Semiperfect CFPF Rings

by

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To my wife Emelia and my daughters Carmelle, Larissa and Teri.

ABSTRACT

The Wedderburn-Artin Theorem (1927) characterised semisimple Artinian rings as finite direct products of matrix rings over division rings. In attempting to generalise Wedderburn's theorem, the natural starting point will be to assume R/RadR is semisimple Artinian. Such rings are called semilocal. They have not been completely characterised to date. If additional conditions are imposed on the radical then more is known about the structure of R. Semiprimary and perfect rings are those rings in which the radical is nilpotent and T-nilpotent respectively. In both these cases the radical is nil, and in rings in which the radical is nil, idempotents lift modulo the radical. Rings which have the latter property are called semiperfect. The characterisation problem of such rings has received much attention in the last few decades.

We study semiperfect rings with a somewhat strong condition arising out of the status of generators in the module categories. More specifically, a ring R is CFPF iff every homomorphic image of R has the property that every finitely generated faithful module over it generates the corresponding module category.

The objective of this thesis is to develop the theory that leads to the complete characterisation of semiperfect right CFPF rings. It will be shown (Theorem 6.3.17) that these rings are precisely finite products of full matrix rings over right duo right VR right σ-cyclic right CFPF rings.

As far as possible theorems proved in Lambek [16] or Fuller and Anderson [12] have not been reproved in this thesis and these texts will serve as basic reference texts.

The basis for this thesis was inspired by results contained in the first two chapters of the excellent LMS publication "FPF Ring Theory" by Carl Faith and Stanley Page [11]. Its results can be traced to the works of G. Azumaya [23], K. Morita [18], Nakayama [20;21], H. Bass [4;5], Carl Faith [8;9;10], S. Page [24;25] and B. Osofsky [22]. Our task is to bring the researcher to the frontiers of FPF ring theory, not so much to present anything new.

A C K N O W L E D G E M E N T S

This thesis could not have been submitted without the assistance of many people.

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C O N T E N T S

NOTATION AND ABBREVIATIONS

Symbol	Description
$\therefore$	therefore
$\ni$	such that
$\forall$	universal quantifier
$\exists$	existential quantifier
$\Rightarrow$	implication
"$\Rightarrow$" , "$\Leftarrow$"	necessity and sufficiency
iff	if and only if
$\rightsquigarrow$	functor mapping
$\approx$	category equivalence (a la Morita)
$\sim$	similarity (a la Morita)
$A \cong B$	A is isomorphic to B
$\longrightarrow$, $f:A \longrightarrow B$, $A \xrightarrow{\phi} B$	mapping
$\phi: A \longrightarrow B \longrightarrow 0$	epimorphism
©	mappings commute
$\langle m_i \rangle_{i \varepsilon I}$	ordered tuple
π , x	cartesian product
$\oplus$	direct sum
o , fg	composition
$M^{(I)}$	direct sum of I copies of M
M^I	direct product of I copies of M
$\ll$	small (superfluous)
$\trianglelefteq$	essential (large)
$E(M)$	injective hull of M

$A^{\perp}$	right annihilator of A
$ann_R(M)$	annihilator in R of M
mod-R	category of right R-modules
$Hom_R(M,N)$	group of R-homomorphisms from M to N
End(M)	endomorphism ring of M
$Tr_N(M)$	trace of M in N
$\mathbb{N}$	set of natural numbers
Z	set of integers
$\lvert F \rvert$	cardinal number of F
J;Rad	Jacobson radical of a ring (module)
$\bar{R}$	factor ring
$\bar{x}$	element of R
B	basic module
R_0	basic ring
e_0	basic idempotent
(f_{ij})	matrix with entries f_{ij}
$(R)_n$	ring of n x n matrices over R
QF	quasi-Frobenius
PF	pseudo-Frobenius
FPF	finitely pseudo-Frobenius
CFPF	completely FPF
prindec	principal indecomposable
VR	"right" valuation ring
p.c. dim (M)	projective cover dimension of M

CHAPTER 1

GENERATORS AND FAITHFUL MODULES, PROJECTIVE MODULES AND INJECTIVE MODULES

1.1 INTRODUCTION

The notion of a generator of mod-R is introduced. The main result of this Chapter is Theorem 1.3.3. We note that all generators of mod-R are faithful objects. We close this chapter by recording some results on projective and injective modules for later use.

1.2 NOTATION AND TERMINOLOGY

Unless otherwise specified, throughout this thesis all rings will be assumed to be associative rings with unity; all modules will be unitary; a "module" will mean a right R-module; mod-R (respectively R-mod) will denote the category of right (respectively left) R-modules; where there is no ambiguity, a "generator of mod-R" will be called a "generator"; R-homomorphisms will be called homomorphisms; where there is no ambiguity we will write Hom(M,N) instead of $Hom_R(M,N)$.

We will make no distinction between $M^{(I)}$ and M^I when I is a finite set since a finite direct product is the same as a finite direct sum.

1.3 GENERATORS OF MOD-R

1.3.1 <u>Definition</u> (generator):

A module M is a generator of mod-R iff for each module N $\exists$ a set I and an epimorphism

$M^{(I)} \longrightarrow N \longrightarrow 0.$

An example of a generator of mod-R is R_R.

To present our main theorem we need to define the trace of a module in another:

1.3.2 <u>Definition</u> (trace of a module):

Let M and N be modules. Then

$$Tr_N(M) = \sum_{h \in Hom(M,N)} Imh$$

The importance of the following theorem lies in its frequent usage in the sequel:

1.3.3 <u>Theorem</u>:

Let M be a module. The following are equivalent:

G_1: M is a generator of mod-R.

G_2: For all modules N $\exists$ an index set I and an epimorphism $M^{(I)} \longrightarrow N \longrightarrow 0$.

G_3: There is a finite integer $n > 0$ and an object Y of mod-R $\ni M^{(n)} \cong R \oplus Y$

G_4: $Tr_R(M) = R$.

<u>Proof</u>:

$G_1 \Longleftrightarrow G_2$: this is just definition 1.3.1.

$G_2 \Rightarrow G_3$: Given G_2, $\exists$ I and an epimorphism $f\colon M^{(I)} \longrightarrow R \longrightarrow 0$, so $\exists\, m = \langle m_i \rangle_{i \in I} \ni f(m) = 1$.

Also, $\exists$ a finite set $F \ni i \notin F \Rightarrow m_i = 0$. For any $r \in R$, $f(mr) = r$ and $i \notin F \Rightarrow \langle m_i r \rangle_i = 0$. So $\exists$ a surjection $fk\colon M^{(F)} \longrightarrow R \longrightarrow 0$ where k is the canonical injection $k\colon M^{(F)} \longrightarrow M^{(I)}$.

Let $fk(m) = 1$ where $m \in M^{(F)}$, and let $\phi\colon R \longrightarrow M^{(F)}$ be defined by $\phi(r) = mr$. Then it easily follows that ϕ is an injection and

$M^{(F)} = \phi(R) \oplus \ker(fk) \cong R \oplus \ker(fk)$. Finally, if $|F| = n$, it is clear that $M^{(F)} \cong M^{(n)}$.

$G_3 \Rightarrow G_4$: Suppose $\exists$ an epimorphism $f\colon M^{(n)} \longrightarrow R \longrightarrow 0$. Let, for each k, $1 \le k \le n$, $i_k\colon M \longrightarrow M^{(n)}$ be canonical. There exists $(m_1, m_2, \ldots, m_n) \in M^{(n)} \ni$ $f((m_1, m_2, \ldots, m_n)) = 1$, so

$1 = f(\sum_{k=1}^{n} i_k(m_k)) = \sum_{k=1}^{n} (fi_k)(m_k)$.

Since $fi_k \in \mathrm{Hom}\,(M,R)\ \forall\, k$, we conclude that $1 \in Tr_R(M)$ i.e. $Tr_R(M) = R$.

$G_4 \Rightarrow G_2$: Suppose $Tr_R(M) = R$. Let N be any module. For each $n \in N$, define $n^* \in \mathrm{Hom}\,(R,N)$ by $n^*(r) = nr$. Let $\{n_j\}_{j \in S}$ be a generating set for N. We claim that $\exists$ an epimorphism

$g\colon M^{(\mathrm{Hom}\,(M,N))} \longrightarrow N \longrightarrow 0$.

Let $I = \mathrm{Hom}\,(M,N) = \{f\colon M \longrightarrow N \mid f \text{ is a homomorphism}\}$. For each $f \in I$, let $M_f = M$. Then from

$$\begin{tikzcd} M_f \arrow[r, "\text{canonical}"] \arrow[dr, "f"'] & \bigoplus_{f \in I} M_f = M^{(I)} \arrow[d, dashed, "\phi"] \\ & N \end{tikzcd}$$

we conclude that $\exists$ a (unique) $\phi\colon M^{(I)} \longrightarrow N$ $\ni \mathrm{Im}\phi = \sum_{f \in I} \mathrm{Im} f = Tr_N(M)$. We need only show that $Tr_N(M) = N$. To this end, let $n \in N$.

Since $\{n_j\}_{j\varepsilon S}$ generates N, $\exists$ a (finite) $F \subseteq S \ni$

$n = \sum_{\alpha\in F} n_\alpha r_\alpha$ for some $\{r_\alpha\}_{\alpha\varepsilon F} \subseteq R$

$= \sum_{\alpha\in F} n^*_\alpha(r_\alpha)$.

Since $Tr_R(M) = R$, $\exists$ $t\varepsilon \mathbb{N}$, $\{m_i\}_{i=1}^{t} \subseteq M$ and

$\{f_i\}_{i=1}^{t} \subseteq Hom(M,R) \ni$

$1 = \sum_{i=1}^{t} f_i(m_i)$.

Thus for each $\alpha\varepsilon F$, $\sum_{i=1}^{t} f_i(m_i r_\alpha) = r_\alpha$.

Hence

$n = \sum_{\alpha\in F} n^*_\alpha(r_\alpha)$

$= \sum_{\alpha\in F} n^*_\alpha \left(\sum_{i=1}^{t} f_i(m_i r_\alpha)\right)$

$= \sum_{\alpha\in F} \sum_{i=1}^{t} (n^*_\alpha f_i)(m_i r_\alpha)$.

But $n^*_\alpha f_i \ \varepsilon\ Hom(M,N)\ \forall\alpha, \forall i$. Hence

$n\varepsilon Tr_N(M)$.

This proves the theorem.

1.4 FAITHFUL MODULES

1.4.1 <u>Definition</u> (faithful module):

A module M is called faithful if $Mr = 0$, $r\varepsilon R \Rightarrow r = 0$.

Clearly R_R is faithful for

$Rr = 0 \Rightarrow 1.r = 0 \Rightarrow r = 0$.

The following will be used often in the sequel:

1.4.2 <u>Proposition</u>:

A cyclic right module R/I is faithful iff I contains no non-zero two-sided ideals.

<u>Proof</u>:

"$\Rightarrow$" : Let R/I be faithful. Let $A \subseteq I$ be a two-sided ideal . We show $A = 0$. Let $a\varepsilon A$.

Then $(R/I)a = 0$. So, since R/I is faithful, $a = 0$.

Hence $A = 0$.

"$\Leftarrow$" : Suppose the only ideal in I is the zero ideal. Let $(R/I)r = 0$. Then $Rr \subseteq I$, so $RrR \subseteq I$. But RrR is a two-sided ideal, hence $RrR = 0$, proving that $r=0$.

$\therefore$ R/I is faithful.

When a module M is faithful its associated ring can be embedded in a direct product of copies of M:

1.4.3 <u>Proposition</u>:

A module M is faithful iff R embeds in M^I for some I.

<u>Proof</u>:

"$\Rightarrow$": Let M be faithful. For each $0 \neq r \varepsilon R$, $\exists\ m_r \varepsilon M \ni m_r r \neq 0$. Let $R^* = R-\{0\}$ and define $m^*_r \varepsilon \operatorname{Hom}(R,M)$ by $m^*_r(s) = m_r s$.

From

$$\begin{tikzcd} M^{R^*} \arrow[rr, "\pi_r"] & & M \\ & R \arrow[ul, dashed, "f"] \arrow[ur, "m^*_r"'] & \end{tikzcd}$$

we conclude that $\exists$ a (unique) $f \varepsilon \operatorname{Hom}(R,M^{R^*})$ $\ni \pi_r f = m^*_r \quad \forall\ 0 \neq r \varepsilon R$. If $x \varepsilon R$ is such that $f(x) = 0$, then $m^*_r(x) = 0\ \forall\ 0 \neq r \varepsilon R$. If $x \neq 0$, we get $m_x x = 0$, contradicting the choice of m_x. Hence $x = 0$ and f is the required embedding.

"$\Leftarrow$" : Suppose for some index set A, that $\phi: R \longrightarrow M^A$ is an embedding. Let $Mr = 0$.

Let $\phi(1) = m$ where $m = \langle m_a \rangle_{a\varepsilon A}\ \varepsilon\ M^A$. So

$\phi(r) = \phi(1.r) = \phi(1).r = mr = \langle m_a r \rangle_{a \varepsilon A} = 0$.

Since ϕ is an embedding, $r = 0$. Hence M is faithful.

1.5 FAITHFUL MODULES AND GENERATORS

It is easy to see that any generator M of mod-R is faithful. For suppose M is a generator. Then we can find $n>0$ and an object Y of mod-R $\ni M^{(n)} \cong R \oplus Y$. Suppose $Mr = 0$, $r \in R$. Then $M^{(n)}r = 0$, so $(R \oplus Y)r = 0$. Thus $Rr = 0$, so $1.r = r = 0$. $\therefore$ M is faithful.

The question now arises: Are all faithful modules generators? The answer is no as the following example shows:
Let p be any prime and let $M = Z_{p\infty}$. Then M is a divisible abelian group, so $nZ_{p\infty} = Z_{p\infty} \neq 0 \ \forall\, n \neq 0$. Hence $Z_{p\infty}$ is faithful as a Z-module. If $Z_{p\infty}$ generates mod-Z, then $\exists\, n \in \mathbb{N}$ and $N_Z \ni$

$$Z_{p\infty}^{(n)} = Z \oplus N_Z.$$

But $Z_{p\infty}$ is torsion, hence so is $Z_{p\infty}^{(n)}$ and therefore Z, a contradiction.

There do however exist rings over which every faithful module is a generator. These rings will be addressed in Chapter 5 (page 43).

1.6 COMPACTLY FAITHFUL MODULES

1.6.1 <u>Definition</u> (compactly faithful modules):
A module M is said to be compactly faithful provided that for some finite integer $n > 0$, R embeds in M^n. By 1.4.3 it is clear that compactly faithful modules are faithful.

Every generator is compactly faithful. For let M be a generator. Then $\exists\, n > 0$ and an object X of mod-R $\ni M^n \cong R \oplus X$. Since R can be naturally embedded in $R \oplus X$, we can embed R in M^n. $\therefore$ M is compactly faithful.

1.7 PROJECTIVE MODULES

1.7.1 Definition (projective module):

Let $\pi : B \longrightarrow A$ be an epimorphism. A module M is called projective in case for any homomorphism $\phi : M \longrightarrow A$ $\exists$ a homomorphism $f : M \longrightarrow B$ $\ni$

$$\begin{tikzcd} M \arrow[r, dashed, "f"] \arrow[dr, "\phi"'] & B \arrow[d, "\pi"] \arrow[dl, phantom, "C"] \\ & A \end{tikzcd}$$

1.7.2 Proposition:

For a module P the following are equivalent:

(a) P is projective.

(b) Every epimorphism $M \longrightarrow P \longrightarrow 0$ splits.

(c) P is isomorphic to a direct summand of M.

Proof: See [16] page 83.

1.7.3 Proposition:

Let $M = \bigoplus_{i \in I} M_i$. Then M is projective iff each M_i is projective.

Proof: See [16] proposition 3 page 82.

It is immediate from 1.7.3 that a direct summand of a projective module is projective.

1.8 INJECTIVE MODULES

1.8.1 <u>Definition</u> (injective module):

Let $k: A \longrightarrow B$ be a monomorphism. A module M is called injective in case for any homomorphism $\phi : A \longrightarrow M$ $\exists$ a homomorphism $f : B \longrightarrow M$ $\ni$

$$\begin{tikzcd} M & B \arrow[l, dashed, "f"'] \\ & A \arrow[u, "k"'] \arrow[ul, "\phi"] \arrow[ul, phantom, shift left=3ex, "C"] \end{tikzcd}$$

1.8.2 <u>Proposition</u>:

For a module M the following are equivalent:

(a) M is injective.

(b) Every monomorphism $k : M \dashrightarrow B$ splits.

(c) If M embeds in B then M is isomorphic to a direct summand of B.

<u>Proof</u>:

The proof is dual to 1.7.2. See [16] page 90.

1.8.3 <u>Proposition</u>:

Let $M = \prod_{i \in I} M_i$. Then M is injective iff each M_i is injective.

<u>Proof</u>:

The proof is dual to 1.7.3. See [16] Proposition 2 page 88.

1.8.4 <u>Definition</u> (essential submodule):

Let N be a module. A submodule M of N is essential (large) in case M has non-zero intersection with every non-zero submodule of N. We will write $M \trianglelefteq N$ to denote that M is an essential submodule of N.

When $M \trianglelefteq N$ and N is injective we call N the <u>injective hull</u> of M. We will denote the injective hull of M (which always exists) by $E(M)$. For other properties of $E(M)$ see [16] page 92.

Among the properties of the injective hull we have the following in (mod-R):

1.8.5 <u>Proposition</u> (properties of the injective hull):

(a) M is injective iff $M = E(M)$.

(b) $M \trianglelefteq N \Rightarrow E(M) = E(N)$.

(c) If $M \subseteq Q$, Q injective, then $Q = E(M) \oplus E'$

(d) If $\bigoplus_{i\in I} E(M_i)$ is injective then

$E(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} E(M_i)$.

<u>Proof</u>: See [12] page 209.

CHAPTER 2

THE MORITA THEOREM

2.1 INTRODUCTION

Morita's study of the category equivalence between mod-R and mod-S for two rings R and S led him to many generator theorems, especially the classical Morita theorem.

Two categories $\mathcal{A}$ and $\mathcal{B}$ are (categorically) equivalent if there exists additive covariant functors $F: \mathcal{A} \rightsquigarrow \mathcal{B}$ and $G: \mathcal{B} \rightsquigarrow \mathcal{A}$ which have the property that FG and GF are isomorphic to the identity functors on the respective categories. The question that was answered by Morita was: When are the module categories of two rings R and S equivalent? (Theorem 2.3.1). We shall state, with brief justification, some of the ring theoretical properties that are shared by rings having equivalent module categories.

We will write $\mathcal{A} \approx \mathcal{B}$ to denote that $\mathcal{A}$ is categorically equivalent to $\mathcal{B}$.

2.2 CATEGORICAL MODULE PROPERTIES

2.2.1 Definition (categorical module property):
A module property is categorical iff it can be defined entirely in terms of modules and morphisms

(i.e. by objects and arrows). Clearly "projective" and "injective" are categorical module properties. There are many others.

It is evident (see [12] section 21) that if an object M (respectively a morphism f) of mod-R has property P, where P is described entirely in terms of modules and morphisms (i.e. by objects and arrows), then for any category equivalence F: mod-R $\rightsquigarrow$ mod-S, F(M) (respectively F(f)) also has property P.

So it follows that under a category equivalence, categorical module properties will be preserved. Since the following properties can each be described entirely in terms of objects and arrows, they are all categorical and hence are preserved under category equivalence:

(split) monomorphism, (split) epimorphism
(split) exact sequence
generator
direct sum (direct product) of modules
projective (injective) modules
finitely presented module
faithful module

The following properties:

semisimple module
finitely generated module
indecomposable module

are also preserved since they arise out of the preservation of exact sequences and direct sums.

Submodule lattices are also preserved under category equivalence (see [12] Proposition 21.7). Hence simple and Artinian modules, being assertions about "inclusions", are also preserved under category equivalence.

2.3 THE MORITA THEOREM

Let R and S be rings. By imposing conditions on R and S we could obtain that mod-R $\approx$ mod-S. Morita's theorem shows that this can be done:

2.3.1 Definition and Theorem (The Morita Theorem, 1958): Let R-mod denote the left-right symmetry of mod-R. Two rings R and S are similar or (Morita) equivalent, written $R \sim S$, in case the following equivalent conditions hold:

S_1: mod-R $\approx$ mod-S

S_2: There exists a finitely generated projective generator (also called a progenerator) P of mod-R $\ni S \cong \mathrm{End}\ (P_R)$

S_3: R-mod $\approx$ S-mod

Proof: see [6] Theorem 4.29.

It is immediately clear from Morita's theorem that $R \sim S$ iff mod-R $\approx$ mod-S. Hence when two rings are similar, the categorical module properties listed in 2.2 possessed by modules over the one ring, will also be possessed by modules over the other ring. These properties we will choose to call Morita invariant "module" properties to avoid confusion with Morita invariant "ring" properties which follow.

2.4 MORITA INVARIANT "RING" PROPERTIES

When two rings are similar certain ring theoretic properties possessed by the one ring will also be possessed by the other. Such a property is called a Morita invariant "ring" property:

2.4.1 Definition (Morita invariant "ring" property): A property P of rings is a Morita invariant "ring" property if it is true that a ring R has property P iff every $S \sim R$ also has property P.

Let P be a ring theoretical property enjoyed by a ring R and let $R \sim S$. If the property P can be described entirely in terms of mod-R, then S enjoys the same property. For example, let R be semisimple and let $R \sim S$. Then every object of mod-R is projective. Hence every object of mod-S is projective, whence S is semisimple. Other Morita invariant "ring" properties are right Artinian, right Noetherian and semiprimitive.

In this thesis we will prove that right "FPF", right "CFPF" and "semiperfect" are Morita invariant "ring" properties.

CHAPTER 3

SEMIPERFECT RINGS

3.1 INTRODUCTION

Semiperfect rings can be characterised in several ways. In this chapter we aim to present homological and internal characterisations of semiperfect rings, the corresponding theorems viz. 3.4 and 3.7 being due to H. Bass [4] and a paper by B. Müller [19]. We cite some well known results which will be used in these proofs so as to afford readability. We also try to complete the groundwork necessary in order to present the basic module and the basic ring of a semiperfect ring in the next chapter.

3.2 NOTATION AND TERMINOLOGY

From now onwards unless otherwise specified, J will denote the Jacobson radical of a ring or module; $\bar{R}$ will denote a factor ring of R and elements of $\bar{R}$ will be denoted by $\bar{x}$.

3.3 PRELIMINARIES TO A HOMOLOGICAL CHARACTERISATION OF SEMIPERFECT RINGS

The results which now follow are needed to follow the homological characterisation theorem of semiperfect rings.

3.3.1 <u>Definition</u> (small submodule):

Let L be an R-submodule of M. Then L is called small (or superfluous) provided $L+X = M \Rightarrow X = M$. We write

$L << M$ to denote that L is small in M.

Relevant properties of small submodules are enumerated in the following proposition.

3.3.2 **Proposition:**

Let M be any module with Jacobson radical $J(M)$. Then

(a) if $K \leq N \leq M$ then $N << M$ iff $K << M$ and $N/K << M/K$

(b) if $K \leq N \leq M$ then $K << N \Rightarrow K << M$

(c) $K_i << M_i$, $i = 1,2,\ldots,n$,

$\Rightarrow \bigoplus_{i=1}^{n} K_i << \bigoplus_{i=1}^{n} M_i$

(d) if M is finitely generated then $MJ << M$

(e) $I << R_R$ iff $I \subseteq J$

(f) $J(M) = \Sigma \{I \mid I << M\}$

(g) if P is projective then $\mathrm{Rad}P = PJ$

Proof: See [12] propositions 5.17, 5.18, 5.20, 15.13, 9.13, 17.10 and [11] corollary (1.4)$'$.

Recall that for each module M there is an injective module $E(M)$ (see 1.8.4) which satisfies the following conditions:

(i) there exists an exact sequence

$0 \longrightarrow M \xrightarrow{\phi} E(M)$

(ii) $\mathrm{Im}\phi$ is an essential submodule of $E(M)$.

See [12] page 207.

The dual question was investigated by H. Bass (1960) - see [12] page 315:

Given any module M does $\exists$ a projective module P $\ni$

(i) $P \xrightarrow{\phi} M \longrightarrow 0$ is an exact sequence

(ii) $\ker\phi << P$.

Bass called P a projective cover, if it existed.

3.3.3 <u>Definition</u> (projective cover):

Let P and M be modules. An epimorphism $\phi: P \longrightarrow M \longrightarrow 0$ is called a projective cover of M in case P is projective and $\ker\phi << P$.

The property of being a projective cover is clearly a categorical module property.

It turns out that modules need not have projective covers:

<u>EXAMPLE</u>

Consider the two-element cyclic group Z_2. As a Z - module, Z_2 has no projective cover. For suppose it did. Let P be such a projective cover. Then $\exists$ an epimorphism $\phi : P \longrightarrow Z_2 \longrightarrow 0$ with $\ker\phi << P$. By 3.3.2 (e),

$$\begin{aligned}\ker \phi &\subseteq \mathrm{Rad}P \\ &= P\mathrm{Rad}Z \text{ by } 3.3.2 \text{ (g)} \\ &= P.0 \\ &= 0\end{aligned}$$

So $Z_2 \cong P$ and hence projective. So $\exists$ a set A $\ni$ $Z^{(A)} \cong Z_2 \oplus X$. Since $Z^{(A)}$ has no elements of finite order, we have the desired contradiction.

3.3.4 <u>Definition</u> (right perfect (semiperfect) ring):

If R is such that every right (respectively right finitely generated) module has a projective cover, R is said to be right perfect (respectively semiperfect).

Perfect rings have characterisations parallel to that of semiperfect rings (see for example [12] Theorem 28.4). We are especially concerned with semiperfect rings. To this end we recall the definition of the socle of a module and a few properties of semisimple modules and rings.

3.3.5 <u>Definition</u> (the socle of a module):

The socle of a module M is the sum of all the minimal submodules of M. It will be denoted by SocM. If no minimal submodules exist, SocM = 0.

3.3.6 <u>Proposition</u> (characterisation of semisimple modules):

For a module M the following are equivalent:

(a) SocM = M

(b) M is the sum of some set of simple modules

(c) M is isomorphic to a direct sum of simple modules

<u>Proof</u>: see [12] Theorem 9.6

3.3.7 <u>Proposition</u> (characterisation of semisimple rings):

The following statements concerning the ring R are equivalent:

(a) Every right module is semisimple

(b) R_R is semisimple

Under these conditions we call R semisimple.

<u>Proof</u>: See [16] proposition 5, page 64.

3.3.8 <u>Proposition</u> (characterisation of semisimple rings):

The following statements concerning the ring R are equivalent:

(a) R is semisimple

(b) R is right Artinian and regular

The equivalence is right-left symmetric.

<u>Proof</u>: See [16] proposition 2 on page 68.

Finally, direct summands and factor rings (modules) of semisimple rings (modules) are semisimple (see [16] page 64).

Using the projective covers of finitely generated and hence also simple modules we can characterise semiperfect rings as the theorem which follows shows:

3.4 **THEOREM (HOMOLOGICAL CHARACTERISATION OF SEMIPERFECT RINGS)**

Let $J = \mathrm{Rad}R$. The following conditions are equivalent on right R-modules:

(a) Every finitely generated module has a projective cover.

(b) Every simple module has a projective cover.

(c) Every simple module is isomorphic to eR/eJ for a suitable idempotent $e \in R$.

Under these conditions we call R semiperfect. In this case R/J is Artinian.

<u>Proof</u>:

(a) $\Rightarrow$ (b): Let S be any simple module. Then S is cyclic and hence has a projective cover by (a).

(b) $\Rightarrow$ (c): Let S be a simple module. Let $p: P \longrightarrow S$ be a projective cover of S. We reserve the right to replace P by any module isomorphic to it as we proceed. There is an exact sequence $0 \longrightarrow I \longrightarrow R \xrightarrow{\phi} S \longrightarrow 0$ with $I \leq R$ a maximal right ideal. Since R_R is projective and p is an epimorphism, $\exists\ u: R \longrightarrow P \ni$

$$\begin{tikzcd} & R \arrow[dl, dashed, "u"'] \arrow[d, "\phi"] & \\ P \arrow[r, "p"'] & S \arrow[r] & 0 \end{tikzcd}$$

Since $p(Imu) = S$, we have $Imu + \ker p = P$. For let $x \in P$. Then $p(x) \in S$. So $\exists\ r \in R \ni \phi(r) = p(x)$. Now $x = u(r) + (x - u(r))$. But $u(r) \in Imu$ and $p(x - u(r)) = p(x) - pu(r) = p(x) - \phi(r) = 0$. So $(x - u(r)) \in \ker p$. Thus $P \subseteq Imu + \ker p$. That $Imu + \ker p \subseteq P$ is clear. So, using the fact that $\ker p << P$, we have that $Imu = P$, so u is an epimorphism. But P is projective, so $\exists$ by 1.7.2 some $X \leq R \ni R = X \oplus \ker u$ where $X \cong P$. So we may assume that $P = eR$ for some idempotent $e \in R$. Now $K = \ker p << P = eR < R \Rightarrow K << R$ by 3.3.2(b), so by 3.3.2(e), $K \subseteq J$. Hence $eK = K \subseteq eJ \leq eR$. Since $eR/K = P/\ker p \cong S$ is simple, we have by maximality of K that $K = eJ$ or $eJ = eR$. But $eJ \subsetneq eR$. For if $eJ = eR$, then $e \in eJ$ so that $x \in J$ exists $\ni e = ex$ and $e(1 - x) = 0$. Since $1 - x$ is a unit, $e = 0$, whence P, so S is zero, a contradiction. Hence $S \cong eR/eJ$.

(c) $\Rightarrow$ (a):

We shall show that (c) implies that R/J is Artinian. First we prove the case for $J = 0$. Suppose $SocR$ is a proper right ideal of R and take a maximal right ideal I with $SocR \subseteq I \subseteq R$.

Then by assumption, $S = R/I$ is isomorphic to eR for some idempotent e. Since eR is a direct summand of R and R_R is projective, eR is also projective, so S is projective, and so by 1.7.2 $R = I \oplus S'$ for some $S' \cong S$. But S' is simple so $S' \neq 0$, and since $S' \subseteq SocR$, $SocR \nsubseteq I$, a contradiction. So we must have $SocR = R$ and hence R is semisimple by 3.3.6 and therefore Artinian by 3.3.8.

In the general case $J(R/J) = 0$. Let M be any simple R/J module. Then M is simple in mod-R as well, so $M \cong eR/eJ$ in mod-R for some idempotent. But then

$M_{R/J} \cong (eR/eJ)_{R/J} \cong ((e+J)(R/J))_{R/J}$

and $e + J$ is an idempotent. From the above, R/J is semisimple and hence Artinian.

We are now ready to prove (c) $\Rightarrow$ (a). Let M be a finitely generated module. Since $J \subseteq \mathrm{ann}_R\,(M/MJ)$, M/MJ can be regarded as an R/J - module, and since R/J is Artinian, M/MJ is semisimple by 3.3.7. Thus by (c), since M/MJ is also finitely generated, there exist finitely many idempotents e_i with $M/MJ \cong \bigoplus_{i=1}^{n} e_iR/e_iJ$ where this is an isomorphism in both mod-R and mod-R/J. Consider the diagram

$$\begin{tikzcd} & \bigoplus_{i=1}^{n} e_iR \arrow[dl, dashed, "u"'] \arrow[d, "p"] & \\ M \arrow[r, "q"'] & M/MJ \arrow[r] & 0 \end{tikzcd}$$

where p and q are the canonical epimorphisms. $\bigoplus_{i=1}^{n} e_iR$ is projective, so u exists. Now q(Imu) = M/MJ, so

M = Imu + ker q (see 3.4 (b) $\Rightarrow$ (c)) = Imu + MJ,

and since MJ << M by 3.3.2(d), M = Imu, so u is an epimorphism. Now ker p = $\bigoplus_{i=1}^{n} e_iJ = (\bigoplus_{i=1}^{n} e_iR)J$, and $\bigoplus_{i=1}^{n} (e_iR)J << \bigoplus_{i=1}^{n} e_iR$ by 3.3.2 (d). So ker p << $\bigoplus_{i=1}^{n} e_iR$. But ker u $\subseteq$ ker p, so ker u << $\bigoplus_{i=1}^{n} e_iR$ by 3.3.2 (a). This shows that u is a projective cover for M. This completes the proof of the theorem.

3.5 "SEMIPERFECT" IS A MORITA INVARIANT "RING" PROPERTY

From 3.4 (a) (or (b)) it is clear that "semiperfect" is a Morita invariant "ring" property. For let R be a semiperfect ring and suppose $R \sim S$. $\exists$ a category equivalence F: mod-R $\rightsquigarrow$ mod-S. Since "finitely generated" and "projective cover" are Morita invariant properties, "semiperfect" is a Morita invariant "ring" property. Hence S is semiperfect.

3.6 PRELIMINARIES TO AN INTERNAL CHARACTERISATION OF SEMIPERFECT RINGS

Using 3.4 (a) one can characterise semiperfect rings internally via projective covers.

3.6.1 <u>Proposition</u>:

Projective covers, when they exist, are unique up to isomorphism.

<u>Proof</u>:

Let $p: P \longrightarrow M$ and $q: Q \longrightarrow M$ be two projective covers of M. Then P must be projective and q must be an epimorphism, so $\exists\ u: P \longrightarrow Q \ni$

$$\begin{tikzcd} & P \arrow[dl, dashed, "u"'] \arrow[d, "p"] \\ Q \arrow[r, "q"'] \arrow[ur, phantom, "C"] & M \arrow[r] & 0 \end{tikzcd}$$

Since $q\ (\mathrm{Im} u) = \mathrm{Im} p = M$, we have that $\mathrm{Im} u + \ker q = Q$ (see 3.4 (b) $\Rightarrow$ (c)). But $\ker q << Q$, so $\mathrm{Im} u = Q$. Hence u is an epimorphism.

Since Q is projective $\exists\ Q' \leq P \ni Q' \cong Q$ and $P = Q' \oplus \ker u$. Now $\ker p << P$ and $\ker u \subsetneq \ker p$, so $\ker u << P$ by 3.3.2 (a), hence $P = Q' \cong Q$.

3.6.2 <u>Proposition</u>:

Finite direct sums of projective covers are projective covers.

<u>Proof</u>:

Let $p_i : P_i \longrightarrow M_i$, $i = 1,2,\ldots,n$, be projective covers. Let $\bigoplus_{i=1}^{n} M_i = N$. For each $i = 1,2, \ldots,n$, we can regard $p_i: P_i \longrightarrow N$. Let $k_i : P_i \longrightarrow \bigoplus_{i=1}^{n} P_i$ be the canonical injection.

$$\begin{tikzcd} \bigoplus_{i=1}^{n} P_i \arrow[rr, dashed, "f"] & & N \\ & P_i \arrow[ul, "k_i"] \arrow[ur, "p_i"'] \arrow[u, phantom, "\mathrm{C}"] & \end{tikzcd}$$

Then $\exists\ f : \bigoplus_{i=1}^{n} P_i \longrightarrow N \ni f \circ k_i = p_i\ \forall i$. So $\mathrm{Im} f = \Sigma\, \mathrm{Im} p_i = N$, so f is an epimorphism. Finally, since $\ker(f) = \ker(\bigoplus_{i=1}^{n} p_i) = \bigoplus_{i=1}^{n} \ker p_i$ and since $\ker p_i << P_i \forall_i$, $\bigoplus_{i=1}^{n} \ker p_i << \bigoplus_{i=1}^{n} P_i$ by 3.3.2 (c). This establishes that $f : \bigoplus_{i=1}^{n} P_i \longrightarrow \bigoplus_{i=1}^{n} M_i$ is a projective cover.

One of the more well-known internal characterisations of semiperfect rings R, is that idempotents lift from R/J to R. More precisely:

3.6.3 <u>Definition</u> (lifting idempotents):
Let I be an ideal in a ring R and let $\bar{g} \varepsilon \bar{R} = R/I$ be an idempotent. We say that $\bar{g}$ can be lifted modulo I in case there is an idempotent $e \varepsilon R \ni \bar{g} = \bar{e}$ i.e. $e - g \varepsilon I$. To say that idempotents lift modulo I means that every idempotent in $\bar{R}$ lifts to one in R.

If I is a nil ideal, idempotents always lift modulo I (see [16] proposition 1, page 72). Also, if idempotents lift modulo $I \leq \mathrm{Rad}\, R$, then finite orthogonal sets of idempotents in $\bar{R} = R/I$ lift to orthogonal sets of idempotents in R. (see [16] proposition 2, page 73.)

When $\bar{R} = R/J$ is a division ring we call R a local ring. Local rings have been characterised:

3.6.4 <u>Proposition</u> (characterisation of local rings):

For a ring R the following conditions are equivalent:

(a) R/J is a division ring

(b) R has a unique maximal right ideal

(c) All non-units of R are contained in a proper ideal

(d) For every $r \in R$, either r or $1 - r$ is a unit.

<u>Proof</u>: See [12] proposition 15.15.

Idempotents can be local:

3.6.5 <u>Definition</u> (local idempotent)):

An idempotent $e \in R$ is local in case eRe is a local ring.

3.6.5 <u>Definition</u> (primitive idempotent):

An idempotent $e \in R$ is called primitive in case $e \neq 0$ and for every pair e_1, e_2 of orthogonal idempotents, $e = e_1 + e_2 \Rightarrow e_1 = 0$ or $e_2 = 0$.

3.6.6 <u>Proposition</u>:

An idempotent $0 \neq e \in R$ is primitive iff eR is an indecomposable right ideal in mod-R.

<u>Proof</u>:

"$\Rightarrow$": Suppose $0 \neq e \in R$ is a primitive idempotent. Let $eR = M \oplus N$. We show that either $M = 0$ or $N = 0$.

$\exists\, m \in M$ and $n \in N \ni e = m+n$, so $em = m^2 + nm$. Now $m \in eR$, so $m = ex$ for some $x \in R$. But then

$em = e^2x = ex = m$, and so $m = m^2 + nm$, so

$m = m^2$ and $nm = 0$. Interchanging m and n gives

$n = n^2$ and $mn = 0$. Thus $\{m,n\}$ is an orthogonal set of idempotents. Since e is primitive either $m = 0$ or $n = 0$. We may suppose that $m = 0$. Let $x \in M$ be any. Then for some $r \in R$, $x = er = (m + n)r = nr \in N$.

$\therefore x \in M \cap N = 0$, so

$x = 0$ and hence $M = 0$.

"$\Leftarrow$" : Suppose eR is indecomposable. Let $e = e_1 + e_2$ with $e_1e_2 = e_2e_1 = 0$. Then $e_1R \cap e_2R = 0$. We show either $e_1 = 0$ or $e_2 = 0$. Let $r \in R$ be any. Then $er \in eR$ and $er = (e_1+e_2)r = e_1r + e_2r \in e_1R + e_2R$, so $eR \subseteq e_1R + e_2R$. On the other hand, let $e_1x + e_2y \in e_1R + e_2R$. Then $e_1x + e_2y = (e_1+e_2)(e_1x + e_2y) = e(e_1x + e_2y) \in eR$, so $e_1R + e_2R \subseteq eR$ and hence $eR = e_1R \oplus e_2R$. Since eR is indecomposable, either $e_1R = 0$ or $e_2R = 0$ i.e. either $e_1 = 0$ or $e_2 = 0$.

3.6.7 <u>Remark</u>: <u>Local idempotents are always primitive.</u> For let e be local. If $eR = M \oplus N$, then the projections onto M and N induce idempotents of $\mathrm{Hom}(eR,eR) \cong eRe$, so the projections are 0 or e. Thus eR is indecomposable. The converse of this assertion fails in general (see [16] page 75).

3.6.8 <u>Proposition</u>:

The following statements about a projective module P are equivalent:

(a) P is the projective cover of a simple module.

(b) PJ is a small, maximal submodule of P.

(c) $\mathrm{End}(P_R)$ is a local ring.

Moreover, if these conditions hold, then $P \cong eR$ for some idempotent $e \in R$.

<u>Proof</u>: See [12] proposition 17.19.

This proposition is used to prove our next result.

3.6.9 <u>Proposition</u>:

The following statements concerning an idempotent $e \in R$ are equivalent:

(a) eR/eJ is simple.

(b) eJ is the unique maximal submodule of eR.

(c) eRe is a local ring.

<u>Proof</u>:

(a) $\Rightarrow$ (b):

Let eR/eJ, $e \in R$ an idempotent, be simple. It is clear that eJ is a maximal submodule of eR. We only have to show that eJ is unique. Since eR is projective, $\mathrm{Rad}(eR) = (eR)J = eJ$ by 3.3.2 (g). eR is cyclic hence finitely generated, so by 3.3.2 (d) $(eR)J = eJ \ll eR$. Let I be any maximal submodule of eR. Then $\mathrm{Rad}(eR) = eJ \subseteq I \subset eR$. But eJ is maximal in eR so $eJ = I$, proving uniqueness.

(b) $\Rightarrow$ (c): Suppose (b). Then $(eR)J = eJ$ is a small maximal submodule of the projective module eR, so by 3.6.8 $\mathrm{End}(eR) \cong eRe$ is a local ring.

(c) $\Rightarrow$ (a): $\mathrm{End}(eR) \cong eRe$, so eRe local $\Rightarrow$ $\mathrm{End}(eR)$ is local. But eR is projective, so by 3.6.8 $(eR)J = eJ$ is a (small) maximal submodule of eR. Hence eR/eJ is simple.

In the theorem which follows, the idea of lifting idempotents and the presence of primitive and local idempotents serve, among others, to provide a nice internal characterisation of semiperfect rings.

3.7 **THEOREM (INTERNAL CHARACTERISATION OF SEMIPERFECT RINGS)**

Let R be a ring with radical J. The following conditions are equivalent:

(a) Every finitely generated right module has a projective cover.

(b) R/J is Artinian and idempotents can be lifted modulo J.

(c) Every primitive idempotent is local and any set of orthogonal idempotents of R is finite.

(d) There are orthogonal local idempotents $e_i (1 \leq i \leq n)$ with $\sum_{i=1}^{n} e_i = 1$.

(e) $R = \bigoplus_{i=1}^{n} e_iR$ where for each $i = 1,2,\ldots,n$, e_iR is indecomposable and $\mathrm{End}(e_iR)$ is a local ring.

Observe that condition (b) implies that the theorem holds in R-mod as well so that the property of being semi perfect is left-right symmetric.

<u>Proof</u>:

(a) $\Rightarrow$ (b):

Since (a) $\Longleftrightarrow$ 3.4, R/J is Artinian. It remains to be shown that idempotents can be lifted modulo J. Idempotents of R/J correspond to decompositions $R/J = A \oplus B$, where A and B are right ideals of R/J, so we have to show that we are able to lift direct decompositions of R/J to direct decompositions of R. Let $R/J = A \oplus B$. Then A and B are cyclic R/J modules and hence cyclic R-modules. Let $p: P \longrightarrow A$ and $q: Q \longrightarrow B$ be projective covers in mod-R. Then by 3.6.2, $P \oplus Q \longrightarrow A \oplus B$ is also a projective cover. Consider the canonical R-epimorphism $\phi: R_R \longrightarrow (R/J)_R$.
$\ker\phi = J \ll R$, so $\phi: R \longrightarrow R/J$ and hence $\phi: R \longrightarrow A \oplus B$ is a projective cover. Since projective covers are R-isomorphic (3.6.1), $R \cong P \oplus Q$, and hence $R = P' \oplus Q'$ where P' and Q' are right ideals of R,

isomorphic to P and Q respectively. The map $R = P' \oplus Q' \xrightarrow{\text{canonical}} A \oplus B = R/J$ takes the idempotent generator of P' (respectively Q') to the idempotent generator of A (respectively B). This proves (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c):
Let e be a primitive idempotent, R/J Artinian and suppose idempotents lift modulo J. We show that eRe is a local ring. This will establish that e is local (see 3.6.5). First we show that $\bar{e} = e + J$ is a primitive idempotent of $\bar{R} = R/J$. Suppose $\bar{e} = \bar{u} + \bar{v}$ where $\bar{u}^2 = \bar{u}$, $\bar{v}^2 = \bar{v}$, $\bar{u}\bar{v} = \bar{v}\bar{u} = \bar{0}$.

Then $\bar{u}(\bar{1} - \bar{e}) = \bar{u} - \bar{u}\bar{e} = \bar{u} - \bar{u}^2 - \bar{u}\bar{v} = \bar{0} = (\bar{1} - \bar{e})\bar{u}$, so that $1 - e$ (which is an idempotent in R) is orthogonal to u modulo J, where u is an idempotent modulo J. By [16] lemma 1 page 73, $\exists$ an idempotent $f \varepsilon R \ni \bar{f} = \bar{u}$ and $f(1 - e) = (1 - e)f = 0$. Hence $f = ef = efe \; \varepsilon \; eRe$. Since e is primitive, the only idempotents of eRe are 0 and e (For if ere is an idempotent for some $r \varepsilon R$, then so is $e - ere$. Now $e = ere + (e - ere)$ and $ere(e - ere) = (e - ere)ere = 0$, so by primitivity of e either $ere = 0$ or $ere = e$). So $f \varepsilon \{0,e\}$, so $\bar{f} = \bar{u} \; \varepsilon \{\bar{0},\bar{e}\}$ and hence either $\bar{u} = \bar{0}$ or $\bar{u} = \bar{e}$, showing that $\bar{e}$ is primitive. Hence $\bar{e}\bar{R}$ is an indecomposable right ideal (3.6.6) of the semisimple ring $\bar{R}$ (by 3.3.8, since $\bar{R}$ is also semiprimitive). But $\bar{e}\bar{R} \leq \bar{R}$, so $\bar{e}\bar{R}$ is itself semisimple. Being indecomposable, $\bar{e}\bar{R}$ is hence simple in mod-$\bar{R}$ and so also in mod-R. It is clear that the natural ring epimorphism $R \longrightarrow \bar{R}$ restricts to an epimorphism of the subrings $eRe \longrightarrow \bar{e}\bar{R}\bar{e}$ with kernel $J \cap eRe = eJe$. Hence $\bar{e}\bar{R}\bar{e} = eRe/eJe \cong \mathrm{End}(\bar{e}\bar{R})$ is a division ring, so eRe is a local ring.

We now only have to prove that any set of orthogonal idempotents is finite. Given any set $\{e_i\}_{i \varepsilon I}$ of orthogonal idempotents of R, we see that $\{\bar{e}_i\}_{i \varepsilon I}$ is a set of orthogonal idempotents of $\bar{R}$. Since $\bar{R}$ is Artinian, $\{\bar{e}_i\}_{i\varepsilon I}$ is finite, for any set of orthogonal idempotents in a semisimple ring is finite. If $\{e_i\}_{i\varepsilon I}$ is infinite then distinct idempotents must exist in R which map onto the same idempotent modulo J. Let $e_i \neq e_j$ be two such idempotents. Then

$\bar{e}_i = \bar{e}_j$, so $e_i - e_j \ \varepsilon\ J$, so

$e_i(e_i - e_j) = e_i \varepsilon J$, so $e_i = 0 = e_j$,

contradicting $e_i \neq e_j$. Hence $\{e_i\}_{i\varepsilon I}$ is finite.

(c) $\Rightarrow$ (d):

It suffices to show that 1 is a sum of primitive idempotents.

First observe that if $e = e_1+e_2$ and $e_2 = e_{21} + e_{22}$ are decompositions of the idempotents e and e_2 into orthogonal idempotents, then $\{e_{21}, e_{22}, e_1\}$ is an orthogonal set. For

$0 = e_1e_2 = e_1(e_{21} + e_{22}) = e_1e_{21} + e_1e_{22}$,

so $0 = (e_1e_{21} + e_1e_{22})e_{21} = e_1e_{21}$. Also,

$0 = e_2e_1 = (e_{21} + e_{22})e_1 = e_{21}e_1 + e_{22}e_1$,

so $0 = e_{21}(e_{21}e_1 + e_{22}e_1) = e_{21}e_1$.

Similarly one shows that $e_{22}e_1 = e_1e_{22} = 0$, and hence $e_1e_{21} = e_{21}e_1 = e_1e_{22} = e_{22}e_1 = 0$.

Now suppose 1 is not the sum of primitive idempotents. Then 1 is not primitive, so $\exists$ an orthogonal decomposition $1 = e_1+e_2$. Let $S_1 = \{e_1,e_2\}$. Now one of e_1 or e_2, say e_1, is not primitive. So $\exists$ an orthogonal decomposition $e_1 = e_{11} + e_{12}$. So $S_2 = \{e_{11},e_{12},e_2\}$ is orthogonal sum 1. Proceeding as above, the process cannot terminate for that would contradict the hypothesis on 1. On the other hand,

$S = \bigcup_{n=1}^{\infty} S_n$ is an infinite orthogonal set which contradicts (c). Hence 1 is a finite sum of primitive, hence local idempotents.

(d) $\Rightarrow$ (e):

By (d), $1 = \sum_{i=1}^{n} e_i$ where the e_i are local, hence primitive. So $R = \sum_{i=1}^{n} e_iR$ and this sum is direct. For pick k, $1 \leq k \leq n$. Then e_kR is a direct summand of R, so $R = e_kR \oplus (1-e_k)R$, so $e_kR \cap \sum_{j\neq k} e_jR = 0$ and hence $\sum_{i=1}^{n} e_iR$ is direct. Since the e_i, $1 \leq i \leq n$, are each primitive, e_iR is indecomposable (by 3.6.6) $\forall i$. Since the e_i, $1 \leq i \leq n$, are local, e_iRe_i is a local ring $\forall i$, hence $\mathrm{End}(e_iR) \cong e_iRe_i$ is local $\forall i$.

(e) $\Rightarrow$ (a):

Since (a) $\Leftrightarrow$ 3.4 (c), it will be sufficient to prove (e) $\Rightarrow$ 3.4 (c). Since for each i, $1 \leq i \leq n$, e_iRe_i is local, e_iJ is the unique maximal submodule of e_iR (3.6.9). Now let S be a simple module. Then for some i, $Se_i \neq 0$. So $\exists\ 0 \neq x \in S \ni xe_i \neq 0$. The map $R \longrightarrow S$ defined by $r \longrightarrow xr$ restricted to e_iR, has image $xe_iR = S$ since S is simple, so S is an epimorphic image of e_iR. But by 3.6.9, e_iJ is the unique maximal submodule of e_iR, so $S \cong e_iR/e_iJ$. This completes the proof of the theorem.

Henceforth we will be working with semiperfect rings. <u>A useful example of a semiperfect ring is a local ring</u>. For if R is local, $\bar{R} = R/J$ is a division ring and hence semisimple. By 3.3.8 $\bar{R}$ is Artinian. The only idempotents of $\bar{R}$ are $\bar{0}$ and $\bar{1}$. For let $\bar{e}$ be an idempotent of $\bar{R}$. Then $\bar{e}(\bar{1} - \bar{e}) = \bar{0}$. Since division rings have no zero divisors $\neq 0$, $\bar{e} \in \{\bar{0}, \bar{1}\}$, hence $\bar{e}$ lifts to an idempotent of R. $\therefore$ by 3.7 (b) R is semiperfect.

3.8 **PROPOSITION (FACTOR RINGS OF SEMIPERFECT RINGS ARE SEMIPERFECT)**

If R is semiperfect then so is every factor ring of R.

Proof:
Given R semiperfect, let $I \leq R$ be any ideal and let $\bar{R} = R/I$. Consider the canonical epimorphism $\phi : R \longrightarrow R \longrightarrow 0$. It is clear that $\sum_{i=1}^{n} e_i = 1$ in $R \Rightarrow \sum_{i=1}^{n} \bar{e}_i = \bar{1}$ in $\bar{R}$, and if the e_i are orthogonal in R then the $\bar{e}_i$ are orthogonal in $\bar{R}$. So we only have to show that local idempotents in R remain local in $\bar{R}$ by 3.7 (d). Let $e \varepsilon R$ be a local idempotent. Then eRe is a local ring and ϕ : eRe $\longrightarrow$ $\overline{eRe}$ = (eRe + I)/I.
But (eRe +I)/I $\cong$ eRe/I $\cap$ eRe = eRe/eIe, so $\overline{eRe}$ is a local ring since any factor ring of a local ring is clearly local. Then since
$\overline{eRe} = \{ere + I \mid r \varepsilon R\} = \{(e + I)(r + I)(e + I) \mid r \varepsilon R\} = \bar{e}\bar{R}\bar{e}$, $\bar{e}\bar{R}\bar{e}$ is a local ring and hence $\bar{e}$ is a local idempotent.

3.9 **IRREDUNDANT CLASS OF REPRESENTATIVES OF INDECOMPOSABLE PROJECTIVE MODULES; OF SIMPLE MODULES**

When a ring is semiperfect we can find an irredundant class of representatives for the simple as well as the projective indecomposable modules in mod-R.

3.9.1 Definition (irredundant class of representatives):
Let $\mathcal{U}$ be a class of R-modules. A class $\mathcal{U}' \subseteq \mathcal{U}$ is a class of representatives (of the isomorphism types) of $\mathcal{U}$ in case each $U \varepsilon \mathcal{U}$ is isomorphic to some $U' \varepsilon \mathcal{U}'$. If in addition, no two elements of $\mathcal{U}'$ are isomorphic, then the class of representatives is said to be irredundant.

The lemma which follows is very important to the sequel:

3.9.2 Exchange lemma:

Let $M_1 \oplus \dots \oplus M_n = A \oplus B$ be a decomposition in mod-R $\ni$ End(A) is a local ring. Then $\exists$ i, $1 \le i \le n$, and an isomorphism $M_i \cong A \oplus X$ for some object X of mod-R. In particular, if M_i is an indecomposable module, i = 1,...,n, then $A \cong M_i$ for some i.

Proof: See [7] Lemma 18.17.

3.9.3 Remark:

When R is semiperfect, $R = \bigoplus_{i=1}^{n} e_iR$ where the e_iR are indecomposable $\forall i$ (3.7(e)). Given any primitive idempotent $0 \neq e \in R$, $eR \cong e_iR$ for some i. For $\bigoplus_{i=1}^{n} e_iR = eR \oplus (1-e)R$, and since R is semiperfect e is local (3.7(c)), so End(eR) $\cong$ eRe is local, so by 3.9.2 for some i, $1 \le i \le n$, $\exists$ X in mod-R $\ni e_iR \cong eR \oplus X$. Since e_iR is indecomposable, either eR = 0 or X = 0. Since $e \neq 0$, $eR \neq 0$, so X = 0 and hence $eR \cong e_iR$.

3.9.4 Definition (primitive module):

A module M is primitive in case $M \cong eR$ for some primitive idempotent $e \in R$.

3.9.3 shows that $\{e_iR\}_{i=1}^{n}$ is a class of representatives for the primitive modules in mod-R. If this class were irredundant it would have to contain m elements, where $m \le n$. The $\{e_i\}_{i=1}^{m}$ are then called a basic set of idempotents.

3.9.5 Definition (basic set of idempotents):

A set of idempotents of a semiperfect ring R is basic in case the e_i, $1 \le i \le m$, are orthogonal and $\{e_1R, \dots, e_mR\}$ is an irredundant class of representatives of the primitive modules in mod-R.

3.9.6 <u>Lemma</u>:

Let e and f be idempotents in a ring R. Then $eR \cong fR$ iff $eR/eJ \cong fR/fJ$, where $J = RadR$.

<u>Proof</u>:

"$\Rightarrow$": Suppose $eR \cong fR$. We have $eR/eJ = eR/eR \cap J \cong (eR+J)/J$. Similarly $fR/fJ \cong (fR+J)/J$. Hence $eR \cong fR \Rightarrow (eR + J)/J \cong (fR + J)/J$, so $eR/eJ \cong fR/fJ$.

"$\Leftarrow$": Suppose $h: eR/eJ \longrightarrow fR/fJ$ is an isomorphism. Consider the natural epimorphisms $\phi: eR \longrightarrow eR/eJ$ and $\phi': fR \longrightarrow fR/fJ$. Since $eJ \subseteq J$, $eJ << R$ (3.3.2 (e)). We show $\ker \phi = eJ << eR$. Let $eR = eJ + L$. We claim $eR = L$. Now

$$eR + (1 - e)R = eJ + L + (1 - e)R$$

$\therefore$ $R = eJ + (L + (1 - e)R)$

Since $eJ << R$, $L + (1 - e)R = R$. Since $L \leq eR$, $L = L \cap eR$. Similarly $eR = eR \cap R$.

But $eR \cap R = eR \cap (L + (1 - e)R)$

$= L + (eR \cap (1 - e)R)$ (by the modular law)

$= L + 0$

$= L$.

Hence $eR = L$ as claimed. Thus, since eR is projective, ϕ is a projective cover. Similarly ϕ' is a projective cover. Consider $h\phi: eR \longrightarrow fR/fJ$. $h\phi$ is an epimorphism. $\ker h\phi = \{x \varepsilon eR \mid h(\phi(x)) = 0\} =$ $= \{x \varepsilon eR \mid \phi(x) = 0\} = eJ = \ker \phi << eR$. Since eR is also projective, $h\phi$ is a projective cover of fR/fJ. But so is ϕ', hence $eR \cong fR$ by 3.6.1.

3.9.7 <u>Proposition</u>:

Let R be a semiperfect ring with radical J. Then for orthogonal primitive idempotents $e_1, \ldots, e_m \varepsilon R$ the following are equivalent:

(a) $\{e_i\}_{i=1}^{m}$ is a basic set of primitive idempotents of R.

(b) $\{e_iR/e_iJ\}_{i=1}^{m}$ is an irredundant class of representatives of the simple modules in mod-R.

(c) $\{e_iR\}_{i=1}^{m}$ is an irredundant class of representatives of the indecomposable projective modules in mod-R.

Proof

(a) $\Rightarrow$ (b):

Let S be any simple module. Then for some idempotent $e \in R$, $S \cong eR/eJ$ by 3.4 (c). Thus $\mathrm{End}(S) \cong eRe/eJe$ (see [12] Corollary 17.12) is a division ring, so eRe is local and so also e. Thus e is primitive, so eR is a primitive right ideal and so by (a), $eR \cong e_iR$ for some i. By our lemma, $eR/eJ \cong e_iR/e_iJ$, so $S \cong e_iR/e_iJ$. Hence $\{e_iR/e_iJ\}_{i=1}^{m}$ is a class of representatives of the simple modules in mod-R, which has to be irredundant. If it is not, then $\exists\ i \neq k \ni e_iR/e_iJ \cong e_kR/e_kJ$, so by our lemma $e_iR \cong e_kR$, contradicting the irredundancy of $\{e_iR\}_{i=1}^{m}$.

(b) $\Rightarrow$ (c):

Let P be any non-zero indecomposable projective module in mod-R. Then since P is projective $\exists$ a set A and $P' \ni R^{(A)} = P \oplus P'$. Since for each i, e_iR is a direct summand of $R^{(A)}$ and it has local endomorphism ring, by 3.9.2, e_iR is a direct summand of either P or P'. We cannot have that e_iR is a direct summand of P' $\forall i$. For then $P = 0$. $\therefore \exists\ i \ni P \cong e_iR \oplus K$ for some object K in mod-R. But P is indecomposable so e_iR or K is 0. Since $e_i \neq 0$, $P \cong e_iR$, proving $\{e_iR\}$ is an irredundant class of indecomposable projective modules.

(c) $\Rightarrow$ (a):

Under the assumption $\{e_i\}_{i=1}^{m}$ is a set of orthogonal primitive idempotents. Let M be any primitive module in mod-R. We only have to show , by 3.9.5, that $M \cong e_i R$ for some i, $1 \leq i \leq m$. Now for some primitive idempotent $e \varepsilon R$, $M \cong eR$. But then eR is indecomposable, and also projective. So by (c) $M \cong eR \cong e_i R$ for some i, $1 \leq i \leq m$.

This completes the proof of the proposition.

CHAPTER 4

THE BASIC MODULE AND THE BASIC RING OF A SEMIPERFECT RING

4.1 INTRODUCTION

A semiperfect ring whose decomposition into a direct sum of indecomposable projective modules contains exactly one copy of each isomorphism type, is called a selfbasic ring. The generators for their module categories are particularly simple: they are precisely those modules for which the ring splits off (Proposition 4.4.1). Every semiperfect ring contains a selfbasic subring to which it is Morita equivalent. The study of semiperfect rings is greatly simplified once this is observed, for in a large number of cases results in selfbasic semiperfect rings can be applied to general semiperfect rings using Morita theory.

In this chapter we present the basic module and the basic ring of a semiperfect ring. We point out that these concepts are described only for semiperfect rings.

4.2 CONSTRUCTION OF THE BASIC MODULE

When R is semiperfect we can decompose R as follows:
$R = \bigoplus_{i=1}^{n} e_iR$ in mod-R where e_i is local $\forall i$.

Renumber so that $\{e_iR\}$, $1 \le i \le m$ form a complete set of non-isomorphic summands. Then $\{e_i\}$, $1 \le i \le m$ is a basic set of idempotents and Proposition 3.9.7 applies. $\{e_iR\}_{i=1}^{m}$ is hence an irredundant class of representatives of the indecomposable projective modules in mod-R.

From now on we will prefer to call the indecomposable projective modules viz. the e_iR, $1 \le i \le m$, the right prindecs and each member of the irredundant class of representatives, an isomorphism class for the right prindecs in mod-R.

4.2.1 <u>Definition</u> (the basic module):

The right ideal $B = e_1R \oplus e_2R \oplus \ldots \oplus e_mR$ of R is called the <u>basic (right) module</u> of R.

Henceforth B will always denote the basic module.

The basic module B of a semiperfect ring is unique up isomorphism. For let B' also be a basic module. Then $B' = f_1R \oplus \ldots \oplus f_kR$ where $\{f_iR\}$ is a complete class of non-isomorphic prindecs of R.

Since each f_i is local, $\exists$ one (and exactly one) $i \ni f_1R \cong e_iR$. Renaming we get $f_1R \cong e_1R$.

A standard argument allows us to conclude that $f_iR \cong e_iR \ \forall i$. So $B' \cong B$.

Thus we can speak of "the" basic module.

4.3 THE BASIC RING

From the basic module $B = e_1R + \ldots + e_mR$ we obtain an orthogonal sum of idempotents viz. $e_1 + e_2 + \ldots + e_m$. This sum is again an idempotent and is called a <u>basic idempotent</u> of R and is denoted by e_0 i.e. $e_0 = e_1 + \ldots + e_m$.

It is now clear that $B = e_0R$. So $\mathrm{End}\,(B_R) = \mathrm{End}((e_0R)_R) \cong e_0Re_0$. This ring is called the <u>basic ring</u> of R and is denoted by R_0. Thus $R_0 = e_0Re_0 \cong \mathrm{End}\,(B_R)$.

Henceforth e_0 and R_0 will denote a basic idempotent and the basic ring respectively. We will be finding it "nicer" to work with R_0 rather than R itself. To work with R_0 instead of R we need to be assured that R_0 is semiperfect whenever R is. To this end we need the following:

4.3.1 <u>Proposition</u>:

The basic module is a progenerator of mod-R.

<u>Proof</u>:

B is clearly finitely generated, and being a direct summand of R, B is also projective. So we only have to show that B is a generator of mod-R. Now every simple module M in mod-R is isomorphic to e_iR/e_iJ for some i (Proposition 3.9.7). Using this fact we can establish that B is a generator of mod-R. Suppose $Tr_R(B) \neq R$. Then $\exists$ a maximal right ideal $I \ni Tr_R(B) \subseteq I$. So R/I is simple, so $\exists$ an epimorphism $g: B \longrightarrow R/I \longrightarrow 0$. Since R/I is simple, $g \neq 0$. Since B is projective, $\exists$ $f: B \longrightarrow R$ $\ni$

$$\begin{tikzcd} & B \arrow[d, "g"] \arrow[dl, dashed, "f"'] & \\ R \arrow[r, "\pi"'] & R/I \arrow[r] & 0 \end{tikzcd}$$

Then $f(B) \subseteq \Sigma\ \mathrm{Im}f = Tr_R(B) \subseteq I$, so $(\pi f)B = 0$. Hence $g = 0$, a contradiction. $\therefore$ we must have $Tr_R(B) = R$ and so B is a generator by G4 of 1.3.3.

4.3.2 <u>Proposition</u>:

A semiperfect ring R is similar to its basic ring R_0. Hence R_0 is semiperfect.

<u>Proof</u>:

B is a progenerator of mod-R and $R_0 \cong \mathrm{End}\,(B_R)$. Hence condition S_2 of the Morita Theorem is satisfied, so $R \sim R_0$. By 3.5, R_0 is semiperfect.

4.3.3 <u>Definition</u> ("selfbasic")

A semiperfect ring is selfbasic in case $e_0 = 1$. When $e_0 = 1$, $B = R$ and $R_0 = R$. Hence it is immediate that a semiperfect ring is selfbasic iff $B = R$ or $R_0 = R$.

4.3.4 <u>Proposition</u>:

The basic ring of a semiperfect ring is selfbasic.

<u>Proof</u>:

Let R be semiperfect. Then $1 = \sum_{i=1}^{n} e_i$ where the e_i is a local idempotent $\forall i$ (see 3.7 (d)). For the basic idempotent e_0 of R we have that $e_0 = \sum_{i=1}^{m} f_i$ for some $m \le n$, where $\{f_i\} \subseteq \{e_i\}$ and $f_iR \cong f_jR$ iff $i = j$. Let S be the basic ring of R. Then $S = e_0Re_0$. We must show that S is selfbasic. Now for each i, $1 \le i \le m$, $f_i = e_0f_ie_0 \in e_0Re_0 = S$. So the identity e_0 of S is a finite sum of orthogonal idempotents of S viz. $e_0 = \sum_{i=1}^{m} f_i$. For each $f_i \in S$, $f_iSf_i = f_i(e_0Re_0)f_i = f_iRf_i$ and the latter is a local ring. So each $f_i \in S$ is a local idempotent. Hence S is semiperfect (3.7 (d)). We claim that e_0 is a basic idempotent of S. Let f_iS and f_jS be any two right prindecs in mod-S.

Suppose $f_iS \cong f_jS$. If we could show $i = j$ we would be done according to the construction in 4.2. Since e_0R is a progenerator in mod-R (see 4.3.1) we have according to [12] page 178 and Theorem 22.1, that the functor

Hom $(e_0R,\_)$: mod-R $\rightsquigarrow$ mod-S defined by

Hom $(e_0R,\_)$: $A_R \longrightarrow Hom_R(e_0R, A_R)$ defines a category equivalence. So under Hom $(e_0R,\_)$, the image of f_iR in mod-R is $Hom_R(e_0R, f_iR)$. Now $Hom_R(e_0R, f_iR) \cong f_iRe_0$ as abelian groups (see [16] lemma 1 page 63) and the isomorphism is an isomorphism of right S - modules as well. Thus

Hom $(e_0R,\_)$: $f_iR \longrightarrow f_iRe_0 = f_iS$.

Hence $f_iS \cong f_jS$ in mod-S $\Rightarrow$ $f_iR \cong f_jR$ in mod-R $\Rightarrow$ $i = j$ as required.

4.4 CONNECTION BETWEEN THE BASIC MODULE AND GENERATORS OF MOD-R

When R is semiperfect its basic module is a direct summand of any generator of mod-R. Thus if R is selfbasic, R_R is a direct summand of any generator of mod-R.

4.4.1 <u>Proposition</u>:

Let R be a semiperfect ring with basic module B and basic ring R_0. Then

(a) $R \sim R_0$.

(b) A module M is a generator of mod-R iff B is isomorphic to a direct summand of M. In particular, if R is selfbasic then M generates mod-R iff $M \cong R \oplus Y$ in mod-R.

<u>Proof</u>:

(a) Let R be semiperfect. Then R R_0 by 4.3.2.

(b) "$\Leftarrow$" :

Suppose $M = B' \oplus X$ where $B' \cong B$ and X is an object of mod-R. Since B is a progenerator and hence a generator, $\exists$ a set I and an epimorphism

$g : B'^{(I)} \longrightarrow K \longrightarrow 0$ for an arbitrary object K of mod-R (see G2 of 1.3.3). Now

$M^{(I)} = B'^{(I)} \oplus K^{(I)}$. Let

$\pi : M^{(I)} \longrightarrow B'^{(I)}$ be the projection map. Then

$g\pi : M^{(I)} \longrightarrow K$ is an epimorphism. Hence M is a generator of mod-R.

"$\Rightarrow$" : Let M be a generator of mod-R. Now

$B = e_1R \oplus \ldots \oplus e_mR \leq R$. $\exists\ n > 0 \ni$

$$M^{(n)} \cong R \oplus K \text{ (see } G_3 \text{ of 1.3.3)}$$

$$= e_1R \oplus C \text{ for some object } C \text{ of mod-}R$$

Hence $M \oplus \ldots \oplus M \cong e_1R \oplus C$. But $\mathrm{End}(e_1R)$ is a local ring, so by the exchange lemma, $M \cong e_1R \oplus X$ for some object X of mod-R.

Thus $M^{(n)} \cong (e_1R)^{(n)} \oplus X^{(n)} = e_2R \oplus Y$ for some object Y of mod-R. Since $\mathrm{End}\ (e_2R)$ is a local ring, e_2R is isomorphic to a direct summand of either e_1R or X. Since e_1R is indecomposable, e_2R is thus isomorphic to a direct summand of X.

Hence $M \cong e_1R \oplus e_2R \oplus D$ for some object D of mod-R, Proceeding in this way we eventually obtain that $M \cong B \oplus X$.

Finally, since R selfbasic $\Rightarrow R = B$, our final statement follows from (b).

4.4.2 <u>Corollary</u>:

Let R be a semiperfect ring with basic module B. Then

(a) An epimorphic image M/I of a R-module M is a generator of mod-R iff $M = B' \oplus C$ for some $I \subset C$ and $B' \cong B$. Then $M/I \cong B \oplus C/I$.

(b) An epimorphic image B/I of B is a generator of mod-R iff $I = 0$. In particular, if R is selfbasic then a cyclic module R/I is a generator of mod-R iff $I = 0$.

Proof:

(a) "$\Rightarrow$" : Suppose M/I is a generator of mod-R. By 4.4.1, $M/I \cong B \oplus X$ for some object X of mod-R. Let $\phi : B \oplus X \longrightarrow M/I$ be this isomorphism. Let $\phi(B) = A/I$ and $\phi(X) = C/I$ for submodules A and C of M. Now

$$\begin{aligned} M/I &= \phi(B \oplus X) \\ &= \phi(B + X) \\ &\subseteq \phi(B) + \phi(X) \\ &= A/I + C/I \\ &\subseteq (A + C)/I \subseteq M/I. \end{aligned}$$

So $M/I = (A + C)/I$, so $M = A + C$. Since ϕ is 1-to-1 $\ker \phi = 0 = B \cap X$. So

$$\begin{aligned} 0 &= \phi(B \cap X) \\ &= \phi(B) \cap \phi(X) \\ &= A/I \cap C/I \\ &= (A \cap C)/I \end{aligned}$$

$\therefore \quad I = A \cap C$

We have done all this to show that there are submodules A and C of $M \ni M = A + C$, $I = A \cap C$. Since ϕ is an isomorphism $A/I \cong B$. Now the sequence $0 \longrightarrow I \longrightarrow A \xrightarrow{\phi} A/I \longrightarrow 0$ is exact. Since $B \cong A/I$ is projective, ϕ is a split epimorphism, so $A = I \oplus B'$ in mod-R, and since $I \subset C$, $M = A + C = B' + C$. This latter sum is direct for $B' \cap C = (B' \cap A) \cap C = B' \cap I = 0$. Thus $M = B' \oplus C$. But $B' \cong A/I \cong B$ and hence $M/I \cong B' \oplus C/I \cong B \oplus C/I$.

"$\Leftarrow$" :

Suppose $M \cong B' \oplus C$ where $I \subset C$ and $B' \cong B$. Then $M/I \cong B \oplus C/I$. Since B is the basic module of R, M/I is a generator of mod-R by 4.4.1.

(b) "$\Rightarrow$" :

Suppose B/I is a generator of mod-R. We must show $I = 0$. Applying (a) we obtain that $B = B' \oplus X$ where $B' \cong B$ and $I \subset X$. If $X \neq 0$, then X has an indecomposable direct summand which is projective since B is. So $\exists\, i,\ 1 \leq i \leq m,\ \ni X \cong e_iR \oplus Y$. But then B contains two non-isomorphic prindecs, a contradiction.

"$\Leftarrow$" : If $I = 0$, $B/I = B$, so B/I is a generator follows from the fact that B is a generator.

If R is selfbasic, $R = B$, so the final statement follows from the preceding one.

CHAPTER 5

SEMIPERFECT FPF RINGS

5.1 INTRODUCTION

The study of classical Frobenius algebras led naturally to a study of QF (quasi Frobenius) rings, first introduced by Ikeda in 1952. QF rings are precisely those rings which are two-sided Artinian and two-sided self-injective. PF (pseudo-Frobenius) rings are generalisations of QF rings and right PF rings have been completely characterised internally by Azumaya [3], Osofsky [22] and Utumi [27] during 1966-1967, to be those rings over which every faithful module is a generator. They also characterised right PF rings as right self-injective semiperfect rings having (finite) essential right socles. Thus, since all right Artinian rings are semiperfect with essential right socles, QF rings are also PF.

Generators in module categories are always faithful (see 1.5). The converse problem viz. describing those rings for which all faithful modules are generators was investigated by Osofsky in 1966. The natural follow-up to this investigation was to characterise those rings for which every finitely generated faithful right as well as left module is a generator of mod-R. Such rings are called FPF (finitely pseudo-Frobenius) rings and to date have not yet been internally characterised.

Parallel to the theory of FPF rings is FP^2F rings. A ring is said to be right FP^2F in case every finitely presented faithful right module is a generator of mod-R. We have chosen to focus from FPF rings to CFPF rings and not from FP^2F rings. It suffices to mention, however, that many of the major theorems in FPF ring theory have counterparts in FP^2F ring theory. Proposition 5.5.1 and 5.5.5 are such examples.

In this chapter we prove that every right FPF ring is right bounded. In seeking a converse to this result a number of important propositions emerge which themselves are affirmations of the difficulty involved in characterising FPF rings.

5.2 NOTATION AND TERMINOLOGY

Since we have chosen to work in mod-R the results in this chapter will be proven for right FPF rings. The results will have natural counterparts in R-mod.

We will denote the right annihilator in R of a module M (respectively an element $x \in M$) by $M^{\perp}$ (respectively $x^{\perp}$).

5.3 RIGHT BOUNDED RINGS

5.3.1 Definition (right duo):

A ring is called right duo in case every right ideal is an ideal (i.e. a two-sided ideal), e.g. all commutative rings are right duo rings. Related to the duo rings are the right bounded (and strongly right bounded) rings:

5.3.2 Definition (right bounded ring):

A ring is right bounded if every essential right ideal contains a non-trivial two-sided ideal.

5.3.3 Proposition

Any right FPF ring R is right bounded.

Proof:

Let R be right FPF and let I be any essential right ideal of R. We must show that I contains a non-trivial two-sided ideal. Suppose R/I is a faithful right R-module. Then since R/I is cyclic and hence finitely generated, R/I is a generator of mod-R. So $\exists$ n > 0 and an object X of mod-R $\ni$ $h: (R/I)^{(n)} \longrightarrow R \oplus X$ is an isomorphism. Let $x_1, \ldots, x_n \varepsilon R$ be such that $h(\bar{x}_1, \ldots, \bar{x}_n) = 1$.

Let $x^{-1}I = \{a\varepsilon R \,/\, xa\varepsilon I\}$. Then $\bigcap_{i=1}^{n} x_i^{-1}I = 0$. For $a\varepsilon \bigcap_{i=1}^{n} x_i^{-1} I \Rightarrow x_i a\varepsilon I \quad \forall i \Rightarrow x_i a + I = 0 \; \forall i \Rightarrow$ $\bar{x}_i a = 0 \; \forall i$. But then $a = h(\bar{x}_1, \ldots, \bar{x}_n).a =$ $= h(\bar{x}_1 a, \ldots, \bar{x}_n a) = h(0) = 0$.

We now show that $x^{-1}I \trianglelefteq R$ for any $x\varepsilon R$. So let $Q \neq 0$ be a right ideal. Fix arbitrary $0 \neq x\varepsilon R$. Then $xQ = 0 \Rightarrow Q \subseteq x^{-1}I \cap Q \Rightarrow x^{-1}I \cap Q \neq 0$.

On the other hand, $xQ \neq 0 \Rightarrow I \cap xQ \neq 0$, so there is an element $y = xq \neq 0$ in $I \cap xQ$ and then $0 \neq q \;\varepsilon\; x^{-1}I \cap Q$, so $x^{-1}I \cap Q \neq 0$. Hence $x^{-1}I \trianglelefteq R$ as was required. Since any finite intersection of essential right ideals is again essential, we have that $0 = \bigcap_{i=1}^{n} x_i^{-1}I$ is essential, a contradiction.

$\therefore$ R/I is not faithful, so I contains a non-trivial two-sided ideal by 1.4.2.

To what extent is the converse of this proposition true? Although there is no direct converse it will be shown (see 5.4.6) that right self-injective strongly right bounded rings are right FPF.

5.4 STRONGLY RIGHT BOUNDED RINGS

5.4.1 <u>Definition</u> (right self-injective ring):
A ring R is right self-injective in case R_R is injective.

5.4.1 <u>Definition</u> (strongly right bounded ring):
A ring is strongly right bounded if every non-zero right ideal contains a non-zero two-sided ideal.

5.4.3 <u>Remarks</u>

(1) When a ring is strongly right bounded every non-zero right ideal I will contain a two-sided ideal which is essential in I. For let $0 \neq I \leq R$ be a right ideal. Let A be the sum of all ideals contained in I. Then A is a non-zero two-sided ideal. Suppose $K \cap A = 0$ for some right ideal K contained in I. If $K \neq 0$ then K contains a non-zero ideal, so $K \cap A \neq 0$, a contradiction. Hence $K = 0$, so $A \trianglelefteq I_R$.

(2) A submodule K of M is essential in M iff for each $0 \neq x \in M \; \exists \; r \in R \ni 0 \neq xr \in K$ (see [12] lemma 5.19)

Recall (see 1.6.1) that every compactly faithful module is faithful. The converse is true in the presence of strong right boundedness:

5.4.4 Proposition:

A finitely generated faithful module M over a strongly right bounded ring R is compactly faithful.

Proof:

Let M be a finitely generated faithful module. $\exists$ $b_1, \ldots, b_n$ in M $\ni$ $M = b_1R + \ldots + b_nR = \sum_{i=1}^{n} b_iR$. We claim $\bigcap_{i=1}^{n} b_i^{\perp} = 0$. Suppose not. Let $\bigcap_{i=1}^{n} b_i^{\perp} = K$. Now $K \neq 0$ is a right ideal, so R strongly right bounded $\Rightarrow$ $\exists$ by the above remark an ideal $I \leq K \ni I \trianglelefteq K$. Let $0 \neq k \in K$. Then we can find $r \in R \ni 0 \neq kr \in I$ by 5.4.3.2. Since I is a two-sided ideal, $R(kr) \subset I \subset b_i^{\perp}$ $\forall i$. Thus $b_iRkr = 0$ $\forall i$, so $Mkr = 0$. Since M is faithful, $kr = 0$, a contradiction. So $K = 0$ as claimed. Finally, consider the homomorphism $\phi: R \longrightarrow M^n$, $n > 0$, defined by $\phi(a) = (b_1a, \ldots, b_na)$. Now $0 = (b_1a, \ldots, b_na) \Rightarrow b_ia = 0$ $\forall i$, $i = 1, \ldots, n$, $\Rightarrow a \in b_i^{\perp}$ $\forall i \Rightarrow a \in \bigcap_{i=1}^{n} b_i^{\perp} = 0$. Thus $a = 0$, so $\ker \phi = 0$. Thus ϕ is an embedding and hence M is compactly faithful by 1.6.1.

Recall (see 1.6.1) that every generator is compactly faithful. In the presence of right self-injectivity the converse is true:

5.4.5 Proposition:

Over a right self-injective ring a compactly faithful module is a generator.

Proof:

Let R be right self-injective.
Let M be a compactly faithful module. Then we can find $n > 0 \ni \phi: R \longrightarrow M^n$ is an embedding. Since

R_R is injective, $\exists$ f: M^n --> R $\ni$

$$\begin{tikzcd} 0 \arrow[r] & R_R \arrow[r, "\phi"] \arrow[d, "I_R"'] \arrow[dr, phantom, "\copyright"] & M^n \arrow[dl, dashed, "f"] \\ & R_R & \end{tikzcd}$$

Hence ϕ is a split monomorphism, so R is isomorphic to a direct summand of M^n. $\therefore$ M is a generator (see G_3 of 1.3.3).

5.4.6 <u>Proposition</u>:
Any right self-injective strongly right bounded ring R is right FPF.

<u>Proof</u>:
Let M be any finitely generated faithful module. Since R is strongly right bounded, M is compactly faithful by 5.4.4. Since R is right self-injective, M is a generator by 5.4.5. $\therefore$ R is right FPF.

5.4.7 <u>Corollary</u>:
Any commutative self-injective ring is FPF.

<u>Proof</u>:
All commutative rings are strongly right bounded.

5.5 SEMIPERFECT RIGHT FPF RINGS

5.5.1 <u>Proposition</u> (right FPF is a Morita invariant "ring" property):

A ring R is right FPF iff every ring S similar to R is right FPF.

<u>Proof</u>:

"$\Leftrightarrow$" : Let R be right FPF and suppose $R \sim S$. Then $\exists$ a category equivalence F: mod-$R \rightsquigarrow$ mod-S. Since "finitely generated", "faithful" and "generator" are Morita invariant "module" properties, right FPF is a Morita invariant "ring" property. Hence R is right FPF iff S is right FPF follows from the fact that $R \sim R$.

Under what conditions is a semiperfect ring right FPF? The following proposition provides a partial answer to this question:

5.5.2 <u>Proposition</u>:

Any semiperfect right self-injective ring R with strongly right bounded basic ring is right FPF.

<u>Proof</u>:

Let R be semiperfect and R_R injective. Then R semiperfect $\Rightarrow R \sim R_0$ by 4.3.2. Since "injective" is a Morita invariant "module" property, R self-injective $\Rightarrow R_0$ is self-injective. Since R_0 is also given to be strongly right bounded, R_0 is right FPF by 5.4.6. Since "right FPF" is a Morita invariant "ring" property by 5.5.1, $R \sim R_0 \Rightarrow$ R is right FPF.

The following example shows that a right FPF ring need not be semiperfect.

5.5.3 **<u>Example</u>**: $F^{\mathbb{N}}$ where F is a field is right FPF but not semiperfect. For suppose F is a field. Then F is a commutative self-injective ring and F is right FPF by 5.4.7. Since commutativity and injectivity are preserved under direct products, $F^{\mathbb{N}}$ is right FPF. Suppose $F^{\mathbb{N}}/\mathrm{Rad}(F^{\mathbb{N}})$ is semisimple. Then

$F^{\mathbb{N}}/\mathrm{Rad}(F^{\mathbb{N}}) = (F/\mathrm{Rad}F)^{\mathbb{N}} = F^{\mathbb{N}}$ is semisimple and hence a right Noetherian ring by 3.3.8. Let

$I_1 = \{< f_1, 0, 0, \ldots >\}$

$I_2 = \{< f_1, f_2, 0, 0, \ldots >\}$

$\vdots \qquad\qquad \vdots$

where $f_i \varepsilon F \ \forall i \ \varepsilon \ \mathbb{N}$. Then for each $k \varepsilon \mathbb{N}$, I_k is an ideal of $F^{\mathbb{N}}$ and $I_1 < I_2 < I_3 < \ldots$, a contradiction. Hence $F^{\mathbb{N}}$ is not semisimple and so cannot be semiperfect.

5.5.4 Definition (uniform module):

A non-zero module M is uniform in case $I \cap K \neq 0$ for any two non-zero submodules I and K of M.

5.5.4.1 Proposition:

The property of being uniform is a Morita invariant "module" property.

Proof:

A module U_R is uniform iff End(E(U)) is indecomposable. Since the properties "injective", "essential", "indecomposable" are all Morita invariant "module" properties, so is "uniform".

5.5.5 Proposition:

If R is a semiperfect right FPF ring, then

(a) the basic ring R_0 is strongly right bounded;

(b) the basic module B is isomorphic to a direct summand of any faithful finitely generated module;

(c) each right prindec eR is a uniform right ideal, hence $R = \bigoplus_{i=1}^{n} e_iR$ is a direct sum of uniform right prindecs e_iR, $i = 1, 2, \ldots, n$.

Proof:

(a) Suppose R is a semiperfect right FPF ring. Assume R is selfbasic. Let $0 \neq I \leq R$ be any right ideal. Then R/I is cyclic and so finitely generated. If R/I is faithful then R/I is a generator, so I = 0 by 4.4.2 (b), a contradiction. Hence R/I is not faithful, so I contains a non-zero two-sided ideal by 1.4.2. Thus R is strongly right bounded if it is selfbasic, semiperfect and right FPF. Since R_0 satisfies all these conditions R_0 is strongly right bounded.

(b) Let M be any finitely generated faithful module. Since R is right FPF, M is a generator of mod-R. That B is isomorphic to a direct summand of M then follows from 4.4.1 (b).

(c) Suppose R is semiperfect, selfbasic. Then $R = \bigoplus_{i=1}^{m} e_iR$ for right prindecs e_iR, $i = 1, \ldots, m$. Let I and K be right R-submodules of say e_1R. Suppose $I \cap K = 0$. We shall show that either I = 0 or K = 0 which will prove that e_1R is uniform.

Let $e_1R/I \oplus e_1R/K \oplus (1 - e_1)R = M$. Then M is finitely generated. We show that M is faithful. Let $Mr = 0$, $r \in R$. Then $(e_1 + I)r = 0$, $(e_1 + K)r = 0$ and $(1-e_1)r = 0$. Hence $e_1r \in I \cap K = 0$ so $e_1r = 0$. Thus $r = (e_1 + (1 - e_1))r = 0$. So M is faithful. Since R is right FPF, M is a generator of mod-R, so R_R and hence e_1R is isomorphic to a direct summand of M i.e. for some object X of mod-R, $e_1R \oplus X \cong M =$

$= e_1R/I \oplus e_1R/K \oplus e_2R \oplus \ldots \oplus e_nR.$

Since $\mathrm{End}(e_1R)$ is local we have by the exchange lemma that e_1R is isomorphic to a direct summand of one of the summands on the right. If e_1R is isomorphic to a direct summand of one of the e_kR, $k \geq 2$, then $e_1R \cong e_kR$ (since the e_kR are indecomposable), contradicting the irredundancy of $\{e_i\}_{i=1}^{m}$. Hence e_1R is isomorphic to a direct summand of e_1R/I or e_1R/K. In the first case let $\phi: e_1R/I \longrightarrow e_1R \oplus X'$ be the isomorphism.

Consider the composite mapping

$$e_1R \xrightarrow{\pi} e_1R/I \xrightarrow{\phi} e_1R \oplus X' \xrightarrow{\pi_1} e_1R.$$

$\pi_1\phi\pi$ is an epimorphism $\ni I \subseteq \ker(\pi_1\phi\pi)$. Next we show that $\ker(\pi_1\phi\pi) = 0$. For projectivity of e_1R ensures $\exists$ $f: e_1R \longrightarrow e_1R \ni$

$$\begin{tikzcd}
& e_1R \arrow[dl, dashed, "f"'] \arrow[d, "1_{e_1R}"] & \\
e_1R \arrow[r, "\pi_1\phi\pi"'] \arrow[ur, phantom, "\copyright", shift right=3ex] & e_1R \arrow[r] & 0
\end{tikzcd}$$

Then $e_1R = \ker(\pi_1\phi\pi) \oplus \mathrm{Im} f$. Since e_1R is indecomposable, either $\ker(\pi_1\phi\pi) = 0$ or $\ker(\pi_1\phi\pi) = e_1R$. In the latter case, $(\pi_1\phi\pi)e_1R = 0$, so $e_1R = 0$ which is impossible, so $\ker(\pi_1\phi\pi) = 0$. Hence $I = 0$. Similarly in the second case we can conclude that $K = 0$. By repeating the above proof we can show that e_iR is uniform $\forall i$, $i = 1, \ldots, m$. We have proved the theorem in the event that R is selfbasic.

In the general case, let R be any semiperfect right FPF ring, and let R_0 be its basic ring. Then R_0

is semiperfect, right FPF, selfbasic so that each of its prindecs is a uniform R_0-module.

Refer to the proof of 4.3.4. We show that the covariant functor
$Hom_R(e_0R,\_) : \text{mod-}R \rightsquigarrow \text{mod-}R_0$ defines a duality between the module categories in which e_iR maps to e_iR_0. As in 4.3.4 $\{e_i\}_{i=1}^{m}$ is a basic set of idempotents of R_0. Hence by the above e_iR_0 is a uniform R_0-module for each i. But the property of being uniform is a Morita invariant "module" property (5.5.4.1). Hence e_iR is uniform in mod-R, for each $1 \leq i \leq m$, proving that all prindecs of R are uniform.

This completes the proof of the proposition.

5.5.6 Proposition (partial converse of 5.5.2):
Any semiperfect right FPF ring with nil radical J is right self-injective.

Proof:
Let R be a semiperfect right FPF ring. First assume that R is selfbasic. Then $R = \bigoplus_{i=1}^{m} e_iR$ for mutually non-isomorphic right prindecs e_iR, $i = 1, \ldots, m$. We show that the e_iR, $i = 1, \ldots, m$, are their own injective hulls. It suffices to prove that $uR + e_iR = e_iR$ for any $0 \neq u \in E(e_iR)$, $i = 1, \ldots, m$. (For then $u \in E(e_iR) \Rightarrow u \in uR + e_iR \Rightarrow u \in e_iR$)

Let $U = uR + e_1R$. Then $U \subset E(e_1R)$. Let $M = U + (1 - e_1)R$. Indeed this sum is direct. For let $K = U \cap (1 - e_1)R$. Then $K \subseteq U \subseteq E(e_1R)$ and $K \cap e_1R = U \cap (1 - e_1)R \cap e_1R = 0$. But $e_1R \vartriangleleft E(e_1R)$, so that $K = 0$. Then M is faithful for $Mr = 0$, $r \varepsilon R$, $\Rightarrow (U + (1 - e_1)R)r = 0 \Rightarrow (uR + R)r = 0 \Rightarrow Rr = 0 \Rightarrow r = 0$. Also, M is finitely generated, so R selfbasic $\Rightarrow M \cong R \oplus X$ for some object X of mod-R by 4.4.1 (b). Thus

$$M = (uR + e_1R) \oplus (1 - e_1)R \cong e_1R \oplus \dots \oplus e_mR \oplus X.$$

Since $\mathrm{End}(e_1R)$ is a local ring, e_1R is isomorphic to a direct summand of one of the summands on the left. Since $e_1R \ncong e_jR$ $\forall j > 1$, e_1R is isomorphic to a direct summand of $U = uR + e_1R$. But e_1R is uniform by 5.5.5, hence $E(e_1R)$ is uniform:

For let $0 \neq A \leq E(e_1R)$ and $0 \neq C \leq E(e_1R)$. Since $e_1R \vartriangleleft E(e_1R)$, $A \cap e_1R \neq 0$. Also $C \cap e_1R \neq 0$. But e_1R is uniform, so $(A \cap e_1R) \cap (C \cap e_1R) \neq 0$. Thus $A \cap C \neq 0$, so $E(e_1R)$ is uniform. Thus $U \subset E(e_1R) \Rightarrow U$ is also uniform. Thus U is indecomposable. Since $0 \neq e_1R$ is isomorphic to a direct summand of U we must have $U \cong e_1R$. Let $D = \mathrm{End}(U_R)$. Then D is a local ring isomorphic to $\mathrm{End}(e_1R) \cong e_1Re_1$. $\mathrm{Rad}(D) = J(D)$ maps onto $J(e_1Re_1) = e_1Je_1 \subseteq J$, so $J(D)$ is nil. Let $f : U \longrightarrow e_1R$ be the stated isomorphism. Since $e_1R \subseteq U$, $f \varepsilon D$. Now $\ker f = 0$, since f is a monomorphism. Hence f is not nilpotent, so $f \notin J(D)$. Hence f is a unit of D End(U) (see 3.6.4), so in particular f is an epimorphism, so $f(U) = U = e_1R$ as required. So e_1R is injective. Similarly, each e_iR, $i = 2, 3, \ldots, m$, is injective. Since a finite direct sum of injective modules is injective (see 1.8.5), $R = \bigoplus_{i=1}^{m} e_iR$ is injective.

Returning to the general case, let R be a semiperfect right FPF ring with nil radical J. Then its basic ring R_0 is selfbasic. Furthermore, "semiperfect" and "right FPF" are Morita invariant "ring" properties, so $R \sim R_0 \Rightarrow R_0$ is also semiperfect and right FPF. Now $J(R_0) = J(e_0Re_0) = e_0Je_0 \subseteq J$, so $J(R_0)$ is nil. Thus by the above, R_0 is self-injective. But "injective" is a Morita invariant "module" property, so $R \sim R_0 \Rightarrow R$ is right self-injective. This completes the proof of the proposition.

We can summarise the major results of this chapter nicely, in the following theorem which characterises semiperfect rings in the case where R/J is Artinian and J is nil.

5.5.7 <u>Proposition</u>:

Let J(= RadR) be nil and R/J Artinian. Then R is semiperfect. Further R is right FPF iff R_0 is strongly right bounded and R is right self-injective.

<u>Proof</u>:

Idempotents modulo nil ideals lift (see 3.6.3) so R is semiperfect.

"$\Rightarrow$" : Follows from 5.5.5 (a) and 5.5.6.

"$\Leftarrow$" : Is 5.5.2.

CHAPTER 6

SEMIPERFECT CFPF RINGS

6.1 INTRODUCTION

The results gathered in the earlier chapters will now be used to make a detailed analysis of semiperfect right CFPF rings. A right CFPF ring is one all of whose factor rings are right FPF. The end result is that semiperfect CFPF rings have a "Wedderburn-type" characterisation of semisimple rings. More specifically, a ring is semiperfect right CFPF iff it is a finite direct product of matrix rings over right CFPF rings which satisfy the following three properties:

(i) Every right ideal is a two-sided ideal (right duo).

(ii) Every finitely generated right module is a direct sum of cyclics (right σ-cyclic).

(iii) The ideal lattice is well ordered (right valuation ring).

Again we lean heavily on the fact that every semiperfect ring is Morita equivalent to a selfbasic ring R_0. In the presence of this condition R_0 - which is also semiperfect and right CFPF - turns out to satisfy the first two of our conditions. The third condition requires more push and so here, the conditions "selfbasic" and "semiperfect" are replaced by "local", a far stronger condition. This presents no problem for we eventually show that such a R_0 is a finite direct product of local rings, each having all the stated properties.

The transition from R_0 to R requires Morita theory, some standard isomorphism theorems and Kaplansky's celebrated theorem that projectives over local rings are free.

Some useful results are encountered en route to the main theorems: e.g. CFPF is a Morita invariant "ring" property and a finite direct product of CFPF rings is CFPF. A general class of right CFPF rings is also described: any right duo ring all of whose factor rings are right self-injective is right CFPF. Thus a commutative ring R for which R/I is injective for every ideal I is CFPF.

6.2 NOTATION AND TERMINOLOGY

In being consistent with our approach, results are only proved for <u>right</u> CFPF rings.

(a_{ij}) will denote a matrix with entries a_{ij} indexed by i and j;

$(R)_n$ will denote the ring of n x n matrices over R.

6.3 SEMIPERFECT RIGHT CFPF RINGS

6.3.1 <u>Definition</u> (right CFPF ring):
A ring R is right CFPF in case every homomorphic image of R is right FPF.

It is immediately clear that R is right CFPF in case every factor ring of R is right FPF.

6.3.2 <u>Definition</u> (completely right self-injective ring)):
A ring is completely right self-injective in case every factor ring is right self-injective.

6.3.3 <u>Proposition</u>:
Any completely right self-injective right duo ring R is right CFPF.

<u>Proof</u>:

Let R satisfy the given conditions and let $I \leq R$ be any ideal. We show that R/I is right FPF. Now R/I is right duo. For let N/I be any right ideal of R/I. Since R is right duo, N/I is a two-sided ideal. Since right duo rings are strongly right bounded, R/I is strongly right bounded. Since R is completely right self-injective, R/I is self-injective. Hence, by 5.4.6, R/I is right FPF.

6.3.4 <u>Proposition</u>:

If R is a semiperfect selfbasic ring then R/A is semiperfect selfbasic for all ideals A of R.

<u>Proof</u>:

Let R be a semiperfect selfbasic ring. Then $\exists$ local orthogonal idempotents $e_1, \ldots, e_m \ni$ $1 = e_1 + \ldots + e_m$ where $e_iR \not\cong e_jR$ for $i \neq j$. Let A be an ideal of R. Then $\bar{R} = R/A$ is semiperfect (by 3.8). Let

$\phi : R \longrightarrow e_1R/e_1A \oplus \ldots \oplus e_mR/e_mA$ be defined by $r \longrightarrow (e_1r + e_1A, \ldots, e_mr + e_mA)$.

Then ϕ is an R epimorphism with kernel A and so $\bar{R} \cong e_1R/e_1A \oplus \ldots \oplus e_mR/e_mA$ in mod-R $\cong \bar{e}_1\bar{R} \oplus \ldots \oplus \bar{e}_m\bar{R}$ as right R/A-modules. Let $\bar{e}_1\bar{R} \oplus \ldots \oplus \bar{e}_m\bar{R} = S$. The $\bar{e}_i$, $1 \leq i \leq m$ are local orthogonal idempotents of S (see 3.8) such that $\bar{1} = \bar{e}_1 + \ldots + \bar{e}_m$. To show that S is selfbasic we show that none of the $\bar{e}_i\bar{R}$ are isomorphic. Suppose $\bar{e}_i\bar{R} \cong \bar{e}_j\bar{R}$, $i,j \leq m$, in mod-$\bar{R}$. Then $e_iR/e_iA \cong e_jR/e_jA$ in mod-R. Since e_iJ (respectively e_jJ) is the unique maximal submodule of e_iR (respectively e_jR) (see 3.6.9), we conclude that $e_iA \subseteq e_iJ$ (respectively $e_jA \subseteq e_jJ$) and since $e_iJ << e_iR$ (respectively $e_jJ << e_jR$) (see 3.3.2 (d)), we have by 3.3.2 (a) that $e_iA << e_iR$ (respectively $e_jA << e_jR$). Hence $e_iR \longrightarrow e_iR/e_iA$ canonically is a projective cover for e_iR/e_iA. Similarly e_jR is a projective cover for $e_iR/e_iA \cong e_jR/e_jA$. Hence by 3.6.1, $e_iR \cong e_jR$, so $i = j$ by definition of the basic module of R (see 4.2). Thus S is selfbasic. Since $\bar{R} \cong S$, $\bar{R}$ is selfbasic.

Under what conditions are right CFPF rings right duo? Our next proposition shows this happens when the ring is semiperfect and selfbasic:

6.3.5 __Proposition__:

If R is a semiperfect, selfbasic, right CFPF ring then R is a right duo ring.

__Proof__:

Let R satisfy the given conditions. Let I be any right ideal of R. We must show that I is a two-sided ideal. Let $A = (R/I)^{\perp}$ in R i.e.
$A = \{r \varepsilon R \,/\, (R/I)r = 0\} = \{r \varepsilon R \,/\, Rr \subseteq I\}$.
Then A is a two-sided ideal in R. Let $r \varepsilon A$. Then $Rr \subseteq I$, so $1r \varepsilon I$ i.e. $r \varepsilon I$. Hence $A \subseteq I$. If D is any two-sided ideal contained in I, then $d \varepsilon D \Rightarrow Rd \subseteq D \subseteq I \Rightarrow d \varepsilon A$, so $D \subseteq A$. Hence A is the largest two-sided ideal of R in I. Since R is right CFPF, R/A is right FPF. R/A is semiperfect and selfbasic by 6.3.4. So the basic ring of R/A viz. R/A is strongly right bounded (by 5.5.5). Now I/A is a right ideal

of R/A. Suppose $I/A \neq 0$. Then we can find a two-sided ideal $C/A \neq 0 \ni C/A \leq I/A$. Then C is a two-sided ideal of R such that $A \leq C \leq I$. By maximality of A, $C \subseteq A$, so $C = A$. Whence $C/A = 0$, a contradiction. $\therefore$ $I/A = 0$, so $I = A$ is a two-sided ideal.

6.3.6 <u>Proposition</u>:

"Right CFPF" is a Morita invariant "ring" property.

<u>Proof</u>:

Let S be a right CFPF ring. Suppose $R \sim S$. Let I be any ideal of R. By [12] proposition 21.11, $\exists$ an ideal I' of $S \ni R/I \sim S/I'$. Since S/I' is right FPF and "right FPF" is a Morita invariant "ring" property, R/I is right FPF. $\therefore$ R is a right CFPF ring.

6.3.7 <u>Corollary</u> to 6.3.5:

Any semiperfect right CFPF ring is similar to a right duo ring.

<u>Proof</u>:

Let R satisfy the given conditions. Then $R \sim R_0 \Rightarrow$ R_0 is semiperfect and right CFPF (by 3.5 and 6.3.5). Since R_0 is also selfbasic (by 4.3.4), we have by 6.3.5 that R_0 is a right duo ring. Then $R \sim R_0 \Rightarrow R$ is similar to right duo ring.

6.3.8 <u>Definition</u> (right valuation ring):

A ring R is a right valuation ring in case the right ideals of R are linearly ordered. We will denote a right valuation ring by right VR.

Clearly the union of all the proper right ideals of R is the unique maximal right ideal of R, so every VR is a local ring.

6.3.9 **Proposition:**

If R is a right CFPF local ring then the right ideals of R are linearly ordered i.e. R is a right VR.

Proof:

Suppose R satisfies the given conditions. Now R is selfbasic since 1 is a local idempotent and R = 1.R. Let A_1 and A_2 be two proper right ideals of R.

Now R local $\Rightarrow$ R is semiperfect (see end of 3.7). Since R is selfbasic and right CFPF, R is a right duo ring (see 6.3.5). Thus A_1 and A_2 are proper two-sided ideals and so is $A = A_1 \cap A_2$. Now R/A_i is a cyclic right R/A-module for each i. Let $M = R/A_1 \oplus R/A_2$. Then $\operatorname{ann}_R(M) = A$ and M is finitely generated over R/A. Furthermore M is faithful as a right $\bar{R} = R/A$-module. But $\bar{R}$ is right FPF, so M is a generator of mod-$\bar{R}$. By 6.3.4 $\bar{R}$ is selfbasic and semiperfect, so we have by 4.4.1 the following decomposition in mod-$\bar{R}$:

$M \cong R/A_1 \oplus R/A_2 \cong \bar{R} \oplus X$ for some object X of mod-$\bar{R}$. Since R is local so is $\bar{R}$. Thus we have by the exchange lemma that $\bar{R}$ is isomorphic to a direct summand of R/A_1 or R/A_2.

But $\operatorname{End}_{R/A}(R/A_1) = \operatorname{End}_{R/A_1}(R/A_1) \cong R/A_1$ as rings and so $\operatorname{End}_{\bar{R}}(R/A_1)$ is local. $\therefore$ R/A_1 is indecomposable as an $\bar{R}$-module ([12] Theorem 5.10). So $\bar{R} = R/A$ is isomorphic to either R/A_1 or R/A_2 as R/A-modules, and hence as R-modules. Thus either $\operatorname{ann}_R(R/A_1) = \operatorname{ann}_R(R/A)$ or

$ann_R(R/A_2) = ann_R(R/A)$ i.e. $A_1 = A$ or $A_2 = A$. Hence $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.

6.3.10 Definition (right σ-cyclic ring):

A ring R is right σ-cyclic in case every finitely generated right R-module decomposes into a direct sum of cyclic modules.

These rings are also referred to as right FGC rings in the literature.

6.3.11 Proposition:

Let R be a semiperfect ring and let P be a finitely generated projective module. Then there exists a finite set $\{e_\alpha R\}_{\alpha \varepsilon F}$ of prindecs of R such that $P = \bigoplus_{\alpha \in F} e_\alpha R$. Further $|F|$ is the same for all such representations.

Proof:

$\exists$ $n \ni R^n \cong P \oplus P'$. Since R^n is a finite direct sum of prindecs each having local endomorphism ring, P is a direct sum of modules $P_\alpha (\alpha \varepsilon I)$ each having local endomorphism ring (Krull-Schmidt Theorem [11] Proposition 1.2A (2)). Since P is finitely generated we may take I to be finite. By Proposition 3.9.7 $\{e_i R\}_{i=1}^{m}$ is an irredundant set of indecomposable projective modules, if $\{e_i\}_{i=1}^{m}$ is a basic set. $\therefore$ $P \cong \bigoplus_{\alpha \in F} e_\alpha R$ where F is finite and $1 \le \alpha \le m$ $\forall \alpha$. Since each $e_\alpha R$ has a local endomorphism ring, $|F|$ is uniquely determined by P, again by the Krull-Schmidt Theorem.

This proposition allows us to make the following definition:

6.3.12 <u>Definition</u> (projective cover dimension of a module):

Let R be a semiperfect ring and let M be a finitely generated right R-module. Let P be the projective cover of M. Then the number of indecomposable summands in any direct sum decomposition of P into a direct sum of indecomposable right R-modules is called the projective cover dimension of M.

We will denote the projective cover dimension of M by p.c.dim(M).

We observe that p.c.dim(M) will be finite, and well-defined since projective covers are unique up to isomorphism.

6.3.13 <u>Proposition</u>:

A semiperfect, selfbasic, right CFPF ring R is right σ-cyclic.

<u>Proof</u>:

Suppose R satisfies the given conditions.
Let M be any finitely generated right R-module. We may assume $M \neq 0$. Let $D = \mathrm{ann}_R(M)$. Then M is finitely generated faithful over R/D which is right FPF. So M is a generator of mod-R/D. Since R/D is selfbasic (see 6.3.4) we have by 4.4.1 the following decomposition in mod-R/D:
$M \cong R/D \oplus X$ for some object X of mod-R/D.
This is a decomposition in mod-R as well, and as R-modules, R/D and X are finitely generated. Let P' in mod-R be any projective cover of R/D and P'' be any projective cover of X. Then
$P' \oplus P'' \longrightarrow R/D \oplus X$ is a projective cover in mod-R (see 3.6.2). Let p.c.dim(X) = m. Then
p.c.dim(M) = p.c.dim(R/D) + p.c.dim(X) $> m$.
Thus if $X \neq 0$ we can split X as we did M. So
$X \cong R/A_1 \oplus X'$ in mod-R/A_1 where
$A_1 = \mathrm{ann}_R(X)$. So $X \cong R/A_1 \oplus X'$ in mod-R, so

$M \cong R/D \oplus R/A_1 \oplus X'$ in mod-R. Since $MD = 0$, $X \leq M \Rightarrow XD = 0$, so $D \subseteq A_1$. Now p.c.dim$(X') < m$. If $X' \neq 0$ we can proceed as above. Since p.c.dim(M) = n is finite, this process must stop after at most n steps, at which stage we will have $M \cong R/D \oplus R/A_1 \oplus \ldots \oplus R/A_m$, where $D \leq A_1 \leq \ldots \leq A_m$, is a direct sum of cyclic modules in mod-R. So R is right σ-cyclic by 6.3.10.

6.3.14 Lemma 1:

Let $R = \prod_{i=1}^{n} R_i$. Then R is right CFPF iff R_i is right CFPF, $i = 1,\ldots,n$.

Proof:

See [11] Theorem 3.4.2.

Lemma 2:

A finite product of semiperfect rings is semiperfect.

Proof:

Let $R = \prod_{i=1}^{n} R_i$ where each R_i is semiperfect. Let $J = \text{Rad}R$ and $J(R_i) = \text{Rad}(R_i)$. Then $J = \prod_{i=1}^{n} J(R_i)$ and $\prod_{i=1}^{n} R_i/J(R_i)$ is a finite direct product of semisimple rings (see 3.7 (b)) and hence semisimple. But $R/J \overset{\phi}{\cong} \prod_{i=1}^{n} R_i/J(R_i)$, so R/J is semisimple. Let $\bar{u}$ be any idempotent of R/J. Then $\phi(\bar{u}) = (\bar{u}_1,\ldots,\bar{u}_n)$ where for each i, $\bar{u}_i$ is an idempotent of $R_i/J(R_i)$. So $\exists$ an idempotent $e_i \in R_i \ni \bar{e}_i = \bar{u}_i$ (see 3.6.3) $\forall i$. Let $e = (e_1,\ldots, e_n) \in R$. Then $e^2 = e$ and $\bar{e} = \bar{u}$. Hence idempotents lift from R/J to R. $\therefore$ R is semiperfect by 3.7 (b).

6.3.15 STRUCTURE THEOREM

A ring R is semiperfect and right CFPF iff R is similar to a finite product of right duo right VR right σ-cyclic right CFPF rings.

Proof:

"$\Rightarrow$" : Suppose R satisfies the given conditions. Assume R is selfbasic. Then $R = \bigoplus_{i=1}^{m} e_iR$ where e_i are local orthogonal idempotents. By 6.3.5, R is right duo. Let $A_1 = e_1^{\perp} = (1 - e_1)R$. Then A_1 is an ideal of R, so $RA_1 \subseteq A_1 = e_1^{\perp}$. Hence $e_1RA_1 = 0$. So in particular $e_1R(1 - e_1) = 0$ i.e. $e_1R = e_1Re_1$. Similarly $e_iR = e_iRe_i$ $\forall i$. Thus $R = \bigoplus_{i=1}^{m} e_iR = \bigoplus_{i=1}^{m} e_iRe_i$. So R is a finite direct sum (= finite product) of local rings e_iRe_i which are right CFPF by Lemma 1. By 6.3.9 each e_iRe_i is a right VR. Now each e_iRe_i being local is semiperfect, selfbasic and CFPF. Thus each e_iRe_i is right duo by 6.3.5 and right σ-cyclic by 6.3.13. $\therefore$ We have R equal to a finite direct product of right duo right VR right σ-cyclic right CFPF rings.

In general, we use the fact that $R \sim R_0$ to deduce that R_0 is semiperfect and right CFPF and since R_0 is selfbasic (see 4.3.4) the result follows.

"$\Leftarrow$" : Let $S = \prod_{i=1}^{n} S_i$ where for each i, $1 \leq i \leq n$, S_i is a right duo right VR right σ-cyclic right CFPF ring. Suppose $R \sim S$. We first show that S is semiperfect and right CFPF. For each i, S_i is a right VR, hence a local ring by 6.3.8 and thus semiperfect (see example at the end of 3.7). By Lemma 2, S is therefore semiperfect. Also, for each i, S_i is right CFPF, so S is right CFPF by Lemma 1. Finally, since "semiperfect" and "right CFPF" are Morita invariant "ring" properties, $R \sim S \Rightarrow R$ is semiperfect and right CFPF.

To present 6.3.17 we need the following lemmas.

6.3.16 <u>Lemma (a)</u>:

Let M be a module. Then for any $n > 0$ $\mathrm{End}(M^n) \cong (\mathrm{End}(M))_n$, the n x n matrix ring over End(M).

<u>Proof</u>: (see [12] Proposition 13.2)

<u>Lemma (b)</u>:

Let $R = R_1 x \ldots x R_m$. Then as matrices $(R)_n \cong (R_1)_n x \ldots x (R_m)_n$.

<u>Proof</u>:

For $r \varepsilon R$, let r_i be the i-th component, $1 \le i \le m$. Consider the map

$$\phi : (a_{ij}) \longrightarrow (((a_{ij})_1), \ldots, ((a_{ij})_n))$$

of $(R)_n$ into $(R_1)_n x \ldots x (R_m)_n$. This is a ring isomorphism.

6.3.17 <u>CHARACTERISATION THEOREM</u> of semiperfect right CFPF rings:

A ring R is semiperfect and right CFPF iff R is a finite product of full matrix rings over right duo right VR right σ -cyclic right CFPF rings.

<u>Proof</u>:

"$\Rightarrow$" : Suppose R is a semiperfect right CFPF ring. Then so is R_0. Since R_0 is also selfbasic R_0 is a finite product of right duo right VR right σ-cyclic right CFPF rings by 6.3.15. So we have $R_0 = R_1 x \ldots x R_m$ where for each i, $1 \leq i \leq m$, R_i is a right duo right VR right σ-cyclic right CFPF ring. By Lemma (b), as matrices, for any n $(R_0)_n \cong (R_1)_n x \ldots x (R_m)_n$.

By [12] corollary 22.7, since $R \sim R_0$, there exists an n and an idempotent matrix $e \varepsilon (R_0)_n \ni R \cong e(R_0)_n e$ as rings.

So $(R_0)_n \cong (R_1)_n x \ldots x (R_m)_n$.

Let 0 denote this isomorphism.

Now e is the identity of $e(R_0)_n e$. Suppose $\phi(e) = (e_1, \ldots, e_m)$. Then each e_i is an idempotent matrix of $(R_i)_n$.

Applying [12] Proposition 7.8 we have for each i, that $e(R_i)_n e \cong e_i (R_i)_n e_i$. So $e(R_0)_n e \cong e_1 (R_1)_n e_1 x \ldots x e_m (R_m)_n e_m$.

Fix i,$1 \leq i \leq m$. By [12] Proposition 4.11, $R_i \cong \mathrm{End}_{R_i}(R_i)$. By Lemma (a), $(\mathrm{End}(R_i))_n \cong \mathrm{End}(R_i{}^n)$. So $e_i (R_i)_n e_i \cong e_i' \mathrm{End}(R_i{}^n) e_i'$ where the mapping $e_i' : R_i{}^n \longrightarrow R_i{}^n$, defined by

$$\begin{bmatrix} r_i \\ \vdots \\ r_{i_n} \end{bmatrix} \longrightarrow e_i \begin{bmatrix} r_i \\ \vdots \\ r_{i_n} \end{bmatrix}$$

is clearly a ring homomorphism. So e_i' is an idempotent of $\mathrm{End}(R_i{}^n)$, so by [12] Proposition 5.9, $e_i'\mathrm{End}(R_i{}^n)e_i' \cong \mathrm{End}(e_i'R_i{}^n)$. Also e_i' induces a decomposition

$R_i{}^n \cong e_i'R_i{}^n \oplus (1 - e_i')R_i{}^n$.

By [12] Corollary 17.3 this means that $e_i'R_i{}^n$ is finitely generated and projective over R_i. Since projective modules over VR rings (which are local rings by 6.3.8) are free, $\exists\ n_i > 0 \ni$ $e_i'R_i{}^n \cong R_i{}^{n_i}$. So $\mathrm{End}(e_i'R_i{}^n) \cong \mathrm{End}(R_i{}^{n_i})$. Thus for each j, $1 \le j \le m$, $\exists\ n_j \ni$

$e(R_0)_n e \cong$

$\cong \mathrm{End}(R_1{}^{n_1}) \times \ldots \times \mathrm{End}(R_m{}^{n_m})$

$\cong (\mathrm{End}(R_1))_{n_1} \times \ldots \times (\mathrm{End}(R_m))_{n_m}$ by Lemma (a)

$\cong (R_1)_{n_1} \times \ldots \times (R_m)_{n_m}$ by [12] Proposition 4.11

= a finite product of full matrix rings over right duo right VR right σ-cyclic right CFPF rings. That R is also such a product follows from the fact that $R \cong e(R_0)_n e$.

"$\Leftarrow$" : Suppose $R = (R_1)_n \times \ldots \times (R_m)_n$ where for each i, $1 \le i \le m$, $(R_i)_n$ is a full matrix ring over R_i with R_i a right duo right VR right σ-cyclic right CFPF ring. By [12] Corollary 22.6, for each i, $(R_i)_n \sim R_i$. But R_i is right CFPF and since "right CFPF" is a Morita invariant "ring" property, $(R_i)_n \sim R_i \Rightarrow (R_i)_n$ is right CFPF for each i. Thus R is right CFPF by Lemma 1. Also for each i, R_i is a right VR, hence local and so semiperfect. Thus $(R_i)_n \sim R_i \Rightarrow (R_i)_n$ is semiperfect for each i ("semiperfect" is a Morita invariant "ring" property) and hence R is semiperfect by Lemma 2.

REFERENCES

[1] Auslander, M. and Goldman, O. "Maximal orders", Trans. Amer. Math. Soc., 97 (1960), 1-24.

[2] Azumaya, G. "A duality theory for injective modules", Amer. J. Math., 81 (1959), 249-278.

[3] ---------------- "Completely faithful modules and self-injective rings", Nagoya J. Math., 17 (1966), 249-78.

[4] Bass, H. "Finitistic dimension and a homological generalisation of semiprimary rings", Trans. Amer. Math. Soc., 95 (1960), 466-488.

[5] ---------------- Algebraic K-Theory, Benjamin, New York, 1968.

[6] Faith, C. Algebra: Rings, Modules and Categories I, Bd. 190, Springer-Verlag, 1973.

[7] ---------------- Algebra II : Ring Theory, Springer-Verlag, 1976.

[8] ---------------- Characterisations of rings by faithful modules, Lecture notes, Math. Dept , Technion, Haifa, 1976.

[9] ---------------- "Semiperfect Prufer and FPF rings", Israel Math. J., 26 (1977), 166-177.

[10] ---------------- "Injective cogenerator rings and a theorem of Tachikawa I, II", _Proc. Amer. Math. Soc._, 60 (1976), 25-30; 62 (1977), 15-18.

[11] Faith, C. and Page, S. _FPF Ring Theory (faithful modules and generators of mod-R)_, London Mathematical Society, Lecture Note Series No. 88, Cambridge Univ. Press, 1984.

[12] Fuller, K. and Anderson, F. _Rings and Categories of Modules_, Graduate texts in Mathematics Vol. 13, New York, Springer-Verlag, 1974.

[13] Hungerford, T.W. _Algebra_, Graduate Texts in Mathematics No. 73, New York, Springer-Verlag, 1974.

[14] Ikeda, M. "A characterisation of quasi-Frobenius rings", _Osaka Math. J._, 4 (1952), 203-210.

[15] Kaplansky, I. "Projective modules", _Ann. Math._, 68 (1958), 372-377.

[16] Lambek, J. _Lectures on Rings and Modules_, 2nd ed., Chelsea Publishing Co., New York, 1976.

[17] Michler, G.O. "Asano orders", _Proc. Lond. Math. Soc._, 19 (1969), 421-43.

[18] Morita, K. "Duality for modules and its applications to the theory of rings with minimum condition", _Sci. Reports_, _Tokyo Kyoiku Daigaku_, 6 (1958), 83-142.

[19] Müller, B. "On semiperfect rings", Illinois J. Math., 14 (1970), 464-467.

[20] Nakayama, T. "On Frobeniusean algebras I,II", Ann. Math., 40 (1939), 611-633.

[21] ---------------- "Note on uniserial and generalised uniserial rings", Proc. Imp. Acad. Tokyo, 16 (1940), 285-289.

[22] Osofsky, B. "A generalisation of quasi-Frobenius rings", J. Algebra, 4 (1966), 373-387.

[23] ---------------- Homological Dimensions of Modules, Regional Conference Series in Mathematics No. 12, Amer. Math. Soc., 1973.

[24] Page, S. "FPF rings and some conjectures of C. Faith", Canada Bull. Math., 1983/84.

[25] ---------------- "Semiperfect FPF rings", Proc. Amer. Math. Soc., 1983/84.

[26] Tachikawa, H. Quasi-Frobenius Rings and Generalisations, Lecture Notes in Mathematics No. 351, Springer-Verlag, New York, 1973.

[27] Utumi, Y. "Self-injective rings" J. Algebra, 6 (1967), 56-64.