ON THE THEORY OF THE FROBENIUS GROUPS

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Abstract

The Frobenius group is an example of a split extension. In this dissertation we study and describe the properties and structure of the group. We also describe the properties and structure of the kernel and complement, two non-trivial subgroups of every Frobenius group. Examples of Frobenius groups are included and we also describe the characters of the group. Finally we construct the Frobenius group $29^2 : \mathsf{SL}(2,5)$ and then compute it's Fischer matrices and character table.

Preface

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The use of the work of others however has been duly acknowledged throughout the dissertation.

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List of Notations

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integer numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{F}$	a field
$\mathbb{F}^*$	multiplicative group of $\mathbb{F}$
$\mathbb{F}_q$	Galois field of q elements
V	vector space
$\dim$	dimension of a vector space
det	determinant of a matrix
tr	trace of a matrix
G	a finite group
$e, 1_G$	identity of G
$\lvert G\rvert$	order of G
$o(g)$	order of $g \in G$
$\cong$	isomorphism of groups
$H \leq G$	H is a subgroup of G
$[G : H]$	index of H in G
$N \trianglelefteq G$	N is a normal subgroup of G
$N \times H$, $\bigotimes$	direct product of groups
$N{:}H$	split extension of N by H
G/N	quotient group
$[g]$, C_g	conjugacy class of g in G
$C_G(g)$	centralizer of $g \in G$
G_x, $\mathrm{Stab}_G(x)$	stabilizer of $x \in X$ when G acts on X
x^G	orbit of $x \in X$
$\lvert \mathrm{Fix}(g)\rvert$	number of elements in a set X fixed by $g \in G$ under the group action
$Aut(G)$	automorphism group of G

$\mathrm{Holo}(G)$	holomorph of G
$[x, y]$	commutator of x and y in G
G'	derived or commutator subgroup of G
$Z(G)$	center of G
D_{2n}	dihedral group consisting of $2n$ elements
$\mathrm{Syl}_p(G)$	set of Sylow $p-$subgroups of G
$\mathbb{Z}_n$	group $\{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$ under addition modulo n
S_n	symmetric group of n objects
A_n	alternating group of n objects
$GL(n, \mathbb{F})$	general linear group over a field $\mathbb{F}$
$GL(n, q)$	finite general linear group over $\mathbb{F}_q$
$SL(n, \mathbb{F})$	special linear group
$\mathrm{Aff}(n, \mathbb{F})$	affine group
χ	character of finite group
χ_ρ	character afforded by a representation ρ of G
$\mathbf{1}$	trivial character
$\deg$	degree of a representation or character
$\mathrm{Irr}(G)$	set of ordinary irreducible characters of G
$\chi\uparrow_H^G$	character induced from subgroup H to G
$\chi\downarrow_H^G$	character restricted from a group G to it's subgroup H
$\rho \uparrow G$	induced representation from subgroup H to group G
$\rho \downarrow G$	restriction of representation ρ of group G to subgroup H
$\chi(1_G)$	degree of character χ
$\overline{\chi(g)}$	conjugate of character value $\chi(g)$
$\widetilde{\chi}$	lift of character χ
$\widehat{\chi}$	character of factor group G/N

$\mathcal{C}(G)$	algebra of class functions of a group G
$\mathbb{F}[G]$	group algebra of a finite group G over a field $\mathbb{F}$
ϕ^g	conjugate class function/character
$I_G(\phi)$	inertia group of a character ϕ
$cf(H)$	set of class functions of group H
$\widehat{S}$	matrix of a representation S
$\langle\ ,\ \rangle$	inner product of class functions or group generated by two elements
$\otimes$	tensor product of representations
$\oplus, \bigoplus$	direct sum
x^g	action of g on x (gx or xg) when group G acts on set X
$x \sim y$	x is equivalent to y or x is conjugate to y
H^g	conjugate of H
$c(G)$	number of conjugacy classes of group G
$\phi(G)$	Frattini subgroup of a group G
$\mathbb{F}_{p,q}$	group of order pq generated by p and q
$\langle x\rangle$	cyclic group generated by x
$\mathrm{Ker}\phi$	kernel of a homomorphism ϕ
$\mathrm{Im}\phi$	image of a function ϕ
(a, b)	greatest common divisor of a and b

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Preliminaries

1.1 Introduction

Generally speaking, any group G which contains a given group H as a subgroup is called an *extension* of H. Here we consider the case in which H is a normal subgroup of G.
A group G having a normal subgroup N can be *factored* into N and G/N. The study of extensions involves the reverse question : Given $N \trianglelefteq G$ and G/N, to what extent can one recapture G?
Otto Schreier first considered the problem of constructing all groups G such that G will have a given normal subgroup N and a given factor group $H \cong G/N$. There is always one such group, since the direct product of N and H has this property.

Note 1.1.1. All groups and all sets on which there is some action in this dissertation are finite.

Definition 1.1.1. *Let G be a non -simple group. Then G is an extension of N by a group H if $N \trianglelefteq G$ and $G/N \cong H$.*

Note 1.1.2. The group H in Definition 1.1.1 need not be a subgroup of G.

Definition 1.1.2. *An extension G of a group N by a group H is said to be split (or a split extension) if $N \trianglelefteq G$, $H \leq G$ such that $G = NH$ and $N \cap H = \{1_G\}$. Thus $G/N = NH/N \cong H/N \cap H = H/\{1_G\} = H$.*

Alternatively we say that N is complemented in G by H or G is the semi- direct product of N by H.

Note 1.1.3. A split extension $\overline{G}$ of N by G will be denoted as $\overline{G} = N : G$ and a non-split extension $\overline{G}$ of N by G will be denoted as $\overline{G} = N{\cdot}G$.

Remark 1.1.1. If G is the semi- direct product of N by H, then every $g \in G$ can be uniquely written in the form $g = nh$ with $n \in N$ and $h \in H$. This representation is unique since if $g = nh = mk$ with $n, m \in N$ and $h, k \in H$, then

$$nh = mk \;\Rightarrow\; m^{-1}n = kh^{-1} \;\Rightarrow\; m^{-1}n \in N \text{ and } m^{-1}n \in H \;\Rightarrow\; m^{-1}n \in N \cap H = \{1_G\} \;\Rightarrow\; m = n.$$

Similarly we can show that $h = k$.

Note 1.1.4. If the subgroup H in Definition 1.1.2 is also normal in G, then $G = N \times H$.

The Frobenius group is an example of a group which is a split extension.
Below is a brief description of the work carried out in this dissertation. In the remaining part of this chapter we describe results about *Permutation groups* since we give our definition of a Frobenius group as a *Permutation group*. This will include definitions and results about permutation groups that we would use later on. A significant part of the chapter is devoted to the theory of *Representations and Characters* which we will use in later chapters. We close the chapter by briefly describing *coset analysis* and the *Fischer matrices*.

In Chapter Two, we define the Frobenius group and give some general properties of the group. In Chapter Three we go into details by looking at the structure of the group. We give some important results about the *kernel* and *complement* of a Frobenius group and also describe ways to construct Frobenius groups.

Chapter Four contains examples of Frobenius groups. Included is a list of Frobenius groups of small order (up to order 32).

In Chapter Five we describe the *Characters of Frobenius groups* and then use these results to construct the character table of the Dihedral group D_{2n}, when n is odd. The method of coset analysis is applied to the Frobenius group verifying results obtained earlier. We also apply the theory of the Fischer matrices to the Frobenius group discovering that the Fischer matrices of the group are very simple.

Chapter Six deals with the group $29^2 : SL(2,5)$ and it's character table. We first describe the construction of the group and then it's character table using the theory we described in earlier chapters. Finally we determine the Fischer matrices for the group $29^2 : SL(2,5)$.

1.2 Permutation Groups

Much of the material covered in this chapter is from Moori [17].

Definition 1.2.1. *([16]). Let* G *be a group and* X *a set. We say that* G ***acts on*** X *if there is a homomorphism* $\rho : G \to S_X$*. Then* $\rho(g) \in S_X \ \forall g \in G$*. The action of* $\rho(g)$on X*, that is* $\rho(g)(x)$*, is denoted by* x^g *for any* $x \in X$*. We say that* G *is a* ***permutation group*** *on* X*.*

Definition 1.2.2. (**Orbits**) *Let* G *be a group that acts on a set* X *and let* $x \in X$*. Then the* **orbit**

of x *under the action of* G *is defined by*

$$x^G := \{x^g \mid g \in G\}.$$

Theorem 1.2.1. *Let* G *be a group that acts on a set* X. *The set of all orbits of* G *on* X *form a partition of* X.

PROOF: Define a relation $\sim$ on X by $x \sim y$ if and only if $x = y^g$ for some $g \in G$. Then $\sim$ is an equivalence relation on X since

$$x \sim y \;\Rightarrow\; x = y^g \;\Rightarrow\; x^{g^{-1}} = y \;\Rightarrow\; y \sim x.$$

And if

$$x = y^g \text{ and } y = z^{g'} \text{ then } x = z^{g'g} \text{ implies that } x \sim z.$$

And $[x] = \{x^g \mid g \in G\} = x^G$. Hence the set of all orbits of G on X partitions X. ∎

Definition 1.2.3. (**Stabilizer**) *If* G *is a group that acts on a set* X *and* $x \in X$ *then the* **stabilizer** *of* x *in* G, *denoted by* G_x *is the set* $G_x = \{g \mid x^g = x\}$. *That is* G_x *is the set of elements of* G *that fixes* x.

Theorem 1.2.2. *Let* G *be a group that acts on a set* X. *Then*

1. G_x *is a subgroup of* G *for each* $x \in X$.
2. $|x^G| = [G : G_x]$, *that is the number of elements in the orbit of* x *is equal to the index of* G_x *in* G.

PROOF: (1) Since $x^{1_G} = x, 1_G \in G_x$. Hence $G_x \neq \emptyset$. Let g, h be two elements of G_x. Then $x^g = x^h = x$. So $(x^g)^{h^{-1}} = (x^h)^{h^{-1}} = x^{1_G} = x$, and therefore $x^{gh^{-1}} = x$. Thus $gh^{-1} \in G_x$.
(2) Since

$$\begin{aligned} x^g = x^h &\iff x = x^{hg^{-1}} \iff hg^{-1} \in G_x \\ &\iff (G_x)g = (G_x)h, \end{aligned}$$

the map $\gamma : x^G \to G/G_x$ given by $\gamma(x^g) = (G_x)g$ is well defined and one-to-one. Obviously γ is onto. Hence there is a one-to-one correspondence between x^G and G/G_x. Thus $|x^G| = |G/G_x|$. ∎

Corollary 1.2.3. *If* G *is a finite group acting on a finite set* X *then* $\forall x \in X$, $|x^G|$ *divides* $|G|$.

PROOF: By Theorem 1.2.2, we have $|x^G| = [G : G_x] = |G|/|G_x|$. Hence $|G| = |x^G| \times |G_x|$. Thus $|x^G|$ divides $|G|$. ∎

Theorem 1.2.4. 1. *If* G *is a finite group, then* $\forall g \in G$ *the number of conjugates of* g *in* G *is equal to* $[G : C_G(g)]$.

2. *If* G *is a finite group and* H *is a subgroup of* G*, then the number of conjugates of* H *in* G *is equal to* $[G : N_G(H)]$.

PROOF: (1) Since G acts on itself by conjugation, using Theorem 1.2.2 we have $|g^G| = [G : G_g]$. But since

$$g^G = \{g^h \mid h \in G\} = \{hgh^{-1} \mid h \in G\} = [g]$$

and

$$G_g = \{h \in G \mid g^h = g\} = \{h \in G \mid hgh^{-1} = g\} = \{h \in G \mid hg = gh\} = C_G(g),$$

we have

$$|g^G| = |[g]| = [G : G_g] = [G : C_G(g)] = \frac{|G|}{|C_G(g)|}.$$

(2) Let G act on the set of it's subgroups by conjugation. Then by Theorem 1.2.2 we have $|H^G| = [G : G_H]$. Since $H^G = \{H^g \mid g \in G\} = \{gHg^{-1} \mid g \in G\} = [H]$ and $G_H = \{g \in G \mid H^g = H\} = \{g \in G \mid gHg^{-1} = H\} = N_G(H)$, we have $|[H]| = |H^G| = [G : G_H] = [G : N_G(H)] = \frac{|G|}{|N_G(H)|}$. ■

Theorem 1.2.5. (**Cauchy − Frobenius**) *Let* G *be a finite group acting on a finite set* X*. Let* n *denote the number of orbits of* G *on* X*. Let* $F(g)$ *denote the number of elements of* X *fixed by* $g \in G$*. Then* $n = \frac{1}{|G|} \sum\limits_{g \in G} F(g)$.

PROOF: Consider $S = \sum_{g \in G} F(g)$. Let $x \in X$. Since there are $|G_x|$ elements in G that fix x, x is counted $|G_x|$ times in S. If $\Delta = x^G$, then $\forall y \in \Delta$ we have $|\Delta| = |x^G| = |y^G| = [G : G_x] = [G : G_y]$. Hence $|G_x| = |G_y|$. Thus Δ contributes $[G : G_x].|G_x|$ to the sum S. But $[G : G_x].|G_x| = |G|$ is independent to the choice of Δ and hence each orbit of G on X contributes $|G|$ to the sum S. Since we have n orbits, we have $S = n|G|$. ■

Definition 1.2.4. (**Transitive Groups**) *Let* G *be a group acting on a set* X*. If* G *has only one orbit on* X*, then we say that* G *is* **transitive** *on* X*, otherwise we say that* G *is* **intransitive** *on* X*. If* G *is transitive on* X*, then* $x^G = X \ \forall x \in X$*. This means that* $\forall x, y \in X, \exists g \in G$ *such that* $x^g = y$.

Note 1.2.1. If G is a finite transitive group acting on a finite set X, then Theorem 1.2.2, part (2) implies that $|x^G| = |X| = |G|/|G_x|$. Hence $|G| = |X|.|G_x|$.

Definition 1.2.5. (**Multiply Transitive Groups**) *Let* G *be a group that acts on a set* X *and let* $|X| = n$ *and* $1 \leq k \leq n$ *be a positive integer. We say that* G *is* $k-$**transitive** *on* X *if for every two ordered* k *- tuples* $(x_1, x_2, \ldots\ldots, x_k)$ *and* $(y_1, y_2, \ldots\ldots, y_k)$ *with* $x_i \neq x_j$ *and* $y_i \neq y_j$ *for* $i \neq j$ *there exists* $g \in G$ *such that* $x_i^g = y_i$ *for* $i = 1, 2, \ldots\ldots, k$.

Theorem 1.2.6. *If* G *is a* k *- transitive group on a set* X *with* $|X| = n$*, then*

$$|G| = n(n-1)(n-2)\ldots\ldots(n-k+1)\left|G_{[x_1, x_2, \ldots\ldots, x_k]}\right|$$

for every choice of k*- distinct* $x_1, x_2, \ldots\ldots, x_k \in X$*, where* $G_{[x_1, x_2, \ldots\ldots, x_k]}$ *denote the set of all elements* $g \in G$ *such that* $x_i^g = x_i, 1 \leq i \leq k$.

PROOF: Let $x_1 \in X$. Then since G is k - transitive, we have

$$|G| = n \times |G_{x_1}| \tag{1.1}$$

and G_{x_1} is $(k-1)$ transitive on $X - \{x_1\}$ (see Theorem 9.7 in Rotman [23]). Choose $x_2 \in X - \{x_1\}$. Then since G_{x_1} is $(k-1)$ - transitive on $X - \{x_1\}$ we have $|G_{x_1}| = |X - \{x_1\}| \times |(G_{x_1})_{x_2}|$, that is $|G_{x_1}| = (n-1) \times |G_{[x_1,x_2]}|$ and $G_{[x_1,x_2]}$ is $(k-2)$ - transitive on $X - \{x_1, x_2\}$.

Now by (1.1) we get that $|G| = n(n-1) \times |G_{[x_1,x_2]}|$. If we continue in this way, we will get

$$|G| = n(n-1)(n-2)\ldots\ldots(n-k+1)\left|G_{[x_1,x_2,\ldots\ldots,x_k]}\right|.$$

■

Theorem 1.2.7. *Let G be a group that acts transitively on a finite set X with $|X| > 1$. Then there exists $g \in G$ such that g has no fixed points.*

PROOF: By Theorem 1.2.5 we have

$$\begin{aligned} 1 = n &= \frac{1}{|G|}\sum_{g\in G} F(g) \\ &= \frac{1}{|G|}\left[F(1_G) + \sum_{g\in G-\{1_G\}} F(g)\right] \\ &= \frac{1}{|G|}\left[|X| + \sum_{g\in G-\{1_G\}} F(g)\right] \end{aligned}$$

If $F(g) > 0$ for all $g \in G$, then we have

$$\begin{aligned} 1 = \frac{1}{|G|}\left[|X| + \sum_{g\in G-\{1_G\}} F(g)\right] &\geq \frac{1}{|G|}\left[|X| + |G| - 1\right] \\ &\geq 1 + \frac{|X|-1}{|G|} > 1. \end{aligned}$$

■

Definition 1.2.6. *([21]).* (**Semiregular** ; **Regular**) *Let X be a nonempty set and let G be a group that acts on X. The permutation group G is said to be* **semiregular** *if $\mathrm{Stab}_G(x) = \{1_G\}$ for all $x \in X$. The permutation group G is said to be* **regular** *if it is both transitive and semiregular.*

1.3 Representation Theory and Characters of Finite Groups

There are two kinds of representations; *permutation* and *matrix*. Cayleys Theorem, which asserts that any group G can be embedded into the Symmetric group S_G, is an example of a permutation representation. We are interested here in matrix representations.

Definition 1.3.1. *Let* G *be a group. Any homomorphism* $\rho : G \to GL(n, \mathbb{F})$*, where* $GL(n, \mathbb{F})$ *is the group consisting of all* $n \times n$ *non-singular matrices is called a* **matrix representation** *or simply a representation of* G*. If* $\mathbb{F} = \mathbb{C}$*, then* ρ *is called an ordinary representation. The integer* n *is called the* **degree** *of* ρ*. Two representations* ρ *and* σ *are said to be* **equivalent** *if there exists* $P \in GL(n, \mathbb{F})$ *such that* $\sigma(g) = P\rho(g)P^{-1}$, $\forall g \in G$.

We will restrict our work to ordinary representations.

Definition 1.3.2. (**Character**) *Let* $\rho : G \to GL(n, \mathbb{C})$ *be a representation of a group* G*. Then* ρ *affords a complex valued function* $\chi_\rho : G \to \mathbb{C}$ *defined by* $\chi_\rho(g) = \mathrm{trace}(\rho(g))$, $\forall g \in G$*. The function* χ_ρ *is called a* ***character*** *afforded by the representation* ρ *of* G *or simply a character of* G*. The integer* n *is called the* **degree** *of* χ_ρ*. If* $n = 1$*, then* χ_ρ *is said to be* **linear***.*

Note 1.3.1. For any group G, consider the function $\rho : G \to GL(1, \mathbb{C})$ given by $\rho(g) = 1$, $\forall g \in G$. It is clear that ρ is a representation of G and $\chi_\rho(g) = 1$, $\forall g \in G$. The character χ_ρ is called the *trivial* character and it may also be denoted by 1.

Definition 1.3.3. (**Class Function**) *If* $\phi : G \to \mathbb{C}$ *is a function that is constant on conjugacy classes of a group* G*, that is* $\phi(g) = \phi(xgx^{-1})$, $\forall x \in G$*, then we say that* ϕ *is a class function.*

Proposition 1.3.1. *A character is a class function.*

PROOF: Immediate since similar matrices have the same trace. ■

Definition 1.3.4. ($\mathbf{F}$ **− Algebra**) *If* F *is a field and* A *is a vector space over* F*, then we say that* A *is an* $\mathbf{F}$ **− Algebra** *if:*

1. A *is a ring with identity,*

2. *for all* $\lambda \in F$ *and* $x, y \in A$*, we have* $\lambda(xy) = \lambda(x)y = x(\lambda y)$.

Definition 1.3.5. (**Group Algebra**) *Let* G *be a finite group and* F *any field. Then by* $F[G]$ *we mean the set of formal sums* $\left\{\sum_{g \in G} \lambda_g . g : \lambda_g \in F\right\}$*. We define the operations on* $F[G]$ *by*

1. $\sum_{g \in G} \lambda_g g \; + \; \sum_{g \in G} \mu_g g := \sum_{g \in G} (\lambda_g + \mu_g) g,$

2. $\lambda\left(\sum_{g \in G} \lambda_g g\right) := \sum_{g \in G} (\lambda\lambda_g) g, \;\; \lambda \in F,$

3. $\left(\sum_{g \in G} \lambda_g g\right) . \left(\sum_{g \in G} \mu_g g\right) := \sum_{g \in G} \left[\sum_{h \in G} \lambda_h \mu_{h^{-1} g}\right] g.$

Under the above operations $F[G]$ *is an* F*-algebra known as the* ***group algebra*** *of* G *over* F*.*

Let G be a group. Now define over the set of class functions of G addition and multiplication of two class functions ψ_1 and ψ_2 by

$$\begin{aligned}(\psi_1+\psi_2)(g) &= \psi_1(g)+\psi_2(g), \ \ \forall g \in G,\\ \psi_1\psi_2(g) &= \psi_1(g)\psi_2(g), \ \ \forall g \in G\end{aligned}$$

Clearly $\psi_1+\psi_2$ and $\psi_1\psi_2$ are class functions of G. Also if $\lambda \in \mathbb{C}$, then $\lambda\psi$ is a class function of G whenever ψ is. Therefore the set of all class functions of G forms an *algebra*, denoted by $\mathcal{C}(G)$. The set of all characters of G forms a *subalgebra* of $\mathcal{C}(G)$.

Proposition 1.3.2. *If χ_ψ and χ_ϕ are two characters of a group G, then so is $\chi_\psi+\chi_\phi$.*

PROOF: Let ψ and ϕ be representations of G affording the characters χ_ψ and χ_ϕ respectively. Define the function ξ on G by $\xi(g) = \begin{pmatrix} \psi(g) & 0 \\ 0 & \phi(g)\end{pmatrix} = \psi(g)\bigoplus\phi(g)$. Clearly ξ is a homomorphism of G with $\chi_\xi = \chi_\psi+\chi_\phi$. ■

Definition 1.3.6. *Let S be a set of $(n\times n)$ matrices over $\mathbb{F}$. We say that S is* **reducible** *if $\exists\ m,k\in\mathbb{N}$, and there exists $P\in GL(n,\mathbb{F})$ such that $\forall A\in S$ we have*

$$PAP^{-1} = \begin{pmatrix} B & 0 \\ C & D\end{pmatrix}$$

where B is an $m\times m$ matrix, D and C are $k\times k$ and $k\times m$ matrices respectively. Here 0 denotes the zero $m\times k$ matrix. If there is no such P, we say that S is **irreducible**. *If $C=0$, the zero $k\times m$ matrix, for all $A\in S$ then we say that S is* **fully reducible**. *We say that S is* **completely reducible** *if $\exists\ P\in GL(n,\mathbb{F})$ such that*

$$PAP^{-1} = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k\end{pmatrix}, \quad \forall A\in S,$$

where each B_i is irreducible.

Definition 1.3.7. *Let $f: G\to GL(n,\mathbb{F})$ be a representation of G over $\mathbb{F}$. Let $S=\{f(g)|g\in G\}$. Then $S\subseteq GL(n,\mathbb{F})$. We say that f is* **reducible**, **fully reducible** *or* **completely reducible** *if S is reducible, fully reducible or completely reducible.*

We state below two important results in representation theory, namely Maschke's Theorem and Schur's Lemma. The proof of both these results can be found in Moori [17].

Theorem 1.3.3. (**Maschke's Theorem**) *Let $\rho: G\to GL(n,\mathbb{F})$ be a representation of a group G. If the characteristic of $\mathbb{F}$ is zero or does not divide $|G|$, then $\rho=\bigoplus_{i=1}^{r}\rho_i$, where ρ_i are irreducible representations of G.*

PROOF: See Moori [17]. ■

Theorem 1.3.4. (**Schur′s Lemma**) *Let ρ and ϕ be two irreducible representations of degree n and m respectively, of a group G over a field $\mathbb{F}$. Assume that there exists an $m \times n$ matrix P such that $P\rho(g) = \phi(g)P$ for all $g \in G$. Then either $P = 0_{m\times n}$ or P is non-singular so that $\rho(g) = P^{-1}\phi(g)P$ (that is ρ and ϕ are equivalent representations of G).*

PROOF: See Moori [17]. ■

Definition 1.3.8. (**Inner Product**) *Let G be a group. Over $\mathcal{C}(G)$ we define an inner product*

$$\langle\ ,\ \rangle : \mathcal{C}(G) \times \mathcal{C}(G) \to \mathbb{C} \text{ by } \langle\psi, \phi\rangle = \frac{1}{|G|} \sum_{g\in G} \psi(g)\overline{\phi(g)},$$

where $\overline{\phi(g)}$ is the complex conjugate of $\phi(g)$.

In the following Proposition we list some properties of characters of a group.

Proposition 1.3.5.

1. *Let χ_ρ be the character afforded by an irreducible representation ρ of a group G. Then $\langle\chi_\rho, \chi_\rho\rangle = 1$*
2. *If χ_ρ and $\chi_{\rho'}$ are irreducible characters of two non equivalent representations of G, then $\langle\chi_\rho, \chi_{\rho'}\rangle = 0$.*
3. *If $\rho \cong \bigoplus_{i=1}^{k} d_i\rho_i$, then $\chi_\rho = \sum_{i=1}^{k} d_i\chi_{\rho_i}$.*
4. *If $\rho \cong \bigoplus_{i=1}^{k} d_i\rho_i$, then $d_i = \langle\chi_\rho, \chi_{\rho_i}\rangle$.*

PROOF: See Moori [17] or James [10]. ■

Proposition 1.3.6. *Let χ_ρ be the character afforded by a representation ρ of a group G. Then ρ is irreducible if and only if $\langle\chi_\rho, \chi_\rho\rangle = 1$.*

PROOF: See James [10]. ■

Let $\mathrm{Irr}(G)$ denote the set of all ordinary irreducible characters of a group G.

Corollary 1.3.7. *The set $\mathrm{Irr}(G)$ forms an orthonormal basis for $\mathcal{C}(G)$ over $\mathbb{C}$.*

PROOF: See James [10]. ■

Note 1.3.2. Corollary 1.3.7 asserts that if ψ is a class function of a group G, then $\psi = \sum_{i=1}^{k} \lambda_i\chi_i$ where $\lambda_i \in \mathbb{C}$ and $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots\ldots, \chi_k\}$. If $\lambda_i \in \mathbb{Z},\ \forall i$, then ψ is called a *generalised character*. Moreover, if $\lambda_i \in \mathbb{N} \cup \{0\}$, then ψ is a character of G.

The following counting result counts the number of irreducible characters of a group.

Theorem 1.3.8. *The number of irreducible characters of a group* G *is equal to the number of conjugacy classes of* G.

PROOF: See James [10] or Moori [17]. ■

Proposition 1.3.9. *The number of linear characters of a group* G *is given by* $|G|/|G'|$*, where* G' *is the derived subgroup of* G.

PROOF: See Moori [16]. ■

1.3.1 The Character Table and Orthogonality Relations

The irreducible characters of a finite group are class functions, and the number of them by Theorem 1.3.8 is equal to the number of conjugacy classes of the group. A table recording the values of all the irreducible characters of the group is called a **character table** of the group.

Definition 1.3.9. (**Character Table**) *The character table of a group* G *is a square matrix whose columns correspond to the conjugacy classes of* G *and whose rows correspond to the irreducible characters of* G.

The character table is a useful tool which can be used to make inferences about the group. For example later on we will show that provided certain conditions are satisfied , we can use a given character table of some group to determine whether the group is Frobenius (see Theorem 5.2.2). The simplicity, normality and solvability as well as the center and commutator of the group can also be determined from the character table.

The following Propositions contains some useful results about the values of the irreducible characters in the character table of a group G.

Proposition 1.3.10.

1. $\chi(1_G) \big| |G|, \ \forall \chi \in \mathrm{Irr}(G)$.
2. $\displaystyle\sum_{i=1}^{|\mathrm{Irr}(G)|} \left(\chi_i(1_G)\right)^2 = |G|$.
3. *If* $\chi \in \mathrm{Irr}(G)$*, then* $\overline{\chi} \in \mathrm{Irr}(G)$*, where* $\overline{\chi}(g) = \overline{\chi(g)}, \ \forall g \in G$.
4. $\chi(g^{-1}) = \overline{\chi(g)}, \ \forall g \in G$. *In particular if* $g^{-1} \in [g]$*, then* $\chi(g) \in \mathbb{R}, \ \forall \chi$.

PROOF: See Moori [17]. ■

The rows and columns of the character table also satisfy orthogonality relations which we state in the next theorem.

Theorem 1.3.11. *Let* $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots\ldots, \chi_k\}$ *and* $\{g_1, g_2, \ldots\ldots, g_k\}$ *be a collection of representatives for the conjugacy classes of a group* G. *For each* $1 \le i \le k$ *let* $C_G(g_i)$ *be the centralizer of* g_i. *Then we have the following:*

1. *The row orthogonality relation:*
   For each $1 \le i, j \le k$,
$$\sum_{r=1}^{k} \frac{\chi_i(g_r)\overline{\chi_j(g_r)}}{|C_G(g_r)|} = \langle \chi_i, \chi_j \rangle = \delta_{ij}.$$

2. *The column orthogonality relation:*
   For each $1 \le i, j \le k$,
$$\sum_{r=1}^{k} \frac{\chi_r(g_i)\overline{\chi_r(g_j)}}{|C_G(g_i)|} = \delta_{ij}.$$

PROOF: (1) Using Proposition 1.3.5(2) we have

$$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \frac{1}{|G|}\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \frac{1}{|G|}\sum_{r=1}^{k} \frac{|G|}{|C_G(g_r)|}\chi_i(g_r)\overline{\chi_j(g_r)} = \sum_{r=1}^{k} \frac{\chi_i(g_r)\overline{\chi_j(g_r)}}{|C_G(g_r)|}.$$

(2) For fixed $1 \le s \le k$, define $\psi_s : G \to \mathbb{C}$ by $\psi_s(g) = \begin{cases} 1 & \text{if } g \in [g_s], \\ 0 & \text{otherwise.} \end{cases}$
It is clear that ψ_s is a class function on G. Since $\mathrm{Irr}(G)$ form an orthonormal basis for $\mathcal{C}(G)$, there exists $\lambda_t's \in \mathbb{C}$ such that $\psi_s = \sum_{t=1}^{k} \lambda_t \chi_t$. Now for $1 \le j \le k$ we have

$$\lambda_j = \langle \psi_s, \chi_j \rangle = \frac{1}{|G|}\sum_{g \in G} \psi_s(g)\overline{\chi_j(g)} = \sum_{t=1}^{k} \frac{\psi_s(g_t)\overline{\chi_j(g_t)}}{|C_G(g_t)|} = \frac{\overline{\chi_j(g_s)}}{|C_G(g_s)|}.$$

Hence $\psi_s = \sum_{j=1}^{k} \frac{\overline{\chi_j(g_s)}}{|C_G(g_s)|}\chi_j$. Thus we have the required formula:

$$\delta_{st} = \psi_s(g_t) = \sum_{j=1}^{k} \frac{\chi_j(g_t)\overline{\chi_j(g_s)}}{|C_G(g_t)|}.$$

■

1.3.2 Tensor Product of Characters

We show in this section that if ψ and χ are characters of a group G, then the product $\chi\psi$ defined by

$$(\chi\psi)(g) = \chi(g).\psi(g), \ \forall g \in G$$

is also a character of G. It is clear that if χ and ψ are class functions on G, then so is $\chi\psi$.

Definition 1.3.10. (**Tensor Product of Matrices − Kronecker Product**)
Let $P = (p_{ij})_{m\times m}$ *and* $Q = (q_{ij})_{n\times n}$ *be two matrices. Define the* $mn \times mn$ *matrix* $P \otimes Q$ *by*

$$P \otimes Q := (p_{ij}Q) = \begin{pmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1m}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2m}Q \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}Q & p_{m2}Q & \cdots & p_{mm}Q \end{pmatrix}$$

Then we have

$$\begin{aligned} \mathrm{trace}(P \otimes Q) &= p_{11}\mathrm{trace}(Q) + p_{12}\mathrm{trace}(Q) + \ldots + p_{mm}\mathrm{trace}(Q) \\ &= \mathrm{trace}(P).\mathrm{trace}(Q) \end{aligned}$$

Definition 1.3.11. (**Tensor Product of Representations**) *Let* T *and* U *be two representations of a group* G*. We define the* ***tensor product*** $T \otimes U$ *by*

$$(T \otimes U)(g) := T(g) \otimes U(g),$$

where $\otimes$ *on the RHS is defined by Definition 1.3.10.*

Theorem 1.3.12. *Let* T *and* U *be two representations of a group* G*. Then*

1. $T \otimes U$ *is a representation of* G*.*

2. $\chi_{T\otimes U} = \chi_T.\chi_U$.

`PROOF:` (1) $\forall g, h \in G$ we have

$$\begin{aligned} (T \otimes U)(gh) &= T(gh) \otimes U(gh) \\ &= (T(g).T(h)) \otimes (U(g).U(h)) \\ &= (T(g) \otimes U(g)).(T(h) \otimes U(h)) \\ &= (T \otimes U)(g).(T \otimes U)(h) \ \text{ by Definition 1.3.11} \end{aligned}$$

(2) $\forall g \in G$

$$\begin{aligned} \chi_{T\otimes U}(g) &= \mathrm{trace}((T \otimes U)(g)) \\ &= \mathrm{trace}(T(g) \otimes U(g)) \\ &= \mathrm{trace}(T(g)).\mathrm{trace}(U(g)) \\ &= \chi_T(g).\chi_U(g) \\ &= (\chi_T.\chi_U)(g). \end{aligned}$$

Hence $\chi_{T\otimes U} = \chi_T.\chi_U$ ∎

Definition 1.3.12. (**Direct Product**) *Let* G *be a group. Let* $G = H \times K$ *be the direct product of* H *and* K*. Let* $T : H \to GL(m, \mathbb{C})$ *and* $U : K \to GL(n, \mathbb{C})$ *be representations of* H *and* K *respectively. Define the* ***direct product*** $T \otimes U$ *as follows: Let* $g \in G$*. Then* g *can be written uniquely in the form* hk*,* $h \in H, k \in K$*. Define*

$$(T \otimes U)(g) := T(h) \otimes U(k)$$

where $\otimes$ *on the RHS is the tensor product given in Definition 1.3.10.*

Also $T \otimes U$ is a representation of degree mn of $G = H \times K$ and

$$(\chi_{T\otimes U})(g) = \chi_T(h).\chi_U(k), \text{where } g = hk.$$

Theorem 1.3.13. *Let* $G = H \times K$ *be the direct product of the groups* H *and* K*. Then the direct product of any irreducible character of* H *and any irreducible character of* K *is an irreducible character of* G*. Moreover, every irreducible character of* G *can be constructed in this way.*

PROOF: Let $\chi_T \in \mathrm{Irr}(H)$ and $\chi_U \in \mathrm{Irr}(K)$. Let $\chi = \chi_{T\otimes U} = \chi_T.\chi_U$. Then χ is a character of G. We claim $\chi \in \mathrm{Irr}(G)$. Let $g \in G$, then $\exists!\ h \in H, k \in K$ such that $g = hk$. So

$$\begin{aligned}
\sum_{g\in G} |\chi(g)|^2 &= \sum_{h\in H}\sum_{k\in K} |\chi_T(h).\chi_U(k)|^2 \\
&= \sum_{h\in H}\sum_{k\in K} |\chi_T(h)|^2.|\chi_U(k)|^2 \\
&= \sum_{h\in H} |\chi_T(h)|^2. \sum_{k\in K} |\chi_U(k)|^2 \\
&= |H|.|K| = |G|.
\end{aligned}$$

Hence $\frac{1}{|G|}\sum |\chi(g)|^2 = 1$; so $\langle \chi, \chi \rangle = 1$. Thus $\chi \in \mathrm{Irr}(G)$. ■

Remark 1.3.1. Suppose that $|\mathrm{Irr}(H)| = r$ and $|\mathrm{Irr}(K)| = s$, then we obtain rs irreducible characters of $G = H \times K$ in this way. If $g_1 = hk$ and $g_2 = h'k'$ are two elements in G, then $g_1 \sim g_2$ if and only if $h \sim h'$ in H and $k \sim k'$ in K. Thus the number of conjugacy classes of G equals the number of conjugacy classes of H times the number of conjugacy classes of K which equals rs. Hence $|\mathrm{Irr}(G)| = rs$.

1.3.3 Lifting of Characters

We present here a method for constructing characters of a group G when G has a normal subgroup N. Assuming that the irreducible characters of the factor group G/N are known, the idea here is to construct characters of G by a process known as *lifting of characters*.

Definition 1.3.13. (**Kernel**) *Let* χ *be a character of a group* G *afforded by a representation* ρ *of* G*. Then*

$$\mathrm{Ker}(\rho) = \mathrm{Ker}(\chi) = \{g \in G \mid \chi(g) = \chi(1_G)\} \trianglelefteq G.$$

Also if $N \leq G$ such that N is an intersection of the kernel of irreducible characters of G, then $N \trianglelefteq G$.

Proposition 1.3.14. *Let G be a group. Let $N \trianglelefteq G$ and $\widetilde{\chi}$ be a character of G/N. The function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \widetilde{\chi}(gN), \forall g \in G$ is a character of G with $\deg(\chi) = \deg(\widetilde{\chi})$. Moreover, if $\widetilde{\chi} \in \mathrm{Irr}(G/N)$, then $\chi \in \mathrm{Irr}(G)$.*

PROOF: Suppose that $\tilde{\rho} : G/N \to GL(n, \mathbb{C})$ is a representation which affords the character $\widetilde{\chi}$. Define the function $\rho : G \to GL(n, \mathbb{C})$ by $\rho(g) = \tilde{\rho}(gN), \forall g \in G$. Then ρ defines a representation on G since

$$\rho(gh) = \tilde{\rho}(ghN) = \tilde{\rho}(gNhN) = \tilde{\rho}(gN)\tilde{\rho}(hN) = \rho(g)\rho(h), \forall g, h \in G.$$

Hence the character χ, which is afforded by ρ, satisfies

$$\chi(g) = \mathrm{trace}(\rho(g)) = \mathrm{trace}(\tilde{\rho}(gN)) = \widetilde{\chi}(gN)\ \forall g \in G.$$

So χ is a character of G. The degree of χ is

$$\deg(\chi) = \chi(1_G) = \widetilde{\chi}(1_G N) = \widetilde{\chi}(N) = \deg(\widetilde{\chi}).$$

Let T be a transversal of N in G. Then

$$\begin{aligned} 1 = \langle \widetilde{\chi}, \widetilde{\chi} \rangle &= \frac{1}{|G/N|} \sum_{gN \in G/N} \widetilde{\chi}(gN)\widetilde{\chi}(gN)^{-1} \\ &= \frac{1}{|G|} \sum_{gN \in G/N} |N|\, \widetilde{\chi}(gN)\widetilde{\chi}(gN)^{-1} \\ &= \frac{1}{|G|} \sum_{g \in T} |N|\, \widetilde{\chi}(gN)\widetilde{\chi}(g^{-1}N) \\ &= \frac{1}{|G|} \sum_{g \in T} |N|\, \chi(g)\chi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi(g^{-1}) \\ &= \langle \chi, \chi \rangle. \end{aligned}$$

■

1.3.4 Induction and Restriction of Characters

Restriction to a Subgroup

Let G be a group, $H \leq G$. If $\rho : G \to GL(n, \mathbb{C})$ is a representation of G, then $\rho \downarrow H : H \to GL(n, \mathbb{C})$ given by $(\rho \downarrow H)(h) = \rho(h), \forall h \in H$, is a representation of H. We say that $\rho \downarrow H$ is the **restriction** of ρ to H. If χ_ρ is the character of ρ, then $\chi_\rho \downarrow H$ is the character of $\rho \downarrow H$. We refer to $\chi_\rho \downarrow H$ as the **restriction** of χ_ρ to H.

Remark 1.3.2. It is clear that $\deg(\rho) = \deg(\rho \downarrow H)$. However, ρ irreducible does not imply (in general) that $\rho \downarrow H$ is irreducible.

Theorem 1.3.15. *Let G be a group, $H \leq G$. Let ψ be a character of H. Then there is an irreducible character χ of G such that $\langle \chi \downarrow H, \psi \rangle_H \neq 0$.*

PROOF: See Moori [17]. ∎

Theorem 1.3.16. *Let G be a group, $H \leq G$. Let $\chi \in \mathrm{Irr}(G)$ and let $\mathrm{Irr}(H) = \{\psi_1, \psi_2, \ldots\ldots, \psi_r\}$. Then $\chi \downarrow H = \sum_{i=1}^{r} d_i \psi_i$, where $d_i \in \mathbb{N} \cup \{0\}$ and $\sum_{i=1}^{r} d_i^2 \leq [G : H]$. $(*)$*
Moreover, we have equality in $(*)$ if and only if $\chi(g) = 0 \ \forall g \in G \backslash H$.

PROOF: We have

$$\sum_{i=1}^{r} d_i^2 = \langle \chi \downarrow H, \chi \downarrow H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h).\overline{\chi(h)}\ .$$

Since χ is irreducible,

$$\begin{aligned} 1 = \langle \chi, \chi \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \chi(g).\overline{\chi(g)} \\ &= \frac{1}{|G|} \sum_{g \in H} \chi(g).\overline{\chi(g)} + \frac{1}{|G|} \sum_{g \in G-H} \chi(g).\overline{\chi(g)} \\ &= \frac{|H|}{|G|} \sum_{i=1}^{r} d_i^2 + K, \end{aligned}$$

where $K = \frac{1}{|G|} \sum_{g \in G-H} \chi(g)\overline{\chi(g)}$. Since $K = \frac{1}{|G|} \sum_{g \in G-H} |\chi(g)|^2$, $K \geq 0$.
Thus

$$\frac{|H|}{|G|} \sum_{i=1}^{r} d_i^2 = 1 - K \leq 1,$$

so

$$\sum_{i=1}^{r} d_i^2 \leq |G|/|H| = [G : H].$$

Also

$$K = 0 \text{ if and only if } |\chi(g)|^2 = 0 \ \forall g \in G - H.$$

Hence $K = 0$ if and only if $\chi(g) = 0, \ \forall g \in G - H$. ∎

Induced Representations

Definition 1.3.14. (**Transversal**) *Let* G *be a group. Let* $H \leq G$*. By a* **right transversal** *of* H *in* G *we mean a set of representatives for the right cosets of* H *in* G*.*

Theorem 1.3.17. *Let* G *be a group. Let* $H \leq G$ *and* T *be a representation of* H *of degree* n*. Extend* T *to* G *by* $T^0(g) = T(g)$ *if* $g \in H$ *and* $T^0(g) = 0_{n\times n}$ *if* $g \notin H$*. Let* $\{x_1, x_2, \ldots\ldots, x_r\}$ *be a right transversal of* H *in* G*. Define* $T \uparrow G$ *by*

$$(T \uparrow G)(g) := \begin{pmatrix} T^0(x_1gx_1^{-1}) & T^0(x_1gx_2^{-1}) & \cdots & T^0(x_1gx_r^{-1}) \\ T^0(x_2gx_1^{-1}) & T^0(x_2gx_2^{-1}) & \cdots & T^0(x_2gx_r^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ T^0(x_rgx_1^{-1}) & T^0(x_rgx_2^{-1}) & \cdots & T^0(x_rgx_r^{-1}) \end{pmatrix} = \big(T^0(x_igx_j^{-1})\big)_{i,j=1} \ , \ \forall g \in G.$$

Then $T \uparrow G$ *is a representation of* G *of degree* nr*.*

PROOF: See Moori [17] ■

Definition 1.3.15. (**Induced Representation/Character**) *The representation* $T \uparrow G$ *defined above is said to be* **induced** *from the representation* T *of* H*. Let* ϕ *be the character afforded by* T*. Then the character afforded by* $T \uparrow G$ *is called the* **induced character** *from* ϕ *and is denoted by* ϕ^G*. If we extend* ϕ *to* G *by* $\phi^0(g) = \phi(g)$ *if* $g \in H$ *and* $\phi^0(g) = 0$ *if* $g \notin H$*, then*

$$\phi^G(g) = \mathrm{trace}\big((T \uparrow G)(g)\big) = \sum_{i=1}^{r} \mathrm{trace}\big(T^0(x_igx_i^{-1})\big) = \sum_{i=1}^{r} \phi^0(x_igx_i^{-1}).$$

Note also that $\phi^G(1_G) = nr = \frac{|G|}{|H|}.\phi(1)$.

Proposition 1.3.18. *The values of the induced character* ϕ^G *are given by*

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G} \phi^0(xgx^{-1}) \ , \forall g \in G.$$

PROOF: See Moori [17]. ■

Proposition 1.3.19. *Let* G *be a group. Let* $H \leq G$*. Assume that* ϕ *is a character of* H *and* $g \in G$*. Let* $[g]$ *denote the conjugacy class of* G *containing* g*.*

1. *If* $H \cap [g] = \emptyset$, then $\phi^G(g) = 0$,
2. *if* $H \cap [g] \neq \emptyset$, then $\phi^G(g) = \big|C_G(g)\big| \displaystyle\sum_{i=1}^{m} \frac{\phi(x_i)}{\big|C_H(x_i)\big|}$,

where $x_1, x_2, \ldots\ldots, x_m$ *are representatives of classes of* H *that fuse to* $[g]$*. That is* $H \cap [g]$ *breaks up into* m *conjugacy classes of* H *with representatives* $x_1, x_2, \ldots\ldots, x_m$*.*

PROOF: By Proposition 1.3.18 we have

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}).$$

If $H \cap [g] = \emptyset$, then $xgx^{-1} \notin H$ for all $x \in G$, so $\phi^0(xgx^{-1}) = 0\ \forall x \in G$ and $\phi^G(g) = 0$. Now suppose that $H \cap [g] \neq \emptyset$. As x runs over G, xgx^{-1} covers $[g]$ exactly $|C_G(g)|$ times, so

$$\begin{aligned}
\phi^G(g) &= \frac{1}{|H|} \times |C_G(g)| \sum_{y\in[g]} \phi^0(y) \\
&= \frac{|C_G(g)|}{|H|} \sum_{y\in[g]\cap H} \phi(y) \\
&= \frac{|C_G(g)|}{|H|} \sum_{i=1}^{m} \left[H : C_H(x_i)\right] \phi(x_i) \\
&= |C_G(g)| \sum_{i=1}^{m} \frac{\phi(x_i)}{|C_H(x_i)|}.
\end{aligned}$$

■

1.3.5 The Frobenius Reciprocity Law

Definition 1.3.16. (**Induced Class Function**) *Let G be a group. Let $H \leq G$ and ϕ be a class function on H. Then the **induced class function** ϕ^G on G is defined by*

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}),$$

where ϕ^0 coincides with ϕ on H and is zero otherwise.

Note also that

$$\begin{aligned}
\phi^G(ygy^{-1}) &= \frac{1}{|H|}\sum_{x\in G}\phi^0(xygy^{-1}x^{-1}) = \frac{1}{|H|}\sum_{x\in G}\phi^0\big((xy)g(xy)^{-1}\big) \\
&= \frac{1}{|H|}\sum_{z\in G}\phi^0(zgz^{-1}) = \phi^G(g).
\end{aligned}$$

Thus ϕ^G is also a class function on G.

Note 1.3.3. Let G be group. If $H \leq G$ and ϕ is a class function on G, then $\phi \downarrow H$ is a class function on H.

Induction and Restriction of characters are related by the following result.

Theorem 1.3.20. (**Frobenius Reciprocity**) *Let G be a group. Let $H \leq G$, ϕ be a class function on H and ψ a class function on G. Then*

$$\langle \phi, \psi \downarrow H\rangle_H = \langle \phi^G, \psi\rangle_G.$$

PROOF:

$$\begin{aligned}\langle \phi^G, \psi\rangle_G &= \frac{1}{|G|}\sum_{g\in G}\phi^G(g)\,.\,\overline{\psi(g)}\\ &= \frac{1}{|G|}\sum_{g\in G}\Big(\frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1})\Big)\,.\,\overline{\psi(g)}\\ &= \frac{1}{|G||H|}\sum_{g\in G}\sum_{x\in G}\phi^0(xgx^{-1})\,.\,\overline{\psi(g)}\ .\end{aligned} \tag{1.2}$$

Let $y = xgx^{-1}$. Then as g runs over G, xgx^{-1} runs through G. Also since ψ is a class function on G, $\psi(y) = \psi(xgx^{-1}) = \psi(g)$. Thus by 1.2 above we have

$$\begin{aligned}\langle \phi^G, \psi\rangle_G &= \frac{1}{|G||H|}\sum_{y\in G}\sum_{x\in G}\phi^0(y)\overline{\psi(y)}\\ &= \frac{1}{|G||H|}\sum_{x\in G}\Big(\sum_{y\in G}\phi^0(y)\overline{\psi(y)}\Big)\\ &= \frac{1}{|G||H|}\,.\,|G|\sum_{y\in G}\phi^0(y)\overline{\psi(y)}\\ &= \frac{1}{|H|}\sum_{y\in H}\phi(y)\overline{\psi(y)} = \langle \phi, \psi\downarrow H\rangle_H\ .\end{aligned}$$

■

1.3.6 Normal Subgroups

Definition 1.3.17. (**Conjugate Class Function/Representation**) *Let G be a group. Let $N \trianglelefteq G$. If ϕ is a class function on N, for each $g \in G$ define $\phi^g(n) = \phi(gng^{-1})$, $n \in N$. The function ϕ^g is said to be **conjugate** to ϕ in G. Also if P is a representation of $N \trianglelefteq G$, the **conjugate representation** is P^g given by $P^g(n) = P(gng^{-1})$.*

Proposition 1.3.21. *Let G be a group. Let $N \trianglelefteq G$ and ϕ, ψ class functions on N. Let $x, y \in G$. Then*

1. *ϕ^x is a class function on N ;*
2. *$(\phi^x)^y = \phi^{xy}$;*
3. *$\langle \phi^x, \psi^y\rangle = \langle \phi, \psi\rangle$;*
4. *$\langle \chi\downarrow N, \phi^x\rangle = \langle \chi\downarrow N, \phi\rangle$ where χ is a class function on G ;*
5. *If ϕ is a character, then so is ϕ^x.*

PROOF: See Moori [17] ■

Proposition 1.3.22. *Let* $g, h \in G$. *Then* $g \sim h$ *if and only if* $\chi(g) = \chi(h)$ *for all characters* χ *of* G.

PROOF: See Moori [17]. ■

Corollary 1.3.23. *If* $\mathrm{Irr}(G) = \{\chi_i \mid i = 1, 2, \ldots, r\}$, then $\cap_{i=1}^{r} \mathrm{Ker}(\chi_i) = \{1_G\}$.

PROOF: If $g \in \cap_{i=1}^{r}\mathrm{Ker}(\chi_i)$, then $\chi_i(g) = \chi_i(1_G)$ $\forall i = 1, 2, \ldots, r$. Hence $\chi(g) = \chi(1_G)$ for all characters χ of G. So $g \sim 1_G$ by Proposition 1.3.22. Thus $g = 1_G$. ■

Theorem 1.3.24. *Let* G *be a group. Let* $N \trianglelefteq G$. *Then there exist some irreducible characters* $\chi_1, \chi_2, \ldots, \chi_s$ *of* G *such that* $N = \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$.

PROOF: Let $\mathrm{Irr}(G/N) = \{\widehat{\chi_1}, \widehat{\chi_2}, \ldots, \widehat{\chi_s}\}$. Then by Corollary 1.3.23, we have

$$\cap_{i=1}^{s}\mathrm{Ker}(\widehat{\chi_i}) = \{1_{G/N}\} = \{N\}.$$

Let χ_i be the lift to G of $\widehat{\chi_i}$ (that is $\chi_i(g) = \widehat{\chi_i}(gN)$, for all $g \in G$). We claim $N = \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$: Since $\chi_i(n) = \widehat{\chi_i}(nN) = \widehat{\chi_i}(N) = \chi_i(1_G)$, we have $n \in \mathrm{Ker}(\chi_i)$ so $N \subseteq \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$. Now let $g \in \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$. Then

$$\widehat{\chi_i}(N) = \chi_i(1_G) = \chi_i(g) = \widehat{\chi_i}(gN) \quad, i = 1, 2, \ldots, s$$

imply that $gN \in \cap_{i=1}^{s}\mathrm{Ker}(\widehat{\chi_i}) = \{N\}$. So $g \in N$ and hence $\cap_{i=1}^{s}\mathrm{Ker}(\chi_i) \subseteq N$. Thus $N = \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$. ■

Definition 1.3.18. *Suppose that* ψ *is a character of a group* G, *and that* χ *is an irreducible character of* G. *We say that* χ *is a* **constituent** *of* ψ *if* $\langle \psi, \chi \rangle \neq 0$. *Thus, the constituents of* ψ *are the irreducible characters* χ_i *of* G *for which the integer* d_i *in the expression* $\psi = d_1\chi_1 + \ldots + d_k\chi_k$ *is non-zero.*

Theorem 1.3.25. (**Clifford Theorem**) *Let* G *be a group. Let* $N \trianglelefteq G$ *and* $\chi \in \mathrm{Irr}(G)$. *Let* ϕ *be an irreducible constituent of* $\chi \downarrow N$ *and let* $\phi_1, \phi_2, \ldots, \phi_k$ *(where* $\phi = \phi_1$*) be the distinct conjugates of* ϕ *in* G. *Then*

$$\chi \downarrow N = e \sum_{i=1}^{k} \phi_i \quad , \quad \text{where } e = \langle \chi \downarrow N, \phi \rangle_N.$$

PROOF: Let $n \in N$. Then

$$\phi^G(n) = \frac{1}{|N|}\sum_{x \in G} \phi^0(xnx^{-1}) = \frac{1}{|N|}\sum_{x \in G} \phi(xnx^{-1}) = \frac{1}{|N|}\sum_{x \in G} \phi^x(n) \quad ,$$

where we have used the fact that $xnx^{-1} \in N$, $\forall x \in G$. Now if $\psi \in \mathrm{Irr}(N)$ and $\psi \notin \{\phi_1, \phi_2, \ldots, \phi_k\}$, then $\langle \sum_{x \in G} \phi^x, \psi \rangle_N = 0$ whence $\langle (\phi^G) \downarrow N, \psi \rangle_N = 0$. Using the Frobenius Reciprocity theorem we get

$$0 = \langle (\phi^G) \downarrow N, \psi \rangle_N = \langle \phi^G, \psi^G \rangle_G$$

and

$$0 \neq \langle \phi, \chi \downarrow N \rangle_N = \langle \phi^G, \chi \rangle_G.$$

Thus

$$\langle \chi, \psi^G \rangle_G = 0 \;\; ; \;\; \text{so } \langle \chi \downarrow N, \psi \rangle_N = 0.$$

Hence

$$\chi \downarrow N = \sum_{i=1}^{k} \langle \chi \downarrow N, \phi_i \rangle_N \phi_i.$$

Now by Proposition 1.3.21(4) we have

$$\langle \chi \downarrow N, \phi_i \rangle_N = \langle \chi \downarrow N, \phi \rangle_N = e \text{ for all } i = 1, 2, \ldots, k.$$

Thus

$$\chi \downarrow N = \sum_{i=1}^{k} e\ \phi_i = e\ \sum_{i=1}^{k} \phi_i.$$

■

Definition 1.3.19. (**Inertia Group**) *Let* G *be a group. Let* $N \trianglelefteq G$ *and let* $\phi \in \mathrm{Irr}(N)$. *Then the* ***inertia group*** *of* ϕ *is defined by*

$$I_G(\phi) := \{g \in G \mid \phi^g = \phi\}.$$

Proposition 1.3.26. *Let* G *be a group. Let* $N \trianglelefteq G$, $\phi \in \mathrm{Irr}(N)$. *Then* $\phi^G \in \mathrm{Irr}(G)$ *if and only if* $I_G(\phi) = N$.

PROOF: Let $g, k \in G$. Then $\phi^g = \phi^k$ if and only if $\phi^{gk^{-1}} = \phi$ if and only if $gk^{-1} \in I_G(\phi)$ if and only if $I_G(\phi).g = I_G(\phi).k$. So if $\{t_1, t_2, \ldots, t_m\}$ is a right transversal for $I_G(\phi)$ in G then $\phi^{t_1}, \phi^{t_2}, \ldots, \phi^{t_m}$ is a complete set of distinct conjugates of ϕ in G. Now for any $g \in G$ we have

$$\phi^G(g) = \frac{1}{|N|}\sum_{x\in G} \phi^0(xgx^{-1}) = \frac{1}{|N|}\sum_{y\in I}\ \sum_{j=1}^{m} \phi^0(yt_jgt_j^{-1}y^{-1}) \;\;, \;\; \text{where } I = I_G(\phi).$$

Thus $\forall\, n \in N$

$$\begin{aligned} (\phi^G \downarrow N)(n) &= \frac{1}{|N|}\sum_{y\in I}\ \sum_{j=1}^{m} \phi(yt_jnt_j^{-1}y^{-1}) \\ &= \frac{1}{|N|}|\ I\ |\ \sum_{j=1}^{m} \phi(t_jnt_j^{-1}) \\ &= [I:N]\ \sum_{j=1}^{m} \phi^{t_j}(n). \end{aligned}$$

(Note: We have used the fact that $yt_jnt_j^{-1}y^{-1} \in N$, $\forall y \in I$, $\forall t_j$). Hence

$$\langle \phi^G \downarrow N, \phi \rangle_N = [I:N]\ \sum_{j=1}^{m} \langle \phi^{t_j}, \phi \rangle_N = [I:N] \;\;,$$

since $\phi^{t_j} \neq \phi$ if $j \neq 1$ and ϕ^{t_j} are irreducible (because ϕ is and by Proposition 1.3.21(3)). Now by the Frobenius Reciprocity theorem we have

$$\langle \phi^G, \phi^G \rangle_G = \langle \phi^G \downarrow N, \phi \rangle_N = [I : N].$$

So ϕ^G is irreducible if and only if $[I : N] = 1$. Hence ϕ^G is irreducible if and only if $N = I_G(\phi)$. ■

Proposition 1.3.27. *Let* G *be group. Assume that* $G = N : H$*. That is* G *is a split extension of* N *by* H*. Let* $\phi \in \text{Irr}(N)$*. Then* $I_G(\phi) = N : I_H(\phi)$*. Hence* $\phi^G \in \text{Irr}(G)$ *if and only if* $I_H(\phi) = \{1_H\}$*.*

`PROOF:`Since $N \leq I_G(\phi)$ and $N \trianglelefteq G$, $N \trianglelefteq I_G(\phi)$. Let $g \in I_G(\phi)$. Then $g \in G$ and $g = nh$ where $n \in N$ and $h \in H$. So

$$\phi = \phi^g = \phi^{nh} = (\phi^n)^h = \phi^h.$$

Hence $h \in I_H(\phi)$; so $I_G(\phi) \subseteq NI_H(\phi)$. Similarly we can show that $NI_H(\phi) \subseteq I_G(\phi)$. Thus $NI_H(\phi) = I_G(\phi)$. Since $I_H(\phi) \subseteq H$ and $H \cap N = \{1_G\}$, $N \cap I_H(\phi) = \{1_G\}$. Thus $I_G(\phi) = N : I_H(\phi)$. Now $I_G(\phi) = N : I_H(\phi) = N$ if and only if $I_H(\phi) = \{1_G\}$. The result now follows by Proposition 1.3.26. ■

1.4 Coset Analysis

The technique works for both split and non-split extensions and was developed and first used by Moori. We use the method described in Mpono [19].

First we define a **lifting**.

Definition 1.4.1. (**Lifting**) *If* $\overline{G}$ *is a split extension of* N *by* G*, then* $\overline{G} = \cup_{g \in G} Ng$*, so* G *may be regarded as a right transversal for* N *in* $\overline{G}$ *(that is, a complete set of right coset representatives of* N *in* $\overline{G}$*). Now suppose* $\overline{G}$ *is any extension of* N *by* G*, not necessarily split, then, since* $\overline{G}/N \cong G$*, there is an onto homomorphism* $\lambda : \overline{G} \to G$ *with kernel* N*. For* $g \in G$ *define a* **lifting** *of* g *to be an element* $\overline{g} \in \overline{G}$ *such that* $\lambda(\overline{g}) = g$*.*

1.4.1 Coset Analysis

Let $\overline{G} = N.G$ where N is an abelian normal subgroup of $\overline{G}$.

- For each conjugacy class $[g]$ in G with representative $g \in G$, we analyze the coset $N\overline{g}$, where $\overline{g}$ is a lifting of g in $\overline{G}$ and $\overline{G} = \cup_{g \in G} N\overline{g}$.
- To each class representative $g \in G$ with lifting $\overline{g} \in \overline{G}$, we define

  $$C_{\overline{g}} = \{x \in \overline{G} \mid x(N\overline{g}) = (N\overline{g})x\}.$$

  Then $C_{\overline{g}}$ is the stabilizer of $N\overline{g}$ in $\overline{G}$ under the action by conjugation of $\overline{G}$ on $N\overline{g}$, and hence $C_{\overline{g}}$ is a subgroup of $\overline{G}$.

- If $\overline{G} = N : G$ then we can identify $C_{\overline{g}}$ with $C_g = \{x \in \overline{G} \mid x(Ng) = (Ng)x\}$, where the lifting of $g \in \overline{G}$ is g itself since $G \leq \overline{G}$.
- The conjugacy classes of $\overline{G}$ will be determined by the action by conjugation of $\overline{G}$, for each each conjugacy class $[g]$ of G, on the elements of $N\overline{g}$.
- To act $\overline{G}$ on the elements of $N\overline{g}$, we first act N and then act $\{\overline{h} \mid h \in C_G(g)\}$ where $\overline{h}$ is a lifting of h in G.
- We describe the action in two steps:

1. The action of N on $N\overline{g}$:
   Let $C_N(\overline{g})$ be the stabilizer of $\overline{g}$ in N. Then for any $n \in N$ we have

$$\begin{aligned} x \in C_N(n\overline{g}) &\iff x(n\overline{g})x^{-1} = n\overline{g}\ , \\ &\iff xnx^{-1}x\overline{g}x^{-1} = n\overline{g}\ , \\ &\iff x\overline{g}x^{-1} = \overline{g}\ ,\ \text{(since N is abelian)} \\ &\iff x \in C_N(\overline{g})\ . \end{aligned}$$

   Thus $C_N(\overline{g})$ fixes every element of $N\overline{g}$. Now let $|C_N(\overline{g})| = k$. Then under the action of N, $N\overline{g}$ splits into k orbits $Q_1, Q_2, \ldots, Q_k$ where $|Q_i| = [N : C_N(\overline{g})] = \frac{|N|}{k}$ for $i \in \{1, 2, \ldots, k\}$.

2. The action of $\{\overline{h} \mid h \in C_N(g)\}$ on $N\overline{g}$:
   Since the elements of $N\overline{g}$ are now in orbits $Q_1, Q_2, \ldots, Q_k$ from step (1) above, we only act $\{\overline{h} \mid h \in C_G(g)\}$ on these k orbits. Suppose that under this action f_j of these orbits $Q_1, Q_2, \ldots, Q_k$ fuse together to form one orbit $\triangle_j$, then the $f_{j's}$ obtained this way satisfy

$$\sum_j f_j = k \ \text{ and } \ |\triangle_j| = f_j \times \frac{|N|}{k} \ .$$

   Thus for $x \in \triangle_j$, we obtain that

$$\begin{aligned} |[x]_{\overline{G}}| &= |\triangle_j| \times |[g]_G| \\ &= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\ &= f_j \times \frac{|\overline{G}|}{k|C_G(g)|}\ . \end{aligned} \tag{1.3}$$

   Thus,

$$\begin{aligned} |C_{\overline{G}}(x)| = \frac{|\overline{G}|}{|[x]_{\overline{G}}|} &= |\overline{G}| \times k\frac{|C_G(g)|}{f_j\ |\overline{G}|} \\ &= \frac{k\ |C_G(g)|}{f_j}\ . \end{aligned} \tag{1.4}$$

1.5 Fischer Matrices

Let $\overline{G}$ be an extension of N by G. Let $\chi_1, \chi_2, \ldots, \chi_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$, and let $\overline{H}_i = I_{\overline{G}}(\chi_i)$ and $H_i = \overline{H}_i/N$. Let ψ_i be an extension of χ_i to $\overline{H}$. Take $\chi_1 = 1_N$, so $\overline{H}_1 = \overline{G}$ and $H_1 = G$. We consider a conjugacy class $[g]$ of G with representative g. Let $X(g) = \{x_1, x_2, \ldots, x_{c(g)}\}$ be representatives of $\overline{G}$-conjugacy classes of elements of the coset $N\overline{g}$. Take $x_1 = \overline{g}$. Let $R(g)$ be a set of pairs (i, y) where $i \in \{1, \ldots, t\}$ such that H_i contains an element of $[g]$, and y ranges over representatives of the conjugacy classes of H_i that fuse to $[g]$. Corresponding to this $y \in H_i$, let $\{y_i\}$ be representatives of conjugacy classes of $\overline{H}_i$ that contain liftings of y.

Definition 1.5.1. *We define the Fischer matrix* $M(g) = \left(a^j_{(i,y)}\right)$ *with columns indexed by* $X(g)$ *and rows indexed by* $R(g)$ (as described above) *by*

$$a^j_{(i,y)} = \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k}) \, . \tag{1.5}$$

where $\sum_k{}'$ is the sum over those k for which y_{l_k} is conjugate to x_j in $\overline{G}$.

Remark 1.5.1. We have kept the theory in this section brief since the Fischer matrices of the Frobenius group are simple. However, Fischer matrices have other special properties which are used in their computations. For a more detailed account see Whitney [25] and Moori and Mpono [18].

2

The Frobenius Group

2.1 Introduction

In the previous chapter we described **Permutation Groups**. The concept of a permutation group is not only interesting in it's own right but can also be used to describe groups in general. In this chapter we will use the definition of permutation groups to introduce a class of permutation groups that are *split* extensions. These groups are the **Frobenius groups**.

2.2 Definition and Preliminaries

Definition 2.2.1. *([14]). Let* G *be a* transitive permutation group *on a set* Ω *with* $|\Omega| > 1$*. Then* G *is said to be a Frobenius Group on* Ω *if:*

1. $G_\alpha \neq \{1_G\}$ *for any* $\alpha \in \Omega$.
2. $G_\alpha \cap G_\beta = \{1_G\}$ *for all* $\alpha, \beta \in \Omega$ *and* $\alpha \neq \beta$.

Note 2.2.1. G_α here is the *stabilizer* of $\alpha \in \Omega$.

Note 2.2.2. Although all our groups in this dissertation are finite, infinite Frobenius groups do exist (see Collins [2]), for an example.

Remark 2.2.1. Although we have defined them as permutation groups, Frobenius groups have numerous equivalent descriptions. The following proposition is one of several characterizations of Frobenius groups.

Proposition 2.2.1. *([6]). A group* G *is a Frobenius group if and only if it has a proper subgroup* $H \neq \{1_G\}$ *such that* $H \cap H^x = \{1_G\}$ *for all* $x \in G - H$.

`PROOF:` Assume G acts on Ω. Take $\alpha \in \Omega$ and let $G_\alpha = H \neq \{1_G\}$.
Now for any $x \in G - H$, $\alpha^x \neq \alpha$ (since if $\alpha^x = \alpha$, then $x \in G_\alpha = H$ which is a contradiction).

Let $1_G \neq y \in H$. We will show that $y \notin H^x$, $\forall x \in G - H$. Since $y \in H$, we have $\alpha^y = \alpha$. Let $\alpha^x = \beta$ for any $x \in G - H$ and $\beta \in \Omega$. Then $\beta = \alpha^x \neq \alpha$. Now if $(\alpha^x)^y = \alpha^x$ then $y \in G_\beta$ contradicting part (2) of definition 2.2.1. Therefore

$$(\alpha^x)^y \neq \alpha^x \Rightarrow \alpha^{xyx^{-1}} \neq \alpha^{1_G} = \alpha \Rightarrow xyx^{-1} \notin G_\alpha$$
$$\Rightarrow y \notin x^{-1}G_\alpha x = x^{-1}Hx = H^{x^{-1}} = H^{x'} \text{ for } x' \in G.$$

So $H \cap H^x = \{1_G\}\ \forall x \in G - H$.

Conversely, set $\Omega = \{xH : x \in G\}$ where $\{1_G\} < H < G$. Then G acts *transitively* on Ω by the *Generalised Cayley Theorem*. First we show that for any $\alpha \in \Omega, G_\alpha \neq 1_G$. Let $\alpha = g_0H$ for some $g_0 \in G$. Then

$$\begin{aligned} G_\alpha &= \{g \in G : gg_0H = g_0H\} = \{g \in G : g_0^{-1}gg_0H = H\} \\ &= \{g \in G : g_0^{-1}gg_0 \in H\} = \{g \in G : g \in g_0Hg_0^{-1}\} \\ &= H^{g_0} \neq \{1_G\} \text{ (since } H \neq \{1_G\}). \end{aligned}$$

So $G_\alpha \neq \{1_G\}$ for any $\alpha \in \Omega$. Now let $\alpha = xH$ and $\beta = yH$ for $\alpha, \beta \in \Omega$ and $x, y \in G$ such that $\alpha \neq \beta$. Then by the above argument $G_\alpha = H^x$ and $G_\beta = H^y$. We know that $H \cap H^t = \{1_G\}\ \forall t \in G \backslash H$. We just need to show that $H^x \cap H^y = \{1_G\}$. So suppose that $g \in H^x$ and $g \in H^y$ for $g \in G$. Then $g = xh'x^{-1} = yh''y^{-1}$ for $h', h'' \in H$, which implies that $h' = x^{-1}yh''y^{-1}x = x^{-1}yh''(x^{-1}y)^{-1}$. Let $w = x^{-1}y$. Then $w \in G \backslash H$ since if $w \in H$, then $x^{-1}y \in H$ which implies that $x^{-1}yH = H$ and hence that $yH = xH$, that is $\alpha = \beta$, which is a contradiction. Thus $h' \in H$ and $h' \in H^w$ implies $h' \in H \cap H^w = \{1_G\}$. Therefore, $g = xh'x^{-1} = x\{1_G\}x^{-1} = 1_G$. Hence, $H^x \cap H^y = \{1_G\}$. ■

Corollary 2.2.2. *([6]). If G is a Frobenius group and $H \leq G$ is the stabilizer of a point then $N_G(H) = H$.*

PROOF: Let $\{1_G\} < H < G$. We know that $H \subseteq N_G(H)$ since $gHg^{-1} = H\ \forall g \in H$. Let $g \in N_G(H)$ and suppose that $g \notin H$. Since $g \in N_G(H), gHg^{-1} = H\ \forall g \in G \backslash H$. But G is a Frobenius group. Therefore $H^g \cap H = \{1_G\}\ \forall g \in G \backslash H$. This implies that $H^g = H = \{1_G\}\ \forall g \in G \backslash H$ contradicting the fact that $H \neq \{1_G\}$. Therefore $g \in H$ and hence $N_G(H) \subseteq H$ and the result follows. ■

Note 2.2.3. ([6]). If G is a Frobenius group on Ω and $H = G_\alpha$ for some $\alpha \in \Omega$ then H is called the **Frobenius Complement** in G. Denote by N^* the set of all $x \in G$ having **no fixed points** in Ω and set $N = N^* \cup \{1_G\}$. Then $N = \big(G \backslash \cup \{H^x : x \in G\}\big) \cup \{1_G\}$ and we call N the **Frobenius Kernel** of G.

In 1901, Frobenius proved that Frobenius **kernels** of Frobenius groups are normal subgroups. We will prove this later on as the Frobenius Theorem after the following two Propositions. It should be mentioned that there is no known proof of Frobenius Theorem which does not use character theory. However, Corradi and Horvath [3] provide a proof of the theorem when the complement H is solvable or has even order and Knapp and Schmid [13] provide a proof that uses character theory but is simpler and more direct.

Proposition 2.2.3. *([6]). Suppose that* G *is a Frobenius group with complement* H*, and suppose that* θ *is an element of the set of class functions of* $H\big(\theta \in cf(H)\big)$ *with* $\theta(1_G) = 0$*, then* $(\theta^G)\downarrow_H = \theta$*.*

PROOF: Note that $\theta^G(1_G) = \frac{|G|}{|H|}\,\theta(1_G) = n \times r$. So $\theta^G(1_G) = [G:H]\,\theta(1_G) = [G:H] \times 0 = 0$. If $1_G \neq y \in H$ then $\theta^G(y) = \frac{1}{|H|}\sum\limits_{t\in G}\theta^0(tyt^{-1})$. Now if $tyt^{-1} \in H$ then $y \in t^{-1}Ht$ and hence $y \in H^{t^{-1}} = H^{t'}$ for $t' \in G$. So $y \in H \cap H^{t'}$. But since $1_G \neq y$ and G is a Frobenius group, $y \in H \cap H^{t'}$ implies that $t' \in H$. Therefore $\theta^0(t'yt'^{-1}) = \theta(t'yt'^{-1}) = \theta(y)$. Hence, $\theta^G(y) = \frac{1}{|H|} \times |H| \times \theta(y) = \theta(y)\ \forall y \in H$. ∎

Proposition 2.2.4. *([6]). Suppose* G *is a Frobenius group with complement* H *and kernel* N*. Then*

1. $|N| = [G:H] > 1$.
2. *If* $K \trianglelefteq G$ *with* $K \cap H = \{1_G\}$ *then* $K \subseteq N$.

PROOF: 1. Since $N_G(H) = H$ by Corollary 2.2.2, there are $[G:H]$ distinct conjugates of H in G. So

$$\begin{aligned}
\left|\cup\{H^x : x \in G\}\right| &= [G:H] \times \big(|H| - 1\big) + 1 \\
&= \frac{|G|}{|H|} \times \big(|H| - 1\big) + 1 \\
&= |G| - \frac{|G|}{|H|} + 1 \\
&= |G| - [G:H] + 1.
\end{aligned}$$

Now since $N = \big(G \setminus \cup\ \{H^x : x \in G\}\big) \cup \{1_G\}$, we have

$$\begin{aligned}
|N| &= |G| - \big(|G| - [G:H] + 1\big) + 1 \\
&= |G| - |G| + [G:H] - 1 + 1 \\
&= [G:H].
\end{aligned}$$

Also $\{1_G\} < H < G$ implies $[G:H] > 1$.
2. Let $1_G \neq k \in K$. Suppose $k \notin N$. Then $k \in H^z$ for some $z \in G\setminus H$. So $k = zhz^{-1}$ for some $h \in H$. So $z^{-1}kz = h \in H$ and since $K \trianglelefteq G$ we have $z^{-1}kz \in K$. Therefore $z^{-1}kz \in H \cap K = \{1_G\}$ and hence $k = 1_G$ which is a contradiction. Therefore $k \in N$ and $K \subseteq N$. ∎

Theorem 2.2.5. *([6]). If* G *is a Frobenius group with complement* H *and kernel* N *then* N *is a normal subgroup of* G*.*

PROOF: We will show that N is the intersection of the kernels of some irreducible characters of G. Take $I_H \neq \phi \in \mathrm{Irr}(H)$, where I_H is the **principle character** of H and set $\theta = \phi - \phi(1_G)I_H$. Then θ is a **generalized character** of H. Also $\theta(1_G) = 0$ and by the Frobenius Reciprocity Theorem,

and Proposition 2.2.3, we have that $\langle\theta^G,\theta^G\rangle = \langle\theta,\theta^G\downarrow_H\rangle = \langle\theta,\theta\rangle$.
Now

$$\begin{aligned}\langle\theta,\theta\rangle &= \langle\phi-\phi(1_G)I_H,\phi-\phi(1_G)I_H\rangle\\ &= \langle\phi,\phi\rangle-\phi(1_G)\langle\phi,I_H\rangle-\phi(1_G)\langle I_H,\phi\rangle \ +(\phi(1_G))^2\langle I_H,I_H\rangle\\ &= 1-0-0+(\phi(1_G))^2\\ &= 1+\phi(1_G)^2.\end{aligned}$$

Again by the Frobenius reciprocity we have that;

$$\begin{aligned}\langle\theta^G,I_G\rangle &= \langle\theta,I_H\rangle\\ &= \langle\phi-\phi(1_G)I_H,I_H\rangle\\ &= \langle\phi,I_H\rangle-\phi(1_G)\langle I_H,I_H\rangle\\ &= 0-\phi(1_G)\times 1\\ &= -\phi(1_G).\end{aligned}$$

So $\langle\theta^G,I_G\rangle = \langle\theta,I_H\rangle = -\phi(1_G)$. Thus if we set $\phi^* = \theta^G+\phi(1_G)I_G$ then ϕ^* is a generalized character of G and we have

$$\begin{aligned}\langle\phi^*,I_G\rangle &= \langle\theta^G+\phi(1_G)I_G,I_G\rangle\\ &= \langle\theta^G,I_G\rangle+\phi(1_G)\langle I_G,I_G\rangle\\ &= -\phi(1_G)+\phi(1_G)\\ &= 0.\end{aligned}$$

Also

$$\begin{aligned}\langle\phi^*,\phi^*\rangle &= \langle\theta^G+\phi(1_G)I_G,\theta^G+\phi(1_G)I_G\rangle\\ &= \langle\theta^G,\theta^G\rangle+\phi(1_G)\langle\theta^G,I_G\rangle+\phi(1_G)\langle I_G,\theta^G\rangle+(\phi(1_G))^2\langle I_G,I_G\rangle\\ &= \langle\theta,\theta\rangle-\phi(1_G)\phi(1_G)-\phi(1_G)\phi(1_G)+(\phi(1_G))^2\\ &= 1+\phi(1_G)^2-\phi(1_G)^2-\phi(1_G)^2+\phi(1_G)^2\\ &= 1.\end{aligned}$$

Therefore either ϕ^* or $-\phi^*\in \mathrm{Irr}(G)$, since if $\phi^*=\sum\lambda_i\chi_i$ for $\lambda\in\mathbb{Z}$ and $\chi_i\in\mathrm{Irr}(G)$ then

$$\begin{gathered}\langle\phi^*,\phi^*\rangle = 1 \ \Rightarrow \ \sum\lambda_i^2 = 1\\ \Rightarrow \ \lambda_i = \pm 1 \text{ or } \lambda_j = 0 \ \forall i\neq j\\ \Rightarrow \ \phi^* = \chi_i \text{ or } -\phi^* = \chi_i.\end{gathered}$$

By Proposition 2.2.3 we have for $y\in H$ that;

$$\begin{aligned}\phi^*(y) &= \theta^G(y)+\phi(1_G)I_G(y) = \theta(y)+\phi(1_G)\times 1\\ &= \theta(y)+\phi(1_G) = \phi(y).\end{aligned}$$

So in particular $\phi^*(1_G) = \phi(1_G) > 0$ and hence $\phi^* \in \mathrm{Irr}(G)$. So for every non-principle $\phi \in \mathrm{Irr}(H)$ we have chosen an extension $\phi^* \in \mathrm{Irr}(G)$.

Now set $K = \bigcap \{\ker\phi^* \ : \ I_H \neq \phi \in \mathrm{Irr}(H)\} \trianglelefteq G$. We want to show that $K = N$. Suppose $y \in H \cap K$. Then $y \in H$ and $y \in K$. So $\phi(y) = \phi^*(y) = \phi^*(1_G)$ (since $y \in \ker\phi^*$, $\forall I_H \neq \phi \in \mathrm{Irr}(H)$).

But $\phi^*(1_G) = \phi(1_G)$ implies that $y \in \ker\phi ; \forall\phi \in \mathrm{Irr}(H)$ and $\phi \neq I_H$. Since $\bigcap_{\phi\in\mathrm{Irr}(H)} \ker\phi = \{1_H\} = \{1_G\}$, (see Moori, Corollary 4.2 [17]), we have that $y = 1_G$. Now since $K \trianglelefteq G$ and $K \cap H = \{1_G\}$, by Proposition 2.2.4 we have $K \subseteq N$. On the other hand if $1_G \neq x \in N$ then $x \notin H^z$ for any $z \in G$. So

$$\begin{aligned} \phi^*(x) \quad &= \theta^G(x) + \phi(1_G)I_G(x) \\ &= 0 + \phi(1_G) = \phi^*(1_G), \end{aligned}$$

since $\theta^G(x) = 0 \ \forall x \in G\backslash H$. This implies that $x \in \ker\phi^*$. So $N \subseteq K$. Thus $K = N \trianglelefteq G$. ■

Corollary 2.2.6. *If G is a Frobenius group with kernel N and complement H then G is a semi-direct product of N by H.*

PROOF: We know that $N = \big(G\backslash \cup \{H^x : x \in G\}\big) \cup \{1_G\}$ and $N \cap H = \{1_G\}$. Also $N \trianglelefteq G$ and $H \leq G$. So

$$|NH| = \frac{|N| \times |H|}{|N \cap H|} = \frac{(G : H) \times |H|}{1} = |G|.$$

Therefore $G = NH$. Thus G is a semi-direct product of N by H. ■

3

Structure of The Frobenius Group

3.1 Introduction

This chapter forms the main part of this thesis. We look here at the structure of the Frobenius group. Some of the results provide us with alternate definitions of the Frobenius group, while others can be used to construct Frobenius groups. We look at the structure of the kernel N and the complement H in greater detail. Lemma 3.2.15 is a useful result and provides some insight into the conjugacy classes of the group. We also give some results about the center, commutator subgroup and Frattini subgroup of a Frobenius group. We end the chapter by briefly mentioning some results about solvability of Frobenius groups.

3.2 Structure

Proposition 3.2.1. *([6]). Suppose that* G *is a Frobenius group with complement* H *and kernel* N*. If* $1_G \neq x \in N$ *then* $C_G(x) \leq N$.

PROOF: Since $C_G(x) \leq G$ we just need to show that $C_G(x) \subseteq N$. First we will show that if $h \in H \cap C_G(x)$ then $h = 1_G$. So suppose that for $h \in H$ and $x \in N, h \in H \cap C_G(x)$. Now $h \in C_G(x)$ implies that $hxh^{-1} = x$. So $h = xhx^{-1} = h^x \in H^x$. Therefore $h \in H \cap H^x = \{1_G\}$ and hence $h = 1_G$. Suppose now that $y \in C_G(x)$. We will show that $y \in N$. Assume that $y \notin N$, then $1_G \neq y \in H^z$ for some $z \in G$. This follows from the definition of the Frobenius kernel, see Note 2.2.3. Now $y \in C_G(x)$ implies that $yxy^{-1} = x$ and since $y \in H^z$ we have $y = zhz^{-1}$ for some $h \in H$. Therefore $z^{-1}yz = h$ and hence $y^{z^{-1}} = y^{z'} = h \in H$ for some $z' \in G$.

Now

$$\begin{aligned} yxy^{-1} &= x \\ \Rightarrow (zhz^{-1})x(zhz^{-1})^{-1} = x &\Rightarrow zh(z^{-1}xz)h^{-1}z^{-1} = x \\ \Rightarrow zh(x^{z^{-1}})h^{-1}z^{-1} = x &\Rightarrow z^{-1}[zh(x^{z^{-1}})h^{-1}z^{-1}]z = z^{-1}xz \\ \Rightarrow h(x^{z^{-1}})h^{-1} = z^{-1}xz &\Rightarrow h(x^{z^{-1}})h^{-1} = x^{z^{-1}} \\ \Rightarrow h \in C_G(x^{z^{-1}}) &\Rightarrow y^{z^{-1}} \in C_G(x^{z^{-1}}). \end{aligned}$$

Therefore $y^{z^{-1}} \in H \cap C_G(x^{z^{-1}})$. Also since $x^{z^{-1}} \in N$ ($\because N \trianglelefteq G$) and $x^{z^{-1}} \neq 1_G$ ($\because x \neq 1_G$), by the first part of the proof, since $y^{z^{-1}} \in H \cap C_G(x^{z^{-1}})$, we have that $y^{z^{-1}} = 1_G$ and hence $y = 1_G$ which is a contradiction. Therefore $y \in N$ and hence $C_G(x) \subseteq N$. ■

Proposition 3.2.2. *([6]). Suppose that G is a Frobenius group with complement H and kernel N. Then $\circ(H) \big| \circ(N) - 1$.*

PROOF: Now G acts by conjugation on N. Restricting this action to H, the complement H acts by conjugation on N. Let $1_G \neq x \in N$. Then $H_x = \{h \in H : x^h = x\} = \{h \in H : hxh^{-1} = x\} = C_H(x)$. Now by Proposition 3.2.1, $C_G(x) \leq N$. Since $C_H(x) \subseteq C_G(x) \leq N$, $C_H(x) \leq N$. But G is a Frobenius group and $H \cap N = \{1_G\}$. Therefore $C_H(x) = \{1_G\}$. Now by the Orbit Stabilizer Theorem, we have that $\left|x^H\right| = [H : H_x]$. So $\left|x^H\right| = \frac{|H|}{|H_x|} = \frac{|H|}{1} = |H|$. Since the H - orbits partition N, $N \backslash \{1_G\}$ is a union of H - orbits each of size $|H|$. Therefore $|N| - 1 = \alpha |H|$ where α is the number of orbits. This implies that $|H| \,\big|\, |N| - 1$. ■

Note 3.2.1. A subgroup H of a group G is called a **Hall** subgroup if $|H|$ and $[G : H]$ are relatively prime. Thus, by Proposition 2.2.4 and the following Corollary, in a Frobenius group the **order of the complement H and the kernel N are always relatively prime.**

Corollary 3.2.3. *The complement H of a Frobenius group G is a Hall subgroup of G and the kernel N is a normal Hall subgroup of G.*

PROOF: By Proposition 3.2.2, we have $|H| \,\big|\, |N| - 1$. So $|H| \times \alpha = |N| - 1$ for some $\alpha \in \mathbb{N}$. So $|N| - \alpha\ |H| = 1$ implies that $(|N|\ ,\ |H|) = 1$. But by Proposition 2.2.4, we have $|N| = [G : H]$. So $\big([G : H]\ ,\ |H|\ \big) = 1$ and hence that H is a Hall subgroup of G. Since $G = NH$ and $|G| = |N| \times |H|$, $\frac{|G|}{|N|} = |H|$. So $[G : N] = |H|$. Hence by above, $\big(|N|\ ,\ [G : N]\ \big) = 1$. This implies that N is a normal Hall subgroup of G. ■

Lemma 3.2.4. *If G is a group and $\mathcal{T}(x) = x^{-1}\ \forall x \in G$, then*

1. *$\mathcal{T}$ is 1-1 from G onto G.*
2. *$\mathcal{T}$ is an automorphism $\Leftrightarrow$ G is abelian.*

PROOF: Easy and omitted. ■

Proposition 3.2.5. *([6]). If G is a Frobenius group with complement H of even order , then the kernel N is abelian.*

PROOF: Since $2\big||H|$, by Cauchy's Theorem there exists $h \in H$ such that $o(h) = 2$. If $1_G \neq x \in N$ then $x \neq x^h \in N$ (since $x^h = x$ implies that $h \in C_G(x) \leq N$ which is a contradiction). So $x^h \neq x$ and hence $x^h x^{-1} \neq 1_G$. Consider now the map $\phi(x) = x^h x^{-1}$ $\forall x \in N$. We will show that ϕ is an automorphism of N which coincides with the map $\psi(z) = z^{-1}$ $\forall z \in N$, and by Lemma 3.2.4 the result will follow.

<u>ϕ is well defined :</u>

Let $x, y \in N$. Then $x = y$ implies that $y^{-1}x = 1_G$. So

$$\begin{aligned} (y^{-1}x)^h &= y^{-1}x = 1_G \\ \Rightarrow\ h(y^{-1}x)h = y^{-1}x &\Rightarrow\ hy^{-1}(hh)xh = y^{-1}x \\ \Rightarrow\ (hy^{-1}h)(hxh) = y^{-1}x &\Rightarrow\ (hyh)^{-1}(hxh) = y^{-1}x \\ \Rightarrow\ (y^h)^{-1}x^h = y^{-1}x &\Rightarrow\ x^h x^{-1} = y^h y^{-1} \\ \Rightarrow\ \phi(x) &= \phi(y). \end{aligned}$$

<u>ϕ is $1-1$:</u>

Suppose $\phi(x) = \phi(y)$ for $x, y \in N$, then

$$\begin{aligned} x^h x^{-1} = y^h y^{-1} &\Rightarrow\ (y^h)^{-1}x^h = y^{-1}x \\ \Rightarrow\ (hyh)^{-1}(hxh) = y^{-1}x &\Rightarrow\ (hy^{-1}h)(hxh) = y^{-1}x \\ \Rightarrow\ hy^{-1}(h^2)xh = y^{-1}x &\Rightarrow\ hy^{-1}xh = y^{-1}x \\ \Rightarrow\ (y^{-1}x)^h = y^{-1}x &\Rightarrow\ y^{-1}x = 1_G, \end{aligned}$$

since $y^{-1}x \in N$ and $h \notin C_G(y^{-1}x)$ by Proposition 3.2.1. Hence $y = x$.

Also ϕ is onto since N is finite. Therefore $N = \{x^h x^{-1} \mid x \in G\}$. Setting $z = x^h x^{-1}$, we have that

$$\begin{aligned} z^h &= (x^h x^{-1})^h = h(x^h x^{-1})h = h(hxh)x^{-1}h \\ &= h^2(xh)x^{-1}h = x(hx^{-1}h) = x(hxh)^{-1} \\ &= x(x^h)^{-1} = (x^h x^{-1})^{-1} = z^{-1}. \end{aligned}$$

Now the map $z \mapsto z^h$ is an automorphism of N. Since $z^h = z^{-1}$, the automorphism $z \mapsto z^h$ is the same as the map $z \mapsto z^{-1}$. Since this map is an automorphism of N, by Lemma 3.2.4 N is abelian. ∎

Proposition 3.2.6. *Suppose that G is a Frobenius group with kernel N and complement H. Let z be an involution in H. Then $x^z = x^{-1}$ $\forall x \in N$.*

PROOF: Since $|H|$ is even, by Proposition 3.2.5 the kernel N is abelian. Note first that if $x \in N$ then $(xz)^2 = xzxz = x(zxz) \in N$, since $N \trianglelefteq G$ implies that $(zxz) \in N$. Now

$$\begin{aligned} z(xz)^2 &= z(xzxz) = \big[(zxz)x\big]z = \big[x(zxz)\big]z \ \text{(since N is abelian)} \\ &= (xzxz)z = (xz)^2 z. \end{aligned}$$

Therefore $(xz)^2 \in C_G(z) \leq H$. Since $(xz)^2 \in N$, we have $(xz)^2 = 1_G$. Now since $(xz)^2 = x(x^z)$, we must have $x^z = x^{-1}$. ■

Note 3.2.2. If $|H|$ is even, then H contains a unique involution z, and therefore z is central. Since, if $z' \in H$ is another involution, then by Proposition 3.2.6 we have

$$\begin{aligned} x^{z'} &= x^{-1} = x^z \\ \Rightarrow z'xz'^{-1} = zxz^{-1} &\Rightarrow (z^{-1}z')x = x(z^{-1}z') \\ \Rightarrow z^{-1}z' &\in C_G(x) \leq N. \end{aligned}$$

Since $H \cap N = \{1_G\}$, $z^{-1}z' = 1_G \;\Rightarrow\; z = z'$.

Theorem 3.2.7. *([6]). A finite group G is a Frobenius group if and only if it has a non-trivial proper normal subgroup N such that if $1_G \neq x \in N$ then $C_G(x) \leq N$.*

PROOF: If G is A Frobenius group, then by Proposition 3.2.1, we have $C_G(x) \leq N$.
Conversely suppose now that a finite group G has a non-trivial proper normal subgroup N such that if $1_G \neq x \in N$ then $C_G(x) \leq N$. First we show that N is a normal Hall subgroup of G. Suppose that N is not a normal Hall subgroup of G. There exists a prime p such that $p\big||N|$ and $p\big|[G:N]$. Let $|G| = p^\alpha q$ and $|N| = p^\beta q'$ with $\alpha > \beta$ and $(p,q) = 1 = (p,q')$. Let P be a **Sylow** p - subgroup of N and let Q be a **Sylow** p - subgroup of G with $\{1_G\} \leq P \leq Q$ and $Q \neq P$. Then $|P| = p^\beta$ and $|Q| = p^\alpha$. Since Q is a non-trivial p - group, the centre of Q is non-trivial.
Clearly $P \leq Q \cap N$. Now $Q \cap N \leq N$ and $Q \cap N \leq Q$. Therefore $Q \cap N$ is a p - subgroup of N. Now since P is a maximal p - subgroup of N we must have that $Q \cap N \leq P$ and hence $P = Q \cap N$. Let $x \in Z(Q)$ with $\circ(x) = p$. Then $xg = gx\ \forall g \in Q$. So $Q \subseteq C_G(x)$. Now if $x \in P$ then $x \in N$ and $C_G(x) \leq N$ implies that $Q \subseteq C_G(x) \subseteq N$ which is a contradiction, since $Q \cap N = P$. Suppose now $x \notin P$. Then for any $1_G \neq y \in P$ we have $y \in Q$ ($\because P \leq Q$) and hence $xy = yx$. Therefore $x \in C_G(y)$. Now $1_G \neq y \in N$ implies that $C_G(y) \leq N$ by Proposition 3.2.1. Hence, $x \in N$, so that $x \in Q \cap N = P$, which is a contradiction. Hence, N must be a normal Hall subgroup of G.
By the Schur-Zassenhaus Theorem there is a complement H to N in G such that $G = NH$ and $N \cap H = \{1_G\}$. Let $x \in G \backslash H$ and suppose that $H \cap H^x \neq \{1_G\}$. Since $G = NH$, we can write $x = nh$ with $x \in N$ and $h \in H$.
Then

$$\begin{aligned} H^x &= H^{nh} = nh(H)(nh)^{-1} \\ &= nh(H)h^{-1}n^{-1} = n(hHh^{-1})n^{-1} = nHn^{-1} = H^n. \end{aligned}$$

So $H \cap H^n \neq \{1_G\}$ and there exists $1_G \neq y \in H \cap H^n$ such that $y \in H$ and $y = nh'n^{-1}$ for some $1_G \neq h' \in H$. So

$$nh'n^{-1} \in H \;\Rightarrow\; (nh'n^{-1})h'^{-1} \in H \;\Rightarrow\; n(h'n^{-1}h'^{-1}) \in H.$$

But $n(h'n^{-1}h'^{-1}) \in N$ since $h'n^{-1}h'^{-1} \in N$ ($\because N \trianglelefteq G$).
Therefore $nh'n^{-1}h'^{-1} \in N \cap H = \{1_G\}$ and hence $nh' = h'n$. This implies that $h' \in C_G(x) \leq N$, which is a contradiction since $h' \neq 1_G$. Therefore $H \cap H^x = \{1_G\}$ $\forall x \in G \backslash H$ and by Proposition 2.2.1, G is a Frobenius group. ∎

Theorem 3.2.8. *([6]).*

1. *Suppose that* $|G| = mn$ *with* $(m, n) = 1$*, that either* $x^n = 1_G$ *or* $x^m = 1_G$ $\forall x \in G$ *and that* $N = \{x \in G : x^n = 1_G\} \trianglelefteq G$*. Then* G *is a Frobenius group with kernel* N*.*

2. *Conversely, if* G *is a Frobenius group with kernel* N *and complement* H*, and if* $|N| = n$ *,* $|H| = m$*, then either* $x^n = 1_G$ *or* $x^m = 1_G$ $\forall x \in G$ *and* $N = \{x \in G : x^n = 1_G\}$*.*

PROOF: (1) First we show that $\big(|N|, m\big) = 1$. So suppose that $\big(|N|, m\big) \neq 1$. Then there exists a prime p such that $p \big| |N|$ and $p \big| m$. But $p \big| |N|$ implies that there exists $x \in N$ such that $o(x) = p$ and hence that $p \big| n$ which contradicts the fact that $(m, n) = 1$. Thus $\big(|N|, m\big) = 1$.
Since $|N| \big| |G| = mn$ and $\big(|N|, m\big) = 1$, we have that $|N| \big| n$. If $n = p^{\alpha}n'$ with $(p, n') = 1$, then $Q \in Syl_p(G)$ implies that $|Q| = p^{\alpha}$. If $1_G \neq x \in Q$ then $x^{p^{\alpha}} = 1_G$ and this implies that $o(x) \big| p^{\alpha}$ and hence $o(x) \big| n$. Now if $x^m = 1_G$ then $o(x) \big| m$ and since $x \neq 1_G$ this implies that $(m, n) \neq 1$ which is a contradiction. Thus $x^n = 1_G$ which implies that $x \in N$. So $Q \subseteq N$. But $Q \in Syl_p(G)$. This implies that $Q \in Syl_p(N)$. Thus for each prime p dividing n there is a Sylow p - subgroup of G in N. So $|N| = p^{\alpha}n''$ with $(p, n'') = 1$. Therefore $n \big| |N|$ and hence $|N| = n$. Since $|G| = mn$ and $(m, n) = 1$, we have that N **is a normal Hall** subgroup of G. By the **Schur Zassenhaus Theorem** there is a complement H to N in G. Therefore $G = NH$ and $N \cap H = \{1_G\}$. The order of H is m. We just need to show that H is a **Frobenius complement**. Suppose now that $H \cap H^x \neq \{1_G\}$ and $x \notin H$. So either $x \in N$ or $x = kh$ for $1_G \neq k \in N$ and $1_G \neq h \in H$. Assume that $x \in N$. Then since $H \cap H^x \neq \{1_G\}$, choose $1_G \neq h \in H \cap H^x$. Then $h = xh'x^{-1}$ for $h' \in H$. Now

$$h = xh'x^{-1} \;\Rightarrow\; x^{-1}hx = h' \;\Rightarrow\; h^{x^{-1}} = h' \;\Rightarrow\; h^{x^{-1}}h^{-1} = h'h^{-1} \in H.$$

Also $h^{x^{-1}}h^{-1} = (x^{-1}hx)h^{-1} = x^{-1}(hxh^{-1}) \in N$ (since $N \trianglelefteq G$).
Therefore

$$h^{x^{-1}}h^{-1} \in H \cap N = \{1_G\} \Rightarrow\; h^{x^{-1}}h^{-1} = 1_G \Rightarrow\; x^{-1}hxh^{-1} = 1_G \Rightarrow\; hx = xh.$$

Since $(m, n) = 1, o(xh) = o(x) \times o(h)$. Let $o(x) = n'$ and $o(h) = m'$, then $n' \big| n$ and $m' \big| m$. Now $o(xh) = n'm'$. Also since $xh \in G$, either $(xh)^n = 1_G$ or $(xh)^m = 1_G$. If $(xh)^n = 1_G$, then $n'm' \big| n$ implies that $n = n'm'k$ for $k \in \mathbb{Z}$. This implies that $m' \big| n$. But $m' \big| m$. This contradicts the fact that $(m, n) = 1$. We get a similar contradiction if $(xh)^m = 1_G$. Assume now that $x \notin N$ and $x = kh$ for $1_G \neq k \in N$ and $1_G \neq h \in H$. Then

$$H^x \;=\; H^{kh} = khH(kh)^{-1} = k(hHh^{-1})k^{-1} = kHk^{-1} = H^k.$$

So $H \cap H^x \neq \{1_G\}$ implies that $H \cap H^k \neq \{1_G\}$. Therefore there exists $y \in H$ such that $y = kh''k^{-1}$ for some $h'' \in H$. Now $yh''^{-1} = k(h''k^{-1}h''^{-1}) \in N \trianglelefteq G$. So $kh''k^{-1}h''^{-1} \in H \cap N = \{1_G\}$. Therefore $kh'' = h''k$ which implies that $h'' \in C_G(k)$ and hence by Proposition 3.2.1 that $h'' \in N$. But this is a contradiction. Thus $H \cap H^x = \{1_G\}\ \forall x \in G \backslash H$ which implies that H is a Frobenius complement of N in G by Proposition 2.2.1.
(2) If $1_G \neq x \in N$ then since $|N| = n, x^n = 1_G$. Also if $1_G \neq y \in H$ then since $|H| = m, y^m = 1_G$. Suppose now $x \in G$ and $x \notin N$ and $x \notin H$. Then $x = kh$ for some $1_G \neq k \in N$ and $1_G \neq h \in H$. Now

$$\begin{aligned}
x^n &= (kh)^n = (kh)(kh)(kh)^{n-2} = (khkh^{-1}hh)(kh)^{n-2} = k(hkh^{-1})h^2(kh)^{n-2} \\
&= kk'h^2(kh)^{n-2} \ \text{(for some } k' \in N \trianglelefteq G) \\
&= k''h^2(kh)^{n-2} \ \text{(for some } k'' \in N) \\
&= k''h^2(kh)(kh)(kh)^{n-4} = k''h^2(khkh^{-1}hh)(kh)^{n-4} \\
&= k''h^2kk'''h^2(kh)^{n-4} \ \text{(for } k''' \in N \trianglelefteq G) \\
&= k''h^2k_2h^2(kh)^{n-4} \ \text{(for } k_2 \in N) \\
&= k''h(hk_2h^{-1})h^3(kh)^{n-4} \\
&= k''hk_3h^3(kh)^{n-4} \ \text{(for } k_3 \in N \trianglelefteq G) \\
&= k''(hk_3h^{-1})h^4(kh)^{n-4} \\
&= k''k_4h^4(kh)^{n-4} \ \text{(for } k_4 \in N \trianglelefteq G) \\
&= k_5h^4(kh)^{n-4} \ \text{(for } k_5 \in N), \ldots\ldots
\end{aligned}$$

continuing in this fashion we find that $x^n = n_0h^n$ for some $n_0 \in N$. Suppose now that $x^n = 1_G$. Then $n_0h^n = 1_G$ which implies that $h^n = n_1$ for some $n_1 \in N$ and hence that $h^n = 1_G$, since $H \cap N = \{1_G\}$. Therefore $o(h)\big|n$ and since $o(h)\big|m$, this implies that $(m, n) \neq 1$ which is a contradiction. Hence $x^n \neq 1_G$. Since $(m, n) = 1$ and $o(x)\big|mn$ and the order of x does not divide n we must have that $o(x)\big|m$. Thus $x^m = 1_G$. Hence, either $x^m = 1_G$ or $x^n = 1_G\ \forall x \in G$.
By definition the Frobenius kernel is $N = \big(G \backslash \cup \{H^x : x \in G\}\big) \cup \{1_G\}$. So if $1_G \neq x \in G$ is in any conjugate of H then $x^m = 1_G$. The remaining $g \in G$ are in the kernel N. Thus $N = \{x \in G : x^n = 1_G\}$. ■

Proposition 3.2.9. *([6]). Suppose* G *is a Frobenius group with kernel* N *and complement* H *and that* $\{1_G\} \neq N_1 \leq N, \{1_G\} \neq H_1 \leq H$, *with* $H_1 \leq N_G(N_1)$. *Then* $G_1 = N_1H_1$ *is a Frobenius group with kernel* N_1 *and complement* H_1.

PROOF: Since $N_1 \trianglelefteq N_G(N_1)$ and $H_1 \leq N_G(N_1)$, $N_1H_1 \leq N_G(N_1) \leq G$. Also $N_1 \trianglelefteq N_G(N_1)$ and $N_1 \leq G_1 = N_1H_1 \leq N_G(N_1)$ implies that $N_1 \trianglelefteq G_1$. Clearly $G_1 = N_1 : H_1$ since $N_1 \cap H_1 \subseteq N \cap H = \{1_G\}$. First we show that $N_1H_1 \cap N = N_1$. Now N_1 is clearly contained in the intersection since $N_1 \subseteq N_1H_1$ and $N_1 \subseteq N$. To show the reverse containment, suppose that $1_G \neq x \in N_1H_1 \cap N$ but $x \notin N_1$. Since $x \in N_1H_1$ and $x \in N$, $x \notin N_1$ implies that $x = 1_G.h$ for some $h \in H_1$ or $x = n'h'$ for

some $n' \in N_1$ and $h' \in H_1$. If $x = 1_G.h$ then $x \in H$ (since $H_1 \leq H$) which is a contradiction since $H \cap N = \{1_G\}$. If $x = n'h'$ then $x \in N$ implies that $n'h' \in N$ and hence that $h' \in N$ which is a contradiction since $H \cap N = \{1_G\}$. Therefore, $N_1 \subseteq N_1H_1 \cap N$ and hence $N_1H_1 \cap N = N_1$.
Now N_1 is a non-trivial normal subgroup of G_1 so that all we need to show is that for $1_G \neq x \in N_1, C_{G_1}(x) \leq N_1$, so by Theorem 3.2.7 we can conclude that G_1 is Frobenius. Since G is Frobenius, for $1_G \neq x \in N_1 \leq N$, we have that $C_G(x) \leq N$ by Proposition 3.2.1. Now $G_1 \leq G$ implies that $C_{G_1}(x) \leq C_G(x) \leq N$. Also since $C_{G_1}(x) \leq G_1$, we have $C_{G_1}(x) \leq G_1 \cap N = N_1H_1 \cap N = N_1$. Hence, by Theorem 3.2.7, G_1 is a Frobenius group. ■

Proposition 3.2.10. *([12]). Let* G *be a Frobenius group with kernel* N *and let* K *be a subgroup of* G*. Then one of the following must occur.*

1. $K \subseteq N$.
2. $K \cap N = \{1_G\}$.
3. K *is a Frobenius group with kernel* $N \cap K$.

PROOF: (1) Let $M = N \cap K$ and assume that neither (1) nor (2) holds. Then $M \neq \{1_G\}$ and $M \neq K$. We have that $M \trianglelefteq K$. Now let $1_G \neq x \in M$, then $x \in N$, so by Proposition 3.2.1, $C_G(x) \subseteq N$. Also $C_K(x) \subseteq C_G(x) \subseteq N$ and $C_K(x) \subseteq K$. So $C_K(x) \subseteq N \cap K = M$. Hence, by Theorem 3.2.7, K is Frobenius with kernel $N \cap K$. ■

Proposition 3.2.11. *([12]). Let* $K \neq \{1_G\}$ *be a subgroup of* G *such that* $K \neq N_G(K)$ *and* $C_G(x) \subseteq K \ \forall\ 1_G \neq x \in K$*. Then* $N_G(K)$ *is a Frobenius group with Frobenius kernel* K*.*

PROOF: It is clear that $K \trianglelefteq N_G(K)$. If $1_G \neq x \in K$ then by hypothesis $C_{N_G(K)}(x) \subseteq C_G(x) \subseteq K$. So by Theorem 3.2.7, $N_G(K)$ is a Frobenius group with kernel K. ■

Proposition 3.2.12. *([6]). Suppose* G *is a Frobenius group with kernel* N *and complement* H*, and that* $K \leq N; K \neq N$ *and* $K \trianglelefteq G$*. Then* G/K *is Frobenius with kernel* N/K*.*

PROOF: By the Correspondence Theorem since $K \leq N$ and $K \trianglelefteq G, N/K \trianglelefteq G/K$. The index of N/K in G/K is $[G/K : N/K]$ and

$$\begin{aligned}
[G/K : N/K] &= |G/K|/|N/K| \\
&= (|G|/|K|) \times (|K|/|N|) \\
&= |G|/|N| \\
&= [G : N] = |H| \text{ by Proposition 2.2.4.}
\end{aligned}$$

The order of N/K is $|N/K| = |N|/|K| = \frac{[G:H]}{|K|}$ by Proposition 2.2.4. Now $[G : H]$ and $|H|$ are relatively prime since H is a Hall subgroup of G. Let $\frac{[G:H]}{|K|} = n$ and $|H| = m$.

Now if p is a prime and $p\big|m$ and $p\big|n$ then $p\big||H|$ and $p\big|[G:H]$. But this is a contradiction because H is a Hall subgroup. Therefore $(m,n)=1$ which implies that N/K is a normal Hall subgroup of G/K. We show now that $N/K=\{xK\in G/K:(xK)^n=1_{G/K}\}$. If $1_{G/K}\neq xK\in G/K$ then since $\big|N/K\big|=n, xK\in N/K$ implies that $(xK)^n=1_{G/K}$. Let $1_{G/K}\neq xK\in G/K$ and $(xK)^n=1_{G/K}$. Suppose now that $(xK)\notin N/K$. Then $x\notin N$ and since G is Frobenius either $x\in H$ or $x=n'h$ for some $1_G\neq n'\in N$ and $1_G\neq h\in H$. If $x\in H$ then since $|H|=m, x^m=1_G$. So $x^mK=K$ implies that $(xK)^m=1_{G/K}$ and hence that $o(xK)\big|m$. But this a contradiction since $o(xK)\big|n$ and $(m,n)=1$. If $x=n'h$ then $x^n=n''h^n$ for some $n''\in N$. (See the proof of Theorem 3.2.8, part(2)). Now $(xK)^n=1_{G/K}$ implies that $x^nK=K$ and hence that $x^n\in K\leq N$. Since $x^n\in N$, $n''h^n\in N$ which implies that $h^n\in N$. Since $h\neq 1_G$, $h^n=1_G$ implies that $o(h)\big|n$ which is a contradiction since $o(h)\big|m$ and $(m,n)=1$. Thus we must have that $N/K=\{xK\in G/K:(xK)^n=1_{G/K}\}$ and by part(1) of Theorem 3.2.8 , G/K is Frobenius with kernel N/K. ■

Proposition 3.2.13. *([6]). If G is abelian and the only characteristic subgroups in G are $\{1_G\}$ and G, then G is elementary abelian.*

PROOF: Let p be a prime divisor of $|G|$. Then $H=\{x\in G:x^p=1_G\}\leq G$. Note first that $H\neq\{1_G\}$ since by **Cauchy's Theorem** there exists $x\in G$ such that $o(x)=p$. So $x\in G$ and $x^p=1_G$ implies that $x\in H$. Also $1_G\in H$ since $1_G^p=1_G$. If $\alpha,\beta\in H$ then $\alpha^p=\beta^p=1_G$.
So

$$\begin{aligned}\alpha^p=\beta^p \quad &\Rightarrow\ \beta^{-p}\alpha^p=1_G\Rightarrow\ (\beta^{-1})^p(\alpha^p)=1_G\\ &\Rightarrow\ (\beta^{-1}\alpha)^p=1_G\ \text{ (since G is abelian)}\end{aligned}$$

So $\beta^{-1}\alpha\in H$ and therefore $H\leq G$.
Let ϕ be any automorphism of G. Since $H\leq G$ and ϕ is an isomorphism, $\phi(H)=\{\phi(h):h\in H\}$ is a subgroup of G. We show that H is a characteristic subgroup of G. Let $h\in H$. Then $h^p=1_G$. Now $\phi(1_G)=1_G$ implies that $\phi(h^p)=1_G$ and hence that $[\phi(h)]^p=1_G$. So $\phi(h)\in H$ and $\phi(H)\leq H$. Since this is true for any automorphism ϕ of G, it is true for ϕ^{-1}. So $\phi^{-1}(H)\leq H$ which implies that $\phi[\phi^{-1}(H)]\leq\phi(H)$ and hence that $H\leq\phi(H)$. So $\phi(H)=H\ \forall\phi\in Aut(G)$. Therefore H is characteristic in G. Since $H\neq\{1_G\}$, and the only characteristic subgroups of G are $\{1_G\}$ and G, we have that $H=G$. So $x^p=1_G\ \forall x\in H$ implies that $x^p=1_G\ \forall x\in G$. By definition this implies that G is an elementary abelian group. ■

Note 3.2.3. A Frobenius group G is said to be **minimal** if no proper subgroup of G is Frobenius.

Remark 3.2.1. Let H be a Frobenius complement of a Frobenius group G. Let $\{1_G,q_1,q_2,\ldots\ldots,q_{n-1}\}$ be a left transversal for H in G. Then $G=H\cup q_1H\cup\ldots\ldots\cup q_{n-1}H$ where $[G:H]=n$. It follows then that the conjugates of H by elements of G are $\{H,H^{q_1},H^{q_2},\ldots\ldots,H^{q_{n-1}}\}$ and no two of them coincide since if $H^{q_1}=H^{q_2}$ then $q_1Hq_1^{-1}=q_2Hq_2^{-1}$ which implies that $(q_2^{-1}q_1)H(q_2^{-1}q_1)^{-1}=H$ and hence that $q_2^{-1}q_1\in N_G(H)=H$ by Corollary 2.2.2.

Therefore $q_2^{-1}q_1H = H$ which implies that $q_2H = q_1H$ which is a contradiction. Also the intersection of any two distinct conjugates of H is trivial since if $x \in H^{q_1} \cap H^{q_2}$ then $x = q_1hq_1^{-1} = q_2h'q_2^{-1}$ for $h, h' \in H$. So $(q_2^{-1}q_1)h(q_2^{-1}q_1)^{-1} = h'$ which implies that $q_2^{-1}q_1 \in H$. Therefore $q_2^{-1}q_1H = H$ which implies that $q_1H = q_2H$ which is a contradiction. Hence any of the conjugates of H satisfies the condition to be a Frobenius complement. Therefore in a Frobenius group G replacing the complement H by any of it's conjugates gives us another representation of G as a Frobenius group.

Theorem 3.2.14. *([6]). If G is a minimal Frobenius group with kernel N and complement H then N is elementary abelian and H has prime order.*

PROOF: If $\{1_G\} < H_1 < H$ then $H_1 < N_G(N) = G$, so by Proposition 3.2.9, $G_1 = NH_1$ is a proper Frobenius subgroup of G contradicting the minimality of G. Hence, H must be of prime order. Let the order of $H = q$ a prime. We will show that N is elementary abelian. Let P be a Sylow p - subgroup of N and let $N' = N_G(P)$. Then $G = NN'$ by the Frattini Argument. Since $|NN'| = |G| = |NH| = |N|\,q$, we have that $q \big| |N'|$. Let $Q_1 \in Syl_q(N')$, then $|Q_1| = q$ and $Q_1 \leq N' = N_G(P)$. So $NQ_1 \leq G$ and $|NQ_1| = \frac{|N||Q_1|}{|N \cap Q_1|} = \frac{|N|q}{1} = |N|\,q$. Therefore $G = NQ_1$. Since Q_1 is also a Sylow q - subgroup of G, it is conjugate to H in G. Therefore by the Remark 3.2.1, $G = NQ_1$ is Frobenius. The minimality of G now implies that $P = N$. If $K \neq \{1_G\}$ is a **characteristic** subgroup of N, then since $N \trianglelefteq G, K \trianglelefteq G$. So applying Proposition 3.2.9 to $G = NH$, we have that $G_1 = KH$ is a Frobenius group. The minimality of G now implies that $K = N$. In particular $Z(N) = N$ which implies that N is abelian. (Since for every group G it's **centre** is characteristic in G). Also $Z(N) \neq \{1_G\}$ since a p - group has a non-trivial centre. By Proposition 3.2.13 now, N is elementary abelian. ∎

The following lemma contains some useful characterisations of Frobenius groups.

Lemma 3.2.15. *([9]). Let $N \trianglelefteq G$, $H \leq G$ with $NH = G$ and $N \cap H = \{1_G\}$. Then the following are equivalent.*

1. $C_G(n) \leq N \;\; \forall 1_G \neq n \in N$.
2. $C_H(n) = \{1_G\} \;\; \forall 1_G \neq n \in N$.
3. $C_G(h) \leq H \;\; \forall 1_G \neq h \in H$.
4. *Every $x \in G \backslash N$ is conjugate to an element of H.*
5. *If $1_G \neq h \in H$, then h is conjugate to every element of Nh.*
6. *H is a Frobenius complement in G.*

PROOF: Note first that

$$G = N \cup Nh_2 \cup Nh_3 \ldots \cup Nh_m \; ,$$

and that

$$H = \{1_G, h_2, h_3, \ldots, h_m\},$$

where the h_i are distinct.
Also

$$Nh_i = \{h_i, n_1 h_i, n_2 h_i, \ldots, n_i h_i\},$$

and each $n_i h_i$ is distinct since if $n_i h_i = n_j h_i$ then $n_i = n_j$.
Also note that

$$\begin{aligned} \forall\, g \in G, \big(Nh_i\big)^g &= g\big(Nh_i\big)g^{-1} = gN(g^{-1}g)h_i g^{-1} = (gNg^{-1})(gh_i g^{-1}) \\ &= Nh_i^g. \end{aligned} \tag{3.1}$$

This is true for any normal subgroup of a group. Since G is a semi-direct product, if $g = mh$ for $m \in N$ and $h \in H$, we have that:

$$\begin{aligned} \big(Nh_i\big)^g = Nh_i^g &= Nh_i^{mh} \quad \text{by (3.1)} \\ &= N\big(mhh_ih^{-1}m^{-1}\big) = Nm\big(hh_ih^{-1}\big)m^{-1} = Nmh'm^{-1} \quad \text{for } h' = hh_ih^{-1} \\ &= Nh'm^{-1} \quad \text{since } m \in N \\ &= Nh'm^{-1}h'^{-1}h' = Nh' \quad \text{since } h'm^{-1}h'^{-1} \in N \\ &= Nhh_ih^{-1} = Nh_i^h = \big(Nh_i\big)^h. \quad \text{by (3.1)} \end{aligned} \tag{3.2}$$

Now we prove the lemma.
$(6) \Rightarrow (1)$
H is a Frobenius complement by definition implies that G is a Frobenius group so (1) then follows by Proposition 3.2.1.

$(1) \Rightarrow (2)$
Let $g \in C_H(n)$, where $1_G \neq n \in N$. Then $gn = ng$ and this implies that $g \in C_G(n) \leq N$. Hence, $g \in H \cap N = \{1_G\}$ and (2) follows.

$(2) \Rightarrow (3)$
Let $g \in C_G(h)$. Then $gh = hg$. Assume that $n \neq 1_G$. Let $g = nh'$ for $n \in N$ and $h' \in H$.
Then

$$\begin{aligned} h &= ghg^{-1} = h^g \Rightarrow h = h^{nh'} \\ &= nh'(h)h'^{-1}n^{-1} = n(h'hh'^{-1})n^{-1} \\ &= nh''n^{-1} = h''^n \quad \text{where } h'' = h'hh'^{-1} \in H. \end{aligned}$$

So

$$\begin{aligned} h &= h''^{n} \Rightarrow nh'' = hn \Rightarrow nh''h^{-1} = hnh^{-1} \\ \Rightarrow nh''' &= n' \text{ where } h''h^{-1} = h''' \in H, \text{ and } hnh^{-1} = n' \in N \\ \Rightarrow h''' &= n^{-1}n' \in N \end{aligned} \tag{3.3}$$

So $h''' \in H \cap N$ and this implies that $h''' = 1_G$ and $n = n'$. Now $hnh^{-1} = n' = n$ implies that $h \in C_G(n) = 1_G$. This contradicts the assumption given in (2) since $h \neq 1_G$. Hence we must have $n = 1_G$ and hence $g = h' \in H$. Thus $C_G(g) \leq H$.

(3) $\Rightarrow$ (4)
We have

$$G = N \cup Nh_2 \cup Nh_3 \ldots\ldots \cup Nh_m.$$

So

$$G \backslash N = Nh_2 \cup Nh_3 \cup Nh_4 \ldots\ldots \cup Nh_m.$$

Since by assumption we have

$$C_G(h_i) \subseteq H \ \forall i = 2, \ldots\ldots, m,$$

we deduce that

$$C_G(h_i) = C_H(h_i) \ \forall i = 2, \ldots\ldots, m,$$

Now

$$\big|[h_i]_H\big| = \big[H : C_H(h_i)\big], \tag{3.4}$$

and

$$\big|[h_i]_G\big| = \big[G : C_G(h_i)\big]. \tag{3.5}$$

Now dividing equation (3.5) by equation (3.4) gives $\frac{\left|[h_i]_G\right|}{\left|[h_i]_H\right|} = \frac{|G|}{|H|}$.

This now implies that

$$\big|[h_i]_G\big| = |N| \times \big|[h_i]_H\big| \ \forall i = 2, \ldots\ldots, m. \tag{3.6}$$

Also if h_i is not conjugate to h_j in H then h_i is not conjugate to h_j in G. From equation (3.6) we have that:

$$\begin{aligned} \Big|\bigcup_{1_G \neq h \in H} [h]_G\Big| &= |N| \times \Big|\bigcup_{1_G \neq h \in H} [h]_H\Big| \text{ (h here is a class representative in } H) \\ &= |N| \times \big(|H| - 1\big) \\ &= |N| \times |H| - |N| \\ &= |G| - |N|. \end{aligned} \tag{3.7}$$

By equation (3.7) now we have that

$$G = N \bigcup \{ \bigcup_{1_G \neq h \in H} [h]_G \}.$$

This implies that if $x \in G \backslash N$ then $x \in [h]_G$ for some $h \in H$ proving (4).

(4) $\Rightarrow$ (5)

Since by (4) nh_i is conjugate to h_j in G for some $h_j \in H$, there exists $g \in G$ with $g = mh$ for $m \in N, h \in H$ such that $(nh_i)^g = h_j$. Now $(nh_i)^g \in (Nh_i)^g = Nh_i^g = Nh_i^h$ by equation 3.1.
So $(nh_i)^g = n'h_i^h$ for some $n' \in N$ implies that $h_j = n'h_i^h$, and hence $n' \in N \cap H = 1_G$. Therefore $h_j = h_i^h$ and hence h_j is conjugate to h_i. Thus nh_i is conjugate to h_i and since this is true for all $n \in N$ the result follows.

(5) $\Rightarrow$ (6)

We need to show that $H \cap H^g = \{1_G\}$ $\forall g \in G \backslash H$. So suppose now that $H \cap H^g \neq \{1_G\}$ with $g \in G, g = nh, n \neq 1_G$. So there is a $1_G \neq h' \in H$ such that $h'^g \in H$. Let $h'^g = h_0$ for some $h_0 \in H$. Now by (5) we have that $h'^g \in Nh'$. So $h'^g = nh'$ for some $n \in N$. Thus we have that $h_0 = h'^g = nh'$. The uniqueness of the representation of each $g \in G$ now implies that $n = 1_G$ which is a contradiction. This now completes the proof. ■

We prove the following Lemma by using Lemma 3.2.15.

Lemma 3.2.16. *Let* G *be a frobenius group with kernel* K *and complement* H. *Then any two non-identity elements of* H *conjugate in* G *are already conjugate in* H.

PROOF: If h_1 is conjugate to h_2 in G ,then there exists $1_G \neq g \in G$ with $g = mh$ where $m \in N, h \in H$ such that $h_1^g = h_2$. Thus

$$\begin{aligned}
gh_1g^{-1} = h_2 \quad &\Rightarrow \quad (mh)h_1(h^{-1}m^{-1}) = h_2 \\
&\Rightarrow \quad m(hh_1h^{-1})m^{-1} = h_2 \;\Rightarrow\; mh'm^{-1} = h_2 \text{ where } h' \in H \\
&\Rightarrow \quad mh'm^{-1}(h'^{-1}h') = h_2 \;\Rightarrow\; m(h'm^{-1}h'^{-1})h' = h_2 \\
&\Rightarrow \quad m'h' = h_2 \text{ where } m' = m(h'm^{-1}h'^{-1}) \in N \\
&\Rightarrow \quad m'h' = 1_G.h_2 \;\Rightarrow\; m' = 1_G \text{ by the uniqueness of the representation of } g \in G \\
&\Rightarrow \quad mh'm^{-1}h'^{-1} = 1_G \;\Rightarrow\; mh' = h'm \;\Rightarrow\; m \in C_G(h') \subseteq H \;\Rightarrow\; m = 1_G.
\end{aligned}$$

Therefore $g = 1_G.h$ which implies that $h_1^h = h_2$ and hence that h_1 is conjugate to h_2 in H. ■

Lemma 3.2.17. *([24]). If* G *is a Frobenius group with kernel* N *and* $K \trianglelefteq G$ *,then either* $K \subseteq N$ *or* $N \subseteq K$.

PROOF: Assume that $K \not\subseteq N$. Let H be a complement of N and let $x \in K \backslash N$.
First we show that $C_G(x) \cap N = \{1_G\}$. Suppose that $C_G(x) \cap N \neq \{1_G\}$ and let $y \in C_G(x) \cap N$. Since

$y \in N$, by Proposition 3.2.1 $C_G(x) \leq N$. Also $y \in C_G(x)$ implies that $xy = yx$. But this implies that $x \in C_G(y) \leq N$ which is a contradiction. Thus $C_G(x) \cap N = \{1_G\}$.
Now $NC_G(x) \leq G$ and $|NC_G(x)| \big| |G|$. So

$$|NC_G(x)|\, k = |G| \;\Rightarrow\; \frac{|C_G(x)|\,|N|}{|C_G(x) \cap N|} k = |G| \;\Rightarrow\; |N|\,|C_G(x)|\, k = |N|\,|H|\,,$$

for some $k \in \mathbb{N}$. This implies that $|C_G(x)| \big| |H|$.
Since $|C_G(x)| \big| |H|$ implies that $|C_G(x)|\, q = |H|$ for some $q \in \mathbb{N}$, and $|G| = |C_G(x)|\,|[x]|$, we have that $|G|\, q = |C_G(x)|\,[x]|\, q = |H|\,|[x]|$. This implies that $|N|\,|H|\, q = |[x]|\,|H|$ and hence that $|N|\, q = |[x]|$. Thus $|N| \big| |[x]|$. Since $[x] \subseteq K$, $|N| \big| |K|$.
Now let $|N| = p^{\alpha} z$, then $|K| = p^{\alpha} k z$ for some $k, z \in \mathbb{N}$. Also since N is a normal Hall subgroup of G, $|G| = p^{\alpha} z'$ for $z' \in \mathbb{N}$. Let $P \in Syl_p(K)$ then $P \in Syl_p(G)$. If $Q \in Syl_p(N)$ then Q is conjugate to P in G which implies that $Q = gPg^{-1}$ for some $g \in G$. So $P = g^{-1}Qg \leq N$ (since $N \trianglelefteq G$). So every Sylow p - subgroup of K such that $p \big| |N|$ is contained in N. Hence, $N \subseteq K$. ■

Note 3.2.4. Let G be a Frobenius group with kernel N and complement H. If $K \trianglelefteq G$ such that $N \subset K$ then $K = N : (H \cap K)$ is a Frobenius group since, $N \subset K$, $H \cap K \neq \{1_G\}$ by Proposition 2.2.4 and by Flavell [5] (Corollary 3.1), $K = (K \cap N) : (H \cap K)$ is a Frobenius group which implies that $K = N : (H \cap K)$ is Frobenius.

Theorem 3.2.18. *([6]). If G is a Frobenius group with kernel N and complement H, then no subgroup of H is Frobenius.*

PROOF: Suppose the result is false. Let G be a counter example of minimal order. Then H itself is Frobenius and minimal. (Since if H has a subgroup H_1 say which is Frobenius, then H will be another counter example of order less than the order of G which is a contradiction). Hence, the kernel K of H is elementary abelian and it's complement Q is cyclic of prime order. We want to show that N is an elementary abelian p - group.
Suppose that p is a prime that divides the order of N. Let P be a Sylow p - subgroup of N and let $N' = N_G(P)$. Then $G = NN'$ by the Frattini argument. So $|G| = \frac{|N||N'|}{|N \cap N'|} = |H|\,|N|$. This implies that $|H| \big| |N'|$. Now $N \cap N' \trianglelefteq N'$ and since $G/N = NN'/N \cong N'/N \cap N'$, $[G : N] = [N' : N \cap N']$. Therefore $\Big([N' : (N \cap N')], |N \cap N'|\Big) = 1$ which implies that $N \cap N'$ is a normal Hall subgroup of N', since if there is a prime q such that $q \big| |N \cap N'|$ and $q \big| [N' : (N \cap N')]$ then $q \big| |N|$ and $q \big| [G : N]$ which is a contradiction since N is a normal Hall subgroup.
By the Schur Zassenhaus Theorem, $N \cap N'$ has a complement L. Therefore $N' = (N \cap N')\, L$ and since $H \cong G/N = N'/N \cap N' \cong L$, $|L| = |H| = [G : N]$. Thus $G = NL$ and L is minimal Frobenius. Since $L \leq N' = N_G(P)$ and $P \trianglelefteq N_G(P), PL \leq N_G(P) \leq G$. We show that PL is Frobenius.
First we show that if $1_G \neq x \in P$, then $C_{PL}(x) \leq PL \cap N$. Now $C_{PL}(x) \subseteq PL$ and since $x \in N$ $(P \subseteq N)$, by Proposition 3.2.1, $C_{PL}(x) \subseteq C_G(x) \subseteq N$. Thus $C_{PL}(x) \subseteq PL \cap N$. We now show that $PL \cap N = N$, and PL is Frobenius will follow from Proposition 3.2.1. Now $P \subseteq PL \cap N$ since $P \subseteq PL$ and $P \subseteq N$. Let $1_G \neq x' \in PL \cap N$, then $x' \in PL$ implies $x' = zl$ with $z \in P$ and $1_G \neq l \in L$. Since

$x' \in N, zl = n$ for some $n \in N$. So $l = z^{-1}n \in N$. Therefore $l \in N$ and $l \in N'$ $(L \leq N')$. Therefore $l \in N \cap N'$. But $(N \cap N') \cap L = \{1_G\}$ since L is the complement of $(N \cap N')$ in N'. Therefore $l = 1_G$ which is a contradiction. Therefore $x' = z$ and hence $x' \in P$. So $PL \cap N \subseteq P$ and $PL \cap N = P$. Now PL is a Frobenius group follows by Proposition 3.2.1. Since PL is Frobenius and we already have that L is minimal Frobenius, PL is another counter example of order less than the order of G. Thus we must have that

$$|PL| = |G| \Rightarrow |PL| = |P|\,|L| = |N|\,|H| \Rightarrow |P| = |N| \Rightarrow P = N.$$

Therefore N is a p group.
Furthermore by Proposition 3.2.12 no non-trivial normal subgroup of G is properly contained in N, because if $R \leq N, R \neq N$ and $R \trianglelefteq G$, then by Proposition 3.2.12, G/R is Frobenius with kernel N/R. Say the complement of N/R is W/R where $W \leq G$. Then by the Third Isomorphism Theorem we have: $W/R \cong \frac{G/R}{N/R} \cong G/N \cong H$. Therefore W/R is Frobenius. But G is the counter example of minimal order and G/R is another counter example and since $\left|G/R\right| < |G|$, we have a contradiction. Therefore $R = \{1_G\}$ which implies that no non-trivial normal subgroup of G is properly contained in N. In particular, since N is a p - group, $\{1_G\} \neq Z(N) = N$ which implies that N is abelian and since N has no non-trivial proper characteristic subgroups, by Proposition 3.2.13, N is an elementary abelian p - group.
We may now view N (written additively) as a vector space over $\mathbb{Z}_p$. The action of H on N by conjugation is a $\mathbb{Z}_p$ representation T, of H on N. First we will show that T is faithfull.
If ρ^h is the automorphism of N which represents conjugation by $h \in H$, then $\rho^h(\alpha) = h\alpha h^{-1}\ \forall \alpha \in N$. For $h \in H$, $T(h) = \rho^h$ and $T(h) = 1$ implies that $\rho^h(\alpha) = \alpha\ \forall \alpha \in N$. Hence $h\alpha h^{-1} = \alpha$ implying that $h\alpha = \alpha h$ and hence $h \in C_G(\alpha)$. Since this is true for all $\alpha \in N, h \in C_G(N)$ and hence $h = 1_G$, implying that T is faithfull. The only T - **invariant** subspaces are 0 and N since N has no other subgroups normal in G. Therefore T is irreducible [6].
Choose a finite extension $\mathbb{F}$ of $\mathbb{Z}_p$ that is a splitting field for both H and K. (Since char($\mathbb{Z}_p$) does not divide the order of H, there is a finite extension $\mathbb{F}$ of $\mathbb{Z}_p$ that is a splitting field for H). Then $T^{\mathbb{F}} \sim S_1 \bigoplus S_2 \bigoplus \ldots\ldots \bigoplus S_l$ with each S_i absolutely irreducible. Say that $S = S_1$ acts on the $\mathbb{F}$ - subspace V of $N^{\mathbb{F}}$. Restricting the action of T to K, write $V = V_1 \bigoplus V_2 \bigoplus \ldots\ldots \bigoplus V_k$ with each V_i an irreducible K - **invariant** subspace. But K is abelian, so each V_i is one dimensional, and if $x \in K$ then $\widehat{S}(x)$ is a diagonal matrix (where $\widehat{S}$ is the matrix representation of S). Combine the subspaces V_i so that $V = W_1 \bigoplus W_2 \bigoplus \ldots\ldots \bigoplus W_u$ where $\widehat{S}(x)$ restricts to a scalar matrix on each W_i, $\forall x \in K$ with different scalars for some x if $i \neq j$. Observe that if $v \in V$ and $\widehat{S}(x)v = \lambda_x v$ for $\lambda_x \in \mathbb{F}$ and $\forall x \in K$, then $v \in W_i$ for some i. Say $Q = \langle y \rangle$ and choose $v \in W_i$. For each $x \in K$ we have $\widehat{S}(x)\widehat{S}(y)v = \widehat{S}(y)\widehat{S}(y^{-1}xy)v = \widehat{S}(y)\lambda_{x^y}v = \lambda_{x^y}\widehat{S}(y)v$. So $\widehat{S}(y)v \in W_j$ for some j. Thus for each i there is some $j = j(i)$ such that $\widehat{S}(y)W_i = W_{j(i)}$. So Q acts as a permutation group on the set $\{W_i\}$. Now Q has no fixed points since a fixed point would be both Q - **invariant** and K - **invariant** as a subspace, hence H - **invariant**, whereas S is irreducible on V. Since Q is cyclic of prime order q, $\widehat{S}(y)$ permutes each Q orbit in $\{W_i\}$ cyclically as a q cycle. Relabel if

necessary so that $\widehat{S}(y)W_1 = W_2, \widehat{S}(y)W_2 = W_3, \ldots\ldots, \widehat{S}(y)W_{q-1} = W_q$. Choose $w \neq 0 \in W_1$ and set $v = w + \widehat{S}(y)w + \widehat{S}(y^2)w + \ldots\ldots$. Then $\widehat{S}(y)v = v \neq 0$, so 1 is an eigenvalue of $\widehat{S}(y)$, hence of $T(y)$. This means that for some $z \in N$ with $z \neq 1$ we have $T(y)z = z^y = z$. But then $y \in C_G(z)$, contradicting Proposition 3.2.1 and proving the theorem. ■

Theorem 3.2.19. *([24]). The Frobenius kernel of a Frobenius group is unique.*

PROOF: Let G be a Frobenius group with Frobenius kernels N and N_1 with H a complement of N. By Lemma 3.2.17, without any loss of generality, we may assume that $N \subseteq N_1$. Let $K = H \cap N_1 \trianglelefteq H$. Since $N \subseteq N_1$ and $N \trianglelefteq G, N \trianglelefteq N_1$. Therefore $NK \leq N_1$. Let $n_1 \in N_1$. Then $n_1 = nh$ for $1_G \neq n \in N$ and $1_G \neq h \in H$, (since $n_1 \in G$). So $h = n^{-1}n_1 \in N_1$ (since $n^{-1} \in N \subseteq N_1$ and $n_1 \in N_1$). So $h \in N_1 \cap H = K$. Since $h \in K, n_1 = nh \in NK$ which implies that $N_1 \subseteq NK$. Therefore $N_1 = NK$. If $1_G \neq x \in K$, then since N_1 is a Frobenius kernel, by Proposition 3.2.1, $C_G(x) \leq N_1$. Also $x \in K$ implies that $x \in H$ and by Lemma 3.2.15, for $1_G \neq x \in H, C_G(x) \subseteq H$. So we have for $1_G \neq x \in K, C_G(x) \subseteq H \cap N_1 = K$. Since $C_H(x) = C_G(x) \subseteq K$ for $1_G \neq x \in K$, we have that H is a Frobenius group with kernel K by Theorem 3.2.7. But this contradicts Theorem 3.2.18. Hence we have $N = N_1$. ■

Proposition 3.2.20. *([6]). Suppose that G is Frobenius with kernel N and complement H, and that p, q are primes in $\mathbb{N}$, not necessarily distinct. If $K \leq H$ and $|K| = pq$ then K is cyclic.*

PROOF: Suppose the result is false. Let G be a counter example of minimal order. So G is Frobenius with complement H and kernel N and there exists $K \leq H$ such that $|K| = p^2$ or pq and K is not cyclic. Since $K \leq H$ with $|K| = p^2$ or pq, by Proposition 3.2.9, NK is Frobenius. Since K is not cyclic, NK is another counterexample of order less than the order of G, which is a contradiction. Therefore $G = NH = NK$ and hence, $H = K$. Therefore $|H| = p^2$ or pq.
Also, if $N_1 \trianglelefteq G$ and $N_1 \leq N$, then N_1H is a Frobenius group by Proposition 3.2.9. But then N_1H is another counterexample of order less than the order of G, which is a contradiction. Thus N is a minimal normal subgroup of G.
If $|H| = pq$, then since H is not cyclic, by Example 2 in the next chapter it is Frobenius. But this contradicts Theorem 3.2.18. Hence, H is of order p^2 and therefore abelian. If $R \in Syl_r(N)$ for r a prime and $N' = N_G(R)$ then $G = NN'$ by the Frattini argument. So $|G| = |N|\,|H| = \frac{|N||N'|}{|N \cap N'|}$. Thus $|H| \mid |N'|$. Now $H \in Syl_p(G)$ since $|G| = |H|\,|N| = p^2m$ where $(m, p) = 1$. Since $|H| \mid |N'|$, some conjugate of H is in N'. Without loss of generality, say H itself is in N'. Applying Proposition 3.2.9 to $G = NN'$, we have that RH is Frobenius. The minimality of G now implies that $G = RH$.
Now since $R \trianglelefteq G$ and $R \leq N$ the minimality of N now implies that $R = N$. Since N is a minimal normal r - group, $N = N_1 \times N_2 \times N_3 \times \ldots\ldots \times N_l$ where the N_i are isomorphic simple groups. This implies that $N = \mathbb{Z}_r \times \mathbb{Z}_r \times \ldots\ldots \times \mathbb{Z}_r$, and thus that N is an elementary abelian r - group.
As in the proof of Theorem 3.2.18, the action of H on N by conjugation determines a faithful irreducible $\mathbb{Z}_r$ representation T of H on N. So choose a finite extension $\mathbb{F}$ of $\mathbb{Z}_r$ that is a splitting field for H so $T^{\mathbb{F}} \sim T_1 \bigoplus T_2 \bigoplus \ldots\ldots \bigoplus T_k$, with $\mathrm{deg} T_i = 1$ $\forall i$. Thus for an appropriate choice of

basis, $\widehat{T}(x)$ is diagonal (where $\widehat{T}$ is the matrix representation of T) $\forall x \in H$, and each diagonal entry of $\widehat{T}_i(x)$ is a pth root of unity in $\mathbb{F}$ (Since $\forall 1_G \neq x \in H$, $[T(x)]^p = I$ (identity matrix) and $T(x)$ is a diagonal matrix. If $\widehat{T}(x) = diag(\epsilon_1, \epsilon_2, \ldots\ldots \epsilon_m)$ where $m = dim\ N$ and $\epsilon_i \in \mathbb{F}$ then since $[T(x)]^p = I$, we have that $\epsilon_i^p = 1$ implies that ϵ_i's are pth roots of unity).
There are at most p distinct pth roots of unity in $\mathbb{F}$, and since $|H| = p^2$, there exists $x, y \in H, x \neq y$ such that $\widehat{T}_1(x) = \widehat{T}_1(y)$, or $\widehat{T}_1(xy^{-1}) = 1_G$. If $z = xy^{-1}$, then this implies that 1 is an eigenvalue of $T(z)$, so there is an eigenvector $u \in N, u \neq 0$ (multiplicatively $u \neq 1$), and $T(z)u = u$ or $u^z = u$. Thus $1_G \neq z \in C_G(u) \cap H$, contradicting Proposition 3.2.7 ■

Lemma 3.2.21. *([6]). Suppose that* $|G| = p^m$ *for some* p *and that* G *has a unique subgroup of order* p. *Then* G *is either cyclic or generalized quaternion.*

PROOF: See Grove [6] page 93. ■

Proposition 3.2.22. *([6]). Suppose* G *is a Frobenius group with complement* H. *Let* $P \in Syl_p(H)$ *then*

1. *If* $p = 2$, *then* P *is cyclic or generalized quaternion.*
2. *If* $p \neq 2$, *then* P *is cyclic.*

PROOF: By Proposition 3.2.20, P contains no noncyclic subgroup of order p^2. Since $Z(P)$ is nontrivial, take $K \leq Z(P)$ with $|K| = p$. If there is another subgroup L of P with $|L| = p$ and $L \neq K$ then $|KL| = \frac{|K||L|}{|K \cap L|} = p^2$. Since KL is abelian, by the Basis Theorem $KL \cong \mathbb{Z}_p \times \mathbb{Z}_p = E_{p^2}$. Therefore KL is noncyclic since E_{p^n} is cyclic if and only if n equals one. But this contradicts Proposition 3.2.20. Thus P has only one subgroup of order p and by Lemma 3.2.21 the result follows. ■

Note 3.2.5. A finite group G is nilpotent if and only if it is the direct product of it's Sylow subgroups.

Proposition 3.2.23. *Frobenius kernels are nilpotent.*

PROOF: See Passman [20] pg 184. ■

The result in Proposition 3.2.23 implies that Frobenius kernels are solvable since every finite nilpotent group is solvable.

3.3 The Center, Commutator and Frattini Subgroups of a Frobenius Group

We describe here briefly the Center, Commutator and the Frattini subgroups of a Frobenius group.

3.3.1 The Center

Lemma 3.3.1. *The center of a Frobenius group is trivial.*

PROOF: Let G be a Frobenius group. Now $Z(G) \leq C_G(x)$ for $x \in G$. Since $C_G(x) \leq N$ $\forall x \in N$, $Z(G) \leq N$. Suppose now that $1_G \neq x \in Z(G)$, then since $Z(G) \leq N, x \notin H$. (Since $H \cap N = \{1_G\}$). Since G is Frobenius, $H^x \cap H = \{1_G\}$ $\forall x \in G \backslash H$. But $x \in Z(G)$ implies that $H^x = H$ which is a contradiction. ■

3.3.2 The Commutator Subgroup

Let G be a Frobenius group with kernel N and complement H.

1. By Lemma 3.2.15, part (5), for all $n \in N$, there exists $g \in G$ such that $h^g = nh$. Hence
$$\begin{aligned} h^g = ghg^{-1} = nh &\Rightarrow ghg^{-1}h^{-1} = n \\ &\Rightarrow [g,h] = n \Rightarrow N \subseteq G'. \end{aligned}$$

2. Also if the complement H has prime order (and hence abelian), then $H \cong G/N$ is abelian and $N \trianglelefteq G$ implies that $G' \subseteq N$. So by (1) above we have that $N = G'$.

3.3.3 The Frattini Subgroup

Let G be a group and let
$$\phi(G) = \bigcap_{M \in \mathcal{M}} M,$$
where M is a **maximal subgroup** of G and $\mathcal{M}$ is the collection of all maximal subgroups of G. Then $\phi(G)$ is called the **Frattini Subgroup** of G. If $G \neq \{1_G\}$ and G is finite, then G certainly has at least one maximal subgroup. Every proper subgroup of G lies in a maximal subgroup. Since any automorphism of G sends a maximal subgroup into a maximal subgroup, the set $\mathcal{M}$ is invariant by any automorphism, and so is $\phi(G)$. This shows that $\phi(G)$ is a characteristic subgroup and since characteristic subgroups are normal, we have that $\phi(G) \trianglelefteq G$. Now if $N \trianglelefteq G$ with G finite, then $N \leq \phi(G)$ if and only if there is no proper subgroup H of G such that $G = NH$ (see Rodrigues [22]). Now if G is a Frobenius group, then since by definition the complement H is a proper subgroup of G, the above result and the result of Lemma 3.2.17 implies that $\phi(G) \leq N$.

Note 3.3.1.
1. If G is a Frobenius group and the order of G is odd, then by the Feit Thompson Theorem, G is solvable.
2. If the complement H of a Frobenius group G is solvable, then G is solvable. (Since, H solvable implies that G/N is solvable. By Proposition 3.2.23, N the Frobenius kernel is solvable. Since G/N is solvable, and $N \trianglelefteq G$ is solvable, G is solvable (see Theorem 4.2.3 in Moori [16])).

3. If the complement H of a Frobenius group G has odd order, then G is solvable. This follows from the Feit Thompson Theorem and (2) above.

4

Examples of Frobenius Groups

We list here with proof some Examples of Frobenius groups. Also we include a list of Frobenius groups of small order (up to 32).

4.1 Examples

Example 1 : The Dihedral Group D_{2q} where q is odd is a Frobenius group.
Let $G = D_{2q}$ with q odd. Then

$$\begin{aligned} G &= \langle a, b : a^q = b^2 = 1_G, bab = a^{-1} \rangle \\ &= \{1_G, a, a^2, \ldots\ldots, a^{q-1}, b, ab, \ldots\ldots, a^{q-1}b\}. \end{aligned}$$

Now $o(a) = q$ and $o(b) = 2$. Let $\langle a \rangle = N$ and $\langle b \rangle = H = \{1_G, b\}$. Since N has index 2 in G, it is normal in G. Now $N \trianglelefteq G, H \leq G$, so $NH \leq G$ and $|NH| = \frac{|N||H|}{|N \cap H|} = 2q$. Therefore $G = N : H$. This implies that H is the complement of N in G. To show that D_{2q} is a Frobenius group, we must show that H is a Frobenius complement in G. We just need to show that $H \cap H^x = \{1_G\} \; \forall x \in G \backslash H$.
Now $x \in G$ implies that $x = a^k$ or $x = a^k b$ for $0 < k \leq q-1$. If $x = a^k$, then $H^x = \{1_G, a^k b a^{-k}\}$. But since $ba^k = a^{-k}b \; \forall k \in \mathbb{N}$, $H^x = \{1_G, a^k(a^k b)\} = \{1_G, a^{2k}b\}$. Suppose now that $H \cap H^x \neq \{1_G\}$ for some $x \in G \backslash H$. Then $H \cap H^x = \{1_G, b\}$. Therefore $a^{2k}b = b$ implies that $a^{2k} = 1_G$. So $q \big| 2k$ and $q \big| k$. Therefore $qk' = k$ for some $k' \in \mathbb{N}$. Hence, $x = a^k = a^{qk'} = 1_G$ which is a contradiction. If $x = a^k b$ then $H^x = \{1_G, (a^k b) b (a^k b)^{-1}\} = \{1_G, a^k b a^{-k}\}$. So by the argument used above we get the contradiction $x = 1_G$. Therefore $H \cap H^x = \{1_G\} \; \forall x \in G \backslash H$. This implies that H is a Frobenius complement in D_{2q}.

Example 2 : If p and q are primes and G is a non-abelian group of order pq, then G is Frobenius. We can assume that p and q are distinct primes, since if $p = q$ then $|G| = p^2$ and G is abelian contradicting the hypothesis. So let $|G| = pq$ with $p > q$. If $q = 2$ then $|G| = 2p$ with p odd. In this case G is either cyclic or $G = D_{2p}$. If G is cyclic then G is abelian contrary to hypothesis. If $G = D_{2p}$ then by Example 1, G is Frobenius.
So assume $|G| = pq, p > q, p \neq 2$. By Cauchy's Theorem G has an element of order p and an

element of order q. Let $a \in G$ such that $o(a) = p$, then $\langle a \rangle = P$ is a Sylow p - subgroup of G. If $b \in G$ such that $o(b) = q$, then $\langle b \rangle = Q$ is a Sylow q - subgroup of G. If n_p is the number of Sylow p - subgroups of G, then $n_p \equiv 1 (\bmod\ p)$ and $n_p | q$. So $n_p = 1$ or $n_p = q$. If $n_p = q$ then $q \equiv 1(\bmod\ p)$ and hence $q - 1 = kp$ for $k \in \mathbb{Z}$. This is not possible since $p > q$. Therefore $n_p = 1$ which implies that $P \trianglelefteq G$. If n_q is the number of Sylow q - subgroups of G, then $n_q \equiv 1(\bmod\ q)$ and $n_q | p$. So $n_q = 1$ or $n_q = p$. If $n_q = 1$, then $Q \trianglelefteq G$. Since $P \cap Q = \{1_G\}$ and $PQ \leq G$, $|PQ| = pq$ and hence $G = P : Q$. So $G \cong P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$. Therefore G is abelian contrary to hypothesis. Therefore $n_q = p$. So there are p Sylow q - subgroups each of order q in G. Now $Q \in Syl_q(G)$ implies that $xQx^{-1} \in Syl_q(G)\ \forall x \in G$. Let $Syl_q(G) = \{Q_i : 1 \leq i \leq p\}$. Since $Q_i \cap Q_j \leq Q_i$ for $i \neq j$, we have that $Q_i \cap Q_j = \{1_G\}$. Therefore $Q \cap Q^x = \{1_G\}\ \forall x \in G \backslash Q$. This implies that Q is a Frobenius complement in G. So if p and q are primes and G is a non-abelian group of order pq, then G is Frobenius. The kernel is the Sylow p - subgroup generated by the element of order p and the complement is the Sylow q - subgroup generated by the element of order q.

Note 4.1.1. (**f.p.f.automorphism**) If G is a group and if $\sigma \in Aut(G)$ fixes only the identity, then σ is called a **fixed point free** automorphism of G.

Example 3 : If H is a non-trivial fixed point free group of automorphisms of a finite group N, then a semi-direct product of N by H is a Frobenius group.
Let $H \leq Aut(N)$. Since H is a fixed point free non-trivial group of automorphisms of N, $\forall 1_G \neq h \in H$ and $\forall 1_G \neq n \in N, n^h \neq n$. Therefore $C_H(n) = \{1_G\}$ and so by Lemma 3.2.15, $G = N : H$ is Frobenius (since H is a Frobenius complement in G).

Example 4 : The semi-direct product $G = \mathbb{Z}_p : \mathbb{Z}_{p-1}$ for p a prime is Frobenius.
Firstly the Aut$(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$. Let $\mathbb{Z}_p = \langle a \rangle$. Each $\alpha \in Aut(\mathbb{Z}_p)$ is determined by $\alpha(a)$. Therefore Aut$(\mathbb{Z}_p) = \{\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{p-1}\}$ where we define $\alpha_i(a) = a^i$ for $i = 1, 2, \ldots\ldots, p-1$. Let $\mathbb{Z}_p^*$ be the multiplicative group of non-zero elements of $\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$.
Define

$$\psi : Aut(\mathbb{Z}_p) \mapsto \mathbb{Z}_p^* \text{ by } \psi(\alpha_i) = \bar{i}$$

Then ψ is an automorphism so that Aut$(\mathbb{Z}_p) \cong \mathbb{Z}_p^*$. Since the non-zero elements of a finite field is a cyclic group, Aut$(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$. We just need to show now that Aut$(\mathbb{Z}_p)$is fixed point free and the result will then follow from Example 3 above. We have defined $\alpha_i : \mathbb{Z}_p \to \mathbb{Z}_p$ by $\alpha_i(a) = a^i$ with $1 \leq i \leq p-1$. If $i = 1$ then $\alpha_1(a) = a$ which implies that $\alpha_1(a^i) = a^i\ \forall\ 1 \leq i \leq p-1$. This implies that $\alpha_1 = 1_{Aut(\mathbb{Z}_p)}$. So if $\alpha_i(a) = a^i$ with $2 \leq i \leq p-1$ then each $\alpha_i \in Aut(\mathbb{Z}_p)$ maps onto a different non-identity element of $\mathbb{Z}_p$. This implies that Aut$(\mathbb{Z}_p)$ is fixed point free. Therefore Z_{p-1} is a non-trivial fixed point free group of automorphisms of $\mathbb{Z}_p$ and by Example 3, the split extension $\mathbb{Z}_p : \mathbb{Z}_{p-1}$ is Frobenius.

Example 5 : If p is a prime, q not necessarily prime and $q | p-1$, then we write $\mathbb{F}_{p,q}$ for the group of order pq with presentation: $\mathbb{F}_{p,q} = \langle a, b | a^p = b^q = 1, b^{-1}ab = a^u \rangle$ where u is an element of

order q in $\mathbb{Z}_p^*$. Then $\mathbb{F}_{p,q}$ is Frobenius.
Let $N = \langle a \rangle$ and $H = \langle b \rangle$. Then $|\mathbf{N}| = p$ and N is cyclic. Also $|H| = q$ and since $q \big| p-1$, $(p, q) = 1$. Let $N \in Syl_p(\mathbb{F}_{p,q})$. By Sylow's theorem $n_p \equiv 1(mod p)$ and $n_p \big| q$. So $n_p = 1 + kp$ and $n_p k' = q$ for $k, k' \in \mathbb{Z}$. So $q = k'(1 + kp)$. But $q < p$. Therefore we must have that $k = 0$. So $n_p = 1$ and $N \trianglelefteq \mathbb{F}_{p,q}$. Therefore $|NH| = \frac{|N||H|}{|N \cap H|} = |N|\,|H| = pq$. So $\mathbb{F}_{p,q} = NH$ and we have that $\mathbb{F}_{p,q}$ is a split extension of N by H. We now show that $H \leq Aut(N)$ and that H is a fixed point free group of automorphisms of N and thus by Example 3 the result will follow.
Now the action of H on N is given by the relation $b^{-1}ab = a^u$ where u is an element of order q in $\mathbb{Z}_{p-1}$. Since $u \in \mathbb{Z}_{p-1}$, $u \in Aut(\mathbb{Z}_p)$. If u fixes $a \in N$ then $b^{-1}ab = a$ or $bab^{-1} = a$. The definition of the multiplication in $\mathbb{F}_{p,q}$ now gives:

$$(ab)(a'b') \quad = \quad aba'b^{-1}bb' = a(ba'b^{-1})bb' = aa'^{b}bb' = (aa')(bb') \ \text{(since } a'^{b} = a'\text{)}.$$

Hence, $N : H = N \times H$ which implies that $\mathbb{F}_{p,q}$ is abelian and hence not Frobenius. Therefore each $u \in \mathbb{Z}_{p-1}$ sends each $a \in N$ to a different , non-identity element of N. Hence, each $b \in H$ induces a fixed point free automorphism of N. Thus by Example 3 above, the semi direct product $\mathbb{F}_{p,q}$ is a Frobenius group.

Example 6 : The Alternating Group A_4 is Frobenius.
We will show that $A_4 = N : H$, where $N \cong V_4$ and $H \cong \mathbb{Z}_3$. Let $N = V_4 = \{1_G, \alpha, \beta, \gamma\}$ where $\alpha = (12)(34)$, $\beta = (13)(24)$ and $\gamma = (14)(23)$, and $H = \langle a \rangle = \langle (123) \rangle$.
Now $V_4 \trianglelefteq A_4$ since V_4 is a union of conjugacy classes of A_4. Therefore $V_4\langle (123) \rangle \leq A_4$. Also it is clear that $V_4 \cap \langle a \rangle = \{1_G\}$ so,

$$|V_4\langle (123)\rangle| = \frac{|V_4| \times |\langle (123)\rangle|}{|V_4 \cap \langle (123)\rangle|} = |V_4| \times |\langle (123)\rangle| = |A_4|$$

So $A_4 = V_4 : H$. The subgroup H generated by a 3-cycle in A_4 is a Sylow 3 - subgroup. By Sylow's Theorem there are four conjugates of H in A_4. That is, $[A_4 : N_{A_4}(H)] = 4$ which implies that $N_{A_4}(H) = H$. Therefore $H^x = H$ if and only if $x \in H$ which implies that $H^x \neq H \ \forall x \in G\backslash H$. Therefore $H^x \cap H = \{1_G\} \ \forall x \in G\backslash H$. Hence H is a Frobenius complement in A_4.

Example 7 : If F is a field, write F^* for it's multiplicative group $F\backslash\{0\}$. Denote by $Aff(F)$ the group (under function composition) of all functions $\tau_{a,b} : F \rightarrow F$ where $a \in F^*, b \in F$ and $\tau_{a,b}(x) = ax + b \ \forall x \in F$ So let $G = Aff(F) = \{\tau_{a,b} : a \in F^*, b \in F\}$. If F is finite, say $|F| = q$ then $|G| = q(q-1)$.
Suppose now that $F = \mathbb{F}_q, q > 2$. Then G acts transitively on F and each $\tau_{1,b}, b \neq 0$ has no fixed points, since if $\tau_{1,b}(x) = x$ for some $x \in F$, then $x + b = x$ implies that $b = 0$ which is a contradiction. If $a \neq 1$ then $\tau_{a,b}$ has a unique fixed point $\frac{b}{1-a}$. The **translation group** $N = \{\tau_{1,b} : b \in F\} \trianglelefteq G$. This group is isomorphic with the additive group of the field F. The subgroup $H = Stab_G(0) = \{\tau_{a,0} : a \in F^*\}$ is isomorphic with the multiplicative group of the field F. So G is a semi-direct product of N by H.
To show that G is Frobenius, we need to show that $C_G(\tau) \leq N \ \forall 1_G \neq \tau \in N$ and the result will

then follow from Theorem 3.2.7. Suppose $1_G \neq \tau \in N, \pi \in G$ and that $\pi\tau\pi^{-1} = \tau$. So $\pi \in C_G(\tau)$. Let $\tau(x) = x + a$ and $\pi(x) = bx + c$, so $\tau^{-1}(x) = x - a$ and $\pi^{-1}(x) = b^{-1}(x - c)$. Now $\pi\tau\pi^{-1} = \tau$ implies that $\pi = \tau\pi\tau^{-1}$. So

$$\pi(x) = \tau\pi\tau^{-1}(x) = \tau\pi(x - a) = \tau(bx - ab + c) = bx - ab + c + a.$$

Thus,

$$bx + c = bx - ab + c + a \;\Rightarrow\; 0 = a(1 - b).$$

Since F has no zero divisors, $a = 0$ or $b = 1$. If $a = 0$ then $\tau(x) = x$ and hence $\tau = 1$ contrary to assumption. Therefore, $b = 1$. Thus, $\pi(x) = x + c$ and $\pi \in N$. Hence, $C_G(\tau) \subseteq N$. The result now follows from Theorem 3.2.7.

4.2 Frobenius Groups of Small Order

We list here the groups of order less than 32. First we make the following notes.

Note 4.2.1.

1. Cyclic groups are not Frobenius since they are abelian and abelian groups are not Frobenius since they have a non-trivial center (see Lemma 3.3.1).

2. p groups are not Frobenius since they have a non-trivial center.

3. S_n is not Frobenius for $n > 4$. Since the only proper normal subgroup of S_n is A_n , if S_n wants to be Frobenius then it must equal a **split extension** of A_n by $\mathbb{Z}_2$ where A_n is the kernel and $\mathbb{Z}_2$ is the complement. But since the order of the kernel and the complement are relatively prime, S_n can be Frobenius if and only if $|A_n|$ is odd. This can only happen if $n = 2$ or $n = 3$. Hence, S_n is not Frobenius for $n > 4$. Now $S_1 = \{1_G\}$ and $S_2 \cong \mathbb{Z}_2$. Also $S_3 \cong D_6$ which is Frobenius by Example 1 and this is the smallest Frobenius group since the group of order six is the smallest non-abelian group (Frobenius groups are non-abelian).
   If $n = 4$ then the normal subgroups of S_4 are S_4, V_4, A_4 and $\{e\}$. Since the kernel is a non-trivial proper subgroup, only V_4 and A_4 can be kernels. However, in a Frobenius group the order of the complement divides the order of the kernel less one (see Proposition 3.2.2). This implies that neither A_4 $(2 \nmid 11)$ nor V_4 $(8 \nmid 3)$ can be a kernel in S_4. Hence S_4 is not Frobenius.

4. A_n is Frobenius if and only if $n = 4$. If $n = 2$ or $n = 3$ then $|A_n| = 1$ or 2 respectively and A_n is not Frobenius since the smallest Frobenius group is S_3. Since A_n is simple if $n \geq 5$, it can't be Frobenius for $n \geq 5$. This leaves $n = 4$ and A_4 is Frobenius by Example 6.

4.2.1 Frobenius Groups of Order < 32

1. The prime integers between 1 and 32 are $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$. For each of these primes, there is precisely one group, the cyclic group of that order. Thus by (1) in the Note 4.2.1, there is no Frobenius group of these orders.

2. There is no Frobenius group of order 4, since there are only two groups of order 4 and both are abelian.

3. There are two groups of order 6. They are S_3 and $\mathbb{Z}_6$. Since $\mathbb{Z}_6$ is abelian, it can't be Frobenius and S_3 we have already mentioned is the smallest Frobenius group.

4. There are five groups of order 8. Three are abelian and two are non-abelian. The non-abelian groups are D_8 and Q_8. Both D_8 and Q_8 have order 2^3 and are therefore **extra special** groups. Thus, if $G = D_8$ then $G' = Z(D_8)$ and $|G'| = |Z(D_8)| = 2$. Similarly, if $H = Q_8$ then $H' = Z(H)$ and $|H'| = |Z(H)| = 2$. Since both these groups have a non-trivial center, they can't be Frobenius (Frobenius groups have a trivial center). Hence, there are no Frobenius groups of order 8.

5. There is no Frobenius group of order 9 since there are only two groups of order 9 and both are abelian.

6. There are two groups of order 10. The abelian group is $\mathbb{Z}_{10}$ and the non-abelian group is D_{10} which is Frobenius by Example 1.

7. There are five groups of order 12. Three of them are non-abelian. They are T, D_{12} and A_4. Both T and D_{12} have non-trivial centers since $Z(T) = Z(D_{12}) = \mathbb{Z}_2$. So neither one of them can be Frobenius. We already know from Example 6 that A_4 is Frobenius.

8. There are two groups of order 14. One is abelian which is $\mathbb{Z}_{14}$ and the other non-abelian which is D_{14}. This group is Frobenius by Example 1.

9. There is only one group of order 15 and this group is cyclic and hence not Frobenius.

10. Of the fourteen groups of order 16, 9 of them are non-abelian. None of them, however, are Frobenius since they have a non-trivial center. We list the nine groups or presentations of them together with their centers (see Humpherys [8]).

    - $G = D_8 \times \mathbb{Z}_2$ and $Z(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$.
    - $G = Q_8 \times \mathbb{Z}_2$ and $Z(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$.
    - $G = D_{16}$ and $Z(G) = \mathbb{Z}_2$.
    - $G = Q_{16}$ and $Z(G) = \mathbb{Z}_2$.
    - $G = \langle x, y : x^8 = y^2 = 1, x^y = x^3\rangle$ and $Z(G) = \langle x^4\rangle$.
    - $G = \langle x, y : x^8 = y^2 = 1, x^y = x^5\rangle$ and $Z(G) = \langle x^2\rangle$.
    - $G = \langle x, y, z : x^4 = y^2 = z^2 = 1,$ x central, $z^y = zx^2\rangle$ and $Z(G) = \langle x\rangle$.
    - $G = \langle x, y : x^4 = y^4 = 1, x^y = x^3\rangle$ and $Z(G) = \langle x^2, y^2\rangle$.
    - $G = \langle x, y, z : x^4 = y^2 = z^2 = 1,$ z central, $x^y = xz\rangle$ and $Z(G) = \langle x^2, z\rangle$.

So there is no Frobenius group of order 16.

11. There are five groups of order 18. Three of them are non-abelian. They are:

    - $G = D_6 \times \mathbb{Z}_3$ with $Z(G) = \mathbb{Z}_3$, hence not Frobenius.
    - D_{18} which is Frobenius by Example 1.
    - The group E with presentation given by:
      $E = \langle x, y, z : x^3 = y^3 = z^2 = 1, xy = yx, zxz^{-1} = x^2, zyz^{-1} = y^2\rangle$. This group is not Frobenius since by using GAP we can show that it has a non-trivial center, $|Z(E)| = 3$.

12. There are five groups of order 20. Three of them are non-abelian. The groups are:

    - D_{20} with $Z(D_{20}) = \mathbb{Z}_2$, hence not Frobenius.
    - Q_{20} with $Z(Q_{20}) = \mathbb{Z}_2$, hence not Frobenius.
    - The group with presentation given by: $\langle x, y : x^5 = y^4 = 1, x^y = x^2\rangle = \mathbb{F}_{5,4}$. This group is Frobenius by Example 5.

13. There are two groups of order 21. The cyclic group $\mathbb{Z}_{21}$ and the non-abelian group with presentation given by: $\langle x, y : x^7 = y^3 = 1, x^y = x^2\rangle = \mathbb{F}_{7,3}$. This group is Frobenius by Example 5.

14. There are two groups of order 22. The non-abelian group is D_{22} which is Frobenius by Example 1. The other group is abelian.

15. There are twelve non-abelian groups of order 24. Eleven of these groups have a non-trivial center and therefore can't be Frobenius. The twelfth group is S_4 which we know by Note 4.2.1, part(3) is not Frobenius. We list the 12 non- abelian groups together with their centers (see Humpherys [8]).

    - $G = \langle x, y : x^3 = 1 = y^8, x^y = x^{-1}\rangle,\ Z(G) = \langle y^2\rangle$.
    - $G = \mathbb{Z}_4 \times D_6,\ Z(G) = \mathbb{Z}_4$.
    - $G = \mathbb{Z}_2 \times Q_{12},\ Z(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$.
    - $G = \mathbb{Z}_2 \times D_{12},\ Z(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$.
    - $G = \mathbb{Z}_2 \times A_4,\ Z(G) = \mathbb{Z}_2$.
    - $G = \mathbb{Z}_3 \times D_8,\ Z(G) = \mathbb{Z}_6$.
    - $G = D_{24},\ Z(G) = \mathbb{Z}_2$.
    - $G = S_4,\ Z(G) = \{1_G\}$.
    - $G = Q_{24},\ Z(G) = \mathbb{Z}_2$.
    - $G = SL(2, 3),\ Z(G) = \mathbb{Z}_2$.
    - $G = \mathbb{Z}_3 \times Q_8,\ Z(G) = \mathbb{Z}_6$.

- $T = \langle x, y, z : x^4 = y^2 = z^3 = 1, yxy^{-1} = x^{-1}, z^x = z^{-1}, z^y = z \rangle$, $Z(T) = \langle x^2 \rangle$.

  Thus there is no Frobenius group of order 24.

16. The two groups of order 25 are both abelian.

17. There are two groups of order 26. The non-abelian group is D_{26} which is Frobenius by Example 1. The other group is $\mathbb{Z}_{26}$.

18. There are five groups of order 27. Two of them are non-abelian. They are:

    - $G = \langle x, y : x^9 = 1 = y^3, x^y = x^2 \rangle$
    - $H = \langle x, y, z : x^3 = 1 = y^3 = z^3,\ z \text{ central},\ x^y = xz \rangle$

    In both cases we have that $G' = Z(G)$ and $|G'| = |Z(G)| = 3$, $H' = Z(H)$ and $|H'| = |Z(H)| = 3$. So both these groups are **extra special** groups and hence not Frobenius.

19. There are four groups of order 28. Two of these four groups are non-abelian. They are:

    - D_{28} with $Z(D_{28}) = \mathbb{Z}_2$, hence not Frobenius.
    - Q_{28} with $Z(Q_{28}) = \mathbb{Z}_4$, hence not Frobenius.

    Thus there is no Frobenius group of order 28.

20. There are four groups of order 30. Three of these groups are non-abelian. They are:

    - $G = \mathbb{Z}_3 \times D_{10}$ with $Z(G) = \mathbb{Z}_3$, hence not Frobenius.
    - $G = \mathbb{Z}_5 \times D_6$ with $Z(G) = \mathbb{Z}_5$, hence not Frobenius.
    - D_{30} which is Frobenius by Example 1.

So for order less than 32, there are exactly ten Frobenius groups. Seven of them are Dihedral groups $D_6, D_{10}, D_{14}, D_{18}, D_{22}, D_{26}$ and D_{30}, one is the alternating group A_4 and the remaining two groups are $\mathbb{F}_{5,4}$ and $\mathbb{F}_{7,3}$.

5

Characters of Frobenius Groups

In this chapter we describe the characters of the Frobenius groups. We will then use the results here to calculate the character table of the **Dihedral group** D_{2n}. The theory introduced here will also be used to calculate the character table of the Frobenius group $G = 29^2 : SL(2,5)$, in the next chapter. Proposition 5.2.1 gives a necessary condition for a group H to be a complement of a Frobenius kernel N in a Frobenius group G. We end the chapter by applying the theory of coset analysis described in Chapter 1 to the Frobenius group and describe the theory to find the Fischer matrices of the Frobenius group.

5.1 Characters of Frobenius Groups

Let $S = \{1, 2, \ldots\ldots, n\}$ and $X = S \times S$. Let σ be a permutation of X and $A = [a_{1,j}]$ an $n \times n$ matrix over a field $\mathbb{F}$. Define $A^\sigma = [b_{i,j}]$ where $b_{i,j} = a_{kl}$ with $(k,l) = (i,j)^\sigma$. Since $A^{\sigma\tau} = (A^\sigma)^\tau$ for any other permutation τ of X, any permutation action on X determines a permutation action on the set of $n \times n$ matrices.

Proposition 5.1.1. *([6]). Suppose that G is a permutation group on $X = S \times S$ as above, $\mathbb{F} \subseteq \mathbb{C}$ and A is an invertible $n \times n$ matrix over $\mathbb{F}$. Suppose further that for each $\sigma \in G$ the matrix A^σ can be obtained from A either by permuting the rows of A or by permuting the columns of A, so G can be viewed either as a permutation group G_r on the set of rows of A or G_c on the set of columns of A. Then the permutation characters θ_r and θ_c of G_r and G_c respectively, are equal.*

PROOF: If $\sigma \in G$ then there are permutation matrices $\sigma(R)$ and $\sigma(C)$ for which $\sigma(R)A = \sigma(A) = A\sigma(C)$. In fact, $\sigma \mapsto \sigma(R)^t$ and $\sigma \mapsto \sigma(C)$ are permutation representations of G_r and G_c respectively. Thus $A^{-1}\sigma(R)A = \sigma(C)$. So trace $\sigma(R)^t$ = trace $\sigma(C)$ $\forall \sigma \in G$ and $\theta_r = \theta_c$. ■

Corollary 5.1.2. *In the setting of Brauer's lemma the number of orbits of G_r and G_c are equal.*

PROOF: Since $\theta_r = \theta_c$ we have that $\langle \theta_r, 1_G \rangle = \langle \theta_c, 1_G \rangle$.

But

$$\theta_r(g) = \theta_c(g)\ \forall g \in G \ \Rightarrow\ \frac{1}{|G|}\sum_{g\in G}\theta_r(g) = \frac{1}{|G|}\sum_{g\in G}\theta_c(g).$$

So the number of orbits of G_r equals the number of orbits of G_c. ■

Proposition 5.1.3. *([6]). If G is a Frobenius group with kernel N and if $\chi_1 \neq \chi \in \mathrm{Irr}(N)$ then χ has inertia group $I_G(\chi) = N$.*

PROOF: If A is the character table of N then G acts on the rows of A by **conjugating characters** and on the columns of A by **conjugating the conjugacy classes** of N. If $x \in G\backslash N, \chi \in \mathrm{Irr}(N)$ and L is a conjugacy class of N, then $\chi^x(l) = \chi(l^x)\ \forall l \in L$. Choose $x \in G\backslash N$ and suppose $L^x = L$ for a conjugacy class L of N. Let $L = [y]$ where $1_G \neq y \in N$. Thus $y^x \in L$, so $y^x = y^n$ for some $n \in N$. So $y^{n^{-1}x} = y$ which implies that $n^{-1}x \in C_G(y)$ and hence that $x \in C_G(y)$. But then $x \in N$ since by Proposition 3.2.1, $C_G(y) \leq N$. This is a contradiction. Therefore $y = 1_G$ and L=$\{1_G\}$. Thus $\theta_c(x) = 1\ \forall x \in G\backslash N$, and so by Brauers Lemma $\theta_r(x) = 1$. But this implies that $\chi^x \neq \chi$ if $\chi_1 \neq \chi \in \mathrm{Irr}(N)$. Hence, if $\chi_1 \neq \chi \in \mathrm{Irr}(N)$, then $\mathrm{Stab}_G(\chi)$ contains elements from N only. This implies that $I_G(\chi) = N$. ■

Note 5.1.1. If G is Frobenius with kernel N and complement H then $G/N \cong H$, so any character of H can be viewed as a character of G/N, hence also as a character of G by **lifting**. In particular, $\mathrm{Irr}(H)$ can be viewed as a subset of $\mathrm{Irr}(G)$ in a natural way.

Theorem 5.1.4. *Suppose that G is Frobenius with complement H and kernel N.*

1. *If $\phi_1 \neq \phi \in \mathrm{Irr}(N)$, then $\phi^G \in \mathrm{Irr}(G)$.*
2. *If $\psi \in \mathrm{Irr}(G)$, then either $N \subset \ker\psi$ or $\psi = \phi^G$ for some irreducible character $\phi_1 \neq \phi$ of N.*
3. *If $\psi \in \mathrm{Irr}(G)$, such that $\ker\psi \not\supset N$ and ρ is the **regular representation** of H, then $\psi\big|_H = n\rho$ where $n \in \mathbb{N}$.*

PROOF: (1) Let $\phi_1 \neq \phi \in \mathrm{Irr}(N)$. Then by Proposition 5.1.3, $I_G(\phi) = N$. But this implies that $\phi^G \in \mathrm{Irr}(G)$ (see Moori, Proposition 5.7. [17]).
(2) Let $\psi\big|_N = \sum a_i\phi_i$ with $\phi_i \in \mathrm{Irr}(N)$. If some $a_i \neq 0$ for $i \neq 1$, then by the Frobenius Reciprocity Theorem we have that $\langle \phi_i^G, \psi\rangle = \langle \phi_i, \psi\big|_N\rangle = a_i \neq 0$ and since by (1) $\phi_i^G \in \mathrm{Irr}(G)$, we have $a_i = 1$ and $\phi_i^G = \psi$. If all $a_i = 0$ for $i \neq 1$ then $\psi\big|_N = a_1\phi_1$, $\psi(x) = a_1\ \forall x \in N$. Hence, $N \subset \ker\psi$. So $\mathrm{Irr}(G) = \mathrm{Irr}(H)\ \cup\ \{\phi^G : \phi_1 \neq \phi \in \mathrm{Irr}(N)\}$.
(3) By part (2), there is an irreducible character ϕ of N such that $\psi = \phi^G$. Now $\phi^G(y) = 0\ \forall y \in H\backslash\{1_G\}$ and also $\phi^G(1_G) = [G:N]\,\phi(1_G) = \rho(1_G)\phi(1_G)$. Thus $\psi\big|_H(y) = \phi^G(y) = \rho(y)\phi(y)\ \forall y \in H$, so that $\psi\big|_H = n\rho$ where $n = \phi(1_G)$ is a positive integer. ■

The following notes are consequences of Theorem 5.1.4

Note 5.1.2. If G is Frobenius with kernel N and complement H then,

1. From Theorem 5.1.4(2) the irreducible characters of G are of 2 types; those with kernel containing N and those induced from non-trivial irreducible characters of N.

2. Also from Theorem 5.1.4(3) the order of H divides the degree of the induced character ϕ^G for $\phi_1 \neq \phi \in \mathrm{Irr}(N)$. That is $|H| \,\big|\, \phi^G(1_G)$. Furthermore, if $\phi_1 \neq \phi \in \mathrm{Irr}(N)$ is a linear character then $|H| = \phi^G(1_G)$.

Theorem 5.1.5. *([6]). Suppose that G is a Frobenius group with kernel N and complement H, and that ϕ, θ are non-trivial irreducible characters of N. Then $\phi^G = \theta^G$ if and only if $\theta \in \mathrm{Orb}_H(\phi) = \Delta_\phi$. Furthermore, $|\Delta_\phi| = |H|$, so G has $\frac{c(N)-1}{|H|}$ distinct irreducible characters of the form $\phi^G, \phi_1 \neq \phi \in \mathrm{Irr}(N)$. (Here $c(N)$ is the number of conjugacy classes of N).*

PROOF: By Theorem 5.1.4, $\theta, \phi \in \mathrm{Irr}(N)$ imply that $\theta^G, \phi^G \in \mathrm{Irr}(G)$. Suppose that $\theta^G = \phi^G$. By Frobenius reciprocity we have that: $\langle \phi^G|_N, \theta\rangle_H = \langle \phi^G, \theta^G\rangle = 1$. So θ is an irreducible constituent of $\phi^G|_N$. So by Clifford Theorem

$$\phi^G|_N = \sum_{\theta_i \in \Delta_\phi} \theta_i. \tag{5.1}$$

Also $\langle \phi^G|_N, \phi\rangle_N = \langle \phi^G, \phi^G\rangle = 1$. So ϕ is an irreducible constituent of $\phi^G|_N$ and by Clifford Theorem

$$\phi^G|_N = \sum_{\phi_i \in \Delta_\phi} \phi_i. \tag{5.2}$$

Now (5.1) and (5.2) imply that $\sum_{\theta_i \in \Delta_\phi} \theta_i = \sum_{\phi_i \in \Delta_\phi} \phi_i$. Therefore some $\theta_i = \phi_j$, which implies that there exists $y \in G$ such that $\theta = \phi^y$. That is $\theta \in \Delta_\phi$. Note that $\theta \in \Delta_\phi$, implies that any conjugate of θ will also be in Δ_ϕ.
So

$$\begin{aligned}
\phi^G(g) &= \frac{1}{|N|}\sum_{x \in G} \phi^0(xgx^{-1}) \\
&= \frac{1}{|N|}\sum_{x \in G} \phi^x(g) \\
&= \frac{1}{|N|}\sum_{x \in G} \theta^{y^{-1}x}(g) \\
&= \frac{1}{|N|}\sum_{z \in G} \theta^z(g) \\
&= \frac{1}{|N|}\sum_{z \in G} \theta^0(zgz^{-1}) = \theta^G(g).
\end{aligned}$$

Thus $\theta^G = \phi^G$. By the Orbit Stabilizer Theorem we have that $|\mathrm{Orb}_H(\phi)||I_H(\phi)| = |H|$. Since $\phi^G \in \mathrm{Irr}(G)$, $I_H(\phi) = 1_G$ which implies that $|\Delta_\phi| = |H|$. ■

Corollary 5.1.6. *If G is Frobenius with complement H and kernel N then $c(G) = c(H) + \frac{c(N)-1}{|H|}$.*

PROOF: This follows from Theorem 5.1.4, part(2) and Theorem 5.1.5. Since $\mathrm{Irr}(G) = \mathrm{Irr}(H) \cup \{\phi^G : \phi_1 \neq \phi \in \mathrm{Irr}(N)\}$ and since the number of irreducible characters equals the number of conjugacy classes, the equation above gives $c(G) = c(H) + \frac{c(N)-1}{|H|}$. ■

5.2 The Character Table of D_{2n} , n odd.

In this section we use the results of the previous section on Frobenius Characters and theory of Frobenius groups to construct the character table of D_{2n}. The result is already well known but we demonstrate here as an example the use of the results on the Frobenius characters to achieve this. The following points are used to construct the character table of D_{2n}.

1. From Example 1 of Section 4.1, we know that $D_{2n} = N : H$ is Frobenius with kernel $N = \mathbb{Z}_n$ and complement $H = \mathbb{Z}_2$. Here $N = \langle a \rangle$ and $H = \langle b \rangle$.

2. Now by Section 3.3.2, $G' = (D_{2n})' = N = \mathbb{Z}_n$. So the number of **linear** characters of D_{2n} equals $[G : G'] = \big[D_{2n} : (D_{2n})'\big] = 2$ (see Moori , Theorem 5.2.21 [17]).

3. By Theorem 5.1.4, $\phi_1 \neq \phi \in \mathrm{Irr}(\mathbb{Z}_n)$ implies that $\phi^G \in \mathrm{Irr}(D_{2n})$. Also by Theorem 5.1.4, $\mathrm{Irr}(D_{2n}) = \mathrm{Irr}(\mathbb{Z}_2) \cup \{\phi^G : \phi_1 \neq \phi \in \mathrm{Irr}(\mathbb{Z}_n)\}$. The number of characters of D_{2n} of the form ϕ^G by Theorem 5.1.5 is equal to $\frac{c(N)-1}{|H|} = \frac{n-1}{2}$. Therefore $|\mathrm{Irr}(D_{2n})| = 2 + \frac{n-1}{2} = \frac{n+3}{2}$.

4. Now by the Note 5.1.2, since $\phi \in \mathrm{Irr}(\mathbb{Z}_n)$ is linear, the degree of the induced character ϕ^G equals $|H| = 2$.

5. Thus D_{2n} has $\frac{n-1}{2}$ characters of degree 2 and two characters of degree 1. This takes care of the number of irreducible characters and their respective degrees.

6. Now the number of conjugacy classes of D_{2n} equals $\frac{n+3}{2}$.

7. By Proposition 3.2.2, the action of $\mathbb{Z}_2$ on $\mathbb{Z}_n$ partitions $\mathbb{Z}_n \backslash \{1_G\}$ into $\mathbb{Z}_2$ orbits each of size $|\mathbb{Z}_2| = 2$. The number of orbits is $\alpha = \frac{|N|-1}{|H|} = \frac{n-1}{2}$. So there are $\frac{n-1}{2}$ conjugacy classes (each orbit represents a conjugacy class, since the action of $\mathbb{Z}_2$ on $\mathbb{Z}_n$ is by **conjugation**) produced by the action of $\mathbb{Z}_2$ on $\mathbb{Z}_n$ each of size 2. Since there are $\frac{n+3}{2}$ conjugacy classes in D_{2n}, there are two remaining conjugacy classes, one a singleton which is the identity conjugacy class and the remaining conjugacy class which has size n. (Because $\frac{n-1}{2} \times 2 + n + 1 = 2n$).

8. Since $\mathbb{Z}_n \trianglelefteq D_{2n}$ and $D_{2n}/\mathbb{Z}_n \cong \mathbb{Z}_2$, the two linear characters of D_{2n} are obtained by lifting the irreducible characters of $D_{2n}/\mathbb{Z}_n$ to D_{2n}. These characters are given by χ_1 and χ_2.

9. Now χ_1 is the identity character and $\chi_1(g) = 1 \ \forall g \in D_{2n}$.

10. $\chi_2(g) = \begin{cases} 1 & \text{when } g \in \mathbb{Z}_n \\ -1 & \text{when } g \notin \mathbb{Z}_n. \end{cases}$

11. Thus the value of χ_2 on each a^r for $1 \leq r \leq \frac{n-1}{2}$ is 1. Here a^r is the representative of the conjugacy class $[a^r]$. This follows from Theorem 5.1.4, part(2) since we have that $N \subseteq \ker\chi_2$. The value of χ_2 on b is -1. Here b is the representative of the remaining conjugacy class $[b]$. Note that $\chi_2(1_G) + \frac{n-1}{2} \times 2 \times 1 + \chi_2(b) \times n = 0 \Rightarrow \chi_2(b) = \frac{1-n-1}{n} = -1$.

12. Now let $\psi_j = \phi_i^G$ for $j = 1, \ldots\ldots, \frac{n-1}{2}$ where $\phi_1 \neq \phi_i \in \mathrm{Irr}(\mathbb{Z}_n)$. Let a^r for $r = 1, \ldots\ldots, \frac{n-1}{2}$ be representatives of the conjugacy classes $[a^r]$. By Proposition 3.2.6, the action of $\mathbb{Z}_2$ on $\mathbb{Z}_n$ by conjugation sends an $x \in \mathbb{Z}_n$ to it's inverse. Thus $x \in [a^r]$ implies that $x^{-1} \in [a^r]$ $\forall r = 1, \ldots\ldots, \frac{n-1}{2}$. Thus $\psi_j(x) = \psi_j(x^{-1})$ $\forall j = 1, \ldots\ldots, \frac{n-1}{2}$ and $\forall x \in \mathbb{Z}_n$. So $\psi_j(x) \in \mathbb{R}$ since $\psi_j(x) = \overline{\psi_j(x)}$ $\forall j = 1, \ldots\ldots, \frac{n-1}{2}$.
Now
$$\psi_j(a^r) = \left|C_{D_{2n}}(a^r)\right| \sum_{i=1}^{n} \frac{\phi(x_i)}{\left|C_{\mathbb{Z}_n}(x_i)\right|} \quad \forall j = 1, \ldots\ldots, \frac{n-1}{2},$$
where x_i are class representatives of Z_n which fuse to form $[a^r]$. But by Lemma 3.2.15, $(3) \Rightarrow (4)$, $\left|C_{D_{2n}}(a^r)\right| = \left|C_{\mathbb{Z}_n}(x_i)\right|$. So
$$\psi_j(a^r) = \sum_{i=1}^{n} \phi(x_i) = \phi(x_i) + \phi(x_i^{-1}) = \phi(x_i) + \overline{\phi(x_i)} \ \forall j = 1, \ldots\ldots, \frac{n-1}{2}.$$
Therefore $\psi_j(a^r) = 2\alpha$ where α is the real part of $e^{\frac{2\pi r i}{n}}$.

13. Finally we have that $\psi_j(b) = 0$, since $\mathbb{Z}_n \cap [b] = \emptyset$.

Using all of the above , we can now construct the character table of the Dihedral group D_{2n} where n is odd.

classes of D_{2n}	$[1]$	$[a^r]$ $(1 \leq r \leq \frac{n-1}{2})$	$[b]$
$\left|C_G(g)\right|$	$2n$	n	2
χ_1	1	1	1
χ_2	1	1	-1
ψ_j $(1 \leq j \leq \frac{n-1}{2})$	2	2α	0

Table 5.1: Character Table of D_{2n}

Proposition 5.2.1. *Let $H \neq \{1_G\}$ be a group. Then there exists a Frobenius group G with Frobenius kernel N and $G/N \cong H$ if and only if there exists an irreducible character χ of H such that for every subgroup K of H with $K \neq \{1_G\}$ we have $\langle \chi\downarrow_K, 1_K\rangle = 0$.*

PROOF: For proof of this result see Feit's book on Characters of Finite Groups, page 136 [4]. ■

By using the result above, it is possible to give a complete classification of groups H which can occur as the complement of a Frobenius kernel N in a Frobenius group G.

We illustrate with the following examples.

Example 1: There is no Frobenius group which has $H = S_3$ as a complement. The character table of S_3 is:

ClassRep	1_{S_3}	(12)	(123)
h_i	1	3	2
χ_1	1	1	1
χ_2	2	0	-1
χ_3	1	-1	1

Table 5.2: Character Table of S_3

Now the subgroups of S_3 excluding the trivial subgroup and S_3 itself are:

$$K_2 = \langle(12)\rangle,\ K_3 = \langle(13)\rangle,\ K_4 = \langle(23)\rangle,\ K_5 = \langle(123)\rangle = A_3.$$

All we need to do is find one subgroup K_i of S_3 which fails to satisfy the condition $\langle \chi_i{\downarrow}, 1_{K_i}\rangle = 0$ for some irreducible character $\chi_i \in \mathrm{Irr}(S_3)$.
We need the character tables of each of the subgroups K_i for $i = 2, 3$ and 4. So we need the character table of $\mathbb{Z}_2$ since each $K_i \cong \mathbb{Z}_2$ for $i = 2, 3, 4$. We will just use the character table of K_2:

Classes	1_{K_2}	(12)
θ_1	1	1
θ_2	1	-1

Table 5.3: Character Table of K_2

Take χ_2 :

$$\chi_2{\downarrow}_{K_2} = \theta_1 + \theta_2 \ \Rightarrow\ \langle \chi_2{\downarrow}_{K_2}, 1_{K_2}\rangle = \langle \theta_1 + \theta_2, \theta_1\rangle = 1.$$

Thus for the irreducible character χ_2, the subgroup K_2 fails to satisfy the condition of Proposition 5.2.1

Now $K_5 = A_3$. The character table is that of $\mathbb{Z}_3$. The character table is:

ClassRep	1_{A_3}	(132)	(123)
h_i	1	3	3
θ_1	1	1	1
θ_2	1	α	$\overline{\alpha}$
θ_3	1	$\overline{\alpha}$	α

Table 5.4: Character Table of K_5

where $\alpha = \frac{-1+\sqrt{3}}{2}$.
Take χ_3 :

$$\chi_3{\downarrow_{K_5}} = \theta_1 \ \Rightarrow \ \langle \theta_1, 1_{K_5} \rangle = \langle \theta_1, \theta_1 \rangle = 1.$$

For the irreducible character χ_3, the subgroup K_5 fails to satisfy the condition of Proposition 5.2.1. Thus there is no irreducible character of S_3 satisfying the condition of the Proposition. Hence, there is no Frobenius group which can have S_3 as a complement.

Example 2: There is no Frobenius group that has $H = V_4 = \{e, a, b, c\}$ as a complement.
Now the character table of V_4 is:

Classes of V_4	[e]	[a]	[b]	[c]
χ_1	1	1	1	1
χ_2	1	-1	-1	1
χ_3	1	-1	1	-1
χ_4	1	1	-1	-1

Table 5.5: Character Table of V_4

The subgroups of V_4 excluding the trivial subgroup and V_4 itself are:

$$K_2 = \{e, a\}, \ K_3 = \{e, b\}, \ K_4 = \{e, c\}.$$

The character table of each K_i for $i = 2, 3, 4$ is:

Classes	[e]	[g]
θ_1	1	1
θ_2	1	-1

Table 5.6: Character Table of K_i for $i = 2, 3, 4$

Take χ_2 :

$$\chi_2\downarrow_{K_2} = \theta_2 \ , \ \chi_2\downarrow_{K_3} = \theta_2 \ , \ \chi_2\downarrow_{K_4} = \theta_1.$$

Thus for the irreducible character χ_2, the subgroup K_4 fails to satisfy the condition of Proposition 5.2.1

Take χ_3 :

$$\chi_3\downarrow_{K_2} = \theta_2 \ , \ \chi_3\downarrow_{K_3} = \theta_1 \ , \ \chi_3\downarrow_{K_4} = \theta_2.$$

For the irreducible character χ_3, the subgroup K_3 fails to satisfy the condition of Proposition 5.2.1

Take χ_4 :

$$\chi_4\downarrow_{K_2} = \theta_1 \ , \ \chi_4\downarrow_{K_3} = \theta_2 \ , \ \chi_4\downarrow_{K_4} = \theta_2.$$

Here the subgroup K_2 fails to satisfy the condition of Proposition 5.2.1.
So none of the nontrivial irreducible characters of V_4 satisfy the condition of Proposition 5.2.1, for every proper subgroup of V_4. Thus there is no Frobenius group which can have V_4 as a complement.

The following theorem illustrates how the character table of a group can be used to determine whether the group is Frobenius or not.

Theorem 5.2.2. *Let G be a group, let $t > 1$ be a proper divisor of $|G|$ and let $K = \cap \ker\chi$, where χ ranges over all irreducible characters of G with t not dividing $\chi(1_G)$. Then the following are equivalent:*

1. *G is a Frobenius group with Frobenius complement of order t.*
2. *$[G : K] = t$ and G is a Frobenius group with kernel K.*
3. *$[G : K] = t$.*

PROOF: See Karpilovsky [12]. ∎

To illustrate the result in Theorem 5.2.2 consider the character table below of the group G with presentation $G = \langle a, b : a^{13} = b^4 = 1, b^{-1}ab = a^5 \rangle$.

g_i	1	a	a^2	a^4	b	b^2	b^3
$\lvert C_G(g_i)\rvert$	52	13	13	13	4	4	4
χ_0	1	1	1	1	1	1	1
χ_1	1	1	1	1	i	-1	$-i$
χ_2	1	1	1	1	-1	1	-1
χ_3	1	1	1	1	$-i$	-1	i
ϕ_1	4	α	β	γ	0	0	0
ϕ_2	4	β	γ	α	0	0	0
ϕ_3	4	γ	α	β	0	0	0

where $\epsilon = e^{\frac{2\pi i}{13}}$ and $\alpha = \epsilon + \epsilon^5 + \epsilon^8 + \epsilon^{12}$, $\beta = \epsilon^2 + \epsilon^3 + \epsilon^{10} + \epsilon^{11}$, $\gamma = \epsilon^4 + \epsilon^6 + \epsilon^7 + \epsilon^9$.

Then applying the result of Theorem 5.2.2 to the character table above we have

$$\langle a \rangle = K = \cap \ker\chi_i \text{ for } i = 0, \ldots, 3.$$

Since $t = [G : K] = 4$, G is a Frobenius group with kernel K.

5.3 Coset analysis applied to the Frobenius Group

We begin by making the following note which applies in this section and the next.

Note 5.3.1. In Chapter 1 in our description of coset analysis, we used the conventional notation for the split extension $\overline{G} = N : G$. In our Frobenius group we have used the notation $G = N : H$. To allow for easy transition between the relevant section of Chapter 1 and this section, we will use the conventional notation.

So let $\overline{G} = N : G$ be a Frobenius group with kernel N and complement G. Since the extension is split, a **lifting** of $g \in \overline{G}$ is g itself since $G \leq \overline{G}$. So $\overline{G} = \cup_{g \in G} Ng$. So in a Frobenius group, for step(1) of coset analysis we have that $C_N(\overline{g}) = C_N(g) = \{1_G\}$ by Lemma 3.2.15. So $k = 1$ here and under the action of N, Ng remains intact. Since $k = 1$, in step(2) we now have that $f_j = 1$ so that $|\Delta_j| = |N|$ and equation (1.3) now implies that: $\left|[x]_{\overline{G}}\right| = |N| . \left|[g]_G\right|$. This is the same result we obtained in Lemma 3.2.15, $(3 \Rightarrow 4)$. Also since $k = f_j = 1$, equation (1.4) now implies that: $\left|C_{\overline{G}}(x)\right| = |C_G(g)|$, which is the same result we obtained in the proof of Lemma 3.2.15. Note that in the equation: $\left|C_{\overline{G}}(x)\right| = |C_G(g)|$, the element x on the left hand side is in Δ_j and the g on the right hand side is in the coset Ng. Since the coset remains intact, $\Delta_j = Ng$ and $x \in Ng$. So we may choose this x to be g. This will give: $\left|C_{\overline{G}}(g)\right| = |C_G(g)|$.

5.4 Fischer Matrices of the Frobenius group

Having defined and described the Fischer matrices in Chapter 1, we now describe the Fischer matrices for a Frobenius group.
Let $\overline{G} = N : G$ be a Frobenius group with kernel N and complement G. By Proposition 5.1.3 we know that $\forall \chi_1 \neq \chi \in Irr(N)$, the inertia group of χ is N. Since $I_{\overline{G}}(\chi_1) = \overline{G}$, in a Frobenius group, the inertia factors are G and $\{1_G\}$ corresponding to the inertia groups $\overline{G}$ and N respectively. Let $X(g)$ and $R(g)$ be as defined in section 1.5 of Chapter 1. We have defined the Fischer matrix $M(g)$ for $g \in G$ as a matrix whose rows are indexed by $R(g)$ and columns by $X(g)$. We now find $X(g)$ and $R(g)$ for a Fischer matrix in a Frobenius group.

If $\underline{g = 1_G}$, then $X(g)$ is made up of the class representatives of the conjugacy classes of $\overline{G}$ that come from N (since $g = 1_G$ implies that $Ng = N$). These are representatives of the $(m+1)$ orbits (m non-trivial orbits and the trivial orbit) of G on N, where $m = \frac{|N|-1}{|G|}$.

For $\underline{g = 1_G}$, the inertia factors H_i for $i = \{1, 2, \ldots\ldots, t\}$ contain [g], where $t = m + 1$ orbits of G on N which is the same as the number of orbits of $\overline{G}$ on $Irr(N)$. The conjugacy classes of the H_i that fuse to [g] is the singleton conjugacy classes containing the identity 1_G. So $y = 1_G$. Thus $R(g)$ contains the $t = m + 1$ ordered pairs $(i, 1)$ where $i \in \{1, 2, \ldots\ldots, t\}$. Therefore the Fischer matrix $M(1_G)$ in a Frobenius group is an $(m + 1) \times (m + 1)$ matrix where m is the number of non-trivial orbits of G on N. Now the entries in this matrix are given by:
$M(1_G) = \left[a^j_{(i,1)}\right] = \psi_i^{\overline{G}}(x_j)$, where $x_j \in X(g)$ for $j = 1, 2, \ldots\ldots, t = m + 1$ and $\psi_i \in Irr(N)$ is a representative of the $t = m + 1$ orbits of $\overline{G}$ on Irr(N).
Note that the $\psi_i^{\overline{G}}$ here is the induction of a character ψ_i which is the extension of $\theta_i \in Irr(N)$ to $I_{\overline{G}}(\theta_i)$. But since $I_{\overline{G}}(\theta_i) = N \ \forall \theta_1 \neq \theta_i \in Irr(N)$, ψ_i is the same as θ_i for each i. So $\psi_i^{\overline{G}}$ is just the induction of $\psi_i \in Irr(N)$ which we know from Theorem 5.1.4 is an irreducible character of $\overline{G}$.

If $\underline{g \neq 1_G}$, then $X(g)$ is made up of the representatives of $\overline{G}$ conjugacy classes of elements of Ng. But by Lemma 3.2.15, every element of Ng is conjugate to g. So the conjugacy class of $\overline{G}$ that contains g will contain Ng. So this entire coset is contained in the conjugacy class of $\overline{G}$ which has g as a representative. The coset therefore contributes to only this conjugacy class of $\overline{G}$ with representative g. Therefore $X(g) = \{g\}$.

For $\underline{g \neq 1_G}$, only $H_1 = G$ contains an element of [g]. So $i = 1$ and $y = g$ only. So $R(g) = \{(1, g)\} \ \forall g \neq 1_G$. Therefore $\forall g \neq 1_G$, the Fischer matrix $M(g)$ is a 1×1 matrix. The entry of this matrix is given by: $M(g) = \left[a^j_{(1,g)}\right] = \psi_1^{\overline{G}}(g) = 1$
<u>In summary then :</u>
In a Frobenius group $\overline{G} = N : G$, the Fischer matrix $M(1_G)$ is an $(m + 1) \times (m + 1)$ matrix where m is the number of non-trivial orbits of G on N, and $M(g) \ \forall g \neq 1_G$ is just 1.

6

The Frobenius Group $29^2 : SL(2,5)$ and it's Character Table

In this chapter we will construct the character table of the Frobenius Group $29^2 : SL(2,5)$ using the theory built in the previous chapter. First we say something about the group $SL(2,5)$ which is the Frobenius complement. We then explain the construction of the group $29^2 : SL(2,5)$ and conclude the chapter by constructing the character table.

6.1 The Group $SL(2,5)$

1. $SL(2,5)$ is a normal subgroup of the General Linear group $GL(2,5)$, of order 120.

2. For q odd, $SL(2,q)$ has $q+4$ conjugacy classes and hence $q+4$ irreducible characters. So $SL(2,5)$ has 9 conjugacy classes and 9 irreducible characters (see Basheer, Section 4.4.1 [1]).

3. The Sylow 2 - subgroups of $SL(2,5)$ are quaternion.

4. The group $SL(2,5)$ is **perfect**, since for $q \geq 5$ and q a prime $SL(2,q)$ is perfect (see Holt[7]). Also Meierfrankenfeld [15] gives two proofs characterizing $SL(2,5)$ as the only perfect Frobenius complement.

5. The group $SL(2,5)$ itself is not Frobenius since it has a non-trivial center, $Z(SL(2,5)) = \{I\ ,\ -I\}$.

Proposition 6.1.1. $SL(2,5)$ *is the unique non-solvable group of order 120 with quaternion Sylow 2 - subgroup. Moreover* $SL(2,5) = \langle x, y, z, : x^3 = y^5 = z^2 = 1, x^z = x, y^z = y, (xy)^2 = z \rangle$.

PROOF: See Passman page 122 [20]. ■

Note 6.1.1. If a group G acts on a set X and $N \trianglelefteq G$, then N is a **regular** normal subgroup if the action of N on X is **regular**. That is, the action of N on X is transitive and $Stab_N(x) = \{1_G\}$

for each $x \in X$. The action of N on X is **semiregular** if $\mathsf{Stab}_N(x) = \{1_G\}$ for each $x \in X$ (see Definition 1.2.6).

Remark 6.1.1. If $G = NH$ is a Frobenius group with kernel N and complement H, then the kernel N is a regular normal subgroup of G (see Note 6.1.1 and Note 2.2.3), and the complement H acts semi-regularly on N (see Lemma 3.2.15 part (2) and Note 6.1.1).

Proposition 6.1.2. $\mathsf{SL}(2,5)$ *is a non-solvable complement.*

PROOF: From Proposition 6.1.1,we have that $\mathsf{SL}(2,5)$ is non-solvable and that

$$\mathsf{SL}(2,5) = \langle x, y, z : x^3 = y^5 = z^2 = 1, x^z = x, y^z = y, (xy)^2 = z \rangle.$$

Let F be a finite field with $\mathsf{char}(F) \notin \{2,3,5\}$ and assume that $\sqrt{5}, \sqrt{-1} \in F$.
Define the matrices $\overline{x}, \overline{y}, \overline{z}$ as follows:

$$\overline{x} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \quad \overline{y} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & \frac{\sqrt{5}+1}{2} \end{pmatrix} \quad \overline{z} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can show that

$$\overline{x}^3 = \overline{y}^5 = \overline{z}^2 = 1,\ \overline{x}^{\overline{z}} = \overline{x},\ \overline{y}^{\overline{z}} = \overline{y} \text{ and } (\overline{xy})^2 = \overline{z}.$$

Thus the map $x \to \overline{x},\ y \to \overline{y},\ z \to \overline{z}$ induces a homomorphism of $\mathsf{SL}(2,5)$ into $\mathsf{Aut}\ V$, where V is the 2 - dimensional vector space over F. In this way $G = \mathsf{SL}(2,5)$ acts on V. If $v \in V$, $g \in G\backslash\{1_G\}$ and $vg = v$, then we show that $v = 0$. It suffices to assume that g has prime order p. If $p = 2$, then $g = z$ and $v = 0$ since $\mathsf{char}(F) \neq 2$. If $p = 3$, then $\langle g \rangle$ is conjugate to $\langle x \rangle$ and hence we can assume that $g = x$. This yields $v = 0$ since $\mathsf{char}(F) \neq 3$. Finally if $p = 5$ then we can assume that $g = y$ and this yields easily $v = 0$ since $\mathsf{char}(F) \neq 5$. Set $L = VG$ and let L act on L/G by permuting the right cosets of G. Then V is a regular normal subgroup and G acts semi-regularly on $V\backslash\{0\}$. Hence L is a Frobenius group and G is a Frobenius complement. ■

Remark 6.1.2. If G is a Frobenius group with kernel N and complement H, then the condition that $|H| \,\big|\, |N| - 1$, (see Proposition 3.2.2), implies that $|N| \equiv 1 \bmod (|H|)$. In the Frobenius group $L = VG$ constructed above in Proposition 6.1.2, $G = \mathsf{SL}(2,5)$ is the complement. Since $|G| = 120$ and $5\big|120$, the congruence relation $|V| \equiv 1 \bmod (|G|)$, is equivalent to $|V| \equiv 1 \pmod 5$ (because $|V| = 1 + k|G| = 1 + k \times 24 \times 5 \Rightarrow |V| \equiv 1 (\bmod 5)$).
Now let $F = \mathsf{GF}(p)$. Since the kernel V is a vector space of dimension two over $\mathsf{GF}(p)$, the order of V as an abelian group is p^2. Therefore for V to be a Frobenius kernel to $G = \mathsf{SL}(2,5)$, we must choose p such that $p^2 \equiv 1 \pmod 5$. Therefore we must find all primes $p \in \overline{a} \in \mathbb{Z}/5\mathbb{Z}$ such that $a^2 = 1$, where $\overline{a}$ is the residue class $\bmod\ 5$.
So we find all p such that $p \in \overline{1} \in \mathbb{Z}_5$ or $p \in \overline{4} \in \mathbb{Z}_5$. We list below all $p < 100$ in these two residue

classes in $\mathbb{Z}_5$. If $p \in \bar{1}$, then $p \in \{11, 31, 41, 61, 71, 91\}$. If $p \in \bar{4}$, then $p \in \{19, 29, 59, 79, 89\}$. Thus, using the construction described in Proposition 6.1.2, for $p < 100$, there are 11 Frobenius groups of this type all having $\mathsf{SL}(2,5)$ as a complement. The smallest such group is a direct product of $\mathsf{SL}(2,5)$ by an abelian group of order 11^2, that is a vector space of dimension 2 over a field of 11 elements.

We will now apply Proposition 6.1.2 to construct the Frobenius group $29^2 : \mathsf{SL}(2,5)$, that is we are choosing $p = 29$ here. Consider $H = \mathsf{SL}(2,5)$ with the following presentation:

$$\langle x, y, z, |x^3 = y^5 = z^2 = 1, x^z = x, y^z = y, (xy)^2 = z\rangle.$$

via

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}, \; y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; z \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices above satisfy the relations in the presentation given and they generate $\mathsf{SL}(2,5)$, so there is a homomorphism onto $\mathsf{SL}(2,5)$.
Now let $F = \mathbb{Z}_{29}$. So $12^2 \equiv -1, 11^2 \equiv 5$. Our choice of $F = \mathbb{Z}_{29}$ stems from the fact it is the first field of characteristic not 2, 3 or 5 in which -1 and 5 are squares.
Now define

$$X = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -12 \\ -12 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ in } \mathsf{SL}(2,29).$$

The matrices X, Y and Z satisfies the relations in the presentation of H. It follows that H maps isomorphically into a subgroup of $\mathsf{SL}(2,29)$ via $x \mapsto X, y \mapsto Y, z \mapsto Z$. Thus we take the point of view that H is that subgroup of $\mathsf{SL}(2,29)$.
Now set $N = \mathbb{Z}_{29} \bigotimes \mathbb{Z}_{29}$, so that $H \leq \mathrm{Aut}(N)$, and finally let G be the resulting semi-direct product of N by H. By Proposition 6.1.2, now $G = 29^2 : \mathsf{SL}(2,5)$ is a Frobenius group.

6.2 The Character Table of $29^2 : \mathsf{SL}(2,5)$

Note 6.2.1. For this section we will use the conventional notation of representing the complement by G and the split extension by $\overline{G}$. So $\overline{G} = N : G \; = \; 29^2 : \mathsf{SL}(2,5)$.

6.2.1 The Characters of $\overline{G}$

1. The number of conjugacy classes of $29^2 : \mathrm{SL}(2,5)$ equals the number of irreducible characters of $29^2 : \mathrm{SL}(2,5)$.

2. By Corollary 5.1.6, the number of conjugacy classes and hence irreducible characters of $29^2 : \mathrm{SL}(2,5)$ equals :

$$c\left(29^2 : \mathrm{SL}(2,5)\right) = c\left(\mathrm{SL}(2,5)\right) + \frac{c\left(29^2\right) - 1}{|\mathrm{SL}(2,5)|} = 9 + \frac{840}{120} = 16.$$

3. By the Note 5.1.2, the sixteen irreducible characters are split into two types; those containing the kernel $N = 29^2$ and those **induced** from non-trivial irreducible characters of 29^2.

4. Also by Theorem 5.1.5, the number of distinct irreducible characters of $\overline{G}$ of the form $\phi^{\overline{G}}$, $\phi_1 \neq \phi \in \mathrm{Irr}(N)$ is given by:

$$\frac{c\,(N) - 1}{|\mathrm{SL}(2,5)|} = \frac{840}{120} = 7.$$

5. The remaining nine irreducible characters of $\overline{G}$ come from G. These nine characters have degrees : 1 , 2 , 2 , 3 , 3 , 4 , 4 , 5 and 6.

6. Since all the irreducible characters of 29^2 are linear, by the Note 5.1.2, we have that $\phi^{\overline{G}}(1_G) = |G| = 120$ for $\phi_1 \neq \phi \in \mathrm{Irr}(29^2)$.

7. So in the character table of $29^2 : \mathrm{SL}(2,5)$ there are sixteen irreducible characters, the first nine are the irreducible characters of $\mathrm{SL}(2,5)$ with degrees : 1 , 2 , 2 , 3 , 3 , 4 , 4 , 5 and 6. The remaining seven characters each of degree 120 are induced from $N = 29^2$.

6.2.2 The Conjugacy Classes of $29^2 : \mathrm{SL}(2,5)$

1. By Proposition 3.2.2, $\mathrm{SL}(2,5)$ acts on $N = 29^2$ partitioning $N \backslash \{1_G\}$ into $\alpha = \frac{|N|-1}{|G|} = 7$ orbits each of size $|G| = 120$. Therefore in $\overline{G}$, $N \backslash \{1_G\}$ splits into seven conjugacy classes each of size 120.

2. Since $29^2 : \mathrm{SL}(2,5)$ has sixteen conjugacy classes, the remaining nine conjugacy classes come from $\mathrm{SL}(2,5)$ by Corollary 5.1.6. But $\mathrm{SL}(2,5)$ has nine conjugacy classes. Thus each conjugacy class of $\mathrm{SL}(2,5)$ gives a conjugacy class of $29^2 : \mathrm{SL}(2,5)$.

3. If $1_{\overline{G}} \neq g \in \overline{G}$ is a representative of $[g]_{\overline{G}}$, then by Lemma 3.2.15, g is conjugate to every element of Ng. Also g is conjugate to g' in G implies that $Ng' \subseteq [g]_{\overline{G}}$ by Lemma 3.2.15. So $[g]_{\overline{G}} = \cup Ng'$ where g' is conjugate to g in G and hence

$$\left|[g]_{\overline{G}}\right| = \left|[g]_G\right| \times |N|.$$

Therefore each conjugacy class $[g]_G$ produces a bigger conjugacy class $[g]_{\overline{G}}$ of size given by the equation above.

6.2.3 Table of Conjugacy Classes of $29^2 : \mathrm{SL}(2,5)$

The following table lists the sixteen conjugacy classes of $29^2 : \mathrm{SL}(2,5)$ together with a representative, the size of each class, the order of each representative and the order of the centralizer of this representative.

Table 6.1: Conjugacy Classes of $\overline{G}$

Class$[\overline{g}]$	$[1]$	$[n_1]$	$[n_2]$	$[n_3]$	$[n_4]$	$[n_5]$	$[n_6]$	$[n_7]$
$\lvert[\overline{g}]\rvert$	1	120	120	120	120	120	120	120
$\circ(\overline{g})$	1	29	29	29	29	29	29	29
$\lvert C_{\overline{G}}(\overline{g})\rvert$	$29^2.120$	29^2	29^2	29^2	29^2	29^2	29^2	29^2

Table 6.1 (continued)

Class$[\overline{g}]$	$[g_1]$	$[g_2]$	$[g_3]$	$[g_4]$	$[g_5]$	$[g_6]$	$[g_7]$	$[g_8]$
$\lvert[\overline{g}]\rvert$	$29^2.12$	$29^2.12$	29^2	$29^2.12$	$29^2.12$	$29^2.20$	$29^2.20$	$29^2.30$
$\circ(\overline{g})$	10	10	2	5	5	3	6	4
$\lvert C_{\overline{G}}(\overline{g})\rvert$	10	10	120	10	10	6	6	4

Note 6.2.2. 1. By Lemma 3.2.15 we have that:

$$\left|[g]_{\overline{G}}\right| = \left|[g]_G\right| \times |N|.$$

Therefore

$$\left|C_{\overline{G}}(g_i)\right| = \frac{\left|\overline{G}\right|}{\left|[g_i]_{\overline{G}}\right|} = \frac{|G| \times |N|}{|[g_i]_G| \times |N|} = \frac{|G|}{|[g_i]_G|} = |C_G(g_i)|.$$

2. Also

$$\begin{aligned}\left|C_{\overline{G}}(n_i)\right| &= \frac{\left|\overline{G}\right|}{|[n_i]|} \;\forall\, i = 1, \ldots\ldots, 7 \\ &= \frac{\left|\overline{G}\right|}{120} = \frac{\left|\overline{G}\right|}{|G|} = |N|.\end{aligned}$$

6.2.4 The Character Table of $SL(2,5)$

We reproduce here the character table of the complement $SL(2,5)$. This character table forms a block of the character table of $29^2 : SL(2,5)$. In the character table we display a representative of each of the nine conjugacy classes and the nine irreducible characters of $SL(2,5)$.

Table 6.2: Character Table of $SL(2,5)$

	$[1]$	$[g_1]$	$[g_2]$	$[g_3]$	$[g_4]$	$[g_5]$	$[g_6]$	$[g_7]$	$[g_8]$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	A	A^*	-2	$-A$	$-A^*$	-1	1	0
χ_3	2	A^*	A	-2	$-A^*$	$-A^*$	-1	1	0
χ_4	3	A	A^*	3	A^*	A	0	0	-1
χ_5	3	A	A^*	3	A	A^*	0	0	1
χ_6	4	-1	-1	4	-1	-1	1	1	0
χ_7	4	1	1	-4	-1	-1	1	-1	0
χ_8	5	0	0	5	0	0	-1	-1	1
χ_9	6	-1	-1	-6	1	1	0	0	0

where $A = \frac{1-\sqrt{5}}{2}$ and $A^* = \frac{1+\sqrt{5}}{2}$.

6.2.5 Construction of the Character Table of $29^2 : SL(2,5)$

We will show here how the character table of $29^2; SL(2,5)$ is completed.

1. If $\widehat{\chi}$ is a character of $\overline{G}/N \cong G$, then by **lifting** of characters we have that $\chi(g) = \widehat{\chi}(gN)$ for $g \in G$. So for $n \in N$,
$$\chi_i(n) = \widehat{\chi_i}(nN) = \widehat{\chi_i}(N) = \widehat{\chi_i}\left(1_{\overline{G}/N}\right) \ \forall i = 2, \ldots\ldots, 9.$$

2. Now $\chi_i(g) = \widehat{\chi_i}(gN)$ for $g \in G$. But by Lemma 3.2.15 $gN \subseteq [g]$ for $g \in G$. So $\widehat{\chi_i}(gN) = \widehat{\chi_i}(g) = \chi_i(g)$ $\forall i = 2, \ldots\ldots, 9$. Therefore the column under each $[g_i]$ in the character table of $29^2 : SL(2,5)$ will be the same as the column under each $[g_i]$ in the character table of $SL(2,5)$.

3. Also $\chi_i(g_i) = 0$ for $i = 10, \ldots\ldots, 16$ and $\forall g_i \in G$ since by Proposition 4.2.4 in Moori [17], we have that $N \cap [g_i]_{\overline{G}} = \emptyset$.

4. By Proposition 4.2.4 in Moori [17], since $N \cap [n_j]_{\overline{G}} \neq \emptyset$,
$$\chi(n_j) = \left|C_{\overline{G}}(n_j)\right| \sum_{i=1}^{m} \frac{\phi(x_i)}{|C_N(x_i)|}, \tag{6.1}$$

where $x_1, x_2, \ldots\ldots, x_m$ are class representatives of classes of N that fuse to $[n_j]_{\overline{G}}$, n_j for $j = 1, \ldots\ldots, 7$ are representatives of the conjugacy classes $[n_1], \ldots\ldots, [n_7]$ and $\chi = \phi^{\overline{G}}$ for $\phi_1 \neq \phi \in \mathrm{Irr}\,(N)$.

5. First we construct the character $\phi \in \mathrm{Irr}\,(N)$ using the **direct product of characters**. Taking the identity character ψ_1 of $\mathbb{Z}_{29}$ and the character ψ_2 of $\mathbb{Z}_{29}$, and using the direct product of characters, we construct the character $\phi = \psi_1.\psi_2$ of $N = 29^2$, where $\phi\,(g) = \psi_1\,(n)\,.\psi_2\,(n')$ and $g = nn'$ for $n, n' \in \mathbb{Z}_{29}$ and $g \in \mathbb{Z}_{29} \bigotimes \mathbb{Z}_{29}$.

6. Since ψ_1 is the identity character of $\mathbb{Z}_{29}$, $\phi\,(g) = \psi_2\,(n')$ and since $\mathbb{Z}_{29}$ has order 29, $\psi_2\,(n')$ is a $29th$ root of unity $\forall n' \in \mathbb{Z}_{29}$.

7. Now N has 29^2 conjugacy classes which we denote as :

$$\begin{array}{cccc} [1a, 1a] & [1a, 29a] & \cdots & [1a, 29ab] \\ [29a, 1a] & [29a, 29a] & \cdots & [29a, 29ab] \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ [29ab, 1a] & [29ab, 29a] & \cdots & [29ab, 29ab] \end{array}$$

The notation here is consistent with that of GAP. Thus the values of χ_2 appears in the character table of $N = 29^2$ as 29 cycles of the 29th roots of unity as follows:

$$\begin{array}{ccccc} 1 & \chi_2(29a) & \chi_2(29b) & \cdots\cdots & \chi_2(29ab) \\ 1 & \chi_2(29a) & \chi_2(29b) & \cdots\cdots & \chi_2(29ab) \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 1 & \chi_2(29a) & \chi_2(29b) & \cdots\cdots & \chi_2(29ab) \end{array}$$

8. Let n_i for $i = 1, \ldots\ldots, 7$ be representatives of the conjugacy classes $[n_1], [n_2], \ldots\ldots, [n_7]$. Then $x \in [n_i]$ implies that $x^{-1} \in [n_i]\ \forall i = 1, 2, \ldots\ldots, 7$. Thus $\chi_j(x) = \chi_j(x^{-1})\ \forall \chi_j$ where $j = 10, \ldots\ldots, 16$. So $\chi_j(x) \in \mathbb{R}$ since $\chi_j(x) = \chi_j(x^{-1}) = \overline{\chi_j(x)}$ for $j = 10, \ldots\ldots, 16$.

9. Now by (6.1) above ,

$$\begin{aligned} \phi^G(n_j) &= \left|C_{\overline{G}}(n_j)\right| \sum_{i=1}^{m} \frac{\phi(x_i)}{|C_N(x_i)|} \\ &= 29^2 \sum_{i=1}^{m} \frac{\phi(x_i)}{29^2} \\ &= \sum_{i=1}^{m} \phi(x_i), \end{aligned}$$

where the x_i are class representatives of the conjugacy classes of N which fuse to give $[n_j]$, where $[n_j]$ for $i = 1, \ldots\ldots, 7$ are class representatives of the classes $[n_1], [n_2], \ldots\ldots, [n_7]$ and

$\phi \in \mathrm{Irr}(N)$ is as constructed in (5) above. Consider now the conjugacy class $[n_1]$ (note that $n_1 \neq 1_N$).

10. The character χ_{10} is one of 7 irreducible characters of the form $\phi^{\overline{G}}$ for $\phi \in \mathrm{Irr}(N)$. So let $\phi_1 = \psi_1 \cdot \psi_2$ be as constructed in (5) above. Then $\psi_{10} = \phi_1^{\overline{G}} = (\psi_1 \cdot \psi_2)^{\overline{G}}$. By (6.1) above we have that:

$$\begin{aligned}\chi_{10}(n_1) &= \sum_{i=1}^{120} \phi(x_i) \\ &= \sum_{i=1}^{120} (\psi_1 \cdot \psi_2)(x_i) \\ &= \sum_{i=1}^{120} \psi_2(x_i),\end{aligned}$$

where the x_i are the 120 elements(vectors) of N which fuse to form $[n_1]$.

11. Since there are 120 elements in each conjugacy class and each element in the conjugacy class containing n_1 has it's inverse also in the class, we can group these 120 elements into 60 pairs. Now $\psi_2\left(x_i^{-1}\right) = \overline{\psi_2(x_i)}$ so $\psi_2(x_i) + \psi_2\left(x_i^{-1}\right) = 2r$ where $\alpha = r + si$ is a 29th root of unity. Using the character table of $\mathbb{Z}_{29}$ we can now find the value of the character ψ_{10} on the conjugacy class $[n_1]$.

12. Listed below are the 120 elements paired with their inverses of the conjugacy class $[n_1]$. The remaining six conjugacy classes are shown in Section A of the Appendix.
$[n_1] = \{(0,1)\&(0,28); (28,0)\&(1,0); (17,23)\&(12,6); (1,28)\&(28,1); (0,12)\&(0,17);$
$(23,17)\&(6,12); (14,6)\&(15,23); (12,11)\&(17,18); (17,0)\&(12,0); (1,15)\&(28,14);$
$(23,23)\&(6,6); (21,14)\&(8,15); (15,28)\&(14,1); (23,12)\&(6,17); (13,9)\&(16,20);$
$(12,17)\&(17,12); (15,1)\&(14,28); (23,14)\&(6,15); (14,22)\&(15,7); (6,21)\&(23,8);$
$(16,15)\&(13,14); (11,23)\&(18,6); (7,13)\&(22,16); (8,14)\&(21,15); (28,16)\&(1,13);$
$(15,15)\&(14,14); (20,23)\&(9,6); (26,10)\&(3,19); (9,24)\&(20,5); (14,20)\&(15,9);$
$(9,7)\&(20,22); (22,8)\&(7,21); (11,21)\&(18,8); (23,3)\&(6,26); (7,26)\&(22,3);$
$(25,9)\&(4,20); (2,22)\&(27,7); (21,27)\&(8,2); (3,21)\&(26,8); (26,11)\&(3,18);$
$(9,23)\&(20,6); (22,4)\&(7,25); (10,7)\&(19,22); (24,25)\&(5,4); (9,2)\&(20,27);$
$(10,21)\&(19,8); (24,3)\&(5,26); (9,26)\&(20,3); (21,26)\&(8,3); (3,9)\&(26,20);$
$(26,22)\&(3,7); (3,17)\&(26,12); (22,9)\&(7,20); (27,10)\&(2,19); (21,24)\&(8,5);$
$(12,9)\&(17,20); (4,26)\&(25,3); (28,21)\&(1,8); (7,1)\&(22,28); (21,22)\&(8,7)\}$

Now evaluating ψ_2 at the second member of each of these ordered pairs, we get :

$$\begin{aligned}\chi_{10}(n_1) &= \sum_{i=1}^{120} \psi_2(x_i) \quad \text{where } x_i \in [n_1] \\ &= 4 + 10\omega_1 + 4\omega_2 + 12\omega_3 + 4\omega_4 + 4\omega_5 \\ &+ 12\omega_6 + 12\omega_7 + 12\omega_8 + 12\omega_9 + 4\omega_{10} \\ &+ 4\omega_{11} + 10\omega_{12} + 4\omega_{13} + 12\omega_{14} = a,\end{aligned}$$

where $\omega_k = \cos\left(\frac{2k\pi}{29}\right)$ for $k = 1, 2, \ldots\ldots, 14$.
Similarly, for the remaining 6 conjugacy classes we get:

$$\begin{aligned}\chi_{10}(n_2) &= 4 + 12\omega_1 + 10\omega_2 + 4\omega_3 + 4\omega_4 + 10\omega_5 \\ &+ 12\omega_6 + 4\omega_7 + 4\omega_8 + 4\omega_9 + 4\omega_{10} \\ &+ 12\omega_{11} + 12\omega_{12} + 12\omega_{13} + 12\omega_{14} = b.\end{aligned}$$

$$\begin{aligned}\chi_{10}(n_3) &= 4 + 12\omega_1 + 12\omega_2 + 12\omega_3 + 10\omega_4 + 12\omega_5 \\ &+ 4\omega_6 + 12\omega_7 + 4\omega_8 + 4\omega_9 + 10\omega_{10} \\ &+ 4\omega_{11} + 12\omega_{12} + 4\omega_{13} + 4\omega_{14} = c.\end{aligned}$$

$$\begin{aligned}\chi_{10}(n_4) &= 4 + 4\omega_1 + 12\omega_2 + 4\omega_3 + 12\omega_4 + 12\omega_5 \\ &+ 12\omega_6 + 4\omega_7 + 10\omega_8 + 10\omega_9 + 12\omega_{10} \\ &+ 4\omega_{11} + 4\omega_{12} + 4\omega_{13} + 12\omega_{14} = d.\end{aligned}$$

$$\begin{aligned}\chi_{10}(n_5) &= 4 + 12\omega_1 + 4\omega_2 + 4\omega_3 + 12\omega_4 + 4\omega_5 \\ &+ 4\omega_6 + 4\omega_7 + 12\omega_8 + 12\omega_9 + 12\omega_{10} \\ &+ 10\omega_{11} + 12\omega_{12} + 10\omega_{13} + 4\omega_{14} = e.\end{aligned}$$

$$\begin{aligned}\chi_{10}(n_6) &= 4 + 4\omega_1 + 12\omega_2 + 10\omega_3 + 4\omega_4 + 12\omega_5 \\ &+ 4\omega_6 + 10\omega_7 + 12\omega_8 + 12\omega_9 + 4\omega_{10} \\ &+ 12\omega_{11} + 4\omega_{12} + 12\omega_{13} + 4\omega_{14} = f.\end{aligned}$$

$$\begin{aligned}\chi_{10}(n_7) &= 4 + 4\omega_1 + 4\omega_2 + 12\omega_3 + 12\omega_4 + 4\omega_5 \\ &+ 10\omega_6 + 12\omega_7 + 4\omega_8 + 4\omega_9 + 12\omega_{10} \\ &+ 12\omega_{11} + 4\omega_{12} + 12\omega_{13} + 10\omega_{14} = g.\end{aligned}$$

13. So for the character χ_{10} we have the following:

Table 6.3: The Values of the Character χ_{10}

Class$[\overline{g}]$	[1]	$[n_1]$	$[n_2]$	$[n_3]$	$[n_4]$	$[n_5]$	$[n_6]$	$[n_7]$
$\lvert[\overline{g}]\rvert$	1	120	120	120	120	120	120	120
$\circ(\overline{g})$	1	29	29	29	29	29	29	29
$\lvert C_{\overline{G}}(\overline{g})\rvert$	$29^2.120$	29^2	29^2	29^2	29^2	29^2	29^2	29^2
χ_{10}	120	a	b	c	d	e	f	g

Table 6.3 (continued)

Class$[\overline{g}]$	$[g_1]$	$[g_2]$	$[g_3]$	$[g_4]$	$[g_5]$	$[g_6]$	$[g_7]$	$[g_8]$
$\lvert[\overline{g}]\rvert$	$29^2.12$	$29^2.12$	29^2	$29^2.12$	$29^2.12$	$29^2.20$	$29^2.20$	$29^2.30$
$\circ(\overline{g})$	10	10	2	5	5	3	6	4
$\lvert C_{\overline{G}}(\overline{g})\rvert$	10	10	120	10	10	6	6	4
χ_{10}	0	0	0	0	0	0	0	0

14. We can now complete the character table of $29^2 : \mathrm{SL}(2,5)$ by completing the values of the irreducible characters χ_i for $i = 11, \ldots\ldots, 16$ by repeating the process described in (5) - (12) above. Each of these irreducible characters are constructed by taking the direct product of irreducible characters of $\mathbb{Z}_{29}$. We list these characters below and then put their values on the conjugacy classes of our group (the computations for each of these characters are shown in Sections C - H of the Appendix) thus completing the character table of the group.

- $\chi_{11} = \phi_2^{\overline{G}} = (\psi_1 \cdot \psi_3)^{\overline{G}}$
- $\chi_{12} = \phi_3^{\overline{G}} = (\psi_1 \cdot \psi_4)^{\overline{G}}$
- $\chi_{13} = \phi_4^{\overline{G}} = (\psi_1 \cdot \psi_5)^{\overline{G}}$
- $\chi_{14} = \phi_5^{\overline{G}} = (\psi_1 \cdot \psi_7)^{\overline{G}}$
- $\chi_{15} = \phi_6^{\overline{G}} = (\psi_1 \cdot \psi_9)^{\overline{G}}$
- $\chi_{16} = \phi_7^{\overline{G}} = (\psi_1 \cdot \psi_{12})^{\overline{G}}$

6.2.6 The Character Table of $29^2 : \mathrm{SL}(2,5)$

Table 6.4: Character Table of $\overline{G} = 29^2 : \mathrm{SL}(2,5)$

Class$[\overline{g}]$	$[1]$	$[n_1]$	$[n_2]$	$[n_3]$	$[n_4]$	$[n_5]$	$[n_6]$	$[n_7]$	$[g_1]$	$[g_2]$	$[g_3]$	$[g_4]$	$[g_5]$	$[g_6]$	$[g_7]$	$[g_8]$
$\lvert[\overline{g}]\rvert$	$[1]$	120	120	120	120	120	120	120	$29^2.12$	$29^2.12$	29^2	$29^2.12$	$29^2.12$	$29^2.20$	$29^2.20$	$29^2.30$
$\circ(\overline{g})$	$[1]$	29	29	29	29	29	29	29	10	10	2	5	5	3	6	4
$\lvert C_{\overline{G}}(\overline{g})\rvert$	$29^2.120$	29^2	29^2	29^2	29^2	29^2	29^2	29^2	10	10	120	10	10	6	6	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	2	2	2	2	2	2	2	2	A	A^*	-2	$-A$	$-A^*$	-1	1	0
χ_3	2	2	2	2	2	2	2	2	A^*	A	-2	$-A^*$	$-A^*$	-1	1	0
χ_4	3	3	3	3	3	3	3	3	A^*	A	3	A^*	A	0	0	-1
χ_5	3	3	3	3	3	3	3	3	A	A^*	3	A	A^*	0	0	-1
χ_6	4	4	4	4	4	4	4	4	-1	-1	4	-1	-1	1	1	0
χ_7	4	4	4	4	4	4	4	4	1	1	-4	-1	-1	1	-1	0
χ_8	5	5	5	5	5	5	5	5	0	0	5	0	0	-1	-1	1
χ_9	6	6	6	6	6	6	6	6	-1	-1	-6	1	1	0	0	0
χ_{10}	120	a	b	c	d	e	f	g	0	0	0	0	0	0	0	0
χ_{11}	120	b	c	d	e	f	g	a	0	0	0	0	0	0	0	0
χ_{12}	120	f	g	a	b	c	d	e	0	0	0	0	0	0	0	0
χ_{13}	120	c	d	e	f	g	a	b	0	0	0	0	0	0	0	0
χ_{14}	120	g	a	b	c	d	e	f	0	0	0	0	0	0	0	0
χ_{15}	120	d	e	f	g	a	b	c	0	0	0	0	0	0	0	0
χ_{16}	120	e	f	g	a	b	c	d	0	0	0	0	0	0	0	0

For values of a,b,c,d,e,f and g see 6.2.5 (12).

6.3 The Fischer Matrices of the Group: $29^2 : \mathrm{SL}(2,5)$

We construct here the Fischer Matrices for our group $29^2 : \mathrm{SL}(2,5)$. As we described in Section 5.4, the Fischer matrices for a Frobenius group comprise of one $(m+1) \times (m+1)$ matrix where m is the number of non-trivial orbits of the action of G on N, and the remaining Fischer matrices are just 1×1 matrices with the sole entry being 1. So we have: $\forall 1_G \neq g \in \overline{G}, M(g) = [1]$.

The Fischer matrix corresponding to the identity element of $\overline{G} = 29^2 : \mathrm{SL}(2,5)$ is a 8×8 matrix given by:

$$M(1_G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 120 & a & b & c & d & e & f & g \\ 120 & b & c & d & e & f & g & a \\ 120 & f & g & a & b & c & d & e \\ 120 & c & d & e & f & g & a & b \\ 120 & g & a & b & c & d & e & f \\ 120 & d & e & f & g & a & b & c \\ 120 & e & f & g & a & b & c & d \end{bmatrix}$$

7

Appendix

7.1 Section A

The remaining six conjugacy classes $[n_i]$ $i = 2, \ldots, 7$ of the group $29^2 : SL(2,5)$.

$[n_2] = \{(0,27)\&(0,2); (2,0)\&(27,0); (24,12)\&(5,17); (27,2)\&(2,27); (0,5)\&(0,24);$
$(12,24)\&(17,5); (1,17)\&(28,12); (5,7)\&(24,22); (24,0)\&(5,0); (27,28)\&(2,1);$
$(12,12)\&(17,17); (16,1)\&(13,28); (28,2)\&(1,27); (12,5)\&(17,24); (3,11)\&(26,18);$
$(5,24)\&(24,5); (28,27)\&(1,2); (12,1)\&(17,28); (1,14)\&(28,15); (17,16)\&(12,13);$
$(26,28)\&(3,1); (7,12)\&(22,17); (15,3)\&(14,26); (13,1)\&(16,28); (2,26)\&(27,3);$
$(28,28)\&(1,1); (18,12)\&(11,17); (6,9)\&(23,20); (11,10)\&(18,19); (1,18)\&(28,11);$
$(11,15)\&(18,14); (14,13)\&(15,16); (7,16)\&(22,13); (12,23)\&(17,6); (15,6)\&(14,23);$
$(8,11)\&(21,18); (25,14)\&(4,15); (16,4)\&(13,25); (23,16)\&(6,13)\&(6,7)\&(23,22);$
$(11,12)\&(18,17); (14,21)\&(15,8); (9,15)\&(20,14); (10,8)\&(19,21); (11,25)\&(18,4);$
$(9,16)\&(20,13); (10,23)\&(19,6); (11,6)\&(18,23); (16,6)\&(13,23); (23,11)\&(6,18);$
$(6,14)\&(23,15); (23,24)\&(6,5); (14,11)\&(15,18); (4,9)\&(25,20); (16,10)\&(13,19);$
$(5,11)\&(24,18); (21,6)\&(8,23); (2,16)\&(27,13); (16,14)\&(13,15); (14,2)\&(15,27)\}.$

$[n_3] = \{(0,4)\&(0,25); (4,25)\&(25,4); (10,5)\&(19,24); (0,19)\&(0,10); (4,0)\&(25,0);$
$(5,10)\&(24,19); (27,24)\&(2,5); (19,15)\&(10,14); (10,0)\&(19,0); (4,2)\&(25,27);$
$(5,5)\&(24,24); (26,27)\&(3,2); (2,25)\&(27,4); (5,19)\&(24,10); (23,7)\&(6,22);$
$(19,10)\&(10,19); (2,4)\&(27,25); (5,27)\&(24,2); (27,1)\&(2,28); (24,26)\&(5,3);$
$(6,2)\&(23,27); (15,5)\&(14,24); (28,23)\&(1,6); (3,27)\&(26,2); (25,6)\&(4,23);$
$(2,2)\&(27,27); (22,5)\&(7,24); (17,11)\&(12,18); (7,9)\&(22,20); (27,22)\&(2,7);$
$(7,28)\&(22,1); (1,3)\&(28,26); (15,26)\&(14,3); (5,12)\&(24,17); (28,17)\&(1,12);$
$(13,7)\&(16,22); (8,1)\&(21,28); (26,21)\&(3,8); (12,26)\&(17,3); (17,15)\&(12,14);$
$(7,5)\&(22,24); (1,16)\&(28,13); (11,28)\&(18,1); (9,13)\&(20,16); (7,8)\&(22,21);$
$(11,26)\&(18,3); (9,12)\&(20,17); (7,17)\&(22,12); (26,17)\&(3,12); (12,7)\&(17,22);$
$(17,1)\&(12,28); (1,7)\&(28,22); (21,11)\&(8,18); (26,9)\&(3,20); (19,7)\&(10,22);$

$(16,17)\&(13,12);(25,26)\&(4,3);(17,19)\&(12,10);(3,28)\&(26,1);(28,4)\&(1,25)\}.$

$[n_4]=\{(0,21)\&(0,8);(8,0)\&(21,0);(9,19)\&(20,10);(21,8)\&(8,21);(0,20)\&(0,9);$
$(19,9)\&(10,20);(4,10)\&(25,19);(20,28)\&(9,1);(9,0)\&(20,0);(21,25)\&(8,4);$
$(19,19)\&(10,10);(6,4)\&(23,25);(25,8)\&(4,21);(19,20)\&(10,9);(12,15)\&(17,14);$
$(20,9)\&(9,20);(25,21)\&(4,8);(19,4)\&(10,25);(4,27)\&(25,2);(10,6)\&(19,23);$
$(17,25)\&(12,4);(28,19)\&(1,10);(2,12)\&(27,17);(23,4)\&(6,25);(8,17)\&(21,12);$
$(25,25)\&(4,4);(14,19)\&(15,10);(24,7)\&(5,22);(15,11)\&(14,18);(4,14)\&(25,15);$
$(15,2)\&(14,27);(27,23)\&(2,6);(28,6)\&(1,23);(19,5)\&(10,24);(2,24)\&(27,5);$
$(3,15)\&(26,14);(13,27)\&(16,2);(6,16)\&(23,13);(5,6)\&(24,23);(24,28)\&(5,1);$
$(15,19)\&(14,10);(27,26)\&(2,3);(7,2)\&(22,27);(11,3)\&(18,26);(15,13)\&(14,16);$
$(7,6)\&(22,23);(11,5)\&(18,24);(15,24)\&(14,5);(6,24)\&(23,5);(5,15)\&(24,14);$
$(24,27)\&(5,2);(5,9)\&(24,20);(27,15)\&(2,14);(16,7)\&(13,22);(6,11)\&(23,18);$
$(20,15)\&(9,14);(26,24)\&(3,5);(8,6)\&(21,23);(23,2)\&(6,27);(2,21)\&(27,8)\}.$

$[n_5]=\{(0,16)\&(0,13);(13,0)\&(16,0);(11,20)\&(18,9);(16,13)\&(13,16);(0,18)\&(0,11);$
$(20,11)\&(9,18);(21,9)\&(8,20);(18,2)\&(11,27);(11,0)\&(18,0);(16,8)\&(13,21);$
$(20,20)\&(9,9);(17,21)\&(12,8);(8,13)\&(21,16);(20,18)\&(9,11);(5,28)\&(24,1);$
$(18,11)\&(11,18);(8,16)\&(21,13);(20,21)\&(9,8);(21,4)\&(8,25);(9,17)\&(20,12);$
$(24,8)\&(5,21);(2,20)\&(27,9);(25,5)\&(4,24);(12,21)\&(17,8);(13,24)\&(16,5);$
$(8,8)\&(21,21);(1,20)\&(28,9);(10,15)\&(19,14);(28,7)\&(1,22);(21,1)\&(8,28);$
$(28,25)\&(1,4);(4,12)\&(25,17);(2,17)\&(27,12);(20,19)\&(9,10);(25,10)\&(4,19);$
$(23,28)\&(6,1);(3,4)\&(26,25);(17,26)\&(12,3);(19,17)\&(10,12);(10,2)\&(19,27);$
$(28,20)\&(1,9);(4,6)\&(25,23);(15,25)\&(14,4);(7,23)\&(22,6);(28,3)\&(1,26);$
$(15,17)\&(14,12);(7,19)\&(22,10);(28,10)\&(1,19);(17,10)\&(12,19);(19,28)\&(10,1);$
$(10,4)\&(19,25);(19,11)\&(10,18);(4,28)\&(25,1);(26,15)\&(3,14);(17,7)\&(12,22);$
$(18,28)\&(11,1);(6,10)\&(23,19);(13,17)\&(16,12);(17,4)\&(12,25);(25,16)\&(4,13)\}.$

$[n_6]=\{(0,26)\&(0,3);(3,0)\&(26,0);(7,18)\&(22,11);(0,22)\&(0,7);(26,3)\&(3,26);$
$(18,7)\&(11,22);(16,11)\&(13,18);(22,25)\&(7,4);(7,0)\&(22,0);(26,13)\&(3,16);$
$(18,18)\&(11,11);(24,16)\&(5,13);(13,3)\&(16,26);(18,22)\&(11,7);(19,2)\&(10,27);$
$(22,7)\&(7,22);(13,26)\&(16,3);((18,16)\&(11,13);(16,21)\&(13,8);(11,24)\&(18,5);$
$(10,13)\&(19,16);(25,18)\&(4,11);(8,19)\&(21,10);(5,16)\&(24,13);(3,10)\&(26,19);$
$(13,13)\&(16,16);(27,18)\&(2,11);(9,28)\&(20,1);(2,15)\&(27,14);(16,27)\&(13,2);$
$(2,8)\&(27,21);(21,5)\&(8,24);(25,24)\&(4,5);(18,20)\&(11,9);(8,9)\&(21,20);$
$(17,27)\&(12,2);(6,8)\&(23,21);(5,23)\&(24,6);(9,5)\&(20,24);(20,4)\&(9,25);$
$(27,11)\&(2,18);(8,12)\&(21,17);(1,21)\&(28,8);(14,17)\&(15,12);(27,6)\&(2,23);$
$(1,5)\&(28,24);(14,9)\&(15,20);(27,20)\&(2,9);(5,20)\&(24,9);(9,27)\&(20,2);$
$(20,8)\&(9,21);(9,22)\&(20,7);(21,2)\&(8,27);(6,28)\&(23,1);(24,15)\&(5,14);$
$(22,2)\&(7,27);(17,9)\&(12,20);(3,24)\&(26,5);(5,8)\&(24,21);(8,26)\&(21,3)\}.$

$[n_7] = \{(0,6)\&(0,23); (23,0)\&(6,0); (15,22)\&(14,7); (6,23)\&(23,6); (0,14)\&(0,15);$
$(26,7)\&(3,22); (22,15)\&(7,14); (14,8)\&(15,21); (15,0)\&(14,0); (6,3)\&(23,26);$
$(22,22)\&(7,7); (10,26)\&(19,3); (3,23)\&(26,6); (22,14)\&(7,15); (20,25)\&(9,4);$
$(14,15)\&(15,14); (3,6)\&(26,23); (22,26)\&(7,3); (26,16)\&(3,13); (7,10)\&(22,19);$
$(9,3)\&(20,26); (8,22)\&(21,7); (13,20)\&(16,9); (19,26)\&(10,3); (23,9)\&(6,20);$
$(3,3)\&(26,26); (4,22)\&(25,7); (11,2)\&(18,27); (25,28)\&(4,1); (26,4)\&(3,25);$
$(25,13)\&(4,16); (16,19)\&(13,10); (8,10)\&(21,19); (22,18)\&(7,11); (13,11)\&(16,18);$
$(5,25)\&(24,4); (12,16)\&(17,13); (10,17)\&(19,12); (18,10)\&(11,19); (11,8)\&(18,21);$
$(25,22)\&(4,7); (16,24)\&(13,5); (2,13)\&(27,16); (28,5)\&(1,24); (25,12)\&(4,17);$
$(2,10)\&(27,19); (28,18)\&(1,11); (25,11)\&(4,18); (10,11)\&(19,18); (18,25)\&(11,4);$
$(11,16)\&(18,13); (18,15)\&(11,14); (16,25)\&(13,4); (17,2)\&(12,27); (10,28)\&(19,1);$
$(14,25)\&(15,4); (24,11)\&(5,18); (23,10)\&(6,19); (19,13)\&(10,16); (13,6)\&(16,23)\}.$

7.2 Section B

7.2.1 Character Values of χ_{10}

Computing the character values for the character χ_{10} for the conjugacy classes $[n_1], [n_2], \ldots, [n_7]$. Now $\chi_{10} = \overline{\phi_1^G} = (\psi_1.\psi_2)$. Also $\chi_{10}(n_i) = \sum_{i=1}^{120} \psi_2(x_i)$ where the x_i are the 120 vectors which fuse to form the conjugacy class $[n_i]$ (n_i a representative of the class).
Now $\psi_2(x_i)$ where $x_i \in \{0, 1, \ldots, 28\}$ equals ψ_2 evaluated at the second entry of the 120 vectors that make up the conjugacy class $[n_i]$. These are the values in the character table of $\mathbb{Z}_{29}$.
Tabulated below are the values of $\psi_2(g)$ for $g \in \mathbb{Z}_{29}$.

Table 7.1: The Values of the Character ψ_2

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
ψ_2	1	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}

Table 7.1 (continued)

g	15	16	17	18	19	20	21	22	23	24	25	26	27	28
ψ_2	$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$	$\overline{\omega}_6$	$\overline{\omega}_7$	$\overline{\omega}_8$	$\overline{\omega}_9$	$\overline{\omega}_{10}$	$\overline{\omega}_{11}$	$\overline{\omega}_{12}$	$\overline{\omega}_{13}$	$\overline{\omega}_{14}$

where $\omega_i = e^{\frac{2k\pi i}{29}}$ and $\overline{\omega}_i$ is the complex conjugate of ω_i.

$\underline{\chi_{10}(n_1):}$

Since $\chi_{10} = \sum_{i=1}^{120} \psi_2(x_i)$, to compute the value $\chi_{10}(n_1)$, run through the conjugacy class $[n_1]$ and count the number of pairs with $0, 1, 2, \ldots, 14$ in the second entry position. Count these digits only once in each pair. Then find the value of $\psi_2(y)$ where $y \in \mathbb{Z}_{29}$ in the table 7.1. Each of these values must then be multiplied by two (except for y=0). For $[n_1]$ there are two pairs with 0, 2, 4, 5, 10, 11 and 13; five pairs with 1 and 12 and six pairs with 3, 6, 7, 8, 9 and 14.
Hence,

$$\begin{aligned}\chi_{10}(n_1) = \;& 2\times 2 + 2\times 5\times \omega_1 + 2\times 2\times \omega_2 + 2\times 6\times \omega_3 + 2\times 2\times \omega_4 + 2\times 2\times \omega_5 \\ & +2\times 6\times \omega_6 + 2\times 6\times \omega_7 + 2\times 6\times \omega_8 + 2\times 6\times \omega_9 + 2\times 2\times \omega_{10} \\ & +2\times 2\times \omega_{11} + 2\times 5\times \omega_{12} + 2\times 2\times \omega_{13} + 2\times 6\times \omega_{14}.\end{aligned}$$

So

$$\begin{aligned}\chi_{10}(n_1) = \;& 4 + 10\omega_1 + 4\omega_2 + 12\omega_3 + 4\omega_4 + 4\omega_5 + 12\omega_6 + 12\omega_7 \\ & +12\omega_8 + 12\omega_9 + 4\omega_{10} + 4\omega_{11} + 10\omega_{12} + 4\omega_{13} + 12\omega_{14} = a.\end{aligned}$$

$\underline{\chi_{10}(n_2):}$

For $[n_2]$, there are two pairs with 0, 3, 4, 7, 8, 9 and 10; five pairs with 2 and 5 and six pairs with 1, 6, 11, 12, 13 and 14.
Hence,

$$\begin{aligned}\chi_{10}(n_2) = \;& 2\times 2 + 2\times 6\times \omega_1 + 2\times 5\times \omega_2 + 2\times 2\times \omega_3 + 2\times 2\times \omega_4 + 2\times 5\times \omega_5 \\ & +2\times 6\times \omega_6 + 2\times 2\times \omega_7 + 2\times 2\times \omega_8 + 2\times 2\times \omega_9 + 2\times 2\times \omega_{10} \\ & +2\times 6\times \omega_{11} + 2\times 6\times \omega_{12} + 2\times 6\times \omega_{13} + 2\times 6\times \omega_{14}.\end{aligned}$$

So

$$\begin{aligned}\chi_{10}(n_2) = \;& 4 + 12\omega_1 + 10\omega_2 + 4\omega_3 + 4\omega_4 + 10\omega_5 + 12\omega_6 + 4\omega_7 \\ & +4\omega_8 + 4\omega_9 + 4\omega_{10} + 12\omega_{11} + 12\omega_{12} + 12\omega_{13} + 12\omega_{14} = b.\end{aligned}$$

$\underline{\chi_{10}(n_3):}$

For $[n_3]$, there are two pairs with 0, 6, 8, 9, 11, 13 and 14; five pairs with 4 and 10 and six pairs with 1, 2, 3, 5, 7 and 12.
Hence,

$$\begin{aligned}\chi_{10}(n_3) = \;& 2\times 2 + 2\times 12\times \omega_1 + 2\times 12\times \omega_2 + 2\times 12\times \omega_3 + 2\times 10\times \omega_4 + 2\times 12\times \omega_5 \\ & +2\times 4\times \omega_6 + 2\times 12\times \omega_7 + 2\times 4\times \omega_8 + 2\times 4\times \omega_9 + 2\times 10\times \omega_{10} \\ & +2\times 4\times \omega_{11} + 2\times 12\times \omega_{12} + 2\times 4\times \omega_{13} + 2\times 4\times \omega_{14}.\end{aligned}$$

So

$$\begin{aligned}\chi_{10}(n_3) = \quad & 4 + 12\omega_1 + 12\omega_2 + 12\omega_3 + 10\omega_4 + 12\omega_5 + 4\omega_6 + 12\omega_7 \\ & +4\omega_8 + 4\omega_9 + 10\omega_{10} + 4\omega_{11} + 12\omega_{12} + 4\omega_{13} + 4\omega_{14} = c.\end{aligned}$$

$\underline{\chi_{10}(n_4):}$

For $[n_4]$, there are two pairs with 0, 1, 3, 7, 11, 12 and 14; five pairs with 8 and 9 and six pairs with 2, 4, 5, 6, 10 and 14.

Hence,

$$\begin{aligned}\chi_{10}(n_4) = \quad & 2\times 2 + 2\times 4\times \omega_1 + 2\times 12\times \omega_2 + 2\times 4\times \omega_3 + 2\times 12\times \omega_4 + 2\times 12\times \omega_5 \\ & +2\times 12\times \omega_6 + 2\times 4\times \omega_7 + 2\times 10\times \omega_8 + 2\times 10\times \omega_9 + 2\times 12\times \omega_{10} \\ & +2\times 4\times \omega_{11} + 2\times 4\times \omega_{12} + 2\times 4\times \omega_{13} + 2\times 12\times \omega_{14}.\end{aligned}$$

So

$$\begin{aligned}\chi_{10}(n_4) = \quad & 4 + 4\omega_1 + 12\omega_2 + 4\omega_3 + 12\omega_4 + 12\omega_5 + 12\omega_6 + 4\omega_7 \\ & +10\omega_8 + 10\omega_9 + 12\omega_{10} + 4\omega_{11} + 4\omega_{12} + 4\omega_{13} + 12\omega_{14} = d.\end{aligned}$$

$\underline{\chi_{10}(n_5):}$

For $[n_5]$, there are two pairs with 0, 2, 3, 5, 6, 7 and 14; five pairs with 11 and 13 and six pairs with 2, 4, 8, 9, 10 and 12.

Hence,

$$\begin{aligned}\chi_{10}(n_5) = \quad & 2\times 2 + 2\times 12\times \omega_1 + 2\times 4\times \omega_2 + 2\times 4\times \omega_3 + 2\times 12\times \omega_4 + 2\times 4\times \omega_5 \\ & +2\times 4\times \omega_6 + 2\times 4\times \omega_7 + 2\times 12\times \omega_8 + 2\times 12\times \omega_9 + 2\times 12\times \omega_{10} \\ & +2\times 10\times \omega_{11} + 2\times 12\times \omega_{12} + 2\times 10\times \omega_{13} + 2\times 4\times \omega_{14}.\end{aligned}$$

So

$$\begin{aligned}\chi_{10}(n_5) = \quad & 4 + 12\omega_1 + 4\omega_2 + 4\omega_3 + 12\omega_4 + 4\omega_5 + 4\omega_6 + 4\omega_7 \\ & +12\omega_8 + 12\omega_9 + 12\omega_{10} + 10\omega_{11} + 12\omega_{12} + 10\omega_{13} + 4\omega_{14} = e.\end{aligned}$$

$\underline{\chi_{10}(n_6):}$

For $[n_6]$, there are two pairs with 0, 1, 4, 6, 10, 12 and 14; five pairs with 3 and 7 and six pairs with 2, 5, 8, 9, 11 and 13.

Hence,

$$\begin{aligned}\chi_{10}(n_6) = \quad & 2\times 2 + 2\times 4\times \omega_1 + 2\times 12\times \omega_2 + 2\times 10\times \omega_3 + 2\times 4\times \omega_4 + 2\times 12\times \omega_5 \\ & +2\times 4\times \omega_6 + 2\times 10\times \omega_7 + 2\times 12\times \omega_8 + 2\times 12\times \omega_9 + 2\times 4\times \omega_{10} \\ & +2\times 12\times \omega_{11} + 2\times 4\times \omega_{12} + 2\times 12\times \omega_{13} + 2\times 4\times \omega_{14}.\end{aligned}$$

So

$$\chi_{10}(n_6) = 4 + 4\omega_1 + 12\omega_2 + 10\omega_3 + 4\omega_4 + 12\omega_5 + 4\omega_6 + 10\omega_7$$
$$+12\omega_8 + 12\omega_9 + 4\omega_{10} + 12\omega_{11} + 4\omega_{12} + 12\omega_{13} + 4\omega_{14} = f.$$

$\underline{\chi_{10}(n_7):}$

For $[n_7]$, there are two pairs with 0, 1, 2, 5, 8, 9 and 12; five pairs with 6 and 14 and six pairs with 3, 4, 7, 10, 11 and 13.

Hence,

$$\chi_{10}(n_7) = 2\times 2 + 2\times 4\times\omega_1 + 2\times 4\times\omega_2 + 2\times 12\times\omega_3 + 2\times 12\times\omega_4 + 2\times 4\times\omega_5$$
$$+2\times 10\times\omega_6 + 2\times 12\times\omega_7 + 2\times 4\times\omega_8 + 2\times 4\times\omega_9 + 2\times 12\times\omega_{10}$$
$$+2\times 12\times\omega_{11} + 2\times 4\times\omega_{12} + 2\times 12\times\omega_{13} + 2\times 10\times\omega_{14}.$$

So

$$\chi_{10}(n_7) = 4 + 4\omega_1 + 4\omega_2 + 12\omega_3 + 12\omega_4 + 4\omega_5 + 10\omega_6 + 12\omega_7$$
$$+4\omega_8 + 4\omega_9 + 12\omega_{10} + 12\omega_{11} + 4\omega_{12} + 12\omega_{13} + 10\omega_{14} = g.$$

7.3 Section C

7.3.1 The Character Values of χ_{11}

Now $\chi_{11} = \phi_1^{\overline{G}} = (\psi_1.\psi_3)$ and $\chi_{11}(n_i) = \sum_{i=1}^{120} \psi_3(x_i)$. To find the values $\chi_{11}(n_i)$, we need the values $\psi_3(g)$ for $g \in \mathbb{Z}_{29}$. They are tabulated below:

Table 7.2: The Values of the Character ψ_3

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
ψ_3	1	ω_2	ω_4	ω_6	ω_8	ω_{10}	ω_{12}	ω_{14}	$\overline{\omega}_{13}$	$\overline{\omega}_{11}$	$\overline{\omega}_9$	$\overline{\omega}_7$	$\overline{\omega}_5$	$\overline{\omega}_3$	$\overline{\omega}_1$

Table 7.2 (continued)

g	15	16	17	18	19	20	21	22	23	24	25	26	27	28
ψ_3	ω_1	ω_3	ω_5	ω_7	ω_9	ω_{11}	ω_{13}	$\overline{\omega}_{14}$	$\overline{\omega}_{12}$	$\overline{\omega}_{10}$	$\overline{\omega}_8$	$\overline{\omega}_6$	$\overline{\omega}_4$	$\overline{\omega}_2$

where $\omega_i = e^{\frac{2k\pi i}{29}}$ and $\overline{\omega}_i$ is the complex conjugate of ω_i.

$\underline{\chi_{11}(n_1):}$

For $[n_1]$ there are two pairs with 0, 2, 4, 5, 10, 11 and 13; five pairs with 1 and 12 and six pairs

with 3, 6, 7, 8, 9 and 14.
Hence,

$$\begin{aligned}\chi_{11}(n_1) = \quad & 2\times 2+2\times 5\times \omega_2+2\times 2\times \omega_4+2\times 6\times \omega_6+2\times 2\times \omega_8+2\times 2\times \omega_{10}\\ & +2\times 6\times \omega_{12}+2\times 6\times \omega_{14}+2\times 6\times \omega_{13}+2\times 6\times \omega_{11}+2\times 2\times \omega_9\\ & +2\times 2\times \omega_7+2\times 5\times \omega_5+2\times 2\times \omega_3+2\times 6\times \omega_1.\end{aligned}$$

So

$$\begin{aligned}\chi_{11}(n_1) = \quad & 4+12\omega_1+10\omega_2+4\omega_3+4\omega_4+10\omega_5+12\omega_6+4\omega_7\\ & +4\omega_8+4\omega_9+4\omega_{10}+12\omega_{11}+12\omega_{12}+12\omega_{13}+12\omega_{14}=b.\end{aligned}$$

$\underline{\chi_{11}(n_2):}$

For $[n_2]$, there are two pairs with 0, 3, 4, 7, 8, 9 and 10; five pairs with 2 and 5 and six pairs with 1, 6, 11, 12, 13 and 14.
Hence,

$$\begin{aligned}\chi_{11}(n_2) = \quad & 2\times 2+2\times 6\times \omega_2+2\times 5\times \omega_4+2\times 2\times \omega_6+2\times 2\times \omega_8+2\times 5\times \omega_{10}\\ & +2\times 6\times \omega_{12}+2\times 2\times \omega_{14}+2\times 2\times \omega_{13}+2\times 2\times \omega_{11}+2\times 2\times \omega_9\\ & +2\times 6\times \omega_7+2\times 6\times \omega_5+2\times 6\times \omega_3+2\times 6\times \omega_1.\end{aligned}$$

So

$$\begin{aligned}\chi_{11}(n_2) = \quad & 4+12\omega_1+12\omega_2+12\omega_3+10\omega_4+12\omega_5+4\omega_6+12\omega_7\\ & +4\omega_8+4\omega_9+10\omega_{10}+4\omega_{11}+12\omega_{12}+4\omega_{13}+4\omega_{14}=c.\end{aligned}$$

$\underline{\chi_{11}(n_3):}$

For $[n_3]$, there are two pairs with 0, 6, 8, 9, 11, 13 and 14; five pairs with 4 and 10 and six pairs with 1, 2, 3, 5, 7 and 12.
Hence,

$$\begin{aligned}\chi_{11}(n_3) = \quad & 2\times 2+2\times 12\times \omega_2+2\times 12\times \omega_4+2\times 12\times \omega_6+2\times 10\times \omega_8+2\times 12\times \omega_{10}\\ & +2\times 4\times \omega_{12}+2\times 12\times \omega_{14}+2\times 4\times \omega_{13}+2\times 4\times \omega_{11}+2\times 10\times \omega_9\\ & +2\times 4\times \omega_7+2\times 12\times \omega_5+2\times 4\times \omega_3+2\times 4\times \omega_1.\end{aligned}$$

So

$$\begin{aligned}\chi_{11}(n_3) = \quad & 4+4\omega_1+12\omega_2+4\omega_3+12\omega_4+12\omega_5+12\omega_6+4\omega_7\\ & +10\omega_8+10\omega_9+12\omega_{10}+4\omega_{11}+4\omega_{12}+4\omega_{13}+12\omega_{14}=d.\end{aligned}$$

$\underline{\chi_{11}(n_4):}$
For $[n_4]$, there are two pairs with 0, 1, 3, 7, 11, 12 and 14; five pairs with 8 and 9 and six pairs with 2, 4, 5, 6, 10 and 14.
Hence,

$$\begin{aligned}\chi_{11}(n_4) = \;& 2\times 2+2\times 4\times\omega_2+2\times 12\times\omega_4+2\times 4\times\omega_6+2\times 12\times\omega_8+2\times 12\times\omega_{10}\\ &+2\times 12\times\omega_{12}+2\times 4\times\omega_{14}+2\times 10\times\omega_{13}+2\times 10\times\omega_{11}+2\times 12\times\omega_9\\ &+2\times 4\times\omega_7+2\times 4\times\omega_5+2\times 4\times\omega_3+2\times 12\times\omega_1.\end{aligned}$$

So

$$\begin{aligned}\chi_{11}(n_4) = \;& 4+12\omega_1+4\omega_2+4\omega_3+12\omega_4+4\omega_5+4\omega_6+4\omega_7\\ &+12\omega_8+12\omega_9+12\omega_{10}+10\omega_{11}+12\omega_{12}+10\omega_{13}+4\omega_{14}=e.\end{aligned}$$

$\underline{\chi_{11}(n_5):}$
For $[n_5]$, there are two pairs with 0, 2, 3, 5, 6, 7 and 14; five pairs with 11 and 13 and six pairs with 2, 4, 8, 9, 10 and 12.
Hence,

$$\begin{aligned}\chi_{11}(n_5) = \;& 2\times 2+2\times 12\times\omega_2+2\times 4\times\omega_4+2\times 4\times\omega_6+2\times 12\times\omega_8+2\times 4\times\omega_{10}\\ &+2\times 4\times\omega_{12}+2\times 4\times\omega_{14}+2\times 12\times\omega_{13}+2\times 12\times\omega_{11}+2\times 12\times\omega_9\\ &+2\times 10\times\omega_7+2\times 12\times\omega_5+2\times 10\times\omega_3+2\times 4\times\omega_1.\end{aligned}$$

So

$$\begin{aligned}\chi_{11}(n_5) = \;& 4+4\omega_1+12\omega_2+10\omega_3+4\omega_4+12\omega_5+4\omega_6+10\omega_7\\ &+12\omega_8+12\omega_9+4\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+4\omega_{14}=f.\end{aligned}$$

$\underline{\chi_{11}(n_6):}$
For $[n_6]$, there are two pairs with 0, 1, 4, 6, 10, 12 and 14; five pairs with 3 and 7 and six pairs with 2, 5, 8, 9, 11 and 13.
Hence,

$$\begin{aligned}\chi_{11}(n_6) = \;& 2\times 2+2\times 4\times\omega_2+2\times 12\times\omega_4+2\times 10\times\omega_6+2\times 4\times\omega_8+2\times 12\times\omega_{10}\\ &+2\times 4\times\omega_{12}+2\times 10\times\omega_{14}+2\times 12\times\omega_{13}+2\times 12\times\omega_{11}+2\times 4\times\omega_9\\ &+2\times 12\times\omega_7+2\times 4\times\omega_5+2\times 12\times\omega_3+2\times 4\times\omega_1.\end{aligned}$$

So

$$\begin{aligned}\chi_{11}(n_6) = \;& 4+4\omega_1+4\omega_2+12\omega_3+12\omega_4+4\omega_5+10\omega_6+12\omega_7\\ &+4\omega_8+4\omega_9+12\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+10\omega_{14}=g.\end{aligned}$$

$\underline{\chi_{11}(n_7):}$

For $[n_7]$, there are two pairs with 0, 1, 2, 5, 8, 9 and 12; five pairs with 6 and 14 and six pairs with 3, 4, 7, 10, 11 and 13.

Hence,

$$\begin{aligned}\chi_{11}(n_7) = \;& 2\times 2 + 2\times 4\times\omega_2 + 2\times 4\times\omega_4 + 2\times 12\times\omega_6 + 2\times 12\times\omega_8 + 2\times 4\times\omega_{10}\\ & +2\times 10\times\omega_{12} + 2\times 12\times\omega_{14} + 2\times 4\times\omega_{13} + 2\times 4\times\omega_{11} + 2\times 12\times\omega_9\\ & +2\times 12\times\omega_7 + 2\times 4\times\omega_5 + 2\times 12\times\omega_3 + 2\times 10\times\omega_1.\end{aligned}$$

So

$$\begin{aligned}\chi_{11}(n_7) = \;& 4 + 10\omega_1 + 4\omega_2 + 12\omega_3 + 4\omega_4 + 4\omega_5 + 12\omega_6 + 12\omega_7\\ & +12\omega_8 + 12\omega_9 + 4\omega_{10} + 4\omega_{11} + 10\omega_{12} + 4\omega_{13} + 12\omega_{14} = a.\end{aligned}$$

7.4 Section D

7.4.1 Character Values of χ_{12}

Now $\chi_{12} = \phi_1^{\overline{G}} = (\psi_1.\psi_4)$ and $\chi_{12}(n_i) = \sum_{i=1}^{120}\psi_4(x_i)$. To find the values $\chi_{12}(n_i)$, we need the values $\psi_4(g)$ for $g \in \mathbb{Z}_{29}$. They are tabulated below:

Table 7.3: The Values of the Character ψ_4

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
ψ_4	1	ω_3	ω_6	ω_9	ω_{12}	$\overline{\omega}_{14}$	$\overline{\omega}_{11}$	$\overline{\omega}_8$	$\overline{\omega}_5$	$\overline{\omega}_2$	ω_1	ω_4	ω_7	ω_{10}	ω_{13}

Table 7.3 (continued)

g	15	16	17	18	19	20	21	22	23	24	25	26	27	28
ψ_4	$\overline{\omega}_{13}$	$\overline{\omega}_{10}$	$\overline{\omega}_7$	$\overline{\omega}_4$	$\overline{\omega}_1$	ω_2	ω_5	ω_8	ω_{11}	ω_{14}	$\overline{\omega}_{12}$	$\overline{\omega}_9$	$\overline{\omega}_6$	$\overline{\omega}_3$

where $\omega_i = e^{\frac{2k\pi i}{29}}$ and $\overline{\omega}_i$ is the complex conjugate of ω_i.

$\underline{\chi_{12}(n_1):}$

For $[n_1]$ there are two pairs with 0, 2, 4, 5, 10, 11 and 13; five pairs with 1 and 12 and six pairs with 3, 6, 7, 8, 9 and 14.

Hence,

$$\begin{aligned}\chi_{12}(n_1) = \quad & 2\times2+2\times5\times\omega_3+2\times2\times\omega_6+2\times6\times\omega_9+2\times2\times\omega_{12}+2\times2\times\omega_{14}\\ & +2\times6\times\omega_{11}+2\times6\times\omega_8+2\times6\times\omega_5+2\times6\times\omega_2+2\times2\times\omega_1\\ & +2\times2\times\omega_4+2\times5\times\omega_7+2\times2\times\omega_{10}+2\times6\times\omega_{13}.\end{aligned}$$

So

$$\begin{aligned}\chi_{12}(n_1) = \quad & 4+4\omega_1+12\omega_2+10\omega_3+4\omega_4+12\omega_5+4\omega_6+10\omega_7\\ & +12\omega_8+12\omega_9+4\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+4\omega_{14}=f.\end{aligned}$$

$\underline{\chi_{12}(n_2):}$

For $[n_2]$, there are two pairs with 0, 3, 4, 7, 8, 9 and 10; five pairs with 2 and 5 and six pairs with 1, 6, 11, 12, 13 and 14.

Hence,

$$\begin{aligned}\chi_{12}(n_2) = \quad & 2\times2+2\times6\times\omega_3+2\times5\times\omega_6+2\times2\times\omega_9+2\times2\times\omega_{12}+2\times5\times\omega_{14}\\ & +2\times6\times\omega_{11}+2\times2\times\omega_8+2\times2\times\omega_5+2\times2\times\omega_2+2\times2\times\omega_1\\ & +2\times6\times\omega_4+2\times6\times\omega_7+2\times6\times\omega_{10}+2\times6\times\omega_{13}.\end{aligned}$$

So

$$\begin{aligned}\chi_{12}(n_2) = \quad & 4+4\omega_1+4\omega_2+12\omega_3+12\omega_4+4\omega_5+10\omega_6+12\omega_7\\ & +4\omega_8+4\omega_9+12\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+10\omega_{14}=g.\end{aligned}$$

$\underline{\chi_{12}(n_3):}$

For $[n_3]$, there are two pairs with 0, 6, 8, 9, 11, 13 and 14; five pairs with 4 and 10 and six pairs with 1, 2, 3, 5, 7 and 12.

Hence,

$$\begin{aligned}\chi_{12}(n_3) = \quad & 2\times2+2\times12\times\omega_3+2\times12\times\omega_6+2\times12\times\omega_9+2\times10\times\omega_{12}+2\times12\times\omega_{14}\\ & +2\times4\times\omega_{11}+2\times12\times\omega_8+2\times4\times\omega_5+2\times4\times\omega_2+2\times10\times\omega_1\\ & +2\times4\times\omega_4+2\times12\times\omega_7+2\times4\times\omega_{10}+2\times4\times\omega_{13}.\end{aligned}$$

So

$$\begin{aligned}\chi_{12}(n_3) = \quad & 4+10\omega_1+4\omega_2+12\omega_3+4\omega_4+4\omega_5+12\omega_6+12\omega_7\\ & +12\omega_8+12\omega_9+4\omega_{10}+4\omega_{11}+10\omega_{12}+4\omega_{13}+12\omega_{14}=a.\end{aligned}$$

$\underline{\chi_{12}(n_4):}$

For $[n_4]$, there are two pairs with 0, 1, 3, 7, 11, 12 and 14; five pairs with 8 and 9 and six pairs

with 2, 4, 5, 6, 10 and 14.
Hence,

$$\begin{aligned}\chi_{12}(n_4) = \;& 2\times 2 + 2\times 4\times \omega_3 + 2\times 12\times \omega_6 + 2\times 4\times \omega_9 + 2\times 12\times \omega_{12} + 2\times 12\times \omega_{14}\\ & +2\times 12\times \omega_{11} + 2\times 4\times \omega_8 + 2\times 10\times \omega_5 + 2\times 10\times \omega_2 + 2\times 12\times \omega_1\\ & +2\times 4\times \omega_4 + 2\times 4\times \omega_7 + 2\times 4\times \omega_{10} + 2\times 12\times \omega_{13}.\end{aligned}$$

So

$$\begin{aligned}\chi_{12}(n_4) = \;& 4 + 12\omega_1 + 10\omega_2 + 4\omega_3 + 4\omega_4 + 10\omega_5 + 12\omega_6 + 4\omega_7\\ & +4\omega_8 + 4\omega_9 + 4\omega_{10} + 12\omega_{11} + 12\omega_{12} + 12\omega_{13} + 12\omega_{14} = b.\end{aligned}$$

$\underline{\chi_{12}(n_5):}$
For $[n_5]$, there are two pairs with 0, 2, 3, 5, 6, 7 and 14; five pairs with 11 and 13 and six pairs with 2, 4, 8, 9, 10 and 12.
Hence,

$$\begin{aligned}\chi_{12}(n_5) = \;& 2\times 2 + 2\times 12\times \omega_3 + 2\times 4\times \omega_6 + 2\times 4\times \omega_9 + 2\times 12\times \omega_{12} + 2\times 4\times \omega_{14}\\ & +2\times 4\times \omega_{11} + 2\times 4\times \omega_8 + 2\times 12\times \omega_5 + 2\times 12\times \omega_2 + 2\times 12\times \omega_1\\ & +2\times 10\times \omega_4 + 2\times 12\times \omega_7 + 2\times 10\times \omega_{10} + 2\times 4\times \omega_{13}.\end{aligned}$$

So

$$\begin{aligned}\chi_{12}(n_5) = \;& 4 + 12\omega_1 + 12\omega_2 + 12\omega_3 + 10\omega_4 + 12\omega_5 + 4\omega_6 + 12\omega_7\\ & +4\omega_8 + 4\omega_9 + 10\omega_{10} + 4\omega_{11} + 12\omega_{12} + 4\omega_{13} + 4\omega_{14} = c.\end{aligned}$$

$\underline{\chi_{12}(n_6):}$
For $[n_6]$, there are two pairs with 0, 1, 4, 6, 10, 12 and 14; five pairs with 3 and 7 and six pairs with 2, 5, 8, 9, 11 and 13.
Hence,

$$\begin{aligned}\chi_{12}(n_6) = \;& 2\times 2 + 2\times 4\times \omega_3 + 2\times 12\times \omega_6 + 2\times 10\times \omega_9 + 2\times 4\times \omega_{12} + 2\times 12\times \omega_{14}\\ & +2\times 4\times \omega_{11} + 2\times 10\times \omega_8 + 2\times 12\times \omega_5 + 2\times 12\times \omega_2 + 2\times 4\times \omega_1\\ & +2\times 12\times \omega_4 + 2\times 4\times \omega_7 + 2\times 12\times \omega_{10} + 2\times 4\times \omega_{13}.\end{aligned}$$

So

$$\begin{aligned}\chi_{12}(n_6) = \;& 4 + 4\omega_1 + 12\omega_2 + 4\omega_3 + 12\omega_4 + 12\omega_5 + 12\omega_6 + 4\omega_7\\ & +10\omega_8 + 10\omega_9 + 12\omega_{10} + 4\omega_{11} + 4\omega_{12} + 4\omega_{13} + 12\omega_{14} = d.\end{aligned}$$

$\underline{\chi_{12}(n_7):}$
For $[n_7]$, there are two pairs with 0, 1, 2, 5, 8, 9 and 12; five pairs with 6 and 14 and six pairs with 3, 4, 7, 10, 11 and 13.
Hence,

$$\begin{aligned}\chi_{12}(n_7) = \;& 2\times 2 + 2\times 4\times\omega_3 + 2\times 4\times\omega_6 + 2\times 12\times\omega_9 + 2\times 12\times\omega_{12} + 2\times 4\times\omega_{14}\\ & +2\times 10\times\omega_{11} + 2\times 12\times\omega_8 + 2\times 4\times\omega_5 + 2\times 4\times\omega_2 + 2\times 12\times\omega_1\\ & +2\times 12\times\omega_4 + 2\times 4\times\omega_7 + 2\times 12\times\omega_{10} + 2\times 10\times\omega_{13}.\end{aligned}$$

So

$$\begin{aligned}\chi_{12}(n_7) = \;& 4 + 12\omega_1 + 4\omega_2 + 4\omega_3 + 12\omega_4 + 4\omega_5 + 4\omega_6 + 4\omega_7\\ & +12\omega_8 + 12\omega_9 + 12\omega_{10} + 10\omega_{11} + 12\omega_{12} + 10\omega_{13} + 4\omega_{14} = e.\end{aligned}$$

7.5 Section E

7.5.1 Character Values of χ_{13}

Now $\chi_{13} = \phi_1^{\overline{G}} = (\psi_1.\psi_5)$ and $\chi_{13}(n_i) = \sum_{i=1}^{120} \psi_5(x_i)$. To find the values $\chi_{13}(n_i)$, we need the values $\psi_5(g)$ for $g \in \mathbb{Z}_{29}$. They are tabulated below:

Table 7.4: The Values of the Character ψ_5

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
ψ_5	1	ω_4	ω_8	ω_{12}	$\overline{\omega}_{13}$	$\overline{\omega}_9$	$\overline{\omega}_5$	$\overline{\omega}_1$	ω_3	ω_7	ω_{11}	$\overline{\omega}_{14}$	$\overline{\omega}_{10}$	$\overline{\omega}_6$	$\overline{\omega}_2$

Table 7.4 (continued)

g	15	16	17	18	19	20	21	22	23	24	25	26	27	28
ψ_5	ω_2	ω_6	ω_{10}	ω_{14}	$\overline{\omega}_{11}$	$\overline{\omega}_7$	$\overline{\omega}_3$	ω_1	ω_5	ω_9	ω_{13}	$\overline{\omega}_{12}$	$\overline{\omega}_8$	$\overline{\omega}_4$

where $\omega_i = e^{\frac{2k\pi i}{29}}$ and $\overline{\omega}_i$ is the complex conjugate of ω_i.

$\underline{\chi_{13}(n_1):}$
For $[n_1]$ there are two pairs with 0, 2, 4, 5, 10, 11 and 13; five pairs with 1 and 12 and six pairs with 3, 6, 7, 8, 9 and 14.

Hence,

$$\begin{aligned}\chi_{13}(n_1) = \;& 2\times 2+2\times 5\times \omega_4+2\times 2\times \omega_8+2\times 6\times \omega_{12}+2\times 2\times \omega_{13}+2\times 2\times \omega_9\\ &+2\times 6\times \omega_5+2\times 6\times \omega_1+2\times 6\times \omega_3+2\times 6\times \omega_7+2\times 2\times \omega_{11}\\ &+2\times 2\times \omega_{14}+2\times 5\times \omega_{10}+2\times 2\times \omega_6+2\times 6\times \omega_2.\end{aligned}$$

So

$$\begin{aligned}\chi_{13}(n_1) = \;& 4+12\omega_1+12\omega_2+12\omega_3+10\omega_4+12\omega_5+4\omega_6+12\omega_7\\ &+4\omega_8+4\omega_9+10\omega_{10}+4\omega_{11}+12\omega_{12}+4\omega_{13}+4\omega_{14}=c.\end{aligned}$$

$\underline{\chi_{13}(n_2):}$

For $[n_2]$, there are two pairs with 0, 3, 4, 7, 8, 9 and 10; five pairs with 2 and 5 and six pairs with 1, 6, 11, 12, 13 and 14.

Hence,

$$\begin{aligned}\chi_{13}(n_2) = \;& 2\times 2+2\times 6\times \omega_4+2\times 5\times \omega_8+2\times 2\times \omega_{12}+2\times 2\times \omega_{13}+2\times 5\times \omega_9\\ &+2\times 6\times \omega_5+2\times 2\times \omega_1+2\times 2\times \omega_3+2\times 2\times \omega_7+2\times 2\times \omega_{11}\\ &+2\times 6\times \omega_{14}+2\times 6\times \omega_{10}+2\times 6\times \omega_6+2\times 6\times \omega_2.\end{aligned}$$

So

$$\begin{aligned}\chi_{13}(n_2) = \;& 4+4\omega_1+12\omega_2+4\omega_3+12\omega_4+12\omega_5+12\omega_6+4\omega_7\\ &+10\omega_8+10\omega_9+12\omega_{10}+4\omega_{11}+4\omega_{12}+4\omega_{13}+12\omega_{14}=d.\end{aligned}$$

$\underline{\chi_{13}(n_3):}$

For $[n_3]$, there are two pairs with 0, 6, 8, 9, 11, 13 and 14; five pairs with 4 and 10 and six pairs with 1, 2, 3, 5, 7 and 12.

Hence,

$$\begin{aligned}\chi_{13}(n_3) = \;& 2\times 2+2\times 12\times \omega_4+2\times 12\times \omega_8+2\times 12\times \omega_{12}+2\times 10\times \omega_{13}+2\times 12\times \omega_9\\ &+2\times 4\times \omega_5+2\times 12\times \omega_1+2\times 4\times \omega_3+2\times 4\times \omega_7+2\times 10\times \omega_{11}\\ &+2\times 4\times \omega_{14}+2\times 12\times \omega_{10}+2\times 4\times \omega_6+2\times 4\times \omega_2.\end{aligned}$$

So

$$\begin{aligned}\chi_{13}(n_3) = \;& 4+12\omega_1+4\omega_2+4\omega_3+12\omega_4+4\omega_5+4\omega_6+4\omega_7\\ &+12\omega_8+12\omega_9+12\omega_{10}+10\omega_{11}+12\omega_{12}+10\omega_{13}+4\omega_{14}=e.\end{aligned}$$

$\underline{\chi_{13}(n_4):}$

For $[n_4]$, there are two pairs with 0, 1, 3, 7, 11, 12 and 14; five pairs with 8 and 9 and six pairs

with 2, 4, 5, 6, 10 and 14.
Hence,

$$\begin{aligned}\chi_{13}(n_4) = \quad & 2\times 2+2\times 4\times \omega_4+2\times 12\times \omega_8+2\times 4\times \omega_{12}+2\times 12\times \omega_{13}+2\times 12\times \omega_9\\ & +2\times 12\times \omega_5+2\times 4\times \omega_1+2\times 10\times \omega_3+2\times 10\times \omega_7+2\times 12\times \omega_{11}\\ & +2\times 4\times \omega_{14}+2\times 4\times \omega_{10}+2\times 4\times \omega_6+2\times 12\times \omega_2.\end{aligned}$$

So

$$\begin{aligned}\chi_{13}(n_4) = \quad & 4+4\omega_1+12\omega_2+10\omega_3+4\omega_4+12\omega_5+4\omega_6+10\omega_7\\ & +12\omega_8+12\omega_9+4\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+4\omega_{14}=f.\end{aligned}$$

$\underline{\chi_{13}(n_5):}$

For $[n_5]$, there are two pairs with 0, 2, 3, 5, 6, 7 and 14; five pairs with 11 and 13 and six pairs with 2, 4, 8, 9, 10 and 12.
Hence,

$$\begin{aligned}\chi_{13}(n_5) = \quad & 2\times 2+2\times 12\times \omega_4+2\times 4\times \omega_8+2\times 4\times \omega_{12}+2\times 12\times \omega_{13}+2\times 4\times \omega_9\\ & +2\times 4\times \omega_5+2\times 4\times \omega_1+2\times 12\times \omega_3+2\times 12\times \omega_7+2\times 12\times \omega_{11}\\ & +2\times 10\times \omega_{14}+2\times 12\times \omega_{10}+2\times 10\times \omega_6+2\times 4\times \omega_2.\end{aligned}$$

So

$$\begin{aligned}\chi_{13}(n_5) = \quad & 4+4\omega_1+4\omega_2+12\omega_3+12\omega_4+4\omega_5+10\omega_6+12\omega_7\\ & +4\omega_8+4\omega_9+12\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+10\omega_{14}=g.\end{aligned}$$

$\underline{\chi_{13}(n_6):}$

For $[n_6]$, there are two pairs with 0, 1, 4, 6, 10, 12 and 14; five pairs with 3 and 7 and six pairs with 2, 5, 8, 9, 11 and 13.
Hence,

$$\begin{aligned}\chi_{13}(n_6) = \quad & 2\times 2+2\times 4\times \omega_4+2\times 12\times \omega_8+2\times 10\times \omega_{12}+2\times 4\times \omega_{13}+2\times 12\times \omega_9\\ & +2\times 4\times \omega_5+2\times 10\times \omega_1+2\times 12\times \omega_3+2\times 12\times \omega_7+2\times 4\times \omega_{11}\\ & +2\times 12\times \omega_{14}+2\times 4\times \omega_{10}+2\times 12\times \omega_6+2\times 4\times \omega_2.\end{aligned}$$

So

$$\begin{aligned}\chi_{13}(n_6) = \quad & 4+10\omega_1+4\omega_2+12\omega_3+4\omega_4+4\omega_5+12\omega_6+12\omega_7\\ & +12\omega_8+12\omega_9+4\omega_{10}+4\omega_{11}+10\omega_{12}+4\omega_{13}+12\omega_{14}=a.\end{aligned}$$

$\underline{\chi_{13}(n_7):}$
For $[n_7]$, there are two pairs with 0, 1, 2, 5, 8, 9 and 12; five pairs with 6 and 14 and six pairs with 3, 4, 7, 10, 11 and 13.
Hence,

$$\begin{aligned}\chi_{13}(n_7) = \quad & 2\times 2+2\times 4\times\omega_4+2\times 4\times\omega_8+2\times 12\times\omega_{12}+2\times 12\times\omega_{13}+2\times 4\times\omega_9\\ & +2\times 10\times\omega_5+2\times 12\times\omega_1+2\times 4\times\omega_3+2\times 4\times\omega_7+2\times 12\times\omega_{11}\\ & +2\times 12\times\omega_{14}+2\times 4\times\omega_{10}+2\times 12\times\omega_6+2\times 10\times\omega_2.\end{aligned}$$

So

$$\begin{aligned}\chi_{13}(n_7) = \quad & 4+12\omega_1+10\omega_2+4\omega_3+4\omega_4+10\omega_5+12\omega_6+4\omega_7\\ & +4\omega_8+4\omega_9+4\omega_{10}+12\omega_{11}+12\omega_{12}+12\omega_{13}+12\omega_{14}=b.\end{aligned}$$

7.6 Section F

7.6.1 The Character Values of χ_{14}

Now $\chi_{14}=\phi_1^{\overline{G}}=(\psi_1.\psi_7)$ and $\chi_{14}(n_i)=\sum_{i=1}^{120}\psi_7(x_i)$. To find the values $\chi_{14}(n_i)$, we need the values $\psi_7(g)$ for $g\in\mathbb{Z}_{29}$. They are tabulated below:

Table 7.5: The Values of the Character ψ_7

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
ψ_7	1	ω_6	ω_{12}	$\overline{\omega}_{11}$	$\overline{\omega}_5$	ω_1	ω_7	ω_{13}	$\overline{\omega}_{10}$	$\overline{\omega}_4$	ω_2	ω_8	ω_{14}	$\overline{\omega}_9$	$\overline{\omega}_3$

Table 7.5 (continued)

g	15	16	17	18	19	20	21	22	23	24	25	26	27	28
ψ_7	ω_3	ω_9	$\overline{\omega}_{14}$	$\overline{\omega}_8$	$\overline{\omega}_2$	ω_4	ω_{10}	$\overline{\omega}_{13}$	$\overline{\omega}_7$	$\overline{\omega}_1$	ω_5	ω_{11}	$\overline{\omega}_{12}$	$\overline{\omega}_6$

where $\omega_i=e^{\frac{2k\pi i}{29}}$ and $\overline{\omega}_i$ is the complex conjugate of ω_i.

$\underline{\chi_{14}(n_1):}$
For $[n_1]$ there are two pairs with 0, 2, 4, 5, 10, 11 and 13; five pairs with 1 and 12 and six pairs with 3, 6, 7, 8, 9 and 14.

Hence,

$$\begin{aligned}\chi_{14}(n_1) = \quad & 2\times 2 + 2\times 5\times \omega_6 + 2\times 2\times \omega_{12} + 2\times 6\times \omega_{11} + 2\times 2\times \omega_5 + 2\times 2\times \omega_1 \\ & +2\times 6\times \omega_7 + 2\times 6\times \omega_{13} + 2\times 6\times \omega_{10} + 2\times 6\times \omega_4 + 2\times 2\times \omega_2 \\ & +2\times 2\times \omega_8 + 2\times 5\times \omega_{14} + 2\times 2\times \omega_9 + 2\times 6\times \omega_3.\end{aligned}$$

So

$$\begin{aligned}\chi_{14}(n_1) = \quad & 4 + 4\omega_1 + 4\omega_2 + 12\omega_3 + 12\omega_4 + 4\omega_5 + 10\omega_6 + 12\omega_7 \\ & +4\omega_8 + 4\omega_9 + 12\omega_{10} + 12\omega_{11} + 4\omega_{12} + 12\omega_{13} + 10\omega_{14} = g.\end{aligned}$$

$\underline{\chi_{14}(n_2):}$

For $[n_2]$, there are two pairs with 0, 3, 4, 7, 8, 9 and 10; five pairs with 2 and 5 and six pairs with 1, 6, 11, 12, 13 and 14.
Hence,

$$\begin{aligned}\chi_{14}(n_2) = \quad & 2\times 2 + 2\times 6\times \omega_6 + 2\times 5\times \omega_{12} + 2\times 2\times \omega_{11} + 2\times 2\times \omega_5 + 2\times 5\times \omega_1 \\ & +2\times 6\times \omega_7 + 2\times 2\times \omega_{13} + 2\times 2\times \omega_{10} + 2\times 2\times \omega_4 + 2\times 2\times \omega_2 \\ & +2\times 6\times \omega_8 + 2\times 6\times \omega_{14} + 2\times 6\times \omega_9 + 2\times 6\times \omega_3.\end{aligned}$$

So

$$\begin{aligned}\chi_{14}(n_2) = \quad & 4 + 10\omega_1 + 4\omega_2 + 12\omega_3 + 4\omega_4 + 4\omega_5 + 12\omega_6 + 12\omega_7 \\ & +12\omega_8 + 12\omega_9 + 4\omega_{10} + 4\omega_{11} + 10\omega_{12} + 4\omega_{13} + 12\omega_{14} = a.\end{aligned}$$

$\underline{\chi_{14}(n_3):}$

For $[n_3]$, there are two pairs with 0, 6, 8, 9, 11, 13 and 14; five pairs with 4 and 10 and six pairs with 1, 2, 3, 5, 7 and 12.
Hence,

$$\begin{aligned}\chi_{14}(n_3) = \quad & 2\times 2 + 2\times 12\times \omega_6 + 2\times 12\times \omega_{12} + 2\times 12\times \omega_{11} + 2\times 10\times \omega_5 + 2\times 12\times \omega_1 \\ & +2\times 4\times \omega_7 + 2\times 12\times \omega_{13} + 2\times 4\times \omega_{10} + 2\times 4\times \omega_4 + 2\times 10\times \omega_2 \\ & +2\times 4\times \omega_8 + 2\times 12\times \omega_{14} + 2\times 4\times \omega_9 + 2\times 4\times \omega_3.\end{aligned}$$

So

$$\begin{aligned}\chi_{14}(n_3) = \quad & 4 + 12\omega_1 + 10\omega_2 + 4\omega_3 + 4\omega_4 + 10\omega_5 + 12\omega_6 + 4\omega_7 \\ & +4\omega_8 + 4\omega_9 + 4\omega_{10} + 12\omega_{11} + 12\omega_{12} + 12\omega_{13} + 12\omega_{14} = b.\end{aligned}$$

$\underline{\chi_{14}(n_4):}$

For $[n_4]$, there are two pairs with 0, 1, 3, 7, 11, 12 and 14; five pairs with 8 and 9 and six pairs

with 2, 4, 5, 6, 10 and 14.
Hence,

$$\begin{aligned}\chi_{14}(n_4) = \;& 2\times 2+2\times 4\times\omega_6+2\times 12\times\omega_{12}+2\times 4\times\omega_{12}+2\times 12\times\omega_5+2\times 12\times\omega_1\\ &+2\times 12\times\omega_7+2\times 4\times\omega_{13}+2\times 10\times\omega_{10}+2\times 10\times\omega_4+2\times 12\times\omega_2\\ &+2\times 4\times\omega_8+2\times 4\times\omega_{14}+2\times 4\times\omega_9+2\times 12\times\omega_3.\end{aligned}$$

So

$$\begin{aligned}\chi_{14}(n_4) = \;& 4+12\omega_1+12\omega_2+12\omega_3+10\omega_4+12\omega_5+4\omega_6+12\omega_7\\ &+4\omega_8+4\omega_9+10\omega_{10}+4\omega_{11}+12\omega_{12}+4\omega_{13}+4\omega_{14}=c.\end{aligned}$$

$\underline{\chi_{14}(n_5):}$

For $[n_5]$, there are two pairs with 0, 2, 3, 5, 6, 7 and 14; five pairs with 11 and 13 and six pairs with 2, 4, 8, 9, 10 and 12.
Hence,

$$\begin{aligned}\chi_{14}(n_5) = \;& 2\times 2+2\times 12\times\omega_6+2\times 4\times\omega_{12}+2\times 4\times\omega_{11}+2\times 12\times\omega_5+2\times 4\times\omega_1\\ &+2\times 4\times\omega_7+2\times 4\times\omega_{13}+2\times 12\times\omega_{10}+2\times 12\times\omega_4+2\times 12\times\omega_2\\ &+2\times 10\times\omega_8+2\times 12\times\omega_{14}+2\times 10\times\omega_9+2\times 4\times\omega_3.\end{aligned}$$

So

$$\begin{aligned}\chi_{14}(n_5) = \;& 4+4\omega_1+12\omega_2+4\omega_3+12\omega_4+12\omega_5+12\omega_6+4\omega_7\\ &+10\omega_8+10\omega_9+12\omega_{10}+4\omega_{11}+4\omega_{12}+4\omega_{13}+12\omega_{14}=d.\end{aligned}$$

$\underline{\chi_{14}(n_6):}$

For $[n_6]$, there are two pairs with 0, 1, 4, 6, 10, 12 and 14; five pairs with 3 and 7 and six pairs with 2, 5, 8, 9, 11 and 13.
Hence,

$$\begin{aligned}\chi_{14}(n_6) = \;& 2\times 2+2\times 4\times\omega_6+2\times 12\times\omega_{12}+2\times 10\times\omega_{11}+2\times 4\times\omega_5+2\times 12\times\omega_1\\ &+2\times 4\times\omega_7+2\times 10\times\omega_{13}+2\times 12\times\omega_{10}+2\times 12\times\omega_4+2\times 4\times\omega_2\\ &+2\times 12\times\omega_8+2\times 4\times\omega_{14}+2\times 12\times\omega_9+2\times 4\times\omega_3.\end{aligned}$$

So

$$\begin{aligned}\chi_{14}(n_6) = \;& 4+12\omega_1+4\omega_2+4\omega_3+12\omega_4+4\omega_5+4\omega_6+4\omega_7\\ &+12\omega_8+12\omega_9+12\omega_{10}+10\omega_{11}+12\omega_{12}+10\omega_{13}+4\omega_{14}=e.\end{aligned}$$

$\underline{\chi_{14}(n_7):}$

For $[n_7]$, there are two pairs with 0, 1, 2, 5, 8, 9 and 12; five pairs with 6 and 14 and six pairs with 3, 4, 7, 10, 11 and 13.

Hence,

$$\begin{aligned}\chi_{14}(n_7) = \; & 2\times 2 + 2\times 4\times\omega_6 + 2\times 4\times\omega_{12} + 2\times 12\times\omega_{11} + 2\times 12\times\omega_5 + 2\times 4\times\omega_1 \\ & +2\times 10\times\omega_7 + 2\times 12\times\omega_{13} + 2\times 4\times\omega_{10} + 2\times 4\times\omega_4 + 2\times 12\times\omega_2 \\ & +2\times 12\times\omega_8 + 2\times 4\times\omega_{14} + 2\times 12\times\omega_9 + 2\times 10\times\omega_3.\end{aligned}$$

So

$$\begin{aligned}\chi_{14}(n_7) = \; & 4 + 4\omega_1 + 12\omega_2 + 10\omega_3 + 4\omega_4 + 12\omega_5 + 4\omega_6 + 10\omega_7 \\ & +12\omega_8 + 12\omega_9 + 4\omega_{10} + 12\omega_{11} + 4\omega_{12} + 12\omega_{13} + 4\omega_{14} = f.\end{aligned}$$

7.7 Section G

7.7.1 Character Values of χ_{15}

Now $\chi_{15} = \phi_1^{\overline{G}} = (\psi_1.\psi_9)$ and $\chi_{15}(n_i) = \sum_{i=1}^{120} \psi_9(x_i)$. To find the values $\chi_{15}(n_i)$, we need the values $\psi_9(g)$ for $g \in \mathbb{Z}_{29}$. They are tabulated below:

Table 7.6: The Values of the Character ψ_9

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
ψ_9	1	ω_8	$\overline{\omega}_{13}$	$\overline{\omega}_5$	ω_3	ω_{11}	$\overline{\omega}_{10}$	$\overline{\omega}_2$	ω_6	ω_{14}	$\overline{\omega}_7$	ω_1	ω_9	$\overline{\omega}_{12}$	$\overline{\omega}_4$

Table 7.6 (continued)

g	15	16	17	18	19	20	21	22	23	24	25	26	27	28
ψ_9	ω_4	ω_{12}	$\overline{\omega}_9$	$\overline{\omega}_1$	ω_7	$\overline{\omega}_{14}$	$\overline{\omega}_6$	ω_2	ω_{10}	$\overline{\omega}_{11}$	$\overline{\omega}_3$	ω_5	ω_{13}	$\overline{\omega}_8$

where $\omega_i = e^{\frac{2k\pi i}{29}}$ and $\overline{\omega}_i$ is the complex conjugate of ω_i.

$\underline{\chi_{15}(n_1):}$

For $[n_1]$ there are two pairs with 0, 2, 4, 5, 10, 11 and 13; five pairs with 1 and 12 and six pairs with 3, 6, 7, 8, 9 and 14.

Hence,

$$\begin{aligned}\chi_{15}(n_1) = \quad & 2\times 2 + 2\times 5\times \omega_8 + 2\times 2\times \omega_{13} + 2\times 6\times \omega_5 + 2\times 2\times \omega_3 + 2\times 2\times \omega_{11}\\ & +2\times 6\times \omega_{10} + 2\times 6\times \omega_2 + 2\times 6\times \omega_6 + 2\times 6\times \omega_{14} + 2\times 2\times \omega_7\\ & +2\times 2\times \omega_1 + 2\times 5\times \omega_9 + 2\times 2\times \omega_{12} + 2\times 6\times \omega_4.\end{aligned}$$

So

$$\begin{aligned}\chi_{15}(n_1) = \quad & 4 + 4\omega_1 + 12\omega_2 + 4\omega_3 + 12\omega_4 + 12\omega_5 + 12\omega_6 + 4\omega_7\\ & +10\omega_8 + 10\omega_9 + 12\omega_{10} + 4\omega_{11} + 4\omega_{12} + 4\omega_{13} + 12\omega_{14} = d.\end{aligned}$$

$\underline{\chi_{15}(n_2):}$

For $[n_2]$, there are two pairs with 0, 3, 4, 7, 8, 9 and 10; five pairs with 2 and 5 and six pairs with 1, 6, 11, 12, 13 and 14.

Hence,

$$\begin{aligned}\chi_{15}(n_2) = \quad & 2\times 2 + 2\times 6\times \omega_8 + 2\times 5\times \omega_{13} + 2\times 2\times \omega_5 + 2\times 2\times \omega_3 + 2\times 5\times \omega_{11}\\ & +2\times 6\times \omega_{10} + 2\times 2\times \omega_2 + 2\times 2\times \omega_6 + 2\times 2\times \omega_{14} + 2\times 2\times \omega_7\\ & +2\times 6\times \omega_1 + 2\times 6\times \omega_9 + 2\times 6\times \omega_{12} + 2\times 6\times \omega_4.\end{aligned}$$

So

$$\begin{aligned}\chi_{15}(n_2) = \quad & 4 + 12\omega_1 + 4\omega_2 + 4\omega_3 + 12\omega_4 + 4\omega_5 + 4\omega_6 + 4\omega_7\\ & +12\omega_8 + 12\omega_9 + 12\omega_{10} + 10\omega_{11} + 12\omega_{12} + 10\omega_{13} + 4\omega_{14} = e.\end{aligned}$$

$\underline{\chi_{15}(n_3):}$

For $[n_3]$, there are two pairs with 0, 6, 8, 9, 11, 13 and 14; five pairs with 4 and 10 and six pairs with 1, 2, 3, 5, 7 and 12.

Hence,

$$\begin{aligned}\chi_{15}(n_3) = \quad & 2\times 2 + 2\times 12\times \omega_8 + 2\times 12\times \omega_{13} + 2\times 12\times \omega_5 + 2\times 10\times \omega_3 + 2\times 12\times \omega_{11}\\ & +2\times 4\times \omega_{10} + 2\times 12\times \omega_2 + 2\times 4\times \omega_6 + 2\times 4\times \omega_{14} + 2\times 10\times \omega_7\\ & +2\times 4\times \omega_1 + 2\times 12\times \omega_9 + 2\times 4\times \omega_{12} + 2\times 4\times \omega_4.\end{aligned}$$

So

$$\begin{aligned}\chi_{15}(n_3) = \quad & 4 + 4\omega_1 + 12\omega_2 + 10\omega_3 + 4\omega_4 + 12\omega_5 + 4\omega_6 + 10\omega_7\\ & +12\omega_8 + 12\omega_9 + 4\omega_{10} + 12\omega_{11} + 4\omega_{12} + 12\omega_{13} + 4\omega_{14} = f.\end{aligned}$$

$\underline{\chi_{15}(n_4):}$

For $[n_4]$, there are two pairs with 0, 1, 3, 7, 11, 12 and 14; five pairs with 8 and 9 and six pairs

with 2, 4, 5, 6, 10 and 14.
Hence,

$$\begin{aligned}\chi_{15}(n_4) = \;& 2\times 2+2\times 4\times \omega_8+2\times 12\times \omega_{13}+2\times 4\times \omega_5+2\times 12\times \omega_3+2\times 12\times \omega_{11}\\ &+2\times 12\times \omega_{10}+2\times 4\times \omega_2+2\times 10\times \omega_6+2\times 10\times \omega_{14}+2\times 12\times \omega_7\\ &+2\times 4\times \omega_1+2\times 4\times \omega_9+2\times 4\times \omega_{12}+2\times 12\times \omega_4.\end{aligned}$$

So

$$\begin{aligned}\chi_{15}(n_4) = \;& 4+4\omega_1+4\omega_2+12\omega_3+12\omega_4+4\omega_5+10\omega_6+12\omega_7\\ &+4\omega_8+4\omega_9+12\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+10\omega_{14}=g.\end{aligned}$$

$\underline{\chi_{15}(n_5):}$

For $[n_5]$, there are two pairs with 0, 2, 3, 5, 6, 7 and 14; five pairs with 11 and 13 and six pairs with 2, 4, 8, 9, 10 and 12.
Hence,

$$\begin{aligned}\chi_{15}(n_5) = \;& 2\times 2+2\times 12\times \omega_8+2\times 4\times \omega_{13}+2\times 4\times \omega_5+2\times 12\times \omega_3+2\times 4\times \omega_{11}\\ &+2\times 4\times \omega_{10}+2\times 4\times \omega_2+2\times 12\times \omega_6+2\times 12\times \omega_{14}+2\times 12\times \omega_7\\ &+2\times 10\times \omega_1+2\times 12\times \omega_9+2\times 10\times \omega_{12}+2\times 4\times \omega_4.\end{aligned}$$

So

$$\begin{aligned}\chi_{15}(n_5) = \;& 4+10\omega_1+4\omega_2+12\omega_3+4\omega_4+4\omega_5+12\omega_6+12\omega_7\\ &+12\omega_8+12\omega_9+4\omega_{10}+4\omega_{11}+10\omega_{12}+4\omega_{13}+12\omega_{14}=a.\end{aligned}$$

$\underline{\chi_{15}(n_6):}$

For $[n_6]$, there are two pairs with 0, 1, 4, 6, 10, 12 and 14; five pairs with 3 and 7 and six pairs with 2, 5, 8, 9, 11 and 13.
Hence,

$$\begin{aligned}\chi_{15}(n_6) = \;& 2\times 2+2\times 4\times \omega_8+2\times 12\times \omega_{13}+2\times 10\times \omega_5+2\times 4\times \omega_3+2\times 12\times \omega_{11}\\ &+2\times 4\times \omega_{10}+2\times 10\times \omega_2+2\times 12\times \omega_6+2\times 12\times \omega_{14}+2\times 4\times \omega_7\\ &+2\times 12\times \omega_1+2\times 4\times \omega_9+2\times 12\times \omega_{12}+2\times 4\times \omega_4.\end{aligned}$$

So

$$\begin{aligned}\chi_{15}(n_6) = \;& 4+12\omega_1+10\omega_2+4\omega_3+4\omega_4+10\omega_5+12\omega_6+4\omega_7\\ &+4\omega_8+4\omega_9+4\omega_{10}+12\omega_{11}+12\omega_{12}+12\omega_{13}+12\omega_{14}=b.\end{aligned}$$

$\underline{\chi_{15}(n_7):}$

For $[n_7]$, there are two pairs with 0, 1, 2, 5, 8, 9 and 12; five pairs with 6 and 14 and six pairs with 3, 4, 7, 10, 11 and 13.

Hence,

$$\begin{aligned}\chi_{15}(n_7) = \quad & 2\times 2+2\times 4\times\omega_8+2\times 4\times\omega_{13}+2\times 12\times\omega_5+2\times 12\times\omega_3+2\times 4\times\omega_{11}\\ & +2\times 10\times\omega_{10}+2\times 12\times\omega_2+2\times 4\times\omega_6+2\times 4\times\omega_{14}+2\times 12\times\omega_7\\ & +2\times 12\times\omega_1+2\times 4\times\omega_9+2\times 12\times\omega_{12}+2\times 10\times\omega_4.\end{aligned}$$

So

$$\begin{aligned}\chi_{15}(n_7) = \quad & 4+12\omega_1+12\omega_2+12\omega_3+10\omega_4+12\omega_5+4\omega_6+12\omega_7\\ & +4\omega_8+4\omega_9+10\omega_{10}+4\omega_{11}+12\omega_{12}+4\omega_{13}+4\omega_{14}=c.\end{aligned}$$

7.8 Section H

7.8.1 Character Values of χ_{16}

Now $\chi_{16}=\phi_1^{\overline{G}}=(\psi_1.\psi_{12})$ and $\chi_{16}(n_i)=\sum_{i=1}^{120}\psi_{12}(x_i)$. To find the values $\chi_{16}(n_i)$, we need the values $\psi_{12}(g)$ for $g\in\mathbb{Z}_{29}$. They are tabulated below:

Table 7.7: The Values of the Character ψ_{12}

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
ψ_{12}	1	ω_{11}	$\overline{\omega}_7$	ω_4	$\overline{\omega}_{14}$	$\overline{\omega}_3$	ω_8	$\overline{\omega}_{10}$	ω_1	ω_{12}	$\overline{\omega}_6$	ω_5	$\overline{\omega}_{13}$	$\overline{\omega}_2$	ω_9

Table 7.7 (continued)

g	15	16	17	18	19	20	21	22	23	24	25	26	27	28
ψ_{12}	$\overline{\omega}_9$	ω_2	ω_{13}	$\overline{\omega}_5$	ω_6	$\overline{\omega}_{12}$	$\overline{\omega}_1$	ω_{10}	$\overline{\omega}_8$	ω_3	ω_{14}	$\overline{\omega}_4$	ω_7	$\overline{\omega}_{11}$

where $\omega_i=e^{\frac{2k\pi i}{29}}$ and $\overline{\omega}_i$ is the complex conjugate of ω_i.

$\underline{\chi_{16}(n_1):}$

For $[n_1]$ there are two pairs with 0, 2, 4, 5, 10, 11 and 13; five pairs with 1 and 12 and six pairs with 3, 6, 7, 8, 9 and 14.

Hence,

$$\begin{aligned}\chi_{16}(n_1) = \quad & 2\times 2+2\times 5\times \omega_{11}+2\times 2\times \omega_7+2\times 6\times \omega_4+2\times 2\times \omega_{14}+2\times 2\times \omega_3 \\ & +2\times 6\times \omega_8+2\times 6\times \omega_{10}+2\times 6\times \omega_1+2\times 6\times \omega_{12}+2\times 2\times \omega_6 \\ & +2\times 2\times \omega_5+2\times 5\times \omega_{13}+2\times 2\times \omega_2+2\times 6\times \omega_9.\end{aligned}$$

So

$$\begin{aligned}\chi_{16}(n_1) = \quad & 4+12\omega_1+4\omega_2+4\omega_3+12\omega_4+4\omega_5+4\omega_6+4\omega_7 \\ & +12\omega_8+12\omega_9+12\omega_{10}+10\omega_{11}+12\omega_{12}+10\omega_{13}+4\omega_{14} = e.\end{aligned}$$

$\underline{\chi_{16}(n_2):}$
For $[n_2]$, there are two pairs with 0, 3, 4, 7, 8, 9 and 10; five pairs with 2 and 5 and six pairs with 1, 6, 11, 12, 13 and 14.
Hence,

$$\begin{aligned}\chi_{16}(n_2) = \quad & 2\times 2+2\times 6\times \omega_{11}+2\times 5\times \omega_7+2\times 2\times \omega_4+2\times 2\times \omega_{14}+2\times 5\times \omega_3 \\ & +2\times 6\times \omega_8+2\times 2\times \omega_{10}+2\times 2\times \omega_1+2\times 2\times \omega_{12}+2\times 2\times \omega_6 \\ & +2\times 6\times \omega_5+2\times 6\times \omega_{13}+2\times 6\times \omega_2+2\times 6\times \omega_9.\end{aligned}$$

So

$$\begin{aligned}\chi_{16}(n_2) = \quad & 4+4\omega_1+12\omega_2+10\omega_3+4\omega_4+12\omega_5+4\omega_6+10\omega_7 \\ & +12\omega_8+12\omega_9+4\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+4\omega_{14} = f.\end{aligned}$$

$\underline{\chi_{16}(n_3):}$
For $[n_3]$, there are two pairs with 0, 6, 8, 9, 11, 13 and 14; five pairs with 4 and 10 and six pairs with 1, 2, 3, 5, 7 and 12.
Hence,

$$\begin{aligned}\chi_{16}(n_3) = \quad & 2\times 2+2\times 12\times \omega_{11}+2\times 12\times \omega_7+2\times 12\times \omega_4+2\times 10\times \omega_{14}+2\times 12\times \omega_3 \\ & +2\times 4\times \omega_8+2\times 12\times \omega_{10}+2\times 4\times \omega_1+2\times 4\times \omega_{12}+2\times 10\times \omega_6 \\ & +2\times 4\times \omega_5+2\times 12\times \omega_{13}+2\times 4\times \omega_2+2\times 4\times \omega_9.\end{aligned}$$

So

$$\begin{aligned}\chi_{16}(n_3) = \quad & 4+4\omega_1+4\omega_2+12\omega_3+12\omega_4+4\omega_5+10\omega_6+12\omega_7 \\ & +4\omega_8+4\omega_9+12\omega_{10}+12\omega_{11}+4\omega_{12}+12\omega_{13}+10\omega_{14} = g.\end{aligned}$$

$\underline{\chi_{16}(n_4):}$
For $[n_4]$, there are two pairs with 0, 1, 3, 7, 11, 12 and 14; five pairs with 8 and 9 and six pairs

with 2, 4, 5, 6, 10 and 14.
Hence,

$$\begin{aligned}\chi_{16}(n_4) = \;& 2\times 2+2\times 4\times \omega_{11}+2\times 12\times \omega_7+2\times 4\times \omega_4+2\times 12\times \omega_{14}+2\times 12\times \omega_3\\ &+2\times 12\times \omega_8+2\times 4\times \omega_{10}+2\times 10\times \omega_1+2\times 10\times \omega_{12}+2\times 12\times \omega_6\\ &+2\times 4\times \omega_5+2\times 4\times \omega_{13}+2\times 4\times \omega_2+2\times 12\times \omega_9.\end{aligned}$$

So

$$\begin{aligned}\chi_{16}(n_4) = \;& 4+10\omega_1+4\omega_2+12\omega_3+4\omega_4+4\omega_5+12\omega_6+12\omega_7\\ &+12\omega_8+12\omega_9+4\omega_{10}+4\omega_{11}+10\omega_{12}+4\omega_{13}+12\omega_{14}=a.\end{aligned}$$

$\underline{\chi_{16}(n_5):}$

For $[n_5]$, there are two pairs with 0, 2, 3, 5, 6, 7 and 14; five pairs with 11 and 13 and six pairs with 2, 4, 8, 9, 10 and 12.
Hence,

$$\begin{aligned}\chi_{16}(n_5) = \;& 2\times 2+2\times 12\times \omega_{11}+2\times 4\times \omega_7+2\times 4\times \omega_4+2\times 12\times \omega_{14}+2\times 4\times \omega_3\\ &+2\times 4\times \omega_8+2\times 4\times \omega_{10}+2\times 12\times \omega_1+2\times 12\times \omega_{12}+2\times 12\times \omega_6\\ &+2\times 10\times \omega_5+2\times 12\times \omega_{13}+2\times 10\times \omega_2+2\times 4\times \omega_9.\end{aligned}$$

So

$$\begin{aligned}\chi_{16}(n_5) = \;& 4+12\omega_1+10\omega_2+4\omega_3+4\omega_4+10\omega_5+12\omega_6+4\omega_7\\ &+4\omega_8+4\omega_9+4\omega_{10}+12\omega_{11}+12\omega_{12}+12\omega_{13}+12\omega_{14}=b.\end{aligned}$$

$\underline{\chi_{16}(n_6):}$

For $[n_6]$, there are two pairs with 0, 1, 4, 6, 10, 12 and 14; five pairs with 3 and 7 and six pairs with 2, 5, 8, 9, 11 and 13.
Hence,

$$\begin{aligned}\chi_{16}(n_6) = \;& 2\times 2+2\times 4\times \omega_{11}+2\times 12\times \omega_7+2\times 10\times \omega_4+2\times 4\times \omega_{14}+2\times 12\times \omega_3\\ &+2\times 4\times \omega_8+2\times 10\times \omega_{10}+2\times 12\times \omega_1+2\times 12\times \omega_{12}+2\times 4\times \omega_6\\ &+2\times 12\times \omega_5+2\times 4\times \omega_{13}+2\times 12\times \omega_2+2\times 4\times \omega_9.\end{aligned}$$

So

$$\begin{aligned}\chi_{16}(n_6) = \;& 4+12\omega_1+12\omega_2+12\omega_3+10\omega_4+12\omega_5+4\omega_6+12\omega_7\\ &+4\omega_8+4\omega_9+10\omega_{10}+4\omega_{11}+12\omega_{12}+4\omega_{13}+4\omega_{14}=c.\end{aligned}$$

$\underline{\chi_{16}(n_7):}$

For $[n_7]$, there are two pairs with 0, 1, 2, 5, 8, 9 and 12; five pairs with 6 and 14 and six pairs with 3, 4, 7, 10, 11 and 13.

Hence,

$$\begin{aligned}\chi_{16}(n_7) = \; & 2\times 2 + 2\times 4\times \omega_{11} + 2\times 4\times \omega_7 + 2\times 12\times \omega_4 + 2\times 12\times \omega_{14} + 2\times 4\times \omega_3 \\ & +2\times 10\times \omega_8 + 2\times 12\times \omega_{10} + 2\times 4\times \omega_1 + 2\times 4\times \omega_{12} + 2\times 12\times \omega_6 \\ & +2\times 12\times \omega_5 + 2\times 4\times \omega_{13} + 2\times 12\times \omega_2 + 2\times 10\times \omega_9.\end{aligned}$$

So

$$\begin{aligned}\chi_{16}(n_7) = \; & 4 + 4\omega_1 + 12\omega_2 + 4\omega_3 + 12\omega_4 + 12\omega_5 + 12\omega_6 + 4\omega_7 \\ & +10\omega_8 + 10\omega_9 + 12\omega_{10} + 4\omega_{11} + 4\omega_{12} + 4\omega_{13} + 12\omega_{14} = d.\end{aligned}$$

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