UNIVERSITY OF KWAZULU-NATAL

NEW CLASSES OF EXACT SOLUTIONS IN RELATIVISTIC ASTROPHYSICS

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As the candidate's supervisor, I have approved this dissertation for submission.

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Abstract

The description of highly compact objects in a general relativistic setting is a problem of present research. The relativistic effects arising from the presence of the electric field and anisotropy are included in our model. We find new exact solutions to the Einstein-Maxwell field equations which are relevant in the description of highly compact stellar objects. Firstly we adopt the approach used by Mafa Takisa and Maharaj (2013b) to obtain a new class of solutions for a different form of the electric field intensity. Our models contain a linear equation of state that is consistent with a quark star. It is interesting that we regain the quark models of Thirukkanesh and Maharaj (2008). Secondly we generalise the charged model of Hansraj and Maharaj (2006) by adding anisotropy to the field equations. Several new classes of exact solutions are found to the Einstein-Maxwell system in terms of Bessel functions and elementary functions. Our solutions contain the neutron star model of Finch and Skea (1989). A physical analysis indicates that the matter distributions are well behaved and regular throughout the stellar structure. Therefore the models found may be useful in modelling a charged anisotropic compact sphere.

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Contents

Chapter 1

Introduction

General relativity is amongst the best theories of modern physics, which originated from simple intuition and emerges as a physical theory of prediction. It plays an important role in understanding the gravitational behaviour of massive bodies in relativistic astrophysics. It has revolutionized the understanding of spacetime and the gravitational field. The interpretation of general relativistic phenomena results in the manifestation of the curvature of the spacetime produced by the presence of massive bodies. General relativity can be summed up in two statements. Firstly spacetime is a curved pseudo-Riemannian manifold with an indefinite metric tensor field. Secondly the relationship between the matter content and the curvature of spacetime is contained in the Einstein field equations. Further details on this development can be found in the work of Blanchet (2005). Shapiro and Teukolsky (1983) pointed out that general relativity models could be utilised in the study of strong gravitational fields and black holes for which the Newtonian gravity models are not suitable. We utilise the Riemann tensor for defining the curvature of spacetime and the energy momentum tensor for matter distributions and the contribution of the electromagnetic field in the presence of charge. The Einstein field equations link the curvature to the total energy momentum. The Bianchy identity ensures conservation of energy momentum.

In relativistic astrophysics we construct mathematical models which describe the behaviour of highly dense objects. These include models of gravitational collapse leading to black holes, neutron stars, pulsars and magnetars, characterised by extremely powerful magnetic fields, and extreme concentrations of energy such as gamma-ray bursts. Exact general relativistic models are relevant to study observed astrophysical processes. In this context spherically symmetric stellar models are important and they are extensively used in several applications. In astrophysics, the collapse of a stellar body can be accurately described by a spherically symmetric gravitational field as mentioned by Shapiro and Teukolsky (1983). In addition spherically symmetric spacetimes have been used in cosmology to study the behaviour and subsequent evolution of the early universe (Krasinski 1997). Stephani (1990) and others have emphasized that in case of high pressures stellar bodies may possess a nonzero charge during the early stages of their evolution. This situation requires an exact solution of the Einstein-Maxwell system of partial differential equations.

Historically, the fundamental exact solutions found in relativistic astrophysics are listed as follows:

- The first exact solution to the Einstein field equations is the Schwarzchild exterior solution.

- The Schwarzchild interior solution describes the interior stellar gravitational field, with constant energy density. For small stars having low pressures, this approach gives an acceptable approximation.

- For a static spherically symmetric charged star, the Reissner-Nordstrom solution describes the exterior gravitational field.

- The Kerr solution is associated with the exterior of the body in rotating motion which is characterised by its mass and angular momentum.

- The Kerr-Newman solution is associated with the exterior of a charged rotating body.

These exact solutions of the Einstein-Maxwell system are essential for describing astrophysical processes.

Our study in this thesis is related to describing the stellar interior with a spherically symmetric anisotropic charged matter distribution. The Einstein-Maxwell system of equations is relevant in describing charged compact objects characterized by a strong gravitational field, such as neutron stars. The investigations of Ivanov (2002) and

Sharma et al (2001) demonstrate that the values of redshifts, luminosities and maximum mass of a compact relativistic star are affected by the presence of electromagnetic field and anisotropy. The works of Mak and Harko (2004), Komathiraj and Maharaj (2007a, 2007b) and Maharaj and Komathiraj (2007) show the significant role of the electromagnetic field in describing the gravitational behaviour of compact stars composed of quark matter. The role of anisotropy has been highlighted by Dev and Gleiser (2002, 2003). In the recent past, several scientists have tried to apply different approaches of finding solutions to the field equations. By a specific choice of the electric field, Hansraj and Maharaj (2006) obtained solutions with isotropic pressures to the Einstein-Maxwell system. These solutions satisfy a barotropic equation of state and contain the Finch and Skea (1989) model. Charged anisotropic models with a linear equation of state were found by Thirukkanesh and Maharaj (2008). Thirukkanesh and Maharaj (2009) found new solutions to the Einstein-Maxwell system with generalised potentials. Mafa Takisa and Maharaj (2013b) utilised a linear equation of state to generate regular solutions of anisotropic spherically symmetric charged distributions which can be related to observed astronomical objects.

The main objective of this thesis is to generate new classes of exact solutions to the Einstein-Maxwell system that are physically acceptable. Two new classes of solutions are obtained. The first class contains the Mafa Takisa and Maharaj (2013b) model. The second class contains the Hansraj and Maharaj (2006) model. Our results reduce to the Mafa Takisa and Maharaj (2013b) model when a parameter related to the electric field is absent. A general class of new solutions is found which generalise the Finch and Skea (1989) stars and their charged analogues due to Hansraj and Maharaj (2006).

In chapter 2, we give details of the Einstein-Maxwell field equations for a static spherically symmetric line element as an equivalent system of differential equations. We provide the relevant background for spacetime curvature and the matter content. For simplification we use a transformation due to Durgapal and Bannerji (1983). With the help of this transformation, a new set of differential equations for the Einstein-Maxwell system is established. A number of physical criteria have to be satisfied for a physically acceptable model.

In chapter 3, we present a new class of exact solutions to the Einstein-Maxwell system. We show that the results of this chapter can regain earlier results. Graphs for electromagnetic and matter variables are presented. Tables of masses for charged and neutral matter are generated showing that the new solutions are physically acceptable.

In chapter 4, another new class of exact solutions to the Einstein-Maxwell system are established. This result is a generalisation of special cases of models previously found. In particular we regain the results of Finch and Skea (1989) and Hansraj and Maharaj (2006). Plots are generated for the matter variables and the electromagnetic quantities which show that the models are well behaved.

In chapter 5, we conclude this thesis with the results of the new exact solutions found.

Chapter 2

Field equations

2.1 Introduction

The theory of general relativity gives a realistic description of a spherically symmetric stellar object. In this chapter we introduce the relevant details necessary to generate a relativistic model for a compact star. We consider the structure of the spherically symmetric spacetimes, relevant differential geometry, the field equations and physical requirements for a relativistic stellar model. More details on these topics can be found in the analyses of Choquet-Bruhat et al (1982), de Felice and Clark (1990), Gron and Hervik (2007), Hawking and Ellis (1973), Misner et al (1973), and Straumann (2004). Curvature and basic spacetime geometry are reviewed in §2.2, and we define the curvature tensor, Ricci tensor and Einstein tensor. In §2.3 we present the matter content whose distribution is considered as a relativistic fluid. We introduce the matter tensor for uncharged matter, and then for charged matter by specifying the contribution of the electromagnetic field tensor to the total energy momentum tensor. The Einstein-Maxwell field equations are analysed in §2.4. We establish first the Einstein field equations for uncharged matter, and then the Einstein-Maxwell equations for charged fluids. The conservation equations of energy momentum for charged and uncharged matter are given. In §2.5 we give the expression of the line element modelling the interior of relativistic star which allows us to generate the connection coefficients, the Ricci tensor, the Ricci scalar, the Einstein tensor and the energy momentum tensor for charged and uncharged fluids. In §2.6 we introduce the linear equation of state and rewrite the Einstein-Maxwell equations in terms of new variables. In §2.7 we define the exterior spacetimes of the star and analyse the criteria required for a relativistic stellar model.

2.2 Curvature

Differentiable manifolds are the most basic structures in differential geometry. In the theory of general relativity, spacetime is defined as a pseudo-Riemannian manifold. The local neighbourhood of a point in spacetime is the same as the open neighbourhood of a point in Euclidean space $\Re^n$. However the global structure of spacetime is different in general from $\Re^n$. Spacetime is a four-dimensional differentiable manifold with the indefinite metric tensor field $\mathbf{g}$. The tensor field $\mathbf{g}$ has the metric signature $(-+++)$ and is symmetric and nonsingular. The condition of nonsingularity permits the metric tensor field to become indefinite. Points in the spacetime manifold are labelled by coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where $x^0 = ct$ is a timelike coordinate and x^1, x^2, x^3 are spacelike coordinates. We use the convention that the speed of light $c = 1$.

The line element in a coordinate basis takes the form

$$ds^2 = g_{ab}dx^a dx^b, \tag{2.1}$$

which gives the infinitesimal distance between neighbouring points in a manifold. Let $\mathbf{U}$ and $\mathbf{V}$ be two parallel vectors fields along any curve such that the inner product $< \mathbf{U}, \mathbf{V} >$ remains constant along the curve. Then we can show that

$$g_{ab;c}U^a V^b \dot{x}^c = 0. \tag{2.2}$$

This result holds at any point for all vectors $\mathbf{U}$, $\mathbf{V}$ and tangent vector $\dot{x}^c$ to the curve. Therefore we find the familiar result

$$g_{ab;c} = 0. \tag{2.3}$$

Hence the covariant derivative of the metric tensor field vanishes. From (2.3) we have

$$\begin{aligned} g_{ab,c} &= \Gamma^d{}_{ac}g_{db} + \Gamma^d{}_{bc}g_{ad}, && (2.4a) \\ g_{bc,a} &= \Gamma^d{}_{ba}g_{dc} + \Gamma^d{}_{ca}g_{bd}, && (2.4b) \end{aligned}$$

$$g_{ca,b} = \Gamma^d{}_{cb} g_{da} + \Gamma^d{}_{ab} g_{cd}. \tag{2.4c}$$

From (2.4) we establish the quantity

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(g_{cd,b} + g_{db,c} - g_{bc,d} \right), \tag{2.5}$$

which are the metric connection coefficients, and they are also known as the Christoffel symbols of the second kind. Commas denote partial differentiation. To generate the result (2.5) we have utilised the fundamental theorem of Riemannian geometry; a unique metric connection preserves the inner product under parallel transport.

The curvature tensor also called the Riemann-Christoffel curvature tensor or the Riemann tensor $\mathbf{R}$ is defined solely in terms of the metric tensor and its derivatives by

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}. \tag{2.6}$$

From (2.6) we obtain several symmetry properties which reduce the number of its independent components:

$$R_{abcd} = -R_{bacd}, \tag{2.7a}$$

$$R_{abcd} = -R_{abdc}, \tag{2.7b}$$

$$R_{abcd} = R_{cdab}, \tag{2.7c}$$

$$R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0. \tag{2.7d}$$

Note from (2.7) we have the property

$$R^a{}_{abc} = 0. \tag{2.8}$$

In addition we can show that

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0, \tag{2.9}$$

which is the Bianchi identity. Semicolons denote covariant differentiation.

The Ricci tensor is obtained by the contraction

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^d{}_{ab}\Gamma^c{}_{dc} - \Gamma^d{}_{ac}\Gamma^c{}_{db}, \end{aligned} \tag{2.10}$$

which is symmetric. Contraction of the Ricci tensor generates the curvature scalar or Ricci scalar defined by

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab}R_{ab}. \end{aligned} \tag{2.11}$$

The Einstein tensor $\mathbf{G}$ is defined in terms of the Ricci tensor and the curvature scalar by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}. \tag{2.12}$$

With the help of the Bianchi identity (2.9), we can show that

$$G^{ab}{}_{;b} = 0, \tag{2.13}$$

which proves that the Einstein tensor has zero divergence.

2.3 Matter content

The barotropic matter distribution is described as a relativistic fluid. We consider a symmetric matter tensor $\mathbf{M}$, which is the energy momentum tensor for uncharged matter, defined by

$$M^{ab} = (\rho + p)\, u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab}. \tag{2.14}$$

In the above p is the isotropic (kinetic) pressure, ρ is the energy density, q^a is the heat flux vector ($q^a u_a = 0$) and π^{ab} is the anisotropic pressure (or stress) tensor ($\pi^{ab}u_a = 0 = \pi^a{}_a$). All theses quantities are measured relative to a comoving fluid four-velocity $\mathbf{u}$ which is unit and timelike ($u^a u_a = -1$). Another equivalent form of the matter tensor $\mathbf{M}$ is

$$M^{ab} = (\rho + p_t)\, u^a u^b + p_t g^{ab} + q^a u^b + q^b u^a + (p_r - p_t)\, X^a X^b, \tag{2.15}$$

where p_r is the radial pressure and p_t is the tangential pressure. In the above $\mathbf{X}$ is a spacelike vector such that $X^a X_a = 1$ and $X^a u_a = 0$. For isotropic matter distributions $p_r = p_t$.

For a perfect fluid $\pi^{ab} = 0 = q^a$, and (2.14) reduces to

$$M^{ab} = (\rho + p)\, u^a u^b + pg^{ab}. \tag{2.16}$$

For relativistic dust, we have

$$M^{ab} = \rho u^a u^b, \tag{2.17}$$

which describes matter with zero pressure.

It is often required that the matter distribution satisfies a barotropic equation of state

$$p_r = p(\rho), \tag{2.18}$$

so that the pressure depends only on the energy density. Particular equations of state with the fluid energy momentum tensor (2.14) are often considered in applications in astrophysics. A simple equation of state, the γ-law equation, that arises is

$$p_r = \gamma\rho, \tag{2.19}$$

where γ is constant. The restriction $0 \leq \gamma \leq 1$ is required to preserve causality. Also widely used is the polytropic equation of state

$$p_r = k\rho^{\frac{n+1}{n}}, \tag{2.20}$$

where k and n are constants.

We now introduce the electromagnetic field tensor $\mathbf{F}$ which can be expressed in terms of the four-potential $\mathbf{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b}. \tag{2.21}$$

The tensor field $\mathbf{F}$ can be written in terms of the electric field $\mathbf{E} = (E^1, E^2, E^3)$ and the magnetic field $\mathbf{B} = (B^1, B^2, B^3)$. We can write explicitly

$$F^{ab} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}.$$

The contribution of the electromagnetic field tensor $\mathbf{E}$ to the total energy momentum tensor is the following

$$E^{ab} = F^a{}_c F^{cb} - \frac{1}{4} g^{ab} F^{cd} F_{cd}. \tag{2.22}$$

Then the total energy momentum for a charged barotropic relativistic fluid is defined as the sum of (2.14) and (2.22) to give

$$T^{ab} = M^{ab} + E^{ab}, \tag{2.23}$$

where $\mathbf{T}$ is the total energy momentum tensor.

2.4 Field equations

Einstein realised that the metric tensor field $\mathbf{g}$ describing the spacetime geometry is related to the amount of gravitating matter $\mathbf{T}$ present nearby. This insight allowed him to formulate the field equations. The field equations of general relativity relate the spacetime curvature to the matter content. The Einstein field equations can be written explicitly as

$$R^{ab} - \frac{1}{2}Rg^{ab} = M^{ab}, \tag{2.24}$$

for neutral matter. We are using units in which the coupling constant is unity ($\frac{8\pi G}{c^4} = 1$). By using (2.13) and (2.24), we get

$$M^{ab}{}_{;b} = 0, \tag{2.25}$$

which is the conservation equation of energy momentum for uncharged matter.

The Einstein-Maxwell field equations for charged matter can be written as

$$\begin{aligned} R^{ab} - \frac{1}{2}Rg^{ab} &= T^{ab} \\ &= M^{ab} + E^{ab}, && (2.26a) \\ F_{ab;c} + F_{bc;a} + F_{ca;b} &= 0, && (2.26b) \\ F^{ab}{}_{;b} &= J^a. && (2.26c) \end{aligned}$$

In the above $\mathbf{J}$ is the four-current defined by

$$J^a = \sigma u^a, \tag{2.27}$$

where σ is the proper charge density. The conservation equation of total energy momentum for charged matter, using (2.13) and (2.26a), becomes

$$T^{ab}{}_{;b} = 0. \tag{2.28}$$

The system (2.26) reduces to Einsteins equation when $\mathbf{E} = 0$ for uncharged matter. It is important to know that (2.26) is a highly nonlinear system of partial differential equations which dictates the behaviour of gravitating systems with an electromagnetic field. We need to solve the system of equations (2.26) to find an exact solution. A particular approach is to stipulate a functional form for the metric tensor field $\mathbf{g}$ and then attempt to integrate the partial differential equations. Another approach is to stipulate particular forms for the matter distribution and the electromagnetic field on physical grounds and then integrate to find the metric tensor field $\mathbf{g}$. Sometimes researchers impose an equation of state, relating the pressure and the energy density, on the gravitating system. For more information on the field equations the reader is referred to Delgaty and Lake (1998), Misner et al (1973) and Narlikar (2002).

2.5 Basic equations

The line element for a static spherically symmetric spacetime, modelling the interior of the relativistic star, has the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.29}$$

The functions $\lambda(r)$ and $\nu(r)$ correspond to the gravitational potentials. The metric tensor field $\mathbf{g}$ therefore has the form

$$g_{ab} = \mathrm{diag}(-e^{2\nu}, e^{2\lambda}, r^2, r^2\sin^2\theta), \tag{2.30}$$

which is diagonal.

By using the formula (2.5), we calculate the connection coefficients with help of the metric tensor to obtain the nonzero components

$$\begin{aligned}
\Gamma^1{}_{11} &= \lambda', \\
\Gamma^1{}_{22} &= -re^{-2\lambda}, \\
\Gamma^1{}_{33} &= -re^{-2\lambda}\sin^2\theta, \\
\Gamma^1{}_{00} &= \nu' e^{2(\nu-\lambda)}, \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta, \\
\Gamma^2{}_{21} &= \frac{1}{r},
\end{aligned}$$

$$
\begin{aligned}
{\Gamma^3}_{31} &= \frac{1}{r}, \\
{\Gamma^3}_{32} &= \cot\theta, \\
{\Gamma^0}_{10} &= \nu'.
\end{aligned}
$$

On substituting the above connection coefficients in (2.10) we generate

$$
\begin{aligned}
R_{00} &= \left(\nu'' + \nu'^2 - \lambda\nu' + \frac{2\nu'}{r}\right) e^{2(\nu-\lambda)}, &\text{(2.31a)}\\
R_{11} &= -\left(\nu'' + \nu'^2 - \lambda'\nu' - \frac{2\lambda'}{r}\right), &\text{(2.31b)}\\
R_{22} &= -\left(1 + r(\nu' - \lambda')\right) e^{-2\lambda} + 1, &\text{(2.31c)}\\
R_{33} &= \sin^2\theta R_{22}, &\text{(2.31d)}
\end{aligned}
$$

which are the components of the Ricci tensor. From (2.31) and (2.11) we obtain

$$
R = 2\left(\frac{1}{r^2} - \left(\nu'' + \nu'^2 - \lambda'\nu' + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{1}{r^2}\right) e^{-2\lambda}\right), \tag{2.32}
$$

which is the Ricci scalar. With the help of the Ricci tensor components (2.31), the Ricci scalar (2.32) and the definition (2.12), we obtain the nonvanishing components

$$
\begin{aligned}
G^{00} &= \frac{1}{r^2} e^{-2\nu} \left(r(1 - e^{2\lambda})\right)', &\text{(2.33a)}\\
G^{11} &= e^{-2\lambda}\left(-\frac{1}{r^2}(1 - e^{2\lambda}) + \frac{2\nu'}{r} e^{-2\lambda}\right), &\text{(2.33b)}\\
G^{22} &= \frac{1}{r^2} e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \lambda'\nu' - \frac{\lambda'}{r}\right), &\text{(2.33c)}\\
G^{33} &= \frac{1}{\sin^2\theta} G^{22}, &\text{(2.33d)}
\end{aligned}
$$

for the Einstein tensor.

It is possible to establish the form of the energy momentum tensor $\mathbf{T}$ with the comoving four-velocity vector $u^a = e^{-\nu}\delta^a_0$ for the line element for a static spherically symmetric spacetime (2.29). For uncharged fluids ($E^{ab} = 0$) the nonvanishing components are given by

$$
\begin{aligned}
T^{00} &= e^{-2\nu}\rho, &\text{(2.34a)}\\
T^{11} &= e^{-2\lambda} p_r, &\text{(2.34b)}\\
T^{22} &= \frac{1}{r^2} p_t, &\text{(2.34c)}\\
T^{33} &= \frac{1}{\sin^2\theta} T^{22}. &\text{(2.34d)}
\end{aligned}
$$

In the case of a charged fluid, it is convenient to make the choice

$$A_a = (\phi(r), 0, 0, 0), \tag{2.35}$$

for the four-potential $\mathbf{A}$. This generates only one nonzero component of the electromagnetic field tensor defined by

$$F_{01} = -\phi'(r),$$

with the contravariant component

$$F^{01} = e^{-(\nu+\lambda)}E(r).$$

We have defined the quantity $E(r) \equiv e^{-(\nu+\lambda)}\phi'(r)$ which denotes the electric field intensity. Thus the proper charge density is given by

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2 E)'. \tag{2.36}$$

The nonzero components of the energy momentum tensor for a charged fluid can then be written as

$$\begin{aligned}
T^{00} &= e^{-2\nu}\left(\rho + \frac{1}{2}E^2\right), && \text{(2.37a)}\\
T^{11} &= e^{-2\lambda}\left(p_r - \frac{1}{2}E^2\right), && \text{(2.37b)}\\
T^{22} &= \frac{1}{r^2}\left(p_t + \frac{1}{2}E^2\right), && \text{(2.37c)}\\
T^{33} &= \frac{1}{\sin^2\theta}T^{22}, && \text{(2.37d)}
\end{aligned}$$

with anisotropic pressure.

Then we obtain the Einstein-Maxwell system of equations

$$\begin{aligned}
\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' &= \rho + \frac{1}{2}E^2, && \text{(2.38a)}\\
-\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} &= p_r - \frac{1}{2}E^2, && \text{(2.38b)}\\
e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) &= p_t + \frac{1}{2}E^2, && \text{(2.38c)}\\
\sigma &= \frac{1}{r^2}e^{-\lambda}(r^2 E)', && \text{(2.38d)}
\end{aligned}$$

from (2.33) and (2.37). From the law of conservation of energy momentum (2.28) we deduce that

$$\frac{dp_r}{dr} = -\frac{1}{r}\left[2(p_r - p_t) + r(\rho + p_r)\frac{d\nu}{dr}\right] + \frac{E}{r^2}\frac{d}{dr}(r^2E), \tag{2.39}$$

in the presence of charge and anisotropy.

When $E = 0$ the system of equations (2.38) governing the behaviour of the gravitational field is reduced to

$$\begin{aligned} \frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' &= \rho, && (2.40a)\\ -\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} &= p_r, && (2.40b)\\ e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) &= p_t, && (2.40c) \end{aligned}$$

for an uncharged body. The conservation equation (2.39) becomes

$$\frac{dp_r}{dr} = -\frac{1}{r}\left[2(p_r - p_t) + r(\rho + p_r)\frac{d\nu}{dr}\right], \tag{2.41}$$

in the presence of anisotropy.

An equivalent form of the Einstein-Maxwell field equations is obtained if we introduce the transformation

$$\begin{aligned} x &= Cr^2, && (2.42a)\\ Z(x) &= e^{-2\lambda(r)}, && (2.42b)\\ A^2y^2(x) &= e^{2\nu(r)}, && (2.42c) \end{aligned}$$

where A and C are constants. This transformation was first used by Durgapal and Bannerji (1983). The line element (2.29) then has the form

$$ds^2 = -A^2y^2(x)dt^2 + \frac{1}{4CxZ(x)}dx^2 + \frac{x}{C}\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.43}$$

in terms of the variable x. The field equations (2.38) become

$$\begin{aligned} \frac{\rho}{C} &= -2\dot{Z} + \frac{1-Z}{x} - \frac{E^2}{2C}, && (2.44a)\\ \frac{p_r}{C} &= 4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} + \frac{E^2}{2C}, && (2.44b)\\ \frac{p_t}{C} &= 4xZ\frac{\ddot{y}}{y} + (4Z + 2x\dot{Z})\frac{\dot{y}}{y} + \dot{Z} - \frac{E^2}{2C}, && (2.44c) \end{aligned}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}(x\dot{E}+E)^2, \tag{2.44d}$$

in terms of the new variables. The conservation equation is

$$\frac{dp_r}{dx} = -\frac{1}{x}\left[p_r - p_t + x(\rho+p_r)\frac{d\nu}{dx}\right] + \frac{E}{x}\frac{d}{dx}(xE), \tag{2.45}$$

which is equivalent to (2.39).

2.6 Linear equation of state

For a physically realistic relativistic star, we expect that the matter distribution should obey a barotropic equation of state of the form (2.18). For a charged anisotropic matter distribution we consider the linear relationship

$$p_r = \alpha\rho - \beta, \tag{2.46}$$

where α and β are constants.

Then the Einstein-Maxwell equations governing the gravitational behaviour of a charged anisotropic sphere, with a linear equation of state, can be written as

$$\frac{\rho}{C} = \frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{2C}, \tag{2.47a}$$

$$p_r = \alpha\rho - \beta, \tag{2.47b}$$

$$p_t = p_r + \Delta, \tag{2.47c}$$

$$\begin{aligned}\Delta = {} & 4CxZ\frac{\ddot{y}}{y} + 2C\left(x\dot{Z} + \frac{4Z}{1+\alpha}\right)\frac{\dot{y}}{y} + \left(\frac{1+5\alpha}{1+\alpha}\right)C\dot{Z} - \frac{C(1-Z)}{x} \\ & + \frac{2\beta}{1+\alpha},\end{aligned} \tag{2.47d}$$

$$\frac{E^2}{2C} = \frac{1-Z}{x} - \left(\frac{1}{1+\alpha}\right)\left(2\alpha\dot{Z} + 4Z\frac{\dot{y}}{y} + \frac{\beta}{C}\right), \tag{2.47e}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}(x\dot{E}+E)^2. \tag{2.47f}$$

The quantity $\Delta = p_t - p_r$ is called the measure of anisotropy and vanishes for isotropic pressures. The nonlinear system of equations (2.47) consists of six independent equations with eight independent variables y, Z, ρ, p_r, p_t, E, σ and Δ. We need to specify two of the quantities involved in the integration process to solve the system (2.47).

The mass of a gravitating object within the stellar radius, obtained from the system (2.47), is important for comparison with observations. The mass contained within a radius x of the sphere is given by the expression

$$m(x) = \frac{1}{4C^{3/2}} \int_0^x \sqrt{\omega}\rho(\omega)\, d\omega. \tag{2.48}$$

This expression is sometimes called the mass function.

2.7 Physical features

The description of the interior spacetime of stellar bodies in relativistic astrophysics requires exact solutions of the Einstein and Einstein-Maxwell field equations. A number of physical criteria have to be satisfied for a physically acceptable model. The matching between any new interior solution to the exterior spacetime of the star is one of the important physical requirements.

The metric tensor field representing the static exterior gravitational field of a massive spherical object is called the Schwarzchild exterior solution. This exterior metric was first obtained in 1916 and has the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.49}$$

where M is the mass of the object as measured by an observer at infinity. In general relativity, this exterior solution is indispensable for the analysis of physical features including length and time, radar sounding, spectral shift, general particle motion, perihelion advance and bending of light. For more details on these topics, the reader is referred to the texts of D'Inverno (1992) and Wald (1984) and Will (1981). For a static charged spherically symmetric body, the line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.50}$$

where Q is related to the total electric charge of the object. The line element (2.50) is called the Reissner-Nordstrom exterior solution. The Reissner-Nordstrom metric (2.50) reduces to the Schwarzchild metric (2.49) when we set $Q = 0$.

As the systems (2.38) and (2.40) generate a variety of exact solutions, it is necessary to establish conditions on the solutions of the Einstein and Einstein-Maxwell systems of equations for a physically reasonable model. On physical grounds we introduce the following criteria:

- The energy density ρ and radial pressure p_r must be finite and positive from the centre to the boundary of stellar body where the radial pressure vanishes.
- The gradients of the energy density $\frac{d\rho}{dr}$ and radial pressure $\frac{dp_r}{dr}$ are negative.
- The speed of sound satisfies $0 \leq \frac{dp_r}{d\rho} \leq 1$ so that the speed of sound is less than the speed of light.
- The exterior metric must correspond to the Reissner-Nordstrom solution when the distribution of matter is charged, and the exterior metric must be the Schwarzchild solution for neutral matter.
- The electrical field is continuous at the boundary for charged solutions.
- The electrical field intensity E and the metric functions $e^{2\lambda}$, $e^{2\nu}$ are positive and nonsingular everywhere in the interior of stellar body.
- The solutions are stable with respect to radial perturbations.

It should be noted that most solutions that have been found do not satisfy the full set of physical criteria given above. It is desirable to generate exact solutions for gravitating fluid spheres which admit all required physical features. For a comprehensive discussion of the physical properties of general relativistic spheres see Delgaty and Lake (1998).

Chapter 3

New solutions I

3.1 Introduction

In this chapter we study a class of models in the Einstein-Maxwell system and obtain a new category of exact solutions. We generalize the solution to the Einstein-Maxwell system, with no singularity in the charge distribution at the centre, obtained by Mafa Takisa and Maharaj (2013b) by generating a new class of exact solutions. The physical features such as the gravitational potentials, electric field intensity, charge distribution and matter distribution are well behaved. These new exact solutions describe a charged relativistic sphere with anisotropic pressures and a linear equation of state. In §3.2, we present the basic assumptions for the physical models. We make a choice for one of the gravitational potentials and the electric field intensity which allow us to integrate the field equations. Three new classes of exact solutions to the Einstein-Maxwell field equations, in terms of elementary functions, are determined in §3.2. The first class is unphysical and the other two classes satisfy the physical criteria established in §2.7. In §3.3 we demonstrate how our exact solutions regain the uncharged anisotropic solutions found by Thirukkanesh and Maharaj (2008), and the charged anisotropic solution found by Mafa Takisa and Maharaj (2013b). In §3.4, we study the physical features of our models and present graphs for the energy density, radial pressure, electric field intensity, charge density and mass. The graphs indicate that the matter and electromagnetic quantities are well behaved. Finally in §3.5 we generate stellar masses and show that our charged general relativistic solutions are of astrophysical importance.

3.2 Physical models

To generate a new class of solutions to the Einstein-Maxwell system requires a choice for one of the gravitational potentials and electric field intensity. In our approach we make the particular choice

$$
\begin{aligned}
Z &= \frac{1+(a-b)\,x}{1+ax}, && (3.1a)\\
\frac{E^2}{C} &= \frac{la^3x^3+sa^2x^2+k(3+ax)}{(1+ax)^2}, && (3.1b)
\end{aligned}
$$

where a, b, s, k, l are real constants. It is important to note that we keep the same form for the gravitational potential first used by Thirukkanesh and Maharaj (2008). The function Z is finite at the centre $x = 0$ and regular in the interior. The function for the electrical field E is a generalised form that contains previous studies as special cases. When $l = s = 0$ we regain the models of Thirukkanesh and Maharaj (2008). If $l = 0$ then the exact models of Mafa Takisa and Maharaj (2013b) are obtained. The function E is finite at the centre and well behaved in the stellar interior.

On substituting (3.1) into (2.47e) we get the first order equation

$$
\begin{aligned}
\frac{\dot{y}}{y} &= \frac{4\alpha b-(1+\alpha)(la^3x^3+sa^2x^2+k(3+ax))}{8(1+ax)\left[1+(a-b)x\right]}\\
&\quad -\frac{\beta(1+ax)}{4C\left[1+(a-b)x\right]}+\frac{(1+\alpha)b}{4\left[1+(a-b)x\right]}. \qquad (3.2)
\end{aligned}
$$

It is necessary to integrate (3.2) to complete the model of a charged gravitating sphere. It is convenient to categorise our solutions in terms of the constant b. We consider, in turn, the following three cases: $b = 0$, $b = a$, $b \neq a$.

3.2.1 The case $b = 0$

When $b = 0$, equation (3.2) assumes the simple form

$$
\frac{\dot{y}}{y} = -\frac{(1+\alpha)(la^3x^3+sa^2x^2+k(3+ax))}{8\,(1+ax)^2}-\frac{\beta}{4C}. \qquad (3.3)
$$

This gives after integration the solution

$$
y = D\,(1+ax)^{(-(3l+k-2s))(1+\alpha)/(8a)}\exp\left[F(x)\right], \qquad (3.4)
$$

where D is a constant of integration and

$$F(x) = \frac{(1+\alpha)\left(4k - 2asx(2+ax) + l(3 + 9ax + 3a^2x^2 - a^3x^3)\right)}{16(1+ax)} - \frac{\beta x}{4C}. \quad (3.5)$$

The energy density $\rho = -\frac{E^2}{2}$ generated by the potential y in (3.4) is negative and the model is unphysical.

3.2.2 The case $b = a$

When $b = a$, the differential equation (3.2) yields the form

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{4\alpha a - (1+\alpha)(la^3x^3 + sa^2x^2 + k(3+ax))}{8(1+ax)} \\ &\quad - \frac{\beta(1+ax)}{4C} + \frac{(1+\alpha)a}{4}. \end{aligned} \quad (3.6)$$

This equation has solution

$$y = D\,(1+ax)^{(4\alpha a - (s+2k)(1+\alpha) + l(1+\alpha))/(8a)} \exp\left[G(x)\right], \quad (3.7)$$

where D is a constant of integration and the function $G(x)$ is given by

$$\begin{aligned} G(x) &= -\frac{x}{16C}\left[2C(s-k)(1+\alpha) - a(c(sx-4)(1+\alpha) + 2\beta x) - 4\beta\right]x^2 \\ &\quad - \frac{l(1+\alpha)}{48ac}\left(11 + 6ax - 3a^2x^2 + 2a^3x^3\right). \end{aligned} \quad (3.8)$$

The energy density ρ generated by the potential y in (3.7) is nonnegative. Then the complete solution to the Einstein-Maxwell field equations (2.47) can be written as

$$e^{2\lambda} = 1 + ax, \quad (3.9a)$$

$$e^{2\nu} = A^2D^2\,(1+ax)^{(4\alpha a - (s+2k)(1+\alpha) + l(1+\alpha))/(4a)} \exp\left[2G(x)\right], \quad (3.9b)$$

$$\frac{\rho}{C} = \frac{-la^3x^3 - sa^2x^2 + (2a-k)(3+ax)}{2(1+ax)^2}, \quad (3.9c)$$

$$p_r = \alpha\rho - \beta, \quad (3.9d)$$

$$p_t = p_r + \Delta, \quad (3.9e)$$

$$\begin{aligned} \Delta &= \frac{1}{16c(1+ax)^3} \\ &\quad \times \left[C^2(k^2x(3+ax)^2(1+\alpha)^2 + a^2x(a^4l^2x^6(1+\alpha)^2\right. \\ &\quad + 2a^3lx^4(-2+sx)(1+\alpha)^2 \\ &\quad - 4(-3 + (8-9\alpha)\alpha + 4sx(2+\alpha)) \end{aligned}$$

$$
\begin{aligned}
& -4ax(2lx(5+3\alpha)-2(2+3\alpha(1+\alpha)) \\
& +sx(6+\alpha(5+3\alpha))) \\
& +a^2x^2((-2+sx)^2(1+\alpha)^2-4lx(8+\alpha(7+3\alpha)))) \\
& +2k(-24+ax(a^3lx^4(1+\alpha)^2 \\
& +a^2x^2(-2+3lx+sx)(1+\alpha)^2 \\
& -2(12+\alpha(5+9\alpha)) \\
& +ax(3sx(1+\alpha)^2-2(7+9\alpha+6\alpha^2))))) \\
& +4cx(1+ax)^2(3k(1+\alpha)+a^3lx^3(1+\alpha) \\
& +a^2x(-2+sx)(1+\alpha)+a(-4-6\alpha+kx(1+\alpha)))\beta \\
& +4x(1+ax)^4\beta^2\big], \qquad (3.9\text{f})
\end{aligned}
$$

$$
\frac{E^2}{C} = \frac{la^3x^3+sa^2x^2+k(3+ax)}{(1+ax)^2}, \qquad (3.9\text{g})
$$

$$
\frac{\sigma^2}{C} = \frac{c(k(6+ax(3+ax))+a^2x^2(2s(2+ax)+alx(5+3ax)))^2}{x(1+ax)^5(k(3+ax)+a^2x^2(s+alx))}. \qquad (3.9\text{h})
$$

The solution (3.9) is an exact solution of the Einstein-Maxwell system when $a = b$.

We can compute the mass function explicitly from (2.48). In this case we can write the mass function in terms of x by

$$
\begin{aligned}
M(x) = \frac{1}{8C^{3/2}}\Bigg[& \frac{l\left(-6a^3x^3+14a^2x^2-70ax-105\right)x^{1/2}}{15a(1+ax)} \\
& -\frac{5\left(6a(k-2a)x+s(-15-10ax+2a^2x^2)\right)x^{1/2}}{15a(1+ax)} \\
& +\left(\frac{7l-5s}{a^{3/2}}\right)\arctan(\sqrt{ax})\Bigg]. \qquad (3.10)
\end{aligned}
$$

We can regain the mass functions of Thirukkanesh and Maharaj (2008) and Mafa Takisa and Maharaj (2013a) from (3.11) as particular cases in the presence of charge. Note that their models necessarily contain an inverse tangent function. In our model when

$$
l = \frac{5}{7}s, \qquad (3.11)
$$

the inverse tangent function is not present in the expression for $m(x)$.

3.2.3 The case $b \neq a$

The most interesting case is when $b \neq a$. When $b \neq a$, the solution of (3.2) can be written in the form

$$y = D(1+ax)^m\left[1+(a-b)x\right]^n \exp\left[F(x)\right], \tag{3.12}$$

where D is a constant of integration. The constants m, n and the function $F(x)$ are given by

$$\begin{aligned}
n &= -\frac{a^3 l(1+\alpha)}{8b} + \frac{1}{8Cb(a-b)^2}\left[a^2 c((s+2K)(1+\alpha) - 4b\alpha)\right. \\
&\quad +abC(2b(1+5\alpha) - 5k(1+\alpha)) \\
&\quad \left. +b^2(3Ck(1+\alpha) - 2bC(1+3x) + 2\beta)\right], \\
m &= \frac{4b\alpha - (s+2k)(1+\alpha) + l(1+\alpha)}{8b}, \\
F(x) &= -\frac{ax}{16C(a-b)^2}\left[(2C(1+\alpha)(s(a-b)+l(b-2a)))\right. \\
&\quad \left. +(4(a-b)\beta + aCl(1+\alpha)(a-b)x)\right].
\end{aligned}$$

As for the case $a = b$, the energy density ρ generated from (3.12) is nonnegative. Then, we can write the exact solution for the system (2.47) as

$$e^{2\lambda} = \frac{1+ax}{1+(a-b)x}, \tag{3.13a}$$

$$\begin{aligned}
e^{2\nu} &= A^2D^2[1+(a-b)x]^{2n}(1+ax)^{2m} \\
&\quad \times \exp\left[-\frac{ax(2C(1+\alpha)(s(a-b)+l(b-2a)))}{8C(a-b)^2}\right. \\
&\quad \left. -\frac{ax(4(a-b)\beta + aCl(1+\alpha)(a-b)x)}{8C(a-b)^2}\right],
\end{aligned} \tag{3.13b}$$

$$\frac{\rho}{C} = -\frac{(k-2b)(3+ax) + a^2x^2(s+alx)}{2(1+ax)^2}, \tag{3.13c}$$

$$p_r = \alpha\rho - \beta, \tag{3.13d}$$

$$p_t = p_r + \Delta, \tag{3.13e}$$

$$\begin{aligned}
\Delta &= \frac{Ck}{8(1+ax)^3(1+(a-b)x)} \\
&\quad \times\left[-24 + x(a^4lx^4(1+\alpha)^2 - 6b(2+\alpha)(-1+3\alpha)x\right. \\
&\quad +a^3x^3(4(-1+\alpha) + (3l+s)(1+\alpha)^2x) \\
&\quad +a^2x^2(3sx(1+\alpha)^2 + 8(-2+3\alpha) - 2bx(-1+\alpha(4+\alpha))) \\
&\quad \left. +2ax(2(-9+5\alpha) + bx(1-3\alpha(7+2\alpha)))\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{Cx}{16(1+ax)^3(1+(a-b)x)}\\
&\times\big[4b^2(3+ax(4+ax)+12\alpha+6\alpha ax(5+ax)+(3+ax)^2\alpha^2)\\
&+a^4s^2x^4(1+\alpha)^2-8a^3lx^2(5+3\alpha)\\
&+a^4lx^3(a^2lx^3(1+\alpha)^2-32(2+\alpha)-8ax(3+\alpha))\\
&+2a^2sx(-8(2+\alpha)+ax(-8(3+\alpha)+ax(-8+alx^2(1+\alpha)^2)))\\
&-4ab(20\alpha+24\alpha ax)\\
&-4a^2bsx^2(-6+\alpha+3\alpha^2+ax(-1+\alpha)(3+\alpha))\\
&-4a^3bx^2(4\alpha+lx(-8+\alpha(-1+3\alpha)+ax(-5+\alpha^2)))\big]\\
&+\frac{\beta x}{4(1+ax)(1+(a-b)x)}\\
&\times\big[(1+\alpha)(k(3+ax)+a^2x^2(s+alx))\\
&-2b(2+3\alpha+ax(1+\alpha))\big]\\
&+\frac{x(1+ax)\beta^2}{C(1+(a-b)x)}+\frac{k^2x(3+ax)^2(1+\alpha)^2}{16(1+ax)^3(1+(a-b)x)}, \qquad (3.13f)
\end{aligned}
$$

$$
\frac{E^2}{C} = \frac{la^3x^3+sa^2x^2+k(3+ax)}{(1+ax)^2}, \qquad (3.13g)
$$

$$
\begin{aligned}
\frac{\sigma^2}{C} &= (1+(a-b)x)\\
&\times\frac{(k(6+ax(3+ax))+a^2x^2(2s(2+ax)+alx(5+3ax)))^2}{x(1+ax)^5(k(3+ax)+a^2x^2(s+alx))}. \qquad (3.13h)
\end{aligned}
$$

This exact solution of the Einstein-Maxwell equations is similar in structure to that in §3.2.2. However note that (3.13) is a new exact solution.

For this solution the mass function is given by

$$
\begin{aligned}
M(x) &= \frac{1}{8C^{3/2}}\left[\frac{l\left(-6a^3x^3+14a^2x^2-70ax-105\right)x^{1/2}}{15a(1+ax)}\right.\\
&-\frac{5\left(6a(k-2b)x+s(-15-10ax+2a^2x^2)\right)x^{1/2}}{15a(1+ax)}\\
&\left.+\left(\frac{7l-5s}{a^{3/2}}\right)\arctan(\sqrt{ax})\right]. \qquad (3.14)
\end{aligned}
$$

This expression contains previously studied mass functions in the presence of charge. As in §3.2.2 note that when

$$
l=\frac{5}{7}s, \qquad (3.15)
$$

the inverse tangent function is absent.

3.3 Known solutions

When we set $s = l = 0$ in (3.1b), then the model (3.13) becomes

$$
\begin{aligned}
e^{2\lambda} &= \frac{1+ax}{1+(a-b)x}, && (3.16\text{a})\\
e^{2\nu} &= A^2D^2[1+(a-b)x]^{2n}(1+ax)^{2m}\exp\left[-\frac{\alpha\beta x}{2C(a-b)}\right], && (3.16\text{b})\\
\frac{\rho}{C} &= \frac{(2b-k)(3+ax)}{2(1+ax)^2}, && (3.16\text{c})\\
p_r &= \alpha\rho-\beta, && (3.16\text{d})\\
p_t &= p_r+\Delta, && (3.16\text{e})\\
\Delta &= \frac{-bC}{(1+ax)}-\frac{bC(1+5\alpha)}{(1+\alpha)(1+ax)^2}+\frac{2\beta}{1+\alpha}+\frac{Cx[1+(a-b)x]}{(1+ax)}\\
&\quad\times\left[4\left(\frac{a^2m(m-1)}{(1+ax)^2}+\frac{2a(a-b)mn}{(1+ax)[1+(a-b)x]}+\frac{(a-b)^2n(n-1)}{[1+(a-b)x]^2}\right)\right.\\
&\quad\left.-\frac{2a\beta(a(m+n)[1+(a-b)x]-bn)}{(a-b)C(1+ax)[1+(a-b)x]}+\frac{a^2\beta^2}{4C^2(a-b)^2}\right]\\
&\quad-\frac{4[1+ax(2+(a-b)x)]-b(5+\alpha)x}{2(a-b)(1+\alpha)(1+ax)^3[1+(a-b)x]}\\
&\quad\times[-4b^2Cn+a^3x(-4C(m+n)+\beta x)+a^2(4C(m+n)(2bx-1)\\
&\quad+\beta(2-bx)x)+a(-4b^2C(m+n)x+\beta+b(4Cm+8Cn-\beta x))], && (3.16\text{f})\\
& && (3.16\text{g})\\
\frac{E^2}{C} &= \frac{k(3+ax)}{(1+ax)^2}, && (3.16\text{h})\\
\frac{\sigma^2}{C} &= \frac{kC(1+(a-b)x)\left[a^2x^2+3ax+6\right]^2}{x(1+ax)^5(3+ax)}, && (3.16\text{i})
\end{aligned}
$$

where the constants m and n are given by

$$
\begin{aligned}
m &= \frac{4\alpha b-2k(1+\alpha)}{8b},\\
n &= \frac{1}{8bC(a-b)^2}\\
&\quad\times\left[a^2C(2k(1+\alpha)-4\alpha b)-abC(5k(1+\alpha)-2b(1+5\alpha))\right.\\
&\quad\left.+b^2(3kC(1+\alpha)-2bC(1+3\alpha)+2\beta)\right].
\end{aligned}
$$

This system of equations corresponds to a charged anisotropic sphere with a linear equation of state. This particular model was first found by Thirukkanesh and Maharaj (2008).

If we set $l = 0$ in (3.1b) then the term la^3x^3 in the electric field is no longer present. The model (3.13) then becomes

$$
\begin{aligned}
e^{2\lambda} &= \frac{1+ax}{1+(a-b)x}, && (3.17a)\\
e^{2\nu} &= A^2D^2[1+(a-b)x]^{2n}(1+ax)^{2m}\exp\left[-\frac{ax[Cs(1+\alpha)+2\beta]}{4C(a-b)}\right], && (3.17b)\\
\frac{\rho}{C} &= \frac{(2b-k)(3+ax)-sa^2x^2}{2(1+ax)^2}, && (3.17c)\\
p_r &= \alpha\rho-\beta, && (3.17d)\\
p_t &= p_r+\Delta, && (3.17e)\\
\Delta &= \frac{-bC}{(1+ax)}-\frac{bC(1+5\alpha)}{(1+\alpha)(1+ax)^2}+\frac{2\beta}{1+\alpha}+\frac{Cx[1+(a-b)x]}{(1+ax)}\\
&\quad\times\left[4\left(\frac{a^2m(m-1)}{(1+ax)^2}+\frac{2a(a-b)mn}{(1+ax)[1+(a-b)x]}+\frac{(a-b)^2n(n-1)}{[1+(a-b)x]^2}\right)\right.\\
&\quad-\frac{a[Cs(1+\alpha)+2\beta](a(m+n)[1+(a-b)x]-bn)}{(a-b)C(1+ax)[1+(a-b)x]}\\
&\quad\left.+\frac{a^2[Cs(1+\alpha)+2\beta]^2}{16C^2(a-b)^2}\right]-\frac{4[1+ax(2+(a-b)x)]-b(5+\alpha)x}{4(a-b)(1+\alpha)(1+ax)^3[1+(a-b)x]}\\
&\quad\times[-8b^2Cn+a^3x(-8C(m+n)+[Cs(1+\alpha)+2\beta]x)\\
&\quad+a^2(8C(m+n)(2bx-1)+[Cs(1+\alpha)+2\beta](2-bx)x)\\
&\quad+a(-8b^2C(m+n)x+[Cs(1+\alpha)+2\beta]\\
&\quad+b(8Cm+16Cn-[Cs(1+\alpha)+2\beta]x))], && (3.17f)\\
\frac{E^2}{C} &= \frac{sa^2x^2+k(3+ax)}{(1+ax)^2}, && (3.17g)\\
\frac{\sigma^2}{C} &= \frac{C(1+(a-b)x)[k(6+ax(3+ax))+2sa^2x^2(2+ax)]^2}{x(1+ax)^5(k(3+ax)+sa^2x^2)}, && (3.17h)
\end{aligned}
$$

where the constants m and n are equivalent to

$$
\begin{aligned}
m &= \frac{4\alpha b-(1+\alpha)(s+2k)}{8b},\\
n &= \frac{1}{8bC(a-b)^2}\\
&\quad\times\left[a^2C((1+\alpha)(s+2k)-4\alpha b)-abC(5k(1+\alpha)-2b(1+5\alpha))\right.\\
&\quad\left.+b^2(3kC(1+\alpha)-2bC(1+3\alpha)+2\beta)\right].
\end{aligned}
$$

This system of equations describes a model of a charged anisotropic sphere with a barotropic linear equation of state. The charged solution (3.17) was first found by Mafa Takisa (2010) and Mafa Takisa and Maharaj (2013b). Our class of solutions is therefore a generalisation of previously known models. It arises because of the additional term with the constant l added in the electric field intensity.

3.4 Physical models

In this chapter, we have presented a new general model of a relativistic astrophysical star, and integrated a differential equation of first order from the Maxwell-Einstein system of field equations. We can show that this solution is physically reasonable. We utilise the exact solution obtained in §3.2.3 when $a \neq b$ for a graphical analysis. The software package Mathematica (Wolfram 2010) was used to generate plots for the matter variables. We made the choices $a = 2$, $b = 2.5$, $\alpha = 0.33$, $\beta = 0.198$, $C = 1$, $l = 1.5$, $s = 2.5$, $k = 0$ for the various parameters. We have made choices of the parameters similar to that in Mafa Takisa and Maharaj (2013b) so that we can be consistent with their analysis. We generated the following graphical plots:

- Figure 3.1: Energy density.
- Figure 3.2: Radial pressure.
- Figure 3.3: Electric field intensity.
- Figure 3.4: Charge density.
- Figure 3.5: Mass.

Figure 3.1 shows that the energy density ρ is positive, finite and strictly decreasing. The radial pressure p_r in Figure 3.2, expressed in terms of ρ in accordance with the equation of state follows a similar evolution as the density energy. In Figure 3.3 we observe the electric field E initially decreases and then increases after reaching a minimum. The charge density σ is a regular and decreasing function in Figure 3.4. In Figure 3.5, the mass function M is continuous, finite and strictly increasing. In the graph plotted the dashed line corresponds to the case $l = 0$, and the solide line corresponds to the case $l \neq 0$, $s \neq 0$. The overall profiles of the matter variables ρ, p_r, E, σ and M in our investigation are similar to those generated by Mafa Takisa and Maharaj (2013b) when $l = 0$. However in the presence of the additional term in the electric field including the parameter $l \neq 0$, we observe that there is some change in the gradients of the respective profiles. The density ρ, radial pressure p_r and mass M, have lower values when $l \neq 0$. The electric field and charge density have higher values when $l \neq 0$. The effect of parameter l is consequently to enhance and strengthen the effect of the electromagnetic field in the relativistic star.

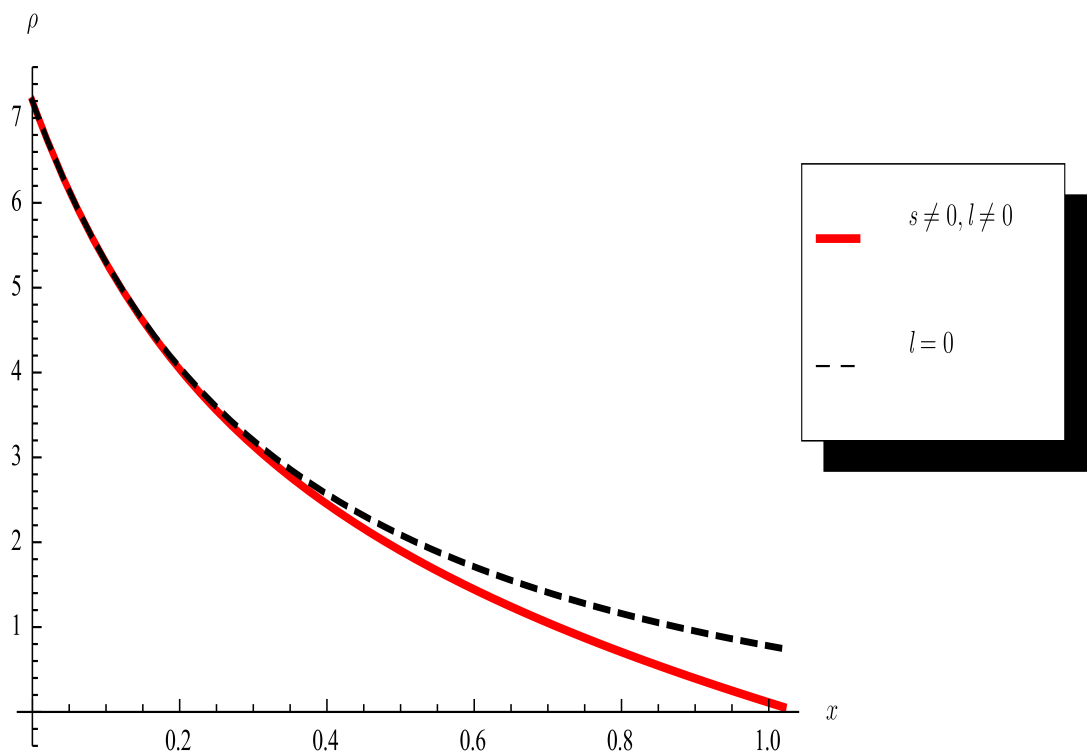

Figure 3.1: Energy density.

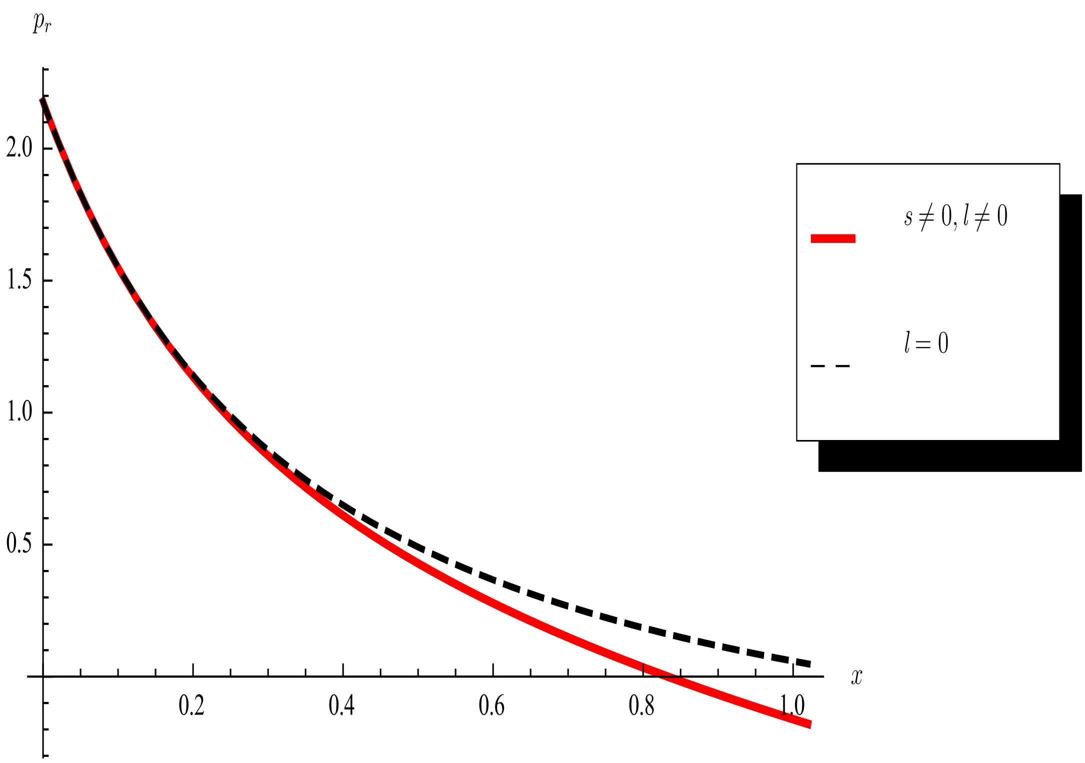

Figure 3.2: Radial pressure.

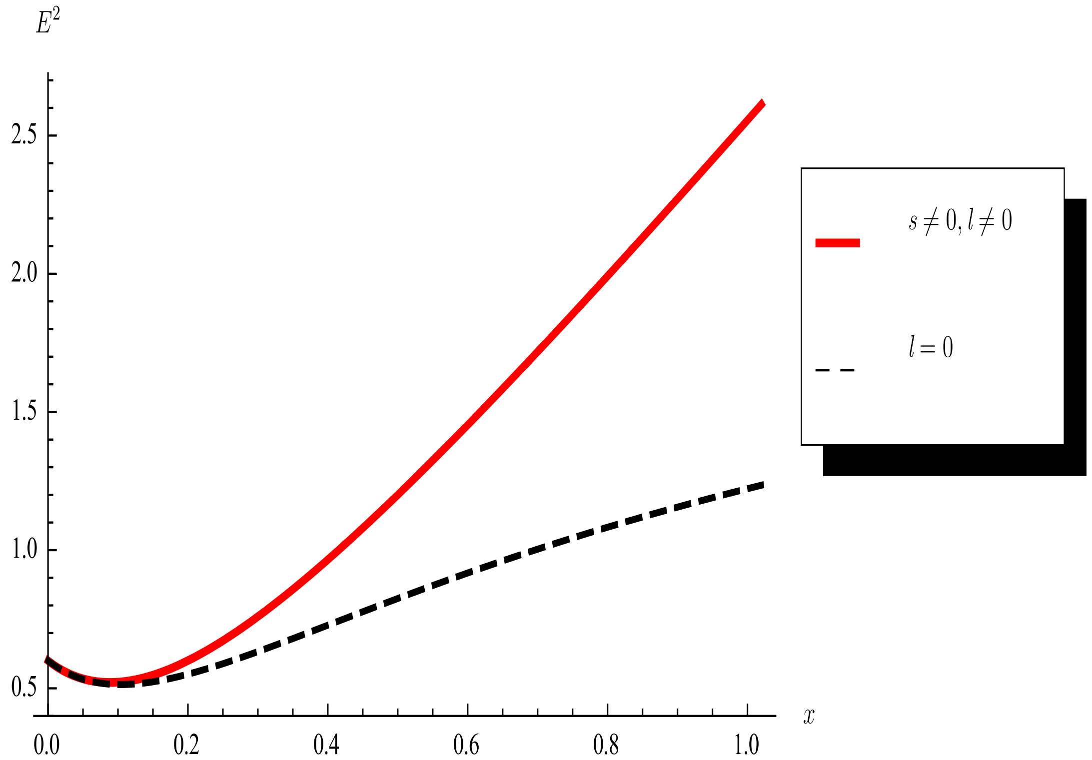

Figure 3.3: Electric field intensity.

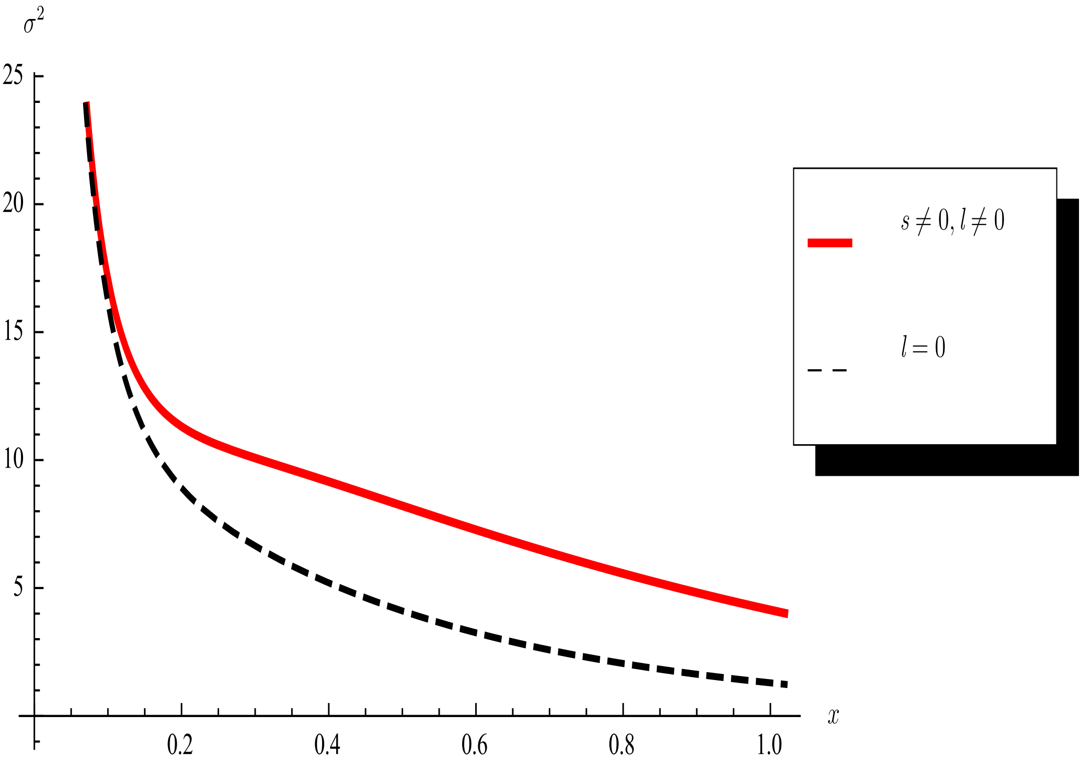

Figure 3.4: Charge density.

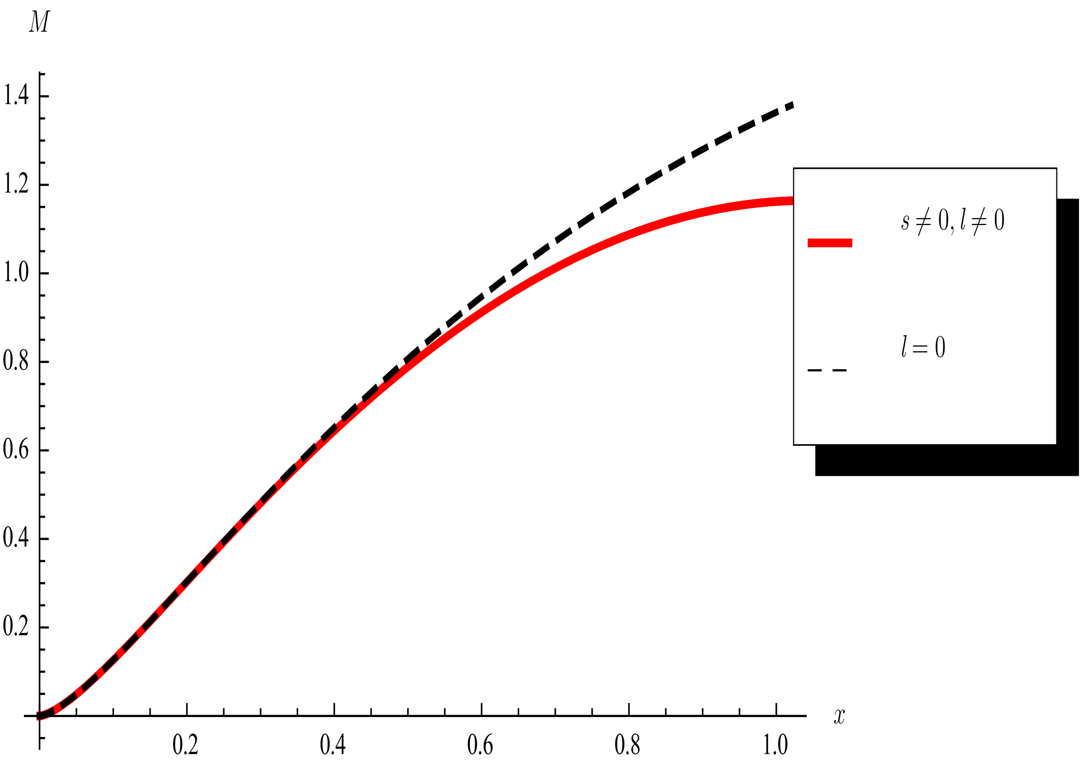

Figure 3.5: Mass function.

3.5 Stellar masses

In this section we generate stellar masses which are consistent with the results of Sharma and Maharaj (2007), Thirukkanesh and Maharaj (2008), and Mafa Takisa and Maharaj (2013b). Therefore the solutions found in this thesis may be applied to realistic stellar astronomical bodies.

We use the transformation

$$\tilde{a} = aR^2, \quad \tilde{b} = bR^2, \quad \tilde{\beta} = \beta R^2, \quad \tilde{k} = kR^2, \quad \tilde{s} = sR^2, \quad \tilde{l} = lR^2. \tag{3.18}$$

This is the same transformation, entended to include the additional parameter l, that was defined by Mafa Takisa and Maharaj (2013b). When $l = 0$ in (3.18) we obtain previous results. With the transformation (3.18) and setting $C = 1$, we can write the energy density (3.13c) and the mass function (3.14) as

$$\rho = \frac{(2\tilde{b} - \tilde{k})(3 + \tilde{a}y) - \tilde{s}\tilde{a}^2y^2 - \tilde{l}\tilde{a}^3y^3}{2R^2(1 + \tilde{a}y)^2}, \tag{3.19a}$$

$$\begin{aligned} M = {} & \frac{\tilde{l}r(-6\tilde{a}^3y^3 + 14\tilde{a}^2y^2 - 70\tilde{a}y - 105)}{120\tilde{a}(1 + \tilde{a}y)} \\ & + \frac{r^3(-3\tilde{k} + 6\tilde{b} + 5\tilde{s})}{12R^2(1 + \tilde{a}y)} + \frac{\tilde{s}r(15 - 2\tilde{a}^2y^2)}{24\tilde{a}(1 + \tilde{a}y)} \\ & + \frac{R(7\tilde{l} - 5\tilde{s})}{8a^{3/2}} \arctan(\sqrt{\tilde{a}y}), \end{aligned} \tag{3.19b}$$

where $y = \frac{r^2}{R^2}$.

If we set $\tilde{l} = 0$, $\tilde{s} \neq 0$, $\tilde{k} \neq 0$, with $E \neq 0$, then (3.19) yields

$$\rho = \frac{(2\tilde{b} - \tilde{k})(3 + \tilde{a}y) - \tilde{s}\tilde{a}^2y^2}{2R^2(1 + \tilde{a}y)^2}, \tag{3.20a}$$

$$\begin{aligned} M = {} & \frac{r^3(-3\tilde{k} + 6\tilde{b} + 5\tilde{s})}{12R^2(1 + \tilde{a}y)} + \frac{\tilde{s}r(15 - 2\tilde{a}^2y^2)}{24\tilde{a}(1 + \tilde{a}y)} \\ & - \frac{5\tilde{s}R}{8a^{3/2}} \arctan(\sqrt{\tilde{a}y}). \end{aligned} \tag{3.20b}$$

These expressions above are related to the solutions in section §3.3 when $l = 0$, found by Mafa Takisa and Maharaj (2013b). If we set $\tilde{l} = 0$, $\tilde{s} = 0$, $\tilde{k} \neq 0$ with $E \neq 0$, we obtain

$$\rho = \frac{(2\tilde{b} - \tilde{k})(3 + \tilde{a}y)}{2R^2(1 + \tilde{a}y)^2}, \tag{3.21a}$$

$$M = \frac{r^3(2\tilde{b} - \tilde{k})}{4R^2(1 + \tilde{a}y)}, \tag{3.21b}$$

which is related to the solutions in §3.3 when $s = l = 0$ found by Thirukkanesh and Maharaj (2008). If we set $\tilde{l} = 0, \tilde{s} = 0, \tilde{k} = 0$ with $E = 0$, we obtain

$$\rho = \frac{\tilde{b}(3 + \tilde{a}y)}{R^2(1 + \tilde{a}y)^2}, \tag{3.22a}$$

$$M = \frac{\tilde{b}r^3}{2R^2(1 + \tilde{a}y)}, \tag{3.22b}$$

which are the solutions attributed to Sharma and Maharaj (2007).

We have produced a variety of stellar masses for particular choices of the parameters $\tilde{a}$ and $\tilde{b}$. These are presented in the following tables:

- Table 3.1: Stellar masses with $l = 0, s = 0$.
- Table 3.2: Stellar masses with $l = 0, s \neq 0$.
- Table 3.3: Stellar masses with $l \neq 0, s \neq 0$.
- Table 3.4: Comparative masses.

We have set $r = 7.07$ km and $R = 43.245$ km which are the values used by Dey et al (1998, 1999). For the electric field we have set $\tilde{k} = 37.403$, $\tilde{s} = 0.137$ and $\tilde{l} = 0.111$.

For $l = 0$, $s = 0$ and $k = 0$ our model reduces to an uncharged stellar body as that in Sharma and Maharaj (2007) in terms of the masses generated. When $l = 0$, $s = 0$ and $k \neq 0$, the masses obtained are similar to that found by Thirukkanesh and Maharaj (2008) for a charged relativistic star model. Note that for $l = 0$, $s \neq 0$ and $k \neq 0$ we find a charged relativistic stellar body and the stellar masses are consistent with the values generated by Mafa Takisa and Maharaj (2013b). To distinguish between the various cases in the presence of the electrical field we let $k \neq 0$, $l = 0$, $s = 0$ in Table 3.1, $k \neq 0$, $l = 0$, $s \neq 0$ in Table 3.2 and $k \neq 0$, $l \neq 0$, $s \neq 0$ in Table 3.3. We have also included the case for $k = 0$, in all tables, so that we can compare with uncharged masses. In Table 3.4 we gather all results and provide a comparative table. It is interesting to observe the effect of the electric field on the masses generated. The introduction of the new parameter l does not appear to appreciably change the mass for the parameter values chosen; a different set of parameters can produce a different profile for the mass.

In particular observe that the parameter values $\tilde{b} = 54.34$ and $\tilde{a} = 53.340$ correspond to the analysis of Dey et al (1998, 1999) for an equation of state with strange matter. The values obtained are consistent with results of observed masses for the X-ray binary pulsar SAX J1808.4-3658. Consequently our new charged general relativistic solutions of the Einstein-Maxwell system are of astrophysical importance.

Table 3.1: Stellar masses with $l = 0, s = 0$

$\tilde{b}$	$\tilde{a}$	$M(M_\odot)$ $k = s = l = 0$ $E = 0$	$M(M_\odot)$ $l = 0, s = 0$ $E \neq 0$
30	23.681	1.175	0.4426
40	36.346	1.298	0.6911
50	48.307	1.396	0.8738
54.34	53.340	1.434	0.9399
60	59.788	1.478	1.0169
70	70.920	1.547	1.1333
80	81.786	1.607	1.2308
90	92.442	1.659	1.3141
100	102.929	1.706	1.3864

Table 3.2: Stellar masses with $l = 0, s \neq 0$

$\tilde{b}$	$\tilde{a}$	$M(M_\odot)$ $k = s = l = 0$ $E = 0$	$M(M_\odot)$ $l = 0, s \neq 0$ $E \neq 0$
30	23.681	1.175	0.4425
40	36.346	1.298	0.6909
50	48.307	1.396	0.8736
54.34	53.340	1.434	0.9396
60	59.788	1.478	1.0165
70	70.920	1.547	1.1329
80	81.786	1.607	1.2303
90	92.442	1.659	1.3136
100	102.929	1.706	1.3860

Table 3.3: Stellar masses with $l \neq 0, s \neq 0$

$\tilde{b}$	$\tilde{a}$	$M(M_\odot)$ $k = s = l = 0$ $E = 0$	$M(M_\odot)$ $l \neq 0, s \neq 0$ $E \neq 0$
30	23.681	1.176	0.4424
40	36.346	1.298	0.6907
50	48.307	1.396	0.8733
54.34	53.340	1.434	0.9393
60	59.788	1.478	1.0162
70	70.920	1.547	1.1324
80	81.786	1.607	1.2297
90	92.442	1.659	1.3129
100	102.929	1.706	1.3850

Table 3.4: Comparative masses

$\tilde{b}$	$\tilde{a}$	$M(M_\odot)$ $k=s=l=0$ $E=0$	$M(M_\odot)$ $l=0, s=0$ $E\neq 0$	$M(M_\odot)$ $l=0, s\neq 0$ $E\neq 0$	$M(M_\odot)$ $(l\neq 0, s\neq 0)$ $E\neq 0$
30	23.681	1.175	0.4426	0.4425	0.4424
40	36.346	1.298	0.6911	0.6909	0.6907
50	48.307	1.396	0.8738	0.8736	0.8733
54.34	53.340	1.434	0.9399	0.9396	0.9393
60	59.788	1.478	1.0169	1.0165	1.0162
70	70.920	1.547	1.1333	1.1329	1.1324
80	81.786	1.607	1.2308	1.2303	1.2297
90	92.442	1.659	1.3141	1.3136	1.3129
100	102.929	1.706	1.3864	1.3860	1.3850

Chapter 4

New solutions II

4.1 Introduction

The Finch and Skea (1989) solution is a model of a highly dense star. It satisfies all physical requirements for a general relativistic stellar configuration and is widely used in the modelling process. Hansraj and Maharaj (2006) found the charged analogue of the Finch-Skea star. In this investigation we extend the Hansraj and Maharaj (2006) model by adding anisotropy to the field equations. We generate the master gravitational equation in §4.2 which is obtained with the help of the Einstein-Maxwell system established in the previous chapter. We make a particular choice for one of the gravitational potentials, the electric field intensity and the anisotropic term. Three classes of solution are possible depending on the quantity $a^2 - \alpha$. In §4.3 we treat the case where $a^2 - \alpha = 0$. In §4.4 we consider the case $a^2 - \alpha > 0$ and we set $a = -1, 1, 3$. For these values of a we find new classes of exact solution to the Einstein-Maxwell system in terms of elementary functions. In §4.5 our study concerns the case $a^2 - \alpha < 0$. As in the previous section we make the choices $a = -1, 1, 3$ and new classes of exact solutions to the Einstein-Maxwell system are obtained in terms of elementary functions. The equation of state is established in §4.6 for a particular model. The other classes of models also admit an equation of state. The physical analysis of the charged anisotropic model is presented in §4.7 with graphs generated for particular parameter values for the electric field.

4.2 The master equation

The Einstein-Maxwell system is given by equations (2.44). From (2.44b) and (2.44c) we can write

$$4xZ\frac{\ddot{y}}{y} + 2x\dot{Z}\frac{\dot{y}}{y} + \dot{Z} - \frac{Z-1}{x} - \frac{E^2}{C} = \frac{\Delta}{C}, \tag{4.1}$$

where $\Delta = p_t - p_r$ is the measure of anisotropy. We can solve the Einstein-Maxwell field equations by choosing specific forms for the gravitational potential Z, the electric field intensity E and anisotropy Δ which are physically reasonable. Therefore we make the choices

$$Z = \frac{1}{1+ax}, \tag{4.2a}$$

$$\frac{E^2}{C} = \frac{(\alpha-\beta)x}{(1+ax)^2}, \tag{4.2b}$$

$$\frac{\Delta}{C} = \frac{\beta x}{(1+ax)^2}, \tag{4.2c}$$

where a, α, β are real constants. The electric field E depends on the real parameters α and β. The form (4.2b) is physically reasonable since E^2 remains regular and positive throughout the sphere if $\alpha > \beta$. In addition the field intensity E becomes zero at the stellar centre and attains a maximum value of $E = \sqrt{(\alpha-\beta)C}/(4a)$ when $r = 1/\sqrt{aC}$. The anisotropy Δ is a decreasing function after reaching a maximum and will have small values close to the stellar boundary. Substitution of (4.2) into (4.1) gives

$$4(1+ax)\ddot{y} - 2a\dot{y} + (a^2-\alpha)y = 0, \tag{4.3}$$

which is the master equation.

There are three categories of solutions in terms of different values of the parameter α. The three cases correspond to

$$a^2-\alpha = 0, \quad a^2-\alpha > 0, \quad a^2-\alpha < 0, \tag{4.4}$$

which generates new models.

4.3 The case $a^2 - \alpha = 0$

With $a^2 - \alpha = 0$, equation (4.3) becomes

$$4(1+ax)\ddot{y} - 2a\dot{y} = 0. \tag{4.5}$$

Equation (4.5) is integrated to give

$$y(x) = \frac{(2+2ax)^{3/2}}{3a}c_1 + c_2, \tag{4.6}$$

where c_1 and c_2 are constants.

The complete solution of the Einstein-Maxwell system is then given by

$$\begin{aligned}
e^{2\lambda} &= 1 + ax, && (4.7a)\\
e^{2\nu} &= A^2\left[\frac{(2+2ax)^{3/2}}{3a}c_1 + c_2\right]^2, && (4.7b)\\
\frac{\rho}{C} &= \frac{6a + 2a^2x + (-\alpha+\beta)x}{2(1+ax)^2}, && (4.7c)\\
\frac{p_r}{C} &= \frac{a}{1+ax} + \frac{(\alpha-\beta)x}{2(1+ax)^2}\\
&\quad + \frac{24ac_1}{4c_1(1+ax)^2 + 3ac_2\sqrt{2(1+ax)}}, && (4.7d)\\
\frac{p_t}{C} &= \frac{-2a + (-\alpha+\beta)x}{2(1+ax)^2}\\
&\quad + \frac{12\sqrt{2}ac_1}{3ac_2\sqrt{1+ax} + 2\sqrt{2}c_1(1+ax)^2}, && (4.7e)\\
\frac{E^2}{C} &= \frac{(\alpha-\beta)x}{(1+ax)^2}, && (4.7f)\\
\frac{\sigma^2}{C} &= \frac{C(\alpha-\beta)(3+ax)^2}{(1+ax)^5}. && (4.7g)
\end{aligned}$$

The line element for this solution (4.7) is given by

$$\begin{aligned}
ds^2 &= -A^2\left[\frac{(2+2ax)^{3/2}}{3a}c_1 + c_2\right]^2 dt^2\\
&\quad + \frac{1+ax}{4Cx}dx^2 + \frac{x}{C}(d\theta^2 + \sin\theta^2 d\phi^2). && (4.8)
\end{aligned}$$

It is interesting to note that when $\alpha = \beta$ then the electric field vanishes and we obtain an uncharged anisotropic model.

4.4 The case $a^2 - \alpha > 0$

When $a^2 - \alpha > 0$ then (4.3) has a more complicated form. However we can transform it to a standard Bessel equation. We can simplify (4.3) with the transformation

$$V = (1+ax)^{\frac{1}{2}}, \tag{4.9a}$$

$$y \;=\; Y(1+ax)^{\frac{2+a}{4}}. \tag{4.9b}$$

Then (4.3) becomes

$$V^2\frac{d^2Y}{dV^2}+V\frac{dY}{dV}+\left(\left(a^2-\alpha\right)V^2-\left(\frac{2+a}{2}\right)^2\right)Y=0, \tag{4.10}$$

where $V=y^{2/(2+a)}Y^{-2/(2+a)}$. Now we use the transformation

$$w=(a^2-\alpha)^{1/2}V, \tag{4.11}$$

to obtain

$$w^2\frac{d^2Y}{dw^2}+w\frac{dY}{dw}+\left(w^2-\left(\frac{a+2}{2}\right)^2\right)Y \;=\; 0, \tag{4.12}$$

which is a Bessel equation of order $\frac{a+2}{2}$. In general the solution of (4.12) is a series. The general solution of (4.12) is a sum of linearly independent Bessel functions $J_{\frac{a+2}{2}}(w)$, of the first kind, and $\mathcal{Y}_{-\frac{a+2}{2}}(w)$, of the second kind, so that

$$Y(w)=b_1J_{\frac{a+2}{2}}(w)+b_2\mathcal{Y}_{-\frac{a+2}{2}}(w), \tag{4.13}$$

and b_1, b_2 are arbitrary constants.

The form of the solution in (4.13) is difficult to use in the modelling process. For specific values of a, when $\frac{a}{2}+1$ is a half-integer, it is possible to write the general solution of (4.13) as a sum of products of Legendre polynomials and trigonometric functions so that elementary functions arise. The solution has a simpler representation when a is an integer. If $a=-1,1,3,...$ then the solution (4.13) can be written as Bessel functions of half-integer order $J_{\frac{1}{2}}$, $J_{-\frac{1}{2}}$, $J_{\frac{3}{2}}$, $J_{-\frac{3}{2}}$, $J_{\frac{5}{2}}$, $J_{-\frac{5}{2}}$,...(Watson 1996). We show that this is possible for the cases $a=-1$, $a=1$, $a=3$.

4.4.1 Model I: $a=-1$

For $a=-1$, the solution (4.13) can be written

$$Y(w)=b_1J_{\frac{1}{2}}(w)+b_2J_{-\frac{1}{2}}(w), \tag{4.14}$$

where

$$J_{\frac{1}{2}}(w) \;=\; \sqrt{\frac{2}{\pi w}}\sin(w), \tag{4.15a}$$

$$J_{-\frac{1}{2}}(w) = -\sqrt{\frac{2}{\pi w}}\cos(w). \tag{4.15b}$$

Then the general solution of (4.3) is given by

$$y(x) = (1-\alpha)^{-1/4}\left[c_1\sin(\sqrt{(1-\alpha)(1-x)}) + c_2\cos(\sqrt{(1-\alpha)(1-x)})\right], \tag{4.16}$$

where $c_1 = \sqrt{\frac{2}{\pi}}b_1$ and $c_2 = -\sqrt{\frac{2}{\pi}}b_2$ are new constants. The complete exact solution of the Einstein-Maxwell system has the form

$$e^{2\lambda} = 1 - x, \tag{4.17a}$$

$$\begin{aligned} e^{2\nu} &= (1-\alpha)^{-1/2}A^2 \\ &\times\left[c_1\sin(\sqrt{(1-\alpha)(1-x)}) + c_2\cos(\sqrt{(1-\alpha)(1-x)})\right]^2, \end{aligned} \tag{4.17b}$$

$$\frac{\rho}{C} = \frac{-6 + x(2+\beta-\alpha)}{2(1-x)^2}, \tag{4.17c}$$

$$\begin{aligned} \frac{p_r}{C} &= \frac{1}{1-x} + \frac{(\alpha-\beta)x}{2(1-x)^2} \\ &-\frac{2(1-\alpha)(c_1 - c_2\tan(\sqrt{(1-\alpha)(1-x)}))}{(1-x)\sqrt{(1-\alpha)(1-x)}(c_2 + c_1\tan(\sqrt{(1-\alpha)(1-x)}))}, \end{aligned} \tag{4.17d}$$

$$\begin{aligned} \frac{p_t}{C} &= \Big[4c_1(-1+\alpha) + c_2\sqrt{(1-\alpha)(1-x)}(2 + x(-2+\alpha+\beta)) \\ &+(c_1\sqrt{(1-\alpha)(1-x)}(2 + x(-2+\alpha+\beta)) \\ &+4c_2(1-\alpha))\tan(\sqrt{(1-\alpha)(1-x)})\Big] \\ &\times\left[2(1-x)^2\sqrt{(1-\alpha)(1-x)}(c_2 + c_1\tan(\sqrt{(1-\alpha)(1-x)}))\right]^{-1}, \end{aligned} \tag{4.17e}$$

$$\frac{E^2}{C} = \frac{(\alpha-\beta)x}{(1-x)^2}, \tag{4.17f}$$

$$\frac{\sigma^2}{C} = \frac{C(\alpha-\beta)(-3+x)^2}{(1-x)^5}. \tag{4.17g}$$

This is a new solution to the Einstein-Maxwell system. The line element for this case is

$$\begin{aligned} ds^2 &= -(\alpha-1)^{-1/2}A^2 \\ &\times\left[c_1\sin(\sqrt{(1-\alpha)(1-x)}) + c_2\cos(\sqrt{(1-\alpha)(1-x)})\right]^2 dt^2 \\ &+\frac{1-x}{4Cx}dx^2 + \frac{x}{C}(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{4.18}$$

4.4.2 Model II: $a = 1$

When $a = 1$ the solution to (4.13) is of the form

$$Y(w) = b_1 J_{\frac{3}{2}}(w) + b_2 J_{-\frac{3}{2}}(w), \tag{4.19}$$

where

$$J_{\frac{3}{2}}(w) = \sqrt{\frac{2}{\pi w}}\left[\frac{\sin(w)}{w} - \cos(w)\right], \tag{4.20a}$$

$$J_{-\frac{3}{2}}(w) = -\sqrt{\frac{2}{\pi w}}\left[\frac{\cos(w)}{w} + \sin(w)\right]. \tag{4.20b}$$

Then the general solution to (4.3) is

$$\begin{aligned} y(x) &= (1-\alpha)^{-3/4}\left[\left(c_2 - c_1\sqrt{(1-\alpha)(1+x)}\right)\cos\left(\sqrt{(1-\alpha)(1+x)}\right)\right. \\ &\quad \left. + \left(c_2\sqrt{(1-\alpha)(1+x)} + c_1\right)\sin\left(\sqrt{(1-\alpha)(1+x)}\right)\right], \end{aligned} \tag{4.21}$$

where we introduced the constants $c_1 = \sqrt{\frac{2}{\pi}} b_1$ and $c_2 = -\sqrt{\frac{2}{\pi}} b_2$. This form of solution is similar to previous studies. With the help of the general solution (4.21), we can write the complete exact charged anisotropic solution of the Einstein-Maxwell system as

$$e^{2\lambda} = 1 + x, \tag{4.22a}$$

$$\begin{aligned} e^{2\nu} &= (1-\alpha)^{-3/2} A^2 \\ &\quad \times \left[\left(c_2 - c_1\sqrt{(1-\alpha)(1+x)}\right)\cos\left(\sqrt{(1-\alpha)(1+x)}\right)\right. \\ &\quad \left. + \left(c_2\sqrt{(1-\alpha)(1+x)} + c_1\right)\sin\left(\sqrt{(1-\alpha)(1+x)}\right)\right]^2, \end{aligned} \tag{4.22b}$$

$$\frac{\rho}{C} = \frac{6 + x(2+\beta-\alpha)}{2(1+x)^2}, \tag{4.22c}$$

$$\begin{aligned} \frac{p_r}{C} &= \left[c_1(2 + (2-\alpha+\beta)x)\sqrt{(1-\alpha)(1+x)}\right. \\ &\quad -c_2(-2 + 4\alpha + (-2+3\alpha+\beta)x) \\ &\quad -c_2(2 + (2-\alpha+\beta)x)\sqrt{(1-\alpha)(1+x)}\tan(\sqrt{(1-\alpha)(1+x)}) \\ &\quad \left.-c_1(-2 + 4\alpha + (-2+3\alpha+\beta)x)\tan(\sqrt{(1-\alpha)(1+x)})\right] \\ &\quad \times \left[2(1+x)^2(c_2 - c_1(\sqrt{(1-\alpha)(1+x)}))\right. \\ &\quad +2c_1(1+x)^2\tan(\sqrt{(1-\alpha)(1+x)}) \\ &\quad \left.+2c_2(1+x)^2\sqrt{(1-\alpha)(1+x)}\tan(\sqrt{(1-\alpha)(1+x)})\right]^{-1}, \end{aligned} \tag{4.22d}$$

$$\frac{p_t}{C} = \left[c_1(2 + (2+\alpha-\beta)x)\sqrt{(1-\alpha)(1+x)}\right.$$

$$
\begin{aligned}
& +c_2(2-4\alpha+(2-3\alpha+\beta)x) \\
& +c_1(2-4\alpha+(2-3\alpha+\beta)x)\tan(\sqrt{(1-\alpha)(1+x)}) \\
& +c_2(-2+(-2-\alpha+\beta)x)\sqrt{(1-\alpha)(1+x)}\tan(\sqrt{(1-\alpha)(1+x)})\Big] \\
& \times\Big[2(1+x)^2(c_2-c_1(\sqrt{(1-\alpha)(1+x)})) \\
& +2c_1(1+x)^2\tan(\sqrt{(1-\alpha)(1+x)}) \\
& +2c_2(1+x)^2\sqrt{(1-\alpha)(1+x)}\tan(\sqrt{(1-\alpha)(1+x)})\Big]^{-1}, \qquad (4.22e)
\end{aligned}
$$

$$
\frac{E^2}{C} = \frac{(\alpha-\beta)x}{(1+x)^2}, \qquad (4.22f)
$$

$$
\frac{\sigma^2}{C} = \frac{C(3+x)^2(\alpha-\beta)}{(1+x)^5}. \qquad (4.22g)
$$

The system (4.22) gives the exact solution of the Einstein-Maxwell system expressed in terms of elementary functions. This is a new solution. We can consider the result (4.22) as a generalisation of the Hansraj and Maharaj (2006) model; when $\beta = 0$ the pressures are isotropic and we regain their model. When $\alpha = 0$ and $\beta = 0$ then we have an uncharged isotropic star which was the model first found by Finch and Skea (1989). We can write the line element in terms of the coordinate x as

$$
\begin{aligned}
ds^2 = & -(1-\alpha)^{-3/2}A^2 \\
& \times\Big[(c_2-c_1\sqrt{(1-\alpha)(1+x)})\cos(\sqrt{(1-\alpha)(1+x)}) \\
& +(c_2\sqrt{(1-\alpha)(1+x)}+c_1)\sin(\sqrt{(1-\alpha)(1+x)})\Big]^2 dt^2 \\
& +\frac{1+x}{4Cx}dx^2+\frac{x}{C}(d\theta^2+\sin^2\theta d\phi^2). \qquad (4.23)
\end{aligned}
$$

The metric (4.23) may be interpreted as the anisotropic, charged generalisation of the Finch and Skea (1989) solution.

4.4.3 Model III: $a = 3$

When $a = 3$ the solution to (4.13) is of the form

$$
Y(w) = b_1 J_{\frac{5}{2}}(w) + b_2 J_{-\frac{5}{2}}(w), \qquad (4.24)
$$

where

$$
J_{\frac{5}{2}}(w) = \sqrt{\frac{2}{\pi w}}\left(\frac{3\sin w}{w^2}-\frac{3\cos w}{w}-\sin w\right), \qquad (4.25a)
$$

$$J_{-\frac{5}{2}}(w) = \sqrt{\frac{2}{\pi w}}\left(-\frac{3\cos w}{w^2} - \frac{3\sin w}{w} + \cos w\right). \tag{4.25b}$$

Then the general solution to (4.3) is

$$\begin{aligned} y(x) &= (9-\alpha)^{-5/4} \\ &\quad \times \Big[(-3c_1\sqrt{(9-\alpha)(1+3x)} \\ &\quad +c_2((9-\alpha)(1+3x)-3))\cos\sqrt{(9-\alpha)(1+3x)} \\ &\quad -(3c_2\sqrt{(9-\alpha)(1+3x)} \\ &\quad +c_1((9-\alpha)(1+3x)-3))\sin\sqrt{(9-\alpha)(1+3x)}\Big], \end{aligned} \tag{4.26}$$

where we have defined $c_1 = a\sqrt{\frac{2}{\pi}}$ and $c_2 = b\sqrt{\frac{2}{\pi}}$ as new constants. The complete exact solution to the Einstein-Maxwell system for this case is thus given by

$$e^{2\lambda} = 1+3x, \tag{4.27a}$$

$$\begin{aligned} e^{2\nu} &= \frac{A^2}{(9-\alpha)^{5/2}} \\ &\quad \times \Big[(-3c_1\sqrt{(9-\alpha)(1+3x)} \\ &\quad +c_2((9-\alpha)(1+3x)-3))\cos\sqrt{(9-\alpha)(1+3x)} \\ &\quad -(3c_2\sqrt{(9-\alpha)(1+3x)} \\ &\quad +c_1((9-\alpha)(1+3x)-3))\sin\sqrt{(9-\alpha)(1+3x)}\Big]^2, \end{aligned} \tag{4.27b}$$

$$\frac{\rho}{C} = \frac{18+(18-\alpha+\beta)\,x}{2(1+3x)^2}, \tag{4.27c}$$

$$\begin{aligned} \frac{p_r}{C} &= 6\,(9-\alpha)\,(1+3x)^{-1} \\ &\quad \times \Big[-c_2 - c_1\sqrt{(9-\alpha)(1+3x)} \\ &\quad +(c_1 - c_2\sqrt{(9-\alpha)(1+3x)})\tan(\sqrt{(9-\alpha)(1+3x)})\Big] \\ &\quad \times \Big[-3c_1\sqrt{(9-\alpha)(1+3x)} + c_2((9-\alpha)(1+3x)-3) \\ &\quad -3c_2\sqrt{(9-\alpha)(1+3x)}\tan(\sqrt{(9-\alpha)(1+3x)}) \\ &\quad -c_1((9-\alpha)(1+3x)-3)\tan(\sqrt{(9-\alpha)(1+3x)})\Big]^{-1} \\ &\quad -\frac{6+(18-\alpha+\beta)x}{12(9-\alpha)(1+3x)}, \end{aligned} \tag{4.27d}$$

$$\begin{aligned} \frac{p_t}{C} &= 6\,(9-\alpha)\,(1+3x)^{-1} \\ &\quad \times \Big[-c_2 - c_1\sqrt{(9-\alpha)(1+3x)} \\ &\quad +(c_1 - c_2\sqrt{(9-\alpha)(1+3x)})\tan(\sqrt{(9-\alpha)(1+3x)})\Big] \end{aligned}$$

$$
\begin{aligned}
& \times\Big[-3c_1\sqrt{(9-\alpha)(1+3x)}+c_2((9-\alpha)(1+3x)-3) \\
& -3c_2\sqrt{(9-\alpha)(1+3x)}\tan(\sqrt{(9-\alpha)(1+3x)}) \\
& -c_1((9-\alpha)(1+3x)-3)\tan(\sqrt{(9-\alpha)(1+3x)})\Big]^{-1} \\
& -\frac{6+(18-\alpha-\beta)x}{12(9-\alpha)(1+3x)}, \qquad (4.27e)
\end{aligned}
$$

$$
\frac{E^2}{C} = \frac{(\alpha-\beta)x}{(1+3x)^2}, \qquad (4.27f)
$$

$$
\frac{\sigma^2}{C} = \frac{9C(\alpha-\beta)(1+x)^2}{(1+3x)^5}. \qquad (4.27g)
$$

This is a new category of exact models for a charged, anisotropic matter distribution. The line element is given by

$$
\begin{aligned}
ds^2 = & -A^2(9-\alpha)^{-5/2} \\
& \times\Big[(-3c_1\sqrt{(9-\alpha)(1+3x)} \\
& +c_2((9-\alpha)(1+3x)-3))\cos\sqrt{(9-\alpha)(1+3x)} \\
& -(3c_2\sqrt{(9-\alpha)(1+3x)} \\
& +c_1((9-\alpha)(1+3x)-3))\sin\sqrt{(9-\alpha)(1+3x)}\Big]^2 dt^2 \\
& +\frac{1+3x}{4Cx}dx^2+\frac{x}{C}(d\theta^2+\sin^2\theta d\phi^2). \qquad (4.28)
\end{aligned}
$$

4.5 The case $a^2-\alpha<0$

We now consider the case $a^2-\alpha<0$ and write the differential equation (4.3) as

$$
4(1+ax)\ddot{y}-2a\dot{y}-(\alpha-a^2)y=0. \qquad (4.29)
$$

Keeping the same transformation (4.9) of §4.4, the equation (4.29) takes the form

$$
V^2\frac{d^2Y}{dV^2}+V\frac{dY}{dV}-\left((\alpha-a^2)V^2+\left(\frac{2+a}{2}\right)^2\right)Y=0, \qquad (4.30)
$$

where $V=y^{2/(2+a)}Y^{-2/(2+a)}$. We cannot use the variable w of §4.4 as $a^2-\alpha<0$. It is important to use a new variable $\tilde{w}$. By taking

$$
\tilde{w}=(\alpha-a^2)^{\frac{1}{2}}V, \qquad (4.31)
$$

equation (4.29) becomes

$$
\tilde{w}^2\frac{d^2Y}{d\tilde{w}^2}+\tilde{w}\frac{dY}{d\tilde{w}}-\left(\tilde{w}^2+\left(\frac{2+a}{2}\right)^2\right)Y=0. \qquad (4.32)
$$

Equation (4.32) is the modified Bessel differential equation of order $\frac{2+a}{2}$. The general solution of (4.32) is a sum of linearly independent modified Bessel functions given by

$$Y(\tilde{w}) = b_1 I_{\frac{a+2}{2}}(\tilde{w}) + b_2 K_{-\frac{a+2}{2}}(\tilde{w}), \tag{4.33}$$

where b_1, b_2 are arbitrary constants. The quantities $I_{\frac{a+2}{2}}(\tilde{w})$, $K_{-\frac{a+2}{2}}(\tilde{w})$ are called modified Bessel functions of the first and second kind respectively. The form of the solution of (4.33) is complicated but can be written in terms of elementary functions when $\frac{a}{2}+1$ is a half-integer. For these parameter values the solution is usually written in terms of hyperbolic functions. For $a = -1, 1, 3, ...$ the solution of (4.32) can be written with the help of modified Bessel functions of half-integer order $I_{\frac{1}{2}}$, $I_{-\frac{1}{2}}$, $I_{\frac{3}{2}}$, $I_{-\frac{3}{2}}$, $I_{\frac{5}{2}}$, $I_{-\frac{5}{2}}$,... We now consider the cases where $a = -1$, $a = 1$ and $a = 3$.

4.5.1 Model I: $a = -1$

When $a = -1$ the solution (4.33) takes the form

$$Y(\tilde{w}) = b_1 I_{\frac{1}{2}}(\tilde{w}) + b_2 I_{-\frac{1}{2}}(\tilde{w}), \tag{4.34}$$

where

$$I_{\frac{1}{2}}(\tilde{w}) = \sqrt{\frac{2}{\pi\tilde{w}}}\sinh(\tilde{w}), \tag{4.35a}$$

$$I_{-\frac{1}{2}}(\tilde{w}) = \sqrt{\frac{2}{\pi\tilde{w}}}\cosh(\tilde{w}). \tag{4.35b}$$

Then the general solution of (4.29) is given by

$$y(x) = (\alpha-1)^{-1/4}\left[c_1\sinh(\sqrt{(\alpha-1)(1-x)}) + c_2\cosh(\sqrt{(\alpha-1)(1-x)})\right], \tag{4.36}$$

where $c_1 = \sqrt{\frac{2}{\pi}}b_1$ and $c_2 = \sqrt{\frac{2}{\pi}}b_2$ are new constants. Then the complete exact solution of the Einstein-Maxwell system is

$$e^{2\lambda} = 1 - x, \tag{4.37a}$$

$$e^{2\nu} = (\alpha-1)^{-1/2}A^2 \times \left[c_1\sinh(\sqrt{(\alpha-1)(1-x)}) + c_2\cosh(\sqrt{(\alpha-1)(1-x)})\right]^2, \tag{4.37b}$$

$$\frac{\rho}{C} = \frac{-6 + x(2+\beta-\alpha)}{2(1-x)^2}, \tag{4.37c}$$

$$
\begin{aligned}
\frac{p_r}{C} &= \frac{1}{1-x} + \frac{(\alpha-\beta)x}{2(1-x)^2} \\
&\quad + \frac{2(1-\alpha)(c_1 + c_2\tanh(\sqrt{(\alpha-1)(1-x)}))}{(1-x)\sqrt{(\alpha-1)(1-x)}(c_2 + c_1\tanh(\sqrt{(\alpha-1)(1-x)}))}, \qquad (4.37d)\\
\frac{p_t}{C} &= \Big[-4c_1(-1+\alpha) + c_2\sqrt{(\alpha-1)(1-x)}(2 + x(-2+\alpha+\beta)) \\
&\quad + (c_1\sqrt{(\alpha-1)(1-x)}(2 + x(-2+\alpha+\beta)) \\
&\quad - 4c_2(-1+\alpha))\tanh(\sqrt{(\alpha-1)(1-x)})\Big] \\
&\quad \times \Big[2(1-x)^2\sqrt{(\alpha-1)(1-x)}(c_2 + c_1\tanh(\sqrt{(\alpha-1)(1-x)}))\Big]^{-1} \qquad (4.37e)\\
\frac{E^2}{C} &= \frac{(\alpha-\beta)x}{(1-x)^2}, \qquad (4.37f)\\
\frac{\sigma^2}{C} &= \frac{C(\alpha-\beta)(-3+x)^2}{(1-x)^5}. \qquad (4.37g)
\end{aligned}
$$

This is a new solution to the Einstein-Maxwell system in terms of hyperbolic functions. The line element for this case is

$$
\begin{aligned}
ds^2 &= -(\alpha-1)^{-1/2}A^2 \\
&\quad \times \Big[c_1\sinh(\sqrt{(\alpha-1)(1-x)}) + c_2\cosh(\sqrt{(\alpha-1)(1-x)})\Big]^2 dt^2 \\
&\quad + \frac{1-x}{4Cx}dx^2 + \frac{x}{C}(d\theta^2 + \sin^2\theta d\phi^2). \qquad (4.38)
\end{aligned}
$$

4.5.2 Model II: $a = 1$

For $a = 1$ the solution (4.33) becomes

$$
Y(\tilde{w}) = b_1 I_{\frac{3}{2}}(\tilde{w}) + b_2 I_{-\frac{3}{2}}(\tilde{w}), \qquad (4.39)
$$

where the modified Bessel functions are given by

$$
\begin{aligned}
I_{\frac{3}{2}}(\tilde{w}) &= \sqrt{\frac{2}{\pi\tilde{w}}}\left[-\frac{\sinh(\tilde{w})}{\tilde{w}} + \cosh(\tilde{w})\right], \qquad (4.40a)\\
I_{-\frac{3}{2}}(\tilde{w}) &= \sqrt{\frac{2}{\pi\tilde{w}}}\left[-\frac{\cosh(\tilde{w})}{\tilde{w}} + \sinh(\tilde{w})\right], \qquad (4.40b)
\end{aligned}
$$

Then the general solution of the equation (4.29) takes the form

$$
\begin{aligned}
y(x) &= (\alpha-1)^{-\frac{3}{4}} \\
&\quad \times \Big[(c_1\sqrt{(\alpha-1)(1+x)} - c_2)\sinh(\sqrt{(\alpha-1)(1+x)}) \\
&\quad + (c_2\sqrt{(\alpha-1)(1+x)} - c_1)\cosh(\sqrt{(\alpha-1)(1+x)})\Big], \qquad (4.41)
\end{aligned}
$$

where $c_1 = b_2\sqrt{\frac{2}{\pi}}$ and $c_2 = b_1\sqrt{\frac{2}{\pi}}$ are new constants. The complete exact solution to the Einstein-Maxwell system for this case can be written as

$$
\begin{aligned}
e^{2\lambda} &= 1 + x, && (4.42a)\\
e^{2\nu} &= (\alpha-1)^{-3/2}A^2 \\
&\quad \times \Big[(c_1\sqrt{(\alpha-1)(1+x)} - c_2)\sinh(\sqrt{(\alpha-1)(1+x)}) \\
&\quad +(c_2\sqrt{(\alpha-1)(1+x)} - c_1)\cosh(\sqrt{(\alpha-1)(1+x)})\Big]^2, && (4.42b)\\
\frac{\rho}{C} &= \frac{6-(\alpha-1)x+(1+\beta)x}{2(1+x)^2}, && (4.42c)\\
\frac{p_r}{C} &= \Big[c_2\sqrt{(\alpha-1)(1+x)}(2+x(2-\alpha+\beta)) \\
&\quad -c_1(-2+4\alpha+x(-2+3\alpha+\beta)) \\
&\quad +(c_1\sqrt{(\alpha-1)(1+x)}(2+x(2-\alpha+\beta)) \\
&\quad -c_2(-2+4\alpha+x(-2+3\alpha+\beta)))\tanh(\sqrt{(\alpha-1)(1+x)})\Big] \\
&\quad \times\Big[2(1+x)^2(c_1-c_2\sqrt{(\alpha-1)(1+x)} \\
&\quad +(c_2-c_1\sqrt{(\alpha-1)(1+x)})\tanh(\sqrt{(\alpha-1)(1+x)}))\Big]^{-1}, && (4.42d)\\
\frac{p_t}{C} &= [c_1(2-4\alpha+x(2-3\alpha+\beta)) \\
&\quad -c_2\sqrt{(\alpha-1)(1+x)}(-2+x(-2+\alpha+\beta)) \\
&\quad +(c_2(2-4\alpha+x(2-3\alpha+\beta)) \\
&\quad -c_1\sqrt{(\alpha-1)(1+x)}(x(-2+\alpha+\beta) \\
&\quad -2))\tanh(\sqrt{(\alpha-1)(1+x)})\Big] \\
&\quad \times\Big[2(1+x)^2(c_1-c_2\sqrt{(\alpha-1)(1+x)} \\
&\quad +(c_2-c_1\sqrt{(\alpha-1)(1+x)})\tanh(\sqrt{(\alpha-1)(1+x)}))\Big]^{-1}, && (4.42e)\\
\frac{E^2}{C} &= \frac{(\alpha-1)x+(1-\beta)x}{(1+ax)^2}, && (4.42f)\\
\frac{\sigma^2}{C} &= \frac{C((1-\beta)+(\alpha-1))(3+ax)^2}{(1+ax)^5}. && (4.42g)
\end{aligned}
$$

Equations (4.42) represent a new solution in terms of hyperbolic functions. This result is a generalisation of the corresponding metric of Hansraj and Maharaj (2006); when $\beta = 0$ the anisotropy vanishes and we regain their model. The line element takes the form

$$ds^2 = -A^2(\alpha-1)^{-\frac{3}{2}}$$

$$\times\Big[(c_1\sqrt{(\alpha-1)(1+x)}-c_2)\sinh(\sqrt{(\alpha-1)(1+x)})$$
$$+(c_2\sqrt{(\alpha-1)(1+x)}-c_1)\cosh(\sqrt{(\alpha-1)(1+x)})\Big]^2 dt^2$$
$$+\frac{1+x}{4Cx}dx^2+\frac{x}{C}(d\theta^2+\sin^2\theta d\phi^2). \qquad (4.43)$$

4.5.3 Model III: $a = 3$

When $a = 3$ we can write the solution (4.33) as

$$Y(\tilde{w}) = b_1 I_{\frac{5}{2}}(\tilde{w}) + b_2 I_{-\frac{5}{2}}(\tilde{w}), \qquad (4.44)$$

where b_1 , b_2 are constants and $I_{\frac{5}{2}}$, $I_{-\frac{5}{2}}$ are modified Bessel functions which may be expressed in terms of hyperbolic functions as

$$I_{\frac{5}{2}}(\tilde{w}) = \sqrt{\frac{2}{\pi\tilde{w}}}\left(\frac{3\sinh(\tilde{w})}{\tilde{w}^2}-\frac{3\cosh(\tilde{w})}{\tilde{w}}+\sinh(\tilde{w})\right), \qquad (4.45a)$$
$$I_{-\frac{5}{2}}(\tilde{w}) = \sqrt{\frac{2}{\pi\tilde{w}}}\left(\frac{3\cosh(\tilde{w})}{\tilde{w}^2}-\frac{3\sinh(\tilde{w})}{w}+\cosh(\tilde{w})\right). \qquad (4.45b)$$

Then the general solution to the differential equation in this case may be written as

$$\begin{aligned} y(x) &= (\alpha-9)^{-5/4} \\ &\times[((3+(\alpha-9)(1+3x))c_1 \\ &-3c_2\sqrt{(\alpha-9)(1+3x)})\sinh(\sqrt{(\alpha-9)(1+3x)}) \\ &+((3+(\alpha-9)(1+3x))c_2 \\ &-3c_1\sqrt{(\alpha-9)(1+3x)})\cosh(\sqrt{(\alpha-9)(1+3x)})\Big], \end{aligned} \qquad (4.46)$$

where $c_1 = b_1\sqrt{\frac{2}{\pi}}$ and $c_2 = b_2\sqrt{\frac{2}{\pi}}$ are new constants. The complete solution to the Einstein-Maxwell equations is given by

$$e^{2\lambda} = 1+3x, \qquad (4.47a)$$
$$\begin{aligned} e^{2\nu} &= A^2(\alpha-9)^{-5/2} \\ &\times[((3+(\alpha-9)(1+3x))c_1 \\ &-3c_2\sqrt{(\alpha-9)(1+3x)})\sinh(\sqrt{(\alpha-9)(1+3x)}) \\ &+((3+(\alpha-9)(1+3x))c_2 \\ &-3c_1\sqrt{(\alpha-9)(1+3x)})\cosh(\sqrt{(\alpha-9)(1+3x)})\Big]^2, \end{aligned} \qquad (4.47b)$$

$$\frac{\rho}{C} = \frac{18+(18-\alpha+\beta)\,x}{2(1+3x)^2}, \tag{4.47c}$$

$$\begin{aligned}
\frac{p_r}{C} &= \frac{-6+(-18+\alpha-\beta)x}{2(1+3x)^2} \\
&\quad \Big[12(1+3x)(\alpha-9)(-c_2+c_1\sqrt{(\alpha-9)(1+3x)} \\
&\quad +(-c_1+c_1\sqrt{(\alpha-9)(1+3x)})\tanh(\sqrt{(\alpha-9)(1+3x)}))\Big] \\
&\quad \times\Big[2(1+3x)^2(-3c_1\sqrt{(\alpha-9)(1+3x)} \\
&\quad +c_2(3+(\alpha-9)(1+3x))+(-3c_2\sqrt{(\alpha-9)(1+3x)} \\
&\quad +c_1(3+(\alpha-9)(1+3x)))\tanh(\sqrt{(\alpha-9)(1+3x)}))\Big]^{-1},
\end{aligned} \tag{4.47d}$$

$$\begin{aligned}
\frac{p_t}{C} &= [-(3c_1(1+3x)(\alpha-9)(-30+4\alpha+x(-54+7\alpha-\beta)) \\
&\quad +c_2\sqrt{(\alpha-9)(1+3x)}(-18(-8+\alpha)+x(-6(-297+\beta) \\
&\quad +\alpha(-354+17\alpha+\beta)+3x(\alpha-9)(-162+17\alpha+\beta)))) \\
&\quad -(3c_2(\alpha-9)(1+3x)(-30+4\alpha+x(-54+7\alpha-\beta)) \\
&\quad +c_1\sqrt{(\alpha-9)(1+3x)}(-18(-8+\alpha)+x(-6(-297+\beta) \\
&\quad +\alpha(-354+17\alpha+\beta) \\
&\quad +3x(\alpha-9)(-162+17\alpha+\beta))))\tanh(\sqrt{(\alpha-9)(1+3x)})\Big] \\
&\quad \times\Big[2(1+3x)^2\sqrt{(\alpha-9)(1+3x)}(3c_1\sqrt{(\alpha-9)(1+3x)} \\
&\quad -c_2(-6+3x(\alpha-9)+\alpha)+(3c_2\sqrt{(\alpha-9)(1+3x)} \\
&\quad -c_1(-6+3x(\alpha-9)+\alpha))\tanh(\sqrt{(\alpha-9)(1+3x)}))\Big]^{-1},
\end{aligned} \tag{4.47e}$$

$$\frac{E^2}{C} = \frac{(9-(9-\alpha)-\beta)\,x}{(1+3x)^2}, \tag{4.47f}$$

$$\frac{\sigma^2}{C} = \frac{9C\,(9-(9-\alpha)-\beta)\,(1+x)^2}{(1+3x)^5}. \tag{4.47g}$$

We have found another class of new solutions which allows for more complex behaviour in the potentials than the earlier cases. For our new solution to the Einstein-Maxwell system given by equations (4.47), the line element has the form

$$\begin{aligned}
ds^2 &= -A^2(\alpha-9)^{-5/2} \\
&\quad \times[((3+(\alpha-9)(1+3x))c_1 \\
&\quad -3c_2\sqrt{(\alpha-9)(1+3x)})\sinh(\sqrt{(\alpha-9)(1+3x)}) \\
&\quad +((3+(\alpha-9)(1+3x))c_2
\end{aligned}$$

$$
\begin{aligned}
&-3c_1\sqrt{(\alpha-9)(1+3x)})\cosh(\sqrt{(\alpha-9)(1+3x)})\Big]^2 dt^2 \\
&+\frac{1+3x}{4Cx}dx^2+\frac{x}{C}(d\theta^2+\sin^2\theta d\phi^2). \qquad (4.48)
\end{aligned}
$$

4.6 Equation of state

An equation of state relating the radial pressure p_r to the energy density ρ is a desirable physical feature in a relativistic stellar model. The expressions for the radial pressure p_r are complicated but all the models found in this chapter admit an equation of state. We illustrate this for the model found in §4.4.2. From equation (4.22c) in §4.4.2 we can establish the expression

$$
x^2+\frac{4\rho-C(2+\beta-\alpha)}{2\rho}x+\frac{\rho-3C}{\rho}=0. \qquad (4.49)
$$

To solve this equation, with distinct real roots, we impose the following condition, where the discriminant of the quadratic equation (4.49) is positive. We have that

$$
\left(\frac{C}{2\rho}(2+\beta-\alpha)\right)^2+\frac{2C}{\rho}(4-\beta+\alpha) \;>\; 0. \qquad (4.50)
$$

Hence the variable x is written in terms of ρ as

$$
x \;=\; -\left[1-\frac{C}{4\rho}(2+\beta-\alpha)\right]+\frac{1}{2}\sqrt{\left(\frac{C}{2\rho}(2+\beta-\alpha)\right)^2+\frac{2C}{\rho}(4-\beta+\alpha)}. \qquad (4.51)
$$

Then from (4.22d) we can write p_r as a function of ρ. Therefore, we have an equation of state of the form

$$
\begin{aligned}
\frac{p_r}{C} \;=\; & [-c_2(-2+4\alpha+(-2+3\alpha+\beta)(-1+F(\rho))) \\
& +c_1(2+(2-\alpha+\beta)(-1+F(\rho)))\sqrt{(1-\alpha)F(\rho)} \\
& -(c_1(-2+4\alpha+(-2+3\alpha+\beta)(-1+F(\rho))) \\
& +c_2(2+(2-\alpha+\beta)(-1+F(\rho)))\sqrt{(1-\alpha)F(\rho)})\tan(\sqrt{(1-\alpha)F(\rho)})\Big] \\
& \times\Big[2(F(\rho))^2(c_2-c_1\sqrt{(1-\alpha)F(\rho)} \\
& +(c_1+c_2\sqrt{(1-\alpha)F(\rho)})\tan(\sqrt{(1-\alpha)F(\rho)}))\Big]^{-1}, \qquad (4.52)
\end{aligned}
$$

where we have set,

$$
F(\rho) \;=\; \frac{1}{2}\sqrt{\left(\frac{C}{2\rho}(2-\alpha+\beta)\right)^2+\frac{2C}{\rho}(4+\alpha-\beta)}+\frac{C}{4\rho}(2-\alpha+\beta).
$$

Consequently the model in §4.4.2 has an equation of state of the general form

$$p_r = p_r(\rho), \tag{4.53}$$

which is barotropic.

Another quantity of physical interest is the speed of sound $\frac{dp_r}{d\rho}$. With the help of (4.52), (4.22d) and (2.42), the expression for the speed of sound in terms of the radial coordinate r becomes

$$\begin{aligned}
\frac{dp_r}{d\rho} &= \frac{2+\alpha-\beta+(2-\alpha+\beta)cr^2}{-10-\alpha+\beta+cr^2(-2-\alpha+\beta)} \\
&\quad -\frac{1}{-10-\alpha+\beta+cr^2(-2-\alpha+\beta)} \\
&\quad \times\Big[(-1+\alpha)(1+cr^2)(2(-2+cr^2(-1+\alpha)+\alpha)(1+\gamma^2) \\
&\quad +2(-1+\gamma^2)\cos(2\sqrt{(1-\alpha)(1+cr^2)}) \\
&\quad +6\gamma\sqrt{(1-\alpha)(1+cr^2)}\cos(2\sqrt{(1-\alpha)(1+cr^2)}) \\
&\quad +(-3\sqrt{(1-\alpha)(1+cr^2)}\sin(2\sqrt{(1-\alpha)(1+cr^2)}) \\
&\quad +\gamma(-4+3\gamma\sqrt{(1-\alpha)(1+cr^2)}))\sin(2\sqrt{(1-\alpha)(1+cr^2)}))\Big] \\
&\quad \times\Big[(-1+\gamma\sqrt{(1-\alpha)(1+cr^2)})\cos(\sqrt{(1-\alpha)(1+cr^2)}) \\
&\quad -(\sqrt{(1-\alpha)(1+cr^2)}+\gamma)\sin(\sqrt{(1-\alpha)(1+cr^2)})\Big]^{-2},
\end{aligned} \tag{4.54}$$

where $\gamma = \frac{c_1}{c_2}$ is a constant. This expression is complicated but it is interesting to note that it is possible to find an analytic expression for the speed of sound. Graphical plots can be generated for $\frac{dp_r}{d\rho}$ as we show in the next section.

4.7 Physical models

Some brief comments about the physical features of the new solutions to the Einstein-Maxwell system are made in this section. In this chapter, we have presented several new models for a relativistic astrophysical star. The underlying equation was the Bessel differential equation which governs the solution of the Maxwell-Einstein system of field equations. The solutions found have matter variables which are regular and well behaved in the interior of the star. As an example we show in this section that the exact solutions found in §4.4.2 are physically reasonable. The matter variables are plotted graphically. The software package Mathematica (Wolfram 2010) was used for the plots with the choice of parameters $a = 1$, $c_1 = 1$, $\alpha = 1/2$, $\gamma = 5$, $C = 1$, $\beta = 1/5$, $c_2 = 1/5$. The following plots were generated:

- Figure 4.1: Energy density.
- Figure 4.2: Radial and Tangential pressure.
- Figure 4.3: Electric field.
- Figure 4.4: Charge density.
- Figure 4.5: Mass function.
- Figure 4.6: Equations of state.
- Figure 4.7: Speed of sound.

Figure 4.1 shows that the density of energy ρ is positive, finite and strictly decreasing. In Figure 4.2 we see that both the tangential and radial pressures are positive and monotonically decreasing functions. In Figure 4.3 the electric field is positive and monotonically increasing and attains a maximum value when $r = 1$. The evolution of the charge density in Figure 4.4 is a decreasing function which is continuous. The mass function is an increasing function with increasing radius in Figure 4.5. We observe that the anisotropy does not affect appreciably the behaviour of mass and for these three cases. We plotted the equation of state for different parameter values in Figure 4.6. We find that the parameter of anisotropy β influences the evolution of the equation of state. In Figure 4.7 the speed of sound satisfies the causality principle $0 \leq \frac{dp_r}{d\rho} \leq 1$ and the speed of sound is less than the speed of light. The plots generated indicate that models found in this chapter are physically reasonable. A detailed study of the physical features such as the luminosity and the relationship to observed astronomical objects will be carried out in future work. Recent treatments of charged relativistic objects have been carried out by Mafa Takisa and Maharaj (2013a, 2013b, 2014a, 2014b), Maharaj and Mafa Takisa (2012), Maharaj et al (2014) and Sunzu et al (2014a, 2014b). In these analyzes the metric was written in terms of Schwarzchild coordinates. Ngubelanga et al (2015a, 2015b) used isotropic coordinates to generate exact solutions to the Einstein-Maxwell system.

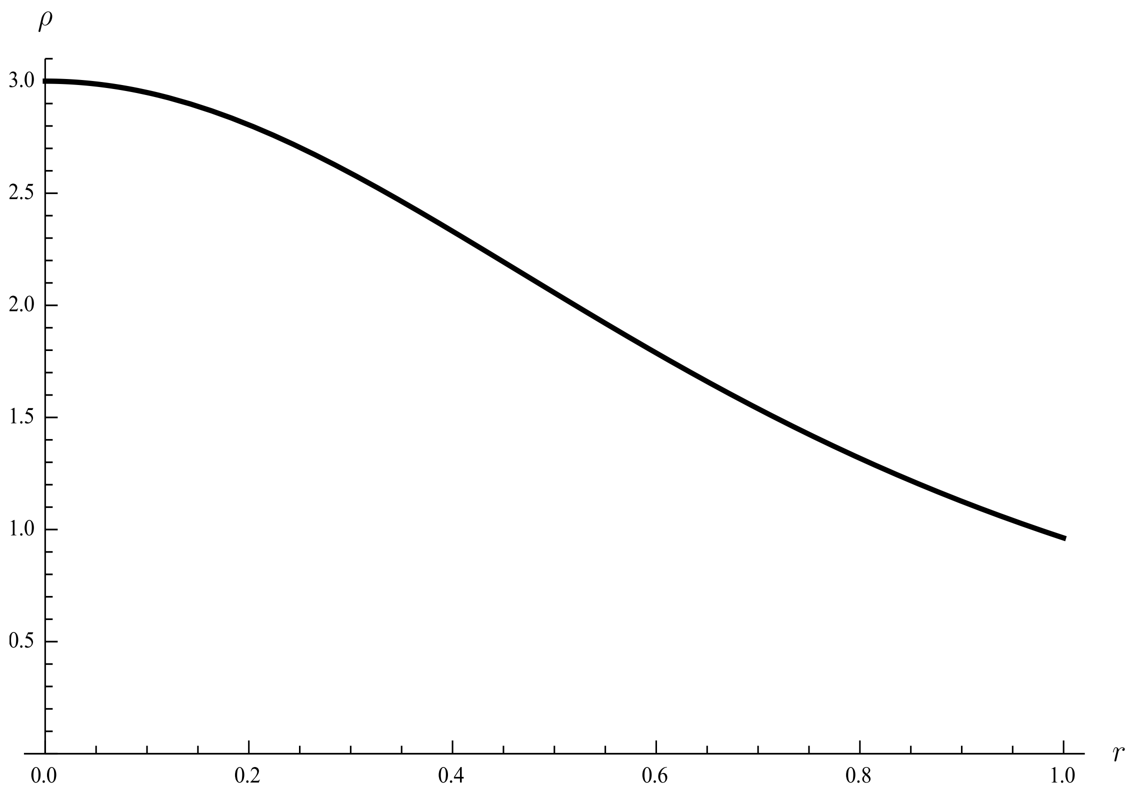

Figure 4.1: Energy density

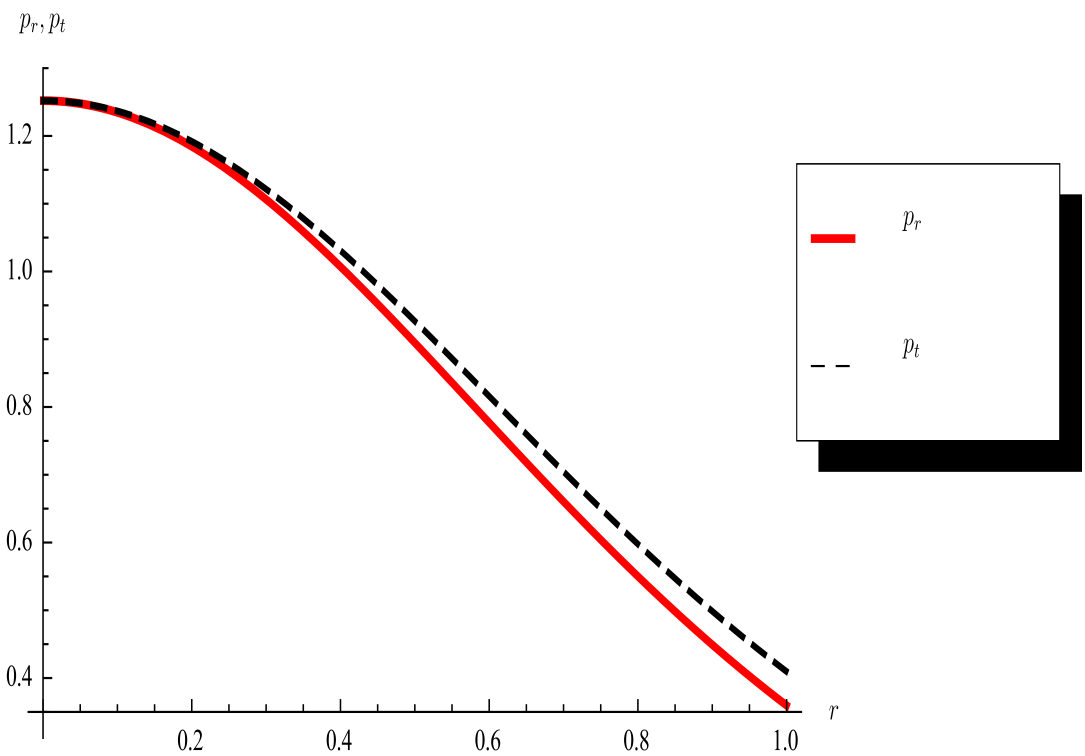

Figure 4.2: Radial and tangential pressures

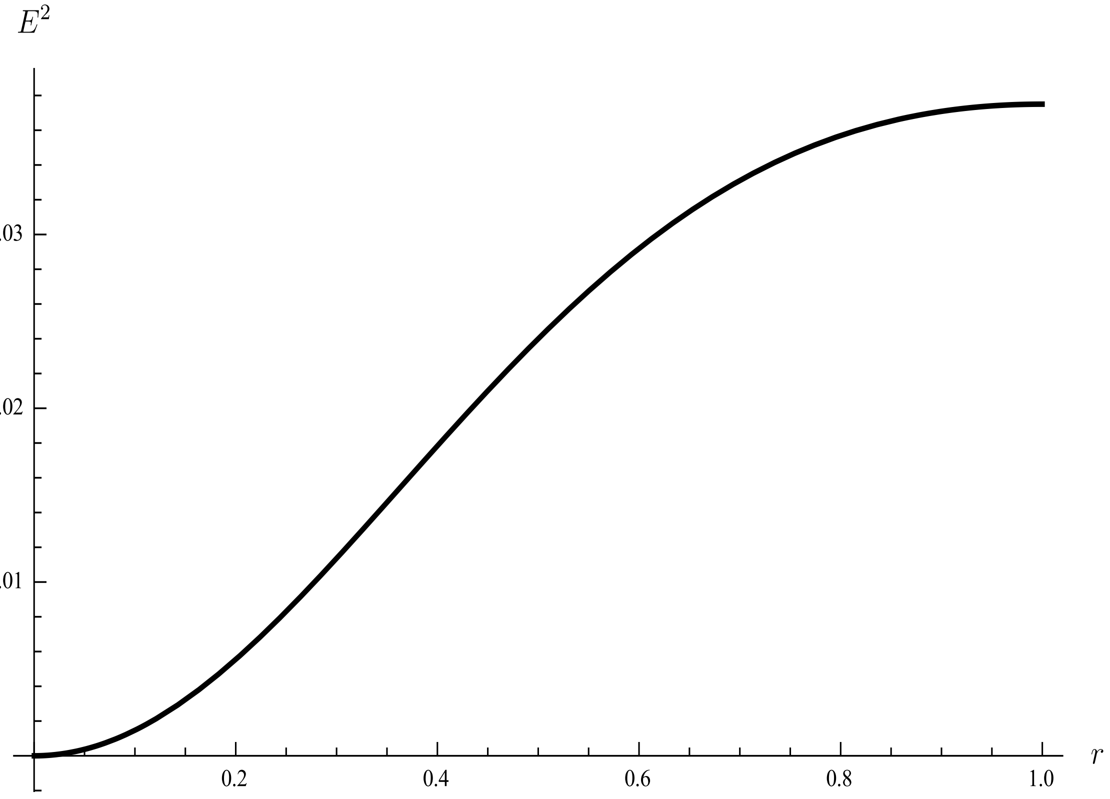

Figure 4.3: Electric field intensity

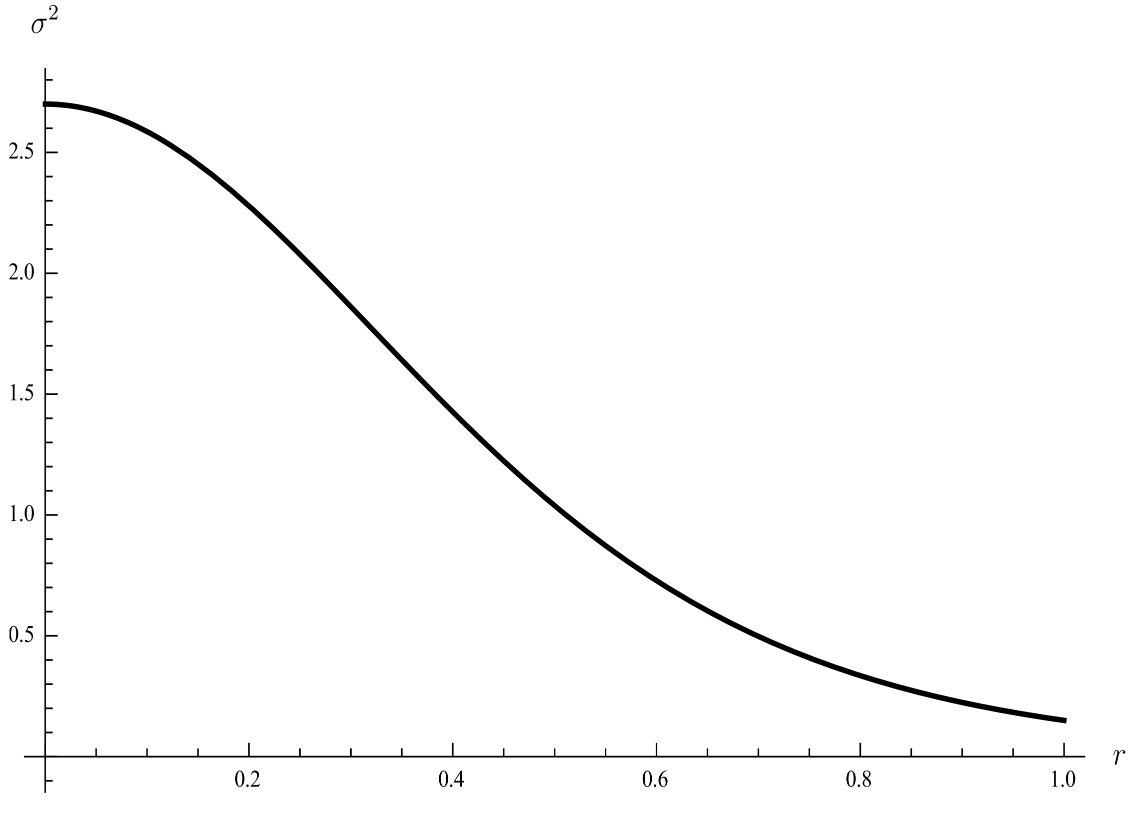

Figure 4.4: Charge density

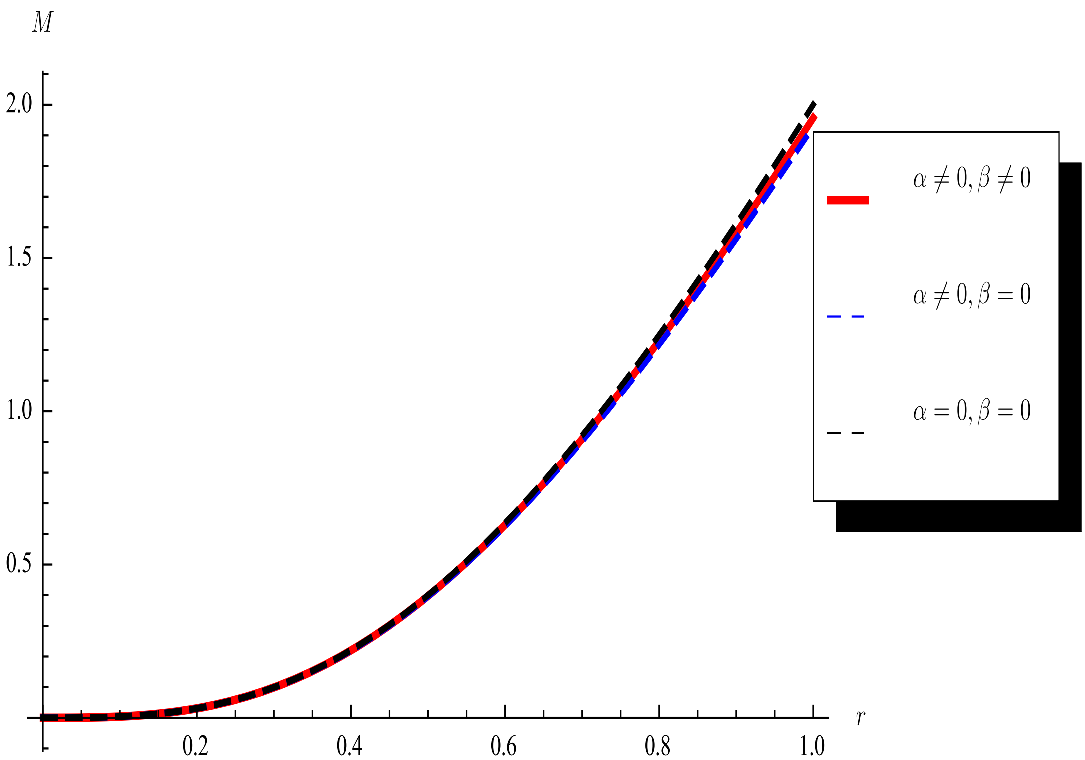

Figure 4.5: Mass function

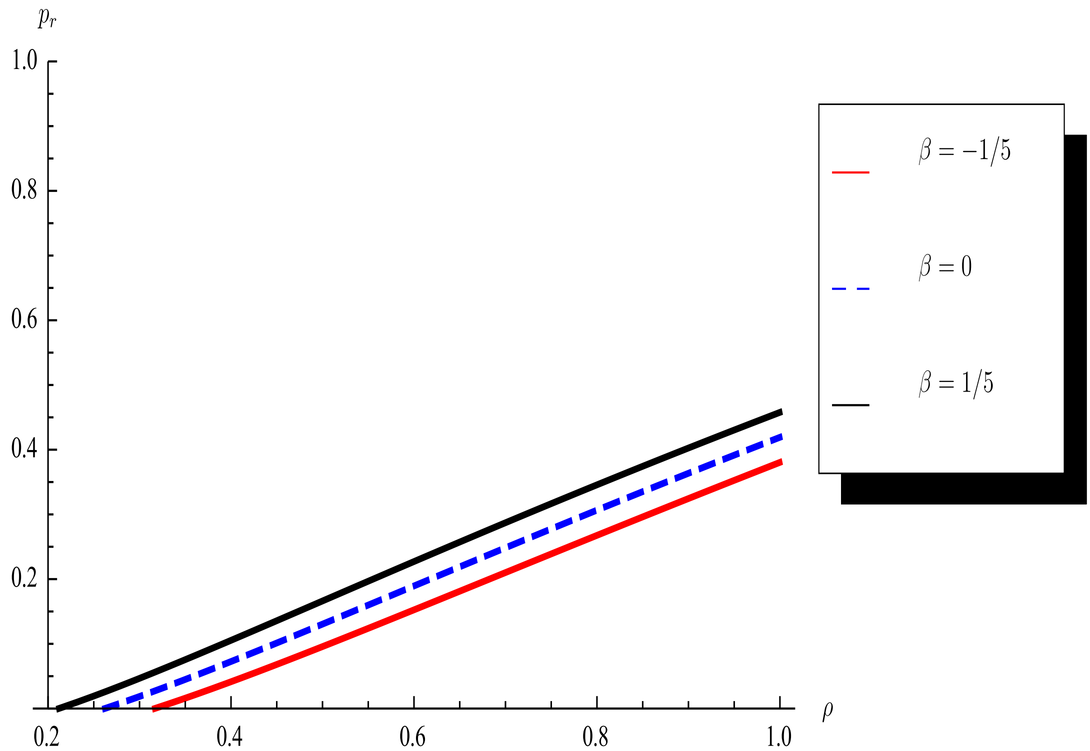

Figure 4.6: Equations of state.

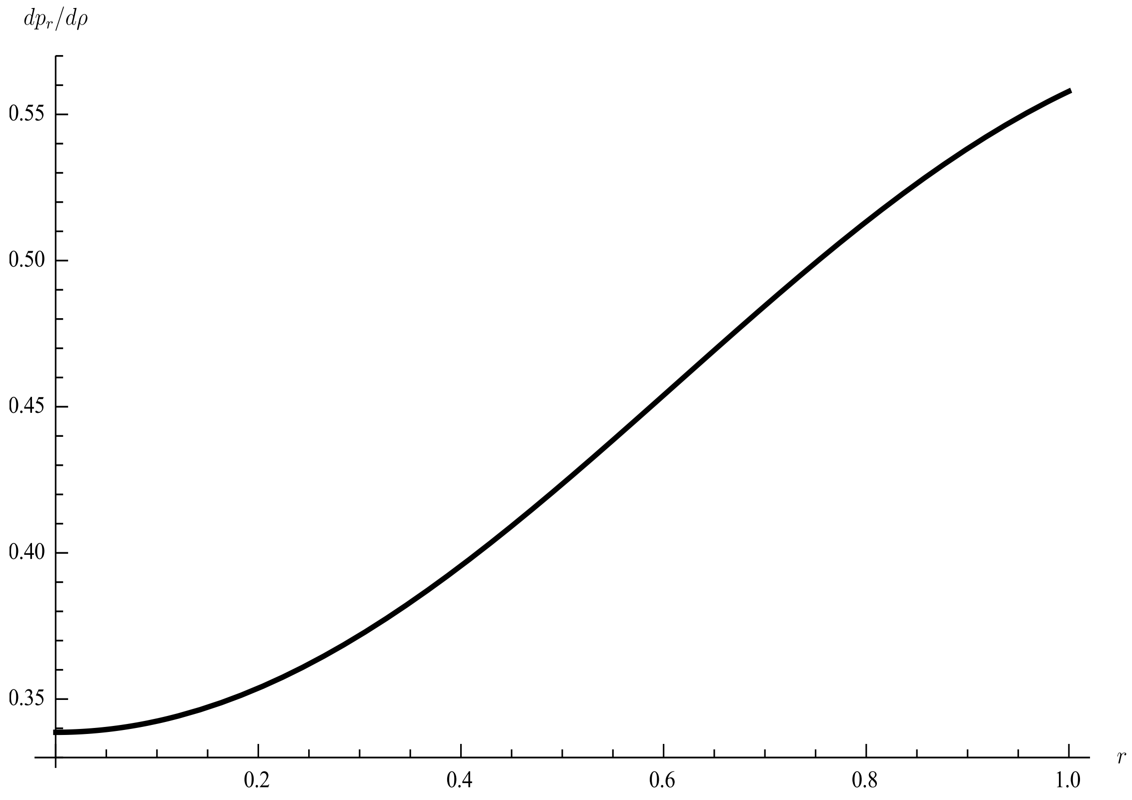

Figure 4.7: Speed of sound.

Chapter 5

Conclusion

The work presented in this thesis is incorporated within the framework of investigation into new exact solutions to the Einstein-Maxwell field equations. Our models are characterised by including the physical effects of the electric field and anisotropy. Two new classes of exact solutions are found by following two different approaches to integrating the field equations. In the first approach we impose a barotropic equation of state. The form of the equation of state chosen between the radial pressure and the energy density is linear. In the second approach the anisotropy leads to a new solution of the field equations. The electric field depends on the parameter describing the anisotropy. These new classes of solutions to the Einstein-Maxwell system of equations obtained in this thesis are physically reasonable in an astrophysical context and serve to model relativistic stars.

The outline of this thesis is the following:

- In chapter 1 we made introductory comments.
- In chapter 2, we reviewed differential geometry applied in general relativity. Our models admit uncharged and charged matter and the line element is static and spherically symmetric. The Einstein field equations for uncharged matter and the Einstein-Maxwell equations for charged fluids were presented. We gave the physical criteria required for a relativistic stellar model.
- In chapter 3 we made the choices

$$Z = \frac{1 + (a-b)\,x}{1 + ax},$$

$$\frac{E^2}{C} = \frac{la^3x^3 + sa^2x^2 + k(3+ax)}{(1+ax)^2},$$

for one of the gravitational potentials and the electric field. These functional forms are a generalisation of models previously studied. We presented new exact solutions to the Einstein-Maxwell equations and showed that they contain the earlier model of Mafa Takisa and Maharaj (2013b). We explicitly regained their metric functions

$$\begin{aligned} e^{2\lambda} &= \frac{1+ax}{1+(a-b)x}, \\ e^{2\nu} &= A^2D^2[1+(a-b)x]^{2n}(1+ax)^{2m} \exp\left[-\frac{ax[Cs(1+\alpha)+2\beta]}{4C(a-b)}\right], \end{aligned}$$

and we studied the physical features and plotted the matter and electrical variables. A comparative table of masses for uncharged and charged matter was established which corresponds to observed astronomical objects. The parameter l does not appear to appreciably change the mass for the parameter values chosen but a different set of parameters can give a different profile for the mass.

- In chapter 4 our choices have the form

$$\begin{aligned} Z &= \frac{1}{1+ax}, \\ \frac{E^2}{C} &= \frac{(\alpha-\beta)x}{(1+ax)^2}, \\ \frac{\Delta}{C} &= \frac{\beta x}{(1+ax)^2}, \end{aligned}$$

for one potential Z, and the electric field and anisotropy. We showed that the underlying equation was a Bessel equation which admits solutions in terms of elementary functions. Several classes of new solutions to the Einstein-Maxwell system were presented for $a = -1, 1, 3$. In particular we generated the line element

$$\begin{aligned} ds^2 &= -(1-\alpha)^{-3/2}A^2 \\ &\times \Big[(c_2 - c_1\sqrt{(1-\alpha)(1+x)})\cos(\sqrt{(1-\alpha)(1+x)}) \\ &+(c_2\sqrt{(1-\alpha)(1+x)} + c_1)\sin(\sqrt{(1-\alpha)(1+x)})\Big]^2 dt^2 \\ &+\frac{1+x}{4Cx}dx^2 + \frac{x}{C}(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned}$$

which contains the Finch and Skea (1989) model for a neutron star and the Hansraj and Maharaj (2006) model for a charged star. These new models satisfy an equation of state in general. A physical analysis is performed for one of new metrics and we show that it is well behaved.

- Our concluding remarks are made in this conclusion.

In summary, two new families of charged anisotropic models have been found. The first new class of exact solutions to the Einstein-Maxwell system of equations contains the results of Mafa Takisa and Maharaj (2013b) and Thirukkanesh and Maharaj (2008). The second new class of exact solutions generalises the corresponding model of Hansraj and Maharaj (2006) and the Finch and Skea (1989) model. Other more interesting cases are contained in this new class which will be the subject of a detailed physical analysis.

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