UNIVERSITY OF KWAZULU-NATAL

NEW ANALYTICAL STELLAR MODELS IN GENERAL RELATIVITY

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New Analytical Stellar Models in General Relativity

by

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ABSTRACT

We present new exact solutions to the Einstein and Einstein-Maxwell field equations that model the interior of neutral, charged and radiating stars. Several new classes of solutions in static spherically symmetric interior spacetimes are found in the presence of charge. These correspond to isotropic matter with a specified electric field intensity. Our solutions are found by choosing different rational forms for one of the gravitational potentials and a particular form for the electric field. The models generated contain results found previously including Finch and Skea (1989) neutron stars, Durgapal and Bannerji (1983) dense stars, Tikekar (1990) superdense stars in the limit of vanishing charge. Then we study the general situation of a compact relativistic object with anisotropic pressures in the presence of the electromagnetic field. We assume the equation of state is linear so that the model may be applied to strange stars with quark matter and dark energy stars. Several new classes of exact solutions are found, and we show that the densities and masses are consistent with real stars. We regain as special cases the Lobo (2006) dark energy stars, the Sharma and Maharaj (2007) strange stars and the realistic isothermal universes of Saslaw *et al* (1996). In addition, we consider relativistic radiating stars undergoing gravitational collapse when the fluid particles are in geodesic motion. We transform the governing equation into Bernoulli, Riccati and confluent hypergeometric equations. These admit an infinite family of solutions in terms of simple elementary functions and special functions. Particular models contain the Minkowski spacetime and the Friedmann dust spacetime as limiting cases. Finally, we model the radiating star with shear, acceleration and expansion in the presence of anisotropic pressures. We obtain several classes of new solutions in terms of arbitrary functions in temporal and radial coordinates by rewriting the junction condition in the form of a Riccati equation. A brief physical analysis indicates that these models are physically reasonable.

To

Our little girl Anika

PREFACE

The study described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban. This thesis was completed under the supervision of Professor S D Maharaj.

The research contained in this thesis represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

S.Thirukkanesh.

March 2009.

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DECLARATION 1 - PLAGIARISM

I, Suntharalingam Thirukkanesh, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

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DECLARATION 2 - PUBLICATIONS

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of contributions of each author to experimental work and writing of each publication)

Publication 1

Thirukkanesh S and Maharaj S D, Charged relativistic sphere with generalised potentials, *Math. Meth. Appl. Sci.*, In press (2008a).

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written my myself with some input from my supervisor.)

Publication 2

Maharaj S D and Thirukkanesh S, Some new static charged spheres, *Nonlinear Analysis*, In press (2008a).

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written my myself with some input from my supervisor.)

Publication 3

Thirukkanesh S and Maharaj S D, Charged anisotropic matter with linear equation of state, *Class. Quantum Grav.* **25**, 235001 (2008b).

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written my myself with some input from my supervisor.)

Publication 4

Maharaj S D and Thirukkanesh S, Generalised isothermal models with strange equation of state, *Pramana - J. Phys.*, In press (2008b).

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Publication 5

Thirukkanesh S and Maharaj S D, Radiating relativistic matter in geodesic motion, *J. Math. Phys.*, Submitted (2008c)

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written my myself with some input from my supervisor.)

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(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written my myself with some input from my supervisor.)

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Contents

Chapter 1

Introduction

The theory of general relativity is an extension of the theory of special relativity by incorporating gravitational effects. The behaviour of the gravitational field is accurately described by the theory of general relativity. The theoretical predictions of general relativity are consistent with the observational and experimental results in astrophysics and cosmology (Davies 1989, Will 1981). In general relativity the gravitational field of an object is contained in the curvature spacetime, and spacetime is taken to be a four-dimensional, differentiable manifold, endowed with a symmetric, nondegenerate metric tensor field. Locally the spacetime geometry of general relativity resembles that of special relativity but globally the geometries differ in that the differentiable manifold is not flat. In general relativity the curvature of spacetime is described by the Riemann tensor. The matter content is represented by the symmetric energy momentum tensor, which in the case of a charge includes the contribution of the electromagnetic field tensor. The total energy momentum tensor is related to the Einstein tensor by the Einstein field equations which satisfy the conservation laws, namely the Bianchi identities. The electromagnetic field is subject to the Maxwell equations. The Einstein and Einstein-Maxwell field equations are systems of highly nonlinear partial differential equations which are difficult to integrate without simplifying assumptions.

Historically the Schwarzschild (1916a) exterior solution is the first exact solution of the Einstein field equations. The exterior Schwarzschild solution is essential for a discussion of the classical tests of general relativity (d'Inverno 1992, Wald 1984, Will 1981). The starting point in the study of fluid spheres is represented by the interior Schwarzschild (1916b) solution from which many of the problems involving spherical symmetry can be modelled. The Schwarzschild interior and exterior solutions match smoothly at the boundary of the star. The Schwarzschild interior solution may be used to model relativistic stars if the variations in the energy density are small. It is a good approximation for small stars in which the pressures are not too large.

Exact models of the Einstein and Einstein-Maxwell system of field equations, for spherically symmetric gravitational fields in static manifolds, are necessary to describe compact spheres in relativistic astrophysics where the gravitational field is strong. Spherically symmetric models are physically significant and are extensively utilised in a variety of applications. In astrophysics, the collapse of a star can be accurately modelled by a spherically symmetric gravitational field (Shapiro and Teukolsky 1983). In cosmology, spherically symmetric spacetimes have been used to model the behaviour and subsequent evolution of the early universe (Krasinski 1997). Under high pressures stars may possess a nonzero charge during the early stages of their evolution (Stephani 2004). Stars may also acquire a net charge through accretion (Shvartsman 1971). The presence of an electric field can also counter the onset of gravitational collapse, as a net charge distribution produces a repulsive Coulomb force. This affects the formation of singularities (Treves and Turolla 1999). The occurrence of charge does have consequences for the cosmic censorship hypothesis (Joshi 1993, Joshi and Dwivedi 1992a, 1992b, 1992c). This conjecture states that any singularity formed by gravitational collapse will always remain hidden behind an event horizon. The presence of charge may affect the formation of naked singularities in gravitational collapse, and exact solutions are helpful in investigations of the cosmic censorship hypothesis. The recent analysis of Ivanov (2002) and Sharma *et al* (2001) show that the presence of the electromagnetic field affects the values of red shifts, luminosities and maximum mass of a compact relativistic object. Gupta and Kumar (2005), Mukherjee (2001), Patel and Koppar (1987) and Tikekar and Singh (1998) demonstrated that it is possible to model charged neutron stars with high densities within acceptable bounds for the surface red shift, luminosity and the total mass. A number of conditions are normally placed on the energy density, the pressure, the electric field intensity and the gravitational potentials to ensure that the model is physically reasonable. However, we should point out that it is rare to find a model that rigorously satisfies all these conditions. For a comprehensive list of exact solutions to the Einstein field equations, many of which may be used to describe the gravitational behaviour of stars, see Delgaty and Lake (1998), Finch and Skea (1998), Krasinski (1997) and Stephani *et al* (2003). Knutsen (1988, 1989, 2000) provides a comprehensive treatment, in particular stellar models, of the conditions to be satisfied for a realistic matter distribution. The interior solutions of the Einstein-Maxwell field equations must match at the boundary to the exterior Reissner-Nordstrom (Nordstrom 1918, Reissner 1916) solution. The exterior Reissner-Nordstrom solution reduces to the Schwarzschild exterior solution in the limit of vanishing electromagnetic field.

In recent years, many researchers have studied anisotropic matter where the radial and tangential pressure components are different: these include the investigations of Bowers and Liang (1974), Chaisi and Maharaj (2005, 2006a), Dev and Gleiser (2002, 2003), Herrera *et al* (2002, 2004a), Maharaj and Chaisi (2006a, 2006b) and Mak and Harko (2002, 2003) among others. They are important for studying relativistic anisotropic spheres and to generate models that permit red shifts higher than the critical red shift of certain isolated objects. Anisotropy also affects the critical mass and stability of highly compact relativistic objects. Mak and Harko (2002) and Sharma and Mukherjee (2002) pointed out that the anisotropic matter appears to be a vital ingredient when modelling boson stars and strange matter with densities higher than neutron stars. Thus far researchers have not investigated the Einstein-Maxwell field equations with a linear equation of state when the matter distribution is anisotropic. The linear equation of state may be applied to model a quark star. Witten (1984) showed that formation of strange matter occur in two ways: the quark-hadron phase transition in the early universe and conversion of neutron stars into strange one at ultrahigh densities. It is expected that strange stars form during the collapse of the core of a massive star after supernova explosion (Cheng *et al* 1998). Drake *et al* (2002) suggested that the X-ray source RXJ1856.5-3754 may be a strange star. The Fermi gas of 3A quarks constitutes a single colour-singlet baryon with baryon number A. This structure of quarks leads to a net positive charge inside the star. Therefore, as a part of this thesis we attempt to model a stellar object in general when anisotropy and charge are present.

In an astrophysical environment, a star usually emits radiation and throws out particles in the process of gravitational collapse. In this situation, the heat flow in the interior of a star should not be neglected so that the interior solution of the gravitational collapse of a radiating star should match to the exterior spacetime described by the Vaidya (1951) solution. Models of relativistic radiating stars are important for the investigation of the cosmic censorship (Goswami and Joshi 2004a, 2004b). Santos (1985) formulated the junction conditions for shear-free collapse, matching the interior metric with the exterior Vaidya metric at the boundary of the star which made it possible to obtain interior solutions. This treatment enabled us to investigate physical features such as surface luminosity, dynamical stability, relaxation effects and temperature profiles. De Oliviera *et al* (1985) proposed a radiating model of an interior static configuration leading to slow gravitational collapse. It was shown earlier that the slowest collapse arises in the case of shear-free fluid interiors (Raychaudhuri 1955). Kolassis *et al* (1988) generated an exact model by assuming geodesic fluid trajectories. In a recent treatment Herrera *et al* (2004b) proposed a model with a vanishing Weyl tensor in a first order approximation without solving the junction condition exactly. Then Maharaj and Covender (2005) and Herrera *et al* (2006) solved the relevant junction condition exactly and generated classes of solutions which contain the Friedmann dust solution as special case. Later Misthry *et al* (2008) obtained several classes of solution by transforming the junction condition to the form of an Abel equation. The most general case involves spacetimes with nonzero shear, acceleration and expansion. The first exact solution with nonzero shear was obtained by Naidu *et al* (2006), considering geodesic motion of fluid particles and then Rajah and Maharaj (2008) obtained two classes of nonsingular solutions which contain the model of Naidu *et al* (2006).

In this thesis, we study spherically symmetric gravitational fields with isotropic and anisotropic matter distributions in the presence of an electromagnetic field. We show that it is possible to generate a variety of exact solutions which are physically reasonable. The linear equation of state, modelling strange matter and quark matter, is shown to be consistent with charge and anisotropy. Also, we consider relativistic radiating stars undergoing gravitational collapse. In particular we seek new exact solutions to the field equations that model the interior of charged, neutral and radiating realistic stars.

This thesis is organised as follows:

- Chapter 1: Introduction.

- Chapter 2: We present a new class of exact solutions to the Einstein-Maxwell system in closed form. This is achieved by choosing a generalised rational form for one of the gravitational potentials and a particular form for the electric field. For specific values of the parameters the new series solution can be written in terms of elementary functions. We regain a number of results found previously including Finch and Skea (1989) neutron stars, Durgapal and Bannerji (1983) dense stars, Tikekar (1990) superdense stars in the limit of vanishing charge.

- Chapter 3: We obtain new exact models for the Einstein-Maxwell system by specifying a particular form for one of the gravitational potentials and the electric field intensity. The condition of pressure isotropy is reduced to a second order linear differential equation. For specific parameter values it is possible to find new exact models in terms of elementary functions. Our model contains a particular charged solution found previously: this suggests that our generalised model could be useful in the study of charged compact objects.

- Chapter 4: In this chapter, we study the general situation of a compact relativistic object with anisotropic pressures in the presence of the electromagnetic field with a linear equation of state. New classes of exact solutions are generated to the Einstein-Maxwell system. Our general model contains anisotropic and isotropic models found previously. We demonstrate that our solutions may be used to model quark stars. The masses and densities correspond to realistic stellar objects.

- Chapter 5: We consider the linear equation of state for matter distributions that may be applied to strange stars with quark matter. In our general approach the compact relativistic body allows for anisotropic pressures in the presence of the electromagnetic field. New exact solutions are found to the Einstein-Maxwell system. A particular case is shown to be regular at the stellar centre. In the isotropic limit we regain the general relativistic isothermal universe. We show that the mass corresponds to values obtained previously for quark stars when anisotropy and charge are present.

- Chapter 6: In this chapter, we study the gravitational behaviour of a spherically symmetric radiating star when the fluid particles are in geodesic motion. We transform the governing equation into a simpler form which allows for a general analytic treatment. We find that Bernoulli, Riccati and confluent hypergeometric equations are possible. These admit solutions in terms of elementary functions and special functions. Particular models contain the Minkowski spacetime and the Friedmann dust spacetime as limiting cases. Our infinite family of solutions contains specific models found previously. For a particular metric we briefly investigate the physical features, derive the temperature profiles and plot the behaviour of the casual and acasual temperatures.

- Chapter 7: We study the behaviour of a relativistic spherically symmetric radiative star with accelerating (or geodesic), expanding and shearing matter distribution. We obtain several classes of new solutions by rewriting the junction condition in the form of a Riccati equation. A pleasing feature of our solutions is the metrics are given for arbitrary functions of the radial coordinate and the temporal coordinate which allows for a wider study of physical features. For a particular metric we investigate the physical properties.

- Chapter 8: The results obtained in this thesis are summarised in the conclusion.

Chapter 2

Charged relativistic sphere with generalised potentials

2.1 Introduction

In this chapter we use the approach of coupling the electromagnetic field tensor to the matter tensor in Einstein's equations such that Maxwell's equations are satisfied. We believe that the qualitative features generated in this charged model should yield results which are physically reasonable. Our objective is to generate a new class of solutions to the Einstein-Maxwell system that satisfies the physical criteria: the gravitational potentials, electric field intensity and matter variables must be finite and continuous throughout the stellar interior. The speed of the sound must be less than the speed of the light, and ideally the solution should be stable with respect to radial perturbations. A barotropic equation of state, linking the isotropic pressure to the energy density, is often assumed to constrain the matter distribution. In addition to these conditions the interior solution must match smoothly at the boundary of the stellar object with the Reissner-Nordstrom exterior spacetime.

In recent years researchers have attempted to introduce a systematic approach to finding solutions to the field equations. Maharaj and Leach (1996) generalised the Tikekar superdense star, Thirukkanesh and Maharaj (2006) generalised the Durgapal and Bannerji neutron star, and Maharaj and Thirukkanesh (2006) generalised the

John and Maharaj (2006) model. These new classes of models were obtained by reducing the condition of pressure isotropy to a recurrence relation with real and rational coefficients which could be solved by mathematical induction, leading to new mathematical and physical insights in the Einstein-Maxwell field equations. We attempt to perform a similar analysis here to the coupled Einstein-Maxwell equations for a general form of the gravitational potentials with charged matter. We find that the generalised condition of pressure isotropy leads to a new recurrence relation which can be solved in general.

We seek new exact solutions to the Einstein-Maxwell field equations, using the systematic series analysis, which may be used to describe the interior relativistic sphere. Our objective is to obtain a general class of exact solutions which contains previously known models as particular cases. This approach produces a number of difference equations, which we demonstrate can be solved explicitly from first principles. We first express the Einstein-Maxwell system of equations for static spherically symmetric line element as an equivalent system using the Durgapal and Bannerji (1983) transformation in §2.2. In §2.3, we choose particular forms for one of the gravitational potentials and the electric field intensity, which reduce the condition of pressure isotropy to a linear second order equation in the remaining gravitational potential. We integrate this generalised condition of isotropy equation using the Frobenius method in §2.4. In general the solution will be given in terms of special functions. However elementary functions are regainable, and in §2.5, we find two categories of solutions in terms of elementary functions by placing certain restriction on the parameters. In §2.6, we regain known charged Einstein-Maxwell models and uncharged Einstein models from our general class of models. In §2.7, we discuss the physical features, plot the matter variables and show that our models are physically reasonable. The results of this chapter have been accepted for publication in Thirukkanesh and Maharaj (2008a)

2.2 The field equations

The gravitational field should be static and spherically symmetric for describing the internal structure of a dense compact relativistic sphere which is charged. For describing such a configuration, we utilise coordinates $(x^a) = (t, r, \theta, \phi)$, such that the generic form of the line element is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.1}$$

The Einstein field equations can be written in the form

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho, \tag{2.2.2a}$$

$$-\frac{1}{r^2}\left(1-e^{-2\lambda}\right) + \frac{2\nu'}{r}e^{-2\lambda} = p, \tag{2.2.2b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p, \tag{2.2.2c}$$

for neutral perfect fluids. The energy density ρ and the pressure p are measured relative to the comoving fluid 4-velocity $u^a = e^{-\nu}\delta^a_0$ and primes denote differentiation with respect to the radial coordinate r. In the system (2.2.2a)-(2.2.2c), we are using units where the coupling constant $\frac{8\pi G}{c^4} = 1$ and the speed of light $c = 1$. This system of equations determines the behaviour of the gravitational field for a neutral perfect fluid source. A different but equivalent form of the field equations can be found if we introduce the transformation

$$x = Cr^2, \quad Z(x) = e^{-2\lambda(r)}, \quad A^2y^2(x) = e^{2\nu(r)}, \tag{2.2.3}$$

so that the line element (2.2.1) becomes

$$ds^2 = -A^2y^2dt^2 + \frac{1}{4CxZ}dx^2 + \frac{x}{C}(d\theta^2 + \sin^2\theta d\phi^2).$$

The parameters A and C are arbitrary constants in (2.2.3). Under the transformation (2.2.3), the system (2.2.2a)-(2.2.2c) has the equivalent form

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C}, \tag{2.2.4a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C}, \tag{2.2.4b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0, \tag{2.2.4c}$$

where dots represent differentiation with respect to x.

In the presence of an electromagnetic field the system (2.2.4a)-(2.2.4c) has to be replaced by the Einstein-Maxwell system of equations. We generate the system

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C}, \tag{2.2.5a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} - \frac{E^2}{2C}, \tag{2.2.5b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2 x}{C}\right)y = 0, \tag{2.2.5c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2, \tag{2.2.5d}$$

where E is the electric field intensity and σ is the charge density. This system of equations governs the behaviour of the gravitational field for a charged perfect fluid source. When $E = 0$ the Einstein-Maxwell equations (2.2.5a)-(2.2.5d) reduce to the uncharged Einstein equations (2.2.4a)-(2.2.4c).

2.3 Choosing Z and E

We seek solutions to the Einstein-Maxwell field equations (2.2.5a)-(2.2.5d) by making explicit choices for the gravitational potential Z and the electric field intensity E on physical grounds. The system (2.2.5a)-(2.2.5d) comprises four equations in the six unknowns Z, y, ρ, p, E and σ. Equation (2.2.5c), called the generalised condition of pressure isotropy, is the master equation in the integration process. In this treatment we specify the gravitational potential Z and electric field intensity E, so that it is possible to integrate (2.2.5c). The explicit solution of the Einstein-Maxwell system (2.2.5a)-(2.2.5d) then follows. We make the particular choice

$$Z = \frac{1+ax}{1+bx} \tag{2.3.1}$$

where a and b are real constants. The function Z is regular at the centre and well behaved in the stellar interior for a wide range of values of a and b. It is important to note that the choice (2.3.1) for Z is physically reasonable. This form for the potential Z contains special cases which correspond to neutron star models, eg. when $a = -\frac{1}{2}$ and $b = 1$ we regain the uncharged dense neutron star of Durgapal and Bannerji (1983). When a is arbitrary and $b = 1$ then Thirukkanesh and Maharaj (2006) and Maharaj and Komathiraj (2007) found charged solutions to the Einstein-Maxwell system. These solutions can be used to model a charged relativistic sphere with desirable physical properties. Consequently the general form (2.3.1) contains known physically acceptable uncharged and charged relativistic stars for particular values of a and b. We seek to study the Einstein-Maxwell system with the choice (2.3.1) in an attempt to find new solutions, and to show explicitly that cases found previously can be placed into our general class of models.

Upon substituting (2.3.1) in equation (2.2.5c) we obtain

$$4(1+ax)(1+bx)\ddot{y} + 2(a-b)\dot{y} + \left[b(b-a) - \frac{E^2(1+bx)^2}{Cx}\right]y = 0. \tag{2.3.2}$$

The differential equation (2.3.2) is difficult to solve in the above form; we first introduce a transformation to obtain a more convenient form. We let

$$1+bx = X, \quad Y(X) = y(x), \quad b \neq 0. \tag{2.3.3}$$

With the help of (2.3.3), (2.3.2) becomes

$$4X\left[aX - (a-b)\right]\frac{d^2Y}{dX^2} + 2(a-b)\frac{dY}{dX} + \left[(b-a) - \frac{E^2X^2}{C(X-1)}\right]Y = 0 \tag{2.3.4}$$

in terms of the new dependent and independent variables Y and X respectively. The differential equation (2.3.4) may be integrated once the electric field E is given. A variety of choices for E is possible but only a few have the desirable features in the stellar interior. Note that the particular choice

$$E^2 = \frac{\alpha C(X-1)}{X^2} = \frac{\alpha Cbx}{(1+bx)^2} \tag{2.3.5}$$

where α is a constant has the advantage of simplifying (2.3.4). The electric field given in (2.3.5) vanishes at the centre of the star, and remains continuous and bounded for all interior points in the star. When $b = 1$ then E^2 reduces to the expression in the treatment of Maharaj and Komathiraj (2007). Thus the choice for E in (2.3.5) is physically reasonable in the study of the gravitational behaviour of charged stars. With the help of (2.3.5) we find that (2.3.4) becomes

$$4X\left[aX-(a-b)\right]\frac{d^2Y}{dX^2} + 2(a-b)\frac{dY}{dX} + \left[(b-a)-\alpha\right]Y = 0. \tag{2.3.6}$$

The differential equation (2.3.6) becomes

$$4X\left[aX-(a-b)\right]\frac{d^2Y}{dX^2} + 2(a-b)\frac{dY}{dX} + (b-a)Y = 0 \tag{2.3.7}$$

when $\alpha = 0$, and there is no charge.

2.4 Solutions

We need to integrate the master equation (2.3.6) to solve the Einstein-Maxwell system (2.2.5a)-(2.2.5d). Two categories of solution are possible when $a = b$ and $a \neq b$.

2.4.1 The case $a = b$

When $a = b$ equation (2.3.6) becomes

$$X^2\frac{d^2Y}{dX^2} - \frac{\alpha}{4a}Y = 0 \tag{2.4.1}$$

which is an Euler-Cauchy equation. The solution of (2.4.1) is

$$Y = \begin{cases} c_1(1+ax)^{(1+\sqrt{1+\alpha/a})/2} + c_2(1+ax)^{(1-\sqrt{1+\alpha/a})/2} & \text{if } a > 0, \\ \\ \sqrt{1+ax}\left[c_1 \sin\left(\sqrt{\frac{a+\alpha}{4a}}\ln(1+ax)\right)\right. \\ \left. +c_2 \cos\left(\sqrt{\frac{a+\alpha}{4a}}\ln(1+ax)\right)\right] & \text{if } a < 0, \end{cases} \tag{2.4.2}$$

where c_1 and c_2 are constants. From (2.2.5a) and (2.3.1) we observe that $\rho = -\frac{E^2}{2}$. We do not pursue this case to avoid negative energy densities. It is interesting to observe that when $a = b = 0$ then it is possible to generate an exact Einstein-Maxwell solution to (2.2.5a)-(2.2.5d), for a different choice of E^2, which contains the Einstein universe as pointed out by Komathiraj and Maharaj (2007a).

2.4.2 The case $a \neq b$

Observe that it is not possible to express the general solution of the master equation (2.3.6) in terms of conventional elementary functions for all values of a, b ($a \neq b$) and α. In general the solution can be written in terms of special functions. It is necessary to express the solution in a simple form so that it is possible to conduct a detailed physical analysis. Hence in this section we attempt to obtain a general solution to the differential equation (2.3.6) in series form. In a subsequent section we show that it is possible to find particular solutions in terms of algebraic functions and polynomials.

We can utilise the method of Frobenius about $X = 0$, since this is a regular singular point of the differential equation (2.3.6). We write the solution of the differential equation (2.3.6) in the series form

$$Y = \sum_{n=0}^{\infty} c_n X^{n+r}, \quad c_0 \neq 0 \tag{2.4.3}$$

where c_n are the coefficients of the series and r is a constant. For an acceptable solution we need to find the coefficients c_n as well as the parameter r. On substituting

(2.4.3) in the differential equation (2.3.6) we have

$$2(a-b)c_0 r[-2(r-1)+1]X^{r-1}+\sum_{n=1}^{\infty}[2(a-b)c_{n+1}(n+r+1)[-2(n+r)+1]$$

$$+c_n\left[4a(n+r)(n+r-1)-(a-b+\alpha)\right]]\,X^{n+r}=0. \tag{2.4.4}$$

For consistency the coefficients of the various powers of X must vanish in (2.4.4). Equating the coefficient of X^{r-1} in (2.4.4) to zero, we find

$$(a-b)c_0 r[2(r-1)-1]=0$$

which is the indicial equation. Since $c_0 \neq 0$ and $a \neq b$, we must have $r=0$ or $r=\frac{3}{2}$. Equating the coefficient of X^{n+r} in (2.4.4) to zero we obtain

$$c_{n+1}=\frac{4a(n+r)(n+r-1)-[a-b+\alpha]}{2(a-b)(n+1+r)[2(n+r)-1]}c_n, \quad n\geq 0 \tag{2.4.5}$$

The result (2.4.5) is the basic difference equation which determines the nature of the solution.

We can establish a general structure for all the coefficients by considering the leading terms. We note that the coefficients $c_1, c_2, c_3, \ldots$ can all be written in terms of the leading coefficient c_0, and this leads to the expression

$$c_{n+1}=\prod_{p=0}^{n}\frac{4a(p+r)(p+r-1)-(a-b+\alpha)}{2(a-b)(p+1+r)[2(p+r)-1]}c_0 \tag{2.4.6}$$

where the symbol $\prod$ denotes multiplication. It is also possible to establish the result (2.4.6) rigorously by using the principle of mathematical induction. We can now generate two linearly independent solutions from (2.4.3) and (2.4.6). For the parameter value $r=0$ we obtain the first solution

$$\begin{aligned}
Y_1 &= c_0\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{4ap(p-1)-(a-b+\alpha)}{2(a-b)(p+1)(2p-1)}X^{n+1}\right] \\
y_1 &= c_0\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{4ap(p-1)-(a-b+\alpha)}{2(a-b)(p+1)(2p-1)}(1+bx)^{n+1}\right].
\end{aligned} \tag{2.4.7}$$

For the parameter value $r = \frac{3}{2}$ we obtain the second solution

$$
\begin{aligned}
Y_2 &= c_0 X^{\frac{3}{2}} \left[1 + \sum_{n=0}^{\infty} \prod_{p=0}^{n} \frac{a(2p+3)(2p+1) - (a-b+\alpha)}{(a-b)(2p+5)(2p+2)} X^{n+1}\right] \\
y_2 &= c_0 (1+bx)^{\frac{3}{2}} \left[1 + \sum_{n=0}^{\infty} \prod_{p=0}^{n} \frac{a(2p+3)(2p+1) - (a-b+\alpha)}{(a-b)(2p+5)(2p+2)} (1+bx)^{n+1}\right].
\end{aligned}
\tag{2.4.8}
$$

Therefore the general solution to the differential equation (2.3.2), for the choice (2.3.5), is given by

$$y = a_1 y_1(x) + b_1 y_2(x) \tag{2.4.9}$$

where a_1 and b_1 are arbitrary constants and y_1 and y_2 are given by (2.4.7) and (2.4.8) respectively. It is clear that the quantities y_1 and y_2 are linearly independent functions. From (2.2.5a)-(2.2.5d) and (2.4.9) the general solution to the Einstein-Maxwell system can be written as

$$e^{2\lambda} = \frac{1+bx}{1+ax}, \tag{2.4.10a}$$

$$e^{2\nu} = A^2 y^2, \tag{2.4.10b}$$

$$\frac{\rho}{C} = \frac{(b-a)(3+bx)}{(1+bx)^2} - \frac{\alpha bx}{2(1+bx)^2}, \tag{2.4.10c}$$

$$\frac{p}{C} = 4\frac{(1+ax)}{(1+bx)}\frac{\dot{y}}{y} + \frac{(a-b)}{(1+bx)} + \frac{\alpha bx}{2(1+bx)^2}, \tag{2.4.10d}$$

$$\frac{E^2}{C} = \frac{\alpha bx}{(1+bx)^2}. \tag{2.4.10e}$$

The result in (2.4.10a)-(2.4.10e) is a new solution to the Einstein-Maxwell field equations. Note that if we set $\alpha = 0$, (2.4.10a)-(2.4.10e) reduce to models for uncharged stars which may contain new solutions to the Einstein field equations (2.2.4a)-(2.2.4c).

2.5 Elementary functions

The general solution (2.4.9) can be expressed in terms of polynomial and algebraic functions. This is possible in general because the series (2.4.7) and (2.4.8) terminate for restricted values of the parameters a, b and α so that elementary functions are possible. Consequently we obtain two sets of general solutions in terms of elementary functions, by determining the specific restriction on the quantity $a - b + \alpha$ for a terminating series. The elementary functions found using this method, can be written as polynomials and polynomials with algebraic functions. We provide the details of the process in the Appendix A; here we present a summary of the results. In terms of the original variable x, the first category of solution can be written as

$$
\begin{aligned}
y = {} & d_1(1+ax)^{\frac{1}{2}}\left[1-(n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+bx)^i\right] \\
& + d_2\,(1+bx)^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i \frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}(1+bx)^i\right] \qquad (2.5.1)
\end{aligned}
$$

for $a - b + \alpha = a(2n + 3)(2n + 1)$, where d_1 and d_2 are arbitrary constants. The second category of solutions can be written as

$$
\begin{aligned}
y = {} & d_3(1+ax)^{\frac{1}{2}}(1+bx)^{\frac{3}{2}} \times \\
& \left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{b-a}\right)^i \frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}(1+bx)^i\right] \\
& + d_4\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+bx)^i\right] \qquad (2.5.2)
\end{aligned}
$$

for $a-b+\alpha = 4an(n-1)$, where d_3 and d_4 are arbitrary constants. It is remarkable to observe that the solutions (2.5.1) and (2.5.2) are expressed completely in terms of elementary functions only. This does not happen often considering the nonlinearity of the gravitational interaction in the presence of charge. We have given our solutions in a simple form: this has the advantage of facilitating the analysis of the physical features of the stellar models. Observe that our approach has combined both the charged and uncharged cases for a relativistic star: when $\alpha = 0$ we obtain the solutions for the uncharged case directly.

2.6 Known solutions

It is interesting to observe that we can regain a number of physically reasonable models from the general class of solutions found in this chapter. These individual models can be generated from the general series solution (2.4.9) or the simplified elementary functions (2.5.1) and (2.5.2). We generate explicitly the following models.

Case I: Hansraj and Maharaj charged stars

For this case we set $a = 0, b = 1$ and $0 \leq \alpha < 1$. Then from (2.4.7) we find that

$$\begin{aligned}
y_1 &= c_0 \left[1 + \sum_{n=0}^{\infty} \prod_{p=0}^{n} \frac{-(1-\alpha)}{2(p+1)(2p-1)} (\sqrt{1+x})^{2n+2}\right] \\
&= c_0 \left(\left[1 - \frac{(\sqrt{(1-\alpha)(1+x)})^2}{2!} + \frac{(\sqrt{(1-\alpha)(1+x)})^4}{4!} \right.\right. \\
&\quad \left. - \frac{(\sqrt{(1-\alpha)(1+x)})^6}{6!} + \ldots \right] + \sqrt{(1-\alpha)(1+x)} \left[\sqrt{(1-\alpha)(1+x)} \right. \\
&\quad \left.\left. - \frac{(\sqrt{(1-\alpha)(1+x)})^3}{3!} + \frac{(\sqrt{(1-\alpha)(1+x)})^5}{5!} - \ldots \right] \right) \\
&= c_0 \cos\sqrt{(1-\alpha)(1+x)} + c_0 \sqrt{(1-\alpha)(1+x)} \sin\sqrt{(1-\alpha)(1+x)}.
\end{aligned}$$

Equation (2.4.8) gives the result

$$
\begin{aligned}
y_2 &= c_0(\sqrt{1+x})^3 \left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{-(1-\alpha)}{(2p+5)(2p+2)}(\sqrt{1+x})^{2n+2}\right] \\
&= \frac{3c_0}{(\sqrt{1-\alpha})^3}\left(\left[\sqrt{(1-\alpha)(1+x)}-\frac{(\sqrt{(1-\alpha)(1+x)})^3}{3!}\right.\right. \\
&\quad \left.+\frac{(\sqrt{(1-\alpha)(1+x)})^5}{5!}-...\right]-\sqrt{(1-\alpha)(1+x)}\left[1\right. \\
&\quad \left.\left.-\frac{(\sqrt{(1-\alpha)(1+x)})^2}{2!}+\frac{(\sqrt{(1-\alpha)(1+x)})^4}{4!}-...\right]\right) \\
&= \frac{3c_0}{(\sqrt{1-\alpha})^3}\left[\sin\sqrt{(1-\alpha)(1+x)}-\sqrt{(1-\alpha)(1+x)}\cos\sqrt{(1-\alpha)(1+x)}\right].
\end{aligned}
$$

Hence the general solution becomes

$$
\begin{aligned}
y &= \left[D_1-D_2\sqrt{(1-\alpha)(1+x)}\right]\cos\sqrt{(1-\alpha)(1+x)} \\
&\quad +\left[D_2+D_1\sqrt{(1-\alpha)(1+x)}\right]\sin\sqrt{(1-\alpha)(1+x)} \qquad (2.6.1)
\end{aligned}
$$

where D_1 and D_2 are new arbitrary constants. The class of charged solutions (2.6.1) is the first category found by Hansraj and Maharaj (2006).

When $a=0, b=1$ and $\alpha=1$ we easily obtain the result

$$y = a_1 + b_1(1+x)^{\frac{3}{2}} \qquad (2.6.2)$$

from (2.4.9). This is the second category of the Hansraj and Maharaj charged solutions.

We now set $a=0, b=1$ and $\alpha>1$. Then from (2.4.7) we obtain

$$
\begin{aligned}
y_1 &= c_0\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{(\alpha-1)}{2(p+1)(2p-1)}(\sqrt{1+x})^{2n+2}\right]\\
&= c_0\left(\left[1+\frac{(\sqrt{(\alpha-1)(1+x)})^2}{2!}+\frac{(\sqrt{(\alpha-1)(1+x)})^4}{4!}\right.\right.\\
&\quad\left.+\frac{(\sqrt{(\alpha-1)(1+x)})^6}{6!}+\ldots\right]-\sqrt{(\alpha-1)(1+x)}\left[\sqrt{(\alpha-1)(1+x)}\right.\\
&\quad\left.\left.+\frac{(\sqrt{(\alpha-1)(1+x)})^3}{3!}+\frac{(\sqrt{(\alpha-1)(1+x)})^5}{5!}+\ldots\right]\right)\\
&= c_0\cosh\sqrt{(\alpha-1)(1+x)}-c_0\sqrt{(\alpha-1)(1+x)}\sinh\sqrt{(\alpha-1)(1+x)}.
\end{aligned}
$$

Equation (2.4.8) gives the result

$$
\begin{aligned}
y_2 &= c_0(\sqrt{1+x})^3\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{(\alpha-1)}{(2p+5)(2p+2)}(\sqrt{1+x})^{2n+2}\right]\\
&= \frac{-3c_0}{(\sqrt{\alpha-1})^3}\left(\left[\sqrt{(\alpha-1)(1+x)}+\frac{(\sqrt{(\alpha-1)(1+x)})^3}{3!}\right.\right.\\
&\quad\left.+\frac{(\sqrt{(\alpha-1)(1+x)})^5}{5!}+\ldots\right]-\sqrt{(\alpha-1)(1+x)}\left[1\right.\\
&\quad\left.\left.+\frac{(\sqrt{(\alpha-1)(1+x)})^2}{2!}+\frac{(\sqrt{(\alpha-1)(1+x)})^4}{4!}+\ldots\right]\right)\\
&= \frac{-3c_0}{(\sqrt{\alpha-1})^3}(\sinh\sqrt{(1-\alpha)(1+x)}-\sqrt{(1-\alpha)(1+x)}\cosh\sqrt{(1-\alpha)(1+x)}).
\end{aligned}
$$

Therefore, the general solution becomes

$$
\begin{aligned}
y &= \left[D_2-D_1\sqrt{(\alpha-1)(1+x)}\right]\sinh\sqrt{(\alpha-1)(1+x)}\\
&\quad+\left[D_1-D_2\sqrt{(\alpha-1)(1+x)}\right]\cosh\sqrt{(\alpha-1)(1+x)} \qquad (2.6.3)
\end{aligned}
$$

where D_1 and D_2 are new arbitrary constants. This is the third category of charged solutions found by Hansraj and Maharaj.

The exact solutions (2.6.1), (2.6.2) and (2.6.3) were comprehensively studied by Hansraj and Maharaj (2006), and it was shown that these solutions correspond to a charged relativistic sphere which is realistic as all conditions for physically acceptability are met. The condition of causality is satisfied and the speed of light is greater than the speed of sound.

Case II: Maharaj and Komathiraj charged stars

If $b = 1$, then (2.5.1) becomes

$$y = d_1(1+ax)^{\frac{1}{2}} \left[1 - (n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{1-a}\right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+x)^i\right]$$

$$+d_2\,(1+x)^{\frac{3}{2}} \left[1 + \frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{1-a}\right)^i \frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}(1+x)^i\right] \quad (2.6.4)$$

for $a - 1 + \alpha = a(2n+1)(2n+3)$. When $b = 1$ then (2.5.2) gives

$$y = d_3(1+ax)^{\frac{1}{2}}(1+x)^{\frac{3}{2}} \left[1 + \frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{1-a}\right)^i \frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}(1+x)^i\right]$$

$$+d_4\left[1 - n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{1-a}\right)^i \frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+x)^i\right] \quad (2.6.5)$$

for $a-1+\alpha = 4an(n-1)$. The two categories of solutions (2.6.4) and (2.6.5) correspond to the Maharaj and Komathiraj (2007) model for a compact sphere in electric fields. The Maharaj and Komathiraj charged stars have a simple form in terms of elementary functions; they are physically reasonable and contain the Durgapal and Bannerji (1983) model and other exact models corresponding to neutron stars.

Case III: Finch and Skea neutron stars

When $\alpha = 0$, we obtain

$$y = \left[D_1 - D_2\sqrt{1+x}\right]\cos\sqrt{1+x} + \left[D_2 + D_1\sqrt{1+x}\right]\sin\sqrt{1+x} \quad (2.6.6)$$

from (2.6.1). Thus, we regain the Finch and Skea (1989) model for a neutron star when the electromagnetic field is absent. The Finch and Skea neutron star model has been shown to satisfy all the physical criteria for an isolated spherically symmetric stellar uncharged source. It is for this reason that this model has been used by many researchers to model the interior of neutron stars.

Case IV: Durgapal and Bannerji neutron stars

If we take $\alpha = 0$ and $n = 0$ then $2a + b = 0$, and we get

$$y = d_1(1+ax)^{\frac{1}{2}}(5-4ax) + d_2(1-2ax)^{\frac{3}{2}}$$

from (2.5.1). If we set $a = -\frac{1}{2}$ (ie., $b = 1$), then it is easy to verify that this equation becomes

$$y = c_1(2-x)^{\frac{1}{2}}(5+2x) + c_2(1+x)^{\frac{3}{2}} \tag{2.6.7}$$

where $c_1 = d_1/3\sqrt{2}$ and $c_2 = d_2$ are new arbitrary constants. Thus we have regained the Durgapal and Bannerji (1983) neutron star model. This model satisfies all physical criteria for acceptability and has been utilised by many researchers to model uncharged neutron stars.

Case V: Tikekar superdense stars

If we take $\alpha = 0$ and $n = 2$ then $7a + b = 0$, and we find

$$y = d_3(1+ax)(1-7ax)^{\frac{3}{2}} + d_4\left[1 + \frac{1}{2}(1-7ax) - \frac{1}{8}(1-7ax)^2\right]$$

from (2.5.2). If we set $a = -1$ (ie., $b = 7$) and let $\tilde{x} = \sqrt{1-x}$ then this equation becomes

$$y = c_1\tilde{x}(1 - \frac{7}{8}\tilde{x}^2)^{\frac{3}{2}} + c_2\left[1 - \frac{7}{2}\tilde{x}^2 + \frac{49}{24}\tilde{x}^4\right] \tag{2.6.8}$$

where $c_1 = d_3 8^{\frac{3}{2}}$ and $c_2 = -d_4/3$ are new arbitrary constants. Thus we have regained the Tikekar (1990) model for superdense neutron star from our general solution. The Tikekar superdense model plays an important role in describing highly dense matter, cold compact matter and core-envelope models for relativistic stars. The Tikekar relativistic star falls into a more general class of models with spheroidal spatial geometry found by Maharaj and Leach (1996); this class can be generalised to include the presence of an electric field as shown by Komathiraj and Maharaj (2007b).

2.7 Physical analysis

In this section we demonstrate that the exact solutions found in this chapter are physically reasonable and may be used to model a charged relativistic sphere. We observe from (2.4.10a) and (2.4.10b) that the gravitational potentials $e^{2\nu}$ and $e^{2\lambda}$ are continuous in the stellar interior and nonzero at the centre for all values of the parameters a, b and α. From (2.4.10c), we can express the variable x in terms of the energy density ρ only as

$$x = \frac{1}{4b}\left\{C[2(b-a)-\alpha]\rho^{-1} \pm \sqrt{C^2[2(b-a)-\alpha]^2\rho^{-2} + 8C[4(b-a)+\alpha]\rho^{-1} - 4}\right\}.$$

Therefore from (2.4.10d), the isotropic pressure p can be expressed in terms of ρ only. Thus all the forms of the solutions presented in this chapter satisfy the barotropic equation of state $p = p(\rho)$ which is a desirable feature. Note that many of the solutions appeared in the literature do not satisfy this property.

To illustrate the graphical behaviour of the matter variables in the stellar interior we consider the particular solution (2.6.1). In this case the line element becomes

$$\begin{aligned} ds^2 &= -A^2\Big[\Big(D_1 - D_2\sqrt{(1-\alpha)(1+r^2)}\Big)\cos\sqrt{(1-\alpha)(1+r^2)} \\ &\quad + \Big(D_2 + D_1\sqrt{(1-\alpha)(1+r^2)}\Big)\sin\sqrt{(1-\alpha)(1+r^2)}\Big]^2 dt^2 \\ &\quad +(1+Cr^2)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{2.7.1}$$

For simplicity we make the choice $A = 1, C = 1, D_1 = 1, D_2 = 4$ in the metric (2.7.1). We choose $\alpha = \frac{1}{2}$ for charged matter and we consider the interval $0 \leq r \leq 1$ to generate the relevant plots. We utilised the software package Mathematica to generate the plots for ρ, p, E^2 and $\frac{dp}{d\rho}$ respectively. The dotted line corresponds to $\alpha = \frac{1}{2}$ and $E^2 \neq 0$; the solid line corresponds to $\alpha = 0$ and $E^2 = 0$. In Figure 2.1, we have plotted the energy density on the interval $0 \leq r \leq 1$. It can be easily seen that the energy densities in both cases are positive and continuous at the centre; it is a monotonically decreasing function throughout the interior of the star from centre to the boundary. In Figure 2.2, we have plotted the behaviour of the isotropic pressure. The pressure p remains regular in the interior and is monotonically decreasing. The role of the electromagnetic field is highlighted in Figures 2.1 and 2.2: the effect of E^2 is to produce smaller values for ρ and p. From Figures 2.1 and 2.2 we observe that the presence of E does not significantly affect ρ but has a much greater influence on p closer to the centre. We believe that this follows directly from our choice (2.3.5) for the electric field intensity; other choices of E generate different profiles as indicated in Komathiraj and Maharaj (2007a). The electric field intensity E^2 is given in Figure 2.3 which is positive, continuous and monotonically increasing. In Figure 2.4, we have plotted $\frac{dp}{d\rho}$ on the interval $0 \leq r \leq 1$ for both charged and uncharged cases. We observe that $\frac{dp}{d\rho}$ is always positive and less than unity. This indicates that the speed of the sound is less than the speed of the light and causality is maintained. Note that the effect of the electromagnetic field is to produce lower values for $\frac{dp}{d\rho}$ and the speed of sound is decreased when $\alpha \neq 0$. Hence we have shown that the solution (2.6.1), for our particular chosen parameter values, satisfies the requirements for a physically reasonable charged body.

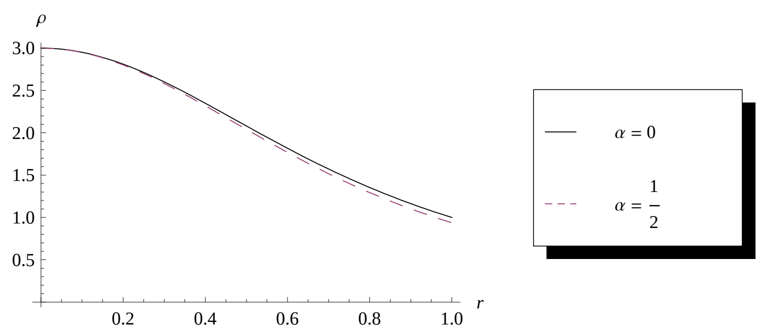

Figure 2.1: Energy density.

For particular parameter values our plots of the behaviour of ρ, p, E^2 and $\frac{dp}{d\rho}$ show that they were physically reasonable. A pleasing feature of our plots is that

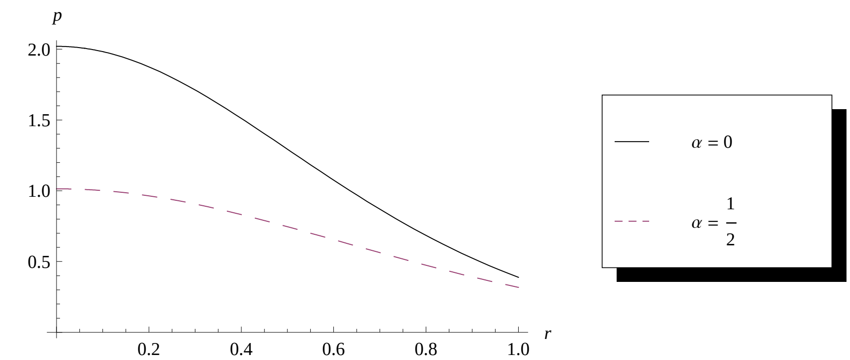

Figure 2.2: Matter pressure.

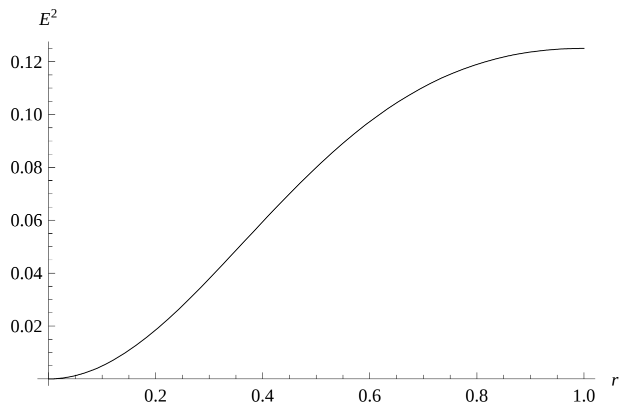

Figure 2.3: Electric field.

we can distinguish between charged and uncharged exact solutions. The presence of charge leads to smaller values of ρ, p and $\frac{dp}{d\rho}$ in the figures generated. This indicates that the presence of charge can dramatically affect the behaviour of the matter and gravitational variables.

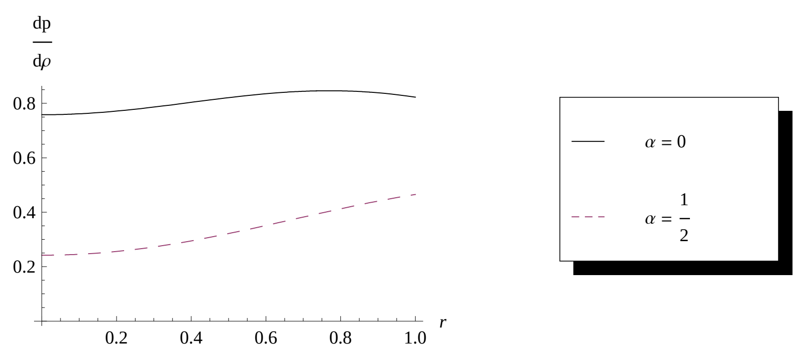

Figure 2.4: $dp/d\rho$.

Chapter 3

Some new static charged spheres

3.1 Introduction

Exact solutions of the Einstein-Maxwell system of field equations, for spherically symmetric gravitational fields in static manifolds, are necessary to describe charged compact spheres in relativistic astrophysics. The solutions to the field equations generated have a number of different applications in relativistic stellar systems. It is for this reason that a number of investigations have been undertaken on the Einstein-Maxwell equations in recent times. A comprehensive review of exact solutions and criteria for physical admissability is provided by Ivanov (2002). A general treatment of nonstatic spherically symmetric solutions to the Einstein-Maxwell system, in the case of vanishing shear was, performed by Wafo Soh and Mahomed (2000) using symmetry methods. The uncharged case was considered by Wafo Soh and Mahomed (1999) who show that all existing solutions arise because of the existence of a Noether point symmetry; the physical relevance of the solutions was investigated by Feroze *et al* (2003). The matching of nonstatic charged perfect fluid spheres to the Reissner-Nordstrom exterior metric was pursued by Mahomed *et al* (2003) who highlighted the role of the Bianchi identities in restricting the number of solutions.

In this chapter, our objective is to find a new class of solutions to the Einstein-Maxwell system that satisfies the physical criteria. We attempt to perform a similar analysis applied in the previous chapter to the coupled Einstein-Maxwell equations by choosing a different rational form for one of the gravitational potentials and a particular form for the charged matter distribution. We obtain a new recurrence relation with real and rational coefficients by simplifying the condition of pressure isotropy, which can then be solved explicitly from first principles. The Einstein Maxwell system of field equations are given in §3.2. In §3.3, we choose specific forms for one of the gravitational potentials and the electric field intensity, which reduce the condition of pressure isotropy to a linear second order equation in the remaining gravitational potential. We integrate this equation using the method of Frobenius in §3.4. In general the solution will be given in terms of elementary functions. We demonstrate that it is possible to find two category of solutions in terms of elementary functions by placing certain restriction on the parameters. We regain known charged Einstein-Maxwell models from our general class of models in §3.5. In §3.6, we discuss the physical features of the solutions found, plot the matter variables, and show that our models are physically reasonable. The results of this chapter have been accepted for publication in Maharaj and Thirukkanesh (2008a).

3.2 Basic equations

We assume that the spacetime is spherically symmetric and static which is consistent with the study of charged compact objects in relativistic astrophysics. In Schwarzschild coordinates (t, r, θ, ϕ) the generic form of the line element is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.2.1}$$

With the help of the the transformations

$$x = Cr^2,\ Z(x) = e^{-2\lambda(r)},\ A^2y^2(x) = e^{2\nu(r)},$$

the Einstein-Maxwell field equations for the line element (3.2.1) can be written in the form as

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C}, \tag{3.2.2a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} - \frac{E^2}{2C}, \tag{3.2.2b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2x}{C}\right)y = 0, \tag{3.2.2c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2, \tag{3.2.2d}$$

where A and C are real constants. For more information on the Einstein-Maxwell system with isotropic pressures see §2.2.

3.3 Choosing potentials

Our objective is to find a new class of solutions to the Einstein-Maxwell system by making explicit choices for the gravitational potential Z and the electric field intensity E. We make the choice for Z as

$$Z(x) = \frac{(1+ax)^2}{1+bx}, \tag{3.3.1}$$

where a and b are real constants. Note that the choice (3.3.1) ensures that the gravitational potential $e^{2\lambda}$ is regular and well behaved in the stellar interior for a wide range of values of the parameters a and b. In addition, when $x = 0$ then $Z = 1$ which ensures that there is no singularity at the stellar centre. A special case of (3.3.1) was studied by Komathiraj and Maharaj (2007a). The choice (3.3.1) does produces charged and uncharged solutions which are necessary for constructing realistic stellar models. On substituting (3.3.1) in (3.2.2c) we obtain

$$4(1+ax)^2(1+bx)\ddot{y} + 2(1+ax)[b(1+ax) - 2(b-a)]\dot{y}$$

$$+\left[(a-b)^2 - \frac{E^2(1+bx)^2}{Cx}\right]y = 0, \tag{3.3.2}$$

which is a second order differential equation.

The differential equation (3.3.2) may be solved if a particular choice of the electric field intensity E is made. For our purpose we set

$$\frac{E^2}{C} = \frac{\alpha a(b-a)x}{(1+bx)^2}, \tag{3.3.3}$$

where α is a constant. The electric field intensity specified in (3.3.3) vanishes at the centre of the sphere; it is continuous and bounded in the stellar interior for wide range of values of x. The quantity E^2 has positive values in the interior of star for relevant choices of the constants α, a and b. Therefore the form given in (3.3.3) is physically reasonable to study the behaviour of charged spheres. With the choice (3.3.3) we can express (3.3.2) in the form

$$4(1+ax)^2\left[b(1+ax)-(b-a)\right]\ddot{y} + 2a(1+ax)\left[b(1+ax)-2(b-a)\right]\dot{y}$$

$$+a(b-a)(b-a-\alpha a)y = 0. \tag{3.3.4}$$

In (3.3.4) we assume that $a \neq 0$ and $a \neq b$ so that the electric field intensity is present. When $\alpha = 0$ there is no charge.

3.4 Solutions

To find the solution of the Einstein-Maxwell system we need to integrate the master equation (3.3.4). We consider two cases on the integration process: $\alpha = \frac{b}{a} - 1$ and $\alpha \neq \frac{b}{a} - 1$.

3.4.1 The case $\alpha = \frac{b}{a} - 1$

In this case the differential equation (3.3.4) becomes

$$2(1+ax)\left[b(1+ax)-(b-a)\right]\ddot{y} + a\left[b(1+ax)-2(b-a)\right]\dot{y} = 0. \tag{3.4.1}$$

Equation (3.4.1) is easily integrable and the solution can be written as

$$y(x) = c_1\left(\sqrt{\frac{a(1+bx)}{b-a}} - \arctan\sqrt{\frac{a(1+bx)}{b-a}}\right) + c_2, \tag{3.4.2}$$

where c_1 and c_2 are constants of integration. Therefore the solution of the Einstein-Maxwell system (3.2.2a)-(3.2.2d) becomes

$$e^{2\lambda} = \frac{1+bx}{(1+ax)^2}, \tag{3.4.3a}$$

$$e^{2\nu} = A^2\left[c_1\left(\sqrt{\frac{a(1+bx)}{b-a}} - \arctan\sqrt{\frac{a(1+bx)}{b-a}}\right) + c_2\right]^2, \tag{3.4.3b}$$

$$\frac{\rho}{C} = \frac{(b-2a)(6+bx)}{2(1+bx)^2} - \frac{a^2x(11+6bx)}{2(1+bx)^2}, \tag{3.4.3c}$$

$$\begin{aligned}\frac{p}{C} &= \frac{(2a-b)(2+bx)}{2(1+bx)^2} + \frac{a^2x(3+2bx)}{2(1+bx)^2} \\ &\quad + \frac{2ac_1(1+ax)\sqrt{\frac{a(1+bx)}{b-a}}}{c_1(1+bx)\left(\sqrt{\frac{a(1+bx)}{b-a}} - \arctan\sqrt{\frac{a(1+bx)}{b-a}}\right) + c_2},\end{aligned} \tag{3.4.3d}$$

$$\frac{E^2}{C} = \frac{(b-a)^2x}{(1+bx)^2}. \tag{3.4.3e}$$

Observe that because of the restrictions $\alpha = \frac{b}{a} - 1$ and $b \neq a$ the charged solution (3.4.3) does not have an uncharged limit. Therefore this solution models a sphere that is always charged and cannot attain a neutral state. Note that the solution (3.4.3) is expressed in a simple form in terms of elementary functions which facilitates a physical analysis of the matter and gravitational variables.

3.4.2 The case $\alpha \neq \frac{b}{a} - 1$

With $\alpha \neq \frac{b}{a} - 1$, equation (3.3.4) is difficult to solve. Consequently we introduce the transformation

$$y = (1+ax)^d U(1+ax), \tag{3.4.4}$$

where U is a function of $(1+ax)$ and d is constant. With the help of (3.4.4), the differential equation (3.3.4) can be written as

$$4(1+ax)^2\left[b(1+ax)-(b-a)\right]\ddot{U}$$

$$+2(1+ax)\left[b(4d+1)(1+ax)-2(2d+1)(b-a)\right]\dot{U}$$

$$+\left[2bd(2d-1)(1+ax)-(b-a)\left(\frac{b}{a}-1-\alpha-4d^2\right)\right]U=0. \quad (3.4.5)$$

Note that there is substantial simplification if we take

$$\frac{b}{a}-1-\alpha=4d^2.$$

Then (3.4.5) becomes

$$2(1+ax)\left[(1+ax)-\frac{(b-a)}{b}\right]\ddot{U}$$

$$+\left[(4d+1)(1+ax)-2(2d+1)\frac{(b-a)}{b}\right]\dot{U}+d(2d-1)U=0, \quad (3.4.6)$$

where $b \neq 0$. We observe that the point $1+ax=\frac{b-a}{b}$ is a regular singular point of the differential equation (3.4.6). Therefore, the solution of the differential equation (3.4.6) can be written in the form of an infinite series by the method of Frobenius:

$$U=\sum_{i=0}^{\infty}c_i\left[(1+ax)-\frac{(b-a)}{b}\right]^{i+r},\ c_0\neq 0, \quad (3.4.7)$$

where c_i are the coefficients of the series and r is the constant. To complete the solution we need to find the coefficients c_i as well as the parameter r explicitly. On substituting (3.4.7) in the differential equation (3.4.6) we have

$$\frac{(b-a)}{b}c_0r(2r-3)\left[(1+ax)-\frac{(b-a)}{b}\right]^{r-1}+\sum_{i=1}^{\infty}\left\{c_i\frac{(b-a)}{b}(i+r)(2i+2r-3)\right.$$

$$\left.+c_{i-1}[(i+r-1)(2i+2r+4d-3)+d(2d-1)]\right\}\left[(1+ax)-\frac{(b-a)}{b}\right]^{i+r-1}=0. \quad (3.4.8)$$

For the consistency the coefficients of various powers of $\left[(1+ax)-\frac{(b-a)}{b}\right]$ must vanish in (3.4.8). Equating the coefficient of $\left[(1+ax)-\frac{(b-a)}{b}\right]^{r-1}$ in (3.4.8) to zero, we find

$$c_0 r(2r-3) = 0.$$

which is the indicial equation. As $c_0 \neq 0$ we must have $r = 0$ or $r = 3/2$. Equating the coefficient of $\left[(1+ax)-\frac{(b-a)}{b}\right]^{i+r-1}$ in (3.4.8) to zero we obtain

$$c_i = \left(\frac{b}{a-b}\right)\frac{[(i+r-1)(2i+2r+4d-3)+d(2d-1)]}{(i+r)(2i+2r-3)}c_{i-1}, \quad i \geq 1. \tag{3.4.9}$$

The difference equation (3.4.9) governs the structure of the solution. We can express the structure for the general coefficient c_i in terms of the leading coefficient c_0 as

$$c_i = \left(\frac{b}{a-b}\right)^i \prod_{p=1}^{i}\frac{[(p+r-1)(2p+2r+4d-3)+d(2d-1)]}{(p+r)(2p+2r-3)}c_0, \tag{3.4.10}$$

where the conventional symbol $\prod$ denotes multiplication. We can verify the result (3.4.10) using mathematical induction.

We can now generate two linearly independent solutions to (3.4.6) with the help of (3.4.7) and (3.4.10). For the parameter value $r = 0$, we obtain the first solution

$$\begin{aligned} U_1 &= c_0\left[1+\sum_{i=1}^{\infty}\left(\frac{b}{a-b}\right)^i\prod_{p=1}^{i}\frac{[(p-1)(2p+4d-3)+d(2d-1)]}{p(2p-3)}\times\right. \\ &\left.\left[(1+ax)-\frac{(b-a)}{b}\right]^i\right]. \end{aligned}$$

For the parameter value $r = 3/2$, we obtain the second solution

$$\begin{aligned} U_2 = c_0\left[(1+ax)-\frac{(b-a)}{b}\right]^{3/2}&\left[1+\sum_{i=1}^{\infty}\left(\frac{b}{a-b}\right)^i\times\right. \\ &\left.\prod_{p=1}^{i}\frac{[(2p+1)(p+2d)+d(2d-1)]}{p(2p+3)}\left[(1+ax)-\frac{(b-a)}{b}\right]^i\right]. \end{aligned}$$

Since the functions U_1 and U_2 are linearly independent we have found the general solution to (3.4.6). Therefore, the solutions to the differential equation (3.3.4) are

$$y_1(x) = c_0(1+ax)^d \left[1 + \sum_{i=1}^{\infty} \left(\frac{b}{a-b}\right)^i \times \right.$$

$$\left. \prod_{p=1}^{i} \frac{[(p-1)(2p+4d-3)+d(2d-1)]}{p(2p-3)} \left[(1+ax) - \frac{(b-a)}{b}\right]^i \right] \quad (3.4.11)$$

and

$$y_2(x) = c_0(1+ax)^d \left[(1+ax) - \frac{(b-a)}{b}\right]^{3/2} \left[1 + \sum_{i=1}^{\infty} \left(\frac{b}{a-b}\right)^i \times \right.$$

$$\left. \prod_{p=1}^{i} \frac{[(2p+1)(p+2d)+d(2d-1)]}{p(2p+3)} \left[(1+ax) - \frac{(b-a)}{b}\right]^i \right]. \quad (3.4.12)$$

Thus the general solution to the differential equation (3.3.2), for the choice of the electric field (3.3.3), is given by

$$y(x) = A_1 y_1(x) + A_2 y_2(x), \quad (3.4.13)$$

where A_1 and A_2 are arbitrary constants and $d^2 = \left(\frac{b}{a} - 1 - \alpha\right)/4$. From (3.4.13) and (3.2.2a)-(3.2.2d), the exact solution of the Einstein-Maxwell system becomes

$$e^{2\lambda} = \frac{1+bx}{(1+ax)^2}, \quad (3.4.14a)$$

$$e^{2\nu} = A^2 y^2, \quad (3.4.14b)$$

$$\frac{\rho}{C} = \frac{(3+bx)(b-2a)}{(1+bx)^2} - \frac{a^2x(5+3bx)}{(1+bx)^2} - \frac{\alpha a(b-a)x}{2(1+bx)^2}, \quad (3.4.14c)$$

$$\frac{p}{C} = 4\frac{(1+ax)^2}{(1+bx)}\frac{\dot{y}}{y} + \frac{a(2+ax)-b}{(1+bx)} + \frac{\alpha a(b-a)x}{2(1+bx)^2}, \quad (3.4.14d)$$

$$\frac{E^2}{C} = \frac{\alpha a(b-a)x}{(1+bx)^2}. \quad (3.4.14e)$$

We believe that this is a new solution to the Einstein-Maxwell system. In general the models in (3.4.14) cannot be expressed in terms of elementary functions as the series in (3.4.11) and (3.4.12) do not terminate. Consequently the solution will be given in terms of special functions. Terminating series are possible for particular values of a and b as we show in the next section.

3.5 Elementary functions

It is possible to generate exact solutions in terms of elementary functions from the series in (3.4.13). This is possible for specific values of the parameters a, b and α so that the series (3.4.11) and (3.4.12) terminate. Consequently two categories of solutions are obtainable in terms of elementary functions by placing restrictions on the quantity $\frac{b}{a}-1-\alpha$. We provide the details of the process in the Appendix B; here we present a summary of the results. We can express the first category of solution, in terms of the variable x, as

$$
\begin{aligned}
y_1(x) = {} & A_1 \frac{1}{(1+ax)^n} \times \\
& \sum_{i=0}^{n} (-1)^{i-1} \left(\frac{b}{b-a}\right)^i \frac{(2i-1)}{(2i)!(2n-2i+1)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i \\
& + A_2 \frac{1}{(1+ax)^n} \left[(1+ax) - \frac{(b-a)}{b}\right]^{3/2} \times \\
& \sum_{i=0}^{n-1} \left(\frac{b}{a-b}\right)^i \frac{(i+1)}{(2i+3)!(2n-2i-2)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i,
\end{aligned}
\tag{3.5.1}
$$

where $\frac{b}{a} - 1 - \alpha = 4n^2$ relates the constants a, b, α and n. The second category of solution, in terms of the variable x, is given by

$$y_2(x) = A_1 \frac{1}{(1+ax)^{n-1/2}} \times$$

$$\sum_{i=0}^{n} (-1)^{i-1} \left(\frac{b}{b-a} \right) \frac{(2i-1)}{(2i)!(2n-2i)!} \left[(1+ax) - \frac{(b-a)}{b} \right]^i$$

$$+A_2 \frac{1}{(1+ax)^{n-1/2}} \left[(1+ax) - \frac{(b-a)}{b} \right]^{3/2} \times$$

$$\sum_{i=0}^{n-2} \left(\frac{b}{a-b} \right)^i \frac{(i+1)}{(2i+3)!(2n-2i-3)!} \left[(1+ax) - \frac{(b-a)}{b} \right]^i, \qquad (3.5.2)$$

where $\frac{b}{a} - 1 - \alpha = 4n(n-1) + 1$ relates the constants a, b, α and n. Thus we have extracted two classes of solutions (3.5.1) and (3.5.2) to the Einstein-Maxwell system in terms of elementary functions from the infinite series solution (3.4.13). This class of solution can be expressed as combinations of polynomials and algebraic functions. The simple form of (3.5.1) and (3.5.2) helps in the study of the physical features of the model.

From our general classes of solutions (3.5.1) and (3.5.2), it is possible to generate particular solutions found for charged stars previously . If we take $b = 1$ and $K = \frac{1-a}{a}$ then it is easy to verify that the equation (3.5.1) becomes

$$y_1(x) = D_1 \left[\frac{K}{K+1+x} \right]^n \sum_{i=0}^{n} (-1)^{i-1} \frac{(2i-1)}{(2i)!(2n-2i+1)!} \left[\frac{1+x}{K} \right]^i$$

$$+D_2 \left[\frac{K}{K+1+x)} \right]^n \left[\frac{1+x}{K} \right]^{3/2} \times$$

$$\sum_{i=0}^{n-1} (-1)^i \frac{(i+1)}{(2i+3)!(2n-2i-2)!} \left[\frac{1+x}{K} \right]^i, \qquad (3.5.3)$$

where $K-\alpha = 4n^2$, $D_1 = \frac{A_1}{(1-a)^n}$ and $D_2 = \frac{A_2}{(1-a)^{n-3/2}}$. Also, equation (3.5.2) becomes

$$y_2(x) = D_1 \left[\frac{K}{K+1+x}\right]^{n-1/2} \sum_{i=0}^{n} (-1)^{i-1} \frac{(2i-1)}{(2i)!(2n-2i)!} \left[\frac{1+x}{K}\right]^i$$

$$+D_2 \left[\frac{K}{K+1+x}\right]^{n-1/2} \left[\frac{1+x}{K}\right]^{3/2} \times$$

$$\sum_{i=0}^{n-2} (-1)^i \frac{(i+1)}{(2i+3)!(2n-2i-3)!} \left[\frac{1+x}{K}\right]^i, \tag{3.5.4}$$

where $K-\alpha = 4n(n-1)+1$, $D_1 = \frac{A_1}{(1-a)^{n-1/2}}$ and $D_2 = \frac{A_2}{(1-a)^{n-2}}$. Thus we have regained the Komathiraj and Maharaj (2007a) charged model; our solutions allow for a wider range of models for charged relativistic spheres. We illustrate this feature with an example involving a specific value for n. For example, suppose that $n = 1$ then $b = (5+\alpha)a$ and we get

$$y = \frac{a_1(7+\alpha+3(5+\alpha)ax) + a_2(1+(5+\alpha)ax)^{3/2}}{1+ax} \tag{3.5.5}$$

from (3.5.1) where a_1 and a_2 are new arbitrary constants. From (3.5.5) and (3.2.2a)-

(3.2.2d) the solution to the Einstein-Maxwell system becomes

$$e^{2\lambda} = \frac{1+(5+\alpha)ax}{(1+ax)^2}, \tag{3.5.6a}$$

$$e^{2\nu} = A^2\left[\frac{a_1(7+\alpha+3(5+\alpha)ax)+a_2(1+(5+\alpha)ax)^{\frac{3}{2}}}{1+ax}\right]^2, \tag{3.5.6b}$$

$$\begin{aligned}\frac{\rho}{C} &= \frac{a(3+\alpha-ax)}{1+(5+\alpha)ax}+\frac{2a(1+ax)\left[3+\alpha-(5+\alpha)ax\right]}{\left[1+(5+\alpha)ax\right]^2} \\ &\quad -\frac{\alpha a^2(4+\alpha)x}{2\left[1+(5+\alpha)ax\right]^2},\end{aligned} \tag{3.5.6c}$$

$$\begin{aligned}\frac{p}{C} &= \frac{2a(1+ax)}{\left[1+(5+\alpha)ax\right]}\times \\ &\quad \frac{\left[4a_1(4+\alpha)+a_2(1+(5+\alpha)ax)^{\frac{1}{2}}\left(13+3\alpha+(5+\alpha)ax\right)\right]}{\left[a_1\left(7+\alpha+3(5+\alpha)ax\right)+a_2\left(1+(5+\alpha)ax\right)^{\frac{3}{2}}\right]} \\ &\quad -\frac{a(3+\alpha-ax)}{1+(5+\alpha)ax}+\frac{\alpha a^2(4+\alpha)x}{2\left[1+(5+\alpha)ax\right]^2},\end{aligned} \tag{3.5.6d}$$

$$\frac{E^2}{C} = \frac{\alpha a^2(4+\alpha)x}{\left[1+(5+\alpha)ax\right]^2}. \tag{3.5.6e}$$

Note that the solution of the form (3.5.6) cannot be regained from Komathiraj and Maharaj (2007a) charged models except for the value of $a = \frac{1}{(5+\alpha)}$. This indicates that our model is a generalisation of the Komathiraj and Maharaj charged models with wider behaviour in the gravitational and electromagnetic fields.

3.6 Physical analysis

In this section, we consider briefly the physical features of the models generated in this chapter. For the pressure to vanish at the boundary $r = R$ in the solution (3.4.14) we require $p(R) = 0$ which gives the condition

$$4(1+aCR^2)^2\left[\frac{\dot{y}}{y}\right]_{r=R}+a(2+aCR^2)-b+\frac{\alpha a(b-a)CR^2}{2(1+bCR^2)} = 0, \tag{3.6.1}$$

where y is given by (3.4.11)-(3.4.13). This will constrain the values of a, b and α. The solution of the Einstein-Maxwell system for $r > R$ is given by the Reissner-Nordstrom metric as

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (3.6.2)$$

where m and q are the total mass and the charge of the star. To match the potentials in (3.4.14) to (3.6.2) generates the relationships between the constants A_1, A_2, a, b and R as follows

$$\left(1 - \frac{2m}{R} + \frac{q^2}{R^2}\right) = A^2[A_1 y_1(R) + A_2 y_2(R)]^2, \quad (3.6.3a)$$

$$\left(1 - \frac{2m}{R} + \frac{q^2}{R^2}\right)^{-1} = \frac{1 + bCR^2}{(1 + aCR^2)^2}. \quad (3.6.3b)$$

The matching conditions (3.6.1) and (3.6.3) place restrictions on the metric coefficients; however there are sufficient free parameters to satisfy the necessary conditions that arise for the model under study. Since these conditions are satisfied by the constants in the solution a relativistic star of radius R is realisable.

From (3.4.14a) and (3.4.14b) we easily observe that the gravitational potentials $e^{2\lambda}$ and $e^{2\nu}$ are continuous and well behaved for wide range of the parameters a and b. From (3.4.14c), the variable x can be expressed solely in terms of the energy density ρ as

$$x = \frac{1}{2b(3a^2C + b\rho)}\Big[b^2C - 2b\rho - 5a^2C - 2abC \pm \sqrt{(a-b)C} \times$$

$$\sqrt{[-27a^2bC + a^3C(25 + 6b\alpha) - b^2(bC + 8\rho) + ab(3bC + 8\rho + 2b\alpha\rho)]}\Big]$$

Hence, from (3.4.14d) the isotropic pressure p can be written as a function of energy density ρ only. Therefore the solutions generated in this chapter satisfy the barotropic equation of state $p = p(\rho)$. Many of the solutions found previously do not satisfy this desirable feature. We illustrate the graphical behaviour of matter variables in the stellar interior for the particular solution (3.5.6). We assume that $a_1 = -4.897, a_2 = C = 1$ and $a = \alpha = 1/4$ for simplicity, and we consider the interval $0 \le r \le 1$. To generate the plots for $\rho, p, E^2, dp/d\rho$ and p versus ρ, we utilised the software package Mathematica. The behaviour of the energy density is plotted in Figure 3.1. It is positive and monotonically decreasing towards the boundary of the stellar object. In Figure 3.2, we have plotted the behaviour of matter pressure p; this is regular, monotonically decreasing and becomes zero at the vacuum boundary of the stellar object. In Figure 3.3, we describe the behaviour of the electric field intensity. It is well behaved and a continuous function. In Figure 3.4, we have plotted the speed of sound $dp/d\rho$. We observe that $0 \leq dp/d\rho \leq 1$ throughout the interior of the stellar object. Therefore the speed of the sound is less than the speed of the light and causality is maintained. In Figure 3.5, we have plotted the pressure p verses the density ρ and we find that this approximates a linear function. This behaviour is to be expected as the gradients of p and ρ have similar profiles in the stellar interior. Thus we have demonstrated that the particular solution satisfies the requirements for a physically reasonable stellar interior in the context of general relativity.

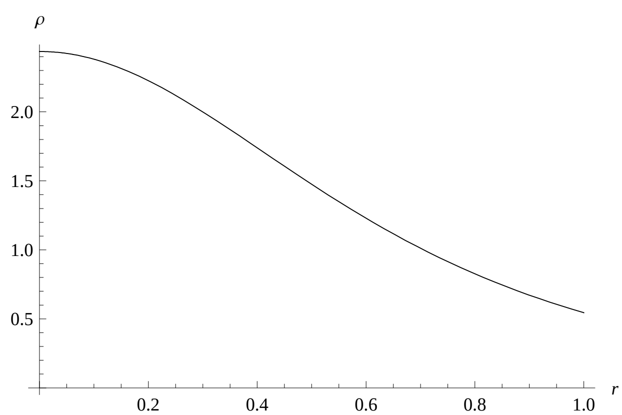

Figure 3.1: Energy density.

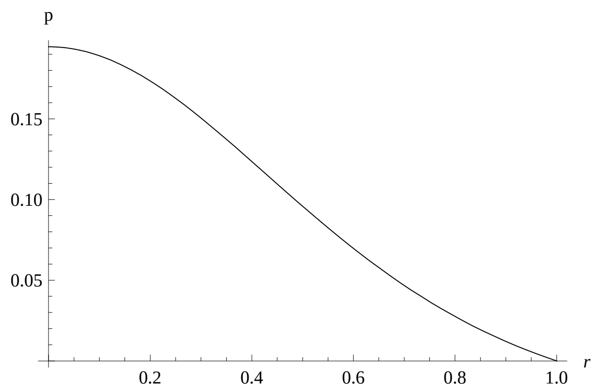

Figure 3.2: Matter pressure.

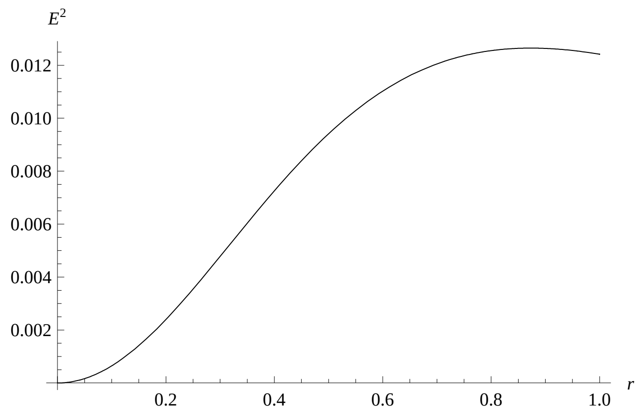

Figure 3.3: Electric field intensity.

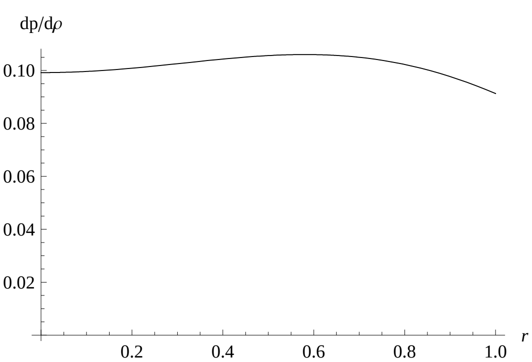

Figure 3.4: Speed of sound $dp/d\rho$.

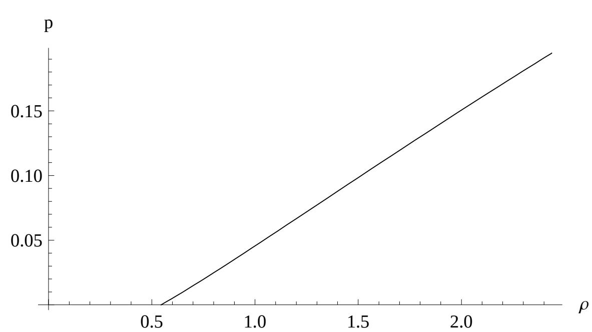

Figure 3.5: Pressure versus density.

Chapter 4

Charged anisotropic matter with linear equation of state

4.1 Introduction

Since the pioneering paper by Bowers and Liang (1974) there have been extensive investigations in the study of anisotropic relativistic matter distributions in general relativity to include the effects of spacetime curvature. The anisotropic interior spacetime matches to the Schwarzschild exterior model. The early work of Ruderman (1972) showed that nuclear matter may be anisotropic in density ranges of 10^{15} gcm^{-3} where nuclear interactions need to be treated relativistically. Note that conventional celestial bodies are not composed purely of perfect fluids so that radial pressures are different from tangential pressures. Anisotropy can be introduced by the existence of a solid stellar core or by the presence of a type 3A superfluid as indicated by Kippenhahn and Weigert (1990). Different kinds of phase transitions (Sokolov 1980) or pion condensation (Sawyer 1972) can generate anisotropy. Binney and Tremaine (1987) have considered anisotropies in spherical galaxies in the context of Newtonian gravitational theory. Herrera and Santos (1995) studied the effects of slow rotation in stars and Letelier (1980) analysed the mixture of two gases, such as ionized hydrogen and electrons, in a framework of a relativistic anisotropic fluid. Weber (1999) showed that strong magnetic fields serve as a vehicle for generating anisotropic pressures inside a compact sphere. Some recent anisotropic models for compact self-gravitating objects with strange matter include the results of Lobo (2006) and Sharma and Maharaj (2007) with a barotropic equation of state. Therefore the study of anisotropic fluid spheres in static spherically symmetric spacetimes is important in relativistic astrophysics.

In recent years there have been several investigations of the Einstein-Maxwell system of equations for static spherically symmetric gravitational fields usually with isotropic pressures to include the effects of the electromagnetic field. The interior spacetime must match at the boundary to the Reissner-Nordstrom exterior model. The models generated can be used to describe charged relativistic bodies in strong gravitational fields such as neutron stars. Many exact solutions have been given by Ivanov (2002) and Thirukkanesh and Maharaj (2006) which satisfy the conditions for a physically acceptable charged relativistic sphere. Charged spheroidal stars have been studied extensively by Komathiraj and Maharaj (2007b), Sharma *et al* (2001), Patel and Koppor (1987), Tikekar and Singh (1998) and Gupta and Kumar (2005). These charged spheroidal models contain uncharged neutron stars in the relevant limit and are consequently relevant in the description of dense astrophysical objects. We point out the particular detailed studies of Sharma *et al* (2006) in cold compact objects, Sharma and Mukherjee (2002) analysis of strange matter and binary pulsars, and Sharma and Mukherjee (2001) analysis of qurk-diquark mixtures in equilibrium in the presence of the electromagnetic field. Charged relativistic matter is also relevant in modeling core-envelope stellar system as shown in the treatments of Thomas *et al* (2005), Tikekar and Thomas (1998) and Paul and Tikekar (2005) in which the stellar core is an isotropic fluid surrounded by a layer of anisotropic fluid. Consequently the study of charged fluid spheres in static spherically symmetric spacetimes is of significance in relativistic astrophysics.

From the above motivation it is clear that both anisotropy and the electromagnetic field are important in astrophysical processes. However previous treatments have largely considered either anisotropy or electromagnetic field separately. The intention of this chapter is to provide a general framework that admits the possibility of tangential pressures with a nonvanishing electric field intensity. We believe that this approach will allow for a richer family of solutions to the Einstein-Maxwell field equations and possibly provide a deeper insight into the behaviour of the gravitational field. On physical grounds we impose a barotropic equation of state which is linear, that relates the radial pressure to the energy density and allows for the existence of strange matter. Our general model will contain strange matter solutions found previously. In this regard we mention the following recent works on strange stars. Mak and Harko (2004) and Komathiraj and Maharaj (2007c) found analytical models in the MIT bag model (Witten 1984) with a strange matter equation of state in the presence of an electromagnetic field. Sharma and Maharaj (2007) generated a class of exact solutions which can be applied to strange stars with quark matter for neutral anisotropic matter. Lobo (2006) found stable dark energy stars which generalise the gravastar model governed by a dark energy equation of state.

The objective of this treatment is to generate exact solutions to the Einstein-Maxwell system, with linear equation of state, that may be utilised to describe a charged anisotropic relativistic body. In §4.2, we express the Einstein-Maxwell system as a new system of differential equations using a coordinate transformation, and then rewrite the system in another form which is easier to analyse. Three classes of new exact solutions to the Einstein-Maxwell system are found in §4.3 in terms of simple elementary functions. We show that particular uncharged anisotropic strange stars found in the past are contained in our general family of solutions. In §4.4, we show that the solutions are physically admissible and plot the matter variables for particular parameter values. We generate values for the mass and central density in §4.5 for charged and uncharged matter. This analysis extends the treatment of Sharma and Maharaj (2007) to include charge, and confirms that the exact solutions found are physically reasonable. The work contained in this chapter has been published in Thirukkanesh and Maharaj (2008b).

4.2 The field equations

Our intention is to model the interior of a dense star. On physical grounds it is necessary for the gravitational field to be static and spherically symmetric. Consequently, we assume that the interior of a spherically symmetric star is described by the line element

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.2.1}$$

in Schwarzschild coordinates $(x^a) = (t, r, \theta, \phi)$. We take the energy momentum tensor for an anisotropic charged imperfect fluid sphere to be of the form

$$T_{ij} = \text{diag}\left(-\rho - \frac{1}{2}E^2, p_r - \frac{1}{2}E^2, p_t + \frac{1}{2}E^2, p_t + \frac{1}{2}E^2\right), \tag{4.2.2}$$

where ρ is the energy density, p_r is the radial pressure, p_t is the tangential pressure and E is the electric field intensity. These quantities are measured relative to the comoving fluid 4-velocity $u^i = e^{-\nu}\delta^i_0$. For the line element (4.2.1) and matter distribution (4.2.2) the Einstein-Maxwell field equations can be expressed as

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho + \frac{1}{2}E^2, \tag{4.2.3a}$$

$$-\frac{1}{r^2}\left(1-e^{-2\lambda}\right) + \frac{2\nu'}{r}e^{-2\lambda} = p_r - \frac{1}{2}E^2, \tag{4.2.3b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p_t + \frac{1}{2}E^2, \tag{4.2.3c}$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)', \tag{4.2.3d}$$

where primes denote differentiation with respect to r and σ is the proper charge density. In the field equations (4.2.3a)-(4.2.3d), we are using units where the coupling constant $\frac{8\pi G}{c^4} = 1$ and the speed of light $c = 1$. The system of equations (4.2.3a)-(4.2.3d) governs the behaviour of the gravitational field for an anisotropic charged imperfect fluid. Note that the system (4.2.3a)-(4.2.3d) becomes

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho, \tag{4.2.4a}$$

$$-\frac{1}{r^2}\left(1-e^{-2\lambda}\right) + \frac{2\nu'}{r}e^{-2\lambda} = p, \tag{4.2.4b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p, \tag{4.2.4c}$$

for matter distributions with isotropic pressures ($p_r = p_t$) in the absence of charge ($E = 0$).

The mass contained within a radius r of the sphere is defined as

$$m(r) = \frac{1}{2}\int_0^r \omega^2 \rho(\omega) d\omega. \tag{4.2.5}$$

A different, but equivalent, form of the field equations is obtained if we introduce a new independent variable x, and define functions y and Z, as follows

$$x = Cr^2, \quad Z(x) = e^{-2\lambda(r)}, \quad A^2y^2(x) = e^{2\nu(r)}, \tag{4.2.6}$$

which was first suggested by Durgapal and Bannerji (1983). In (4.2.6), the quantities A and C are arbitrary constants. Under the transformation (4.2.6), the system (4.2.3a)-(4.2.3d) becomes

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C}, \tag{4.2.7a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p_r}{C} - \frac{E^2}{2C}, \tag{4.2.7b}$$

$$4xZ\frac{\ddot{y}}{y} + (4Z + 2x\dot{Z})\frac{\dot{y}}{y} + \dot{Z} = \frac{p_t}{C} + \frac{E^2}{2C}, \tag{4.2.7c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2, \tag{4.2.7d}$$

where dots denote differentiation with respect to the variable x. The mass function (4.2.5) becomes

$$m(x) = \frac{1}{4C^{3/2}}\int_0^x \sqrt{w}\rho(w)dw \tag{4.2.8}$$

in terms of the new variables in (4.2.6).

For a physically realistic relativistic star we expect that the matter distribution should satisfy a barotropic equation of state $p_r = p_r(\rho)$. For our purposes we assume the linear equation of state

$$p_r = \alpha\rho - \beta, \tag{4.2.9}$$

where α and β are constants. Then it is possible to write the system (4.2.7a)-(4.2.7d) in the simpler form

$$\frac{\rho}{C} = \frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{2C}, \tag{4.2.10a}$$

$$p_r = \alpha\rho - \beta, \tag{4.2.10b}$$

$$p_t = p_r + \Delta, \tag{4.2.10c}$$

$$\begin{aligned} \Delta &= 4CxZ\frac{\ddot{y}}{y} + 2C\left[x\dot{Z} + \frac{4Z}{(1+\alpha)}\right]\frac{\dot{y}}{y} \\ &\quad + \frac{(1+5\alpha)}{(1+\alpha)}C\dot{Z} - \frac{C(1-Z)}{x} + \frac{2\beta}{(1+\alpha)}, \end{aligned} \tag{4.2.10d}$$

$$\frac{E^2}{2C} = \frac{1-Z}{x} - \frac{1}{(1+\alpha)}\left[2\alpha\dot{Z} + 4Z\frac{\dot{y}}{y} + \frac{\beta}{C}\right], \tag{4.2.10e}$$

$$\frac{\sigma^2}{C} = 4\frac{Z}{x}(x\dot{E} + E)^2, \tag{4.2.10f}$$

where the quantity $\Delta = p_t - p_r$ is the measure of anisotropy in this model. In the system (4.2.10a)-(4.2.10f), there are eight independent variables $(\rho, p_r, p_t, \Delta, E, \sigma, y, Z)$ and only six independent equations. This suggests that it is possible to specify two of the quantities involved in the integration process. The resultant system will remain highly nonlinear but it may be possible to generate exact solutions.

4.3 Generating exact models

We must make physically reasonable choices for any two of the independent variables and then solve the system (4.2.10a)-(4.2.10f) to generate exact models. In this chapter, we choose forms for the gravitational potential Z and electric field intensity E. We make the specific choices

$$Z = \frac{1+(a-b)x}{1+ax}, \tag{4.3.1}$$

$$\frac{E^2}{C} = \frac{k(3+ax)}{(1+ax)^2}, \tag{4.3.2}$$

where a, b and k are real constants. The gravitational potential Z is regular at the origin and well behaved in the stellar interior for a wide range of values for the parameters a and b. The electric field intensity is continuous, bounded and a decreasing function from the origin to the boundary of the sphere. Therefore the forms chosen in (4.3.1)-(4.3.2) are physically reasonable. On substituting (4.3.1) and (4.3.2) in (4.2.10e) we obtain

$$\begin{aligned}\frac{\dot{y}}{y} = {} & \frac{(1+\alpha)b}{4\left[1+(a-b)x\right]} + \frac{\alpha b}{2(1+ax)\left[1+(a-b)x\right]} \\ & - \frac{\beta(1+ax)}{4C\left[1+(a-b)x\right]} - \frac{(1+\alpha)k(3+ax)}{8(1+ax)\left[1+(a-b)x\right]},\end{aligned} \tag{4.3.3}$$

which is a linear equation in the gravitational potential y. For the integration of equation (4.3.3) it is convenient to consider three cases: $b = 0, a = b$ and $a \neq b$.

4.3.1 The case $b = 0$

When $b = 0$, (4.3.3) becomes

$$\frac{\dot{y}}{y} = -\frac{\beta}{4C} - \frac{(1+\alpha)k(3+ax)}{8(1+ax)^2} \tag{4.3.4}$$

with solution

$$y = D(1+ax)^{\frac{-k(1+\alpha)}{a}} \exp\left[\frac{2k(1+\alpha)}{a(1+ax)} - \frac{\beta x}{4C}\right], \tag{4.3.5}$$

where D is the constant of integration. We observe that $\rho = -\frac{E^2}{2}$ for this case which we do not consider further to avoid negative energy densities.

4.3.2 The case $a = b$

When $a = b$, (4.3.3) becomes

$$\frac{\dot{y}}{y} = \frac{(1+\alpha)a}{4} + \frac{\alpha a}{2(1+ax)} - \frac{\beta(1+ax)}{4C} - \frac{(1+\alpha)k(3+ax)}{8(1+ax)}. \tag{4.3.6}$$

On integrating (4.3.6) we get

$$y = D(1+ax)^{\frac{2a\alpha-(1+\alpha)k}{4a}} \exp[F(x)], \tag{4.3.7}$$

where $F(x) = \frac{x}{8C}\left[-kC(1+\alpha) - 2\beta + a(2C(1+\alpha) - \beta x)\right]$ and D is the constant of integration. Then we can generate an exact model for the system (4.2.10a)-(4.2.10f) as follows

$$e^{2\lambda} = 1+ax, \tag{4.3.8a}$$

$$e^{2\nu} = A^2D^2(1+ax)^{\frac{2a\alpha-k(1+\alpha)}{2a}}\exp[2F(x)], \tag{4.3.8b}$$

$$\frac{\rho}{C} = \frac{(2a-k)}{2}\frac{(3+ax)}{(1+ax)^2}, \tag{4.3.8c}$$

$$p_r = \alpha\rho-\beta, \tag{4.3.8d}$$

$$p_t = p_r+\Delta, \tag{4.3.8e}$$

$$\begin{aligned}\Delta = {} & \frac{1}{16C(1+ax)^3}\left\{C^2\left[k^2(1+\alpha)^2x(3+ax)^2\right.\right.\\ & +4a^2x(3-8\alpha+9\alpha^2+a^2(1+\alpha)^2x^2+2ax(2+3\alpha+3\alpha^2))\\ & \left.-4k(12+a^3(1+\alpha)^2x^3+a^2x^2(7+9\alpha+6\alpha^2)+ax(12+5\alpha+9\alpha^2))\right]\\ & -4Cx(1+ax)^2[(1+\alpha)(2a^2x-3k)-a\beta(k(1+\alpha)-6\alpha-4)]\\ & \left.+4\beta^2x(1+ax)^4\right\},\end{aligned} \tag{4.3.8f}$$

$$\frac{E^2}{C} = \frac{k(3+ax)}{(1+ax)^2}, \tag{4.3.8g}$$

in terms of elementary functions.

The solution (4.3.8a)-(4.3.8g) may be used to model a charged anisotropic star with a linear equation of state. In this case the mass function is

$$m(x) = \frac{(2a-k)x^{3/2}}{4C^{3/2}(1+ax)}, \tag{4.3.9}$$

which is similar to forms used in other investigations. The gravitational potentials and matter variables are continuous and well behaved in the stellar interior. Note that when $k = 0$ the model (4.3.8a)-(4.3.8g) reduces to a solution for uncharged anisotropic stars. Equation (4.3.8f) yields

$$\begin{aligned}\Delta &= \frac{1}{4C(1+ax)^3}\left\{C^2a^2x\left[3-8\alpha+9\alpha^2+a^2(1+\alpha)^2x^2+2ax(2+3\alpha+3\alpha^2)\right]\right.\\ &\quad \left.-2Cx(1+ax)^2[(1+\alpha)a^2x+a\beta(3\alpha+2)]+\beta^2x(1+ax)^4\right\} \qquad (4.3.10)\end{aligned}$$

when $k = 0$ so that the model is necessarily anisotropic with $\Delta \neq 0$ in general even in the simpler case of uncharged matter. Some treatments of the physical properties of anisotropic spheres in general relativity include the investigations of Dev and Gleiser (2002, 2003), Mak and Harko (2002, 2003), Chaisi and Maharaj (2005, 2006a) and Maharaj and Chaisi (2006a) with $\Delta \neq 0$.

4.3.3 The case $a \neq b$

On integrating (4.3.3) we get

$$y = D(1+ax)^l[1+(a-b)x]^n \exp\left[\frac{-a\beta x}{4C(a-b)}\right] \qquad (4.3.11)$$

where D is the constant of integration, and l and n are given by

$$\begin{aligned}l &= \frac{2\alpha b-(1+\alpha)k}{4b},\\ n &= \frac{1}{8bC(a-b)^2}\left[2a^2C(k(1+\alpha)-2\alpha b)-abC(5k(1+\alpha)-2b(1+5\alpha))\right.\\ &\quad \left.+b^2(3kC(1+\alpha)-2bC(1+3\alpha)+2\beta)\right].\end{aligned}$$

Then we can generate an exact model for the system (4.2.10a)-(4.2.10f) in the form

$$e^{2\lambda} = \frac{1+ax}{1+(a-b)x}, \tag{4.3.12a}$$

$$e^{2\nu} = A^2D^2(1+ax)^{2l}[1+(a-b)x]^{2n}\exp\left[\frac{-a\beta x}{2C(a-b)}\right], \tag{4.3.12b}$$

$$\frac{\rho}{C} = \frac{(2b-k)}{2}\frac{(3+ax)}{(1+ax)^2}, \tag{4.3.12c}$$

$$p_r = \alpha\rho - \beta, \tag{4.3.12d}$$

$$p_t = p_r + \Delta, \tag{4.3.12e}$$

$$\begin{aligned}
\Delta &= \frac{-bC}{(1+ax)} - \frac{bC(1+5\alpha)}{(1+\alpha)(1+ax)^2} + \frac{2\beta}{(1+\alpha)} \\
&\quad + \frac{Cx[1+(a-b)x]}{(1+ax)}\left[4\left(\frac{a^2l(l-1)}{(1+ax)^2} + \frac{2a(a-b)ln}{(1+ax)[1+(a-b)x]}\right.\right. \\
&\quad \left.\left. + \frac{(a-b)^2n(n-1)}{[1+(a-b)x]^2}\right) - \frac{2a\beta(a(l+n)[1+(a-b)x]-bn)}{(a-b)C(1+ax)[1+(a-b)x]} + \frac{a^2\beta^2}{4C^2(a-b)^2}\right] \\
&\quad - \frac{4[1+ax(2+(a-b)x)] - b(5+\alpha)x}{2(a-b)(1+\alpha)(1+ax)^3[1+(a-b)x]} \times \\
&\quad \left[-4b^2Cn + a^3x(-4C(l+n)+\beta x) + a^2(4C(l+n)(2bx-1) + \beta(2-bx)x)\right. \\
&\quad \left. + a(-4b^2C(l+n)x + \beta + b(4Cl + 8Cn - \beta x))\right],
\end{aligned} \tag{4.3.12f}$$

$$\frac{E^2}{C} = \frac{k(3+ax)}{(1+ax)^2}, \tag{4.3.12g}$$

in terms of elementary functions.

Therefore we have generated a second class of solutions (4.3.12a)-(4.3.12g) that models a charged anisotropic star with a linear equation of state. The mass function has the form

$$m(x) = \frac{(2b-k)x^{3/2}}{4C^{3/2}(1+ax)}. \tag{4.3.13}$$

The form of the mass function (4.3.13) represents an energy density which is monotonically decreasing in the stellar interior and remains finite at the centre $x = 0$. It is physically reasonable and has been used in the past to study the properties of isotropic fluid spheres: Matese and Whitman (1980) generated equilibrium configurations in general relativity, Finch and Skea (1989) studied neutron star models and Mak and Harko (2003) analysed anisotropic relativistic stars with this form of mass function. Lobo (2006) demonstrated that (4.3.13) is consistent with stable dark energy stars which generalises the gravastar model of Mazur and Mottola (2004). It was then shown that large stability regions exist close to the event horizon thereby making it difficult to distinguish dark energy stars from black holes. Sharma and Maharaj (2007) found a new class of exact solutions to Einstein equations that can be applied to strange stars with quark matter with this mass distribution. Consequently the mass function (4.3.13) is of astrophysical importance in the description of compact objects.

It is interesting to observe that for particular parameter values we can regain uncharged anisotropic and isotropic models ($k = 0$) from our general solution (4.3.12a)-(4.3.12g). We regain the following particular cases of physical interest:

(i) Sharma and Maharaj model

If we set $\beta = \alpha\rho_s$ then

$$p_r = \alpha(\rho - \rho_s),$$

where ρ_s is the density at the surface $r = s$. Thus we regain the equation of state of Sharma and Maharaj (2007). Then by setting $C = 1$ and $A^2D^2 = B$ we find that the line element is of the form

$$\begin{aligned} ds^2 &= -B(1+ar^2)^{\alpha}[1+(a-b)r^2]^{\gamma} \exp\left(\frac{-a\beta r^2}{2(a-b)}\right) dt^2 \\ &\quad + \frac{1+ax}{1+(a-b)x} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{4.3.14}$$

where

$$\gamma = \frac{5ab\alpha - 2a^2\alpha - 3b^2\alpha + ab - b^2 + b\beta}{2(a-b)^2}.$$

The line element (4.3.14) corresponds to the uncharged anisotropic model of Sharma and Maharaj (2007). They showed that this solution may be used to describe compact objects such as strange stars with a linear equation of state with quark matter.

(ii) Lobo model

If we set $\beta = 0$ then

$$p_r = \alpha\rho$$

and we regain the equation of state studied by Lobo (2006). Then on setting $a = 2b, C = 1$ and $A^2D^2 = 1$ we generate the metric

$$ds^2 = -(1+br^2)^{(1-\alpha)/2}(1+2br^2)^{\alpha}dt^2 + \left(\frac{1+2br^2}{1+br^2}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.3.15)$$

The line element (4.3.15) corresponds to the uncharged anisotropic model of Lobo (2006). We point out that the line element (4.3.15) serves as an interior solution with $\alpha < -\frac{1}{3}$ which may be matched to an exterior Schwarzschild solution in a model for dark energy stars. Lobo (2006) proved that stability regions exist for dark energy stars by selecting particular values of α in a graphical analysis.

(iii) Isotropic models

In general $\Delta \neq 0$ so that the model remains anisotropic. However for particular parameter values we can show that $\Delta = 0$ in the relevant limit in the general solution (4.3.12a)-(4.3.12g). If we set $a = 0$ and $b = 1$ then we obtain

$$\begin{aligned} l &= \frac{\alpha}{2} \\ n &= \frac{1}{4C}[\beta - (1+3\alpha)C] \\ \Delta &= \frac{x}{4C(1-x)}[\beta - 3(1+\alpha)C][\beta - (1+3\alpha)C]. \end{aligned} \quad (4.3.16)$$

Two different cases arise as a consequence of (4.3.16) if we set $\Delta = 0$.

In the first case, we observe that when $\beta = 0$ and $\alpha = -1$ then $\Delta = 0$. The equation of state becomes $p_r(= p_t) = -\rho$. For this case the line element becomes

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (4.3.17)$$

where we have set $A = D = 1$ and $C = \frac{1}{R^2}$. The metric (4.3.17) corresponds to the familiar isotropic uncharged de Sitter model.

In the second case, we see that when $\beta = 0$ and $\alpha = -\frac{1}{3}$ then $\Delta = 0$. The equation of state becomes $p_r(= p_t) = -\frac{1}{3}\rho$. For this case the line element becomes

$$ds^2 = -A^2 dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (4.3.18)$$

where we have set $D = 1$ and $C = \frac{1}{R^2}$. The metric (4.3.18) corresponds to the well known isotropic uncharged Einstein model.

4.4 Physical analysis

The solutions found in this chapter may be connected to the Einstein-Maxwell equations for the exterior of our source. We need to match the Reissner-Nordstrom exterior spacetime

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

to the interior spacetime (4.2.1) across the boundary $r = R$. This generates the conditions

$$\begin{aligned} 1 - \frac{2m}{R} + \frac{q^2}{R^2} &= A^2 y^2(CR^2) \\ \left(1 - \frac{2m}{R} + \frac{q^2}{R^2}\right)^{-1} &= \frac{1 + aCR^2}{1 + (a-b)CR^2} \end{aligned}$$

which relates the constants a, b, A, C, D, α and β. This demonstrates that the continuity of the metric coefficients across the boundary of the star $r = R$ is easily satisfied as there are sufficient number of free parameters. If there is a surface layer of charge then the pressure may be nonzero which would place restrictions on the function ν through the matching conditions at the boundary. However the number of free parameters available easily satisfies the necessary conditions that arise for a particular model under investigation.

We now briefly consider the physical behaviour of the models generated in §4.3.3 for the case $a \neq b$. From the explicit forms (4.3.12a) and (4.3.12b) we can easily see that the gravitational potentials $e^{2\nu}$ and $e^{2\lambda}$ are continuous, well behaved and nonsingular at the origin. The energy density ρ is continuous and monotonically decreasing from the centre to the boundary of the star, which is a necessary condition for a realistic model. The radial pressure p_r also has the same feature because ρ and p_r are linked by a linear equation of state. The tangential pressure p_t is also nonsingular at the origin and continuous for a wide range of the parameters a, b and k. To maintain the usual casuality condition we must place the restriction that $0 \leq \alpha \leq 1$ if we require $\frac{dp_r}{d\rho} \leq 1$. However note that our models do allow for $\alpha < 0$ in the case of anisotropic dark energy stars. The form chosen for electric field intensity E is physically reasonable and describes a decreasing function .

With the help of a particular example we can demonstrate the above features graphically. Figures 4.1-4.4 represent the energy density, the radial pressure, the tangential pressure and the electric field intensity, respectively. To plot the graphs we choose the parameters $a = 3, b = 2.15, \alpha = 0.33, \beta = \alpha\rho_s = 0.198, C = 1$ and $k = 0.2$, where ρ_s is the density at the boundary $r = s = 1.157$. Note that our choice of $\alpha = 0.33$ ensures that both the radial pressure and the tangential pressure for the neutral sphere vanish at the boundary. We observe from Figures 4.1-4.4 that the matter variables ρ, p_r, p_t and E have the appropriate features to describe a compact relativistic sphere. Solid lines represent uncharged matter and dashed lines include the effect of charge in Figures 4.1-4.3. We observe that the effect of E is to produce lower values for ρ, p_r and p_t when compared to the case of uncharged matter. In Figure 4.5 we have plotted the measure of anisotropy Δ for the same parameter values used above. Note that the effect of the electromagnetic field is to increase the magnitude of Δ which affects the behaviour of p_t.

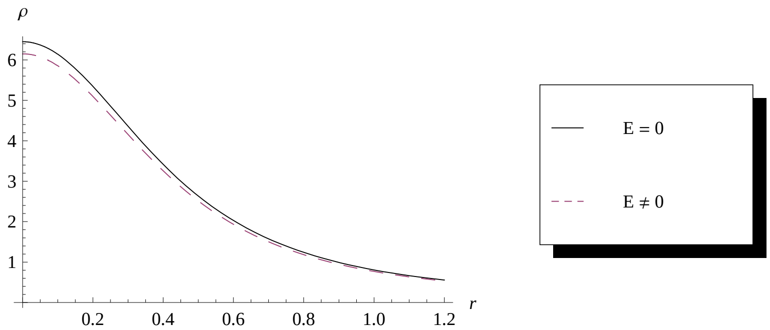

Figure 4.1: Energy density.

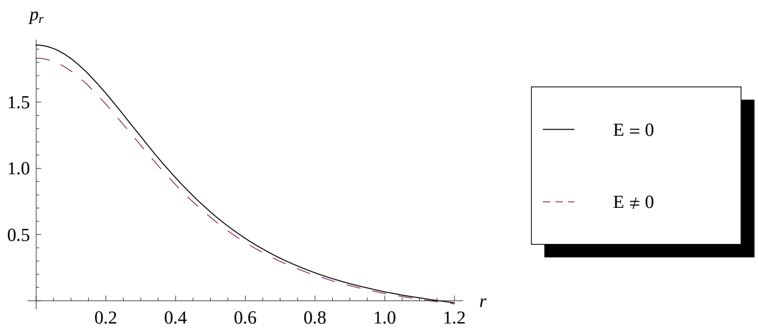

Figure 4.2: Radial pressure.

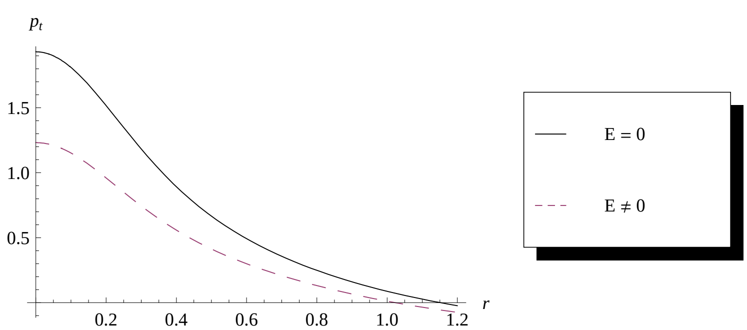

Figure 4.3: Tangential pressure.

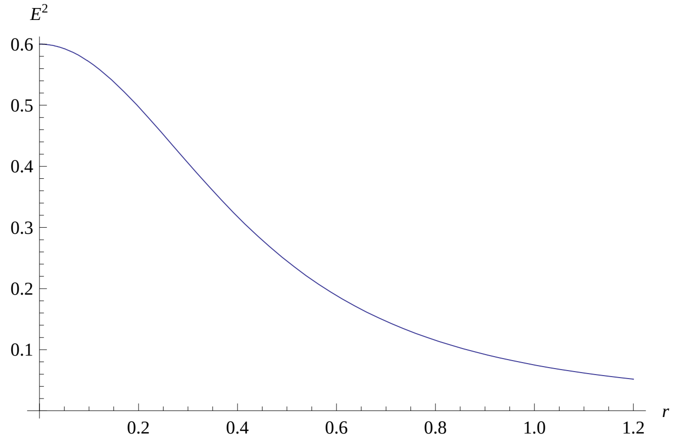

Figure 4.4: Electric field intensity.

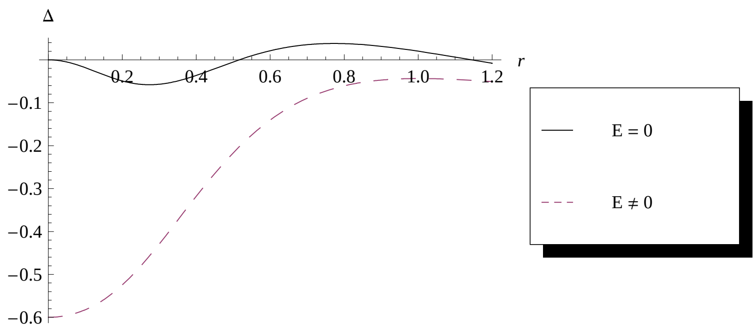

Figure 4.5: Measure of anisotropy.

4.5 Stellar structure

In this section we show that the solutions generated in this chapter can be used to describe realistic compact objects. In particular we seek to compare our results with those of Sharma and Maharaj (2007) since they regain values for the stellar mass agreeable with observations. To achieve consistency with Sharma and Maharaj (2007) we introduce the transformations

$$\tilde{a} = aR^2,\ \tilde{b} = bR^2,\ \tilde{\beta} = \beta R^2,\ \tilde{k} = kR^2.$$

Under these transformations the energy density becomes

$$\rho = \frac{(2\tilde{b} - \tilde{k})(3 + \tilde{a}y)}{2R^2(1 + \tilde{a}y)^2}, \tag{4.5.1}$$

and the mass contained within a radius s has the form

$$M = \frac{(2\tilde{b} - \tilde{k})s^3/R^2}{4(1 + \tilde{a}s^2/R^2)}, \tag{4.5.2}$$

where we have set $C = 1$ and $y = \frac{r^2}{R^2}$. When $\tilde{k} = 0$ (or $E = 0$), (4.5.1) and (4.5.2) reduce to the expressions of Sharma and Maharaj (2007):

$$\rho = \frac{\tilde{b}(3 + \tilde{a}y)}{R^2(1 + \tilde{a}y)^2}, \quad M = \frac{\tilde{b}s^3/R^2}{2(1 + \tilde{a}s^2/R^2)} \tag{4.5.3}$$

which gives the density ρ and mass M of an uncharged star of radius s.

If we choose $\tilde{a} = 53.34, \tilde{b} = 54.34, R = 43.245$ km and $s = 7.07$ km then we can produce an uncharged model ($\tilde{k} = 0$) with mass $M = 1.433M_\odot$ and central density $\rho_c = 4.672 \times 10^{15}\text{gcm}^{-3}$. The corresponding value of $\alpha = 0.437$ is obtained by requiring that the anisotropy vanishes at the boundary. To simplify comparison with Sharma and Maharaj (2007) we have used the same values of $\tilde{a}, \tilde{b}, R$ and s; however our value for α is a correction. It should be noted that these results are consistent with the equation of state for strange matter formulated by Dey *et al* (1998). This has astrophysical significance as their model has been used to describe the X-ray binary pulsar SAX J1808.4-3658. When the charge is nonzero we set $\tilde{k} = 37.403$ and then we obtain the mass $M = 0.940M_\odot$ and central density $\rho_c = 3.064 \times 10^{15}\text{gcm}^{-3}$. The values for M and ρ_c generalise the figures of Sharma and Maharaj (2007) to include the effect of the electromagnetic field. Choosing different set of values for the parameters will produce different results as shown in Table 4.1. Note that the values presented in Table 4.1 correspond to a star of radius $s = 7.07$km. The value of $\tilde{k} = 37.403$ is selected, in generating Table 4.1, so that the density and mass of the Sharma and Maharaj (2007) analysis is regained for uncharged matter. Furthermore, the value of $\tilde{k} = 37.403$ with $E = 0$ generates a star of mass $1.433M_\odot$ which is the same as the strange star model of Dey *et al* (1998). With this value of $\tilde{k}$ we find that the star has mass $0.940M_\odot$ in the presence of charge so that the stellar core has a lower density which represents a weaker field. This is consistent as the effect of the electromagnetic field is repulsive.

We observe that the values for the mass in the presence of charge ($E \neq 0$) is always less than the uncharged case. The central density of the charged sphere is also less than the uncharged case. Sharma and Maharaj (2007) showed that anisotropy affects the mass and central densities of massive objects. We have shown that the inclusion of the electromagnetic field also affects M and ρ_c. Both anisotropy and

$\tilde{b}$	$\tilde{a}$	α	ρ_c ($\times 10^{15}$ gcm^{-3}) $E = 0$	M ($M_\odot$) $E = 0$	ρ_c ($\times 10^{15}$ gcm^{-3}) $E \neq 0$	M ($M_\odot$) $E \neq 0$
30	23.681	0.401	2.579	1.175	0.971	0.443
40	36.346	0.400	3.439	1.298	1.831	0.691
50	48.307	0.424	4.298	1.396	2.691	0.874
54.34	53.340	0.437	4.671	1.433	3.064	0.940
60	59.788	0.457	5.158	1.477	3.550	1.017
70	70.920	0.495	6.017	1.546	4.410	1.133
80	81.786	0.537	6.877	1.606	5.269	1.231
90	92.442	0.581	7.737	1.659	6.129	1.314
100	102.929	0.627	8.596	1.705	6.989	1.386
183	186.163	1.083	15.730	1.959	14.124	1.759

Table 4.1: Central density and mass for different anisotropic stellar models for neutral and charged bodies

charge are physical quantities that affect the range of degenerate states in our model. For the calculation of mass and central density we have set $s = 7.07$ km, $R = 43.245$ km, $\tilde{k} = 37.403$ and $\rho_s = 1.17119 \times 10^{15}$gcm^{-3} for the uncharged case.

Chapter 5

Generalised isothermal models with strange equation of state

5.1 Introduction

In an early and seminal treatment the existence of quark matter in a stellar configuration in hydrostatic equilibrium was suggested by Itoh (1970). Subsequently the analysis of strange stars consisting of quark matter has been considered in a number of investigations. Strange stars are likely to form in the period of collapse of the core regions of a massive star after a supernova explosion which was pointed out by Cheng *et al* (1998). The core of a neutron star or proto-neutron star is a suitable environment for conventional barotropic matter to convert into strange quark matter. Regions of low temperatures and sufficiently high temperatures are required for a first or second order phase transition which results in deconfined quark matter. Another possibility suggested by Cheng and Dai (1996) to explain the formation of a strange star is the accretion of sufficient mass in a rapidly spinning dense star in X-ray binaries which undergoes a phase transition. The behaviour of matter at ultrahigh densities for quark matter is not well understood: in an attempt to study the physics researchers normally restrict their attention to the MIT bag model (see the treatments of Chodos *et al* (1974), Farhi and Jaffe (1984) and Witten (1984)).

The strange matter equation of state is taken to be

$$p = \frac{1}{3}(\rho - 4B) \tag{5.1.1}$$

where ρ is the energy density, p is the pressure and B is the bag constant. The vacuum pressure B is the bag model equilibrates the pressure and stabilises the system; the constant B determines the quark confinement. The studies of Bombaci (1997), Dey *et al* (1998), Li *et al* (1995, 1999a, 1999b), Pons *et al* (2002), Usov (2004) and Xu *et al* (1999, 2001) directed at particular compact astronomical objects suggest that these could be strange stars composed of quark matter with equation of state (5.1.1).

Mak and Harko (2004) found an exact general relativistic model of a quark star that admits a conformal Killing vector. This was shown by Komathiraj and Maharaj (2007c) to be a part of a more general class of exact analytical models in the presence of the electromagnetic field with isotropic pressures. The role of anisotropy was investigated by Lobo (2006), Mak and Harko (2002) and Sharma and Maharaj (2007) for strange stars with quark matter with neutral anisotropic distributions. It is our intention to study the Einstein-Maxwell system with a linear equation of state with anisotropic pressures; this treatment would be applicable to a strange stars which are charged and anisotropic which is the most general case. The field equations are given in §5.2. A new exact solution, in terms of simple elementary functions, is given in §5.3. In §5.4, we demonstrate that it is possible to find a particular model which is nonsingular at the stellar origin. The limit of vanishing anisotropy is studied in §5.5, and we regain the isothermal universes studied previously. In §5.6, we consider the physical features of the new solutions, plot the matter variables for particular parameter values and show that the quark star mass is consistent with earlier treatments. The results of this chapter have been accepted for publication in Maharaj and Thirukkanesh (2008b).

5.2 Basic equations

It is our intention to model the interior of a dense realistic star with a general matter distribution. On physical grounds we can take the gravitational field to be static and spherically symmetric. Consequently, we assume that the gravitational field of the stellar interior is represented by the line element

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.2.1}$$

in Schwarzschild coordinates $(x^a) = (t, r, \theta, \phi)$. We consider the general case of a matter distribution with both anisotropy and charge. Therefore we take the energy momentum tensor for the interior to be an anisotropic charged imperfect fluid; this is represented by the form

$$T_{ij} = \text{diag}\left(-\rho - \frac{1}{2}E^2, p_r - \frac{1}{2}E^2, p_t + \frac{1}{2}E^2, p_t + \frac{1}{2}E^2\right). \tag{5.2.2}$$

With the help of the transformations

$$x = Cr^2, \quad Z(x) = e^{-2\lambda(r)}, \quad A^2y^2(x) = e^{2\nu(r)},$$

the Einstein-Maxwell field equations can be written in the form

$$\frac{\rho}{C} = \frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{2C}, \tag{5.2.3a}$$

$$p_r = \alpha\rho - \beta, \tag{5.2.3b}$$

$$p_t = p_r + \Delta, \tag{5.2.3c}$$

$$\begin{aligned} \Delta = {} & 4CxZ\frac{\ddot{y}}{y} + 2C\left[x\dot{Z} + \frac{4Z}{(1+\alpha)}\right]\frac{\dot{y}}{y} + \frac{(1+5\alpha)}{(1+\alpha)}C\dot{Z} \\ & -\frac{C(1-Z)}{x} + \frac{2\beta}{(1+\alpha)}, \end{aligned} \tag{5.2.3d}$$

$$\frac{E^2}{2C} = \frac{1-Z}{x} - \frac{1}{(1+\alpha)}\left[2\alpha\dot{Z} + 4Z\frac{\dot{y}}{y} + \frac{\beta}{C}\right], \tag{5.2.3e}$$

$$\frac{\sigma^2}{C} = 4\frac{Z}{x}(x\dot{E} + E)^2, \tag{5.2.3f}$$

where A, C, α and β are constant. For more information on the Einstein-Maxwell system with anisotropic pressures see §4.2.

The definition of mass function takes the form

$$m(x) = \frac{1}{4C^{3/2}}\int_0^x \sqrt{w}\rho(w)dw. \tag{5.2.4}$$

5.3 New solutions

In this chapter, we choose different physically reasonable forms for the gravitational potential Z and electric field intensity E and then integrate the system (5.2.3a)-(5.2.3f) to generate exact models. We make the specific choices

$$Z = \frac{1}{a+bx^n}, \tag{5.3.1}$$

$$\frac{E^2}{C} = \frac{2k(d+2x)}{a+bx^n}, \tag{5.3.2}$$

where a, b, d, n and k are real constants. The potential Z is regular at the origin and continuous in the stellar interior for a wide range of values for the parameters a, b and n. The electric field intensity E is a bounded and decreasing function from the origin to the surface of the sphere. Therefore the forms chosen in (5.3.1)-(5.3.2) are physically acceptable. These specific choices for Z and E simplify the integration process. Equation (5.2.3e) can be written as

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{(a-1)(1+\alpha)}{4x} + \frac{\alpha}{2}\frac{bnx^{n-1}}{(a+bx^n)} + \frac{(1+\alpha)b}{4}x^{n-1} \\ &\quad -\frac{\beta}{4C}(a+bx^n) - \frac{(1+\alpha)k}{4}(d+2x) \end{aligned} \tag{5.3.3}$$

where we have used (5.3.1) and (5.3.2). This has the advantage of being a first order linear equation in the gravitational potential y.

Equation (5.3.3) can be integrated in closed form to give

$$y = Dx^{\frac{(a-1)(1+\alpha)}{4}}(a+bx^n)^{\frac{\alpha}{2}}\exp[F(x)], \tag{5.3.4}$$

where we have defined

$$F(x) = -\frac{\beta x}{4C}\left[a + \frac{bx^n}{n+1}\right] + \frac{(1+\alpha)}{4}\left[\frac{bx^n}{n} - k(dx+x^2)\right]$$

and D is a constant of integration. Now from (5.3.1), (5.3.2) and (5.3.4) we can generate an exact model for the system (5.2.3a)-(5.2.3f) as follows

$$e^{2\lambda} = a + bx^n, \tag{5.3.5a}$$

$$e^{2\nu} = A^2D^2x^{\frac{(a-1)(1+\alpha)}{2}}(a+bx^n)^\alpha \exp[2F(x)], \tag{5.3.5b}$$

$$\frac{\rho}{C} = \frac{(a-1)+bx^n}{x(a+bx^n)} + \frac{2bnx^{n-1}}{(a+bx^n)^2} - \frac{k(d+2x)}{(a+bx^n)}, \tag{5.3.5c}$$

$$p_r = \alpha\rho - \beta, \tag{5.3.5d}$$

$$p_t = p_r + \Delta, \tag{5.3.5e}$$

$$\begin{aligned}\Delta = \frac{1}{4}\Bigg\{ & \frac{4C(1-a-bx^n)}{x(a+bx^n)} - \frac{4bCn(1+5\alpha)x^{n-1}}{(1+\alpha)(a+bx^n)^2} + \frac{8\beta}{(1+\alpha)} \\ & -\frac{2[4a-bx^n(n(1+\alpha)-4)]^2}{(1+\alpha)x(a+bx^n)^3}\Big(a^2(\beta x - C(1+\alpha)) + bx^n\big(C\big[(1+\alpha)\times \\ & (1-bx^n+dkx+2kx^2) - 2n\alpha\big] + b\beta x^{n+1}\big) + a\big[2b\beta x^{n+1} \\ & +C(1+\alpha)(1+dkx+2kx^2-2bx^n)\big]\Big) + \frac{Cx}{(a+bx^n)}\Bigg[\frac{1}{x^2}\Big((a-1)(1+\alpha) \\ & \left(\frac{4bn\alpha x^n}{(a+bx^n)} + (a(1+\alpha)-5-\alpha)\right) + \frac{4bn\alpha x^n(2a(n-1)+(n\alpha-2)bx^n)}{(a+bx^n)^2}\Big) \\ & -\frac{2[a(1+\alpha)(a-1+bx^n)+bx^n(2n\alpha-(1+\alpha))]}{Cx^2(a+bx^n)}\big(\beta x(a+bx^n) \\ & +C(1+\alpha)(k(dx+2x^2)-bx^n)\big) + 4\big((1+\alpha)(b(n-1)x^{n-2}-2k) \\ & -\frac{bn\beta x^{n-1}}{C} + \frac{[C(1+\alpha)(k(d+2x)-bx^{n-1})+\beta(a+bx^n)]^2}{4C^2}\big)\Bigg]\Bigg\}, \end{aligned} \tag{5.3.5f}$$

$$\frac{E^2}{C} = \frac{2k(d+2x)}{a+bx^n}. \tag{5.3.5g}$$

The equations (5.3.5a)-(5.3.5g) represent an exact solution to the Einstein-Maxwell system (5.2.3a)-(5.2.3f) for a charged imperfect fluid with the linear equation of state $p_r = \alpha\rho - \beta$. Exact solutions with the equation of state $p_r = \alpha\rho - \beta$ have been used to model compact objects such as strange stars, as shown by Sharma and Maharaj (2007), and dark energy stars which are stable, as demonstrated by Lobo (2006). The exact solution (5.3.5a)-(5.3.5g) may be regarded as a generalisation of an isothermal universe model as we indicate in §5.5.

5.4 A nonsingular model

The solution (5.3.5a)-(5.3.5g) admits singularities at the stellar centre in general The singularity may be avoided for particular parameter values. If we set $a = 1$ and $n = 1$ then we generate the line element

$$ds^2 = -A^2D^2(1+bx)^\alpha \exp[2F(x)]dt^2 + (1+bx)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (5.4.1)$$

where $F(x) = -\frac{\beta x}{8C}(2+bx) + \frac{(1+\alpha)}{4}[bx - k(dx + x^2)]$, with energy density

$$\frac{\rho}{C} = \frac{b(3+bx)}{(1+bx)^2} - \frac{k(d+2x)}{(1+bx)}. \quad (5.4.2)$$

When $k = 0$ then the mass function (5.2.4) has the form

$$m(x) = \frac{bx^{3/2}}{2C^{3/2}(1+bx)}. \quad (5.4.3)$$

The expression of the mass function given in (5.4.3) represents an energy density which is monotonically decreasing in the interior of the uncharged sphere and has a finite value at the centre $x = 0$. This is physically reasonable and similar mass profiles appear in the treatments of general relativistic equilibrium configurations of Matese and Whitman (1980), neutron star models of Finch and Skea (1989), anisotropic stellar solutions of Mak and Harko (2003), and the dark energy stars of Lobo (2006). The gravitational potentials for charged imperfect fluid solution corresponding to the line element (5.4.1) are nonsingular at the origin. The energy density ρ, the measure of anisotropy Δ, and associated quantities p_r and p_t are also regular at the centre for our choice of E. Note that the quantity Δ vanishes at the centre and is a continuous function in the stellar interior. Hence the model (5.3.5a)-(5.3.5g) admits a particular case corresponding to the parameter values $a = 1$ and $n = 1$ which is regular and well behaved at the origin.

5.5 Isotropic models

It is possible to consider the special case of isotropic pressures with $p_r = p_t$ in the uncharged limit for neutral matter. When $k = 0$ ($E = 0$) equation (5.3.5f) becomes

$$\begin{aligned}
\Delta \;=\; & \frac{1}{4}\left\{\frac{4C(1-a-bx^n)}{x(a+bx^n)} - \frac{4bCn(1+5\alpha)x^{n-1}}{(1+\alpha)(a+bx^n)} + \frac{8\beta}{(1+\alpha)}\right. \\
& -\frac{2[4a-(n(1+\alpha)-4)bx^n]}{(1+\alpha)x(a+bx^n)^3}\Big[a^2(\beta x - C(1+\alpha)) - bx^n\left(C\left((1+\alpha)(bx^n-1)\right.\right. \\
& \left.\left.+2n\alpha\right) - b\beta x^{n+1}\right) + a\left(C(1+\alpha)(1-2bx^2) + 2b\beta x^{n+1}\right)\Big] \\
& +\frac{C}{x(a+bx^n)}\left[4b(n-1)(1+\alpha)x^n + (a-1)(1+\alpha)(a(1+\alpha)-5-\alpha)\right. \\
& +\frac{4(a-1)bn\alpha(1+\alpha)x^n}{(a+bx^n)} + \frac{4bn\alpha[2a(n-1)+b(n\alpha-2)x^n]x^n}{(a+bx^n)^2} - \frac{4bn\beta x^{n+1}}{C} \\
& +\frac{[a\beta x - b(C(1+\alpha)-\beta x)x^n]^2}{C^2} + \frac{2}{C(a+bx^n)}\times \\
& \left.\left.(a(1+\alpha)(a-1+bx^n) + (2n\alpha-(1+\alpha))bx^n)\left((C(1+\alpha)-\beta x)bx^n - a\beta x\right)\right]\right\}.
\end{aligned} \tag{5.5.1}$$

Equation (5.5.1) shows that the model remains anisotropic even for the uncharged case in general. However for particular parameter values we can show that $\Delta = 0$ in the relevant limit in the general solution (5.3.5a)-(5.3.5g). If we set $b = 0$ and $\beta = 0$ then (5.5.1) becomes

$$\Delta = \frac{C(a-1)}{4ax}\left\{a(1+\alpha)^2 - \left[4\alpha + (1+\alpha)^2\right]\right\}.$$

From the above equation we easily observe that when $a = 1$ and $a = 1 + \frac{4\alpha}{(1+\alpha)^2}$ the measure of anisotropy Δ vanishes. When $a = 1$ we note from (5.3.5c) that $\rho = 0$ since $b = k = 0$. Consequently we cannot regain an isotropic model when $a = 1$; to avoid vanishing energy densities we must have $a \neq 1$ when $\Delta = 0$. When $a = 1 + \frac{4\alpha}{(1+\alpha)^2}$ we obtain the expressions

$$ds^2 = -Br^{\frac{4\alpha}{1+\alpha}}dt^2 + \left[1 + \frac{4\alpha}{(1+\alpha)^2}\right] dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.5.2}$$

$$\rho = \frac{4\alpha}{[4\alpha + (1+\alpha)^2]r^2}, \quad p_r(= p_t) = \alpha\rho, \tag{5.5.3}$$

where we have set $A^2 D^2 a^\alpha = B$ and $C = 1$.

The above solution was obtained by Saslaw *et al* (1996) in their investigation of general relativistic isothermal universes. Since $\rho \propto r^{-2}$ we may interpret (5.5.2) as a relativistic cosmological metric since there is an analogy that can be made with the well known Newtonian solution as pointed out by Chandrasekhar (1939). In the Newtonian case the density ρ is finite in the core regions and decreases according to r^{-2} in the rest of the model. The total mass and region of the isothermal model are infinite. Therefore the line element (5.5.2) may be interpreted as a relativistic inhomogeneous universe where the nonzero pressure balances gravity. Saslaw *et al* (1996) point out that (5.5.2) may be viewed as the asymptotic state of the Einstein-de Sitter cosmological model, in an expansion-free state as $t \to \infty$, where the hierarchial distribution of matter has clustered over large scales. As $t \to \infty$ the Einstein-de Sitter model tends to the line element (5.5.2) and the universe evolves into an isothermal static sphere given by the exact solution (5.5.2) with equation of state $p_r(= p_t) = \alpha\rho$. To move from the pressure-free Einstein-de Sitter model to the static isothermal metric requires a phase transition for condensation through clustering of galaxies. It is interesting to observe that Chaisi and Maharaj (2006b) and Maharaj and Chaisi (2006b) have obtained generalised anisotropic static isothermal spheres by utilising a known isotropic metric to produce a new anisotropic solution of the Einstein field equations. Govender and Govinder (2004) have found simple nonstatic generalisations of isothermal universes which describe an isothermal sphere of galaxies in quasi-hydrostatic equilibrium with heat dissipation driving the system to equilibrium. Hence the imperfect relativistic fluid, in the presence of electromagnetic field, with the linear equation of state $p_r = \alpha\rho - \beta$, is a generalisation of the conventional isothermal model with a clear physical basis.

5.6 Physical analysis

We observe that the exact solution (5.3.5a)-(5.3.5g) may be singular at the origin in general. The solution should be used to describe the gravitational field of the envelope in the outer regions of a quark star or a dark energy star with equation of state $p_r = \alpha\rho - \beta$. To avoid the singularity at the centre another solution is required to model the stellar core. Examples of core-envelope models in general relativity are provided by Thomas *et al* (2005), Tikekar and Thomas (1998) and Paul and Tikekar (2005). We observe that if $a = 1$ then the imperfect charged solution (5.3.5a)-(5.3.5g) admits the line element

$$ds^2 = -A^2D^2(1+bx^n)^\alpha \exp[2F(x)]dt^2 + (1+bx^n)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (5.6.1)$$

where $F(x) = -\frac{\beta x}{4C}\left[1 + \frac{bx^n}{n+1}\right] + \frac{(1+\alpha)}{4}\left[\frac{bx^n}{n} - k(dx + x^2)\right]$. It is clear from (5.6.1) that the gravitational potentials are nonsingular at the origin for all values of n. If $n = 1$ then there is no singularity in the energy density ρ and the model is nonsingular throughout the stellar interior. However if $n \neq 1$ then the gravitational potential potentials (5.6.1) may continue to be well behaved but singularities may appear in the matter variables at the origin $r = 0$.

By considering a particular example we can demonstrate graphically that the matter variables are well behaved outside the origin. Figures 6.1-6.5 represent the energy density, the radial pressure, the tangential pressure, the electric field intensity and the measure of anisotropy, respectively. Note that solid lines represent uncharged matter ($E = 0$) and dashed lines include the effect of charged matter $E \neq 0$. To plot the graphs we choose the parameters $n = 2, a = 1, b = 40, k = 2, d = 1, \alpha = \frac{1}{3}$ and $\beta = 0.3569$, and the stellar boundary is set at $r = 1$. From Figures 5.1 and 5.2 we see that both the energy density ρ and the radial pressure p_r are continuous throughout the interior, increasing from the centre to $r = 0.32$ and then decreasing. Note that the radial pressure is zero at the boundary $r = 1$ for the uncharged case $E = 0$. We observe from Figure 5.3 that the tangential press p_t is continuous and well behaved in the interior regions. From Figure 5.4 we observe that the electric field intensity E is decreasing smoothly throughout the stellar interior. We can observe from Figure 5.5 that the measure of anisotropy is continuous throughout the stellar interior. The behaviour of Δ outside the centre is likely to correspond to physically realistic matter in the presence of the electromagnetic field. Figure 5.5 has a profile similar to the anisotropic boson stars studied by Dev and Gleiser (2002) and the compact anisotropic relativistic spheres of Chaisi and Maharaj (2005). From the figures we can see that the effect of electric field intensity E is to produce lower values for ρ, p_r, p_t and Δ.

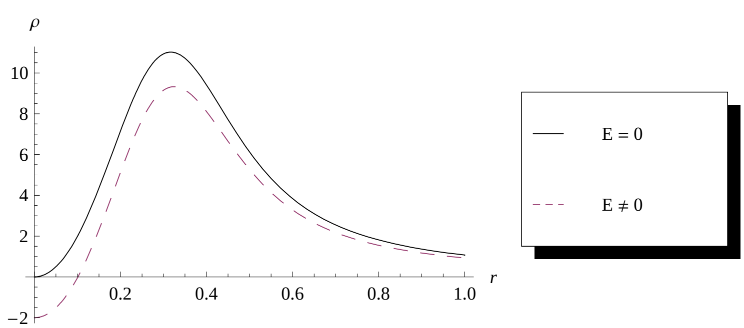

Figure 5.1: Energy density.

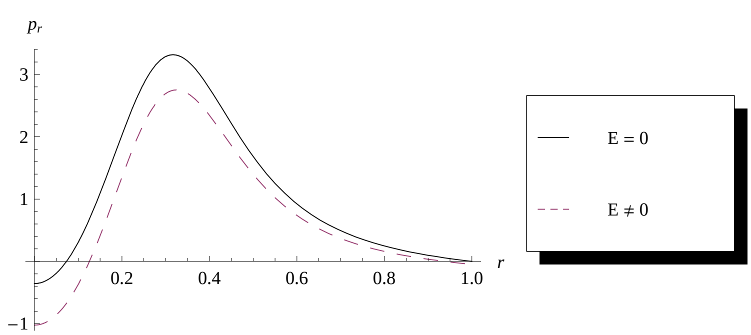

Figure 5.2: Radial pressure.

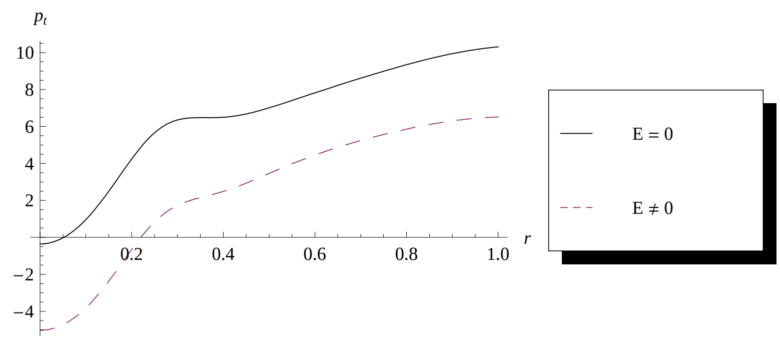

Figure 5.3: Tangential pressure.

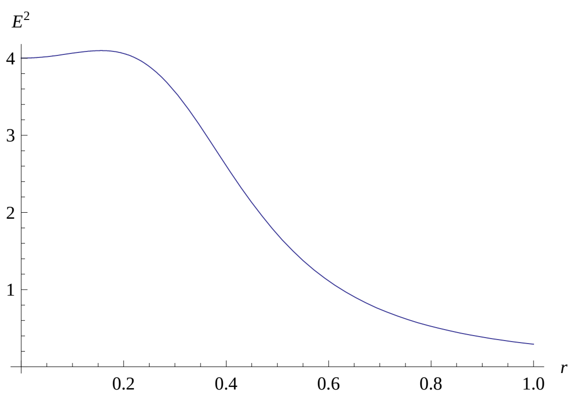

Figure 5.4: Electric field intensity.

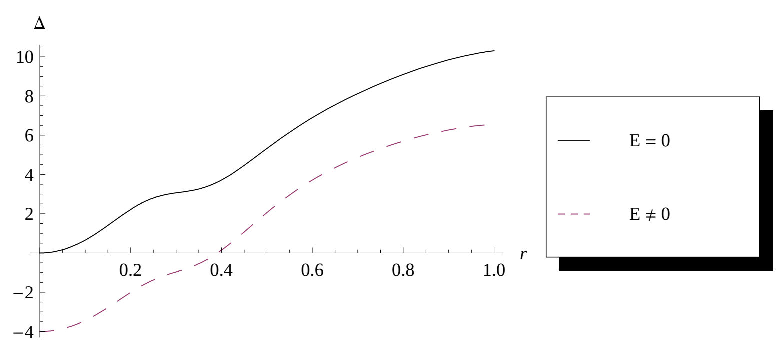

Figure 5.5: Measure of anisotropy.

We now show that the solutions generated in this chapter can be used to describe realistic compact objects for the case $n = 1$ as an example from §5.3. In our model, when $n = 1$, the parameters b has the dimension of length^{-2}, k has the dimension of length^{-4} and d has the dimension of length^2. For simplicity, we introduce the transformations

$$\tilde{b} = bR^2,\ \tilde{k} = kR^4,\ \tilde{d} = dR^{-2},$$

where R is a parameter which has the dimension of a length. Under these transformations the energy density becomes

$$\rho = \frac{1}{R^2}\left[\frac{\tilde{b}}{1+\tilde{b}y} + \frac{2\tilde{b}}{(1+\tilde{b}y)^2} - \frac{\tilde{k}(\tilde{d}+2y)}{1+\tilde{b}y}\right],$$

where we have set $C = 1$ and $y = \frac{r^2}{R^2}$. Then the mass contained within a radius s has the form

$$M = \frac{1}{2}\left\{\frac{\tilde{b}\frac{s^3}{R^2}}{1+\tilde{b}\frac{s^2}{R^2}} - \frac{2\tilde{k}}{3\tilde{b}}\frac{s^3}{R^2} + \tilde{k}(2-\tilde{b}\tilde{d})\left[\frac{s}{\tilde{b}^2} - \frac{R}{\tilde{b}^{5/2}}\arctan\left[\sqrt{\tilde{b}}\frac{s}{R}\right]\right]\right\}.$$

For simplicity we set $\tilde{b}\tilde{d} = 2$ so that these expressions reduce to

$$\rho = \frac{1}{R^2}\left[\frac{\tilde{b}}{1+\tilde{b}y} + \frac{2\tilde{b}}{(1+\tilde{b}y)^2} - \frac{2\tilde{k}}{\tilde{b}}\right], \tag{5.6.2}$$

$$M = \frac{1}{2}\left[\frac{\tilde{b}}{1+\tilde{b}\frac{s^2}{R^2}} - \frac{2\tilde{k}}{3\tilde{b}}\right]\frac{s^3}{R^2}, \tag{5.6.3}$$

which are simple forms. It is now easy to calculate the density and mass for particular parameter values from (5.6.2) and (5.6.3). For example, when $s = 7.07$km, $R = 1$km, $\tilde{b} = 0.03$ and $\tilde{k} = 0.00045$, we obtain the mass $M_{E=0} = 1.436M_{\odot}$ for uncharged matter $M_{E\neq 0} = 0.240M_{\odot}$ for charged matter. Note that the value of the mass for uncharged matter is consistent with the strange star models previously found by Sharma and Maharaj (2007) and Dey *et al* (1998). We have shown that the inclusion of the electromagnetic field affects the value for the mass M. It is also possible to to relate our results to other treatments. If we set $s = 9.46$km, $R = 1$km, $\tilde{b} = 0.35$ and $\tilde{k} = 0.00045$, then we obtain the mass $M_{E=0} = 3.103M_{\odot}$ for uncharged matter and $M_{E\neq 0} = 2.858M_{\odot}$ for charged matter. These values for the mass are similar to charged quark stellar models generated by Mak and Harko (2004) which describe a unique static charged configuration of quark matter admitting a one-parameter group of conformal symmetries. We have shown that the presence of anisotropy and charge in the matter distribution yields masses which are consistent with other investigations.

The plots of the matter variables indicate that the solution may be used to model quark stars, at least in the envelope if singularities are present at the origin. We calculated the mass in a special case ($a = 1, n = 1$) and showed that this value is consistent with strange matter distributions of Dey *et al* (1998), Mak and Harko (2004) and Sharma and Maharaj (2007). Ours is a particular solution of the Einstein-Maxwell system with the nice feature of containing the isothermal universe in the isotropic limit. A more comprehensive study of other possible solutions admitted, with the strange matter equation of state, is likely to produce other interesting results.

Chapter 6

Radiating relativistic matter in geodesic motion

6.1 Introduction

Relativistic models of radiating stars are useful in the investigation of cosmic censorship hypothesis, gravitational collapse with dissipation, formation of superdense matter, dynamical stability of radiating matter and temperature profiles in the context of irreversible thermodynamics. The general model, incorporating all necessary physical requirements and variables, is complicated and difficult to solve; the treatments of Herrera *et al* (2004a) and Di Prisco *et al* (2007) involving physically meaningful charged spherically symmetric collapse with shear and dissipation illustrate the complexity of the processes. To solve the field equations, and to find tractable forms for the gravitational and matter variables, we need to make simplifying assumptions. De Oliviera *et al* (1985) proposed a radiating model in which an initial static configuration leads to collapse. This approach may be adapted to describe the end state of collapse as shown by Govender *et al* (2003). In a recent treatment Herrera *et al* (2004b) proposed a model in which the form of Weyl tensor was highlighted when studying radiative collapse with an approximate solution. Maharaj and Govender (2005), Herrera *et al* (2006) and Misthry *et al* (2008) showed that it is possible to solve the field equations and boundary conditions exactly in this scenario. For recent treatments involving collapse with equations of state and formation of black holes see Goswami and Joshi (2004a, 2004b).

A useful approach in understanding the effects of dissipation is due to Kolassis *et al* (1988) in which the fluid trajectories are assumed to be geodesic. In the limit, in the absence of heat flow, the interior Friedmann dust solution was regained. This solution formed the basis for many investigations involving the physical behaviour such as the rate of collapse, surface luminosity and temperature profiles. These include the analytic model of radiating spherical gravitational collapse with neutrino flux by Grammenos and Kolassis (1992), the model describing realistic astrophysical processes with heat flow by Tomimura and Nunes (1993), and models undergoing collapse with heat flow as a possible mechanism for gamma-ray bursts by Zhe *et al* (2008). Herrera *et al* (2002) considered geodesic fluid spheres in coordinates which are not comoving but with anisotropic pressures. Govender *et al* (1998) showed that the behaviour of the temperature in casual thermodynamics for geodesic motion produces higher central temperatures than the Eckart theory. The first exact solution with shear, satisfying the boundary conditions, was obtained by Naidu *et al* (2006) by considering geodesic fluid trajectories. Later Rajah and Maharaj (2008) extended this treatment and obtained classes of models which are nonsingular at the centre.

It is clear that the assumption of geodesic motion is physically acceptable and has been used by other investigators in attempts to describe realistic astrophysical processes. Here we attempt to perform a systematic treatment on the governing equation at the boundary for shear-free collapse by assuming the geodesic motion of the fluid particle. Our intention is to show that the nonlinear boundary condition may be analysed systematically to produce an infinite family of exact solutions. In §6.2, we present the model governing the description of a radiating star using the Einstein field equations together with the junction conditions. We show that it is possible to transform the junction condition to a Bernoulli equation and a Riccati equation. Solutions are obtained in terms of elementary functions in §6.3. In §6.4, we show that the boundary condition, under relevant assumptions, can be written in the form of a confluent hypergeometric equation. We demonstrate that an infinite family of solutions in terms of elementary functions are possible. In §6.5, we obtain the explicit form for the causal temperature using the truncated form of the Maxwell-Cattaneo heat transport equation for a particular metric. This illustrates that the simple forms for the gravitational potentials obtained in this chapter are physically plausible. The results of this chapter have been submitted for publication in Thirukkanesh and Maharaj (2008c).

6.2 The model

We analyse a spherically symmetric relativistic radiating star undergoing shear-free gravitational collapse. This assumption is reasonable when modelling a radiating star in relativistic astrophysics. If we suppose that the particle trajectories are geodesic then the acceleration vanishes. Then the line element, for the matter distribution interior to the boundary of the radiating star, is given by

$$ds^2 = -dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{6.2.1}$$

where $B = B(r,t)$ is the only surviving metric function. The energy momentum tensor including radiation for the interior spacetime is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{6.2.2}$$

where the energy density ρ, the pressure p and the heat flow vector $\mathbf{q}$ are measured relative to the timelike fluid 4-velocity $u^a = \delta^a_0$. The heat flow vector takes the form $q^a = (0, q, 0, 0)$ since $\mathbf{q} \cdot \mathbf{u} = 0$ for heat flow which is radially directed.

The nonzero components of Einstein field equations, for the line element (6.2.1) and the energy momentum tensor (6.2.2), can be written as

$$\rho = 3\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right), \tag{6.2.3a}$$

$$p = -2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + \frac{2}{r}\frac{B'}{B}\right), \tag{6.2.3b}$$

$$p = -2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{1}{B^2}\left(\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{1}{r}\frac{B'}{B}\right), \tag{6.2.3c}$$

$$q = -\frac{2}{B^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2}\right), \tag{6.2.3d}$$

where dots and primes denote differentiation with respect to time t and r respectively. Equating (6.2.3b) and (6.2.3c) we obtain the condition

$$\left(\frac{1}{B}\right)'' = \frac{1}{r}\left(\frac{1}{B}\right)' \tag{6.2.4}$$

which is the condition of pressure isotropy. Equation (6.2.4) is integrable and we obtain

$$B = \frac{d}{C_2(t) - C_1(t)r^2} \tag{6.2.5}$$

where $C_1(t)$ and $C_2(t)$ are functions of time, and d is a constant. As the functional form for the potential B is specified the matter variables ρ, p and q are known quantities, and the system (6.2.3) has been solved in principle.

The interior spacetime (6.2.1) has to be matched across the boundary $r = b$ to the exterior Vaidya spacetime

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right)dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi), \tag{6.2.6}$$

where $m(v)$ denotes the mass of the star as measured by an observer at infinity. The hypersurface at the boundary is denoted by Σ. The matching of the line elements (6.2.1) and (6.2.6), and matching of the extrinsic curvature at the surface of the star, leads to a set of equations. The boundary conditions at Σ have the form

$$dt = \left[\left(1 - \frac{2m}{R} + 2\frac{dR}{dv}\right)^{1/2} dv\right]_\Sigma, \tag{6.2.7a}$$

$$(rB)_\Sigma = R_\Sigma, \tag{6.2.7b}$$

$$p_\Sigma = (qB)_\Sigma, \tag{6.2.7c}$$

$$[m(v)]_\Sigma = \left[\frac{r^3}{2}\left(\dot{B}^2 B - \frac{B'^2}{B}\right) - r^2 B'\right]_\Sigma, \tag{6.2.7d}$$

where the subscript means that the relevant quantities are evaluated on Σ.

From (6.2.3), (6.2.5) and (6.2.7c) we generate the condition

$$\begin{aligned} -4db(\dot{C}_1 C_2 - C_1 \dot{C}_2)(C_1 b^2 - C_2) - 4C_1 C_2 (C_1 b^2 - C_2)^2 \\ -2d^2(\ddot{C}_1 b^2 - \ddot{C}_2)(C_1 b^2 - C_2) + 5d^2(\dot{C}_1 b^2 - \dot{C}_2)^2 &= 0. \end{aligned} \tag{6.2.8}$$

Effectively (6.2.8) results from the nonvanishing of the pressure gradient across the hypersurface Σ. Equation (6.2.8) governs the dynamical evolution of shear-free radiating stars in which fluid trajectories are geodesic. To complete the description in this particular radiating model we need to explicitly solve the differential equation (6.2.8).

6.3 Generating analytic solutions

A particular solution to (6.2.8) was found by Kolassis *et al* (1988) by inspection. We show that it is possible to transform (6.2.8) into familiar differential equations which admit solutions in closed form. Our method is a more systematic approach in solving equation (6.2.8). In this approach we let

$$C_1 b^2 - C_2 = u(t). \tag{6.3.1}$$

On substituting (6.3.1) into (6.2.8) we can write

$$4bdu^2\dot{C}_1 + 4(u^2 - bd\dot{u})uC_1 - 4b^2u^2C_1^2 = d^2(2u\ddot{u} - 5\dot{u}^2). \tag{6.3.2}$$

Equation (6.3.2) is simpler than (6.2.8) and can be viewed as a first order differential equation in the variable C_1. In general, (6.3.2) is a Riccati equation (in C_1), and is difficult solve in the above form without simplifying assumptions. For the integration of (6.3.2), in terms of elementary functions, we consider the following two cases:

6.3.1 Bernoulli equation

We set

$$2u\ddot{u} - 5\dot{u}^2 = 0 \tag{6.3.3}$$

so that the function u is given by

$$u = \alpha \quad \text{or} \quad u = \beta(t + \gamma)^{-2/3}, \tag{6.3.4}$$

where α, β and γ are real constants. With the assumption (6.3.3), (6.3.2) becomes

$$4bdu^2\dot{C}_1 + 4(u^2 - bd\dot{u})uC_1 - 4b^2u^2C_1^2 = 0. \tag{6.3.5}$$

Equation (6.3.5) is nonlinear but is a Bernoulli equation which can be linearised in general.

When $u = \alpha$, equation (6.3.5) becomes

$$\dot{C}_1 + \frac{\alpha}{bd}C_1 - \frac{b}{d}C_1^2 = 0 \tag{6.3.6}$$

which is a Bernoulli equation with constant coefficients. The solution of (6.3.6) is given by

$$C_1 = \frac{\alpha}{b^2 - \exp\left(\frac{\alpha(t+e)}{bd}\right)},$$

where e is the constant of integration. Consequently the remaining function C_2 has the form

$$C_2 = \frac{\alpha \exp\left(\frac{\alpha(t+e)}{bd}\right)}{b^2 - \exp\left(\frac{\alpha(t+e)}{bd}\right)}.$$

Hence the interior line element (6.2.1) has the specific form

$$ds^2 = -dt^2 + \frac{d^2}{\alpha^2}\left[\frac{b^2 - \exp\left(\frac{\alpha(t+e)}{bd}\right)}{r^2 - \exp\left(\frac{\alpha(t+e)}{bd}\right)}\right]^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \qquad (6.3.7)$$

in terms of exponential functions. We believe that this is a new solution to the Einstein field equations for a radiating star. It is interesting to observe that if we set $\alpha = d$ when $t \to \infty$ (or large values of the constant e) then (6.3.7) becomes the flat Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

which is a limiting case.

When $u = \beta(t+\gamma)^{-2/3}$, (6.3.5) becomes

$$\dot{C}_1 + \left[\frac{\beta}{bd}(t+\gamma)^{-2/3} + \frac{2}{3}(t+\gamma)^{-1}\right] C_1 - \frac{b}{d}C_1^2 = 0 \qquad (6.3.8)$$

which is also a Bernoulli equation with variable coefficients. The solution of (6.3.8) is given by

$$C_1 = \frac{\beta}{\left[b^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)\right]}(t+\gamma)^{-2/3}$$

where f is the constant of integration. Consequently the remaining function C_2 is given by

$$C_2 = \frac{-\beta^2 f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)}{\left[b^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)\right]}(t+\gamma)^{-2/3}.$$

Hence the interior line element (6.2.1) takes the particular form

$$ds^2 = -dt^2 + \frac{d^2}{\beta^2}\left[\frac{b^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)}{r^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)}\right]^2 (t+\gamma)^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \qquad (6.3.9)$$

If we set

$$\gamma = 0, d = \left(\frac{M}{6}\right)^{1/3} b, f = \frac{3}{ab^2}, \beta = -\frac{b^2}{3}$$

then (6.3.9) becomes

$$ds^2 = -dt^2 + \frac{9\left(\frac{M}{6}\right)^{2/3}}{b^2}\left[\frac{1 - ab^2 \exp\left(\frac{6t}{M}\right)^{1/3}}{1 - ar^2 \exp\left(\frac{6t}{M}\right)^{1/3}}\right]^2 t^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

which was first found by Kolassis *et al* (1988). Here we have shown that their model found by inspection arises naturally as a solution of a Bernoulli equation. It is easy to see that for large values of the constant f we obtain

$$ds^2 = -dt^2 + t^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

from (6.3.9). This corresponds to the Friedmann metric when the fluid is in the form of dust with vanishing heat flux.

6.3.2 Riccati equation

If we set

$$u^2 - db\dot{u} = 0 \tag{6.3.10}$$

then the function u is given by

$$u = -bd(t+a)^{-1} \tag{6.3.11}$$

where a is a constant. In this case equation (6.3.2) becomes

$$4bd\dot{C}_1 - 4b^2C_1^2 + d^2(t+a)^{-2} = 0, \tag{6.3.12}$$

which is an inhomogeneous Riccati equation. The solution of equation (6.3.12) is given by

$$C_1 = \frac{-d\left[1-\sqrt{2}+(1+\sqrt{2})g(t+a)^{\sqrt{2}}\right]}{2b\left[1+g(t+a)^{\sqrt{2}}\right]}(t+a)^{-1}, \tag{6.3.13}$$

where g is the constant of integration. Consequently the remaining function has the form

$$C_2 = bd\left\{1 - \frac{\left[1-\sqrt{2}+(1+\sqrt{2})g(t+a)^{\sqrt{2}}\right]}{2\left[1+g(t+a)^{\sqrt{2}}\right]}\right\}(t+a)^{-1}.$$

Hence the interior metric (6.2.1) has the specific form

$$ds^2 = -dt^2 + \frac{d^2(t+a)^2}{\left[C_1(r^2-b^2)(t+a)-bd\right]^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{6.3.14}$$

which is written in terms of C_1. We believe that (6.3.14) is a new solution for a radiating star whose particles are constrained to travel on geodesics. The simple form of (6.3.14) will assist in studying the physical features of our model. The solution (6.3.14) arises in a natural way once we realise that the underlying dynamical equation (6.2.8) at the boundary is a Riccati equation.

6.4 Special functions

The solutions found in the previous sections all have power law forms for the quantity u. It is possible that other solutions in terms of elementary functions or special functions may exist with a power law representation for u. Consequently, in this section, we attempt to generate a general class of solutions to the model (6.2.8) by assuming

$$u = \alpha(t+a)^n. \tag{6.4.1}$$

On substituting (6.4.1) into (6.3.2) we obtain

$$(t+a)^2\dot{C}_1 + \left[\frac{\alpha}{db}(t+a)^{n+1} - n\right](t+a)C_1 - \frac{b}{d}(t+a)^2C_1^2 = -\frac{d}{4b}n(3n+2). \tag{6.4.2}$$

The nonlinear equation (6.4.2) is a Riccati equation and it is difficult to solve the equation in the above form. If we introduce a transformation

$$\frac{b}{d}C_1 = -\frac{\dot{U}}{U} \tag{6.4.3}$$

then (6.4.2) becomes the second order linear differential equation

$$(t+a)^2\ddot{U} + \left[\frac{\alpha}{db}(t+a)^{n+1} - n\right](t+a)\dot{U} - \frac{n(3n+2)}{4}U = 0 \tag{6.4.4}$$

in the function U with variable coefficients. We can transform (6.4.4) to simpler form if we let

$$\psi = (t+a)^{n+1},\ W = U\psi^{-k},\ k = \frac{(n+1) \pm \sqrt{4n(n+1)+1}}{2(n+1)}. \tag{6.4.5}$$

Then (6.4.4) becomes

$$(n+1)\psi\frac{d^2W}{d\psi^2} + \left[\frac{\alpha}{bd}\psi + 2k(n+1)\right]\frac{dW}{d\psi} + \frac{\alpha k}{bd}W = 0. \tag{6.4.6}$$

If we let

$$X = \frac{-\alpha\psi}{bd(n+1)},\ Y(X) = W(\psi)$$

then (6.4.6) has the equivalent form

$$X\frac{d^2Y}{dX^2} + (2k - X)\frac{dY}{dX} - kY = 0. \tag{6.4.7}$$

Observe that (6.4.7) is the confluent hypergeometric equation with solution in terms of special functions in general.

Note that the solution of (6.4.7) can be written in terms of

$$\begin{aligned} Y &= \mathcal{J}\left(k, 2k; X\right), \\ W &= \mathcal{J}\left(k, 2k; \frac{-\alpha\psi}{bd(n+1)}\right) \end{aligned}$$

where $\mathcal{J}$ are Kummer functions. In general the solution of the equation (6.4.6) can be written in terms of the Kummer series. Observe that when $k > 0$ we can write

$$\begin{aligned} \tilde{W} &= \mathcal{J}(k, 2k; X) \\ &= \frac{\Gamma(2k)}{[\Gamma(k)]^2}\int_0^1 e^{X\tau}\left[\tau(1-\tau)\right]^{k-1} d\tau \end{aligned} \tag{6.4.8}$$

as a particular solution of the differential equation (6.4.6) where $\Gamma(z) = \int_0^\infty e^{-\tau}\tau^{z-1}d\tau$ is the gamma function. From (6.4.8) we note that the solution can be expressed in terms of elementary functions for all natural numbers k. Consequently the differential equation (6.4.2) admits solutions in terms of elementary functions when k is a natural number.

6.4.1 Particular metrics

We can regain previous cases from the general form (6.4.8). We illustrate this feature for particular values of k. When $k = 1$, we obtain $n = 0$ or $n = -2/3$. For this case the particular solution of the equation (6.4.6) becomes

$$\tilde{W} = \frac{e^X - 1}{X}, \quad X = \frac{-\alpha\psi}{bd(n+1)} \tag{6.4.9}$$

with the help of (6.4.8).

When $n = 0$, from (6.4.5) and (6.4.9) we can easily see that

$$\tilde{U} = \frac{bd}{\alpha}\left[1 - \exp\left(\frac{-\alpha(t+a)}{bd}\right)\right]$$

is a particular solution of the equation (6.4.4). Then with the help of (6.4.3) we find that

$$\tilde{C}_1 = \frac{\alpha}{b^2\left[1 - \exp\left(\frac{\alpha(t+a)}{bd}\right)\right]} \tag{6.4.10}$$

is a particular solution of (6.4.2) which is given by

$$\dot{C}_1 + \frac{\alpha}{bd}C_1 - \frac{b}{d}{C_1}^2 = 0. \tag{6.4.11}$$

The general solution of (6.4.11) becomes

$$C_1 = \frac{\alpha D}{b^2\left[D + \exp\left(\frac{\alpha(t+a)}{bd}\right)\right]},$$

where D is an arbitrary constant. Consequently the interior metric (6.2.1) has the specific form

$$ds^2 = -dt^2 + \frac{b^4 d^2}{\alpha^2}\left[\frac{D + \exp\left(\frac{\alpha(t+a)}{bd}\right)}{Dr^2 + b^2\exp\left(\frac{\alpha(t+a)}{bd}\right)}\right]^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{6.4.12}$$

in terms of exponential functions. Note that the line element (6.4.12) reduces to the metric (6.3.7) if we set $D = -b^2$.

When $n = -2/3$, from (6.4.5) and (6.4.9) we observe that

$$\tilde{U} = \frac{bd}{3\alpha}\left[1 - \exp\left(\frac{-3\alpha(t+a)^{1/3}}{bd}\right)\right]$$

is a particular solution of the equation (6.4.4). Hence with the help of (6.4.3) we obtain

$$\tilde{C}_1 = \frac{\alpha(t+a)^{2/3}}{b^2\left[1 - \exp\left(\frac{3\alpha(t+a)^{1/3}}{bd}\right)\right]} \tag{6.4.13}$$

as a particular solution of (6.4.2) which has the form

$$\dot{C}_1 + \left[\frac{\alpha}{bd}(t+a)^{-2/3} + \frac{2}{3}(t+a)^{-1}\right]C_1 - \frac{b}{d}{C_1}^2 = 0. \tag{6.4.14}$$

The general solution of (6.4.14) becomes

$$C_1 = \frac{\alpha D(t+a)^{2/3}}{b^2\left[D + \exp\left(\frac{3\alpha(t+a)^{1/3}}{bd}\right)\right]},$$

where D is an arbitrary constant. Consequently the interior metric (6.2.1) takes the particular form

$$ds^2 = -dt^2 + \frac{b^4 d^2}{\alpha^2}\left[\frac{D + \exp\left(\frac{3\alpha(t+a)^{1/3}}{bd}\right)}{Dr^2 + \exp\left(\frac{3\alpha(t+a)^{1/3}}{bd}\right)}\right]^2 (t+a)^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{6.4.15}$$

Note that the line element (6.4.15) reduces to the line element (6.3.9) if we set $D = \frac{b^2}{\alpha f}$.

6.4.2 A new solution

It is possible to generate an infinite family of new solutions from the general form (6.4.8) by specifying values for the parameter k. These may correspond to new solutions for a radiating sphere which are not accelerating. We illustrate this process by taking $k = 2$ (so that $n = -2$ or $n = -4/5$) in (6.4.8). We consider only the case $n = -2$ as the integration procedure is same for other values of k (or n) . For this case the particular solution of the equation (6.4.6) becomes

$$\tilde{W} = \frac{6}{X^3}\left[2 + X + (X-2)e^X\right], \quad X = \frac{-\alpha\psi}{bd(n+1)}. \tag{6.4.16}$$

When $n = -2$, from (6.4.5) and (6.4.16) we observe that

$$\tilde{U} = \frac{6b^2d^2}{\alpha^3}\left[[2bd(t+a) + \alpha] - [2bd(t+a) - \alpha]\exp\left(\frac{\alpha}{bd(t+a)}\right)\right]$$

is a particular solution of the equation (6.4.4). Hence with the help of (6.4.3) we obtain

$$\tilde{C}_1 = \frac{2b^2d^2(t+a)^2 - [2bd(t+a)(bd(t+a) - \alpha) + \alpha^2]\exp\left(\frac{\alpha}{bd(t+a)}\right)}{b^2\left[(2bd(t+a) - \alpha)\exp\left(\frac{\alpha}{bd(t+a)}\right) - (2bd(t+a) + \alpha)\right](t+a)^2} \tag{6.4.17}$$

is a particular solution of (6.4.2) which has the form

$$\dot{C}_1 + \left[\frac{\alpha}{bd}(t+a)^{-2} + 2(t+a)^{-1}\right]C_1 - \frac{b}{d}{C_1}^2 = -\frac{2d}{b}(t+a)^{-2}. \tag{6.4.18}$$

The general solution of (6.4.18) becomes

$$C_1 = -\frac{\left[[\alpha^2 + 2bd(t+a)(bd(t+a)-\alpha)]\exp\left(\frac{\alpha}{bd(t+a)}\right) + 2b^2d^2D(t+a)^2\right]}{b^2\left[D(2bd(t+a)+\alpha) + (2bd(t+a)-\alpha)\exp\left(\frac{\alpha}{bd(t+a)}\right)\right](t+a)^2}, \quad (6.4.19)$$

where D is an arbitrary constant. Consequently the interior metric (6.2.1) has the specific form

$$ds^2 = -dt^2 + \frac{d^2}{[C_1(r^2-b^2) + \alpha(t+a)^{-2}]^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad (6.4.20)$$

where C_1 is given by (6.4.19). Hence we have found a new solution to the boundary condition (6.2.8) by specifying a particular value for the parameter k. This process can be repeated for other values of k and an infinite family of solutions are possible in which the gravitational potentials can be expressed in terms of elementary functions.

6.5 Physical analysis

The simple forms of the gravitational potentials found in this chapter permit a detailed study of the physical features of a radiating star. In this study we consider the particular line element (6.4.12) and set $\alpha = bd$ and $a = 0$ to obtain

$$ds^2 = -dt^2 + b^2\left[\frac{D+\exp(t)}{Dr^2 + b^2\exp(t)}\right]^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad (6.5.1)$$

for simplicity. For the metric (6.5.1) the matter variables can be written as

$$\rho = \frac{3D\exp(t)\left\{\exp(t)\left[Db^4 + 6b^2Dr^2 + Dr^4 + 4b^4\exp(t)\right] + 4D^2r^4\right\}}{[D+\exp(t)]^2\left[Dr^2 + b^2\exp(t)\right]^2}, \quad (6.5.2a)$$

$$p = \frac{D\exp(t)}{[D+\exp(t)]^2\left[Dr^2 + b^2\exp(t)\right]^2}\times$$

$$\left\{\exp(t))\left[2b^2\exp(t)(r^2-3b^2) - 2Db^2r^2 - 3D(r^4+b^4)\right] + 2D^2r^2(b^2-3r^2)\right\}, \quad (6.5.2b)$$

$$q = \frac{4Dr\exp(t)}{[D+\exp(t)]^2}. \quad (6.5.2c)$$

When $D = 0$ then (6.5.1) becomes the Minkowski metric with $\rho = p = q = 0$. The matter variables are expressed in simple analytic forms which facilitate the analysis of the physical behaviour. From (6.5.2) we have that at the centre of the sphere

$$\rho_0 = \frac{3D\,[D + 4\exp(t)]}{[D + \exp(t)]^2}$$

$$p_0 = -\frac{3D[D + 2\exp(t)]}{[D + \exp(t)]^2}$$

$$q_0 = 0$$

so that ρ_0 and p_0 have finite values at the centre $r = 0$ with vanishing heat flux q_0. The gravitational potentials in (6.5.1) are finite at the centre and nonsingular in the stellar interior. The quantities ρ, p and q are well behaved and regular in the interior of the sphere, at least in regions close to the centre. At later times as $t \to \infty$ we note that $q \propto r$ so that the magnitude of the heat flux depends linearly on the radial coordinate.

Next we briefly consider the relativistic effect of casual temperature of this model. The Maxwell-Cattaneo heat transport equation, in the absence of rotation and viscous stresses, is given by

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa \left(h_a{}^b \nabla_b T + T\dot{u}_a \right), \tag{6.5.3}$$

where $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space, T is the local equilibrium temperature, κ (≥ 0) is the thermal conductivity and τ (≥ 0) is the relaxation time. Equation (6.5.3) reduces to the acausal Fourier heat transport equation when $\tau = 0$. For the line element (6.2.1), the casual transport equation (6.5.3) can be written as

$$T(t, r) = -\frac{1}{\kappa} \int \left[\tau (q\dot{B}) B + qB^2 \right] dr \tag{6.5.4}$$

for geodesic motion. Martinez (1996), Govender *et al* (1998) and Di Prisco *et al* (1996) have demonstrated that the relaxation time τ on the thermal evolution, plays a significant role in the latter stages of collapse. For the line element (6.5.1), (6.5.4) becomes

$$
\begin{aligned}
T(t,r) &= \frac{\tau b^2 \exp(t)\left\{2D^2r^2 - b^2\exp(t)\left[\exp(t) - D\right]\right\}}{\kappa\left[\exp(t) + D\right]\left[b^2\exp(t) + Dr^2\right]^2} \\
&\quad + \frac{2b^2\exp(t)}{\kappa\left[b^2\exp(t) + Dr^2\right]} + h(t),
\end{aligned} \tag{6.5.5}
$$

where $h(t)$ is a function of integration. For simplicity we assumed that τ and κ are constant. The function $h(t)$ may be related to the central temperature $T_c(t)$ by

$$
h(t) = T_c(t) - \frac{\tau\left[D - \exp(t)\right]}{\kappa\left[D + \exp(t)\right]} - \frac{2}{\kappa}. \tag{6.5.6}
$$

From (6.5.5) and (6.5.6) the temperature can be written as

$$
\begin{aligned}
T(t,r) &= T_c(t) - \frac{\tau Dr^2\left\{Dr^2\left[D - \exp(t)\right] - 2b^2\exp(t)\right\}}{\kappa\left[D + \exp(t)\right]\left[Dr^2 + b^2\exp(t)\right]^2} \\
&\quad - \frac{2Dr^2}{\kappa\left[Dr^2 + b^2\exp(t)\right]}.
\end{aligned} \tag{6.5.7}
$$

When $\tau = 0$, we can regain the acausal (Eckart) temperature profiles from (6.5.7). In Figure 6.1, we plot the casual (solid line) and acasual (dashed line) temperatures against the radial coordinate on the interval $0 \le r \le 5$ for particular parameter values ($\kappa = \tau = 1, b = 5, D = 70$ and $h(t) = 0$) on the spacelike hypersurface $t = 1$. We observe that the temperature is monotonically decreasing from centre to the boundary in both casual and acasual cases. It is clear that the casual temperature is greater than the acasual temperature throughout the stellar interior. At the boundary Σ we have

$$
T(t, r_\Sigma)_{\text{casual}} \simeq T(t, r_\Sigma)_{\text{acasual}}.
$$

Our figures have been generated by assuming constant values for the parameters τ and κ. Changing the values of the relaxation time and the thermal conductivity would produce different gradients for the curves but the result would not change qualitatively.

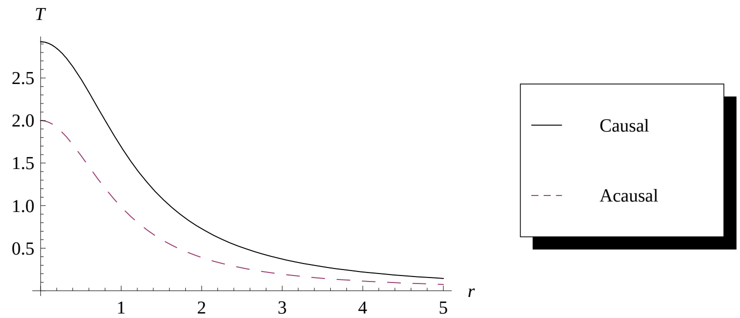

Figure 6.1: Temperature T versus radial coordinate r ($\tau = 1$).

Chapter 7

Radiating collapse with anisotropic pressures

7.1 Introduction

A star usually emits radiation and throws out particles during gravitational collapse. Therefore the heat flow in the interior of the star must be taken into account so that the interior solution of the radiating star should match to the Vaidya (1951) exterior metric at the boundary. The investigation of the gravitational behaviour of a collapsing star depends on the formulation of the junction conditions matching the interior metric with the exterior Vaidya metric across the boundary of the star. In the past, investigations in radiating collapse have focussed on shear-free spacetimes with isotropic pressures (see the treatments of Herrera *et al* (2004a), Herrera *et al* (2006), Maharaj and Govender (2005)and Misthry *et al* (2006)). This scenario can be generalised to include anisotropic pressures in the presence of shear for fluid particles travelling on geodesics or particles experiencing acceleration. The first analytic solution for anisotropic pressures with shear was obtained by Naidu *et al* (2006), by considering geodesic fluid trajectories. Rajah and Maharaj (2008) generalised the Naidu *et al* (2006) model to be nonsingular at the centre. The general situation requires a model which is expanding, accelerating and shearing. Noguiera and Chan (2004) attempted such a study but found that they had to utilise numerical techniques to make progress. Our objective here is to show that is possible to model such processes exactly.

In this chapter we attempt to perform a systematic treatment on the governing equation at the boundary of the radiating star. In §7.2, we present the field equations. The junction condition are given in §7.3. In §7.4, we generate the differential equation governing the gravitational behaviour of a radiating, shearing and accelerating sphere. The master equation is rewritten in the form of a Riccati equation. In §7.5, we consider the situation where the fluid particles are travelling on geodesics and obtain two classes of solutions which contain models found previously. In §7.6, we consider the most general case of expanding, shearing and accelerating radiative collapse and obtain three classes of solutions in terms of arbitrary functions on the radial and temporal coordinates. In §7.7, we briefly investigate the physical features of the model generated and present the casual and acasual temperature profiles.

7.2 Field equations

The most general form for the interior space time of a spherically symmetric collapsing star with nonzero shear and accelerating fluid particle is given by the line metric

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7.2.1}$$

where A, B and Y are functions of both the temporal coordinate t and radial coordinate r. The fluid 4-velocity vector $\mathbf{u}$ is given by $u^a = \frac{1}{A}\delta^a_0$ which is comoving. For the line element (7.2.1) the 4-acceleration, the expansion scalar Θ and the magnitude of the shear scalar σ are given by

$$\dot{u}^a = \left(0, \frac{A'}{AB^2}, 0, 0\right), \tag{7.2.2a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right), \tag{7.2.2b}$$

$$\sigma = -\frac{1}{3A}\left(\frac{\dot{B}}{B} - \frac{\dot{Y}}{Y}\right), \tag{7.2.2c}$$

where primes and dots denote the differentiation with respect to r and t respectively. The energy momentum tensor for the interior matter distribution is described by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{7.2.3}$$

where p is the isotropic pressure, ρ is the density of the fluid, π_{ab} is the stress tensor and q_a is the heat flux vector. The stress tensor has the form

$$\pi_{ab} = (p_r - p_t)\left(n_a n_b - \frac{1}{3}h_{ab}\right), \tag{7.2.4}$$

where p_r is the radial pressure, p_t is the tangential pressure and $\mathbf{n}$ is a unit radial vector given by $n^a = \frac{1}{B}\delta^a_1$. The isotropic pressure

$$p = \frac{1}{3}(p_r + 2p_t) \tag{7.2.5}$$

relates the radial pressure and the tangential pressure.

For the line element (7.2.1) and matter distribution (7.2.3) the coupled Einstein field equations become

$$\begin{aligned} \rho &= \frac{2}{A^2}\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{1}{A^2}\frac{\dot{Y}^2}{Y^2} \\ &\quad -\frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'}{B}\frac{Y'}{Y}\right), \end{aligned} \tag{7.2.6a}$$

$$\begin{aligned} p_r &= \frac{1}{A^2}\left(-2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y}\right) \\ &\quad +\frac{1}{B^2}\left(\frac{Y'^2}{Y^2} + 2\frac{A'}{A}\frac{Y'}{Y}\right) - \frac{1}{Y^2}, \end{aligned} \tag{7.2.6b}$$

$$\begin{aligned} p_t &= -\frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} - \frac{\dot{A}}{A}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) \\ &\quad +\frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + \frac{A'}{A}\frac{Y'}{Y} - \frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right), \end{aligned} \tag{7.2.6c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{Y}'}{Y} + \frac{\dot{B}}{B}\frac{Y'}{Y} + \frac{A'}{A}\frac{\dot{Y}}{Y}\right), \tag{7.2.6d}$$

where the heat flux $q^a = (0, q, 0, 0)$ has only the nonvanishing radial component. The system of equations (7.2.6a)-(7.2.6d) governs the general situation in describing matter distributions with anisotropic pressures in the presence of heat flux for a spherically symmetric relativistic stellar object. The system of field equations (7.2.6a)-(7.2.6d) describes the nonlinear gravitational interaction for a shearing matter distribution which is expanding and accelerating. From (7.2.6a)-(7.2.6d), we observe that if forms for the gravitational potentials A, B and Y are known, then the expressions for the matter variables ρ, p_r, p_t and q follow immediately.

7.3 Junction conditions

The Vaidya exterior spacetime of radiating star is given by

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7.3.1}$$

where $m(v)$ denotes the mass of the fluid as measured by an observer at infinity. The line element (7.3.1) describes coherent null radiation which is flowing in the radial direction relative to the hypersurface Σ which represents the boundary of the star. The matching of the interior spacetime (7.2.1) with the exterior spacetime (7.3.1) leads to the following set of junction conditions on the hypersurface:

$$A(R_\Sigma, t)dt = \left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \tag{7.3.2a}$$

$$Y(R_\Sigma, t) = R_\Sigma(v), \tag{7.3.2b}$$

$$m(v)_\Sigma = \left[\frac{Y}{2}\left(1 + \frac{\dot{Y}^2}{A^2} - \frac{Y'^2}{B^2}\right)\right]_\Sigma, \tag{7.3.2c}$$

$$(p_r)_\Sigma = (qB)_\Sigma. \tag{7.3.2d}$$

The nonvanishing radial pressure at the boundary Σ leads to the additional equation (7.3.2d), which has to be satisfied with the system of field equations (7.2.6a)-(7.2.6d). The junction condition for shear-free spacetimes was first derived by Santos (1985) and later it was extended by Glass (1989) and Maharaj and Govender (2000) to incorporate spacetimes with nonzero shear.

7.4 The master equation

Substituting (7.2.6b) and (7.2.6d) in (7.3.2d) we obtain

$$2Y\ddot{Y} + \dot{Y}^2 - 2\left(\frac{\dot{A}}{A} + \frac{A'}{B}\right)Y\dot{Y} + 2\frac{A}{B}Y\dot{Y}'$$

$$-2\frac{A}{B^2}\left(A' + \dot{B}\right)YY' - \frac{A^2}{B^2}Y'^2 + A^2 = 0. \tag{7.4.1}$$

The equation (7.4.1) governs the gravitational behaviour of the radiating anisotropic star with nonzero shear, acceleration and expansion. As the equation (7.4.1) is highly nonlinear, it is difficult to solve without simplifying assumptions. Equation (7.4.1) contains the three gravitational functions $A(r,t), B(r,t)$ and $Y(r,t)$. Therefore, to find a solution we have to specify any two functions. For convenience we rewrite (7.4.1) in the form of a Riccati equation in the gravitational potential B as

$$\begin{aligned} \dot{B} &= \left[\frac{\ddot{Y}}{AY'} + \frac{\dot{Y}^2}{2AYY'} - \frac{\dot{A}}{A^2}\frac{\dot{Y}}{Y'} + \frac{A}{2YY'}\right]B^2 \\ &\quad + \left[\frac{\dot{Y}'}{Y'} - \frac{A'}{A}\frac{\dot{Y}}{Y'}\right]B - \left[A' + \frac{AY'}{2Y}\right]. \end{aligned} \tag{7.4.2}$$

From (7.4.2) it is clear that to solve the Riccati equation (7.4.2) we should specify the potentials $A(r,t)$ and $Y(r,t)$ as demonstrated in the following sections. When $A = 1$ then fluid particles are geodesic and (7.4.2) becomes

$$\dot{B} = \left[\frac{\ddot{Y}}{Y'} + \frac{\dot{Y}^2}{2YY'} + \frac{1}{2YY'}\right]B^2 + \frac{\dot{Y}'}{Y'}B - \frac{Y'}{2Y}. \tag{7.4.3}$$

The case of geodesic motion was previously studied by Naidu *et al* (2006) and Rajah and Maharaj (2008).

7.5 Geodesic motion with anisotropic pressures

In previous investigations it was assumed that the potential $Y(r,t)$ is separable. In this section we demonstrate that it is possible to find solutions systematically without assuming separable forms for $Y(r,t)$. If we introduce the transformation

$$B = ZY' \tag{7.5.1}$$

then equation (7.4.3) becomes

$$\dot{Z} = \frac{1}{2Y}\left[FZ^2 - 1\right], \tag{7.5.2}$$

where we have set

$$F = 2Y\ddot{Y} + \dot{Y}^2 + 1.$$

Observe that equation (7.5.2) is integrable if we set F as a constant or function of r only. We demonstrate that this technique leads to new solutions in the following sections.

7.5.1 Analytic solution I

If we set $F = 1$ then the function Y is given by

$$Y(r,t) = [R_1(r)t + R_2(r)]^{2/3}, \tag{7.5.3}$$

where $R_1(r)$ and $R_2(r)$ are any functions of r only. For this case equation (7.5.2) becomes

$$\dot{Z} = \frac{1}{2[R_1(r)t + R_2(r)]^{2/3}}\left[Z^2 - 1\right]. \tag{7.5.4}$$

On integrating (7.5.4) we obtain

$$Z = \frac{1 + f(r)\exp\left[3(R_1t + R_2)^{1/3}/R_1\right]}{1 - f(r)\exp\left[3(R_1t + R_2)^{1/3}/R_1\right]}, \tag{7.5.5}$$

where $f(r)$ is a function of integration. Hence from (7.5.1), (7.5.3) and (7.5.5) we get

$$B = \frac{2}{3}\left[\frac{1 + f(r)\exp\left[3(R_1t + R_2)^{1/3}/R_1\right]}{1 - f(r)\exp\left[3(R_1t + R_2)^{1/3}/R_1\right]}\right]\frac{[R_1't + R_2']}{[R_1t + R_2]^{1/3}}. \tag{7.5.6}$$

Therefore the line element (7.2.1) takes the particular form

$$ds^2 = -dt^2 + \frac{4}{9}\left[\frac{1+f(r)\exp\left[3(R_1t+R_2)^{1/3}/R_1\right]}{1-f(r)\exp\left[3(R_1t+R_2)^{1/3}/R_1\right]}\right]^2 \frac{[R_1't+R_2']^2}{[R_1t+R_2]^{2/3}}dr^2$$

$$+[R_1(r)t+R_2(r)]^{4/3}(d\theta^2+\sin^2\theta d\phi^2). \quad (7.5.7)$$

From (7.5.7) observe that the line element is given in terms of arbitrary functions $R_1(r), R_2(r)$ and $f(r)$ so that it is possible to generate an infinite number of exact solutions.

It is interesting to see that for particular choices of the arbitrary functions we regain models found previously. If we set

$$R_1 = R^{3/2} \text{ and } R_2 = aR^{3/2}$$

then the line element (7.5.7) reduces to

$$ds^2 = -dt^2$$

$$+(t+a)^{4/3}\left\{R'^2\left[\frac{1+f(r)\exp\left[3(t+a)^{1/3}/R\right]}{1-f(r)\exp\left[3(t+a)^{1/3}/R\right]}\right]^2 dr^2 + R^2(d\theta^2+\sin^2\theta d\phi^2)\right\}. \quad (7.5.8)$$

The line element (7.5.8) corresponds to the first category of the Rajah and Maharaj (2008) models for anisotropic radiating star with shear. If we set $a=0$ and $R=r$ then (7.5.8) reduces to

$$ds^2 = -dt^2 + t^{4/3}\left\{\left[\frac{1+f(r)\exp\left[3t^{1/3}/r\right]}{1-f(r)\exp\left[3t^{1/3}/r\right]}\right]^2 dr^2 + R^2(d\theta^2+\sin^2\theta d\phi^2)\right\}. \quad (7.5.9)$$

The metric (7.5.9) was first found by Naidu *et al* (2006) in their analysis of pressure anisotropy and heat dissipation in a spherically symmetric radiating star undergoing gravitational collapse. Note that when

$$R_1 = r^{3/2}, R_2 = 0 \text{ and } f(r) = 0$$

the line element (7.5.7) reduces to

$$ds^2 = -dt^2 + t^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{7.5.10}$$

The line element (7.5.10) corresponds to the Friedmann metric when the fluid is in the form of dust with vanishing heat flux. If we set

$$R_1 = 0 \text{ and } R_2 = r^{3/2}$$

then the line element (7.5.7) takes the form

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

which corresponds to flat Minkowski spacetime. Note that the Minkowski spacetime cannot be regained from the Rajah and Maharaj (2008) model due to the separable form chosen for the potential Y in their treatment.

7.5.2 Analytic solution II

If we set $F = 1 + R_1^2(r)$ then the function Y is given by

$$Y(r,t) = R_1(r)t + R_2(r), \tag{7.5.11}$$

where $R_1(r)$ and $R_2(r)$ are any functions of r only. For this case equation (7.5.2) becomes

$$\dot{Z} = \frac{[R_1^2+1]}{2[R_1 t + R_2]}\left[Z^2 - \frac{1}{[R_1^2+1]}\right]. \tag{7.5.12}$$

The solution of (7.5.12) can be written as

$$Z = \frac{1}{\sqrt{R_1^2+1}}\left[\frac{1 + g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}{1 - g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}\right], \tag{7.5.13}$$

where $g(r)$ is the function of integration. Hence from (7.5.1), (7.5.11) and (7.5.13) we get

$$B = \frac{1}{\sqrt{R_1^2+1}}\left[\frac{1 + g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}{1 - g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}\right][R_1' t + R_2']. \tag{7.5.14}$$

Therefore the line element (7.2.1) takes the particular form

$$ds^2 = -dt^2 + \frac{1}{[R_1^2+1]}\left[\frac{1+g(r)[R_1t+R_2]^{\sqrt{R_1^2+1}/R_1}}{1-g(r)[R_1t+R_2]^{\sqrt{R_1^2+1}/R_1}}\right]^2 [R_1't+R_2']^2dr^2$$
$$+[R_1(r)t+R_2(r)]^2(d\theta^2+\sin^2\theta d\phi^2) \tag{7.5.15}$$

in terms of arbitrary functions $R_1(r), R_2(r)$ and $g(r)$. Therefore again we can generate an infinite number of exact solution to (7.4.3).

Note that when

$$R_1 = R \text{ and } R_2 = aR$$

the line element (7.5.15) reduces to

$$ds^2 = -dt^2 + (t+a)^2\left\{\frac{R'^2}{[R^2+1]}\left[\frac{1+h(r)[t+a]^{\sqrt{R^2+1}/R}}{1-h(r)[t+a]^{\sqrt{R^2+1}/R}}\right]^2 dr^2 + R^2(d\theta^2+\sin^2\theta d\phi^2)\right\}, \tag{7.5.16}$$

where we have defined the new arbitrary function $h(r) = g(r)R^{\sqrt{R^2+1}/R}$. The line element (7.5.16) corresponds to the second category of the Rajah and Maharaj (2008) models for anisotropic radiating star with shear. If we set

$$R_1 = 0, R_2 = r \text{ and } g(r) = 0$$

the line element (7.5.15) reduces to the flat Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2).$$

7.6 Accelerating motion with anisotropic pressures

In this section we consider the more general case with the gravitational potential A being any function of r and t. In the integration of equation (7.4.2) we consider three cases.

7.6.1 Bernoulli equation

Observe that equation (7.4.2) reduces to a Bernoulli equation if we set

$$A' + \frac{AY'}{2Y} = 0.$$

If we take

$$Y(r,t) = R(r)C_1(t) \tag{7.6.1}$$

then

$$A(r,t) = \frac{\alpha}{\sqrt{R(r)}}C_2(t), \tag{7.6.2}$$

where α is a real constant, $C_1(t)$ and $C_2(t)$ are functions of t, and $R(r)$ is a function of r. Substituting (7.6.1) and (7.6.2) into (7.4.2) we get

$$\dot{B} - \frac{R^{3/2}}{\alpha R' C_1}\left[\frac{\ddot{C}_1}{C_2} + \frac{\dot{C}_1^{\,2}}{2C_1C_2} - \frac{\dot{C}_1\dot{C}_2}{C_2^2} + \frac{\alpha^2 C_2}{2R^3C_1}\right]B^2 - \frac{3}{2}\frac{\dot{C}_1}{C_1}B = 0, \tag{7.6.3}$$

which is a Bernoulli equation in the variable B.

On integrating (7.6.3) we obtain

$$B = \frac{-\alpha R' C_1^{3/2}}{R^{3/2}\int\left[\frac{\ddot{C}_1}{C_2} + \frac{\dot{C}_1^{\,2}}{2C_1C_2} - \frac{\dot{C}_1\dot{C}_2}{C_2^2} + \frac{\alpha^2 C_2}{2R^3C_1}\right]\sqrt{C_1}dt - \alpha R' g(r)}, \tag{7.6.4}$$

where $g(r)$ is a function of integration. Therefore the functions

$$A = \frac{\alpha}{\sqrt{R}}C_2, \tag{7.6.5a}$$

$$B = \frac{-\alpha R' C_1^{3/2}}{R^{3/2}\int\left[\frac{\ddot{C}_1}{C_2} + \frac{\dot{C}_1^{\,2}}{2C_1C_2} - \frac{\dot{C}_1\dot{C}_2}{C_2^2} + \frac{\alpha^2 C_2}{2R^3C_1}\right]\sqrt{C_1}dt - \alpha R' g(r)} \tag{7.6.5b}$$

$$Y = RC_1, \tag{7.6.5c}$$

satisfy the junction condition (7.4.2). From (7.6.5) we can generate an infinite family of solutions for different choices of the arbitrary functions $C_1(t), C_2(t)$ and $R(r)$. In principle the quantity B can be evaluated if the functions $C_1(t)$ and $C_2(t)$ are specified.

7.6.2 Inhomogeneous Riccati equation

Note that equation (7.4.2) becomes an inhomogeneous Riccati equation if we set

$$\frac{\dot{Y}'}{Y'} - \frac{A'}{A}\frac{\dot{Y}}{Y'} = 0.$$

If we take

$$Y(r,t) = R(r)C(t) \tag{7.6.6}$$

then

$$A(r,t) = \alpha \dot{Y}, \tag{7.6.7}$$

where α is a real constant and $R(r)$ and $C(t)$ are arbitrary functions of r and t respectively. Substituting (7.6.6) and (7.6.7) into (7.4.2) we get

$$\dot{B} = \frac{(1+\alpha^2)}{2\alpha R'}\frac{\dot{C}}{C^2}B^2 - \frac{3}{2}\alpha R'\dot{C}, \tag{7.6.8}$$

which is an inhomogeneous Riccati equation.

It is difficult to solve the equation (7.6.8) in the above form; we first introduce a transformation to obtain a more convenient form. Note that the transformation

$$B = wC$$

leads to a separable equation:

$$\frac{C}{\dot{C}}\dot{w} = \frac{(1+\alpha^2)}{2\alpha R'}w^2 - w - \frac{3}{2}\alpha R'. \tag{7.6.9}$$

In general it is difficult to solve the equation (7.6.9). However it is possible to integrate (7.6.9) for $\alpha = -2$. When $\alpha = -2$ the equation (7.6.9) can be written as

$$\frac{\dot{w}}{(5w - 6R')(w + 2R')} = \frac{-1}{4R'}\frac{\dot{C}}{C}.$$

On integrating the above equation we obtain

$$w = \frac{2R'[3C^4 + f(r)]}{5C^4 - f(r)},$$

where $f(r)$ is a function of integration. Hence, we get the result

$$B = \frac{2R'C[3C^4 + f(r)]}{5C^4 - f(r)}. \tag{7.6.10}$$

Therefore the solution to the equation (7.4.2) becomes

$$A = -2R\dot{C}, \tag{7.6.11a}$$

$$B = \frac{2R'C[3C^4 + f(r)]}{5C^4 - f(r)}, \tag{7.6.11b}$$

$$Y = RC. \tag{7.6.11c}$$

We can observe from (7.6.11) that it is possible to generate an infinite number of models for different choices of the arbitrary functions $R(r)$, $f(r)$ and $C(t)$.

7.6.3 Linear equation

Note that equation (7.4.2) becomes a linear equation if we set

$$\frac{\ddot{Y}}{AY'} + \frac{\dot{Y}^2}{2AYY'} - \frac{\dot{A}}{A^2}\frac{\dot{Y}}{Y'} + \frac{A}{2YY'} = 0.$$

For this case it is difficult to find a general relationship between the potentials Y and A similar to previous cases in this chapter. However it possible to find particular functions that satisfy this condition. For example, if we take

$$Y(r,t) = R(r)[\alpha + \beta t - \gamma^2 t^2], \tag{7.6.12a}$$

$$A(r,t) = 2\gamma R(r)\sqrt{\alpha + \beta t - \gamma^2 t^2}, \tag{7.6.12b}$$

where α, β and γ is a real constant and $R(r)$ is a functions of r only, then (7.4.2) becomes

$$\dot{B} + 3\gamma R'\sqrt{\alpha + \beta t - \gamma^2 t^2} = 0. \tag{7.6.13}$$

On integrating (7.6.13) we get

$$B = \frac{3R'}{8\gamma^2}\Big[2\gamma(\beta - 2\gamma^2 t)\sqrt{\alpha + \beta t - \gamma^2 t^2}$$

$$-(\beta^2 + 4\alpha\gamma^2)\arctan\left[\frac{2\gamma^2 t - \beta}{2\alpha\sqrt{\alpha + \beta t - \gamma^2 t^2}}\right]\Big] + g(r),$$

where $g(r)$ is a function of integration. Therefore the functions

$$A = 2\gamma R\sqrt{\alpha + \beta t - \gamma^2 t^2}, \tag{7.6.14a}$$

$$B = \frac{3R'}{8\gamma^2}\Big[2\gamma(\beta - 2\gamma^2 t)\sqrt{\alpha + \beta t - \gamma^2 t^2}$$

$$-(\beta^2 + 4\alpha\gamma^2)\arctan\left[\frac{2\gamma^2 t - \beta}{2\alpha\sqrt{\alpha + \beta t - \gamma^2 t^2}}\right]\Big] + g(r), \tag{7.6.14b}$$

$$Y = R(\alpha + \beta t - \gamma^2 t^2), \tag{7.6.14c}$$

satisfy the junction condition (7.4.2).

7.7 Physical analysis

The simple forms of the solutions found in this chapter facilitate a study of physical behaviour. In this section we briefly consider the physical features of the solution generated in §7.6.2. For the gravitational potentials obtained in (7.6.11), the line element (7.2.1) becomes

$$ds^2 = -4R^2\dot{C}^2 dt^2 + \left[\frac{2R'C[3C^4 + f(r)]}{5C^4 - f(r)}\right]^2 dr^2 + R^2C^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7.7.1}$$

In the study of the physical features we take $C(t) = (t+a)^n$ and $f(r) = k$, where a, n and k are real constants, for simplicity. In this case the kinematical quantities become

$$\dot{u}^a = \left(0, \frac{[5(t+a)^{4n} - k]^2}{4RR'[3(t+a)^{4n} + k]^2(t+a)^{2n}}, 0, 0\right), \tag{7.7.2a}$$

$$\Theta = \frac{[3k^2 + 26k(t+a)^{4n} - 45(t+a)^{8n}]}{2R[-k^2 + 2k(t+a)^{4n} + 15(t+a)^{8n}](t+a)^n}, \tag{7.7.2b}$$

$$\sigma = \frac{16k(t+a)^{3n}}{3R[k^2 - 2k(t+a)^{4n} - 15(t+a)^{8n}]}. \tag{7.7.2c}$$

The matter variables become

$$\rho = \frac{[-3k^3 - 43k^2(t+a)^{4n} + 15k(t+a)^{8n} + 95(t+a)^{12n}]}{2R^2[3(t+a)^{4n}+k]^2[5(t+a)^{4n}-k](t+a)^{2n}} \qquad (7.7.3a)$$

$$p_r = \frac{1}{4R^2(t+a)^{2n}}\left[\frac{3[5(t+a)^{4n}-k]^2}{[3(t+a)^{4n}+k]^2} - 5\right] \qquad (7.7.3b)$$

$$p_t = \frac{4[-13k^2 - 45k^2(t+a)^{4n} - 95k(t+a)^{8n} + 25(t+a)^{12n}]}{R^2[k^2 - 2k(t+a)^{4n} - 15(t+a)^{8n}]^2} \qquad (7.7.3c)$$

$$q = \frac{[k^3 + 25k^2(t+a)^{4n} - 165k(t+a)^{8n} + 75(t+a)^{12n}]}{4R^2R'[3(t+a)^{4n}+k]^3(t+a)^{3n}} \qquad (7.7.3d)$$

The expressions in (7.7.2a)-(7.7.2c) and (7.7.3a)-(7.7.3d) are given in terms of simple elementary functions which will assist in a physical analysis. The total luminosity for an observer rest at infinity is given by the equation

$$L_\infty = -\left(\frac{dm}{dv}\right)_\Sigma = \frac{(p_r)_\Sigma}{2}\left[Y^2\left(\frac{Y'}{B} + \dot{Y}\right)^2\right]_\Sigma , \qquad (7.7.4)$$

where $\frac{dm}{dv} \leq 0$ as $L_\infty \geq 0$. Therefore the total luminosity L_∞ can be easily obtainable from (7.7.3b), (7.6.11b) and (7.6.11c).

Next we briefly consider the relativistic effect of casual temperature of our model. The Maxwell-Cattaneo heat transport equation, in the absence of rotation and viscous stress in the truncated version, is given by

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa\left(h_a{}^b \nabla_b T + T\dot{u}_a\right), \qquad (7.7.5)$$

where τ is the relaxation time, κ is the thermal conductivity and $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space. Equation (7.7.5) reduces to the acausal Fourier heat transport equation when $\tau = 0$. For the line element (7.2.1) the casual transport equation (7.7.5) can be written as

$$T(t,r) = -\frac{1}{\kappa A}\int \left[\tau(\dot{qB})B + AqB^2\right] dr. \qquad (7.7.6)$$

Martinez (1996), Govender *et al* (1998) and Di Prisco *et al* (1996) has demonstrated that the relaxation time τ on the thermal evolution, plays a significant role in the latter stages of collapse. Rajah and Maharaj (2008) and Naidu *et al* (2006) showed that for geodesic motion in the presence of shear stress, the relaxation time decreases as the collapse proceeds and the central temperature increases. For our model, when $R(r) = r + b$, (7.7.6) becomes

$$T(t,r) = \frac{\tau[-k^3 + 15k^2(t+a)^{4n} - 315k(t+a)^{8n} + 45(t+a)^{12n}]}{\kappa(r+b)^2[3(t+a)^{4n}+k]^2[5(t+a)^{4n}-k](t+a)^{2n}} + \frac{[k^2+30k(t+a)^{4n} - 15(t+a)^{8n}]\ln[r+b]}{\kappa(r+b)[-k^2+2k(t+a)^{4n}+15(t+a)^{8n}](t+a)^n} + h(t), \quad (7.7.7)$$

where $h(t)$ is a function of integration and we set τ and κ to be constant. We have expressed the temperature T in a simple analytic form in terms of powers of t. The function $h(t)$ may be related to the central temperature $T_c(t)$ as

$$h(t) = T_c(t) - \frac{\tau[-k^3 + 15k^2(t+a)^{4n} - 315k(t+a)^{8n} + 45(t+a)^{12n}]}{\kappa b^2[3(t+a)^{4n}+k]^2[5(t+a)^{4n}-k](t+a)^{2n}} - \frac{[k^2+30k(t+a)^{4n} - 15(t+a)^{8n}]\ln[b]}{\kappa b[-k^2+2k(t+a)^{4n}+15(t+a)^{8n}](t+a)^n}, \quad (7.7.8)$$

Note that Herrera *et al* (2006) also assumed the same relation when they studied shear-free spacetimes. From (7.7.7) and (7.7.8) the temperature can be written as

$$T(t,r) = T_c(t) - \frac{\tau r(r+2b)}{\kappa b^2(r+b)^2}\frac{[-k^3 + 15k^2(t+a)^{4n} - 315k(t+a)^{8n} + 45(t+a)^{12n}]}{[3(t+a)^{4n}+k]^2[5(t+a)^{4n}-k](t+a)^{2n}} - \frac{1}{\kappa}\left[\frac{\ln[b]}{b} - \frac{\ln[r+b]}{(r+b)}\right]\frac{[k^2+30k(t+a)^{4n} - 15(t+a)^{8n}]}{[-k^2+2k(t+a)^{4n}+15(t+a)^{8n}](t+a)^n}. \quad (7.7.9)$$

When $\tau = 0$, we can regain the acausal (Eckart) temperature profiles from (7.7.9). In Figure 7.1, we plot the casual (solid line) and acasual (dashed line) temperatures against the radial coordinate on the interval $0 \leq r \leq 2$ for particular parameter values ($a = 1, b = 0.1, k = 1, n = 2, t = 3, \kappa = 2, \tau = 10$ and $h(t) = 0$). We can easily observe that the temperature is monotonically decreasing from centre to boundary in both casual and acasual cases. It is clear that the casual temperature is greater than the acasual temperature throughout the stellar interior. At the boundary $\sum$:

$$T(t, r_{\sum})_{\text{casual}} \simeq T(t, r_{\sum})_{\text{acasual}}.$$

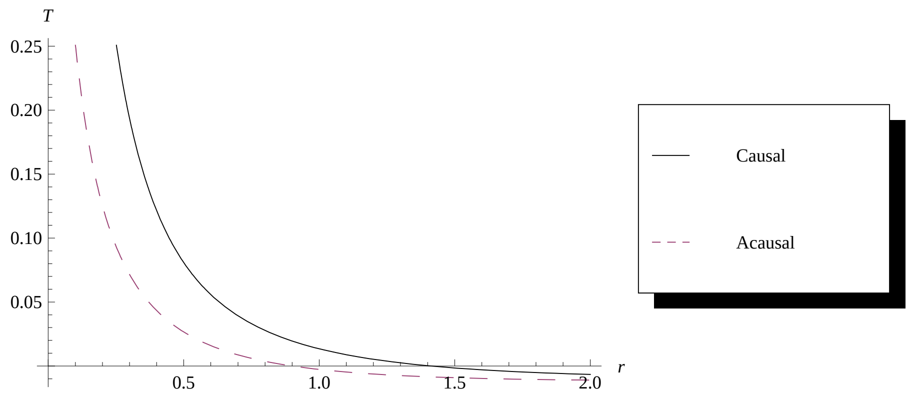

Figure 7.1: Temperature T vs radial coordinate r ($\tau = 10$).

Chapter 8

Conclusion

The goal of this thesis was to generate new classes of exact solutions to the Einstein and Einstein-Maxwell field equations which may be used to model realistic stellar objects in spherically symmetric spacetimes. By a consideration of the nonlinear structure of the Einstein field equations, which describes the gravitational behaviour of the interior of a relativistic star, we sought to introduce suitable transformations to simplify the master equation into an integrable form. Several new families of solutions were obtained to the Einstein-Maxwell system by specifying different, but physically reasonable forms, for one of the gravitational potentials and the electric field intensity. In addition, we examined the more general situation of the Einstein-Maxwell system with anisotropic pressures. New classes of exact solutions were generated which may be useful to study strange stars with a linear equation of state. Then we studied relativistic radiating stars under going gravitational collapse and a number of new exact solutions were obtained. We showed that these solutions met the necessary physical requirements and generated causal temperature profiles.

We now present an overview of the main results achieved during the course of this project:

- The objective of Chapter 2 was to study the Einstein-Maxwell system in the presence of isotropic pressures. For the the gravitational potential

$$Z = \frac{1 + ax}{1 + bx}$$

and the electric field

$$E^2 = \frac{\alpha C b x}{(1+bx)^2}$$

the condition of pressure isotropy reduced to the differential equation

$$4X\left[aX-(a-b)\right]\frac{d^2Y}{dX^2}+2(a-b)\frac{dY}{dX}+\left[(b-a)-\alpha\right]Y=0,$$

where $X = 1 + bx$ and $Y(X) = y(x)$, and a, b and α are real constants. In general the solution of the above equation was given in a series form; in addition we obtained two families of solutions in terms of elementary functions for specific values of the parameters a, b and α. The solutions found satisfy a barotropic equation of state $p = p(\rho)$. We regained the charged solutions of Hansraj and Maharaj (2006) and Maharaj and Komathiraj (2007); the uncharged neutron star models of Finch and Skea (1989), Durgapal and Bannerji (1983) and Tikekar (1990) arise as special cases of our general class. We plotted the behaviour of ρ, p, E^2 and $\frac{dp}{d\rho}$, for particular parameter values, and verified that the models were physically viable. Graphs were plotted to distinguish between the behaviour of the charged and uncharged models.

- The intention of Chapter 3 was to investigate the Einstein-Maxwell system in the presence of isotropic pressures for the potential

$$Z(x) = \frac{(1+ax)^2}{1+bx}$$

and the electric field intensity

$$E^2 = \frac{\alpha a(b-a)Cx}{(1+bx)^2}.$$

In this case the condition of pressure isotropy reduced to

$$4(1+ax)^2\left[b(1+ax)-(b-a)\right]\ddot{y}+2a(1+ax)\left[b(1+ax)-2(b-a)\right]\dot{y}$$

$$+a(b-a)(b-a-\alpha a)y=0.$$

Two categories of solutions are possible corresponding to $\alpha = \frac{b}{a} - 1$ and $\alpha \neq \frac{b}{a} - 1$. When $\alpha = \frac{b}{a} - 1$ the general solution is expressible in terms of elementary functions only. When $\alpha \neq \frac{b}{a}-1$, and using the method of Frobenius, we showed that a series solution exists. We demonstrated that two classes of solutions can be extracted in terms of elementary functions from the general class to the above equation. Note that if $\alpha = 0$ then we regain uncharged solutions of the Einstein field equations. The simple form of the exact solutions, in terms of polynomials and algebraic functions, facilitates the analysis of the physical features of a charged sphere. The solutions found satisfy a barotropic equation of state. We showed that our model is a generalisation of the charged solutions found previously by Komathiraj and Maharaj (2007a). For particular parameter values we plotted the behaviour of ρ, p, E^2 and $\frac{dp}{d\rho}$ and showed that they were physically reasonable.

- In Chapter 4, we derived a general framework for the Einstein-Maxwell system with linear equation of state that models the interior of a dense star as an anisotropic charged imperfect fluid sphere. To solve the system we made the assumptions

$$\begin{aligned} p_r &= \alpha\rho - \beta, \\ Z &= \frac{1+(a-b)x}{1+ax}, \\ E^2 &= \frac{kC(3+ax)}{(1+ax)^2}. \end{aligned}$$

  The equation of state may be applicable to strange stars with quark matter and dark energy stars. Three new classes of exact solutions were generated to this system. Our class of solutions contain the uncharged anisotropic models of Sharma and Maharaj (2007) and Lobo (2006); the uncharged isotropic models of de Sitter and Einstein were regained as special cases from our general class of solutions. We showed that the masses and densities for stellar objects are consistent with the results of Dey *et at* (1998) and Sharma and Maharaj (2007) in the limit of vanishing charge. Consequently these solutions may be useful in the investigation of stellar bodies such as SAX J1808.4-3658.

- The objective of Chapter 5 was to study the Einstein-Maxwell system of equations with a strange matter equation of state for anisotropic matter distributions. To solve the field equations we made the assumptions

$$p_r = \alpha\rho - \beta,$$

$$Z = \frac{1}{a + bx^n},$$

$$E^2 = \frac{2kC(d + 2x)}{a + bx^n}.$$

  A new class of exact solutions was obtained to the system of nonlinear Einstein-Maxwell equations. We regained the general relativistic isothermal universe of Saslaw *et al* (1996), which is an extension of the conventional Newtonian isothermal universe with density $\rho \propto r^{-2}$, when the anisotropy vanishes. The plots of the matter variables indicate that the solution may be used to model quark stars, at least in the envelope if singularities are present at the origin. We calculated the stellar mass for particular parameters values ($a = 1, n = 1$) and showed that this value is consistent with strange matter distributions of Dey *et al* (1998), Mak and Harko (2004) and Sharma and Maharaj (2007). Ours is a particular solution of the Einstein-Maxwell system with the interesting feature of containing the isothermal universe in the isotropic limit.

- In Chapter 6, we investigated the simple situation of a shear-free metric with particles travelling on geodesic trajectories. The governing equation of the boundary condition of the stellar model was transformed to the Riccati equation

$$4bdu^2\dot{C}_1 + 4(u^2 - bd\dot{u})uC_1 - 4b^2u^2C_1^2 = d^2(2u\ddot{u} - 5\dot{u}^2).$$

  Under certain assumptions this equation was reduced to a Bernoulli equation that admits two solutions in terms of elementary functions. The first solution contains the Minkowski spacetime as a limiting case while the second solution corresponds to the Kolassis *et al* (1988) model with the Friedmann dust spacetime as the limiting case. Then we transformed the Riccati equation into the confluent hypergeometric equation

$$X\frac{d^2Y}{dX^2} + (2k - X)\frac{dY}{dX} - kY = 0$$

that admits solution in terms of special functions namely the Kummer functions, in general. For particular parameter values in the special function we demonstrated that an infinite family of solutions, in terms of elementary functions, are possible. The simple form of the solutions makes it possible to study the physical features of the model and to find an analytic form for the causal temperature.

- The aim of Chapter 7 was to generate new analytical solutions that model the interior of a spherically symmetric radiative star undergoing gravitational collapse with an accelerating, expanding and shearing matter distribution. We rewrote the junction condition in the form of the Riccati equation

$$\begin{aligned}\dot{B} &= \left[\frac{\ddot{Y}}{AY'} + \frac{\dot{Y}^2}{2AYY'} - \frac{\dot{A}}{A^2}\frac{\dot{Y}}{Y'} + \frac{A}{2YY'}\right]B^2 \\ &\quad + \left[\frac{\dot{Y}'}{Y'} - \frac{A'}{A}\frac{\dot{Y}}{Y'}\right]B - \left[A' + \frac{AY'}{2Y}\right].\end{aligned}$$

We found two classes of exact solutions when the fluid particles are travelling on geodesics which contain the models of Naidu *et al* (2006), Rajah and Maharaj (2008), the Friedmann dust metric and the Minkowski spacetime. Three classes of accelerating, expanding and shearing solutions were generated by transforming the master equation into a Bernoulli equation, an inhomogeneous Riccati equation and a linear equation. These solutions depend on arbitrary functions of the temporal coordinate and the radial coordinate which allow for an infinite family of solutions. For a particular metric we investigated the physical features. We derived the temperature profiles in general and plotted the behaviour of the casual and acasual temperatures for particular parameter values.

Appendix A

In Appendix A we derive the elementary functions (2.5.1) and (2.5.2).

On substituting $r = 0$ into equation (2.4.5) we find that

$$c_{i+1} = \frac{4ai(i-1) - ((a-b)+\alpha)}{(a-b)(2i+2)(2i-1)} c_i, \quad i \geq 0. \tag{A.0.1}$$

If we set $a-b+\alpha = 4an(n-1)$, where n is a fixed integer and we assume that $a \neq 0$, then $c_{n+1} = 0$. It is easy to see that the subsequent coefficients $c_{n+2}, c_{n+3}, c_{n+4}, \ldots$ vanish and equation (A.0.1) has the solution

$$c_i = -n(n-1)\left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!} c_0 \; , 1 \leq i \leq n. \tag{A.0.2}$$

Then from equation (2.4.3) (when $r = 0$) and (A.0.2) we generate

$$Y_1 = c_0 \left[1 - n(n-1) \sum_{i=1}^{n} \left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!} X^i \right] \tag{A.0.3}$$

where $a - b + \alpha = 4an(n-1)$.

On substituting $r = \frac{3}{2}$ into (2.4.5), we get

$$c_{i+1} = \frac{a(2i+3)(2i+1) - ((a-b)+\alpha)}{(a-b)(2i+5)(2i+2)} c_i, \quad i \geq 0. \tag{A.0.4}$$

If we set $a-b+\alpha = a(2n+3)(2n+1)$, where n is a fixed integer and we assume that $a \neq 0$, then $c_{n+1} = 0$. Also we see that the subsequent coefficients $c_{n+2}, c_{n+3}, c_{n+4}, \ldots$ vanish and equation (A.0.4) is solved to give

$$c_i = \left(\frac{4a}{b-a}\right)^i \frac{3(2i+2)(n+i+1)!}{(n+1)(n-i)!(2i+3)!} c_0, \qquad 1 \leq i \leq n. \tag{A.0.5}$$

Then from equations (2.4.3) (when $r = \frac{3}{2}$) and (A.0.5) we generate

$$Y_1 = c_0 X^{\frac{3}{2}} \left[1 + \frac{3}{(n+1)} \sum_{i=1}^{n} \left(\frac{4a}{b-a} \right)^i \frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!} X^i \right] \tag{A.0.6}$$

where $a - b + \alpha = a(2n+3)(2n+1)$. The elementary functions (A.0.3) and (A.0.6) comprise the first solution of the differential equation (2.3.6) for appropriate values of $a - b + \alpha$.

We take the second solution of (2.3.6) to be of the form

$$Y_2 = [aX - (a-b)]^{\frac{1}{2}} u(X)$$

where $u(X)$ is an arbitrary function. On substituting Y_2 into (2.3.6) we obtain

$$4X\,[aX - (a-b)]\,\ddot{u} - 2\,[2aX + (a-b)]\,\dot{u} - [2a - b + \alpha]\,u = 0 \tag{A.0.7}$$

where dots denote differentiation with respect to X. We write the solution of the differential equation (A.0.7) in the series form

$$u = \sum_{n=0}^{\infty} c_n X^{n+r}\,, \quad c_0 \neq 0. \tag{A.0.8}$$

On substituting (A.0.8) into the differential equation (A.0.7) we find

$$\begin{aligned} 2(a-b)c_0 r[-2(r-1)+1]X^{r-1} - \sum_{n=0}^{\infty} \big(2(a-b)c_{n+1}(n+1+r)[2(n+r)-1] \\ -c_n[4a(n+r)^2 - (2a-b+\alpha)]\big)\,X^{n+r} \;=\; 0. \end{aligned} \tag{A.0.9}$$

Setting the coefficient of X^{r-1} in (A.0.9) to zero we find

$$(a-b)c_0 r[2(r-1)-1] = 0.$$

which is the indicial equation. Since $c_0 \neq 0$ and $a \neq b$ we must have $r = 0$ or $r = \frac{3}{2}$. Equating the coefficient of X^{n+r} in (A.0.9) to zero we find that

$$c_{n+1} = \frac{4a(n+r)^2 - (2a-b+\alpha)}{2(a-b)(n+r+1)[2(n+r)-1]} c_n \tag{A.0.10}$$

We establish a general structure for the coefficients by considering the leading terms.

On substituting $r = 0$ in equation (A.0.10) we obtain

$$c_{i+1} = \frac{4ai^2 - (2a - b + \alpha)}{(a-b)(2i+2)(2i-1)} c_i. \tag{A.0.11}$$

We assume that $a - b + \alpha = a(2n+3)(2n+1)$ where n is a fixed integer. Then $c_{n+2} = 0$ from (A.0.11). Consequently the remaining coefficients $c_{n+3}, c_{n+4}, c_{n+5}, ...$ vanish and equation (A.0.11) has the solution

$$c_i = -(n+1)\left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!} c_0, \quad 1 \leq i \leq n+1. \tag{A.0.12}$$

Then from the equations (A.0.8) (when $r = 0$) and (A.0.12) we find

$$u = c_0 \left[1 - (n+1) \sum_{i=1}^{n+1} \left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!} X^i \right].$$

Hence we generate the result

$$Y_2 = c_0 \left[aX - (a-b)\right]^{\frac{1}{2}} \left[1 - (n+1) \sum_{i=1}^{n+1} \left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!} X^i \right] \tag{A.0.13}$$

where $a - b + \alpha = a(2n+3)(2n+1)$.

On substituting $r = \frac{3}{2}$ into equation (A.0.10) we obtain

$$c_{i+1} = \frac{a(2i+3)^2 - (2a - b + \alpha)}{(a-b)(2i+5)(2i+2)} c_i. \tag{A.0.14}$$

We assume that $a - b + \alpha = 4an(n-1)$ where n is a fixed integer. Then $c_{n-1} = 0$ from (A.0.14). Consequently the remaining coefficients $c_n, c_{n+1}, c_{n+2}, ...$ vanish and (A.0.14) can be solved to obtain

$$c_i = \left(\frac{4a}{b-a}\right)^i \frac{3(2i+2)(n+i)!}{n(n-1)(2i+3)!(n-i-2)!} c_0, \quad i \leq n-2. \tag{A.0.15}$$

Then from the equations (A.0.8) (when $r = \frac{3}{2}$) and (A.0.15) we have

$$u = c_0 X^{\frac{3}{2}} \left[1 + \frac{3}{n(n-1)} \sum_{i=1}^{n-2} \left(\frac{4a}{b-a}\right)^i \frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!} X^i \right].$$

Hence we generate the result

$$Y_2 = c_0 \left[aX - (a-b)\right]^{\frac{1}{2}} X^{\frac{3}{2}} \left[1 + \frac{3}{n(n-1)} \sum_{i=1}^{n-2} \left(\frac{4a}{b-a}\right)^i \frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!} X^i \right] \tag{A.0.16}$$

where $a-b+\alpha=4an(n-1)$. The functions (A.0.13) and (A.0.16) generate the second solution of the differential equation (2.3.6).

The solutions found can be written in terms of two classes of elementary functions. We have the first category of solutions

$$\begin{aligned} Y &= D_1\left[aX-(a-b)\right]^{\frac{1}{2}}\left[1-(n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}X^i\right] \\ &+D_2X^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}X^i\right] \end{aligned} \tag{A.0.17}$$

for $a-b+\alpha=a(2n+3)(2n+1)$, where D_1 and D_2 are arbitrary constants. In terms of x the solution (A.0.17) becomes

$$\begin{aligned} y = &\ d_1(1+ax)^{\frac{1}{2}}\left[1-(n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+bx)^i\right] \\ +d_2 &\ (1+bx)^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}(1+bx)^i\right] \end{aligned} \tag{A.0.18}$$

where $d_1=D_1\sqrt{b}$ and $d_2=D_2$ are new arbitrary constants. The second category of solutions is given by

$$\begin{aligned} Y &= D_1\left[aX-(a-b))\right]^{\frac{1}{2}}X^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}X^i\right] \\ &+D_2\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}X^i\right] \end{aligned} \tag{A.0.19}$$

for $a-b+\alpha=4an(n-1)$, where D_1 and D_2 are arbitrary constants. In terms of x the solution (A.0.19) becomes

$$\begin{aligned} y &= d_3(1+ax)^{\frac{1}{2}}(1+bx)^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}(1+bx)^i\right] \\ &+d_4\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+bx)^i\right] \end{aligned} \tag{A.0.20}$$

where $d_3=D_1\sqrt{b}$ and $d_4=D_2$ are new arbitrary constants.

Appendix B

In the Appendix B we derive the elementary functions (3.5.1) and (3.5.2).

On substituting $r = 0$ in (3.4.9) and setting $d = -n$, we obtain

$$c_i = \left(\frac{b}{a-b}\right)\frac{(n-i+1)(2n-2i+3)}{i(2i-3)}c_{i-1}, \ i \geq 1 \tag{B.0.1}$$

where n is a fixed integer. Then from (B.0.1) we observe that $c_{n+1} = 0$. It is easy to see that the subsequent coefficients $c_{n+2}, \ c_{n+3}, \ c_{n+4}, \ . \ . \ .$ vanish and equation (B.0.1) has the solution

$$c_i = (-1)^{i-1}\left(\frac{b}{b-a}\right)^i \frac{(2i-1)(2n+1)!}{(2i)!(2n-2i+1)!}c_0, \ 0 \leq i \leq n. \tag{B.0.2}$$

Then from (3.4.7) (when $r = 0$) and (B.0.2), we generate

$$U_1 = c_0 \sum_{i=0}^{n}(-1)^{i-1}\left(\frac{b}{b-a}\right)^i \frac{(2i-1)(2n+1)!}{(2i)!(2n-2i+1)!}\left[(1+ax) - \frac{(b-a)}{b}\right]^i \tag{B.0.3}$$

where $d = -n$.

On substituting $r = 0$ in (3.4.9) and setting $d = \frac{1}{2} - n$ we obtain

$$c_i = \left(\frac{b}{a-b}\right)\frac{(n-i+1)(2n-2i+1)}{i(2i-3)}c_{i-1}, \ i \geq 1 \tag{B.0.4}$$

where n is fixed integer. Then from (B.0.4) we observe that $c_{n+1} = 0$. Therefore the subsequent coefficients $c_{n+2}, \ c_{n+3}, \ c_{n+4}, \ . \ . \ .$ vanish and the equation (B.0.4) is solved to give

$$c_i = (-1)^{i-1}\left(\frac{b}{b-a}\right)^i \frac{(2i-1)(2n)!}{(2i)!(2n-2i)!}c_0, \ 0 \leq i \leq n. \tag{B.0.5}$$

Then from (3.4.7) (when $r = 0$) and (B.0.5), we generate

$$U_1 = c_0 \sum_{i=0}^{n} (-1)^{i-1} \left(\frac{b}{b-a}\right)^i \frac{(2i-1)(2n)!}{(2i)!(2n-2i)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i \tag{B.0.6}$$

where $d = \frac{1}{2} - n$. The elementary functions (B.0.3) and (B.0.6) comprise the first solution of the differential equation (3.4.6) for appropriate values of d.

On substituting $r = 3/2$ into (3.4.9) and setting $d = -n$, we get

$$c_i = \left(\frac{b}{a-b}\right) \frac{(n-i)(2n-2i-1)}{i(2i+3)} c_{i-1}, \; i \geq 1 \tag{B.0.7}$$

where n is fixed integer. From (B.0.7) we observe that $c_n = 0$. Therefore the subsequent coefficients $c_{n+1}, \; c_{n+2}, \; c_{n+3}, \; . \; . \; .$ vanish and the equation (B.0.7) is solved to give

$$c_i = 6 \left(\frac{b}{a-b}\right)^i \frac{(i+1)(2n-2)!}{(2i+3)!(2n-2i-2)!} c_0, \; 0 \leq i \leq n-1. \tag{B.0.8}$$

From (3.4.7) (when $b = 3/2$) and (B.0.8) we generate

$$U_2 = 6c_0 \left[(1+ax) - \frac{(b-a)}{b}\right]^{3/2} \times$$

$$\sum_{i=0}^{n-1} \left(\frac{b}{a-b}\right)^i \frac{(i+1)(2n-2)!}{(2i+3)!(2n-2i-2)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i \tag{B.0.9}$$

where $d = -n$.

On substituting $r = 3/2$ in (3.4.9) and setting $d = \frac{1}{2} - n$, we obtain

$$c_i = \left(\frac{b}{a-b}\right) \frac{(n-i-1)(2n-2i-1)}{i(2i+3)} c_{i-1}, \; i \geq 1 \tag{B.0.10}$$

where n is fixed integer. From (B.0.10) we observe that $c_{n-1} = 0$. Therefore the subsequent coefficients $c_n, \; c_{n+1}, \; c_{n+2}, \; . \; . \; .$ vanish and the equation (B.0.10) has the solution

$$c_i = 6 \left(\frac{b}{a-b}\right)^i \frac{(i+1)(2n-3)!}{(2i+3)!(2n-2i-3)!} c_0, \; 0 \leq i \leq n-2 \tag{B.0.11}$$

From (3.4.7) (when $r = 3/2$) and (B.0.11) we generate

$$U_2 = 6c_0 \left[(1+ax) - \frac{(b-a)}{b}\right]^{3/2} \times$$

$$\sum_{i=0}^{n-2} \left(\frac{b}{a-b}\right)^i \frac{(i+1)(2n-3)!}{(2i+3)!(2n-2i-3)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i \tag{B.0.12}$$

where $d = \frac{1}{2} - n$. The functions (B.0.9) and (B.0.12) generate the second solution of the differential equation (3.4.6) for appropriate values of d.

The solutions found can be written in terms of two classes of elementary functions. We have the first category of solutions

$$y(x) = A_1 \frac{1}{(1+ax)^n} \sum_{i=0}^{n} (-1)^{i-1} \left(\frac{b}{b-a}\right)^i \frac{(2i-1)}{(2i)!(2n-2i+1)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i$$

$$+A_2 \frac{1}{(1+ax)^n} \left[(1+ax) - \frac{(b-a)}{b}\right]^{3/2} \times$$

$$\sum_{i=0}^{n-1} \left(\frac{b}{a-b}\right)^i \frac{(i+1)}{(2i+3)!(2n-2i-2)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i \tag{B.0.13}$$

where $\frac{b}{a} - 1 - \alpha = 4n^2$ relates the constants a, b, α and n. The second category of solutions is given by

$$y(x) = A_1 \frac{1}{(1+ax)^{n-1/2}} \sum_{i=0}^{n} (-1)^{i-1} \left(\frac{b}{b-a}\right) \frac{(2i-1)}{(2i)!(2n-2i)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i$$

$$+A_2 \frac{1}{(1+ax)^{n-1/2}} \left[(1+ax) - \frac{(b-a)}{b}\right]^{3/2} \times$$

$$\sum_{i=0}^{n-2} \left(\frac{b}{a-b}\right)^i \frac{(i+1)}{(2i+3)!(2n-2i-3)!} \left[(1+ax) - \frac{(b-a)}{b}\right]^i \tag{B.0.14}$$

where $\frac{b}{a} - 1 - \alpha = 4n(n-1) + 1$ relates the constants a, b, α and n.

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