mathbf{A},\mathcal{K},\mathfrak{B}_*)} a$ iff $\forall\,[\alpha\in\mathsf{Con}^{\mathcal{K}}(\mathbf{A})]\;\forall\,[b\in\mathsf{uni}(\mathbf{A})]\;\mathfrak{B}^{\mathbf{A}}_b\,[A]\subseteq\alpha\rightarrow\mathfrak{B}^{\mathbf{A}}_b[\![a]\!]\subseteq\alpha$.
□

We now focus on the $\mathfrak{a}$-deductive system $D_\forall(\mathcal{K},\mathfrak{B}_*)$. The following characterization of $D_\forall(\mathcal{K},\mathfrak{B}_*)$-consequence in terms of $\mathcal{K}$-equational consequence $\models_{\mathcal{K}}$, follows at once from the previous corollary together with Lemma 1.457 on page 88. It is the quantification over all terms $q\in\mathsf{Tm}$ in the following characterization that motivates the subscript '$\forall$' in the notation '$D_\forall(\mathcal{K},\mathfrak{B}_*)$'.

Proposition 12.4 $P\vdash_{D_\forall(\mathcal{K},\mathfrak{B}_*)} p$ iff $\forall\,[q\in\mathsf{Tm}]\;\mathfrak{B}^{\approx}_q\,[P]\models_{\mathcal{K}}\mathfrak{B}^{\approx}_q[\![p]\!]$. □

While $D_\forall(\mathcal{K},\mathfrak{B}_*)$ is generally neither finitary nor structural, we shall now demonstrate that this logic is *finitely structural* (see Definition 6.14 on page 225). Consequently $D_\forall(\mathcal{K},\mathfrak{B}_*)$ can be *soundly approximated* by a coarsest sentential calculus (see Proposition 6.34 on page 230).

Proposition 12.5 $D_\forall(\mathcal{K},\mathfrak{B}_*)$ is finitely structural.

Proof. Suppose that $P\vdash_{D_\forall(\mathcal{K},\mathfrak{B}_*)} p$ for some finite set of terms P. Let σ be any substitution. (*We must show that* $\sigma\,[P]\vdash_{D_\forall(\mathcal{K},\mathfrak{B}_*)}\sigma(p)$.) Let $q\in\mathsf{Tm}$. (*By Proposition 12.4, we must show that* $\mathfrak{B}^{\approx}_q\,[\sigma\,[P]]\models_{\mathcal{K}}\mathfrak{B}^{\approx}_q(\sigma(p))$.) Let x be a variable not occurring in the variables of $P\cup\{p\}$. Since $P\vdash_{D_\forall(\mathcal{K},\mathfrak{B}_*)} p$, $\mathfrak{B}^{\approx}_x\,[P]\models_{\mathcal{K}}\mathfrak{B}^{\approx}_x(p)$, by Proposition 12.4. Let ρ be the $\mathbf{Tm}$-substitution mapping $x\mapsto q$ and agreeing with σ on all other variables. By structurality of $\models_{\mathcal{K}}$, $\mathfrak{B}^{\approx}_{\rho(x)}\,[\rho\,[P]]\models_{\mathcal{K}}\mathfrak{B}^{\approx}_{\rho(x)}(\rho(p))$, i.e., $\mathfrak{B}^{\approx}_q\,[\sigma\,[P]]\models_{\mathcal{K}}\mathfrak{B}^{\approx}_q(\sigma(p))$, as required. ⋄

We now consider a deductive system closely related to $D_\forall(\mathcal{K},\mathfrak{B}_*)$. Our aim is to identify $\mathcal{K}$ deductions of the form $\mathfrak{B}_z(P)\models_{\mathcal{K}}\mathfrak{B}_z(P)$, but for which the role of z is 'free'; that is, the variable z must be replaceable in the deduction by any other variable without effecting the validity of the deduction over $\mathcal{K}$. To this end, we introduce the following notion of a *meta-quantifier* and *meta-variable*.

Definition 12.6 (Meta-Quantifiers and Meta-Variables) We write $\mathsf{\forall}[\mathsf{z}]\;\mathfrak{B}^{\approx}_{\mathsf{z}}\,[P]\models_{\mathcal{K}}\mathfrak{B}^{\approx}_{\mathsf{z}}[\![p]\!]$ iff, for all $z\in\mathsf{V}$, $\mathfrak{B}^{\approx}_z\,[P]\models_{\mathcal{K}}\mathfrak{B}^{\approx}_z[\![p]\!]$; we call $\mathsf{\forall}$ a **meta-quantifier** and z a **meta-variable**. The scope of the meta-quantifier is the entire expression to the right of the meta-variable deliminators; scope may be controlled by parenthesis in the natural manner. $\square$

Definition 12.7 (The Deductive System $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$) Let $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$ be the $\mathfrak{a}$-deductive system defined by consequence relation

$$P \vdash_{D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)} p \text{ iff } \forall[\mathsf{z}]\, \mathfrak{B}^{\approx}_{\mathsf{z}}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{\mathsf{z}}[\![p]\!]. \tag{12.1}$$

$\square$

Proof. (*We need to show that this definition well defines the consequence relation of a logic.*) $\boxed{\text{(6.1)}}$ Suppose that $p \in P$. For any $z \in \mathsf{V}$, $\mathfrak{B}^{\approx}_{z}[\![p]\!] \subseteq \mathfrak{B}^{\approx}_{z}[P]$, and so $\mathfrak{B}^{\approx}_{z}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{z}[\![p]\!]$. $\boxed{\text{(6.2)}}$ Suppose that $Q \subseteq P$ and $Q \vdash_{D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)} p$. For each $z \in \mathsf{V}$, $\mathfrak{B}^{\approx}_{z}[Q] \subseteq \mathfrak{B}^{\approx}_{z}[P]$ and $\mathfrak{B}^{\approx}_{z}[Q] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{z}[\![p]\!]$, so $\mathfrak{B}^{\approx}_{z}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{z}[\![p]\!]$. Hence $P \vdash_{D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)} p$. $\boxed{\text{(6.3)}}$ Suppose that $Q \vdash_{D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)} p$ and for all $q \in Q$, $P \vdash_{D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)} q$. For each $z \in \mathsf{V}$, $\mathfrak{B}^{\approx}_{z}[Q] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{z}[\![p]\!]$ and, for all $q \in Q$, $\mathfrak{B}^{\approx}_{z}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{z}[\![q]\!]$, and hence $\mathfrak{B}^{\approx}_{z}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{z}[\![p]\!]$. So $P \vdash_{D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)} p$. $\diamond$

A comparison of Proposition 12.4 with (12.1) demonstrates that $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$ is finer than $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$.

Proposition 12.8 $D_{\forall}(\mathcal{K}, \mathfrak{B}_*) \preceq D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$. $\square$

Not only is $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$ finer than $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$, any *finitary* deductive system finer than $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$ is also finer than $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$. Consequently, if $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$ is finitary then so is $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$.

Lemma 12.9 If $\mathcal{D}$ is a finitary deductive system and $\mathcal{D} \preceq D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$, then $\mathcal{D} \preceq D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$.

Proof. (*By Proposition 12.4, it suffices to show that if $P \vdash_{\mathcal{D}} p$ then $\forall\, [q \in \mathsf{Tm}]\, \mathfrak{B}^{\approx}_{q}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{q}[\![p]\!]$.*) Suppose that $P \vdash_{\mathcal{D}} p$. Let $q \in \mathsf{Tm}$. By assumed finitariness, there exists $P' \subseteq P$ such that $P' \vdash_{\mathcal{D}} p$. Since $\mathcal{D} \preceq D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$, $P' \vdash_{D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)} p$, and so by definition, $\forall[\mathsf{z}]\, \mathfrak{B}^{\approx}_{\mathsf{z}}[P'] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{\mathsf{z}}[\![p]\!]$. Let z be a variable not occurring in any of the terms of P' nor occurring in p. Then $\mathfrak{B}^{\approx}_{z}[P'] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{z}[\![p]\!]$. Let σ be the substitution mapping $z \mapsto q$ and fixing all other variables. By the structurality of $\models_{\mathcal{K}}$ and the fact that σ is a homomorphism, $\mathfrak{B}^{\approx}_{\sigma(z)}[\sigma[P']] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{\sigma(z)}[\![\sigma(p)]\!]$. Since $\sigma(z) = q$, $\sigma[P'] = P'$ and $\sigma(p) = p$, $\mathfrak{B}^{\approx}_{q}[P'] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{q}[\![p]\!]$. Hence $\mathfrak{B}^{\approx}_{q}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{q}[\![p]\!]$. $\diamond$

Consequently, if $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$ is finitary, then $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*) \preceq D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$, and hence $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*) = D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$, by Proposition 12.8, in which case, since $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$ is finitely structural, these equal logics are structural and hence sentential calculi.

Corollary 12.10 If $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$ is finitary then $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*) = D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$, in which case this logic is structural and hence a *sentential* calculus. $\square$

The deductive system $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$ is also finitely structural.

Proposition 12.11 $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$ is finitely structural.

Proof. Suppose that $P \vdash_{D_{\forall_z}(\mathcal{K},\mathfrak{B}_*)} p$ for some finite set of terms P. Let σ be any substitution. (*We must show that* $\sigma[P] \vdash_{D_{\forall_z}(\mathcal{K},\mathfrak{B}_*)} \sigma(p)$.) Let $z \in \mathsf{V}$. (*We must show that* $\mathfrak{B}^{\approx}_z[\sigma[P]] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![\sigma(p)]\!]$.) Let x be a variable not occurring in the variables of $P \cup \{p\}$. Since $P \vdash_{D_{\forall_z}(\mathcal{K},\mathfrak{B}_*)} p$, $\mathfrak{B}^{\approx}_x[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_x[\![p]\!]$. Let ρ be the **Tm**-substitution mapping $x \mapsto z$ and agreeing with σ on all other variables. By structurality of $\models_{\mathcal{K}}$, $\mathfrak{B}^{\approx}_{\rho(x)}[\rho[P]] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{\rho(x)}[\![\rho(p)]\!]$, i.e., $\mathfrak{B}^{\approx}_z[\sigma[P]] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![\sigma(p)]\!]$, as required. $\diamond$

We now introduce the sentential 1-calculus $S(\mathcal{K},\mathfrak{B}_*)$. The importance of this family of logics lies in the number of sentential 1-calculi that it encompasses. Included within this family are the logics $S(\mathcal{K},\tau)$ of [BR99] (and hence the assertional logics), where τ is a unary system of equations, defined in Example 2.92 on page 107, and these logics $S(\mathcal{K},\tau)$ encompass *all* algebraizable logics [BR99],[BP89a]. Also included amongst the logics $S(\mathcal{K},\mathfrak{B}_*)$, are the sentential calculi $S(\mathfrak{a},\mathsf{cos})$ of cosets and $S(\mathcal{K},\mathsf{mem})$ of relative cosets, as well as the sentential calculi $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ of idempotent $\boldsymbol{u}$-cosets. Further, the theory of idempotent $\boldsymbol{u}$-cosets specializes to the theory of $\boldsymbol{u}$-cosets of [Bar95], which unifies/includes Ursini's theory of ideals in universal algebra [Urs94] and Aglianò's theory of cosets in universal algebra [AU87],[AU92].

Definition 12.12 (The Sentential Calculus $S(\mathcal{K},\mathfrak{B}_*)$) Let $S(\mathcal{K},\mathfrak{B}_*)$ denote the structural and finitary deductive system axiomatized by all inference-rules (and axioms) $P \vdash p$, such that

$$\forall[\mathsf{z}]\ \mathfrak{B}^{\approx}_{\mathsf{z}}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{\mathsf{z}}[\![p]\!]. \tag{12.2}$$

$\square$

Remark 12.13 Let $P \cup \{p\}$ be a *finite* set of terms, let z be any variable *not occurring* in the terms of $P \cup \{p\}$ (such a z must exist since $P \cup \{p\}$ is a finite set). Then

$$\forall[\mathsf{z}]\ \mathfrak{B}^{\approx}_{\mathsf{z}}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{\mathsf{z}}[\![p]\!] \text{ iff } \models_{\mathcal{K}} \bigwedge \mathfrak{B}^{\approx}_{z}[P] \rightarrow \bigwedge \mathfrak{B}^{\approx}_{z}[\![p]\!], \tag{12.3}$$

which amounts to $\mathcal{K}$'s satisfaction of *finitely* many (in fact, $\mathsf{card}(\mathfrak{B})$) *quasi-identities*; in this case the expression $\forall[\mathsf{z}]\ \mathfrak{B}^{\approx}_{\mathsf{z}}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_{\mathsf{z}}[\![p]\!]$ may be viewed as a *finite schema* of quasi-identities. $\square$

Consequently, since P is finite in (12.2), $\mathcal{K}$'s satisfaction of (12.2) is equivalent to $\mathcal{K}$'s satisfaction of *finitely* many quasi-identities (by Remark 12.13). Note that by definition, $S(\mathcal{K},\mathfrak{B}_*)$ is the propositional (sentential in this case) approximation $\mathcal{D}_{|\mathsf{p}}$ of $D_{\forall_{\mathsf{z}}}(\mathcal{K},\mathfrak{B}_*)$ (see Definition 6.33 on page 230). Hence, by Proposition 6.34 on page 230, since $D_{\forall_{\mathsf{z}}}(\mathcal{K},\mathfrak{B}_*)$ is finitely structural, $S(\mathcal{K},\mathfrak{B}_*)$ is a coarsest sentential calculus finer than $D_{\forall_{\mathsf{z}}}(\mathcal{K},\mathfrak{B}_*)$. So by Lemma 12.9, $S(\mathcal{K},\mathfrak{B}_*)$ is a coarsest sentential calculus finer than $D_{\forall}(\mathcal{K},\mathfrak{B}_*)$. We formalize these arguments for ease of future reference.

Proposition 12.14 $S(\mathcal{K},\mathfrak{B}_*) \preceq D_{\forall}(\mathcal{K},\mathfrak{B}_*) \preceq D_{\forall_{\mathsf{z}}}(\mathcal{K},\mathfrak{B}_*)$. Further, if $\mathcal{S}$ is a sentential calculus with $\mathcal{S} \preceq D_{\forall_{\mathsf{z}}}(\mathcal{K},\mathfrak{B}_*)$, then $\mathcal{S} \preceq S(\mathcal{K},\mathfrak{B}_*)$.

Corollary 12.15 $S(\mathcal{K},\mathfrak{B}_*)$ is the coarsest sentential calculus finer than $D_{\forall_{\mathsf{z}}}(\mathcal{K},\mathfrak{B}_*)$ (resp. $D_{\forall}(\mathcal{K},\mathfrak{B}_*)$). $\square$

We rephrase these results in forms more suitable to our needs in the subsequent chapters.

Proposition 12.16 $S(\mathcal{K}, \mathfrak{B}_*)$ is the coarsest sentential calculus $\mathcal{S}$ satisfying

$$P \vdash_{\mathcal{S}} p \text{ implies } \forall\, [q \in \mathsf{Tm}]\, \mathfrak{B}_q^{\approx}\, [P] \models_{\mathcal{K}} \mathfrak{B}_q^{\approx}[\![p]\!], \tag{12.4}$$

and $S(\mathcal{K}, \mathfrak{B}_*)$ is the coarsest sentential calculus $\mathcal{S}$ satisfying

$$P \vdash_{\mathcal{S}} p \text{ implies } \forall\, [z \in \mathsf{V}]\, \mathfrak{B}_z^{\approx}\, [P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]. \tag{12.5}$$

□

Observe that (12.4) is equivalently rephrased as the requirement that for each term q, the translation $\mathfrak{B}_q$ be *continuous* from $S(\mathcal{K}, \mathfrak{B}_*)$ to $S^2(\Theta^{\mathcal{K}})$; similarly for (12.5).

Corollary 12.17 $S(\mathcal{K}, \mathfrak{B}_*)$ is the coarsest sentential calculus $\mathcal{S}$ such that for each term q (resp. each variable z), $\mathfrak{B}_q$ (resp. $\mathfrak{B}_z$) is c-continuous from $\mathcal{S}$ to $S^2(\Theta^{\mathcal{K}})$. □

We note that the logics $S(\mathcal{K}, \mathfrak{B}_*)$, $D_{\forall_\mathsf{z}}(\mathcal{K}, \mathfrak{B}_*)$ and $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$ all agree with respect to *finite* consequences. More precisely, we have the following.

Corollary 12.18 For *finite* $P \cup \{p\}$, the following conditions are equivalent.

1. $P \vdash_{S(\mathcal{K}, \mathfrak{B}_*)} p$.
2. $\forall[\mathsf{z}]\, \mathfrak{B}_\mathsf{z}^{\approx}\, [P] \models_{\mathcal{K}} \mathfrak{B}_\mathsf{z}^{\approx}[\![p]\!]$.
3. $\forall\, [q \in \mathsf{Tm}]\, \mathfrak{B}_q^{\approx}\, [P] \models_{\mathcal{K}} \mathfrak{B}_q^{\approx}[\![p]\!]$.
4. $P \vdash_{D_{\forall_\mathsf{z}}(\mathcal{K}, \mathfrak{B}_*)} p$.
5. $P \vdash_{D_{\forall}(\mathcal{K}, \mathfrak{B}_*)} p$.

□

So $S(\mathcal{K}, \mathfrak{B}_*)$ is a 'sound' approximation of $D_{\forall_\mathsf{z}}(\mathcal{K}, \mathfrak{B}_*)$ (and $D_{\forall_\mathsf{z}}(\mathcal{K}, \mathfrak{B}_*)$) that 'completely captures' the finite deductions of $D_{\forall_\mathsf{z}}(\mathcal{K}, \mathfrak{B}_*)$ (and $D_{\forall_\mathsf{z}}(\mathcal{K}, \mathfrak{B}_*)$), and is the coarsest sentential calculus to do this. We now show that in general we cannot do any better, since, as demonstrated by the following counter example, $D_{\forall_\mathsf{z}}(\mathcal{K}, \mathfrak{B}_*)$ is generally neither finitary nor structural.

Counter Example 12.19 (Generally, $D_{\forall_\mathsf{z}}(\mathcal{K}, \mathfrak{B}_*)$ is neither Structural nor Finitary.)

> Let $\mathcal{K}$ be any quasivariety and $\boldsymbol{u}$ a unary term that is *not* $\mathcal{K}$-idempotent. Consider the binary system $\boldsymbol{u}(x, y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$.
>
> **Theorem 12.20** $D_{\forall_\mathsf{z}}(\mathcal{K}, \boldsymbol{u}_*)$ is neither finitary nor structural.
>
> *Proof.* Choose any variable $w \in \mathsf{V}$. Then $\mathsf{V} \vdash_{D_{\forall_\mathsf{z}}(\mathcal{K}, \boldsymbol{u}_*)} \boldsymbol{u}(w)$ because, for any $z \in \mathsf{V}$, we have
>
> $$\begin{aligned} \{v \approx \boldsymbol{u}(z) : v \in \mathsf{V}\} &\models_{\mathcal{K}} \{w \approx \boldsymbol{u}(z), z \approx \boldsymbol{u}(z)\} \\ &\models_{\mathcal{K}} \boldsymbol{u}(w) \approx \boldsymbol{u}(\boldsymbol{u}(z)) \approx \boldsymbol{u}(z). \end{aligned}$$
>
> $\boxed{\text{Non-Finitary}}$ To refute finitariness of $D_{\forall_\mathsf{z}}(\mathcal{K}, \boldsymbol{u}_*)$, it therefore suffices to show that there is no finite subset V' of V with $V' \vdash_{D_{\forall_\mathsf{z}}(\mathcal{K}, \boldsymbol{u}_*)} \boldsymbol{u}(w)$. Suppose otherwise. Then, for such V', we

have $V' \cup \{w\} \vdash_{D_{\forall_z}(\mathcal{K},\boldsymbol{u}_*)} \boldsymbol{u}(w)$. We may choose a variable $z \in \mathsf{V} - (V' \cup \{w\})$ and obtain $\{v \approx \boldsymbol{u}(z) : v \in V' \cup \{w\}\} \models_{\mathcal{K}} \boldsymbol{u}(w) \approx \boldsymbol{u}(z)$. Substituting $\boldsymbol{u}(z)$ for all variables in $V' \cup \{w\}$, we get $\models_{\mathcal{K}} \boldsymbol{u}(\boldsymbol{u}(z)) \approx \boldsymbol{u}(z)$, a contradiction. $\boxed{\text{Non-Structural}}$ Let σ be the substitution such that $\sigma(v) = w$ for all $v \in \mathsf{V}$. Since $\sigma[\mathsf{V}] = \{w\}$ and $\sigma(\boldsymbol{u}(w)) = \boldsymbol{u}(w)$, to refute structurality, it is enough to show that $\{\sigma(\boldsymbol{u}(w))\} \nvdash_{D_{\forall_z}(\mathcal{K},\boldsymbol{u}_*)} \boldsymbol{u}(w)$. But since $\{w\}$ is a finite subset of V, we have already established this. Thus, $D_{\forall_z}(\mathcal{K},\boldsymbol{u}_*)$ is neither structural nor finitary. $\diamond$

$\square$

The following characterization of $S(\mathcal{K},\mathfrak{B}_*)$ shall prove useful in the parametrized algebraization of logics pursued in the next three chapters. Observe that the translation $\mathfrak{B}_z^{\approx}$ maps each variable base $\mathfrak{B}_y/\bot_{\mathcal{K}}$ to the pivot $[y \Rsh z]$; in particular, $\mathfrak{B}_z^{\approx}$ maps each variable base $\mathfrak{B}_z/\bot_{\mathcal{K}}$ to the pivot $[z \Rsh z]$. Note that $\models_{\mathcal{K}} [z \Rsh z]$.

Proposition 12.21 $S(\mathcal{K},\mathfrak{B}_*)$ is the coarsest sentential calculus $\mathcal{S}$ such that, for any terms $P \cup p$ and variable z,

$$Z,\ P \vdash_{\mathcal{S}} p \quad \text{implies} \quad \mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!], \tag{12.6}$$

where $Z \subseteq \mathfrak{B}_z/\bot_{\mathcal{K}}$ is any realization of the pivots from z; for example $Z = \mathfrak{B}_z/\bot_{\mathcal{K}}$.

Proof. Suppose that $Z,\ P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$. Then by Proposition 12.16, $\mathfrak{B}_z^{\approx}[Z] \cup \mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]$. Since $Z \subseteq \mathfrak{B}_z/\bot_{\mathcal{K}}$ and $\models_{\mathcal{K}} [z \Rsh z]$, by Remark 9.40 on page 323, $\models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[Z]$; hence $\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]$. Let $\mathcal{S}$ be a sentential calculus satisfying (12.6). Suppose that $P \vdash_{\mathcal{S}} p$. Then certainly $Z \vdash_{\mathcal{S}} p$. So by assumption (12.6), $\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]$. So by Proposition 12.16, $\mathcal{S} \preceq S(\mathcal{K},\mathfrak{B}_*)$. $\diamond$

In the case that the binary system *pivots finitarily*, the implication (12.6) strengthens to an equivalence (in the case that $Z = \mathfrak{B}_z/\bot_{\mathcal{K}}$). Observe that in this case, by prepending the *variable base* to the deduction of the left-hand-side, the need to quantify over *all* variables disappears on the right-hand-side. In Theorem 13.22 on page 399, we shall establish a converse to this result, namely that if $S(\mathcal{K},\mathfrak{B}_*)$ satisfies (12.7), then $\mathfrak{B}_*$ *pivots finitarily* for $\mathcal{K}$.

Proposition 12.22 Suppose that $\mathfrak{B}_*$ *pivots finitarily* for $\mathcal{K}$. Then $S(\mathcal{K},\mathfrak{B}_*)$ is the coarsest sentential 1-calculus $\mathcal{S}$ such that for any set $P \cup \{p\}$ of terms and any variable z,

$$\mathfrak{B}_z/\bot_{\mathcal{K}},\ P \vdash_{\mathcal{S}} p \quad \text{iff} \quad \mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]. \tag{12.7}$$

Proof. $\boxed{\text{(12.7) is valid}}$ $\boxed{\Rightarrow}$ By (12.6). $\boxed{\Leftarrow}$ Suppose $\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]$. Since $\mathfrak{B}_*$ pivots finitarily for $\mathcal{K}$, there exists a *finite* subset $Z \subseteq_{\mathsf{f}} \mathfrak{B}_z/\bot_{\mathcal{K}}$ and a *finite* subset $P' \subseteq_{\mathsf{f}} P$ such that, *for all* variables y, $\models_{\mathcal{K}} \bigwedge\mathfrak{B}_y^{\approx}[Z]\ \mathsf{and} \bigwedge\mathfrak{B}_y^{\approx}[P'] \rightarrow \bigwedge\mathfrak{B}_y^{\approx}[\![p]\!]$, and hence $Z \cup P' \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$, by Corollary 12.18. Since $Z \subseteq_{\mathsf{f}} \mathfrak{B}_z/\bot_{\mathcal{K}}$ and $P' \subseteq P$, $\mathfrak{B}_z/\bot_{\mathcal{K}},\ P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$. $\boxed{\text{Maximality}}$ Follows trivially, since (12.7) implies (12.6). $\diamond$

Formula (12.7) is the key to our theory of *parameterized* algebraization. Observe that if $\mathfrak{B}$ is essentially $\mathcal{K}$-unary, the z drops out of this formula, yielding

$$\mathfrak{B}/\bot_{\mathcal{K}},\ P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p \quad \text{iff} \quad \mathfrak{B}^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}[\![p]\!]. \tag{12.8}$$

In Example 12.60, we shall show that in this case $\mathfrak{B}/\perp_{\mathcal{K}}$ are all $S(\mathcal{K}, \mathfrak{B}_*)$-theorems, and so, in the essentially unary case, formula (12.8) is equivalent to

$$P \vdash_{S(\mathcal{K}, \mathfrak{B}_*)} p \quad \text{iff} \quad \mathfrak{B}^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}[\![p]\!]. \tag{12.9}$$

Replacing $S(\mathcal{K}, \mathfrak{B}_*)$ with an arbitrary sentential 1-calculus $\mathcal{S}$ and $\mathfrak{B}$ with a unary system τ in (12.9), we obtain precisely the definition that $\mathcal{K}$ is an *algebraic semantics* for $\mathcal{S}$ with *defining equations* τ (see Definition 2.110 on page 111). We shall take (12.7) as our definition that $\mathcal{K}$ is a *parameterized algebraic semantics* for $\mathcal{S}$ with *parameterized defining equations* $\mathfrak{B}_*$.

While we do not see how to replace the $\mathfrak{B}_z/\perp_{\mathcal{K}}$ in (12.7) with an arbitrary realization of the pivot from z, as was possible in (12.6), in the case that $\mathfrak{B}_*$ *pivots finitarily* for $\mathcal{K}$ and has *finite* pivots, $\mathfrak{B}_z/\perp_{\mathcal{K}}$ may be replaced in (12.7) by a *finite* realization of the pivot from z. We require the following lemma.

Lemma 12.23 If $\mathfrak{B}_*$ has *finite pivots* then $Z \dashv\vdash_{S(\mathcal{K}, \mathfrak{B}_*)} \mathfrak{B}_z/\perp_{\mathcal{K}}$, where Z is any *finite* realization of the pivot from z.

Proof. Since $Z \subseteq \mathfrak{B}_z/\perp_{\mathcal{K}}$, $\mathfrak{B}_z/\perp_{\mathcal{K}} \vdash_{S(\mathcal{K}, \mathfrak{B}_*)} Z$. Let $p \in \mathfrak{B}_z/\perp_{\mathcal{K}}$. Since for all variables y, $\mathfrak{B}_y^{\approx}[Z] \models_{\mathcal{K}} [z \upharpoonleft y]$, for all variables y, $\mathfrak{B}_y^{\approx}[Z] \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[p]$. So by Corollary 12.18, $Z \vdash_{S(\mathcal{K}, \mathfrak{B}_*)} p$. Hence $Z \vdash_{S(\mathcal{K}, \mathfrak{B}_*)} \mathfrak{B}_z/\perp_{\mathcal{K}}$. ◇

The following strengthening of Proposition 12.22, follows immediately from that proposition together with the previous lemma.

Corollary 12.24 Suppose that $\mathfrak{B}_*$ *pivots finitarily* for $\mathcal{K}$ and has *finite* pivots. Then $S(\mathcal{K}, \mathfrak{B}_*)$ is the coarsest sentential 1-calculus $\mathcal{S}$ such that for any set $P \cup \{p\}$ of terms and any variable z,

$$Z,\, P \vdash_{\mathcal{S}} p \quad \text{iff} \quad \mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!], \tag{12.10}$$

where Z is any *finite* realization of the pivot from z. □

We now consider another characterization of $S(\mathcal{K}, \mathfrak{B}_*)$, in terms of a process that we informally refer to as '**shifting**'.

Definition 12.25 (Freeing Variables by Shifting) Let $\rhd_{\mathsf{v}_0}$ be the substitution mapping $\mathsf{v}_j \mapsto \mathsf{v}_{j+1}$. For each $\mathsf{v}_i \in \mathsf{V}$ with $i > 0$, let $\rhd_{\mathsf{v}_i}$ be the substitution mapping $\mathsf{v}_j \mapsto \mathsf{v}_{j+1}$, for $j < i - 1$, mapping $\mathsf{v}_{i-1} \mapsto \mathsf{v}_{i+1}$, and mapping $\mathsf{v}_j \mapsto \mathsf{v}_{j+2}$, for $j > i$. For a term p and variable x, let $\lhd_x^p$ be the substitution with $\lhd_y^p(\rhd_x(y)) = y$, for all variables other than x, and with $\lhd_x^p(x) = p$. We write $\lhd_x$ for $\lhd_x^x$. □

Remark 12.26 $\lhd_x \rhd_x$ is the identity substitution.

Theorem 12.27 Let z be any variable. Then $S(\mathcal{K}, \mathfrak{B}_*)$ is axiomatized by all rules (and axioms) $P \vdash p$, such that

$$\mathfrak{B}_z^{\approx}[\rhd_z[P]] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\rhd_z(p)]\!]. \tag{12.11}$$

Proof. Let $\mathcal{S}$ be the sentential calculus axiomatized by the rules and axioms mentioned in the statement of this theorem.

$\boxed{\mathcal{S} \preceq S(\mathcal{K}, \mathfrak{B}_*)}$ Let $P \vdash p$ be an $\mathcal{S}$ rule (or axiom). Then $\mathfrak{B}_z^{\approx}[\rhd_z[P]] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\rhd_z(p)]\!]$. Since z does not occur in any of the terms of $\rhd_z[P] \cup \{\rhd_z(p)\}$, $\rhd_z[P] \vdash \rhd_z(p)$ is a $S(\mathcal{K}, \mathfrak{B}_*)$ rule (or axiom) by Remark 12.13. By structurality of $S(\mathcal{K}, \mathfrak{B}_*)$, $\lhd_z[\rhd_z[P]] \vdash_{S(\mathcal{K},\mathfrak{B}_*)} \lhd_z(\rhd_z(p))$, so by Remark 12.26, $P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$. So $\mathcal{S} \preceq S(\mathcal{K}, \mathfrak{B}_*)$ by Proposition 6.31 on page 229. $\boxed{S(\mathcal{K}, \mathfrak{B}_*) \preceq \mathcal{S}}$ Let $P \vdash p$ be an $S(\mathcal{K}, \mathfrak{B}_*)$ rule (or axiom). By structurality, $\rhd_z[P] \vdash_{S(\mathcal{K},\mathfrak{B}_*)} \rhd_z(p)$. So by Corollary 12.15, $\mathfrak{B}_z^{\approx}[\rhd_z[P]] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\rhd_z(p)]\!]$. Hence $P \vdash p$ is an $\mathcal{S}$ rule (or axiom). So $S(\mathcal{K}, \mathfrak{B}_*) \preceq \mathcal{S}$ by Proposition 6.31. $\diamond$

Corollary 12.28 Let z be any variable and $P \cup p \subseteq \mathsf{Tm}$. Then

$$P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p \text{ iff } \mathfrak{B}_z^{\approx}[\rhd_z[P]] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\rhd_z(p)]\!]. \tag{12.12}$$

Proof. $\boxed{\Rightarrow}$ Suppose that $P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$. By structurality, $\rhd_z[P] \vdash_{S(\mathcal{K},\mathfrak{B}_*)} \rhd_z(p)$. So by Corollary 12.15, $\mathfrak{B}_z^{\approx}[\rhd_z[P]] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\rhd_z(p)]\!]$. $\boxed{\Leftarrow}$ Suppose that $\mathfrak{B}_z^{\approx}[\rhd_z[P]] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\rhd_z(p)]\!]$. By finitariness of $\models_{\mathcal{K}}$, there exists a finite subset $I \subseteq \mathfrak{B}_z^{\approx}[\rhd_z[P]]$, such that $I \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\rhd_z(p)]\!]$, and so there exists a finite subset $P' \subseteq_{\mathsf{f}} P$, with $I \subseteq \mathfrak{B}_z^{\approx}[\rhd_z[P']]$, such that $\mathfrak{B}_z^{\approx}[\rhd_z[P']] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\rhd_z(p)]\!]$. So by Theorem 12.27, $P' \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$, and hence $P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$. $\diamond$

Open Problem 12.29 Prove that $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$ is generally neither finitary nor structural. Of course, by Proposition 12.5, $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$ is finitely structural, and so if it is finitary then it must be structural, by Remark 6.15. (Also see Open Problem 9.94 on page 334.)

Open Problem 12.30 When is $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$ structural? Is the structurality in any way related to $\mathfrak{B}_*$ pivoting for $\mathcal{K}$?

Open Problem 12.31 When does $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*) = S(\mathcal{K}, \mathfrak{B}_*)$?

Open Problem 12.32 Does $(S(\mathcal{K}, \mathfrak{B}_*))_{|\mathcal{K}} = S(\mathcal{K}, \mathfrak{B}_*)$?

Open Problem 12.33 Are the logics $D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$ and $D_{\forall}(\mathcal{K}, \mathfrak{B}_*)$ generally distinct? If so, when do they coincide?

12.2 Model Theory

We now turn to model theoretic concerns. The following result shows that the $S(\mathcal{K}, \mathfrak{B}_*)$-filters of $\mathbf{A}$ include all parametric solution sets of $\mathfrak{B}_*$ on $\mathbf{A}$ modulo $\mathcal{K}$. The converse is not generally true [BR99, E 6.1].

Proposition 12.34 Let $\mathbf{A}$ be any $\mathfrak{a}$-algebra and $\mathcal{S}$ any sentential calculus such that $\mathcal{S} \preceq D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$. Then $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \subseteq \mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \subseteq \mathsf{Fi}_{S(\mathcal{K},\mathfrak{B}_*)}(\mathbf{A}) \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$.

Proof. Since $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) = \{\bigcap \mathcal{A} : \emptyset \neq \mathcal{A} \subseteq \mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})\}$ and since filters form a closed system, it suffices to show that $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \subseteq \mathsf{Fi}_{S(\mathcal{K},\mathfrak{B}_*)}(\mathbf{A}) \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$. $\boxed{\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \subseteq \mathsf{Fi}_{S(\mathcal{K},\mathfrak{B}_*)}(\mathbf{A})}$ Let $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a \in \mathsf{uni}(\mathbf{A})$. Suppose that $P' \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$ and that $\mathsf{i}' : \mathbf{Tm} \to \mathbf{A}$ with $\mathsf{i}'[P'] \subseteq \mathfrak{B}^{\mathbf{A}}_a/\alpha$. Now $P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$ for some finite subset P of P'. Let y be a variable not occurring in $P \cup \{t\}$. Then $\mathfrak{B}^{\approx}_y[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_y[\![t]\!]$. Let i be the interpretation mapping $y \mapsto a$ and agreeing with i' on all other variables. Then $\mathsf{i}[P] = \mathsf{i}'[P] \subseteq \mathfrak{B}^{\mathbf{A}}_a/\alpha$, and so $\mathfrak{B}^{\mathbf{A}}_a[\mathsf{i}'[P]] \subseteq \alpha$. Since $\mathbf{A}/\alpha \in \mathcal{K}$, we deduce that $\mathfrak{B}^{\mathbf{A}}_a[\![\mathsf{i}(t)]\!] \subseteq \alpha$, i.e., $\mathsf{i}'(t) = \mathsf{i}(t) \in \mathfrak{B}^{\mathbf{A}}_a/\alpha$. $\boxed{\mathsf{Fi}_{S(\mathcal{K},\mathfrak{B}_*)}(\mathbf{A}) \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})}$ By Corollary 12.15 on page 378, $\mathcal{S} \preceq S(\mathcal{K},\mathfrak{B}_*)$, and hence $\mathsf{Fi}_{S(\mathcal{K},\mathfrak{B}_*)}(\mathbf{A}) \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$ by Proposition 7.18. $\diamond$

Consequently the logics $U_{\forall}(\mathbf{A},\mathcal{K},\mathfrak{B}_*)$ are all models of $S(\mathcal{K},\mathfrak{B}_*)$ (see Remark 7.16 on page 256).

Corollary 12.35 $U_{\forall}(\mathbf{A},\mathcal{K},\mathfrak{B}_*)$ is a model of $S(\mathcal{K},\mathfrak{B}_*)$. $\square$

Since for sentential calculi, filters on the term algebra coincide with theories, the $S(\mathcal{K},\mathfrak{B}_*)$-theories must include all normals (and parameterized solutions) of $\mathfrak{B}_*$ on the term algebra. In particular, all term bases (and hence all variable bases) are theories of $S(\mathcal{K},\mathfrak{B}_*)$.

Corollary 12.36 $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{Tm}) \subseteq \mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{Tm}) \subseteq \mathsf{Th}(S(\mathcal{K},\mathfrak{B}_*))$. $\square$

With respect to the theory of parameterized algebraization, developed in the next three chapters, not all the filters of $S(\mathcal{K},\mathfrak{B}_*)$ are 'well-behaved'. We shall now isolate the 'well-behaved' filters, which we call $\langle\mathcal{K},\mathfrak{B}_b\rangle$-*cosets*. The reader is urged to distinguish this notion of coset, defined for all binary systems $\mathfrak{B}$, from the notion of an *idempotent* $\langle\mathcal{K},\boldsymbol{u}\rangle$-*coset*, introduced in Example 9.63 on page 328. To the end of identifying these 'well-behaved' filters, we first consider a mechanism for 'completely' or 'totally' evaluating a set of terms in an algebra, with a single variable fixed. This tool plays a key role in the theory of 'parametrized algebraization' developed in the later chapters.

Definition 12.37 (Total Evaluation with z Fixed) Suppose P is a set of terms, z a variable, $\mathbf{A}$ an algebra and $b \in \mathsf{uni}(\mathbf{A})$. We define

$$\mathsf{E}^{\mathbf{A}}_{z:b}[P] = \bigcup\{\mathsf{i}[P] : \mathsf{i} \in \mathsf{hom}(\mathbf{Tm},\mathbf{A}) \text{ and } \mathsf{i}(z) = b\}, \tag{12.13}$$

which we call the **total evaluation of P with $z = b$.** $\square$

The total evaluation of a *variable-base* of $\mathfrak{B}_*$ is independent of the variable. More precisely, we have the following.

Lemma 12.38 For any $\mathfrak{a}$-algebra $\mathbf{A}$ and $b \in \mathsf{uni}(\mathbf{A})$, $\mathsf{E}^{\mathbf{A}}_{y:b}[\mathfrak{B}_y/\bot_{\mathcal{K}}] = \mathsf{E}^{\mathbf{A}}_{z:b}[\mathfrak{B}_z/\bot_{\mathcal{K}}]$, for any variables y and z.

Proof. Assume that $y, z \in \mathsf{V}$ are distinct. Let $p \in \mathfrak{B}_y/\bot_{\mathcal{K}}$. We may assume, without loss of generality, that $p = p(y, z, x_0, \ldots, x_{l-1})$ where $\{y, z\} \cap \{x_0, \ldots, x_{l-1}\} = \emptyset$. Let $\mathsf{i} \in \mathsf{hom}(\mathbf{Tm},\mathbf{A})$ with $\mathsf{i}(y) = b$, so $\mathsf{i}(p) = p^{\mathbf{A}}(b, \mathsf{i}(z), \mathsf{i}(x_0), \ldots, \mathsf{i}(x_{l-1}))$. Let σ be the transposition $(y\,z)$. By (2) of Proposition 9.34 on page 322, $p(z, y, \vec{x}) = \sigma(p) \in \mathfrak{B}_z/\bot_{\mathcal{K}}$. Let $\mathsf{i}' \in \mathsf{hom}(\mathbf{Tm},\mathbf{A})$ be the homomorphism determined by $\mathsf{i}'(z) = b$, $\mathsf{i}'(y) = \mathsf{i}(z)$ and $\mathsf{i}'(v) = \mathsf{i}(v)$ for all $v \in \mathsf{V} - \{z, y\}$. Then $\mathsf{i}(p) = p^{\mathbf{A}}(b, \mathsf{i}(z), \mathsf{i}(x_0), \ldots, \mathsf{i}(x_{l-1})) = \mathsf{i}'(\sigma(p)) \in$

$\mathsf{E}^{\mathbf{A}}_{z:b}[\mathfrak{B}_z/\bot_{\mathcal{K}}]$. Thus, $\mathsf{E}^{\mathbf{A}}_{y:b}[\mathfrak{B}_y/\bot_{\mathcal{K}}] \subseteq \mathsf{E}^{\mathbf{A}}_{z:b}[\mathfrak{B}_z/\bot_{\mathcal{K}}]$. The reverse inclusion follows by symmetry. $\diamond$

Since the total evaluation of a variable-base is independent of the variable, we can identify these evaluations without reference to a variable.

Definition 12.39 (Pivotal Points) For any $\mathfrak{a}$-algebra $\mathbf{A}$ and $b \in \mathsf{uni}(\mathbf{A})$, we define $\mathsf{Pv}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) = \mathsf{E}^{\mathbf{A}}_{z:b}[\mathfrak{B}_z/\bot_{\mathcal{K}}]$, where z is any variable, the definition being independent of the particular choice of variable by the previous lemma. We call $\mathsf{Pv}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ the *pivotal points of* $\mathfrak{B}_*$ *at* b *modulo* $\mathcal{K}$ (or simply the **pivitals at** b where unambiguous). $\square$

For our purposes, the 'best behaved' $S(\mathcal{K}, \mathfrak{B}_*)$-filters of an algebra $\mathbf{A}$ are those containing the pivitals at some point b.

Definition 12.40 ($\langle \mathcal{K}, \mathfrak{B}_* \rangle$-Cosets) Let $b \in \mathsf{uni}(\mathbf{A})$ be given. We define the set of **$\langle \mathcal{K}, \mathfrak{B}_b \rangle$-cosets of $\mathbf{A}$** to be

$$\mathsf{Cos}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) \doteq \mathsf{Fi}_{S(\mathcal{K},\mathfrak{B}_*)}(\langle \mathbf{A}, \mathsf{Pv}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) \rangle) = \{F \in \mathsf{Fi}_{S(\mathcal{K},\mathfrak{B}_*)}(\mathbf{A}) \,:\, \mathsf{Pv}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) \subseteq F\}.$$

The corresponding algebraic lattice $\mathbf{Fi}_{S(\mathcal{K},\mathfrak{B}_*)}(\langle \mathbf{A}, \mathsf{Pv}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) \rangle)$ is then written as $\mathbf{Cos}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$. Let $\mathsf{Cos}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) = \bigcup_{b \in \mathsf{uni}(\mathbf{A})} \mathsf{Cos}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$, the members of which are called the **$\langle \mathcal{K}, \mathfrak{B}_* \rangle$-cosets of $\mathbf{A}$**. $\square$

Recall that by Proposition 12.34, all the parameterized solutions $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ are $S(\mathcal{K}, \mathfrak{B}_*)$-filters on $\mathbf{A}$. We shall now show that in restricting focus to those $S(\mathcal{K}, \mathfrak{B}_*)$-filters that are $\mathfrak{B}_*$-cosets, we still encompass all parameterized solutions.

Proposition 12.41 For $\mathfrak{a}$-algebra $\mathbf{A}$ and $b \in \mathsf{uni}(\mathbf{A})$, $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) \subseteq \mathsf{Cos}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$. Consequently, $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \subseteq \mathsf{Cos}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$.

Proof. Let z be any variable. (*By Proposition 12.34, it suffices to show that* $\mathsf{E}^{\mathbf{A}}_{z:b}[\mathfrak{B}_z/\bot_{\mathcal{K}}] \subseteq \mathfrak{B}^{\mathbf{A}}_b/\alpha$.) Let $\mathsf{i} \in \mathsf{hom}(\mathbf{Tm}, \mathbf{A})$ with $\mathsf{i}(z) = b$. Suppose $p \in \mathfrak{B}_z/\bot_{\mathcal{K}}$. Then for each $\langle \delta, \epsilon \rangle \in \mathfrak{B}$, $\mathcal{K}$ satisfies $\delta(p, z) \approx \varepsilon(p, z)$ so, by Lemma 1.457 on page 88, $\langle \delta^{\mathbf{A}}(\mathsf{i}(p), b), \varepsilon^{\mathbf{A}}(\mathsf{i}(p), b) \rangle \in \|\emptyset\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$. Thus, $\mathsf{i}(p) \in \mathfrak{B}^{\mathbf{A}}_b/\bot^{\mathcal{K}}_{\mathbf{A}}$. This shows that $\mathsf{i}[\mathfrak{B}_z/\bot_{\mathcal{K}}] \subseteq \mathfrak{B}^{\mathbf{A}}_b/\bot^{\mathcal{K}}_{\mathbf{A}} \subseteq \mathfrak{B}^{\mathbf{A}}_b/\alpha$. $\diamond$

12.3 Examples

We now identify some interesting cases. We shall see in the next three chapters, that our theory of parameterized algebraization may only be applied to a logic $S(\mathcal{K}, \mathfrak{B}_*)$ if $\mathfrak{B}_*$ *pivots finitarily* in $\mathcal{K}$. Our most general family of binary systems with this property are the *separable* binary systems (see Example 9.52 on page 326).

Example 12.42 (The Logics of Separable Binary Systems)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras.

Definition 12.43 (The Logics of Separable Binary Systems) The sentential 1-calculus $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are n-indexed families of unary terms, is the logic $S(\mathcal{K}, \mathfrak{B}_*)$, where $\mathfrak{B}(x, y) = \{\langle \boldsymbol{u}_i(x), \boldsymbol{v}_i(y) \rangle : i \in n\}$, and is called the **separable**

logic determined by $\boldsymbol{U}$, $\boldsymbol{V}$ and $\mathcal{K}$, or just the **separable logic** where unambiguous. The separable logic $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$ is called **trivial** if $\mathfrak{B}(x,y) = \{\langle \boldsymbol{u}_i(x), \boldsymbol{v}_i(y)\rangle : i \in n\}$ has trivial variable roots, otherwise it is called **non-trivial**. $\square$

Recall that if a separable system $\mathfrak{B}$ is non-trivial, then it has *finite pivots*, realized by any singleton $\{q\}$ where $q \in \mathfrak{B}_z/\bot_{\mathcal{K}}$, i.e., $\models_{\mathcal{K}} \bigwedge_{i\in n} \boldsymbol{u}_i(q) \approx \boldsymbol{u}'_i(z)$, *symmetric pivots* and $\mathfrak{B}_*$ *pivots finitarily* in $\mathcal{K}$ (see Corollary 9.57 on page 326). Consequently, the following characterization of $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$ follows at once from Corollary 12.24.

Corollary 12.44 Suppose that $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$ is nontrivial and $q \in \mathfrak{B}_z/\bot_{\mathcal{K}}$. Then $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$ is the coarsest sentential 1-calculus $\mathcal{S}$ such that for any set $P \cup \{p\}$ of terms and any variable z,

$$q,\ P \vdash_{\mathcal{S}} p \quad \text{iff} \quad \mathfrak{B}^{\approx}_z[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]. \tag{12.14}$$

Remark 12.45 If $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$ is nontrivial and $q \in \mathfrak{B}_z/\bot_{\mathcal{K}}$, then $q \dashv\vdash_{S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)} \mathfrak{B}_z/\bot_{\mathcal{K}}$.

$\square$

While separable logics may be trivial, one situation where this is never the case occurs when $\boldsymbol{U} = \{x\}$ and $\boldsymbol{V} = \{\boldsymbol{u}(y)\}$, where $\boldsymbol{u}$ is $\mathcal{K}$-unary. We now identify these logics, which we call the logics of *identified membership*. This logic is tightly coupled with the notion of the relative congruences classes of an algebra $\mathbf{A}$ at points $\boldsymbol{u}^{\mathbf{A}}(a)$, although this relationship is not as strong as it is in the case that $\boldsymbol{u}$ is $\mathcal{K}$-idempotent.

Example 12.46 (The Logics $S(\mathcal{K}, \boldsymbol{u})$ of Identified Membership)

Let $\boldsymbol{u}$ be a unary term, *not* necessarily idempotent over quasivariety $\mathcal{K}$. Let $\boldsymbol{u}(x,y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$.

Definition 12.47 (The Logics of Identified Membership) The logic $S(\mathcal{K}, \boldsymbol{u}_*)$, is denoted by $S(\mathcal{K}, \boldsymbol{u})$ and is called the **logic of identified membership** (determined by $\mathcal{K}$ and $\boldsymbol{u}$) or just the **$\boldsymbol{u}$-membership logic** (determined by $\mathcal{K}$). We refer to $\langle \mathcal{K}, \boldsymbol{u}_*\rangle$-cosets as **$\langle \mathcal{K}, \boldsymbol{u}\rangle$-cosets**. $\square$

There is a potential clash between our notation '$S(\mathcal{K}, 0)$', denoting the *assertional logic*, of Example 2.92 on page 107, and our notation '$S(\mathcal{K}, \boldsymbol{u})$', denoting the logic of identified membership, in the case that $\boldsymbol{u}$ is equal to a constant 0. This ambiguity is avoided as these logics will be shown to be equivalent (see Theorem 12.54 of Example 12.53).

Warning 12.48 The notion of a *$\langle \mathcal{K}, \boldsymbol{u}\rangle$-coset* is distinct from that of an *idempotent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-coset*, the former being defined for any unary term $\boldsymbol{u}$, while the latter is only defined for those unary terms idempotent over $\mathcal{K}$ (see Example 9.63 on page 328).

Clearly the logics of identified membership are *separable logics*, and as separable logics they are *non-trivial*, by Remark 9.59 on page 327. By definition, $S(\mathcal{K}, \boldsymbol{u})$ is axiomatized by all inference-rules (and axioms) $P \vdash p$, such that $\forall[\mathsf{z}]\ P \approx \boldsymbol{u}(\mathsf{z}) \models_{\mathcal{K}} p \approx \boldsymbol{u}(\mathsf{z})$.

Observe that the $\mathcal{K}$-constantness of $\boldsymbol{u}$ is discernible in the associated logic of identified membership.

Proposition 12.49 The following conditions are equivalent.

1. $S(\mathcal{K}, \boldsymbol{u})$ has at least one theorem.
2. $\boldsymbol{u}$ is a constant term over $\mathcal{K}$.

□

Since $\boldsymbol{u}(z) \in \boldsymbol{u}_z/\bot_{\mathcal{K}}$, the following result obtains immediately from Remark 12.45, Corollary 12.44, Proposition 12.16 and Corollary 12.18.

Corollary 12.50 For unary term $\boldsymbol{u}$, the following are all valid.

1. $\boldsymbol{u}(z) \dashv\vdash_{S(\mathcal{K},\boldsymbol{u})} \boldsymbol{u}_z/\bot_{\mathcal{K}}$.
2. $S(\mathcal{K}, \boldsymbol{u})$ is the coarsest sentential 1-calculus $\mathcal{S}$ such that for any set $P \cup \{p\}$ of terms and any variable z,
$$\boldsymbol{u}(z), P \vdash_{\mathcal{S}} p \text{ iff } P \approx \boldsymbol{u}(z) \models_{\mathcal{K}} p \approx \boldsymbol{u}(z). \tag{12.15}$$
3. $S(\mathcal{K}, \boldsymbol{u})$ is the coarsest sentential 1-calculus $\mathcal{S}$ such that for any set $P \cup \{p\}$ of terms
$$P \vdash_{S(\mathcal{K},\boldsymbol{u})} p \text{ implies } \forall[\mathsf{z}]\ P \approx \boldsymbol{u}(z) \models_{\mathcal{K}} p \approx \boldsymbol{u}(z). \tag{12.16}$$
4. For *finite* $P \cup \{t\} \subseteq \mathsf{Tm}$, we have
$$P \vdash_{S(\mathcal{K},\boldsymbol{u})} p \text{ iff } \forall[\mathsf{z}] \models_{\mathcal{K}} \left[\bigwedge\nolimits_{q\in P} q \approx \boldsymbol{u}(z)\right] \to p \approx \boldsymbol{u}(z). \tag{12.17}$$

□

We now consider the $\langle\mathcal{K}, \boldsymbol{u}\rangle$-cosets. The following result follows by Proposition 12.41 and (4) of Remark 9.59 on page 327.

Corollary 12.51 For an $\mathfrak{a}$-algebra $\mathbf{A}$, $\{\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!] \,:\, \alpha \in \mathsf{Con}_{\mathcal{K}}(\mathbf{A})\} \subseteq \mathsf{Cos}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A})$.

Remark 12.52 The following are valid.

1. The $\langle\mathcal{K}, \boldsymbol{u}\rangle$-cosets of an algebra $\mathbf{A}$ determined by b are just the $S(\mathcal{K}, \boldsymbol{u})$-filters of $\mathbf{A}$ that contain $p^{\mathbf{A}}(b_0, \ldots, b_{n-1}, b)$, whenever $n \in \omega$, $\models_{\mathcal{K}} p(x_0, \ldots, x_n) \approx \boldsymbol{u}(x_n)$ and $\vec{b} \in \mathbf{A}$.
2. When $\mathbf{A}$ belongs to the variety generated by $\mathcal{K}$, therefore, these are just the $S(\mathcal{K}, \boldsymbol{u})$-filters of $\mathbf{A}$ containing $\boldsymbol{u}^{\mathbf{A}}(a)$, so a $\langle\mathcal{K}, \boldsymbol{u}\rangle$-coset of $\mathbf{A}$ is just an $S(\mathcal{K}, \boldsymbol{u})$-filter of $\mathbf{A}$ not disjoint from $\boldsymbol{u}^{\mathbf{A}}[\mathsf{uni}(\mathbf{A})]$.

□

Recall the definition of the logics of *idempotent $\boldsymbol{u}$-cosets* given in Example 9.63. We shall now show that the *logics of identified membership* encompass the *sentential calculus of idempotent $\boldsymbol{u}$-cosets*, and hence the assertional logics of [BR99].

Example 12.53 (The Idempotent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-Coset Logics)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras, $\boldsymbol{u}$ a unary term *idempotent* over $\mathcal{K}$ and $\mathbf{A}$ be an $\mathfrak{a}$-algebra, not necessarily in $\mathcal{K}$. Let $\boldsymbol{u}(x, y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$.

The following result relates the logics $U_{\forall}(\mathbf{A}, \mathcal{K}, \boldsymbol{u}_*)$ to the logics $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$, $U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ and $U_{\mathsf{c}}(\mathbf{A}, 0\text{-}\mathsf{cos}^{\mathcal{K}})$. Since we have opted to 'force' the empty-set to be a $\boldsymbol{u}$-coset in the case that $\boldsymbol{u}$ is not a $\mathcal{K}$-constant, we need to take care with respect to the empty-set, in this case. This is *not* a problem when the algebra is the term algebra or a $\mathcal{K}$-free algebra, since the empty-set is a $D_\forall(\mathcal{K}, \boldsymbol{u}_*)$-theory and a $\boldsymbol{D}_\forall(\mathcal{K}, \boldsymbol{u}_*)$-theory. This issue is of no concern when 0 is a $\mathcal{K}$-constant. Recall the definition of the essentially unary binary translation $\mathbf{0}(x)$ associated with a $\mathcal{K}$-constant 0 (see Example 9.63 on page 328).

Theorem 12.54 1. For unary term $\boldsymbol{u}$ *idempotent* over $\mathcal{K}$, $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}}) = D_\forall(\mathcal{K}, \boldsymbol{u}_*) = D_{\forall_{\mathsf{z}}}(\mathcal{K}, \boldsymbol{u}_*) = S(\mathcal{K}, \boldsymbol{u})$, $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}}) = \boldsymbol{D}_\forall(\mathcal{K}, \boldsymbol{u}_*)$ and $\mathsf{Th}(U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})) = \mathsf{Th}(U_\forall(\mathbf{A}, \mathcal{K}, \boldsymbol{u}_*)) \cup \{\emptyset\}$.

2. For $\mathcal{K}$-constant 0, $S_{\mathsf{c}}(0\text{-}\mathsf{cos}^{\mathcal{K}}) = D_\forall(\mathcal{K}, \mathbf{0}_*) = D_{\forall_{\mathsf{z}}}(\mathcal{K}, \mathbf{0}_*) = S(\mathcal{K}, 0)$, $\boldsymbol{S}_{\mathsf{c}}(0\text{-}\mathsf{cos}^{\mathcal{K}}) = \boldsymbol{D}_\forall(\mathcal{K}, \mathbf{0}_*)$ and $U_{\mathsf{c}}(\mathbf{A}, 0\text{-}\mathsf{cos}^{\mathcal{K}}) = U_\forall(\mathbf{A}, \mathcal{K}, \mathbf{0}_*)$.

Proof. $\boxed{(1)}$ By definitions and Corollary 9.71 on page 329, $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}}) = D_\forall(\mathcal{K}, \boldsymbol{u}_*)$, $\boldsymbol{S}_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}}) = \boldsymbol{D}_\forall(\mathcal{K}, \boldsymbol{u}_*)$ and $\mathsf{Th}(U_{\mathsf{i}}(\mathbf{A}, \boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})) = \mathsf{Th}(U_\forall(\mathbf{A}, \mathcal{K}, \boldsymbol{u}_*)) \cup \{\emptyset\}$. So by Corollary 12.10, $D_\forall(\mathcal{K}, \boldsymbol{u}_*) = D_{\forall_{\mathsf{z}}}(\mathcal{K}, \boldsymbol{u}_*)$, and by Corollary 12.15, $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}}) = S(\mathcal{K}, \boldsymbol{u})$. $\boxed{(2)}$ Follows immediately from definitions and (1). ◇

In particular, the *assertional logic* $S(\mathcal{K}, 0)$ and the 0-*membership logic* $S(\mathcal{K}, 0)$ coincide, and hence ambiguity is avoided.

The following result, relating *idempotent* $\langle\mathcal{K}, \boldsymbol{u}\rangle$-*cosets* and $\langle\mathcal{K}, \boldsymbol{u}\rangle$-*cosets*, follows from definitions, the previous theorem and Proposition 12.34.

Corollary 12.55 For any algebra $\mathbf{A}$, not necessarily in $\mathcal{K}$, every *non-empty idempotent* $\langle\mathcal{K}, \boldsymbol{u}\rangle$-*coset* is a $\langle\mathcal{K}, \boldsymbol{u}\rangle$-*coset.*

□

We briefly consider the special case of the membership logic.

Example 12.56 (The Membership Logic)

Let $\mathcal{K}$ be a quasivariety of $\mathfrak{a}$-algebras. Since the membership logic $S(\mathcal{K}, \mathsf{mem})$ coincides with the idempotent $\langle\mathcal{K}, y\rangle$-coset logic $S_{\mathsf{i}}(y\text{-}\mathsf{cos}^{\mathcal{K}})$ for any variable y (see Example 9.63 on page 328), it follows from Theorem 12.54, that the membership logic is equivalent to the logic $S(\mathcal{K}, y)$ of identified membership, for any variable y.

Corollary 12.57 $S(\mathcal{K}, y) = S(\mathcal{K}, \mathsf{mem})$. □

In the case of the membership logics, we can strengthen the condition of *coarsestness* in (2) of Corollary 12.50 to one of *uniqueness* in the case that the quasivariety $\mathcal{K}$ is non-trivial.

Theorem 12.58 Let $\mathcal{K}$ be a quasivariety. The membership logic $S(\mathcal{K}, \mathsf{mem})$ is the *coarsest* sentential 1-calculus $\mathcal{S}$ such that for any set $P \cup \{t\}$ of terms and any variable z,

$$z, P \vdash_{\mathcal{S}} t \;\text{ iff }\; P \approx z \models_{\mathcal{K}} t \approx z. \tag{12.18}$$

Further, if $\mathcal{K}$ is a *nontrivial* quasivariety, then the membership logic $S(\mathcal{K}, \mathsf{mem})$ is the *unique* sentential 1-calculus satisfying (12.18).

Proof. That $\mathcal{S}$ and $\mathcal{K}$ satisfy (12.18) follows from (2) of Corollary 12.50. Suppose a sentential calculus $\mathcal{S}$ satisfies (12.18) (for all P, t, z). If $P \vdash_{\mathcal{S}} p$ then, for all $z \in V$, we have $z, P \vdash_{\mathcal{S}} p$, whence $P \approx z \models_{\mathcal{K}} t \approx z$. By (2) of Corollary 12.50, $S(\mathcal{K}, \mathsf{mem})$ is the largest sentential calculus with this property, so $\mathcal{S} \subseteq S(\mathcal{K}, \mathsf{mem})$. It remains only to prove that $S(\mathcal{K}, \mathsf{mem}) \subseteq \mathcal{S}$.

Suppose $P' \vdash_{S(\mathcal{K},\mathsf{mem})} p$ and choose a finite $P \subseteq P'$ such that $P \vdash_{S(\mathcal{K},\mathsf{mem})} p$. Note that $P \neq \emptyset$, as $\mathcal{K}$ is nontrivial. Let z be a variable not occurring in $P \cup \{t\}$. Now $P \approx z \models_{\mathcal{K}} t \approx z$ (by definition of $S(\mathcal{K}, \mathsf{mem})$), so $z, P \vdash_{\mathcal{S}} p$. Choosing any $g \in P$ and substituting g for z, we obtain $P \vdash_{\mathcal{S}} p$, whence $P' \vdash_{\mathcal{S}} p$. $\diamond$

The following results follow immediately from Proposition 9.86 on page 333 and Corollary 12.55.

Remark 12.59 The $\langle \mathcal{K}, y\rangle$-cosets of $\mathbf{A}$ are just the *nonempty* $S(\mathcal{K}, \mathfrak{B}_*)$-filters of $\mathbf{A}$.

A protoalgebraic sentential calculus (in the sense of [BP86]) always has theorems, unless it is the almost trivial calculus for which $x \vdash y$ (see, e.g. [BP92, Theorem 13.2]). It follows from Proposition 12.49, that $S(\mathcal{K}, \boldsymbol{u})$ is *not* protoalgebraic unless $\boldsymbol{u}$ is constant over $\mathcal{K}$. Thus, only a trivial quasivariety can have a protoalgebraic membership logic. Consequently the membership logic of (non-trivial) quasivarieties $\mathcal{K}$, can *never* have an *equivalent algebraic semantics* and so can *never be algebraizable* in the sense of [BP89a]. This was the primary motivation for our development of the theory of *parameterized algebraization*. We shall show that a quasivariety is relatively congruence *regular* iff its *membership logic is* $\langle\{z\}, z\rangle$*-algebraizable*, in which case this quasivariety is the (unique) $\langle\{z\}, z\rangle$*-equivalent algebraic semantics* for its membership logic; this reflects the fact that a quasivariety is relatively *point regular* iff its *assertional logic is algebraizable* (in which case this quasivariety is the (unique) equivalent algebraic semantics its assertional logic). For example, the *fully congruence regular* variety of quasigroups (see Counter Example 3.1 on page 122) is the (unique) $\langle\{z\}, z\rangle$-equivalent algebraic semantics for its membership logic. Recall that by the aforementioned counter example, the variety of quasigroups cannot be the equivalent algebraic semantics for any sentential 1-calculus.

Despite this and its 1-deductive character, the membership logic of a quasivariety $\mathcal{K}$ interprets much of the (2-deductive) equational consequence relation $\models_{\mathcal{K}}$, and *all* of $\models_{\mathcal{K}}$ when $\mathcal{K}$ is a variety. Indeed, identities $s \approx p$ of a quasivariety $\mathcal{K}$ correspond exactly to inferences $s \vdash p$ of $S(\mathcal{K}, \mathsf{mem})$, and the quasi-equational theory of a *variety* is reducible (e.g., via Mal'cev's Lemma) to such identities. In particular, any Mal'cev condition applicable to a variety is discernible in its membership logic. (Congruence permutability of $\mathcal{K}$, for instance, is the existence of a ternary term p for which $y \vdash_{S(\mathcal{K},\mathsf{mem})} t(x, x, y), t(y, x, x)$.)

We shall see later that in a quasivariety $\mathcal{K}$ for which $\mathcal{K}$-congruences are fully regular (e.g., the variety of quasigroups), the nonempty $S(\mathcal{K}, \mathsf{mem})$-filters of any algebra $\mathbf{A}$ are precisely the $\mathcal{K}$-congruence classes of $\mathbf{A}$ (see Corollary 15.20 on page 428). It can be shown that distinct quasivarieties of this kind have distinct membership logics.

$\square$

Recall Example 2.92 on page 107, where we defined the sentential 1-calculus $S(\mathcal{K}, \tau)$ [BR99], determined by a unary system of equations $\tau(x)$ and a quasivariety $\mathcal{K}$. In the next example, we shall show that the logics $S(\mathcal{K}, \mathfrak{B}_*)$ encompass the logics $S(\mathcal{K}, \tau)$ of [BR99]. Note that the logics $S(\mathcal{K}, \tau)$ encompass the *assertional logics*, which we have already shown to by encompassed by the logics of identified membership.

Example 12.60 ($S(\mathcal{K}, \tau)$)

We may consider a unary system τ as a $\mathcal{K}$-unary binary system $\tau(x,y)$ (see Example 9.97 on page 334). In this case,

$$\forall[\mathsf{z}]\ \tau_{\mathsf{z}}^{\approx}[P] \models_{\mathcal{K}} \tau_{\mathsf{z}}^{\approx}[\![p]\!] \text{ iff } \tau^{\approx}[P] \models_{\mathcal{K}} \tau^{\approx}[\![p]\!]. \tag{12.19}$$

Consequently, $S(\mathcal{K},\tau) \equiv S(\mathcal{K},\tau_*)$. Conversely, suppose that $\mathfrak{B}$ is essentially $\mathcal{K}$-unary, in the sense of Example 9.97. Then by definition, for each $\langle\delta,\epsilon\rangle \in \mathfrak{B}$, there exist unary $\delta', \varepsilon' \in \mathsf{Tm}$ such that $\mathcal{K}$ satisfies $\delta(x,y) \approx \delta'(x)$ and $\varepsilon(x,y) \approx \varepsilon'(x)$. Then $\tau = \{\langle\delta',\epsilon'\rangle : \langle\delta,\epsilon\rangle \in \mathfrak{B}\}$ is a unary system of equations and $S(\mathcal{K},\tau) \equiv S(\mathcal{K},\mathfrak{B}_*)$. Further, these constructions are mutually inverse.

Let $\mathfrak{B}$ be a binary system of equations that is *essentially $\mathcal{K}$-unary*, and let $\mathfrak{B}'$ be a unary realization of $\mathfrak{B}$ in $\mathcal{K}$. For any algebra $\mathbf{A}$, any $a \in \mathsf{uni}(\mathbf{A})$ and any $\mathcal{K}$-congruence α of $\mathbf{A}$, we have $\mathfrak{B}'^{\mathbf{A}}_a/\alpha = \mathfrak{B}'^{\mathbf{A}}/\alpha$. In this case, all of $\mathbf{A}$'s $S(\mathcal{K},\mathfrak{B}_*)$- (i.e., $S(\mathcal{K},\mathfrak{B}')$-) filters are $\langle\mathcal{K},\mathfrak{B}_*\rangle$-cosets (in [BR99] these are called the $\mathcal{K},\mathfrak{B}$**-ideals** of $\mathbf{A}$).

□

We now show how the sentential calculi $S_*(\mathcal{K},\mathsf{id})$, $S_*(\mathcal{K},\mathsf{fi})$, $S_0(\mathcal{K},\mathsf{id})$ and $S_1(\mathcal{K},\mathsf{fi})$ of lattice ideals and filters (introduced in Example 6.98 on page 246 and further characterized in Example 8.66 on page 296) arise as logics of parameterized solutions to binary equations.

Example 12.61 (Logics of Ideals, Filters and Convexities of Lattices)

Recall Example 9.102 on page 335, where we defined the binary systems $\triangle(x,y) = \{\langle x \vee y, y\rangle\}$ and $\triangledown(x,y) = \{\langle x \wedge y, y\rangle\}$. We now aim to show that for a quasivariety $\mathcal{K}$ of lower unbounded lattice expansions the sentential calculus $S_*(\mathcal{K},\mathsf{id})$ is equivalent to the sentential calculus $S(\mathcal{K},\triangle_*)$. We begin by showing that these logics agree on all deductions from non-empty sets of formulae.

Theorem 12.62 Let $\mathcal{K}$ be a quasivariety of lower unbounded lattice expansions, $P \cup \{p\} \subseteq \mathsf{Tm}$ with $P \neq \emptyset$ and let z be any variable. The following conditions are equivalent.

1. $P \vdash_{S_*(\mathcal{K},\mathsf{id})} p$.
2. $\rhd_z[P] \leq z \models_{\mathcal{K}} \rhd_z(p) \leq z$.
3. $\triangle_z^{\approx}[\rhd_z[P]] \models_{\mathcal{K}} \triangle_z^{\approx}[\![\rhd_z(p)]\!]$.
4. $P \vdash_{S(\mathcal{K},\triangle_*)} p$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Suppose that $P \vdash_{S_*(\mathcal{K},\mathsf{id})} p$. By finitariness, there exists $\emptyset \neq \{p_1,\ldots,p_n\} \subseteq_{\mathsf{f}} P$ with $\{p_1,\ldots,p_n\} \vdash_{S_*(\mathcal{K},\mathsf{id})} p$. By Theorem 8.75 of Example 8.66 on page 296, $\{\overline{p_1},\ldots,\overline{p_n}\} \vdash_{\boldsymbol{S}_*(\mathcal{K},\mathsf{id})} \overline{p}$. Hence $\overline{p} \in \|\{\overline{p_1},\ldots,\overline{p_n}\}\|^{\mathbf{F}_{\mathcal{K}}}_{\mathsf{fi}_\diamond}$. So $\overline{p} \leq^{\mathbf{F}_{\mathcal{K}}} \overline{p_1} \vee^{\mathbf{F}_{\mathcal{K}}} \ldots \vee^{\mathbf{F}_{\mathcal{K}}} \overline{p_n}$. Hence by Lemma 1.457 on page 88,

$$\models_{\mathcal{K}} p \leq p_1 \vee \ldots \vee p_n.$$

By $\models_{\mathcal{K}}$ structurality,

$$\models_{\mathcal{K}} \rhd_z(p) \leq \rhd_z(p_1) \vee \ldots \vee \rhd_z(p_n). \tag{i}$$

Since it is generally true that

$$\rhd_z(p_1) \leq z, \ldots, \rhd_z(p_n) \leq z \models_{\mathcal{K}} \rhd_z(p_1) \vee \ldots \vee \rhd_z(p_n) \leq z \tag{ii}$$

it follows form (ii) and (i) that $\rhd_z(p_1) \leq z, \ldots, \rhd_z(p_n) \leq z \models_{\mathcal{K}} \rhd_z(p) \leq z$. $\boxed{(2)\Rightarrow(1)}$ Suppose that $\rhd_z[P] \leq z \models_{\mathcal{K}} \rhd_z(p) \leq z$. By finitariness of $\models_{\mathcal{K}}$, for some finite $\{p_1,\ldots,p_n\} \subseteq P$,

$$\rhd_z[\{p_1,\ldots,p_n\}] \leq z \models_{\mathcal{K}} \rhd_z(p) \leq z.$$

Since z does not occur in any of the terms $\rhd_z(p_1), \ldots, \rhd_z(p_n), \rhd_z(p)$, by structurality of $\models_{\mathcal{K}}$,

$$\rhd_z[\{p_1, \ldots, p_n\}] \leq (\rhd_z(p_1) \vee \ldots \vee \rhd_z(p_n)) \models_{\mathcal{K}} \rhd_z(p) \leq (\rhd_z(p_1) \vee \ldots \vee \rhd_z(p_n)) \,. \qquad \text{(i)}$$

Applying the **Tm** endomorphism $\lhd_z$ to (i), by structurality of $\models_{\mathcal{K}}$, we obtain

$$\{p_1, \ldots, p_n\} \leq p_1 \vee \ldots \vee p_n \models_{\mathcal{K}} p \leq p_1 \vee \ldots \vee p_n. \qquad \text{(ii)}$$

Since the left-hand-side of (ii) is always true, we have

$$\models_{\mathcal{K}} p \leq p_1 \vee \ldots \vee p_n. \qquad \text{(iii)}$$

So by (iii) and (6.15) of Definition 6.110 on page 248, $p_1, \ldots, p_n \vdash p$ is a $S_*(\mathcal{K}, \mathsf{id})$-rule, and hence $\{p_1, \ldots, p_n\} \vdash_{S_*(\mathcal{K},\mathsf{id})} p$. So $P \vdash_{S_*(\mathcal{K},\mathsf{id})} p$. $\boxed{(2)\Leftrightarrow(3)}$ By Remark 9.104 on page 335. $\boxed{(3)\Leftrightarrow(4)}$ By Corollary 12.28. $\diamond$

We now show that the sentential calculus $S_*(\mathcal{K}, \mathsf{id})$ is equivalent to the sentential calculus $S(\mathcal{K}, \triangle_*)$. In the light of the previous theorem and the fact that $S_*(\mathcal{K}, \mathsf{id})$ has no theorems (see Definition 6.110 on page 248), it suffices to prove that $S(\mathcal{K}, \triangle_*)$ has no theorems.

Corollary 12.63 $S_*(\mathcal{K}, \mathsf{id}) \equiv S(\mathcal{K}, \triangle_*)$ for any quasivariety $\mathcal{K}$ of lower-unbounded lattice expansions.

Proof. By Theorem 12.62, it suffices to show that $S(\mathcal{K}, \triangle_*)$ has no theorems. Suppose, to the contrary, that $\vdash_{S(\mathcal{K},\triangle_*)} p$. Let z be a variable not occurring in p. Then by Corollary 12.28, $\models_{\mathcal{K}} \triangle_z^{\approx}[\![\rhd_z(p)]\!]$, i.e., $\models_{\mathcal{K}} \rhd_z(p) \leq z$. Since the variable z does not occur in the term p, $\models_{\mathcal{K}} p \leq z$. Let $x_1, \ldots, x_n$ be the variables occurring in p, $\mathbf{P} \in \mathcal{K}$ and $\vec{a} \in \mathsf{uni}(\mathbf{P})$. For any $b \in \mathsf{uni}(\mathbf{P})$, $p^{\mathbf{P}}(\vec{a}) \leq b$. So every member of $\mathcal{K}$ is lower-bounded, which contradicts the assumption that $\mathcal{K}$ be a quasivariety of lower-unbounded lattice expansions (since there must exist at least one member of $\mathcal{K}$ that is not lower-bounded). $\diamond$

We omit the proofs of the following, which are similar to the proofs of Theorem 12.62 and Corollary 12.63.

Theorem 12.64 $S_*(\mathcal{K}, \mathsf{fi}) \equiv S(\mathcal{K}, \triangledown_*)$ for any quasivariety $\mathcal{K}$ of upper-unbounded lattice expansions.

Theorem 12.65 $S_0(\mathcal{K}, \mathsf{id}) \equiv S(\mathcal{K}, \triangle_*)$ for any quasivariety $\mathcal{K}$ of 0-lattice expansions.

Theorem 12.66 $S_1(\mathcal{K}, \mathsf{fi}) \equiv S(\mathcal{K}, \triangledown_*)$ for any quasivariety $\mathcal{K}$ of 1-lattice expansions.

$\square$

Chapter 13

Parameterized Algebraic Semantics

Consider the membership logic $S(\mathcal{K}, \mathsf{mem})$ of a quasivariety $\mathcal{K}$. By Theorem 12.58, of Example 12.47 on page 385,

$$\{z\} \cup P \vdash_{S(\mathcal{K},\mathsf{mem})} t \ \text{ iff } \ P \approx z \models_{\mathcal{K}} t \approx z. \tag{13.1}$$

As a consequence, certain deductions in $S(\mathcal{K}, \mathsf{mem})$, namely those of the form $\{z\} \cup P \vdash_{S(\mathcal{K},\mathsf{mem})} t$, may be tested for validity in the quasi-equational theory of $\mathcal{K}$. This property of $S(\mathcal{K}, \mathsf{mem})$ and $\mathcal{K}$ is similar in spirit to the relationship between a sentential calculus $\mathcal{S}$ and an algebraic semantics $\mathcal{K}$ for that calculus, although in the case of an algebraic semantics, *all* deductions may be interpreted and tested for validity in the quasi-equational theory of $\mathcal{K}$. In this chapter we shall explore a weaker notion of an algebraic semantics, based on relationships between a logic and a quasivariety motivated by the form of (13.1). We shall treat $\{z\}$ as a *parameter*, and introduce a notion of a *parametrized algebraic semantics*.

In $\oint$13.1 we define the notion of a parametrized algebraic semantics and establish some basic properties of logics satisfying this property. In $\oint$13.2 we characterize the relationship between a logic and its parametrized algebraic semantics, and in $\oint$13.3 consider a number of examples of logics, introduced earlier, that have no algebraic semantics, but for which some parametrized algebraic semantics exists. We also show how the standard notion of an algebraic semantics obtains from the parametrized version by taking the parameter to be the empty-set. As a consequence, the results we obtain for parametrized algebraic semantics generalize those for algebraic semantics, and most of the well-known results of algebraic semantics derive easily from the theory developed in this chapter.

13.1 Definition

We shall introduce two notions of a *parameterized* algebraic semantics. The first of these two notions, introduced next, explicates the parameter as a pair $\langle X, z\rangle$, where X is a set of terms and z a variable. In the second formulation, introduced shortly, the parameter is a binary system $\mathfrak{B}$. While the two notions are equivalent, both forms have certain advantages in particular circumstances.

The following definition is motivated by Proposition 12.22 on page 380, which states that if $\mathfrak{B}_*$ *pivots finitarily* for $\mathcal{K}$. Then $S(\mathcal{K}, \mathfrak{B}_*)$ is the coarsest sentential 1-calculus $\mathcal{S}$ such that for any set $P \cup \{p\}$ of terms and any variable z,

$$\mathfrak{B}_z/\!\perp_{\mathcal{K}},\ P \vdash_{\mathcal{S}} p \quad \text{iff} \quad \mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]. \tag{13.2}$$

The reader may be disconcerted by the absence of X from the right hand side of (13.3) in the following definition. One should view (13.3) as a generalization of (13.2).

Definition 13.1 ($\langle X, z\rangle$-Algebraic Semantics) Let X be a set of terms and z a variable. A quasivariety $\mathcal{K}$ is called an **$\langle X, z\rangle$-algebraic semantics** for a sentential calculus $\mathcal{S}$ if there exists a binary system of equations $\mathfrak{B}$ such that, for any $P \cup \{t\} \subseteq \mathsf{Tm}$,

$$X, P \vdash_{\mathcal{S}} t \quad \text{iff} \quad \mathfrak{B}_z\,[P] \models_{\mathcal{K}} \mathfrak{B}_z[\![t]\!]. \tag{13.3}$$

In this case, we refer to $\mathfrak{B}_*$ as **$\langle X, z\rangle$-defining equations for $\mathcal{S}$ and $\mathcal{K}$** or just **parameterized defining equations for $\mathcal{S}$ and $\mathcal{K}$** or even just **defining equations for $\mathcal{S}$ and $\mathcal{K}$** where unambiguous. □

Towards the aim of establishing our alternative formulation of a parameterized algebraic semantics, we shall now demonstrate that the notion of an $\langle X, z\rangle$-algebraic semantics (essentially) depends only on the *parameterized defining equations* $\mathfrak{B}_*$, and *not* on X, *nor* on the variable z. Recall the definition of p-invariance given in Definition 1.337 on page 64. Recall further, that by Remark 9.36 on page 322, the variable base $\mathfrak{B}_z/\!\perp_{\mathcal{K}}$ is z-invariant.

Proposition 13.2 Let $\mathcal{S}$ be a sentential calculus and $\mathcal{K}$ a quasivariety.

1. $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with parameterized defining equations $\mathfrak{B}_*$ iff $\mathcal{K}$ is a $(\|X\|_{\mathcal{S}}, z)$-algebraic semantics for $\mathcal{S}$ with defining equations $\mathfrak{B}_*$. In this case $\|X\|_{\mathcal{S}} = \mathfrak{B}_z/\!\perp_{\mathcal{K}}$, which is z-invariant, and for every variable y, $\mathcal{K}$ is a $(\mathfrak{B}_y/\!\perp_{\mathcal{K}}, y)$-algebraic semantics for $\mathcal{S}$ with parameterized defining equations $\mathfrak{B}_*$.

2. If $\mathcal{K}$ is a $\langle \mathfrak{B}_z/\!\perp_{\mathcal{K}}, z\rangle$-algebraic semantics for $\mathcal{S}$ with parameterized defining equations $\mathfrak{B}_*$, then for every variable y, $\mathcal{K}$ is a $\langle \mathfrak{B}_y/\!\perp_{\mathcal{K}}, y\rangle$-algebraic semantics for $\mathcal{S}$ with parameterized defining equations $\mathfrak{B}_*$.

3. If $\mathcal{K}$ is a $\langle \mathfrak{B}_z/\!\perp_{\mathcal{K}}, z\rangle$-algebraic semantics for $\mathcal{S}$ with parameterized defining equations $\mathfrak{B}_*$, then $\mathcal{S} \preceq S(\mathcal{K}, \mathfrak{B}_*) \preceq D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$.

Proof. $\boxed{(1)}$ The first assertion is obvious. Suppose that $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ and let $T = \|X\|_{\mathcal{S}}$. By (13.3), $T \vdash_{\mathcal{S}} t$ iff $\models_{\mathcal{K}} \mathfrak{B}_z[\![t]\!]$ iff $t \in \mathfrak{B}_z/\!\perp^{\mathcal{K}}$. So $T = \mathfrak{B}_z/\!\perp^{\mathcal{K}}$ (which is z-invariant by Remark 9.36 on page 322). Let y be any variable and σ the transposition $(y\,z)$. Since σ is an involution of $\mathbf{Tm}$, $\mathfrak{B}_y/\!\perp^{\mathcal{K}} = \sigma[\mathfrak{B}_z/\!\perp^{\mathcal{K}}]$ by (2) of Proposition 9.34 on page 322, and $\mathfrak{B}_y/\!\perp^{\mathcal{K}}, P \vdash_{\mathcal{S}} t$ [iff] $\mathfrak{B}_z/\!\perp^{\mathcal{K}}, \sigma[P] \vdash_{\mathcal{S}} \sigma(t)$ [iff] $\mathfrak{B}_z\,[\sigma[P]] \models_{\mathcal{K}} \mathfrak{B}_z[\![\sigma(t)]\!]$ [iff] $\mathfrak{B}_y[P] \models_{\mathcal{K}} \mathfrak{B}_y[\![t]\!]$. So $\mathcal{K}$ is a $(\mathfrak{B}_y/\!\perp^{\mathcal{K}}, y)$-algebraic semantics for $\mathcal{S}$ with $(\mathfrak{B}_y/\!\perp^{\mathcal{K}}, y)$-defining equations $\mathfrak{B}_*$. $\boxed{(2)}$ Follows immediately from (1). $\boxed{(3)}$ It suffices by Corollary 12.15 on page 378, to show that $\mathcal{S} \preceq D_{\forall_z}(\mathcal{K}, \mathfrak{B}_*)$. Suppose that $P \vdash_{\mathcal{S}} p$. Let y be any variable. Certainly, $\mathfrak{B}_y/\!\perp^{\mathcal{K}}, \mathsf{Tm} \vdash_{\mathcal{S}} p$. By (2), $\mathcal{K}$ is a $\langle \mathfrak{B}_y/\!\perp_{\mathcal{K}}, y\rangle$-algebraic semantics for $\mathcal{S}$ with defining equations $\mathfrak{B}_*$, and so $\mathfrak{B}_y(P) \models_{\mathcal{K}} \mathfrak{B}_y(p)$. Since the variable y was chosen arbitrarily, $P \vdash_{D_{\forall_z}(\mathcal{K},\mathfrak{B}_*)} p$. $\diamond$

Thus our definition of $\langle X, z\rangle$-algebraic semantics depends not on the precise value in $2^{\mathrm{Tm}} \times \mathsf{V}$ of the parameter $\langle X, z\rangle$ but only on its equivalence-class modulo identification of such pairs with their images under all transpositions of variables. This observation justifies the following alternative formulation of the notion of a parameterized algebraic semantics.

Definition 13.3 ($\mathfrak{B}_*$-Algebraic Semantics) For a binary system of equations $\mathfrak{B}$, we call $\mathcal{K}$ a **$\mathfrak{B}_*$-algebraic semantics** for $\mathcal{S}$ with **parameterized defining equations** $\mathfrak{B}_*$, if $\mathcal{K}$ is a $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z\rangle$-algebraic semantics for $\mathcal{S}$ for some variable z. $\square$

The following result follows at once from (3) of the previous proposition.

Corollary 13.4 If $\mathcal{K}$ a **$\mathfrak{B}_*$-algebraic semantics** for $\mathcal{S}$ then $\mathcal{S} \preceq S(\mathcal{K}, \mathfrak{B}_*)$.

Discussion 13.5 ($\langle X, z\rangle$ vs. $\mathfrak{B}_*$) It is possible to recast all results on $\langle X, z\rangle$-algebraic semantics in terms of $\mathfrak{B}_*$-algebraic semantics. At times, for example in the next chapter on *parametrized protoalgebraicity*, entities in the roles of X and z shall arise naturally in contexts where it is inconvenient, unnatural or unnecessary to presuppose a system of binary equations $\mathfrak{B}$ in the framework. For this reason we have used the parameter $\langle X, z\rangle$ as a starting point.

This approach also makes certain concepts and results from the (parameterless) theory of algebraization more conspicuous as specializations. Clearly, a particular specification of $\langle X, z\rangle$ is independent of the type $\mathfrak{a}$ just in case $X \subseteq \mathsf{V}$. In fact, only *two* such *individual* specifications, viz., $X = \emptyset$ and $X = \{z\}$, interpret our definition of $\langle X, z\rangle$-algebraic semantics non-trivially (the trivial case being $X = \mathsf{V}$), in view of the z-invariance of $\|X\|^{\mathsf{Tm}}_{\mathsf{fi}_{\mathcal{S}}}$. The *empty case* encompasses the notion of *algebraic semantics* given in [BP89a] (see Example 13.29). By allowing also the *single variable case*, our definition contains a purely *logical* generalization of the notion of having an algebraic semantics: specifically, this case draws *membership logics* into its scope (see Example 13.34). Using $\mathfrak{B}$ rather than $\langle X, z\rangle$ as parameter in our definition would have obscured the fact that possession of an (unparameterized) algebraic semantics is a *language-independent* property for sentential calculi.

For *particular* values of $X \not\subseteq \mathsf{V}$, our definition can be tested only within linguistic contexts where it is meaningful. Nevertheless, certain 'second order' specifications of values for $\langle X, z\rangle$, not encompassed by the cases $X \subseteq \{z\}$, are also language-independent. One of these is the requirement that X consist of just one unary term u in the variable z. The significance of this condition in the case where $\{\langle x, u(y)\rangle\}$ is the $\langle X, z\rangle$-defining equations will be explored in $\oint$15.6, but is anticipated in Example 13.34.

$\square$

Discussion 13.6 (Quantifying Existentially over $\mathfrak{B}$) It is tempting to quantify existentially over $\mathfrak{B}$ (or over $\langle X, z\rangle$) at the start of the definition of a parametrized algebraic semantics; speaking only of a *parameterized algebraic semantics* with no explicit reference to a specific parameter. This would, however, *trivialize* its meaning, since *every* quasivariety is a $\mathfrak{B}_*$-algebraic semantics for every sentential calculus in the case that $\mathfrak{B} = \{\langle x, x\rangle\}$, and *every* quasivariety is a $\langle X, z\rangle$-algebraic semantics for every sentential calculus in the case that $X = \mathsf{V}$. $\square$

In our paper [BR03], a number of results were phrased in terms of $\mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X\rangle)$, where as in this text, we phrase the same results (equivalently) in terms of the filtration logic $\mathcal{S}_{:X}$; note that $\mathsf{Th}(\mathcal{S}_{:X}) = \mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X\rangle)$; we tend to invoke the latter notation in *proofs*, given that this notion is more familiar to those acquainted with algebraic logic. The reason for this is so that we can draw closer parallels with the analogous results from the standard theory, since in that case these results are phrased in terms of $\mathcal{S}$ and not $\mathbf{Fi}_{\mathcal{S}}(\mathbf{Tm})$, and further, so that we can explicate a *fundamental problem* that we continually have to find solutions to while developing the theory of parameterized algebraization. We shall now briefly discuss this problem.

It has become clear to us that the role of the terms X in the parameter $\langle X, z\rangle$, is to describe the theories of $\mathcal{S}$ that are 'well-behaved'. Suppose that $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with parameterized defining equations $\mathfrak{B}_*$. In this case, the filtration logic $\mathcal{S}_{:X} = \mathcal{S}_{:\mathfrak{B}_z/\perp^{\mathcal{K}}}$ *would* have $\mathcal{K}$ as a *non-parameterized* algebraic semantics, *except* for the fact that this logic is generally *not structural* (it is finitary and defined on the term algebra) and hence is generally *not sentential*; the homomorphic pre-image of a theory in $\mathcal{S}_{:X}$, while certainly an $\mathcal{S}$-theory (by structurality), need not be an $\mathcal{S}_{:X}$-theory, since the homomorphic pre-image need not contain X. Clearly, the only situation where $\mathcal{S}_{:X}$ is structural, occurs when the theory generated by X is *fully invariant*, and, as we shall see, in this case the parameterized defining equations $\mathfrak{B}_*$ are *essential $\mathcal{K}$-unary*, $\mathcal{S}_{:X} = \mathcal{S}_{:\mathfrak{B}_z/\perp^{\mathcal{K}}} = \mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}$, for all variables z and y, and $\mathcal{K}$ is an *algebraic semantics* for $\mathcal{S}_{:X}$ with defining equations $\tau(x) = \{\langle \delta(x,x), \epsilon(x,x)\rangle \ :\ \langle \delta, \epsilon\rangle \in \mathfrak{B}\}$ (see Theorem 13.30 of Example 13.29). In all other cases, $\mathcal{S}_{:\mathfrak{B}_z/\perp^{\mathcal{K}}} \neq \mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}$ for distinct variable z and y; in such cases, for each variable y, each filtration logic $\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}$ has $\mathcal{K}$ as an 'algebraic semantics' with 'unary defining equations $\mathfrak{B}_z(x)$', where $\mathfrak{B}_z(x)$ is viewed as a unary system in x with z fixed (quotations since these filtration logics are not *sentential*). In the following result we formalize the latter observation without explicitly mentioning the words 'algebraic semantics'; the result follows trivially from Definition 13.1, Proposition 13.2 and Remark 6.11 on page 225. (In $\oint$17, using a notion of equivalent logics over constructs, we shall be able to give more rigour to this observation, since we will be able to treat $\mathcal{S}_{:\mathfrak{B}_z/\perp^{\mathcal{K}}}$ as a *propositional* logic in the construct consisting of the single language $\mathbf{Tm}$ together with all endomorphisms of the term algebra that fix z; from this perspective $\mathcal{S}_{:\mathfrak{B}_z/\perp^{\mathcal{K}}}$ is indeed structural since $\mathfrak{B}_z/\perp^{\mathcal{K}}$ is z-invariant; see Example 17.60 on page 488.)

Proposition 13.7 $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with parameterized defining equations $\mathfrak{B}_*$ iff, for each variable y,

$$P \vdash_{\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}} t \quad \text{iff} \quad \mathfrak{B}_y\,[P] \models_{\mathcal{K}} \mathfrak{B}_y[\![t]\!]. \tag{13.4}$$

Recall Definition 9.23 on page 319, where we defined the $\mathfrak{B}_a$-class $\mathfrak{B}_a^{\mathbf{A}}/\alpha$, where $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, and recall that $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A})$ denotes the set of all $\mathfrak{B}_a$-classes which is an *algebraic* closed system. Recall further that $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) = \{\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_b}(\mathbf{A}) \ :\ a \in \mathsf{uni}(\mathbf{A})\}$, the members of which are (also) called *principal $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-normals.* Generally this set does not form a closed system. By closing $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$ under non-empty (possibly infinite) intersections, we obtained the set $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A})$, which does form a closed system (but generally not an algebraic closed system), the members of which we called $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-*normals.* The following result follows immediately from (3) of Proposition 13.2 and Proposition 12.34 on page 382.

Proposition 13.8 Let $\mathcal{K}$ be an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ and let $\mathbf{A}$ be an algebra not necessarily in $\mathcal{K}$. Then $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \subseteq \mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$, and for each $a \in \mathsf{uni}(\mathbf{A})$, $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A}) \subseteq \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathfrak{B}^{\mathbf{A}}_a / \bot^{\mathbf{A}}_{\mathcal{K}}\rangle)$. $\square$

We end this section by remarking that every quasivariety $\mathcal{K}$ is always 'halfway' to being a $\mathfrak{B}_*$-algebraic semantics for $S(\mathcal{K}, \mathfrak{B}_*)$, for any system $\mathfrak{B}$ of binary equations, since if $\mathfrak{B}_z/\bot_{\mathcal{K}}, P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$ then it is always true that $\mathfrak{B}^{\approx}_z[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]$.

Remark 13.9 $\mathfrak{B}_z/\bot_{\mathcal{K}}, P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$ implies $\mathfrak{B}^{\approx}_z[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]$.

Proof. Suppose that $\mathfrak{B}_z/\bot_{\mathcal{K}}, P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$. By Corollary 12.15 on page 378,

$$\forall[\mathsf{z}]\ \mathfrak{B}^{\approx}_z[\mathfrak{B}_z/\bot_{\mathcal{K}}] \cup \mathfrak{B}^{\approx}_z[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!].$$

In particular, $\mathfrak{B}^{\approx}_z[\mathfrak{B}_z/\bot_{\mathcal{K}}] \cup \mathfrak{B}^{\approx}_z[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]$, and since $\models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\mathfrak{B}_z/\bot_{\mathcal{K}}]$, we have $\mathfrak{B}^{\approx}_z[P] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]$. $\diamond$

13.2 Characterizations

Our aim is to establish a characterization of the property of having an $\langle X, z\rangle$-algebraic semantics, that is similar in spirit to Corollary 2.108 on page 111, and which specializes to Corollary 2.108 in the case that X is the empty-set. Recall that by Definition 2.107 on page 111, for $\alpha \subseteq \mathsf{Fm}(\mathcal{S})^2$ and substitution σ,

$$\sigma_{\mathcal{K}}(\alpha) = \left\| \underset{\rightarrow}{\sigma}[\alpha] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}.$$

Definition 13.10 ($\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{\mathbf{A},a}$) Define a function $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{\mathbf{A},a}(\cdot) : \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathfrak{B}^{\mathbf{A}}_a/\bot^{\mathcal{K}}_{\mathbf{A}}\rangle) \to \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ by $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{\mathbf{A},a}(F) = \left\| \mathfrak{B}^{\mathbf{A}}_a(F) \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$. We drop the subscript $\mathbf{A}$ when $\mathbf{A} = \mathbf{Tm}$. $\square$

Remark 13.11 It is generally true (i.e., without any assumption of $\mathfrak{B}_*$-algebraic semantics) that for all $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ and $a \in \mathsf{uni}(\mathbf{A})$, $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{\mathbf{A},a}(\mathfrak{B}_a/\alpha) \subseteq \alpha$. $\square$

Recall that conventionally the discourse (and results) of continuous functions and continuous translations applies to logics (see Convention 6.13 on page 225 and Definition 6.2 on page 223). Viewing $\mathfrak{B}^{\mathbf{A}}_a$ as a translation from $\mathsf{F}_{\mathcal{S}}(\langle \mathbf{A}, \mathfrak{B}^{\mathbf{A}}_a/\bot^{\mathcal{K}}_{\mathbf{A}}\rangle)$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{\mathbf{A},a}$ is simply $(\mathfrak{B}^{\mathbf{A}}_a)^{\star}$ as defined in Definition 5.1 on page 175, but with a restricted domain. Further, from this perspective and by definition, $\mathfrak{B}^{\mathbf{A}}_a/\cdot$ is $(\mathfrak{B}^{\mathbf{A}}_a)^{\blacktriangleleft}(\cdot)$, also defined in the aforementioned definition.

Suppose that $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ and $\mathbf{A}$ is an algebra not necessarily in $\mathcal{K}$. By Corollary 9.28 on page 320, $\mathfrak{B}^{\mathbf{A}}_a$ is *strictly continuous* from the finitary closed system $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A})$ to $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. Since $\mathfrak{B}^{\mathbf{A}}_a$ is (strictly) continuous from $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A})$ into $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, and since $\mathsf{Sol}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A}) \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$, by the Proposition 13.8, it follows by Remark 5.22 on page 183 that $\mathfrak{B}^{\mathbf{A}}_a$ is *continuous* from $\mathsf{F}_{\mathcal{S}}(\langle \mathbf{A}, \mathfrak{B}^{\mathbf{A}}_a/\bot^{\mathbf{A}}_{\mathcal{K}}\rangle)$ (and from $\mathsf{F}_{\mathcal{S}}(\mathbf{A})$) into $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. Consequently, together with Theorem 5.21 on page 182 and Theorem 5.40 on page 186, the following useful result obtains[1].

[1] The reader still unconvinced by our motivation for developing the unifying theory of continuous translations between closed systems, should note that in our paper [BR03], the proof of this result is half a page, as is the

Proposition 13.12 Let $\mathcal{K}$ be an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ and let $\mathbf{A}$ be an algebra not necessarily in $\mathcal{K}$. $\mathsf{N}^{\mathcal{K}}_{\mathfrak{B}_*}(\mathbf{A}) \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$. For each $a \in \mathsf{uni}(\mathbf{A})$, $\mathfrak{B}^{\mathbf{A}}_a$ is continuous from $\mathsf{F}_{\mathcal{S}}(\mathbf{A})$ into $\mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, and hence

1. $\mathfrak{B}^{\mathbf{A}}_a/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \rightarrow_{\blacktriangle} \mathbf{Fi}^{\mathcal{S}}(\langle \mathbf{A}, \mathfrak{B}^{\mathbf{A}}_a/\perp^{\mathcal{K}}_{\mathbf{A}}\rangle)$;
2. For $A \subseteq \mathsf{uni}(\mathbf{A})$ $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{\mathbf{A},a}(\|A\|^{\langle \mathbf{A},\perp^{\mathbf{A}}_{\mathcal{K}}\rangle}_{\mathsf{fi}_{\mathcal{S}}}) = \left\| \mathfrak{B}^{\mathbf{A}}_a[A] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{A}}}$.
3. $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{\mathbf{A},a} : \mathbf{Fi}^{\mathcal{S}}(\langle \mathbf{A}, \mathfrak{B}^{\mathbf{A}}_a/\perp^{\mathcal{K}}_{\mathbf{A}}\rangle) \rightarrow_{\blacktriangledown} \mathbf{Con}^{\mathcal{K}}(\mathbf{A})$.
4. For every $F \in \mathbf{Fi}^{\mathcal{S}}(\langle \mathbf{A}, \mathfrak{B}^{\mathbf{A}}_a/\perp^{\mathcal{K}}_{\mathbf{A}}\rangle)$, $\mathfrak{B}^{\mathbf{A}}_a/\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{\mathbf{A},a}(F) = F$.

We shall now demonstrate that when $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$, then for any variable y, the translation $\mathfrak{B}_y$ is strictly continuous from the filtration logic $\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}$ to the closed system $\mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})$ of relative congruences on the term algebra.

Theorem 13.13 Suppose that $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$. Then for any variable y, $\mathfrak{B}_y$ is *strictly* continuous from $\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}$ into $\mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})$. Consequently the following statements are all valid.

1. For every variable y and $\alpha \in \mathfrak{B}^{\mathcal{S},\mathcal{K}}_y[\mathsf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})]$, $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_y(\mathfrak{B}_y/\alpha) = \alpha$.
2. For every variable y, $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_y : \mathbf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}) \cong \mathfrak{B}^{\mathcal{S},\mathcal{K}}_y\left[\mathbf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})\right] \triangleleft_{\blacktriangledown} \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ with inverse isomorphism $(\mathfrak{B}_y/\cdot)_{|\mathfrak{B}^{\mathcal{S},\mathcal{K}}_y\left[\mathsf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})\right]}$.
3. The lattice $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_y\left[\mathbf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})\right]$ is algebraic and compact in $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$.

Proof. By the Proposition 13.12, $\mathfrak{B}_y$ is continuous from $\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}$ into $\mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})$. (*We shall show that $\mathfrak{B}_y$ is consequence reflecting by showing that for all $T \in \mathsf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})$, $\mathfrak{B}_y/\mathfrak{B}^{\mathcal{S},\mathcal{K}}_y(T) = T$. This suffices by equivalent condition (7) of Proposition 5.71 on page 193.*) Now, $p \in \mathfrak{B}_y/\mathfrak{B}^{\mathcal{S},\mathcal{K}}_y(T)$ [iff] $\mathfrak{B}_y[\![p]\!] \subseteq \|\mathfrak{B}_y[T]\|_{\Theta^{\mathcal{K}}_{\mathsf{Tm}}}$ [iff] $\mathfrak{B}_y[T] \models_{\mathcal{K}} \mathfrak{B}_y[\![p]\!]$ [iff] $T \vdash_{\mathcal{S}} p$ (by Proposition 13.2, since $\mathfrak{B}_y/\perp^{\mathcal{K}} \subseteq T$) [iff] $p \in T$. (*The remaining consequents follow from Proposition 5.71 on page 193, Theorem 5.73 on page 195, and Theorem 5.111 on page 205.*) ◇

Definition 13.14 (The Lattice $\boldsymbol{\mathfrak{B}}_y[\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}]$) For brevity, we shall denote the algebraic lattice $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_y\left[\mathbf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})\right]$ by $\boldsymbol{\mathfrak{B}}_y[\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}]$. □

Note that because $\mathfrak{B}_y[\mathsf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})]$ is not generally closed under intersections, $\boldsymbol{\mathfrak{B}}_y[\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}]$ need not be a sublattice of $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$.

Recall that *commutivity with substitutions* plays an important role in the theory of *algebraic semantics* (see Corollary 2.108 on page 111). The analogous notion in the theory of $\langle X, z\rangle$-algebraic semantics is commutivity with substitutions *modulo* X.

analogous (algebraic semantics) proof in [BP89a]. A number of other results in this chapter and the following two chapters, have had their proofs similarly dramatically shortened. We shall not repeat this point again, since the reader can deduce where such reductions have occurred by noting the references to results in ∮5.

Definition 13.15 (Commutation with Substitutions modulo X) For a substitution σ and terms X, we say that a function $f : \mathsf{Th}(\mathcal{S}_{:X}) \to \mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})$ **commutes with σ modulo X** if, whenever $T \in \mathsf{Th}(\mathcal{S}_{:X})$, we have $f(\sigma^{\mathcal{S}}_{X}(T)) = \sigma_{\mathcal{K}}(f(T))$. □

We now aim to establish the primary result of this chapter, which is a parameterized analogue of Corollary 2.108 on page 111 [BP89a]. We have already established that if $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}$ then, for every variable y, $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{y}$ induces an isomorphism from the theory lattice of $\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}$ onto a join complete subsemilattice of the relative congruence lattice $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ that is compact in $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$; we shall now show that $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{y}$ commutes with substitutions modulo $\mathfrak{B}_y/\perp^{\mathcal{K}}$ that *fix* y.

Lemma 13.16 Let $\mathcal{K}$ be an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}$. $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{y}$ commutes with substitutions modulo $\mathfrak{B}_y/\perp^{\mathcal{K}}$ that fix y.

Proof. Suppose that $T \in \mathsf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})$ and that σ is a substitution that fixes y. Then

$$\begin{aligned}\sigma_{\mathcal{K}}(\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{y}(T)) &= \sigma_{\mathcal{K}}\big(\|\mathfrak{B}_y(T)\|_{\Theta^{\mathcal{K}}_{\mathsf{Tm}}}\big) = \|\sigma[\mathfrak{B}_y(T)]\|_{\Theta^{\mathcal{K}}_{\mathsf{Tm}}} = \|\mathfrak{B}_y(\sigma[T])\|_{\Theta^{\mathcal{K}}_{\mathsf{Tm}}}\\ &= \mathfrak{B}^{\mathcal{S},\mathcal{K}}_{y}(\|\sigma[T]\|_{\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}}}) = \mathfrak{B}^{\mathcal{S},\mathcal{K}}_{y}(\sigma^{\mathcal{S}}_{\mathfrak{B}_y/\perp^{\mathcal{K}}}(T)), \quad \text{by (2).}\end{aligned}$$

◇

We now establish the main result of the chapter, characterizing parameterized algebraic semantics in the spirit of the characterization of algebraic semantics given in [BP89a] (see Corollary 2.108 of our text). In ∮17, we shall establish this result from a non-parameterized theory of equivalent logics over constructs (see Example 17.60 on page 488). The reader who may feel that our notion of a $\mathfrak{B}_*$-algebraic semantics is in someway an arbitrary 'trick' would do well to consider (2) of this result, since this notion is completely determined by (and characterizes) an isomorphism from $\mathbf{Th}(\mathcal{S}_{:X})$ onto a join-complete subsemilattice of $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ that is compact in $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ and which commutes with surjective substitutions (modulo X) that fix z. In fact, we discovered the notion of parameterized algebraic semantics from proving (2) of this theorem, after discovering such a relationship between the membership logic and $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ (with $X = \{z\}$).

Theorem 13.17 Let $\mathcal{S}$ be a sentential calculus and $\mathcal{K}$ a quasivariety.

1. If $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ then for each variable y, $\mathfrak{B}_y/\perp_{\mathcal{K}}$ is a y-invariant $\mathcal{S}$-theory, $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{y}$ commutes with substitutions (modulo $\mathfrak{B}_y/\perp_{\mathcal{K}}$) that fix y, and $\mathfrak{B}^{\mathcal{S},\mathcal{K}}_{y}$ maps $\mathbf{Th}(\mathcal{S}_{:\mathfrak{B}_y/\perp^{\mathcal{K}}})$ isomorphically onto a join-complete subsemilattice of $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ which is compact in $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$.

2. Conversely, suppose there exists a variable z, a set of terms X generating a z-invariant $\mathcal{S}$-theory T, and a function f that maps $\mathbf{Th}(\mathcal{S}_{:X})$ isomorphically onto a join-complete subsemilattice of $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ that is compact in $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$. Suppose also that f commutes with surjective substitutions (modulo X) that fix z. Then there exists a binary system of equations $\mathfrak{B}$ such that $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$, with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$, and $\mathsf{f} = \mathfrak{B}^{\mathcal{S},\mathcal{K}}_{z}$.

Proof. $\boxed{(1)}$ By Remark 9.36 on page 322, Theorem 13.13 and Lemma 13.16. $\boxed{(2)}$ Since $\|\{x\}\|_{\mathcal{S}:T}$ is a compact element of $\mathbf{Th}(\mathcal{S}_{:T})$, $\mathsf{f}(\|\{x\}\|_{\mathcal{S}:T})$ is finitely generated, by $\{\langle p_0, q_0\rangle, \ldots, \langle p_{n-1}, q_{n-1}\rangle\}$, say. Let σ be any surjective substitution fixing x and z and mapping any other variables that occur in the terms $\{p_0, q_0, \ldots, p_{n-1}, q_{n-1}\}$ to x. Let $\mathfrak{B}(x, z) = \underrightarrow{\sigma}[\{\langle p_0, q_0\rangle, \ldots, \langle p_{n-1}, q_{n-1}\rangle\}]$. $\boxed{\mathsf{f} = \mathfrak{B}_z^{\mathcal{S},\mathcal{K}}}$ Now T is, by assumption, σ-invariant, so by Proposition 2.46 on page 101,

$$\sigma_T^{\mathcal{S}}(\|\{x\}\|_{\mathcal{S}:T}) = \|\sigma(x)\|_{\mathcal{S}:T} = \|\{x\}\|_{\mathcal{S}:T}\,.$$

By assumption, f commutes with σ (modulo T) so

$$\begin{aligned}
\mathsf{f}(\|\{x\}\|_{\mathcal{S}:T}) &= \mathsf{f}(\sigma_T^{\mathcal{S}}(\|\{x\}\|_{\mathcal{S}:T})) \\
&= \sigma_{\mathcal{K}}(\mathsf{f}(\|\{x\}\|_{\mathcal{S}:T})) \\
&= \sigma_{\mathcal{K}}(\|\{\langle p_i, q_i\rangle : i < n\}\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}) \\
&= \|\sigma[\{\langle p_i, q_i\rangle : i < n\}]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} \\
&= \|\mathfrak{B}_z(x)\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}.
\end{aligned}$$

Let $t \in \mathsf{Tm}$, and consider any surjective substitution ρ that takes x to t and fixes z. By assumption, T is a ρ-invariant $\mathcal{S}$-theory, modulo which f commutes with ρ. Using Proposition 2.46, we have

$$\begin{aligned}
\mathsf{f}(\|\{t\}\|_{\mathcal{S}:T}) &= \mathsf{f}(\|\rho[\{x\}]\|_{\mathcal{S}:T}) \\
&= \mathsf{f}(\rho_T^{\mathcal{S}}(\|\{x\}\|_{\mathcal{S}:T})) \\
&= \rho_{\mathcal{K}}(\mathsf{f}(\|\{x\}\|_{\mathcal{S}:T})) \\
&= \rho_{\mathcal{K}}(\|\mathfrak{B}_z(x)\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}) \\
&= \|\rho[\mathfrak{B}_z(x)]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} \\
&= \|\mathfrak{B}_z[\![t]\!]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}.
\end{aligned}$$

So, for $P \subseteq \mathsf{Tm}$, $\mathsf{f}(\|P\|_{\mathcal{S}:T}) = \mathsf{f}(\blacktriangledown^{\mathsf{Th}(\mathcal{S}:T)}_{g\in P} \|\{g\}\|_{\mathcal{S}:T}) = \blacktriangledown^{\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})}_{g\in P} \mathsf{f}(\|\{g\}\|_{\mathcal{S}:T}) = \blacktriangledown^{\mathbf{Con}^{\mathcal{K}}(\mathsf{Tm})}_{g\in P} \|\mathfrak{B}_z(g)\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} = \|\mathfrak{B}_z[P]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}$, and so $\mathfrak{B}_z^{\mathcal{S},\mathcal{K}}[\|P\|_{\mathcal{S}:T}] = \mathsf{f}(\left\|\|P\|_{\mathcal{S}:T}\right\|_{\mathcal{S}:T}) = \mathsf{f}(\|P\|_{\mathcal{S}:T})$. Consequently, $\mathsf{f} = \mathfrak{B}_z^{\mathcal{S},\mathcal{K}}$. $\boxed{(T, z)\text{-algebraic semantics}}$ Now, for $P \cup \{t\} \subseteq \mathsf{Tm}$, we have

$$\begin{aligned}
\mathfrak{B}_z\,[P] \models_{\mathcal{K}} \mathfrak{B}_z[\![t]\!] \ &\text{iff } \|\mathfrak{B}_z[\![t]\!]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} \subseteq \|\mathfrak{B}_z\,[P]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} \\
&\text{iff } \mathsf{f}(\|\{t\}\|_{\mathcal{S}:T}) \subseteq \mathsf{f}(\|P\|_{\mathcal{S}:T}) \\
&\text{iff } \|\{t\}\|_{\mathcal{S}:T} \subseteq \|P\|_{\mathcal{S}:T} \\
&\text{iff } \|\{t\} \cup T\|_{\mathcal{S}} \subseteq \|P \cup T\|_{\mathcal{S}} \\
&\text{iff } \|\{t\}\|_{\mathcal{S}} \subseteq \|P \cup T\|_{\mathcal{S}} \\
&\text{iff } P \vdash_{\mathcal{S}} t.
\end{aligned}$$

Thus, $\mathcal{K}$ is a (T, z)-algebraic semantics for $\mathcal{S}$ with (T, z)-defining equations $\mathfrak{B}_*$. $\diamond$

As was pointed out in [BP89a, Theorem 2.7], not every deductive system has an algebraic semantics, essentially because such deductive systems are forced to admit inferences of a certain form. Although every sentential calculus has an $\langle X, z\rangle$-algebraic semantics for *some* X and z (e.g., $X = \mathsf{Tm}$ and any $z \in \mathsf{V}$), an analogue of the aforementioned restriction holds in our context also.

Proposition 13.18 Let $\mathcal{K}$ be an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B} = \{\langle \delta_i, \varepsilon_i\rangle \,:\, i < n\}$. If $i < n$ and $p \in \mathsf{Tm}$ then $X, p, \delta_i(p, z) \vdash_{\mathcal{S}} \varepsilon_i(p, z)$.

Proof. $\big(\boldsymbol{\delta}(p,z) \approx \boldsymbol{\varepsilon}(p,z)\big) \cup \big(\boldsymbol{\delta}(\delta_i(p,z),z) \approx \boldsymbol{\varepsilon}(\delta_i(p,z),z)\big) \models_{\mathcal{K}} \boldsymbol{\delta}(\varepsilon_i(p,z),z) \approx \boldsymbol{\varepsilon}(\varepsilon_i(p,z),z)$, for each $i < n$. $\diamond$

On the other hand, certain important conditions, automatically satisfied in the case of an algebraic semantics $\mathcal{K}$, are forced on $\models_{\mathcal{K}}$ whenever $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics. Recall the notion that $\mathfrak{B}_*$ *pivots* in $\mathcal{K}$, given in Definition 9.41 on page 323, and note the characterizations of this property given in Proposition 9.43 on page 324.

Proposition 13.19 If $\mathcal{K}$ is a $\mathfrak{B}_*$-algebraic semantics for $\mathcal{S}$ then $\mathfrak{B}_*$ pivots in $\mathcal{K}$, i.e., for any variables z and y and any $P \cup \{t\} \subseteq \mathsf{Tm}$,

$$\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![t]\!] \text{ implies } [z \Rsh y] \cup \mathfrak{B}_y^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[\![t]\!]. \tag{13.5}$$

Proof. Suppose that $\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![t]\!]$. Then $\mathfrak{B}_z/\bot_{\mathcal{K}}, P \vdash_{\mathcal{S}} t$, since, by assumption and Proposition 13.2, $\mathcal{K}$ is a $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z\rangle$-defining equations $\mathfrak{B}_*$. So certainly, $\mathfrak{B}_y/\bot_{\mathcal{K}}, \mathfrak{B}_z/\bot_{\mathcal{K}}, P \vdash_{\mathcal{S}} t$, and so $[z \Rsh y] \cup \mathfrak{B}_y^{\approx}[P]) \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[\![t]\!]$, since $\mathcal{K}$ is also a $(\mathfrak{B}_y/\bot_{\mathcal{K}}, y)$-algebraic semantics for $\mathcal{S}$ by Proposition 13.2. $\diamond$

Definition 13.20 ($\mathfrak{B}_*$-Deductive Quasivarieties) Let $\mathcal{K}$ be a quasivariety and $\mathfrak{B}$ a system of binary equations. We say that $\mathcal{K}$ is **$\mathfrak{B}_*$-deductive** if $\mathcal{K}$ is a $\mathfrak{B}_*$-algebraic semantics of some sentential calculus $\mathcal{S}$ with defining equations $\mathfrak{B}_*$. $\square$

We now give necessary and sufficient conditions for $\mathcal{K}$ to be $\mathfrak{B}_*$-deductive.

Corollary 13.21 If $\mathcal{K}$ is $\mathfrak{B}_*$-deductive then $\mathfrak{B}_*$ pivots in $\mathcal{K}$.

Theorem 13.22 Let $\mathcal{K}$ be a quasivariety, $\mathfrak{B}$ a system of binary equations and z a variable. The following conditions are equivalent.

1. $\mathcal{K}$ is a $(\mathfrak{B}_z/\bot_{\mathcal{K}}, z)$-algebraic semantics for $S(\mathcal{K}, \mathfrak{B}_*)$ with $(\mathfrak{B}_z/\bot_{\mathcal{K}}, z)$-defining equations $\mathfrak{B}$.

2. $\mathcal{K}$ is $\mathfrak{B}_*$-deductive.

3. $\mathfrak{B}_*$ pivots *finitarily* for $\mathcal{K}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Trivial. $\boxed{(2)\Rightarrow(3)}$ Let $\mathcal{S}$ be a sentential calculus for which $\mathcal{K}$ is a $\mathfrak{B}_*$-algebraic semantics. Assume that $\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]$. Since $\mathcal{K}$ is a $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z\rangle$-algebraic semantics for $\mathcal{S}$, we have $\mathfrak{B}_z/\bot_{\mathcal{K}}, P \vdash_{\mathcal{S}} p$. By finitariness of $\mathcal{S}$, there exists a finite subset $Z \subseteq_{\mathsf{f}} \mathfrak{B}_z/\bot_{\mathcal{K}}$ and a finite subset $P' \subseteq_{\mathsf{f}} P$ such that $Z, P' \vdash_{\mathcal{S}} p$. Since $\mathcal{K}$ is a $\langle \mathfrak{B}_y/\bot_{\mathcal{K}}, y\rangle$-algebraic semantics for $\mathcal{S}$, for all variables y, and certainly $\mathfrak{B}_y/\bot_{\mathcal{K}}, Z, P' \vdash_{\mathcal{S}} p$, we have $\mathfrak{B}_y^{\approx}[Z] \cup \mathfrak{B}_y^{\approx}[P'] \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[\![p]\!]$. Since y is an arbitrary variable, $\forall[\mathsf{z}]\, \mathfrak{B}_{\mathsf{z}}^{\approx}[Z] \cup \mathfrak{B}_{\mathsf{z}}^{\approx}[P'] \models_{\mathcal{K}} \mathfrak{B}_{\mathsf{z}}^{\approx}[\![p]\!]$. $\boxed{(3)\Rightarrow(1)}$ By Proposition 12.22 on page 380. $\diamond$

Note 13.23 ($\mathcal{K}$ is *not* always a $(\mathfrak{B}_z/\bot^{\mathcal{K}}, z)$-algebraic semantics for $S(\mathcal{K}, \mathfrak{B}_*)$) The equivalent conditions of Theorem 13.22 *can* fail (i.e., $\mathcal{K}$ is not always a $(\mathfrak{B}_z/\bot^{\mathcal{K}}, z)$-algebraic semantics for $S(\mathcal{K}, \mathfrak{B}_*)$). To see this, recall Example 9.113 on page 338, which demonstrated that for the binary system $\mathfrak{B} = \{\langle x \oplus y, y\rangle\}$, $\mathfrak{B}_*$ does not pivot in the quasivariety LM of all polrims[2]. Consequently, in this case, LM is not always a $(\mathfrak{B}_z/\perp^{\mathcal{K}}, z)$-algebraic semantics for $S(\mathsf{LM}, \mathfrak{B}_*)$. $\square$

Corollary 13.24 If $\mathcal{K}$ is $\mathfrak{B}_*$-deductive and $\mathcal{K}$ is a $\mathfrak{B}_*$-algebraic semantics for $\mathcal{S}$, then, for all variables z and terms $P \cup \{p\}$, we have $\mathfrak{B}_z/\perp^{\mathcal{K}}, P \vdash_{\mathcal{S}} p$ iff $\mathfrak{B}_z/\perp^{\mathcal{K}}, P \vdash_{S(\mathcal{K},\mathfrak{B}_*)} p$. $\square$

The following result follows from Theorem 13.22 together with Remark 9.49 on page 325.

Corollary 13.25 If $\mathfrak{B}_*$ has finite pivots in $\mathcal{K}$ *and* $\mathfrak{B}_*$ pivots in $\mathcal{K}$, then $\mathcal{K}$ is $\mathfrak{B}_*$-deductive. $\square$

Open Problem 13.26 ($\mathfrak{B}_*$-Deductive does not imply $\mathfrak{B}_*$ has finite pivots in $\mathcal{K}$) Prove that the implication of the previous corollary cannot be strengthed to an equivalence. This amounts to showing that for a $\mathfrak{B}_*$-deductive quasivariety $\mathcal{K}$, $\mathfrak{B}_*$ need not have finite pivots in $\mathcal{K}$.

Open Problem 13.27 (Pivoting does not imply $\mathfrak{B}_*$-Deductive) Show that if $\mathfrak{B}_*$ pivots in $\mathcal{K}$ then $\mathcal{K}$ need not be $\mathfrak{B}_*$-deductive. This amounts to demonstrating that if $\mathfrak{B}_*$ pivots in $\mathcal{K}$ it need not follow that $\mathfrak{B}_*$ pivot finitarily in $\mathcal{K}$.

Open Problem 13.28 Find conditions on $\mathcal{K}$ under which the $\mathfrak{B}_*$-deductivity of $\mathcal{K}$ is equivalent to a quasi-Mal'cev condition.

13.3 Examples

The first example of this section helps to distinguish the notion of algebraic semantics (as in [BP89a]) in the context of $\langle X, z\rangle$-algebraic semantics (see Definition 2.105 on page 111).

Example 13.29 (Standard Algebraic Semantics)

Theorem 13.30 Let $\mathcal{K}$ be an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B} = \{\langle \delta_i, \varepsilon_i\rangle : i < n\}$ and let $\tau(x) = \{\langle \delta_i(x, x), \varepsilon_i(x, x)\rangle : i < n\}$. Then the following conditions are equivalent.

1. For every variable y, $\|X\|_{\mathcal{S}} = \mathfrak{B}_y/\perp_{\mathcal{K}} = \mathfrak{B}_z/\perp_{\mathcal{K}}$, $\|X\|_{\mathcal{S}}$ is fully invariant and $\mathfrak{B}_z^{\mathcal{S},\mathcal{K}}$ commutes with all substitutions (modulo $\mathfrak{B}_z/\perp_{\mathcal{K}}$).
2. $\mathfrak{B}_z^{\mathcal{S},\mathcal{K}}$ commutes with all surjective substitutions (modulo $\mathfrak{B}_z/\perp_{\mathcal{K}}$).
3. $\mathcal{K}$ satisfies $\mathfrak{B}_z^{\approx}[\![x]\!] \leftrightarrow \mathfrak{B}_x^{\approx}[\![x]\!]$.
4. $\mathcal{S}_{:X}$ is a sentential calculus and $\mathcal{K}$ is an algebraic semantics for $\mathcal{S}_{:X}$ with defining equations τ.

Proof. $\boxed{(1)\Rightarrow(2)}$ Trivial. $\boxed{(2)\Rightarrow(3)}$ Take distinct variables x, v and let σ be the transposition (zv). Let $F = \|\{x\}\|_{\mathsf{fi}_{\mathcal{S}}}^{\langle \mathsf{Tm}, \mathfrak{B}_z/\perp^{\mathcal{K}}\rangle}$, so $\sigma^{\mathcal{S}}_{\mathfrak{B}_z/\perp^{\mathcal{K}}}(F) = \|\{\sigma(x)\}\|_{\mathsf{fi}_{\mathcal{S}}}^{\langle \mathsf{Tm}, \mathfrak{B}_z/\perp^{\mathcal{K}}\rangle} = F$. Now $\|\mathfrak{B}_z[\![x]\!]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} = \mathfrak{B}_z^{\mathcal{S},\mathcal{K}}(F)$ (by (2)), so

$$\|\mathfrak{B}_z[\![x]\!]\|_{\Theta^{\mathcal{K}}_{\mathsf{Tm}}} = \mathfrak{B}_z^{\mathcal{S},\mathcal{K}}(\sigma^{\mathcal{S}}_{\mathfrak{B}_z/\perp^{\mathcal{K}}}(F)) = \sigma_{\mathcal{K}}(\mathfrak{B}_z^{\mathcal{S},\mathcal{K}}(F)) = \left\| \sigma \left[\|\mathfrak{B}_z[\![x]\!]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} \right] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}$$
$$= \|\sigma[\mathfrak{B}_z[\![x]\!]]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} = \|\mathfrak{B}_v[\![x]\!]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}.$$

[2]We would like to thank James Raftery for this example.

By Lemma 1.457 on page 88, we have $\mathfrak{B}_z^{\approx}[\![x]\!] \dashv\models_{\mathcal{K}} \mathfrak{B}_v^{\approx}[\![x]\!]$. Replacing v by x, we obtain (3).

$\boxed{\text{(3)}\Rightarrow\text{(1)}}$ Let $T = \|X\|_{\mathcal{S}}$, let σ be any substitution and $t \in T$. From $X \vdash_S t$ we infer that $\models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![t]\!]$, hence also $\models_{\mathcal{K}} \mathfrak{B}_{\sigma(z)}^{\approx}[\![\sigma(t)]\!]$. By (3), therefore, $\models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![\sigma(t)]\!]$, whence $X \vdash_S \sigma(t)$. Thus, $\sigma(t) \in T$, so T is fully invariant. It follows immediately that $T = \mathfrak{B}_z/\bot^{\mathcal{K}} = \mathfrak{B}_y/\bot^{\mathcal{K}}$ for any variable y. Finally, let $R \in \mathsf{Th}(\mathcal{S}_{:\mathfrak{B}_z/\bot^{\mathcal{K}}})$. We have

$$\begin{aligned}
&\sigma_{\mathcal{K}}(\mathfrak{B}_z^{\mathcal{S},\mathcal{K}}(R)) = \|\sigma[\mathfrak{B}_z[R]]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} = \left\| \mathfrak{B}_{\sigma(z)}(\sigma[R]) \right\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} \\
&= \blacktriangledown^{\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})}_{s\in\sigma[R]} \left\| \mathfrak{B}_{\sigma(z)}[\![s]\!] \right\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} = \blacktriangledown^{\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})}_{s\in\sigma[R]} \|\mathfrak{B}_z[\![s]\!]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} \quad \text{(by (3))} \\
&= \mathfrak{B}_z^{\mathcal{S},\mathcal{K}}(\|\sigma[R]\|^{\langle \mathbf{Tm}, \mathfrak{B}_z/\bot^{\mathcal{K}}\rangle}_{\mathsf{fi}_{\mathcal{S}}}) \quad \text{(by (2))} = \mathfrak{B}_z^{\mathcal{S},\mathcal{K}}(\sigma^{\mathcal{S}}_{\mathfrak{B}_z/\bot^{\mathcal{K}}}(R)).
\end{aligned}$$

$\boxed{\text{(1) and (3)} \Rightarrow \text{(4)}}$ Since $\|X\|_{\mathcal{S}}$ is fully invariant by assumption (1), $\mathcal{S}_{:X}$ is structural and hence a sentential calculus. Then $P \vdash_{\mathcal{S}_{:X}} p$ [iff, by Remark 6.11 on page 225] $X, P \vdash_{\mathcal{S}} p$ [iff] $\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]$ [iff by (3)] $\tau^{\approx}[P] \models_{\mathcal{K}} \tau^{\approx}[\![p]\!]$. $\boxed{\text{(2)}\Rightarrow\text{(2)}}$ By Corollary 2.108 on page 111.

◇

Corollary 13.31 Given a quasivariety $\mathcal{K}$ and a sentential calculus $\mathcal{S}$ over the same language, the following conditions are equivalent.

1. $\mathcal{K}$ is an algebraic semantics for $\mathcal{S}$ in the sense of [BP89a].
2. There exists a system of binary equations $\mathfrak{B}$ such that, for any variable z, $\mathcal{K}$ is an $(\emptyset, z)$-algebraic semantics for $\mathcal{S}$ with $(\emptyset, z)$-defining equations $\mathfrak{B}_*$ and $\mathfrak{B}_z^{\mathcal{S},\mathcal{K}}$ commutes with all surjective substitutions modulo $\mathfrak{B}_z/\bot_{\mathcal{K}}$ (or equivalently, modulo $\emptyset$).
3. There exists a system of binary equations $\mathfrak{B} = \{\langle\delta_i, \varepsilon_i\rangle : i < n\}$ such that, for any variable z, $\mathcal{K}$ is an $(\emptyset, z)$-algebraic semantics for $\mathcal{S}$ with $(\emptyset, z)$-defining equations $\mathfrak{B}_*$ and $\mathcal{K}$ satisfies $\mathfrak{B}_z^{\approx}[\![x]\!] \leftrightarrow \mathfrak{B}_x^{\approx}[\![x]\!]$.

□

Recall Example 12.42 on page 384, where we introduced the separable logics $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$. In the following example, we demonstrate that a non-trivial separable logic $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$ always has $\mathcal{K}$ as a $\mathfrak{B}_*$-semantics, where $\mathfrak{B}(x, y) = \{\langle \boldsymbol{u}_i(x), \boldsymbol{v}_i(y)\rangle : i \in n\}$. The importance of this example lies in the fact that the non-trivial separable logics encompass the logics of *identified membership*, which in turn encompass the logics of *idempotent $\boldsymbol{u}$-cosets*, which encompass the *membership logics*. So in all these cases, the determining quasivariety forms a parametrized algebraic semantics for each of these logics.

Example 13.32 (The Logics of Separable Systems)

Consider a non-trivial separable logic $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$ and the binary system $\mathfrak{B}(x, y) = \{\langle \boldsymbol{u}_i(x), \boldsymbol{v}_i(y)\rangle : i \in n\}$. The following result follows at once from Corollary 12.44 on page 385.

Corollary 13.33 $\mathcal{K}$ is a $\mathfrak{B}_*$-semantics for $S(\mathcal{K}, \boldsymbol{U}_x \approx_n \boldsymbol{V}_y)$.

□

Since the logics of identified membership, introduced in Example 12.47 on page 385, are *non-trivial* separable logics, these must always have their determining quasivariety as a parameterized algebraic semantics.

Example 13.34 (Logics of Identified Membership)

Consider a quasivariety $\mathcal{K}$ and a $\mathcal{K}$-unary term $\boldsymbol{u}$. Let $\boldsymbol{u}(x,y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$. The following result follows from (2) of Corollary 12.50 on page 386.

Corollary 13.35 $\mathcal{K}$ is a $\langle\{\boldsymbol{u}(z)\}, z\rangle$-algebraic semantics for its u-membership logic $S(\mathcal{K}, \boldsymbol{u})$, with $\langle\{\boldsymbol{u}(z)\}, z\rangle$-defining equations $\boldsymbol{u}_*$. □

Recall that for any algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$,

$$\{\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!] \,:\, a \in \mathsf{uni}(\mathbf{A}), \alpha \in \mathsf{Con}_{\mathcal{K}}(\mathbf{A})\} \subseteq \mathsf{Cos}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A}) \tag{13.6}$$

(see by Corollary 12.51 on page 386). We shall now use the fact that $\mathcal{K}$ is a $\langle\{\boldsymbol{u}(z)\}, z\rangle$-algebraic semantics for $S(\mathcal{K}, \boldsymbol{u})$ to prove that

$$\mathsf{Cos}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A}) \subseteq \{\alpha[\![\boldsymbol{u}(a)]\!] \,:\, \alpha \in \mathsf{Con}(\mathbf{A})\}. \tag{13.7}$$

Note that the congruences on the right-hand-side are *non-relative*.

Corollary 13.36 $\mathsf{Cos}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A}) \subseteq \{\alpha[\![\boldsymbol{u}(a)]\!] \,:\, \alpha \in \mathsf{Con}(\mathbf{A})\}$, for any algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$.

Proof. Let

$$Y \in \mathsf{Cos}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A}) = \mathsf{Fi}_{S(\mathcal{K},\boldsymbol{u}_*)}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[\boldsymbol{u}_z/\bot_{\mathcal{K}}]\rangle).$$

We use Corollary 1.356 on page 68 to show that $Y \in \mathsf{PN}_{\boldsymbol{u}_a}(\mathbf{A})$. Suppose that $b, c \in Y$ and that q is a unary polynomial of $\mathbf{A}$ with $q(b) \in Y$. For some term $t(x, \vec{w})$ and some $\vec{d} \in \mathsf{uni}(\mathbf{A})$, we have $q(a') = t^{\mathbf{A}}(a', \vec{d})$ for all $a' \in \mathbf{A}$. Since $\mathcal{K} \models \big[x \approx \boldsymbol{u}(z) \text{ and } y \approx \boldsymbol{u}(z) \text{ and } t(x,\vec{w}) \approx \boldsymbol{u}(z)\big] \rightarrow t(y, \vec{w}) \approx \boldsymbol{u}(z)$, we have $\boldsymbol{u}_z/\bot_{\mathcal{K}}, x, y, t(x,\vec{w}) \vdash_{S(\mathcal{K},\boldsymbol{u}_*)} t(y,\vec{w})$, since, by Corollary 13.35, $\mathcal{K}$ is a $(\{\boldsymbol{u}(z)\}, z)$-algebraic semantics for $S(\mathcal{K}, \boldsymbol{u})$ $(= S(\mathcal{K}, \boldsymbol{u}_*))$ with $(\{\boldsymbol{u}(z)\}, z)$-defining equation $x \approx \boldsymbol{u}(y)$. Since $\mathsf{E}^{\mathbf{A}}_{z:a}[\boldsymbol{u}_z/\bot_{\mathcal{K}}] \subseteq Y$ and $b, c, t^{\mathbf{A}}(b, \vec{d}) = q(b) \in Y$, we have $q(c) = t^{\mathbf{A}}(c, \vec{d}) \in Y$.
◇

So by (13.6) and (13.7), if $\mathcal{K}$ is a *variety* and $\mathbf{A} \in \mathcal{K}$, then the $\langle\mathcal{K}, \boldsymbol{u}_*\rangle$-cosets at a are precisely the congruence classes containing $\boldsymbol{u}^{\mathbf{A}}(a)$.

Corollary 13.37 If $\mathcal{K}$ is a variety then $\mathsf{Cos}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A}) = \{\alpha[\![\boldsymbol{u}(a)]\!] \,:\, \alpha \in \mathsf{Con}(\mathbf{A})\}$, for all $\mathbf{A} \in \mathcal{K}$ and $a \in \mathsf{uni}(\mathbf{A})$.

We briefly highlight the cases for the logic $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ of idempotent $\langle\mathcal{K}, \boldsymbol{u}\rangle$-cosets, the membership logic and the assertional logic $S(\mathcal{K}, 0)$, all of which are encompassed by the logics of identified membership.

Corollary 13.38 If $\boldsymbol{u}$ is idempotent over $\mathcal{K}$ then $\mathcal{K}$ is a $\langle\{\boldsymbol{u}(z)\}, z\rangle$-algebraic semantics for the *idempotent $\boldsymbol{u}$-coset logic* $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$, with $\langle\{\boldsymbol{u}(z)\}, z\rangle$-defining equation $\boldsymbol{u}$.

Corollary 13.39 $\mathcal{K}$ is always a $\langle\{z\}, z\rangle$-algebraic semantics for its membership logic $S(\mathcal{K}, \mathsf{mem})$ with $\langle\{z\}, z\rangle$-defining equations $\{\langle x, y\rangle\}$. If $\mathcal{K}$ is a variety then, for any $\mathbf{A} \in \mathcal{K}$ the $\langle\mathcal{K}, y\rangle$-cosets of $\mathbf{A}$ are precisely the congruence classes of $\mathbf{A}$.

Corollary 13.40 [BR99] If $\mathcal{K}$ is a variety with constant term 0 then, for each $\mathbf{A} \in \mathcal{K}$, the $S(\mathcal{K}, 0)$-filters of $\mathbf{A}$ are precisely the congruence classes of $\mathbf{A}$ containing $0^{\mathbf{A}}$.

□

Open Problem 13.41 Does there exist a $\mathfrak{B}_*$-deductive quasivariety $\mathcal{K}$ such that $\sigma[\mathfrak{B}_z/\bot^{\mathcal{K}}] \not\models_{\mathcal{K}} \mathfrak{B}_z/\bot^{\mathcal{K}}$, where σ is the substitution mapping all variables to z?

Chapter 14

Parameterized Protoalgebraicity

We have noted that for a non-trivial quasivariety $\mathcal{K}$, the membership logic $S(\mathcal{K}, \mathsf{mem})$ fails to be protoalgebraic. It is useful to underpin the forthcoming parametrized abstraction of algebraizability with a weaker parametrized logical notion generalizing protoalgebraicity, which asks for the monotonicity of suitable *restrictions* of the Leibniz operator. The most primitive such notion, introduced in ∮14.1, requires only a set of terms X as a *parameter*. When a logic $\mathcal{S}$ has this property and, in addition, X generates a *z-invariant* $\mathcal{S}$-theory for some variable z, it will make sense to add z as a parameter, in which case we speak of *$\langle X, z\rangle$-protoalgebraicity*. In this case, the theories containing X satisfy a *relative* form of *structurality*, that is, if T is a theory containing X, then the theory generated by the image of T, under a substitution that *fixes* z, also contains X. Several characterizations of $\langle X, z\rangle$-protoalgebraicity are given, all of which specialize to well-known characterizations of protoalgebraicity in the case that X is the empty-set.

Algebraizable sentential calculi satisfy a property *stronger* than protoalgebraicity; they are *equivalential* [BP89a]. In ∮14.2 we introduce the natural parametrized version of this property, and logics which satisfy this property are termed *$\langle X, z\rangle$-equivalential*. In ∮15, we shall show that all sentential calculi that are $\langle X, z\rangle$-algebraizable (a notion defined in that chapter) are *$\langle X, z\rangle$-equivalential* and consequently *$\langle X, z\rangle$-protoalgebraic*.

In ∮14.3 we introduce the notion of a sentential calculus being *almost protoalgebraic*, which is the requirement that the logic have *no theorems* and that the Leibniz operator be inclusion preserving when restricted to *non-empty* theories. This property is valuable in that it requires *no parameter* in its formulation. We show that for logics with no theorems, the condition of almost-protoalgebraicity is equivalent to the property of $\langle\{z\}, z\rangle$-protoalgebraicity.

Finally, in ∮14.4 we apply the theory of parametrized protoalgebraicity to the logics introduced earlier in this text that have no theorems.

14.1 Protoalgebraicity at X

Unlike parameterized algebraic semantics, parameterized protoalgebraicity does not have a $\mathfrak{B}_*$ variant, since protoalgebraicity is a purely logical notion and not defined in terms of the relationship to some quasivariety. While we are most interested in $\langle X, z\rangle$-protoalgebraicity, we begin with a more general notion of X-protoalgebraicity, where X is a set of terms. Recall that except for the almost-trivial logic, a logic without theorems cannot be protoalgebraic, since the Leibniz operator always maps the empty set to the largest congruence, and as such cannot preserve the order of theories. The parameter X in the definition of X-protoalgebraicity, serves to specify for which theories we want the Leibniz operator to be order preserving, namely those above (i.e., including) X. In this way we can avoid problems such as the empty set when it is a theory.

Definition 14.1 (Protoalgebraicity at X) Let $\mathcal{S}$ be a sentential calculus and X a set of terms. We shall say that $\mathcal{S}$ is **X-protoalgebraic** if $\Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle} : \mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X\rangle) \rightarrow_{\leq} \mathbf{Con}(\mathbf{Tm})$. □

Clearly, $\emptyset$-protoalgebraicity is simply protoalgebraicity in the sense of [BP89a] (see Definition 2.132 on page 116 of our text). Further, $\mathcal{S}$ is X-protoalgebraic iff $\mathcal{S}$ is $\|X\|_{\mathcal{S}}$-protoalgebraic.

The equivalent conditions of the following lemma, while not concerned with X-protoalgebraicity as it stands, will turn out, under certain circumstances, to be equivalent to X-protoalgebraicity. This result is best interpreted as characterizing a parameterized version of the *filter correspondence property* (see Definition 2.132 on page 116). The reader is urged to recall the definition of the *total evaluation* $\mathsf{E}^{\mathbf{A}}_{z:a}[X]$ *with z fixed at a* (see Definition 12.37 on page 383).

Lemma 14.2 Let $\mathcal{S}$ be a sentential calculus, z a variable and X a set of terms. Then the following conditions are equivalent.

1. For every algebra $\mathbf{A}$ and each $a \in \mathsf{uni}(\mathbf{A})$, $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle} : \mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle) \rightarrow_{\leq} \mathbf{Con}(\mathbf{A})$.

2. For every algebra $\mathbf{A}$, $\alpha \in \mathsf{Con}(\mathbf{A})$, $a \in A$ and $F, H \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$ with $F \subseteq H$, α is compatible with H whenever α is compatible with F.

3. For any surjective homomorphism of algebras $h : \mathbf{A} \to \mathbf{B}$, $b \in \mathsf{uni}(\mathbf{B})$, $F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$ and $H \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{B}, \mathsf{E}^{\mathbf{B}}_{z:b}[X]\rangle)$, we have $F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H] = h^{-1}[\|h[F] \cup H\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}]$.

4. For every algebra $\mathbf{A}$, $a \in \mathsf{uni}(\mathbf{A})$ and $F \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$, if $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F)$ then $b \in \|\{c\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $c \in \|\{b\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Suppose α is compatible with F. Then $\alpha \subseteq \Omega^{\mathbf{A}}(F) \subseteq \Omega^{\mathbf{A}}(H)$, by (1). Let $b, c \in \mathsf{uni}(\mathbf{A})$ with $b \in H$ and $\langle b, c\rangle \in \alpha$. Then $\langle b, c\rangle \in \Omega^{\mathbf{A}}(H)$, which is compatible with H. It follows that $c \in H$, so α is compatible with H. $\boxed{(2)\Rightarrow(3)}$ Since $\mathcal{S}$-filterhood is preserved under inverse homomorphic images (Proposition 7.24 on page 257), $F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H] \subseteq h^{-1}[\|h[F] \cup H\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}]$. We prove the converse inclusion. Since h is surjective, there exists $a \in h^{-1}[\{b\}]$. We show that $\mathsf{E}^{\mathbf{A}}_{z:a}[X] \subseteq h^{-1}[H]$. Let $e \in \mathsf{hom}(\mathbf{Tm}, \mathbf{A})$ with $e(\vec{z}) = a$. Then $h \circ e \in \mathsf{hom}(\mathbf{Tm}, \mathbf{B})$ and $h(e(z)) = b$, hence $(h \circ e)[X] \subseteq H$ by assumption. Thus $e[X] \subseteq h^{-1}[H]$, as required. Since $\ker h$ is compatible with $h^{-1}[H]$, it follows from (2) that $\ker h$ is compatible with $F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H]$, so $h^{-1}[h[F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H]]] = F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H]$. Also, by Proposition 7.25 on page 258, $h[F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H]] \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{B})$. By the surjectivity of h, therefore, $\|h[F] \cup H\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}} \subseteq h[F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H]]$, whence $h^{-1}[\|h[F] \cup H\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}] \subseteq h^{-1}[h[F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H]]] = F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H]$. $\boxed{(3)\Rightarrow(4)}$ Suppose $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F)$. Let $\mathbf{B} = \mathbf{A}/\ \Omega^{\mathbf{A}}(F)$ and consider the canonical homomorphism $h : \mathbf{A} \to \mathbf{B}$. Since $\ker h$ is compatible with F, we have $H \doteq F/\ \Omega^{\mathbf{A}}(F) = h[F] \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{B})$, by Proposition 7.25 on page 258. We show that $\mathsf{E}^{\mathbf{A}/\Omega^{\mathbf{A}}(F)}_{z:a/\Omega^{\mathbf{A}}(F)}[X] \subseteq H$. Let $e : \mathbf{Tm} \to \mathbf{A}/\ \Omega^{\mathbf{A}}(F)$ with $e(z) = a/\ \Omega^{\mathbf{A}}(F)$. For each $v \in \mathsf{V}$, pick a representative $v^* \in e(v)$, and let $e' \in \mathsf{hom}(\mathbf{Tm}, \mathbf{A})$ be determined by the rules $e'(v) = v^*$ for $z \neq v \in \mathsf{V}$, and $e'(z) = a$. By assumption, $e'[X] \subseteq F$ and, since $e = h \circ e'$, we have $e[X] \subseteq F/\ \Omega^{\mathbf{A}}(F)$, as required. From $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F)$, we infer

$b \in h^{-1}[\{h(c)\}] \subseteq h^{-1}[\|h[\|\{c\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}] \cup H\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}]$. By (3), $b \in \|\{c\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}} \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H]$ and, since $\Omega^{\mathbf{A}}(F)$ is compatible with F, we have $h^{-1}[H] = h^{-1}[h[F]] = F$, as required. By symmetry, $c \in \|\{b\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}} \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} F$.
$\boxed{(4)\Rightarrow(1)}$ Let $F, H \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$ with $F \subseteq H$. Suppose $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F)$ and $b \in H$. By (4), $c \in \|\{b\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}} \subseteq H$. Thus, $\Omega^{\mathbf{A}}(F)$ is compatible with H. This forces $\Omega^{\mathbf{A}}(F) \subseteq \Omega^{\mathbf{A}}(H)$. $\diamond$

The proof of the following characterization of X-protoalgebraicity is very similar to (and slightly easier than) that of the previous lemma, and will therefore be omitted here. We shall see shortly that the equivalent conditions of this theorem imply the equivalent conditions of the previous lemma. While we have not found a counter example, it is our intuition that generally the converse is not valid.

Theorem 14.3 Let $\mathcal{S}$ be a sentential calculus and X a set of terms. The following conditions are equivalent.

1. $\mathcal{S}$ is X-protoalgebraic.
2. For every $\alpha \in \mathsf{Con}(\mathbf{Tm})$ and $U, W \in \mathsf{Th}(\mathcal{S}_{:X})$ with $U \subseteq W$, α is compatible with W whenever α is compatible with U.
3. For any algebra $\mathbf{B}$, any surjective homomorphism $h : \mathbf{Tm} \to \mathbf{B}$, any $\mathcal{S}$-theory U and any $F \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{B}, h[X]\rangle)$, we have $U \vee^{\mathcal{S}} h^{-1}[F] = h^{-1}[\|h[U] \cup F\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}]$.
4. For every $U \in \mathsf{Th}(\mathcal{S}_{:X})$, we have $U, p \vdash_{\mathcal{S}} q$ and $U, q \vdash_{\mathcal{S}} p$ whenever $\langle p, q\rangle \in \Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle}(U)$.

$\square$

We are mostly interested in the following X-protoalgebraic deductive systems.

Definition 14.4 ($\langle X, z\rangle$-Protoalgebraicity) Let $\mathcal{S}$ be an X-protoalgebraic sentential calculus and z a variable. If X generates a z-invariant $\mathcal{S}$-theory, we shall say that $\mathcal{S}$ is **$\langle X, z\rangle$-protoalgebraic**. $\square$

The following result includes a characterization of X-protoalgebraicity in the case that the $\mathcal{S}$-theory generated by X is z-invariant for some variable z. For a finite set of ternary terms $\mathbf{\Delta}$ and variables x and y, consider the conditions:

$$X \vdash_{\mathcal{S}} \mathbf{\Delta}(x, x, z), \tag{14.1}$$

$$X, y, \mathbf{\Delta}(x, y, z) \vdash_{\mathcal{S}} x; \quad \text{and} \tag{14.2}$$

$$y, \mathbf{\Delta}(x, y, z) \vdash_{\mathcal{S}} x. \tag{14.3}$$

Theorem 14.5 Let $\mathcal{S}$ be a sentential calculus, X a set of terms and z a variable. Consider the following conditions.

1. There exists a finite set of ternary terms $\mathbf{\Delta}$ such that (14.1) and (14.3) are satisfied.
2. There exists a finite set of ternary terms $\mathbf{\Delta}$ such that (14.1) and (14.2) are satisfied.
3. $\mathcal{S}$ is X-protoalgebraic.
4. The conditions of Lemma 14.2 are satisfied.

Generally, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If the $\mathcal{S}$-theory generated by X is z-invariant then all four conditions are equivalent.

Proof. $\boxed{(1) \Rightarrow (2)}$ Trivial. $\boxed{(2) \Rightarrow (3)}$ Assuming (2)'s hypotheses, we prove condition (4) of Theorem 14.3. First note that there is a finite $X' \subseteq X$ and variables v, w not occurring in $X' \cup \{z\}$ such that $X' \vdash_{\mathcal{S}} \Delta_i(v, v, z)$ for $i < m$ and

$$X', w, \Delta_0(v, w, z), \ldots, \Delta_{m-1}(v, w, z) \vdash_{\mathcal{S}} v$$

(by the finitariness and structurality of $\mathcal{S}$). Suppose $U \in \mathsf{Th}(\mathcal{S}_{:X})$ and $\langle p, q\rangle \in \Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle}(U)$. Let $i < m$. By structurality of $\mathcal{S}$ and the above, $X' \vdash_{\mathcal{S}} \Delta_i(p, p, z)$. Since $X' \subseteq U \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{Tm})$, we infer that $\Delta_i(p, p, z) \in U$. It follows by compatibility that $\Delta_i(p, q, z) \in U$, so $q, \Delta_i(p, q, z) \in \|U \cup \{q\}\|_{\mathcal{S}}$. Since $i < m$ was arbitrary, (14.2) gives $p \in \|U \cup \{q\}\|_{\mathcal{S}}$, whence $U, q \vdash_{\mathcal{S}} p$. By symmetry, $U, p \vdash_{\mathcal{S}} q$. $\boxed{(3) \Rightarrow (4)}$ Let $\mathcal{S}$ be X-protoalgebraic; we prove condition (1) of Lemma 14.2. Let $\mathbf{A}$ be an algebra and $F, H \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$ with $F \subseteq H$. Suppose, to the contrary, that there exists $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F) - \Omega^{\mathbf{A}}(H)$. By the remarks following Lemma 1.355, there exist $k \in \omega$, distinct variables $x_0, x_1, \ldots, x_{k+1} \in \mathsf{V} - \{z\}$, a term $t(x_0, x_2, x_3, \ldots, x_{k+1})$ and, for each $i < k$, an element $d_i \in \mathsf{uni}(\mathbf{A})$, such that

$$t^{\mathbf{A}}(b, \vec{d}) \in H \quad \text{and} \quad t^{\mathbf{A}}(c, \vec{d}) \notin H. \tag{i}$$

Let $h : \mathbf{Tm} \to \mathbf{A}$ such that $h(z) = a$, $h(x_0) = b$, $h(x_1) = c$ and $h(x_{2+i}) = d_i$, for $i < k$. Now $h^{-1}[F]$ and $h^{-1}[H]$ are $\mathcal{S}$-theories with $h^{-1}[F] \subseteq h^{-1}[H]$. Since $h : \mathbf{Tm} \to \mathbf{A}$ takes z to a, we have $h[X] \subseteq F$, i.e., $X \subseteq h^{-1}[F]$. By the X-protoalgebraicity of $\mathcal{S}$, $\Omega^{\mathbf{Tm}}(h^{-1}[F]) \subseteq \Omega^{\mathbf{Tm}}(h^{-1}[H])$. Since $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F)$, it follows from Lemma 16.21 on page 447 that

$$\begin{aligned}\langle t^{\mathbf{Tm}}(x_0, x_2, x_3, \ldots, x_{k+1}), t^{\mathbf{Tm}}(x_1, x_2, x_3, \ldots, x_{k+1})\rangle &\in h^{-1}[\Omega^{\mathbf{A}}(F)] \\ &\subseteq \Omega^{\mathbf{Tm}}(h^{-1}[F]).\end{aligned}$$

By (i), $t^{\mathbf{Tm}}(x_0, x_2, x_3, \ldots, x_{k+1}) \in h^{-1}[H]$, so $t^{\mathbf{Tm}}(x_1, x_2, x_3, \ldots, x_{k+1}) \in h^{-1}[H]$. Thus $t^{\mathbf{A}}(c, \vec{d}) \in H$, contradicting (i). $\square$

Now let $T = \|X\|_{\mathcal{S}}$ be z-invariant.

$\boxed{(4) \Rightarrow (1)}$ Assume that T is z-invariant and that $\mathcal{S}, X$ and z satisfy the equivalent conditions of Lemma 14.2. Let x and y be distinct variables with $z \notin \{x, y\}$. If η is the substitution sending y to x and fixing all other variables then $G \doteq \{t \in \mathsf{Tm} : X \vdash_{\mathcal{S}} \eta(t)\}$ is an $\mathcal{S}$-theory, by structurality of $\mathcal{S}$. It is easy to see that $\mathsf{E}^{\mathbf{Tm}}_{z:z}[X] \subseteq G$ and that $\langle x, y\rangle \in \Omega^{\mathbf{Tm}}(G)$. By Lemma 14.2(4), therefore, $y, G \vdash_{\mathcal{S}} x$. Then, since $\mathcal{S}$ is finitary, there are terms $\Delta_0(x, y, z, u_0, \ldots, u_k), \ldots, \Delta_{m-1}(x, y, z, u_0, \ldots, u_k) \in G$ such that $y, \Delta_0(x, y, z, \vec{u}), \ldots, \Delta_{m-1}(x, y, z, \vec{u}) \vdash_{\mathcal{S}} x$. (Here $\vec{u}$ are variables distinct from x, y and z.) Then $y, \Delta_0(x, y, z, z, \ldots, z), \ldots, \Delta_{m-1}(x, y, z, z, \ldots, z) \vdash_{\mathcal{S}} x$. We have $X \vdash_{\mathcal{S}} \Delta_j(x, x, z, \vec{u})$ for all $j < m$. By applying the substitution that sends $u_0, \ldots, u_k$ to z and fixes all other variables, we obtain $X \vdash_{\mathcal{S}} \Delta_j(x, x, z, z, \ldots, z)$ for each $j < m$. $\diamond$

Thus, when X generates a z-invariant $\mathcal{S}$-theory, we may use (14.3) and (14.2) interchangeably in the (restricted) characterization of X-protoalgebraicity given by the above result. Note, however, that the value in V of the meta-variable z is immaterial (by structurality) in (14.3) but *not* in (14.2).

Remark 14.6 When X generates a *fully* invariant $\mathcal{S}$-theory (i.e., the generated theory is closed under *all* substitutions), the ternary terms in Theorem 14.5(1) may be replaced by binary terms.

Remark 14.7 $\mathcal{S}$ is $\emptyset$-protoalgebraic iff $\mathcal{S}$ is protoalgebraic in the sense of Blok and Pigozzi [BP86],[BP88],[BP92] (see Definition 2.132 on page 116). Since the fully invariant set of all $\mathcal{S}$-theorems is generated as an $\mathcal{S}$-theory by $\emptyset$, Lemma 14.2 and Theorems 14.3 and 14.5 generalize certain results from these papers (see Theorem 2.135 on page 117, formulated for sentential 1-calculi and Corollary 2.138 on page 118).

It follows at once from conditions (1) and (3) of the previous theorem that if $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic then $\mathcal{S}$ is $(\sigma[X], y)$-protoalgebraic, where y is *any* variable and σ is the transposition (yz).

For an arbitrary sentential calculus $\mathcal{S}$ and an algebra $\mathbf{A}$, the closure operator $\|\cdot\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ need have no simple internal characterization. As the next result shows, when $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic, however, the generation of certain filters becomes relatively easy to describe.

Corollary 14.8 Let $\mathcal{S}$ be an $\langle X, z\rangle$-protoalgebraic sentential calculus, $\mathbf{A}$ an algebra, $Y \subseteq \mathsf{uni}(\mathbf{A})$, $c \in \mathsf{uni}(\mathbf{A})$ and $F = \|Y\|^{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:c}[X]\rangle}_{\mathsf{fi}_{\mathcal{S}}}$. Let H be the set of all $b \in \mathsf{uni}(\mathbf{A})$ such that for some $G \cup \{t\} \subseteq \mathsf{Tm}$ and some $\bar{a} \in \mathsf{uni}(\mathbf{A})^{\omega}$, we have $G \vdash_{\mathcal{S}} t$ and $G^{\mathbf{A}}(\bar{a}) \subseteq Y \cup \|\mathsf{E}^{\mathbf{A}}_{z:c}[X]\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $t^{\mathbf{A}}(\bar{a}) = b$. Then $F = H$.

Proof. Clearly $H \subseteq F$, so we need only show that H is an $\mathcal{S}$-filter of $\mathbf{A}$. Let $G \vdash_{\mathcal{S}} t$ and let $\bar{a} \in \mathsf{uni}(\mathbf{A})^{\omega}$ with $G^{\mathbf{A}}(\bar{a}) \subseteq H$. By finitariness of $\mathcal{S}$, we may assume that $G = \{p_i : i < k\}$ for some $k \in \omega$. For each $i < k$, there exist $G_i \cup \{s_i\} \subseteq \mathsf{Tm}$ and $\bar{b}_i \in \mathsf{uni}(\mathbf{A})^{\omega}$ with $G_i \vdash_{\mathcal{S}} s_i$ such that $G_i^{\mathbf{A}}(\bar{b}_i) \subseteq Y \cup \|\mathsf{E}^{\mathbf{A}}_{z:c}[X]\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $s_i^{\mathbf{A}}(\bar{b}_i) = p_i^{\mathbf{A}}(\bar{a})$. We may assume that for distinct $i, j < k$, the sets of variables occurring, respectively, in $G_i \cup \{s_i\}$, in $G_j \cup \{s_j\}$ and in $G \cup \{t\}$ are mutually disjoint and omit z. By condition (3) of the previous theorem, $\left(\bigcup_{i<k} G_i\right), \{\Delta_j(p_i, s_i, z) : i < k, j < m\} \vdash_{\mathcal{S}} t$. There is a homomorphism $e : \mathbf{Tm} \to \mathbf{A}$ which, for each $i < k$, sends a variable $\mathbf{v}_r$ to $\bar{b}_i(r)$ [resp. $\bar{a}(r)$] if it is used by $G_i \cup \{s_i\}$ [resp. $G \cup \{t\}$], and which sends z to c. Using such a homomorphism e, we obtain $\{\Delta_j^{\mathbf{A}}(p_i^{\mathbf{A}}(\bar{a}), s_i^{\mathbf{A}}(\bar{b}_i), c) : i < k, j < m\} \subseteq \|\mathsf{E}^{\mathbf{A}}_{z:c}[X]\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $\bigcup_{i<k} G_i^{\mathbf{A}}(\bar{b}_i) \subseteq Y \cup \|\mathsf{E}^{\mathbf{A}}_{z:c}[X]\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$. Thus, $t^{\mathbf{A}}(\bar{a}) \in H$. $\diamond$

Corollary 14.9 Let $\mathcal{S}$ be an $\langle X, z\rangle$-protoalgebraic sentential calculus, $\mathbf{A}$ an algebra and $a \in \mathsf{uni}(\mathbf{A})$. Then $\Omega^{\mathcal{S}}_{\mathbf{A}}[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)]$ is closed under arbitrary intersections.

Proof. Let $\{F_i : i \in I\} \subseteq \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$. By Theorem 14.5 and Lemma 14.2, $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle}$ preserves order, hence $\Omega^{\mathbf{A}}(\bigcap_I F_i) \subseteq \bigcap_I \Omega^{\mathbf{A}}(F_i)$. For the reverse inclusion, note that $\bigcap_I \Omega^{\mathbf{A}}(F_i) \in \mathsf{Con}(\mathbf{A})$, and that $\bigcap_I \Omega^{\mathbf{A}}(F_i)$ is compatible with $\bigcap_I F_i$; the result following by Proposition 16.10 on page 443. $\diamond$

14.2 $\langle X, z\rangle$-Equivalential Logics

We turn now to consider conditions stronger than $\langle X, z\rangle$-protoalgebraicity.

Definition 14.10 (Implication and Equivalence Terms, $\langle X, z\rangle$-Equivalential) A family $\mathbf{\Delta}$ of ternary terms is called a **system of $\langle X, z\rangle$-implication terms for $\mathcal{S}$** if, for all variables x and y,

$$X \vdash_{\mathcal{S}} \mathbf{\Delta}(x,x,z) \quad \text{and} \tag{Rlx}$$

$$X, y, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} x; \tag{Det}$$

and it is called a **system of $\langle X, z\rangle$-equivalence terms for** $\mathcal{S}$ if, in addition, for all variables x and y, any $l \in \omega$ and any $(l+1)$-ary term $t(u, \vec{v})$, we have

$$X, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} \mathbf{\Delta}\big(t(x, v_0, \ldots, v_{l-1}), t(y, v_0, \ldots, v_{l-1}), z\big). \tag{Sub}$$

A sentential calculus $\mathcal{S}$ is called $\langle X, z\rangle$**-equivalential** if it has a *finite* system of $\langle X, z\rangle$-equivalence terms. $\square$

The following corollary follows from the previous definition and Theorem 14.5.

Corollary 14.11 Let X be a set of terms and z a variable.

1. If $\mathcal{S}$ is $\langle X, z\rangle$-equivalential then $\mathcal{S}$ is X-protoalgebraic.
2. If $\mathcal{S}$ is $\langle X, z\rangle$-equivalential and X generates a z-invariant $\mathcal{S}$-theory, then $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic.
3. When X generates a z-invariant $\mathcal{S}$-theory, $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic iff it has a finite system of $\langle X, z\rangle$-implication terms.

Remark 14.12 Under the assumptions of (2) of the previous corollary, X may be dropped from (Det)).

We now characterize the property that a sentential calculus have a system of $\langle X, z\rangle$-equivalence terms under the assumption that the theory generated by X is z-invariant. Equivalent condition (2) shows how to 'evaluate' the Leibniz operator for certain theories.

Theorem 14.13 Let $\mathcal{S}$ be a sentential calculus, let $X \subseteq \mathsf{Tm}$ generate a z-invariant $\mathcal{S}$-theory T and let $\mathbf{\Delta} = \{\Delta_i : i \in I\}$ be a family of ternary terms. The following conditions are equivalent.

1. $\mathbf{\Delta}$ is a system of $\langle X, z\rangle$-equivalence terms for $\mathcal{S}$.
2. For every algebra $\mathbf{A}$, $a \in \mathsf{uni}(\mathbf{A})$ and $F \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$,

$$\Omega^{\mathbf{A}}(F) = \{\langle b, c\rangle \in \mathsf{uni}(\mathbf{A})^2 : \mathbf{\Delta}^{\mathbf{A}}(b,c,a) \subseteq F\}. \tag{14.4}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ Let $\alpha = \{\langle b, c\rangle \in \mathsf{uni}(\mathbf{A})^2 : \mathbf{\Delta}^{\mathbf{A}}(b,c,a) \subseteq F\}$. We show first that α is a congruence of $\mathbf{A}$ that is compatible with F. Let x and y be distinct variables with $z \notin \{x, y\}$. Let $b \in \mathsf{uni}(\mathbf{A})$ and consider any homomorphism $e : \mathbf{Tm} \to \mathbf{A}$ with $e(z) = a$ and $e(x) = b$. Since $e[X] \subseteq F$, it follows from (Rlx) that $\mathbf{\Delta}^{\mathbf{A}}(b,b,a) \subseteq F$. Thus, α is reflexive. Suppose that $\mathbf{\Delta}^{\mathbf{A}}(b,c,a) \subseteq F$. Then for each $\Delta_k \in \mathbf{\Delta}$, we have $\mathbf{\Delta}^{\mathbf{A}}(\Delta^{\mathbf{A}}_k(c,b,a), \Delta^{\mathbf{A}}_k(c,c,a), a) \subseteq F$ (by (Sub)) and, since $\mathbf{\Delta}^{\mathbf{A}}(c,c,a) \subseteq F$, we infer $\Delta^{\mathbf{A}}_k(c,b,a) \in F$ from (Det). Thus, α is symmetric. In a similar manner one shows that α is transitive. Now (Sub) and transitivity may be used to show that α is compatible with fundamental operations of $\mathbf{A}$, so $\alpha \in \mathsf{Con}(\mathbf{A})$. Suppose $\langle b, c\rangle \in \alpha$ with $b \in F$. Let $e : \mathbf{Tm} \to \mathbf{A}$ be any homomorphism with $e(z) = a$, $e(x) = c$ and

$e(y) = b$. Then $e[X] \cup \mathbf{\Delta}^{\mathbf{A}}(c,b,a) \cup \{b\} \subseteq F$, whence $c \in F$, by (Det). Consequently, $\alpha \subseteq \Omega^{\mathbf{A}}(F)$. For the converse inclusion, suppose $\langle b,c\rangle \in \Omega^{\mathbf{A}}(F)$. Let $\Delta_k \in \mathbf{\Delta}$. Now $\langle \Delta_k^{\mathbf{A}}(b,b,a), \Delta_k^{\mathbf{A}}(b,c,a)\rangle \in \Omega^{\mathbf{A}}(F)$. By the compatibility of $\Omega^{\mathbf{A}}(F)$ with F, we have $\Delta_k^{\mathbf{A}}(b,c,a) \in F$. Thus, $\langle b,c\rangle \in \alpha$. $\boxed{(2)\Rightarrow(1)}$ Let x and y be variables other than z. Since T is z-invariant, $\mathsf{E}^{\mathbf{Tm}}_{z:z}[T] = T$, hence $T \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, \mathsf{E}^{\mathbf{Tm}}_{z:z}[X]\rangle)$. By assumption, $\Omega^{\mathbf{Tm}}(T) = \{\langle p,q\rangle \in \mathsf{Tm}^2 : \mathbf{\Delta}(p,q,z) \subseteq T\}$. Since $\langle x,x\rangle \in \Omega^{\mathbf{Tm}}(T)$, we have $\mathbf{\Delta}(x,x,z) \subseteq T$, hence $X \vdash_{\mathcal{S}} \mathbf{\Delta}(x,x,z)$. Let $U = \|X \cup \{y\} \cup \mathbf{\Delta}(x,y,z)\|_{\mathcal{S}}$. Then $\langle x,y\rangle \in \Omega^{\mathbf{Tm}}(U)$, whereupon compatibility requires that $x \in U$, proving (Det). Let $W = \|X \cup \mathbf{\Delta}(x,y,z)\|_{\mathcal{S}}$ and let $t(u,\vec{v})$ be any $(l+1)$-ary term, $l \in \omega$. Now for each $i < l$, we have $\langle x,y\rangle \in \Omega^{\mathbf{Tm}}(W)$, whence $\langle t(x,\vec{v}), t(y,\vec{v})\rangle \in \Omega^{\mathbf{Tm}}(W)$, and (Sub) follows from (14.4). ◇

When it is tedious or difficult to establish condition (Sub) directly, the following result is useful.

Proposition 14.14 Let X generate a z-invariant $\mathcal{S}$-theory T and let $\mathbf{\Delta}$ be a family of ternary terms. If

$$X, \mathbf{\Delta}(u_0,v_0,z), \ldots, \mathbf{\Delta}(u_{l-1},v_{l-1},z) \vdash_{\mathcal{S}} \mathbf{\Delta}(\star(\vec{u}), \star(\vec{v}), z) \tag{Sub'}$$

for every $l \in \omega$, every l-ary fundamental operation symbol $\star$ and any variables $\vec{u}, \vec{v}$, then $\mathcal{S}, X$ and z satisfy (Sub).

Thus, a system $\mathbf{\Delta}$ of $\langle X,z\rangle$-implication terms for $\mathcal{S}$ is a system of $\langle X,z\rangle$-equivalence terms for $\mathcal{S}$ iff $\mathcal{S}, X$ and z satisfy (Sub').

Proof. We prove the first assertion by induction on the number of operation symbols occurring *explicitly* in the term $t(u,\vec{v})$ from (Sub). Suppose $\mathcal{S}, X$ and z satisfy (Sub'). If $t(u,\vec{v})$ is a projection or a constant, the result follows easily. Suppose that $r \in \omega$, that $t(u,\vec{v}) = f(t_0(u,\vec{v}), \ldots, t_r(u,\vec{v}))$ and that $X, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} \mathbf{\Delta}(t_i(x,\vec{v}), t_i(y,\vec{v}), z)$, for all $i \leq r$. By finitariness and structurality of $\mathcal{S}$, we may assume without loss of generality that z, x, y and $\vec{v}$ are distinct. Then by structurality, the z-invariance of T and (Sub'),

$$\begin{aligned} X, \mathbf{\Delta}(t_0(x,\vec{v}), t_0(y,\vec{v}), z), \ldots, \mathbf{\Delta}(t_r(x,\vec{v}), t_r(y,\vec{v}), z) \\ \vdash_{\mathcal{S}} \mathbf{\Delta}\big(f(t_0(x,\vec{v}), \ldots, t_r(x,\vec{v})), f(t_0(y,\vec{v}), \ldots, t_r(y,\vec{v})), z\big). \end{aligned}$$

In the last assertion it suffices, by the above, to show that the existence of a system $\mathbf{\Delta}$ of $\langle X,z\rangle$-equivalence terms for $\mathcal{S}$ forces $\mathcal{S}, X$ and z to satisfy (Sub'). Let $U = \|\{\mathbf{\Delta}(u_i,v_i,z) : i < l\}\|^{\langle \mathbf{Tm}, X\rangle}_{\mathsf{fi}_{\mathcal{S}}}$. By Theorem 14.13, $\langle u_i, v_i\rangle \in \Omega^{\mathbf{Tm}}(U)$ for each $i < l$, whence $\langle f(\vec{u}), f(\vec{v})\rangle \in \Omega^{\mathbf{Tm}}(U)$. A further application of Theorem 14.13 proves (Sub'). ◇

Recall the definition of a $\sqcup$-preserving function between orders (see Definition 1.180 on page 40), which is a function that preserves directed joins. We note once again, that in the literature such functions are called *continuous* [BP89a]. We have avoided this usage of the term 'continuous' given our very different usage of this term with regard to functions and translations between closed systems.

Corollary 14.15 If $\mathcal{S}$ is $\langle X,z\rangle$-equivalential and X generates a z-invariant $\mathcal{S}$ theory, then for each algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$,

$$\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle} : \mathbf{Fi}^{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle) \rightarrow_{\sqcup} \mathbf{Con}(\mathbf{A}). \tag{14.5}$$

Proof. Suppose that $\mathbf{\Delta}$ is a finite system of $\langle X, z\rangle$-equivalence terms for $\mathcal{S}$. Let $\mathbf{A}$ be an algebra and $a \in \mathsf{uni}(\mathbf{A})$. Let $\{F_j : j \in J\} \subseteq \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$ be upward directed by inclusion and let $F = \bigcup_J F_j$. Since $\mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$ is algebraic, $F \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$. By Lemma 14.2 and Theorem 14.5, $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle}$ preserves order, so $\bigcup_J \Omega^{\mathbf{A}}(F_j) \subseteq \Omega^{\mathbf{A}}(F)$. Let $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F)$. By the previous theorem, $\mathbf{\Delta}^{\mathbf{A}}(b, c, a) \subseteq F$. Since $\mathbf{\Delta}$ is a finite family and F an upward directed union, $\mathbf{\Delta}^{\mathbf{A}}(b, c, a) \subseteq F_j$ for some $j \in J$, hence $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F_j)$. $\diamond$

14.3 Almost-Protoalgebraic Logics

We now consider the situation where a sentential calculus fails to have theorems, yet the Leibniz relation is order preserving with respect to non-empty theories. We shall call such logics *almost-protoalgebraic*; only the presence of the empty theory is preventing (full) protoalgebraicity.

Definition 14.16 (Almost-Protoalgebraic Logics) We shall say that a sentential calculus $\mathcal{S}$ is **almost-protoalgebraic** if $\mathcal{S}$ has no theorems and $\Omega^{\mathcal{S}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Con}(\mathbf{Tm})$ is order-preserving when restricted to *non-empty* sets. $\square$

Since $\{z\}$ always generates a z-invariant $\mathcal{S}$-theory, the following corollary follows (almost directly) from Theorem 14.5.

Corollary 14.17 Let z be any variable. Then $\mathcal{S}$ is almost protoalgebraic iff $\mathcal{S}$ is $\langle \{z\}, z\rangle$-protoalgebraic and has no theorems.

Proof. $\boxed{\Rightarrow}$ Suppose that $\mathcal{S}$ is almost protoalgebraic. Then it is certainly $\langle \{z\}, z\rangle$-protoalgebraic, by Theorem 14.5, and, by definition, has no theorems. $\boxed{\Leftarrow}$ Conversely, suppose that $\mathcal{S}$ is $\langle \{z\}, z\rangle$-protoalgebraic and has no theorems. Let T and R by two $\mathcal{S}$-theories with $T \subseteq R$. By assumption, $\emptyset \neq T$. Let $p \in T$. Then $\mathsf{E}^{\mathbf{Tm}}_{z:p}[\{z\}] = \bigcup\{\sigma[\{z\}] : \sigma \in \mathsf{Sub}(\mathbf{Tm}) \text{ and } \sigma(z) = p\} = \{p\} \subseteq T \subseteq R$. By equivalent condition (1) of Lemma 14.2, together with Theorem 14.5, and the fact that $\mathcal{S}$-theories and $\mathcal{S}$-filters on $\mathbf{Tm}$ coincide since $\mathcal{S}$ is structural (see Theorem 7.48 on page 263), $\Omega^{\mathcal{S}}(T) \subseteq \Omega^{\mathcal{S}}(R)$. $\diamond$

The following characterizations of almost-protoalgebraicity, derive easily from Lemma 14.2 and Theorem 14.3, together with the previous corollary.

Theorem 14.18 For a sentential calculus $\mathcal{S}$ *without theorems*, the following conditions are equivalent.

1. $\mathcal{S}$ is almost-protoalgebraic.
2. For every algebra $\mathbf{A}$ the map $\Omega^{\mathcal{S}}_{\mathbf{A}} : \mathsf{Fi}_{\mathcal{S}}(\mathbf{A}) \to \mathbf{Con}(\mathbf{A})$ is order preserving when restricted to non-empty sets.
3. For every algebra $\mathbf{A}$, $\alpha \in \mathsf{Con}(\mathbf{A})$ and $F, H \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$ with $\emptyset \neq F \subseteq H$, α is compatible with H whenever α is compatible with F.
4. For any surjective homomorphism of algebras $h : \mathbf{A} \to \mathbf{B}$, $\emptyset \neq F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$ and $\emptyset \neq H \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{B})$, we have $F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})} h^{-1}[H] = h^{-1}[\|h[F] \cup H\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}]$.

5. For every algebra $\mathbf{A}$, $\emptyset \neq F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{A})$, if $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F)$ then $b \in \|\{c\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $c \in \|\{b\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$.

6. For every $\alpha \in \mathsf{Con}(\mathbf{Tm})$ and $U, W \in \mathsf{Th}(\mathcal{S})$ with $\emptyset \neq U \subseteq W$, α is compatible with W whenever α is compatible with U.

7. For any algebra $\mathbf{B}$, any surjective homomorphism $h : \mathbf{Tm} \to \mathbf{B}$, any non-empty $\mathcal{S}$-theory U and any $\emptyset \neq F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{B})$, we have $U \vee^{\mathcal{S}} h^{-1}[F] = h^{-1}[\|h[U] \cup F\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{S}}}]$.

8. For every $\emptyset \neq U \in \mathsf{Th}(\mathcal{S})$, we have $U, p \vdash_{\mathcal{S}} q$ and $U, q \vdash_{\mathcal{S}} p$ whenever $\langle p, q\rangle \in \Omega^{\mathcal{S}}(U)$.

9. There exist $m \in \omega$ and ternary terms $\Delta_0, \dots, \Delta_{m-1}$ such that

$$z \vdash_{\mathcal{S}} \Delta_i(x, x, z), \quad \text{for } i < m; \tag{14.6}$$

$$y, \Delta_0(x, y, z), \dots, \Delta_{m-1}(x, y, z) \vdash_{\mathcal{S}} x. \tag{14.7}$$

□

14.4 Examples

Recall the definition of the *equational logic* $S(\mathcal{K}, \approx)$ determined by a quasivariety $\mathcal{K}$ (see Example 8.38 on page 291). In the following example we show that this logic is almost protoalgebraic iff the quasivariety $\mathcal{K}$ is trivial.

Example 14.19 (Equational Logics)

Note that the logic $S(\mathcal{K}, \approx)$ has no theorems.

Proposition 14.20 $S(\mathcal{K}, \approx)$ is almost-protoalgebraic iff $\mathcal{K}$ is trivial.

Proof. Suppose that $S(\mathcal{K}, \approx)$ satisfies Equations (14.6) and (14.7). By (14.7) and Corollary 8.40 of Example 8.38 on page 291, either $\models_{\mathcal{K}} y \approx x$, in which case the result follows, or there exists a single ternary term Δ, such that $\models_{\mathcal{K}} \Delta(x, y, z) \approx x$, and so $\models_{\mathcal{K}} \Delta(x, x, z) \approx x$ by structurality. But by (14.6), $\models_{\mathcal{K}} \Delta(x, x, z) \approx z$, hence $\models_{\mathcal{K}} z \approx x$, and again the result follows. ◇

□

In the next example, we consider the *subuniverse logic* $S(\mathcal{K}, \mathsf{su})$ determined by a quasivariety $\mathcal{K}$ (see Example 5.48 on page 188 and Example 8.51 on page 293) of $\mathfrak{a}$-algebras where the type $\mathfrak{a}$ has *no constant symbols*. By assuming that $\mathfrak{a}$ has no constant symbols, the logic $S(\mathcal{K}, \mathsf{su})$ has no theorems, and as such is a candidate for almost-protoalgebraicity. We shall characterize the almost-protoalgebraicity of $S(\mathcal{K}, \mathsf{su})$ and relate this condition to subuniverse coherence (see ∮10.1).

Example 14.21 (Subuniverse Logics)

Let $\mathfrak{a}$ be a type of algebras *without constant symbols* and $\mathcal{K}$ an $\mathfrak{a}$-quasivariety. Then certainly, $S(\mathcal{K}, \mathsf{su})$ *has no theorems.* Let $\mathcal{V}$ be the variety generated by $\mathcal{K}$.

Proposition 14.22 The following conditions are equivalent.

1. $S(\mathcal{K},\mathsf{su})$ is almost-protoalgebraic.

2. There exists a positive integer n, ternary terms $\Delta_1,\dots,\Delta_m$, unary terms $\boldsymbol{u}_1,\dots,\boldsymbol{u}_m$, and an $(m+1)$-ary term q, such that,

$$\models_{\mathcal{K}} \boldsymbol{u}_i(z) \approx \Delta_i(x,x,z), \quad \text{for each } i\in n, \text{ and} \tag{14.8}$$

$$\models_{\mathcal{K}} x \approx q(y,\Delta_0(x,y,z),\dots,\Delta_{n-1}(x,y,z)). \tag{14.9}$$

3. $S(\mathcal{V},\mathsf{su})$ is almost-protoalgebraic.

4. For every $\mathbf{A}\in\mathcal{V}$, $\alpha\in\mathsf{Con}(\mathbf{A})$ and $B,C\in\mathsf{Su}(\mathbf{A})$, if $\emptyset\neq B\subseteq C$ and α is compatible with B, then α is compatible with C.

5. If $\alpha\in\mathsf{Con}(\mathbf{F}_{\mathcal{V}})$ and $B,C\in\mathsf{Su}(\mathbf{F}_{\mathcal{V}})$, if $\emptyset\neq B\subseteq C$ and α is compatible with B, then α is compatible with C.

Proof. Recall that by (8.9) of Example 8.51 on page 293, $P\vdash_{S(\mathcal{K},\mathsf{su})} p$ iff $\overline{[P]}\vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}\overline{p}$. $\boxed{(1)\Rightarrow(2)}$ By Theorem 14.18, there exists a finite set $\boldsymbol{\Delta}$ of ternary terms such that $z\vdash_{S(\mathcal{K},\mathsf{su})}\boldsymbol{\Delta}(x,x,z)$ and $y,\boldsymbol{\Delta}(x,y,z)\vdash_{S(\mathcal{K},\mathsf{su})} x$. So $\overline{z}\vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}\overline{[\boldsymbol{\Delta}(x,x,z)]}$ and $\overline{y},\overline{[\boldsymbol{\Delta}(x,y,z)]}\vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}\overline{x}$. The result follows by Remark 6.81 of Example 6.79 on page 242, together with Lemma 1.457 on page 88. $\boxed{(2)\Rightarrow(1)}$ So $\boldsymbol{u}_i^{\mathbf{F}_{\mathcal{K}}}(\overline{z})=\Delta_i^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{x},\overline{z})$, for each i, and $\overline{x}=q^{\mathbf{F}_{\mathcal{K}}}(\overline{y},\Delta_0^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{y},\overline{z}),\dots,\Delta_{n-1}^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{y},\overline{z}))$. Hence $\Delta_i^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{x},\overline{z})\in\|\overline{z}\|_{\mathsf{su}}^{\mathbf{F}_{\mathcal{K}}}$, and $\overline{x}\in\left\|\{\overline{y},\Delta_0^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{y},\overline{z}),\dots,\Delta_{n-1}^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{y},\overline{z})\}\right\|_{\mathsf{su}}^{\mathbf{F}_{\mathcal{K}}}$, by Theorem 1.344 on page 65. Hence, by definition, $\overline{z}\vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}\Delta_i^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{x},\overline{z})$, for each i, and $\overline{y},\Delta_0^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{y},\overline{z}),\dots,\Delta_{n-1}^{\mathbf{F}_{\mathcal{K}}}(\overline{x},\overline{y},\overline{z})\vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})}\overline{x}$. Hence $z\vdash_{S(\mathcal{K},\mathsf{su})}\Delta_i(x,x,z)$, for each i, and $y,\Delta_0(x,y,z),\dots,\Delta_{n-1}(x,y,z)\vdash_{S(\mathcal{K},\mathsf{su})} x$. The result follows by Theorem 14.18. $\boxed{(2)\Leftrightarrow(3)}$ Since $\mathcal{K}$ and $\mathcal{V}$ satisfy precisely the same identities, the result follows by the arguments as (1)⇒(2) and (2)⇒(1). $\boxed{(3)\Rightarrow(4)}$ Follows by Theorem 14.18 and the fact that the archology $\mathfrak{A}(\mathcal{V},\mathsf{su})$ is maximal, by Proposition 8.92 of Example 8.89 on page 301. $\boxed{(4)\Rightarrow(5)}$ Trivial. $\boxed{(5)\Rightarrow(3)}$ Let T and R be $S(\mathcal{V},\mathsf{su})$-theories and $\alpha\in\mathsf{Con}(\mathbf{Tm})$ such that $\emptyset\neq T\subseteq R$ and α compatible with T. Suppose that $p\in R$ and $p\,\alpha\,q$. By Proposition 1.358 on page 68, $\boldsymbol{\alpha}=\{\langle\overline{r_1},\overline{r_2}\rangle : \langle r_1,r_2\rangle\in\alpha\}$ is a congruence on $\mathbf{F}_{\mathcal{V}}$. Further $\overline{[T]},\overline{[R]}\in\mathsf{Su}(\mathbf{F}_{\mathcal{V}})$ (by (8.2) of Example 8.51), $\emptyset\neq\overline{[T]}\subseteq\overline{[R]}$, and certainly $\boldsymbol{\alpha}$ is compatible with $\overline{[T]}$, and hence is compatible with $\overline{[R]}$, by assumption. Since $\overline{p}\in\overline{[R]}$ and $\overline{p}\;\boldsymbol{\alpha}\;\overline{q}$, $\overline{q}\in\overline{[R]}$. Hence $q\in\underline{\left[\overline{[R]}\right]}=R$, by Corollary 8.54 of Example 8.51). The result follows by Theorem 14.18. ◇

Open Problem 14.23 We have not been able to establish that the following necessary conditions are indeed equivalent to the conditions of the previous result.

1. For any $\mathbf{A},\mathbf{B}\in\mathcal{V}$, $h:\mathbf{A}\twoheadrightarrow\mathbf{B}$, $\emptyset\neq F\in\mathsf{Su}(\mathbf{A})$ and $\emptyset\neq H\in\mathsf{Su}(\mathbf{B})$, we have $\left\|F\cup h^{-1}[H]\right\|_{\mathsf{su}}^{\mathbf{A}}=h^{-1}[\|h[F]\cup H\|_{\mathsf{su}}^{\mathbf{B}}]$.
2. For any algebra $\mathbf{B}\in\mathcal{V}$, $h:\mathbf{F}_{\mathcal{V}}\twoheadrightarrow\mathbf{B}$, any $\emptyset\neq\boldsymbol{T}\in\mathsf{Su}(\mathbf{F}_{\mathcal{V}})$, and any $\emptyset\neq F\in\mathsf{Su}(\mathbf{B})$, we have $\left\|\boldsymbol{T}\cup h^{-1}[F]\right\|_{\mathsf{su}}^{\mathbf{F}_{\mathcal{V}}}=h^{-1}[\|h[\boldsymbol{T}]\cup F\|_{\mathsf{su}}^{\mathbf{B}}]$.

Recall the definitions of *subalgebra $\langle\mathcal{K},\boldsymbol{u}_1,\dots,\boldsymbol{u}_n\rangle$-coherence* and *subalgebra $\mathcal{K}$-coherence*, given in ∮10.1, as well as the characterizations of these conditions, given in Theorem 10.2 and Corollary 10.3, respectively. A comparison of equivalent condition (2) of the previous proposition with equivalent condition (2) of Theorem 10.2, demonstrates that if $S(\mathcal{K},\mathsf{su})$ is *almost-protoalgebraic*, then there exist *some* unary terms $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$, such that $\mathcal{K}$ is *subalgebra $\langle\mathcal{K},\boldsymbol{u}_1,\dots,\boldsymbol{u}_n\rangle$-coherent.*

Corollary 14.24 If $S(\mathcal{K}, \mathsf{su})$ is almost-protoalgebraic then $\mathcal{K}$ is subalgebra $\langle \mathcal{K}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent, for some unary terms $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$. □

While we have not found a counter-example yet, it appears that the converse is not generally true. If one compares (14.8) of the previous proposition with (10.1) of Theorem 10.2, in the former, there is a one-to-one correspondence between the $\boldsymbol{u}_i$'s and the Δ_i's, while in the latter, the correspondence between the $\boldsymbol{u}_i$'s and the Δ_i's is weaker, given the role of the selection function in (10.1). If, however, $\mathcal{K}$ is $\mathcal{K}$-*coherent*, then, by Corollary 10.3, $S(\mathcal{K}, \mathsf{su})$ *is almost-protoalgebraic.*

Corollary 14.25 If $\mathcal{K}$ is subalgebra $\mathcal{K}$-coherent then $S(\mathcal{K}, \mathsf{su})$ is almost-protoalgebraic.

Open Problem 14.26 Show that if $\mathcal{K}$ is subalgebra $\langle \mathcal{K}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$-coherent, for some unary terms $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$, $S(\mathcal{K}, \mathsf{su})$ need not be almost-protoalgebraic.

Open Problem 14.27 Show that if $S(\mathcal{K}, \mathsf{su})$ is almost-protoalgebraic $\mathcal{K}$ need not be subalgebra $\mathcal{K}$-coherent.

□

Just as characterizing the protoalgebraicity of the sentential 1-calculi $S(\mathcal{K}, \tau)$ is an important problem in algebraic logic, characterizing the $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z \rangle$-protoalgebraicity of $S(\mathcal{K}, \mathfrak{B}_*)$ is important in parameterized algebraic logic. This is the aim of the next example. Recall that $S(\mathcal{K}, \tau)$ is protoalgebraic precisely when $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \tau \rangle$-classes (see Example 2.140 on page 118). We shall (analogously) relate the $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z \rangle$-protoalgebraicity of $S(\mathcal{K}, \mathfrak{B}_*)$ with the condition that $\mathcal{K}$ have $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes (see ∮10.3).

Example 14.28 (Characterizing $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z \rangle$-Protoalgebraicity of $S(\mathcal{K}, \mathfrak{B}_*)$)

Since $\mathfrak{B}_z/\bot_{\mathcal{K}}$ is z-invariant, the following characterization of the protoalgebraicity of $S(\mathcal{K}, \mathfrak{B}_*)$, follows at once from Theorem 14.5.

Corollary 14.29 The following conditions are equivalent.

1. $S(\mathcal{K}, \mathfrak{B}_*)$ is $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z \rangle$-protoalgebraic.
2. There exists a finite set of ternary terms $\boldsymbol{\Delta}$ and variables x and y, satisfying
$$\mathfrak{B}_z/\bot_{\mathcal{K}} \vdash_{\mathcal{S}} \boldsymbol{\Delta}(x, x, z), \tag{14.10}$$
$$\mathfrak{B}_z/\bot_{\mathcal{K}}, y, \boldsymbol{\Delta}(x, y, z) \vdash_{\mathcal{S}} x. \tag{14.11}$$
3. There exists a finite set of ternary terms $\boldsymbol{\Delta}$ and variables x and y, satisfying (14.10) and
$$y, \boldsymbol{\Delta}(x, y, z) \vdash_{\mathcal{S}} x. \tag{14.12}$$

Lemma 14.30 If $S(\mathcal{K}, \mathfrak{B}_*)$ is $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z \rangle$-protoalgebraic, then the following statements are all valid.

1. The $S(\mathcal{K}, \mathfrak{B}_*)$-filters of every algebra are $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-coherent.
2. The $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-cosets of every algebra are $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-coherent.
3. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.

4. $\mathcal{K}$ has weakly $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.

Proof. $\boxed{\scriptstyle (1)}$ Let $\mathbf{A}$ be any algebra, F an $S(\mathcal{K}, \mathfrak{B}_*)$-filter of $\mathbf{A}$, $a \in \mathsf{uni}(\mathbf{A})$ and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$ with $\mathfrak{B}_a^{\mathbf{A}} / \alpha \subseteq F$. By Proposition 12.41 on page 384, $\mathsf{E}^{\mathbf{A}}_{z:a}[\mathfrak{B}_z / \bot_{\mathcal{K}}] \subseteq \mathfrak{B}_a^{\mathbf{A}} / \alpha \subseteq F$. By Proposition 9.31 on page 321, α is compatible with $\mathfrak{B}_a^{\mathbf{A}} / \alpha$, which is an $S(\mathcal{K}, \mathfrak{B}_*)$-filter of $\mathbf{A}$ by Proposition 12.34 on page 382. Now α is compatible with F by Theorem 14.5 and (2) of Lemma 14.2. Thus, if $\langle b, c \rangle \in \alpha$ with $c \in F$, then $b \in F$. $\boxed{\scriptstyle (2)}$ Follows from (1) since, by definition, $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-cosets are $S(\mathcal{K}, \mathfrak{B}_*)$-filters. $\boxed{\scriptstyle (3)}$ Follows from (2) and Proposition 12.41 on page 384. $\boxed{\scriptstyle (4)}$ Follows from (3) and Remark 10.12 on page 348. $\diamond$

The relationship between the $\langle \mathfrak{B}_z / \bot_{\mathcal{K}}, z \rangle$-protoalgebraicity of $S(\mathcal{K}, \mathfrak{B}_*)$ and $\mathcal{K}$ having $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes, is tightest when $\mathfrak{B}_*$ *pivots finitarily.*

Theorem 14.31 Let $\mathfrak{B}$ be a binary system of equations such that $\mathfrak{B}_*$ *pivots finitarily* for quasivariety $\mathcal{K}$ and let z be any variable. Then the following conditions are equivalent.

1. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.
2. $\mathcal{K}$ has weakly $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-classes.
3. $S(\mathcal{K}, \mathfrak{B}_*)$ is $\langle \mathfrak{B}_z / \bot_{\mathcal{K}}, z \rangle$-protoalgebraic.
4. The $S(\mathcal{K}, \mathfrak{B}_*)$-filters of every algebra are $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-coherent.
5. The $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-cosets of every algebra are $\langle \mathcal{K}, \mathfrak{B}_* \rangle$-coherent.

Proof. $\boxed{\scriptstyle (1)\Leftrightarrow(2)}$ By Remark 10.12 on page 348 and Theorem 10.14 on page 348. $\boxed{\scriptstyle (2)\Rightarrow(3)}$ By Theorem 10.10 on page 347, there exists a finite set $\mathbf{\Delta}$ of ternary terms such that

$$\models_{\mathcal{K}} \mathfrak{B}_z^{\approx} [\mathbf{\Delta}(x, x, z)], \quad \text{and}$$
$$\models_{\mathcal{K}} \bigwedge \mathfrak{B}_z^{\approx} [\![y]\!] \;\mathsf{and}\; \bigwedge \mathfrak{B}_z^{\approx} [\mathbf{\Delta}(x, y, z)] \rightarrow \mathfrak{B}_z^{\approx} [\![x]\!].$$

In other words,

$$\models_{\mathcal{K}} \mathfrak{B}_z^{\approx} [\mathbf{\Delta}(x, x, z)], \quad \text{and}$$
$$\mathfrak{B}_z^{\approx} [\![y]\!] \cup \mathfrak{B}_z^{\approx} [\mathbf{\Delta}(x, y, z)] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx} [\![x]\!].$$

Since $\mathfrak{B}$ pivots for quasivariety $\mathcal{K}$, there exists a finite subset $Z \subseteq_{\mathsf{f}} \mathfrak{B}_z / \bot_{\mathcal{K}}$, such that, for all variables w,

$$\mathfrak{B}_w^{\approx} [Z] \models_{\mathcal{K}} \mathfrak{B}_w^{\approx} [\mathbf{\Delta}(x, x, z)], \quad \text{and}$$
$$\mathfrak{B}_w^{\approx} [Z] \cup \mathfrak{B}_w^{\approx} [\![y]\!] \cup \mathfrak{B}_w^{\approx} [\mathbf{\Delta}(x, y, z)] \models_{\mathcal{K}} \mathfrak{B}_w^{\approx} [\![x]\!].$$

Since the variable w is arbitrary,

$$\forall[\mathsf{z}]\; \mathfrak{B}_{\mathsf{z}}^{\approx} [Z] \models_{\mathcal{K}} \mathfrak{B}_{\mathsf{z}}^{\approx} [\mathbf{\Delta}(x, x, z)], \quad \text{and}$$
$$\forall[\mathsf{z}]\; \mathfrak{B}_{\mathsf{z}}^{\approx} [Z] \cup \mathfrak{B}_{\mathsf{z}}^{\approx} [\![y]\!] \cup \mathfrak{B}_{\mathsf{z}}^{\approx} [\mathbf{\Delta}(x, y, z)] \models_{\mathcal{K}} \mathfrak{B}_{\mathsf{z}}^{\approx} [\![x]\!].$$

Since Z is finite, by definition, $Z \vdash \mathbf{\Delta}(x, x, z)$ and $Z, y, \mathbf{\Delta}(x, y, z) \vdash x$ are $S(\mathcal{K}, \mathfrak{B}_*)$-rules, hence

$$\mathfrak{B}_z / \bot_{\mathcal{K}} \vdash_{S(\mathcal{K}, \mathfrak{B}_*)} \mathbf{\Delta}(x, x, z), \quad \text{and}$$
$$\mathfrak{B}_z / \bot_{\mathcal{K}}, y, \mathbf{\Delta}(x, y, z) \vdash_{S(\mathcal{K}, \mathfrak{B}_*)} x.$$

The result follows by Corollary 14.29. $\boxed{\scriptstyle (3)\Rightarrow(4)\Rightarrow(5)\Rightarrow(1)}$ The proof proceeds as in the proof of Lemma 14.30. $\diamond$

$\square$

Example 14.32 (The Logics of Identified Membership Logics)

Let $\boldsymbol{u}$ be a $\mathcal{K}$-unary term and let $\boldsymbol{u}(x,y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$. Recall that in this case, $\boldsymbol{u}_*$ pivots *finitarily* for $\mathcal{K}$, by Proposition 9.61 of Example 9.58 on page 327. Recall further, that $\|u(z)\|_{S(\mathcal{K},\boldsymbol{u})} = \|\boldsymbol{u}_z/\bot_{\mathcal{K}}\|_{S(\mathcal{K},\boldsymbol{u})}$, by Corollary 12.50 of Example 12.47 on page 385. This observation permits the slightly simpler conditions of (5) of the following result.

Corollary 14.33 The following conditions are equivalent.

1. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-classes.
2. $S(\mathcal{K}, \boldsymbol{u})$ is $\langle\{u(z)\}, z\rangle$-protoalgebraic.
3. The $S(\mathcal{K}, \boldsymbol{u})$-filters of every algebra are $\langle \mathcal{K}, \boldsymbol{u}\rangle$-coherent.
4. The $\langle \mathcal{K}, \boldsymbol{u}\rangle$-cosets of every algebra are $\langle \mathcal{K}, \boldsymbol{u}\rangle$-coherent.
5. There exists a finite set of ternary terms $\boldsymbol{\Delta}$ and variables x and y, satisfying

$$\boldsymbol{u}(z) \vdash_{\mathcal{S}} \boldsymbol{\Delta}(x,x,z), \tag{14.13}$$
$$\boldsymbol{u}(z), y, \boldsymbol{\Delta}(x,y,z) \vdash_{\mathcal{S}} x. \tag{14.14}$$

6. There exists a finite set of ternary terms $\boldsymbol{\Delta}$ and variables x and y, satisfying (14.13) and

$$y, \boldsymbol{\Delta}(x,y,z) \vdash_{\mathcal{S}} x. \tag{14.15}$$

$\square$

Example 14.34 (The Idempotent u-Coset Logics and Others)

Let $\boldsymbol{u}$ be an unary term *idempotent* over $\mathcal{K}$. In the case that $\boldsymbol{u}$ is *not $\mathcal{K}$-constant*, we may add an extra equivalent condition to the characterization of $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ being $\langle\{\boldsymbol{u}(z)\}, z\rangle$-protoalgebraic, namely that $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ be **almost protoalgebraic**. This follows since, by Remark 9.85 on page 333, $\{p\} \vdash_{S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})} \boldsymbol{u}(p)$. Note that we have to avoid the $\mathcal{K}$-constant case because of the definition of almost-protoalgebraic.

Corollary 14.35 For non $\mathcal{K}$-constant $\boldsymbol{u}$, the following conditions are equivalent.

1. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \boldsymbol{u}\rangle$-classes.
2. $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is $\langle\{\boldsymbol{u}(z)\}, z\rangle$-protoalgebraic.
3. $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$ is *almost*-protoalgebraic.
4. The $S_{\mathsf{i}}(\boldsymbol{u}\text{-}\mathsf{cos}^{\mathcal{K}})$-filters of every algebra are $\mathcal{K}$-coherent.
5. There exists a finite set of ternary terms $\boldsymbol{\Delta}$ and variables x and y, satisfying

$$z \vdash_{\mathcal{S}} \boldsymbol{\Delta}(x,x,z), \tag{14.16}$$
$$z, y, \boldsymbol{\Delta}(x,y,z) \vdash_{\mathcal{S}} x. \tag{14.17}$$

6. There exists a finite set of ternary terms $\boldsymbol{\Delta}$ and variables x and y, satisfying (14.16) and

$$y, \boldsymbol{\Delta}(x,y,z) \vdash_{\mathcal{S}} x. \tag{14.18}$$

$\square$

Example 14.36 (The Membership Logic)

The following characterization of the $\langle\{z\}, z\rangle$-protoalgebraicity of the membership logic $S(\mathcal{K}, \mathsf{mem})$ is an immediate special case of Corollary 14.33. Note that we are not using Corollary 14.35, since the quasivariety $\mathcal{K}$ may be trivial, in which case the term z is $\mathcal{K}$-constant.

Corollary 14.37 The following conditions are equivalent.

1. $\mathcal{K}$ has $\mathcal{K}$-coherent $\mathcal{K}$-classes.
2. $S(\mathcal{K}, \mathsf{mem})$ is $\langle\{z\}, z\rangle$-protoalgebraic.
3. The $S(\mathcal{K}, \mathsf{mem})$-filters of every algebra are $\mathcal{K}$-coherent.
4. There exists a finite set of ternary terms $\boldsymbol{\Delta}$ and variables x and y, satisfying

$$z \vdash_{\mathcal{S}} \boldsymbol{\Delta}(x, x, z), \tag{14.19}$$

$$z, y, \boldsymbol{\Delta}(x, y, z) \vdash_{\mathcal{S}} x. \tag{14.20}$$

5. There exists a finite set of ternary terms $\boldsymbol{\Delta}$ and variables x and y, satisfying (14.19) and

$$y, \boldsymbol{\Delta}(x, y, z) \vdash_{\mathcal{S}} x. \tag{14.21}$$

□

Example 14.38 ($S(\mathcal{K}, \tau)$)

In the case that τ is $\mathcal{K}$-unary, then the standard characterization of the protoalgebraicity of the logic $S(\mathcal{K}, \tau)$ of [BR99] follows. Note that in this case $\tau_z/\bot_{\mathcal{K}} = \tau/\bot_{\mathcal{K}} = \mathsf{Thm}(S(\mathcal{K}, \tau))$. Consequently, $\langle \tau_z/\bot_{\mathcal{K}}, z\rangle$-protoalgebraicity is simply protoalgebraicity. So Theorem 2.143 on page 119 obtains immediately.

□

Chapter 15

Parameterized Equivalent Algebraic Semantics

In this chapter we complete our theory of *parametrized algebraization*, establishing a notion that a sentential calculus have a *parametrized equivalent algebraic semantics*.

In $\oint$15.1 we introduce the notion of an *equivalent* $\langle X, z\rangle$*-algebraic semantics*, and establish the basic properties of such logics. In $\oint$15.2, we show that every logic having an equivalent $\langle X, z\rangle$-algebraic semantics must be $\langle X, z\rangle$-protoalgebraic, and in $\oint$15.3 we show that, if a logic has an $\langle X, z\rangle$-algebraic semantics, this $\langle X, z\rangle$-algebraic semantics is *unique*. A number of characterizations are established in $\oint$15.4, including characterizations in terms of the Leibniz operator, characterizations in terms of the existence of formulae that must be satisfied by the logics, and a characterization in terms of conditions that a quasivariety must satisfy in order for there to exist a logic with this quasivariety as its parametrized equivalent algebraic semantics, the latter condition being a form of regularity. These results are generalizations of, and specialize to, the standard characterizations of algebraicity in the literature of algebraizable logics.

A number of examples are presented in $\oint$15.5. Of particular importance is an example showing that a quasivariety is *relatively regular* precisely when its membership logic is $\langle\{z\}, z\rangle$algebraizable.

Convention 15.1 We remind the reader that unless specified to the contrary $\mathcal{S}$ denotes a sentential 1-calculus, X a set of terms, z a variable and $\mathcal{K}$ a quasivariety of algebras (over the same language $\mathfrak{a}$), all fixed but arbitrary.

Convention 15.2 Let $\mathfrak{B}$ be a system of binary equations, $\mathbf{\Delta}$ a family of ternary terms, $\mathbf{A}$ an algebra, $P \subseteq \mathsf{uni}(\mathbf{A})$ and $a, b, c, d \in \mathsf{uni}(\mathbf{A})$. For a set Σ of equations and a term p, $\mathbf{\Delta}(\Sigma, p)$ abbreviates $\{\Delta(q, r, p) \ : \ \Delta \in \mathbf{\Delta}, q \approx r \in \Sigma\}$. We write $\mathbf{\Delta}(q \approx r, p)$ for $\mathbf{\Delta}(\{q \approx r\}, p)$. For $\alpha \subseteq \mathsf{uni}(\mathbf{A})^2$ and $a \in \mathsf{uni}(\mathbf{A})$, we write $\mathbf{\Delta}^{\mathbf{A}}(\alpha, a)$ for $\{\Delta^{\mathbf{A}}(b, c, a) \ : \ \Delta \in \mathbf{\Delta}, \langle b, c\rangle \in \alpha\}$, dropping the superscript '$\mathbf{A}$' in the case that $\mathbf{A}$ is the term algebra on V.

15.1 Definition

We now introduce the definition of a $\langle X, z\rangle$-equivalent algebraic semantics. Observe that the *binary* equivalence formulae $\mathbf{\Delta}$ involved in the definition of an algebraic semantics (see Definition 2.110 on page 111) have been replaced by *ternary* terms; this is to 'accommodate' the variable z. Further, observe that the equational consequence $\models_{\mathcal{K}}$ is *fully* interpreted in the sentential consequence relation $\vdash_{\mathcal{S}}$. Note that while the following definition has been formulated minimally, we shall shortly show this usage to be compatible with our existing terminology (see lemmas 15.5 and 15.7).

Definition 15.3 ($\langle X, z\rangle$-Equivalent Algebraic Semantics) We shall call $\mathcal{K}$ an **$\langle X, z\rangle$-equivalent algebraic semantics for** $\mathcal{S}$ if there exist a *binary* system $\mathfrak{B}$ and a finite family $\mathbf{\Delta}$ of *ternary* terms such that, for any set $\Sigma \cup \{r \approx s\}$ of equations and any $t \in \mathsf{Tm}$,

$$\Sigma \models_{\mathcal{K}} r \approx s \text{ iff } X, \mathbf{\Delta}(\Sigma, z) \vdash_{\mathcal{S}} \mathbf{\Delta}(r \approx s, z), \quad \text{and} \tag{15.1}$$

$$X, \mathbf{\Delta}(\mathfrak{B}_z^{\approx}[\![t]\!], z) \dashv\vdash_{\mathcal{S}} X, t. \tag{15.2}$$

In this case $\mathfrak{B}_*$ and $\mathbf{\Delta}$ are called, respectively, a set of **$\langle X, z\rangle$-defining equations**, and a set of **$\langle X, z\rangle$-equivalence terms** for $\mathcal{S}$ and $\mathcal{K}$, and we say that $\mathcal{S}$ is **$\langle X, z\rangle$-equivalent** to $\mathcal{K}$. □

Remark 15.4 We may replace X by $\|X\|_{\mathcal{S}}$ throughout the definition of $\langle X, z\rangle$-equivalent algebraic semantics without changing its meaning.

Due to the minimality of the previous definition, it is not clear that an $\langle X, z\rangle$-equivalent algebraic semantics is an $\langle X, z\rangle$-algebraic semantics. In the following result, we characterize the notion of an $\langle X, z\rangle$-equivalent algebraic semantics in terms of an $\langle X, z\rangle$-algebraic semantics and a $\models_{\mathcal{K}}$ analogue of (15.2). Notice that (15.3) amounts to a full 'untranslation' of any equation, while (15.2) 'untranslates' only 'in the context of X'. Similarly, while the equational consequence $\models_{\mathcal{K}}$ is *fully* interpreted in the sentential consequence relation $\vdash_{\mathcal{S}}$ in (15.1), only *some* $\vdash_{\mathcal{S}}$ consequences are interpreted in $\models_{\mathcal{K}}$, namely those 'prefixed' by X (implicit in the definition of $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$, see (13.3)). The alert reader will notice that (15.3) is precisely the quasi-Mal'cev condition characterizing the $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-regularity of $\mathcal{K}$ (see Theorem 11.18 on page 366).

Lemma 15.5 Given a system of binary equations $\mathfrak{B}$ and a finite family $\mathbf{\Delta}$ of ternary terms, the following conditions are equivalent.

1. $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ satisfying

$$\mathfrak{B}_z^{\approx}[\mathbf{\Delta}(x, y, z)] \dashv\models_{\mathcal{K}} x \approx y. \tag{15.3}$$

2. (15.1) and (15.2) hold for $\mathcal{S}$ and $\mathcal{K}$.

In particular, an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with respect to the same defining equations.

Proof. $\boxed{(1)\Rightarrow(2)}$ By (15.3) and (13.3), $\Sigma \models_{\mathcal{K}} r \approx s$ [iff] $\mathfrak{B}_z^{\approx}[\mathbf{\Delta}(\Sigma, z)] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\mathbf{\Delta}(r, s, z)]$ [iff] $X, \bigcup_{g \approx h \in \Sigma} \mathbf{\Delta}(g, h, z) \vdash_{\mathcal{S}} \mathbf{\Delta}(r, s, z)$. Also, by (15.3), we have $\mathfrak{B}_z^{\approx}[\mathbf{\Delta}(\mathfrak{B}_z[\![t]\!], z)] \dashv\models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![t]\!]$ so, by

(13.3), $X, \mathbf{\Delta}(\mathfrak{B}_z[\![t]\!], z) \dashv\vdash_{\mathcal{S}} X, p$. $\boxed{(2)\Rightarrow(1)}$ By (15.2) and (15.1), $X, P \vdash_{\mathcal{S}} p$ iff $X, \mathbf{\Delta}(\mathfrak{B}_z[P], z) \vdash_{\mathcal{S}} \mathbf{\Delta}(\mathfrak{B}_z[\![t]\!], z)$ iff $\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]$. By (15.2), $X, \mathbf{\Delta}(\mathfrak{B}_z[\mathbf{\Delta}(x,y,z)], z) \dashv\vdash_{\mathcal{S}} X, \mathbf{\Delta}(x,y,z)$, so by (15.1), $\mathfrak{B}_z^{\approx}[\mathbf{\Delta}(x,y,z)] \mathrel{=\!\mid\!\models}_{\mathcal{K}} x \approx y$. ◇

Under the equivalent conditions of Lemma 15.5, $\|X\|_{\mathcal{S}} = \mathfrak{B}_z/\bot_{\mathcal{K}}$, $\|X\|_{\mathcal{S}}$ is z-invariant and, for every variable y, $\mathcal{K}$ is a $(\mathfrak{B}_y/\bot_{\mathcal{K}}, y)$-equivalent algebraic semantics for $\mathcal{S}$; the $\langle X, z\rangle$-defining equations and equivalence terms serve the same purpose for the new parameters (see Proposition 13.2). Consequently, the definition of $\langle X, z\rangle$-equivalent algebraic semantics may be reformulated in terms of $\mathfrak{B}$ and $\mathbf{\Delta}$ only.

Definition 15.6 ($\mathfrak{B}_*$-Equivalent Algebraic Semantics) For a binary system $\mathfrak{B}$, we shall call $\mathcal{K}$ a **$\mathfrak{B}_*$-equivalent algebraic semantics for** $\mathcal{S}$ if there exists a finite family $\mathbf{\Delta}$ of *ternary* terms such that, for any set $\Sigma \cup \{r \approx s\}$ of equations, any $t \in \mathsf{Tm}$ and all variables y,

$$\Sigma \models_{\mathcal{K}} r \approx s \text{ iff } \mathfrak{B}_y/\bot_{\mathcal{K}}, \mathbf{\Delta}(\Sigma, y) \vdash_{\mathcal{S}} \mathbf{\Delta}(r \approx s, y), \quad \text{and} \tag{15.4}$$

$$\mathfrak{B}_y/\bot_{\mathcal{K}}, \mathbf{\Delta}(\mathfrak{B}_y^{\approx}[\![t]\!], y) \dashv\vdash_{\mathcal{S}} \mathfrak{B}_y/\bot_{\mathcal{K}}, t. \tag{15.5}$$

□

15.2 Relationships to $\langle X, z\rangle$-Protoalgebraicity

The following two lemmas list conditions forced upon $\mathcal{S}$ when it has an $\langle X, z\rangle$-equivalent algebraic semantics; these will contribute to a characterization of logics possessing such a semantics (Theorem 15.14). Those listed in the first lemma generalize parts of a characterization of algebraizable logics given in [BP89a]; the extra conditions of the latter lemma have no such antecedent, but are automatically satisfied when $X = \emptyset$, $\mathcal{S}$ is algebraizable and $\mathfrak{B}$ and $\mathbf{\Delta}$ augment the defining equations and equivalence terms of $\mathcal{S}$ by fictitious last coordinates. The reader is urged to recall Definition 14.10 on page 409.

Lemma 15.7 Let $\mathcal{K}$ be an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-equivalence terms $\mathbf{\Delta}$. Then $\mathcal{S}$ satisfies (Rlx), (Det), (Sub),

$$X, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} \mathbf{\Delta}(y,x,z); \tag{Sym}$$

$$X, \mathbf{\Delta}(x,y,z), \mathbf{\Delta}(y,w,z) \vdash_{\mathcal{S}} \mathbf{\Delta}(x,w,z). \tag{Trn}$$

Consequently, $\mathcal{S}$ is $\langle X, z\rangle$-equivalential and, since the $\mathcal{S}$-theory generated by X is z-invariant, $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic.

Proof. We prove (Det) as an illustration. Observe that $\mathfrak{B}_z^{\approx}[\![y]\!] \cup \{x \approx y\} \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![x]\!]$ so, by (15.3), $\mathfrak{B}_z^{\approx}[\![y]\!] \cup \mathfrak{B}_z^{\approx}[\mathbf{\Delta}(x,y,z)] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![x]\!]$, whence $X, y, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} x$ (by (13.3)). ◇

Lemma 15.8 Let $\mathcal{K}$ be an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-equivalence terms $\mathbf{\Delta}$ and $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$. Then

$$X, \mathbf{\Delta}(x,y,z) \dashv\vdash_{\mathcal{S}} X, \mathbf{\Delta}(\mathfrak{B}_w[\mathbf{\Delta}(x,y,w)], z), \tag{Ex1}$$

and whenever $\{g(z,y,\vec{u}) : g \in P\} \cup \{t(z,y,\vec{u})\} \subseteq \mathsf{Tm}$ and σ is the transposition (yz),

$$X, P \vdash_{\mathcal{S}} p \text{ iff } X, \mathbf{\Delta}(\mathfrak{B}_y[\sigma[P]], z) \vdash_{\mathcal{S}} \mathbf{\Delta}(\mathfrak{B}_y[\![\sigma(t)]\!], z), \tag{Ex2}$$

(where the range of $\vec{u} \in \mathsf{V}^\omega$ is understood to exclude y and z).

Proof. Interpret the result of replacing z by w in (15.3) as two instances of the left hand side of (15.1). (Ex1) is the conjunction of corresponding instances of the right hand side of (15.1).
Now $X, P \vdash_{\mathcal{S}} p$ is equivalent, by (13.3), to $\mathfrak{B}_z^{\approx}[P] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\![p]\!]$ which, by the structurality of $\models_{\mathcal{K}}$ and the invertibility of σ, is equivalent to $\mathfrak{B}_y^{\approx}[\sigma[P]] \models_{\mathcal{K}} \mathfrak{B}_y^{\approx}[\![\sigma(t)]\!]$. By (15.3), this is equivalent to $\mathfrak{B}_z^{\approx}[\mathbf{\Delta}(\mathfrak{B}_y[\sigma[P]], z)] \models_{\mathcal{K}} \mathfrak{B}_z^{\approx}[\mathbf{\Delta}(\mathfrak{B}_y[\![\sigma(t)]\!], z)]$ and, by (13.3), to $X, \mathbf{\Delta}(\mathfrak{B}_y[\sigma[P]], z) \vdash_{\mathcal{S}} \mathbf{\Delta}(\mathfrak{B}_y[\![\sigma(t)]\!], z)$, as required by (Ex2). $\diamond$

15.3 Uniqueness

Recall that if a sentential calculus has an equivalent algebraic semantics, then this equivalent algebraic semantics is unique [BP89a] (see Corollary 2.115 on page 113 of our text). The following result shows that (for fixed X and z) an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ (if such exists) is *unique*. Of course, a sentential calculus may have different $\langle X, z\rangle$-equivalent algebraic semantics for different values of X (in fact, *every* sentential calculus has the trivial quasivariety as its *unique* $\langle \mathsf{Tm}, z\rangle$-equivalent algebraic semantics); this reflects the earlier observation that when $\mathcal{K}$ is an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$, while all of the equational consequence $\models_{\mathcal{K}}$ is interpreted in the $\mathcal{S}$-consequence $\vdash_{\mathcal{S}}$, only some of the $\mathcal{S}$-consequence $\vdash_{\mathcal{S}}$ is interpreted in $\models_{\mathcal{K}}$, namely those consequences 'in the context of X'.

Theorem 15.9 Let $\mathcal{K}$ and $\mathcal{K}'$ be two $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ with respective $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ and $\mathfrak{B}'_*$, and $\langle X, z\rangle$-equivalence terms $\mathbf{\Delta}$ and $\mathbf{\Delta}'$. Then $X, \mathbf{\Delta}(x,y,z) \dashv\vdash_{\mathcal{S}} X, \mathbf{\Delta}'(x,y,z)$, $\mathcal{K} = \mathcal{K}'$ and $\mathfrak{B}_z^{\approx}(x) \dashv\models_{\mathcal{K}} \mathfrak{B}'^{\approx}_z(x)$.

Proof. By (Sub), $X, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} \mathbf{\Delta}(\Delta'_j(x,y,z), \Delta'_j(y,y,z), z)$ for each $\Delta'_j \in \mathbf{\Delta}'$ and, since $X \vdash_{\mathcal{S}} \mathbf{\Delta}'(y,y,z)$ (by (Rlx)), $X, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} \mathbf{\Delta}'(x,y,z)$, by (Det). By symmetry, $X, \mathbf{\Delta}'(x,y,z) \vdash_{\mathcal{S}} \mathbf{\Delta}(x,y,z)$. Now, by (15.1), $\Sigma \models_{\mathcal{K}} r \approx s$ iff $\Sigma \models_{\mathcal{K}'} r \approx s$, so $\mathcal{K} = \mathcal{K}'$. Finally, by (15.2), $X, \mathbf{\Delta}(\mathfrak{B}_z[\![x]\!], z) \dashv\vdash_{\mathcal{S}} X, \mathbf{\Delta}'(\mathfrak{B}'_z[\![x]\!], z)$, hence $X, \mathbf{\Delta}(\mathfrak{B}_z[\![x]\!], z) \dashv\vdash_{\mathcal{S}} X, \mathbf{\Delta}(\mathfrak{B}'_z[\![x]\!], z)$, from which we infer, by (15.1), $\mathfrak{B}_z^{\approx}[\![x]\!] \dashv\models_{\mathcal{K}} \mathfrak{B}'^{\approx}_z[\![x]\!]$. $\diamond$

15.4 Characterizations

We now present a number of characterizations of the property that a logic have a parametrized equivalent algebraic semantics, and the property that a quasivariety be a parametrized equivalent algebraic semantics of some logic.

15.4.1 Via the Leibniz Operator

Recall that $\mathcal{K}$ is the equivalent algebraic semantics for S iff for each algebra $\mathbf{A}$, the Leibniz operator induces an isomorphism from the $\mathcal{S}$-filter lattice on $\mathbf{A}$ onto the $\mathcal{K}$-relative congruence lattice on $\mathbf{A}$, and equivalently, the Leibniz operator induces an isomorphism from the theory lattice onto the relative congruence lattice on the term algebra [BP89a] (see Theorem 2.117 on page 113 of our text). We now aim to obtain an analogous characterization of a sentential 1-calculus having an $\langle X, z\rangle$-equivalent algebraic semantics.

The difficulty is in isolating which filters we aim to have isomorphic to the relative congruences. In the theory version of this result it is clearer. Certainly we should expect the lattice of all theories containing X to be isomorphic to the relative congruence lattice on the term algebra, and it ought to be true that for each variable y, the lattice of theories containing (the theory) $\mathfrak{B}_y/\bot_{\mathcal{K}}$ is also isomorphic to the relative congruence lattice on the term algebra.

In the following result, we show that if $\mathcal{K}$ is an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$, then for each algebra $\mathbf{A}$ and point a in the universe of this algebra, the Leibniz operator induces an isomorphism from the lattice of all $\mathcal{S}$-filters on $\mathbf{A}$ that contain the solution $\mathfrak{B}_a^{\mathbf{A}}/\bot_{\mathbf{A}}^{\mathcal{K}}$ onto the relative congruence lattice on $\mathbf{A}$. We show further, that in this case, $\mathfrak{B}_a^{\mathbf{A}}/\bot_{\mathbf{A}}^{\mathcal{K}}$ is in fact a filter and this filter is precisely the filter generated by the total evaluation $\mathsf{E}_{y:a}^{\mathbf{A}}[\mathfrak{B}_y/\bot_{\mathcal{K}}]$ of $\mathfrak{B}_y/\bot_{\mathcal{K}}$ with y fixed at a, for *any* variable y.

Theorem 15.10 Let $\mathcal{K}$ be the $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ and $\langle X, z\rangle$-equivalence terms $\mathbf{\Delta}$. For any algebra $\mathbf{A}$, any $a \in \mathsf{uni}(\mathbf{A})$ and any variable y, we have $\mathfrak{B}_a^{\mathbf{A}}/\bot_{\mathbf{A}}^{\mathcal{K}} = \|\mathsf{E}_{z:a}^{\mathbf{A}}[X]\|_{\mathsf{fi}_{\mathcal{S}}}^{\mathbf{A}} = \|\mathsf{E}_{y:a}^{\mathbf{A}}[\mathfrak{B}_y/\bot_{\mathcal{K}}]\|_{\mathsf{fi}_{\mathcal{S}}}^{\mathbf{A}}$, and

$$\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \mathfrak{B}_a^{\mathbf{A}}/\bot_{\mathbf{A}}^{\mathcal{K}}\rangle} : \mathbf{Fi}^{\mathcal{S}}(\langle \mathbf{A}, \mathfrak{B}_a^{\mathbf{A}}/\bot_{\mathbf{A}}^{\mathcal{K}}\rangle) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{A}),$$

with inverse isomorphism $(\mathfrak{B}_a^{\mathbf{A}}/\cdot)_{|\mathbf{Con}^{\mathcal{K}}(\mathbf{A})}$.

Proof. Let $F = \|\mathsf{E}_{z:a}^{\mathbf{A}}[X]\|_{\mathsf{fi}_{\mathcal{S}}}^{\mathbf{A}}$. We first show that $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, F\rangle} : \mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, F\rangle) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{A})$, with inverse isomorphism $(\mathfrak{B}_a^{\mathbf{A}}/-)_{|\mathsf{Con}^{\mathcal{K}}(\mathbf{A})}$. By Lemma 15.7, $\mathcal{S}$ is $\langle X, z\rangle$-equivalential, so $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, F\rangle}$ is order preserving (by Corollary 14.15 on page 411). Let $H \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, F\rangle)$. Then

$$\Omega^{\mathbf{A}}(H)= \{\langle b, c\rangle : \mathbf{\Delta}^{\mathbf{A}}(b, c, a) \subseteq H\}, \tag{15.6}$$

by Theorem 14.13 on page 410. By (15.2), $b \in H$ iff $\mathbf{\Delta}^{\mathbf{A}}(\mathfrak{B}_a^{\mathbf{A}}[\![b]\!], a) \subseteq H$, iff $\mathfrak{B}_a^{\mathbf{A}}[\![b]\!] \subseteq \Omega^{\mathbf{A}}(H)$ (by (15.6)), hence $\mathfrak{B}_a^{\mathbf{A}}/\Omega^{\mathbf{A}}(H) = H$. We shall show that $\Omega^{\mathbf{A}}(H) \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$. Suppose that $\Sigma \models_{\mathcal{K}} r \approx s$ and $\{\langle g^{\mathbf{A}}(\bar{a}), h^{\mathbf{A}}(\bar{a})\rangle : g \approx h \in \Sigma\} \subseteq \Omega^{\mathbf{A}}(H)$. We may assume without loss of generality that Σ is finite and that the variables occurring in $\Sigma \cup \{r \approx s\}$ are among $\vec{x} = x_0, \dots, x_{l-1}$, where $z \notin \{x_0, \dots, x_{l-1}\}$. By (15.6), $\mathbf{\Delta}^{\mathbf{A}}(g^{\mathbf{A}}(\vec{a}), h^{\mathbf{A}}(\vec{a}), a) \subseteq H$ (where $\vec{a} = a_0, \dots, a_{l-1}$). Choose $e \in \mathsf{hom}(\mathbf{Tm}, \mathbf{A})$ with $e(z) = a$ and $e(x_i) = a_i$ for $i < l$. By (15.1), $X, \{\mathbf{\Delta}(g(\vec{x}), h(\vec{x}), z) : g \approx h \in \Sigma\} \vdash_{\mathcal{S}} \mathbf{\Delta}(r(\vec{x}), s(\vec{x}), z)$, and since $e[X] \cup \mathbf{\Delta}^{\mathbf{A}}(g^{\mathbf{A}}(\vec{a}), h^{\mathbf{A}}(\vec{a}), a) \subseteq H$, we have $\mathbf{\Delta}^{\mathbf{A}}(r^{\mathbf{A}}(\vec{a}), s^{\mathbf{A}}(\vec{a}), a) \subseteq H$. By (15.6), $\langle r^{\mathbf{A}}(\bar{a}), s^{\mathbf{A}}(\bar{a})\rangle \in \Omega^{\mathbf{A}}(H)$. Since $\mathfrak{B}_a^{\mathbf{A}}/-$ is order preserving, it remains to show, for any given $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{A})$, that $\Omega^{\mathbf{A}}(\mathfrak{B}_a^{\mathbf{A}}/\alpha)= \alpha$. Since $\mathfrak{B}_z/\bot_{\mathcal{K}} = \|X\|_{\mathcal{S}}$, we have $\mathsf{E}_{z:a}^{\mathbf{A}}[X] \subseteq \mathfrak{B}_a^{\mathbf{A}}/\bot_{\mathbf{A}}^{\mathcal{K}} \subseteq \mathfrak{B}_a^{\mathbf{A}}/\alpha$, by Proposition 12.41 on page 384. By (15.6) and (15.3), $\langle b, c\rangle \in \Omega^{\mathbf{A}}(\mathfrak{B}_a^{\mathbf{A}}/\alpha)$ iff $\mathbf{\Delta}^{\mathbf{A}}(b, c, a) \subseteq \mathfrak{B}_a^{\mathbf{A}}/\alpha$ iff $\langle b, c\rangle \in \alpha$.
Finally, $\Omega^{\mathbf{A}}(F)= \bot_{\mathbf{A}}^{\mathcal{K}}$, since F is the least element of $\mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, F\rangle)$, so $F = \mathfrak{B}_a^{\mathbf{A}}/\Omega^{\mathbf{A}}(F) = \mathfrak{B}_a^{\mathbf{A}}/\bot_{\mathbf{A}}^{\mathcal{K}}$. The rest follows from Lemma 12.38 on page 383 and Proposition 12.41. ⋄

The following result gives necessary and sufficient conditions on the Leibniz operators of algebras, for $\mathcal{K}$ to be the $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$.

Theorem 15.11 The following conditions are equivalent (for fixed X, z).

1. $\mathcal{K}$ is the $\langle X, z\rangle$-equivalent algebraic semantics for S.
2. $\|X\|_{\mathcal{S}}$ is z-invariant and, for each algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$, $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle}$: $\mathbf{Fi}^{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{A})$.
3. $\|X\|_{\mathcal{S}}$ is z-invariant and $\Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle} : \mathbf{Th}(\mathcal{S}_{:X}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$.
4. $\|X\|_{\mathcal{S}}$ is z-invariant and there exists an isomorphism $f : \mathbf{Th}(\mathcal{S}_{:X}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$ that commutes with surjective substitutions (modulo X) that fix z.

Proof. $\boxed{(1)\Rightarrow(2)}$ Follows at once from Theorem 15.10. $\boxed{(2)\Rightarrow(3)}$ Since the $\mathcal{S}$-theory T generated by X is z-invariant, it follows that $\mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, \mathsf{E}^{\mathbf{Tm}}_{z:z}[X]\rangle) = \mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X\rangle)$. $\boxed{(3)\Rightarrow(4)}$ (*We need to show that* $\Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, T\rangle}$ *commutes with surjective substitutions (modulo T) that fix z.*) Let σ be such a substitution and $U \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, T\rangle)$. $\boxed{\text{Claim: } \sigma\left[\Omega^{\mathbf{Tm}}(U)\right] \subseteq \Omega^{\mathbf{Tm}}(\sigma^{\mathcal{S}}_T(U)).}$ From $T \subseteq U \subseteq \sigma^{-1}[\sigma^{\mathcal{S}}_T(U)]$, (3), Lemma 2.68 on page 104 and the surjectivity of σ, we infer $\Omega^{\mathbf{Tm}}(U) \subseteq \Omega^{\mathbf{Tm}}(\sigma^{-1}[\sigma^{\mathcal{S}}_T(U)]) = \sigma^{-1}[\Omega^{\mathbf{Tm}}(\sigma^{\mathcal{S}}_T(U))]$, establishing this claim. $\Box$ Now $\sigma_{\mathcal{K}}(\Omega^{\mathbf{Tm}}(U)) = \big\|\sigma[\Omega^{\mathbf{Tm}}(U)]\big\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} \subseteq \big\|\Omega^{\mathbf{Tm}}(\sigma^{\mathcal{S}}_T(U))\big\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} = \Omega^{\mathbf{Tm}}(\sigma^{\mathcal{S}}_T(U))$, since $\Omega^{\mathbf{Tm}}(\sigma^{\mathcal{S}}_T(U)) \in \mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})$ by assumption. To prove the reverse inclusion, note that there exists, by assumption, $W \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, T\rangle)$ such that $\Omega^{\mathbf{Tm}}(W) = \sigma_{\mathcal{K}}(\Omega^{\mathbf{Tm}}(U))$. Now $\Omega^{\mathbf{Tm}}(U) \subseteq \sigma^{-1}[\sigma_{\mathcal{K}}(\Omega^{\mathbf{Tm}}(U)$ $)] = \sigma^{-1}[\Omega^{\mathbf{Tm}}(W)] = \Omega^{\mathbf{Tm}}(\sigma^{-1}[W])$, by Lemma 2.68, since σ is surjective. Then $\sigma^{-1}[W] \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, T\rangle)$, by Proposition 2.46, since T is σ-invariant. Thus, $U \subseteq \sigma^{-1}[W]$, i.e., $\sigma[U] \subseteq W$, since $\Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle}$ is an isomorphism. Now $\sigma^{\mathcal{S}}_T(U) = \|\sigma[U]\|^{\langle \mathbf{Tm}, T\rangle}_{\mathsf{fi}_{\mathcal{S}}} \subseteq \|W\|^{\langle \mathbf{Tm}, T\rangle}_{\mathsf{fi}_{\mathcal{S}}} = W$, since $W \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, T\rangle)$. It follows that $\Omega^{\mathbf{Tm}}(\sigma^{\mathcal{S}}_T(U)) \subseteq \Omega^{\mathbf{Tm}}(W) = \sigma_{\mathcal{K}}(\Omega^{\mathbf{Tm}}(U))$. $\boxed{(4)\Rightarrow(1)}$ By Theorem 13.17, $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$, say; $T \doteq \|X\|_{\mathcal{S}} = \mathfrak{B}_z/\bot_{\mathcal{K}}$ and $f = \mathfrak{B}_z$. Let $g = f^{-1}$. For $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{Tm})$ and a surjective substitution σ that fixes z, we have $g(\sigma_{\mathcal{K}}(\alpha)) = \sigma^{\mathcal{S}}_T(g(\alpha))$. Let x and y be distinct variables other than z, let $\alpha = \|\{\langle x, y\rangle\}\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}$ and set $U = g(\alpha)$. Since α is compact in $\mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$, U is compact in $\mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, T\rangle)$, so $U = \|\{p_0, \ldots, p_{m-1}\}\|^{\langle \mathbf{Tm}, T\rangle}_{\mathsf{fi}_{\mathcal{S}}}$, for some $m \in \omega$ and $p_0, \ldots, p_{m-1} \in \mathsf{Tm}$. We may assume that $p_j = p_j(x, y, z, u_0, \ldots, u_{l-1})$, for suitable $u_0, \ldots, u_{l-1} \in \mathsf{V}$ and all $j < m$, and that $\{x, y, z\} \cap \{u_0, \ldots, u_{l-1}\} = \emptyset$. Let σ be any surjective substitution fixing x, y and z, such that $\sigma(u_i) = x$ for $i < l$. For each $j < m$, define $\Delta_j(x, y, z) = p_j(x, y, z, x, \ldots, x) = \sigma(p_j)$. Now, $\sigma_{\mathcal{K}}(\alpha) = \sigma_{\mathcal{K}}(\|\{\langle x, y\rangle\}\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}) = \|\{\sigma(\langle x, y\rangle)\}\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} = \alpha$. Since σ fixes z, T is σ-invariant, whence (using Proposition 2.46 on page 101), $U = g(\sigma_{\mathcal{K}}(\alpha)) = \sigma^{\mathcal{S}}_T(g(\alpha)) = \sigma^{\mathcal{S}}_T(U) = \sigma^{\mathcal{S}}_T(\|\{p_0, \ldots, p_{m-1}\}\|^{\langle \mathbf{Tm}, T\rangle}_{\mathsf{fi}_{\mathcal{S}}}) = \|\sigma[\{p_0, \ldots, p_{m-1}\}]\|^{\langle \mathbf{Tm}, T\rangle}_{\mathsf{fi}_{\mathcal{S}}}$. Thus

$$\begin{aligned}\|\{\langle x, y\rangle\}\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}} &= \mathfrak{B}_z(g(\alpha)) = \mathfrak{B}_z(\|\{\Delta_j(x, y, z) : j < m\}\|^{\langle \mathbf{Tm}, T\rangle}_{\mathsf{fi}_{\mathcal{S}}}) \\ &= \blacktriangledown_{j<m}\|\mathfrak{B}_z[\![\Delta_j(x, y, z)]\!]\|_{\Theta^{\mathcal{K}}_{\mathbf{Tm}}}\end{aligned}$$

(by Lemma 13.16 on page 397). By Lemmas 1.457 on page 88 and 15.5, $\mathcal{K}$ is a (T, z)-equivalent algebraic semantics for $\mathcal{S}$ with (T, z)-equivalence terms $\Delta_0, \ldots, \Delta_{m-1}$ and (T, z)-defining equations $\mathfrak{B}_*$. $\diamond$

Our results specialize to the logics $S(\mathcal{K}, \mathfrak{B}_*)$ determined by a system of binary equations $\mathfrak{B}$ as follows.

Theorem 15.12 For any system of binary equations $\mathfrak{B}$ and variable z, the following conditions on a quasivariety $\mathcal{K}$ are equivalent.

1. $\mathcal{K}$ is the $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z\rangle$-equivalent algebraic semantics of $S(\mathcal{K}, \mathfrak{B}_*)$.

2. $\mathcal{K}$ is the $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z\rangle$-equivalent algebraic semantics of some sentential calculus.

3. There exists a finite set $\mathbf{\Delta}$ of ternary terms such that $\mathcal{K}$ satisfies (13.5) and (15.3).

Proof. $\boxed{(1)\Rightarrow(2)}$ Immediate. $\boxed{(2)\Rightarrow(3) \text{ and } (3)\Rightarrow(1)}$ Follow from Theorem 13.22 and Lemma 15.5. $\diamond$

15.4.2 Internal to a Logic

If $\mathcal{S}$ has an $\langle X, z\rangle$-equivalent algebraic semantics $\mathcal{K}$, then $\mathcal{K}$ is unique and the defining equations are equivalent modulo $\mathcal{K}$, by Theorem 15.9. Consequently, we can sensibly introduce the notion that a sentential calculus be $\langle X, z\rangle$-*algebraizable*, without any reference to a specific quasivariety nor a specific set of defining equations.

Definition 15.13 ($\langle X, z\rangle$-Algebraizable) We say that $\mathcal{S}$ is **$\langle X, z\rangle$-algebraizable** if $\mathcal{S}$ has an $\langle X, z\rangle$-equivalent algebraic semantics. $\square$

The following result characterizes this (parametrized) logical property internally (i.e., without apriori reference to a quasivariety).

Theorem 15.14 For any (fixed) set X of terms and variable z, the following conditions on a sentential 1-calculus $\mathcal{S}$ are equivalent, where $\mathcal{K}$ is the quasivariety generated by $\{\mathbf{Tm}/\Omega_{\mathbf{Tm}}(T) : T \in \mathsf{Th}(\mathcal{S}_{:X})\}$.

1. $\mathcal{S}$ is $\langle X, z\rangle$-algebraizable.

2. $\|X\|_{\mathcal{S}}$ is z-invariant and there exist a binary system of equations $\mathfrak{B}$ and a finite set of ternary terms $\mathbf{\Delta}$ such that $\mathcal{S}$ satisfies (15.2), (Rlx), (Sym), (Trn), (Ex1) and (Sub').

3. $\|X\|_{\mathcal{S}}$ is z-invariant, $\mathcal{S}$ has a finite system of $\langle X, z\rangle$-congruence terms $\mathbf{\Delta}$ and there exists a binary system of equations $\mathfrak{B}$ such that $\mathcal{S}$ satisfies (Ex1) and (15.2).

4. $\|X\|_{\mathcal{S}}$ is z-invariant and $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle} : \mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{A})$, for every $\mathfrak{a}$-algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$.

5. $\|X\|_{\mathcal{S}}$ is z-invariant and $\Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle} : \mathbf{Th}(\mathcal{S}_{:X}) \cong \mathbf{Con}^{\mathcal{K}}(\mathbf{Tm})$.

In this case, for every $\mathfrak{a}$-algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$, $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle}$ has the same range as $\Omega^{\mathcal{S}}_{\langle \mathbf{A}, h[\mathsf{E}^{\mathbf{A}}_{z:a}[X]]\rangle}$, for every involution h of $\mathbf{A}$. In particular, $\Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle}$ has the same range as $\Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, \sigma[X]\rangle}$ for every involution σ of $\mathbf{Tm}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Follows from the definitions and Lemmas 15.7 and 15.8.
$\boxed{(2)\Rightarrow(3)}$ By Proposition 14.14, we need only show (Det). By (Sym), $\|X\|_{\mathcal{S}}, \mathbf{\Delta}(x, y, z) \vdash_S \mathbf{\Delta}(y, x, z)$. For each $\langle \delta, \epsilon\rangle \in \mathfrak{B}$, we may therefore conclude from (Sub) that

$$\|X\|_{\mathcal{S}}, \mathbf{\Delta}(x, y, z) \vdash_{\mathcal{S}} \mathbf{\Delta}(\delta(x, z), \delta(y, z), z) \cup \mathbf{\Delta}(\epsilon(y, z), \epsilon(x, z), z).$$

By (15.2),

$$\|X\|_{\mathcal{S}}, y \vdash_{\mathcal{S}} \mathbf{\Delta}(\mathfrak{B}_z[\![y]\!], z),$$

so by (Trn),

$$\|X\|_{\mathcal{S}}, y, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} \mathbf{\Delta}(\mathfrak{B}_z[\![x]\!], z).$$

Finally, by (15.2), $\|X\|_{\mathcal{S}}, \mathbf{\Delta}(\mathfrak{B}_z[\![x]\!], z) \vdash_{\mathcal{S}} x$, hence the result.

$\boxed{(3)\Rightarrow(4)}$ Let $F \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$. By (15.2) and Theorem 14.13 on page 410,

$$b \in F \text{ iff } \mathbf{\Delta}^{\mathbf{A}}(\mathfrak{B}^{\mathbf{A}}_a(b), a) \subseteq F \text{ iff } \mathfrak{B}^{\mathbf{A}}_a(b) \subseteq \Omega^{\mathbf{A}}(F). \tag{15.7}$$

$\boxed{\mathcal{K} \text{ is an } \langle X,z\rangle\text{-algebraic semantics for } \mathcal{S} \text{ with defining equations } \mathfrak{B}_*}$ Suppose that $X, G \vdash_{\mathcal{S}} p$, let $\mathbf{A} = \mathbf{Tm}/\Omega^{\mathbf{Tm}}(T')$, where T' is an $\mathcal{S}$-theory containing X, and suppose that $\mathfrak{B}_z[G] \subseteq \Omega^{\mathbf{Tm}}(T')$. Since $\|X\|_{\mathcal{S}}$ is z-invariant, we may use (15.7) to infer that $G \subseteq T'$, and hence $p \in T'$. By (15.7) again, $\mathfrak{B}_z[\![p]\!] \subseteq \Omega^{\mathbf{Tm}}(T')$. This shows that $\mathfrak{B}^{\approx}_z[G] \models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\![p]\!]$. Conversely, suppose that $\mathfrak{B}_z[G] \models_{\mathcal{K}} \mathfrak{B}_z[\![p]\!]$. Let $T' = \|G\|^{\langle \mathbf{Tm}, X\rangle}_{\mathsf{fi}_{\mathcal{S}}}$. Since $G \subseteq T'$, it must follow by the z-invariance of $\|X\|_{\mathcal{S}}$ together with (15.7), that $\mathfrak{B}_z[G] \subseteq \Omega^{\mathbf{Tm}}(T')$. Since $\mathbf{Tm}/\Omega^{\mathbf{Tm}}(T') \in \mathcal{K}$, it follows that $\mathfrak{B}_z[\![p]\!] \subseteq \Omega^{\mathbf{Tm}}(T')$. By (15.7), $\mathbf{\Delta}(\mathfrak{B}_z[\![p]\!], z) \subseteq T'$. Now by assumption (15.2), $p \in T'$, i.e., $X, G \vdash_{\mathcal{S}} p$ as required. Hence $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with defining equations $\mathfrak{B}_*$. □

$\boxed{\mathcal{K} \text{ satisfies (15.3)}}$ Since $\mathbf{\Delta}$ is a system of $\langle X, z\rangle$-equivalence terms, we have $X \vdash_{\mathcal{S}} \mathbf{\Delta}(x,x,z)$, and since we have already established that $\mathcal{K}$ is an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with defining equations $\mathfrak{B}_*$, $\models_{\mathcal{K}} \mathfrak{B}^{\approx}_z[\mathbf{\Delta}(x,x,z)]$. We prove the converse deduction model theoretically. Let $\mathbf{A} \in \mathcal{K}$, i.e., $\mathbf{A} = \mathbf{Tm}/\Omega^{\mathbf{Tm}}(T')$, where T' is an $\mathcal{S}$-theory containing X, and suppose that $\mathfrak{B}_r[\mathbf{\Delta}(p,q,r)] \subseteq \Omega^{\mathbf{Tm}}(T')$. (*We must show that* $\langle p, q\rangle \in \Omega^{\mathbf{Tm}}(T')$.) Since (i) $\Omega^{\mathbf{Tm}}(T')$ identifies $\mathbf{\Delta}(\delta(\mathbf{\Delta}(p,q,r)), \epsilon(\mathbf{\Delta}(p,q,r)), z)$ with $\mathbf{\Delta}(\delta(\mathbf{\Delta}(p,q,r)), \delta(\mathbf{\Delta}(p,q,r)), z)$, for each $\langle \delta, \epsilon\rangle \in \mathfrak{B}$, (ii) $\mathbf{\Delta}(\delta(\mathbf{\Delta}(p,q,r)), \delta(\mathbf{\Delta}(p,q,r)), z) \subseteq T'$, for each $\langle \delta, \epsilon\rangle \in \mathfrak{B}$, and (iii) $\Omega^{\mathbf{Tm}}(T')$ is compatible with T', we have $\mathbf{\Delta}(\mathfrak{B}_r[\mathbf{\Delta}(p,q,r)], z) \subseteq T'$. So by (Ex1), $\mathbf{\Delta}(p,q,r) \subseteq T'$. So by Theorem 14.13 on page 410 (with z in the role of a), $\langle p, q\rangle \in \Omega^{\mathbf{Tm}}(T')$. Hence $\mathcal{K}$ satisfies (15.3). □

So by Lemma 15.5, $\mathcal{K}$ is an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$, and hence the result follows by Theorem 15.11.

$\boxed{(4)\Rightarrow(5)}$ Let $\mathbf{A} = \mathbf{Tm}$ and $a = z$. The result follows from (4) since $\|X\|_{\mathcal{S}}$ is z-invariant. $\boxed{(5)\Rightarrow(1)}$ Follows from Theorem 15.11.

$\boxed{\text{Additional Assertion}}$ Let $\mathbf{A}$ be an algebra, $a \in \mathsf{uni}(\mathbf{A})$ and h an involution of $\mathbf{A}$.

$\boxed{\Omega_{\mathbf{A}}\left[h\left[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)\right]\right] \subseteq \Omega_{\mathbf{A}}\left[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)\right]}$

(*We first show that* $\Omega_{\mathbf{A}}\left[h\left[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)\right]\right] \subseteq \Omega_{\mathbf{A}}\left[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)\right]$.) Let $F \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$. Let

$$F' = \{b \in \mathsf{uni}(\mathbf{A}) : \mathbf{\Delta}^{\mathbf{A}}(\mathfrak{B}^{\mathbf{A}}_{h(a)}(h(b)), a) \subseteq F\}.$$

(*We first show that* $\mathsf{E}^{\mathbf{A}}_{z:a}[\|X\|_{\mathcal{S}}] \subseteq F'$.) Suppose $X \vdash_{\mathcal{S}} r(z, x_0, \ldots, x_{l-1})$ and let $e \in \mathsf{hom}(\mathbf{Tm}, \mathbf{A})$ with $e(z) = a$. We show that $r^{\mathbf{A}}(a, e(x_0), \ldots, e(x_{l-1})) \in F'$. Let $y \in \mathsf{V} - \{z, x_0, \ldots, x_{l-1}\}$. By (Ex2), $X \vdash_{\mathcal{S}} \mathbf{\Delta}(\mathfrak{B}_y(r(y, \vec{x})), z)$. Consider any $f \in \mathsf{hom}(\mathbf{Tm}, \mathbf{A})$ with $f(y) = h(a)$, $f(z) = a$ and $f(x_i) = h(e(x_i))$ for all $i < l$. Now $f[X] \subseteq F$, hence

$$\mathbf{\Delta}^{\mathbf{A}}(\mathfrak{B}^{\mathbf{A}}_{h(a)}(h(r^{\mathbf{A}}(a, e(x_0), \ldots, e(x_{l-1})))), a) = f\left[\mathbf{\Delta}(\mathfrak{B}_y(r(y, \vec{x})), z)\right] \subseteq F.$$

Thus $e(r) = r^{\mathbf{A}}(a, e(x_0), \ldots, e(x_{l-1})) \in F'$. (*Next we show that* F' *is an* $\mathcal{S}$*-filter of* $\mathbf{A}$*, hence* $F' \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[\|X\|_{\mathcal{S}}]\rangle) \subseteq \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$.) Suppose that $P \vdash_{\mathcal{S}} p$ and that $P^{\mathbf{A}}(\bar{b}) \subseteq F'$, where $\bar{b} \in \mathsf{uni}(\mathbf{A})^{\omega}$. By the structurality and finitariness of $\mathcal{S}$, we may assume without loss of generality that for some $l, k \in \omega$, $P = \{g_i(x_0, \ldots, x_{k-1}) : i < l\}$, $t = t(x_0, \ldots, x_{k-1})$ and $z \in \mathsf{V} - \{x_0, \ldots, x_{k-1}\}$. Then for each $i < l$, $\mathbf{\Delta}^{\mathbf{A}}(\mathfrak{B}^{\mathbf{A}}_{h(a)}(h(g^{\mathbf{A}}_i(\vec{b}))), a) \subseteq F$. Let $y \in \mathsf{V} - \{z, x_0, \ldots, x_{k-1}\}$. Since $X, \{g_i(\vec{x}) : i < l\} \vdash_{\mathcal{S}} p$, it follows from (Ex2) that $\|X\|_{\mathcal{S}}, \bigcup_{i<l} \mathbf{\Delta}(\mathfrak{B}_y(g_i(\vec{x})), z) \vdash_{\mathcal{S}} \mathbf{\Delta}(\mathfrak{B}_y(t(\vec{x})), z)$. Thus, $\mathbf{\Delta}^{\mathbf{A}}(\mathfrak{B}^{\mathbf{A}}_{h(a)}(h(t^{\mathbf{A}}(\vec{b}))), a) \subseteq F$, so $t^{\mathbf{A}}(\vec{b}) \in F'$, as required.

By (14.4) and (Ex1), $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F')$ [iff] $\mathbf{\Delta}^{\mathbf{A}}(b, c, a) \subseteq F'$ [iff] $\mathbf{\Delta}^{\mathbf{A}}(\mathfrak{B}^{\mathbf{A}}_{h(a)}(h(\mathbf{\Delta}^{\mathbf{A}}(b,c,a))), a) \subseteq F$ [iff] $\mathbf{\Delta}^{\mathbf{A}}(h(b), h(c), a) \subseteq F$ [iff] $\langle h(b), h(c)\rangle \in \Omega^{\mathbf{A}}(F)$ [iff] $\langle b, c\rangle \in h[\Omega^{\mathbf{A}}(F)] = \Omega^{\mathbf{A}}(P)$. Since $F' \in$

$\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$, we have the desired inclusion.

$\boxed{\Omega_{\mathbf{A}}\left[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)\right] \subseteq \Omega_{\mathbf{A}}\left[h\left[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)\right]\right]}$ Let $F \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$. Consider $h[F]$ and notice that since h is an involution, $h\left[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)\right] = \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, h\left[\mathsf{E}^{\mathbf{A}}_{z:a}[X]\right]\rangle)$. By the inclusion established previously, we may chose $G \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)$ such that $\Omega^{\mathbf{A}}(h[F]) = \Omega^{\mathbf{A}}(G)$. So by Lemma 2.68 on page 104, $\Omega^{\mathbf{A}}(F) = \Omega^{\mathbf{A}}(h[G])$. Since $h\left[\mathsf{E}^{\mathbf{A}}_{z:a}[X]\right] \subseteq h[G]$, we conclude that $\Omega^{\mathbf{A}}(F) \in \Omega_{\mathbf{A}}\left[h\left[\mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[X]\rangle)\right]\right]$.
$\diamond$

Remark 15.15 By comparing (5) of the above theorem with (3) of Theorem 2.118 on page 113, we see immediately that $(\emptyset, z)$-algebraizability is just algebraizability (in the sense of [BP89a]).

15.4.3 Characterizations Internal to a Quasivariety

We shall now demonstrate the relationship between the property that $\mathcal{K}$ be an $\langle X, z\rangle$-equivalent algebraic semantics of some sentential calculus and the purely universal algebraic notion that $\mathcal{K}$ be $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-regular, defined in Definition 11.16 on page 365 and characterized in Theorem 11.18 on page 366 and Corollary 11.20 on page 367.

If one compares (11.6) of Theorem 11.18 on page 366, for the case of a single $\mathfrak{B}$, with (15.3) of Lemma 15.5, it is immediately clear that if $\mathcal{K}$ is an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ then $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-regular. Conversely, if $\mathcal{K}$ is $\mathfrak{B}_*$-*deductive* (equivalently, $\mathfrak{B}_*$ pivots *finitarily* for $\mathcal{K}$) and $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-regular, then $\mathcal{K}$ is an $\langle X, z\rangle$-equivalent algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$.

Recall the definition of the $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-cosets $\mathsf{Cos}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A})$ at a, given in Definition 12.40 on page 384, and in particular recall that $\mathsf{Cos}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A})$ forms an algebraic closed system and that we denote the associated algebraic lattice by $\mathbf{Cos}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A})$.

Definition 15.16 ($\langle \mathcal{K}, \mathfrak{B}_*\rangle$-Coset Determination) We say that $\mathcal{K}$ is **$\langle \mathcal{K}, \mathfrak{B}_*\rangle$-coset determined** if $\mathfrak{B}^{\mathbf{A}}_a/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Cos}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A})$, for every algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$. $\square$

The following result follows from Theorem 11.18 on page 366, Lemma 15.5, and Theorem 15.10.

Theorem 15.17 Suppose the quasivariety $\mathcal{K}$ is $\mathfrak{B}_*$-*deductive.* The following conditions are equivalent (where z is an arbitrary variable).

1. $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-regular.

2. $\mathcal{K}$ is the $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z\rangle$-equivalent algebraic semantics of $S(\mathcal{K}, \mathfrak{B}_*)$.

3. $\mathcal{K}$ is the $\langle \mathfrak{B}_z/\bot_{\mathcal{K}}, z\rangle$-equivalent algebraic semantics of *some* sentential 1-calculus.

4. $\mathcal{K}$ is $\langle \mathcal{K}, \mathfrak{B}_*\rangle$-coset determined.

In this case, $\mathsf{PN}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A}) = \mathsf{Cos}^{\mathcal{K}}_{\mathfrak{B}_a}(\mathbf{A})$, for every algebra $\mathbf{A}$ and every $a \in \mathsf{uni}(\mathbf{A})$. $\square$

15.5 Examples

Example 15.18 (Regularity Conditions in Universal Algebra)

Let $\boldsymbol{u}$ be a unary term over $\mathcal{K}$ and $\mathfrak{B}(x,y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$. By Corollary 13.35 on page 402, $\mathcal{K}$ is *always* a $\langle\{\boldsymbol{u}(z)\}, z\rangle$-algebraic semantics for its logic $S(\mathcal{K}, \boldsymbol{u})$ of identified membership, and hence is always $\mathfrak{B}_*$-deductive. As such, the hypothesis of Theorem 15.17 is always satisfied, and so $\mathcal{K}$ is $\langle\mathcal{K}, \boldsymbol{u}\rangle$-regular precisely when $\mathcal{K}$ is the $\langle\{\boldsymbol{u}(z)\}, z\rangle$-equivalent algebraic semantics of $S(\mathcal{K}, \mathfrak{B}_*)$. Recall that with each $\mathcal{K}$-unary term $\boldsymbol{u}$, we associate the binary system $\boldsymbol{u}(x,y) = \{\langle x, \boldsymbol{u}(y)\rangle\}$.

Corollary 15.19 For $\mathcal{K}$-unary term $\boldsymbol{u}$, the following conditions are equivalent (where z is an arbitrary variable).

1. $\mathcal{K}$ is $\langle\mathcal{K}, \boldsymbol{u}\rangle$-regular.
2. $\mathcal{K}$ is the $\langle\{\boldsymbol{u}(z)\}, z\rangle$-equivalent algebraic semantics of $S(\mathcal{K}, \boldsymbol{u})$ with defining equations $\boldsymbol{u}$.
3. $\mathcal{K}$ is the $\langle\{\boldsymbol{u}(z)\}, z\rangle$-equivalent algebraic semantics of *some* sentential 1-calculus.
4. $\boldsymbol{u}^{\mathbf{A}}(a)/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Cos}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A})$, for every algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$.

In this case, for any algebra $\mathbf{A} \in \mathcal{K}$, $\mathsf{Cos}^{\mathcal{K}}_{\boldsymbol{u}_a}(\mathbf{A}) = \{\alpha[\![\boldsymbol{u}^{\mathbf{A}}(a)]\!] \,:\, \alpha \in \mathsf{Con}_{\mathcal{K}}(\mathbf{A})\}$. □

For a variable y, let $\boldsymbol{y}(x,y) = \{\langle x, y\rangle\}$.

Corollary 15.20 The following conditions are equivalent (where z is an arbitrary variable).

1. $\mathcal{K}$ is relatively regular.
2. $\mathcal{K}$ is the $\langle\{z\}, z\rangle$-equivalent algebraic semantics of the membership logic $S(\mathcal{K}, \mathsf{mem})$ with defining equations $\boldsymbol{y}$.
3. $\mathcal{K}$ is the $\langle\{z\}, z\rangle$-equivalent algebraic semantics of *some* sentential 1-calculus.
4. $a/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Cos}^{\mathcal{K}}_{\boldsymbol{y}_a}(\mathbf{A})$, for every algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$.

In this case, for any algebra $\mathbf{A} \in \mathcal{K}$, $\mathsf{Cos}^{\mathcal{K}}_{\boldsymbol{y}_a}(\mathbf{A}) = \{\alpha[\![a]\!] \,:\, \alpha \in \mathrm{BiCon}^{\mathcal{K}}\,\mathbf{A}\}$. □

Recall that by Corollary 13.39 on page 402, if $\mathcal{K}$ is a variety then, for any $\mathbf{A} \in \mathcal{K}$ the $\langle\mathcal{K}, \boldsymbol{y}_*\rangle$-cosets of $\mathbf{A}$ are precisely the congruence classes of $\mathbf{A}$. Consequently, when $\mathcal{K}$ is a variety *or* when $\mathcal{K}$ is a relatively regular quasivariety, the definition of $\langle\mathcal{K}, \boldsymbol{y}_*\rangle$-coset provides a complete internal (syntactic) characterization of '$\mathcal{K}$-congruence class'. The assumptions that $\mathcal{K}$ is a variety *or* $\mathcal{K}$ is a relatively regular quasivariety may be unified by the condition that the range of the Leibniz operator $\Omega^{\mathcal{S}}_{\langle\mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:a}[\boldsymbol{y}_z/\perp^{\mathcal{K}}]\rangle}$ consists of $\mathcal{K}$-congruences for every $a \in \mathbf{A}$. A similar unification for the hypothesis of the following result and the hypothesis of Corollary 13.40 on page 402 appears in [BR99].

Corollary 15.21 [BR99] For $\mathcal{K}$-constant term 0, the following conditions are equivalent.

1. $\mathcal{K}$ is relatively point regular at 0.
2. $\mathcal{K}$ is the equivalent algebraic semantics of the assertional logic $S(\mathcal{K}, 0)$.
3. $0^{\mathbf{A}}/\cdot : \mathbf{Con}^{\mathcal{K}}(\mathbf{A}) \cong \mathbf{Cos}^{\mathcal{K}}_{\mathbf{0}_a}(\mathbf{A})$, for every algebra $\mathbf{A}$ and $a \in \mathsf{uni}(\mathbf{A})$.

In this case, for any algebra $\mathbf{A}$, the $\langle\mathcal{K}, 0\rangle$-cosets of $\mathbf{A}$ are precisely the $\mathcal{K}$-congruence classes of $\mathbf{A}$ containing $0^{\mathbf{A}}$. □

Open Problem 15.22 Notice that there is not analagous condition 3. in the previous corollary. Why not? Is the missing condition '$\mathcal{K}$ is the equivalent algebraic semantics of *some* sentential 1-calculus with defining equations $\mathbf{0}$'? What is to be made of this?

The following characterization of relative 1-subregularity follows at once from definition and Corollary 15.19.

Corollary 15.23 $\mathcal{K}$ is relatively 1-subregular iff there exists a unary term u such that $\mathcal{K}$ is the $\langle\{\boldsymbol{u}(z)\}, z\rangle$-equivalent algebraic semantics of its u-membership logic $S(\mathcal{K}, \boldsymbol{u})$.

□

Open Problem 15.24 Show that there exists a $\langle\mathcal{K}, \mathfrak{B}_*\rangle$-regular quasivariety $\mathcal{K}$ that is not $\mathfrak{B}_*$-deductive[1].

15.6 Parametrized algebraization: zones of application

Three potential 'zones' of applicability of the parametrized algebraization process described here, corresponding to three kinds of value for the parameter $\langle X, z\rangle$.

Recall that the significant *individual* values of $\langle X, z\rangle$ that are *language-independent* are (1) $X = \emptyset$ and (2) $X = \{z\}$. The former is the zone of the Blok-Pigozzi theory of algebraization; the latter the zone including membership logics: for these, the defining quasivariety shall turn out to be an $\langle X, z\rangle$-equivalent algebraic semantics just when its relative congruences are fully regular (see Corollary 15.20). The unification of these two zones was one of our primary objectives.

Our third zone is the language-independent *class of* values of $\langle X, z\rangle$ specified by the second order condition that X consist of unary terms in z (in which case, z-invariance is immediate). In particular, the logical property of being $\langle\{\boldsymbol{u}(z)\}, z\rangle$-algebraizable with defining equation $x \approx u(z)$ *for some* unary term u has a significant algebraic counterpart: see Corollary 15.23. An instance of this phenomenon falling *outside* our first two zones is Example 11.28.

Any remaining zones of application must correspond to cases where X is not exhausted by unary terms in z. Meaningful instances of the results of this section are to be found in cases where X is a *fully invariant* theory generated *as such* by a nonempty set of terms in which z does *not* occur: these cases constitute a fourth zone of application. The next example, which illustrates this, adds nothing essential to our knowledge of the logics discussed in it but shows that our definitions make sense in this fourth zone. For clarification regarding the logics and quasivarieties cited, see [BP89a, Th 5.9, 5.10 and 5.11].

Example 15.25

For *distinct* variables x, y, z, the (non-algebraizable) implicational fragment BCI of Girard's linear logic is (T, z)-algebraizable if T is the *fully invariant* BCI-theory generated by the term $x \supset (y \supset x)$ (the *'weakening'* axiom), i.e., T is the set of theorems of Meredith's logic BCK. Iséki's quasivariety BCK of all BCK-algebras is the unique (T, z)-equivalent algebraic

[1] We had an example of such a quasivariety, but due to circumstances beyond our control, this example has been lost.

semantics of BCI. The (T, z)-defining equations and equivalence terms are $\{\langle \delta(x, y), \epsilon(x, y) \rangle\} = \{\langle x, x \supset x \rangle\}$ and $\{\Delta_0(x, y, z), \Delta_1(x, y, z)\} = \{x \supset y, y \supset x\}$.[2]

□

Our definitions permit consideration of a fifth zone, in which z *occurs* in non-unary terms that are not theorems and that generate X as a fully invariant theory, but we know of no natural or instructive instances of this phenomenon and have not pursued it here.

[2] There is a similar result in which the 'mingle' axiom $x \supset (x \supset x)$ and a different quasivariety play the roles of 'weakening' and BCK.

Part VI

Protoalgeabricity and Equivalence in Constructural Abstract Logic

As noted in the introduction, the *secondary aim* of this text is to analyse our theory of *parameterized* algebraization, by which we mean, to give some sort of explanation of the results from some other *non-parameterized* perspective. To this end, we develop a theory of protoalgebraicity and equivalence for logics over constructs. This theory is in the spirit of the standard sentential theory and has no parameters. We then apply this theory to obtain many of our parameterized results, by considering the construct consisting of only the term algebra and all endomorphisms that fix z, and the construct of all algebras, each with a fixed but arbitrary designated point, and only those homomorphisms that map designated points to designated points. In the case of $\langle X, z\rangle$-protoalgebraicity, we consider the logic determined by all $\mathcal{S}$-theories that contain X, which, while not a sentential calculus, is a structural and finitary calculus with respect to the constructs just described (under the additional assumption that the theory generated by X is invariant with respect to substitutions that fix z). Clearly the parameter still exists, it has merely been devolved to the morphisms defining the construct.

In $\oint 16$, we characterize logics over constructs satisfying the filter correspondence property, and introduce a notion of a *Leibniz equivalence relation* for logics over constructs. With the aid of this Leibniz equivalence relation, we introduce the concept of a protoalgebraic logic over a signature $\mathfrak{s}$, although we only consider $\mathfrak{s}$-structural global logics, i.e., $\mathfrak{s}$-calculi. This notion of protoalgebraicity encompasses that standard notion of protoalgebraic sentential calculi [BP89a], and since we make no assumption regarding finitariness, our notion also encompasses protoalgebraic structural $\mathfrak{a}$-calculi. An important result is a characterization of protoalgebraicity in terms of the existence of formulae and deductions that must be satisfied involving the formula. Since this characterization is in the spirit of the analogous characterization of protoalgebraic sentential *one*-calculi given in [BP89a] (see Corollary 2.138 on page 118), we are able to derive a new characterization of protoalgebraic sentential n-calculi that is in the spirit of the aforementioned characterization and far simpler than the characterization of protoalgebraic sentential n-calculi given in [Pal03] (see Theorem 2.137 on page 118). Recall that in Part 4, as an auxiliary program we aimed to complete the theory developed in [BR99], by extending these result from sentential *one*-calculi to sentential n-calculi more generally, and that while we indeed achieve an analogous relationship between algebraizable sentential n-calculi and $\langle\mathcal{K}, \mathfrak{N}\rangle$-regularity, and that we successfully introduce and characterized the notion that $\mathcal{K}$ has $\langle\mathcal{K}, \mathfrak{N}\rangle$-coherent $\mathfrak{N}$-classes, we were unable to characterize the protoalgebraicity of the sentential n-calculus $S^n(\mathcal{K}, \mathfrak{N})$ in terms of $\mathcal{K}$ having $\langle\mathcal{K}, \mathfrak{N}\rangle$-coherent $\mathfrak{N}$-classes. We shall now be able to close this auxiliary program by proving this result using our new characterization of protoalgebraic sentential n-calculi.

We must note that very recently, a notion of **protoalgebraic π-institutions** has appeared [Vou07a], [Vou06] (and the unpublished [Vou07b] and [Vou07c]). Since our definition of the Leibniz relation and theirs is very different (both abstract the standard Leibniz relation), while their notion of protoalgebraicity is defined in a manner analogous to ours, unless the two notions of Leibniz relation coincide, their results and our analogous results may in fact be distinct; while still both generalizing the standard results. Since we have not had the time to analyse these papers deeply, we have adopted the approach of referencing these papers in any of our results that mirror in form results from those papers. Some of our key results, vital with respect to our aim of explaining $\langle X, z\rangle$-protoalgebraicity from the perspective of protoalgebraic logics over constructs, have not been obtained in the aforementioned papers. While in [Vou06] it is shown that a generalization of equivalence formulae satisfied by a π-institution implies N-protoalgebraicity (this is their notion of protoalgebraicity), they are generally unable to establish a converse; they claim that generally a converse is impossible to obtain. We on the other hand, have achieved such a characterization, as mentioned in the previous paragraph (we are at an advantage though, since our notions are *only* developed with respect to global calculi, while their theory is developed for arbitrary (i.e., not necessarily term) π-institutions).

In $\oint 17$ we introduce the notion of equivalent logics, where the logics lie in different constructs, by means of category isomorphisms between the constructs. We use this theory to provide an alternative perspective of parameterized algebraic and equivalent semantics. While at the time of the development of this theory the theory of deductively equivalent π-institutions [Vou03] was not published, we have subsequently analysed the relationship between our theory and that of [Vou03], and where our results specialize those in the literature we have duely referenced. This specialization *only* occurs for *global* logics and *term* π-institutions. Since this is the case for the logic described above, some of our theory of parameterized algebraization obtains from the theory of deductively equivalent term π-institutions.

During the process of attempting to reconcile our theory with that of [Vou03], we discovered that our theory and theirs coincides only for *term* π-institutions (which, in our case, are analogues of logics over a free object). Further, there is a clear dichotomy in [Vou03], where in one direction the theory 'works' generally, while in the other direction, the theory 'works' only for *term* π-institutions. Consequently, the analogous *Blok-Pigozzi theorems* (in this case, theorems relating interpretations and theory-category functors) only pertain for term π-institutions. Our results, on the other hand, pertain generally; not just for global logics. In analyzing why our results hold generally and those of [Vou03] do not, it became clear that the problem lies in the notion of *naturality*, which is implicit in the definition of a translation between π-institutions (implicit in the demand for a categorical *natural transformation*). This notion of naturality is purely *syntactic*, that is, it depends only on the signatures of the institutions and not the *logic* (encoded as closure operators). We have termed this notion *syntactic naturality.* It is precisely this syntactic naturality that is lost in the move from a *translation* to a *theory-functor*, and as such, a (*syntactically natural*) translation *cannot* be recovered from a theory-functor except in the *term* case. We have taken the opportunity to develop (up to interpretation) and suggest (in the case of deductive equivalence) a more general theory of equivalence between π-institutions, one which is based on a notion of *logical naturality* rather than *syntactic naturality.* We shall show that logically naturality is encoded in the associated theory-category functors and can be recovered from these functors. We further show that syntactically natural translations are logically natural, and that *logically natural* translations from *term* π-institutions give rise to logically equivalent *syntactically natural* translations, thus explaining why the *Blok-Pigozzi theorems* obtained in [Vou03] work for term π-institutions. We should note that it is claimed in [GF05] (this 'paper' is an abstract and no preprint or published proof has appeared) that the result of Voutsadakis results can be extended to *multi-term* π-institutions, but even this result, if it is true, does not pertain to arbitrary π-institutions.

With regard to the third aim of this text, which is the unification of as many arguments from algebraic logic (and its relatives) under the banner of continuous translations between closed systems, we also consider the notion that a logic in one construct *models* a logic in another construct. This concept of model has nothing to do with the notion that one logic model another logic in the same construct; that form of model, which we studied in ∮7, is a generalization of a *matrix-model* of a sentential calculus; we replaced matrix-models with logic-models so as to locate this notion of model within the framework of continuous functions and their products and quotient. The notion of model considered in ∮17.4, is a weak form of a logic in one construct being a *semantics* for a logic in *another* construct.

Recall that a sentential calculus $\mathcal{S}_2$ is called a *formal semantics* for sentential calculus $\mathcal{S}_1$, both of the same type but with possible different signatures, if there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that

$$\Gamma \vdash_{\mathcal{S}_1} \phi \text{ iff } \boldsymbol{\tau}[\Gamma] \vdash_{\mathcal{S}_2} \boldsymbol{\tau}[\![\phi]\!] \tag{15.8}$$

(see Definition 2.95 on page 108). In the case that $\mathcal{S}_2$ is a quasivariety $\mathcal{K}$ (more precisely $\mathcal{S}_2 = S^2(\Theta^{\mathcal{K}})$), this is the definition of an algebraic semantics. Recall further the *Blok-Pigozzi theorem* that characterizes formal semantics in terms of *isomorphisms* from the theory lattice of $\mathcal{S}_1$ onto a *join-complete subsemilattice* of the theory lattice of $\mathcal{S}_2$ that is *compact* in the theory lattice of $\mathcal{S}_2$ and which *commutes* (see Theorem 2.96 on page 109). This property of commutivity plays a key role in the theory of algebraic logics. Noting that (15.8) amounts to the requirement that the formal translation $\boldsymbol{\tau}$ be strictly continuous from (the theory closed system of) $\mathcal{S}_1$ to (the theory closed system of) $\mathcal{S}_2$, we shall consider weakening this condition from strict continuity to just continuity. In the language of this chapter, interpreted for sentential calculi, $\mathcal{S}_2$ is called a *formal model* for sentential calculus $\mathcal{S}_1$, if there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that

$$\Gamma \vdash_{\mathcal{S}_1} \phi \text{ implies } \boldsymbol{\tau}[\Gamma] \vdash_{\mathcal{S}_2} \boldsymbol{\tau}[\![\phi]\!]. \tag{15.9}$$

We shall show that a translation from $\mathcal{S}_1$ to $\mathcal{S}_2$ is a formal model iff $\tau^\star$ commutes (thereby characterizing this important property of commutivity); the associated *Blok-Pigozzi theorem* characterizes formal models in terms of commuting join-preserving functions between the theory lattices. While such a notion as been introduced in Categorical Abstract Algebraic Logic (CAAL), called semi-interpretation in [Vou03], the direction in which it has been analysed is *not as a weak form of interpretation* (as we do), but rather as a precursor to the development of a model theory in the spirit of the *matrix model-theory* of sentential calculi [Vou07b]; in particular, no Blok-Pigozzi theorem is obtained, and questions concerning the theory-category functor are not considered (we believe this is because these questions have not even been asked in AAL). In our text we characterize this notion in the spirit of a weak form of interpretablity and as a consequence, obtain a *Blok-Pigozzi theorem* characterizing weak-interpretation in CAAL. Such a characterization is new in CAAL (and, in fact, in AAL).

We have had personal communication with Voutsadakis, and it is his belief that the questions that we are asking in this part are interesting and possibly important in CAAL. We shall motivate the importance later (in particular, see the introductory narrative of ∮17.4).

Chapter 16

Leibniz Equivalence and Protoalgebraicity

The primary aim of this chapter is to develop a *non-parameterized* theory of protoalgebraic logics over constructs and then to use this non-parameterized theory to provide an alternative explanation of our theory of $\langle X, z\rangle$-protoalgebraicity. Three related notions are considered, namely the *filter correspondence property*, the *Leibniz equivalence relation* (together with the notion of a *reduced matrix model*) and *protoalgebraicity*, at the level of discourse of logics over constructs. We shall only consider *structural* logics over *global* languages, that is $\mathfrak{s}$-*calculi*.

Of these three notions, the least controversial is the filter correspondence property, since it is naturally phrasable in the discourse. In ∮16.1 we define this notion, and provide characterizations. We have already noted that interpretations between languages are continuous between the closed systems of $\mathcal{D}$-filters on matrices of these languages (see Proposition 7.24 on page 257). As we shall see, the filter correspondence property is effectively the requirement that all reductive matrix-homomorphisms be *strictly* continuous between the closed systems of $\mathcal{D}$-filters on these matrices. Consequently, the characterizations presented in this section derive fairly directly from the theory of strictly continuous functions developed in ∮5.3.3. For sentential calculi, the filter correspondence property is equivalent to the condition of protoalgebraicity (see Theorem 2.135 on page 117 and [BP86]). In this section we only present those characterizations of the filter correspondence property that we have been able to obtain without appeal to any notion of Leibniz equivalence. Note that, in these characterizations, where notions of congruence are used in the standard characterization for sentential calculi, we replace these with the kernels of morphisms, which we have found to suffice, and which certainly coincide with congruences in the sentential case. We also characterize the (apparently) weaker condition of filter correspondence between filters on the global language (on the 'left') and an arbitrary matrix (on the 'right'). We have not been able to establish the equivalence of these to notions without appeal to the theory of Leibniz equivalence and protoalgebraicity developed subsequently.

In ∮16.2 we propose a definition of *Leibniz equivalence*, taking as our starting point the characterization given in Theorem 1.494 on page 92, which in many texts is the definition of this relation. It is clear that our notion coincides with the standard one for the case that $\mathfrak{s}$ is a type of algebras. While it is easily shown that the Leibniz relation as defined in ∮16.2 is an *equiva-*

lence relation that is *compatible* with the determining subset, and that *contains* the kernel of any morphism whose kernel is compatible with the determining subset, the notion of a construct does not have the imperative strength to force this relation itself to be the kernel of a morphism. To this end, in order to successfully develop a theory of protoalgebraicity at this level of discourse, we introduce the notion that an $\mathfrak{s}$-calculus be *Leibniz interpretable*, which is the requirement that the Leibniz relation determined by any filter of that calculus on any $\mathfrak{s}$-language, be the kernel of some surjective morphism. This property is trivially true for sentential calculi, since the Leibniz relation over algebras is always a congruence, without any appeal to a sentential calculus. In such cases we say that the signature $\mathfrak{s}$ is Leibniz interpretable.

Under the assumption of Leibniz interpretablity, we are able to successfully develop a theory of *reduced models* ($\oint$16.6) and *protoalgebraicity* ($\oint$16.4), We characterize the latter condition and prove its equivalence to the filter correspondence property, and to the weakened notion of filter correspondence mentioned earlier; while the equivalence of protoalgebraicity and the filter correspondence property require Leibniz interpretablity, *much* of our theory of protoalgebraicity does *not* require this assumption. It should be noted that this is a 'flat' theory of protoalgebraicity, since the distinction between sentential 1-calculi and sentential n-calculi has been devolved to the construct. As a consequence, we have been able to discover a simpler and more natural internal characterization of protoalgebraic n-deductive systems than that of [Pal03], and which is more in the spirit of the characterization of protoalgebraic sentential 1-calculi.

Recall that in $\oint$7.3 we introduced the notion of a maximal model, and that we provided a sufficient condition for this property, and promised to show that this condition was *necessary* for protoalgebraic logics. This is the topic of $\oint$16.5.

We also consider a number of examples. The primary example concerns $\langle X, z\rangle$-protoalgebraicity; we are able to establish a number of results from $\oint$14 using the machinery developed in this chapter, by devolving the parameter $\langle X, z\rangle$ to the construct by suitably restricting the homomorphisms. Another important example establishes the aforementioned characterization of protoalgebraic sentential n-calculi, which is then used to characterize the protoalgebraicity of the sentential n-calculus $S^n(\mathcal{K}, \mathfrak{N})$, of solutions to an n-ary system of equations $\mathfrak{N}$ modulo a quasivariety $\mathcal{K}$, in terms of $\mathcal{K}$ having coherent $\langle \mathcal{K}, \mathfrak{N}\rangle$-classes, which generalizes the analogous result for unary systems of [BR99] to n-ary systems. We also take the opportunity to characterize the protoalgebraicity of some of the sentential 1-calculi introduced subsequent to $\oint$2, namely the subuniverse logic $S(\mathcal{K}, \mathsf{su})$ in the case that the type of algebras under consideration has at least one constant symbol (and hence $S(\mathcal{K}, \mathsf{su})$ has theorems), and the lattice ideal and filter logics $S_0(\mathcal{K}, \mathsf{id})$ and $S_1(\mathcal{K}, \mathsf{fi})$, both of which have theorems.

Convention 16.1 (Chapter Conventions) Throughout this chapter, $\mathcal{D}$ shall denoted an arbitrary $\mathfrak{s}$-calculus (i.e, global $\mathfrak{s}$-structural logic) with language $\mathbf{G}$, where $\mathbf{G}$ is an $\mathfrak{s}$-free language with ω variables V.

16.1 Filter Correspondence

We begin by considering the filter correspondence property. For sentential calculi, the filter correspondence property is equivalent to the condition of protoalgebraicity (see Theorem 2.135 on page 117 and [BP89a]). The filter correspondence property is analogous to the well-known (congruence) *correspondence theorem* of universal algebras. In fact, Elgueta has successfully extended the notion of protoalgebraicity from sentential calculi (and matrices) to structures more generally, thereby developing various correspondence theorems for structures [Elg98].

Definition 16.2 (Filter Correspondence Property) We say that $\mathcal{D}$ has the **filter correspondence property** if, for all $\mathcal{D}$-matrices $\mathbf{M}$ and $\mathbf{N}$, every reductive $\mathfrak{s}$-matrix homomorphism f from $\mathbf{M}$ onto $\mathbf{N}$, and every $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$, $f^{-1}\left[\|f[F]\|^{\mathbf{N}}_{\mathsf{fi}_{\mathcal{D}}}\right] = F$, i.e., $f^{-1}\left[f^{\mathcal{D}}(F)\right] = F$. $\square$

In the following result we present a number of characterizations of the filter correspondence property. These are the characterizations that we have been able to establish without appeal to any restrictions on a logic or its signature. The reader is urged to compare the characterizations below with those of Theorem 2.135 on page 117. Note that in the analogues of the equivalent conditions of that theorem that involve congruences, we have replaced *congruences* with *kernels of morphisms*, which, for algebras, amounts to the same thing.

Theorem 16.3 The following conditions are equivalent.

1. $\mathcal{D}$ has the filter correspondence property.
2. For all $\mathcal{D}$-matrices $\mathbf{M}$ and $\mathbf{N}$, every reductive morphism from $\mathbf{M}$ onto $\mathbf{N}$ is strictly continuous from $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$.
3. For all $\mathcal{D}$-matrices $\mathbf{M}$ and $\mathbf{N}$, every reductive morphism from $\mathbf{M}$ onto $\mathbf{N}$ is consequence reflecting from $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$.
4. For every $\mathfrak{s}$-language $\mathbf{A}$, surjective $\mathfrak{s}$-morphism f from $\mathbf{A}$, and $F, G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$ with $F \subseteq G$, if $\equiv_f$ is compatible with F then $\equiv_f$ is compatible with G.
5. If $f : \mathbf{A} \twoheadrightarrow_{\mathfrak{s}} \mathbf{B}$, $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$ and $G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{B})$, then $F \vee^{\mathbf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G] = f^{-1}\left[\|f[F] \cup G\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{D}}}\right]$.
6. For all $\mathcal{D}$-matrices $\mathbf{M}$ and $\mathbf{N}$, $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$ and $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$, $f^{-1}\left[\|f[F]\|^{\mathbf{lg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{D}}}\right] = F$.
7. $f^{\mathcal{D}} : \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M}) \Rightarrow f^{\mathcal{D}}\left[\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})\right] \subseteq \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$ with inverse $f^{-1}[\cdot]_{|f^{\mathcal{D}}[\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})]}$, for all $\mathcal{D}$-matrices $\mathbf{M}$ and $\mathbf{N}$, and every $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$.
8. $f^{\mathcal{D}} : \mathbf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M}) \cong \mathbf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$ with inverse $f^{-1}[\cdot]_{|\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})}$, for all $\mathcal{D}$-matrices $\mathbf{M}$ and $\mathbf{N}$, and every $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathcal{D}$-matrices and $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$. By Proposition 7.24 on page 257, every $g : \mathbf{lg}(\mathbf{M}) \to_{\mathfrak{s}} \mathbf{lg}(\mathbf{N})$ is continuous from $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{lg}(\mathbf{M}))$ into $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{lg}(\mathbf{N}))$, i.e., $g^{-1}[G] \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{lg}(\mathbf{M}))$ for all $G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{lg}(\mathbf{N}))$ (by Theorem 5.40 on page 186). Let $G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$. So certainly $f^{-1}[G] \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{lg}(\mathbf{M}))$. Since $\mathsf{D}_{\mathbf{N}} \subseteq G$ and f is reductive, $\mathsf{D}_{\mathbf{M}} = f^{-1}[\mathsf{D}_{\mathbf{N}}] \subseteq f^{-1}[G]$, so $f^{-1}[G] \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$. So f is continuous from $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$, by Theorem 5.40. Further, for all $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$, since $f^{-1}\left[\|f[F]\|^{\mathbf{N}}_{\mathsf{fi}_{\mathcal{D}}}\right] = F$, by assumption, so f is consequence reflecting from $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$, by (8) of Proposition 5.71 on page 193. Consequently, f is strictly continuous from $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$. $\boxed{(2)\Rightarrow(3)}$ Trivial. $\boxed{(3)\Rightarrow(1)}$ By (8) of Proposition 5.71 on page 193. $\boxed{(1)\Rightarrow(4)}$ Assume $f : \mathbf{A} \twoheadrightarrow_{\mathfrak{s}} \mathbf{B}$, and $F, G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$ with $F \subseteq G$ and $\equiv_f$ compatible with F. (*It suffices to show that* $f^{-1}[f[G]] = G$, *in which case* $\equiv_f$ *is compatible with* G *by Remark 1.72 on page 25.*) By (1.46) of Table 1.2 on page 21, $G \subseteq f^{-1}[f[G]]$. (*We prove* *the converse inclusion.*) Since f is surjective and $\equiv_f$ is compatible with F, $f[F]$ is a $\mathcal{D}$-filter on $\mathbf{B}$, by Proposition 7.25 on page 258, and so $\langle \mathbf{B}, f[F]\rangle$ is a $\mathcal{D}$-matrix. By assumption, $\langle \mathbf{A}, F\rangle$ is a $\mathcal{D}$-matrix. Since f is surjective and $\equiv_f$ is compatible with F, $f^{-1}[f[F]] = F$, by Remark 1.72. So f is reductive from $\langle \mathbf{A}, F\rangle$ to $\langle \mathbf{B}, f[F]\rangle$. Since $F \subseteq G$ and $G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$, $G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\langle \mathbf{A}, F\rangle)$. So by assumption (1), $f^{-1}[f[G]] \subseteq f^{-1}\left[\|f[G]\|^{\langle \mathbf{B}, f[F]\rangle}_{\mathsf{fi}_{\mathcal{D}}}\right] = G$. $\boxed{(4)\Rightarrow(5)}$ Since $\mathcal{D}$-filterhood is preserved under inverse morphic images (Proposition 7.24 on page 257), $F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G] \subseteq f^{-1}[\|f[F] \cup G\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{D}}}]$ and $f^{-1}[G]$ is a $\mathcal{D}$-filter of $\mathbf{A}$. Since $\ker f$ is compatible with $f^{-1}[G] \subseteq F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]$, it follows from assumption (4) that $\ker f$ is compatible with $F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]$, so $f^{-1}[f[F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]]] = F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]$, by Remark 1.72. Since $\ker f$ is compatible with $F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]$ and f is surjective, $f[F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]] \in \mathsf{Fi}_{\mathcal{D}}(\mathbf{B})$, by Proposition 7.25 on page 258. By the surjectivity of f, therefore, $\|f[F] \cup G\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{D}}} \subseteq f[F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]]$, whence $f^{-1}[\|f[F] \cup G\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{D}}}] \subseteq f^{-1}[f[F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]]] = F \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[G]$. $\boxed{(5)\Rightarrow(6)}$ Suppose that $\mathbf{M}$ and $\mathbf{N}$ are $\mathcal{D}$-matrices, $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$ and $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$. Since $\mathsf{D}_{\mathbf{M}} \subseteq F$, by reduction, $\mathsf{D}_{\mathbf{N}} = f[\mathsf{D}_{\mathbf{M}}] \subseteq f[F]$. By assumption (5), reduction and the fact that $\mathsf{D}_{\mathbf{N}} \subseteq f[F]$ and $\mathsf{D}_{\mathbf{M}} \subseteq F$, $F = F \vee^{\mathbf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})} \mathsf{D}_{\mathbf{M}} = F \vee^{\mathbf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})} f^{-1}[\mathsf{D}_{\mathbf{N}}] = f^{-1}\left[\|f[F] \cup \mathsf{D}_{\mathbf{N}}\|^{\mathbf{lg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{D}}}\right] = f^{-1}\left[\|f[F]\|^{\mathbf{lg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{D}}}\right]$. $\boxed{(6)\Rightarrow(1)}$ Suppose that $\mathbf{M}$ and $\mathbf{N}$ are $\mathcal{D}$-matrices, $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$ and $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{M})$. Since $\mathsf{D}_{\mathbf{M}} \subseteq F$, by reduction, $\mathsf{D}_{\mathbf{N}} = f[\mathsf{D}_{\mathbf{M}}] \subseteq f[F]$, so $\|f[F]\|^{\mathbf{lg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{D}}} = \|f[F] \cup \mathsf{D}_{\mathbf{N}}\|^{\mathbf{lg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{D}}} = \|f[F]\|^{\mathbf{N}}_{\mathsf{fi}_{\mathcal{D}}}$, by (3) of Remark 7.75 on page 269. So $f^{-1}\left[\|f[F]\|^{\mathbf{N}}_{\mathsf{fi}_{\mathcal{D}}}\right] = f^{-1}\left[\|f[F]\|^{\mathbf{lg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{D}}}\right] = F$, by assumption (6). $\boxed{(3)\Leftrightarrow(7)}$ By Proposition 5.71 on page 193. $\boxed{(2)\Rightarrow(8)}$ (*By Theorem 5.73 on page 195, it suffices to prove surjectivity.*) Let G be a $\mathcal{D}$-filter of $\mathbf{N}$. Since f is surjective, $f\left[f^{-1}[G]\right] = G$ and $f^{-1}[G]$ is a $\mathcal{D}$-filter of $\mathbf{M}$, by Proposition 7.76 on page 270. So $f^{\mathcal{D}}(f^{-1}[G]) = \|f\left[f^{-1}[G]\right]\|^{\mathbf{N}}_{\mathsf{fi}_{\mathfrak{L}}} = \|G\|^{\mathbf{N}}_{\mathsf{fi}_{\mathfrak{L}}} = G$. $\boxed{(8)\Rightarrow(7)}$ Trivial. $\diamond$

The proof of the following lemma is very similar to the proofs of the previous theorem and corollary, effectively replacing the left hand filters etc., with $\mathcal{D}$-theories etc., and simplifying some expressions; as such we omit the proof. Note that $\mathcal{D}$-theories and $\mathcal{D}$-filters on $\mathbf{G}$ coincide, by our assumption that $\mathcal{D}$ is structural (see Theorem 7.48 on page 263). Recall the definition of the *filtration logic* $\mathsf{L}_{:\Gamma}$ of a logic L by $\Gamma \subseteq \mathsf{Fm}(\mathsf{L})$, given in Definition 6.10 on page 224. Observe that if $\mathcal{D}$ has the filter correspondence property then the equivalent conditions of the following lemma are valid.

Lemma 16.4 The following conditions are equivalent.

1. For $\mathcal{D}$-theory T, $\mathcal{D}$-matrix $\mathbf{N}$ and $f : \langle \mathbf{G}, T\rangle \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$, $f^{-1}\left[\|f[T]\|^{\mathbf{N}}_{\mathsf{fi}_{\mathcal{D}}}\right] = T$.
2. For $\mathcal{D}$-theory T, $\mathcal{D}$-matrix $\mathbf{N}$ and $f : \langle \mathbf{G}, T\rangle \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$, $f^{-1}\left[\|f[T]\|^{\mathbf{lg}(\mathbf{N})}_{\mathsf{fi}_{\mathcal{D}}}\right] = T$.
3. For every $\mathcal{D}$-theory T and $\mathcal{D}$-matrix $\mathbf{N}$, every reductive morphism from $\langle \mathbf{G}, T\rangle$ onto $\mathbf{N}$ is strictly continuous from $\mathsf{L}_{:T}$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$.
4. For every $\mathcal{D}$-theory T and $\mathcal{D}$-matrix $\mathbf{N}$, every reductive morphism from $\langle \mathbf{G}, T\rangle$ onto $\mathbf{N}$ is consequence reflecting from $\mathsf{L}_{:T}$ onto $\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$.
5. For every surjective $\mathfrak{s}$-interpretation f from $\mathbf{G}$ and $\mathcal{D}$-theories T and R with $T \subseteq R$, if $\equiv_f$ is compatible with T then $\equiv_f$ is compatible with R.
6. If $f : \mathbf{G} \twoheadrightarrow_{\mathfrak{s}} \mathbf{B}$, $T \in \mathsf{Th}(\mathcal{D})$ and $G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{B})$, then $T \vee^{\mathbf{Th}(\mathcal{D})} f^{-1}[G] = f^{-1}\left[\|f[T] \cup G\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{D}}}\right]$.
7. $f^{\mathcal{D}} : \mathsf{Th}(\mathcal{D}_{:T}) \Rightarrow f^{\mathcal{D}}[\mathsf{Th}(\mathcal{D}_{:T})] \subseteq \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$ with inverse $f^{-1}[\cdot]_{|f^{\mathcal{D}}[\mathsf{Th}(\mathcal{D}_{:T})]}$, for $\mathcal{D}$-theory T, $\mathcal{D}$-matrix $\mathbf{N}$ and $f : \langle \mathbf{G}, T\rangle \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$.

8. $f^{\mathcal{D}} : \mathbf{Th}(\mathcal{D}_{:T}) \cong \mathbf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})$ with inverse $f^{-1}[\cdot]_{|\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{N})}$, for every $\mathcal{D}$-theory T, $\mathcal{D}$-matrix $\mathbf{N}$ and $f : \langle \mathbf{G}, T\rangle \twoheadrightarrow^{\mathsf{r}}_{\mathfrak{s}} \mathbf{N}$.

16.2 Leibniz Equivalence

In this section we introduce two notions of *Leibniz equivalence* in a manner that is free of any algebraic baggage such as terms, polynomials or congruence relations. In ∮16.2.1, we introduce the first notion, which we term the *flat* Leibniz equivalence relation, and which is defined in terms of a signature only. We have taken the characterization of the Leibniz relation given in Theorem 1.494 on page 92 as the starting point for our definition. The Leibniz relation that we define is easily seen to coincide, in the case of algebras, with the standard notion of the Leibniz relation on an algebra determined by a subset of the universe of the algebra, and which proves to be a useful tool in the algebraization of 1-deductive systems, since it provides a mapping from sets of points to congruence relations.

When algebraizing n-deductive systems, one requires a mapping from sets of n-tuples to congruence relations, i.e. special sets of pairs of *points*, and *not* sets of pairs of n-tuples. Recall that in our program to 'flatten' out the vector nature of n-deductive systems, given a signature $\mathfrak{s}$ and natural n, we consider logics modulo the signature $\underrightarrow{\mathfrak{s}}_{[n]}$. In this case, our *flat* Leibniz relation maps sets of n-tuples to sets of pairs of n-tuples, so, at first glance, it appears that this notion will not suffice as a general notion of Leibniz equivalence. We shall see that this is in fact not the case.

To this end we define, in ∮16.2.2, what we call the *root* Leibniz equivalence relation, in terms of a signature $\mathfrak{s}$ and a natural n, which is a mapping from sets of n-tuples from the universe of an $\mathfrak{s}$-object, to an equivalence relation on the universe of the object. We then show that the flat Leibniz relations on a $\underrightarrow{\mathfrak{s}}_{[n]}$-object are precisely the promotions (i.e., n-powers) of the root Leibniz relations determined by sets of n-tuples over $\mathfrak{s}$-objects.

There appear to be five[1] main uses for the Leibniz equivalence relation (and the Leibniz operator) in the theory of algebraic logic. The first use is to factor matrices so as to provide *reduced matrix models*. The second is to define and analyse the condition of *protoalgebraicity*, which is a necessary precondition for algebraization. The third use, is to realize an *isomorphism* from the theories and filters of an n-deductive system to the theories and filters (which are congruences) of an equational 2-deductive system. The fourth and possibly most important use, stems from the fact that characterizations of meta-logical properties in terms of the Leibniz operator are usually much easier to *falsify* than their definitions are. Fifthly, Leibniz congruences help to construct the algebraic counterpart of a logic, which is a class of pure algebras invariant under the equivalence of consequence relations. We shall show that in our general constructural setting, the first two uses of the Leibniz relation can be sensibly considered. We have not yet explored the others at this level of generality.

While our notion of the Leibniz relation, like the standard algebraic Leibniz relation, is indeed an equivalence relation, compatible with the determining set, and containing the kernels of all morphisms compatible with the determining set, there does not appear to be enough 'power' in the notion of a construct, to ensure that our Leibniz relation is indeed the kernel of some

[1]We would like to thank James Raftery for completing this enumeration.

morphism. In order to achieve the standard results in our more general setting, we introduce the notion that a logic $\mathcal{D}$ be *Leibniz interpretable* (defined in ∮16.3), be which we mean that the Leibniz relation determined by a filter be the kernel of some surjective morphism. Of course for n-deductive systems over algebras this is always the case. This is the only place where our constructural notion of logics does not *directly* achieve the standard results of Blok and Pigozzis' theory.

Note that while in Part III it was most convenient to view the elements of an arbitrary $\mathfrak{s}$-language $\mathbf{A}$ as *formulae*, in this chapter it is more convenient to view the elements as *points*; consequently we shall tend to write $\mathsf{uni}(\mathbf{A})$ rather than $\mathsf{Fm}(\mathbf{A})$ (by definition $\mathsf{Fm}(\mathbf{A}) = \mathsf{uni}(\mathbf{A})$). Only the elements of *global* languages will still be viewed as formulae.

16.2.1 The Flat Leibniz Equivalence Relation

The following definition of the *flat Leibniz relation* takes the characterization of the standard Leibniz relation given in Theorem 1.494 on page 92 as the starting point. Given an $\mathfrak{s}$-language $\mathbf{A}$ and $A \subseteq \mathsf{uni}(\mathbf{A})$, we shall identify two points from $\mathbf{A}$ if they cannot be distinguished by interpretations of global formulae into A.

Definition 16.5 (The Flat Leibniz Equivalence Relation) For an $\mathfrak{s}$-language $\mathbf{A}$, distinct variables $x_1, \dots, x_n \in \mathsf{V}$, $a_1, \dots, a_n \in \mathsf{uni}(\mathbf{A})$ and a function $f : (\mathsf{V} - \{x_1, \dots, x_n\}) \to \mathsf{uni}(\mathbf{A})$, let $f_{\frac{x_1,\dots,x_n}{a_1,\dots,a_n}}$ denote the unique interpretation of $\mathbf{G}$ into $\mathbf{A}$ extending f and mapping x_i to a_i for each $1 \leq i \leq n$.

With each $\mathfrak{s}$-language $\mathbf{A}$ and $A \subseteq \mathsf{uni}(\mathbf{A})$, we associate the binary relation $\Omega^{\mathbf{A}}_{\mathfrak{s}}(A)$ on $\mathsf{uni}(\mathbf{A})$, defined by $a\ \Omega^{\mathbf{A}}_{\mathfrak{s}}(A)\ b$ iff, for every $\phi \in \mathsf{Fm}(\mathbf{G})$, every $x \in \mathsf{V}$ and every function $f : (\mathsf{V} - \{x\}) \to \mathsf{uni}(\mathbf{A})$, $f_{\frac{x}{a}}(\phi) \in A$ iff $f_{\frac{x}{b}}(\phi) \in A$. We write $\Omega^{\mathbf{A}}(A)$ for $\Omega^{\mathbf{A}}_{\mathfrak{s}}(A)$ where the signature $\mathfrak{s}$ is unambiguously understood. □

The following equivalent formulation is simpler in that the requirement to test over *all* variables has been replaced by the requirement to test for any *one* variable.

Remark 16.6 *$a\ \Omega^{\mathbf{A}}(A)\ b$ iff, for some $x \in \mathsf{V}$, every $\phi \in \mathsf{Fm}(\mathbf{G})$, and every function $f : (\mathsf{V} - \{x\}) \to \mathsf{uni}(\mathbf{A})$, $f_{\frac{x}{a}}(\phi) \in A$ iff $f_{\frac{x}{b}}(\phi) \in A$.*

Proof. $\boxed{\Rightarrow}$ Trivial. $\boxed{\Leftarrow}$ Assume that for some $x \in \mathsf{V}$, every $\phi \in \mathsf{Fm}(\mathbf{G})$, and every function $g : (\mathsf{V} - \{x\}) \to \mathsf{uni}(\mathbf{A})$, $g_{\frac{x}{a}}(\phi) \in A$ iff $g_{\frac{x}{b}}(\phi) \in A$. Let $\phi \in \mathsf{Fm}(\mathbf{G})$, $y \in \mathsf{V}$ and $f : (\mathsf{V} - \{y\}) \to \mathsf{uni}(\mathbf{A})$, with $f_{\frac{y}{a}}(\phi) \in A$. (*It suffices to show that $f_{\frac{y}{b}}(\phi) \in A$.*) Consider the $\mathbf{G}$-substitution σ mapping $x \mapsto y$, $y \mapsto x$, and fixing all other variables, and the function $g : (\mathsf{V} - \{x\}) \to \mathsf{uni}(\mathbf{A})$ mapping $y \mapsto f(x)$ and $z \mapsto f(z)$ for all $z \in \mathsf{V} - \{x, y\}$.

$\boxed{\text{Claim: for all } c,\ g_{\frac{x}{c}}\sigma = f_{\frac{y}{c}}}$ By composition of morphisms, $g_{\frac{x}{c}}\sigma : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$ and $f_{\frac{y}{c}} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$, so by the freedom of $\mathbf{G}$, it suffices to show that these function agree on all variables; $g_{\frac{x}{c}}\sigma(x) = g_{\frac{x}{c}}(y) = g(y) = f(x) = f_{\frac{y}{c}}(x)$, $g_{\frac{x}{c}}\sigma(y) = g_{\frac{x}{c}}(x) = c = f_{\frac{y}{c}}(y)$, and for all $z \in \mathsf{V} - \{x, y\}$, $g_{\frac{x}{c}}\sigma(z) = g_{\frac{x}{c}}(z) = f(z) = f_{\frac{y}{c}}(z)$. □

So by this claim, $g_{\frac{x}{a}}(\sigma(\phi)) = g_{\frac{x}{a}}\sigma(\phi) = f_{\frac{y}{a}}(\phi) \in A$, so by assumption (and the claim) $f_{\frac{y}{b}}(\phi) = g_{\frac{x}{b}}\sigma(\phi) = g_{\frac{x}{b}}(\sigma(\phi)) \in A$. ◇

We shall now demonstrate that the flat Leibniz relation $\Omega^{\mathbf{A}}(A)$ is an *equivalence relation* that is *compatible* with A and contains the kernel of any morphism from $\mathbf{A}$ whose kernel is compatible with A. The only important property of the standard Leibniz relation that we are missing, is that $\Omega^{\mathbf{A}}(A)$ need not be the kernel of some morphism from $\mathbf{A}$; recall that in the standard case, the Leibniz relation is always a *congruence*.

Proposition 16.7 Let $\mathbf{A}$ be an $\mathfrak{s}$-language and $A \subseteq \mathsf{uni}(\mathbf{A})$.

1. $\Omega^{\mathbf{A}}(A)$ is an equivalence relation on $\mathsf{uni}(\mathbf{A})$.
2. $\Omega^{\mathbf{A}}(A)$ is compatible with A.
3. If $g : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$ and $\equiv_g$ is compatible with A, then $\equiv_g \subseteq \Omega^{\mathbf{A}}(A)$.

Proof. $\boxed{(1)}$ Easy. $\boxed{(2)}$ Suppose that $a \in A$ and $a\, \Omega^{\mathbf{A}}(A)\, b$. Let $x \in \mathsf{V}$ and any $f : (\mathsf{V} - \{x\}) \rightarrow \mathsf{uni}(\mathbf{A})$. Now $f_{\frac{x}{a}}(x) = a \in A$, and so $b = f_{\frac{x}{b}}(x) \in A$. $\boxed{(3)}$ Let $g : \mathbf{A} \rightarrow_{\mathfrak{s}} \mathbf{B}$ with $\equiv_g$ compatible with A. Suppose that $a \equiv_g b$, i.e., $g(a) = g(b)$. Let $\phi \in \mathsf{Fm}(\mathbf{G})$, $x \in \mathsf{V}$ and $f : (\mathsf{V} - \{x\}) \rightarrow \mathsf{uni}(\mathbf{A})$, with $f_{\frac{x}{a}}(\phi) \in A$.
$\boxed{\text{Claim: } gf_{\frac{x}{a}} = gf_{\frac{x}{b}}}$ For any variable y other than x, $g(f_{\frac{x}{a}}(y)) = g(f(y)) = g(f_{\frac{x}{b}}(y))$, and $g(f_{\frac{x}{a}}(x)) = g(a) = g(b) = g(f_{\frac{x}{b}}(x))$; so by $\mathfrak{s}$-freedom of $\mathbf{G}$, $gf_{\frac{x}{a}} = gf_{\frac{x}{b}}$. $\square$
So $g(f_{\frac{x}{a}}(\phi)) = g(f_{\frac{x}{b}}(\phi))$, i.e., $f_{\frac{x}{a}}(\phi) \equiv_g f_{\frac{x}{b}}(\phi)$. By assumed compatibility of $\equiv_g$ with A, $f_{\frac{x}{a}}(\phi) \in A$ iff $f_{\frac{x}{b}}(\phi) \in A$. Hence $a\, \Omega^{\mathbf{A}}(A)\, b$. $\diamond$

In the following example we show that the flat Leibniz relation $\Omega^{\mathbf{A}}_{\mathfrak{a}}(A)$ coincides with the standard Leibniz relation $\Omega^{\mathbf{A}}(A)$, where $\mathbf{A}$ is an $\mathfrak{a}$-algebra and $A \subseteq \mathsf{uni}(\mathbf{A})$.

Example 16.8 (The Flat Leibniz Relation in $\mathfrak{a}$)

Let $\mathfrak{a}$ be a type of algebras, $\mathbf{A}$ be an $\mathfrak{a}$-algebra and $A \subseteq \mathsf{uni}(\mathbf{A})$.

Proposition 16.9 The following conditions are equivalent.

1. $a\, \Omega^{\mathbf{A}}(A)\, b$.
2. For every term $p(x, \vec{y})$ and $\vec{c} \in \mathsf{uni}(\mathbf{A})$, $p^{\mathbf{A}}(a, \vec{c}) \in A$ iff $p^{\mathbf{A}}(b, \vec{c}) \in A$.

Proposition 16.10 [BP89a] $\Omega^{\mathbf{A}}(A)$ is the largest congruence on $\mathbf{A}$ compatible with A.

$\square$

For an $\mathfrak{a}$-algebra $\mathbf{A}$ and $\mathtt{A} \subseteq \mathsf{uni}(\mathbf{A})^n$, the standard Leibniz relation $\Omega^{\mathbf{A}}(\mathtt{A})$ is an equivalence relation on $\mathsf{uni}(\mathbf{A})$. If we consider $\mathbf{A}^n$ as a language of the signature $\underrightarrow{\mathfrak{a}}_{[n]}$, the flat Leibniz relation $\Omega^{\mathbf{A}^n}_{\underrightarrow{\mathfrak{a}}_{[n]}}(\mathtt{A})$ is an equivalence relation on $\mathsf{uni}(\mathbf{A})^n$. So the standard $\Omega^{\mathbf{A}}(\mathtt{A})$ and our $\Omega^{\mathbf{A}^n}_{\underrightarrow{\mathfrak{a}}_{[n]}}(\mathtt{A})$ are incomparable. In ∮16.2.2, we will show that this is not really a problem, since $\underrightarrow{\Omega^{\mathbf{A}}(\mathtt{A})}_{[n]} = \Omega^{\mathbf{A}^n}_{\underrightarrow{\mathfrak{a}}_{[n]}}(\mathtt{A})$, and in most uses of $\Omega^{\mathbf{A}}(\mathtt{A})$, such as defining reduced matrices and defining/characterizing protoalgebraicity and the filter correspondence property, these use-cases implicitly invoke $\underrightarrow{\Omega^{\mathbf{A}}(\mathtt{A})}_{[n]}$.
In the following example we characterize $\Omega^{\mathbf{A}^n}_{\underrightarrow{\mathfrak{a}}_{[n]}}(\mathtt{A})$.

Example 16.11 (The Flat Leibniz Relation in $\underrightarrow{\mathfrak{a}}_{[n]}$)

Let $\mathfrak{a}$ be a type of algebras and $\underrightarrow{\mathfrak{a}}_{[n]}$ the n-power construct. Let $\mathbf{A}$ be an $\mathfrak{a}$-algebra and $\mathtt{A} \subseteq \mathsf{uni}(\mathbf{A})^n$.

Remark 16.12 $\mathtt{a}\ \Omega^{\mathbf{A}}_{\rightarrow}(\mathtt{A})\ \mathtt{b}$ iff for every $\langle p_1(x_1, \ldots, x_n, \vec{y}), \ldots, p_n(x_1, \ldots, x_n, \vec{y})\rangle \in (\mathsf{Tm})^n$, and $\vec{c} \in \mathsf{uni}(\mathbf{A})$, $\langle p_1^{\mathbf{A}}(\mathtt{a}, \vec{c}), \ldots, p_n^{\mathbf{A}}(\mathtt{a}, \vec{c})\rangle \in \mathtt{A}$ iff $\langle p_1^{\mathbf{A}}(\mathtt{b}, \vec{c}), \ldots, p_n^{\mathbf{A}}(\mathtt{a}, \vec{c})\rangle \in \mathtt{A}$.

$\square$

16.2.2 The Root Leibniz Equivalence Relation

The following definition of the *root Leibniz relation*, which *pertains only to vector signatures* $\underrightarrow{\mathfrak{s}}_{[n]}$ where $\mathfrak{s}$ is a signature, is introduced to explain the relationship between our *flat* Leibniz relation and the standard when applied to subsets of $\mathsf{uni}(\mathbf{A})^n$; we aim to show that in such cases, our flat Leibniz relation coincides with the promotion of the standard Leibniz relation. The proof of this result is very technical, and is not required in the sequel, other than to obtain standard results from our results concerning the flat Leibniz relation.

Definition 16.13 (The Root Leibniz Equivalence Relation) With each $\mathfrak{s}$-language $\mathbf{A}$, integer n and $\mathtt{A} \subseteq \mathsf{uni}(\mathbf{A})^n$, we associate the binary relation $\mathbf{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})$ on $\mathsf{uni}(\mathbf{A})$, defined by $a\ \mathbf{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})\ b$ iff, for every $\langle \phi_1, \ldots, \phi_n \rangle \in \mathsf{Fm}(\mathbf{G})^n$, every $x \in \mathsf{V}$ and every function $f : (\mathsf{V} - \{x\}) \rightarrow \mathsf{uni}(\mathbf{A})$, $\langle f_{\frac{x}{a}}(\phi_1), \ldots, f_{\frac{x}{a}}(\phi_n) \rangle \in \mathtt{A}$ iff $\langle f_{\frac{x}{b}}(\phi_1), \ldots, f_{\frac{x}{b}}(\phi_n) \rangle \in \mathtt{A}$. $\square$

Remark 16.14 $\mathbf{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})$ is an equivalence relation on $\mathsf{uni}(\mathbf{A})$. $\square$

The proof of the following result is similar to the proof of Remark 16.6.

Remark 16.15 $a\ \mathbf{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})\ b$ iff, for some $x \in \mathsf{V}$, every $\langle \phi_1, \ldots, \phi_n \rangle \in \mathsf{Fm}(\mathbf{G})^n$, and every function $f : (\mathsf{V} - \{x\}) \rightarrow \mathsf{uni}(\mathbf{A})$, $\langle f_{\frac{x}{a}}(\phi_1), \ldots, f_{\frac{x}{a}}(\phi_n) \rangle \in \mathtt{A}$ iff $\langle f_{\frac{x}{b}}(\phi_1), \ldots, f_{\frac{x}{b}}(\phi_n) \rangle \in \mathtt{A}$. $\square$

In the next result we establish the relationship between the flat Leibniz relation and the root Leibniz relation.

Theorem 16.16 $\underrightarrow{\mathbf{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})} = \Omega^{\mathbf{A}}_{\underrightarrow{\mathfrak{s}}_{[n]}}(\mathtt{A})$.

Proof. $\boxed{\subseteq}$ Suppose that $\mathtt{a}\ \underrightarrow{\mathbf{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})}\ \mathtt{b}$, i.e., for each $0 \leq i \leq n-1$, $\mathsf{a}_{(i)}\ \mathbf{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})\ \mathsf{b}_{(i)}$. We shall show that $\mathtt{a}\ \Omega^{\mathbf{A}}_{\underrightarrow{\mathfrak{s}}_{[n]}}(\mathtt{A})\ \mathtt{b}$, proceeding inductively on the co-ordinates. $\boxed{\text{Base Case}}$ The proof that $\mathtt{a}\ \Omega^{\mathbf{A}}_{\underrightarrow{\mathfrak{s}}_{[n]}}(\mathtt{A})\ \langle \mathsf{b}_{(0)}, \mathsf{a}_{(1)}, \ldots, \mathsf{a}_{(n-1)} \rangle$, is similar to the proof of the inductive step, and so is omitted. $\boxed{\text{Inductive Hypothesis}}$ Assume that $\mathtt{a}\ \Omega^{\mathbf{A}}_{\underrightarrow{\mathfrak{s}}_{[n]}}(\mathtt{A})\ \langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m)}, \mathsf{a}_{(m+1)}, \ldots, \mathsf{a}_{(n-1)} \rangle$. $\boxed{\text{Inductive Step}}$ We show that $\mathtt{a}\ \Omega^{\mathbf{A}}_{\underrightarrow{\mathfrak{s}}_{[n]}}(\mathtt{A})\ \langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m+1)}, \mathsf{a}_{(m+2)}, \ldots, \mathsf{a}_{(n-1)} \rangle$, by showing that $\langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m)}, \mathsf{a}_{(m+1)}, \ldots, \mathsf{a}_{(n-1)} \rangle\ \Omega^{\mathbf{A}}_{\underrightarrow{\mathfrak{s}}_{[n]}}(\mathtt{A})\ \langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m+1)}, \mathsf{a}_{(m+2)}, \ldots, \mathsf{a}_{(n-1)} \rangle$, which suffices since $\Omega^{\mathbf{A}}_{\underrightarrow{\mathfrak{s}}_{[n]}}(\mathtt{A})$ is transitive, the required result following from the induction hypothesis. Consider the $\underrightarrow{\mathbf{G}}$-variable $\langle \mathsf{v}_0, \ldots, \mathsf{v}_{n-1} \rangle$. Let $\langle \phi_1, \ldots, \phi_1 \rangle \in \mathsf{Fm}(\mathbf{G})^n$, $\boldsymbol{f} : (\mathsf{Var}_{\underrightarrow{\mathfrak{s}}_{[n]}}(\underrightarrow{\mathbf{G}}) - \{\langle \mathsf{v}_0, \ldots, \mathsf{v}_{n-1} \rangle\}) \rightarrow \mathsf{uni}(\mathbf{A})^n$, with

$$\boldsymbol{f}_{\frac{\langle \mathsf{v}_0, \ldots, \mathsf{v}_{n-1} \rangle}{\langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m)}, \mathsf{a}_{(m+1)}, \ldots, \mathsf{a}_{(n-1)} \rangle}}(\langle \phi_1, \ldots, \phi_n \rangle) \in \mathtt{A}.$$

(*It suffices to show that* $\boldsymbol{f}_{\frac{\langle \mathsf{v}_0, \ldots, \mathsf{v}_{n-1} \rangle}{\langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m+1)}, \mathsf{a}_{(m+2)}, \ldots, \mathsf{a}_{(n-1)} \rangle}}(\langle \phi_1, \ldots, \phi_1 \rangle) \in \mathtt{A}$.)

Let

$$g = \left(\sqrt[n]{\boldsymbol{f}_{\frac{\langle \mathsf{v}_0, \ldots, \mathsf{v}_{n-1} \rangle}{\langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m)}, \mathsf{a}_{(m+1)}, \ldots, \mathsf{a}_{(n-1)} \rangle}}} \right)_{|\mathsf{V} - \{\mathsf{v}_{m+1}\}},$$

which is a function from $\mathsf{V} - \{\mathsf{v}_{m+1}\}$ into $\mathsf{uni}(\mathbf{A})$.

$\boxed{\text{Claim: for any } a \in \mathsf{uni}(\mathbf{A}),\ g_{\underrightarrow{\frac{\mathsf{v}_{m+1}}{a}}} = \boldsymbol{f}_{\frac{\langle \mathsf{v}_0, \ldots, \mathsf{v}_{n-1} \rangle}{\langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m)}, a, \mathsf{a}_{(m+2)}, \ldots, \mathsf{a}_{(n-1)} \rangle}}}$

Both functions are $\underrightarrow{\mathfrak{s}}_{[n]}$-morphisms from $\underrightarrow{\mathbf{G}}$ into $\underrightarrow{\mathbf{A}}$, so it suffices to show that

$$g_{\frac{\mathsf{v}_{m+1}}{a}} = \sqrt[n]{\boldsymbol{f}_{\frac{\langle \mathsf{v}_0, \ldots, \mathsf{v}_{n-1} \rangle}{\langle \mathsf{b}_{(0)}, \ldots, \mathsf{b}_{(m)}, a, \mathsf{a}_{(m+2)}, \ldots, \mathsf{a}_{(n-1)} \rangle}}}.$$

Since these are both $\mathfrak{s}$-morphisms from $\mathbf{G}$ into $\mathbf{A}$, it suffices, by $\mathfrak{s}$-freedom of $\mathbf{G}$, to show that they coincide on variables. By the definition of g, these functions coincide on all variables $\mathsf{V} - \{\mathsf{v}_{m+1}\}$. Further,

$$g_{\frac{\mathsf{v}_{m+1}}{a}}(\mathsf{v}_{m+1}) = a = \sqrt[n]{\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\langle \mathsf{b}_{(0)},\ldots,\mathsf{b}_{(m)},a,\mathsf{a}_{(m+2)},\ldots,\mathsf{a}_{(n-1)}\rangle}}}(\mathsf{v}_{m+1}),$$

since

$$\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\langle \mathsf{b}_{(0)},\ldots,\mathsf{b}_{(m)},a,\mathsf{a}_{(m+2)},\ldots,\mathsf{a}_{(n-1)}\rangle}}(\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle) = \langle \mathsf{b}_{(0)},\ldots,\mathsf{b}_{(m)},a,\mathsf{a}_{(m+2)},\ldots,\mathsf{a}_{(n-1)}\rangle.$$

This establishes the claim. $\square$

So

$$\langle g_{\frac{\mathsf{v}_{m+1}}{\mathsf{a}_{(m+1)}}}(\phi_1),\ldots,g_{\frac{\mathsf{v}_{m+1}}{\mathsf{a}_{(m+1)}}}(\phi_n)\rangle = \boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\langle \mathsf{b}_{(0)},\ldots,\mathsf{b}_{(m)},\mathsf{a}_{(m+1)},\ldots,\mathsf{a}_{(n-1)}\rangle}}(\langle \phi_1,\ldots,\phi_n\rangle) \in \mathtt{A}.$$

Hence, since $\mathsf{a}_{(m+1)}\ \boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})\ \mathsf{b}_{(m+1)}$ (as $\mathtt{a}\ \underrightarrow{\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{s}}(\mathtt{A})}\ \mathtt{b}$ by assumption),

$$\langle g_{\frac{\mathsf{v}_{m+1}}{\mathsf{b}_{(m+1)}}}(\phi_1),\ldots,g_{\frac{\mathsf{v}_{m+1}}{\mathsf{b}_{(m+1)}}}(\phi_n)\rangle \in \mathtt{A}.$$

So

$$\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\langle \mathsf{b}_{(0)},\ldots,\mathsf{b}_{(m+1)},\mathsf{a}_{(m+2)},\ldots,\mathsf{a}_{(n-1)}\rangle}}(\langle \phi_1,\ldots,\phi_1\rangle) = \langle g_{\frac{\mathsf{v}_{m+1}}{\mathsf{b}_{(m+1)}}}(\phi_1),\ldots,g_{\frac{\mathsf{v}_{m+1}}{\mathsf{b}_{(m+1)}}}(\phi_n)\rangle \in \mathtt{A}.$$

$\boxed{\supseteq}$ Suppose that $\mathtt{a}\ \Omega^{\mathbf{A}}_{\underrightarrow{\mathfrak{s}}[n]}(\mathtt{A})\ \mathtt{b}$. Let $0 \le i \le n-1$. (*We must show that* $\mathsf{a}_{(i)}\ \boldsymbol{\Omega}^{\mathbf{A}}(\mathtt{A})\ \mathsf{b}_{(i)}$.) Let $\langle \phi_1,\ldots,\phi_n\rangle \in \mathsf{Fm}(\mathbf{G})^n$, $f : (\mathsf{V} - \{\mathsf{v}_0\}) \to \mathsf{uni}(\mathbf{A})$ with

$$\langle f_{\frac{\mathsf{v}_0}{\mathsf{a}_{(i)}}}(\phi_1),\ldots,f_{\frac{\mathsf{v}_0}{\mathsf{a}_{(i)}}}(\phi_n)\rangle \in \mathtt{A}.$$

(*It suffices to show that* $\langle f_{\frac{\mathsf{v}_0}{\mathsf{b}_{(i)}}}(\phi_1),\ldots,f_{\frac{\mathsf{v}_0}{\mathsf{b}_{(i)}}}(\phi_n)\rangle \in \mathtt{A}$.) Consider the $\underrightarrow{\mathbf{G}}$-variable $\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle$. Define $g : \mathsf{V} - \{\mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\} \to \mathsf{uni}(\mathbf{A})$ by

$$g(\mathsf{v}_{n+j}) = f(\mathsf{v}_{j+1}),$$

for $j > 0$. Define $\boldsymbol{f} : \mathsf{Var}_{\underrightarrow{\mathfrak{s}}[n]}(\underrightarrow{\mathbf{G}}) - \{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle\} \to \mathsf{uni}(\mathbf{A})^n$ by

$$\boldsymbol{f} = \underrightarrow{g}_{|(\mathsf{Var}_{\underrightarrow{\mathfrak{s}}[n]}(\underrightarrow{\mathbf{G}}) - \{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle\})}.$$

For any $\mathsf{c} \in \mathsf{uni}(\mathbf{A})^n$, $\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}} = \underrightarrow{h_{\mathsf{c}}}$, where h_{c} is the unique interpretation of $\mathbf{G}$ into $\mathbf{A}$ with

$$h_{\mathsf{c}}(\mathsf{v}_j) = \begin{cases} \mathsf{c}_{(j)} & ; \quad 0 \le j \le n-1, \\ g(\mathsf{v}_j) = f(\mathsf{v}_{j-n+1}) & ; \quad \text{otherwise.} \end{cases}$$

Let σ be the $\mathbf{G}$-substitution with $\sigma(\mathsf{v}_0) = \mathsf{v}_i$, and $\sigma(\mathsf{v}_j) \mapsto \mathsf{v}_{n+j-1}$, for $j > 0$.

$\boxed{\text{Claim: For } \mathsf{c} \in \mathsf{uni}(\mathbf{A})^n,\ \boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}}\underrightarrow{\sigma} = \underrightarrow{f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}}}$

Since $\sigma : \mathbf{G} \to_{\mathfrak{s}} \mathbf{G}$ and $f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$, by definition of the construct $\underrightarrow{\mathfrak{s}}[n]$, $\underrightarrow{\sigma} : \underrightarrow{\mathbf{G}} \to_{\underrightarrow{\mathfrak{s}}[n]} \underrightarrow{\mathbf{G}}$ and $\underrightarrow{f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}} : \underrightarrow{\mathbf{G}} \to_{\underrightarrow{\mathfrak{s}}[n]} \underrightarrow{\mathbf{A}}$. Since $\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}} : \underrightarrow{\mathbf{G}} \to_{\underrightarrow{\mathfrak{s}}[n]} \underrightarrow{\mathbf{A}}$ and $\underrightarrow{\sigma} : \underrightarrow{\mathbf{G}} \to_{\underrightarrow{\mathfrak{s}}[n]} \underrightarrow{\mathbf{G}}$, by composition of morphisms, $\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}}\underrightarrow{\sigma} : \underrightarrow{\mathbf{G}} \to_{\underrightarrow{\mathfrak{s}}[n]} \underrightarrow{\mathbf{A}}$. By the $\underrightarrow{\mathfrak{s}}[n]$-freedom of $\underrightarrow{\mathbf{G}}$, it suffices to show that $\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}}\underrightarrow{\sigma}$ and $\underrightarrow{f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}}$

coincide on variables.

$$
\begin{aligned}
\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}}\underset{\rightarrow}{\sigma}(\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle) &= \boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}}(\underset{\rightarrow}{\sigma}(\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle))\\
&= \boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}}(\langle \sigma(\mathsf{v}_0),\ldots,\sigma(\mathsf{v}_{n-1})\rangle)\\
&= \boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathsf{c}}}(\langle \mathsf{v}_i,\mathsf{v}_n,\ldots,\mathsf{v}_{2n-2}\rangle)\\
&= \langle h_{\mathsf{c}}(\mathsf{v}_i),h_{\mathsf{c}}(\mathsf{v}_n),\ldots,h_{\mathsf{c}}(\mathsf{v}_{2n-2})\rangle\\
&= \langle \mathsf{c}_{(i)},g(\mathsf{v}_n),\ldots,g(\mathsf{v}_{2n-2})\rangle\\
&= \langle \mathsf{c}_{(i)},f(\mathsf{v}_1),\ldots,f(\mathsf{v}_{n-1})\rangle\\
&= \langle f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}(\mathsf{v}_0),f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}(\mathsf{v}_1),\ldots,f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}(\mathsf{v}_{n-1})\rangle\\
&= \underset{\longrightarrow}{f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}}(\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle),
\end{aligned}
$$

and for $j > 0$,

$$
\begin{aligned}
\boldsymbol{f}_{\frac{\langle \mathsf{v}_{jn},\ldots,\mathsf{v}_{jn+n-1}\rangle}{\mathsf{c}}}\underset{\rightarrow}{\sigma}(\langle \mathsf{v}_{jn},\ldots,\mathsf{v}_{jn+n-1}\rangle) &= \boldsymbol{f}_{\frac{\langle \mathsf{v}_{jn},\ldots,\mathsf{v}_{jn+n-1}\rangle}{\mathsf{c}}}(\langle \sigma(\mathsf{v}_{jn}),\ldots,\sigma(\mathsf{v}_{jn+n-1})\rangle)\\
&= \boldsymbol{f}_{\frac{\langle \mathsf{v}_{jn},\ldots,\mathsf{v}_{jn+n-1}\rangle}{\mathsf{c}}}(\langle \mathsf{v}_{jn+n-1},\mathsf{v}_{jn+n},\ldots,\mathsf{v}_{jn+2n-2}\rangle)\\
&= \langle h_{\mathsf{c}}(\mathsf{v}_{jn+n-1}),\ldots,h_{\mathsf{c}}(\mathsf{v}_{jn+2n-2})\rangle\\
&= \langle g(\mathsf{v}_{jn+n-1}),\ldots,g(\mathsf{v}_{jn+2n-2})\rangle\\
&= \langle f(\mathsf{v}_{jn+n-1-n+1}),\ldots,f(\mathsf{v}_{jn+2n-2-n+1})\rangle\\
&= \langle f(\mathsf{v}_{jn}),\ldots,f(\mathsf{v}_{jn+n-1})\rangle\\
&= \langle f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}(\mathsf{v}_{jn}),\ldots,f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}(\mathsf{v}_{jn+n-1})\rangle\\
&= \underset{\longrightarrow}{f_{\frac{\mathsf{v}_0}{\mathsf{c}_{(i)}}}}(\langle \mathsf{v}_{jn},\ldots,\mathsf{v}_{jn+n-1}\rangle).
\end{aligned}
$$

This establishes the claim. □

So

$$
\begin{aligned}
\boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathtt{a}}}(\langle \sigma(\phi_1),\ldots,\sigma(\phi_n)\rangle) &= \boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathtt{a}}}\underset{\rightarrow}{\sigma}(\langle \phi_1,\ldots,\phi_n\rangle)\\
&= \underset{\longrightarrow}{f_{\frac{\mathsf{v}_0}{\mathtt{a}_{(i)}}}}(\langle \phi_1,\ldots,\phi_n\rangle)\\
&= \langle f_{\frac{\mathsf{v}_0}{\mathtt{a}_{(i)}}}(\phi_1),\ldots,f_{\frac{\mathsf{v}_0}{\mathtt{a}_{(i)}}}(\phi_n)\rangle\\
&\in \mathtt{A}.
\end{aligned}
$$

Since $\mathtt{a}\ \Omega^{\mathbf{A}}_{\underset{\rightarrow}{\mathfrak{s}}[n]}(\mathtt{A})\ \mathtt{b}$,

$$
\begin{aligned}
\langle f_{\frac{\mathsf{v}_0}{\mathtt{b}_{(i)}}}(\phi_1),\ldots,f_{\frac{\mathsf{v}_0}{\mathtt{b}_{(i)}}}(\phi_n)\rangle &= \underset{\longrightarrow}{f_{\frac{\mathsf{v}_0}{\mathtt{b}_{(i)}}}}(\langle \phi_1,\ldots,\phi_n\rangle)\\
&= \boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathtt{b}}}\underset{\rightarrow}{\sigma}(\langle \phi_1,\ldots,\phi_n\rangle)\\
&= \boldsymbol{f}_{\frac{\langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle}{\mathtt{b}}}(\langle \sigma(\phi_1),\ldots,\sigma(\phi_n)\rangle)\\
&\in \mathtt{A}.
\end{aligned}
$$

◇

16.3 Leibniz Interpretability

While we have not sought a counter example, we have no reason to suspect that the flat Leibniz relation need be the kernel of a morphism. Some of the theory developed in the sequel requires that this be the case.

Definition 16.17 (Leibniz Interpretability) We say that $\mathcal{D}$ is **Leibniz interpretable in $\mathfrak{s}$** if, for every $\mathfrak{s}$-object $\mathbf{A}$ and every $\mathcal{D}$-filter F of $\mathbf{A}$, there exists a *surjective* $\mathfrak{s}$-interpretation, denoted by $[\,\Omega^{\mathbf{A}}(F)\,](\cdot)$, from $\mathbf{A}$ onto some $\mathfrak{s}$-language, denoted $[\,\Omega^{\mathbf{A}}(F)\,](\mathbf{A})$, with kernel $\Omega^{\mathbf{A}}(F)$. We say that *signature* $\mathfrak{s}$ is **Leibniz interpretable** if, for every $\mathfrak{s}$-object $\mathbf{A}$ and every subset $A \subseteq \mathsf{uni}(\mathbf{A})$, there exists a surjective $\mathfrak{s}$-interpretation from $\mathbf{A}$ with kernel $\Omega^{\mathbf{A}}(A)$. So as to avoid ambiguity, we introduce no special notions for Leibniz interpretable signatures. $\square$

Remark 16.18 If $\mathfrak{s}$ is Leibniz interpretable then so is any $\mathfrak{s}$-deductive system.

Remark 16.19 If $\mathcal{D}$ is Leibniz interpretable and F is a $\mathcal{D}$-filter of $\mathbf{A}$ then $[\,\Omega^{\mathbf{A}}(F)\,]^{-1}\big[[\,\Omega^{\mathbf{A}}(F)\,]\,[F]\big] = F$.

Proof. $[\,\Omega^{\mathbf{A}}(F)\,](\cdot)$ is surjective onto $\mathsf{uni}([\,\Omega^{\mathbf{A}}(F)\,](\mathbf{A}))$ and $\equiv_{[\Omega^{\mathbf{A}}(F)]}$ is compatible with F, since $\equiv_{[\Omega^{\mathbf{A}}(F)]} = \Omega^{\mathbf{A}}(F)$, and $\Omega^{\mathbf{A}}(F)$ is compatible with F by Proposition 16.7. So the result follows by Remark 1.72 on page 25. $\diamond$

Example 16.20 ($\underset{\rightarrow}{\mathfrak{a}}{}_{[n]}$ is Leibniz Interpretable) [BP89a]

> If $\mathbf{A}$ is an $\mathfrak{a}$-algebra and $\mathtt{A} \subseteq \mathsf{uni}(\mathbf{A})^n$, then $\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})$ on $\mathsf{uni}(\mathbf{A})$ is a congruence on $\mathbf{A}$ [BP89a]. Consequently, the canonical homomorphism $\mathfrak{q}_{\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})}$ is a surjective homomorphism of $\mathbf{A}$ onto $\mathbf{A}/\,\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})$ with $\equiv_{\mathfrak{q}_{\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})}} = \mathfrak{q}_{\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})}$. So $\underset{\longrightarrow}{\mathfrak{q}_{\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})}}$ is a surjective $\underset{\rightarrow}{\mathfrak{a}}{}_{[n]}$-morphism from $\underset{\rightarrow}{\mathbf{A}}$ onto $\underset{\longrightarrow}{\mathbf{A}/\,\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})}$ with $\equiv_{\underset{\longrightarrow}{\mathfrak{q}_{\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})}}} = \underset{\longrightarrow}{\boldsymbol{\Omega}^{\mathbf{A}}_{\mathfrak{a}}(\mathtt{A})} = \Omega^{\underset{\rightarrow}{\mathbf{A}}}_{\underset{\rightarrow}{\mathfrak{a}}{}_{[n]}}(\mathtt{A})$, the final equality following by Theorem 16.16.

$\square$

The following result, when interpreted for *surjective* morphisms, is weaker than the analogous result for sentential calculi (see Lemma 2.68 on page 104), which asserts that if f is a *surjective* (matrix) homomorphism from $\mathbf{M}$ onto $\mathbf{N}$ and $F \in \mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$, then $\Omega^{\mathcal{S}}_{\mathbf{M}}(f^{-1}[F]) = f^{-1}\left[\Omega^{\mathcal{S}}_{\mathbf{N}}(F)\right]$. That result is key in proving the implication (1) implies (11) in Theorem 2.135 on page 117 characterizing protoalgebraic sentential calculi. While we have not been able to establish equality for surjective morphisms, we have found that this weaker result, by not requiring surjectivity, in fact yields a simpler proof of the aforementioned implication, since the inclusion is all that is required in the proof (see the proof of Theorem 16.28). The standard proof (see [BP89a] or [vA95]) requires two cases, first a countable argument is presented, and then a proof by contradiction for the uncountable case. By not requiring surjectivity, this countable/uncountable distinction is avoided.

Lemma 16.21 If $\mathcal{D}$ is Leibniz interpretable, then $\underset{\rightarrow}{f}^{-1}\left[\Omega^{\mathbf{A}}(F)\right] \subseteq \Omega^{\mathbf{G}}(f^{-1}[F])$, for any $f : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$ and $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$.

Proof. By Leibniz interpretability, there exists $g : \mathbf{A} \twoheadrightarrow_{\mathfrak{s}} \mathbf{B}$ for some $\mathbf{B}$ with $\equiv_g = \Omega^{\mathbf{A}}(F)$. $\boxed{\text{Claim: } \equiv_{gf} \text{ is compatible with } f^{-1}[F]}$ Suppose that $\phi \in f^{-1}[F]$ and $\phi \equiv_{gf} \psi$, i.e., $f(\phi) \in F$ and $gf(\phi) = gf(\psi)$. Hence $f(\phi) \equiv_g f(\psi)$. Since $f(\phi) \in F$ and $\equiv_g = \Omega^{\mathbf{A}}(F)$ is compatible with F, $f(\psi) \in F$. Hence $\psi \in f^{-1}[F]$. □

So by (3) of Proposition 16.7, $\equiv_{gf} \subseteq \Omega^{\mathbf{G}}(f^{-1}[F])$. Let $\langle \phi, \psi \rangle \in \underset{\to}{f}^{-1}\left[\Omega^{\mathbf{A}}(F)\right] = \underset{\to}{f}^{-1}[\equiv_g]$. Then $\langle f(\phi), f(\psi) \rangle \in \equiv_g$. So $g(f(\phi)) = g(f(\psi))$. So $gf(\phi) = gf(\psi)$, hence $\langle \phi, \psi \rangle \in \equiv_{gf} \subseteq \Omega^{\mathbf{G}}(f^{-1}[F])$. ◇

The following useful result follows from (3) of Proposition 16.7.

Remark 16.22 If $\mathcal{D}$ is Leibniz interpretable, $T, R \in \mathsf{Th}(\mathcal{D})$ and $\Omega_{\mathfrak{s}}(T)$ is compatible with R, then $\Omega_{\mathfrak{s}}(T) \subseteq \Omega_{\mathfrak{s}}(R)$.

16.4 Protoalgebraicity

In this section we shall develop a theory of protoalgebraicity for calculi over constructs. We shall show that the standard characterizations of protoalgebraic sentential calculi can be achieved at this level of discourse, although in some cases we require that the calculus be Leibniz interpretable; in particular, if a logic is Leibniz interpretable then the filter correspondence property and protoalgebraicity coincide. We are even able to obtain a characterization in terms of the existence of formulae and the satisfaction of certain consequences involving these formulae, which is similar in spirit to the analogous characterization of protoalgebraic sentential 1-calculi (see Corollary 2.138 on page 118); this result does *not* require that the calculus be Leibniz interpretable. From this result, we obtain a new characterization of protoalgebraic sentential n-calculi that is simpler than that of [Pal03], which we use to characterize the protoalgebraicity of the logics $S^n(\mathcal{K}, \mathfrak{N})$ in terms of $\mathcal{K}$ having coherent $\langle \mathcal{K}, \mathfrak{N} \rangle$-classes, thereby generalizing the analogous sentential 1-calculi results of [BR99] to sentential n-calculi more generally.

Recall the definition of the *point proto-Leibniz relation* $\simeq^{\mathbb{C}}(A)$, associated with a closed system $\mathbb{C}$ and a set $A \subseteq \mathsf{uni}(\mathbb{C})$, given in ∮4.2.4, and note in particular, Proposition 4.69 on page 154, which states that $a \simeq (A)\, b$ iff $A \cup \{a\} \vdash b$ and $A \cup \{b\} \vdash a$.

Definition 16.23 (The Filter Proto-Leibniz Relation) For any $\mathfrak{s}$-language $\mathbf{A}$ and $A \subseteq \mathsf{uni}(\mathbf{A})$, we write $\simeq^{\mathfrak{s},\mathcal{D}}_{\mathbf{A}}(A)$ for $\simeq^{\mathsf{Th}(\mathsf{F}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A}))}(A)$, writing $\simeq^{\mathcal{D}}_{\mathbf{A}}(A)$ for $\simeq^{\mathfrak{s},\mathcal{D}}_{\mathbf{A}}(A)$ whenever the signature $\mathfrak{s}$ is understood, and drop the subscript $\mathbf{A}$ in the case that $\mathbf{A} = \mathbf{G}$. □

Remark 16.24

$$
\begin{aligned}
a \simeq^{\mathcal{D}}_{\mathbf{A}}(A)\, b \qquad &\text{iff} \qquad A \cup \{a\} \Vdash^{\mathbf{B}}_{\mathcal{D}} b \quad \text{and} \quad A \cup \{b\} \Vdash^{\mathbf{B}}_{\mathcal{D}} a \\
&\text{iff} \qquad b \in \|\{c\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}} \quad \text{and} \quad c \in \|\{b\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}
\end{aligned}
\tag{16.1}
$$

Remark 16.25 Since $\mathcal{D}$ is structural by assumption, $\mathcal{D}$-filters on $\mathbf{G}$ and $\mathcal{D}$-theories coincide, hence

$$\phi \simeq^{\mathcal{D}}_{\mathbf{G}}(\Gamma)\, \psi \quad \leftrightarrow \quad \Gamma \cup \{\phi\} \vdash \psi \text{ and } \Gamma \cup \{\psi\} \vdash \phi. \tag{16.2}$$

Definition 16.26 (Protoalgebraicity) $\mathcal{D}$ is called protoalgebraic if, for all $T \in \mathsf{Th}(\mathcal{D})$, $\Omega^{\mathbf{G}}(T) \subseteq \simeq^{\mathcal{D}}_{\mathbf{G}}(T)$, i.e., $\phi\, \Omega^{\mathbf{G}}(T)\, \psi$ implies $T \cup \{\phi\} \vdash_{\mathcal{D}} \psi$ and $T \cup \{\psi\} \vdash_{\mathcal{D}} \phi$. □

We now aim to characterize protoalgebraic calculi *internally*, that is, in terms of the existence of formulae and consequences that must be satisfied, in the spirit of Theorem 2.137 on page 118 and Corollary 2.138. The characterization that we shall present is more in the spirit of the latter result, characterizing protoalgebraic 1-sentential calculi, than the former. From this result, we shall obtain a new characterization of protoalgebraic n-sentential calculi, that is simpler than that of Theorem 2.137, and closer in form to that of Corollary 2.138.

First we require a mechanism for substituting one variable with another in a formula. This task is trivial for terms of algebras, which only depend on a finite number of variables and defined constructively from these variables. For formulae of constructs more generally, we need to employ substitutions. The following definition simplifies the notational burden of this task.

Definition 16.27 ($\phi[x_1, \ldots, x_{i-1}, {x_i}^*, x_{i+1}, \ldots, x_n]$) Let $\phi \in \mathsf{Fm}(\mathbf{G})$ and $x_1, \ldots, x_n$ distinct $\mathfrak{s}$-variables of $\mathbf{G}$. By $\phi[x_1, \ldots, x_{i-1}, {x_i}^*, x_{i+1}, \ldots, x_n]$, we mean the $\mathbf{G}$-formula $\sigma(\phi)$, where σ is the $\mathfrak{s}$-substitution of $\mathbf{G}$ fixing all variables in $\{x_1, \ldots, x_n\} - \{x_i\}$ and mapping all other variables to x_i. For $\Gamma \subseteq \mathsf{Fm}(\mathbf{G})$, we write $\Gamma[x_1, \ldots, x_{i-1}, {x_i}^*, x_{i+1}, \ldots, x_n]$ for $\{\phi[x_1, \ldots, x_{i-1}, {x_i}^*, x_{i+1}, \ldots, x_n] : \phi \in \Gamma\}$. In this notation, *precisely one* of the variables must be 'starred'. $\square$

Note that the following result does *not* depend on Leibniz interpretability.

Theorem 16.28 The following conditions are equivalent.

1. $\mathcal{D}$ is protoalgebraic.
2. The exists $\Delta \subseteq \mathsf{Fm}(\mathcal{D})$ such that
$$\vdash_{\mathcal{D}} \Delta[{\mathsf{v}_0}^*] \tag{16.3}$$
$$\mathsf{v}_1 \cup \Delta[{\mathsf{v}_0}^*, \mathsf{v}_1] \vdash_{\mathcal{D}} \mathsf{v}_0. \tag{16.4}$$
3. If $T, R \in \mathsf{Th}(\mathcal{D})$ and $T \subseteq R$ then $\Omega(T)$ is compatible with R.
4. For every $\mathfrak{s}$-language $\mathbf{A}$ and $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$, $\Omega^{\mathbf{A}}(F) \subseteq \simeq^{\mathcal{D}}_{\mathbf{A}}(F)$; i.e., if $a\ \Omega^{\mathbf{A}}(F)\ b$ then $a \in \|\{b\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}$ and $b \in \|\{a\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}$.
5. For every $\mathfrak{s}$-language $\mathbf{A}$, if $F, G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$ and $F \subseteq G$ then $\Omega^{\mathbf{A}}(F)$ is compatible with G.

Further, if $\mathcal{D}$ is *finitary*, then Δ may be chosen to be a *finite* set of formulae in (2).

Proof.

$\boxed{(1)\Rightarrow(2)}$ Let σ be the substitution mapping all variables to v_0, ρ the substitution fixing v_1 and mapping all other variables to v_0, and ι is the substitution sending v_1 to v_0 and fixing all other variables. Let $T = \{\phi \in \mathsf{Fm}(\mathbf{G}) : \vdash_{\mathcal{D}} \iota(\phi)\}$.

$\boxed{\text{Claim: } T \text{ is a } \mathcal{D}\text{-theory}}$ Suppose that $T \vdash_{\mathcal{D}} \phi$. Then by structurality, $\iota[T] \vdash_{\mathcal{D}} \iota(\phi)$. By construction, $\vdash_{\mathcal{D}} \iota[T]$, so $\vdash_{\mathcal{D}} \iota(\phi)$. So by definition, $\phi \in T$. $\square$

$\boxed{\text{Claim: } \langle \mathsf{v}_0, \mathsf{v}_1 \rangle \in \Omega(T).}$ Let $x \in \mathsf{V}$, $x \notin \{\mathsf{v}_0, \mathsf{v}_1\}$, $\phi \in \mathsf{Fm}(\mathbf{G})$, and $f : (\mathsf{V} - \{x\}) \to \mathsf{Fm}(\mathbf{G})$, with $f_{\frac{x}{\mathsf{v}_0}}(\phi) \in T$. (*It suffices to show that* $f_{\frac{x}{\mathsf{v}_1}}(\phi) \in T$.) By the definition of T, $\vdash_{\mathcal{D}} \iota(f_{\frac{x}{\mathsf{v}_0}}(\phi))$.

$\boxed{\text{Claim: } \iota f_{\frac{x}{\mathsf{v}_0}} = \iota f_{\frac{x}{\mathsf{v}_1}}}$ $\iota f_{\frac{x}{\mathsf{v}_0}}(x) = \iota(\mathsf{v}_0) = \mathsf{v}_0 = \iota(\mathsf{v}_1) = \iota f_{\frac{x}{\mathsf{v}_1}}(x)$, and for any other variable y, $\iota f_{\frac{x}{\mathsf{v}_0}}(y) = \iota(f(y)) = \iota f_{\frac{x}{\mathsf{v}_1}}(y)$, so equality follows by $\mathfrak{s}$-freedom of $\mathbf{G}$. $\square$

So, since $\vdash_{\mathcal{D}} \iota(f_{\frac{x}{\mathsf{v}_0}}(\phi))$, $\vdash_{\mathcal{D}} \iota(f_{\frac{x}{\mathsf{v}_1}}(\phi))$ and hence $f_{\frac{x}{\mathsf{v}_1}}(\phi) \in T$. $\square$

Since T is a theory and $\langle \mathsf{v}_0, \mathsf{v}_1 \rangle \in \Omega(T)$, by protoalgebraicity, $\mathsf{v}_1, T \vdash_{\mathcal{D}} \mathsf{v}_0$. Let $\Delta \subseteq T$ such that $\mathsf{v}_1, \Delta \vdash_{\mathcal{D}} \mathsf{v}_0$, noting that if $\mathcal{D}$ is finitary then Δ may be chosen to be a finite set. So $\mathsf{v}_1 \cup \Delta \vdash_{\mathcal{D}} \mathsf{v}_0$. By structurality, $\rho(\mathsf{v}_1) \cup \rho[\Delta] \vdash_{\mathcal{D}} \rho(\mathsf{v}_0)$, i.e., $\mathsf{v}_1, \Delta(\mathsf{v}_0{}^*, \mathsf{v}_1) \vdash_{\mathcal{D}} \mathsf{v}_0$. Since $\Delta \subseteq T$, $\vdash_{\mathcal{D}} \iota[\Delta]$, and so by structurality, $\vdash_{\mathcal{D}} \sigma[\iota[\Delta]]$.

$\boxed{\text{Claim: } \sigma\iota = \sigma}$ $\sigma\iota(\mathsf{v}_1) = \sigma(\mathsf{v}_0) = \mathsf{v}_0 = \sigma(\mathsf{v}_1)$, and for any other variable y, $\sigma\iota(y) = \sigma(y)$, so equality follows by $\mathfrak{s}$-freedom of $\mathbf{G}$. $\square$

So $\vdash_{\mathcal{D}} \sigma[\Delta]$, i.e. $\vdash_{\mathcal{D}} \Delta(\mathsf{v}_0{}^*)$.

$\boxed{(2)\Rightarrow(4)}$ Let $F \in \mathsf{Fi}_{\mathcal{D}}(\mathbf{A})$ and suppose that $a\ \Omega^{\mathbf{A}}(\mathcal{D})\ b$. Let $\mathbf{\Delta}$ be as in (2) of Theorem 16.28. Let $f : \mathsf{V} - \{\mathsf{v}_1\} : \to \mathsf{uni}(\mathbf{A})$ such that $f(\mathsf{v}_i) = a$ for all $i \neq 1$. Let $\Delta \in \mathbf{\Delta}$. Let σ be the substitution mapping all variables to v_0. Since F is a filter and $f_{\frac{\mathsf{v}_1}{a}}$ is an interpretation of $\mathbf{G}$ into $\mathbf{A}$, it follows from (16.3) that $f_{\frac{\mathsf{v}_1}{a}}(\sigma(\Delta)) \in F$. Let ρ be the substitution fixing v_1 and mapping all other variables to v_0.

$\boxed{\text{Claim: } f_{\frac{\mathsf{v}_1}{a}}\sigma = f_{\frac{\mathsf{v}_1}{a}}\rho}$ For $i \neq 1$, $f_{\frac{\mathsf{v}_1}{a}}(\sigma(\mathsf{v}_i)) = f_{\frac{\mathsf{v}_1}{a}}(\mathsf{v}_0) = f_{\frac{\mathsf{v}_1}{a}}(\rho(\mathsf{v}_i))$, and $f_{\frac{\mathsf{v}_1}{a}}(\sigma(\mathsf{v}_1)) = f_{\frac{\mathsf{v}_1}{a}}(\mathsf{v}_0) = f(\mathsf{v}_0) = a = f_{\frac{\mathsf{v}_1}{a}}(\mathsf{v}_1) = f_{\frac{\mathsf{v}_1}{a}}(\rho(\mathsf{v}_1))$; the result follows by $\mathfrak{s}$-freedom of $\mathbf{G}$. $\square$

So $f_{\frac{\mathsf{v}_1}{a}}(\rho(\Delta)) \in F$. Since $a\ \Omega^{\mathbf{A}}(\mathcal{D})\ b$, $f_{\frac{\mathsf{v}_1}{b}}(\rho(\Delta)) \in F$. Hence $f_{\frac{\mathsf{v}_1}{b}}[\mathsf{v}_1 \cup \rho[\mathbf{\Delta}]] \subseteq F \cup \{b\}$, and so by (16.4), $a = f(\mathsf{v}_0) = f_{\frac{\mathsf{v}_1}{b}}(\mathsf{v}_0) \in \|F \cup \{b\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}$. The outstanding assertion follows symmetrically.

$\boxed{(4)\Rightarrow(5)}$ Assume that $F, G \in \mathsf{Fi}_{\mathcal{D}}(\mathbf{A})$, $F \subseteq G$, $a \in G$ and $a\ \Omega^{\mathbf{A}}(F)\ b$. By assumption (4), $b \in \|\{a\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}$. Since $\{a\} \cup F \subseteq G$ and since G is a filter, $b \in \|\{a\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}} \subseteq G$. The outstanding assertion follows symmetrically. $\boxed{(5)\Rightarrow(3)}$ By assumed structurality, $\mathcal{D}$-filters of $\mathbf{G}$ coincide with $\mathcal{D}$-theories (by Theorem 7.48 on page 263), so the result follows trivially. $\boxed{(3)\Rightarrow(1)}$ Let $T \in \mathsf{Th}(\mathcal{D})$ and suppose that $\phi\ \Omega(T)\ \psi$. Since $T \subseteq \|T \cup \{\phi\}\|_{\mathcal{D}}$, $\Omega(T)$ is compatible with $\|T \cup \{\phi\}\|_{\mathcal{D}}$ by assumption (3), and since $\phi \in \|T \cup \{\phi\}\|_{\mathcal{D}}$ and $\phi\ \Omega(\|T \cup \{\phi\}\|_{\mathcal{D}})\ \psi$, $\psi \in \|T \cup \{\phi\}\|_{\mathcal{D}}$. Hence $T \cup \{\phi\} \vdash_{\mathcal{D}} \psi$. Symmetrically, $T \cup \{\psi\} \vdash_{\mathcal{D}} \phi$. $\diamond$

Filter generation for protoalgebraic sentential calculi has a simple characterization (see Theorem 2.139 on page 118). The following result generalizes Theorem 2.139 to our more general context. For another generalization of Theorem 2.139, see Corollary 14.8 on page 409.

Theorem 16.29 Let $\mathcal{D}$ be a protoalgebraic $\mathfrak{s}$-calculus, $\mathbf{A}$ an $\mathfrak{s}$-language and $Y \subseteq \mathsf{uni}(\mathbf{A})$, then

$$\|Y\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}} = \{\mathsf{i}(\phi)\ :\ \mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{G}, \mathbf{A}), \Gamma \vdash_{\mathcal{D}} \phi, \mathsf{i}[\Gamma] \subseteq Y \cup \|\emptyset\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}\}. \tag{16.5}$$

Proof. Let $F = \{\mathsf{i}(\phi) : \mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{G}, \mathbf{A}), \Gamma \vdash_{\mathcal{D}} \phi, \mathsf{i}[\Gamma] \subseteq Y \cup \|\emptyset\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}\}$. By the definition of a filter, $F \subseteq \|Y\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}$. (*Hence it suffices to show that F is a $\mathcal{D}$-filter of* $\mathbf{A}$.) Suppose that $\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{G}, \mathbf{A})$, $\Gamma \vdash_{\mathcal{D}} \phi$ and $\mathsf{i}[\Gamma] \subseteq F$. (*We must show that* $\mathsf{i}(\phi) \in F$.)

Since $\mathsf{i}[\Gamma] \subseteq F$, for each $\psi \in \Gamma$, there exist $\Gamma_\psi \cup \{\phi_\psi\} \subseteq \mathsf{Fm}(\mathbf{G})$ and $\mathsf{i}_\psi \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{G}, \mathbf{A})$ with $\Gamma_\psi \vdash_{\mathcal{D}} \phi_\psi$ such that $\mathsf{i}_\psi[\Gamma_\psi] \subseteq Y \cup \|\emptyset\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}$ and $\mathsf{i}_\psi(\phi_\psi) = \mathsf{i}(\psi)$. Let $\{V\} \cup \{V_\psi\ :\ \psi \in \Gamma\}$ be a partition of V where V and each V_ψ are infinite. Suppose that $V = \{x_0, x_1, \ldots\}$ and $V_\psi = \{y_0^\psi, y_1^\psi, \ldots\}$, for each $\psi \in \Gamma$. Let σ be the substitution such that $\sigma(\mathsf{v}_i) = x_i$, for each $i \in \omega$, and, for each $\psi \in \Gamma$, let σ_ψ be the substitution such that $\sigma_\psi(\mathsf{v}_i) = y_i^\psi$, for each $i \in \omega$. By structurality, $\sigma[\Gamma] \vdash_{\mathcal{D}} \sigma(\phi)$ and, for each $\psi \in \Gamma$, $\sigma_\psi[\Gamma_\psi] \vdash_{\mathcal{D}} \sigma_\psi(\phi_\psi)$.

Let ρ be the substitution fixing v_1 and mapping all other variables to v_0. For each $\psi \in \Gamma$, let ρ_ψ be the substitution mapping $\mathsf{v}_1 \mapsto \sigma_\psi(\phi_\psi)$ and mapping all other variables to $\sigma(\psi)$. Then, for each $\psi \in \Gamma$, by (16.4) and structurality, $\rho_\psi(\mathsf{v}_1) \cup \rho_\psi\rho[\Delta] \vdash_{\mathcal{D}} \rho_\psi(\mathsf{v}_0)$, i.e., $\sigma_\psi(\phi_\psi) \cup \rho_\psi\rho[\Delta] \vdash_{\mathcal{D}} \sigma(\psi)$, and since $\sigma_\psi[\Gamma_\psi] \vdash_{\mathcal{D}} \sigma_\psi(\phi_\psi)$, $\sigma_\psi[\Gamma_\psi] \cup \rho_\psi\rho[\Delta] \vdash_{\mathcal{D}} \sigma(\psi)$. So

$$\bigcup_{\psi \in \Gamma} (\sigma_\psi[\Gamma_\psi] \cup \rho_\psi\rho[\Delta]) \vdash_{\mathcal{D}} \sigma[\Gamma] \vdash_{\mathcal{D}} \sigma(\phi). \tag{16.6}$$

Let j be the interpretation of $\mathbf{G}$ into $\mathbf{A}$ such that $\mathsf{j}(x_i) = \mathsf{i}(\mathsf{v}_i)$, for each $i \in \omega$, and $\mathsf{j}(y_i^\psi) = \mathsf{i}_\psi(\mathsf{v}_i)$, for each $\psi \in \Gamma$ and $i \in \omega$.

$\boxed{\text{Claim: } \mathsf{j}\sigma = \mathsf{i}}$ $\mathsf{j}(\sigma(\mathsf{v}_i)) = \mathsf{j}(x_i) = \mathsf{i}(\mathsf{v}_i)$, so result follows by $\mathfrak{s}$-freedom of $\mathbf{G}$. $\boxed{\text{Claim: } \mathsf{j}\sigma_\psi = \mathsf{i}_\psi}$ $\mathsf{j}(\sigma_\psi(\mathsf{v}_i)) = \mathsf{j}(y_i^\psi) = \mathsf{i}_\psi(\mathsf{v}_i)$, so result follows by $\mathfrak{s}$-freedom of $\mathbf{G}$. □

Let $\psi \in \Gamma$. By the previous claim, $\mathsf{j}[\sigma_\psi[\Gamma_\psi]] = \mathsf{i}_\psi[\Gamma_\psi] \subseteq Y \cup \|\emptyset\|^{\mathbf{A}}_{\mathsf{fi}_\mathcal{D}}$. Let ι be the substitution mapping all variables to v_0.

$\boxed{\text{Claim: } \mathsf{j}\rho_\psi\rho = \mathsf{j}\rho_\psi\iota}$ $\mathsf{j}(\rho_\psi(\rho(\mathsf{v}_1))) = \mathsf{j}(\rho_\psi(\mathsf{v}_1)) = \mathsf{j}(\sigma_\psi(\phi_\psi)) = \mathsf{i}_\psi(\phi_\psi) = \mathsf{i}(\psi) = \mathsf{j}(\sigma(\psi)) = \mathsf{j}(\rho_\psi(\mathsf{v}_0)) = \mathsf{j}(\rho_\psi(\iota(\mathsf{v}_1)))$, otherwise $\mathsf{j}(\rho_\psi(\rho(\mathsf{v}_i))) = \mathsf{j}(\rho_\psi(\mathsf{v}_0)) = \mathsf{j}(\rho_\psi(\iota(\mathsf{v}_i)))$, so result follows by $\mathfrak{s}$-freedom of $\mathbf{G}$. □

By (16.4) and structurality, $\vdash_\mathcal{D} \rho_\psi[\iota[\Delta]]$. So $\mathsf{j}\rho_\psi\rho[\Delta] = \mathsf{j}[\rho_\psi[\iota[\Delta]]] \subseteq \|\emptyset\|^{\mathbf{A}}_{\mathsf{fi}_\mathcal{D}}$.

Hence,

$$\mathsf{j}\left[\bigcup_{\psi\in\Gamma}\left(\sigma_\psi[\Gamma_\psi] \cup \rho_\psi\rho[\Delta]\right)\right] \subseteq Y \cup \|\emptyset\|^{\mathbf{A}}_{\mathsf{fi}_\mathcal{D}},$$

and so by the first claim, (16.6) and the definition of F, $\mathsf{i}(\phi) = \mathsf{j}(\sigma(\phi)) \in F$. ◇

We now aim to relate protoalgebraicity and the filter correspondence property. We also consider the property that the Leibniz relation be order preserving with respect to theories and filters.

Lemma 16.30 The following conditions are equivalent.

1. $\Omega(\cdot)_{|\mathsf{Th}(\mathcal{D})}$ is $\subseteq$-preserving.
2. $\mathcal{D}$ is protoalgebraic and, for all $T, R \in \mathsf{Th}(\mathcal{D})$ with $T \subseteq R$, if $\Omega(T)$ is compatible with R then $\Omega(T) \subseteq \Omega(R)$.

Proof. $\boxed{(1)\Rightarrow(2)}$ (*It suffices to prove protoalgebraicity. We shall show that equivalent condition (3) of Theorem 16.28 is valid.*) Suppose that $T, R \in \mathsf{Th}(\mathcal{D})$, $T \subseteq R$, $\phi \in R$ and $\phi\ \Omega(T)\ \psi$. Then by assumption, $\phi\ \Omega(R)\ \psi$, and since $\Omega(R)$ is compatible with R (by (2) of Proposition 16.7) and $\phi\ \Omega(R)\ \psi$ and $\phi \in R$, $\psi \in R$. $\boxed{(2)\Rightarrow(1)}$ Suppose that $T, R \in \mathsf{Th}(\mathcal{D})$ such that $T \subseteq R$. By assumed protoalgebraicity and equivalent condition (3) of Theorem 16.28, $\Omega(T)$ is compatible with R, and so by assumption, $\Omega(T) \subseteq \Omega(R)$. ◇

By a similar argument, we have the following.

Lemma 16.31 The following conditions are equivalent.

1. For all $\mathfrak{s}$-languages $\mathbf{A}$, $\Omega^{\mathbf{A}}(\cdot)_{|\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})}$ is $\subseteq$-preserving.
2. $\mathcal{D}$ is protoalgebraic and, for all $\mathfrak{s}$-languages $\mathbf{A}$, $F, G \in \mathsf{Fi}_\mathcal{D}(\mathbf{A})$ with $F \subseteq G$, if $\Omega^{\mathbf{A}}(F)$ is compatible with G then $\Omega^{\mathbf{A}}(F) \subseteq \Omega^{\mathbf{A}}(G)$.

□

Clearly the equivalent conditions of the latter lemma imply the equivalent conditions of the former. We have not been able to establish or disprove the converse implication.

Open Problem 16.32 Do the equivalent conditions of the former lemma imply the equivalent conditions of the latter?

Lemma 16.33 1. If the equivalent conditions of Lemma 16.30 are satisfied, then so are the equivalent conditions of Lemma 16.4.

2. If the equivalent conditions of Lemma 16.31 are satisfied, then $\mathcal{D}$ has the filter correspondence property.

Proof. We shall prove (2), the proof of (1) being similar. We prove that equivalent condition (4) of Theorem 16.3 is valid. Let $\mathbf{A}$ be an $\mathfrak{s}$-language, f a surjective $\mathfrak{s}$-morphism from $\mathbf{A}$ and $F, G \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$ with $F \subseteq G$ and $\equiv_f$ compatible with F. (*We must show that* $\equiv_f$ *is compatible with* G.) Let $b, c \in \mathsf{uni}(\mathbf{A})$ with $b \in G$ and $\langle b, c\rangle \in\, \equiv_f$. (*We must show that* $c \in G$.) By (3) of Proposition 16.7 together with (1) of Corollary 16.31, $\langle b, c\rangle \in\, \equiv_f\, \subseteq \Omega^{\mathbf{A}}(F) \subseteq \Omega^{\mathbf{A}}(G)$. Since $\Omega^{\mathbf{A}}(G)$ is compatible with G, by (2) of Proposition 16.7, $c \in G$. $\diamond$

In the case that $\mathcal{D}$ is Leibniz interpretable, we have a coincidence of protoalgebraicity, the filter correspondence property, the equivalent conditions of Lemma 16.4, the equivalent conditions of Lemma 16.30 and the equivalent conditions of Lemma 16.31 (formalized in Theorem 16.35). The key to this coincidence is the following observation which follows immediately from (2) and (3) of Proposition 16.7.

Remark 16.34 If $\mathcal{D}$ is Leibniz interpretable, then, for all $T, R \in \mathsf{Th}(\mathcal{D})$ with $T \subseteq R$, if $\Omega(T)$ is compatible with R then $\Omega(T) \subseteq \Omega(R)$.

Theorem 16.35 If $\mathcal{D}$ is Leibniz interpretable, the following conditions are equivalent.

1. For every $\mathfrak{s}$-language $\mathbf{A}$, $\Omega^{\mathbf{A}}(\cdot)_{|\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})}$ is $\subseteq$-preserving.

2. $\mathcal{D}$ has the filter correspondence property.

3. For every $\mathfrak{s}$-language $\mathbf{A}$ and $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{A})$, $\Omega^{\mathbf{A}}(F) \subseteq \simeq^{\mathcal{D}}_{\mathbf{A}}(F)$; i.e., if $b\ \Omega^{\mathbf{A}}(F)\ c$ then $b \in \|\{c\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}$ and $c \in \|\{b\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}$.

4. $\Omega(\cdot)_{|\mathsf{Th}(\mathcal{D})}$ is $\subseteq$-preserving.

5. The equivalent conditions of Lemma 16.4 are satisfied.

6. $\mathcal{D}$ is protoalgebraic.

Proof. $\boxed{(1)\Rightarrow(2)}$ By (2) of Lemma 16.33. $\boxed{(2)\Rightarrow(3)}$ Suppose $\langle b, c\rangle \in \Omega^{\mathbf{A}}(F)$. (*We show that* $b \in \|\{c\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}} \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} F$*, the proof that* $c \in \|\{b\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}} \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} F$ *being symmetrical.*) Since $\mathcal{D}$ is Leibniz interpretable, there exists $f : \mathbf{A} \twoheadrightarrow_{\mathfrak{s}} \mathbf{B}$, for some $\mathfrak{s}$-language $\mathbf{B}$, with $\equiv_f\, = \Omega^{\mathbf{A}}(F)$. Since $\Omega^{\mathbf{A}}(F)$ is compatible with F by (2) of Proposition 16.7, $\ker f$ is compatible with F. Since f is surjective and $\ker f$ is compatible with F, $f[F] \in \mathsf{Fi}_{\mathcal{D}}(\mathbf{B})$, by Proposition 7.25. From $\langle b, c\rangle \in\, \equiv_f$, we infer $b \in f^{-1}[\{f(c)\}] \subseteq f^{-1}[\|f[\|\{c\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}}] \cup f[F]\|^{\mathbf{B}}_{\mathsf{fi}_{\mathcal{D}}}] \overset{(i)}{=} \|\{c\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}} \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} f^{-1}[f[F]] \overset{(ii)}{=} \|\{c\}\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{D}}} \vee^{\mathbf{Fi}_{\mathcal{D}}(\mathbf{A})} F$, where *(i)* follows by equivalent condition (5) of Theorem 16.3, and *(ii)* follows by Remark 1.72 on page 25 and the compatibility of $\equiv_f$ with F. $\boxed{(3)\Rightarrow(1)}$ By Remark 16.34 and Corollary 16.31. $\boxed{(4)\Rightarrow(5)\ \Rightarrow(6)\Rightarrow(4)}$ The proof is similar to and easier than the proof of (1)⇒(2) ⇒(3)⇒(1) and as such we omit it. Note that by our assumption that $\mathcal{D}$ is $\mathfrak{s}$-structural, $\mathcal{D}$-theories and $\mathcal{D}$-filters on $\mathbf{G}$ coincide, Theorem 7.48 on page 263. $\boxed{(3)\Leftrightarrow(6)}$ By Theorem 16.28. $\diamond$

16.4.1 Examples

The following example demonstrates that the standard characterization of protoalgebraic sentential 1-calculi obtain from Theorem 16.28 (see Corollary 2.138 on page 118).

Example 16.36 (Protoalgebraic Sentential 1-Calculi)

Corollary 16.37 [BP89a] For a sentential 1-calculus $\mathcal{P}$, the following conditions are equivalent.

1. $\mathcal{P}$ is protoalgebraic.
2. There exist terms $\Delta_1(x, y), \Delta_m(x, y)$, for some $m > 0$, such that

$$\vdash_{\mathcal{D}} \Delta_i(x, x), \quad \text{for all } i \leq m, \text{ and} \tag{16.7}$$

$$y \cup \{\Delta_i(x, y) \, : \, i \leq m\} \vdash_{\mathcal{D}} x. \tag{16.8}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ By finitariness and Theorem 16.28 on page 449, there exist terms $\Delta'_i(x, y, \ldots, \mathsf{v}_k), \ldots, \Delta'_m(x, y, \ldots, \mathsf{v}_k)$, for some $m > 0$ and $k > 1$, such that

$$\vdash_{\mathcal{D}} \Delta'_i(x, x, \ldots, x), \quad \text{for all } i \leq m, \text{ and}$$

$$y \cup \{\Delta'_i(x, y, x, \ldots, x) \, : \, i \leq m\} \vdash_{\mathcal{D}} x.$$

For each $i \leq m$, define $\Delta_i(x, y) = \Delta'_i(x, y, x, \ldots, x)$. Then $\mathcal{P}$ and $\Delta_1, \ldots, \Delta_m$ satisfy (16.7) and (16.8). $\boxed{(2)\Rightarrow(1)}$ It is simple to show that the terms of (2) satisfy (16.3) and (16.4) of Theorem 16.28. $\diamond$

$\square$

In the next example, we characterize protoalgebraic sentential n-calculi via Theorem 16.28, obtaining a simpler characterization than that of [Pal03] (see Corollary 2.138 on page 118), and which is more in the spirit of the analogous characterization of sentential 1-calculi, described above. We shall apply this result in the subsequent example, to show that the sentential n-calculus $S^n(\mathcal{K}, \mathfrak{N})$ is protoalgebraic iff $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle \mathcal{K}, \mathfrak{N} \rangle$-classes; this result is new.

Example 16.38 (Protoalgebraic Sentential n-Calculi)

Let $\mathcal{S}$ be a sentential n-calculus with signature $\mathfrak{a}$, where $\mathfrak{a}$ is a type of algebras. Recall that we view $\mathcal{S}$ as a $\underset{\rightarrow}{\mathfrak{a}}_{[n]}$-calculus. From this perspective, we take the variables of $\mathcal{S}$ to be $\{\langle \mathsf{v}_{kn}, \ldots, \mathsf{v}_{kn+n-1} \rangle \, : \, k \geq 0\}$, where each v_i is a 1-variable. In the following theorem we apply Theorem 16.28 to the case of sentential n-calculi. Note that all variables in this result are 1-variables and *not* n-variables.

Corollary 16.39 The following conditions are equivalent.

1. $\mathcal{S}$ is protoalgebraic.
2. For all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, and $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, $f^{\mathcal{S}} : \mathsf{Fi}_{\mathcal{S}}(\mathbf{M}) \Rightarrow f^{\mathcal{S}}[\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})] \subseteq \mathsf{Fi}_{\mathcal{S}}(\mathbf{N})$ with inverse $\underset{\rightarrow}{f}^{-1}[\cdot]_{|f^{\mathcal{S}}[\mathsf{Fi}_{\mathcal{S}}(\mathbf{M})]}$.
3. For all $\mathcal{S}$-matrices $\mathbf{M}$ and $\mathbf{N}$, and $f : \mathbf{M} \twoheadrightarrow^{\mathsf{r}} \mathbf{N}$, $f^{\mathcal{S}} : \mathbf{Fi}_{\mathcal{S}}(\mathbf{M}) \cong \mathbf{Fi}_{\mathcal{S}}(\mathbf{N})$, with inverse $\underset{\rightarrow}{f}^{-1}[\cdot]_{|\mathsf{Fi}_{\mathcal{S}}(\mathbf{N})}$.

4. There exist n-formulae $\mathbf{\Delta}_1(x_1,\ldots,x_n,y_0,\ldots,y_n),\ldots,\mathbf{\Delta}_m(x_1,\ldots,x_n,y_0,\ldots,y_n)$, for some $m>1$ and $k>1$, such that

$$\vdash_{\mathcal{S}} \mathbf{\Delta}_i(x_1,\ldots,x_n,x_0,\ldots,x_n), \quad \text{for all } 1\le i\le k, \quad \text{and} \tag{16.9}$$

$$\{\langle y_1,\ldots,y_n\rangle\}\cup\{\mathbf{\Delta}_i(x_1,\ldots,x_n,y_1,\ldots,y_n) \,:\, i\le m\}\vdash_{\mathcal{S}} \langle x_0,\ldots,x_{n-1}\rangle. \tag{16.10}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ By finitariness and Theorem 16.28, there exist n-formulae $\mathbf{\Delta}'_1(\mathsf{v}_0,\ldots,\mathsf{v}_{kn-1}),\ldots,\mathbf{\Delta}'_m(\mathsf{v}_0,\ldots,\mathsf{v}_{kn-1})$, for some $m>1$ and $k>1$, such that

$$\vdash_{\mathcal{S}} \mathbf{\Delta}'_i(\mathsf{v}_0,\ldots,\mathsf{v}_{n-1},\mathsf{v}_0,\ldots,\mathsf{v}_{n-1},\ldots,\mathsf{v}_0,\ldots,\mathsf{v}_{n-1}), \quad \text{for all } 1\le i\le k, \quad \text{and}$$

$$\begin{aligned}&\{\langle \mathsf{v}_n,\ldots,\mathsf{v}_{2n-1}\rangle\}\\ &\cup\{\mathbf{\Delta}'_i(\mathsf{v}_0,\ldots,\mathsf{v}_{n-1},\mathsf{v}_n,\ldots,\mathsf{v}_{2n-1},\mathsf{v}_0,\ldots,\mathsf{v}_{n-1},\ldots,\mathsf{v}_0,\ldots,\mathsf{v}_{n-1}) \,:\, i\le m\}\\ &\quad\vdash_{\mathcal{S}} \langle \mathsf{v}_0,\ldots,\mathsf{v}_{n-1}\rangle.\end{aligned}$$

Define $\mathbf{\Delta}_i(x_1,\ldots,x_n,y_0,\ldots,y_n) = \mathbf{\Delta}'_i(x_1,\ldots,x_n,y_1,\ldots,y_n,x_1,\ldots,x_n,\ldots,x_1,\ldots,x_n)$, for $i\le m$. Then $\mathcal{S}$ and $\mathbf{\Delta}_1,\ldots,\mathbf{\Delta}_m$ satisfy (16.9) and (16.10). $\boxed{(2)\Rightarrow(1)}$ Follows easily by Theorem 16.28. $\diamond$

$\square$

As an application of the previous result, we now show that the sentential n-calculus $S^n(\mathcal{K},\mathfrak{N})$ (of solutions to an n-ary system of equations $\mathfrak{N}$) is protoalgebraic iff $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes. The proof is immediate. We do not see an easy proof of this result from the standard characterization of sentential n-calculi given in [Pal03].

Example 16.40 ($S^n(\mathcal{K},\mathfrak{N})$)

Let $\mathfrak{a}$-be a type of algebras, $\mathcal{K}$ a quasivariety of $\mathfrak{a}$-algebras and $\mathfrak{N}(x_1,\ldots,x_n)$ an n-ary system of equations. Recall that by Proposition 9.8 on page 314,

$$\Gamma \vdash_{S^n(\mathcal{K},\mathfrak{N})} \phi \text{ iff } \mathfrak{N}^{\approx}[\Gamma]\models_{\mathcal{K}} \mathfrak{N}^{\approx}[\![\phi]\!]. \tag{16.11}$$

Observe that interpreting (16.9) and (16.10) for the case that $\mathcal{S}=S^n(\mathcal{K},\mathfrak{N})$ and applying (16.11), we obtain (10.7) and (10.8) of Theorem 10.7 on page 345 (characterizing the property that $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes). So the following result obtains immediately.

Corollary 16.41 The following conditions are equivalent.

1. $S^n(\mathcal{K},\mathfrak{N})$ is protoalgebraic.
2. $\mathcal{K}$ has $\mathcal{K}$-coherent $\langle\mathcal{K},\mathfrak{N}\rangle$-classes.

$\square$

In the next example, we employ Corollary 16.39 to show that $S^2(\mathfrak{a},\equiv)$ is protoalgebraic.

Example 16.42 ($S^2(\mathfrak{a},\equiv)$ **is Protoalgebraic)**

Let $\Delta_1(x_1,x_2,y_1,y_2)=\langle x_1,y_1\rangle$ and $\Delta_2(x_1,x_2,y_1,y_2)=\langle x_2,y_2\rangle$. Then $\Delta_1(x_1,x_2,x_1,x_2)=\langle x_1,x_1\rangle$ and $\Delta_2(x_1,x_2,x_1,x_2)=\langle x_2,x_2\rangle$, so by reflexivity,

$$\vdash_{S^2(\mathfrak{a},\equiv)} \Delta_1(x_1,x_2,x_1,x_2),\Delta_2(x_1,x_2,x_1,x_2).$$

Further, $\Delta_1(x_1, x_2, y_1, y_2) = \langle x_1, y_1\rangle$ and $\Delta_2(x_1, x_2, y_1, y_2) = \langle x_2, y_2\rangle$, and so by symmetry and transitivity,

$$\langle y_1, y_2\rangle, \Delta_1(x_1, x_2, y_1, y_2), \Delta_2(x_1, x_2, y_1, y_2) \vdash_{S^2(\mathfrak{a},\equiv)} \langle x_1, x_2\rangle,$$

so $S^2(\mathfrak{a}, \equiv)$ is protoalgebraic by Corollary 16.39.

$\square$

We now turn to the primary example of this chapter, in which we use the machinery of protoalgebraic logics over constructs to provide an alternative perspective on $\langle X, z\rangle$-protoalgebraicity.

Example 16.43 ($\langle X, z\rangle$-Protoalgebraicity)

Let $\mathcal{S}$ be a sentential 1-calculus, $X \subseteq \mathsf{Fm}(\mathcal{S})$ and z a variable such that X generates a *z-invariant* $\mathcal{S}$-theory. Let $\mathfrak{s}$ be the signature consisting of the single language $\mathbf{Tm}$ together with all endomorphisms of $\mathbf{Tm}$ that fix z and consider the $\mathbf{Tm}$-logic $\mathcal{S}_{:X}$ as an $\mathfrak{s}$-logic. Recall that generally $\mathcal{S}_{:X}$ is not sentential since it is not generally $\mathfrak{a}$-structural; the reason why we have to restrict the homomorphisms is to ensure $\mathfrak{s}$-structurality.

Remark 16.44 $\mathcal{S}_{:X}$ is $\mathfrak{s}$-structural.

Proof. By the z-invariance of the $\mathcal{S}$-theory generated by X. $\diamond$

Remark 16.45 $\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{S}_{:X}}(\mathbf{Tm}) = \mathsf{Th}(\mathcal{S}_{:X}) = \mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{Tm}, X\rangle)$.

Proof. The first equality follows by $\mathfrak{s}$-structurality and Theorem 7.48 on page 263, while the second equality follows by definition. $\diamond$

Remark 16.46 The $\mathfrak{s}$-variables of $\mathcal{S}_{:X}$ are $\mathsf{V} - \{z\}$. $\square$

The following observation follows by Remark 6.11 on page 225.

Remark 16.47 $P \vdash_{\mathcal{S}} p$ iff $X, P \vdash_{\mathcal{S}_{:X}} p$.

We now characterize the flat Leibniz relation $\Omega_{\mathfrak{s}}(P)$.

Proposition 16.48 $p\ \Omega_{\mathfrak{s}}(P)\ q$ iff, for some variable $x \in \mathsf{V} - \{z\}$, every term $s(z, x, \vec{y})$ (where $\vec{y} \in \mathsf{V} - \{z, x\}$) and $\vec{r} \in \mathsf{Tm}$, $s(z, p, \vec{r}) \in P$ iff $s(z, p, \vec{r}) \in P$.

Proof. $p\ \Omega_{\mathfrak{s}}(P)\ q$ [iff] for some variable $x \in \mathsf{V} - \{z\}$, every term s, and every function $f : (\mathsf{V} - \{z, x\}) \to \mathsf{Tm}$, $f_{\frac{x}{p}}(s) \in P$ iff $f_{\frac{x}{q}}(s) \in P$ [iff] for some variable $x \in \mathsf{V} - \{z\}$, every term $s(z, x, \vec{y})$ (where $\vec{y} \in \mathsf{V} - \{z, x\}$) and $\vec{r} \in \mathsf{Tm}$, $s(z, p, \vec{r}) \in P$ iff $s(z, p, \vec{r}) \in P$. $\diamond$

Interestingly, the standard Leibniz relation $\Omega^{\mathbf{Tm}}(P)$ coincides with the flat Leibniz relation $\Omega_{\mathfrak{s}}(P)$. Recall that $\Omega^{\mathbf{Tm}}(P) = \Omega_{\mathfrak{a}}(P)$ by Example 16.8.

Proposition 16.49 $\Omega_{\mathfrak{a}}(P) = \Omega_{\mathfrak{s}}(P)$.

Proof. Assume that $p\ \Omega_{\mathfrak{a}}(P)\ q$. Let $x \in \mathsf{V} - \{z\}$, $s(z,x,\vec{y})$ a term (where $\vec{y} \in \mathsf{V} - \{z,x\}$) and $\vec{r} \in \mathsf{Tm}$, such that $s(z,p,\vec{r}) \in P$. Since $p\ \Omega_{\mathfrak{a}}(P)\ q$, $s(z,q,\vec{r}) \in P$. Conversely, assume that $p\ \Omega_{\mathfrak{s}}(P)\ q$. Let $s(x,\vec{y})$ be a term (where $\vec{y} \in \mathsf{V} - \{x\}$) and $\vec{r} \in \mathsf{Tm}$, such that $s(p,\vec{r}) \in P$. We may assume, without loss of generality, that $z \notin \{x,\vec{y}\}$. Let $s'(z,x,\vec{y}) = s(x,\vec{y})$. Then $s'(z,p,\vec{r}) = s(p,\vec{r}) \in P$. Since $p\ \Omega_{\mathfrak{s}}(P)\ q$, $s(q,\vec{r}) = s'(z,q,\vec{r}) \in P$. ⋄

The following result follows from the previous proposition together with Remark 16.22.

Corollary 16.50 For all $T, R \in \mathsf{Th}(\mathcal{S}_{:X})$, if $\Omega_{\mathfrak{s}}(T)$ is compatible with R then $\Omega_{\mathfrak{s}}(T) \subseteq \Omega_{\mathfrak{s}}(R)$. □

We now use (only) the machinery of this chapter to characterize the $\mathfrak{s}$-protoalgebraicity of $\mathcal{S}_{:X}$. Since the last of these characterizations is that $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic, we have obtained a characterization of the $\langle X, z\rangle$-protoalgebraicity of $\mathcal{S}$ independently of the theory of ∮14.

Corollary 16.51 If $\|X\|_{\mathcal{S}}$ is z-invariant, the following conditions are equivalent.

1. $\mathcal{S}_{:X}$ is $\mathfrak{s}$-protoalgebraic.
2. There exists a finite set of ternary terms $\mathbf{\Delta}$ such that

$$\vdash_{\mathcal{S}_{:X}} \mathbf{\Delta}(x,x,z) \quad \text{and} \tag{16.12}$$

$$y, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}_{:X}} x. \tag{16.13}$$

3. If $T, R \in \mathsf{Th}(\mathcal{S})$ and $X \subseteq T \subseteq R$ then $\Omega_{\mathfrak{a}}(T)$ is compatible with R.
4. $\Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle} : \mathbf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X\rangle) \to_{\leq} \mathbf{Con}(\mathbf{Tm})$.
5. $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic.
6. There exists a finite set of ternary terms $\mathbf{\Delta}$ such that

$$X \vdash_{\mathcal{S}} \mathbf{\Delta}(x,x,z) \quad \text{and} \tag{16.14}$$

$$X, y, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} x. \tag{16.15}$$

7. There exists a finite set of ternary terms $\mathbf{\Delta}$ such that (16.14) holds and

$$y, \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} x. \tag{16.16}$$

8. For every $U \in \mathsf{Fi}_{\mathcal{S}}(\langle \mathbf{Tm}, X\rangle)$, we have $U, p \vdash_{\mathcal{S}} q$ and $U, q \vdash_{\mathcal{S}} p$ whenever $\langle p, q\rangle \in \Omega^{\mathcal{S}}_{\langle \mathbf{Tm}, X\rangle}(U)$.

Proof. $\boxed{(1)\Leftrightarrow(2)}$ By Theorem 16.28, the fact that z is not an $\mathfrak{s}$-variable of $\mathbf{Tm}$, and the fact that all $\mathfrak{s}$-substitutions on $\mathbf{Tm}$ fix z. $\boxed{(1)\Leftrightarrow(3)}$ By Theorem 16.28, the definition of $\mathcal{S}_{:X}$, Remark 16.45 and Proposition 16.49. $\boxed{(1)\Leftrightarrow(4)}$ By Lemma 16.30, together with Corollary 16.50, and Proposition 16.49 and Remark 16.45. $\boxed{(4)\Leftrightarrow(5)}$ By Definition 14.1 on page 406. $\boxed{(5)\Leftrightarrow(6)}$ Definitional. $\boxed{(6)\Rightarrow(7)}$ By finitariness, there exists $\mathbf{\Delta}'(x,y,z,v_1,\ldots,v_n) \subseteq_{\mathsf{f}} X$, such that $y, \mathbf{\Delta}'(x,y,z,v_1,\ldots,v_n), \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} x$. Let σ be the substitution with $\sigma[\{v_1,\ldots,v_n\}] = \{z\}$ and fixing all other variables. By structurality, $\sigma(y), \sigma[\mathbf{\Delta}'(x,y,z,v_1,\ldots,v_n)], \sigma[\mathbf{\Delta}(x,y,z)] \vdash_{\mathcal{S}} \sigma(x)$, i.e., $y, \mathbf{\Delta}'(x,y,z,z,\ldots,z), \mathbf{\Delta}(x,y,z) \vdash_{\mathcal{S}} x$. Further, by the z invariance of the theory generated by X, $X \vdash_{\mathcal{S}} \sigma[X]$, and hence $X \vdash_{\mathcal{S}} \mathbf{\Delta}'(x,y,z,z,\ldots,z)$. So the terms $\mathbf{\Delta}'(x,y,z,z,\ldots,z) \cup \mathbf{\Delta}(x,y,z)$ satisfy (16.14) and (16.16). $\boxed{(7)\Rightarrow(6)}$ Trivial. $\boxed{(1)\Leftrightarrow(8)}$ By the definition that $\mathcal{S}_{:X}$ be $\mathfrak{s}$-protoalgebraic, together with Proposition 16.49 and Remark 16.45. $\diamond$

The previous result is deficient in that while it has established the 'global' or 'logical' characterizations, it has not established the 'local' or 'model-theoretic' characterizations. We now show how these too may be obtained by the machinery of this chapter.

For each $\mathfrak{a}$-algebra $\mathbf{A} \neq \mathbf{Tm}$, pick an element $\mathsf{d}_{\mathbf{A}} \in \mathsf{uni}(\mathbf{A})$, and let $\mathsf{d}_{\mathbf{Tm}} = z$. Consider the construct $\mathfrak{t}$ consisting of all $\mathsf{type}(\mathcal{S})$-algebras together with all homomorphisms between algebras $\mathbf{A}$ and $\mathbf{B}$ that map $\mathsf{d}_{\mathbf{A}}$ to $\mathsf{d}_{\mathbf{B}}$.

Observe that $\mathfrak{t}$-substitutions and $\mathfrak{s}$-substitutions of $\mathbf{Tm}$ coincide. Hence $\mathcal{S}_{:X}$ is $\mathfrak{t}$-structural.

Lemma 16.52 If $\|X\|_{\mathcal{S}}$ is z-invariant, then the following are all valid.

1. $\mathsf{Fi}^{\mathfrak{t}}_{\mathcal{S}:X}(\mathbf{A}) = \mathsf{Fi}^{\mathfrak{t}}_{\mathcal{S}:X}(\mathsf{E}^{\mathbf{A}}_{z:\mathsf{d}_{\mathbf{A}}}[X])$.
2. $\mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:\mathsf{d}_{\mathbf{A}}}[X]\rangle) = \mathsf{Fi}^{\mathfrak{t}}_{\mathcal{S}:X}(\mathbf{A})$.
3. $\mathsf{Fi}^{\mathfrak{t}}_{\mathcal{S}:X}(\mathbf{Tm}) = \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{S}:X}(\mathbf{Tm}) = \mathsf{Th}(\mathcal{S}_{:X}) = \mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{Tm}, X\rangle)$.
4. $a\ \Omega^{\mathbf{A}}_{\mathfrak{t}}(A)\ b$ iff, for some variable $x \in \mathsf{V} - \{z\}$, every term $p(z, x, \vec{y})$ (where $\vec{y} \in \mathsf{V} - \{z, x\}$) and $\vec{c} \in \mathsf{Tm}$, $p(\mathsf{d}_{\mathbf{A}}, a, \vec{c}) \in A$ iff $p(\mathsf{d}_{\mathbf{A}}, a, \vec{c}) \in A$.
5. $\Omega^{\mathbf{A}}_{\mathfrak{a}}(A) = \Omega^{\mathbf{A}}_{\mathfrak{t}}(A)$.
6. For all $\mathfrak{s}$-languages $\mathbf{A}$, $F, G \in \mathsf{Fi}^{\mathfrak{t}}_{\mathcal{S}:X}(\mathbf{A})$ with $F \subseteq G$, if $\Omega^{\mathbf{A}}_{\mathfrak{t}}(F)$ is compatible with G then $\Omega^{\mathbf{A}}_{\mathfrak{t}}(F) \subseteq \Omega^{\mathbf{A}}_{\mathfrak{t}}(G)$.

Proof. $\boxed{1}$ Let $F \in \mathsf{Fi}^{\mathfrak{t}}_{\mathcal{S}:X}(\mathbf{A})$. Let i be a homomorphism from $\mathbf{Tm}$ into $\mathbf{A}$ such that $\mathsf{i}(z) = \mathsf{d}_{\mathbf{A}}$. Then i is a $\mathfrak{t}$-morphism from $\mathbf{Tm}$ into $\mathbf{A}$, and since $X \subseteq \mathsf{Thm}(\mathcal{S}_{:X})$ and $F \in \mathsf{Fi}^{\mathfrak{t}}_{\mathcal{S}:X}(\mathbf{A})$, we have $\mathsf{i}[X] \subseteq F$. $\boxed{2}\boxed{\subseteq}$ Let $F \in \mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:\mathsf{d}_{\mathbf{A}}}[X]\rangle)$. Suppose that $P \vdash_{\mathcal{S}:X} p$, $\mathsf{i} : \mathbf{Tm} \rightarrow_{\mathfrak{t}} \mathbf{A}$ and $\mathsf{i}[P] \subseteq F$. Then $X, P \vdash_{\mathcal{S}} p$, i is a homomorphism from $\mathbf{Tm}$ into $\mathbf{A}$ and $\mathsf{i}[X \cup P] \subseteq F$, since $\mathsf{i}(z) = \mathsf{d}_{\mathbf{A}}$ and so $\mathsf{i}[X] \subseteq \mathsf{E}^{\mathbf{A}}_{z:\mathsf{d}_{\mathbf{A}}}[X] \subseteq F$. Hence $\mathsf{i}(p) \in F$. $\boxed{\supseteq}$ Let $F \in \mathsf{Fi}^{\mathfrak{t}}_{\mathcal{S}:X}(\mathbf{A})$. Suppose that $P \vdash_{\mathcal{S}} p$, $f : \mathbf{Tm} \rightarrow_{\mathfrak{a}} \mathbf{A}$ and $f[P] \subseteq F$. (*We must show that* $f(p) \in F$.) Let $\{x_1, x_2, \ldots\}$ be the variables other than z, let σ be the $\mathfrak{a}$-substitution determined by $\sigma(z) = z$, $\sigma(x_1) = z$, and $\sigma(x_n) = \sigma(x_{n-1})$, for $n > 1$, and the $f' : \mathbf{Tm} \rightarrow_{\mathfrak{a}} \mathbf{A}$ determined by $f'(z) = \mathsf{d}_{\mathbf{A}}$, $f'(x_1) = f(z)$, and $f'(x_n) = f(x_{n-1})$, for $n > 1$. Note that $f'\sigma = f$ and $f' : \mathbf{Tm} \rightarrow_{\mathfrak{t}} \mathbf{A}$. By structurality, $\sigma[P] \vdash_{\mathcal{S}} \sigma(p)$, hence $X, \sigma[P] \vdash_{\mathcal{S}} \sigma(p)$ and so $\sigma[P] \vdash_{\mathcal{S}:X} \sigma(p)$. Since $f'[\sigma[P]] = f[P] \subseteq F$, $f(p) = f'(\sigma(p)) \in F$. $\boxed{3}$ By definitions and Remark 16.45. $\boxed{4}$ $a\ \Omega_{\mathfrak{t}}(A)\ b$ [iff] for some variable $x \in \mathsf{V} - \{z\}$, every term p, and every function $f : (\mathsf{V} - \{z, x\}) \rightarrow \mathsf{uni}(\mathbf{A})$, $f_{\frac{x}{a}}(p) \in A$ iff $f_{\frac{x}{b}}(p) \in A$ [iff] for some variable $x \in \mathsf{V} - \{z\}$, every term $p(z, x, \vec{y})$ (where $\vec{y} \in \mathsf{V} - \{z, x\}$) and $\vec{c} \in \mathsf{Tm}$, $p(\mathsf{d}_{\mathbf{A}}, a, \vec{c}) \in A$ iff $p(\mathsf{d}_{\mathbf{A}}, a, \vec{c}) \in A$. $\boxed{5}$ Assume that $a\ \Omega^{\mathbf{A}}_{\mathfrak{a}}(A)\ b$. Let $x \in \mathsf{V} - \{z\}$, $p(z, x, \vec{y})$ a term (where $\vec{y} \in \mathsf{V} - \{z, x\}$) and $\vec{c} \in \mathsf{Tm}$, such that $p(\mathsf{d}_{\mathbf{A}}, a, \vec{c}) \in A$. Since $a\ \Omega^{\mathbf{A}}_{\mathfrak{a}}(A)\ b$, $p(\mathsf{d}_{\mathbf{A}}, b, \vec{c}) \in A$. Conversely, assume that $a\ \Omega^{\mathbf{A}}_{\mathfrak{t}}(A)\ b$. Let $p(x, \vec{y})$ be a term (where $\vec{y} \in \mathsf{V} - \{x\}$) and $\vec{c} \in \mathsf{Tm}$, such that $p(a, \vec{c}) \in A$. We may assume, without loss of generality, that $z \notin \{x, \vec{y}\}$. Let $p'(z, x, \vec{y}) = p(x, \vec{y})$. Then $p'(\mathsf{d}_{\mathbf{A}}, a, \vec{c}) = p(a, \vec{c}) \in A$. Since $a\ \Omega^{\mathbf{A}}_{\mathfrak{t}}(A)\ b$, $p(b, \vec{c}) = p'(\mathsf{d}_{\mathbf{A}}, b, \vec{c}) \in A$. $\boxed{(6)}$ Follows from (5), since this property holds for $\Omega^{\mathbf{A}}_{\mathfrak{a}}(\cdot)$. $\diamond$

Since $\mathfrak{t}$-substitutions and $\mathfrak{s}$-substitutions of $\mathbf{Tm}$ coincide, $\mathcal{S}_{:X}$ is $\mathfrak{s}$-protoalgebraic iff $\mathcal{S}_{:X}$ is $\mathfrak{t}$-protoalgebraic iff (by Corollary 16.51) $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic. Consequently, the following result obtains from Theorem 16.28.

Corollary 16.53 If $\|X\|_{\mathcal{S}}$ is z-invariant, then the following conditions are equivalent.

1. $\mathcal{S}_{:X}$ is $\mathfrak{t}$-protoalgebraic.

2. $\mathcal{S}_{:X}$ is $\mathfrak{s}$-protoalgebraic.
3. $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic.
4. For every algebra $\mathbf{A}$ and $F \in \mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:\mathsf{d}_{\mathbf{A}}}[X]\rangle)$, if $a\ \Omega^{\mathbf{A}}(F)\ b$ then $a \in \|\{b\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $b \in \|\{a\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$.
5. For every algebra $\mathbf{A}$, if $F, G \in \mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:\mathsf{d}_{\mathbf{A}}}[X]\rangle)$ and $F \subseteq G$ then $\Omega^{\mathbf{A}}(F)$ is compatible with G.
6. For every algebra $\mathbf{A}$, $\Omega^{\mathbf{A}}(\cdot)_{|\mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:\mathsf{d}_{\mathbf{A}}}[X]\rangle)} : \mathbf{Fi}_{\mathcal{S}}^{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:\mathsf{d}_{\mathbf{A}}}[X]\rangle} \to_{\subseteq} \mathbf{Con}(\mathbf{A})$.

□

Since equivalent condition (3) of the previous result is independent of $\mathfrak{t}$, and the 'designated points' $\mathsf{d}_{\mathbf{A}}$ were chosen arbitrarily in the definition of $\mathfrak{t}$, we obtain the following characterizations of $\langle X, z\rangle$-protoalgebraicity.

Corollary 16.54 The following conditions are equivalent.

1. $\mathcal{S}$ is $\langle X, z\rangle$-protoalgebraic.
2. For every algebra $\mathbf{A}$, $c \in \mathsf{uni}(\mathbf{A})$ and $F \in \mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:c}[X]\rangle)$, if $a\ \Omega^{\mathbf{A}}(F)\ b$ then $a \in \|\{b\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$ and $b \in \|\{a\} \cup F\|^{\mathbf{A}}_{\mathsf{fi}_{\mathcal{S}}}$.
3. For every algebra $\mathbf{A}$, $c \in \mathsf{uni}(\mathbf{A})$, if $F, G \in \mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:c}[X]\rangle)$ and $F \subseteq G$ then $\Omega^{\mathbf{A}}(F)$ is compatible with G.
4. For every algebra $\mathbf{A}$, $c \in \mathsf{uni}(\mathbf{A})$, $\Omega^{\mathbf{A}}(\cdot)_{|\mathsf{Fi}^{\mathfrak{a}}_{\mathcal{S}}(\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:c}[X]\rangle)} : \mathbf{Fi}_{\mathcal{S}}^{\langle \mathbf{A}, \mathsf{E}^{\mathbf{A}}_{z:c}[X]\rangle} \to_{\subseteq} \mathbf{Con}(\mathbf{A})$.

We have not been able to establish whether or not $\mathcal{S}_{:X}$ is Leibniz interpretable (with respect to $\mathfrak{t}$) and so cannot invoke the full power of Theorem 16.35. Given all the equivalent conditions of Theorem 14.5 on page 407 (including parameterized analogues of the filter correspondence property) either $\mathcal{S}_{:X}$ *is* Leibniz interpretable (with respect to $\mathfrak{t}$) *or* Theorem 16.35 is valid independently of Leibniz interpretability.

Open Problem 16.55 Is $\mathcal{S}_{:X}$ Leibniz interpretable (with respect to $\mathfrak{t}$)?

Open Problem 16.56 Is Leibniz interpretability a necessary assumption in the formulation of Theorem 16.35?

□

We take the opportunity to consider the protoalgebraicity of some of the sentential 1-calculi introduced subsequently to ∮2 which have theorems.

Example 16.57 (Subuniverse Logics with Constants)

Let $\mathfrak{a}$ be a type of algebras *with at least one constant symbol* and $\mathcal{K}$ an $\mathfrak{a}$-quasivariety. Then, $S(\mathcal{K}, \mathsf{su})$ *has theorems*. Let $\mathcal{V}$ be the variety generated by $\mathcal{K}$.

Proposition 16.58 The following conditions are equivalent.

1. $S(\mathcal{K}, \mathsf{su})$ is protoalgebraic.
2. There exist a positive integer n, binary terms $\Delta_1, \ldots, \Delta_m$, equationally definable constant terms $\mathbf{0}_1, \ldots, \mathbf{0}_m$ and an $(m+1)$-ary term q, such that,

$$\models_{\mathcal{K}} \mathbf{0}_i \approx \Delta_i(x, x), \quad \text{for each } i \in n, \text{ and} \tag{16.17}$$

$$\models_{\mathcal{K}} x \approx q(y, \Delta_0(x, y), \ldots, \Delta_{n-1}(x, y)). \tag{16.18}$$

3. $S(\mathcal{V}, \mathsf{su})$ is protoalgebraic.

4. For every $\mathbf{A} \in \mathcal{V}$, $\alpha \in \mathsf{Con}(\mathbf{A})$ and $B, C \in \mathsf{Su}(\mathbf{A})$, if $B \subseteq C$ and α is compatible with B, then α is compatible with C.

5. If $\alpha \in \mathsf{Con}(\mathbf{F}_{\mathcal{V}})$ and $B, C \in \mathsf{Su}(\mathbf{F}_{\mathcal{V}})$, if $B \subseteq C$ and α is compatible with B, then α is compatible with C.

Proof. $\boxed{(1)\Rightarrow(2)}$ By Corollary 16.37, there exists a finite set $\mathbf{\Delta}$ of binary terms such that $\vdash_{S(\mathcal{K},\mathsf{su})} \mathbf{\Delta}(x,x)$ and $y, \mathbf{\Delta}(x,y) \vdash_{S(\mathcal{K},\mathsf{su})} x$. So $\vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \overline{[\mathbf{\Delta}(x,x)]}$ and $\overline{y}, \overline{[\mathbf{\Delta}(x,y)]} \vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \overline{x}$. The result follows by Remark 6.81 of Example 6.79 on page 242, together with Lemma 1.457 on page 88. $\boxed{(2)\Rightarrow(1)}$ So $\mathbf{0}_i^{\mathbf{F}_{\mathcal{K}}} = \Delta_i^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{x})$, for each i, and $\overline{x} = q^{\mathbf{F}_{\mathcal{K}}}(\overline{y}, \Delta_0^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{y}), \ldots, \Delta_{n-1}^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{y}))$. Hence $\Delta_i^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{x}) \in \|\emptyset\|_{\mathsf{su}}^{\mathbf{F}_{\mathcal{K}}}$, and $\overline{x} \in \left\| \{\overline{y}, \Delta_0^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{y}), \ldots, \Delta_{n-1}^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{y})\} \right\|_{\mathsf{su}}^{\mathbf{F}_{\mathcal{K}}}$, by Corollary 16.37 on page 453. Hence, by definition, $\vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \Delta_i^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{x})$, for each i, and $\overline{y}, \Delta_0^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{y}), \ldots, \Delta_{n-1}^{\mathbf{F}_{\mathcal{K}}}(\overline{x}, \overline{y}) \vdash_{\boldsymbol{S}(\mathcal{K},\mathsf{su})} \overline{x}$. Hence $\vdash_{S(\mathcal{K},\mathsf{su})} \Delta_i(x,x)$, for each i, and $y, \Delta_0(x,y), \ldots, \Delta_{n-1}(x,y) \vdash_{S(\mathcal{K},\mathsf{su})} x$. The result follows by Theorem 2.135. $\boxed{(2)\Leftrightarrow(3)}$ Since $\mathcal{K}$ and $\mathcal{V}$ satisfy precisely the same identities, the result follows by the same arguments as (1)⇒(2) and (2)⇒(1). $\boxed{(3)\Rightarrow(4)}$ Follows by Theorem 2.135 and the fact that the archology $\mathfrak{A}(\mathcal{V}, \mathsf{su})$ is maximal, by Proposition 8.92 of Example 8.89 on page 301. $\boxed{(4)\Rightarrow(5)}$ Trivial. $\boxed{(5)\Rightarrow(3)}$ Let T and R be $S(\mathcal{V}, \mathsf{su})$-theories and $\alpha \in \mathsf{Con}(\mathbf{Tm})$ such that $T \subseteq R$ and α compatible with T. Suppose that $p \in R$ and $p \,\alpha\, q$. By Proposition 1.358 on page 68, $\boldsymbol{\alpha} = \{\langle \overline{r_1}, \overline{r_2} \rangle : \langle r_1, r_2 \rangle \in \alpha\}$ is a congruence on $\mathbf{F}_{\mathcal{V}}$. Further $\overline{[T]}, \overline{[R]} \in \mathsf{Su}(\mathbf{F}_{\mathcal{V}})$ (by (8.2) of Example 8.51 on page 293), $\overline{[T]} \subseteq \overline{[R]}$, and certainly $\boldsymbol{\alpha}$ is compatible with $\overline{[T]}$, and hence is compatible with $\overline{[R]}$, by assumption. Since $\overline{p} \in \overline{[R]}$ and $\overline{p} \,\boldsymbol{\alpha}\, \overline{q}$, $\overline{q} \in \overline{[R]}$. Hence $q \in \underline{\left[\overline{[R]}\right]} = R$, by Corollary 8.54 of Example 8.51). The result follows by Theorem 2.135. $\diamond$

Open Problem 16.59 We have not been able to establish the equivalence of the following conditions with those of the previous result. Note that they are certainly necessary conditions.

1. For any $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$, $F \in \mathsf{Su}(\mathbf{A})$ and $H \in \mathsf{Su}(\mathbf{B})$, we have $\left\| F \cup h^{-1}[H] \right\|_{\mathsf{su}}^{\mathbf{A}} = h^{-1}[\|h[F] \cup H\|_{\mathsf{su}}^{\mathbf{B}}]$.

2. For any algebra $\mathbf{B} \in \mathcal{V}$, $h : \mathbf{F}_{\mathcal{V}} \twoheadrightarrow \mathbf{B}$, any $\boldsymbol{T} \in \mathsf{Su}(\mathbf{F}_{\mathcal{V}})$, and any $F \in \mathsf{Su}(\mathbf{B})$, we have $\left\| \boldsymbol{T} \cup h^{-1}[F] \right\|_{\mathsf{su}}^{\mathbf{F}_{\mathcal{V}}} = h^{-1}[\|h[\boldsymbol{T}] \cup F\|_{\mathsf{su}}^{\mathbf{B}}]$.

Recall the definitions of *subalgebra point* $\langle \mathcal{K}, \mathbf{0}_1, \ldots, \mathbf{0}_n \rangle$*-coherence* given in ∮10.1, as well as the characterization of this condition given in Corollary 10.4. A comparison of equivalent condition (2) of the previous proposition with equivalent condition (2) of Corollary 10.4, demonstrates that if $S(\mathcal{K}, \mathsf{su})$ is *protoalgebraic*, then there exists *some* equationally definable constant terms $\mathbf{0}_1, \ldots, \mathbf{0}_n$, such that $\mathcal{K}$ is *subalgebra point* $\langle \mathcal{K}, \mathbf{0}_1, \ldots, \mathbf{0}_n \rangle$*-coherent*.

Corollary 16.60 If $S(\mathcal{K}, \mathsf{su})$ is protoalgebraic then $\mathcal{K}$ is subalgebra point $\langle \mathcal{K}, \mathbf{0}_1, \ldots, \mathbf{0}_n \rangle$-coherent, for some $\mathcal{K}$-constant terms $\mathbf{0}_1, \ldots, \mathbf{0}_n$. □

While we have not found a counter-example yet, it appears that the converse is not generally true. If one compares (16.17) of the previous proposition with (10.5) of Corollary 10.4, in the former, there is a one-to-one correspondence between the $\mathbf{0}_i$'s and the Δ_i's, while in the latter, the correspondence between the $\mathbf{0}_i$'s and the Δ_i's is weaker, given the role of the selection function in (10.5).

□

Example 16.61 (Bounded Lattice Ideals and Filters)

Definition 16.62 (Ideal and Filter Coherence) For a quasivariety $\mathcal{K}$ of 0-lattice (resp. 1-lattice) expansions, we say that $\mathcal{K}$ is ideal $\mathcal{K}$-coherent (resp. filter $\mathcal{K}$-coherent) if every ideal (resp. filter) of a member of $\mathcal{K}$ is $\langle \mathcal{K}, 0\rangle$-coherent (resp. $\langle \mathcal{K}, 1\rangle$-coherent). $\square$

Proposition 16.63 The following conditions are equivalent.

1. $S_0(\mathcal{K}, \mathsf{id})$ is protoalgebraic.
2. There exist terms $\Delta_1(x,y), \Delta_m(x,y)$, for some $m > 0$, such that

$$\models_{\mathcal{K}} \Delta_i(x,x) \approx 0, \quad \text{for all } i \leq m, \text{ and} \tag{16.19}$$

$$\models_{\mathcal{K}} y \leq x \vee \Delta_1(x,y) \vee \Delta_m(x,y). \tag{16.20}$$

3. $\mathcal{K}$ is ideal $\mathcal{K}$-coherent.

Proof. $\boxed{(1) \Leftrightarrow (2)}$ By Corollary 16.37, Theorem 8.75 on page 297, Remark 6.102 on page 247 and Remark 6.103. $\boxed{(2) \Rightarrow (3)}$ Let $\mathbf{P} \in \mathcal{K}$, $\mathsf{I} \in \mathsf{Id}_{\diamond}(\mathbf{P})$, and $\alpha \in \mathsf{Con}^{\mathcal{K}}(\mathbf{P})$ with $\alpha[\![0^{\mathbf{P}}]\!] \subseteq \mathsf{I}$. Suppose that $a \in \mathsf{I}$ and $a \,\alpha\, b$. For each $1 \leq i \leq m$, $\Delta_i^{\mathbf{P}}(b,b)\,\alpha\,0^{\mathbf{P}}$, by (16.19), hence $\Delta_i^{\mathbf{P}}(a,b)\,\alpha\,0^{\mathbf{P}}$, since $a\,\alpha\,b$, and hence $\Delta_i^{\mathbf{P}}(a,b) \in \mathsf{I}$. So $a, \Delta_1^{\mathbf{P}}(a,b), \ldots, \Delta_m^{\mathbf{P}}(a,b) \in \mathsf{I}$, hence $a \vee^{\mathbf{P}} \Delta_1^{\mathbf{P}}(a,b) \ldots \vee^{\mathbf{P}} \Delta_m^{\mathbf{P}}(a,b) \in \mathsf{I}$. By (16.20), $b \leq^{\mathbf{P}} a \vee^{\mathbf{P}} \Delta_1^{\mathbf{P}}(a,b) \ldots \vee^{\mathbf{P}} \Delta_m^{\mathbf{P}}(a,b)$, hence $b \in \mathsf{I}$. $\boxed{(3) \Rightarrow (2)}$ Let $\mathbf{F}$ be the $\mathcal{K}$-free algebra generated by $\overline{x}$ and $\overline{y}$, let $\alpha = \|\langle \overline{x}, \overline{y}\rangle\|_{\Theta^{\mathcal{K}}_{\mathbf{F}}}$, let $\mathsf{I} = \left\|\alpha[\![\overline{0}]\!] \cup \{\overline{x}\}\right\|^{\mathbf{F}}_{\mathsf{id}_{\diamond}}$. Since $\alpha[\![\overline{0}]\!] \subseteq \mathsf{I}$, $\overline{x} \in \mathsf{I}$ and $\overline{x}\,\alpha\,\overline{y}$, by assumption, $\overline{y} \in \mathsf{I} = \left\|\alpha[\![\overline{0}]\!] \cup \{\overline{x}\}\right\|^{\mathbf{F}}_{\mathsf{id}_{\diamond}}$. So $\alpha[\![\overline{0}]\!] \cup \{\overline{x}\} \vdash^{\mathbf{F}}_{\mathsf{id}_{\diamond}} \overline{y}$. By Remark 4.93 of Example 4.88 on page 158, there exist $\overline{\Delta_1}, \ldots, \overline{\Delta_m} \in \alpha[\![\overline{0}]\!]$ such that $\overline{y} \leq^{\mathbf{F}} \overline{x} \vee^{\mathbf{F}} \overline{\Delta_1} \vee^{\mathbf{F}} \overline{\Delta_m}$. The result follows by Lemma 1.457 on page 88. $\diamond$

The equivalent conditions of the previous result are too strong to be satisfied by any non-trivial *quasivariety* of 0-lattices. In order to find an example satisfying this condition, we need to have a 'typed' notion of *complementation*.

Proposition 16.64 If $\mathcal{K}$ is a quasivariety of distributed 0-complemented-lattice expansions, then $S_{0'}(\mathcal{K}, \mathsf{id})$ is protoalgebraic.

Proof. Define $\Delta(x,y) = x \wedge y'$. Then by (1.109), $\models_{\mathcal{K}} x \wedge x' \approx 0$, and hence $\models_{\mathcal{K}} \Delta(x,x) \approx 0$. Let $\mathbf{P} \in \mathcal{K}$ and $a, b \in \mathsf{uni}(\mathbf{P})$. Then $b \vee \Delta^{\mathbf{P}}(a,b) = b \vee (a \wedge b') \overset{\text{dst}}{=} (b \vee a) \wedge (b \vee b') \geq (b \vee a) \wedge (b) \overset{\text{dst}}{=} (b \wedge b) \vee (a \wedge b) \overset{\text{idp}}{=} b \vee (a \wedge b) \overset{\text{abs}}{=} b$. Since $\mathbf{P}$, a and b are arbitrary, $\models_{\mathcal{K}} y \leq y \vee \Delta(x,y)$. The result follows by Proposition 16.63. $\diamond$

Open Problem 16.65 What about modular 0-complemented lattices. Alternatively, is a converse achievable? That is, does $S_{0'}(\mathcal{K}, \mathsf{id})$ being protoalgebraic imply distributivity?

We leave it to the reader to formulate the upper-bounded dual of these results.

$\square$

16.5 Maximal Models

Recall the notion of maximal modellability defined in ∮7.3, and in particular the sufficient condition of Corollary 7.54. As promised, we shall show that this condition is necessary for protoalgebraic structural deductive systems.

Lemma 16.66 For a (structural) protoalgebraic $\mathfrak{s}$-calculus $\mathcal{D}$ and a set $\mathfrak{M}$ of $\mathfrak{s}$-logics, the following conditions are equivalent.

1. For each $\mathsf{M} \in \mathfrak{M}$, $\mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{lg}(\mathsf{M})) \subseteq \mathsf{Th}(\mathsf{M})$.

2. For each $\mathsf{M} \in \mathfrak{M}$, if $\mathsf{G} \vdash_{\mathsf{M}} \mathsf{g}$, then there exists $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathcal{D})$ and $\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathcal{D}, \mathsf{M})$, with $\Gamma \vdash_{\mathcal{D}} \phi$, $\mathsf{i}\,[\Gamma] \subseteq \mathsf{G} \cup \|\emptyset\|^{\mathbf{lg}(\mathsf{M})}_{\mathsf{fi}_{\mathcal{D}}}$ and $\mathsf{i}(\phi) = \mathsf{g}$.

3. For each $\mathsf{M} \in \mathfrak{M}$ and $F \in \mathsf{Fi}^{\mathfrak{s}}_{\mathcal{D}}(\mathbf{lg}(\mathsf{M}))$, if $F \vdash_{\mathsf{M}} \mathsf{g}$, then there exists $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathcal{D})$ and $\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathcal{D}, \mathsf{M})$, with $\Gamma \vdash_{\mathcal{D}} \phi$, $\mathsf{i}\,[\Gamma] \subseteq F$ and $\mathsf{i}(\phi) = \mathsf{g}$.

Proof. $\boxed{(1)\Rightarrow(2)}$ Let $\mathsf{M} \in \mathfrak{M}$, and suppose that $\mathsf{G} \vdash_{\mathsf{M}} \mathsf{g}$. Since by assumption, $\|\mathsf{G}\|^{\mathbf{lg}(\mathsf{M})}_{\mathsf{fi}_{\mathcal{D}}}$ is an M-theory, $\mathsf{g} \in \|\mathsf{G}\|^{\mathbf{lg}(\mathsf{M})}_{\mathsf{fi}_{\mathcal{D}}}$. The result follows by Theorem 16.29. $\boxed{(2)\Rightarrow(3)}$ Follows since any filter contains the minimum filter. $\boxed{(3)\Rightarrow(1)}$ By Lemma 7.53 on page 264. ⋄

Corollary 16.67 Let $\mathcal{D}$ be a protoalgebraic $\mathfrak{s}$-calculus and $\mathfrak{M}$ a set of $\mathfrak{s}$-logics. Then $\mathfrak{M}$ constitutes a maximal model of $\mathcal{D}$ iff $\mathfrak{M}$ constitutes a model of $\mathcal{D}$ and the equivalent conditions of the previous lemma hold.

16.6 Reduced Matrix Models

For the sake of completeness, we show that a theory of reduced matrix models can be developed under the assumption of Leibniz interpretability.

Convention 16.68 Throughout this section, we assume that $\mathcal{D}$ is Leibniz interpretable.

Definition 16.69 (Reduced Matrices) Let $\mathbf{M}$ be a $\mathcal{D}$-matrix. We define a function $\Omega^{\mathbf{M}}_{\mathcal{D}}(\cdot)$ from $\mathsf{Fi}_{\mathcal{D}}(\mathbf{M})$ by $\Omega^{\mathbf{M}}_{\mathcal{D}}(F)$=$\Omega^{\mathbf{M}}_{\mathfrak{s}}(F)$. We call $\mathbf{M}$ **reduced** if $\Omega^{\mathbf{M}}_{\mathcal{D}}(\mathsf{D}_{\mathbf{M}})$ is the identity relation on $\mathsf{uni}(\mathbf{M})$. The class of all reduced matrix models of $\mathcal{D}$ is denoted by $\mathsf{MMod}_*(\mathcal{D})$. The matrix $\langle [\ \Omega^{\mathbf{lg}(\mathbf{M})}(\mathsf{D}_{\mathbf{M}})\ ](\mathbf{lg}(\mathbf{M})), [\ \Omega^{\mathbf{lg}(\mathbf{M})}(\mathsf{D}_{\mathbf{M}})\ ]\,[\mathsf{D}_{\mathbf{M}}]\rangle$ is denoted by $\mathbf{M}^*$. □

Proposition 16.70 If $\mathbf{M}$ is a $\mathcal{D}$-matrix then $\mathbf{M}^*$ is a reduced $\mathcal{D}$-matrix.

Proof. Let $\mathbf{A} = \mathbf{lg}(\mathbf{M})$, $\mathbf{B} = [\ \Omega^{\mathbf{lg}(\mathbf{M})}(\mathsf{D}_{\mathbf{M}})\ ](\mathbf{lg}(\mathbf{M}))$ and $f = [\ \Omega^{\mathbf{lg}(\mathbf{M})}(\mathsf{D}_{\mathbf{M}})\ ](\cdot)$.
$\boxed{\mathbf{M}^* \text{ is a } \mathcal{D}\text{-matrix}}$ f is a surjective $\mathfrak{s}$-morphism from $\mathbf{A}$ onto $\mathbf{B}$ whose kernel is compatible with $\mathsf{D}_{\mathbf{M}}$, since $\equiv_f = \Omega^{\mathbf{A}}(\mathsf{D}_{\mathbf{M}})$ which is compatible with $\mathsf{D}_{\mathbf{M}}$ by Proposition 16.7. So by Proposition 7.25 on page 258, $f\,[\mathsf{D}_{\mathbf{M}}]$ is a $\mathcal{D}$-filter on $\mathbf{B}$. $\boxed{\mathbf{M}^* \text{ is reduced}}$ Suppose that $f(a)\ \Omega^{\mathbf{M}^*}(f\,[\mathsf{D}_{\mathbf{M}}])\ f(b)$. (*We must show that* $f(a) = f(b)$; *since* $\equiv_f = \Omega^{\mathbf{A}}(\mathsf{D}_{\mathbf{M}})$, *it suffices to show that* $a\ \Omega^{\mathbf{A}}(\mathsf{D}_{\mathbf{M}})\ b$.)
Suppose that $\phi \in \mathsf{Fm}(\mathbf{G})$, $g : (\mathsf{V} - \{x\}) \to \mathsf{uni}(\mathbf{A})$ and $g_{\frac{x}{a}}(\phi) \in \mathsf{D}_{\mathbf{M}}$. (*It suffices to show that* $g_{\frac{x}{b}}(\phi) \in \mathsf{D}_{\mathbf{M}}$.)
Define $h : (\mathsf{V} - \{x\}) \to \mathsf{uni}(\mathbf{B})$ by $h(y) = f(g(y))$. $\boxed{\text{Claim: For all } c \in \mathsf{uni}(\mathbf{A}),\ h_{\frac{x}{f(c)}} = f\left(g_{\frac{x}{c}}\right).}$ $h_{\frac{x}{f(c)}}(x) = f(c) =$

$f\left(g_{\frac{x}{c}}(x)\right) = fg_{\frac{x}{c}}(x)$, and for variable $y \neq x$, $h_{\frac{x}{f(c)}}(y) = h(y) = f(g(y)) = f\left(g_{\frac{x}{c}}(y)\right) = fg_{\frac{x}{c}}(y)$. So equality follows by $\mathfrak{s}$-freedom of $\mathbf{G}$. □

So $h_{\frac{x}{f(a)}}(\phi) = f\left(g_{\frac{x}{a}}(\phi)\right) \in f[\mathsf{D}_{\mathbf{M}}]$, since $g_{\frac{x}{a}}(\phi) \in \mathsf{D}_{\mathbf{M}}$. Hence $f\left(g_{\frac{x}{b}}(\phi)\right) = h_{\frac{x}{f(b)}}(\phi) \in f[\mathsf{D}_{\mathbf{M}}]$, since $f(a)\ \Omega^{\mathbf{M}^*}(f[\mathsf{D}_{\mathbf{M}}])\ f(b)$. So $g_{\frac{x}{b}}(\phi) \in f^{-1}[f[\mathsf{D}_{\mathbf{M}}]] = \mathsf{D}_{\mathbf{M}}$, by Remark 1.72 on page 25 and the compatibility of $\equiv_f$ with $\mathsf{D}_{\mathbf{M}}$ (since $\equiv_f = \Omega^{\mathbf{A}}(\mathsf{D}_{\mathbf{M}})$ which is compatible with $\mathsf{D}_{\mathbf{M}}$ by Proposition 16.7). ◇

Theorem 16.71 $\mathsf{MMod}_*(\mathcal{D})$ is a matrix semantics for $\mathcal{D}$.

Proof. Let $\mathbf{M} \in \mathsf{MMod}(\mathcal{D})$ and suppose that $\Gamma \models^{\mathbf{M}^*}_{\mathbf{G}} \phi$. (*Since* $\mathsf{MMod}_*(\mathcal{D}) \subseteq \mathsf{MMod}(\mathcal{D})$, *by Proposition 16.70, and* $\mathsf{MMod}(\mathcal{D})$ *constitutes a semantics for* $\mathcal{D}$, *by Theorem 7.72 on page 269, it suffices to show that* $\Gamma \models^{\mathbf{M}}_{\mathbf{G}} \phi$.) Let $\mathbf{A} = \mathbf{lg}(\mathbf{M})$, $\mathbf{B} = [\ \Omega^{\mathbf{A}}(\mathsf{D}_{\mathbf{M}})\ ](\mathbf{A})$ and $f = [\ \Omega^{\mathbf{A}}(\mathsf{D}_{\mathbf{M}})\ ](\cdot)$. Let $\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{G}, \mathbf{A})$ with $\mathsf{i}[\Gamma] \subseteq \mathsf{D}_{\mathbf{M}}$. (*By Proposition 7.70 on page 268, it suffices to show that* $\mathsf{i}(\phi) \in \mathsf{D}_{\mathbf{M}}$.) Consider $f\mathsf{i} \in \mathsf{Int}_{\mathfrak{s}}(\mathbf{G}, \mathbf{B})$. Since $\mathsf{i}[\Gamma] \subseteq \mathsf{D}_{\mathbf{M}}$, $f\mathsf{i}[\Gamma] \subseteq f[\mathsf{D}_{\mathbf{M}}] = \mathsf{D}_{\mathbf{M}^*}$. Since $\Gamma \models^{\mathbf{M}^*}_{\mathbf{G}} \phi$, $f\mathsf{i}(\phi) \in \mathsf{D}_{\mathbf{M}^*} = f[\mathsf{D}_{\mathbf{M}}]$. So $\mathsf{i}(\phi) \in f^{-1}[f[\mathsf{D}_{\mathbf{M}}]] = \mathsf{D}_{\mathbf{M}}$, by Remark 16.19. ◇

Chapter 17

Equivalent Logics

The chapter has three aims. The first aim is to explain the theory of parameterized algebraic semantics and parameterized equivalent algebraic semantics from an alternative non-parameterized perspective. To this end, we develop a theory of *semantics* and *equivalent semantics* for logics in *different* constructs; this theory has no notion of a parameter. The manner in which we locate $\langle X, z\rangle$-semantics and $\langle X, z\rangle$-equivalent semantics within this theory is similar to the approach taken in Example 16.43 on page 455, although the morphisms permitted in $\mathcal{K}$ must also be restricted.

The second aim of this chapter, stems from the second objective of this text, namely to unify as many of the arguments in algebraic logic under the banner of continuous translations between closed systems. Given that the notion of a *semantics* can be characterized in terms of *strict continuity*, we consider a weaker notion than semantics, which we call *model*, and which is based on *continuity* rather than strict continuity. We characterize this notion in terms of commutivity with substitutions; hence, looking in the other direction, we obtain a characterization of the important notion of commutivity independently of any other preconditions. We are also able to obtain a *Blok-Pigozzi theorem* for modellability. This result is new, even when interpreted for sentential calculi. Consequently, the *Blok-Pigozzi theorem* obtained is unlike all the others in the literature; modellability is characterized by commuting join-preserving homomorphisms. The idea of studying such homomorphisms was suggested to us by Blok (personal communication 1999/2000).

The third aim of this chapter is to tighten the results concerning interpretations and deductive equivalence between π-institutions. As noted in the introduction to Part VI, *Blok-Pigozzi theorems* characterizing interpretations and deductive equivalence between π-institutions have only been obtained for term (and, we believe, multi-term) π-institutions. By removing the implicit notion of syntactic naturality in the definition of a translation, and relocating it as logical naturality in the definition of an interpretation, we are able to obtain a full *Blok-Pigozzi theorem* characterizing interpretations between arbitrary π-institutions, and we do this in a manner that the results of [Vou03] obtain directly in the case that the π-institutions are term. For the sake of completeness, we also develop the theory of semi-interpretations between π-institutions (which generalize our notion of a model of a logic across constructs). Since the direction in which we analyse semi-interpretation has not been considered in the literature of CAAL (just as the sentential interpretation of our notion of model has not been considered in AAL) the results we obtain for semi-interpretation are entirely novel.

As an indication of the success of our program of unifying logical arguments under the umbrella of continuous translations between closed systems, we invoke much of the machinery of $\oint 5$. In fact all three of the *Blok-Pigozzi theorems* obtained in this chapter (for models, semantics and equivalent semantics) have pure (that is, logic free) analogues in that chapter, and one half of each of these theorems was obtained by elementary arguments alone.

We briefly outline the contents of the chapter.

In $\oint 17.1$ we consider *category* functors and *category* isomorphisms between signatures of logics, noting that while the substitutions of two sentential calculi of the same type but with different dimension can be put into natural one-to-one correspondence, there is no *natural* one-to-one correspondence between the formulae, and as such, a *construct* isomorphism is inappropriate.

In $\oint 17.2$ we introduce the notion of a *translation* between languages in different constructs and between logics in different constructs. Particular attention is paid to the distinction between syntactic naturality and logical naturality; the former being defined only in terms of the signatures and a functor between them, while the latter makes additional usage of the logics involved. Both forms of naturality involve commutivity with substitutions across the signature functor. In $\oint 17.2.1$ we associate a logic with each translation τ from a 'source' language to a 'target' logic on a familiar language (i.e., a language related to the 'source' language by a functor), defined on the 'source' language. This logic plays the same role in the theory of models, semantics and equivalent semantics, as the logic $S^n(\mathcal{K}, \mathfrak{N})$ plays in the theory of algebraic semantics and equivalent algebraic semantics.

The final three sections concern, in increasing strength, *models*, *semantics* and *equivalent semantics* of logics in different constructs. In the discourse of CAAL, these notions are called *semi-interpretation*, *interpretation* and *deductive equivalence*. We have chosen the terms 'semantics' and 'equivalent semantics' to mirror the terms 'algebraic semantics' and 'equivalent algebraic semantics'; in this case 'model' is the appropriate term for our weakening of 'semantics', although the reader is once again cautioned against confusing this term with 'matrix-model' or our notion that a logic in one construct be a 'model' of another logic in the *same* construct, as introduced in $\oint 7$.

Section $\oint 17.3$ is concerned with *models* of logics between constructs and *semi-interpretations* between π-institutions. We begin with models in $\oint 17.3.1$, paying close attention to syntactic naturality and logical naturality. With the results developed here as motivation, we develop the theory of semi-interpretations between π-institutions in $\oint 17.3.2$.

In $\oint 17.4$ we deal with *semantics* of logics between constructs and *interpretations* between π-institutions. With the previous section as motivation, we develop the theory of interpretations first, in $\oint 17.4.1$, and then obtain our result for model of logics between constructs in $\oint 17.4.2$ as a special case. The reason that we adopted to reverse approach in the previous section was the need to motivate our move from syntactic naturality to logical naturality. In this section, we also show how much of our theory of $\langle X, z\rangle$-semantics can be obtained from the machinery developed in this section, by judicious choice of constructs.

The final section $\oint 17.5$, deals with *equivalent semantics* of logics in different constructs. We do not develop an improved version of *deductive equivalence* between π-institutions, for reasons of time constraints and since we wish to gauge the community's reaction to our dropping of syntactic naturality and replacing it with logical naturality. We do, however, pose the question as an open problem with a conjecture (which we are almost certain is true). As an example, we explain $\langle X, z\rangle$-equivalent algebraic semantics from the perspective of equivalent semantics of logics in different constructs, obtaining many of the results from $\oint$15.

Convention 17.1 (Structurality) All logics considered in this chapter are assumed to be structural with respect to their signature.

Remark 17.2 By our assumption that an $\mathfrak{s}$-logic L be structural, the L-filters of L coincide with the theories of L, by Theorem 7.48 on page 263.

Convention 17.3 For an $\mathfrak{s}$-logic L and a substitution σ on L, we shall write $\sigma^{\star}(\cdot)$ for $\sigma_{\mathfrak{s}}^{\mathsf{L}}(\cdot)$, the latter being defined in Definition 7.13 on page 255, which, by the previous remark, is the function from $\mathfrak{P}(\mathsf{Fm}(\mathsf{L}))$ into $\mathsf{Th}(\mathsf{L})$ defined by $\sigma^{\star}(\Gamma) = \|\sigma\,[\Gamma]\|_{\mathsf{L}}$.

17.1 Familiar and Isomorphic Signatures of Logics

Observe that the relationship between the signature $\underset{\rightarrow[n]}{\mathfrak{a}}$ of sentential n-calculi and the signature $\underset{\rightarrow[m]}{\mathfrak{a}}$ of sentential m-calculi cannot generally be described by a *construct isomorphism* since the formulae of $\mathbf{Tm}^n$ and $\mathbf{Tm}^m$ cannot generally be put into useful one-to-one correspondence; this relationship is better described by a *category isomorphism*.

Definition 17.4 (Familiar and Isomorphic Signatures of Logics) Let $\mathfrak{s}$ and $\mathfrak{t}$ be signatures of logics (over constructs). We call $\mathfrak{t}$ familiar to $\mathfrak{s}$ if there exists a *category* functor from $\mathfrak{s}$ into $\mathfrak{t}$, typically denoted $\cdot^{>}$. We call $\mathfrak{s}$ and $\mathfrak{t}$ **isomorphic**, if they are isomorphic as *categories*; in this case, we typically denote the functor from $\mathfrak{s}$ onto $\mathfrak{t}$ by $\cdot^{>}$ and denote the inverse functor by $\cdot^{<}$. $\square$

Convention 17.5 Throughout this chapter, unless specified to the contrary, $\mathfrak{s}$ and $\mathfrak{t}$ shall denote arbitrary signatures with $\mathfrak{t}$ familiar to $\mathfrak{s}$.

In the following example, we show that for a signature $\mathfrak{s}$, the signatures $\underset{\rightarrow[n]}{\mathfrak{s}}$ and $\underset{\rightarrow[m]}{\mathfrak{s}}$ are isomorphic.

Example 17.6

> Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are isomorphic signatures and that n and m are positive non-zero integers. Consider the signatures $\underset{\rightarrow[n]}{\mathfrak{s}}$ and $\underset{\rightarrow[m]}{\mathfrak{s}}$, where $\mathfrak{s}$ is a signature of logics such that $\underset{\rightarrow[n]}{\mathfrak{s}}$ and $\underset{\rightarrow[m]}{\mathfrak{s}}$ are well-defined. Then $\underset{\rightarrow[n]}{\mathbf{A}}{}^{>} = \underset{\rightarrow[m]}{\mathbf{A}}$ and $\underset{\rightarrow[n]}{\mathsf{i}}{}^{>} = \underset{\rightarrow[m]}{\mathsf{i}}$ defines a category isomorphism from $\underset{\rightarrow[n]}{\mathfrak{s}}$ to $\underset{\rightarrow[m]}{\mathfrak{s}}$ with inverse isomorphism defined by $\underset{\rightarrow[m]}{\mathbf{A}}{}^{<} = \underset{\rightarrow[n]}{\mathbf{A}}$ and $\underset{\rightarrow[m]}{\mathsf{i}}{}^{<} = \underset{\rightarrow[n]}{\mathsf{i}}$.

$\square$

Consequently, for a type of algebras $\mathfrak{a}$, viewed as a signature, the signatures $\underset{\rightarrow[n]}{\mathfrak{a}}$ and $\underset{\rightarrow[m]}{\mathfrak{a}}$ are isomorphic.

Example 17.7

Suppose that n and m are positive non-zero integers and $\mathfrak{a}$ is a type of algebras. By the previous example, the signature $\underrightarrow{\mathfrak{a}}_{[n]}$ of sentential n-calculi is isomorphic to the signature $\underrightarrow{\mathfrak{a}}_{[m]}$ of sentential m-calculi, under the category isomorphism described in that example. Notice that these signatures are *not* isomorphic as *constructs*.

□

The following example is introduced towards our aim of explaining the theory of *parameterized* algebraic and equivalent algebraic semantics from an alternative non-parameterized perspective, using the machinery developed in this chapter.

Example 17.8

Let $\mathcal{S}$ be a sentential 1-calculus of type $\mathfrak{a}$, $\mathcal{K}$ an $\mathfrak{a}$-quasivariety, $X \subseteq \mathsf{Fm}(\mathcal{S})$ and z a variable. As in Example 16.43 on page 455, let $\mathfrak{s}$ be the signature consisting of the single language $\mathbf{Tm}$ together with all endomorphisms of $\mathbf{Tm}$ that fix z. Recall that the $\mathfrak{s}$-variables of $\mathbf{Tm}$ are all the variables other than z. By Example 17.6, the signatures $\mathfrak{s}$ and $\underrightarrow{\mathfrak{s}}_{[2]}$. are isomorphic. Let $\cdot^{>}$ and $\cdot^{<}$ be the functors described in the aforementioned example.

□

17.2 Translations

We now introduce the notion of a translation from one logic to another, the definition of which obtains from the notion of a (concrete) translation introduced in Definition 5.17 on page 180. The primary notion makes no use of a functor from the source logic to the target logic. In order to locate the theory developed subsequently within the framework of equivalent π-institutions [Vou03], we isolate a special family of translations which we term *natural*, where the definition of naturality depends on a functor. We use the word 'natural' so as to draw parallels with the *natural transformation* in the definition of a translation from one π-institution to another (see Definition 17.30). Essentially, a natural translation is a translation that commutes with substitutions at a purely linguistic level, i.e., independently of the logics over the languages involved. There is another notion of commutivity that plays a role in the theory of equivalent sentential calculi and equivalent π-institutions; this notion of commutivity invokes not only the language but the logics as well. We find it useful to refer to the former natural commutivity as *syntactic commutivity* and the latter version as *logical commutivity*. We shall argue in this chapter, that the requirement of syntactic commutivity, i.e., the positioning of naturality within the definition of a translation between π-institutions is misplaced, and that it is this misplacing that leads the theory of equivalent π-institutions only fully working for *term* π-institutions. While we do not develop such a theory of equivalence at the π-institutional level, since we only require a notion of equivalence between logics over constructs, we shall show that, in the constructural case, the burden of commutivity can be moved from the *syntactic* level to the *logical* level, without effecting fundamentally altering the theory, since for global logics (of which term π-institutions are a special case) the two theories coincide, with the advantage of a fully developed theory of equivalence of *non-global* logics. We shall suggest, in a series of open problems, how such a theory might obtain for equivalence between (not necessarily term) π-institutions.

Definition 17.9 (Translations) Let $\mathbf{A}$ be an $\mathfrak{s}$-language and $\mathbf{B}$ a $\mathfrak{t}$-language. A (concrete) translation τ from $\mathsf{Fm}(\mathbf{A})$ to $\mathsf{Fm}(\mathbf{B})$ (i.e, a binary relationship from $\mathsf{Fm}(\mathbf{A})$ to $\mathsf{Fm}(\mathbf{B})$, see Definition 5.17) is called a **translation from A to B**, denoted $\tau : \mathbf{A} \multimap \mathbf{B}$, if $\mathbf{B} = \mathbf{A}^{>}$. A translation τ from $\mathbf{A}$ to $\mathbf{B}$ is called $\cdot^{>}$**-natural** (or just **natural** where unambiguous), if, for all $\mathfrak{s}$-substitutions σ of $\mathbf{A}$ and all $\phi \in \mathsf{Fm}(\mathbf{A})$, $\tau[\![\sigma(\phi)]\!] = \sigma^{>}[\tau[\![\phi]\!]]$. Let L be an $\mathfrak{s}$-logic and M a $\mathfrak{t}$-logic. When we call τ a **translation from L to M**, denoted $\tau : \mathsf{L} \multimap \mathsf{M}$, we mean that τ is a translation from $\mathbf{lg}(\mathsf{L})$ to $\mathbf{lg}(\mathsf{M})$, which implicitly implies that $\mathbf{lg}(\mathsf{M}) = \mathbf{lg}(\mathsf{L})^{>}$. □

Convention 17.10 Conventionally, logics shall inherit the notions of closed systems with regards to translations, that is, we may write L instead of $\mathsf{Th}(\mathsf{L})$ in notations introduced in $\oint$10. In particular, for a translation τ from L to M, we may write $\tau^{\star}(\cdot)$, which is the function from $\mathfrak{P}(\mathsf{Fm}(\mathsf{L}))$ into $\mathsf{Th}(\mathsf{L})$ defined by $\tau^{\star}(\Gamma) = \|\tau[\Gamma]\|_{\mathsf{M}}$, and may write $\tau^{\blacktriangleleft}(\cdot)$, which is the function from $\mathfrak{P}(\mathsf{Fm}(\mathsf{M}))$ into $\mathfrak{P}(\mathsf{Fm}(\mathsf{M}))$ defined by $\tau^{\blacktriangleleft}(\Phi) = \overleftarrow{\tau}\lfloor\Phi\rfloor$ (see Definition 5.1 on page 175).

We now introduce a means of identifying two translations between the same logics as equivalent modulo these logics.

Definition 17.11 (Logically Equivalent Translations) Let $\tau_1 : \mathsf{L} \multimap \mathsf{M}$ and $\tau_2 : \mathsf{L} \multimap \mathsf{M}$. We call τ_1 and τ_2 **logically equivalent** if ${\tau_1}^{\star} = {\tau_2}^{\star}$. □

We now aim to isolate a special family of translations determined by a set of formulae of a 'target' global language, beginning first with a technical construction and a few observations regarding this construction.

Definition 17.12 (The Interpretations $\mathsf{e}_{\phi}^{\mathfrak{s},\mathbf{A}}$) Assume that $\mathfrak{s}$ has a global language $\mathbf{G}$. For each $\mathfrak{s}$-language $\mathbf{A}$ and $\phi \in \mathsf{Fm}(\mathbf{A})$, let $\mathsf{e}_{\phi}^{\mathfrak{s},\mathbf{A}}$ denote the unique interpretation of $\mathbf{G}$ into $\mathbf{A}$ mapping all $\mathbf{G}$-variables to ϕ. We shall write $\mathsf{e}_{\phi}^{\mathbf{A}}$ for $\mathsf{e}_{\phi}^{\mathfrak{s},\mathbf{A}}$ and e_{ϕ} for $\mathsf{e}_{\phi}^{\mathbf{G}}$, wherever unambiguous. Note that any use of the notation $\mathsf{e}_{\phi}^{\mathfrak{s},\mathbf{A}}$ shall conventionally implicitly imply the assumption of this definition. □

Lemma 17.13 The following are all valid.

1. $\mathsf{e}_{\sigma(\phi)}^{\mathbf{A}} = \sigma\mathsf{e}_{\phi}^{\mathbf{A}}$, for all $\sigma : \mathbf{A} \to_{\mathfrak{s}} \mathbf{A}$ and $\phi \in \mathsf{Fm}(\mathbf{A})$.

2. $\mathsf{e}_{\mathsf{i}(\phi)}^{\mathbf{A}} = \mathsf{i}\mathsf{e}_{\phi}^{\mathbf{G}}$, for all $\mathsf{i} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$ and $\phi \in \mathsf{Fm}(\mathbf{G})$.

3. $\mathsf{e}_{\phi}^{\mathbf{A}} = \mathsf{e}_{\phi}^{\mathbf{A}}\mathsf{e}_{x}^{\mathbf{G}}$ and consequently $(\mathsf{e}_{\phi}^{\mathbf{A}})^{>} = (\mathsf{e}_{\phi}^{\mathbf{A}})^{>}(\mathsf{e}_{x}^{\mathbf{G}})^{>}$, for all $\mathfrak{s}$-languages $\mathbf{A}$, $\phi \in \mathsf{Fm}(\mathbf{A})$ and $x \in \mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$.

Proof. $\boxed{(1)}$ Since $\sigma\mathsf{e}_{\phi}^{\mathbf{A}} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$ and $\mathsf{e}_{\sigma(\phi)}^{\mathbf{A}} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$, it suffices, by $\mathfrak{s}$-freedom of $\mathbf{G}$, to show that these morphisms agree on all variables. For any $x \in \mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$, $\mathsf{e}_{\sigma(\phi)}^{\mathbf{A}}(x) = \sigma(\phi) = \sigma(\mathsf{e}_{\phi}^{\mathbf{A}}(x)) = \sigma\mathsf{e}_{\phi}^{\mathbf{A}}(x)$. $\boxed{(2)}$ Since $\mathsf{i}\mathsf{e}_{\phi}^{\mathbf{G}} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$ and $\mathsf{e}_{\mathsf{i}(\phi)}^{\mathbf{A}} : \mathbf{G} \to_{\mathfrak{s}} \mathbf{A}$, it suffices, by $\mathfrak{s}$-freedom of $\mathbf{G}$, to show that these morphisms agree on all variables. For any $x \in \mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$, $\mathsf{e}_{\mathsf{i}(\phi)}^{\mathbf{A}}(x) = \mathsf{i}(\phi) = \mathsf{i}(\mathsf{e}_{\phi}^{\mathbf{G}}(x)) = \mathsf{i}\mathsf{e}_{\phi}^{\mathbf{G}}(x)$. $\boxed{(3)}$ By (2) and definition, $\mathsf{e}_{\phi}^{\mathbf{A}}\mathsf{e}_{x}^{\mathbf{G}} = \mathsf{e}_{\mathsf{e}_{\phi}^{\mathbf{A}}(x)}^{\mathbf{A}} = \mathsf{e}_{\phi}^{\mathbf{A}}$, and hence, since $\cdot^{>}$ is a functor, $(\mathsf{e}_{\phi}^{\mathbf{A}})^{>} = (\mathsf{e}_{\phi}^{\mathbf{A}}\mathsf{e}_{x}^{\mathbf{G}})^{>} = (\mathsf{e}_{\phi}^{\mathbf{A}})^{>}(\mathsf{e}_{x}^{\mathbf{G}})^{>}$. ⋄

Definition 17.14 (Formal Translation) Assume that $\mathfrak{s}$ has a global language $\mathbf{G}$, that $\mathfrak{t}$ has a global language $\mathbf{H} = \mathbf{G}^{>}$. A $\cdot^{>}$**-formal translation** (or just **formal translation** where unambiguous) $\boldsymbol{\tau}$ from *signature* $\mathfrak{s}$ to *signature* $\mathfrak{t}$, is a set of $\mathbf{H}$-formulae. With each $\cdot^{>}$*-formal* translation $\boldsymbol{\tau}$ to $\mathbf{H}$ and each $\mathfrak{s}$-language $\mathbf{A}$, define a binary relationship $\boldsymbol{\tau}^{\mathbf{A}}_{>}$ from $\mathbf{A}$ to $\mathbf{A}^{>}$ by $\boldsymbol{\tau}^{\mathbf{A}}_{>}[\![\phi]\!] = \mathsf{e}^{\mathbf{A}}_{\phi}{}^{>}[\boldsymbol{\tau}]$, for each $\phi \in \mathsf{Fm}(\mathbf{A})$, which we call the **translation induced by $\boldsymbol{\tau}$ and $\mathbf{A}$**. Whenever we mention formal translations in a *general* context, we shall assume that $\mathbf{G}$ and $\mathbf{H}$ are as assumed in the previous definition, *even if these logics are denoted by different symbols*. Note that the latter notation unambiguously describes both signatures, since these signatures are encoded in the functor $\cdot^{>}$. We tend to drop this subscript functor from this notion wherever unambiguous. A formal translation is called *finitary* if it is a finite set. We denote arbitrary formal translations with emboldened versions of symbols denoting arbitrary translations. Let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be two $\cdot^{>}$-formal translations. We say that $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are **syntactically $\cdot^{>}$-equivalent**, or just **syntactically equivalent** when unambiguous, if $\boldsymbol{\tau}_1^{\mathbf{G}} = \boldsymbol{\tau}_2^{\mathbf{G}}$. □

In the next result, we show that formal translations induce *natural* translations between familiar languages.

Proposition 17.15 If $\boldsymbol{\tau}$ is a $\cdot^{>}$-formal translation from $\mathfrak{s}$ to $\mathfrak{t}$, then, for each $\mathfrak{s}$-language $\mathbf{A}$, $\boldsymbol{\tau}^{\mathbf{A}}$ is a $\cdot^{>}$-*natural* translation from $\mathbf{A}$ to $\mathbf{A}^{>}$.

Proof. We need to establish naturality. Let σ be an $\mathfrak{s}$-substitution of $\mathbf{A}$ and $\phi \in \mathsf{Fm}(\mathbf{A})$. By (1) of Lemma 17.13, $\boldsymbol{\tau}^{\mathbf{A}}[\![\sigma(\phi)]\!] = \mathsf{e}^{\mathbf{A}}_{\sigma(\phi)}{}^{>}[\boldsymbol{\tau}] = (\sigma\mathsf{e}^{\mathbf{A}}_{\phi})^{>}[\boldsymbol{\tau}] = \sigma^{>}\left[\mathsf{e}^{\mathbf{A}}_{\phi}{}^{>}[\boldsymbol{\tau}]\right] = \sigma^{>}\left[\boldsymbol{\tau}^{\mathbf{A}}[\![\phi]\!]\right]$. ◇

Proposition 17.16 If $\cdot^{>}$-formal translations $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are syntactically equivalent, then $\boldsymbol{\tau}_1^{\mathbf{A}} = \boldsymbol{\tau}_2^{\mathbf{A}}$, for every $\mathfrak{s}$-language $\mathbf{A}$.

Proof. Let $\mathbf{A}$ be an $\mathfrak{s}$-language and $\phi \in \mathsf{Fm}(\mathbf{A})$. Consider any variable $x \in \mathsf{Var}_{\mathfrak{s}}(\mathbf{G})$. Then by (3) of Lemma 17.13 and the assumption that $\boldsymbol{\tau}_1^{\mathbf{G}} = \boldsymbol{\tau}_2^{\mathbf{G}}$, $\boldsymbol{\tau}_1^{\mathbf{A}}[\![\phi]\!] = \mathsf{e}^{\mathbf{A}}_{\phi}{}^{>}[\boldsymbol{\tau}_1] = \left((\mathsf{e}^{\mathbf{A}}_{\phi})^{>}(\mathsf{e}^{\mathbf{G}}_{x})^{>}\right)[\boldsymbol{\tau}_1] = (\mathsf{e}^{\mathbf{A}}_{\phi})^{>}\left[(\mathsf{e}^{\mathbf{G}}_{x})^{>}[\![\boldsymbol{\tau}_1]\!]\right] = (\mathsf{e}^{\mathbf{A}}_{\phi})^{>}\left[\boldsymbol{\tau}_1^{\mathbf{G}}[\![x]\!]\right] = (\mathsf{e}^{\mathbf{A}}_{\phi})^{>}\left[\boldsymbol{\tau}_2^{\mathbf{G}}[\![x]\!]\right] = (\mathsf{e}^{\mathbf{A}}_{\phi})^{>}\left[(\mathsf{e}^{\mathbf{G}}_{x})^{>}[\![\boldsymbol{\tau}_2]\!]\right] = \left((\mathsf{e}^{\mathbf{A}}_{\phi})^{>}(\mathsf{e}^{\mathbf{G}}_{x})^{>}\right)[\boldsymbol{\tau}_2] = \mathsf{e}^{\mathbf{A}}_{\phi}{}^{>}[\boldsymbol{\tau}_2] = \boldsymbol{\tau}_2^{\mathbf{A}}[\![\phi]\!]$. ◇

Recall the definition of a formal $\langle n, m\rangle$-translation given in Definition 2.95 on page 108. In the following example we show that *formal $\langle n, m\rangle$-translations* coincide with *formal translation* (as defined above), when 'evaluated' from $\underrightarrow{\mathbf{Tm}}_{[n]}$. As a consequence, the theory developed in this chapter specializes to and, in the case of ∮17.3, enriches, the Block and Pigozzi theory of equivalent sentential calculi (see ∮2.5).

Example 17.17 (Formal $\langle n, m\rangle$-Translations)

Let $\mathfrak{a}$ be a type of algebras and n and m non-zero naturals. By Example 17.7, $\underrightarrow{\mathfrak{a}}_{[n]}$ and $\underrightarrow{\mathfrak{a}}_{[m]}$ are isomorphic; let $\cdot^{>}$ and $\cdot^{<}$ denote the associated isomorphic functors. Let $\boldsymbol{\tau}$ be a formal $\langle n, m\rangle$-translation. In other words, $\boldsymbol{\tau}$ is a finite set of m-tuples of terms in n-variables. So $\boldsymbol{\tau}$ is a finite set of $\underrightarrow{\mathfrak{a}}_{[m]}$-formulae and hence $\boldsymbol{\tau}$ is a finitary $\cdot^{>}$-formal translation from $\underrightarrow{\mathfrak{a}}_{[n]}$ to $\underrightarrow{\mathfrak{a}}_{[m]}$. Conversely, suppose that $\boldsymbol{\pi}$ is a finitary $\cdot^{>}$-formal translation from $\underrightarrow{\mathfrak{a}}_{[n]}$ to $\underrightarrow{\mathfrak{a}}_{[m]}$. Let ρ be the $\mathbf{Tm}$-endomorphism mapping $\mathsf{v}_{ni+j} \mapsto \mathsf{v}_j$, for $i = 0, 1, \ldots$ and $1 \leq j \leq n$. Let

$$\boldsymbol{\pi}' = \{\langle \rho(q_1), \ldots, \rho(q_m)\rangle \;:\; \langle q_1, \ldots, q_m\rangle \in \boldsymbol{\pi}\},$$

which is a formal $\langle n,m\rangle$-translation (and a finitary $\cdot^{>}$-formal translation from $\underset{\rightarrow[n]}{\mathfrak{a}}$ to $\underset{\rightarrow[m]}{\mathfrak{a}}$).

Proposition 17.18 $\boldsymbol{\tau}_{>}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]} = \boldsymbol{\tau}^{\mathbf{Tm}}$, $\boldsymbol{\pi}_{>}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]} = \boldsymbol{\pi}'^{\mathbf{Tm}}$ and $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'^{\mathbf{Tm}}$ are syntactically equivalent.

Proof. By renaming the variables, we may view $\boldsymbol{\tau}$ as a finite set of $\underset{\rightarrow[m]}{\mathfrak{a}}$-formulae in the *single* $\underset{\rightarrow[n]}{\mathfrak{a}}$-variable $\langle \mathsf{v}_1,\ldots,\mathsf{v}_n\rangle$. Suppose that

$$\boldsymbol{\tau} = \{\langle q_1^k(\mathsf{v}_1,\ldots,\mathsf{v}_n),\ldots,q_m^k(\mathsf{v}_1,\ldots,\mathsf{v}_n)\rangle \ : \ 1\leq k\leq l\}.$$

Consider an arbitrary $\underset{\rightarrow[n]}{\mathfrak{a}}$-formula $\langle p_1,\ldots,p_n\rangle$. Let σ be the $\mathbf{Tm}$-endomorphism mapping $\mathsf{v}_{i+j}\mapsto p_j$, for $i=0,1,\ldots$ and $1\leq j\leq n$. Then

$$\begin{aligned}
\boldsymbol{\tau}_{>}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]}[\![\langle p_1,\ldots,p_n\rangle]\!] &= \mathsf{e}_{\langle p_1,\ldots,p_n\rangle}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]\ >}[\boldsymbol{\tau}] = \underset{\rightarrow[m]}{\sigma}[\boldsymbol{\tau}]\\
&= \{\langle q_1^k(p_1,\ldots,p_n),\ldots,q_m^k(p_1,\ldots,p_n)\rangle \ : \ 1\leq k\leq l\}\\
&= \{\langle q_1(p_1,\ldots,p_n),\ldots,q_m(p_1,\ldots,p_n)\rangle \ : \ \langle q_1,\ldots,q_m\rangle\in\boldsymbol{\tau}\}\\
&= \boldsymbol{\tau}^{\mathbf{Tm}}[\![\langle p_1,\ldots,p_n\rangle]\!],
\end{aligned}$$

and so $\boldsymbol{\tau}_{>}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]} = \boldsymbol{\tau}^{\mathbf{Tm}}$.

Since $\boldsymbol{\pi}$ is finitary, we may assume that

$$\begin{aligned}
\boldsymbol{\pi} = \{&\langle q_1^k(\mathsf{v}_1,\ldots,\mathsf{v}_n,\mathsf{v}_{n+1},\ldots,\mathsf{v}_{2n},\ldots,\mathsf{v}_{ln+1},\ldots,\mathsf{v}_{(l+1)n}),\\
&\ldots,q_m^k(\mathsf{v}_1,\ldots,\mathsf{v}_n,\mathsf{v}_{n+1},\ldots,\mathsf{v}_{2n},\ldots,\mathsf{v}_{ln+1},\ldots,\mathsf{v}_{(l+1)n})\rangle \ : \ 1\leq k\leq l\}.
\end{aligned}$$

Then

$$\begin{aligned}
\boldsymbol{\pi}_{>}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]}[\![\langle p_1,\ldots,p_n\rangle]\!] &= \mathsf{e}_{\langle p_1,\ldots,p_n\rangle}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]\ >}[\boldsymbol{\pi}] = \underset{\rightarrow[m]}{\sigma}[\boldsymbol{\pi}]\\
&= \begin{array}{l}\{\langle q_1^k(p_1,\ldots,p_n,p_1,\ldots,p_n,\ldots,p_1,\ldots,p_n),\\ \quad\ldots,q_m^k(p_1,\ldots,p_n,p_1,\ldots,p_n,\ldots,p_1,\ldots,p_n)\rangle \ : \ 1\leq k\leq l\}\end{array}\\
&= \{\langle \rho(q_1)(p_1,\ldots,p_n),\ldots,\rho(q_m)(p_1,\ldots,p_n)\rangle \ : \ \langle q_1,\ldots,q_m\rangle\in\boldsymbol{\pi}\}\\
&= \boldsymbol{\pi}'^{\mathbf{Tm}}[\![\langle p_1,\ldots,p_n\rangle]\!],
\end{aligned}$$

and so $\boldsymbol{\pi}_{>}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]} = \boldsymbol{\pi}'^{\mathbf{Tm}}$. Finally,

$$\begin{aligned}
\boldsymbol{\pi}_{>}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]}[\![\langle p_1,\ldots,p_n\rangle]\!] &= \begin{array}{l}\{\langle q_1^k(p_1,\ldots,p_n,p_1,\ldots,p_n,\ldots,p_1,\ldots,p_n),\\ \quad\ldots,q_m^k(p_1,\ldots,p_n,p_1,\ldots,p_n,\ldots,p_1,\ldots,p_n)\rangle \ : \ 1\leq k\leq l\}\end{array}\\
&= \mathsf{e}_{\langle p_1,\ldots,p_n\rangle}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]\ >}[\{\langle \rho(q_1),\ldots,\rho(q_m)\rangle \ : \ \langle q_1,\ldots,q_m\rangle\in\boldsymbol{\pi}\}]\\
&= \mathsf{e}_{\langle p_1,\ldots,p_n\rangle}^{\overset{\mathbf{Tm}}{\longrightarrow}[n]\ >}[\boldsymbol{\pi}'] = \boldsymbol{\pi}'^{\overset{\mathbf{Tm}}{\longrightarrow}[n]}_{>}[\![\langle p_1,\ldots,p_n\rangle]\!],
\end{aligned}$$

and so $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'^{\mathbf{Tm}}$ are syntactically equivalent. ◇

□

Recall the notion that a quasivariety $\mathcal{K}$ be a $\langle X,z\rangle$-algebraic semantics of a sentential 1-calculus $\mathcal{S}$, given in Definition 13.1 on page 392, and recall in particular the notion of a binary system of equations $\mathfrak{B}$ being $\langle X,z\rangle$-defining equations for $\mathcal{S}$ and $\mathcal{K}$. Towards our aim of realizing the theory of parameterized algebraization from within the context of logics over constructs, in the following example we show how (the evaluation of) $\langle X, z\rangle$-defining equations for $\mathcal{S}$ and $\mathcal{K}$ coincide with formal translations (formal with respect to suitably defined signatures in the spirit of Example 16.43 on page 455).

Example 17.19

Let $\mathcal{S}$ be a sentential 1-calculus of type $\mathfrak{a}$ and z a variable. As in Example 16.43 on page 455, let $\mathfrak{s}$ be the signature consisting of the single language $\mathbf{Tm}$ together with all endomorphisms of $\mathbf{Tm}$ that fix z. Recall that the $\mathfrak{s}$-variables of $\mathbf{Tm}$ are all the variables other than z. By Example 17.6, the signatures $\mathfrak{s}$ and $\underrightarrow{\mathfrak{s}}_{[2]}$. are isomorphic. Let $\cdot^{>}$ and $\cdot^{<}$ be the functors described in the aforementioned example.

Let $\mathfrak{B}(x, z)$ be a binary system of $\mathfrak{a}$-equations. Certainly $\mathfrak{B}$ is a finitary $\cdot^{>}$-formal translation from $\mathfrak{s}$ to $\underrightarrow{\mathfrak{s}}_{[2]}$. Conversely, suppose that $\boldsymbol{\tau}$ is a finitary $\cdot^{>}$-formal translation from $\mathfrak{s}$ to $\underrightarrow{\mathfrak{s}}_{[2]}$, i.e., $\boldsymbol{\tau}$ is a finite set of pairs of terms. Consider any variable x distinct from z. Let ρ_x be the $\mathbf{Tm}$-endomorphism fixing z and mapping all other variables to x. Let ${}_x\boldsymbol{\tau} = \{\langle \rho_x(\delta), \rho_x(\epsilon)\rangle \;:\; \langle \delta, \epsilon\rangle \in \boldsymbol{\tau}\}$, which is a binary system of equations (and a finitary $\cdot^{>}$-formal translation from $\mathfrak{s}$ to $\underrightarrow{\mathfrak{s}}_{[2]}$).

Proposition 17.20 $\mathfrak{B}_z = \mathfrak{B}_{>}^{\mathbf{Tm}}$, $\boldsymbol{\tau}_{>}^{\mathbf{Tm}} = ({}_x\boldsymbol{\tau})_z$ and $\boldsymbol{\tau}$ and ${}_x\boldsymbol{\tau}$ are syntactically equivalent.

Proof. Let $p \in \mathsf{Tm}$ and let σ be the unique $\mathfrak{s}$-substitution mapping all $\mathfrak{s}$-variables to p. In other words, σ is the $\mathfrak{a}$-substitution of $\mathbf{Tm}$ fixing z and mapping all other variables to p. Then

$$\begin{aligned}
\mathfrak{B}_z[\![p]\!] &= \{\langle \delta(p, z), \epsilon(p, z)\rangle \;:\; \langle \delta(x, z), \epsilon(x, z)\rangle \in \mathfrak{B}\} \\
&= \{\langle \delta(\sigma(x), \sigma(z)), \epsilon(\sigma(x), \sigma(z))\rangle \;:\; \langle \delta(x, z), \epsilon(x, z)\rangle \in \mathfrak{B}\} \\
&= \{\langle \sigma(\delta(x, z)), \sigma(\epsilon(x, z))\rangle \;:\; \langle \delta(x, z), \epsilon(x, z)\rangle \in \mathfrak{B}\} \\
&= \{\underrightarrow{\sigma}(\langle \delta(x, z), \epsilon(x, z)\rangle) \;:\; \langle \delta(x, z), \epsilon(x, z)\rangle \in \mathfrak{B}\} \\
&= \underrightarrow{\sigma}[\{\langle \delta(x, z), \epsilon(x, z)\rangle \;:\; \langle \delta(x, z), \epsilon(x, z)\rangle \in \mathfrak{B}\}] \\
&= \underrightarrow{\sigma}[\mathfrak{B}] = \sigma^{>}[\mathfrak{B}] = \mathsf{e}_p{}^{>}[\mathfrak{B}] = \mathfrak{B}_{>}^{\mathbf{Tm}}[\![p]\!],
\end{aligned}$$

and so $\mathfrak{B}_z = \mathfrak{B}_{>}^{\mathbf{Tm}}$. Since $\boldsymbol{\tau}$ is finitary, we may assume that $\boldsymbol{\tau} = \{\langle \delta_i(x_1, \ldots, x_m, z), \epsilon_i(x_1, \ldots, x_m, z)\rangle \;:\; i \in n\}$, for some natural n and variables $x_1, \ldots, x_m$ distinct from z. Then

$$\begin{aligned}
({}_x\boldsymbol{\tau})_z[\![p]\!] &= \{\langle \delta_i(p, \ldots, p, z), \epsilon_i(p, \ldots, p, z)\rangle \;:\; i \in n\} \\
&= \{\langle \sigma(\delta_i(x_1, \ldots, x_m, z)), \sigma(\epsilon_i(x_1, \ldots, x_m, z))\rangle \;:\; i \in n\} \\
&= \{\underrightarrow{\sigma}(\langle \delta_i(x_1, \ldots, x_m, z), \epsilon_i(x_1, \ldots, x_m, z)\rangle) \;:\; i \in n\} \\
&= \underrightarrow{\sigma}[\{\langle \delta_i(x_1, \ldots, x_m, z), \epsilon_i(x_1, \ldots, x_m, z)\rangle \;:\; i \in n\}] \\
&= \underrightarrow{\sigma}[\boldsymbol{\tau}] = \sigma^{>}[\boldsymbol{\tau}] = \mathsf{e}_p{}^{>}[\boldsymbol{\tau}] = \boldsymbol{\tau}_{>}^{\mathbf{Tm}}[\![p]\!],
\end{aligned}$$

and hence $\boldsymbol{\tau}_{>}^{\mathbf{Tm}} = ({}_x\boldsymbol{\tau})_z$. Finally,

$$\begin{aligned}
\boldsymbol{\tau}_{>}^{\mathbf{Tm}}[\![p]\!] = ({}_x\boldsymbol{\tau})_z[\![p]\!] &= \underrightarrow{\sigma}[\{\langle \delta_i(x, \ldots, x, z), \epsilon_i(x, \ldots, x, z)\rangle \;:\; i \in n\}] \\
&= \underrightarrow{\sigma}[({}_x\boldsymbol{\tau})] = \sigma^{>}[({}_x\boldsymbol{\tau})] = \mathsf{e}_p{}^{>}[({}_x\boldsymbol{\tau})] = ({}_x\boldsymbol{\tau})_{>}^{\mathbf{Tm}}[\![p]\!],
\end{aligned}$$

and so $\boldsymbol{\tau}$ and ${}_x\boldsymbol{\tau}$ are syntactically equivalent. $\diamond$

We now consider translations going in the other direction. Let $\boldsymbol{\Delta}(x, y, z)$ be a finite set of ternary terms. Certainly $\boldsymbol{\Delta}(\mathsf{v}_1, \mathsf{v}_2, z)$ is a finitary $\cdot^{<}$-formal translation from $\underrightarrow{\mathfrak{s}}_{[2]}$ to $\mathfrak{s}$.

We may assume, without loss of generality, that $\mathsf{V} = \{z, \mathsf{v}_1, \mathsf{v}_2, \ldots\}$, i.e., $z = \mathsf{v}_0$. Conversely, suppose that $\boldsymbol{\pi}$ is a finitary $\cdot^{>}$-formal translation from $\underrightarrow{\mathfrak{s}}_{[2]}$ to $\mathfrak{s}$, i.e., $\boldsymbol{\pi}$ is a finite set of terms. Note that the variables of $\underrightarrow{\mathfrak{s}}_{[2]}$ may be taken to be $\{\langle \mathsf{v}_{1+i}, \mathsf{v}_{2+i}\rangle \;:\; i = 0, 1, \ldots\}$ (see Example 6.54 on page 239).

Consider any $\underrightarrow{\mathfrak{s}}_{[2]}$-variable $\langle x, y\rangle$. Let $\rho_{\langle x,y\rangle}$ be the $\mathbf{Tm}$-endomorphism such that $\rho_{\langle x,y\rangle}(z) = z$ and, for all $i = 0, 1, \ldots$, $\rho_{\langle x,y\rangle}(\mathsf{v}_{1+i}) = x$ and $\rho_{\langle x,y\rangle}(\mathsf{v}_{2+i}) = y$. Then $\underrightarrow{\rho_{\langle x,y\rangle}}$ is a $\underrightarrow{\mathfrak{s}}_{[2]}$-endomorphism of $\mathbf{Tm}^2$. Let ${}_{\langle x,y\rangle}\boldsymbol{\pi} = \{\rho_{\langle x,y\rangle}(p) \;:\; p \in \boldsymbol{\pi}\}$, which is a finite set of ternary terms in variables x, y and z (and a finitary $\cdot^{<}$-formal translation from $\underrightarrow{\mathfrak{s}}_{[2]}$ to $\mathfrak{s}$).

The proof of the following result is similar to that of the previous proposition.

Proposition 17.21 $\boldsymbol{\Delta}(p, q, z) = \boldsymbol{\Delta}^{\mathbf{Tm}^2}_{>}[\![\langle p, q\rangle]\!]$, $\boldsymbol{\pi}^{\mathbf{Tm}^2}_{>}[\![\langle p, q\rangle]\!] = {}_{\langle x,y\rangle}\boldsymbol{\pi}(p, q, z)$ and $\boldsymbol{\pi}$ and ${}_{\langle x,y\rangle}\boldsymbol{\pi}$ are syntactically equivalent.

□

17.2.1 Logics Induced by Translations

Given a translation from a language to a logic, we shall show how a logic can be induced on the 'source' language. Generally this induced logic is not structural. To the end of establishing a necessary condition for the induced logic to be structural, we introduce a notion of logical naturality that depends only on the 'target' logic.

Definition 17.22 (Logics Induced by Translations) Let $\mathbf{A}$ be an $\mathfrak{s}$-language, M a $\mathfrak{t}$-logic with language $\mathbf{A}^{>}$ and τ a translation from $\mathbf{A}$ to $\mathbf{A}^{>}$. We say that τ is **$\langle \cdot^{>}, \mathsf{M}\rangle$-natural** if $\|\sigma^{>}[\tau[\Gamma]]\|_{\mathsf{M}} = \|\tau[\sigma[\Gamma]]\|_{\mathsf{M}}$, for all $\Gamma \subseteq \mathsf{Fm}(\mathbf{A})$. We denote the (possible non-structural) logic $L(\mathbf{A}, \tau^{\blacktriangleleft}[\mathsf{Th}(\mathsf{M})])$ by $L^{\mathbf{A}}_{\cdot>}(\mathsf{M}, \tau)$. □

By definition,

$$\mathsf{Th}(L^{\mathbf{A}}_{\cdot>}(\mathsf{M}, \tau)) = \{\tau^{\blacktriangleleft}(T) \;:\; T \in \mathsf{Th}(\mathsf{M})\} \tag{17.1}$$

and by Proposition 5.85 on page 198,

$$\Gamma \vdash_{L^{\mathbf{A}}_{\cdot>}(\mathsf{M},\tau)} \phi \text{ iff } \tau[\phi] \vdash_{\mathsf{M}} \tau[\![\phi]\!]. \tag{17.2}$$

Proposition 17.23 If τ is $\langle \cdot^{>}, \mathsf{M}\rangle$-natural then $L^{\mathbf{A}}_{\cdot>}(\mathsf{M}, \tau)$ is $\mathfrak{s}$-structural.

Proof. Suppose that $\Gamma \vdash_{L^{\mathbf{A}}_{\cdot>}(\mathsf{M},\tau)} \phi$ and let σ be an $\mathfrak{s}$-substitution of $\mathbf{A}$. By (17.2), $\tau[\phi] \vdash_{\mathsf{M}} \tau[\![\phi]\!]$, and so by assumed $\mathfrak{t}$-structurality of M, $\sigma^{>}[\tau[\phi]] \vdash_{\mathsf{M}} \sigma^{>}[\tau[\![\phi]\!]]$, i.e., $\sigma^{>}[\tau[\![\phi]\!]] \subseteq \|\sigma^{>}[\tau[\phi]]\|_{\mathsf{M}}$. Consequently, $\|\sigma^{>}[\tau[\![\phi]\!]]\|_{\mathsf{M}} \subseteq \|\sigma^{>}[\tau[\phi]]\|_{\mathsf{M}}$. Hence by assumed $\langle \cdot^{>}, \mathsf{M}\rangle$-naturality, $\|\tau[\![\sigma(\phi)]\!]\|_{\mathsf{M}} \subseteq \|\tau[\sigma[\Gamma]]\|_{\mathsf{M}}$. So $\tau[\![\sigma(\phi)]\!] \subseteq \|\tau[\sigma[\Gamma]]\|_{\mathsf{M}}$. i.e., $\tau[\sigma[\Gamma]] \vdash_{\mathsf{M}} \tau[\![\sigma(\phi)]\!]$. So by (17.2), $\sigma[\Gamma] \vdash_{L^{\mathbf{A}}_{\cdot>}(\mathsf{M},\tau)} \sigma(\phi)$. ◇

Open Problem 17.24 Can the implication of the previous result be strengthened to an equivalence.

17.3 $\cdot^{>}$-Models

Recall the definition that sentential calculus $\mathcal{S}_2$ be a formal semantics for sentential calculus $\mathcal{S}_1$, given in Definition 2.95 on page 108 [BP89a], and in particular, note the characterization of this property in terms of (amongst others) commutivity with substitutions, given in Theorem 2.96. Since the property that a quasivariety be an algebraic semantics for a sentential calculus is merely a special case of a formal semantics, this property of commutivity features significantly in the theory of algebraizable logics. Further, a natural analogue of commutivity plays an important role in the theory of quasi-equivalent and hence deductively equivalent π-institutions [Vou03]. The primary aim of this section is to characterize this property of commutivity *independently* of the other characteristics of a formal semantics.

By definition, a sentential calculus $\mathcal{S}_2$ is a formal semantics for sentential calculus $\mathcal{S}_1$ if there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that, for all $\Gamma \cup \phi \in \mathsf{Fm}(\mathcal{S}_1)$,

$$\Gamma \vdash_{\mathcal{S}_1} \phi \text{ iff } \boldsymbol{\tau}[\Gamma] \vdash_{\mathcal{S}_2} \boldsymbol{\tau}[\![\phi]\!]. \tag{17.3}$$

Recalling the definition of a *strict continuous* translation between closed systems, it follows at once that $\mathcal{S}_2$ is a formal semantics for $\mathcal{S}_1$ iff there exists a formal translation τ from $\mathcal{S}_1$ to $\mathcal{S}_2$ such that the binary relationship $\boldsymbol{\tau}^{\mathbf{Tm}}$ is a strictly continuous translation from the closed system of theories of $\mathcal{S}_1$ to the closed system of theories of $\mathcal{S}_2$. In this section, we shall show that the property of commutivity with substitutions is equivalent to the binary relationship $\boldsymbol{\tau}^{\mathbf{Tm}}$ being continuous, i.e.,

$$\Gamma \vdash_{\mathcal{S}_1} \phi \text{ implies } \boldsymbol{\tau}[\Gamma] \vdash_{\mathcal{S}_2} \boldsymbol{\tau}[\![\phi]\!]. \tag{17.4}$$

Observe the analogue between condition (17.4) and condition (2.5) defining a matrix *model* (see Definition 2.36 on page 100). Note that while many of the later results of this chapter are special cases of results from CAAL [Vou03] and generalizations of results from AAL [BP89a], the results from this section are new. We note that in [Vou05], a notion of a semi-interpretation was introduced which corresponds directly to the notion of model considered in this section, but that the theory developed in that paper is not developed as a weak form of interpretation (as defined in [Vou03]), but rather as a precursor to a notion of model of a π-institution as a generalization of a matrix-model of a sentential calculus; a theory more recently developed in [Vou07b]. The focus in [Vou05] is entirely different to the focus of this section, and none of the results obtained here appear in [Vou05] or [Vou07b]. This is possibly because the results we obtain are new even in AAL. Consequently, in $\oint$17.3.2, we have developed the theory of semi-interpretations between π-institutions as a weak form of interpretation in the spirit of [Vou03] rather than [Vou05].

In $\oint$17.3.1 we develop a theory of a logic being a $\cdot^{>}$-model of some other logic, and characterize this relationship in terms of commutivity with substitutions. The reader should be clear to distinguish this notion from the notion of model developed in $\oint$7; in the latter case the logics lie in the same construct and the notion of model is in the spirit of a matrix model of a sentential calculus (see $\oint$2.3.2); in the former case, the logics lie in different constructs, related by a functor $\cdot^{>}$, and the notion of model is a weak form of the notion of a formal-semantics (see $\oint$2.5) from which the notion of an algebraic-semantics is derived (see $\oint$2.6). With this constructural theory as motivation, and in particular the motivation for replacing syntactic naturality with logic naturality, we develop, in $\oint$17.3.2, the more general theory of semi-interpretations between π-institutions.

17.3.1 $\cdot^{>}$-Models

We introduce three notions of *models*, the first two notions are based on *continuity*, the other is based on *commutivity*. By relating these three notions, we shall arrive at a characterization of *commutivity as continuity*. In addition, we introduce the notion of a *formal model*.

Definition 17.25 (Models and Commuting Translations) Let L be an $\mathfrak{s}$-logic and M a familiar $\mathfrak{t}$-logic. We call M a **weak-model** (resp. **natural $\cdot^{>}$-model**) of L if there exists a translation $\tau : \mathsf{Fm}(\mathsf{L}) \multimap \mathsf{Fm}(\mathsf{M})$ (resp. $\cdot^{>}$-natural translation $\tau : \mathsf{Fm}(\mathsf{L}) \multimap \mathsf{Fm}(\mathsf{M})$) such that τ is continuous from L to M, i.e., for all $\Gamma \cup \Phi \subseteq \mathsf{Fm}(\mathsf{L})$,

$$\Gamma \vdash_{\mathsf{L}} \Phi \rightarrow \tau[\Gamma] \vdash_{\mathsf{M}} \tau[\Phi], \tag{17.5}$$

in which case we call τ a **modelling translation** (resp. **natural $\cdot^{>}$-modelling translation**).

We say that translation $\tau : \mathsf{L} \multimap \mathsf{M}$ **$\cdot^{>}$-commutes with** substitution $\sigma \in \mathsf{Sub}_{\mathfrak{s}}(\mathsf{L})$ if, for all $\phi \in \mathsf{Fm}(\mathsf{L})$, $\tau^{\star}(\sigma^{\star}(\Gamma)) = \sigma^{>\star}(\tau^{\star}(\Gamma))$, and say that function $\mathsf{f} : \mathsf{Th}(\mathsf{L}) \rightarrow \mathsf{Th}(\mathsf{M})$ **$\cdot^{>}$-commutes with** σ if $\sigma^{>\star}(\mathsf{f}(T)) = \mathsf{f}(\sigma^{\star}(T))$, for all $T \in \mathsf{Th}(\mathsf{L})$. We say that τ or f **$\cdot^{>}$-commute**, if they commute with all substitutions. We call M a **$\cdot^{>}$-model** of L if there exists a (possibly unnatural) translation $\tau : \mathsf{Fm}(\mathsf{L}) \multimap \mathsf{Fm}(\mathsf{M})$ such that τ $\cdot^{>}$-commutes, in which case we call τ a **$\cdot^{>}$-modelling translation** (this terminology is justified by (3) of Theorem 17.26 below).

Let $\mathcal{D}$ be an $\mathfrak{s}$-calculus (i.e., global and structural) with (global) language $\mathbf{G}$, and suppose that $\mathcal{E}$ is a $\mathfrak{t}$-calculus with language (global) $\mathbf{G}^{>}$. We call $\mathcal{E}$ a **formal $\cdot^{>}$-model** of $\mathcal{D}$ if there exists a *formal* translation $\boldsymbol{\tau} : \mathfrak{s} \multimap \mathfrak{t}$ such that $\mathcal{E}$ is a weak-model of $\mathcal{D}$ with modelling translation $\boldsymbol{\tau}^{\mathbf{G}}$, in which case we call $\boldsymbol{\tau}$ a **formal $\cdot^{>}$-modelling translation** from $\mathcal{D}$ to $\mathcal{E}$. Conventionally, any results pertaining to formal $\cdot^{>}$-models implicitly imply the existence of $\mathfrak{s}$-global $\mathbf{G}$ with $\mathbf{G}^{>}$ being $\mathfrak{t}$-global. $\square$

Note that the notion of a weak-model does not make use of the functor $\cdot^{>}$, while *all* of the other notions of model do. In the case of a formal $\cdot^{>}$-model, the use of the functor occurs in the definition of the $\cdot^{>}$-formal translation. While the notion of a weak-model is of little value in its own right, it serves as a useful bridging step between the theory of continuous translations developed in $\oint$5 and the logical theory of models developed in this chapter. The reader is urged to recall the various characterizations of continuity given in Theorem 5.21 on page 182, Theorem 5.40 on page 186 and Proposition 5.105 on page 204, as well as the statement of Theorem 5.108 on page 204.

We now present the main result of this section, characterizing weak-models, $\cdot^{>}$-models and formal $\cdot^{>}$-models, and locating natural $\cdot^{>}$-models as special $\cdot^{>}$-models. Note that this result does not characterize natural $\cdot^{>}$-models, except in the global case, in which case, natural $\cdot^{>}$-models, $\cdot^{>}$-models and formal $\cdot^{>}$-models, coincide, modulo logical equivalence of translations. Informally, we view statement (2) as the converse of (1), (4) as the converse of (3), and (8) as the converse of (7). Note that (1) and (2) serve to fully characterize weak-models, (3) and (4) fully characterize $\cdot^{>}$-models and (6), (7) and (8) fully characterize formal $\cdot^{>}$-models. While we do not have a counter-example showing that an analogous characterization of natural $\cdot^{>}$-models is impossible generally, it is our intuition that this is indeed the case. The fundamental problem is obtaining a *natural* translation, i.e., realizing *syntactic* commutation, from the *logically* commuting $\blacktriangledown$-homomorphism, this information being lost in the move from τ to $\tau^{\star}$, since the latter encodes only *logical* commutation and not *syntactic* commutation.

Theorem 17.26 The following statements are all valid.

1. If M is a weak-model of L with weak-modelling translation τ then τ satisfies

$$\tau^{\star}{}_{|\mathsf{Th}(\mathsf{L})} : \mathbf{Th}(\mathsf{L}) \to_{\blacktriangledown} \mathbf{Th}(\mathsf{M}). \tag{17.6}$$

2. Suppose that $\mathsf{f} : \mathbf{Th}(\mathsf{L}) \to_{\blacktriangledown} \mathbf{Th}(\mathsf{M})$. Let τ be any translation satisfying

$$\forall\,[\phi \in \mathsf{Fm}(\mathsf{L})]\ \ \|\tau[\![\phi]\!]\|_{\mathsf{M}} = \mathsf{f}(\|\{\phi\}\|_{\mathsf{L}}); \tag{17.7}$$

for example τ defined by

$$\forall\,[\phi \in \mathsf{Fm}(\mathsf{L})]\ \tau[\![\phi]\!] = \mathsf{f}(\|\{\phi\}\|_{\mathsf{L}}). \tag{17.8}$$

Then M is a weak-model of L with weak-modelling translation τ satisfying

$$\tau^{\star}{}_{|\mathsf{Th}(\mathsf{L})} = \mathsf{f}. \tag{17.9}$$

3. If M is a $\cdot^{>}$-model of L with $\cdot^{>}$-modelling translation τ, then M is a weak-model of L with weak-modelling translation τ; consequently, (17.6) is valid and $\tau^{\star}{}_{|\mathsf{Th}(\mathsf{L})}$ $\cdot^{>}$-commutes.

4. Suppose that $\mathsf{f} : \mathbf{Th}(\mathsf{L}) \to_{\blacktriangledown} \mathbf{Th}(\mathsf{M})$ such that f $\cdot^{>}$-commutes. Let τ be any (*possibly unnatural*) translation satisfying (17.7); for example τ defined by (17.8). Then M is a $\cdot^{>}$-model of L with $\cdot^{>}$-modelling translation τ satisfying (17.9).

5. If M is a natural $\cdot^{>}$-model of L with natural $\cdot^{>}$-modelling translation τ, then M is a $\cdot^{>}$-model of L with $\cdot^{>}$-modelling translation τ.

6. Suppose that $\mathcal{D}$ is a (global) $\mathfrak{s}$-calculus with (global) language $\mathbf{G}$, $\mathcal{E}$ is a (global) $\mathfrak{t}$-calculus and $\mathbf{lg}(\mathcal{E}) = \mathbf{G}^{>}$. If $\mathcal{E}$ is a $\cdot^{>}$-model of $\mathcal{D}$ with $\cdot^{>}$-modelling translation τ, then $\mathcal{E}$ is a formal $\cdot^{>}$-model of $\mathcal{D}$ with formal $\cdot^{>}$-modelling translation $\boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is any $\cdot^{>}$-formal translation satisfying

$$\|\boldsymbol{\tau}\|_{\mathcal{E}} = \|\tau[\![x]\!]\|_{\mathcal{E}}\,, \tag{17.10}$$

where x is any (fixed) $\mathbf{G}$-variable; in this case, τ and $\boldsymbol{\tau}^{\mathbf{G}}$ are logically equivalent, i.e., $\tau^{\star} = \boldsymbol{\tau}^{\mathbf{G}\star}$; one such formal translation is given by

$$\boldsymbol{\tau} = \tau[\![x]\!]. \tag{17.11}$$

7. If $\mathcal{E}$ is a formal $\cdot^{>}$-model of $\mathcal{D}$ with formal $\cdot^{>}$-modelling translation $\boldsymbol{\tau}$, then $\mathcal{E}$ is a natural $\cdot^{>}$-model of $\mathcal{D}$ with natural $\cdot^{>}$-modelling translation $\boldsymbol{\tau}^{\mathbf{G}}$; consequently $\boldsymbol{\tau}^{\mathbf{G}}$ satisfies (17.6) and $\boldsymbol{\tau}^{\mathbf{G}\star}{}_{|\mathsf{Th}(\mathcal{D})}$ $\cdot^{>}$-commutes.

8. Suppose that $\mathcal{D}$ is a (global) $\mathfrak{s}$-calculus with (global) language $\mathbf{G}$, $\mathcal{E}$ is a (global) $\mathfrak{t}$-calculus and $\mathbf{lg}(\mathcal{E}) = \mathbf{G}^{>}$. Suppose further, that $\mathsf{f} : \mathbf{Th}(\mathcal{D}) \to_{\blacktriangledown} \mathbf{Th}(\mathcal{E})$ such that f $\cdot^{>}$-commutes. Then $\mathcal{E}$ is a formal $\cdot^{>}$-model of $\mathcal{D}$ with formal $\cdot^{>}$-modelling translation $\boldsymbol{\tau}$ and $\boldsymbol{\tau}^{\mathbf{G}}_{>}$ satisfies (17.9), where $\boldsymbol{\tau}$ is any formal translation satisfying

$$\|\boldsymbol{\tau}\|_{\mathcal{E}} = \mathsf{f}\,[\|x\|_{\mathcal{D}}]\,, \tag{17.12}$$

where x is any (fixed) $\mathbf{G}$-variable; one such formal translation is given by

$$\boldsymbol{\tau} = \mathsf{f}\,[\|x\|_{\mathcal{D}}]\,. \tag{17.13}$$

Proof. $\boxed{(1)}$ Follows immediately from equivalent condition (12) of Theorem 5.40 on page 186. $\boxed{(2)}$ By Theorem 5.108 on page 204. $\boxed{(3)}$ Since the identity maps are substitutions and category functors preserve identities, $\tau^\star(\|\Gamma\|) = \tau^\star(\|\mathsf{id}\,[\Gamma]\|) = \tau^\star(\|\mathsf{id}^>\,[\Gamma]\|) = \tau^\star(\mathsf{id}^{>\star}(\Gamma)) = \mathsf{id}^\star(\tau^\star(\Gamma)) = \tau^\star(\Gamma)$, so the result follows by (5) of Theorem 5.21. $\boxed{(4)}$ By (2), τ is continuous and ${\tau^\star}_{|\mathsf{Th}(\mathsf{L})} = \mathsf{f}$. By the continuity of τ and (5) of Theorem 5.21, $\sigma^\star(\tau^\star(\Gamma)) = \sigma^\star(\tau^\star(\|\Gamma\|)) = \sigma^\star({\tau^\star}_{|\mathsf{Th}(\mathsf{L})}(\|\Gamma\|)) = {\tau^\star}_{|\mathsf{Th}(\mathsf{L})}(\sigma^{>\star}(\|\Gamma\|)) = \tau^\star(\sigma^{>\star}(\|\Gamma\|)) = \tau^\star(\sigma^{>\star}(\Gamma))$, the final equality following by (7.28) of Table 7.2 on page 274 since M is assumed to be $\mathfrak{t}$-structural. $\boxed{(5)}$ $\tau^\star(\sigma^\star(\Gamma)) = \tau^\star(\|\sigma\,[\Gamma]\|_{\mathsf{L}}) \overset{(i)}{=} \tau^\star(\sigma\,[\Gamma]) = \|\tau\,[\sigma\,[\Gamma]]\|_{\mathsf{M}} = \left\|\bigcup_{\phi\in\Gamma}\tau[\![\sigma(\phi)]\!]\right\|_{\mathsf{M}} \overset{(ii)}{=} \left\|\bigcup_{\phi\in\Gamma}\sigma^>\,[\tau[\![\phi]\!]]\right\|_{\mathsf{M}} \overset{(iii)}{=} \sigma^{>\star}\left(\left\|\bigcup_{\phi\in\Gamma}\tau[\![\phi]\!]\right\|_{\mathsf{M}}\right) = \sigma^{>\star}\left(\|\tau\,[\Gamma]\|_{\mathsf{M}}\right) = \sigma^{>\star}\left(\tau^\star(\Gamma)\right)$, where *(i)* follows by assumed continuity of τ and (5) of Theorem 5.21, *(ii)* follows by naturality, and *(iii)* follows by $\mathfrak{t}$-structurality of M and (7.28) and (7.23) of Table 7.2 on page 274. $\boxed{(6)}$ Note that by (3), τ is continuous from $\mathcal{D}$ to $\mathcal{E}$, and that by Proposition 17.15, $\boldsymbol{\tau}^{\mathbf{G}}$ is a *natural* translation. We shall repeatedly make implicit reference to equivalent condition (5) of Theorem 5.21 characterizing continuity. Observe that for $\phi \in \mathsf{Fm}(\mathbf{G})$, $\|\tau[\![\phi]\!]\|_{\mathcal{E}} = \tau^\star(\{\phi\}) \overset{(i)}{=} \tau^\star(\|\{\phi\}\|_{\mathcal{E}}) = \tau^\star(\left\|\mathsf{e}^{\mathbf{G}}_\phi[\![x]\!]\right\|_{\mathcal{E}}) = \tau^\star(\mathsf{e}^{\mathbf{G}\star}_\phi(\{x\})) \overset{(ii)}{=} \mathsf{e}^{\mathbf{G}>\star}_\phi(\tau^\star(\{x\})) = \mathsf{e}^{\mathbf{G}>\star}_\phi(\|\tau[\![x]\!]\|_{\mathcal{E}}) = \left\|\mathsf{e}^{\mathbf{G}>}_\phi\,[\|\tau[\![x]\!]\|_{\mathcal{E}}]\right\|_{\mathcal{E}} = \left\|\mathsf{e}^{\mathbf{G}>}_\phi\,[\|\boldsymbol{\tau}\|_{\mathcal{E}}]\right\|_{\mathcal{E}} \overset{(iii)}{=} \left\|\mathsf{e}^{\mathbf{G}>}_\phi\,[\boldsymbol{\tau}]\right\|_{\mathcal{E}} = \left\|\boldsymbol{\tau}^{\mathbf{G}}[\![\phi]\!]\right\|_{\mathcal{E}}$, where *(i)* follows by the continuity of τ, *(ii)* follows by assumed commutivity of τ, and *(iii)* follows by structurality and (7.28) of Table 7.2 on page 274. Hence $\left\|\boldsymbol{\tau}^{\mathbf{G}}\,[\|\Gamma\|_{\mathcal{D}}]\right\|_{\mathcal{E}} = \left\|\tau\,[\|\Gamma\|_{\mathcal{D}}]\right\|_{\mathcal{E}} \overset{\text{(i)}}{=} \|\tau\,[\Gamma]\|_{\mathcal{E}} = \left\|\boldsymbol{\tau}^{\mathbf{G}}\,[\Gamma]\right\|_{\mathcal{E}}$, where *(i)* follows by the continuity of τ. Hence $\boldsymbol{\tau}^{\mathbf{G}}$ is continuous. $\boxed{(7)}$ By definition, $\mathcal{E}$ is a weak model of $\mathcal{D}$ with weak modelling translation $\boldsymbol{\tau}^{\mathbf{G}}$, and by Proposition 17.15, $\boldsymbol{\tau}^{\mathbf{G}}$ is $\cdot^>$-natural. $\boxed{(8)}$ For each $\phi \in \mathsf{Fm}(\mathcal{D})$, $\left\|\boldsymbol{\tau}^{\mathbf{G}}_>[\![\phi]\!]\right\|_{\mathcal{E}} = \left\|(\mathsf{e}^{\mathbf{G}}_\phi)^>\,[\boldsymbol{\tau}]\right\|_{\mathcal{E}} \overset{\text{(i)}}{=} \left\|(\mathsf{e}^{\mathbf{G}}_\phi)^>\,[\|\boldsymbol{\tau}\|_{\mathcal{E}}]\right\|_{\mathcal{E}} = (\mathsf{e}^{\mathbf{G}}_\phi)^{>\star}(\|\boldsymbol{\tau}\|_{\mathcal{E}}) = (\mathsf{e}^{\mathbf{G}}_\phi)^{>\star}(\mathsf{f}(\|x\|_{\mathcal{D}})) \overset{\text{(ii)}}{=} \mathsf{f}(\mathsf{e}^{\mathbf{G}\star}_\phi(\|x\|_{\mathcal{D}})) \overset{\text{(iii)}}{=} \mathsf{f}(\mathsf{e}^{\mathbf{G}\star}_\phi(\{x\})) = \mathsf{f}(\left\|\{\mathsf{e}^{\mathbf{G}}_\phi(x)\}\right\|_{\mathcal{E}}) = \mathsf{f}(\|\{\phi\}\|_{\mathsf{L}})$, where (i) and (iii) follow by structurality and (ii) follows by $\cdot^>$-commutivity. So f and the translation $\boldsymbol{\tau}^{\mathbf{G}}_>$ satisfy the requirements of statement (2) of this theorem, and hence $\mathcal{E}$ is a weak-model of $\mathcal{D}$ with weak-modelling translation $\boldsymbol{\tau}^{\mathbf{G}}_>$ and ${\boldsymbol{\tau}^{\mathbf{G}\star}_>}_{|\mathsf{Th}(\mathcal{D})} = \mathsf{f}$, which suffices. $\diamond$

Consequently, if $\mathcal{D}$ is a (global) $\mathfrak{s}$-calculus with (global) language $\mathbf{G}$, $\mathcal{E}$ is a (global) $\mathfrak{t}$-calculus and $\mathbf{lg}(\mathcal{E}) = \mathbf{G}^>$, then the notions of formal $\cdot^>$-model, natural $\cdot^>$-model and $\cdot^>$-model coincide, modulo logical equivalence of the respective modeling translations.

Note that the previous result effectively characterizes the property of commutivity implicit in the theory of formally equivalent sentential calculi and the theory of algebraizable sentential calculi. We formalize this observation in the following result.

Corollary 17.27 A formal $\langle n, m\rangle$-translation $\boldsymbol{\tau}$ from a sentential n-calculus $\mathcal{S}_1$ to a sentential m-calculus $\mathcal{S}_2$ commutes with substitutions iff, for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}(\mathcal{S}_1)$,

$$\Gamma \vdash_{\mathcal{S}_1} \phi \text{ implies } \boldsymbol{\tau}\,[\Gamma] \vdash_{\mathcal{S}_2} \boldsymbol{\tau}[\![\phi]\!]. \tag{17.14}$$

As noted, *syntactic* naturality is too strong a condition to impose on translations generally. Syntactic naturality needs to be replaced with commutivity with substitutions, which may be viewed as a *logical* naturality modulo the two logics under translation. In the next result we demonstrate that when analysing translations between logics in the context of weak-modellability, we need only require *logical* naturality modulo the *target* logic; the source logic may be treated syntactically. Recall the notion of a translation being $\langle\cdot^>, \mathsf{M}\rangle$-natural, given in Definition 17.22.

Proposition 17.28 Let M be a weak-model of L with weak-modelling translation τ. Then M is a $\cdot^>$-model of L with $\cdot^>$-modelling translation τ iff τ is $\langle\cdot^>, \mathsf{M}\rangle$-natural.

Proof. $\boxed{\Rightarrow}$ By $\|\sigma^{>}[\tau[\Gamma]]\|_{\mathsf{M}} = \left\|\sigma^{>}\left[\|\tau[\Gamma]\|_{\mathsf{L}}\right]\right\|_{\mathsf{M}} = \left\|\tau\left[\|\sigma[\Gamma]\|_{\mathsf{L}}\right]\right\|_{\mathsf{M}} = \|\tau[\sigma[\Gamma]]\|_{\mathsf{M}}$, where the first equality follows by structurality, the second by commutivity, and the third follows by the continuity of τ.
$\boxed{\Leftarrow}$ $\left\|\sigma^{>}\left[\|\tau[\Gamma]\|_{\mathsf{L}}\right]\right\|_{\mathsf{M}} = \|\sigma^{>}[\tau[\Gamma]]\|_{\mathsf{M}} = \|\tau[\sigma[\Gamma]]\|_{\mathsf{M}} = \left\|\tau\left[\|\sigma[\Gamma]\|_{\mathsf{L}}\right]\right\|_{\mathsf{M}}$, where the first equality follows by structurality, the second by $\langle\cdot^{>},\mathsf{M}\rangle$-naturality and the third follows by the continuity of τ. $\diamond$

The importance of this observation lies in the fact that when analysing $\cdot^{>}$-modellability, we always have a *structural* logic, namely $L^{\mathbf{A}}_{\cdot^{>}}(\mathsf{M},\tau)$, against which to compare the source logic; the logic $L^{\mathbf{A}}_{\cdot^{>}}(\mathsf{M},\tau)$ plays an analogous role to that played by the logic $S^{n}(\mathcal{K},\mathfrak{N})$ in the theory of algebraizable sentential calculi (see ∮9.1.2). For example, we have the following result, which follows immediately from the previous proposition together with Proposition 5.86 on page 198.

Corollary 17.29 Let L and M be familiar logics and let τ be a $\langle\cdot^{>},\mathsf{M}\rangle$-natural translation from L to M. Then M is a $\cdot^{>}$-model of L with $\cdot^{>}$-modelling translation τ iff $\mathsf{L} \preceq L^{\mathbf{lg}(\mathsf{L})}_{\cdot^{>}}(\mathsf{M},\tau)$. $\square$

17.3.2 Semi-Interpretations between π-Institutions

With the previous results as motivation, we shall develop the theory of *semi-interpretations between π-institutions*. This theory is novel in two aspects. Firstly, we give treatment to semi-interpretations that is in the spirit of [Vou03], i.e., as a weak form of interpretation between π-institutions, rather than in the spirit of [Vou05]; in the latter paper, semi-interpretations are analyzed as a precursor to the development of a theory of model in the spirit of a *matrix model* of a sentential calculus. In this sense, the results obtained are entirely new in the field of Categorical Abstract Algebraic Logic (CAAL). Secondly, we replace (the implicit notion of) *syntactic naturality* with *logical naturality*; the benefits of this will become apparent in the next section, where we apply the same change to the notion of *interpretation*, thereby obtaining sharper results than those in [Vou03].

We begin by considering a notion of translation from one π-institution to another that is a weaker variant of the notion of translation as given in [Vou03]. Note that what we call a *natural translation* is called a *translation* in [Vou03].

Definition 17.30 (Translations between π-Institutions) Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions. A **translation of $\mathcal{I}$ in $\mathcal{J}$** is a pair $\langle\mathsf{F},\tau.\rangle : \mathcal{I} \multimap \mathcal{J}$, where $\mathsf{F} : \mathbf{Sign}_{\mathcal{I}} \to \mathbf{Sign}_{\mathcal{J}}$ (is a functor) and, for each $\mathfrak{s} \in \mathsf{Sign}_{\mathcal{I}}$, $\tau_{\mathfrak{S}}$ is a function from $\mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$ into $\mathfrak{P}(\mathsf{Sent}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S})))$. For $\langle\mathsf{F},\tau.\rangle : \mathcal{I} \multimap \mathcal{J}$, each $\tau_{\mathfrak{S}}$ may be viewed as a (concrete) translation from $\mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$ to $\mathsf{Sent}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S}))$. In keeping with the primary discourse of this text, we shall adopt binary relationship notation with respect to $\tau_{\mathfrak{S}}$.

A translation $\langle\mathsf{F},\tau.\rangle : \mathcal{I} \multimap \mathcal{J}$ is called **natural** if, for each $\sigma : \mathfrak{S} \to \mathfrak{T}$ and $\phi \in \mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$, $\overline{\mathsf{F}(\sigma)}\,[\tau_{\mathfrak{S}}[\![\phi]\!]] = \tau_{\mathfrak{T}}[\![\overline{\sigma}(\phi)]\!]$; i.e., $\tau : \mathsf{SEN}_{\mathcal{I}} \to \mathcal{P}\mathsf{SEN}_{\mathcal{J}}\mathsf{F}$ (is a natural transformation) [Vou03], and is called **logically-natural** if, for all $\sigma : \mathfrak{S} \to_{\mathsf{Sign}_{\mathcal{I}}} \mathfrak{T}$ and $\Gamma \in \mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$,

$$\left\|\overline{\mathsf{F}(\sigma)}\left[\|\tau_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \subseteq \left\|\tau_{\mathfrak{T}}\left[\|\overline{\sigma}[\Gamma]\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}. \tag{17.15}$$

Let $\langle\mathsf{F},\tau.\rangle : \mathcal{I} \multimap \mathcal{J}$ and $\langle\mathsf{F},\pi.\rangle : \mathcal{I} \multimap \mathcal{J}$. We call $\langle\mathsf{F},\tau.\rangle$ and $\langle\mathsf{F},\pi.\rangle$ **logically equivalent** if $\|\tau_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})} = \|\pi_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})}$, for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$ and $\Gamma \subseteq \mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$. We call $\langle\mathsf{F},\tau.\rangle$ **finitary** if $\tau_{\mathfrak{S}}$ is finitary for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, i.e., $\tau_{\mathfrak{S}}[\![\phi]\!]$ is finite for all $\phi \in \mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$. $\square$

Remark 17.31 Translation $\langle \mathsf{F}, \tau^{.}\rangle$ is logically-natural iff, for all $\sigma : \mathfrak{S} \rightarrow_{\mathsf{Sign}_{\mathcal{I}}} \mathfrak{T}$ and $\Gamma \in \mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$,

$$\overline{\mathsf{F}(\sigma)}\left[\|\tau_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right] \subseteq \left\|\tau_{\mathfrak{T}}\left[\|\overline{\sigma}[\Gamma]\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}.$$

Remark 17.32 A natural translation is logically-natural.

Proof. $\left\|\overline{\mathsf{F}(\sigma)}\left[\|\tau_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} = \left\|\overline{\mathsf{F}(\sigma)}[\tau_{\mathfrak{S}}[\Gamma]]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} = \|\tau_{\mathfrak{T}}[\overline{\sigma}[\Gamma]]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \subseteq \left\|\tau_{\mathfrak{T}}\left[\|\overline{\sigma}[\Gamma]\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}$. $\diamond$

Remark 17.33 $\langle \mathsf{F}, \tau^{.}\rangle$ and $\langle \mathsf{F}, \pi^{.}\rangle$ are logically equivalent iff $\|\tau_{\mathfrak{S}}[\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})} = \|\pi_{\mathfrak{S}}[\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})}$, for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$ and $\phi \in \mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$.

Proof. $\boxed{\Rightarrow}$ Trivial. $\boxed{\Leftarrow}$ $\|\tau_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})} = \left\|\bigcup_{\phi\in\Gamma}\tau_{\mathfrak{S}}[\![\phi]\!]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})} = \left\|\bigcup_{\phi\in\Gamma}\|\tau_{\mathfrak{S}}[\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})}\right\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})} = \left\|\bigcup_{\phi\in\Gamma}\|\pi_{\mathfrak{S}}[\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})}\right\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})} = \left\|\bigcup_{\phi\in\Gamma}\pi_{\mathfrak{S}}[\![\phi]\!]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})} = \|\pi_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathcal{I})}$. $\diamond$

Recall the definition of a *term* π-institution and, in particular, the definition of the substitution $\sigma_{\langle\mathfrak{S},\phi\rangle}$ (see Definition 6.44 on page 234).

Definition 17.34 (Formal Translations) Let $\mathcal{I}$ be a term π-institution with source signature - variable pair $\langle\mathfrak{A}, x\rangle$ and $\mathcal{J}$ a π-institution. A $\langle\mathfrak{A}, x\rangle$**-formal translation from** $\mathcal{I}$ **to** $\mathcal{J}$ is a pair $\langle \mathsf{F}, \boldsymbol{\tau}\rangle$ with $\mathsf{F} : \mathsf{Sign}_{\mathcal{I}} \to \mathsf{Sign}_{\mathcal{J}}$ and $\boldsymbol{\tau} \subseteq \mathsf{Sent}(\mathsf{F}(\mathfrak{A}))$. Conventionally, any mention of a $\langle\mathfrak{A}, x\rangle$-formal translation from $\mathcal{I}$ to $\mathcal{J}$ implicitly implies that $\mathcal{I}$ is a term π-institution with source signature - variable pair $\langle\mathfrak{A}, x\rangle$. With each $\langle\mathfrak{A}, x\rangle$-formal translation $\langle \mathsf{F}, \boldsymbol{\tau}\rangle$ from $\mathcal{I}$ to $\mathcal{J}$ we associate the *natural* translation $\langle \mathsf{F}, \boldsymbol{\tau}^{.}\rangle : \mathcal{I} \multimap \mathcal{J}$ defined by $\boldsymbol{\tau}_{\mathfrak{S}}[\![\phi]\!] = \overline{\mathsf{F}(\sigma_{\langle\mathfrak{S},\phi\rangle})}[\boldsymbol{\tau}]$, for each $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$ and $\phi \in \mathsf{Sent}(\mathfrak{S})$. $\square$

Proof. (*We need to establish naturality.*) Let $\rho : \mathfrak{S} \to \mathfrak{T}$. $\overline{\mathsf{F}(\rho)}[\boldsymbol{\tau}_{\mathfrak{S}}[\![\phi]\!]] = \overline{\mathsf{F}(\rho)}\left[\overline{\mathsf{F}(\sigma_{\langle\mathfrak{S},\phi\rangle})}[\![\boldsymbol{\tau}]\!]\right] = \left(\overline{\mathsf{F}(\rho)}\,\overline{\mathsf{F}(\sigma_{\langle\mathfrak{T},\phi\rangle})}\right)[\boldsymbol{\tau}] = \overline{\mathsf{F}(\rho)\mathsf{F}(\sigma_{\langle\mathfrak{T},\phi\rangle})}[\boldsymbol{\tau}] = \overline{\mathsf{F}(\rho\sigma_{\langle\mathfrak{T},\phi\rangle})}[\boldsymbol{\tau}] \overset{\text{(i)}}{=} \overline{\mathsf{F}(\sigma_{\langle\mathfrak{T},\overline{\rho}(\phi)\rangle})}[\boldsymbol{\tau}] = \boldsymbol{\tau}_{\mathfrak{T}}[\![\overline{\rho}(\phi)]\!]$, where (i) follows by (6.6). $\diamond$

In the following example, we demonstrate that translations between familiar logics over constructs may be faithfully realized as translations between the associated π-institutions. The reader is urged to recall Example 6.46 on page 234.

Example 17.35

Let L and M be familiar logics and consider the associated π-institutions $\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}$ and $\mathcal{I}^{\mathfrak{s}}_{\mathsf{M}}$. Let τ be a translation from L to M. Let F be the functor from $\mathsf{Sign}_{\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}}$ to $\mathsf{Sign}_{\mathcal{I}^{\mathfrak{t}}_{\mathsf{M}}}$ induced by the functor $\cdot^{>}$, and $\tau'_{\mathbf{lg}(\mathsf{L})} : \mathsf{Fm}(\mathbf{A}) \to \mathfrak{P}(\mathsf{Fm}(\mathbf{B}))$ mapping $\tau'_{\mathbf{lg}(\mathsf{L})}(\phi) = \tau[\![\phi]\!]$.

Remark 17.36 $\langle \mathsf{F}, \tau'^{.}\rangle$ is a translation from $\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}$ and $\mathcal{I}^{\mathfrak{s}}_{\mathsf{M}}$. $\langle \mathsf{F}, \tau'^{.}\rangle$ is natural iff τ is $\cdot^{>}$-natural. $\square$

Conversely, suppose that $\langle \cdot^{>}, \tau^{.}\rangle$ is a translation from $\mathcal{I}^{\mathfrak{s}}_{\mathsf{L}}$ to $\mathcal{I}^{\mathfrak{s}}_{\mathsf{M}}$.

Remark 17.37 $\tau_{\mathbf{lg}(\mathsf{L})}$ is a translation from L to M. $\tau_{\mathbf{lg}(\mathsf{L})}$ is $\cdot^{>}$-natural iff $\langle \cdot^{>}, \tau^{.}\rangle$ is natural. $\square$

Clearly these constructions constitute a mutually inverting one-to-one correspondence.

□

We now introduce various notions of *semi-interpretation*. The weakest, simply called *continuous*, plays the same role as a *weak-model* in the analogous theory for constructs developed in the previous sub-section. The notion of a *natural semi-interpretation* coincides with the notion called semi-interpretation in [Vou05], and generalizes the notion of a *natural model* in the analogous theory for constructs. What we call a *semi-interpretation* does not yet exist in the theory of CAAL; it generalizes our earlier notion of *model* to the setting of CAAL and is based on commutivity.

Definition 17.38 (Semi-Interpretations of π-Institutions) Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions and $\langle \mathsf{F}, \tau.\rangle$ a translation from $\mathcal{I}$ to $\mathcal{J}$. We call $\langle \mathsf{F}, \tau.\rangle$ **continuous** if, for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, $\tau_{\mathfrak{S}}$ is continuous from the closed system $\mathsf{Th}_{\mathcal{I}}(\mathfrak{S})$ to the closed system $\mathsf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S}))$, i.e., for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Sent}(\mathfrak{S})$,

$$\Gamma \vdash_{\mathfrak{S}} \phi \text{ implies } \tau_{\mathfrak{S}}\,[\Gamma] \vdash_{\mathsf{F}(\mathfrak{S})} \tau_{\mathfrak{S}} [\![\phi]\!]. \tag{17.16}$$

A **natural semi-interpretation** is a natural continuous translation [Vou05]. We call $\langle \mathsf{F}, \tau.\rangle$ a **semi-interpretation** if, for all $\sigma : \mathfrak{S} \to_{\mathsf{Sign}_{\mathcal{I}}} \mathfrak{T}$ and $\Gamma \in \mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$,

$$\left\| \overline{\mathsf{F}(\sigma)} \left[\|\tau_{\mathfrak{S}}\,[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} = \left\| \tau_{\mathfrak{T}} \left[\|\overline{\sigma}\,[\Gamma]\|^{\mathcal{I}}_{\mathfrak{T}} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}. \tag{17.17}$$

□

Remark 17.39 Every semi-interpretation is logically-natural. □

Theorem 17.40 Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions and $\langle \mathsf{F}, \tau.\rangle$ a translation from $\mathcal{I}$ to $\mathcal{J}$.

1. If $\langle \mathsf{F}, \tau.\rangle$ is a natural semi-interpretation then $\langle \mathsf{F}, \tau.\rangle$ is a semi-interpretation.

2. If $\langle \mathsf{F}, \tau.\rangle$ is a semi-interpretation then $\langle \mathsf{F}, \tau.\rangle$ is continuous.

3. If $\mathcal{I}$ is term with source signature - variable pair $\langle \mathfrak{A}, x\rangle$ and $\langle \mathsf{F}, \tau.\rangle$ is a semi-interpretation, then $\langle \mathsf{F}, \boldsymbol{\tau}.\rangle$ is a natural semi-interpretation and $\langle \mathsf{F}, \tau.\rangle$ and $\langle \mathsf{F}, \boldsymbol{\tau}.\rangle$ are logically equivalent, where $\boldsymbol{\tau} \subseteq \mathsf{Sent}(\mathsf{F}(\mathfrak{A}))$ is any $\langle \mathfrak{A}, x\rangle$-formal translation (from $\mathcal{I}$ to $\mathcal{J}$) satisfying

$$\|\boldsymbol{\tau}\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{A})} = \|\tau_{\mathfrak{A}} [\![x]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{A})}\,; \tag{17.18}$$

one such $\langle \mathfrak{A}, x\rangle$-formal translation is given by

$$\boldsymbol{\tau} = \tau_{\mathfrak{A}} [\![x]\!]; \tag{17.19}$$

in particular, $\langle \mathsf{F}, \tau.\rangle$ is logically equivalent to a *natural* semi-interpretation.

Proof.
$\boxed{(1)}$ $\left\| \tau_{\mathfrak{T}} \left[\|\overline{\sigma}\,[\Gamma]\|^{\mathcal{I}}_{\mathfrak{T}} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \stackrel{(i)}{=} \|\tau_{\mathfrak{T}}\,[\overline{\sigma}\,[\Gamma]]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} = \left\| \bigcup_{\phi \in \Gamma} \tau_{\mathfrak{T}} [\![\overline{\sigma}\,[\phi]]\!] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \stackrel{(ii)}{=} \left\| \bigcup_{\phi \in \Gamma} \overline{\mathsf{F}(\sigma)}\,[\tau_{\mathfrak{S}} [\![\phi]\!]] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \stackrel{(iii)}{=}$ $\left\| \overline{\mathsf{F}(\sigma)} \left[\left\| \bigcup_{\phi \in \Gamma} \tau_{\mathfrak{S}} [\![\phi]\!] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} = \left\| \overline{\mathsf{F}(\sigma)} \left[\|\tau_{\mathfrak{S}}\,[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}$, where *(i)* follows by assumed continuity of $\tau_{\mathfrak{T}}$, *(ii)* follows by assumed naturality and *(iii)* follows by structurality. $\boxed{(2)}$ Let $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$. (*We must show that* $\tau_{\mathfrak{S}}$ *is continuous from the closed system* $\mathsf{Th}_{\mathcal{I}}(\mathfrak{S})$ *to the closed system* $\mathsf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S}))$.) Let 1

denote the identity $\mathbf{Sign}_{\mathcal{I}}$-morphism of $\mathfrak{S}$. Since $\mathsf{SEN}_{\mathcal{I}} : \mathbf{Sign}_{\mathcal{I}} \to \mathbf{Set}$, $\overline{1} \doteq \mathsf{SEN}_{\mathcal{I}}(1)$ must be the *identity function* on $\mathsf{Sent}(\mathfrak{S}) \doteq \mathsf{SEN}_{\mathcal{I}}(\mathfrak{S})$. Since $\mathsf{F} : \mathbf{Sign}_{\mathcal{I}} \to \mathbf{Sign}_{\mathcal{J}}$, $\mathsf{F}(1)$ must be the identity $\mathbf{Sign}_{\mathcal{I}}$-morphism of $\mathsf{F}(\mathfrak{S})$, and, as argued previously, $\overline{\mathsf{F}(1)} \doteq \mathsf{SEN}_{\mathcal{J}}(\mathsf{F}(1))$ must be the *identity function* on $\mathsf{Sent}(\mathsf{F}(\mathfrak{S})) \doteq \mathsf{SEN}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S}))$. Hence, $\left\| \tau_{\mathfrak{S}} \left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \left\| \tau_{\mathfrak{S}} \left[\| \overline{1} \left[\Gamma\right] \|^{\mathcal{I}}_{\mathfrak{S}} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \overset{(i)}{=} \left\| \overline{\mathsf{F}(1)} \left[\|\tau_{\mathfrak{S}} \left[\Gamma\right]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \left\| \|\tau_{\mathfrak{S}} \left[\Gamma\right]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \|\tau_{\mathfrak{S}} \left[\Gamma\right]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}$, so the result follows by (5) of Theorem 5.21. $\boxed{(3)}$ Since $\langle \mathsf{F}, \boldsymbol{\tau}. \rangle$ is natural, it suffices, by (1), to show that $\langle \mathsf{F}, \boldsymbol{\tau}. \rangle$ is continuous. Let $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$. For $\phi \in \mathsf{Sent}(\mathfrak{S})$, $\|\boldsymbol{\tau}_{\mathfrak{S}} [\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \left\| \overline{\mathsf{F}(\sigma_{\langle \mathfrak{S}, \phi\rangle})} \left[\boldsymbol{\tau}\right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \overset{\text{(i)}}{=} \left\| \overline{\mathsf{F}(\sigma_{\langle \mathfrak{S}, \phi\rangle})} \left[\|\boldsymbol{\tau}\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{A})} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \left\| \overline{\mathsf{F}(\sigma_{\langle \mathfrak{S}, \phi\rangle})} \left[\|\tau_{\mathfrak{A}} [\![x]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{A})} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \overset{\text{(ii)}}{=} \left\| \tau_{\mathfrak{S}} \left[\| \{ \overline{\sigma_{\langle \mathfrak{S}, \phi\rangle}}(x)\} \|^{\mathcal{I}}_{\mathfrak{S}} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \overset{(6.5)}{=} \left\| \tau_{\mathfrak{S}} \left[\|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \overset{\text{(iii)}}{=} \|\tau_{\mathfrak{S}} [\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}$, where (i) follows by structurality, (ii) by commutivity and (iii) by continuity of $\tau_{\mathfrak{S}}$. Hence, and by continuity of $\tau_{\mathfrak{S}}$, for $\Gamma \subseteq \mathsf{Sent}(\mathfrak{S})$, $\left\| \boldsymbol{\tau}_{\mathfrak{S}} \left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \left\| \tau_{\mathfrak{S}} \left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \|\tau_{\mathfrak{S}} \left[\Gamma\right]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \|\boldsymbol{\tau}_{\mathfrak{S}} \left[\Gamma\right]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}$. Consequently, $\boldsymbol{\tau}_{\mathfrak{S}}$ is continuous. So $\langle \mathsf{F}, \boldsymbol{\tau}_{\mathfrak{S}} \rangle$ is a natural semi-interpretation and hence, by (1), a semi-interpretation. Logical equivalence follows by Remark 17.33 and the already established fact that $\|\boldsymbol{\tau}_{\mathfrak{S}} [\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \|\tau_{\mathfrak{S}} [\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}$. ◇

We now aim to characterize the existence of a semi-interpretation from $\mathcal{I}$ to $\mathcal{J}$ in terms of the existence of a functor from $\mathbf{TH}(\mathcal{I})$ to $\mathbf{TH}(\mathcal{J})$.

Definition 17.41 (Signature Respecting Theory Functors) [Vou03] Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions. We call $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ **signature-respecting** if there exists $\mathsf{F}^\flat : \mathbf{Sign}_{\mathcal{I}} \to \mathbf{Sign}_{\mathcal{J}}$ such that $\mathsf{SIG}_{\mathcal{J}}\mathsf{F} = \mathsf{F}^\flat\mathsf{SIG}_{\mathcal{I}}$, in which case the functor $\mathsf{F}^\flat$ is unique [Vou03]. □

The proof of the following characterization of signature-respecting $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ contains a construction of the unique functor $\mathsf{F}^\flat$.

Proposition 17.42 If $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ then, F is signature-respecting iff

1. $\mathsf{SIG}(\mathsf{F}(\langle \mathfrak{S}, T\rangle)) = \mathsf{SIG}(\mathsf{F}(\langle \mathfrak{S}, R\rangle))$, for all $\langle \mathfrak{S}, T\rangle, \langle \mathfrak{S}, R\rangle \in \mathsf{Th}(\mathcal{I})$, and
2. $\mathsf{SIG}(\sigma) = \mathsf{SIG}(\rho)$ implies $\mathsf{SIG}(\mathsf{F}(\sigma)) = \mathsf{SIG}(\mathsf{F}(\rho))$.

Proof. $\boxed{\Rightarrow}$ Trivial. $\boxed{\Leftarrow}$ Define $\mathsf{F}^\flat(\mathfrak{S}) = \mathsf{SIG}(\mathsf{F}(\langle \mathfrak{S}, \mathsf{Sent}(\mathfrak{S})\rangle))$, for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$. With each $\sigma : \mathfrak{S} \to_{\mathbf{Sign}_{\mathcal{I}}} \mathcal{J}$, we associate the morphism $\sigma^* : \langle \mathfrak{S}, \mathsf{Sent}(\mathfrak{S})\rangle \to_{\mathbf{TH}(\mathcal{I})} \langle \mathfrak{T}, \mathsf{Sent}(\mathfrak{T})\rangle$ with $\mathsf{SIG}(\sigma^*) = \sigma$ (this morphism must exist). Define $\mathsf{F}^\flat(\sigma) = \mathsf{SIG}(\mathsf{F}(\sigma^*))$. Since $\mathsf{F}(\sigma^*) : \mathsf{F}(\langle \mathfrak{S}, \mathsf{Sent}(\mathfrak{S})\rangle) \to_{\mathbf{TH}(\mathcal{J})} \mathsf{F}(\langle \mathfrak{T}, \mathsf{Sent}(\mathfrak{T})\rangle)$, $\mathsf{SIG}(\mathsf{F}(\sigma^*)) : \mathsf{SIG}(\mathsf{F}(\langle \mathfrak{S}, \mathsf{Sent}(\mathfrak{S})\rangle)) \to_{\mathcal{J}} \mathsf{SIG}(\mathsf{F}(\langle \mathfrak{T}, \mathsf{Sent}(\mathfrak{T})\rangle))$, i.e., $\mathsf{F}^\flat(\sigma) : \mathsf{F}^\flat(\mathfrak{S}) \to_{\mathbf{Sign}_{\mathcal{J}}} \mathsf{F}^\flat(\mathfrak{T})$. It is easily shown that $\mathsf{F}^\flat : \mathbf{Sign}_{\mathcal{I}} \to \mathbf{Sign}_{\mathcal{J}}$. For all $\langle \mathfrak{S}, T\rangle \in \mathsf{Th}(\mathcal{I})$, $\mathsf{F}^\flat(\mathsf{SIG}(\langle \mathfrak{S}, T\rangle)) = \mathsf{F}^\flat(\mathfrak{S}) = \mathsf{SIG}(\mathsf{F}(\langle \mathfrak{S}, \mathsf{Sent}(\mathfrak{S})\rangle)) = \mathsf{SIG}(\mathsf{F}(\langle \mathfrak{S}, T\rangle))$, where the final equality follows by assumption (1). For all $\sigma : \langle \mathfrak{S}, T\rangle \to \langle \mathfrak{T}, R\rangle$, $\mathsf{F}^\flat(\mathsf{SIG}(\sigma)) = \mathsf{SIG}(\mathsf{F}((\mathsf{SIG}(\sigma))^*)) = \mathsf{SIG}(\mathsf{F}(\sigma))$, where the final equality follows by assumption (2) and the fact that $\mathsf{SIG}(\sigma) = \mathsf{SIG}(\mathsf{SIG}(\sigma))^*)$. Hence F is signature preserving. ◇

Remark 17.43 If $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ is signature-respecting, then

$$\mathsf{SIG}(\mathsf{F}(\langle \mathfrak{S}, T\rangle)) = \mathsf{F}^\flat(\mathfrak{S}). \tag{17.20}$$

□

In the following example, we show that there exist $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ for which condition (1) of Proposition 17.42 fails. That is, there can exist two theories $\langle \mathfrak{S}, T\rangle$ and $\langle \mathfrak{S}, R\rangle$ with the *same* signature, but which map under F to theories with *differing* signatures.

Example 17.44

Let $\mathcal{I}$ be the π-institution where $\mathbf{Sign}_{\mathcal{I}}$ is a category with a single object $\mathfrak{S}$ and only an identity morphism, such that $\mathsf{Sent}(\mathfrak{S}) = \{a\}$ and $\mathsf{Th}(\mathcal{I}) = \{\langle \mathfrak{S}, \emptyset\rangle, \langle \mathfrak{S}, \{a\}\rangle\}$. Let $\mathcal{J}$ be the π-institution where $\mathbf{Sign}_{\mathcal{J}}$ is a category with two objects $\mathfrak{T}_1$ and $\mathfrak{T}_2$, with the only morphisms being the identity morphisms on each object, such that $\mathsf{Sent}(\mathfrak{T}_1) = \{b_1\}$, $\mathsf{Sent}(\mathfrak{T}_2) = \{b_2\}$, $\mathsf{Th}(\mathcal{J}) = \{\langle \mathfrak{T}_1, \{b_1\}\rangle, \langle \mathfrak{T}_2, \{b_2\}\rangle\}$. Define $\mathsf{F}(\langle \mathfrak{S}, \emptyset\rangle) = \langle \mathfrak{T}_1, \{b_1\}\rangle$ and $\mathsf{F}(\langle \mathfrak{S}, \{a\}\rangle) = \langle \mathfrak{T}_2, \{b_2\}\rangle$. The only $\mathbf{TH}(\mathfrak{S})$-morphisms are the identity morphisms $\mathsf{id}_{\langle \mathfrak{S}, \emptyset\rangle}$ and $\mathsf{id}_{\langle \mathfrak{S}, \{a\}\rangle}$. Define $\mathsf{F}(\mathsf{id}_{\langle \mathfrak{S}, \emptyset\rangle})$ to be the identity morphism on $\langle \mathfrak{T}_1, \{b_1\}\rangle$ and $\mathsf{F}(\mathsf{id}_{\langle \mathfrak{S}, \{a\}\rangle})$ to be the identity morphism on $\langle \mathfrak{T}_2, \{b_2\}\rangle$. Certainly, $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$.

□

Given that there exist such functors, there is a problem in the definition of monotonicity and join-preservation in Definition 5.1 of [Vou03], which are defined without an assumption of signature-preservation. While such a definition of monotonicity may make set-theoretic sense, it clearly makes no semantic sense. In the case of join-preservation, the definition does not make set-theoretic sense, since the union may combine sentences from different signatures, and so the closure cannot be defined. These problems disappear if signature-preservation is assumed. Fortunately, the theory developed in [Vou03] all takes place in a signature-preserving context, and as such no problems occur as a result of this omission. We rectify this omission in the following definition. We introduce an additional property, which we call *logical-naturality*, that is weaker than the property of commutivity with substitutions and which is the theory-functor analogue of logically-natural interpretations.

Definition 17.45 (Monotonic, Join-Respecting and Commuting Functors) Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions and $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ be a *signature respecting* functor. We call F **monotonic** (**strongly monotonic**) if, for all $\langle \mathfrak{S}, T\rangle, \langle \mathfrak{S}, R\rangle \in \mathsf{Th}(\mathcal{I})$,

$$T \subseteq R \text{ implies } \text{ (iff) } \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, T\rangle\big)\Big) \subseteq \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, R\rangle\big)\Big) \tag{17.21}$$

join-respecting if, for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$ and $\Gamma \subseteq \mathsf{Sent}(\mathfrak{S})$,

$$\mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\rangle\big)\Big) = \left\| \bigcup_{\phi \in \Gamma} \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}}\rangle\big)\Big) \right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{S})}, \tag{17.22}$$

and say that F is **logically-natural** (resp. **commutes with substitutions** [Vou03]) if, for all $\sigma : \mathfrak{S} \to_{\mathsf{Sign}_{\mathcal{I}}} \mathfrak{T}$ and $\langle \mathfrak{S}, T\rangle \in \mathsf{Th}(\mathcal{I})$, $\mathsf{THS}_{\mathcal{J}}\big(\mathsf{F}^\flat(\sigma)\big)\Big(\mathsf{F}\big(\langle \mathfrak{S}, T\rangle\big)\Big) \subseteq$ (resp. $=$) $\mathsf{F}\Big(\mathsf{THS}_{\mathcal{I}}(\sigma)\big(\langle \mathfrak{S}, T\rangle\big)\Big)$, i.e.,

$$\left\| \overline{\mathsf{F}^\flat(\sigma)} \left[\mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, T\rangle\big)\Big)\right] \right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{T})} \subseteq \text{ (resp. } =) \ \mathsf{th}\Big(\mathsf{F}\Big(\Big\langle \mathfrak{T}, \|\overline{\sigma}\,[T]\|^{\mathcal{I}}_{\mathfrak{T}}\Big\rangle\Big)\Big). \tag{17.23}$$

For each $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, let $\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}} : \mathsf{Th}_{\mathcal{I}}(\mathfrak{S}) \to \mathsf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S})^\flat)$ defined by $\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}(T) = \mathsf{th}(\mathsf{F}(\langle \mathfrak{S}, T\rangle))$, which is a well-defined function by signature-preservation. □

Lemma 17.46 Let $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ be a *signature respecting* functor.

1. F is (strongly) monotonic iff, for each $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, $\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}$ is (strictly) $\subseteq$-order preserving from $\mathbf{Th}_{\mathcal{I}}(\mathfrak{S})$ into $\mathbf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S})^{\flat})$.

2. F is join-preserving iff, for each $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, $\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}} : \mathbf{Th}_{\mathcal{I}}(\mathfrak{S}) \to_{\blacktriangledown} \mathbf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S})^{\flat})$; in which case F is monotonic.

Proof. $\boxed{(1)}$ Trivial due to assumption of signature-preservation. $\boxed{(2)}$ $\boxed{\Rightarrow}$ For $\mathcal{T} \subseteq \mathsf{Th}_{\mathcal{I}}(\mathfrak{S})$,

$$\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}(\blacktriangledown \mathcal{T}) = \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}\left(\left\| \bigcup \mathcal{T} \right\|^{\mathcal{I}}_{\mathfrak{S}}\right) = \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \left\| \bigcup \mathcal{T} \right\|^{\mathcal{I}}_{\mathfrak{S}} \rangle\big)\Big) \overset{(17.22)}{=} \left\| \bigcup_{\phi \in \bigcup \mathcal{T}} \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}} \rangle\big)\Big) \right\|^{\mathcal{J}}_{\mathsf{F}^{\flat}(\mathfrak{S})}$$

$$= \left\| \bigcup_{T \in \mathcal{T}} \bigcup_{\phi \in T} \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}} \rangle\big)\Big) \right\|^{\mathcal{J}}_{\mathsf{F}^{\flat}(\mathfrak{S})} = \left\| \bigcup_{T \in \mathcal{T}} \left\| \bigcup_{\phi \in \Gamma} \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}} \rangle\big)\Big) \right\|^{\mathcal{J}}_{\mathsf{F}^{\flat}(\mathfrak{S})} \right\|^{\mathcal{J}}_{\mathsf{F}^{\flat}(\mathfrak{S})}$$

$$\overset{(17.22)}{=} \left\| \bigcup_{T \in \mathcal{T}} \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \|T\|^{\mathcal{I}}_{\mathfrak{S}} \rangle\big)\Big) \right\|^{\mathcal{J}}_{\mathsf{F}^{\flat}(\mathfrak{S})} = \left\| \bigcup_{T \in \mathcal{T}} \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, T \rangle\big)\Big) \right\|^{\mathcal{J}}_{\mathsf{F}^{\flat}(\mathfrak{S})}$$

$$= \left\| \bigcup_{T \in \mathcal{T}} \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}(T) \right\|^{\mathcal{J}}_{\mathsf{F}^{\flat}(\mathfrak{S})} = \blacktriangledown \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}[\mathcal{T}].$$

$\boxed{\Leftarrow}$

$$\mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}} \rangle\big)\Big) = \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}(\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}) = \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}\left(\left\| \bigcup_{\phi \in \Gamma} \{\phi\} \right\|^{\mathcal{I}}_{\mathfrak{S}}\right) = \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}\left(\left\| \bigcup_{\phi \in \Gamma} \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}} \right\|^{\mathcal{I}}_{\mathfrak{S}}\right)$$

$$= \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}\left(\blacktriangledown_{\phi \in \Gamma} \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}}\right) = \blacktriangledown_{\phi \in \Gamma} \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}(\|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}}) = \left\| \bigcup_{\phi \in \Gamma} \mathsf{th}\Big(\mathsf{F}\big(\langle \mathfrak{S}, \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}} \rangle\big)\Big) \right\|^{\mathcal{J}}_{\mathsf{F}^{\flat}(\mathfrak{S})}.$$

$\diamond$

In [Vou03], a functor $\mathsf{I}^{\sharp} : \mathsf{TH}(\mathcal{I}) \to \mathsf{TH}(\mathcal{J})$ is associated with each *natural* translation $\mathsf{I} = \langle \mathsf{F}, \tau. \rangle : \mathcal{I} \multimap \mathcal{J}$ (just called a translation in that text). In that definition, the property of naturality plays an important role in ensuring that the images of theory-morphisms are indeed theory-morphisms. Since we work with translations that are not necessarily natural, we have isolated the weakest condition on a translation under which this definition remains valid. This turns out to be logical-naturality, in particular equivalent condition (3) of the following characterization of logical-naturality.

Lemma 17.47 For $\langle \mathsf{F}, \tau. \rangle : \mathcal{I} \multimap \mathcal{J}$, the following conditions are equivalent.

1. $\langle \mathsf{F}, \tau. \rangle$ is logically-natural.

2. For all $\sigma : \langle \mathfrak{S}, T \rangle \to_{\mathsf{TH}(\mathcal{I})} \langle \mathfrak{T}, R \rangle$,

$$\left\| \overline{\mathsf{F}(\sigma)}\left[\|\tau_{\mathfrak{S}}[T]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \right] \right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \subseteq \|\tau_{\mathfrak{T}}[R]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}. \tag{17.24}$$

3. For all $\sigma : \langle \mathfrak{S}, T \rangle \to_{\mathsf{TH}(\mathcal{I})} \langle \mathfrak{T}, R \rangle$,

$$\overline{\mathsf{F}(\sigma)}\left[\|\tau_{\mathfrak{S}}[T]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \right] \subseteq \|\tau_{\mathfrak{T}}[R]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}. \tag{17.25}$$

Proof. $\boxed{(1)\Rightarrow(2)}$ If $\sigma : \langle\mathfrak{S}, T\rangle \rightarrow_{\mathsf{TH}(\mathcal{I})} \langle\mathfrak{T}, R\rangle$, then by definition, $\overline{\sigma}\,[T] \subseteq R$, and hence $\left\|\overline{\mathsf{F}(\sigma)}\left[\|\tau_{\mathfrak{S}}\,[T]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \overset{\text{(i)}}{=} \left\|\overline{\mathsf{F}(\sigma)}\,[\tau_{\mathfrak{S}}\,[T]]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \overset{(17.15)}{\subseteq} \|\tau_{\mathfrak{T}}\,[\overline{\sigma}\,[T]]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \subseteq \|\tau_{\mathfrak{T}}\,[R]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}$, where (i) follows by structurality. $\boxed{(2)\Rightarrow(1)}$ Let $\sigma : \mathfrak{S} \rightarrow_{\mathsf{Sign}_{\mathcal{I}}} \mathfrak{T}$ and $\Gamma \in \mathsf{Sent}_{\mathcal{I}}(\mathfrak{S})$. Since $\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right] \subseteq \left\|\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{I}}_{\mathfrak{T}}$, there exists $\sigma' : \left\langle\mathfrak{S}, \|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right\rangle : \rightarrow_{\mathsf{TH}(\mathcal{I})}\left\langle\mathfrak{T}, \left\|\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{I}}_{\mathfrak{T}}\right\rangle$, with $\mathsf{SIG}(\sigma') = \sigma$. Hence $\left\|\overline{\mathsf{F}(\sigma)}\left[\|\tau_{\mathfrak{S}}\,[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \subseteq \left\|\overline{\mathsf{F}(\sigma)}\left[\left\|\tau_{\mathfrak{S}}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} = \left\|\overline{\mathsf{F}(\sigma')}\left[\left\|\tau_{\mathfrak{S}}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \overset{(17.24)}{\subseteq} \left\|\tau_{\mathfrak{T}}\left[\left\|\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} = \left\|\tau_{\mathfrak{T}}\left[\|\overline{\sigma}\,[\Gamma]\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}$, where the final equality follows by structurality. $\boxed{(2)\Leftrightarrow(3)}$ Trivial. $\diamond$

The following definition comes from [Vou03] with the apriori assumption of *naturality* on the translation replaced with *logical-naturality*.

Definition 17.48 ($\mathsf{I}^\sharp$) With each *logically-natural* translation $\mathsf{I} = \langle\mathsf{F}, \tau.\rangle : \mathcal{I} \multimap \mathcal{J}$, we associate the *signature-respecting* functor $\mathsf{I}^\sharp : \mathsf{TH}(\mathcal{I}) \to \mathsf{TH}(\mathcal{J})$, with $\mathsf{I}^{\sharp^\flat} = \mathsf{F}$, defined by

$$\mathsf{I}^\sharp\big(\langle\mathfrak{S}, T\rangle\big) = \big\langle\mathsf{F}(\mathfrak{S}), \|\tau_{\mathfrak{S}}\,[T]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\big\rangle, \tag{17.26}$$

and for all $\sigma : \langle\mathfrak{S}, T\rangle \rightarrow_{\mathsf{TH}(\mathcal{I})} \langle\mathfrak{T}, R\rangle$, $\mathsf{I}^\sharp(\sigma) : \mathsf{I}^\sharp\big(\langle\mathfrak{S}, T\rangle\big) \rightarrow_{\mathsf{TH}(\mathcal{I})} \mathsf{I}^\sharp\big(\langle\mathfrak{T}, R\rangle\big)$ is the unique theory-morphism such that $\mathsf{SIG}\left(\mathsf{I}^\sharp(\sigma)\right) = \mathsf{F}\,(\sigma)$ (recall that conventionally we may write σ for $\mathsf{SIG}_{\mathcal{I}}(\sigma)$ and so conventionally $\mathsf{F}\,(\sigma) = \mathsf{F}\,(\mathsf{SIG}_{\mathcal{I}}(\sigma))$). $\square$

Proof. Since SIG is faithful [Vou03], the definition of $\mathsf{I}^\sharp(\sigma)$ is well-defined, provided it can be shown that for all $\sigma : \langle\mathfrak{S}, T\rangle \rightarrow_{\mathsf{TH}(\mathcal{I})} \langle\mathfrak{T}, R\rangle$, $\mathsf{I}^\sharp(\sigma) : \big\langle\mathsf{F}(\mathfrak{S}), \|\tau_{\mathfrak{S}}\,[T]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\big\rangle \rightarrow \big\langle\mathsf{F}(\mathfrak{T}), \|\tau_{\mathfrak{T}}\,[T]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}\big\rangle$, i.e., $\overline{\mathsf{I}^\sharp(\sigma)}\left[\|\tau_{\mathfrak{S}}\,[T]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right] \subseteq \|\tau_{\mathfrak{T}}\,[R]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})}$, which is precisely (17.25). That $\mathsf{I}^\sharp$ is a signature-respecting functor with with $\mathsf{I}^{\sharp^\flat} = \mathsf{F}$, follows precisely as in page 298 of [Vou03]; the proof makes no use of logical-naturality (nor naturality). $\diamond$

We now provide the characterization of semi-interpretations in terms of special theory-category functors. Recall that natural translations and semi-interpretations are both logically-natural, by Remark 17.32 and Remark 17.39, respectively.

Theorem 17.49 Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions.

1. If $\mathsf{F} : \mathsf{TH}(\mathcal{I}) \to \mathsf{TH}(\mathcal{J})$ is a signature-respecting and join-respecting functor that commutes with substitutions, then any translation $\langle\mathsf{F}^\flat, \tau.\rangle : \mathcal{I} \multimap \mathcal{J}$ satisfying

$$\forall\,[\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}, \phi \in \mathsf{Sent}(\mathfrak{S})] \quad \|\tau_{\mathfrak{S}}[\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{S})} = \mathsf{th}\bigg(\mathsf{F}\left(\langle\mathfrak{S}, \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}}\rangle\right)\bigg) \tag{17.27}$$

is a semi-interpretation satisfying

$$\forall\,[\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}, T \in \mathsf{Th}_{\mathcal{I}}(\mathfrak{S})] \quad \tau_{\mathfrak{S}}\,[T] = \mathsf{th}\bigg(\mathsf{F}\,(\langle\mathfrak{S}, T\rangle)\bigg); \tag{17.28}$$

all such translations are logically-equivalent; one such translation is defined by

$$\forall\,[\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}, \phi \in \mathsf{Sent}(\mathfrak{S})] \quad \tau_{\mathfrak{S}}[\![\phi]\!] = \mathsf{th}\bigg(\mathsf{F}\left(\langle\mathfrak{S}, \|\{\phi\}\|^{\mathcal{I}}_{\mathfrak{S}}\rangle\right)\bigg). \tag{17.29}$$

2. If $\mathsf{I} : \mathcal{I} \multimap \mathcal{J}$ is a semi-interpretation, then $\mathsf{I}^\sharp : \mathsf{TH}(\mathcal{I}) \to \mathsf{TH}(\mathcal{J})$ is a signature-respecting and join-respecting functor that commutes with substitutions.

3. Suppose that $\mathcal{I}$ is a term π-institution with source signature - variable pair $\langle \mathfrak{A}, x\rangle$. If $\mathsf{F} : \mathsf{TH}(\mathcal{I}) \to \mathsf{TH}(\mathcal{J})$ is a signature-respecting and join-respecting functor that commutes with substitutions, then, for any $\langle \mathfrak{A}, x\rangle$-formal translation $\langle \mathsf{F}^\flat, \boldsymbol{\tau}\rangle$ (from $\mathcal{I}$ to $\mathcal{J}$) satisfying

$$\|\boldsymbol{\tau}\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{A})} = \mathsf{th}\Big(\mathsf{F}\left(\langle \mathfrak{A}, \|\{x\}\|^{\mathcal{I}}_{\mathfrak{A}}\rangle\right)\Big), \tag{17.30}$$

$\langle \mathsf{F}^\flat, \boldsymbol{\tau}.\rangle$ is a *natural* semi-interpretation from $\mathcal{I}$ to $\mathcal{J}$; all such $\langle \mathfrak{A}, x\rangle$-formal translations result in logically-equivalent $\langle \mathsf{F}^\flat, \boldsymbol{\tau}.\rangle$; one such $\langle \mathfrak{A}, x\rangle$-formal translation is defined by

$$\boldsymbol{\tau} = \mathsf{th}\Big(\mathsf{F}\left(\langle \mathfrak{A}, \|\{x\}\|^{\mathcal{I}}_{\mathfrak{A}}\rangle\right)\Big). \tag{17.31}$$

Proof. $\boxed{(1)}$ By (2) of Lemma 17.46 and Theorem 5.108 on page 204, $\langle \mathsf{F}^\flat, \tau.\rangle$ is continuous and satisfies (17.28). Hence,

$$\begin{aligned}
\left\|\overline{\mathsf{F}^\flat(\sigma)}\left[\|\tau_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{T})} &\overset{\text{(i)}}{=} \left\|\overline{\mathsf{F}^\flat(\sigma)}\left[\left\|\tau_{\mathfrak{S}}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{T})}\\
&\overset{(17.28)}{=} \left\|\overline{\mathsf{F}^\flat(\sigma)}\left[\left\|\mathsf{th}\Big(\mathsf{F}\left(\langle \mathfrak{S}, \|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\rangle\right)\Big)\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{T})}\\
&= \left\|\overline{\mathsf{F}^\flat(\sigma)}\left[\mathsf{th}\Big(\mathsf{F}\left(\langle \mathfrak{S}, \|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\rangle\right)\Big)\right]\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{T})}\\
&\overset{(17.23)}{=} \mathsf{th}\left(\mathsf{F}\left(\left\langle \mathfrak{T}, \left\|\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{I}}_{\mathfrak{T}}\right\rangle\right)\right)\\
&\overset{(17.28)}{=} \left\|\mathsf{th}\left(\mathsf{F}\left(\left\langle \mathfrak{T}, \left\|\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{I}}_{\mathfrak{T}}\right\rangle\right)\right)\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{T})}\\
&= \left\|\tau_{\mathfrak{T}}\left[\left\|\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{T})} \overset{\text{(ii)}}{=} \left\|\tau_{\mathfrak{T}}\left[\|\overline{\sigma}[\Gamma]\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}^\flat(\mathfrak{T})},
\end{aligned}$$

where (i) and (ii) follow by structurality. Logical equivalence follows by Remark 17.33 and definition. $\boxed{(2)}$ Suppose that $\mathsf{I} = \langle \mathsf{F}, \tau.\rangle$. Since semi-interpretations are logically-natural, $\mathsf{I}^\sharp$ is well-defined and signature-respecting, with $\mathsf{I}^{\sharp^\flat} = \mathsf{F}$. Since semi-translations are continuous (by (2) of Theorem 17.40), $\mathsf{I}^\sharp$ is join-respecting (by definition, (2) of Lemma 17.46 and equivalent condition (12) of Theorem 5.40 on page 186). Finally, $\left\|\overline{\mathsf{I}^{\sharp^\flat}(\sigma)}\left[\mathsf{th}\Big(\mathsf{I}^\sharp(\langle \mathfrak{S}, \|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\rangle)\Big)\right]\right\|^{\mathcal{J}}_{\mathsf{I}^{\sharp^\flat}(\mathfrak{T})} = \left\|\overline{\mathsf{F}(\sigma)}\left[\mathsf{th}\Big(\mathsf{I}^\sharp(\langle \mathfrak{S}, \|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\rangle)\Big)\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \overset{(17.26)}{=} \left\|\overline{\mathsf{F}(\sigma)}\left[\left\|\tau_{\mathfrak{S}}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \overset{(17.17)}{=} \left\|\tau_{\mathfrak{T}}\left[\|\overline{\sigma}[\Gamma]\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \overset{\text{(i)}}{=} \left\|\tau_{\mathfrak{T}}\left[\left\|\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{I}}_{\mathfrak{T}}\right]\right\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{T})} \overset{(17.26)}{=} \mathsf{th}\left(\mathsf{I}^\sharp\left(\left\langle \mathfrak{T}, \left\|\overline{\sigma}\left[\|\Gamma\|^{\mathcal{I}}_{\mathfrak{S}}\right]\right\|^{\mathcal{I}}_{\mathfrak{T}}\right\rangle\right)\right)$, where (i) follows by structurality. $\boxed{(3)}$ Follows from (1) together with (3) of Theorem 17.40. $\diamond$

17.4 $\cdot^{>}$-Semantics

In this section, we consider the property that a $\mathfrak{t}$-calculus be a $\cdot^{>}$-*semantics* for an $\mathfrak{s}$-calculus, which is a stronger notion than a $\cdot^{>}$-*model*, generalizing the notion that a sentential calculus be a *formal semantics* for another (see Definition 2.95 on page 108) and hence generalizing the notion of an *algebraic semantics* in the sense of [BP89a]. We shall use this theory to explain our theory of *parameterized semantics*.

With the previous section as motivation, we shall develop the theory of *interpretations* between π-institutions first, since this specializes to the theory $\cdot^{>}$-*semantics* of calculi over constructs. Note that the notion of an interpretation between π-institutions was first introduced in [Vou03]. Because of the implicit *syntactic naturality* in the definition of a translation in that text, a full *Blok-Pigozzi theorem*, i.e., in this case, a theorem relating *interpretations* and signature-respecting, strongly monotonic and join-respecting *theory-functors* that commute with substitutions, is only obtained under the assumption that the 'source' π-institution be term; while generally interpretations give rise to such theory-functors, the converse in not generally true. It is clear to us, from our independently developed theory of $\cdot^{>}$-semantics for logics over constructs, that this problem stems from the fact that the *syntactic naturality*, *implicit* in the definition of a translation (in that text), is lost in the move from an interpretation to a theory-functor; a syntactically natural translation cannot be recovered from the theory-functor, except in the case of term (and multi-term [GF05], see below) π-institutions. What is not lost, however, is *logical naturality*. Recall from the previous section, that we have broken with the nomenclature of [Vou03], in that what we call a translation differs from their notion with the same name, in that we have removed the requirement, that we term *syntactic naturality*, from the definition; our *natural translation* coincides with their *translation*; consequently what we call a *natural interpretation* coincides with their *interpretation*. In this section we shall show that the aforementioned Blok-Pigozzi theorem obtains for all π-institutions with respect to our notion of interpretation.

We shall now briefly motivate the significance of this result. Interpretability is a precursor to deductive equivalence, where *interpretablity* in CAAL is to *formal semantics* in AAL (and hence algebraic semantics) as *deductive equivalence* is to *formal equivalence* (and hence equivalent algebraic semantics). As is the case in AAL, deductive equivalence in CAAL is developed from (and strengthens) interpretation; effectively deductive equivalence requires interpretations in both direction and in addition a requirement of 'mutual untranslation', i.e., applying the interpretation in one direction and then applying the reverse interpretation to the result, must yield a result that is *logically equivalent* to the starting point. Due to the role of *two* interpretations, the term requirement is compounded; the appropriate Blok-Pigozzi theorem only obtains when *both* π-institutions are term. This is indeed unfortunate, since it is clear from [Vou98] and [Vou02] that the reason the theory of deductive equivalence was developed in the first place was as a tool to classify certain π-institutions as *algebraic* [Vou02]; a π-institution is called algebraizable if it is deductively equivalent to an $\langle \mathbf{L}, \boldsymbol{\Xi}, \mathbf{Q} \rangle$-algebraic π-institution [Vou02]. Unfortunately, the latter class of π-institutions are generally not *term*, in particular, the equational π-institution $\mathcal{EQ}$ is not term [Vou03]. For reasons of time constraints and the need to gauge the community's response to our idea of replacing syntactic naturality with logical naturality, we have not developed our theory all the way to deductive equivalence between π-institutions; we do, however, develop such a theory of *equivalent* $\langle \cdot^{>}, \cdot^{<} \rangle$-*semantics* for logics over constructs (see $\oint$17.5), leaving the establishment of the analogous Blok-Pigozzi theorem for logical natural deductive equivalences between arbitrary π-institutions as an open problem.

We must note that José Gil-Férez claims to have extended Voutsadakis' result from term

π-institutions to *multi-term* π-institutions, although only an abstract claiming this result exists [GF05]. Observe that the term requirement on both π-institutions has been generalized to the requirement that both π-institutions be multi-term; it is our understanding that just as term π-institutions are like free objects with at least one variable, multi-term π-institutions are like free objects with multiple and possible infinite variables. We are certain that unless syntactic naturality is replaced with logical naturality, no general result will be obtained, without some sort of global variable-like condition. While currently global logics are most studied by logicians, non-global logics arise ubiquitously; we encountered numerous non-global logics in the earlier parts of this text, to name just one such example (of personal interest to us), the logic $U(\mathbf{A}, \mathsf{cos}^{\mathcal{K}})$ of relative cosets on an arbitrary algebra. A major source on non-global logics is the logics of individual computer systems, in fact more generally, the logics of any system (a motor vehicle engine for example) with no term that generically represents all other terms of the system; this phrasing is to be taken as informal.

17.4.1 Interpretations between π-Institutions

As with semi-interpretations, we introduce three variants of interpretations. The first is simply called *strictly continuous*, and is based purely on the notion of a strictly continuous translation between closed systems. The second, called a *natural interpretation*, coincides with the notion called interpretation in [Vou03]. The third notion, simply called *interpretation*, which is the notion that we are advocating as a 'better translation' is simply a $\vdash$-strict semi-interpretation; which is strictly continuous by (2) of Theorem 17.40.

Definition 17.50 (Interpretations of π-Institutions) Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions and $\langle \mathsf{F}, \tau.\rangle$ a translation from $\mathcal{I}$ to $\mathcal{J}$. We call $\langle \mathsf{F}, \tau.\rangle$ **strictly continuous** (resp. a **natural interpretation** [Vou03], **interpretation**) if $\langle \mathsf{F}, \tau.\rangle$ is continuous (resp. a natural semi-interpretation, semi-interpretation) and for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, $\tau_{\mathfrak{S}}$ is $\vdash$-reflecting from the closed system $\mathsf{Th}_{\mathcal{I}}(\mathfrak{S})$ to the closed system $\mathsf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S}))$, i.e., for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Sent}(\mathfrak{S})$,

$$\tau_{\mathfrak{S}}[\Gamma] \vdash_{\mathsf{F}(\mathfrak{S})} \tau_{\mathfrak{S}}[\![\phi]\!] \text{ implies } \Gamma \vdash_{\mathfrak{S}} \phi. \tag{17.32}$$

$\square$

Remark 17.51 $\langle \mathsf{F}, \tau.\rangle$ is strictly continuous from $\mathcal{I}$ to $\mathcal{J}$ iff, for all $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, $\tau_{\mathfrak{S}}$ is strictly continuous from the closed system $\mathsf{Th}_{\mathcal{I}}(\mathfrak{S})$ to the closed system $\mathsf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S}))$, i.e., for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Sent}(\mathfrak{S})$,

$$\Gamma \vdash_{\mathfrak{S}} \phi \text{ iff } \tau_{\mathfrak{S}}[\Gamma] \vdash_{\mathsf{F}(\mathfrak{S})} \tau_{\mathfrak{S}}[\![\phi]\!]. \tag{17.33}$$

We show that natural interpretations are interpretations and interpretations are strictly continuous. Further, interpretations between term π-institutions are logically equivalent to natural interpretations, which can be realized as formal translations.

Theorem 17.52 Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions and $\langle \mathsf{F}, \tau.\rangle$ a translation from $\mathcal{I}$ to $\mathcal{J}$.

1. If $\langle \mathsf{F}, \tau.\rangle$ is a natural interpretation then $\langle \mathsf{F}, \tau.\rangle$ is an interpretation.

2. If $\langle \mathsf{F}, \tau.\rangle$ is an interpretation then $\langle \mathsf{F}, \tau.\rangle$ is strictly continuous.

3. If $\mathcal{I}$ is term with source signature - variable pair $\langle\mathfrak{A}, x\rangle$ and $\langle\mathsf{F}, \tau.\rangle$ is an interpretation, then $\langle\mathsf{F}, \boldsymbol{\tau}.\rangle$ is a natural interpretation and $\langle\mathsf{F}, \tau.\rangle$ and $\langle\mathsf{F}, \boldsymbol{\tau}.\rangle$ are logically equivalent, where $\boldsymbol{\tau} \subseteq \mathsf{Sent}(\mathsf{F}(\mathfrak{A}))$ is any $\langle\mathfrak{A}, x\rangle$-formal translation (from $\mathcal{I}$ to $\mathcal{J}$) satisfying (17.18); one such $\langle\mathfrak{A}, x\rangle$-formal translation is given by (17.19); in particular, $\langle\mathsf{F}, \tau.\rangle$ is logically equivalent to a *natural* semi-interpretation.

Proof. $\boxed{\text{(1) and (2)}}$ The proof follows from (1) and (2) of Theorem 17.40, together with the previous remark. $\boxed{\text{(3)}}$ By (3) of Theorem 17.52, it suffices to show that $\boldsymbol{\tau}.$ satisfies (17.32). Suppose that $\boldsymbol{\tau}_{\mathfrak{S}}[\Gamma] \vdash_{\mathsf{F}(\mathfrak{S})} \boldsymbol{\tau}_{\mathfrak{S}}[\![\phi]\!]$. Then $\|\boldsymbol{\tau}_{\mathfrak{S}}[\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \subseteq \|\boldsymbol{\tau}_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}$. Since $\langle\mathsf{F}, \tau.\rangle$ and $\langle\mathsf{F}, \boldsymbol{\tau}.\rangle$ are logically equivalent by (3) of Theorem 17.52, we have $\|\tau_{\mathfrak{S}}[\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \|\boldsymbol{\tau}_{\mathfrak{S}}[\![\phi]\!]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} \subseteq \|\boldsymbol{\tau}_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})} = \|\tau_{\mathfrak{S}}[\Gamma]\|^{\mathcal{J}}_{\mathsf{F}(\mathfrak{S})}$. Hence $\tau_{\mathfrak{S}}[\Gamma] \vdash_{\mathsf{F}(\mathfrak{S})} \tau_{\mathfrak{S}}[\![\phi]\!]$. So by assumption and (2), $\Gamma \vdash_{\mathfrak{S}} \phi$. $\diamond$

The following observation follows immediately from Lemma 17.46, together with Remark 1.174 on page 40 and Remark 1.190 on page 42.

Remark 17.53 If $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ is a signature-respecting functor, then F is strongly monotonic and join-respecting iff for each $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, $\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}} : \mathbf{Th}_{\mathcal{I}}(\mathfrak{S}) \cong \mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}[\mathbf{Th}_{\mathcal{I}}(\mathfrak{S})] \lhd_{\blacktriangledown} \mathbf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S})^{\flat})$.

We now provide a Blok-Pigozzi theorem relating interpretations between *arbitrary* π-institutions with signature-respecting, strongly monotonic and join-respecting theory-functors that commute with substitutions. Note that while we cite [Vou03] in statement (3), the result as phrased is deeper than the analogous result in [Vou03].

Theorem 17.54 Let $\mathcal{I}$ and $\mathcal{J}$ be π-institutions.

1. If $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ is a signature-respecting, strongly monotonic and join-respecting functor that commutes with substitutions, then any translation $\langle\mathsf{F}^{\flat}, \tau.\rangle : \mathcal{I} \multimap \mathcal{J}$ satisfying (17.27) is an interpretation satisfying (17.28) and
$$\forall\,[\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}] \quad \tau_{\mathfrak{S}}{}^{\blacktriangleleft}{}_{|\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}[\mathsf{Th}_{\mathcal{I}}(\mathfrak{S})]} = (\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}})^{-1} \tag{17.34}$$
(this inverse being well-defined by the previous remark); all such translations are logically-equivalent; one such translation is defined by (17.29); if in addition, $\mathcal{I}$ and $\mathcal{J}$ are both finitary and $\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}[\mathbf{Th}_{\mathcal{I}}(\mathfrak{S})]$ is compact in $\mathbf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S})^{\flat})$, then $\langle\mathsf{F}^{\flat}, \tau.\rangle$ may be chosen to be finitary.

2. If $\mathsf{I} : \mathcal{I} \multimap \mathcal{J}$ is an interpretation, then $\mathsf{I}^{\sharp} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ is a signature-respecting, strongly monotonic and join-respecting functor that commutes with substitutions; if in addition $\mathcal{I}$ and $\mathcal{J}$ are both finitary and I is finitary, then, for each $\mathfrak{S} \in \mathsf{Sign}_{\mathcal{I}}$, $(\mathsf{I}^{\sharp})^{\mathfrak{S}}_{\mathsf{th}}[\mathbf{Th}_{\mathcal{I}}(\mathfrak{S})]$ is compact in $\mathbf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S})^{\flat})$.

3. [Vou03] Suppose that $\mathcal{I}$ is a term π-institution with source signature - variable pair $\langle\mathfrak{A}, x\rangle$. If $\mathsf{F} : \mathbf{TH}(\mathcal{I}) \to \mathbf{TH}(\mathcal{J})$ is a signature-respecting, strongly monotonic and join-respecting functor that commutes with substitutions, then, for any $\langle\mathfrak{A}, x\rangle$-formal translation $\langle\mathsf{F}^{\flat}, \boldsymbol{\tau}\rangle$ (from $\mathcal{I}$ to $\mathcal{J}$) satisfying (17.30), $\langle\mathsf{F}^{\flat}, \boldsymbol{\tau}.\rangle$ is a *natural* interpretation from $\mathcal{I}$ to $\mathcal{J}$; all such translations $\langle\mathsf{F}^{\flat}, \boldsymbol{\tau}.\rangle$ that arise in this manner are logically-equivalent; one such $\langle\mathfrak{A}, x\rangle$-formal translation is defined by (17.31); if in addition, $\mathcal{I}$ and $\mathcal{J}$ are both finitary and $\mathsf{F}^{\mathfrak{S}}_{\mathsf{th}}[\mathbf{Th}_{\mathcal{I}}(\mathfrak{S})]$ is compact in $\mathbf{Th}_{\mathcal{J}}(\mathsf{F}(\mathfrak{S})^{\flat})$, then $\boldsymbol{\tau}$ may be chosen to be a finite set.

Proof. $\boxed{(1)}$ Follows by (1) of Theorem 17.49, Remarks 17.51 and 17.53, together with Theorem 5.110 on page 205. The additional finitary assertion follows from (2) of Theorem 5.111 on page 205. $\boxed{(2)}$ Follows by (2) of Theorem 17.49, Remarks 17.51 and 17.53, together with equivalent condition (8) of Theorem 5.73 on page 195. The additional finitary assertion follows from (1) of Theorem 5.111 on page 205. $\boxed{(3)}$ Follows by (1) together with (3) of Theorem 17.49 and (3) of Theorem 17.52. $\diamond$

Corollary 17.55 [Vou03] If $\mathsf{I} : \mathcal{I} \multimap \mathcal{J}$ is an *natural* interpretation, then $\mathsf{I}^\sharp : \mathsf{TH}(\mathcal{I}) \to \mathsf{TH}(\mathcal{J})$ is a signature-respecting, strongly monotonic and join-respecting functor that commutes with substitutions.

17.4.2 $\cdot^{>}$-Semantics

We briefly introduce the various notions of $\cdot^{>}$-semantics, which are essentially the notions developed in ∮17.3.1 strengthened in the same manner that interpretations between π-institutions strengthens semi-interpretations. In the light of Example 17.35, the theory of $\cdot^{>}$-semantics is a special case of the theory of interpretations between π-institutions.

Definition 17.56 ($\cdot^{>}$-Semantics) Let L be an $\mathfrak{s}$-logic and M a familiar $\mathfrak{t}$-logic. We call M a **weak-semantics** (resp. **$\cdot^{>}$-semantics**, **natural $\cdot^{>}$-semantics**) for L if M is a weak-model (resp. $\cdot^{>}$-model, natural $\cdot^{>}$-model) with weak-modelling translation (resp. $\cdot^{>}$-modelling translation, natural $\cdot^{>}$-modelling translation) τ such that τ is $\vdash$-reflecting from $\mathsf{Th}(\mathsf{L})$ to $\mathsf{Th}(\mathsf{M})$, in which case we call τ a **weak-semantic translation** (resp. **$\cdot^{>}$-semantic translation**, **natural $\cdot^{>}$-semantic translation**). We define the notions of a **formal $\cdot^{>}$-semantics** and a **formal $\cdot^{>}$-semantic translation** analogously. $\square$

The following result follows at once from Theorems 17.52 and 17.54.

Corollary 17.57 Let L be an $\mathfrak{s}$-logic and M a familiar $\mathfrak{t}$-logic.

1. If M is a weak-semantics for L with weak-semantic translation τ then

$$\tau^{\star}{}_{|\mathsf{Th}(\mathsf{L})} : \mathbf{Th}(\mathsf{L}) \cong \tau\left[\mathbf{Th}(\mathsf{L})\right] \lhd_{\blacktriangledown} \mathbf{Th}(\mathsf{M}) \quad \text{and} \tag{17.35}$$

$$\tau^{\star}{}_{|\mathsf{Th}(\mathsf{L})}{}^{-1} = \tau^{\blacktriangleleft}{}_{|\tau^{\star}[\mathsf{Th}(\mathsf{L})]}; \tag{17.36}$$

   if in addition, L and M are both finitary and τ is finitary, then $\tau\left[\mathbf{Th}(\mathsf{L})\right]$ is compact in $\mathbf{Th}(\mathsf{M})$.

2. If M is a $\cdot^{>}$-semantics for L with $\cdot^{>}$-semantic translation τ, then M is a weak-semantics for L with weak-semantic translation τ; consequently, (17.35) and (17.36) are valid and $\tau^{\star}{}_{|\mathsf{Th}(\mathsf{L})}$ $\cdot^{>}$-commutes.

3. If M is a natural $\cdot^{>}$-semantics for L with natural $\cdot^{>}$-semantic translation τ, then M is a $\cdot^{>}$-semantics for L with $\cdot^{>}$-semantic translation τ.

4. Suppose that $\mathcal{D}$ is a (global) $\mathfrak{s}$-calculus with (global) language $\mathbf{G}$, $\mathcal{E}$ is a (global) $\mathfrak{t}$-calculus and $\mathbf{lg}(\mathcal{E}) = \mathbf{G}^{>}$. If $\mathcal{E}$ is a $\cdot^{>}$-semantics for $\mathcal{D}$ with $\cdot^{>}$-semantic translation τ, then $\mathcal{E}$ is a

formal $\cdot^{>}$-semantics for $\mathcal{D}$ with formal $\cdot^{>}$-semantic translation $\boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is any $\cdot^{>}$-formal translation satisfying (17.10), where x is any (fixed) $\mathbf{G}$-variable; in this case, τ and $\boldsymbol{\tau}^{\mathbf{G}}$ are logically equivalent; one such formal translation is given by (17.11).

5. If $\mathcal{E}$ is a formal $\cdot^{>}$-semantics for $\mathcal{D}$ with formal $\cdot^{>}$-semantic translation $\boldsymbol{\tau}$, then $\mathcal{E}$ is a natural $\cdot^{>}$-semantics for $\mathcal{D}$ with natural $\cdot^{>}$-semantic translation $\boldsymbol{\tau}^{\mathbf{G}}$; consequently $\boldsymbol{\tau}^{\mathbf{G}}$ satisfies (17.35) and (17.36), and ${\boldsymbol{\tau}^{\mathbf{G}}}^{\star}{}_{|\mathsf{Th}(\mathcal{D})}$ $\cdot^{>}$-commutes.

6. Suppose that $\mathsf{f} : \mathbf{Th}(\mathsf{L}) \cong \mathsf{f}\,[\mathbf{Th}(\mathsf{L})] \triangleleft_{\blacktriangledown} \mathbf{Th}(\mathsf{M})$ (resp. and f $\cdot^{>}$-commutes). Let τ be any translation satisfying (17.7); for example τ defined by (17.8). Then M is a weak-semantics (resp. $\cdot^{>}$-semantics) for L with weak-semantic (resp. $\cdot^{>}$-semantic) translation τ satisfying (17.9) and
$$\tau^{\blacktriangleleft}{}_{|\mathsf{f}[\mathsf{Th}(\mathsf{L})]} = \mathsf{f}^{-1}. \tag{17.37}$$
If in addition, L and M are both finitary and $\tau\,[\mathbf{Th}(\mathsf{L})]$ is compact in $\mathbf{Th}(\mathsf{M})$, then τ may be chosen to be finitary.

7. [Vou03] Suppose that $\mathcal{D}$ is a (global) $\mathfrak{s}$-calculus with (global) language $\mathbf{G}$, $\mathcal{E}$ is a (global) $\mathfrak{t}$-calculus and $\mathbf{lg}(\mathcal{E}) = \mathbf{G}^{>}$. Suppose further, that $\mathbf{Th}(\mathcal{D}) \cong \mathsf{f}\,[\mathbf{Th}(\mathcal{D})] \triangleleft_{\blacktriangledown} \mathbf{Th}(\mathcal{E})$ and f $\cdot^{>}$-commutes. Then $\mathcal{E}$ is a formal $\cdot^{>}$-semantics for $\mathcal{D}$ with formal $\cdot^{>}$-semantic translation $\boldsymbol{\tau}$ and $\boldsymbol{\tau}^{\mathbf{G}}_{>}$ satisfies (17.9) and (17.37), where $\boldsymbol{\tau}$ is any formal translation satisfying (17.12), where x is any (fixed) $\mathbf{G}$-variable; one such formal translation is given by (17.13). If in addition, $\mathcal{D}$ and $\mathcal{E}$ are both finitary and $\tau\,[\mathbf{Th}(\mathcal{D})]$ is compact in $\mathbf{Th}(\mathcal{E})$, then $\boldsymbol{\tau}$ may be chosen to be finitary.

□

Corollary 17.58 Let L and M be familiar logics and let τ be a $\langle \cdot^{>}, \mathsf{M}\rangle$-natural translation from L to M. Then M is a $\cdot^{>}$-semantics of L with $\cdot^{>}$-semantic translation τ iff $\mathsf{L} = L^{\mathbf{lg}(\mathsf{L})}_{\cdot^{>}}(\mathsf{M}, \tau)$. □

Open Problem 17.59 It seems to us that the assumption that τ be a $\langle \cdot^{>}, \mathsf{M}\rangle$-natural translation from L to M can be dropped and that $\cdot^{>}$-semantics can be weakened to merely weak-semantics, since strict continuity completely describes the consequence relation of $\mathcal{S}$.

We now consider the first of our primary examples, in which we use the unparameterized theory of $\cdot^{>}$-semantics for logics of constructs to explain our theory of $\langle X, z\rangle$-algebraic semantics. We must note that all the logics considered in this example are global with respect to their appropriate signatures, and hence induce term π-institutions; as such this example can be developed entirely with the machinery of [Vou03].

Example 17.60 ($\langle X, z\rangle$-Algebraic Semantics)

Let $\mathcal{S}$, z, $\mathfrak{s}$, $\cdot^{>}$ and $\cdot^{<}$ be as in Example 17.19. Recall that the $\mathfrak{s}$-variables of $\mathbf{Tm}$ are all the variables other than z.

Let $\mathcal{K}$ be an $\mathfrak{a}$-quasivariety, $X \subseteq \mathsf{Fm}(\mathcal{S})$. By Proposition 13.2 on page 392, if $\mathcal{K}$ is a $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ then $\|X\|_{\mathcal{S}} = \mathfrak{B}_z/\bot_{\mathcal{K}}$, which is z-invariant. Consequently, we shall assume, without loss of generality, that $\|X\|_{\mathcal{S}}$ is z-invariant. As in Example 16.43 on page 455, consider the $\mathbf{Tm}$-logic $\mathcal{S}_{:X}$. By Remark 16.44 on page 455 and the assumption that $\|X\|_{\mathcal{S}}$ is z-invariant, $\mathcal{S}_{:X}$ is $\mathfrak{s}$-structural.

We shall consider the filtration logic $\mathcal{S}_{:X}$ as a propositional $\mathfrak{s}$-calculus and we shall consider the relative congruence logic $S^2(\Theta^{\mathcal{K}})$ as a $\underset{\rightarrow}{\mathfrak{s}}_{[2]}$-deductive system. Certainly $S^2(\Theta^{\mathcal{K}})$ is finitary, and since $S^2(\Theta^{\mathcal{K}})$ is $\underset{\rightarrow}{\mathfrak{a}}_{[2]}$-structural and $\underset{\rightarrow}{\mathfrak{s}}_{[2]}$ is a subconstruct of $\underset{\rightarrow}{\mathfrak{a}}_{[2]}$, $S^2(\Theta^{\mathcal{K}})$ is $\underset{\rightarrow}{\mathfrak{s}}_{[2]}$-structural and hence a propositional $\underset{\rightarrow}{\mathfrak{s}}_{[2]}$-calculus.

The following result, characterizing $\mathcal{K}$ being an $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ in terms of $S^2(\Theta^{\mathcal{K}})$ being a formal $\cdot^{>}$-semantics for $\mathcal{S}_{:X}$, follows easily from Remark 16.47 on page 455 and Proposition 17.20.

Proposition 17.61 Let x be any variable distinct from z, $\mathfrak{B}(x, z)$ be a binary system of equations, $\boldsymbol{\tau}$ a $\cdot^{>}$-formal translation from $\mathfrak{s}$ to $\underset{\rightarrow}{\mathfrak{s}}_{[2]}$.

1. $\mathcal{K}$ is a $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ iff $S^2(\Theta^{\mathcal{K}})$ is a formal $\cdot^{>}$-semantics for $\mathcal{S}_{:X}$ with formal $\cdot^{>}$-semantic translation $\mathfrak{B}$.
2. $S^2(\Theta^{\mathcal{K}})$ is a formal $\cdot^{>}$-semantics for $\mathcal{S}_{:X}$ with formal $\cdot^{>}$-semantic translation $\boldsymbol{\tau}$ iff $\mathcal{K}$ is a $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $({}_x\boldsymbol{\tau})_*$.

Proof. $\boxed{(1)}$ $\boxed{\Rightarrow}$ $P \vdash_{\mathcal{S}_{:X}} p$ [iff, by Remark 16.47 on page 455] $X, P \vdash_{\mathcal{S}} p$ [iff, by assumption and (13.3)] $\mathfrak{B}_z[P] \models_{\mathcal{K}} \mathfrak{B}_z[\![t]\!]$ [iff, by Proposition 17.20] $\mathfrak{B}^{\mathbf{Tm}}_{>}[P] \models_{\mathcal{K}} \mathfrak{B}^{\mathbf{Tm}}_{>}[\![t]\!]$ [iff] $\mathfrak{B}^{\mathbf{Tm}}_{>}[P] \vdash_{S^2(\Theta^{\mathcal{K}})} \mathfrak{B}^{\mathbf{Tm}}_{>}[\![t]\!]$. $\boxed{\Leftarrow}$ $X \cup P \vdash_{\mathcal{S}} p$ [iff] $P \vdash_{\mathcal{S}_{:X}} p$ [iff, by assumption and (17.33)] $\mathfrak{B}^{\mathbf{Tm}}_{>}[P] \vdash_{S^2(\Theta^{\mathcal{K}})} \mathfrak{B}^{\mathbf{Tm}}_{>}[\![t]\!]$ [iff] $\mathfrak{B}^{\mathbf{Tm}}_{>}[P] \models_{\mathcal{K}} \mathfrak{B}^{\mathbf{Tm}}_{>}[\![t]\!]$ [iff, by Proposition 17.20] $\mathfrak{B}_z[P] \models_{\mathcal{K}} \mathfrak{B}_z[\![t]\!]$. $\boxed{(2)}$ $\boxed{\Rightarrow}$ $X \cup P \vdash_{\mathcal{S}} p$ [iff] $P \vdash_{\mathcal{S}_{:X}} p$ [iff, by assumption and (17.33)] $\tau^{\mathbf{Tm}}_{>}[P] \vdash_{S^2(\Theta^{\mathcal{K}})} \tau^{\mathbf{Tm}}_{>}[\![t]\!]$ [iff] $\tau^{\mathbf{Tm}}_{>}[P] \models_{\mathcal{K}} \tau^{\mathbf{Tm}}_{>}[\![t]\!]$ [iff, by Proposition 17.20] $({}_x\boldsymbol{\tau})_z[P] \models_{\mathcal{K}} ({}_x\boldsymbol{\tau})_z[\![t]\!]$. $\boxed{\Leftarrow}$ $P \vdash_{\mathcal{S}_{:X}} p$ [iff, by Remark 16.47 on page 455] $X, P \vdash_{\mathcal{S}} p$ [iff, by assumption and (13.3)] $({}_x\boldsymbol{\tau})_z[P] \models_{\mathcal{K}} ({}_x\boldsymbol{\tau})_z[\![t]\!]$ [iff, by Proposition 17.20] $\tau^{\mathbf{Tm}}_{>}[P] \models_{\mathcal{K}} \tau^{\mathbf{Tm}}_{>}[\![t]\!]$ [iff] $\tau^{\mathbf{Tm}}_{>}[P] \vdash_{S^2(\Theta^{\mathcal{K}})} \tau^{\mathbf{Tm}}_{>}[\![t]\!]$. $\diamond$

Consequently, Theorem 13.17 on page 397 (with *surjectivity* dropped) follows from Corollary 17.57. Since we are dealing only with global logics and formal translations (and hence natural translations), some of our theory of $\langle X, z\rangle$-algebraic semantics [BR03] is merely a special case of the theory of interpretations between π-institutions developed in [Vou03].

□

Open Problem 17.62 (Commutivity with *Surjective* Substitutions)

We have been unable to prove the following result, which replaces commutivity by substitutions with commutivity by *surjective* substitutions. A theorem of this nature would be important for two reasons. Firstly, it would more closely generalize the analogous result from the standard theory of algebraizable logics. Secondly, if such a result where valid, we would more closely encompass our parametrized theory of algebraization (see the previous example). The primary problem is that the substitutions $\mathsf{e}^{\mathbf{G}}_{\psi}$ that we have been using to 'evaluate' formal translations are not surjective. This can be partially overcome, yet one problem still remains. We give an attempted proof, indicating where the problem occurs.

Definition 17.63 (Finitary Free Objects) Let $\mathbf{F}$ be $\mathfrak{s}$-freely generated by V. We say that $\mathbf{F}$ is **finitary** if, for each $\boldsymbol{p} \in \mathsf{uni}(\mathbf{F})$ there exists a finite $X_{\boldsymbol{p}} \subseteq_{\mathsf{f}} V$, such that for any two $\mathbf{F}$-endomorphisms f and g that agree on $X_{\boldsymbol{p}}$, $f(\boldsymbol{p}) = g(\boldsymbol{p})$. We shall call such a set $X_{\boldsymbol{p}}$ **essential variables** for $\boldsymbol{p}$. □

Conjecture 17.64 Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ have *images* and $\mathbf{G}$ and $\mathbf{H}$ are both *finitary*. Suppose that $\mathcal{D}$ and $\mathcal{E}$ are both *finitary* and f is an isomorphism of $\mathbf{Th}(\mathcal{D})$ onto some $\blacktriangledown$-sublattice of $\mathbf{Th}(\mathcal{E})$ that is compact in $\mathbf{Th}(\mathcal{E})$ and f commutes with all *surjective* $\mathcal{D}$-substitutions. Then there exists finitary formal semantic translation $\boldsymbol{\tau}$, such that $\mathcal{E}$ is a formal $\mathfrak{t}$-semantics for $\mathcal{D}$ with formal semantic translation $\boldsymbol{\tau}$, ${\boldsymbol{\tau}^{\mathbf{G}^\star}}_{|\mathsf{Th}(\mathcal{D})} = \mathsf{f}$ and ${\boldsymbol{\tau}^{\mathbf{G}^\blacktriangleleft}}_{|\mathsf{f}[\mathsf{Th}(\mathcal{D})]} = \mathsf{f}^{-1}$.

Proof. Since $\|\{x\}\|_{\mathcal{D}}$ is a compact element of $\mathbf{Th}(\mathcal{D})$, $\mathsf{f}(\|\{x\}\|_{\mathcal{D}})$ is a finitely generated $\mathcal{E}$-theory. Let $\boldsymbol{\tau}$ be a finite set of $\mathbf{H}$-formulae such that $\|\boldsymbol{\tau}\|_{\mathcal{E}} = \mathsf{f}(\|\{x\}\|_{\mathcal{D}})$. For each $\mathbf{G}$-formula ψ and $\phi \in \boldsymbol{\tau}$, let X^ψ_ϕ be a set of dependent variables for the $\mathbf{H}$-formula $\mathsf{e}^{\mathbf{G}>}_\psi(\phi)$. Let $X^\psi = \bigcup_{\phi\in\boldsymbol{\tau}} X^\psi_\phi$, which is a finite set of $\mathbf{H}$-variables. Since $\mathfrak{t}$ is assumed to have images, there exists a *surjective* $\mathbf{H}$-substitution $\mathsf{d}^{\boldsymbol{\tau}}_\psi$ mapping each $y \in X^\psi$ to $\mathsf{e}^{\mathbf{G}>}_\psi(y)$, by Remark 1.239 on page 48. So for each $\phi \in \boldsymbol{\tau}$, since $\mathsf{d}^{\boldsymbol{\tau}}_\psi$ and $\mathsf{e}^{\mathbf{G}>}_\psi$ agree on the dependent variables X^ψ_ϕ of ϕ, $\mathsf{d}^{\boldsymbol{\tau}}_\psi(\phi) = \mathsf{e}^{\mathbf{G}>}_\psi(\phi)$. So $\mathsf{d}^{\boldsymbol{\tau}}_\psi[\boldsymbol{\tau}] = \mathsf{e}^{\mathbf{G}>}_\psi[\boldsymbol{\tau}]$.

Now $\boldsymbol{\tau}^{\mathbf{G}^\star}[\![\psi]\!] = \left\|\mathsf{e}^{\mathbf{G}>}_\psi[\boldsymbol{\tau}]\right\|_{\mathcal{E}} = \left\|\mathsf{d}^{\boldsymbol{\tau}}_\psi[\boldsymbol{\tau}]\right\|_{\mathcal{E}} = \left\|\mathsf{d}^{\boldsymbol{\tau}}_\psi\left[\|\boldsymbol{\tau}\|_{\mathcal{E}}\right]\right\|_{\mathcal{E}} = \left\|\mathsf{d}^{\boldsymbol{\tau}}_\psi\left[\mathsf{f}(\|\{x\}\|_{\mathcal{D}})\right]\right\|_{\mathcal{E}} = \left\|\mathsf{d}^{\boldsymbol{\tau}\,<>}_\psi\left[\mathsf{f}(\|\{x\}\|_{\mathcal{D}})\right]\right\|_{\mathcal{E}} = \mathsf{d}^{\boldsymbol{\tau}\,<>\star}_\psi(\mathsf{f}(\|\{x\}\|_{\mathcal{D}})) \overset{(i)}{=} \mathsf{f}(\mathsf{d}^{\boldsymbol{\tau}\,<\star}_\psi\left[\|\{x\}\|_{\mathcal{D}}\right]) = \mathsf{f}(\left\|\mathsf{d}^{\boldsymbol{\tau}\,<}_\psi\left[\|\{x\}\|_{\mathcal{D}}\right]\right\|_{\mathcal{D}}) = \mathsf{f}(\left\|\{\mathsf{d}^{\boldsymbol{\tau}\,<}_\psi(x)\}\right\|_{\mathcal{D}}) \overset{(ii)}{=} \mathsf{f}(\|\{\psi\}\|_{\mathcal{D}})$.

If this where true, the proof would complete in the usual manner, i.e., invoking the $\blacktriangledown$ of f. $\diamond$

Unfortunately, we see no reason why (ii) should be valid, or how to overcome this problem. Further, (i) requires $\mathsf{d}^{\boldsymbol{\tau}\,<}_\psi$ to be surjective, and while surjections in constructs are epimorphisms and category isomorphic functors preserve epimorphisms, epimorphisms need not be surjections [Ada83, 110]. The second of these problems can be 'overcome' by assuming that the category isomorphism preserve surjective substitutions, which it certainly does for the sentential cases and in our case.

It seems that one may have to develop a stronger notion of isomorphic *constructs*, one that does not force the matching objects of the two constructs to have the same universe, but which still relates points of the universes in some way, or at least the free generators of the free objects in some way.

We note that the *Blok-Pigozzi theorem* of [Vou03] also suffers from this deficiency; there is no commutivity with *surjective* substitutions analogue.

17.5 Equivalent $\langle\cdot^>, \cdot^<\rangle$-Semantics

In this section we develop the theory of one logic being an equivalent semantics for another. Due to time constraints, we have not developed this theory at the institutional level. Where results are special cases of the theory of deductively equivalent π-institutions developed in [Vou03], we have duely referenced these results. At the end of this section, as an open problem, we suggest how the Blok-Pigozzi theorems pertaining to deductive equivalence can be generalized from term π-institutions to arbitrary π-institutions by replacing the implicit notion of syntactic commutivity with logical commutivity.

Convention 17.65 (Isomorphic Signatures) Throughout this section, unless specified to the contrary, $\mathfrak{s}$ and $\mathfrak{t}$ shall denote fixed but arbitrary isomorphic signatures and we shall denote the isomorphic functor from $\mathfrak{s}$ onto $\mathfrak{t}$ by $\cdot^>$ and denote the inverse functor by $\cdot^<$.

The reader is urged to recall the material of ∮5.3.5 and ∮5.4.4, and in particular the notion of mutually untranslating translations.

Definition 17.66 (Equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-Semantics) Let L be an $\mathfrak{s}$-logic and let M be a familiar $\mathfrak{t}$-logic. We call M an **equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics** (resp. a **natural equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics**, **formal equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics**) for L if there exists a (resp. $\cdot^{>}$-natural, $\cdot^{>}$-formal) translation $\tau : \mathsf{L} \multimap \mathsf{M}$ and a (resp. $\cdot^{<}$-natural, $\cdot^{<}$-formal) translation $\pi : \mathsf{M} \multimap \mathsf{L}$, such that M is a (resp. natural, formal) $\cdot^{>}$-model of L with (resp. natural, formal) $\cdot^{>}$-modelling translation τ, L is a (resp. natural, formal) $\cdot^{<}$-model of M with (resp. natural, formal) $\cdot^{<}$-modelling translation π, and τ and π are mutually untranslating, in which case we call $\langle \tau, \pi \rangle$ **$\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations** (resp. **natural $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations**, **formal $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations**). □

The following result follows from Theorem 17.26, Corollary 17.26, Theorem 5.101 on page 202, Theorem 5.102 on page 203 and Theorem 5.132 on page 210. We note that statement (6) rephrased for *normal* equivalent semantics and statement (7) rephrased for *formal* semantics are special cases of results from [Vou03].

Corollary 17.67

1. If $\mathcal{E}$ is a formal equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics for $\mathcal{D}$ with formal $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations $\langle \boldsymbol{\tau}, \boldsymbol{\pi} \rangle$, then $\mathcal{E}$ is a normal equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics for $\mathcal{D}$ with normal $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations $\langle \boldsymbol{\tau}^{\mathbf{lg}(\mathcal{D})}, \boldsymbol{\pi}^{\mathbf{lg}(\mathcal{E})} \rangle$.
2. If M is a normal equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics for L with normal $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations $\langle \tau, \pi \rangle$, then M is an equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics for L with $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations $\langle \tau, \pi \rangle$.
3. Suppose that $\mathcal{D}$ is a (global) $\mathfrak{s}$-calculus with (global) language $\mathbf{G}$, $\mathcal{E}$ is a (global) $\mathfrak{t}$-calculus and $\mathbf{lg}(\mathcal{E}) = \mathbf{G}^{>}$. If $\mathcal{E}$ is an equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics for $\mathcal{D}$ with $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations $\langle \tau, \pi \rangle$, then $\mathcal{E}$ is a formal equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics for $\mathcal{D}$ with formal $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations $\langle \boldsymbol{\tau}, \boldsymbol{\pi} \rangle$, for some $\langle \boldsymbol{\tau}, \boldsymbol{\pi} \rangle$ with $\boldsymbol{\tau}^{\mathbf{lg}(\mathcal{D})}$ logically equivalent to τ and $\boldsymbol{\pi}^{\mathbf{lg}(\mathcal{E})}$ logically equivalent to π; the construction of such $\langle \boldsymbol{\tau}, \boldsymbol{\pi} \rangle$ is described in (6) of Theorem 17.26.
4. Let $\tau : \mathsf{L} \multimap \mathsf{M}$ and $\pi : \mathsf{M} \multimap \mathsf{L}$. The following conditions are equivalent.
   (a) M is an (resp. natural, formal) equivalent $\langle \cdot^{>}, \cdot^{<} \rangle$-semantics for L with (resp. natural, formal) $\langle \cdot^{>}, \cdot^{<} \rangle$-equivalence translations $\langle \tau, \pi \rangle$.
   (b) M is a (resp. natural, formal) $\cdot^{>}$-semantics for L with (resp. natural, formal) $\cdot^{>}$-semantic translation τ, L a (resp. natural, formal) $\cdot^{<}$-semantics for M with (resp. natural, formal) $\cdot^{<}$-semantic translation π, and τ and π are mutually untranslating.
   (c) M is a (resp. natural, formal) $\cdot^{>}$-semantics for L with (resp. natural, formal) $\cdot^{>}$-semantic translation τ and τ untranslates π.
   (d) L is a (resp. natural, formal) $\cdot^{<}$-semantics for M with (resp. natural, formal) $\cdot^{<}$-semantic translation π and π untranslates τ.

5. If M is an **equivalent** $\langle \cdot^{>}, \cdot^{<}\rangle$**-semantics** for L with $\langle \cdot^{>}, \cdot^{<}\rangle$-equivalence translations $\langle \tau, \pi\rangle$, then

   (a) $\pi^{\blacktriangleleft}{}_{|\mathsf{Th}(\mathsf{L})} = \tau^{*}{}_{|\mathsf{Th}(\mathsf{L})} : \mathbf{Th}(\mathsf{L}) \cong \mathbf{Th}(\mathsf{M})$ with inverse isomorphism $\pi^{\blacktriangleleft}{}_{|\mathsf{Th}(\mathsf{L})} = \tau^{*}{}_{|\mathsf{Th}(\mathsf{L})}$;

   (b) $\tau^{\blacktriangleleft}{}_{|\mathsf{Th}(\mathsf{M})} = \pi^{*}{}_{|\mathsf{Th}(\mathsf{M})} : \mathbf{Th}(\mathsf{M}) \cong \mathbf{Th}(\mathsf{L})$ with inverse isomorphism $\pi^{\blacktriangleleft}{}_{|\mathsf{Th}(\mathsf{L})} = \tau^{*}{}_{|\mathsf{Th}(\mathsf{L})}$;

   (c) $\tau^{*}{}_{|\mathsf{Th}(\mathsf{L})}$ and $\pi^{*}{}_{|\mathsf{Th}(\mathsf{M})}$ both commute.

6. Suppose that $\mathsf{f} : \mathbf{Th}(\mathsf{L}) \cong \mathbf{Th}(\mathsf{M})$ and f $\cdot^{>}$-commutes. Let $\tau : \mathsf{L} \multimap \mathsf{M}$ be a translation such that $\|\tau[\![\phi]\!]\|_{\mathsf{M}} = \mathsf{f}(\|\{\phi\}\|_{\mathsf{L}})$, for each $\phi \in \mathsf{Fm}(\mathsf{L})$, and let $\pi : \mathsf{M} \multimap \mathsf{L}$ be a translation such that $\|\pi[\![\psi]\!]\|_{\mathsf{L}} = \mathsf{f}^{-1}(\|\{\psi\}\|_{\mathsf{M}})$, for each $\psi \in \mathsf{Fm}(\mathsf{M})$. Then

   (a) M is an **equivalent** $\langle \cdot^{>}, \cdot^{<}\rangle$**-semantics** for L with $\langle \cdot^{>}, \cdot^{<}\rangle$-equivalence translations $\langle \tau, \pi\rangle$;

   (b) $\tau^{\star}{}_{|\mathsf{Th}(\mathsf{L})} = \pi^{\blacktriangleleft}{}_{|\mathsf{Th}(\mathsf{L})} = \mathsf{f}$;

   (c) $\tau^{\blacktriangleleft}{}_{|\mathsf{Th}(\mathsf{M})} = \pi^{\star}{}_{|\mathsf{Th}(\mathsf{M})} = \mathsf{f}^{-1}$.

   One such realization of τ and π is given by $\tau[\![\phi]\!] = \mathsf{f}(\|\{\phi\}\|_{\mathsf{L}})$, for each $\phi \in \mathsf{Fm}(\mathsf{L})$, $\pi[\![\psi]\!] = \mathsf{f}^{-1}(\|\{\psi\}\|_{\mathsf{M}})$, for each $\psi \in \mathsf{Fm}(\mathsf{M})$.

   Further, if both L and M are finitary, then both τ and π may be chosen so as to be finitary.

Note that it is possible that $\mathsf{f} : \mathbf{Th}(\mathsf{L}) \cong \mathbf{Th}(\mathsf{M})$ and f $\cdot^{>}$-commutes and for L to be finitary, yet M may fail to be finitary; i.e., finitariness is not generally preserved by equivalence of consequence relations [Her93].

We now turn to the second of our primary examples of this chapter, in which we use the non-parameterized theory of this chapter to explain $\langle X, z\rangle$-equivalent algebraic semantics.

Example 17.68 ($\langle X, z\rangle$-Equivalent Algebraic Semantics)

Let the context be as in Example 17.19 and Example 17.60.

The proof of the following result follows from Proposition 17.61 together with Proposition 17.21.

Proposition 17.69 Let x and y be distinct variables distinct from z, $\mathfrak{B}(x,z)$ be a binary system of equations, $\Delta(x,y,z)$ a finite set of ternary terms, $\boldsymbol{\tau}$ a $\cdot^{>}$-formal translation from $\mathfrak{s}$ to $\underset{\rightarrow}{\mathfrak{s}}{}^{[2]}$ and $\boldsymbol{\pi}$ a $\cdot^{<}$-formal translation from $\underset{\rightarrow}{\mathfrak{s}}{}^{[2]}$ to $\mathfrak{s}$.

1. $\mathcal{K}$ is an equivalent $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $\mathfrak{B}_*$ and $\langle X, z\rangle$-equivalence formulae $\boldsymbol{\Delta}$ iff $S^2(\Theta^{\mathcal{K}})$ is a formal equivalent $\cdot^{>}$-semantics for $\mathcal{S}_{:X}$ with formal $\cdot^{>}$-equivalence translations $\langle \mathfrak{B}, \boldsymbol{\Delta}\rangle$.
2. $S^2(\Theta^{\mathcal{K}})$ is a formal equivalent $\cdot^{>}$-semantics for $\mathcal{S}_{:X}$ with formal $\cdot^{>}$-equivalence translations $\langle \boldsymbol{\tau}, \boldsymbol{\pi}\rangle$ iff $\mathcal{K}$ is an equivalent $\langle X, z\rangle$-algebraic semantics for $\mathcal{S}$ with $\langle X, z\rangle$-defining equations $({}_{x}\boldsymbol{\tau})_*$ and $\langle X, z\rangle$-equivalence formulae ${}_{\langle x,y\rangle}\boldsymbol{\pi}$.

□

Consequently, the equivalence of conditions (1) and (4) of Theorem 15.11 on page 424 (with *surjectivity* dropped) follows from Corollary 17.67. Again, since we are dealing only with global logics and formal translations (and hence natural translations), some of our theory of equivalent $\langle X, z\rangle$-algebraic semantics [BR03] follows from the theory of deductively equivalent π-institutions as developed in [Vou03].

$\square$

Open Problem 17.70 Characterize formal equivalent semantics in terms of isomorphisms between the filter lattices of isomorphic languages. Use this result, together with the signature $\mathfrak{t}$ defined in Example 16.43 on page 455, to capture more characterizations of $\langle X, z\rangle$-equivalent semantics from $\oint$15.

Open Problem 17.71 Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are isomorphic. For a translation τ from an $\mathfrak{s}$-language $\mathbf{A}$ into a logic M whose language is isomorphic to $\mathbf{A}$, call M τ-regular if, for all $T, R \in \mathsf{Th}(\mathsf{M})$, $\tau^{\blacktriangleleft}(F) = G^{\blacktriangleleft}$ implies $F = G$.

> **Conjecture 17.72** $L^{\mathbf{A}}_{\cdot^>}(\mathsf{M}, \tau)$ is a $\langle \cdot^>, \cdot^<\rangle$-semantics for M with $\langle \cdot^>, \cdot^<\rangle$-equivalence translations $\langle \tau, \pi\rangle$ iff M is τ-regular.

If that is not true, try formal semantics. If it is true, extend the definition and result to deductively equivalent π-institutions, and then characterize the protoalgebraicity of $L^{\mathbf{A}}_{\cdot^>}(\mathsf{M}, \tau)$ in terms of a suitable notion of M having $\langle \mathsf{M}, \tau\rangle$-coherent $\tau^{\blacktriangleleft}$-classes; these would be $\tau^{\blacktriangleleft}(T)$ for each $T \in \mathsf{M}$.

Open Problem 17.73 *Rephrase the definition of deductively equivalent* π-institutions by dropping the implicit requirement that the translations involved be syntactically natural, by adopting the notion of translation given in this chapter. Prove the following result.

> **Conjecture 17.74** Arbitrary π-institutions are deductively equivalent iff there exists a signature-respecting adjoint equivalence between the theory-categories.

Open Problem 17.75 The notion of a π-institution is a half-hearted attempt at a categorical formulation of a multi-signature logic, given that (1) $\mathsf{SEN}_{\mathcal{I}} : \mathbf{Sign}_{\mathcal{I}} \to \mathbf{Set}$ and the latter category is concrete, and (2) a closure operator is not a categorical notion; it is *concrete* (i.e., a construct) and set theoretic (or elementary in the light of $\oint$4). Consequently, a π-institution can be converted into a family of logics over constructs; similar to an archology; structurality becomes continuity between each of the logics over the construct.

Find a *purely* categorical abstraction of a logic; the endomorphisms would correspond to σ^* where σ is a substitution in the concrete sense, and commutivity would be an important defining feature. The objects are the theories. Or perhaps the other way round. There must be *no* closure operators or consequence relations. If such a pure categorical abstraction can be found, are there any theorems to prove?

Part VII

Appendices, Bibliography, Glossaries and Index

Appendix A

Motivating the term Continuity

In this appendix we provide a partial justification for our usage of the term *continuous*, introduced in $\oint 5$. We introduce the notion of a space, defined as in topology, but without the requirement that the universe be open, nor that open sets be closed under finite intersections. It is easily seen that spaces are in one-to-one correspondence with closed systems. The theory is developed following topology as closely as possible. We introduce a notion of convergence that coincides with convergence in topological spaces in the case that the space is topological. We then show that a *function* between spaces is continuous (as defined in $\oint 5$) iff it preserves convergence.

We have attempted to extend this result to translations but have encountered problems, essentially because the pole of a translation consists of points and not a point. Interestingly, we shall show that a translation is continuous between spaces iff the *reduced* pre-image of open sets is open, and so a *function* is continuous between spaces iff the pre-image of open sets is open; as is the case in topology.

A.1 Spaces

In the following definition we purposely define a space to follow as closely with the standard topological definition.

Definition A.1 (Space) A space $\mathbb{A}$ is uniquely determined by a collection $\mathsf{uni}(\mathbb{A})$, called the **universe** and a collection $\mathsf{op}(\mathbb{A}) \subseteq \mathfrak{P}(\mathsf{uni}(\mathbb{A}))$, the members of which are called **open**, such that, $\mathsf{op}(\mathbb{A})$ forms an open-system over $\mathsf{uni}(\mathbb{A})$, i.e.,

1. $\emptyset \in \mathsf{op}(\mathbb{A})$, and
2. $\mathcal{A} \subseteq \mathsf{op}(\mathbb{A})$ implies $\bigcup \mathcal{A} \in \mathsf{op}(\mathbb{A})$.

The collection of all spaces with universe $\mathcal{A}$ is denoted by $\mathsf{Space}(A)$. The $\subseteq$-ordered collection $\langle \mathsf{op}(\mathbb{A}), \subseteq \rangle$ is denoted by $\mathbf{op}(\mathbb{A})$, which by definition is a complete lattice. We call $\bigcup \mathsf{op}(\mathbb{A})$ the **plain** of $\mathbb{A}$, which we denote by $\mathfrak{p}_{\mathbb{A}}$, and call $\overset{\mathsf{uni}(\mathbb{A})}{\neg} \mathfrak{p}_{\mathbb{A}}$ the **constraint** of $\mathbb{A}$, which we denote by $\Bbbk_{\mathbb{A}}$. A point is called **constrained** or *a* **constraint point** if it lies in $\Bbbk_{\mathbb{A}}$, and is called **plain** or **unconstrained** if it lies in $\mathfrak{p}_{\mathbb{A}}$. We call U a **neighbourhood** of a, denoted by $a \Subset_{\mathbb{A}} U$ or $U \Supset_{\mathbb{A}} a$, if there exists open O with $a \in O \subseteq U$. The collection of all neighbourhoods of a point a is denoted by $\mathsf{Nbh}_{\mathbb{A}}(a)$. We say that A is **proximate to** b (or that b is a **consequence of** A), denoted by $A \vdash_{\mathbb{A}} b$ or $b \dashv_{\mathbb{A}} A$, if every neighbourhood of b **meets** (i.e., has non-empty intersection with) A. For $B \subseteq \mathsf{uni}(\mathbb{A})$, we define $A \vdash_{\mathbb{A}} B$ iff $A \vdash_{\mathbb{A}} b$ for all $b \in B$. By the **closure** of A, we mean the collection $\|A\|_{\mathbb{A}} \subseteq \mathsf{uni}(\mathbb{A})$ defined by $\|A\|_{\mathbb{A}} = \{b \in \mathsf{uni}(\mathbb{A}) : A \vdash_{\mathbb{A}} b\}$. We call $G \subseteq \mathsf{uni}(\mathbb{A})$ **closed** if it contains all points proximate to it, i.e., $\|G\|_{\mathbb{A}} \subseteq G$. The collection of all closed collections of $\mathbb{A}$ is denoted by $\mathsf{cl}_{\mathbb{A}}$, and the $\subseteq$-ordered collection $\langle \mathsf{cl}_{\mathbb{A}}, \subseteq \rangle$ is denoted by $\mathbf{cl}_{\mathbb{A}}$. Define $|A|_{\mathbb{A}} = \{a \in A : a \Subset_{\mathbb{A}} A\}$, which we call the **interior** of A. Wherever unambiguous, we tend to drop the subscript '$\mathbb{A}$' from these notations and tend to write 'op' and 'cl' for '$\mathsf{op}(\mathbb{A})$' and '$\mathsf{cl}_{\mathbb{A}}$' respectively (etc.).

We say that space $\mathbb{A}$ is **finer** (**coarser**) than space $\mathbb{B}$, denoted by $\mathbb{A} \preceq \mathbb{B}$ (resp. $\mathbb{A} \succeq \mathbb{B}$), iff $\mathsf{uni}(\mathbb{A}) = \mathsf{uni}(\mathbb{B})$ and $\mathsf{op}(\mathbb{A}) \supseteq \mathsf{op}(\mathbb{B})$ (resp. $\mathsf{op}(\mathbb{A}) \subseteq \mathsf{op}(\mathbb{B})$). $\square$

Clearly topological spaces are spaces.

Remark A.2 The plain is the largest open set.

For topological spaces, the plain is the universe; this is not true for spaces generally.

Convention A.3 (Complements) So as to ease the notational burden, when working with a space $\mathbb{A}$, we shall write $\neg A$ for $\overset{\mathsf{uni}(\mathbb{A})}{\neg} A$, where $A \subseteq \mathsf{uni}(\mathbb{A})$.

While we have purposely defined the above notions in a manner that reflects the standard topological definitions, we now show that 'spaces are really just closed systems'. Consequently all the results of concrete closed systems pertain to spaces.

Theorem A.4 The fourth column of Table A.1 characterizes the proximity/consequence relation, the closure operator, the closed collections and interior operator of a space. Table A.2 on page 500 describes the fundamental relationships between the various associates of a space.

Proof. (*We prove some of these results as examples, leaving the rest to the reader.*)
(*We first show that row 3 column 4 of Table A.1 is valid.*)
$\boxed{\text{The characterization of neighbourhoods is valid}}$ $\boxed{a \Subset U \subseteq V \text{ implies } a \Subset V}$ Suppose that $a \Subset U$ and $U \subseteq V$. So there exists $O \in \mathsf{op}$, with $a \in O \subseteq U$. But then, $a \in O \subseteq V$; so $a \Subset V$. $\boxed{\forall\,[a \Subset U]\,\exists\,[a \Subset V \subseteq U]\,\forall\,[b \in V]\,\exists\,[b \Subset W \subseteq V]}$ Suppose that $a \Subset U$. Let $O = \bigcup\{P \in \mathsf{op} : P \subseteq U\} \in \mathsf{op}$. Since $a \Subset U$, there exists $P \in \mathsf{op}$ with $a \in P \subseteq U$. Hence, $a \in O \subseteq U$. Letting $V = O$ (for notational convenience), we *certainly* have that $a \in O \subseteq V$, and so $a \Subset V$, and in particular, $a \Subset V \subseteq U$. Let $b \in V$. Trivially, $b \in O$. So by the definition of O, there exists $P \in \mathsf{op}$ with $b \in P \subseteq U$. But, $P \subseteq V$: if $c \in P$, then $c \in P \subseteq U$, and so $c \in O = V$, again by the definition of O. So $b \in P \subseteq V$. Setting, $W = P$ (again for notational convenience), we *certainly* have $b \in P \subseteq W$, and so $b \Subset W$. In particular, we have $b \Subset W \subseteq V$. Next we show that if $\Subset$ is defined as in row 3 column 4 of Table A.1, then the formula

$$O \in \mathsf{op} \text{ iff } \forall\,[a \in O]\,\exists\,[a \Subset U]\,U \subseteq O$$

defines the open collections of a space.
$\boxed{O \in \mathsf{op} \text{ iff } \forall\,[a \in O]\,\exists\,[a \Subset U]\,U \subseteq O \text{ defines the open collections of a space.}}$ The empty-set is open since the quantification is vacuous. Let $\mathcal{A}$ be a non-empty collection of 'open collections'. For each $a \in \bigcup\mathcal{A}$: $a \in O$, for some $\emptyset \neq O \in \mathcal{A}$; hence, there exists U such that $a \Subset U \subseteq O$; thus $a \Subset U \subseteq \bigcup\mathcal{A}$. So $\bigcup\mathcal{A}$ is 'open'. Next we show that these operations are mutually inverse. Let $\mathsf{sp}(\star)$ denote the map from 'neighbourhood relations', characterized by row 3 column 4 of Table A.1, to spaces defined by $O \in \mathsf{op}$ iff $\forall\,[a \in O]\,\exists\,[a \Subset U]\,U \subseteq O$.

Name	Notation	Definition	Characterization
open collections	op		$\emptyset \in \mathsf{op}$ $\mathcal{A} \subseteq \mathsf{op} \to \bigcup \mathcal{A} \in \mathsf{op}$
neighbourhood relation	$\Subset$	$a \Subset U \leftrightarrow \exists\,[O \in \mathsf{op}]\; a \in O \subseteq U$	$a \Subset U \subseteq V$ implies $a \Subset V$, $\forall\,[a \Subset U]\;\exists\,[a \Subset V \subseteq U]\;\forall\,[b \in V]\;\exists\,[b \Subset W \subseteq V]$
proximity relation	$\vdash$	$A \vdash b \leftrightarrow \forall\,[b \Subset U]\; U \cap A \neq \emptyset$	$a \in A \to A \vdash a$ $B \subseteq A$ and $B \vdash a \to A \vdash a$ if $B \vdash c$ and $\forall\,[b \in B]\; A \vdash b$, then $A \vdash c$
closure operator	$\lVert\cdot\rVert$	$\lVert A\rVert = \{b \,:\, A \vdash b\}$	$A \subseteq \lVert A\rVert$ $A \subseteq A' \to \lVert A\rVert \subseteq \lVert A'\rVert$, $\lVert\, \lVert A\rVert\, \rVert = \lVert A\rVert$
closed collections	cl	$A \in \mathsf{cl} \leftrightarrow \lVert A\rVert \subseteq A$	$\mathsf{uni} \in \mathsf{cl}$ $\emptyset \neq \mathcal{A} \subseteq \mathsf{cl} \to \bigcap \mathcal{A} \in \mathsf{cl}$
interior operator	$\lvert\cdot\rvert$	$\lvert A\rvert = \{a \in A \,:\, a \Subset A\}$	$A \supseteq \lvert A\rvert$ $A \subseteq A' \to \lvert A\rvert \subseteq \lvert A'\rvert$, $\lvert(\lvert A\rvert)\rvert = \lvert A\rvert$

Table A.1: Fundamental Definitions and Characterizations (see Theorem A.4 on page 498)

Consider a 'neighbourhood relation' $\Supset$.

$\boxed{\Supset_{\mathsf{sp}(\Supset)} = \Supset}$ $\boxed{\Supset_{\mathsf{sp}(\Supset)} \subseteq \Supset}$ Suppose that $U \Supset_{\mathsf{sp}(\Supset)} a$. Then there exists $O \in \mathsf{op}(\mathsf{sp}(\Supset))$, such that $a \in O \subseteq U$. Since $a \in O \in \mathsf{op}(\mathsf{sp}(\Supset))$, there exists V, with $a \Subset V \subseteq O$. But then, $a \Subset V \subseteq O \subseteq U$. Hence, $a \Subset U$. $\boxed{\Supset_{\mathsf{sp}(\Supset)} \supseteq \Supset}$ Suppose that $a \Subset U$. Then there exists V such that $a \Subset V \subseteq U$ and for all $b \in V$ there exists W with $b \Subset W \subseteq V$. For notational convenience, let $O = V$. Clearly $a \in O \subseteq U$. Let $b \in O = V$. Then there exists W with $b \Subset W \subseteq V = O$. Hence $O \in \mathsf{sp}(\Supset)$. Since $a \in O \subseteq U$, we have $a \Subset_{\mathsf{sp}(\Supset)} U$. Consider a space $\mathbb{A}$.

$\boxed{\mathsf{sp}(\Supset_{\mathbb{A}}) = \mathbb{A}}$ $\boxed{\mathsf{sp}(\Supset_{\mathbb{A}}) \subseteq \mathbb{A}}$ Let $O \in \mathsf{op}(\mathsf{sp}(\Supset))$. By definition, for each $a \in O$, $\exists [U_a] a \Subset U_a \subseteq O$. Hence, there exists $O_a \in \mathsf{op}(\mathbb{A})$, with $a \in O_a \subseteq U_a$. So $O = \bigcup_{a \in O} O_a \in \mathsf{op}(\mathbb{A})$. $\boxed{\mathsf{sp}(\Supset_{\mathbb{A}}) \supseteq \mathbb{A}}$ Let $O \in \mathsf{op}(\mathbb{A})$. For each $a \in O$: since $a \in O \subseteq O$, $a \Subset_{\mathbb{A}} O$, hence $a \Subset_{\mathbb{A}} O \subseteq O$. Consequently, $O \in \mathsf{sp}(\Supset)$. Next we show that the proximity relation satisfies the conditions of row 3 column 4 of Table A.1. $\boxed{\text{Row 3 Column 4 is valid}}$ $\boxed{a \in A \to A \vdash a}$ Suppose that $a \in A$. If a is constrained then it has no neighbourhood, in which case, every neighbourhood of a meets A. Otherwise, a has neighbourhoods, in which case they all meet A, since $a \in A$. $\boxed{B \subseteq A \text{ and } B \vdash a \to A \vdash a}$ Suppose that $B \subseteq A$ and $B \vdash a$. Then every neighbourhood of a meets B and hence meets A. (This quantification may be vacuous but the result stands.) $\boxed{\text{if } B \vdash c \text{ and } \forall\,[b \in B]\; A \vdash b, \text{then } A \vdash c}$ Suppose that $B \vdash c$ and $\forall\,[b \in B]\; A \vdash b$. Suppose that $c \Subset_{\mathbb{A}} U$. Then there exists an open collection O with $c \subseteq O \subseteq U$. By definition, $c \Subset_{\mathbb{A}} O$, and hence O meets B. Let $b \in O \cap B$. Then since $b \Subset_{\mathbb{A}} O$, O meets A. Hence U meets A. Since row 3 column 4 of Table A.1 characterizes point-consequence relations, the equivalence of the characterizations of row 3 column 4, row 4 column 4 and row 5 column 4, follow from the results of $\oint 4$ and definitions. The proof of the equivalence of closed systems and open systems is very simple via complementation. We leave the rest to the reader. $\diamond$

Convention A.5 (Determining Spaces by Closed Systems) Consequent to Theorem A.4, when we speak of a closed system $\mathbb{C}$ determining a space $\mathbb{A}$, we mean the space $\mathbb{A}$ with $\mathsf{cl}_{\mathbb{A}} = \mathsf{cl}_{\mathbb{C}}$.

Remark A.6 Since $\mathsf{cl}_{\mathbb{A}}$ is a closed system, $\mathbf{cl}_{\mathbb{A}}$ is a complete lattice, $\blacktriangle^{\mathbf{cl}_{\mathbb{A}}} \emptyset = \mathsf{uni}(\mathbb{A})$, $\blacktriangle^{\mathbf{cl}_{\mathbb{A}}} \mathcal{C} = \bigcap \mathcal{C}$, for $\emptyset \neq \mathcal{C} \subseteq \mathsf{cl}_{\mathbb{A}}$, and for $\mathcal{C} \subseteq \mathsf{cl}_{\mathbb{A}}$, $\blacktriangledown^{\mathbf{cl}_{\mathbb{A}}} \mathcal{C} = \bigcap \{G \in \mathsf{cl}_{\mathbb{A}} \;:\; \bigcup \mathcal{C} \subseteq G\} = \lVert \bigcup \mathcal{C} \rVert_{\mathbb{A}} =$

$\Rsh$	op	$\Subset$	$\vdash$
op		Definitional	$A \vdash a \leftrightarrow a \in O \in \mathsf{op} \to O \cap A \neq \emptyset$
$\Subset$	$O \in \mathsf{op}$ iff $\forall [a \in O] \exists [a \Subset U] U \subseteq O$		Definitional
$\vdash$	$O \in \mathsf{op} \leftrightarrow \vdash [\neg O] \cap O = \emptyset$	$a \Subset U \leftrightarrow \neg U \not\vdash a$	
$\Vert\cdot\Vert$	$O \in \mathsf{op} \leftrightarrow O = \neg \Vert \neg O \Vert$	$a \Subset U \leftrightarrow a \notin \Vert \neg U \Vert$	$A \vdash b \leftrightarrow b \in \Vert A \Vert$
cl	$\mathsf{op} = \{\neg G \ : \ G \in \mathsf{cl}\}$	$a \Subset U \leftrightarrow \exists [\neg U \subseteq G \in \mathsf{cl}] \, a \notin G$	$A \vdash a \leftrightarrow (A \subseteq G \in \mathsf{cl} \to a \in G)$
$\vert\cdot\vert$	$\mathsf{op} = \{\vert A \vert \ : \ A \subseteq \mathsf{uni}\}$	$a \Subset U \leftrightarrow a \in \vert U \vert$	$A \vdash a \leftrightarrow a \notin \vert \neg A \vert$

$\Rsh$	$\Vert\cdot\Vert$	cl	$\vert\cdot\vert$
op	$a \in \Vert A \Vert \leftrightarrow a \in O \in \mathsf{op} \to O \cap A \neq \emptyset$	$\mathsf{cl} = \{\neg O \ : \ O \in \mathsf{op}\}$	$\vert A \vert = \bigcup\{O \in \mathsf{op} \ : \ O \subseteq A\}$
$\Subset$	$a \in \Vert A \Vert \leftrightarrow \forall [a \Subset U] \, U \cap A \neq \emptyset$	$G \in \mathsf{cl}$ iff $\forall [a \in \neg G] \, \exists [a \Subset U] \, U \cap G = \emptyset$	Definitional
$\vdash$	Definitional	$G \in \mathsf{cl}$ iff $G \vdash a \to a \in G$	$a \in \vert A \vert \leftrightarrow \neg A \not\vdash a$
$\Vert\cdot\Vert$		Definitional	$\vert A \vert = \neg \Vert \neg A \Vert$
cl	$\Vert A \Vert = \bigcap\{G \in \mathsf{cl} \ : \ A \subseteq G\}$		$a \in \vert A \vert \leftrightarrow \neg A \subseteq G \in \mathsf{cl} \to a \notin G$
$\vert\cdot\vert$	$\Vert A \Vert = \neg \vert \neg A \vert$	$G \in \mathsf{cl}$ iff $G = \neg \vert \neg G \vert$	

Table A.2: Fundamental Relationships (see Theorem A.4 on page 498)

$\Vert \bigcup\{\Vert A \Vert_{\mathbb{A}} \ : \ A \in \mathcal{A}\} \Vert_{\mathbb{A}}$ (see Remark 4.46).

Proposition A.7 $\blacktriangledown^{\mathbf{op}(\mathbb{A})} \mathcal{A} = \bigcup \mathcal{A}$, $\blacktriangle^{\mathbf{op}(\mathbb{A})} \emptyset = \mathfrak{p}_{\mathbb{A}}$, and, for $\emptyset \neq \mathcal{O} \subseteq \mathsf{op}(\mathbb{A})$,

$$\overset{\mathbf{op}(\mathbb{A})}{\blacktriangle} \mathcal{A} = \bigcup\{O \in \mathsf{op}(\mathbb{A}) \ : \ O \subseteq \bigcap \mathcal{A}\} = \left| \bigcap \mathcal{A} \right|_{\mathbb{A}} = \left| \bigcap \{ |A|_{\mathbb{A}} \ : \ A \in \mathcal{A}\} \right|_{\mathbb{A}}.$$

Remark A.8 $(\overset{\mathsf{uni}(\mathbb{A})}{\neg} \cdot)_{|\mathsf{cl}_{\mathbb{A}}} : \mathbf{cl}_{\mathbb{A}} \cong \mathbf{op}(\mathbb{A})^{\mathsf{d}}$ with inverse isomorphism $(\overset{\mathsf{uni}(\mathbb{A})}{\neg} \cdot)_{|\mathsf{op}(\mathbb{A})}$.

Definition A.9 (Special Spaces) We call a space $\mathbb{A}$ **plain** or **unconstrained**, or even **open**, if $\mathfrak{p}_{\mathbb{A}} = \mathsf{uni}(\mathbb{A})$, otherwise we call the space **constrained**. We call $\mathbb{A}$ **effectively-constrained** if $\emptyset$ is closed *and* is covered by a unique closed collection. We call $\mathbb{A}$ **discrete**, **trivial** and **indiscrete** (or **almost-trivial**), if $\mathsf{cl}_{\mathbb{A}} = \mathfrak{P}(\mathsf{uni}(\mathbb{A}))$, $\mathsf{cl}_{\mathbb{A}} = \{\mathsf{uni}(\mathbb{A})\}$ and $\mathsf{cl}_{\mathbb{A}} = \{\emptyset, \mathsf{uni}(\mathbb{A})\}$, respectively. The discrete, trivial and indiscrete spaces on X, are denoted by $\mathbb{S}(X, \bot)$, $\mathbb{S}(X, \top)$ and $\mathbb{S}(X, \top_{\emptyset})$, respectively. We call $\mathbb{A}$ **absolute** (or **context free**) if $\mathsf{op}(\mathbb{A})$ is closed under finite non-empty intersections, otherwise we call it **relative** (or **context sensitive**). A **completely-absolute** space is a space whose open sets are closed under arbitrary non-empty intersections. Plain absolute spaces are called **topological**. A space is called **clopen** if every open collection is closed, and is called **finitary** if its closed collections form an algebraic closed system. $\square$

Proposition A.10 For a space $\mathbb{A}$, the following conditions are equivalent.

1. $\mathbb{A}$ is context-free.

2. The intersection of two open collections is open.

3. $\mathsf{cl}_{\mathbb{A}}$ is closed under finite non-empty unions.

4. The union of two closed collections is closed.

5. For every $a \in \mathsf{uni}(\mathbb{A})$, if $a \Subset_{\mathbb{A}} U$ and $a \Subset_{\mathbb{A}} V$, then $a \Subset_{\mathbb{A}} U \cap V$.

Proof. We prove (2)⇔(5) as an example.
$\boxed{\scriptstyle (2)\Rightarrow(5)}$ Suppose that $a \Subset_{\mathbb{A}} U_1$ and $a \Subset_{\mathbb{A}} U_2$. So there exist open collections O_1 and O_2, with $a \in O_1 \subseteq U_1$ and $a \in O_2 \subseteq U_2$. By assumption (2), $O_1 \cap O_2$ is an open collection, and certainly contains a. Further, $O_1 \cap O_2 \subseteq U_1 \cap U_2$. Hence $a \Subset_{\mathbb{A}} U_1 \cap U_2$. $\boxed{\scriptstyle (5)\Rightarrow(2)}$ Let O_1 and O_2 be two open collections. If $O_1 \cap O_2 = \emptyset$, then certainly $O_1 \cap O_2$ is open. Otherwise, $O_1 \cap O_2 \neq \emptyset$. (*By Theorem A.4, it suffices to show that every point in $O_1 \cap O_2$ has a neighbourhood contained in $O_1 \cap O_2$.*) Let $a \in O_1 \cap O_2$. Then $a \Subset_{\mathbb{A}} O_1$ and $a \Subset_{\mathbb{A}} O_2$, and hence by assumption (5), $a \Subset_{\mathbb{A}} O_1 \cap O_2$, and certainly $O_1 \cap O_2 \subseteq O_1 \cap O_2$. ⋄

A.2 Continuous Translations

Recall the definition of a continuous translation between (concrete) closed systems, and recall that we showed that a translation is continuous iff the *reduced* pre-image of every closed set is closed. We now show that a translation is continuous iff the (*standard*) pre-image of every open set is open. Of course we are conflating closed systems and spaces; as such, this result informs the theory of closed systems.

Theorem A.11 For a translation τ from $\mathbb{A}$ to $\mathbb{B}$, the following conditions are equivalent.

1. τ is continuous from $\mathbb{A}$ into $\mathbb{B}$.
2. $\overleftarrow{\tau}\,[O] \in \mathsf{op}(\mathbb{A})$, for all $O \in \mathsf{op}(\mathbb{B})$.
3. $\overleftarrow{\tau}\,[|C|] \subseteq |\overleftarrow{\tau}\,[|C|]| \quad (\forall\, [C \subseteq \mathsf{uni}(\mathbb{B})]\,)$.
4. $\overleftarrow{\tau}\,[|C|] = |\overleftarrow{\tau}\,[|C|]| \quad (\forall\, [C \subseteq \mathsf{uni}(\mathbb{B})]\,)$.
5. $|\overleftarrow{\tau}\,[P]| = \overleftarrow{\tau}\,[P] \qquad (\forall\, [P \in \mathsf{op}(\mathbb{B})]\,)$.

Proof. $\boxed{\scriptstyle (1)\Rightarrow(2)}$ Let $P \in \mathsf{op}(\mathbb{B})$. Then $\neg P \in \mathsf{cl}_{\mathbb{B}}$. By assumption and (3) of Theorem 5.40 on page 186, $\overleftarrow{\tau}\,\lfloor\neg P\rfloor \in \mathsf{cl}_{\mathbb{A}}$, and hence $\neg\overleftarrow{\tau}\,\lfloor\neg P\rfloor \in \mathsf{op}(\mathbb{A})$. Since τ is grounded, it follows, by (5) of Lemma 5.15 on page 179, $\neg\overleftarrow{\tau}\,\lfloor\neg P\rfloor = \overleftarrow{\tau}\,[\neg\neg P] = \overleftarrow{\tau}\,[P]$, as required. $\boxed{\scriptstyle (2)\Rightarrow(1)}$ Dual to the proof of (1)⇒(2). $\boxed{\scriptstyle (2)\Rightarrow(3)}$ By assumption (2), $\overleftarrow{\tau}\,\left[|B|_{\mathbb{B}}\right]$ is open and *certainly* is contained in itself, and so, by maximality, $\overleftarrow{\tau}\,\left[|B|_{\mathbb{B}}\right] \subseteq \left|\overleftarrow{\tau}\,\left[|B|_{\mathbb{B}}\right]\right|_{\mathbb{A}}$. $\boxed{\scriptstyle (3)\Rightarrow(4)}$ The required converse inclusion is *always* true since interior operators are $\subseteq$-decreasing. $\boxed{\scriptstyle (4)\Rightarrow(5)}$ Let $P \in \mathsf{op}(\mathbb{B})$. Since P is open, $|P| = P$, so $|\overleftarrow{\tau}\,[P]| = \left|\overleftarrow{\tau}\,\left[|P|\right]\right| \overset{(i)}{=} \overleftarrow{\tau}\,\left[|P|\right] = \overleftarrow{\tau}\,[P]$, the equality (i) following by assumption (4). ⋄

Corollary A.12 A *function* is continuous from space $\mathbb{A}$ into $\mathbb{B}$ iff the inverse image of open sets are open.

We end this section with an interesting observation. Recall from topology that a function is called open (resp. closed) if the image of an open set (resp. closed set) is open (resp. closed), and recall that an open function need not be closed. Note that while the reduced pre-image of a function coincides with the functional pre-image, this is not the case for reduced images of functions.

Proposition A.13 If f is open then, for every closed set G, $f\lfloor G\rfloor$ is closed, and dually for closed functions. The same is true for open and closed translations.

A.3 Accumulation and Convergence

To the end of justifying our usage of the term continuity, we now consider the notions of *accumulation* and *convergence*. In the next chapter, we shall characterize continuous function in terms of preservation of convergence. The notion of convergence that we introduce is in essence the notion of convergent *filter-bases* in topology [Eng68], but without the *filter*. This is unavoidable since open collections of spaces are not closed under finite intersection. While the reader may find this entirely unsatisfactory, the point is that we shall show that continuous functions (as defined in $\oint$5), preserve convergence. So all *interesting* convergences are preserved.

Definition A.14 (Accumulation) We call b an **accumulation of** A, denoted by $A \dashrightarrow_{\mathbb{A}} b$ or $b \dashleftarrow_{\mathbb{A}} A$, if every neighbourhood of b meets $A - \{b\}$. $\square$

Definition A.15 (Convergence) Let $\mathcal{A} \subseteq \mathfrak{P}(\mathsf{uni}(\mathbb{A}))$ with $\emptyset \notin A$. We say that $\mathcal{A}$ **converges to** a, denoted by $\mathcal{A} \rightsquigarrow_{\mathbb{A}} a$, if

$$\forall\, [a \Subset_{\mathbb{A}} U]\; \exists\, [B \in \mathcal{A}]\; B \subseteq U. \tag{A.1}$$

$\square$

Remark A.16 In the previous definition, it *is permissible* for $\mathcal{A} = \emptyset$.

Remark A.17 If $a \in \Bbbk_{\mathbb{A}}$, then every $\mathcal{A} \subseteq \mathfrak{P}(\mathsf{uni}(\mathbb{A}))$ converges to a, provided $\emptyset \notin \mathcal{A}$, since a has no neighbourhoods.

Remark A.18 If $a \in \mathfrak{p}_{\mathbb{A}}$ and $\mathcal{A} \rightsquigarrow_{\mathbb{A}} a$ then $\mathcal{A} \neq \emptyset$.

Proof. Since $a \in \mathfrak{p}_{\mathbb{A}}$, $a \Subset \mathfrak{p}_{\mathbb{A}}$. Since $A \rightsquigarrow_{\mathbb{A}} a$, *there exists* $B \in \mathcal{A}$ such that, $B \subseteq U$. So $\mathcal{A} \neq \emptyset$. $\diamond$

Remark A.19 $\emptyset$ converges to a iff $a \in \Bbbk_{\mathbb{A}}$.

Proof. $\boxed{\Rightarrow}$ Suppose that $\emptyset$ converges to a. Then by Remark A.18, $a \notin \mathfrak{p}_{\mathbb{A}}$. Hence $a \in \Bbbk_{\mathbb{A}}$. $\boxed{\Leftarrow}$ Suppose that $a \in \Bbbk_{\mathbb{A}}$. Then by Remark A.17, $\emptyset$ converges to a. $\diamond$

Remark A.20 If $\emptyset \notin \mathcal{A} \subseteq \mathcal{A}' \subseteq \mathfrak{P}(\mathsf{uni}(\mathbb{A}))$, then, $\mathcal{A} \rightsquigarrow_{\mathbb{A}} a$ implies $\mathcal{A}' \rightsquigarrow_{\mathbb{A}} a$.

The following result characterizes consequence/nearness in terms of convergence.

Theorem A.21 (Characterizing Proximity as Convergence) For $B \cup \{a\} \subseteq \mathsf{uni}(\mathbb{A})$, the following conditions are equivalent.

1. $B \vdash_{\mathbb{A}} a$.

2. $\{U \cap B \ : \ a \Subset_{\mathbb{A}} U\}$ converges to a.

3. There exists $\emptyset \notin \mathcal{A} \subseteq \mathfrak{P}(B)$ such that $\mathcal{A}$ converges to a.

4. $\mathfrak{P}(B) - \{\emptyset\}$ converges to a.

Proof. $\boxed{(1)\Rightarrow(2)}$ If $a \in \Bbbk_{\mathbb{A}}$, then $\{U \cap B \ : \ a \Subset_{\mathbb{A}} U\} = \emptyset$, which converges to a, by Remark A.17. Otherwise, $a \in \mathfrak{p}_{\mathbb{A}}$ and hence $\emptyset \notin \{U \cap B \ : \ a \Subset_{\mathbb{A}} U\}$. Hence $B \cap \mathfrak{p}_{\mathbb{A}} \neq \emptyset$. Suppose that $a \Subset V$. By assumption (1), $V \cap B \neq \emptyset$. Setting $A = V \cap B$, we have $\emptyset \neq A \in \{U \cap B \ : \ U \in \mathsf{Nbh}(a)\}$ with $A \subseteq V$. $\boxed{(2)\Rightarrow(3)}$ Trivial. $\boxed{(3)\Rightarrow(4)}$ By Remark A.20. $\boxed{(4)\Rightarrow(1)}$ If $a \in \Bbbk_{\mathbb{A}}$, then certainly $B \vdash_{\mathbb{A}} a$. Otherwise $a \in \mathfrak{p}_{\mathbb{A}}$. Let $a \Subset_{\mathbb{A}} U$. Since $\mathfrak{P}(B) - \{\emptyset\}$ converges to a, there exists $A \in \mathfrak{P}(B) - \{\emptyset\}$ with $A \subseteq U$. So $U \cap B \neq \emptyset$. Hence $B \vdash_{\mathbb{A}} a$. ◇

We end this section by noting that for *context-free* spaces, such as topological spaces, the convergent collections characterizing consequence in the previous result may be sharpened to *filter*-bases (see [Eng68] for a definition).

Proposition A.22 If $\mathbb{A}$ is *context-free* then $\{U \cap B \ : \ a \Subset_{\mathbb{A}} U\}$ is closed under *finite non-empty intersections.*

Proof. Let $\mathcal{A} = \{U \cap B \ : \ a \Subset_{\mathbb{A}} U\}$. Let $U_1 \cap B, U_2 \cap B \in \mathcal{A}$. If either $U_1 \cap B$ or $U_2 \cap B$ are empty, then their intersection is empty and a member $\mathcal{A}$. Otherwise both are non-empty. By Proposition A.10, $a \Subset_{\mathbb{A}} U_1 \cap U_2$. So $(U_1 \cap B) \cap (U_2 \cap B) = (U_1 \cap U_2) \cap B \in \mathcal{A}$. So $\mathcal{A}$ is closed under binary intersections, and hence, by induction, is closed under finite non-empty intersections. ◇

Corollary A.23 [Eng68] For a *context-free* space $\mathbb{A}$, $B \cup \{a\} \subseteq \mathsf{uni}(\mathbb{A})$, the following conditions are equivalent.

1. $B \vdash_{\mathbb{A}} a$.

2. There exists a *filter-base* $\mathcal{A} \subseteq \mathfrak{P}(B)$ such that $\mathcal{A}$ converges to a.

Open Problem A.24 Formulate the analgous result for finitary spaces.

A.4 Continuous Functions

We now characterize continuous functions (as defined in ∮5) in terms of preservation of convergence.

Theorem A.25 Let $\mathbb{A}$ and $\mathbb{B}$ be spaces and f a function from $\mathsf{uni}(\mathbb{A})$ into $\mathsf{uni}(\mathbb{B})$. The following conditions are equivalent.

1. f is continuous from $\mathbb{A}$ into $\mathbb{B}$.

2. $\mathcal{A} \rightsquigarrow a$ implies $f\{\mathcal{A}\} \rightsquigarrow f(a)$, for all $\emptyset \notin \mathcal{A} \subseteq \mathfrak{P}(\mathsf{uni}(\mathbb{A}))$ and $a \in \mathsf{uni}(\mathbb{A})$.

Proof. $\boxed{\scriptstyle (1)\Rightarrow(2)}$ Assume that $\mathcal{A} \rightsquigarrow a$. Suppose that $f(a) \Subset V$. (We must show that there exists $X \in \mathcal{A}$ with $f[X] \subseteq V$.) By assumption, there exists $a \Subset U$ with $f[U] \subseteq V$. Since $\mathcal{A} \rightsquigarrow a$, there exists $X \in \mathcal{A}$ with $X \subseteq U$. So $f[X] \subseteq f[U] \subseteq V$. $\boxed{\scriptstyle (2)\Rightarrow(1)}$ Suppose that $A \vdash_{\mathbb{A}} a$. By equivalent condition (3) of Theorem A.21 on page 502, there exists $\emptyset \notin \mathcal{A} \subseteq \mathfrak{P}(A)$ such that $\mathcal{A} \rightsquigarrow_{\mathbb{A}} a$. So by assumption (2), $f\{\mathcal{A}\} \rightsquigarrow_{\mathbb{B}} f(a)$. Since $\mathcal{A} \subseteq \mathfrak{P}(A)$, $f\{\mathcal{A}\} \subseteq f[A]$, and so by equivalent condition (3) of Theorem A.21, $f[A] \vdash_{\mathbb{B}} f(a)$. $\diamond$

Generally, continuous functions do not preserve accumulations (not even between topological spaces), i.e., $A \dashrightarrow a$ does not necessarily imply that $f[A] \dashrightarrow f(a)$.

Counter Example A.26 ($A \dashrightarrow a \not\Rightarrow f[A] \dashrightarrow f(a)$)

> Let $\mathbb{A}$ be any topological space such that $A \dashrightarrow a$. Let $\mathbb{B}$ be the topological space with singleton universe $\{b\}$ and f the function from $\mathsf{uni}(\mathbb{A})$ onto $\{b\}$ mapping all points to b. Then it is not true that $f[A] \dashrightarrow f(a)$.

$\square$

Open Problem A.27 Is the term continuity justified for translations. One immediate problem that arise is that the poles of a translation are sets (functional poles are points). Further, our initial invesigations inform us that convergence is 'best preserved' by the *reduced*-image of translations. It appears to us that one requires a notion of a neighbourhood of a *set* (as opposed to a point) and a notion of convergence to a *set*; such notions are (partially) developed in the next appendix.

Appendix B

Elementary Interior and Elementary Spaces

In this appendix we show that topology may be given an elementary underpinning.

B.1 Elementary Interior

B.1.1 Elementary Interior Operators

Elementary interior operators are formulated dually to elementary closure operators.

Definition B.1 (Elementary Interior Operators) The **type of elementary interior operators**, denoted $\mathsf{type}(\mathsf{eio})$, has a binary relation symbol $\leq$ and a unary operation symbol $|\cdot|$. An **elementary space** is a $\mathsf{type}(\mathsf{esp})$-structure $\mathfrak{o} = \langle \mathsf{uni}_{\mathsf{e}}(\mathfrak{o}); \leq^{\mathfrak{o}}; |\cdot|_{\mathfrak{o}} \rangle$ whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{o}}$ and called the **underlying order**, and such that $\mathfrak{o}$ satisfies the (further) axioms

$$\text{(order-preserving)} \qquad x \leq y \longrightarrow |x| \leq |y| \,, \tag{B.1}$$

$$\text{(decreasing)} \qquad |x| \leq x \qquad \text{and} \tag{B.2}$$

$$\text{(idempotent)} \qquad ||x|| \approx |x| \,, \tag{B.3}$$

in which case we call $|\cdot|_{\mathfrak{o}}$ the **interior operator** and write $\mathsf{uni}_{\mathsf{e}}(\mathfrak{o})$ for $\mathsf{uni}(\mathbf{P}_{\mathfrak{o}})$ which we call the **elementary universe** (or just **universe** when unambiguous). When we call $\mathfrak{o}$ an **elementary interior operator on order P**, we mean that $\mathbf{P}_{\mathfrak{o}} = \mathbf{P}$. For an order $\mathbf{P}$ and an operator $|\cdot|$ on $\mathsf{uni}_{\mathsf{e}}(\mathbf{P})$, when we say that $|\cdot|$ determines an (elementary) interior operator on $\mathbf{P}$, or say that $\langle \mathbf{P}; |\cdot| \rangle$ is an (elementary) interior operator, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; |\cdot| \rangle$ is an elementary interior operator. Let $\mathsf{EIO}(\mathbf{P})$ denote the set of (elementary) interior operators on order $\mathbf{P}$. $\square$

Lemma B.2 Let u be an idempotent operator on the universe of order $\mathbf{P}$. If u is decreasing then u is order-preserving iff $\mathsf{u} : \mathbf{P} \rightarrow_{\blacktriangle} \mathsf{u}[\mathbf{P}]$.

Proof. Dual to Lemma 4.3 on page 134. $\diamond$

The following corollary to the previous lemma characterizes the elementary interior operators on a given order.

Corollary B.3 Let $|\cdot|$ be an operator on the universe of order $\mathbf{P}$. Then $|\cdot|$ determines an elementary interior operator on $\mathbf{P}$ iff $|\cdot|$ is decreasing, idempotent and $|\cdot| : \mathbf{P} \to_{\blacktriangle} \mathsf{u}[\mathbf{P}]$.

B.1.2 Elementary Open Systems

Dual to elementary closed systems are elementary open systems.

Definition B.4 (Elementary Open Systems) The **type of elementary open systems**, denoted $\mathsf{type}(\mathsf{eos})$, has a binary relation symbol $\leq$ and a unary relation symbol op. An **elementary open system** is a $\mathsf{type}(\mathsf{eos})$-structure whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{o}}$ and called the **underlying order**, and is such that $\mathfrak{o}$ satisfies the axiom

$$\forall[x]\, \exists[z]\, (z \text{ is } \mathsf{op}\, \mathsf{and}\, z \leq x\, \mathsf{and} (\forall[y]\, y \text{ is } \mathsf{op}\, \mathsf{and}\, y \leq x \longrightarrow y \leq z)), \tag{B.4}$$

in which case we call $\mathsf{op}(\mathfrak{o})\cdot$ the associated **open relation** and write $\mathsf{uni}_{\mathsf{e}}(\mathfrak{o})$ for $\mathsf{uni}(\mathbf{P}_{\mathfrak{o}})$ which we call the **elementary universe** (or just **universe** when unambiguous). We tend to conflate the unary relation $\mathsf{op}(\mathfrak{o})$ with the set $\{c \;:\; c \text{ is } \mathsf{op}(\mathfrak{o})\}$, hence writing either $a \,\mathsf{is}\, \mathsf{op}(\mathfrak{o})$ or $a \in \mathsf{op}(\mathfrak{o})$, as appropriate. When we call $\mathfrak{o}$ an **elementary open system on order P**, we mean that $\mathbf{P}_{\mathfrak{o}} = \mathbf{P}$. For an order $\mathbf{P}$ and a unary relation op on $\mathsf{uni}(\mathbf{P})$, when we say that op determines an (elementary) open system on $\mathbf{P}$, or say that $\langle \mathbf{P}; \mathsf{op}\rangle$ is an (elementary) open system, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; \mathsf{op}\rangle$ is an elementary interior operator. Let $\mathsf{EOS}(\mathbf{P})$ denote the set of (elementary) open systems on order $\mathbf{P}$. The suborder of $\mathbf{P}_{\mathfrak{o}}$ induced by $\mathbf{P}_{\mathfrak{o}}$ on (the set) $\mathsf{op}(\mathfrak{o})$ is denoted by (emboldened) $\mathbf{op}(\mathfrak{o})$. □

Proposition B.5 Let $\mathbf{P}$ be an order and op a unary relation on $\mathsf{uni}(\mathbf{P})$. Then op determines an elementary open system on $\mathbf{P}$ iff

$$\forall\, [a \in \mathsf{uni}(\mathbf{P})]\;\; \blacktriangledown\big(\langle a]_{\mathbf{P}_{\mathfrak{o}}} \cap \mathsf{op}\big) \text{ exists and is}\, \mathsf{op}\,. \tag{B.5}$$

□

Consequently, if $\mathfrak{o}$ is an elementary open system then, for each $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{o})$, there exists a (unique) $\leq^{\mathfrak{o}}$-least open point above a. We highlight some useful consequences for ease of later reference, first introducing facilitating notation.

Definition B.6 (Open-Cover) Let $\mathfrak{o}$ be an elementary open system. With each $a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{o})$, we associate the set $\mathsf{support}_{\mathfrak{o}}(a)$, which we call the **open-cover** of a, defined by

$$\mathsf{support}_{\mathfrak{o}}(a) = \langle a]_{\mathbf{P}_{\mathfrak{o}}} \cap \mathsf{op}(\mathfrak{o}) \doteq \{b \,\mathsf{is}\, \mathsf{op}(\mathfrak{o}) \;:\; b \leq a\}.$$

□

Corollary B.7 For an elementary open system $\mathfrak{o}$ the following formulae are valid.

$$\blacktriangledown \mathsf{support}_{\mathfrak{o}}(a) \text{ exists,} \tag{B.6}$$

$$\blacktriangledown \mathsf{support}_{\mathfrak{o}}(a) \text{ is } \mathsf{op}(\mathfrak{o}), \tag{B.7}$$

$$a \geq \blacktriangledown \mathsf{support}_{\mathfrak{o}}(a), \tag{B.8}$$

$$\mathsf{support}_{\mathfrak{o}}(a) = \mathsf{support}_{\mathfrak{o}}(\blacktriangledown \mathsf{support}_{\mathfrak{o}}(a)), \tag{B.9}$$

$$\blacktriangledown \mathsf{support}_{\mathfrak{o}}(a) \in \mathsf{support}_{\mathfrak{o}}(a), \tag{B.10}$$

$$a \text{ is } \mathsf{op}(\mathbf{O}) \text{ iff } a = \blacktriangledown \mathsf{support}_{\mathfrak{o}}(a) \quad \text{and} \tag{B.11}$$

$$\blacktriangledown \mathsf{support}_{\mathfrak{o}}(a) = \overset{\mathbf{op}(\mathfrak{o})}{\blacktriangledown} \mathsf{support}_{\mathfrak{o}}(a). \tag{B.12}$$

Example B.8 (The Discrete Open System on P)

For any ordered set $\mathbf{P}$, $\mathsf{uni}(\mathbf{P})$ determines an open system on $\mathbf{P}$. We call this open system the **discrete open system** on $\mathbf{P}$.

□

Definition B.9 (Associating Elementary Open Systems and Interior Operators) With each elementary interior operator $\mathfrak{o}$, we associate the elementary open system $\mathsf{eos}(\mathfrak{o})$ on $\mathbf{P}_{\mathfrak{o}}$, for which we tend to write $\mathsf{op}(\mathfrak{o})$ for $\mathsf{op}(\mathsf{eos}(\mathfrak{o}))$, determined by

$$a \text{ is } \mathsf{op}(\mathfrak{o}) \text{ iff } |a|_{\mathfrak{o}} = a. \tag{B.13}$$

With each elementary open system $\mathbf{O}$, we associate the elementary interior operator $\mathsf{eio}(\mathbf{O})$ on $\mathbf{P}_{\mathbf{O}}$, for which we tend to write $|\cdot|_{\mathbf{O}}$ for $|\cdot|_{\mathsf{eio}(\mathbf{O})}$, determined by

$$|a|_{\mathbf{O}} = \blacktriangledown \mathsf{support}_{\mathfrak{o}}(a), \tag{B.14}$$

this operator being well-defined by (B.6). □

Proof. Dual to proof of Definition B.9 on page 507. ◇

Proposition B.10 For order $\mathbf{P}$, $\mathsf{eos}(\cdot)$ and $\mathsf{eio}(\cdot)$ define mutually inverse bijections between $\mathsf{EIO}(\mathbf{P})$ and $\mathsf{ECS}(\mathbf{P})$.

Proof. Dual to proof of Proposition 4.11 on page 137. ◇

Convention B.11 (Conflating Elementary Interior Operators and Open Systems) Consequent to the previous definition and proposition we shall (tend to) syntactically conflate elementary interior operators and open systems, and, as such, treat (B.13) and (B.14) as properties of these conflated structures.

Remark B.12 Rephrasing (B.13) in the light of the previous results,

$$\mathsf{op}(\mathfrak{o}) = \{|a|_{\mathfrak{o}} \;:\; a \in \mathsf{uni}_{\mathsf{e}}(\mathfrak{o})\}. \tag{B.15}$$

Remark B.13 $|a|_{\mathfrak{o}}$ is the greatest $\mathfrak{o}$-open c below a (by (B.14) and (B.10)). This property is referred to as the **maximality property of elementary interior operators** or simply **maximality** where unambiguous. Further, by (B.13) and (B.2),

$$a \,\text{is}\, \mathsf{op}(\mathfrak{o}) \text{ iff } |a|_{\mathfrak{o}} \geq a. \tag{B.16}$$

Corollary B.14 If $\mathfrak{o}$ is an elementary open system and $A \subseteq \mathsf{uni_e}(\mathfrak{o})$ such that $\blacktriangle A$ exists, then $\blacktriangle^{\mathbf{op}(\mathfrak{o})}\{|a|_{\mathfrak{o}} \,:\, a \in A\}$ exists and

$$|\blacktriangle A|_{\mathfrak{o}} = \overset{\mathbf{op}(\mathfrak{o})}{\blacktriangle}\{|a|_{\mathfrak{o}} \,:\, a \in A\}. \tag{B.17}$$

Proof. Dual to Corollary 4.15 on page 137. $\diamond$

Proposition B.15 If $\mathbf{O}$ is an elementary open system then $\mathbf{op}(\mathbf{O}) \lhd_{\blacktriangledown} \mathbf{P_O}$.

Proof. Dual to Proposition 4.16 on page 138. $\diamond$

B.1.3 Elementary Neighbourhood Relations

Note that the elementary neighbourhood relation is *not* the dual of the elementary consequence relation. As we shall soon see, in the concrete setting it is the (reversed) consequence relation.

Definition B.16 (Elementary Neighbourhood Relations) The **type of elementary neighbourhood relations**, denoted $\mathsf{type}(\mathsf{enr})$, has a binary relation symbol $\leq$ and a binary relation symbol $\Supset$, which we often write in reversed form $\Subset$. An **elementary neighbourhood relation** is a $\mathsf{type}(\mathsf{enr})$-structure $\mathfrak{o} = \langle \mathsf{uni_e}(\mathfrak{o}); \Supset_{\mathfrak{o}}; \leq_{\mathfrak{o}}\rangle$ whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{o}}$ and called the **underlying order**, and such that $\mathfrak{o}$ satisfies the (further) axioms

$$y \Supset x \longrightarrow y \geq x, \tag{B.18}$$

$$z \geq y \Supset x \longrightarrow z \Supset x, \tag{B.19}$$

$$\begin{aligned} \forall[x]\exists[y] \quad & x \Supset y \text{ and } (\forall[z]\, x \Supset z \longrightarrow y \geq z) \\ & \text{and } (\forall[w]\, x \Supset w \longrightarrow y \Supset w) \\ & \text{and } (\forall[v]\, y \geq v \longrightarrow y \Supset v) \end{aligned} \tag{B.20}$$

in which case we call $\Supset_{\mathfrak{o}}$ (and $\Subset_{\mathfrak{o}}$) the **neighbourhood relation** and write $\mathsf{uni_e}(\mathfrak{o})$ for $\mathsf{uni}(\mathbf{P}_{\mathfrak{o}})$ which we call the **elementary universe** (or just **universe** when unambiguous). When we call $\mathfrak{o}$ an **elementary neighbourhood relation on order P**, we mean that $\mathbf{P}_{\mathfrak{o}} = \mathbf{P}$. For an order $\mathbf{P}$ and an operator $\Supset$ on $\mathsf{uni_e}(\mathbf{P})$, when we say that $\Supset$ determines a (elementary) neighbourhood relation on $\mathbf{P}$, or say that $\langle \mathbf{P}; \Supset\rangle$ is a (elementary) neighbourhood relation, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; \Supset\rangle$ is an elementary neighbourhood relation. Let $\mathsf{ENR}(\mathbf{P})$ denote the set of (elementary) interior operators on order $\mathbf{P}$. $\square$

Proposition B.17 A binary relation $\ni$ on the universe of an order $\mathbf{P}$ determines an elementary neighbourhood relation on $\mathbf{P}$ iff it satisfies (B.18) and, for all $a \in \mathsf{uni}(\mathbf{P})$,

$$\blacktriangledown \ni [\![a]\!] \text{ exists,} \tag{B.21}$$

$$a \ni (\blacktriangledown \ni [\![a]\!]), \tag{B.22}$$

$$a \ni b \longrightarrow (\blacktriangledown \ni [\![a]\!]) \ni b, \tag{B.23}$$

$$(\blacktriangledown \ni [\![a]\!]) \geq b \longrightarrow (\blacktriangledown \ni [\![a]\!]) \ni b, \tag{B.24}$$

$$a \leq b \text{ implies } \ni_{\mathfrak{o}} [\![a]\!] \subseteq \ni_{\mathfrak{o}} [\![b]\!]. \tag{B.25}$$

Warning B.18 Condition (B.21) is so fundamental that we may refer to it without explicit reference.

We enumerate some properties of elementary neighbourhood relations that prove useful in the sequel.

Proposition B.19 For an elementary neighbourhood relation $\mathfrak{o}$, the following formulae are all valid.

$$\blacktriangledown \ni [\![a]\!] \leq a, \tag{B.26}$$

$$\ni [\![a]\!] = \ni [\![\blacktriangledown \ni [\![a]\!]]\!], \quad \text{i.e.,} \quad a \ni b \text{ iff } (\blacktriangledown \ni [\![a]\!]) \ni b, \tag{B.27}$$

$$\blacktriangledown \ni [\![\blacktriangledown \ni [\![a]\!]]\!] = \blacktriangledown \ni [\![a]\!], \tag{B.28}$$

$$a \ni_{\mathfrak{o}} b \text{ iff } (\blacktriangledown \ni_{\mathfrak{o}} [\![a]\!]) \geq b. \tag{B.29}$$

Proof. $\boxed{\text{(B.26)}}$ Since $a \ni (\blacktriangledown \ni [\![a]\!])$ by (B.21), it follows by (B.18) that $a \geq (\blacktriangledown \ni [\![a]\!])$. $\boxed{\text{(B.27)}}$ Since $(\blacktriangledown \ni [\![a]\!]) \leq a$ by (B.26), it follows by (B.25) that $\ni_{\mathfrak{o}} [\![(\blacktriangledown \ni [\![a]\!])]\!] \subseteq \ni_{\mathfrak{o}} [\![a]\!]$. Conversely, let $b \in \ni_{\mathfrak{o}} [\![a]\!]$, i.e., $a \ni_{\mathfrak{o}} b$. So by (B.23), $(\blacktriangledown \ni [\![a]\!]) \ni_{\mathfrak{o}} b$, i.e., $b \in \ni_{\mathfrak{o}} [\![\blacktriangledown \ni [\![a]\!]]\!]$. $\boxed{\text{(B.28)}}$ Follows trivially from (B.21) and (B.27). $\boxed{\text{(B.29)}}$ $\boxed{\Rightarrow}$ Suppose that $a \ni_{\mathfrak{o}} b$. Then by (B.27), $(\blacktriangledown \ni_{\mathfrak{o}} [\![a]\!]) \ni_{\mathfrak{o}} b$, and so by (B.18), $\blacktriangledown \ni_{\mathfrak{o}} [\![a]\!] \geq b$. $\boxed{\Leftarrow}$ Suppose that $(\blacktriangledown \ni_{\mathfrak{o}} [\![a]\!]) \geq b$. Then by (B.26) and (B.24), $a \geq (\blacktriangledown \ni_{\mathfrak{o}} [\![a]\!]) \ni_{\mathfrak{o}} b$, and so by (B.19), $a \ni_{\mathfrak{o}} b$.
$\diamond$

Definition B.20 (Associating Elementary Neighbourhood Relations and Interior Operators) With each elementary interior operator $\mathfrak{o}$, we associate the elementary neighbourhood relation $\mathsf{enr}(\mathfrak{o})$ on $\mathbf{P}_{\mathfrak{o}}$, for which we tend to write $\ni_{\mathfrak{o}}$ for $\ni_{\mathsf{enr}(\mathfrak{o})}$, determined by

$$u \ni_{\mathfrak{o}} a \text{ iff } |u|_{\mathfrak{o}} \geq a. \tag{B.30}$$

With each elementary neighbourhood relation $\mathfrak{o}$, we associate the elementary interior operator $\mathsf{eio}(\mathfrak{o})$ on $\mathbf{P}_{\mathfrak{o}}$, for which we tend to write $|\cdot|_{\mathfrak{o}}$ for $|\cdot|_{\mathsf{eio}(\mathfrak{o})}$, determined by

$$|a|_{\mathfrak{o}} = \blacktriangledown \ni_{\mathfrak{o}} [\![a]\!], \tag{B.31}$$

this operator being well-defined by (B.30). □

Proof. $\boxed{\mathsf{enr}(\mathfrak{o}) \text{ is an elementary neighbourhood relation}}$ Let $\mathfrak{o}$ be an elementary interior operator. We shall invoke Proposition B.17. $\boxed{\text{Claim 1: } a \ni_{\mathsf{enr}(\mathfrak{o})} |a|_{\mathfrak{o}}, \text{ i.e., } |a|_{\mathfrak{o}} \in \ni_{\mathsf{enr}(\mathfrak{o})} [\![a]\!]}$ $a \ni_{\mathsf{enr}(\mathfrak{o})} |a|_{\mathfrak{o}}$ [iff] $|a|_{\mathfrak{o}} \geq |a|_{\mathfrak{o}}$ [iff] true.

$\boxed{\text{Claim 2: } \blacktriangledown \Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket = |a|_{\mathfrak{o}}}$ (*By Claim 1, it suffices to show that* $|a|_{\mathfrak{o}}$ *is an upper bound of* $\Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket$*.*) Suppose that $a \Supset_{\mathsf{enr}(\mathfrak{o})} b$. Then by definition, $b \leq |a|_{\mathfrak{o}}$. $\boxed{\text{(B.21)}}$ By Claim 2. $\boxed{\text{(B.22)}}$ $a \Supset (\blacktriangledown \Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket)$ [iff by Claim 2] $a \Supset_{\mathsf{enr}(\mathfrak{o})} |a|_{\mathfrak{o}}$ [iff] $|a|_{\mathfrak{o}} \geq |a|_{\mathfrak{o}}$ [iff] true. $\boxed{\text{(B.23)}}$ Suppose that $a \Supset_{\mathsf{enr}(\mathfrak{o})} b$, i.e., $|a|_{\mathfrak{o}} \geq b$. By idempotence, $\big||a|_{\mathfrak{o}}\big|_{\mathfrak{o}} \geq b$. So by Claim 2, $\big|\blacktriangledown \Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket\big|_{\mathfrak{o}} \geq b$, i.e., $(\blacktriangledown \Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket) \Supset_{\mathsf{enr}(\mathfrak{o})} b$. $\boxed{\text{(B.24)}}$ Suppose that $(\blacktriangledown \Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket) \geq b$. Then by Claim 2, $|a|_{\mathfrak{o}} \geq b$. By idempotence, $\big||a|_{\mathfrak{o}}\big|_{\mathfrak{o}} \geq b$, and so by Claim 2, $\big|(\blacktriangledown \Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket)\big|_{\mathfrak{o}} \geq b$, i.e., $(\blacktriangledown \Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket) \Supset_{\mathsf{enr}(\mathfrak{o})} b$. $\boxed{\text{(B.25)}}$ Suppose that $a \leq b$ and $a \Supset_{\mathsf{enr}(\mathfrak{o})} c$, i.e., $|a|_{\mathfrak{o}} \geq c$. By order preservation of elementary interior operators, $|a|_{\mathfrak{o}} \leq |b|_{\mathfrak{o}}$. Hence $|b|_{\mathfrak{o}} \geq c$, i.e., $b \Supset_{\mathsf{enr}(\mathfrak{o})} c$. $\boxed{\text{(B.18)}}$ Suppose that $a \Supset_{\mathsf{enr}(\mathfrak{o})} b$, i.e., $|a|_{\mathfrak{o}} \geq b$. By the decreasing nature of elementary interior operators, $a \geq |a|_{\mathfrak{o}}$. Hence $a \geq b$. $\boxed{\mathsf{eio}(\mathfrak{o}) \text{ is an elementary interior operator}}$ Let $\mathfrak{o}$ be an elementary neighbourhood relation. $\boxed{\text{Order preserving}}$ $a \leq b$ [implies, by (B.25)] $\Supset_{\mathfrak{o}} \llbracket a \rrbracket \subseteq \Supset_{\mathfrak{o}} \llbracket b \rrbracket$ [implies] $\blacktriangledown \Supset_{\mathfrak{o}} \llbracket a \rrbracket \leq \blacktriangledown \Supset_{\mathfrak{o}} \llbracket b \rrbracket$ [implies] $|a|_{\mathsf{eio}(\mathfrak{o})} \leq |b|_{\mathsf{eio}(\mathfrak{o})}$. $\boxed{\text{Decreasing}}$ By (B.21), $a \Supset_{\mathfrak{o}} (\blacktriangledown \Supset_{\mathfrak{o}} \llbracket a \rrbracket) = |a|_{\mathsf{eio}(\mathfrak{o})}$, and so by (B.18), $a \geq |a|_{\mathsf{eio}(\mathfrak{o})}$. $\boxed{\text{Idempotent}}$ $\Big||a|_{\mathsf{eio}(\mathfrak{o})}\Big|_{\mathsf{eio}(\mathfrak{o})} = |\blacktriangledown \Supset_{\mathfrak{o}} \llbracket a \rrbracket|_{\mathsf{eio}(\mathfrak{o})} = \blacktriangledown \Supset_{\mathfrak{o}} \llbracket \blacktriangledown \Supset_{\mathfrak{o}} \llbracket a \rrbracket \rrbracket \overset{(B.28)}{=} \blacktriangledown \Supset_{\mathfrak{o}} \llbracket a \rrbracket = |a|_{\mathsf{eio}(\mathfrak{o})}$. $\diamond$

Proposition B.21 For order $\mathbf{P}$, $\mathsf{eio}(\cdot)$ and $\mathsf{enr}(\cdot)$ define mutually inverse bijections between $\mathsf{ENR}(\mathbf{P})$ and $\mathsf{EIO}(\mathbf{P})$.

Proof. $\boxed{\mathsf{eio}(\mathsf{enr}(\mathfrak{o})) = \mathfrak{o}}$ By definition and Claim 2 of the proof of the previous definition, $|a|_{\mathsf{eio}(\mathsf{enr}(\mathfrak{o}))} = \blacktriangledown \Supset_{\mathsf{enr}(\mathfrak{o})} \llbracket a \rrbracket = |a|_{\mathfrak{o}}$. $\boxed{\mathsf{enr}(\mathsf{eio}(\mathfrak{o})) = \mathfrak{o}}$ $a \Supset_{\mathsf{enr}(\mathsf{eio}(\mathfrak{o}))} b$ [iff] $|a|_{\mathsf{eio}(\mathfrak{o})} \geq b$ [iff] $(\blacktriangledown \Supset_{\mathfrak{o}} \llbracket a \rrbracket) \geq b$ [iff by (B.29)] $a \Supset_{\mathfrak{o}} b$. $\diamond$

Convention B.22 (Conflating Elementary Interior Operators and Neighbourhood Relations) Consequent to the previous definition and proposition we shall (tend to) syntactically conflate elementary interior operators and neighbourhood relations, and, as such, treat (B.30) and (B.31) as properties of these conflated structures.

Corollary B.23 For an elementary open system $\mathfrak{o}$, the following formulae are all valid.

$$a \Subset_{\mathfrak{o}} b \text{ iff } \exists\,[o \in \mathsf{op}(\mathfrak{o})]\; a \leq o \leq b, \tag{B.32}$$

$$a \leq o \in \mathsf{op}(\mathfrak{o}) \text{ implies } a \Subset_{\mathfrak{o}} o, \tag{B.33}$$

$$a \leq b \Subset_{\mathfrak{o}} u \text{ implies } a \Subset_{\mathfrak{o}} u, \tag{B.34}$$

$$a \in \mathsf{op}(\mathfrak{o}) \text{ iff } a \Subset_{\mathfrak{o}} a. \tag{B.35}$$

Proof. $\boxed{\text{(B.32)}}$ $\boxed{\Rightarrow}$ Suppose that $a \Subset_{\mathfrak{o}} b$. Then by (B.30) and $|\cdot|_{\mathfrak{o}}$-decreasingness, $a \leq |b|_{\mathfrak{o}} \leq b$, which suffices since $|b|_{\mathfrak{o}}$ is open by (B.15). $\boxed{\Leftarrow}$ Suppose that $o \in \mathsf{op}(\mathfrak{o})$ and $a \leq o \leq b$. By maximality, $o \leq |b|_{\mathfrak{o}} \leq b$, and hence $a \leq |b|_{\mathfrak{o}}$. So by (B.30), $a \Subset_{\mathfrak{o}} b$. $\boxed{\text{(B.33)}}$ If $a \leq o \in \mathsf{op}(\mathfrak{o})$ then by (B.13), $a \leq o = |a|_{\mathfrak{o}}$, and so by (B.30), $a \Subset_{\mathfrak{o}} o$. $\boxed{\text{(B.34)}}$ Assume that $a \leq b \Subset_{\mathfrak{o}} u$. Then by (B.30), $b \leq |u|_{\mathfrak{o}}$. Hence $a \leq |u|_{\mathfrak{o}}$, and so by (B.30), $a \Subset_{\mathfrak{o}} u$. $\boxed{\text{(B.35)}}$ $a \in \mathsf{op}(\mathfrak{o})$ [iff] $|a|_{\mathfrak{o}} = a$ [iff] $a \leq |a|_{\mathfrak{o}}$ [iff] $a \Subset_{\mathfrak{o}} a$. $\diamond$

Open Problem B.24 What is the (closure) dual formulation of the elementary neighbourhood relation?

B.1.4 Elementary Co-Consequence Relations

As mentioned earlier, the elementary neighbourhood relation is *not* the dual formulation of the elementary consequence relation. Dual to elementary consequence relations we have, what we have termed, elementary *co-consequence* relations. We have included the proofs of certain results as justification of this duality.

Definition B.25 (Elementary Co-Consequence Relations) The **type of elementary co-consequence relations**, denoted $\mathsf{type}(\mathsf{ecc})$, has a binary relation symbol $\leq$ and a binary relation symbol $\propto$. An **elementary co-consequence relation** is a $\mathsf{type}(\mathsf{ecc})$-structure $\mathfrak{c}$ whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{c}}$ and called the **underlying order**, and is such that $\mathfrak{c}$ satisfies the axioms

$$\text{(order-preservation)} \qquad y \leq x \longrightarrow y \propto x, \tag{B.36}$$

$$\text{(transitivity)} \qquad x \propto y \,\mathsf{and}\, y \propto z \longrightarrow x \propto z \qquad \text{and} \tag{B.37}$$

$$\text{(limit)} \qquad \forall[x]\,\exists[y]\, x \propto y \,\mathsf{and} (\forall[z]\, x \propto z \longrightarrow z \geq y), \tag{B.38}$$

in which case we call $\propto_{\mathfrak{c}}$ the associated **co-consequence relation**. When we call $\mathfrak{c}$ an **elementary co-consequence relation on order P**, we mean that $\mathbf{P}_{\mathfrak{c}} = \mathbf{P}$. For an order $\mathbf{P}$ and a binary relation $\propto$ on $\mathsf{uni}(\mathbf{P})$, when we say that $\propto$ determines a (elementary) co-consequence relation on $\mathbf{P}$, or say that $\langle \mathbf{P}; \propto \rangle$ is a (elementary) co-consequence relation, we mean that $\langle \mathsf{uni}(\mathbf{P}); \leq^{\mathbf{P}}; \propto \rangle$ is an elementary co-consequence relation. Let $\mathsf{ECC}(\mathbf{P})$ denote the set of (elementary) co-consequence relations on order $\mathbf{P}$. □

Proposition B.26 Let $\mathbf{P}$ be an order and $\propto$ a binary relation on the universe of $\mathbf{P}$. Then $\propto$ determines a co-consequence relation on $\mathbf{P}$ iff (B.36) and (B.37) hold and

$$\blacktriangle \propto [\![a]\!] \text{ exists and } a \propto (\blacktriangle \propto [\![a]\!]). \tag{B.39}$$

□

We enumerate some basic properties of elementary co-consequence relations.

Remark B.27 Let $\mathfrak{c}$ be an elementary co-consequence relation. The following formulae are all valid.

$$\text{(pre-down-preserving)} \qquad a \propto b \text{ and } a \geq c \text{ implies } c \propto b, \tag{B.40}$$

$$\text{(post-up-preserving)} \qquad a \propto b \text{ and } c \geq b \text{ implies } a \propto c, \tag{B.41}$$

$$\text{(reflexive)} \qquad a \propto a, \tag{B.42}$$

$$a \geq \blacktriangle \propto [\![a]\!], \tag{B.43}$$

$$(\blacktriangle \propto [\![a]\!]) \propto a \quad \text{and} \tag{B.44}$$

$$a \propto b \text{ iff } b \geq \blacktriangle \propto [\![a]\!]. \tag{B.45}$$

Proof. $\boxed{\text{(B.40) and (B.41)}}$ By order-preservation and transitivity. $\boxed{\text{(B.42)}}$ By order-preservation and transitivity. $\boxed{\text{(B.43)}}$ By (B.39), $\blacktriangle \propto [\![a]\!]$ exists, and by (already established) reflexivity, $a \in \propto [\![a]\!]$; hence $a \geq \blacktriangle \propto [\![a]\!]$. $\boxed{\text{(B.44)}}$ By order-preservation and (B.43). $\boxed{\text{(B.45)}}$ $\boxed{\Rightarrow}$ $a \propto b$ [implies] $b \in \propto [\![a]\!]$ [implies] $b \geq \blacktriangle \propto [\![a]\!]$ $\boxed{\Leftarrow}$ $b \geq \blacktriangle \propto [\![a]\!]$ [implies by order-preservation] $\blacktriangle \propto [\![a]\!] \propto b$ [implies by (B.39) and transitivity] $a \propto b$. ◇

Definition B.28 (Associating Co-Consequence Relations and Interior Operators) With each elementary co-consequence relation $\mathfrak{o}$, we associate the elementary interior operator $\mathsf{eio}(\mathfrak{o})$, for which we tend to abbreviate $|a|_{\mathsf{eio}(\mathfrak{o})}$ by $|a|_{\mathfrak{o}}$, defined by

$$|a|_{\mathfrak{o}} = \blacktriangle \propto_{\mathfrak{o}} [\![a]\!], \tag{B.46}$$

this operator being well-defined by (B.39). With each elementary interior operator $\mathfrak{o}$, we associate the elementary co-consequence relation $\mathsf{ecc}(\mathfrak{o})$ in $\mathbf{P}_{\mathfrak{o}}$, for which we tend to abbreviate $\propto_{\mathsf{ecc}(\mathfrak{o})}$ by $\propto_{\mathfrak{o}}$, where

$$a \propto_{\mathfrak{o}} b \leftrightarrow b \geq |a|_{\mathfrak{o}}\,. \tag{B.47}$$

$\square$

Proof. $\boxed{\text{eio}(\mathfrak{o}) \text{ is an elementary interior operator}}$ $\boxed{\text{Order preserving}}$ $a \geq b$ [implies, by order-preservation of $\propto_{\mathfrak{o}}$] $b \propto_{\mathfrak{o}} a$ [implies by $\propto_{\mathfrak{o}}$-transitivity] $\propto_{\mathfrak{o}}[\![a]\!] \subseteq \propto_{\mathfrak{o}}[\![b]\!]$ [implies] $\blacktriangle\propto_{\mathfrak{o}}[\![a]\!] \geq \blacktriangle\propto_{\mathfrak{o}}[\![b]\!]$ [implies] $|a|_{\mathsf{eio}(\mathfrak{o})} \geq |b|_{\mathsf{eio}(\mathfrak{o})}$. $\boxed{\text{Increasing}}$ (B.42) [implies] $a \in \propto_{\mathfrak{o}}[\![a]\!]$ [implies] $a \geq \blacktriangle\propto_{\mathfrak{o}}[\![a]\!]$ [implies] $a \geq |a|_{\mathsf{eio}(\mathfrak{o})}$. $\boxed{\text{Idempotent}}$ (B.39) [implies] $a \propto_{\mathfrak{o}} (\blacktriangle\propto_{\mathfrak{o}}[\![a]\!])$ [implies] $a \propto_{\mathfrak{o}} |a|_{\mathsf{eio}(\mathfrak{o})}$ [implies by $\propto_{\mathfrak{o}}$-transitivity] $\propto_{\mathfrak{o}}[\![|a|_{\mathsf{eio}(\mathfrak{o})}]\!] \subseteq \propto_{\mathfrak{o}}[\![a]\!]$ [implies] $\blacktriangle\propto_{\mathfrak{o}}[\![|a|_{\mathsf{eio}(\mathfrak{o})}]\!] \geq \blacktriangle\propto_{\mathfrak{o}}[\![a]\!]$ [implies] $\Big||a|_{\mathsf{eio}(\mathfrak{o})}\Big|_{\mathsf{eio}(\mathfrak{o})} \geq |a|_{\mathsf{eio}(\mathfrak{o})}$, which suffices. $\boxed{\text{ecc}(\mathfrak{o}) \text{ is an elementary co-consequence relation}}$ We use Proposition B.26. $\boxed{\text{Order-preservation (B.36)}}$ $a \geq b$ [implies by $|\cdot|_{\mathfrak{o}}$-decreasingness] $a \geq b \geq |b|_{\mathfrak{o}}$ [implies] $b \propto_{\mathsf{ecc}(\mathfrak{o})} a$. $\boxed{\text{Transitivity (B.37)}}$ $a \propto_{\mathsf{ecc}(\mathfrak{o})} b$ and $c \propto_{\mathsf{ecc}(\mathfrak{o})} a$ [implies] $b \geq |a|_{\mathfrak{o}}$ and $a \geq |c|_{\mathfrak{o}}$ [implies by order-preservation and idempotence] $b \geq |a|_{\mathfrak{o}}$ and $|a|_{\mathfrak{o}} \geq |c|_{\mathfrak{o}}$ [implies] $b \geq |c|_{\mathfrak{o}}$ [implies] $c \propto_{\mathsf{ecc}(\mathfrak{o})} b$. $\boxed{\text{Claim: } \blacktriangle\propto_{\mathsf{ecc}(\mathfrak{o})}[\![a]\!] = |a|_{\mathfrak{o}}}$ Suppose that $b \in \propto_{\mathsf{ecc}(\mathfrak{o})}[\![a]\!]$, i.e., $a \propto_{\mathsf{ecc}(\mathfrak{o})} b$. By definition, $b \geq |a|_{\mathfrak{o}}$. So $|a|_{\mathfrak{o}}$ is a lower bound of $\propto_{\mathsf{ecc}(\mathfrak{o})}[\![a]\!]$. Suppose that c is a lower bound of $\propto_{\mathsf{ecc}(\mathfrak{o})}[\![a]\!]$, i.e., if $a \propto_{\mathsf{ecc}(\mathfrak{o})} b$ then $b \geq c$, i.e., if $b \geq |a|_{\mathfrak{o}}$ then $b \geq c$. Certainly, $|a|_{\mathfrak{o}} \geq |a|_{\mathfrak{o}}$, hence $|a|_{\mathfrak{o}} \geq c$, which suffices. $\boxed{\text{(B.39)}}$ (*In the light of the previous claim, it suffices to show that* $a \propto_{\mathsf{ecc}(\mathfrak{o})} |a|_{\mathfrak{o}}$.) $a \propto_{\mathsf{ecc}(\mathfrak{o})} |a|_{\mathfrak{o}}$ [iff] $|a|_{\mathfrak{o}} \geq |a|_{\mathfrak{o}}$ [iff] true. $\diamond$

Proposition B.29 For order $\mathbf{P}$, $\mathsf{eio}(\cdot)$ and $\mathsf{ecc}(\cdot)$ define mutually inverse bijections, between $\mathsf{ENR}(\mathbf{P})$ and $\mathsf{ECO}(\mathbf{P})$.

Proof. (*It suffices to prove that* $\mathsf{ecc}(\cdot)$ *is injective and* $\mathsf{ecc}(\mathsf{eio}(\mathfrak{o})) = \mathfrak{o}$.)
$\boxed{\text{ecc}(\cdot) \text{ is injective}}$ Suppose that $\mathfrak{o}, \mathfrak{p} \in \mathsf{EIO}(\mathbf{P})$ and $\mathsf{ecc}(\mathfrak{o}) = \mathsf{ecc}(\mathfrak{p})$, i.e., $\propto_{\mathsf{ecc}(\mathfrak{o})} = \propto_{\mathsf{ecc}(\mathfrak{p})}$. Now, $|a|_{\mathfrak{o}} \geq |a|_{\mathfrak{p}}$ [iff] $a \propto_{\mathsf{ecc}(\mathfrak{p})} |a|_{\mathfrak{o}}$ [iff] $a \propto_{\mathsf{ecc}(\mathfrak{o})} |a|_{\mathfrak{o}}$ [iff] $|a|_{\mathfrak{o}} \geq |a|_{\mathfrak{o}}$ [iff] true. So $|a|_{\mathfrak{o}} \geq |a|_{\mathfrak{p}}$. Symmetrically, $|a|_{\mathfrak{p}} \geq |a|_{\mathfrak{o}}$. Hence $|\cdot|_{\mathfrak{o}} = |\cdot|_{\mathfrak{p}}$, i.e., $\mathfrak{o} = \mathfrak{p}$. $\boxed{\text{ecc}(\text{eio}(\mathfrak{o})) = \mathfrak{o}}$ Let $\mathfrak{o} \in \mathsf{ECC}(\mathbf{P})$. $a \propto_{\mathsf{ecc}(\mathsf{eio}(\mathfrak{o}))} b$ [iff] $b \geq |a|_{\mathsf{eio}(\mathfrak{o})}$ [iff] $b \geq \blacktriangle\propto_{\mathfrak{o}}[\![a]\!]$ [iff by (B.45)] $a \propto_{\mathfrak{o}} b$. $\diamond$

Convention B.30 (Conflating Co-Consequence Relations and Interior Operators) In the light of the previous definition and proposition, we shall tend to syntactically conflate elementary co-consequence relations and elementary interior operators and (hence) elementary open systems, and hence treating (B.46) and (B.47) as properties of these conflated structures.

Corollary B.31 For an elementary interior operator $\mathfrak{o}$,

$$|a|_{\mathfrak{o}} = |b|_{\mathfrak{o}} \text{ iff } a \propto_{\mathfrak{o}} b \text{ and } b \propto_{\mathfrak{o}} a, \tag{B.48}$$

$$a \propto_{\mathfrak{o}} |a|_{\mathfrak{o}}, \tag{B.49}$$

$$|a|_{\mathfrak{o}} \propto_{\mathfrak{o}} a, \tag{B.50}$$

$$a \,\text{is}\, \mathsf{op}(\mathfrak{c}) \text{ iff } a \propto_{\mathfrak{c}} b \to b \geq a \quad \text{and} \tag{B.51}$$

$$a \propto_{\mathfrak{c}} b \text{ iff } \forall\, [o \,\text{is}\, \mathsf{op}(\mathfrak{c})]\; a \geq o \to b \geq o. \tag{B.52}$$

Proof.

$\boxed{\text{(B.48)}}$ $\boxed{\Rightarrow}$ Suppose that $|a|_{\mathfrak{o}} = |b|_{\mathfrak{o}}$. Since elementary interior operators are decreasing, $b \geq |b|_{\mathfrak{o}} = |a|_{\mathfrak{o}}$ and $a \geq |a|_{\mathfrak{o}} = |b|_{\mathfrak{o}}$, i.e., $b \geq |a|_{\mathfrak{o}}$ and $a \geq |b|_{\mathfrak{o}}$. So by (B.47), $a \propto_{\mathfrak{o}} b$ and $b \propto_{\mathfrak{o}} a$. $\boxed{\Leftarrow}$ Suppose that $a \propto_{\mathfrak{o}} b$ and $b \propto_{\mathfrak{o}} a$. Then by (B.47), $b \geq |a|_{\mathfrak{o}}$ and $a \geq |b|_{\mathfrak{o}}$. Since elementary interior operators are order-preserving and idempotent, $|b|_{\mathfrak{o}} \geq |a|_{\mathfrak{o}}$ and $|a|_{\mathfrak{o}} \geq |b|_{\mathfrak{o}}$. Hence $|a|_{\mathfrak{o}} = |b|_{\mathfrak{o}}$. $\boxed{\text{(B.49)}}$ $a \propto_{\mathfrak{o}} |a|_{\mathfrak{o}}$ [iff by (B.47)] $|a|_{\mathfrak{o}} \geq |a|_{\mathfrak{o}}$ [iff] true. $\boxed{\text{(B.50)}}$ $|a|_{\mathfrak{o}} \propto_{\mathfrak{o}} a$ [iff by (B.47)] $a \geq \big||a|_{\mathfrak{o}}\big|_{\mathfrak{o}}$ [iff by idempotence] $a \geq |a|_{\mathfrak{o}}$ [iff by decreasingness] true. $\boxed{\text{(B.51)}}$ $\boxed{\Rightarrow}$ Assume that $a \,\text{is}\, \mathsf{op}(\mathfrak{c})$ and $a \propto_{\mathfrak{c}} b$. Then by (B.13), $|a|_{\mathfrak{c}} = a$, and by (B.47), $b \geq |a|_{\mathfrak{c}}$. Hence $b \geq a$. $\boxed{\Leftarrow}$ Assume that $a \propto_{\mathfrak{c}} b \to b \geq a$. Since by (B.49), $a \propto_{\mathfrak{c}} |a|_{\mathfrak{c}}$, by assumption $|a|_{\mathfrak{c}} \geq a$. So by (B.16), $a \,\text{is}\, \mathsf{op}(\mathfrak{c})$. $\boxed{\text{(B.52)}}$ $\boxed{\Rightarrow}$ Suppose that $a \propto_{\mathfrak{c}} b$. Then by (B.47), $b \geq |a|_{\mathfrak{c}}$. Let $o \,\text{is}\, \mathsf{op}(\mathfrak{c})$ such that $a \geq o$. Then $|a|_{\mathfrak{c}} \overset{(B.1)}{\geq} |o|_{\mathfrak{c}} \overset{(B.13)}{=} o$. Hence $b \geq o$. $\boxed{\Leftarrow}$ Since $a \overset{(B.2)}{\geq} |a|_{\mathfrak{c}} \overset{(B.15)}{\in} \mathsf{op}(\mathfrak{c})$, by assumption, $b \geq |a|_{\mathfrak{c}}$. So by (B.47), $a \propto_{\mathfrak{c}} b$. ◇

B.2 Concrete Interior

We turn now to concrete considerations. While the definitions and results of $\oint$A certainly inform this section, the definitions of $\oint$A and $\oint$B must be treated as independent.

Definition B.32 (Concrete Interior) An elementary interior operator (resp. open system, neighbourhood relation, co-consequence relation) $\mathfrak{o}$ is called **complete** if the underlying order is a complete lattice, and is called **concrete** if $\mathbf{P}_{\mathfrak{o}} = \mathfrak{P}(A)$ for some (unique) *non-empty* set A; in the latter case, we write $\mathsf{uni}(\mathfrak{o})$ for A, which we call the **universe** (and never use the term 'universe' synonymously for 'elementary universe'), and call $\mathfrak{o}$ an **interior operator over** A (resp. **open system over** A, **neighbourhood relation over** A, **co-consequence relation over** A) or just an **interior operator** (resp. **open system**, **neighbourhood relation**, **co-consequence relation**). Arbitrary (concrete) open systems are denoted by $\mathbb{O}$, $\mathbb{P}$ and $\mathbb{Q}$, with the usual adornments, and use of this symbolism shall imply that the open systems are concrete. By our convention of conflating open systems, interior operators, etc., we may speak of $\mathbb{O}$ being an interior operator, etc. The set of all open systems over A is denoted by $\mathsf{OpSys}(A)$. □

In the case of *complete* open systems, the necessary condition of Proposition B.15 is sufficient, as demonstrated by the following characterization of *complete* open systems.

Proposition B.33 Let $\mathbf{P}$ be a *complete lattice* and $\mathsf{op} \subseteq \mathsf{uni}(\mathbf{P})$. The following conditions are equivalent.

1. op determines an elementary open system on $\mathbf{P}$.

2. $\mathbf{P}_{|_{\mathsf{op}}} \lhd_{\blacktriangledown} \mathbf{P}$.

3. $\forall\, [A \subseteq \mathsf{op}]\ \blacktriangledown^{\mathbf{P}} A \in \mathsf{op}$.

4. $0^{\mathbf{P}} \in \mathsf{op}$ and $\forall\, [\emptyset \neq A \subseteq \mathsf{op}]\ \blacktriangledown A \in \mathsf{op}$.

Proof. Dual to the proof of Proposition 4.43 on page 149. $\diamond$

Corollary B.34 $\mathcal{A} \subset \mathfrak{P}(X)$ determines the open sets of a concrete open system iff $\emptyset \in \mathcal{A}$ and for all $\mathfrak{A} \subseteq \mathcal{A}$, $\cup\mathfrak{A} \in \mathcal{A}$.

We enumerate a few results about (concrete) open systems.

Proposition B.35 Let $\mathbb{O}$ be an open system. Then $\mathbf{op}(\mathfrak{O})$ is a complete lattice, with

$$\overset{\mathbf{op}(\mathfrak{O})}{\blacktriangle} \mathcal{A} = \begin{cases} \mathsf{uni}(\mathbb{O}) & ; \quad \mathcal{A} = \emptyset, \\ \bigcap \mathcal{A} & ; \quad \text{otherwise} \end{cases} \quad \text{and} \tag{B.53}$$

$$\overset{\mathbf{op}(\mathfrak{O})}{\blacktriangledown} \mathcal{A} = \bigcap \left\{ O \in \mathsf{op}(\mathbb{O}) \ : \ \bigcup \mathcal{A} \subseteq O \right\}, \tag{B.54}$$

for all $\mathcal{A} \subseteq \mathsf{op}(\mathbb{O})$.

Remark B.36 For an open system $\mathbb{O}$,

$$\overset{\mathbf{op}(\mathfrak{O})}{\blacktriangledown} \mathcal{A} = \left| \bigcup \mathcal{A} \right|_{\mathfrak{O}} = | \bigcup |[\mathcal{A}]|_{\mathbb{O}}|_{\mathbb{O}} = | \bigcup \{ |A|_{\mathbb{O}} \ : \ A \in \mathcal{A} \} |_{\mathbb{O}}. \tag{B.55}$$

B.3 Elementary Spaces

Definition B.37 (Inverlutions) Order inverting involutions on an order are called **inverlution operators** or **inverlutions**. $\square$

Remark B.38 Inverlutions are characterizable elementarily.

Definition B.39 (Elementary Spaces) The **type of elementary spaces**, denoted $\mathsf{type}(\mathsf{esp})$, has a binary relation symbol $\leq$, a unary relation symbol op and a unary operation symbol $\neg$. An **elementary space** is a $\mathsf{type}(\mathsf{esp})$-structure $\mathfrak{s} = \langle \mathsf{uni}_{\mathsf{e}}(\mathfrak{s}); \leq^{\mathfrak{s}}; \mathsf{op}(\mathfrak{s})\cdot, \overset{\mathfrak{s}}{\neg} \rangle$ whose $\leq$-reduct is an order, denoted $\mathbf{P}_{\mathfrak{s}}$ and called the **underlying order**, such that $\langle \mathsf{uni}_{\mathsf{e}}(\mathfrak{s}); \leq^{\mathfrak{s}}; \mathsf{op}(\mathfrak{s})\cdot \rangle$ is an open system and $\overset{\mathfrak{s}}{\neg}$ is an inverlution on $\mathbf{P}_{\mathfrak{s}}$ called the **inverlution operator**. We write $\mathsf{uni}_{\mathsf{e}}(\mathfrak{s})$ for $\mathsf{uni}(\mathbf{P}_{\mathfrak{s}})$ which we call the **elementary universe** (or just **universe** when unambiguous). When we call $\mathfrak{s}$ an **elementary space on order P**, we mean that $\mathbf{P}_{\mathfrak{s}} = \mathbf{P}$. Let $\mathsf{ESP}(\mathbf{P})$ denote the set of (elementary) spaces on order $\mathbf{P}$. Conventionally, elementary spaces inherit all the notions of open systems. $\square$

Note that in order for an ordered-set $\mathbf{P}$ to admit an inverlution, $\mathbf{P}$ must be order-isomorphic to its dual-order $\mathbf{P}^{\mathsf{d}}$, and so the orders admitting inverlutions are self dual orders. Consequently the underlying order of an elementary space is self-dual.

Proposition B.40 If $\mathfrak{s}$ is an elementary space then $\overset{\mathfrak{s}}{\neg}[\mathsf{op}(\mathfrak{s})]$ determines an elementary closed system on $\mathbf{P}_{\mathfrak{s}}$.

Convention B.41 (Inheriting the Notions of Closure) Consequent to the previous proposition, elementary spaces shall inherit all the notions of elementary closed systems and all the elementary structures of closure that we conflate with elementary closed systems.

Proposition B.42 Let $\mathfrak{s}$ be an elementary space.

$$o \in \mathsf{op}(\mathfrak{s}) \leftrightarrow \overset{\mathfrak{s}}{\neg} o \in \mathsf{cl}_{\mathfrak{s}}, \tag{B.56}$$

$$g \in \mathsf{cl}_{\mathfrak{s}} \leftrightarrow \overset{\mathfrak{s}}{\neg} g \in \mathsf{op}(\mathfrak{s}) \quad \text{and} \tag{B.57}$$

$$a \vdash b \leftrightarrow \forall [o \in \mathsf{op}]\, b \not\leq \neg o \longrightarrow a \not\leq \neg o. \tag{B.58}$$

We leave it to the reader to establish other interesting relationships between closure notions and interior notions in elementary spaces (see Table A.1 on page 499 and Table A.2 on page 500).

Open Problem B.43 We can see two means of defining elementary topological spaces. In the first approach, an elementary topological space is an elementary space such that the order is bottomed and topped, the bottom and top are open points and the join of any two open points exists and is open. This definition is elementary. Alternatively we could require that the order be a 01-*lattice*, 0 and 1 are open and the join of any two open points is open. Which of these two approaches is better? It is our intuition that the second is better, in that the notion that an open point meets an arbitrary point can be well-defined; i.e., $o \wedge b \neq 0$.

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Appendix C

Index of Open Problems

Appendix D

Glossary of Closed Systems, Formal Systems and Logics

D.1 Closed Systems

D.2 Formal Systems

D.3 Sentential Calculi

D.4 Logics over Constructs

D.5 Archologies

Appendix E

General Glossary

Glossary

$\doteq$ 15
Set 15
$\langle a, b\rangle_{(0)}$ 15
$\langle a, b\rangle_{(1)}$ 15
$A \times B$ 16
A^2 16
$r_{[0]}$ 16
$r_{[1]}$ 16
$\oplus$ 16
$\mathsf{card}(A)$ 16
$A \subseteq_{\mathsf{f}} B$ 16
ω 16
r 16
$\mathsf{do}(r)$ 16
$\mathsf{co}(r)$ 16
D_r 16
$a \ r \ b$ 16
$\mathsf{Bship}(A, B)$ 17
$\mathsf{gr}(r)$ 17
$\mathsf{rg}(r)$ 17
${}_A\blacksquare_B$ 17
${}_A\emptyset_B$ 17
$r_{:s}$ 17
$r_{|s|}$ 17
$r_{|C}$ 17
$r_{D|}$ 17
$r \circ s$ 17
$\overleftarrow{r}$ 17
$\not{r}$ 17
f 18
$A \to B$ 18
B^A 18
$f : \mathcal{F}$ 18
$f : A \to B$ 18
$a \mapsto b$ 18
${}_B{\hookrightarrow}_A$ 18
f^{-1} 19
gf 19
$\underset{\to}{f}_{[2]}$ 19
$\underset{\to}{f}$ 19
$r[\![a]\!]$ 19
$r_{[\![]\!]}(\cdot)$ 19
$r[A]$ 19
$r_{[]}(\cdot)$ 19
$\mathsf{Pole}(r)$ 19
$\mathsf{Image}(r)$ 19
$\mathsf{Prepole}(r)$ 19
$\mathsf{PreIm}(r)$ 19
$f^{-1}[\cdot]$ 20
$\mathsf{BRel}(A)$ 20
α 20
$\mathsf{uni}(\alpha)$ 20
$\alpha_{(0)}$ 20
$\alpha_{(1)}$ 20
$\blacksquare_A$ 20
$\emptyset_A$ 20
$=_A$ 20
$\alpha_{|C|}$ 22
$\alpha \circ^n \beta$ 22
$\underset{\to}{f}[\alpha]$ 22
$f\{\mathcal{A}\}$ 22
$\mathsf{Op}(A)$ 23
$\mathsf{fixed\text{-}points}_{\mathsf{u}} B$ 23
$\mathsf{idempotent}_{\mathsf{u}}(B)$ 23
$\mathsf{idempotent\text{-}operations}(A)$ 23
$\mathsf{permutations}(\mathrm{A})$ 23
$\mathsf{ER}(A)$ 23
$\mathsf{EC}_B(\alpha)$ 23
$\mathsf{EC}(\alpha)$ 23
$\coprod \alpha$ 23
$A/\ \alpha$ 23
$\mathsf{q}_\alpha(\cdot)$ 23
$\mathsf{uni}(\kappa)$ 24
$\equiv_\kappa$ 24
q_κ 24
$\equiv_f$ 24
$\coprod f$ 24
q_f 24
M 25
$\mathsf{uni}(\mathsf{M})$ 25
D_{M} 25
$\langle \mathsf{uni}(\mathsf{M}), \mathsf{D}_{\mathsf{M}}\rangle$ 25
$\langle \mathsf{D}_{\mathsf{M}}\rangle$ 25
$\langle A, \mathsf{a}\rangle$ 25
N 25
$\mathsf{M} \lhd_{\mathsf{D}} \mathsf{N}$ 25
$\mathsf{M} \lhd \mathsf{N}$ 25
$\mathsf{M}_{|_A}$ 25
$\mathsf{M} \twoheadrightarrow^{\mathsf{r}} \mathsf{N}$ 25
$\mathsf{M}/\ \alpha$ 26
$[\frac{J}{b}](\mathsf{a})$ 27
$\mathsf{a}_{|J}$ 27
$[\cdot = \bullet]_{\mathrm{A}^{\mathrm{I}}}$ 27
$[\cdot = \bullet]$ 27
$\underset{\to}{r}_{[I]}$ 27
$\underset{\to}{r}$ 27
$\prod f$ 28
$\mho^{\boldsymbol{F}}_{\mathcal{A}}$ 28
$\boldsymbol{R}$ 29
$\mathsf{uni}(\boldsymbol{R})$ 29
$\mathsf{idx}(\boldsymbol{R})$ 29
$\mathsf{D}_{\boldsymbol{R}}$ 29
$\underset{\to}{\mathsf{uni}}(\boldsymbol{R})$ 29
$\langle I, A, \mathtt{A}\rangle$ 29
$\mathsf{RIm}_I(A)$ 29
$\langle \mathsf{uni}(\boldsymbol{R}), \mathsf{D}_{\boldsymbol{R}}\rangle$ 29
M 29
$\mathsf{idx}(\mathsf{M})$ 29
$\mathsf{dim}(\mathsf{M})$ 29
$\mathsf{ar}(\mathsf{M})$ 29
$\langle \mathsf{uni}(\mathsf{M}), \mathsf{D}_{\mathsf{M}}\rangle$ 29
R 29
$\mathsf{a} \in R$ 29
$\prod f$ 29
$\underset{\to}{R}_{[\mathsf{idx}(\boldsymbol{R})]}$ 30
$\underset{\to}{R}$ 30
$\boldsymbol{R}_{\mathsf{u}}$ 30
$\boldsymbol{R} \lhd_{\mathsf{D}} \boldsymbol{Q}$ 30
$\boldsymbol{R}_{|_A}$ 30
$\boldsymbol{f}$ 30
$\boldsymbol{R}/\ \alpha$ 31
O 32
$O(a_1, \ldots, a_n)$ 32
$O(\langle a_1, \ldots, a_n\rangle)$ 32
$\mathsf{Op}_{[n]}(A)$ 32
$\mathsf{BOp}(A)$ 32

Appendix F

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