CONTINUOUS SYMMETRIES

OF

DIFFERENCE EQUATIONS

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This dissertation is submitted to the School of Mathematical Sciences, Faculty of Science and agriculture, University of KwaZulu-Natal, Durban, in fulfillment of the requirements for the degree of Master of Science.

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As the candidate's supervisor, I have approved this dissertation for submission.

Signed: Pr. K S Govinder December 2011

Abstract

We consider the study of symmetry analysis of difference equations. The original work done by Lie about a century ago is known to be one of the best methods of solving differential equations.

Lie's theory of difference equations on the contrary, was only first explored about twenty years ago. In 1984, Maeda [42] constructed the similarity methods for difference equations. Some work has been done in the field of symmetries of difference equations for the past years. Given an ordinary or partial differential equation (PDE), one can apply Lie algebra techniques to analyze the problem. It is commonly known that the number of independent variables can be reduced after the symmetries of the equation are obtained. One can determine the optimal system of the equation in order to get a reduction of the independent variables. In addition, using the method, one can obtain new solutions from known ones. This feature is interesting because some differential equations have apparently useless trivial solutions, but applying Lie symmetries to them, more interesting solutions are obtained.

The question arises when it happens that our equation contains a discrete quantity. In other words, we aim at investigating steps to be performed when we have a difference equation. Doing so, we find symmetries of difference equations and use them to linearize and reduce the order of difference equations. In this work, we analyze the work done by some researchers in the field and apply their results to some examples.

This work will focus on the topical review of symmetries of difference equations and going through that will enable us to make some contribution to the field in the near future.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

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May the Lord be glorified for His grace upon me in my studies and ministry duties.

Declaration 1 - Plagiarism

I, Bienvenue Feugang Nteumagne, declare that

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Signed

Dedication

TO MY DEAR MOTHER

Now go, write it before them in a table, and note it in a book, that it may be for the time to come for ever and ever. **Isaiah 30: 7-8**

TO GOD BE THE GLORY

Preface

Will there ever be a complete Universe? A Universe where there will be nothing to seek? A Universe where everything has been discovered? Will the inhabitants of the world ever rest from exploring? Oh how we would have wished it had been the case! Unfortunately, the infinite dimension of the Universe brings about an infinite variety of problems.

About a century ago, a Norwegian Mathematician S Lie in his research discovered an efficient method for solving differential equations. This theory is a very popular tool and owes its popularity to the fact that it takes complex problems and simplifies them into easier ones through transformations in the variables involved. Once the easier problems are solved, the inverse transformations are carried out to achieve solution of the original ones.

After a century of application of this method to differential equations, a question only arose decades ago as to whether this " powerful" method could be used to solve difference equations.

As we embark on this journey of gathering the information that is already available in this field of study, we hope to discover new things and work towards the completion of our Scientific Universe.

B F Nteumagne

December 2011

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Chapter 1

Introduction

1.1 The importance of differential equations

Differential equations (DEs) arise in many areas of science and technology. In particular they arise whenever a deterministic relationship involving quantities that change in a uniformly or nonuniformly continuous way and their rates of change [33, 60] in space and time is known or formulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as time varies. Newton's laws [12] allow one to relate the position, velocity, acceleration and various forces acting on the body and state this relation as a differential equation for the unknown position of the body as a function of time. In some cases, this differential equation (called an equation of motion) may be solved explicitly and is given by the equation

$$m\ddot{x} = F(x, \dot{x}, t), \tag{1.1}$$

where m is the mass of the moving body, $x(t)$ its position at each time t, $F(x, \dot{x}, t)$ represents the sum of all external forces acting on the body and the '$\,\ddot{}\,$' represents derivative with respect to t. Equation (1.1) is a second order ordinary differential equation (ODE).

An example of modeling a real world problem using DEs is the determination of the velocity of a ball falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the deceleration due to air resistance. Gravity is constant but air resistance may be modeled as proportional to the ball's velocity [12]. This means the ball's acceleration, which is the derivative of its velocity, depends on the velocity. Finding the velocity as a function of time involves solving a differential equation of the same form as equation (1.1) above.

Additionally, in an ecosystem constituted by rabbits and foxes for example, we have a predator-prey interaction between the two species - the foxes feed on the rabbits. The population r and f of theses two species rabbits and foxes respectively at any time t can be modeled using the Lotka-Volterra equations

$$\begin{aligned} \dot{r} &= r(a - bf) \\ \dot{f} &= f(c - dr), \end{aligned} \tag{1.2}$$

where a, b, c and d are the parameters of interaction between the two species. The Lotka-Volterra model (1.2) is a system of scalar first order nonlinear ODEs in the unknowns r and f.

It is also possible to model the diffusion of heat in a body as follows: Let t be the time it takes for heat to diffuse to a distance x from the heat source and $u(x, t)$ the heat at time t and at the distance x from the source. Then, one of the well known linear partial differential equations (PDEs) is the one-dimensional heat equation given by

$$u_t - u_{xx} = 0. \tag{1.3}$$

Finally, let t, x and y be the independent variables which are the coordinates of the model and u, v and ϕ the dependent variables, u and v are the velocity components of the flow and ϕ is the preasure exerted on the fluid. Then the Navier-Stokes equations for two dimensional flows are given by

$$\begin{aligned} u_t + uu_x + vu_y &= -\phi_x + u_{xx} + u_{yy} \\ v_t + uv_x + vv_y &= -\phi_y + v_{xx} + v_{yy} \\ u_x + v_y &= 0. \end{aligned} \tag{1.4}$$

The first two are known as the Navier-Stokes equations and the latter is the continuity equation, and all three of them form a system of nonlinear PDEs. This equation has been thoroughly studied using Lie symmetry analysis.

DEs are mathematically studied from several different perspectives [34]. We are mostly concerned with their solutions, i.e., the set of functions that satisfy the equation. Only the simplest DEs admit solutions given by explicit formulas. However, some properties of solutions of a given DE may be determined without finding their exact form. If an exact formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by DEs, while many numerical methods have been developed to determine solutions with a given degree of accuracy. A century ago, the Norwegian scientist Sophus Lie developed a method that tackles the problem of complexity in DEs. Given a complicated DE, his method allows us to transform this into a simple DE, the solution of which is sometimes known, and the reverse transformation leads to solution of the original problem.

1.2 Short historical background on Lie groups

Sophus Lie himself considered the winter of 1873 to 1874 as the birth date of his theory of continuous groups. Hawkins, however, suggests that it was "Lie's prodigious research activity during the four-year period from the fall of 1869 to the fall of 1873" that led to the theory's creation [59]. Some of Lie's early ideas were developed in collaboration with Felix Klein. Lie met with Klein every day from October 1869 through 1872: in Berlin from the end of October 1869 to the end of February 1870, and in Paris, Göttingen and Erlangen in the subsequent two years [59]. Lie stated that all of the principal results were obtained by 1884. However, during the 1870s all his papers (except the very first note) were published in Norwegian journals, which impeded recognition of the work throughout the rest of Europe [21]. In 1884 a young German mathematician, Friedrich Engel, came to work with Lie on a systematic treatise to expose his theory of continuous groups [59]. From this effort resulted the three-volume *Theorie der Transformationsgruppen*, published in 1888, 1890, and 1893 [59]. Lie's ideas did not stand in isolation from the rest of mathematics. In fact, his interest in the geometry of DEs was first motivated by the work of Carl Gustav Jacobi, on the theory of PDEs of first order and on the equations of classical mechanics [4]. Much of Jacobi's work was published posthumously in the 1860s, generating enormous interest in France and Germany [21].

Lie's *idée fixée* was to develop a theory of symmetries of DEs that would accomplish for them what Evariste Galois had done for algebraic equations: namely, to classify them in terms of group theory [59]. Additional impetus to consider continuous groups came from ideas of Bernhard Riemann, on the foundations of geometry, and their further development in the hands of Klein [59]. Thus three major themes in 19th century mathematics were combined by Lie in creating his new theory: the idea of symmetry, as exemplified by Galois through the algebraic notion of a group; geometric theory and the explicit solutions of DEs of mechanics, worked out by Poisson and Jacobi; and the new understanding of geometry that emerged in the works of Plücker, Möbius, Grassmann and others, and culminated in Riemann's revolutionary vision of the subject.

Although today Sophus Lie is rightfully recognized as the creator of the theory of continuous groups, a major stride in the development of their structure theory, which was to have a profound influence on subsequent development of mathematics, was made by Wilhelm Killing, who in 1888 published the first paper in a series entitled *Die Zusammensetzung der stetigen endlichen Transformationsgruppen* (The composition of continuous finite transformation groups) [21]. The work of Killing, later refined and generalized by Élie Cartan, led to classification of semi-simple Lie algebras, Cartan's theory of symmetric spaces, and Hermann Weyl's description of representations of compact and semi-simple Lie groups using highest weights [21]. Weyl brought the early period of the development of the theory of Lie groups to fruition, for not only did he classify irreducible representations of semi-simple Lie groups and connect the theory of groups with quantum mechanics, but he also put Lie's theory itself on firmer footing by clearly enunciating the distinction between Lie's infinitesimal groups (i.e., Lie algebras) and the Lie groups proper, and began investigations of topology of Lie groups [4] . The theory of Lie groups was systematically reworked in modern mathematical language in a monograph by Claude Chevalley [21].

Lie symmetry analysis of DEs is a systematic way of finding exact solutions of ordinary and PDEs. It permeates many mathematical models and in particular those formulated in terms of DEs. The mathematical discipline that embodies and synthesizes symmetries of DEs is called Lie group theory. A symmetry is a change, a transformation that leaves an object invariant or apparently unchanged [48]. Generally, an object needs not have only one symmetry, but many symmetries. The collection of symmetries of an object has a beautiful internal structure, it forms a group.

Quantification of symmetry indeed turns out to be a very important aspect of Lie groups. For instance, we know that a square has fewer symmetries than does the circle, but more than a triangle. The formal definition of symmetry allows for quantification.

Felix Klein's Erlangen program of 1872 pronounced that Geometry (at that time) was symmetry [59]. Klein is reported to have said that: "Geometrical properties are characterized by their invariance under groups of transformation."[4]. It was Lie who discovered the theory of transformation groups. Moreover, he introduced groups into geometry. He tackled fundamental problems and his first paper was on geometry. Lie's theory of transformation groups provided a synthesis. He said: "My theory of invariants of all continuous groups embraces all theories of invariants hitherto noted." [59].

1.3 Difference equations and applications

Mathematical computations are frequently based on equations that allow us to compute the value of a function recursively from a given set of values. Such an equation is called "difference equation" (ΔEs) or "recurrence equation". Problems that involve discrete variables often lead to mathematical models involving ΔEs [19]. In economics for example, some financial information (such as savings, national income, government spending, interest rate movements) are only available on a quarterly, semi-annually, or yearly basis. Models that will best describe variations in such variables will have to be designed in terms of ΔEs, since the time variable is discrete. The study of ΔEs has received significant attention in the past few decades [38–42]. This is due to their many applications in real-life problems. ΔEs have many areas of application such as in mathematics, physics, chemistry, astrophysics, economics, finance and social sciences, (see for example [45]) just to mention a few. The construction of models with discrete time dependence appeals to ΔEs. Many phenomenon are well modeled by use of ΔEs. Population modeling cannot receive a realistic investigation unless models are designed in terms of ΔEs: When modeling the population of a species, one can only have a whole number of participants in the population. This is one of the several applications of ΔEs.

In economics, the national income, is modeled by use of the expression

$$Y_t = C_t + I_t + G_t, \tag{1.5}$$

where Y_t is the amount of the gross domestic product or national income at time t, C_t the monetary value of the consumer expenditure on consumption goods and services, I_t the monetary value of the aggregate spending on long term investment and G_t the monetary value of the total government spending. Each component is modeled separately and the final model is given by the second order non-homogeneous ordinary difference equation (OΔE)

$$Y(t+2) - Y(t+1) + \beta Y(t) = 1, \tag{1.6}$$

where β is known as the marginal propensity to consume, i.e., the slope of the consumption function [19]. The general approach used to solve this equation has always assumed that the time variable t is continuous, which is unrealistic because in general, all the components of the national income are recorded every quarter, or each year. The time variable must be treated as a discrete variable if one needs to construct a model that reflects the real situation.

Additionally, consider the heat transfer in a room which is only accessible once a day. A rod is placed in the room to measure the heat from the heat source to the door. It is obvious that our data will give us a daily temperature t (discrete) on the distance x (continuous) from the source. This scenario can be modeled using the system

$$\Delta_t u - u_{xx} = 0, \tag{1.7}$$

where

$$\Delta_t u = \frac{u(t+h) - u(t)}{h}. \tag{1.8}$$

Such an equation is called differential-difference equation (DΔE).

In finance, the first order OΔE

$$A_{k+1} - (1+i)A_k + (k+1)R = 0 \tag{1.9}$$

models the amount of an annuity that pays an amount of kR after a time interval of k years at an interest rate i. It is easy to show that the accumulated amount will be given by the formula [45]

$$A_k = \frac{R}{i}[(1+i)S_{ki} - k], \tag{1.10}$$

where $S_{ki} = \frac{(1+i)^k - 1}{i}$ is the accumulated amount of 1 unit invested for k years at i. Furthermore, in economics and finance, simple and compound interest are modeled by use of ΔEs. We are able to determine the value an amount of money place under a fixed interest into a bank after a period by use of ΔEs. The simple interest law is written in the following OΔE

$$D_{k+1} - D_k - rD_0 = 0, \tag{1.11}$$

where D_k is the amount available at time k years, D_0 is the initial available amount and r is the interest rate [45]. This equation is classified as a linear first-order non-homogeneous OΔE with constant coefficients. Solutions to such can be easily found using recurrence relations. In labor management, the dynamical equations that describe the negotiating process between labor and management are given by

$$\begin{aligned} M_{k+1} &= M_k + \alpha(L_k - M_k) \\ L_{k+1} &= L_k - \beta(L_k - M_k), \end{aligned} \tag{1.12}$$

where M_k and L_k denote the management offer and labor demand respectively [5, 45]. This is a system of OΔEs and direct methods can be used to solve them explicitly. In biological modeling we observe that single species populations are modeled by the OΔE [45]

$$N_{k+1} = fN_k, \tag{1.13}$$

where N_k is the population size at time k and f the reproduction rate, while a red blood cell production model can be described by the equation [45]

$$R_{k+2} - (1-f)R_{k+1} - \gamma R_k = 0, \tag{1.14}$$

where R_k is the number of white blood cells produced at time k, f is the fraction of red blood cell removed by the spleen, and γ the production constant. This equation will allow us to know how many red blood cells are produced each day.

ΔEs also find applications in physics. For instance, models of systems in physical sciences provide important insight into the working of the natural world. The construction of the associated discrete models in general relies on the discrete nature and how the various properties of these systems are both measured and analyzed. Hence, the discrete mathematical formulation of the problem is often an exact reflection of the actual experimental procedures used to define the system of interest, which is that of the various time-scales that occur in the experimental analysis and mathematical formulation of physical systems. These results are then applied to the development of Newton's law of cooling.

Considering a compact object, such as a glass of hot water, located in a "quiet" room. We are interested in determining how the temperature of the object changes as time evolves.

Let:

T_0 = initial temperature of the object;

T_R = temperature of the room;

Δt =time between temperature increments;

T_s = time-constant of the system;

T_k = temperature of the body at time $t_k = k(\Delta t)$.

Then the rate of cooling is found to be proportional to the difference between the temperature of the object and the temperature of the room [45], i.e.,

$$\frac{T_{k+1} - T_k}{\Delta t} \propto (T_k - T_R). \tag{1.15}$$

In mathematical physics, one may discretize the Navier-Stokes DEs to seek numerical solutions. An example would be the consideration of the numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. The normal strategic first step to attempt to solving this problem would be the Navier-Stokes equations. In continuous-time, this system is given by (1.4). The method of finite-difference provides a system of ΔEs

$$
\begin{aligned}
\frac{u_{m,n,p+1}-u_{m,n,p}}{\Delta t} + u_{m,n,p}\frac{u_{m+1,n,p}-u_{m,n,p}}{\Delta x} + v_{m,n,p}\frac{u_{m,n+1,p}-u_{m,n,p}}{\Delta y} \\
= -\frac{\phi_{m+1,n,p}-\phi_{m,n,p}}{\Delta x} + \frac{u_{m+2,n,p}-2u_{m+1,n,p}+u_{m,n,p}}{(\Delta x)^2} \\
+\frac{u_{m,n+2,p}-2u_{m,n+1,p}+u_{m,n,p}}{(\Delta y)^2} \\
\frac{v_{m,n,p+1}-v_{m,n,p}}{\Delta t} + u_{m,n,p}\frac{v_{m+1,n,p}-v_{m,n,p}}{\Delta x} + v_{m,n,p}\frac{v_{m,n+1,p}-v_{m,n,p}}{\Delta y} \\
= -\frac{\phi_{m,n+1,p}-\phi_{m,n,p}}{\Delta y} + \frac{v_{m+2,n,p}-2v_{m+1,n,p}+v_{m,n,p}}{(\Delta x)^2} \\
+\frac{v_{m,n+2,p}-2v_{m,n+1,p}+v_{m,n,p}}{(\Delta y)^2} \\
\frac{u_{m+1,n,p}-u_{m,n,p}}{\Delta x} + \frac{v_{m,n+1,p}-v_{m,n,p}}{\Delta y} = 0,
\end{aligned}
\tag{1.16}
$$

which need be solved.

1.4 Outline

We wish to study Lie's theory of extended groups applied to ΔEs. This method could provide a great tool to solve nonlinear problems when applied to ΔEs. We will review the work done on continuous symmetries of ΔEs, both ordinary and partial difference equations. Thereafter we will apply this method to some interesting equations. We wish to illustrate the usefulness of Lie analysis in ΔEs as it has proved its efficiency for DEs. After defining the key concepts in this field, we solve some equations of interest using Lie symmetry analysis. We first deal with the known field of Lie symmetry analysis of DEs in Chapter 2. Secondly, we investigate the work done in continuous symmetries of OΔEs in Chapter 3. Thirdly, we apply Lie's theory to (PΔEs) in Chapter 4. In Chapter 5 we summarize our work and discuss some open problems in the field.

Chapter 2

Symmetries of Differential Equations

2.1 Introduction

Once the symmetries of a system are obtained, one can deduce new solutions from known ones using these symmetries. Additionally, we can classify the families of equations into equivalence classes. This is achieved via the construction of types of equations that admit a prescribed group of transformations. We can also linearize equations by invertible transformations. In the case of ODEs, we can reduce the order using the admitted symmetries. Once the symmetries of a PDE are obtained, reduction of the PDE via a combination of the number of independent variables is possible. Finally, solutions via Lie's theory constitutes a benchmark for testing numerical algorithms.

Conserved quantities are very important in mathematical physics. They provide information on the properties of the solutions of differential equations [30]. Conservation of momentum, energy and mass play a crucial role in the analysis of solutions of PDEs and the application of symmetries of differential equations to determine these quantities is very important.

Definition 2.1.1 *An* equivalence transformation *of a function is a reversible transformation of independent, dependent or both variables that preserves its form . We say that such functions are invariants* [48]. ■

Example 2.1.2 *Consider the differential equation given by*

$$y'' = \frac{y'}{x} + \frac{4}{x^3}y^2. \tag{2.1}$$

The transformation

$$\bar{x} = \frac{1}{x}, \qquad \bar{y} = \frac{y}{x^2} \tag{2.2}$$

leaves (2.1) invariant, and so is an equivalence transformation of (2.1). The same is true of

$$\bar{x} = ax \qquad \bar{y} = ay, \tag{2.3}$$

where a is a real constant. ■

Example 2.1.3 *An equation of the form*

$$u_{tx} + a(t,x)u_t + b(t,x)u_x + c(t,x)u = 0 \tag{2.4}$$

was discovered by Laplace [31, 37] to have

$$h = a_t + ab - c, \qquad k = b_x + ab - c \tag{2.5}$$

as invariants. These are called the Laplace invariants. *They are invariant under linear homogeneous transformations of the dependent variable*

$$\bar{u} = \sigma(t,x)u, \qquad \sigma(t,x) \neq 0. \tag{2.6}$$

The equations

$$u_{tx} - u_t + u_x - u = 0 \tag{2.7}$$

and

$$\bar{u}_{\bar{t}\bar{x}} = 0 \tag{2.8}$$

have the same Laplace invariant $h = 0 = k$. They can therefore be transformed into each other [44]. *In fact, the transformation*

$$\begin{aligned} \bar{t} &= t \\ \bar{x} &= x \\ \bar{u} &= u\exp(t-x) \end{aligned} \tag{2.9}$$

transforms (2.7) into (2.8). ■

Definition 2.1.4 *A* symmetry *is a change, a transformation, that leaves an object invariant or apparently unchanged [48].* ■

Definition 2.1.5 *A family G of transformations* [48]

$$T_a : \bar{x^i} = f^i(x, u, a);\ \bar{u^\alpha} = \phi^\alpha(x, u, a),\ i = 1, ..., n;\ \alpha = 1, ..., m \tag{2.10}$$

where a is a real parameter which continuously ranges in values from a neighborhood $\mathbb{D} \subset \mathbb{R}$ of $a = 0$ and f^i, ϕ^α are differentiable functions, is a continuous one-parameter (local) Lie group of transformations *if the following properties are satisfied:*

- *Closure: If T_a, $T_b \in G$, and $a, b \in \mathbb{D}' \subset \mathbb{D}$, then*

$$T_aT_b = T_c \in G,\ c = \phi(a, b) \in \mathbb{D} \tag{2.11}$$

- *Identity: $\forall\ a \in \mathbb{D}' \subset \mathbb{D}$ and $T_a \in G, \exists\ T_0 \in G$ such that*

$$T_0T_a = T_aT_0 = T_a \in G \tag{2.12}$$

- *Inverses: For $T_a \in G, a \in \mathbb{D}' \subset \mathbb{D},\ \exists\ T_a^{-1} = T_{a^{-1}} \in G,\ a^{-1} \in \mathbb{D}$ such that*

$$T_aT_a^{-1} = T_a^{-1}T_a = T_0. \tag{2.13}$$

The associative property follows from the first property. ■

Consider the DE

$$E^\sigma(x, u, u_{(1)}, ..., u_{(n)}) = 0, \qquad \sigma = 1, ..., m, \tag{2.14}$$

where $u_{(j)}$ is the jth derivative of the dependent variable u. If a family of transformations satisfying definition 2.1.5 of a group G are symmetries of the equation (2.14), then G is called a *symmetry group* of (2.14) and (2.14) is said to *admit* or *possess* G as a group [48]. According to Lie's theory, the construction of a one-parameter group G is equivalent to the determination of the first order approximation of the corresponding infinitesimal transformations

$$\bar{x^i} \approx x^i + a\xi^i(x, u),\ \bar{u^\alpha} \approx u^\alpha + a\eta^i(x, u), \tag{2.15}$$

where the "≈" stands for the approximation due to the truncation of the Taylor expansions of the transformations (2.10) and

$$\xi^i(x,u) = \left.\frac{\partial f^i(x,u,a)}{\partial a}\right|_{a=0}, \qquad \eta^\alpha(x,u) = \left.\frac{\partial \phi^\alpha(x,u,a)}{\partial a}\right|_{a=0}. \tag{2.16}$$

If X is the symbol for the infinitesimal transformations, then equation (2.15) is generated by

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{2.17}$$

X is known as the infinitesimal operator or generator of the group G of symmetries. In what follows, we write the functions ξ and η and all the coefficients of the infinitesimal generator without their arguments. The generator X tells us how the variables x^i and u^α transform. An interest in the transformation of the derivatives of the u^α's is also of importance. This transformation is given by the the generator of the prolonged group $G^{(n)}$, where n is the highest order of derivative of u^α of interest. This prolongation is given by

$$X^{[n]} = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \zeta^\alpha_{x_i}\frac{\partial}{\partial u^\alpha_{x_i}} + \cdots + \zeta^\alpha_{x_i x_j \ldots x_n}\frac{\partial}{\partial u^\alpha_{x_1 \ldots x_n}} \tag{2.18}$$

The coefficients of the summation are calculated using the total derivative operator [29]

$$D_{x_i} = \frac{\partial}{\partial x_i} + u^\alpha_{x_i}\frac{\partial}{\partial u^\alpha} + \cdots \tag{2.19}$$

Hence the recursive formulas are:

$$\begin{aligned} \zeta^\alpha_{x_i} &= D_{x_i}\eta^\alpha - \left(D_{x_i}\xi^j\right)u^\alpha_{x_j}, \\ \zeta^\alpha_{x_i x_k} &= D_{x_k}\zeta^\alpha_{x_i} - \left(D_{x_k}\xi^j\right)u^\alpha_{x_i x_j}, \\ \zeta^\alpha_{x_i x_k x_l} &= D_{x_l}\zeta^\alpha_{x_i x_k} - (D_{x_k}\xi^j)u^\alpha_{x_i x_k x_j} \end{aligned} \tag{2.20}$$

and so on, where we have used the Einstein summation convention over repeated indexes.

Definition 2.1.6 *A function F is said to be invariant under the symmetry X if* [48]

$$XF = 0. \tag{2.21}$$

■

Note: If F depends on derivatives, then we need to act on F with the nth extension of X, where n is the highest derivative in F.

Theorem 2.1.7 *Equation (2.14) admits X as symmetry iff* [48]

$$X^{[n]}E\big|_{E=0} = 0, \tag{2.22}$$

where $X^{[n]}$ is the n^{th} prolonged operator of the generator X and n the order of the DE. ■

2.2 Calculation of symmetries of differential equations

Example 2.2.1 *Consider the second order ODE*

$$y'' + \frac{y'}{x} + \exp(y) = 0. \tag{2.23}$$

We work out the generator of symmetry

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{2.24}$$

if any, admitted by (2.23)*. The second prolongation of X is*

$$X^{[2]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}, \tag{2.25}$$

where

$$\zeta_x = \eta_x + y'\eta_y - y'\xi_x - y'^2\xi_y \tag{2.26}$$

and

$$\begin{aligned}\zeta_{xx}\big|_{\left(y''=-\frac{y'}{x}-\exp(y)\right)} &= \eta_{xx} + 2y'\eta_{xy} + y'^2\eta_{yy} - \left(\frac{y'}{x} + \exp(y)\right)\eta_y - y'\xi_{xx} \\ &- 2y'^2\xi_{xy} - y'^3\xi_{yy} + 2\left(\frac{y'}{x} + \exp(y)\right)(\xi_x + \xi_y y') - \left(\frac{y'}{x} + \exp(y)\right)\xi_y - y'\eta_y.\end{aligned} \tag{2.27}$$

The invariance condition is

$$X^{[2]}\left(y'' + \frac{y'}{x} + \exp(y)\right)\Bigg|_{\left(y''=-\frac{y'}{x}-\exp(y)\right)} = 0. \tag{2.28}$$

This gives

$$\zeta_{xx}\big|_{\left(y''=-\frac{y'}{x}-\exp(y)\right)} + \frac{1}{x}\zeta_x + \exp(y)\eta - \frac{y'}{x^2}\xi \;=\; 0. \tag{2.29}$$

After expansion of the ζs *and replacement of* y'' *by* $-\frac{y'}{x} - \exp(y)$*, we get*

$$\begin{aligned} \eta_{xx} + 2y'\eta_{xy} + y'^2\eta_{yy} - \left(\frac{y'}{x} + \exp(y)\right)\eta_y - y'\xi_{xx} \\ - 2y'^2\xi_{xy} - y'^3\xi_{yy} + 2\left(\frac{y'}{x} + \exp(y)\right)(\xi_x + \xi_y y') - \left(\frac{y'}{x} + \exp(y)\right)\xi_y \\ + \frac{1}{x}\left(\eta_x + y'\eta_y - y'\eta_y - y'\xi_x - y'^2\xi_y\right) + \eta\exp(y) - \frac{1}{x^2}y'\xi \;&=\; 0. \end{aligned} \tag{2.30}$$

Splitting this equation via the coefficients of powers of y' *gives rise to the system*

$$\begin{aligned} y'^3: &\quad & \xi_{yy} &= 0, \\ y'^2: &\quad & \eta - 2\xi_{xy} + \frac{2}{x}\xi_y &= 0, \\ y': &\quad & 2\eta_{xy} + 3\exp(y)\xi_y - \xi_{xx} + \frac{1}{x}\xi_x - \frac{1}{x^2}\xi &= 0, \\ y'^0: &\quad & \eta_{xx} - \exp(y)\eta_y + 2\exp(y)\xi_x + \frac{1}{x}\eta_x + \exp(y)\eta &= 0. \end{aligned} \tag{2.31}$$

From this we get the coefficients of the symmetry generators

$$\xi = C_2 x\log x - C_2 x + \frac{1}{2}C_1 x, \qquad \eta = -2C_2\log x - C_1, \tag{2.32}$$

where C_1 *and* C_2 *are constants.*

This gives rise to the two symmetry generators

$$X_1 = \frac{1}{2}x\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \qquad X_2 = (x\log x - x)\frac{\partial}{\partial x} - 2\log x\frac{\partial}{\partial y}, \tag{2.33}$$

for appropriate choice of the constants. ■

Example 2.2.2 *The following system may arise from the study of three interacting and co-habiting species [14]:*

$$\begin{aligned} x'' + M\frac{x(t)}{\sqrt{(x(t)^2 + y(t)^2 + z(t)^2)^3}} &= 0 \\ y'' + M\frac{x(t)}{\sqrt{(x(t)^2 + y(t)^2 + z(t)^2)^3}} &= 0 \\ z'' + M\frac{x(t)}{\sqrt{(x(t)^2 + y(t)^2 + z(t)^2)^3}} &= 0. \end{aligned} \tag{2.34}$$

This system is very unwieldy and too complicated for us to calculate the symmetries by hand. We make use of Sym *[13, 14] to obtain the following results for the coefficients of the infinitesimal transformations*

$$\begin{aligned}
\xi(t,x,y,z) &= C_4 + tC_5 \\
\eta_1(t,x,y,z) &= -yC_1 - zC_2 + \frac{2xC_5}{3} \\
\eta_2(t,x,y,z) &= xC_1 - zC_3 + \frac{2yC_5}{3} \\
\eta_2(t,x,y,z) &= xC_2 + yC_3 + \frac{2zC_5}{3},
\end{aligned} \tag{2.35}$$

where

$$X = \xi\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial x} + \eta_2\frac{\partial}{\partial y} + \eta_3\frac{\partial}{\partial z} \tag{2.36}$$

is the symmetry of the system and the C_i's are arbitrary constants. ■

This is not the only method for finding symmetries of differential equations. There are several techniques that yield different types of symmetries. Among others, we distinguish the evolutionary symmetry method [48].

Consider equation (2.14). For an evolutionary symmetry, we assume the form

$$X_e = Q\frac{\partial}{\partial u}, \tag{2.37}$$

where the characteristic $Q = \eta - \tau\frac{\partial u}{\partial t} - \xi\frac{\partial u}{\partial x}$. The nth prolongation of X_e is given by

$$X_e^{[n]} = \Sigma_{k=0}^{n} D_k Q\frac{\partial}{\partial u_k}, \tag{2.38}$$

where D_k denotes the kth total derivative of the characteristic Q and u_k, the kth partial derivative of u. The overdetermined system of determining equations that allows us to find the characteristic Q is given by imposing the symmetry condition

$$X_e^{[n]}E\big|_{E=0} = 0. \tag{2.39}$$

Once the the system is solved, the characteristic Q is obtained and hence the symmetries of the equation.

Example 2.2.3 *Consider the one-dimensional heat equation given by*

$$u_t - u_{xx} = 0, \tag{2.40}$$

with evolutionary symmetry generator of the form

$$X_e = Q\frac{\partial}{\partial u}. \tag{2.41}$$

The second prolongation formula of X_e is given by

$$X_e^{[2]} = D_j Q \frac{\partial}{\partial u_j}, \tag{2.42}$$

with summation over the dummy index j.
We therefore get

$$D_t Q - D_{xx} Q|_{u_{xx}=u_t} = 0. \tag{2.43}$$

After expanding, we obtain

$$\begin{aligned}
\eta_t + u_t\eta_u - u_t\tau_t + 2u_t^2\tau_u - 2u_{tt}\tau - u_x\xi_t - u_x u_t\xi_u - 2\xi u_{tx} & \\
- \eta_{xx} - 2u_x\eta_{ux} - u_t\eta_u - u_x^2\eta_{uu} - u_t\tau_{xx} & \\
- 2u_t u_x\tau_{ux} - u_t^2\tau_u - u_t - u_x^2\tau_{uu} - 2u_{tx}\tau_x - 2u_{tx}u_x\tau_u - u_{tt}\tau & \\
- u_x\xi_{xx} - 2u_t u_x\xi_{ux} - u_x u_t\xi_u - u_x^3\xi_{uu} - u_{tx}\xi - 2u_t\xi_x - 3u_t u_x\xi_u &= 0.
\end{aligned} \tag{2.44}$$

Separating with respect to the powers of the derivatives of u, we get:

$$\begin{aligned}
u_x u_{tx}: &\quad \tau_u = 0\\
u_{tx}: &\quad \tau_x = 0\\
u_t^2: &\quad \tau_u = 0\\
u_x^2 u_t: &\quad \tau_u u = 0\\
u_t u_x: &\quad \tau_{ux} + \xi_u = 0\\
u_t: &\quad \tau_t - \tau_{xx} - \xi_x = 0\\
u_x^3: &\quad \xi_{uu} + \xi_t - 2\eta_{ux} = 0\\
1: &\quad \eta_t = \eta_{xx}
\end{aligned} \tag{2.45}$$

Solving this overdetermined system gives us

$$\begin{aligned}
\xi &= \alpha_1 + \alpha_2 x + \alpha_3 t + 4\alpha_5 x t, && (2.46)\\
\tau &= (\alpha_5 - \alpha_4 x - 2\alpha_5 t - \alpha_5 x^2)u + p(x,t) && (2.47)\\
\eta &= (\alpha_5 - \alpha_4 x - 2\alpha_5 t - \alpha_5 x^2)u + p(x,t) && (2.48)
\end{aligned}$$

where α_i, β_j*, are arbitrary constants and* $p(x,t)$ *an arbitrary solution of the heat equation. This gives rise to the six symmetries of the heat equation*

$$\begin{aligned}
X_e = {} & \big((\alpha_5 - \alpha_4 x - 2\alpha_5 t - \alpha_5 x^2)u + p(x,t) - (\alpha_5 - \alpha_4 x - 2\alpha_5 t - \alpha_5 x^2)uu_t - p(x,t)u_t \\
& - (\alpha_1 + \alpha_2 x + \alpha_3 t + 4\alpha_5 x t)\, u_x\big) \frac{\partial}{\partial u}. \qquad (2.49)
\end{aligned}$$

Writing this in the traditional way, we obtain

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x} && (2.50)\\
X_2 &= \frac{\partial}{\partial t} && (2.51)\\
X_3 &= u\frac{\partial}{\partial u} && (2.52)\\
X_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} && (2.53)\\
X_5 &= 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} && (2.54)\\
X_6 &= 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u} && (2.55)
\end{aligned}$$

plus the infinite-dimensional symmetry

$$X_p = p(x,t)\frac{\partial}{\partial u}, \qquad (2.56)$$

which exists because the equation is linear. The commutation table of the above symmetries is given in Table 2.1 below:

Table 2.1: *Lie bracket of the admitted symmetry algebra*

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	X_1	$-X_3$	$2X_5$
X_2	0	0	0	$2X_2$	$2X_1$	$4X_4 - 2X_3$
X_3	0	0	0	0	0	0
X_4	$-X_1$	$-2X_2$	0	0	X_5	$2X_6$
X_5	X_3	$-2X_1$	0	$-X_5$	0	0
X_6	$-2X_5$	$2X_3 - 4X_4$	0	$-2X_6$	0	0

where $[X_i, X_j] = X_iX_j - X_jX_i$, for $i = 1, ..., 6$. ■

2.3 Uses of symmetries of differential equations

2.3.1 Reduction of order

The main idea is that one can apply a symmetry to the equation to reduce the order of equation from order n, say, to order $n-1$. It is well known that one can then apply the symmetries successively to the equation to reduce the equation, hopefully from order n to $n-1$, then $n-2$ down to order 0, and this gives, by reversing the transformation into the original variable the general solution of the ODE.

In general, if

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{2.57}$$

is a symmetry, then the reduction variables associated with (2.57) are obtained by solving the corresponding Lagrange system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\zeta_x}. \tag{2.58}$$

Theorem 2.3.1 *If equation (2.14) admits symmetries X_1 and X_2 with*

$$[X_1, X_2] = \lambda X_1, \tag{2.59}$$

then reduction via X_1 will result in X_2 (transformed) being a symmetry of the reduced equation [48]. ■

Example 2.3.2 *The symmetries of (2.23) were previously calculated and given in (2.33). We observe that*

$$[X_1, X_2] = X_1 \in \{X_1, X_2\}, \tag{2.60}$$

and so, we reduce the order of our ODE using X_1. The first prolongation of X_1 is

$$X_1^{[1]} = \frac{1}{2}x\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - \frac{1}{2}y'\frac{\partial}{\partial y'}. \tag{2.61}$$

The Lagrange characteristic system is given by

$$2\frac{dx}{x} = -\frac{dy}{1} = -2\frac{dy'}{y'} \tag{2.62}$$

and hence the invariants are

$$u = x^2 \exp(y) \tag{2.63}$$

$$v = xy'. \tag{2.64}$$

A second order invariant can be expressed in terms of u, v, and dv/du and our ODE (2.23) becomes the first order ODE

$$\frac{dv}{du} = \frac{-1}{2+v}. \tag{2.65}$$

The latter equation admits

$$X_2 = (u \log u - u)\frac{\partial}{\partial u} - 2\log u\frac{\partial}{\partial v}. \tag{2.66}$$

The solution of (2.65) can easily be found to be

$$2v + \frac{1}{2}v^2 + u = A, \tag{2.67}$$

where A is a constant. Writing the latter equation in terms of original variables gives the reduced first order ODE

$$2xy' + \frac{1}{2}x^2y'^2 + x^2e^y = A. \tag{2.68}$$

To simplify this equation further, one may again use $z = x^2e^y$. We notice that

$$y' = \frac{z'x - 2z}{zx}. \tag{2.69}$$

Substituting y' into (2.68) gives

$$(xz')^2 = -2z^3 + 4z^2 + 2z^2 A, \tag{2.70}$$

which is of separable form. In fact, separating the variables in this equation and solving yields

$$x = \frac{\sqrt{2+A-z} - \sqrt{2+A}}{\sqrt{2+A-z} + \sqrt{2+A}} \exp\left(\frac{1}{\sqrt{4+2A}}\right), \qquad 2+A-z \geq 0, \tag{2.71}$$

and

$$x = \exp\left(\sqrt{\frac{-2}{2+A}} \arctan\sqrt{\frac{z-2-A}{2+A}}\right), \qquad z-2-A > 0. \tag{2.72}$$

■

2.3.2 First integrals of ordinary differential equations

Let the equation

$$E(x, y, y', y'') = 0 \tag{2.73}$$

have the symmetry

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. \tag{2.74}$$

Definition 2.3.3 *A first integral of E is a function $I = f(x, y, y')$ (where f depends on y' nontrivially) such that [48]*

$$\left.\frac{dI}{dx}\right|_{E=0} = 0. \tag{2.75}$$

■

We can extend the definition to an nth order equation in an obvious manner.

To find a first integral of E admitting (2.74), we also invoke Definition 2.1.6:

$$X^{[1]} f = 0. \tag{2.76}$$

This is equivalent to

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta_x\frac{\partial f}{\partial y'} = 0, \tag{2.77}$$

where ζ_x is defined in (2.20). In practice, we impose (2.76) and then (2.75). The associated Lagrange system is given by

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\zeta_x} \left(= \frac{df}{0} \right). \tag{2.78}$$

This system yields two characteristics namely α and ρ such that

$$I = f(x, y, y') = g(\alpha, \rho). \tag{2.79}$$

The function f satisfies (2.75) if

$$\left.\frac{dI}{dx}\right|_{E=0} = 0, \tag{2.80}$$

implying that

$$\alpha' \frac{\partial g}{\partial \alpha} + \rho' \frac{\partial g}{\partial \rho} = 0. \tag{2.81}$$

This admits the Lagrange system

$$\frac{\mathrm{d}\alpha}{\alpha'} = \frac{d\rho}{\rho'} \left(= \frac{dg}{0} \right). \tag{2.82}$$

We have the characteristic p and

$$I = f(x, y, y') = h(p), \tag{2.83}$$

where h is an arbitrary function, usually chosen to be the identity mapping.

Example 2.3.4 *The first order prolongation of symmetry X_1 of equation (2.23) is given by*

$$X_1^{[1]} = \frac{1}{2} x \frac{\partial}{\partial x} - \frac{\partial}{\partial y} - \frac{1}{2} y' \frac{\partial}{\partial y'}. \tag{2.84}$$

We impose (2.76) and obtain

$$\frac{1}{2} x \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} - \frac{1}{2} y' \frac{\partial f}{\partial y'} = 0. \tag{2.85}$$

This implies that

$$2\frac{dx}{x} = \frac{dy}{-1} = 2\frac{dy'}{y'}, \tag{2.86}$$

and gives rise to the characteristics

$$\alpha = x^2 \exp(y), \qquad \beta = xy'. \tag{2.87}$$

Therefore

$$I = f(x, y, y') = g(\alpha, \beta). \tag{2.88}$$

Imposing (2.75), we have

$$\frac{\partial g}{\partial \alpha} - (2+\beta)\frac{\partial g}{\partial \beta} = 0 \tag{2.89}$$

and the corresponding Lagrange system is

$$\frac{d\alpha}{1} = \frac{g\beta}{-(2+\beta)}. \tag{2.90}$$

The characteristic is

$$p = \alpha + 2\beta + \frac{1}{2}\beta^2. \tag{2.91}$$

Thus the function h is given by

$$h(p) = h(x^2\exp(y) + 2xy' + \frac{1}{2}(xy')^2). \tag{2.92}$$

Hence a first integral of (2.23) is taken to be

$$f(x, y, y') = \frac{1}{2}(xy')^2 + 2xy' + x^2\exp(y). \tag{2.93}$$

■

Since these determining equations are linear and homogeneous, their solutions form a vector field L [3, 48].

2.3.3 Transformation of equations

It is a well known established result that given an ODE, it is possible to transform this equation into a simpler one, e.g., an equation for which the solution is known. Firstly, both equations must admit the same Lie algebra of symmetries. Indeed the symmetries must conform to the same realization of the admitted Lie algebra.

Definition 2.3.5 *If*

$$X_1 = \xi_1^i\frac{\partial}{\partial x^i} + \eta_1^\alpha\frac{\partial}{\partial u^\alpha} \tag{2.94}$$

and

$$X_2 = \xi_2^i \frac{\partial}{\partial x^i} + \eta_2^\alpha \frac{\partial}{\partial u^\alpha} \tag{2.95}$$

satisfy (2.22) then their commutator is given by

$$\begin{aligned} [X_1, X_2] &= X_1X_2 - X_2X_1 \\ &= (X_1(\xi_2^i) - X_2(\xi_1^i))\frac{\partial}{\partial x^i} + (X_1(\eta_2^\alpha) - X_2(\eta_1^\alpha))\frac{\partial}{\partial u^\alpha} \end{aligned} \tag{2.96}$$

and satisfies the following properties:

- **Bilinearity.** *If* $X_1, X_2, X_3 \in L$, *then*

  $$[\alpha X_1 + \beta X_2, X_3] = \alpha[X_1, X_3] + \beta[X_2, X_3], \tag{2.97}$$

  where α, β *are scalars.*

- **Skew-symmetry.** *If* $X_1, X_2 \in L$ *then*

  $$[X_1, X_2] = -[X_2, X_1]. \tag{2.98}$$

- **Jacobi Identity.** *If* $X_1, X_2, X_3 \in L$, *then*

  $$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0. \tag{2.99}$$

Hence we say that the vector space L *of all solutions of the determining equations forms a* Lie algebra *which generates a multi-parameter group admitted by (2.14). By* multi-parameter *or more precisely* r-parameter, *we mean a group generated by transformations* $T_{\underline{a}}$ *as above where* $\underline{a} = (a_1, a_2, ..., a_r)$. ■

Example 2.3.6 *The free particle equation*

$$Y'' = 0 \tag{2.100}$$

has eight symmetries which form the Lie algebra $s\ell(3, \Re)$, *as does the equation*

$$y'' - 2y' + y = 0. \tag{2.101}$$

Since $s\ell(3,\Re)$ has only one realization, these two equations belong to the same equivalence class. Moreover, these Lie algebras have the same dimensions and their elements have a direct correspondence. It is well known that this can be used to find a transformation of (2.101) into (2.100). In fact, (2.101) possesses the symmetry

$$V = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{2.102}$$

and (2.100) possesses

$$U = \frac{\partial}{\partial Y}. \tag{2.103}$$

We wish to find a transformation

$$X = F(x, y), \quad Y = G(x, y) \tag{2.104}$$

to transform (2.101) into (2.100). Operating on (2.104) with (2.102) yields

$$(F_x + G_x)\frac{\partial}{\partial X} + (F_y + G_y)\frac{\partial}{\partial Y} = \frac{\partial}{\partial Y}. \tag{2.105}$$

We obtain the system of PDEs

$$\begin{aligned} F_x + G_x &= 0 \\ F_y + G_y &= 1 \end{aligned} \tag{2.106}$$

and hence the solution

$$F = f(y\exp(-x)), \quad G = x + g(y\exp(-x)) \tag{2.107}$$

for the functions F and G. We take $f(y\exp(-x)) = y\exp(-x)$ and $g = 0$ to obtain the simplest transformation. This transformation takes (2.101) into (2.100). ■

In general, all linear second order ODEs have eight symmetries and admit the Lie algebra $s\ell(3,\Re)$. They can therefore be transformed into the free particle ODE by a point transformation. We can also linearize any second order ODE that has eight symmetries by point transformations. In 1883, Lie stated a compatibility condition for the problem of linearization map for two connected symmetries. Later in 1987 the issue was revised by Sarlet, Mahomed and Leach [56] and in 1989 for linearizable ODEs admitting two unconnected generators of symmetry by Mahomed and Leach [28]. In 1990, Mahomed and Leach showed that an nth order ODE is linearizable via a point transformation if and only if it admits an n-dimensional Abelian Lie algebra [43].

2.3.4 Group-invariant solutions of differential equations

Definition 2.3.7 *Let G be a transformation group acting on a manifold M. An* invariant *of G is a real-valued function [48]*

$$I : M \to R \tag{2.108}$$

which satisfies

$$I(g - x) = I(x) \tag{2.109}$$

for all transformations g. ■

After calculating the symmetries of the equations, one may use any linear combination of these to reduce the number of independent or dependent variables of the equation. Consider the system (2.14) admitting G as symmetry group. If G has a complete set of invariants

$$y^i = \eta^i(x, u) \text{ and } \quad w^j = \zeta^j(x, u), \tag{2.110}$$

where the y^is are the new independent variables and the w^js the new dependent variables. These invariants can be used to reduce the PDE by reducing the number of variables by one. We explain the method below.

Consider the symmetry

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{2.111}$$

say, for which we compute the invariants by the method of characteristics. We have

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha}. \tag{2.112}$$

Then once the characteristics or invariants y^i and w^j are obtained, these must satisfy the condition that w must be a solution whenever u is. This condition gives rise to a differential equation in w. The later has one variable less than the original equation. We explain this in an example.

Example 2.3.8 *Consider the combination of translation operators of the heat equation (2.40)*

$$\frac{\partial}{\partial t} + C\frac{\partial}{\partial x}. \tag{2.113}$$

The characteristic system is

$$\frac{dt}{1} = \frac{dx}{C} = \frac{du}{0}. \tag{2.114}$$

This gives rise to the invariants

$$\gamma = x - Ct, \ \ \textit{and} \ \alpha = u. \tag{2.115}$$

Hence

$$u = h(x - Ct), \tag{2.116}$$

where h satisfies the second order ODE

$$h'' + Ch' = 0. \tag{2.117}$$

Equation (2.117) admits

$$h(\gamma) = C_1 \exp(-C\gamma) + C_2 \tag{2.118}$$

as solution. We therefore write the solution for the heat equation by inverting (2.116) to obtain

$$u(x,t) = C_1 \exp(-C(x - Ct)) + C_2, \tag{2.119}$$

which is commonly known as the traveling wave solution. ■

The solution (2.119) was obtained by just taking a linear combination of the symmetries of the equation. However, one can choose a minimal combination, which yields transformations such that any other linear combination will be isomorphic to the set of such minimal linear combinations. Such a set is called an *optimal system* [47, 48]. The optimal system is obtained by a series of adjoint maps. Since we do not utilize optimal systems in this thesis, we do not discuss this any further.

2.3.5 Conservation laws

There are several methods for finding conservation laws for differential equations. We investigate some of these method and apply them to some well known examples.

Consider the nth order DE of k independent variables and N dependent variables (2.14) which can be assumed to have maximum rank and can be solved locally.

Definition 2.3.9 *A conserved vector of (2.14) is an n-tuple $T = (T^1, T^2, ..., T^n)$ such that*

$$D_i T^i = 0 \tag{2.120}$$

for all solution of (2.14) [48]. ■

Note: i) We call local conservation laws those that are free from integral terms.
ii) There are also trivial conservation laws. The first kind is the one for which the vector T vanishes for all solutions of (2.14). For example, Naz *et al* [46] established that

$$\begin{aligned} T^1 &= \sqrt{Cu}\cos(\sqrt{Cu})[v_x - u] \\ T^2 &= C\sin(\sqrt{Cu})[v_x - u] + \sqrt{Cu}\cos(\sqrt{Cu})[Cxu - \frac{1}{u^s}u_x v_x - v_t] \end{aligned} \tag{2.121}$$

forms a trivial conservation law for the system

$$\begin{aligned} v_x &= u \\ v_t &= \left(\frac{1}{u}\right)_x + Cxu, C \geq 0, \end{aligned} \tag{2.122}$$

since T^1 and T^2 vanish for all solutions of the system. The second kind of trivial conservation law is the one that vanishes identically for arbitrary functions, not only for solutions of the system (2.14). For example,

$$D_t(u_x) + D_x(u_t) = 0. \tag{2.123}$$

holds for all smooth functions $u = g(x, t)$ satisfying [46]

$$u_{tx} = 0. \tag{2.124}$$

We seek non-trivial conservation laws.

Direct method

This method consists of deriving the determining equations given by

$$D_i T^i \mid_{E_\alpha=0} = 0, \tag{2.125}$$

where $T = (T^1, ..., T^n)$, n being the dimension of the problem. This method is reported to originate from Laplace in the 1790's [46] and is able to determine all the local conservation laws of the equation.

Example 2.3.10 *We seek the conservation laws for the Maxwellian distribution, given by*

$$u_{tx} + u^2 = 0 \tag{2.126}$$

using the direct method. The determining equations are given by considering the condition

$$\left(D_t T^1 + D_x T^2\right)\Big|_{u_{tx}=-u^2} = 0, \tag{2.127}$$

which expands to

$$\left(T^1_t + T^1_u u_t + T^1_{u_t} u_{tt} + T^1_{u_x} u_{tx} + T^2_x + T^2_u u_x + T^2_{u_x} u_{xx} + T^2_{u_t} u_{tx}\right)\Big|_{u_{tx}+u^2=0} = 0. \tag{2.128}$$

We now substitute $u_{tx} = -u^2$ into equation (2.128). Furthermore, for simplicity, we assume the forms

$$T^1 = a(t,x,u)\frac{u_x^2}{2} + b(t,x,u),\ T^2 = c(t,x,u)\frac{u_t^2}{2} + d(t,x,u). \tag{2.129}$$

Then

$$\frac{1}{2}c_u u_t^2 u_x + \frac{1}{2}a_u u_t u_x^2 + \frac{1}{2}c_x u_t^2 + \frac{1}{2}a_t u_x^2 + (b_u - cu^2)u_t + (d_u - au^2)u_x + (d_u - au^2)u_x + b_t + d_x = 0. \tag{2.130}$$

Splitting this with respect to the derivatives of u leads to

$$\begin{aligned} u_x u_t^2: &\quad c_u = 0 \\ u_t u_x^2: &\quad a_u = 0 \\ u_t^2: &\quad c_x = 0 \\ u_x^2: &\quad a_t = 0 \\ u_t: &\quad b_u - cu^2 = 0 \\ u_x: &\quad b_u - au^2 = 0 \\ 1: &\quad b_t + d_x = 0. \end{aligned} \tag{2.131}$$

This gives rise to the conservation laws

$$\begin{aligned} T^1 &= -\frac{1}{2}xu_x^2 + \frac{1}{3}tu^3, \ T^2 = \frac{1}{2}tu_t^2 - \frac{1}{3}xu^3 \\ T^1 &= \frac{1}{3}u^3, \ T^2 = \frac{1}{2}u_t^2, \\ T^1 &= \frac{1}{2}u_x^2, \ T^2 = \frac{1}{3}u^3. \end{aligned} \tag{2.132}$$

■

Noether's approach

In this approach, the conservation laws are computed by use of Noether's theorem [48].

Definition 2.3.11 Euler's operator *[30] is defined as*

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \Sigma_{s\geq 1}(-1)^s D_{i_1}...D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1...i_s}}, \tag{2.133}$$

where D is defined as in (2.19). ■

Definition 2.3.12 *The* Noether operators *associated with the Lie-Bäcklund operator X are*

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \Sigma_{s\geq 1}(-1)^s D_{i_1}...D_{i_s}(W^\alpha)\frac{\delta}{\delta u^\alpha_{i_1...i_s}}, \ i = 1, ..., n \tag{2.134}$$

with $W^\alpha = \eta^\alpha - \xi^i u^\alpha$. ■

Definition 2.3.13 *Suppose there exists a function $L(x, u, u_{(1)}, ..., u_{(s)})$ such that (2.14) is equivalent to*

$$\frac{\delta L}{\delta u^\alpha} = 0, \alpha = 1, 2, ..., N. \tag{2.135}$$

Then L is called a Lagrangian *of equation (2.14), and (2.135) are the corresponding Euler-Lagrange differential equations [35] .* ■

Definition 2.3.14 *A* Lie-Bäcklund operator *X is a Noether symmetry generator associated with a given Lagrangian L of (2.135) if there is a vector $B = (B^1, ..., B^n)$, such that*

$$X(L) + LD_i(\xi^i) = D_i(B^i) \tag{2.136}$$

is satisfied [48]. ■

Definition 2.3.15 *The* Noether conservation vector *is given by [35]*

$$\begin{aligned} T^i &= B^i - N^i L \\ &= B^i - \xi^i L - W^\alpha \frac{\delta L}{\delta u_i^\alpha} \Sigma_{s\geq 1} D_{i_1,\dots,i_s}(W^\alpha) \frac{\delta L}{\delta u^\alpha_{i_1,\dots,i_s}}, \end{aligned} \tag{2.137}$$

which is the conserved vector for the Euler-Lagrange equation (2.135). ■

In this approach, we compute L and substitute in equation (2.136) to find the Noether symmetry generators. Thereafter, we use equation (2.137) to generate the conserved vector T. The characteristics W^α of the symmetry generator are then the characteristics of the conservation law.

Another variational method is the characteristic method [48]. Here we need to solve

$$D_i T^i = Q^\alpha E_\alpha, \tag{2.138}$$

where Q^α are the characteristics which are also called the multipliers which make the equation exact. This approach involves the variational derivative of (2.138):

$$\frac{\delta}{\delta u^\beta}(Q^\alpha E_\alpha) = 0, \tag{2.139}$$

for arbitrary functions $u(x^1, x^2, \ldots, x^n)$. All the multipliers can be calculated using (2.139) for which the equation can be expressed as a local conservation law. In the variational approach on space of solutions of the DE, the variational derivative of (2.138) is computed on the space of solutions of the DE, i.e.,

$$\frac{\delta}{\delta u^\beta}(Q^\alpha E_\alpha) \mid_{E_\alpha=0}= 0. \tag{2.140}$$

This approach does not necessarily lead to a conservation law, but to adjoint symmetries. Given a Lie-Bäcklund operator, a conservation vector T is obtained from the relation

$$X(T^i) + D_k(\xi^k)T^i - D_k(\xi^i)T^k = 0. \tag{2.141}$$

It is also possible to construct conserved vectors directly by using the multiplier, which is locally expressed in a standard Cauchy-Kovalevskaya form [48].

2.4 Summary

In conclusion, Lie symmetry analysis of DEs is indeed a great tool in the hands of the scientist seeking exact solutions for complex problems. We have reviewed the work on symmetry analysis in DEs. The physical interpretation of the theoretical models is important when transformations are made on equations. With conservation laws, some physical quantities such as mass, energy, and others are taken care of while solutions are being found. We are now in a position to investigate the use of Lie groups in ΔEs.

Chapter 3

Symmetries of Differential-Difference and Ordinary Difference Equations

3.1 Introduction

We now investigate the construction of symmetries of difference equations. By studying the local structure of the set of solutions, we derive the method to systematically determine one-parameter groups of symmetries in closed form. We wish to use these groups to achieve successive reductions of order and calculate first integrals and conservation laws for difference equations much as we achieved for differential equations.

3.2 Notation and definitions

Definition 3.2.1 *A* shift operator S_σ *is any function that satisfies [19]*

$$S_\sigma u(t) = u(t+\sigma). \tag{3.1}$$

■

Definition 3.2.2 *Let $q \in \mathbb{N}$. The function u is said to be q-*periodic *if [19]*

$$u(t+q) = u(t) \tag{3.2}$$

for a given $q \in \mathbb{N}$. ∎

Definition 3.2.3 *A function* u_σ *is said to be* totally unit periodic *on* Q *if [19]*

$$\forall \sigma \in Q, \;\; u(t+\sigma) = u(t). \tag{3.3}$$

∎

Definition 3.2.4 *An equation of the form*

$$E(l, x_i, u_{(l+j)x_i^m}), \quad j = 0, 1, 2, \ldots, \quad m = 0, 1, 2, \ldots, \tag{3.4}$$

where l *is the discrete independent variable,* x_i *are the continuous independent variables and* $u_{(l+j)x_i^m}$ *represents the simultaneous* j *shifts of the dependent variable* u_l *and its* m*th derivatives with respect to the continuous variable* x_i *is a D∆E.* ∎

Definition 3.2.5 *A functional relation that takes on the values of* $u : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{C}$ *at different points of the discrete independent variable* n *may be written in the following form*

$$E(n, u_n, u_{n+1}, u_{n+2}, \ldots) \;\; = \;\; 0, \tag{3.5}$$

where n *is the discrete independent variable and* u_{n+j} *the* j*th shift of the dependent variable* u_n *(alternatively, the value of* u *at point* $(n+j)$*) and is called an O∆E [40].* ∎

Note

i. We assume that all grid points are a normalized distance of one apart.
ii. For DEs and D∆Es, n refers to the highest derivative. However, for purely ∆Es, n is the discrete independent variable and k refers to the highest shift.

3.3 Determination of symmetries of differential-difference equations

To find the symmetries of D∆Es, we consider a symmetry generator of the form

$$X = \xi^i(l, x_i, u_l)\frac{\partial}{\partial x_i} + \eta(l, x_i, u_l)\frac{\partial}{\partial u_l}. \tag{3.6}$$

We do not consider continuous derivatives operating on the discrete variable l and on the shifts of u_l. As for DEs the condition for (3.4) to admit a symmetry is

$$X^{[n]}E\big|_{E=0} = 0 \tag{3.7}$$

and we proceed as in the case of DEs.

Example 3.3.1 *Consider the Toda system [38]*

$$\Delta_l = u_{lxt} - e^{u_{l-1}-u_l} + e^{u_l-u_{l+1}} = 0. \tag{3.8}$$

In this equation, x and t are the continuous independent variables. The symmetry generator (3.6) then becomes

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u_l}. \tag{3.9}$$

The second prolonged group generator, taking into account only those terms that do not vanish is given by

$$X^{[2]} = \eta\frac{\partial}{\partial u_l} + \zeta_{tx}\frac{\partial}{\partial u_{ltx}} \tag{3.10}$$

and the invariance condition is given by

$$X^{[2]}\Delta_l\big|_{\Delta_l=0} = 0, \tag{3.11}$$

which implies

$$\zeta_{tx} + \eta\left(e^{u_{l-1}-u_l} - e^{u_l-u_{l+1}}\right)\big|_{\Delta_l=0} = 0, \tag{3.12}$$

where

$$\begin{aligned}
\zeta_{tx} \quad = \quad & \eta_{tx} + u_{lt}\eta_{u_lx} + u_{ltx}\eta_{u_l} + u_l x\eta_{u_lt} + u_{lx}u_{lt}\eta_{u_lu_l} - u_{ltt}\tau_x \\
& -u_{ltt}u_{lx}\tau_{u_l} - u_{lt}\tau_{tx} - u_{lt}^2\tau_{u_{lx}} \\
& -u_{lt}u_{ltx}\tau_{u_l} - u_{lx}u_{lt}^2\tau_{u_lu_l} - u_{ltx}\xi_x \\
& -u_{ltx}u_{lx}\xi_{u_l} - u_{lx}(\xi_{tx} + u_{lt}\xi_{u_lx} \\
& +u_{ltx}\xi_{u_l} + u_{lx}\xi_{u_lt} + u_{lt}u_{lx}\xi_{u_lu_l}) \\
& -u_{lt}\tau_t - u_{lt}^2\tau_{u_lu_l} - u_{lxx}\xi_t - u_{lxx}u_{lt}\xi_{u_l}
\end{aligned} \tag{3.13}$$

which leads to

$$\begin{aligned}
\eta_{tx} + u_{lt}\eta_{u_l x} + u_{ltx}\eta_{u_l} + u_{lx}\eta_{u_l t} \\
+ u_{lx}u_{lt}\eta_{u_l u_l} - u_{ltt}\tau_x - u_{ltt}u_{lx}\tau_{u_l} \\
- u_{lt}\tau_{tx} - u_{lt}^2\tau_{u_l x} - u_{lt}u_{ltx}\tau_{u_l} \\
- u_{lx}u_{lt}^2\tau_{u_l u_l} - u_{ltx}\xi_x - u_{ltx}u_{lx}\xi_{u_l} \\
- u_{lx}(\xi_{tx} + u_{lt}\xi_{u_l x} + u_{ltx}\xi_{u_l} + u_{lx}\xi_{u_l t} \\
+ u_{lt}u_{lx}\xi_{u_l u_l}) - u_{lt}\tau_t - u_{lt}^2\tau_{u_l u_l} \\
- u_{lxx}\xi_t - u_{lxx}u_{lt}\xi_{u_l} + \eta\left(e^{u_{l-1}-u_l} + e^{u_l - u_{l+1}}\right) &= 0. \qquad (3.14)
\end{aligned}$$

Separating with respect to the derivatives $u_{ltx}u_{lt}$, $u_{ltt}u_{lx}$, u_{lxx},u_{ltt}, $u_{lx}u_{lt}$, u_{lx} *and* u_{lt} *gives the over-determined system*

$$\begin{aligned}
u_{lxx}u_{lt}: &\quad \xi_{u_l} = 0, \\
u_{ltt}u_{lx}: &\quad \tau_{u_l} = 0, \\
u_{lxx}: &\quad \xi_t = 0, \\
u_{ltt}: &\quad \tau_x = 0, \qquad (3.15)\\
u_{lx}u_{lt}: &\quad \eta_{u_l u_l} = 0, \\
u_{lx}: &\quad \eta_{u_l t} = 0, \\
u_{lt}: &\quad \eta_{u_l x} - \tau_t = 0,
\end{aligned}$$

giving

$$\xi = \xi(x),\ \tau = A(l)t + K(l),\ \eta = A(l)xu_l + C(l) + B(l,t,x). \qquad (3.16)$$

The remaining terms constitute

$$\begin{aligned}
B_{tx}(l,t,x) + (A(l)u_l x + C(l) + B(l,t,x)) \\
\times\left(e^{u_{l-1}-u_l} + e^{u_l - u_{l+1}}\right) &= 0 \qquad (3.17)
\end{aligned}$$

Equation (3.17) is solved and we obtain the symmetry algebra

$$\begin{aligned}
X_1(f) &= f(t)\frac{\partial}{\partial t} + f'(t)l\frac{\partial}{\partial u_l}, \\
X_2(g) &= g(x)\frac{\partial}{\partial x} + g'(x)\frac{\partial}{\partial u_l}, \\
X_3(k) &= k(t)\frac{\partial}{\partial u_l}, \\
X_4(h) &= h(x)\frac{\partial}{\partial u_l},
\end{aligned} \tag{3.18}$$

where the functions h, f, g and k are arbitrary C^∞ functions. This corresponds to the results in [38]. ■

3.4 Symmetry reduction of differential-difference equations

In addition to the above symmetries, DΔEs also possess discrete symmetries. We determined continuous symmetries above. They constitute the continuous subgroup of the entire group of symmetries, which consists of the continuous and discrete symmetries. We focus here only on continuous symmetries. However, we note that when we use a continuous symmetry to reduce the order of DΔEs, we obtain purely ΔEs, and when we use discrete subgroups on our DΔEs, they yield purely DEs [38].

Example 3.4.1 *Consider the symmetry*

$$X_1(f) = f(t)\frac{\partial}{\partial t} + \dot{f}l\frac{\partial}{\partial u_l} \tag{3.19}$$

with the invariants $u - l\log f(t)$ and x. Therefore we have

$$u_l(x,t) = \bar{u}_l(x) + l\log f(t). \tag{3.20}$$

Equation (3.8) becomes

$$\bar{u}_{l-1}(,x) - \bar{u}_l(x) = \bar{u}_l(x) - \bar{u}_{l+1}(x) \tag{3.21}$$

with solution

$$\bar{u}_l = a(x)l + b(x), \tag{3.22}$$

where a and b are arbitrary functions of x [38]. Thus

$$u = a(x)l + b(x) + l\log f(t) \tag{3.23}$$

is a solution to (3.8). ■

3.5 Determination of symmetries of ordinary difference equations

Contrary to the case of differential equations where we could define a symmetry vector field X as

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u_n^\alpha}, \tag{3.24}$$

our independent variable is discrete and therefore, we cannot apply the continuous derivative operator to it. We redefine our vector field as

$$X = Q(n, u_n)\frac{\partial}{\partial u_n}. \tag{3.25}$$

We treat the independent variable n in the equation as a parameter for our symmetry calculation. Hydon [24] proposes a method that enables us to find symmetries using the so-called local symmetry condition [24]. Consider an OΔE of the form

$$u_{n+k} = \omega(n, u_n, u_{n+1}, \ldots, u_{n+k-1}),\ \frac{\partial \omega}{\partial u_n} \neq 0. \tag{3.26}$$

Lemma 3.5.1 *If $k \in \mathbb{Z}$, the transformation generated by the kth shift S^k is a trivial symmetry of a general OΔE.* ■

Note that if k is negative, then [24]

$$S^k = (S^{-1})^{-k}. \tag{3.27}$$

Lemma 3.5.2 *Every order-preserving symmetry is equivalent to a vertical symmetry [24],*

$$\tilde{\Gamma} : (n, \phi^1, \ldots, \phi^k) \longmapsto (n, \tilde{\phi}^1, \ldots, \tilde{\phi}^k), \tag{3.28}$$

where the ϕ^is represent the first integrals of the equation. ■

The proof of this lemma is easy and straightforward as it is based on the fact that the unique construction of $\tilde{\Gamma}$ is

$$\tilde{\Gamma} = S^{n-\hat{n}(n)}\Gamma. \tag{3.29}$$

Given Lemma 3.5.2, we only consider vertical symmetries henceforth. Accordingly, we seek symmetries Γ with $n = \hat{n}(n)$. In terms of the original variables,

$$\Gamma : (n, u_n, \ldots, u_{n+k-1}) \longrightarrow (n, \hat{u}_n, \ldots, \hat{u}_{n+k-1}). \tag{3.30}$$

The action of Γ on u_{n+k} is determined by its action on u_n. Suppose

$$\hat{u}_n = g(n, u_n, \ldots, u_{n+k-1}) = G(n, \phi^1, \ldots, \phi^k). \tag{3.31}$$

Then the set of solutions of the OΔE (3.26) is

$$\hat{u}_{n+k} = G(n+k, \phi^1, \ldots, \phi^k) = S^k \hat{u}_n,\ \ k = 1, \ldots, n. \tag{3.32}$$

These conditions are similar to the prolongation formula for the so-called dynamical symmetries of ODEs, which reflect the necessity of contact conditions to be satisfied on the set of solutions. The symmetry condition for equation (3.26) is

$$\hat{u}_{n+k} = \omega(n, \hat{u}_n, \ldots, \hat{u}_{n+k-1}) \tag{3.33}$$

whenever (3.26) holds.

We need to linearize the symmetry condition about the identity. In doing so, we seek one parameter (local) transformation of the form

$$\hat{u}_n = u_n + \epsilon Q(n, u_n, \ldots, u_{n+k-1}) + O(\epsilon^2). \tag{3.34}$$

Q is called the characteristic of the resulting one-parameter group. From the prolongation formula (3.32), we obtain

$$\hat{u}_{n+h} = u_{n+h} + \epsilon S^h Q + O(\epsilon^2), h = 1, \ldots, k. \tag{3.35}$$

Expanding u_{n+h} to first order in ϵ results in what is known as the Local Symmetry Condition (LSC) [24]

$$S^k Q - X\omega = 0, \tag{3.36}$$

where

$$X = Q\frac{\partial}{\partial u_n} + SQ\frac{\partial}{\partial u_{n+1}} + \cdots + S^{k-1}Q\frac{\partial}{\partial u_{n+k-1}}. \tag{3.37}$$

Note: When these symmetry generators X are written in terms of first integrals they exhibit the form

$$X = F^1(\phi^1, \ldots, \phi^k)\frac{\partial}{\partial \phi^1} + \cdots + F^k(\phi^1, \ldots, \phi^k)\frac{\partial}{\partial \phi^k}, \tag{3.38}$$

since each ϕ^i is a function of $\Phi(\phi^1, \ldots, \phi^k)$ only. From equations (3.38) and (3.37), it is easy to see that X and S commute as operators on functions. In fact, given any sufficiently smooth function

$$g(n, u_n, \ldots, u_{n+k-1}) = G(n, \Phi), \tag{3.39}$$

(3.38) implies that

$$S(XG) = S\left(F^i(\Phi)\frac{\partial G}{\partial \phi^i}(n, \Phi)\right) = F^i(\Phi)\frac{\partial G}{\partial \phi^i}(n+1, \Phi) = X(SG). \tag{3.40}$$

Therefore,

$$S(Xg) = X(Sg). \tag{3.41}$$

We will use (3.41) to derive symmetry reduction for OΔEs.

Example 3.5.3 *Consider the OΔE [24]*

$$u_{n+2} = \frac{u_n u_{n+1}}{2u_n - u_{n+1}}. \tag{3.42}$$

This is a second order ordinary difference equation. We seek point symmetries, whose characteristics are of the form $Q = Q(n, u_n)$. The LSC becomes

$$S^2 Q - X\left(\frac{u_n u_{n+1}}{2u_n - u_{n+1}}\right) = 0 \tag{3.43}$$

i.e.

$$Q(n+2, \omega) - \left[Q\frac{\partial}{\partial u_n} + SQ\frac{\partial}{\partial u_{n+1}}\right]\left(\frac{u_n u_{n+1}}{2u_n - u_{n+1}}\right) = 0. \tag{3.44}$$

Simplifying this equation gives rise to

$$Q(n+2,\omega) - \frac{2u_n^2}{(2u_n - u_{n+1})^2} Q(n+1, u_{n+1}) + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2} Q(n, u_n) = 0. \tag{3.45}$$

The problem with (3.45) lies in the fact that Q takes three separate pairs of arguments. However, taking u_{n+1} as a function of n, u_n and ω, we observe that

$$\frac{\partial u_{n+1}}{\partial u_n} = -\frac{\frac{\partial \omega}{\partial u_n}}{\frac{\partial \omega}{\partial u_{n+1}}} = \frac{u_{n+1}^2}{2u_n^2}. \tag{3.46}$$

We therefore apply the operator

$$L = \frac{\partial}{\partial u_n} + \frac{u_{n+1}^2}{2u_n^2} \frac{\partial}{\partial u_{n+1}} \tag{3.47}$$

to (3.45) and obtain the expression

$$\begin{aligned} -\frac{u_{n+1}^2}{(2u_n - u_{n+1})^2} Q'(n+1, u_{n+1}) + \frac{2u_{n+1}}{(2u_n - u_{n+1})^2} Q(n+1, u_{n+1}) & \\ + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2} Q'(n, u_n) - \frac{2u_{n+1}^2}{u_n(2u_n - u_{n+1})^2} Q(n, u_n) &= 0, \end{aligned} \tag{3.48}$$

where the $'$ denotes the derivative with respect to the continuous variable argument. We therefore get

$$-Q'(n+1, u_{n+1}) + \frac{2}{u_{n+1}} Q(n+1, u_{n+1}) + Q'(n, u_n) - \frac{2}{u_n} Q(n, u_n) = 0. \tag{3.49}$$

By isolating the terms in u_n, we get

$$\frac{d}{du_n}\left(Q'(n, u_n) - \frac{2}{u_n} Q(n, u_n)\right) = 0, \tag{3.50}$$

where n is just a parameter. Integrating once yields

$$Q'(n, u_n) - \frac{2}{u_n} Q(n, u_n) = A(n), \tag{3.51}$$

while a second integration gives

$$Q(n, u_n) = A(n) u_n + B(n) u_n^2. \tag{3.52}$$

When we substitute (3.52) into (3.49), the terms involving $B(n)$ and $B(n+1)$ all vanish and we are left with

$$A(n+1) = A(n). \tag{3.53}$$

The solution to this OΔE is

$$A(n) = C_1. \tag{3.54}$$

$B(n)$ is obtained by substituting (3.52) into (3.45). After simplification, we obtain the OΔE

$$B(n+2) - 2B(n+1) + B(n) = 0, \tag{3.55}$$

which admits the solution

$$B(n) = C_2 n + C_3. \tag{3.56}$$

The general expression for the characteristic $Q(n, u_n)$ is thus

$$Q(n, u_n) = C_1 u_n + (C_2 n + C_3) u_n^2. \tag{3.57}$$

In summary, (3.42) admits the three symmetry generators

$$\begin{aligned} X_1 &= u_n \frac{\partial}{\partial u_u} \\ X_2 &= n u_n^2 \frac{\partial}{\partial u_n} \\ X_3 &= u_n^2 \frac{\partial}{\partial u_n}. \end{aligned} \tag{3.58}$$

■

This method is successfully used to determine the symmetries of a general form of OΔEs.

3.6 Uses of symmetries of ordinary difference equations

We can use the symmetries of OΔEs to compute the first integrals of the equation, reduce its order, derive physically important solutions from known trivial ones, or linearize nonlinear OΔEs. We tackle these applications in turn.

3.6.1 First integrals of ordinary difference equations

Consider the OΔE

$$u_{n+k} = \omega(u_n, u_{n+1}, \dots, u_{n+k-1}). \tag{3.59}$$

Suppose this equation admits the symmetry generator (3.25). A first integral of (3.59) admitting (3.25) is obtained by solving

$$X^{[k-1]}f = 0, \tag{3.60}$$

where $X^{[k-1]}$ represents the $(k-1)th$ prolonged operator and f represents a first integral of (3.59). Unlike in the case of ODEs where we had to invoke definition 2.1.6, the discrete analogue for OΔEs is

$$I_{n+1} = I_n\big|_{u_{n+k}=\omega}\,. \tag{3.61}$$

Solving (3.61) is no longer as easy as in the case of ODEs. We illustrate this via an example.

Example 3.6.1 *Consider the free particle OΔE*

$$u_{n+2} - 2u_{n+1} + u_n = 0. \tag{3.62}$$

Here, we wish to find "autonomous" first integrals only. As a result, we take f to be $f(u_n, u_{n+1})$ and we do not impose (3.60), but rather go directly to (3.61). This means that we now have

$$f(u_n, u_{n+1}) = f(u_{n+1}, u_{n+2}). \tag{3.63}$$

To avoid confusion, we will set $g(u_{n+1}, u_{n+2}) = f(u_{n+1}, u_{n+2})$ in our subsequent calculations. We also set

$$u_{n+2} = 2u_{n+1} - u_n = \alpha \tag{3.64}$$

for convenience and so

$$f(u_n, u_{n+1}) = g(u_{n+1}, \alpha). \tag{3.65}$$

Differentiating (3.65) separately with respect to u_n and u_{n+1} gives

$$\frac{\partial f}{\partial u_n} = -\frac{\partial g}{\partial \alpha} \tag{3.66}$$

and

$$\frac{\partial f}{\partial u_{n+1}} = \frac{\partial g}{\partial u_{n+1}} + 2\frac{\partial g}{\partial \alpha} \tag{3.67}$$

respectively. Hence

$$\frac{\partial^2 g}{\partial \alpha^2} = 0, \tag{3.68}$$

giving

$$g(u_{n+1},\alpha) = g_0(u_{n+1}) + g_1(u_{n+1})\alpha \tag{3.69}$$

and so,

$$\frac{\partial f}{\partial u_n} = -g_1. \tag{3.70}$$

Thus

$$f(u_n,u_{n+1}) = -g_1(u_{n+1})u_n + f_0(u_{n+1}). \tag{3.71}$$

Equation (3.67) now becomes

$$\begin{aligned}\frac{\partial f}{\partial u_{n+1}} &= \frac{dg_0}{du_{n+1}} + \alpha\frac{dg_1}{du_{n+1}} + 2g_1 \\ &= -\frac{dg_1}{du_{n+1}}u_n + \frac{df_0}{du_{n+1}}.\end{aligned} \tag{3.72}$$

This leads to

$$\frac{d\left(f_0 - g_0\right)}{du_{n+1}} = \left(u_n + \alpha\right)\frac{dg_1}{du_{n+1}} + 2g_1, \tag{3.73}$$

implying that

$$\frac{dg_1}{du_{n+1}} = 0 \tag{3.74}$$

and so

$$g_1 = C_1, \tag{3.75}$$

a constant. Integrating (3.73) now yields

$$f_0(u_{n+1}) = 2C_1u_{n+1} + g_0 + C_2, \tag{3.76}$$

where C_2 is an arbitrary constant and so we have

$$f(u_n,u_{n+1}) = -C_1u_n + 2C_1u_{n+1} + g_0 \tag{3.77}$$

and

$$g(u_{n+1},u_{n+2}) = g_0 + C_1u_{n+2}, \tag{3.78}$$

where we have ignored the additive constant C_2. Substituting back in (3.65) and recalling that $g(u_{n+1},u_{n+2}) = f(u_{n+1},u_{n+2})$ requires

$$g_0(u_{n+1}) = -C_1u_{n+1}. \tag{3.79}$$

Therefore,

$$f(u_n, u_{n+1}) = C_1(u_{n+1} - u_n). \tag{3.80}$$

We conclude that a first integral of (3.62) is

$$I = u_{n+1} - u_n, \tag{3.81}$$

or any function of I. ■

Thus one can use symmetries of OΔEs to obtain first integrals much like the case of ODEs.

3.6.2 Reduction of order of ordinary difference equations

Consider the system of q coupled "first-order" difference equations

$$u_i(n+1) = F_i(n, u_1(n), u_2(n), \ldots, u_q(n)), \quad i = 1, \ldots, q. \tag{3.82}$$

Theorem 3.6.2 *Assume that this system is invariant under the one-parameter infinitesimal evolutionary point transformation [50]*

$$\begin{aligned} n^* &= n \\ u_i^*(n) &= u_i(n) + \epsilon v(n, u(n)), \;\; i = 1, \ldots, q \end{aligned} \tag{3.83}$$

where $u = (u_1, \ldots, u_q)$. Then (3.82) reduces to $q-1$ OΔEs. ■

The proof of this theorem is analogous to that of reduction of order by canonical coordinates for ODEs. Firstly, any infinitesimal point symmetry generator

$$X_e = Q_i(n, u_n^i)\frac{\partial}{\partial u_n^i} \tag{3.84}$$

can be transformed into

$$\begin{aligned} y^* &= y \\ w_i^*(y^*) &= w_i(y), \quad i = 1, \ldots, q-1 \\ w_i^*(y^*) &= w_q(y) + \epsilon, \end{aligned} \tag{3.85}$$

by introducing a canonical variable

$$y = \eta(n,u), \quad w = \zeta(n,u). \tag{3.86}$$

For ODEs, the above condition is sufficient to prove reduction of order (see for example [48]), as any transformation of (3.86) takes an ODE into another ODE. An OΔE however is not always transformed into another OΔE by a general transformation as given in (3.86). A sufficient condition for this to happen is that the transformation must be of the form [48]

$$y = n, \ w = \zeta(n,u). \tag{3.87}$$

In other words, the symmetry needs to be a point evolutionary symmetry transformation since (3.83) yields (3.86). We then know that in terms of n and w, (3.82) takes the form

$$w_i(n+1) = H_i(n, w_1(n), \ldots, w_{q-1}(n)), \ i = 1, 2, \ldots, q \tag{3.88}$$

$$w_q(n) = H_q(n, w_1(n), \ldots, w_{q-1}(n)). \tag{3.89}$$

Note that (3.89) is decoupled from (3.88) and can be trivially solved for w_q in terms of $n, w_1(n), \ldots, w_{q-1}(n)$. Note that successive reduction of the number of equations by one results in the resolution of the entire system.

Corollary 3.6.3 *A system that possesses one evolutionary symmetry generator possesses infinitely many such generators [42]. That is because (3.88) and (3.89) are invariant under the infinite-dimensional symmetry group generated by the infinitesimal transformations*

$$\begin{aligned} n^* &= n, \\ w_i^*(n) &= w_i(n), \ i = 1, 2, \ldots, q-1 \\ w_q^*(n) &= w_q(n) + \epsilon\lambda(n), \end{aligned} \tag{3.90}$$

where λ is an arbitrary unit periodic function. ■

This also explains why the above reduction of order will not necessarily work for ordinary differential-difference equations or for difference equations with incommensurate spans.

Example 3.6.4 *We found the characteristics of symmetry generators of (3.42) to be linear combinations of*

$$Q_1 = u_n, \qquad Q_2 = nu_n^2, \qquad Q_3 = u_n^2. \tag{3.91}$$

Consider the symmetry

$$X = u_n \frac{\partial}{\partial u_n}. \tag{3.92}$$

The resulting first order prolongation is

$$X^{[1]} = u_n \frac{\partial}{\partial u_n} + u_{n+1} \frac{\partial}{\partial u_{n+1}}. \tag{3.93}$$

This gives the invariant

$$v_n = \frac{u_{n+1}}{u_n}. \tag{3.94}$$

Substituting in (3.42) gives the condition for v, namely

$$v_{n+1} = \frac{2}{1 - 2v_n}, \tag{3.95}$$

which has order one. Therefore the order of the equation has been reduced by one. Note that from any given starting value v_0, one can find the value of v_k for any k. ■

3.6.3 Linearization of ordinary difference equations

Theorem 3.6.5 *A nonlinear OΔE is linearizable iff it admits a factorisable symmetry of the form [9]*

$$X = A(n)G(u_n) \frac{\partial}{\partial u_n}. \tag{3.96}$$

■

We prove this result for the family of second order OΔEs of the form

$$u_{n+2} = F([u]), \tag{3.97}$$

where

$$F([u]) = F(u_n, u_{n+1}). \tag{3.98}$$

If X is a symmetry of (3.97), then

$$X^{[2]}(u_{n+2} - F([u])) \mid_{u_{n+2}=F([u])} = 0. \tag{3.99}$$

We assume a symmetry generator of the form

$$X = A(n)G(u_n)\frac{\partial}{\partial u_n}. \tag{3.100}$$

Then (3.99) becomes

$$A(n+2)G(F([u])) = A(n+1)G(u_{n+1})\frac{\partial F([u])}{\partial u_{n+1}} + A(n)G(u_n)\frac{\partial F([u])}{\partial u_n}. \tag{3.101}$$

Assuming $A \neq 0$, we separate (3.101) into

$$G(u_n)\frac{\partial F([u])}{\partial u_n} = -\frac{A(n+1)}{A(n)}G(u_{n+1})\frac{\partial F([u])}{\partial u_{n+1}} + \frac{A(n+2)}{A(n)}G(F([u])). \tag{3.102}$$

Differentiating with respect to n we obtain

$$\left(\frac{A(n+1)}{A(n)}\right)' G(u_{n+1})\frac{\partial F([u])}{\partial u_{n+1}} = \left(\frac{A(n+2)}{A(n)}\right)' G(F([u])). \tag{3.103}$$

If A is not unit-periodic, we have

$$\frac{\left(\frac{A(n+2)}{A(n)}\right)'}{\left(\frac{A(n+1)}{A(n)}\right)'} = \frac{G(u_{n+1})}{G(F([u]))}\frac{\partial F([u])}{\partial u_{n+1}} = \Gamma_1, \tag{3.104}$$

where Γ_1 is a separation constant. Rearranging the second equation, we obtain

$$\frac{\frac{\partial F([u_n])}{\partial u_{n+1}}}{G(F([u]))} = \frac{\Gamma_1}{G(u_{n+1})}. \tag{3.105}$$

Hence, integrating with respect to u_{n+1} gives

$$\int^{F([u])} \frac{d\varsigma}{G(\varsigma)} = \Gamma_1 \int^{u_{n+1}} \frac{d\varsigma}{G(\varsigma)} + g(u_n). \tag{3.106}$$

Replacing $G(u_{n+1})\frac{\partial F([u_n])}{\partial u_{n+1}}$ via (3.105) in (3.102), separating and integrating, we obtain (taking (3.106)) into account)

$$\int^{F([u])} \frac{d\varsigma}{G(\varsigma)} = \Gamma_1 \int^{u_{n+1}} \frac{d\varsigma}{G(\varsigma)} + \Gamma_2 \int^{u_n} \frac{d\varsigma}{G(\varsigma)} + \Gamma_3, \tag{3.107}$$

where Γ_2 and Γ_3 are also constants. If we let

$$w_n = \int^{u_n} \frac{d\varsigma}{G(\varsigma)}, \tag{3.108}$$

then (3.107) reduces to

$$w_{n+2} = \Gamma_1 w_{n+1} + \Gamma_2 w_n + \Gamma_3, \tag{3.109}$$

which is linear.

Example 3.6.6 *Equation (3.42) was shown in example 3.5.3 to possess the symmetry*

$$X_2 = n u_n^2 \frac{\partial}{\partial u_n} \tag{3.110}$$

which is factorisable with $A = n$ and $G = u_n^2$.

The homogenizing variable w is given by setting

$$w_n = \int^{u_n} \frac{d\varsigma}{\varsigma^2} = -\frac{1}{u_n} \tag{3.111}$$

in (3.108). Equation (3.42) now becomes

$$w_{n+2} - 2w_{n+1} + w_n = 0 \tag{3.112}$$

with solution

$$w_n = n\delta_n + \lambda_n. \tag{3.113}$$

Thus

$$u_n = -\frac{1}{n\delta_n + \lambda_n}, \tag{3.114}$$

where δ and λ are unit periodic functions, is the general solution of (3.42). ■

Note that for this example, some authors have used an alternate method of finding symmetries by taking the Laurent series expansions (see for example [54]). Such a method, (repeated in [50]) is particularly limited as it is not always obvious to recognize the pattern when we expand the Laurent series. One could rather find all the point symmetries of the equation using the LSC [24] and choose the evolutionary and factorisable ones to perform linearization.

We realize that linearization of OΔEs is possible when we apply conditions given in the literature [9, 50, 54]. We have also applied Lie groups to reduce the order of OΔEs. It would be interesting to investigate the applicability of this linearization technique in PΔEs in detail.

Chapter 4

Symmetries of Partial Difference Equations

4.1 Introduction

Integrable systems are those that, although highly nontrivial and nonlinear, are amenable to exact and rigorous techniques for their solvability. They can take many shapes or forms: nonlinear evolution equations, PDEs and ODEs and ΔEs, Hamiltonian many-body systems, quantum systems and spin models in statistical mechanics. A large number of mathematical techniques have been developed to disentangle the rich structures behind these systems [26]. In this chapter we confine ourselves to integrable PΔEs. They have been classified in terms of the Adler-Bobenko-Suris (ABS) series. The discrete potential Korteweg-de Vries (dKdV) equation is a popular example of such a system. We will investigate its symmetries in detail later.

4.2 Determination of symmetries of partial difference equations

4.2.1 Method

Consider the equation

$$\omega(m, n, u_{00}, u_{10}, u_{01}, u_{11}) = 0, \tag{4.1}$$

where m and n are the independent variables, u_{00} is the dependent variable denoting u_{mn} and the u_{ij}'s denote the forward shifts i times in the m direction and j times in the n direction, i.e., $u_{ij} = u(m+i, n+j) = S_m^i S_n^j u_{00}$. We use the method of the LSC (see section 3.3) for PΔEs. We consider the symmetry in the form

$$X = Q(m, n, u_{-10}, u_{0-1}, u_{00}, u_{10}, u_{01}) \frac{\partial}{\partial u_{00}}. \tag{4.2}$$

We require

$$X^{[2]}\omega\big|_{\omega=0} = 0, \tag{4.3}$$

with

$$X^{[2]} = Q\frac{\partial}{\partial u_{00}} + (S_m Q)\frac{\partial}{\partial u_{10}} + (S_n Q)\frac{\partial}{\partial u_{01}} + (S_m S_n Q)\frac{\partial}{\partial u_{11}}. \tag{4.4}$$

Lie point symmetries of ΔEs restrict the arguments of Q to $Q = Q(m, n, u_{00})$. However, considering higher symmetries is more interesting [41]. We assume symmetries that depend upon the values of m and n on a square centered at u_{00} as shown in Figure 4.1 [52].

With this representation, we can use Q as in (4.2). Such a symmetry is termed a *five-point symmetry* [52]. The LSC (4.4) becomes

$$Q\omega_{u_{00}} + S_m Q\omega_{u_{10}} + S_n Q\omega_{u_{01}} + S_m S_n Q\omega_{u_{11}} = 0. \tag{4.5}$$

Let $\bar{u}_{ij}$ be the result of solving (4.1) for u_{ij}. Then (4.5) can be written as

$$\begin{aligned} & Q(\bar{u}_{00}(u_{00}, u_{-11}, u_{01}), \bar{u}_{00}(u_{1-1}, u_{00}, u_{10}), u_{00}, u_{10}, u_{01})\omega_{u_{00}} \\ & + Q(u_{00}, u_{1-1}, u_{10}, u_{20}, u_{11})\omega_{u_{10}} + Q(u_{-11}, u_{00}, u_{01}, u_{11}, u_{02})\omega_{u_{01}} \\ & + Q(u_{01}, u_{10}, u_{11}, \bar{u}_{11}(u_{10}, u_{20}, u_{11}), \bar{u}_{11}(u_{01}, u_{11}, u_{02}))\omega_{u_{1,1}} \quad = \quad 0, \end{aligned} \tag{4.6}$$

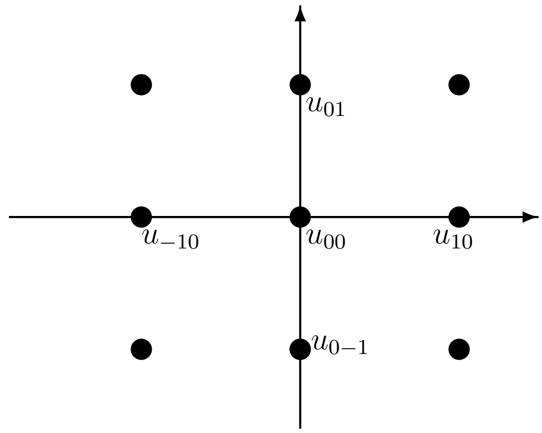

Figure 4.1: Form of a five-point symmetry.

where we have suppressed the m and n dependencies. As before (in section 3.5), Q has varied arguments. To make progress we differentiate (4.6) with respect to u_{-11} and u_{1-1} and obtain

$$\frac{\partial^2}{\partial u_{-11}\partial u_{1-1}} Q(\bar{u}_{00}(u_{00}, u_{-11}, u_{01}), \bar{u}_{00}(u_{1-1}, u_{00}, u_{10}), u_{00}, u_{10}, u_{01}) = 0. \tag{4.7}$$

Integrating yields Q as a sum of functions in a simpler form, i.e.,

$$Q(m, n, u_{-10}, u_{0-1}, u_{00}, u_{10}, u_{01}) = Q^m(m, n, u_{-10}, u_{00}, u_{10}) + Q^n(m, n, u_{0-1}, u_{00}, u_{01}), \tag{4.8}$$

where Q^m is the contribution from the m direction and Q^n from the n direction of the two dimensional map. Substituting Q in (4.6) we obtain

$$\begin{aligned}(Q^m(\bar{u}_{00}(u_{00}, u_{-11}, u_{01}), u_{00}, u_{10}) + Q^n(\bar{u}_{00}(u_{-11}, u_{00}, u_{01}), u_{00}, u_{01}))\omega_{u_{00}} & \\ + (Q^m(u_{00}, u_{10}, u_{20}) + Q^n(u_{1-1}, u_{01}, u_{11}))\omega_{u_{10}} & \\ + (Q^m(u_{-11}, u_{01}, u_{11}) + Q^n(u_{00}, u_{01}, u_{02}))\omega_{u_{01}} & \\ + (Q^m(u_{01}, u_{11}, \bar{u}_{11}(u_{10}, u_{20}, u_{11})) + Q^n(u_{10}, u_{11}, \bar{u}_{11}(u_{01}, u_{11}, u_{02})))\omega_{u_{11}} &= 0. \end{aligned} \tag{4.9}$$

Differentiating with respect to u_{20} we obtain

$$\omega_{u_{10}} \frac{\partial}{\partial u_{20}} Q^m(u_{00}, u_{10}, u_{20}) + \omega_{u_{11}} \frac{\partial}{\partial u_{20}} Q^m(u_{01}, u_{11}, \bar{u}_{11}(u_{10}, u_{20}, u_{11})) = 0. \tag{4.10}$$

The arguments of Q^m are still varied. We divide (4.10) by $\omega_{u_{10}}$ and differentiate the resulting equation with respect to u_{01} to obtain

$$\frac{\partial}{\partial u_{01}} \left(\frac{\omega_{u_{11}}}{\omega_{u_{10}}} \frac{\partial}{\partial u_{20}} Q^m(u_{01}, u_{11}, \bar{u}_{11}(u_{10}, u_{20}, u_{11})) \right) = 0. \tag{4.11}$$

Differentiation of (4.9) with respect to u_{02} gives rise to the expression

$$\omega_{u_{01}} \frac{\partial}{\partial u_{20}} Q^n(u_{00}, u_{01}, u_{02}) + \omega_{u_{11}} \frac{\partial}{\partial u_{02}} Q^n(u_{10}, u_{11}, \bar{u}_{11}(u_{01}, u_{11}, u_{02})) = 0. \tag{4.12}$$

We divide (4.12) by $\omega_{u_{01}}$ and differentiate with respect to u_{10} to obtain

$$\frac{\partial}{\partial u_{10}} \left(\frac{\omega_{u_{11}}}{\omega_{u_{01}}} \frac{\partial}{\partial u_{02}} Q^n(u_{10}, u_{11}, \bar{u}_{11}(u_{01}, u_{11}, u_{02})) \right) = 0. \tag{4.13}$$

We need to solve (4.11) and (4.13) for Q^m and Q^n respectively. Note that so far, we have differentiated the determining equations (4.9) twice. This has created a hierarchy of functional-differential equations that every five-point symmetry must satisfy. The unknown functions Q^m and Q^n can be found completely by going up the hierarchy, a step at a time, to determine more constraints. As the constraints are solved sequentially, further information is gained about these functions. At the highest stage, the determining equation is satisfied and the only remaining unknowns are the constants that multiply each symmetry component.

In the case of the dKdV equation [1]

$$\omega = (u_{11} - u_{00})(u_{10} - u_{01}) + \beta - \alpha, \tag{4.14}$$

this method gives rise to the symmetries [52]:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} \\
X_2 &= \frac{\partial}{\partial u_{00}} \\
X_3 &= (-1)^{m+n} \frac{\partial}{\partial u_{00}} \\
X_4 &= u_{00} \frac{\partial}{\partial u_{00}} + 2\alpha \frac{\partial}{\partial \alpha} + 2\beta \frac{\partial}{\partial \beta} \\
X_5 &= (-1)^{m+n} u_{00} \frac{\partial}{\partial u_{00}} \\
X_6 &= \frac{1}{u_{10} - u_{-10}} \frac{\partial}{\partial u_{00}} \\
X_7 &= \frac{1}{u_{01} - u_{0-1}} \frac{\partial}{\partial u_{00}} \\
X_8 &= \frac{m}{u_{10} - u_{-10}} \frac{\partial}{\partial u_{00}} - \frac{\partial}{\partial \alpha} \\
X_9 &= \frac{n}{u_{01} - u_{0-1}} \frac{\partial}{\partial u_{00}} - \frac{\partial}{\partial \beta}.
\end{aligned} \tag{4.15}$$

It is a simple matter to verify that they satisfy (4.9).

4.3 Uses of symmetries of partial difference equations

Given a system and its symmetries, we hope to be able to use these symmetries to analyze the equation. We state and prove the liearization theorem for PΔEs.

Definition 4.3.1 *A function $u : \mathbb{C}^k \to \mathbb{C}$ is totally periodic on P if*

$$u_{(\alpha)} = u \quad \forall \alpha \in P \tag{4.16}$$

■

Definition 4.3.2 *A function $F : E \to \mathbb{C}$ has a maximum rank if there is a generator X of symmetry on E such that [48]*

$$X(F) \mid_{F=0} = 0. \tag{4.17}$$

■

We are now in a position to state and prove the linearization theorem.

Theorem 4.3.3 *Assume that an autonomous, meromorphic PΔE of maximum rank is given by*

$$F([u])|_P = 0, \tag{4.18}$$

where $F : J^P_\Delta(E) \to \mathbb{C}$ and $J^P_\Delta(E)$ is the discrete P-jet space on the manifold $M(\mathbb{C})$. If (4.18) has a factorizable evolutionary symmetry generator

$$X(n,u) = A(x)G(u)\frac{\partial}{\partial u} \tag{4.19}$$

with $A \in M(\mathbb{C})$ minimally P-periodic defined on the manifold M and $n = (n^1, n^2, \ldots, n^k)$, then in the coordinates (m, w) given by

$$\begin{aligned} m(n) &= n \\ w(n,u) &= \int^u \frac{d\zeta}{G(\zeta)}, \end{aligned} \tag{4.20}$$

the PΔE is linear with constant coefficients [50]. ■

Proof

The equation under consideration is of Pth order and so, the Pth prolongation of X is given by

$$X^P = \Sigma_{j=0}^{M-1} G(u_{(\alpha(j))}) \frac{\partial}{\partial u_{(\alpha(j))}}. \tag{4.21}$$

Imposing the invariance condition yields

$$\Sigma_{j=0}^{M-1} A(n+\alpha) G(u_{(\alpha)}) \left. \frac{\partial F[u]}{\partial u_{(\alpha)}} \right|_{F([u])=0} = 0. \tag{4.22}$$

Assume that $\frac{\partial F([u])}{\partial u_{(\alpha(M-1))}} \neq 0$. (We could always choose P in such a way that this is possible together with maximum rank.) Equation (4.22) becomes

$$\Sigma_{k=0}^{M-1} A(n+\alpha(k)) G(u_{(\alpha(k))}) \left. \frac{\partial F([u])}{\partial u_{(\alpha(k))}} \right|_{F([u])=0} = 0. \tag{4.23}$$

Splitting the first term and dividing it by $A(n+\alpha(1)) \neq 0$ makes the left hand side of (4.23) independent of n. We have

$$-G(u_{(\alpha(1))}) \left. \frac{\partial F([u])}{\partial u_{(\alpha(1))}} \right|_{F([u])=0} = \Sigma_{k=2}^{M} \frac{A(n+\alpha(k))}{A(n+\alpha(1))} G(u_{(\alpha(k))}) \left. \frac{\partial F([u])}{\partial u_{(\alpha(k))}} \right|_{F([u])=0} = 0. \tag{4.24}$$

The autonomy of F allows the n^j to remain independent of $u_{(\alpha)}$. Thus differentiating (4.24) with respect to any of the n^j makes F to vanish on the left hand side. Additionally, since A is totally periodic on P, there exists at least one generator of symmetry X on $\mathbb{C}^n$ with which we can operate without making the right hand side zero as well. Hence, (4.23) reduces to

$$\Sigma_{k=1}^{M-1} A_{\alpha(p);n}(n) G(u_{(\alpha(k))}) \left. \frac{\partial F([u])}{\partial u_{(\alpha(k))}} \right|_{F([u])=0} = 0. \tag{4.25}$$

It has been established (see for example [9]) that $\mu(p)$ can be chosen so that

$$A^p_{\alpha(p)\mu(p)} \neq 0, \quad \forall p < M-1. \tag{4.26}$$

The process of dividing and differentiating can continue until we have completely separated variables and (4.25) reduces to

$$- \left. \frac{A^{M-2}_{\alpha(M-1);\mu(M-2)}(n)}{A^{M-2}_{\alpha(M-2);\mu(M-2)}(n)} \right|_{F([u])=0} = \left. \frac{G(u_{(\alpha(M-2))}) \frac{\partial F([u])}{\partial u_{(\alpha(M-2))}}}{G(u_{(\alpha(M-1))}) \frac{\partial F([u])}{\partial u_{(\alpha(M-1))}}} \right|_{F([u])=0}. \tag{4.27}$$

Therefore each side of (4.27) is equal to a constant, Γ_{M-1} say, commonly known as the separation constant. The right hand side of equation (4.27) evaluates to

$$\frac{1}{G(u_{(\alpha(M-1))})}\frac{\partial u_{(\alpha(M-1))}}{\partial u_{(\alpha(M-2))}} = \Gamma_{M-1}\frac{1}{G(u_{(\alpha(M-2))})}. \tag{4.28}$$

Integrating (4.28) with respect to $u_{(\alpha(M-1))}$ results in

$$\int^{u_{(\alpha(M-1))}} \frac{d\zeta}{G(\zeta)} = \Gamma_{M-1}\int^{u_{(\alpha(M-2))}} \frac{d\zeta}{G(\zeta)} + \Omega_{\alpha(M-2)}([u]), \tag{4.29}$$

where $\Omega_{\alpha(M-2)}([u])$ is the constant of integration and hence does not depend on $u_{(\alpha(M-2))}$. Considering now that

$$\Sigma_{k=M-3}^{M-1} A^{M-3}_{\alpha(k);\mu(M-3)}(n) G(u_{(\alpha(k))}) \left.\frac{\partial F([u])}{\partial u_{(\alpha(k))}}\right|_{F([u])=0} = 0, \tag{4.30}$$

we apply the separation of variables to equation (4.27) and substitute

$$\Gamma_{M-1}G(u_{(\alpha(M-1))})\frac{\partial F([u])}{\partial u_{(\alpha(M-1))}} = G(u_{(\alpha(M-2))})\frac{\partial F([u])}{\partial u_{(\alpha(M-2))}} \tag{4.31}$$

into (4.30). This yields

$$\begin{aligned}
\Gamma_{M-2} &= \left.\frac{\Gamma_{M-1}A^{M-3}_{\alpha(M-2);\mu(M-3)}(n) + A^{M-3}_{\alpha(M-1);\mu(M-3)}(n)}{A^{M-3}_{\alpha(M-3);\mu(M-3)}(n)}\right|_{F([u])=0} \\
&= \left.\frac{G(u_{(\alpha(M-3))})\frac{\partial F([u])}{\partial u_{(\alpha(M-3))}}}{G(u_{(\alpha(M-1))})\frac{\partial F}{\partial u_{(\alpha(M-1))}}([u])}\right|_{F([u])=0}.
\end{aligned} \tag{4.32}$$

We integrate the right hand side of (4.32) and obtain

$$\int^{u_{(\alpha(M-1))}} \frac{d\zeta}{G(\zeta)} = \Gamma_{M-2}\int^{u_{(\alpha(M-3))}} \frac{d\zeta}{G(\zeta)} + \Omega_{\alpha(M-3)}([u]). \tag{4.33}$$

The process of differentiating and separating continues and we get at the end of the chain of separation the sequence

$$\begin{aligned}
\int^{u_{\alpha(M-1)}} \frac{d\zeta}{G(\zeta)} &= \Gamma_{M-1}\int^{u_{\alpha(M-3)}} \frac{d\zeta}{G(\zeta)} + \Omega_{\alpha(M-2)} \\
&\vdots \\
\int^{u_{\alpha(M-1)}} \frac{d\zeta}{G(\zeta)} &= \Gamma_{1}\int^{u_{\alpha(1)}} \frac{d\zeta}{G(\zeta)} + \Omega_{\alpha(0)}([u]),
\end{aligned} \tag{4.34}$$

where the function $\Omega_{\alpha(j)}$, is taken as the constant of integration as it is independent of $u_{(\alpha(j))}$ for all j. If we define

$$w_{\alpha(j)} = \int^{u_{\alpha(j)}} \frac{d\zeta}{G(\zeta)}, \tag{4.35}$$

the above sequence yields

$$w_{M-1} - \Sigma_{k=0}^{M-2}\Gamma_{k+1}w_{\alpha(k)} - \Gamma_0 \mid_{F=0} = 0. \tag{4.36}$$

This is a linear, constant coefficient PΔE resulting from the nonlinear one originally considered.■

Example 4.3.4 *For the nonlinear PΔE [9, 54]*

$$\begin{aligned}&\left(u_{20}\sqrt{1+u_{11}^2} + u_{11}\sqrt{1+u_{20}^2}\right)\left(\sqrt{(1+u_{01}^2)(1+u_{00}^2)} + u_{00}u_{01}\right)\\ -&\left(u_{01}\sqrt{1+u_{00}^2} + u_{00}\sqrt{1+u_{00}^2} + u_{00}\sqrt{1+u_{01}^2}\right)\left(\sqrt{(1+u_{20}^2)(1+u_{11}^2)}u_{11}u_{20}\right) = 0\end{aligned} \tag{4.37}$$

where u_{ij} indicates the simultaneous shifts i times in the first independent variable and j times in the second. In this case, the set P is given by

$$P = \{(0,0),(0,1),(1,0),(1,1),(2,0)\} \tag{4.38}$$

and it can be shown that equation (4.37) admits the evolutionary factoraizable symmetry

$$X(x,y,u) = (1 + 2^x3^y\sin(\pi y) + 2^x\sin(\pi x) + 2^y)\sqrt{1+u^2}\frac{\partial}{\partial u}. \tag{4.39}$$

Note that the factor

$$A(x) = 1 + 2^x3^y\sin(\pi y) + 2^x\sin(\pi x) + 2^y \tag{4.40}$$

is minimally $P-$periodic. By Theorem 4.3.3, the transformation

$$w(u) = \int^u \frac{d\zeta}{G(\zeta)} = \operatorname{arcsinh}(u) \tag{4.41}$$

can be used to linearize the PΔE (4.37) to the linear PΔE

$$w_{20} + w_{11} - w_{01} - w_{00} = 0. \tag{4.42}$$

■

4.4 Conservation laws of partial difference equations

A technique for obtaining conservation laws of PΔE was investigated by Hydon [26]. The method has an important feature in that it does not use symmetries and is not based on Noether's approach. As such, the system under analysis does not have to admit a Lagrangian or Hamiltonian. It is important to note that the equation need not be integrable as the conservation of conserved quantities must be maintained after discretization of a DE [26].

Consider a PΔE of the form

$$E(u_{00}, u_{10}, u_{01}, u_{11}) = 0, \tag{4.43}$$

where u_{00} is the dependent variable denoting u_{mn} and u_{ij} is the denotes the forward shifts i times in the m direction and j times in the n direction. That is $u_{ij} = u_{mn} = S_m^i S_n^j u_{00}$. In this work, we will rewrite (4.43) as

$$u_{11} = z(u_{00}, u_{10}, u_{01}), \tag{4.44}$$

or

$$u_{10} = \Omega(u_{00}, u_{01}, u_{11}). \tag{4.45}$$

A conservation law of equation (4.43) is an expression of the form

$$(S_m - id)T^1 + (S_n - id)T^2 = 0 \tag{4.46}$$

that is satisfied by all solutions of (4.43). T^1 and T^2 are the components of the conservation law T and id is the identity mapping [46].

Conservation laws that depend only on u_{00}, u_{01} and u_{10} are known as 3-point conservation laws. In this case we have

$$T^1 = T^1(m, n, u_{00}, u_{01}), \quad T^2 = T^2(m, n, u_{00}, u_{10}). \tag{4.47}$$

Substituting (4.47) into (4.46) yields

$$T^1(m+1, n, u_{10}, u_{11}) - T^1(m, n, u_{00}, u_{01}) + T^2(m, n+1, u_{01}, u_{11}) - T^2(m, n, u_{00}, u_{10}) = 0 \tag{4.48}$$

and replacing u_{11} by z yields

$$T^1(m+1, n, u_{10}, z) - T^1(m, n, u_{00}, u_{01}) + T^2(m, n+1, u_{01}, z) - T^2(m, n, u_{00}, u_{10}) = 0. \tag{4.49}$$

In order to eliminate terms dependent on z, we apply the differential operators

$$D_1 = \frac{\partial}{\partial u_{01}} - \frac{z_{u_{01}}}{z_{u_{00}}} \frac{\partial}{\partial u_{00}}, \quad D_2 = \frac{\partial}{\partial u_{10}} - \frac{z_{u_{10}}}{z_{u_{00}}} \frac{\partial}{\partial u_{00}}, \tag{4.50}$$

which commute, to (4.49), where $z_{u_{ij}} = \frac{\partial z}{\partial u_{ij}}$. Note that z is invariant under the operators D_1 and D_1. We then have

$$D_1 D_2 (T^1(m, n, u_{00}, u_{01}) + T^2(m, n, u_{00}, u_{10})) = 0. \tag{4.51}$$

To make progress, we first eliminate one of the components, T^2 for example, and obtain a condition for T^1 only. This expression is obtainable via differentiation of (4.51) several times with respect to specific variables. This PDE in T^1 is then separated with respect to the powers of u_{11} and a system of determining equations is obtained. We get additional information on T^1 by substituting (4.45) into (4.48). This gives

$$T^1(m+1, n, \Omega, u_{11}) - T^1(m, n, u_{00}, u_{01}) + T^2(m, n+1, u_{01}, u_{11}) - T^2(m, n, u_{00}, \Omega) = 0. \tag{4.52}$$

To eliminate terms depending on Ω we use the differential operators

$$D_3 = \frac{\partial}{\partial u_{01}} - \frac{\Omega_{u_{01}}}{\Omega_{u_{00}}} \frac{\partial}{\partial u_{00}}, \; D_4 = \frac{\partial}{\partial u_{11}} - \frac{\Omega_{u_{11}}}{\Omega_{u_{00}}} \frac{\partial}{\partial u_{00}}. \tag{4.53}$$

Equation (4.52) becomes

$$D_3 D_4 (-T^1(m, n, u_{00}, u_{01}) + T^2(m, n+1, u_{01}, u_{11})) = 0. \tag{4.54}$$

The process follows exactly as the one we just described, and yields another expression for $T^1(m, n, u_{00}, u_{01})$ which is typically different from the one resulting from the substitution of (4.44) above. We are then able to determine T^1. Notice that if we eliminated T^1 instead of T^2 in (4.51), we would have obtained a differential equation in T^2 (by differentiating several times, this round, with respect to u_{01}). The obtained expression would then be separated with respect to the powers of u_{11} and would have yielded an overdetermined system of PDEs in T^2. However, it is possible to obtain T^2 directly from (4.46) after we have T^1 using

$$T^2 = -(S_n - id)^{-1}(S_m - id)\, T^1. \tag{4.55}$$

We illustrate the details in an example.

Example 4.4.1 *The dKdV equation (4.14) for $\beta - \alpha = -1$ can be written in either of the forms*

$$u_{11} = z,\ z = \frac{1}{u_{01} - u_{10}} + u_{00} \tag{4.56}$$

or

$$u_{10} = \Omega,\ \Omega = \frac{1}{u_{11} - u_{00}} + u_{01}. \tag{4.57}$$

As we seek 3-point conservation laws, we restrict ourselves to the T^i's defined via (4.47) and proceed as in equations (4.48) to (4.49). To get rid of $T^1(m+1, n, u_{10}, z)$, we apply the operator D_1 to (4.49) to obtain

$$T^1_{u_{01}}(m,n,u_{00},u_{01}) + T^2_{u_{01}}(m,n+1,u_{01},z) - \frac{1}{(u_{01}-u_{10})^2}\left(T^1_{u_{00}}(m,n,u_{00},u_{01}) - T^2_{u_{00}}(m,n,u_{00},u_{10})\right) = 0. \tag{4.58}$$

We now eliminate the term $T^2(m, n+1, u_{01}, z)$ by applying D_2 to (4.58). We obtain

$$T^1_{u_{00}^2} + T^2_{u_{00}^2} - (u_{10} - u_{01})^2 (T^1_{u_{00}u_{01}} - T^2_{u_{00}u_{10}}) - 2(u_{10} - u_{01})(T^1_{u_{00}} + T^2_{u_{00}}) = 0, \tag{4.59}$$

which is the equivalent of (4.51). Differentiating (4.59) three times with respect to u_{01} yields

$$T^1_{u_{00}^2 u_{01}^3} - (u_{10} - u_{01})^2 T^1_{u_{00}u_{01}^4} + 4(u_{10} - u_{01}) T^1_{u_{00}u_{01}^3} = 0, \tag{4.60}$$

which is a functional equation that can be separated into an overdetermined system. To do so, we notice that T^1 is independent of u_{10}. Hence one could separate (4.60) with respect to the powers of u_{10} to obtain the system

$$\begin{array}{rrcl} u_{10}^2: & T^1_{u_{00}u_{01}^4} & = & 0 \\ u_{10}: & 2u_{01}T^1_{u_{00}u_{01}^4} + 4T^1_{u_{00}u_{01}^3} & = & 0 \\ 1: & T^1_{u_{00}^2u_{01}^3} - u_{01}^2 T^1_{u_{00}u_{01}^4} - 4u_{01}T^1_{u_{00}u_{01}^3} & = & 0. \end{array} \tag{4.61}$$

To obtain further information on T^1, we now substitute (4.57) into (4.48). We obtain

$$T^1(m+1, n, \Omega, u_{11}) - T^1(m, n, u_{00}, u_{01}) + T^2(m, n+1, u_{01}, u_{11}) - T^2(m, n, u_{00}, \Omega) = 0. \tag{4.62}$$

Differentiating with respect to u_{00} and u_{11} keeping Ω fixed gives

$$T^1_{u_{01}^2} - \tilde{T}^2_{u_{01}^2} - (u_{11} - u_{00})^2 (T^1_{u_{00}u_{01}} + \tilde{T}^2_{u_{01}u_{11}}) - 2(u_{11} - u_{00})(T^1_{u_{00}} - \tilde{T}^2_{u_{01}}) = 0, \tag{4.63}$$

with $\tilde{T}^2 = T^2(m, n+1, u_{00}, u_{11})$*. To eliminate* $\tilde{T}^2$ *and its derivatives, we differentiate (4.63) three times with respect to* u_{01} *to obtain*

$$T^1_{u_{00}^3 u_{01}^2} - (u_{11} - u_{00})^2 T^1_{u_{00}^4 u_{01}} + 4(u_{11} - u_{00}) T^1_{u_{00}^3 u_{01}} = 0. \tag{4.64}$$

Equation (4.64) separates with respect to the powers of u_{11} *into*

$$\begin{aligned} u_{11}^2 : \quad & T^1_{u_{00}^4 u_{01}} &= \quad 0 \\ u_{11} : \quad & 2u_{00} T^1_{u_{00}^4 u_{01}} + 4 T^1_{u_{00}^3 u_{01}} &= \quad 0 \\ 1 : \quad & T^1_{u_{00}^3 u_{01}^2} - u_{00}^2 T^1_{u_{00}^4 u_{01}} - 4u_{00} T^1_{u_{00}^3 u_{01}} &= \quad 0. \end{aligned} \tag{4.65}$$

We solve this system together with (4.61) to obtain

$$T^1 = c_1 u_{00} u_{01} + c_2 u_{00}^2 u_{01} + c_3 u_{00} u_{01}^2 + c_4 u_{00}^2 u_{01}^2 + f_1 + f_2, \tag{4.66}$$

with the arbitrary functions

$$c_i = c_i(m, n) \tag{4.67}$$

and

$$f_1 = f_1(m, n, u_{01}), \quad f_2 = f_2(m, n, u_{00}). \tag{4.68}$$

We now focus on determining T^2*. Substituting (4.66) into (4.59) and differentiating twice with respect to* u_{01} *yields*

$$c_1 + c_2 u_{00} + 2c_3 u_{10} + c_4(2 + 4c_4 u_{00} u_{10}) + T^2_{u_{00} u_{10}} = 0. \tag{4.69}$$

Solving this equation gives

$$T^2 \quad = \quad -(c_1 + 2c_4) u_{00} u_{10} - c_2 u_{00}^2 u_{10} - c_3 u_{00} u_{10}^2 - c_4 u_{00}^2 u_{10}^2 + g_1 + g_2, \tag{4.70}$$

where $g_1 = g_1(m, n, u_{10}), \quad g_2 = g_2(m, n, u_{00})$*. It remains to determine the functions* c_i*,* f_j *and* g_j*. They can be found completely by going up the hierarchy, a step at a time, to determine more constraints. Without loss of generality, the trivial conservation law*

$$\begin{aligned} T^1_0 &= (S_n - id) f_2 \\ T^2_0 &= -(S_m - id) f_2 \end{aligned} \tag{4.71}$$

can be added to T^1 and T^2 respectively to remove the term $f_2(m,n,u_{00})$ [26]. The same thing can be done to remove the term $g_2(m,n,u_{00})$.

This implies that

$$T^2 = -(c_1+2c_4)u_{00}u_{10} - c_2u_{00}^2u_{10} - c_3u_{00}u_{10}^2 - c_4u_{00}^2u_{10}^2 + g_1, \tag{4.72}$$

and

$$T^1 = c_1u_{00}u_{01} + c_2u_{00}^2u_{01} + c_3u_{00}u_{01}^2 + c_4u_{00}^2u_{01}^2 + f_1. \tag{4.73}$$

Substituting (4.72) and (4.73) into (4.58) gives

$$\begin{aligned}
c_1(m+1,n)u_{10}u_{11} + c_2(m+1,n)u_{10}^2u_{11} + c_3(m+1,n)u_{10}u_{11}^2 + c_4(m+1,n)u_{10}^2u_{11}^2 \\
+ f_1(m+1,n,u_{11}) - c_1(m,n)u_{00}u_{01} - c_2(m,n)u_{00}^2u_{01} - c_3(m,n)u_{00}u_{01}^2 \\
- c_4(m,n)u_{00}^2u_{01}^2 - f_1(m,n,u_{01}) - (c_1(m,n+1) + 2c_4(m,n+1))u_{01}u_{11} \\
- c_2(m,n+1)u_{01}^2u_{11} - c_3(m,n+1)u_{01}u_{11}^2 - c_4(m,n+1)u_{01}^2u_{11}^2 + g_1(m,n+1,u_{11}) \\
+ (c_1(m,n) + 2c_4(m,n))u_{00}u_{10} + c_2(m,n)u_{00}^2u_{10} + c_3(m,n)u_{00}u_{10}^2 + c_4(m,n)u_{00}^2u_{10}^2 - g_1(m,n,u_{10}) &= 0.
\end{aligned} \tag{4.74}$$

Differentiating (4.74) 3 times with respect to u_{11} and solving the resulting equation, we obtain

$$f_1(m+1,n,u_{11}) = -g_1(m,n+1,u_{11}) + f_3(m,n)u_{11} + f_4(m,n). \tag{4.75}$$

We substitute the expression of f_1 into T_1 and back into (4.74), taking (4.56) into account, then we solve this equation by separation of variables and obtain

$$\begin{aligned}
c_1(m,n+1) + c_1(m,n) &= 2c_4(m,n+1) \\
c_1(m+1,n) + c_1(m,n) &= 2c_4(m,n) \\
c_4(m+1,n) + c_4(m,n) &= 0 \\
c_4(m,n+1) + c_4(m,n) &= 0 \\
c_2(m+1,n) - c_2(m,n+1) &= 0 \\
c_2(m,n) &= -\frac{1}{3}c_3(m,n+1) - \frac{2}{3}c_3(m+1,n) \\
c_3(m,n+1) - c_3(m+1,n) &= 0 \\
c_3(m,n+1) - c_3(m,n-1) &= 0,
\end{aligned} \tag{4.76}$$

with solution

$$c_3(m,n) = (-1)^{m-1}(C_3(-1)^{(m+n)} + (-1)^{(n-m)/2}D_3), \tag{4.77}$$

$$c_4(m,n) = (-1)^{m+n}C_4. \tag{4.78}$$

Note that we also have

$$c_2(m,n) = -c_3(m,n-1). \tag{4.79}$$

Hence we have

$$\begin{aligned}
c_1(m,n) &= 2(-1)^{m+1}\left(-1-(-1)^n\right)C_4 + (-1)^{n+m}C_1 \\
c_2(m,n) &= (-1)^m(C_2(-1)^{(n+m-1)} + (-1)^{(n-m-1)/2}D_3) \\
g_1(m,n,u_{10}) &= (-1)^{(m+n)}G_1 + G_2 \\
f_3(m,n) &= (-1)^{m+n}F_3 \\
f_4(m,n) &= (-1)^{m+n}F_4.
\end{aligned} \tag{4.80}$$

This process leads us to the conserved vectors

$$\begin{aligned}
T_1 &= 2(-1)^{m+1}\left(-1-(-1)^n\right)C_4 + (-1)^{n+m}C_1u_{00}u_{01} + (-1)^{m+n}C_4u_{00}^2u_{01}^2 \\
&\quad +(-1)^m(C_2(-1)^{(n+m-1)} + (-1)^{(n-m-1)/2}D_3)u_{00}^2u_{01} \\
&\quad +(-1)^m(C_3(-1)^{(n+m)} + (-1)^{(n-m)/2}D_3)u_{00}u_{01}^2 \\
&\quad +(-1)^{m+n}F_4 + (-1)^{m+n}F_3u_{01} + (-1)^{(m+n)}G_1u_{01} + G_2 \\
T_2 &= -2(-1)^{m+1}\left(-1-(-1)^n\right)C_4u_{00}u_{10} - (-1)^{n+m}C_1u_{00}u_{10} - (-1)^{m+n}C_4u_{00}^2u_{10}^2 \\
&\quad -(-1)^m(C_2(-1)^{(n+m-1)} + (-1)^{(n-m-1)/2}D_3)u_{00}^2u_{10} \\
&\quad -(-1)^{m-1}(C_3(-1)^{(n+m)} + (-1)^{(n-m)/2}D_3)u_{00}u_{10}^2.
\end{aligned} \tag{4.81}$$

With specific choices of the constants C_i, D_3, F_i and G_i, one may write the components of the conserved quantities. We may therefore have the following: For $C_1 = C_4 = D_3 = F_4 = G_1 = G_2 = 0, C_2 = -F_3 = -C_3 = 1$, we have

$$\begin{aligned}
T^1 &= u_{00}u_{01}^2 - u_{00}^2u_{01} + u_{00} - u_{01}, \\
T^2 &= u_{00}^2u_{10} - u_{00}u_{10}^2.
\end{aligned} \tag{4.82}$$

For $C_1 = F_4 = G_1 = G_2 = 0, C_2 = C_3 = F_3 = 1,$

$$\begin{aligned} T^1 &= (-1)^{m+n+1}\left(u_{00}u_{01}^2 + u_{00}^2u_{01} - u_{00} - u_{01}\right), \\ T^2 &= (-1)^{m+n}\left(u_{00}^2u_{10} + u_{00}u_{10}^2\right). \end{aligned} \tag{4.83}$$

For $C_1 = -2C_3 = -4F_4 = -2, C_2 = C_4 = F_3 = G_1 = G_2 = 0,$

$$\begin{aligned} T^1 &= (-1)^{m+n+1}\left(u_{00}u_{01}^2 - 2u_{00}u_{01} + \frac{1}{2}\right), \\ T^2 &= (-1)^{m+n}\left(u_{00}u_{10}^2 - 2u_{10}u_{00}\right). \end{aligned} \tag{4.84}$$

For $C_1 = -2F_4 = 1, C_2 = C_3 = C_4 = F_3 = G_1 = G_2 = 0,$

$$\begin{aligned} T^1 &= (-1)^{m+n+1}\left(u_{00}u_{01} - \frac{1}{2}\right), \\ T^2 &= (-1)^{m+n}\left(u_{00}u_{10}\right). \end{aligned} \tag{4.85}$$

These correspond to the ones given in [51]. ■

The method illustrated above is called the three-point conservation law method because the functions T^i are restricted to depend on m, n, u_{00}, u_{01} and u_{10} only. Relaxing this condition to allow T^1 to depend on variables $m, n, u_{-11}, u_{-10}, u_{00}, u_{0-1}$ and T^2 on $m, n, u_{-10}, u_{00}, u_{0-1}, u_{1-1}$ gives rise to five-point conservation laws [51]. The method uses the same technique, but involves more lengthy calculations. However, it has been shown [51] that the five-point conservation laws approach yields three additional conservation laws. One can find even more conservation laws by considering seven-point or nine-point conservation laws. However, the calculations become too complex. Gardner method allows us to systematically construct additional conservation laws using the symmetries of the equation. It has also been proved that the symmetry method yields infinitely many conservation laws for the dKdV equation. This method seams to be more interesting than the direct method. However, since we are interested in the understanding of finding conservation laws of difference equations, we prefer to build on the basic and simple method first and the more involving ones will constitute our future work.

As a final comment, we observe that Hydon [25] discussed multisymplectic conservation laws. This approach was applied to complex valued differential equations and differential-difference equations.

Chapter 5

Conclusion

5.1 Summary

We introduced the topic of differential equations in Chapter 1. Some examples of models described in terms of DEs were briefly discussed. Thereafter, we motivated our interest in discrete systems by giving some applications of ΔEs. A short background history of Lie symmetries was also provided in this Chapter.

In Chapter 2, we introduced the concept of Lie symmetry analysis of DEs by defining some key concepts. Thereafter, we gave the general algorithm for finding symmetries of DEs (ordinary and partial). Additionally, the uses of symmetries of DEs were discussed: Reduction of order was illustrated via an example. We also calculated first integrals and discussed the concept of transformation of equations and illustrated the processes by use of examples. For PDEs, we were able to find group-invariant solutions and introduced the concept of optimal systems. We reviewed the work on conservation laws, and an explanatory example was given to illustrate one of the methods. Other methods were also listed.

In Chapter 3, we introduced the application of Lie symmetry analysis to DΔEs and defined some key concepts. Thereafter, we showed that the method for finding symmetries of DΔEs is similar to that of DEs. As an example, we constructed the solutions of the Toda system using its symmetries. For OΔEs, we realized that the method for finding symmetries of DEs could not just be simply extrapolated to the purely discrete case. In this case, we only consider continuous transformations of the dependent variable. The uses of symmetries of OΔEs were discussed. We introduced first integrals of OΔEs and we were able to reduce the order of an OΔE. We also linearized OΔEs using a special type of symmetries $-$ the factorizable symmetries.

In Chapter 4, we applied symmetry analysis to PΔEs. We outlined the method and verified the results for the dKdV equation. The uses of symmetries of PΔEs were discussed and we linearized a highly nonlinear PΔE using its factorizable symmetry. We also studied the conservation laws of PΔEs and illustrated the direct method via a special case of the discrete KdV equation.

5.2 Observations

A comparative study shows that just like in the case of differential equations, in difference equations we can transform solutions into solutions or reduce difficult problems to simple ones. However, we have seen that the point symmetry methods used in differential equations need to be modified in difference equations. Given the fact that the independent variable is now discrete, we are no longer able to apply the continuous differential operator to it in the symmetry vector field (This excludes differential-difference as some independent variables are continuous [26].).

For differential equations, either the equations are already known and group theory is used to solve them, or the symmetries of the problem at hand are known and are used to build the theoretical model - the symmetries precede the equations. This is also applicable for difference equations, but we note some challenges: The physical process described may be discrete and the lattices involved may be real physical objects. In linear theories such as quantum mechanics, or quantum field theory on a lattice, generalized point symmetries are most appropriate. This is also true for nonlinear problems on given fixed lattices [24]. They may mainly be used for the same motive as in the continuous case - to identify integrable systems on lattices. This method can also be used to generate more interesting solutions from known trivial ones. One of the interesting features of point symmetries of differential equations is that dilations particularly appear as generalized symmetries of difference equations [16].

For nonlinear difference equations in physics, we have in mind the situation when the processes are actually continuous and are described in terms of differential equations. The problem is then discretized so that it can be solved (see for example [17]). Since we then have to choose the lattice, we can do so to preserve symmetry. Difference equations and lattices form parts of the difference scheme and the lattice is just part of the solution of the scheme. Therefore we can restrict to point transformations, but this will act the same time on the lattice and the solutions.

5.3 Open problems

The concept of symmetry adopted lattices [15, 16, 39] and the use of Lie point symmetry for linearization condition for difference equations (see for example [9, 54]) still need attention. Numerical methods for differential equations have also been left out so far - making use of their symmetry properties or the treatment of asymptotic symmetries for difference equations.

The use of Umbral calculus constitutes interest for further studies in the field of symmetries of difference equations. Furthermore, in calculating the symmetries of difference equations (during the process of linearization) in Chapter 3, we faced the difficulty of not being able to recognize the pattern easily for the Laurent series approximations. The development of some computer algebra (see for example Sym [13, 14], Lie [22], etc) would help bypass the complexity issue in the calculation of symmetries and reduction of order. They may help create new packages for discrete systems.

Moreover, while the classification of admitted symmetries has received exhaustive attention in DEs, this aspect has not received much attention so far in the discrete case. For example, it is unclear what is the symmetry group of linear second order OΔEs. Indeed, it is unknown whether all linear second order OΔEs belong to the same equivalent class. The same issues apply for PΔEs.

Finally, we note that the work on conservation laws has focused on equations with only two independent variables. Although Hydon [26] states in his conclusion that the method can be used for n independent variables, no practical work has thus far been carried out to the best of our knowledge. The extension of this theory and its generalization may be of interest for further research. Additionally, there are several methods for finding conservation laws of DEs which could be modified to find those of ΔEs. Kara and Mahomed discovered a method for finding conservation laws via symmetries and Lagrangian [35]. The quest for a discrete analogue of these methods and the development of new ones intrinsic to ΔEs is the subject for future work.

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