THE PRELIMINARY GROUP CLASSIFICATION OF THE EQUATION $\mathbf{u_{tt} = f(x, u_x)u_{xx} + g(x, u_x)}$

by

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Abstract

We study the class of partial differential equations $u_{tt} = f(x, u_x)u_{xx} + g(x, u_x)$, with arbitrary functions $f(x, u_x)$ and $g(x, u_x)$, from the point of view of group classification. The principal Lie algebra of infinitesimal symmetries admitted by the whole class is three-dimensional. We use the method of preliminary group classification to obtain a classification of these equations with respect to a one-dimesional extension of the principal Lie algebra and then a countable-dimensional subalgebra of their equivalence algebra. Each of these equations admits an additional infinitesimal symmetry. L.V. Ovsiannikov [9] has proposed an algorithm to construct efficiently the optimal system of an arbitrary decomposable Lie algebra. We use this algorithm to construct an optimal system of subalgebras of all dimensionalities (from one-dimensional to six- dimensional) of a seven-dimensional solvable Lie algebra.

Declaration

I declare that the contents of this dissertation is the result of my own work except where due reference has been made. It has not been submitted before for any degree to any other institution.

O K Narain
March 1995

TO MY PARENTS

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Contents

Chapter 1

Introduction

Sophus Lie was the first person to consider the problem of classification of partial differential equations according to their symmetries. Lie's algorithm for finding the symmetry group of a differential equation or system of differential equations can be found in the literature, in particular [4] - [7].

Ames *et al.* [10] investigated the group properties and associated Lie algebra of the quasilinear hyperbolic equations of the form

$$u_{tt} = f(u_x)u_{xx}.$$

The investigation was continued by Torrisi *et al.* [11] to include equations of the form

$$u_{tt} = f(x, u_x)u_{xx}.$$

In this dissertation our goal is to get sufficiently acquainted with the literature on this subject and to gain a deeper understanding of classification and research methods. To do this we set out to give a detailed review of papers [1] - [3] which deal with the equation

$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x), \tag{1.1}$$

where f and g are arbitrary functions of their arguments.

Other papers written on this subject include [13] - [17].

We do not claim originality in this study, but our contribution is the provision of details. At the end of this exercise we have a rich classification of this equation. This study and classification are important because these equations feature prominently in many physical problems, namely, non-linear wave equations involving non-homogeneous processes, non-linear telegraph equation, equations of the flow of a one-dimensional gas, etc..

The classification problem of equation (1.1) reduces to the classification of the subalgebras of an equivalence algebra. For each subalgebra of the full Lie algebra there corresponds a set of group-invariant solutions of the given system of partial differential equations. The problem of classifying all subalgebras of the Lie algebra L up to similarity is the problem of constructing the optimal system of subalgebras θL and this plays a very important role in the group analysis of differential equations.

The presence of arbitrary functions in equation (1.1) does not allow us to make profitable use of computer packages in the various symbolic languages, such as REDUCE or MACSYMA.

Ibragimov *et al.* [8] suggested the method of preliminary group classification. The essence of this method is to look for extensions of the principal Lie algebra admitted by a class of differential equations among elements of its equivalence algebra. The limitation of this method is that it can carry out the classification only relative to the finite-dimensional subalgebras of the full algebra of equivalence transformations.

Ovsiannikov [9] has proposed an algorithm which enables the optimal systems of arbitrary decomposable Lie algebra to be efficiently constructed.

Using Lie-point symmetries we demonstrate the application of these two methods to construct the optimal system of subalgebras θL of equation (1.1).

In Chapter 2 we construct the principal Lie algebra and the equivalence transformations of equation (1.1).

In Chapter 3 using the method of preliminary group classification we obtain a classification of equation (1.1) with respect to a one-dimensional subalgebra of their equivalence algebra. Each of these equations admits an additional infinitesimal symmetry beyond the principal Lie algebra.

In Chapter 4 we obtain a classification of equation (1.1) with respect to a countable-dimensional subalgebra of their equivalence algebra. Again, each of these equations admits an additional infinitesimal symmetry.

In Chapter 5 by using Ovsiannikov's algorithm we construct the optimal system θL for all dimensionalities, namely, $\theta L_7 = \bigcup_{1\leq k\leq 6} \theta_k(L_7)$. The arbitrariness in the process of the construction of the optimal solution is minimized by normalizing the optimal system.

Finally, in Appendices A - D we tabulate some of the results obtained.

Chapter 2

The Equivalence Transformations

2.1 The Principal Lie Algebra

In this section we wish to determine the Lie algebra admitted by the equation (1.1) for arbitrary functions f and g. We call this the *principal Lie algebra* of the equation (1.1) and will denote it by $L_{\mathcal{P}}$.

Geometrically the equation (1.1) can be interpreted as a surface in the $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$ - space. The corresponding nonlinear group action on the (t, x, u) - space translates into a linear infinitesimal action of this algebra on the same space. The generators of the group which are elements of $L_{\mathcal{P}}$ are of the form:

$$\vec{X} = \xi_1(t,x,u)\frac{\partial}{\partial t} + \xi_2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2.1}$$

These represent local vector fields on the (t, x, u) - space. $L_{\mathcal{P}}$ will be completely determined if we can find the coefficients ξ_1, ξ_2 and η in (2.1).

Since the surface $u_{tt} - fu_{xx} - g = 0$ is a second order differential equation, the infinitesimal action of $\vec{X}$ needs to be prolonged to the second order, namely,

$$\vec{X}^{(2)} = \vec{X} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}} \tag{2.2}$$

where

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_t D_t(\xi_1) - u_x D_t(\xi_2), \\ \zeta_2 &= D_x(\eta) - u_t D_x(\xi_1) - u_x D_x(\xi_2), \\ \zeta_{11} &= D_t(\zeta_1) - u_{tt} D_t(\xi_1) - u_{tx} D_t(\xi_2), \\ \zeta_{12} &= D_x(\zeta_1) - u_{xt} D_x(\xi_1) - u_{xx} D_x(\xi_2), \\ \zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\xi_1) - u_{xx} D_x(\xi_2) \end{aligned} \tag{2.3}$$

and the *total derivatives* D_t and D_x are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \ldots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \ldots. \end{aligned} \tag{2.4}$$

The generator $\vec{X}^{(2)}$ is thus a local vector field extending $\vec{X}$ onto the $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$ - space.

The invariance condition for equation (1.1) is

$$\vec{X}^{(2)}[u_{tt} - f(x, u_x) u_{xx} - g(x, u_x)] = 0 \tag{2.5}$$

restricted to the surface $u_{tt} - fu_{xx} - g = 0$. This condition yields the following equation

$$\zeta_{11} - \zeta_{22} f - u_{xx}(\xi_2 f_x + \zeta_2 f_{u_x}) - \xi_2 g_x - \zeta_2 g_{u_x} = 0. \tag{2.6}$$

From the linear independence of the variables u^0 and u_{xx} we obtain the following determining equations:

$$\zeta_{11} - \zeta_{22} f - \xi_2 g_x - \zeta_2 g_{u_x} = 0, \tag{2.7}$$

$$\xi_2 f_x + \zeta_2 f_{u_x} = 0. \tag{2.8}$$

Since these equations are true for arbitrary f and g, it follows that

$$\xi_2 = 0, \quad \zeta_2 = 0. \tag{2.9}$$

Equation (2.7) then becomes

$$\zeta_{11} - \zeta_{22} f = 0. \tag{2.10}$$

In the case of arbitrary f it follows that

$$\zeta_{11} = \zeta_{22} = 0. \tag{2.11}$$

From equations (2.3), (2.9) and (2.11) we obtain

$$D_t(\eta) - u_t D_t(\xi_1) = \zeta_1, \tag{2.12}$$

$$D_x(\eta) - u_t D_t(\xi_1) = 0, \tag{2.13}$$

$$D_t(\zeta_1) - u_{tt} D_t(\xi_1) = 0, \tag{2.14}$$

$$-u_{tx} D_x(\xi_1) = 0. \tag{2.15}$$

Equation (2.15) gives

$$u_{tx}\left(\frac{\partial \xi_1}{\partial x}+u_x\frac{\partial \xi_1}{\partial u}\right)=0. \tag{2.16}$$

By the independence of u_{tx} and $u_{tx}u_x$ we have

$$\frac{\partial \xi_1}{\partial x}=\frac{\partial \xi_1}{\partial u}=0. \tag{2.17}$$

Equation (2.13) gives

$$\frac{\partial \eta}{\partial x}+u_x\frac{\partial \eta}{\partial u}-u_t\left(\frac{\partial \xi_1}{\partial x}+u_x\frac{\partial \xi_1}{\partial u}\right)=0 \tag{2.18}$$

and hence by independence arguments

$$\frac{\partial \eta}{\partial x}=\frac{\partial \eta}{\partial u}=0. \tag{2.19}$$

From (2.12) and (2.14) we obtain

$$\frac{\partial^2 \eta}{\partial t^2}-u_t\frac{\partial^2 \xi_1}{\partial t^2}-u_{tt}\left(2\frac{\partial \xi_1}{\partial t}+u_x\frac{\partial \xi_1}{\partial u}\right)=0 \tag{2.20}$$

and hence

$$\frac{\partial \xi_1}{\partial t}=\frac{\partial^2 \xi_1}{\partial t^2}=\frac{\partial^2 \eta}{\partial t^2}=0. \tag{2.21}$$

Solving equations (2.17), (2.19) and (2.21) yields

$$\xi_1=c_1,\ \xi_2=0 \text{ and } \eta=c_2+c_3t, \tag{2.22}$$

where c_1, c_2 and c_3 are arbitrary constants.

The generator (2.1) then takes the form

$$\vec{X} = c_1 \frac{\partial}{\partial t} + (c_2 + c_3 t) \frac{\partial}{\partial u}. \tag{2.23}$$

The basis vectors for the principal Lie algebra $L_{\mathcal{P}}$ are therefore

$$\vec{X}_1 = \frac{\partial}{\partial t}, \ \vec{X}_2 = \frac{\partial}{\partial u}, \ \vec{X}_3 = t\frac{\partial}{\partial u}. \tag{2.24}$$

2.2 The Equivalence Transformations

In this section we construct a subgroup E_c of the group of all equivalence transformations E of the equation (1.1). In particular, we will construct the generators of the Lie algebra of the subgroup E_c.

By an *equivalence transformation* we mean a nondegenerate change of the variable t, x and u, which takes any equation of the form (1.1) to an equation of the same form. In general, after the transformation, the functions $f(x, u_x)$ and $g(x, u_x)$ may be different. The method to construct E_c was suggested by Ovsiannikov [4] and is termed the Lie infinitesimal criterion.

Since the functions f and g in equation (1.1) vary during the action of E_c, let us replace these with local variables f^1 and f^2 respectively. We now wish to determine the infinitesimal generator $\vec{Y}$ of the group E:

$$\begin{aligned} \vec{Y} &= \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \mu^k \frac{\partial}{\partial f^k} \\ &= \vec{X} + \mu^k \frac{\partial}{\partial f^k}, k = 1, 2, \end{aligned} \tag{2.25}$$

where $\vec{X}$ is as in equation (2.1). The dependence of f^k and μ^k are as follows: $f^k = f^k(t,x,u,u_t,u_x)$ and $\mu^k = \mu^k(t,x,u,u_t,u_x,f^1,f^2)$. Equation (1.1) thus takes the form of the system:

$$\begin{aligned} & u_{tt} - f^1 u_{xx} - f^2 = 0, \\ & f^k_t = f^k_u = f^k_{u_t} = 0, \;\; k = 1,2. \end{aligned} \tag{2.26}$$

The action of $\vec{Y}$ extends to that of

$$\tilde{\vec{Y}} = \vec{X}^{(2)} + \omega_1^k \frac{\partial}{\partial f_t^k} + \omega_0^k \frac{\partial}{\partial f_u^k} + \omega_{01}^k \frac{\partial}{\partial f_{u_t}^k}, \tag{2.27}$$

where $\vec{X}^{(2)}$ is as in equation (2.2) and

$$\omega_a^k = \tilde{D}_a(\mu^k) - f_t^k \tilde{D}_a(\xi_1) - f_x^k \tilde{D}_a(\xi_2) - f_u^k \tilde{D}_a(\eta) - f_{u_t}^k \tilde{D}_a(\zeta_1) - f_{u_x}^k \tilde{D}_a(\zeta_2), \tag{2.28}$$

where

(i) $a = t, u, u_t$,

(ii) $\omega_1^k = \omega_t^k,\; \omega_0^k = \omega_u^k,\; \omega_{01}^k = \omega_{u_t}^k$,

and

$$\tilde{D}_a = \frac{\partial}{\partial a} + f_a^k \frac{\partial}{\partial f^k}.$$

From $f_t^k = f_u^k = f_{u_t}^k = 0$, it follows that $\tilde{D}_t = \dfrac{\partial}{\partial t}$, $\tilde{D}_u = \dfrac{\partial}{\partial u}$ and $\tilde{D}_{u_t} = \dfrac{\partial}{\partial u_t}$.

By simplifying (2.28) for the various values of a we obtain

$$\begin{aligned} \omega_1^k &= \mu_t^k - f_x^k(\xi_2)_t - f_{u_x}^k(\zeta_2)_t, \\ \omega_0^k &= \mu_u^k - f_x^k(\xi_2)_u - f_{u_x}^k(\zeta_2)_u, \\ \omega_{01}^k &= \mu_{u_t}^k - f_{u_x}^k(\zeta_2)_{u_t}. \end{aligned} \tag{2.29}$$

The invariance conditions

$$\begin{gathered}\tilde{\tilde{\vec{Y}}}(u_{tt} - f^1 u_{xx} - f^2) = 0,\\ \tilde{\tilde{\vec{Y}}}(f_t^k) = \tilde{\tilde{\vec{Y}}}(f_u^k) = \tilde{\tilde{\vec{Y}}}(f_{u_t}^k) = 0,\ \ k = 1, 2,\end{gathered} \tag{2.30}$$

restricted to the surface $u_{tt} - f^1 u_{xx} - f^2 = 0$, yield

$$\omega_1^k = \omega_0^k = \omega_{01}^k = 0,\ \ k = 1, 2$$

and hence also

$$\begin{aligned} \mu_t^k - f_x^k(\xi_2)_t - f_{u_x}^k(\zeta_2)_t &= 0,\\ \mu_u^k - f_x^k(\xi_2)_u - f_{u_x}^k(\zeta_2)_u &= 0,\\ \mu_{u_t}^k - f_{u_x}^k(\zeta_2)_{u_t} &= 0.\end{aligned} \tag{2.31}$$

Since equations (2.31) must hold for every f^1 and f^2, we obtain:

$$\begin{gathered}\mu_t^k = \mu_u^k = \mu_{u_t}^k = 0,\ \ k = 1, 2,\\ (\xi_2)_t = (\xi_2)_u = 0,\\ (\zeta_2)_t = (\zeta_2)_u = (\zeta_2)_{u_t} = 0.\end{gathered} \tag{2.32}$$

Equations (2.32) yield:

$$\begin{aligned} &\mu^k = \mu^k(x, u_x, f^1, f^2),\\ &\xi_2 = \xi_2(x).\end{aligned}$$

We now have from equations (2.3)

$$\begin{aligned}\zeta_1 &= \frac{\partial\eta}{\partial t} + u_t\left(\frac{\partial\eta}{\partial u}\right) - u_t\left(\frac{\partial\xi_1}{\partial t} + u_t\frac{\partial\xi_1}{\partial u}\right),\\ \zeta_2 &= \frac{\partial\eta}{\partial x} + u_x\frac{\partial\eta}{\partial u} - u_t\left(\frac{\partial\xi_1}{\partial x} + u_x\frac{\partial\xi_1}{\partial u}\right) - u_x\frac{\partial\xi_2}{\partial x}.\end{aligned} \tag{2.33}$$

From equations (2.32) and (2.33) we have

$$\begin{aligned}(\zeta_2)_t &= \frac{\partial^2\eta}{\partial t\partial x} + u_{tx}\left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi_2}{\partial x}\right) + u_x\frac{\partial^2\eta}{\partial t\partial u} - u_{tt}\left(\frac{\partial\xi_1}{\partial x} + u_x\frac{\partial\xi_1}{\partial u}\right)\\ &\quad -u_t\left(\frac{\partial^2\xi_1}{\partial t\partial x} + u_{tx}\frac{\partial\xi_1}{\partial u} + u_x\frac{\partial^2\xi_1}{\partial t\partial u}\right) = 0,\\ (\zeta_2)_u &= \frac{\partial^2\eta}{\partial u\partial x} + u_x\frac{\partial^2\eta}{\partial u^2} - u_t\left(\frac{\partial^2\xi_1}{\partial u\partial x} + u_x\frac{\partial^2\xi_1}{\partial u^2}\right) = 0.\end{aligned} \tag{2.34}$$

From equations (2.33) and independence arguments we obtain

$$\frac{\partial\xi_1}{\partial x} = \frac{\partial\xi_1}{\partial u} = 0$$

and

$$\frac{\partial^2\eta}{\partial u^2} = \frac{\partial^2\eta}{\partial t\partial x} = \frac{\partial^2\eta}{\partial u\partial x} = \frac{\partial^2\eta}{\partial t\partial u} = \frac{\partial\eta}{\partial u} - \frac{\partial\xi_2}{\partial x} = 0.$$

We therefore have

$$\begin{aligned}\xi_1 &= \xi_1(t),\\ \xi_2 &= \xi_2(x),\\ \eta &= c_1u + F(x) + H(t),\\ \mu^k &= \mu^k(x, u_x, f^1, f^2).\end{aligned} \tag{2.35}$$

The invariance condition (2.29) yields

$$\zeta_{11} - \mu^1 u_{xx} - \zeta_{22} f^1 - \mu^2 = 0. \tag{2.36}$$

Using equations (2.3), (2.33), (2.35) and $u_{tt} = f^1 u_{xx} + f^2$ we have

$$\begin{aligned} \zeta_1 &= H'(t) + c_1 u_t - (\xi_1)' u_t, \\ \zeta_2 &= F'(x) + c_1 u_x - (\xi_2)' u_x, \\ \zeta_{11} &= H''(t) + (c_1 - 2(\xi_1)')(f^1 u_{xx} + f^2) - (\xi_1)'' u_t, \\ \zeta_{22} &= F''(x) + [c_1 - 2(\xi_2)'] u_{xx} - (\xi_2)'' u_x. \end{aligned} \tag{2.37}$$

From (2.36) and (2.37) it follows that

$$(\xi_1)'' u_t + \{[c_1 - 2(\xi_1)'] f^1 - \mu^1 - [c_1 - 2(\xi_2)'] f^1\} u_{xx} +$$

$$[c_1 - 2(\xi_1)'] f^2 + H'' - f^1 F'' + f^1 u_x (\xi_2)'' - \mu^2 = 0. \tag{2.38}$$

From the independence of u^0, u_t, u_x, and u_{xx} we obtain the following determining equations:

$$(\xi_1)'' = 0, \tag{2.39}$$

$$[c_1 - 2(\xi_1)'] f^1 - \mu^1 - [c_1 - 2(\xi_2)'] f^1 = 0, \tag{2.40}$$

$$[c_1 - 2(\xi_1)'] f^2 + H'' - f^1 F'' + f^1 u_x (\xi_2)'' - \mu^2 = 0. \tag{2.41}$$

Equation (2.39) gives $\xi_1 = c_2 t + c_3$, where c_2 and c_3 are arbitrary constants. Let $\xi_2 = \varphi(x)$, where $\varphi(x)$ is an arbitrary function of x.

From (2.40) we obtain

$$\mu^1 = 2(\varphi' - c_2)f^1. \tag{2.42}$$

Differentiating (2.41) with respect to t we get $H''' = 0$ and hence

$$H = c_4t^2 + c_5t + c_6, \tag{2.43}$$

where c_4, c_5 and c_6 are arbitrary constants.

Therefore from (2.41) we have

$$\mu^2 = (c_1 - 2c_2)f^2 + 2c_4 + (\varphi'' u_x - F'')f^1. \tag{2.44}$$

Altogether we have

$$\begin{aligned}
\xi_1 &= c_2t + c_3, \\
\xi_2 &= \varphi(x), \\
\eta &= c_1u + F(x) + c_4t^2 + c_5t, \\
\mu^1 &= 2(\varphi' - c_2)f^1, \\
\mu^2 &= (c_1 - 2c_2)f^2 + 2c_4 + (\varphi'' u_x - F'')f^1,
\end{aligned} \tag{2.45}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants and $\varphi(x)$ and $F(x)$ are arbitrary functions. The constant c_6 has been incorporated into the function $F(x)$.

The infinite-dimensional subgroup E_c of the equivalence transformations has a Lie algebra generated by the following infinitesimal generators:

$$\vec{Y}_1 = \frac{\partial}{\partial t},$$

$$
\begin{aligned}
\vec{Y}_2 &= \frac{\partial}{\partial u}, \\
\vec{Y}_3 &= t\frac{\partial}{\partial u}, \\
\vec{Y}_4 &= x\frac{\partial}{\partial u}, \\
\vec{Y}_5 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \\
\vec{Y}_6 &= t\frac{\partial}{\partial t} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}, \\
\vec{Y}_7 &= t^2\frac{\partial}{\partial u} + 2\frac{\partial}{\partial g}, \\
\vec{Y}_8 &= u\frac{\partial}{\partial u} + g\frac{\partial}{\partial g}, \\
\vec{Y}_\varphi &= \varphi\frac{\partial}{\partial x} + 2\varphi' f\frac{\partial}{\partial f} + \varphi'' u_x\frac{\partial}{\partial g}, \\
\vec{Y}_F &= F\frac{\partial}{\partial u} - F'' f\frac{\partial}{\partial g}.
\end{aligned}
\tag{2.46}
$$

The vector $\vec{Y}_2$ is obtained by setting $F = 1$ in $\vec{Y}_F$ and it is included above because it is part of the principal Lie algebra $L_{\mathcal{P}}$.

The following reflections

$$
\begin{aligned}
t &\longmapsto -t, \\
x &\longmapsto -x, \\
u &\longmapsto -u, \\
g &\longmapsto -g
\end{aligned}
\tag{2.47}
$$

are included in the group E_c obtained by integrating the vector fields (2.46).

Chapter 3

The ten-dimensional subalgebra

3.1 The method of Preliminary Group Classification

In this section we briefly sketch the method of preliminary group classification. In this method we will use any finite or countable-dimensional subalgebra of the algebra $L_{\mathcal{E}}$, constructed in Chapter 2. Later on in Chapter 4, use will be made of a countable-dimensional subalgebra. For now let us select a ten-dimensional subalgebra L_{10} of $L_{\mathcal{E}}$ whose generators are as follows:

$$
\begin{aligned}
\vec{Y}_1 &= \frac{\partial}{\partial t}, \\
\vec{Y}_2 &= \frac{\partial}{\partial u}, \\
\vec{Y}_3 &= t\frac{\partial}{\partial u},
\end{aligned}
$$

$$
\begin{aligned}
\vec{Y}_4 &= \frac{\partial}{\partial x}, \\
\vec{Y}_5 &= x\frac{\partial}{\partial u}, \\
\vec{Y}_6 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \\
\vec{Y}_7 &= -\frac{t}{2}\frac{\partial}{\partial t} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \\
\vec{Y}_8 &= \frac{t^2}{2}\frac{\partial}{\partial u} + \frac{\partial}{\partial g}, \\
\vec{Y}_9 &= u\frac{\partial}{\partial u} + g\frac{\partial}{\partial g}, \\
\vec{Y}_{10} &= \frac{x^2}{2}\frac{\partial}{\partial u} - f\frac{\partial}{\partial g}.
\end{aligned}
\tag{3.1}
$$

Since the functions f and g have the following dependence on the variables: $f = f(x, u_x)$ and $g = g(x, u_x)$, we have to prolong the generators (3.1) to ones including the variable u_x.

Therefore $\vec{Y}_i$ needs to be prolonged to

$$\tilde{Y}_i = \vec{Y}_i + \eta^{(1)}\frac{\partial}{\partial u_x},$$

where

$$\eta^{(1)} = D_x\eta - (D_x\xi_2)u_x.$$

Hence the extensions are:

$$\tilde{Y}_1 = \frac{\partial}{\partial t},$$

$$
\begin{aligned}
\tilde{Y}_2 &= \frac{\partial}{\partial u}, \\
\tilde{Y}_3 &= t\frac{\partial}{\partial u}, \\
\tilde{Y}_4 &= \frac{\partial}{\partial x}, \\
\tilde{Y}_5 &= x\frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}, \\
\tilde{Y}_6 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x}, \\
\tilde{Y}_7 &= -\frac{t}{2}\frac{\partial}{\partial t} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \\
\tilde{Y}_8 &= \frac{t^2}{2}\frac{\partial}{\partial u} + \frac{\partial}{\partial g}, \\
\tilde{Y}_9 &= u\frac{\partial}{\partial u} + g\frac{\partial}{\partial g} + u_x\frac{\partial}{\partial u_x}, \\
\tilde{Y}_{10} &= \frac{x^2}{2}\frac{\partial}{\partial u} - f\frac{\partial}{\partial g} + x\frac{\partial}{\partial u_x}.
\end{aligned}
\tag{3.2}
$$

By taking the projections of the generators (3.2) on the (x, u_x, f, g) - space we obtain the following nonzero projections:

$$
\vec{Z}_i = pr(\tilde{Y}_{i+3}) \quad i = 1, 2, \ldots, 7 \tag{3.3}
$$

or

$$
\begin{aligned}
\vec{Z}_1 &= \frac{\partial}{\partial x}, \\
\vec{Z}_2 &= \frac{\partial}{\partial u_x}, \\
\vec{Z}_3 &= x\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u_x}, \\
\vec{Z}_4 &= f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g},
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\vec{Z}_5 &= \frac{\partial}{\partial g}, \\
\vec{Z}_6 &= g\frac{\partial}{\partial g} + u_x\frac{\partial}{\partial u_x}, \\
\vec{Z}_7 &= x\frac{\partial}{\partial u_x} - f\frac{\partial}{\partial g}.
\end{aligned}
$$

We denote by L_7 the algebra whose basis is the set of generators (3.4).

By the *preliminary group classification* we will mean the classification of all nonequivalent equations of the form (1.1) with respect to a given equivalence group E_c. It is worthwhile to note that E_c is not necessarily the largest equivalence group but it can be any subgroup of the group of all equivalence transformations.

This method was proposed in [8] and it is applied when an equivalence group is generated by a finite dimensional Lie algebra $L_{\mathcal{E}}$. The essence of the method is the determination of all the equivalence classes of subalgebras of $L_{\mathcal{E}}$ of various dimensions; under conjugation or similarity. As regards to equations of the form (1.1) the following propositions contain the essence of the method:

Proposition 1 *Let L_m be an m-dimensional subalgebra of L_7. Let $\mathbf{Z}^{(i)}$ ($i = 1, 2, \ldots, m$) be a basis of L_m and $\mathbf{Y}^{(i)}$ be the elements of the algebra L_{10} such that $\mathbf{Z}^{(i)} = pr(\tilde{Y}^{(i)})$, i.e., if*

$$
\mathbf{Z}^{(i)} = \sum_{\alpha=1}^{7} e_i^{\alpha} \vec{Z}_{\alpha} \tag{3.5}
$$

then by (3.1) - (3.3) we have

$$
\mathbf{Y}^{(i)} = \sum_{\alpha=1}^{7} e_i^{\alpha} \vec{Y}_{i+3}. \tag{3.6}
$$

If the functions $f = \Phi(x, u_x)$ *and* $g = \Gamma(x, u_x)$ *are invariant with respect to the algebra* L_m *then the equation*

$$u_{tt} = \Phi(x, u_x)u_{xx} + \Gamma(x, u_x) \tag{3.7}$$

admits the generators

$$\mathbf{X}^{(i)} = \text{ projection of } \mathbf{Y}^{(i)} \text{ onto the } (t, x, u) - \text{space.}$$

Proposition 2 *Let equation (3.7) and the equation*

$$u_{tt} = \Phi'(x, u_x)u_{xx} + \Gamma'(x, u_x) \tag{3.8}$$

be constructed according to Proposition 1 via subalgebras L_m *and* L'_m *respectively. If* L_m *and* L'_m *are similar subalgebras in* L_{10}*, then the equations (3.7) and (3.8) are equivalent with respect to the equivalence group* G_{10} *generated by* L_{10}*.*

From these propositions it follows that the problem of the preliminary group classification of equation (1.1) with respect to the finite-dimensional subalgebra L_{10} of $L_{\mathcal{E}}$ is reduced to the algebraic problem of constructing the nonsimilar subalgebras of L_7 or determining the optimal system of subalgebras.

3.2 The adjoint group for the algebra L_7

Here we wish to construct the adjoint group of L_7. Before doing that, we give some definitions and explain some terms.

Let G be a Lie group and L its Lie algebra. For each element $T \in G$ there exists an inner automorphism $T_a \longmapsto TT_aT^{-1}$ of G. Each group automorphism induces a Lie algebra automorphism. The set of all automorphisms of L induced from the inner automorphisms of G form a local Lie group called the *group of inner automorphisms* of L or the *adjoint group* of L which we denote by G^A.

The Lie algebra of G^A is the adjoint algebra L^A (or $ad\ L$) of the algebra L defined as follows:

For each $X \in L$, the linear mapping:

$$ad_X : L \longrightarrow L$$

defined by $ad_X(\xi) = [\xi, X]$ is an automorphism of the algebra L. Since the above map also satisfies the product rule for differentiation of the algebra L, it is called the *inner derivation* of L. The set L^A of all inner derivations together with the bracket $[ad_X, ad_Y] = ad_{[X,Y]}$ is a Lie algebra, called the *adjoint algebra* of L. Clearly the adjoint algebra L^A is the Lie algebra of the adjoint group G^A.

Two subalgebras in L are *conjugate* or *similar* if there exists an element of G^A which maps one subalgebra into the other. The collection of all pairwise nonconjugate $m-$dimensional subalgebras is called an optimal system of order m in L and is denoted by $\theta_m L$. Since we will be determining an optimal system of order one, we will show that every element of L_7 is conjugate to one of various canonical forms.

We now wish to construct the adjoint group of the algebra L_7. Let us

	$\vec{Z}_1$	$\vec{Z}_2$	$\vec{Z}_3$	$\vec{Z}_4$	$\vec{Z}_5$	$\vec{Z}_6$	$\vec{Z}_7$
$\vec{Z}_1$	0	0	$\vec{Z}_1$	0	0	0	$\vec{Z}_2$
$\vec{Z}_2$	0	0	$\vec{Z}_2$	0	0	$\vec{Z}_2$	0
$\vec{Z}_3$	$-\vec{Z}_1$	$-\vec{Z}_2$	0	0	0	0	0
$\vec{Z}_4$	0	0	0	0	$-\vec{Z}_5$	0	0
$\vec{Z}_5$	0	0	0	$\vec{Z}_5$	0	$\vec{Z}_5$	0
$\vec{Z}_6$	0	$-\vec{Z}_2$	0	0	$-\vec{Z}_5$	0	$-\vec{Z}_7$
$\vec{Z}_7$	$-\vec{Z}_2$	0	0	0	0	$\vec{Z}_7$	0

Table 3.1: Commutators of L_7

denote the elements of *ad* L_7 by the letter A. The generators of *ad* L_7 are

$$\vec{A}_\alpha = \sum_{\beta=1}^{7} \left[\vec{Z}_\alpha, \vec{Z}_\beta\right] \frac{\partial}{\partial \vec{Z}_\beta}, \qquad \alpha = 1, 2, \ldots, 7. \tag{3.9}$$

The commutation table of L_7 is given in Table 3.1.

Using Table 3.1 and equation (3.9) we obtain the following generators:

$$\begin{aligned}
\vec{A}_1 &= \vec{Z}_1 \frac{\partial}{\partial \vec{Z}_3} + \vec{Z}_2 \frac{\partial}{\partial \vec{Z}_7}, \\
\vec{A}_2 &= \vec{Z}_2 \frac{\partial}{\partial \vec{Z}_3} + \vec{Z}_2 \frac{\partial}{\partial \vec{Z}_6}, \\
\vec{A}_3 &= -\vec{Z}_1 \frac{\partial}{\partial \vec{Z}_1} - \vec{Z}_2 \frac{\partial}{\partial \vec{Z}_2}, \\
\vec{A}_4 &= -\vec{Z}_5 \frac{\partial}{\partial \vec{Z}_5}, \qquad\qquad\qquad (3.10) \\
\vec{A}_5 &= \vec{Z}_5 \frac{\partial}{\partial \vec{Z}_4} + \vec{Z}_5 \frac{\partial}{\partial \vec{Z}_6},
\end{aligned}$$

$$\vec{A}_6 = -\vec{Z}_2\frac{\partial}{\partial\vec{Z}_2} - \vec{Z}_5\frac{\partial}{\partial\vec{Z}_5} - \vec{Z}_7\frac{\partial}{\partial\vec{Z}_7},$$
$$\vec{A}_7 = -\vec{Z}_2\frac{\partial}{\partial\vec{Z}_1} + \vec{Z}_7\frac{\partial}{\partial\vec{Z}_6}.$$

By letting

$$\vec{A}_i = \xi_i^1\frac{\partial}{\partial\vec{Z}_1} + \xi_i^2\frac{\partial}{\partial\vec{Z}_2} + \dots + \xi_i^7\frac{\partial}{\partial\vec{Z}_7}$$

and solving the initial value problem

$$\frac{d\vec{Z}'_k}{da_i} = \xi_i^k \quad \text{with } \vec{Z}'_k = \vec{Z}_k \text{ when } a_i = 0 \ (k = 1,2,\dots,7 \text{ and } i = 1,2,\dots,7)$$

we obtain the one-parameter groups of linear transformations.

For example taking $\vec{A}_1$ we obtain

$$\vec{Z}'_1 = \vec{Z}_1,\ \vec{Z}'_2 = \vec{Z}_2,\ \vec{Z}'_3 = \vec{Z}_3 + a_1\vec{Z}_1,\ \vec{Z}'_4 = \vec{Z}_4,$$
$$\vec{Z}'_5 = \vec{Z}_5,\ \vec{Z}'_6 = \vec{Z}_6,\ \vec{Z}'_7 = \vec{Z}_7 + a_1\vec{Z}_2,$$

where $a_1 \in \Re$.[1]

Therefore in the adjoint group of L_7 we have

$$M_1(a_1) = \begin{Vmatrix} 1 & 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & a_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

[1] $\Re$ is the set of real numbers and $\Re^+$ is the set of positive real numbers.

where $a_1 \in \Re$.

Similarly for $A_2, A_3, \ldots, A_7$ we have

$$M_2(a_2) = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_2 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|,$$

$$M_3(a_3) = \left\| \begin{matrix} a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|,$$

$$M_4(a_4) = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|,$$

$$M_5(a_5) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 & 1 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

$$M_6(a_6) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_6 \end{Vmatrix},$$

$$M_7(a_7) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_7 & 1 \end{Vmatrix},$$

where $a_2, a_5, a_7 \in \Re$ and $a_3, a_4, a_6 \in \Re^+$.

Let $M = \prod_{\alpha=1}^{7} M_\alpha(a_\alpha)$. Then

$$M = \left\| \begin{array}{ccccccc} a_3 & 0 & a_1a_3 & 0 & 0 & 0 & 0 \\ -a_3a_7 & a_3a_6 & a_2a_3a_6 - a_1a_3a_7 & 0 & 0 & a_2a_3a_6 & a_1a_3a_6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5a_6 & a_4a_6 & a_5a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_7 & a_6 \end{array} \right\| .$$

For each $\mathbf{Z} \in L_7$ we have

$$\mathbf{Z} = \sum_{i=1}^{7} e^i \vec{Z}_i \equiv \left(e^1, e^2, \ldots, e^7\right). \tag{3.11}$$

Let $\mathbf{e} = (e^1, e^2, \ldots, e^7)$, $\bar{\mathbf{e}} = (\bar{e}^1, \bar{e}^2, \ldots, \bar{e}^7)$ and $\bar{\mathbf{e}} = M\mathbf{e}$.

Then the components of $\bar{\mathbf{e}}$ are:

$$\begin{aligned} \bar{e}^1 &= a_3\left(e^1 + a_1e^3\right), \\ \bar{e}^2 &= a_3\left[-a_7e^1 + a_6e^2 + \left(a_2a_6 - a_1a_7\right)e^3 + a_2a_6e^6 + a_1a_6e^7\right], \\ \bar{e}^3 &= e^3, \\ \bar{e}^4 &= e^4, \\ \bar{e}^5 &= a_6\left(a_5e^4 + a_4e^5 + a_5e^6\right), \\ \bar{e}^6 &= e^6, \\ \bar{e}^7 &= a_7e^6 + a_6e^7. \end{aligned} \tag{3.12}$$

These transformations give rise to the adjoint group elements of the algebra L_7. The reflections (2.47) give rise to the following transformations:

$$\vec{Z}_1 \longmapsto -\vec{Z}_1, \quad \vec{Z}_2 \longmapsto -\vec{Z}_2, \tag{3.13}$$

$$\vec{Z}_2 \longmapsto -\vec{Z}_2, \quad \vec{Z}_5 \longmapsto -\vec{Z}_5, \quad \vec{Z}_7 \longmapsto -\vec{Z}_7. \tag{3.14}$$

3.3 Optimal system of order one

In the next section we will be extending the algebra $L_{\mathcal{P}}$ by one-dimensional subalgebras of L_7. We therefore need to construct the optimal system of one-dimensional subalgebras of L_7. This is carried out as follows:

(i) By using $M \in G^A$ and reflections (3.13) and (3.14) we map $\mathbf{e} = (e^1, e^2, \ldots, e^7)$ to as simple a form $\bar{\mathbf{e}} = (\bar{e}^1, \bar{e}^2, \ldots, \bar{e}^7)$ as possible.

(ii) We will then divide the vectors obtained into nonequivalent classes. In any class we select a representative which has the simplest possible form.

For $M \in G^A$ the mapping

$$\bar{\mathbf{e}} = M\mathbf{e}$$

leaves the components e^3, e^4 and e^6 invariant in (3.12). Thus we need to seek all the possibilities for e^3, e^4 and e^6 and in each case simplify the other components of $\mathbf{e}$ by the transformations (3.12).

CASE 1 : $e^3 \neq 0,\ e^4 \neq 0,\ e^6 \neq 0$

By substituting

$$a_1 = -\frac{e^1}{e^3},\ a_6 = 1,\ a_7 = -\frac{e^7}{e^6} \tag{3.15}$$

in (3.12) we obtain

$$\bar{e}^1 = a_3\left(e^1 - \frac{e^1}{e^3}e^3\right) = 0, \quad \bar{e}^7 = -\frac{e^7}{e^6}e^6 + e^7. \tag{3.16}$$

From (3.15) and (3.16) and by keeping the other parameters arbitrary any vector $\mathbf{e}$ is transformed to $\bar{\mathbf{e}} = (0, \bar{e}^2, \bar{e}^3, \bar{e}^4, \bar{e}^5, \bar{e}^6, 0)$, provided Case 1 is valid.

We can further simplify the vector $\bar{\mathbf{e}}$ by means of the transformations (3.12) by putting $a_1 = a_7 = 0$. Hence the components of the vector $\mathbf{e}$ are transformed to the vector $\bar{\mathbf{e}}$ having components:

$$\begin{aligned}
\bar{e}^1 &= 0, \\
\bar{e}^2 &= a_3 \left[e^2 + a_2 \left(e^3 + e^6 \right) \right], \\
\bar{e}^3 &= e^3, \\
\bar{e}^4 &= e^4, \\
\bar{e}^5 &= \left[a_5 \left(e^4 + e^6 \right) + a_4 e^5 \right], \\
\bar{e}^6 &= e^6, \\
\bar{e}^7 &= 0.
\end{aligned} \tag{3.17}$$

From the components of $\bar{\mathbf{e}}$ we can distinguish the following four subcases:

SUBCASE 1 : $e^3 + e^6 \neq 0, \quad e^4 + e^6 \neq 0$

By putting

$$a_2 = \frac{-e^2}{e^3 + e^6}, \tag{3.18}$$

$$a_4 = 1, \quad a_5 = \frac{-e^5}{e^4 + e^6} \tag{3.19}$$

we get $\bar{e}^2 = 0, \ \bar{e}^5 = 0$ and thus

$$\bar{\mathbf{e}} = (0, 0, e^3, e^4, 0, e^6, 0). \tag{3.20}$$

The vector (3.20) can be written in the form

$$\bar{\mathbf{e}} = (0, 0, \alpha, \beta, 0, 1, 0), \quad \alpha \neq 0, -1, \ \beta \neq 0, -1 \tag{3.21}$$

using the fact that any infinitesimal generator can be defined up to a constant factor.

SUBCASE 2 : $e^3 + e^6 \neq 0, \quad e^4 + e^6 = 0$

Substitution for a_2 using (3.18) in (3.17) yields $\bar{e}^2 = 0, \ \bar{e}^4 = -e^6, \ \bar{e}^5 = a_4 e^5$.

Thus

$$\bar{\mathbf{e}} = (0, 0, e^3, -e^6, a_4 e^5, e^6, 0).$$

Here we have either $e^5 = 0$ or $e^5 \neq 0$.

For $e^5 \neq 0$: By using the factor $a_4 = \dfrac{e^6}{e^5}$ and the reflection (3.13) we obtain

$$\bar{\mathbf{e}} = (0, 0, e^3, -e^6, e^6, e^6, 0). \tag{3.22}$$

Again using the fact that any infinitesimal generator can be defined up to a constant factor we write vector (3.22) in the form

$$\bar{\mathbf{e}} = (0, 0, \alpha, -1, 1, 1, 0), \ \alpha \neq 0, -1. \tag{3.23}$$

For $e^5 = 0$ we obtain

$$\bar{\mathbf{e}} = (0, 0, \alpha, -1, 0, 1, 0), \ \alpha \neq 0, -1. \tag{3.24}$$

SUBCASE 3 : $e^3 + e^6 = 0, \quad e^4 + e^6 \neq 0$

Substitution of (3.19) in (3.17) leads to

$$\bar{\mathbf{e}} = (0, a_3e^2, -e^6, e^4, 0, e^6, 0).$$

Following the same procedure as in Subcase 2 we obtain from this vector two different vectors:

$$\bar{\mathbf{e}} = (0, 1, -1, \beta, 0, 1, 0), \ \beta \neq 0, -1, \tag{3.25}$$

$$\bar{\mathbf{e}} = (0, 0, -1, \beta, 0, 1, 0), \ \beta \neq 0, -1. \tag{3.26}$$

SUBCASE 4 : $e^3 + e^6 = 0, \quad e^4 + e^5 = 0$

Here vector (3.17) yields

$$\bar{\mathbf{e}} = (0, a_3e^2, -e^6, -e^6, a_4e^5, e^6, 0).$$

Using arbitrary positive factors for a_3, a_4 and the reflections (3.13) and (3.14) we obtain from this vector the following four vectors:

$$\bar{\mathbf{e}} = (0, 1, -1, -1, 1, 1, 0), \tag{3.27}$$

$$\bar{\mathbf{e}} = (0, 0, -1, -1, 1, 1, 0), \tag{3.28}$$

$$\bar{\mathbf{e}} = (0, 1, -1, -1, 0, 1, 0), \tag{3.29}$$

$$\bar{\mathbf{e}} = (0, 0, -1, -1, 0, 1, 0). \tag{3.30}$$

In summary, for Case 1, any vector $\mathbf{e}$ is equivalent to vectors (3.21) and (3.23) - (3.30). Using equation (3.11) we see that these vectors give rise to the following nonequivalent generators:

$$\begin{aligned}
&\alpha\vec{Z}_3 + \beta\vec{Z}_4 + \vec{Z}_6, \ \alpha \neq 0, \ \beta \neq 0, \\
&\alpha\vec{Z}_3 - \vec{Z}_4 + \vec{Z}_5 + \vec{Z}_6, \ \alpha \neq 0, \\
&\vec{Z}_2 - \vec{Z}_3 + \beta\vec{Z}_4 + \vec{Z}_6, \ \beta \neq 0, \\
&\vec{Z}_2 - \vec{Z}_3 - \vec{Z}_4 + \vec{Z}_5 + \vec{Z}_6.
\end{aligned} \tag{3.31}$$

The restriction on the parameters α and β is changed in order to present the generators in a compact form. For example, the vector (3.29) is included in vector (3.24) if the condition $\beta \neq -1$ is cancelled.

Similarly the analysis of the other cases yields the following nonequivalent generators:

CASE 2 : $e^3 \neq 0, \ e^4 \neq 0, \ e^6 = 0$

$$\begin{aligned}
&\alpha\vec{Z}_3 + \vec{Z}_4 + \vec{Z}_7, \ \alpha \neq 0, \\
&\alpha\vec{Z}_3 + \vec{Z}_4, \ \alpha \neq 0.
\end{aligned} \tag{3.32}$$

CASE 3 : $e^3 \neq 0, \ e^4 = 0, \ e^6 \neq 0$

$$\begin{aligned}
&\vec{Z}_2 - \vec{Z}_3 + \vec{Z}_6, \\
&\alpha\vec{Z}_3 + \vec{Z}_6, \ \alpha \neq 0.
\end{aligned} \tag{3.33}$$

CASE 4 : $e^3 \neq 0, \ e^4 = 0, \ e^6 = 0$

$$\begin{aligned}
&\vec{Z}_3, \ \vec{Z}_3 + \vec{Z}_5, \ \vec{Z}_3 + \vec{Z}_7, \\
&\vec{Z}_3 + \vec{Z}_5 + \vec{Z}_7, \ \vec{Z}_3 + \vec{Z}_5 - \vec{Z}_7.
\end{aligned} \tag{3.34}$$

CASE 5 : $e^3 = 0,\ e^4 \neq 0,\ e^6 \neq 0$

$$\begin{gathered}\vec{Z}_4 + \vec{Z}_5 - \vec{Z}_6, \\ \vec{Z}_1 - \vec{Z}_4 + \vec{Z}_5 + \vec{Z}_6.\end{gathered} \tag{3.35}$$

CASE 6 : $e^3 = 0,\ e^4 \neq 0,\ e^6 = 0$

$$\begin{gathered}\vec{Z}_4,\ \vec{Z}_1 + \vec{Z}_4,\ \vec{Z}_2 + \vec{Z}_4, \\ \vec{Z}_4 + \vec{Z}_7,\ \vec{Z}_1 + \vec{Z}_4 + \vec{Z}_7.\end{gathered} \tag{3.36}$$

CASE 7 : $e^3 = 0,\ e^4 = 0,\ e^6 \neq 0$

$$\vec{Z}_6,\ \vec{Z}_1 + \vec{Z}_6. \tag{3.37}$$

CASE 8 : $e^3 = 0,\ e^4 = 0,\ e^6 = 0$

$$\begin{gathered}\vec{Z}_1,\ \vec{Z}_2,\ \vec{Z}_5,\ \vec{Z}_7, \\ \vec{Z}_1 + \vec{Z}_5,\ \vec{Z}_1 + \vec{Z}_7, \\ \vec{Z}_2 + \vec{Z}_5,\ \vec{Z}_5 + \vec{Z}_7,\ \vec{Z}_5 - \vec{Z}_7, \\ \vec{Z}_1 + \vec{Z}_5 + \vec{Z}_7,\ \vec{Z}_1 + \vec{Z}_5 - \vec{Z}_7.\end{gathered} \tag{3.38}$$

Altogether from (3.31) - (3.38) we have the following optimal system of one-dimensional subalgebras of L_7 (α and β are arbitrary constants) :

$$\mathbf{Z}^{(1)} = \vec{Z}_1, \quad \mathbf{Z}^{(2)} = \vec{Z}_2, \quad \mathbf{Z}^{(3)} = \vec{Z}_3, \quad \mathbf{Z}^{(4)} = \vec{Z}_4 + \alpha\vec{Z}_3,$$

$$\mathbf{Z}^{(5)} = \vec{Z}_5, \quad \mathbf{Z}^{(6)} = \vec{Z}_6 + \alpha\vec{Z}_3 + \beta\vec{Z}_4, \quad \mathbf{Z}^{(7)} = \vec{Z}_7,$$

$$\mathbf{Z}^{(8)} = \vec{Z}_1 + \vec{Z}_4, \quad \mathbf{Z}^{(9)} = \vec{Z}_1 + \vec{Z}_5, \quad \mathbf{Z}^{(10)} = \vec{Z}_1 + \vec{Z}_6 + \beta\vec{Z}_4,$$

$$\mathbf{Z}^{(11)} = \vec{Z}_1 + \vec{Z}_7, \quad \mathbf{Z}^{(12)} = \vec{Z}_2 + \vec{Z}_4, \quad \mathbf{Z}^{(13)} = \vec{Z}_2 + \vec{Z}_5,$$

$$\mathbf{Z}^{(14)} = \vec{Z}_3 + \vec{Z}_5, \quad \mathbf{Z}^{(15)} = \vec{Z}_3 + \vec{Z}_7, \quad \mathbf{Z}^{(16)} = \vec{Z}_5 + \vec{Z}_7,$$

$$\mathbf{Z}^{(17)} = \vec{Z}_5 - \vec{Z}_7, \quad \mathbf{Z}^{(18)} = \vec{Z}_1 + \vec{Z}_4 + \vec{Z}_7, \quad \mathbf{Z}^{(19)} = \vec{Z}_1 + \vec{Z}_5 + \vec{Z}_7,$$

$$\mathbf{Z}^{(20)} = \vec{Z}_1 + \vec{Z}_5 - \vec{Z}_7, \quad \mathbf{Z}^{(21)} = \alpha\vec{Z}_3 + \vec{Z}_4 + \vec{Z}_7,$$

$$\mathbf{Z}^{(22)} = \vec{Z}_3 + \vec{Z}_5 + \vec{Z}_7, \quad \mathbf{Z}^{(23)} = \vec{Z}_3 + \vec{Z}_5 - \vec{Z}_7,$$

$$\mathbf{Z}^{(24)} = \vec{Z}_1 - \vec{Z}_4 + \vec{Z}_5 + \vec{Z}_6, \quad \mathbf{Z}^{(25)} = \vec{Z}_2 - \vec{Z}_3 + \beta\vec{Z}_4 + \vec{Z}_6,$$

$$\mathbf{Z}^{(26)} = \alpha\vec{Z}_3 - \vec{Z}_4 + \vec{Z}_5 + \vec{Z}_6, \quad \mathbf{Z}^{(27)} = \vec{Z}_2 - \vec{Z}_3 - \vec{Z}_4 + \vec{Z}_5 + \vec{Z}_6,$$

3.4 Equations admitting an extension of algebra $L_{\mathcal{P}}$ by one

To obtain all nonequivalent equations (1.1) admitting an extension by one of the principal Lie algebra $L_{\mathcal{P}}$ we apply Propositions 1 and 2 to the optimal system obtained in the previous section. For each subalgebra in the optimal system we obtain equations of the form (1.1) such that they admit, together with three basic generators of $L_{\mathcal{P}}$, also a fourth generator $\vec{X}_4$. Whenever these extensions occur, we list the corresponding functions f and g and the additional generator $\vec{X}_4$.

To illustrate the method we choose the following examples from our optimal system:

(a) Consider $\mathbf{Z}^{(24)}$:

$$
\begin{aligned}
\mathbf{Z}^{(24)} &= \vec{Z}_1 - \vec{Z}_4 + \vec{Z}_5 + \vec{Z}_6 \\
&= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u_x} - f \frac{\partial}{\partial f} + \frac{\partial}{\partial g}.
\end{aligned}
$$

Invariants are found from the subsidiary equations:

$$
\frac{dx}{1} = \frac{du_x}{u_x} = -\frac{df}{f} = \frac{dg}{1}.
$$

From these equation we obtain:

(i)

$$
\frac{dx}{1} = \frac{du_x}{u_x}
$$

$$
\Rightarrow I_1 = e^{-x} u_x. \tag{3.39}
$$

(ii)

$$
\frac{dx}{1} = -\frac{df}{f}
$$

$$
\Rightarrow I_2 = e^{x} f. \tag{3.40}
$$

(iii)

$$
dx = dg
$$

$$
\Rightarrow I_3 = g - x. \tag{3.41}
$$

where I_k, $k = 1, 2$ and 3, are the labels for the characteristics.

By applying Proposition 1 we can take the invariance equations in the form

$$I_2 = \Phi(I_1) \text{ and } I_3 = \Gamma(I_1). \tag{3.42}$$

Let $\lambda = I_1 = e^{-x}u_x$. From (3.40) and (3.42) we have

$$e^x f = \Phi(\lambda) \Rightarrow f = e^{-x}\Phi(\lambda).$$

From (3.41) and (3.42) we have

$$g - x = \Gamma(\lambda) \Rightarrow g = \Gamma(\lambda) + x.$$

In terms of equation (3.11) the subalgebra $\mathbf{Z}^{(24)}$ is equivalent to the vector $\mathbf{e} = (1, 0, 0, -1, 1, 1, 0)$. Applying equation (3.6) to the subalgebra $\mathbf{Z}^{(24)}$ we obtain

$$\begin{aligned}\mathbf{Y}^{(24)} &= \vec{Y}_4 - \vec{Y}_7 + \vec{Y}_8 + \vec{Y}_9 \\ &= \frac{t}{2}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \left(\frac{t^2}{2} + u\right)\frac{\partial}{\partial u} - f\frac{\partial}{\partial f} + \frac{\partial}{\partial g}.\end{aligned}$$

By taking the projection of $\mathbf{Y}^{(24)}$ onto the (t, x, u) - space we obtain the additional generator $\vec{X}_4$ of the subalgebra $\mathbf{Z}^{(24)}$, namely,

$$\vec{X}_4 = t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} + \left(t^2 + 2u\right)\frac{\partial}{\partial u}.$$

Hence the equation

$$u_{tt} = e^{-x}\Phi(e^{-x}u_x)u_{xx} + \Gamma(e^{-x}u_x) + x$$

admits the four-dimensional algebra L_4 with generators

$$\vec{X}_1 = \frac{\partial}{\partial t},\ \vec{X}_2 = \frac{\partial}{\partial u},\ \vec{X}_3 = t\frac{\partial}{\partial u} \text{ and } \vec{X}_4 = t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} + \left(t^2 + 2u\right)\frac{\partial}{\partial u}.$$

(b) Consider $\mathbf{Z}^{(5)} = \vec{Z}_5 = \dfrac{\partial}{\partial g}$:

Invariants of this subalgebra are

$$I_1 = x, \ I_2 = u_x, \ I_3 = f. \tag{3.43}$$

In this case there are no invariant equations of the form (3.7) *i.e.*, the invariants (3.43) cannot be solved with respect to the functions f and g.

Proceeding in a similar manner we perform the calculations for the other subalgebras in our optimal system. In Appendix A we give the result of the preliminary group classification of equation (1.1) admitting an extension of the principal Lie algebra $L_{\mathcal{P}}$ by one dimension. There are 29 nonequivalent equations in this list.

Chapter 4

The countable-dimensional subalgebra

In this chapter we consider the equivalence transformations not contained in L_{10}. We will investigate a countable-dimensional subalgebra $L_\#$ of the infinite-dimensional equivalence algebra $L_\mathcal{E}$ or rather a countable number of n-dimensional extensions L_n of L_{10}. We then proceed with the method of preliminary group classification for the equation (1.1) with respect to the subalgebra $L_\#$.

4.1 The countable-dimensional subalgebra $L_\#$

In this section we obtain a countable-dimensional subalgebra $L_\#$ of $L_\mathcal{E}$.

After extending the generators (2.46) onto the (u, t, x, u_x, f, g) - space we get the following full equivalence algebra $L_\mathcal{E}$ given by the following generators:

$$\begin{aligned}
\vec{Y}_1 &= \frac{\partial}{\partial t}, \\
\vec{Y}_2 &= t\frac{\partial}{\partial u},
\end{aligned}$$

$$
\begin{aligned}
\vec{Y}_3 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x}, \\
\vec{Y}_4 &= -\frac{t}{2}\frac{\partial}{\partial t} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \\
\vec{Y}_5 &= \frac{t^2}{2}\frac{\partial}{\partial u} + \frac{\partial}{\partial g}, \\
\vec{Y}_\varphi &= \varphi(x)\frac{\partial}{\partial x} + 2\varphi'(x)f\frac{\partial}{\partial f} + \varphi''(x)u_x f\frac{\partial}{\partial g} - u_x\varphi'(x)\frac{\partial}{\partial u_x}, \\
\vec{Y}_F &= F(x)\frac{\partial}{\partial u} - F''(x)f\frac{\partial}{\partial g} + F'(x)\frac{\partial}{\partial u_x}.
\end{aligned}
\tag{4.1}
$$

where $\varphi(x)$ and $F(x)$ are arbitrary functions.

Taking the projections of generators (4.1) onto the $(x,\ u_x,\ f,\ g)$ - space we obtain the following non-zero projections:

$$
\begin{aligned}
\vec{Z}_1 &= pr(\vec{Y}_3) &&= x\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u_x}, \\
\vec{Z}_2 &= pr(\vec{Y}_4) &&= f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \\
\vec{Z}_3 &= pr(\vec{Y}_5) &&= \frac{\partial}{\partial g}, \\
\vec{V}_\varphi &= pr(\vec{Y}_\varphi) &&= \varphi(x)\frac{\partial}{\partial x} + 2\varphi'(x)f\frac{\partial}{\partial f} + \varphi''(x)u_x f\frac{\partial}{\partial g} - u_x\varphi'(x)\frac{\partial}{\partial u_x}, \\
\vec{W}_F &= pr(\vec{Y}_F) &&= -F''(x)f\frac{\partial}{\partial g} + F'(x)\frac{\partial}{\partial u_x}.
\end{aligned}
\tag{4.2}
$$

The table of commutators of $L_{\mathcal{E}}$ are given in Table 4.1.

From Table 4.1 we see that we obtain subalgebras of dimension $n+5$, $n \geq 1$, by taking the following functions of φ and F:

$$
\varphi = 1,\ x;
$$
$$
F = x,\ \frac{1}{2}x^2, \ldots, \frac{1}{n}x^n.
$$

	$\vec{Z}_1$	$\vec{Z}_2$	$\vec{Z}_3$	$\vec{V}_\psi$	$\vec{W}_G$
$\vec{Z}_1$	0	0	0	$\vec{V}_{x\psi'-\psi}$	$\vec{W}_{xG'-2G}$
$\vec{Z}_2$	0	0	$-\vec{Z}_3$	0	0
$\vec{Z}_3$	0	$\vec{Z}_3$	0	0	0
$\vec{V}_\varphi$	$\vec{V}_{\varphi-x\varphi'}$	0	0	$\vec{V}_{\psi\varphi'-\varphi\psi'}$	$\vec{W}_{\varphi G'}$
$\vec{W}_F$	$\vec{W}_{2F-xF'}$	0	0	$-\vec{W}_{\psi F'}$	0

Table 4.1: Commutators of $L_\mathcal{E}$

We denote their corresponding generators by $\vec{V}_1$, $\vec{V}_2$ and $\vec{W}_1$, $\vec{W}_2$, $\ldots, \vec{W}_n$ respectively.

The subalgebras L_{n+5} are contained in the countable-dimensional subalgebra $L_\#$ which corresponds to the choice of F as an analytic function of x. The table of commutators of L_{n+5} are given in Table 4.2.

4.2 The adjoint algebra $L^*_\#$

In this section we construct the adjoint algebra $L^*_\#$ which generates the group of inner automorphisms of the algebra $L_\#$. Similarly to equation (3.9) in Section 3.2, each row of Table 4.2 can be considered as the coordinates of the infinitesimal generator of the adjoint algebra $L^*_\#$.

Our problem essentially now is to find all classes of the generators:

$$\mathbf{Z} = e^1\vec{Z}_1 + e^2\vec{Z}_2 + e^3\vec{Z}_3 + e^4\vec{V}_1 + e^5\vec{V}_2 + e^6\vec{W}_1 + e^7\vec{W}_2 + \sum_{i=1}^{\infty} e^i\vec{W}_i \qquad (4.3)$$

	$\vec{Z}_1$	$\vec{Z}_2$	$\vec{Z}_3$	$\vec{V}_1$	$\vec{V}_2$	$\vec{W}_1$	$\vec{W}_2$	$\cdots$	$\vec{W}_n$
$\vec{Z}_1$	0	0	0	$-\vec{V}_1$	0	$-\vec{W}_1$	0	$\cdots$	$\frac{n-2}{n}\vec{W}_n$
$\vec{Z}_2$	0	0	$-\vec{Z}_3$	0	0	0	0	$\cdots$	0
$\vec{Z}_3$	0	$\vec{Z}_3$	0	0	0	0	0	$\cdots$	0
$\vec{V}_1$	$\vec{V}_1$	0	0	0	$\vec{V}_1$	0	$\vec{W}_1$	$\cdots$	$\vec{W}_{n-1}$
$\vec{V}_2$	0	0	0	$-\vec{V}_1$	0	$\vec{W}_1$	$\vec{W}_2$	$\cdots$	$\vec{W}_n$
$\vec{W}_1$	$\vec{W}_1$	0	0	0	$-\vec{W}_1$	0	0	$\cdots$	0
$\vec{W}_2$	0	0	0	$-\vec{W}_1$	$-\vec{W}_2$	0	0	$\cdots$	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vec{W}_n$	$-\frac{n-2}{n}\vec{W}_n$	0	0	$-\vec{W}_{n-1}$	$-\vec{W}_n$	0	0	$\cdots$	0

Table 4.2: Commutators of L_{n+5}

nonequivalent with respect to the group of inner automorphisms. We will investigate the automorphisms $M_i(a_i)$ which correspond to the generators $\vec{A}_i$ $(i \geq 8)$ since $1 \leq i \leq 7$ have been dealt with in Chapter 3.

The automorphism $M_i(a_i)$ for $i \geq 8$ can be expressed as follows:

For $n \geq 3$ we have

$$\vec{A}_{n+5} = -\frac{n-2}{n}\vec{W}_n\frac{\partial}{\partial \vec{Z}_1} - \vec{W}_{n-1}\frac{\partial}{\partial \vec{V}_1} - \vec{W}_n\frac{\partial}{\partial \vec{V}_2}. \tag{4.4}$$

The one-parameter group of linear transformation is obtained by solving the equations:

$$\frac{d\vec{Z}_1'}{da_{n+5}} = -\frac{n-2}{n}\vec{W}_n, \quad \frac{d\vec{V}_1'}{da_{n+5}} = -\vec{W}_{n-1}, \quad \frac{d\vec{V}_2'}{da_{n+5}} = -\vec{W}_n \tag{4.5}$$

subject to the initial conditions $\vec{Z}_1' = \vec{Z}_1$, $\vec{V}_1' = \vec{V}_1$, $\vec{V}_2' = \vec{V}_2$ when $a_{n+5} = 0$.

Thus $\vec{A}_{n+5}$ generates the following one-parameter group of linear transformations:

$$\begin{gathered} \vec{Z}_1' = \vec{Z}_1 - \frac{n-2}{n}\vec{W}_n a_{n+5}, \; \vec{Z}_2' = \vec{Z}_2, \; \vec{Z}_3' = \vec{Z}_3, \\ \vec{V}_1' = \vec{V}_1 - \vec{W}_{n-1}a_{n+5}, \; \vec{V}_2' = \vec{V}_2 - \vec{W}_n a_{n+5}, \\ \vec{W}_1' = \vec{W}_1, \; \vec{W}_2' = \vec{W}_2, \; \vec{W}_n' = \vec{W}_n, \end{gathered} \tag{4.6}$$

where $n \geq 3$. The transformation (4.6) can be represented by the following matrix:

$$M_{n+5}(a_{n+5}) = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & -a_{n+5} & 0 & \dots & 1 & 0 \\ -\frac{n-2}{n}a_{n+5} & 0 & 0 & 0 & -a_{n+5} & \dots & 0 & 1 \end{matrix} \right\| .$$

From the above matrix we obtain the following transformation of components:

$$\begin{aligned} \bar{e}^i &= e^i, \quad i \neq n+4,\ n+5, \\ \bar{e}^{n+4} &= e^{n+4} - a_{n+5}e^4, \\ \bar{e}^{n+5} &= e^{n+4} - a_{n+5}e^5 - \frac{n-2}{n}a_{n+5}e^1, \end{aligned} \tag{4.7}$$

where $n \geq 3$.

Only the automorphism $M_8(a_8)$ changes the component e^7 of equation (4.3) and this occurs only when the component $e^4 \neq 0$. The optimal system of one-dimensional subalgebras obtained in Section 3.3 has four vectors with $e^4 \neq 0$, namely,

$$\begin{aligned} \mathbf{Z}^{(11)} &= \vec{V}_1 + \vec{W}_2, \\ \mathbf{Z}^{(19)} &= \vec{Z}_2 + \vec{V}_1 + \vec{W}_2, \\ \mathbf{Z}^{(20)} &= \vec{Z}_3 + \vec{V}_1 + \vec{W}_2, \\ \mathbf{Z}^{(21)} &= \vec{Z}_3 + \vec{V}_1 - \vec{W}_2. \end{aligned}$$

These vectors have $e^1 = e^5 = 0$. Thus $M_8(a_8)$ changes only e^7 and now $\bar{e}^7$

can be annulled by putting $a_8 = \dfrac{e^7}{e^4}$ in (4.7) as follows:

$$\begin{aligned}\bar{e}^7 &= e^7 - a_8 e^4\\ &= e^7 - \frac{e^7}{e^4}e^4\\ &= 0.\end{aligned}$$

Therefore the subalgebras $\mathbf{Z}^{(11)}$, $\mathbf{Z}^{(19)}$, $\mathbf{Z}^{(20)}$ and $\mathbf{Z}^{(21)}$ are equivalent to the subalgebras $\mathbf{Z}^{(1)}$, $\mathbf{Z}^{(8)}$ and $\mathbf{Z}^{(9)}$. We have thus reduced the number of one-dimensional subalgebras obtained in Chapter 3 by four.

As a result of this the optimal system of one-dimensional subalgebras of L_7 relative to the adjoint algebra $L^*_{\#}$ written in the form:

$$\mathbf{Z} = e^1\vec{Z}_1 + e^2\vec{Z}_2 + e^3\vec{Z}_3 + e^4\vec{V}_1 + e^5\vec{V}_2 + e^6\vec{W}_1 + e^7\vec{W}_2$$

are

$$\tilde{Z}^{(1)} = \vec{V}_1, \quad \tilde{Z}^{(2)} = \vec{W}_1, \quad \tilde{Z}^{(3)} = \vec{Z}_1, \quad \tilde{Z}^{(4)} = \vec{Z}_1 + \alpha\vec{Z}_2,$$

$$\tilde{Z}^{(5)} = \vec{Z}_3, \quad \tilde{Z}^{(6)} = (\tfrac{1}{2} + \alpha)\vec{Z}_1 + (1+\beta)\vec{Z}_2 - \tfrac{1}{2}\vec{V}_2,$$

$$\tilde{Z}^{(7)} = \vec{W}_2, \quad \tilde{Z}^{(8)} = \vec{Z}_2 + \vec{V}_1, \quad \tilde{Z}^{(9)} = \vec{Z}_3 + \vec{V}_1,$$

$$\tilde{Z}^{(10)} = \tfrac{1}{2}\vec{Z}_1 + (1+\beta)\vec{Z}_2 + \vec{V}_1 - \tfrac{1}{2}\vec{V}_2,$$

$$\tilde{Z}^{(11)} = \vec{Z}_2 + \vec{W}_1, \quad \tilde{Z}^{(12)} = \vec{Z}_3 + \vec{W}_1, \quad \tilde{Z}^{(13)} = \vec{Z}_1 + \vec{Z}_3,$$

$$\tilde{Z}^{(14)} = \vec{Z}_1 + \vec{W}_2, \quad \tilde{Z}^{(15)} = \vec{Z}_3 + \vec{W}_2, \quad \tilde{Z}^{(16)} = \vec{Z}_3 - \vec{W}_2,$$

$$\tilde{Z}^{(17)} = \alpha\vec{Z}_1 + \vec{Z}_2 + \vec{W}_2, \quad \tilde{Z}^{(18)} = \vec{Z}_1 + \vec{Z}_3 + \vec{W}_2,$$

$$\tilde{Z}^{(19)} = \vec{Z}_1 + \vec{Z}_3 - \vec{W}_2, \quad \tilde{Z}^{(20)} = \tfrac{1}{2}\vec{Z}_1 + \vec{Z}_3 + \vec{V}_1 - \tfrac{1}{2}\vec{V}_2,$$

$$\tilde{Z}^{(21)} = -\tfrac{1}{2}\vec{Z}_1 + (1+\beta)\vec{Z}_2 - \tfrac{1}{2}\vec{V}_2 + \vec{W}_1,$$

$$\tilde{Z}^{(22)} = (\tfrac{1}{2} + \alpha)\vec{Z}_1 + \vec{Z}_3 - \tfrac{1}{2}\vec{V}_2, \quad \tilde{Z}^{(23)} = -\tfrac{1}{2}\vec{Z}_1 + \vec{Z}_3 - \tfrac{1}{2}\vec{V}_2 + \vec{W}_1.$$

4.3 One-dimensional Optimal System of subalgebras of $L_{\#}$

In this section we will construct the one-dimensional optimal system of subalgebras of $L_{\#}$. Using the chain of transformations (4.7) we simplify and then divide any vectors of the form:

$$\mathbf{Z}^{[i]} = \tilde{Z}^{(i)} + \sum_{i=1}^{n} e^{5+i}\vec{W}_i \tag{4.8}$$

into nonequivalent classes.

For the vectors $\mathbf{Z}^{[1]}$, $\mathbf{Z}^{[3]}$, $\mathbf{Z}^{[4]}_{\alpha\neq 0}$, $\mathbf{Z}^{[8]}$, $\mathbf{Z}^{[9]}$, $\mathbf{Z}^{[13]}$, $\mathbf{Z}^{[14]}$, $\mathbf{Z}^{[17]}_{\alpha\neq 0}$, $\mathbf{Z}^{[18]}$, $\mathbf{Z}^{[19]}$, $\mathbf{Z}^{[21]}$, $\mathbf{Z}^{[23]}$ and $\mathbf{Z}^{[6]}$, $\mathbf{Z}^{[22]}$ with $\alpha \neq \frac{1}{n-2}, n \geq 3$ the transformation (4.7) only changes the components $\bar{e}^{n+4}$ and $\bar{e}^{n+5}$ as follows:

$$\bar{e}^{n+4} = e^{n+4} - a_{n+5}e^4, \quad \bar{e}^i = e^i \quad i \neq n+4,\ n \geq 3,\ \bar{e}^4 \neq 0;$$

$$\begin{aligned}
\bar{e}^{n+5} &= e^{n+5} - a_{n+5}e^5 - \tfrac{n-2}{n}e^1 \\
&= e^{n+5} + a_{n+5}\left(-e^5 - \tfrac{n-2}{n}e^1\right) \\
&= e^{n+5} + a_{n+5}\xi(e^1, e^5), \\
\bar{e}^i &= e^i \quad i \neq n+5,\ n \geq 3.
\end{aligned}$$

Using the appropriate factors for a_i the components e^i ($i \geq 8$) become zero and the components e^i ($i \leq 7$) remain unchanged. We need to perform only a finite number of transformations to annul the components e^i, provided that the sum in equation (4.3) is finite.

The vector $\mathbf{Z}^{[6]}$ with $\alpha = \frac{1}{n-2}$, $n \geq 3$, simplifies as follows:

$$\begin{aligned}
\mathbf{Z}^{[6]} &= (\tfrac{1}{2}+\alpha)\vec{Z}_1 + (1+\beta)\vec{Z}_2 - \tfrac{1}{2}\vec{V}_2 + \sum_{i=1}^{n} e^{5+i}\vec{W}_i \\
&= (\tfrac{1}{2}+\tfrac{1}{n-2})\vec{Z}_1 + (1+\beta)\vec{Z}_2 - \tfrac{1}{2}\vec{V}_2 + \sum_{i=1}^{n} e^{5+i}\vec{W}_i \\
&= \tfrac{n}{2}\tfrac{1}{n-2}\vec{Z}_1 + (1+\beta)\vec{Z}_2 - \tfrac{1}{2}\vec{V}_2 + \sum_{i=1}^{n} e^{5+i}\vec{W}_i.
\end{aligned}$$

Therefore $\mathbf{Z}^{[6]}$ can be written as

$$\mathbf{Z}^{[6]} = \frac{n}{2}\vec{Z}_1 + (n-2)(1+\beta)\vec{Z}_2 - \frac{n-2}{2}\vec{V}_2 + \mu\vec{W}_n, \quad n \geq 3.$$

Similarly vector $\mathbf{Z}^{[22]}$ with $\alpha = \frac{1}{n-2}$, $n \geq 3$ can be simplified to the form:

$$\mathbf{Z}^{[22]} = \frac{n}{2}\vec{Z}_1 + (n-2)\vec{Z}_3 - \frac{n-2}{2}\vec{V}_2 + \mu\vec{W}_n, \quad n \geq 3.$$

In the case of vectors $\mathbf{Z}^{[10]}$ and $\mathbf{Z}^{[20]}$ their components include $e^1 = \frac{1}{2}$, $e^4 = 1$, $e^5 = -\frac{1}{2}$, $e^6 = 0$, $e^7 = 0$. Thus the transformation (4.7) only changes the components of $\mathbf{Z}^{[10]}$ and $\mathbf{Z}^{[20]}$ as follows:

$$\begin{aligned}
\bar{e}^{n+4} &= e^{n+4} - a_{n+5}e^4 \\
&= e^{n+4} - a_{n+5},
\end{aligned}$$

$$\begin{aligned}
\bar{e}^{n+5} &= e^{n+5} - a_{n+5}e^5 - \tfrac{n-2}{n}e^1 \\
&= e^{n+5} - a_{n+5}(-\tfrac{1}{2} + \tfrac{n-2}{n}\tfrac{1}{2}) \\
&= e^{n+5} + \tfrac{1}{n}a_{n+5},
\end{aligned}$$

and

$$\bar{e}^i = e^i \quad i \neq n+4, n+5, \quad n \geq 3.$$

We can annul the last component e^N of equation (4.8) by the transformation M_N and then use the transformations M_{N-1}, $M_{N-2}, \ldots, M_8$ to bring the vectors $\mathbf{Z}^{[10]}$ and $\mathbf{Z}^{[20]}$ to the form:

$$\mathbf{Z}^{[i]} = \tilde{Z}^{(i)} + \delta \vec{W}_2, \quad i = 10,\ 20.$$

Thus vectors $\mathbf{Z}^{[10]}$ and $\mathbf{Z}^{[20]}$ are similar to $\tilde{Z}^{(10)}$ and $\tilde{Z}^{(20)}$ respectively.

Finally, the vectors $\mathbf{Z}^{[2]}$, $\mathbf{Z}^{[5]}$, $\mathbf{Z}^{[7]}$, $\mathbf{Z}^{[11]}$, $\mathbf{Z}^{[12]}$, $\mathbf{Z}^{[15]}$ and $\mathbf{Z}^{[16]}$ are unchanged by the transformation (4.7) since their components include $e^1 = e^4 = e^5 = 0$.

Thus the optimal system of one-dimensional subalgebras of $L_{\#}$ which we have now constructed is as follows:

$$\tilde{Z}^{[1]} = \vec{V}_1, \quad \tilde{Z}^{[2]} = \vec{W}_{F(x)}, \quad \tilde{Z}^{[3]} = \vec{Z}_1 + \alpha \vec{Z}_2,$$

$$\tilde{Z}^{[4]} = \vec{Z}_3 + \vec{W}_{F(x)}, \quad \tilde{Z}^{[5]} = (\frac{1}{2} + \alpha)\vec{Z}_1 + (1+\beta)\vec{Z}_2 - \frac{1}{2}\vec{V}_2,$$

$$\tilde{Z}^{[6]} = \frac{n}{2}\vec{Z}_1 + (n-2)(1+\beta)\vec{Z}_2 - \frac{n-2}{2}\vec{V}_2 + \mu \vec{W}_n, \quad n \geq 3,$$

$$\tilde{Z}^{[7]} = \vec{Z}_2 + \vec{V}_1, \quad \tilde{Z}^{[8]} = \vec{Z}_3 + \vec{V}_1,$$

$$\tilde{Z}^{[9]} = \frac{1}{2}\vec{Z}_1 + (1+\beta)\vec{Z}_2 + \vec{V}_1 - \frac{1}{2}\vec{V}_2,$$

$$\tilde{Z}^{[10]} = \vec{Z}_2 + \vec{W}_{F(x)}, \quad \tilde{Z}^{[11]} = \vec{Z}_1 + \vec{Z}_3,$$

$$\tilde{Z}^{[12]} = \vec{Z}_1 + \vec{W}_2, \quad \tilde{Z}^{[13]} = \alpha \vec{Z}_1 + \vec{Z}_2 + \vec{W}_2,$$

$$\tilde{Z}^{[14]} = \vec{Z}_1 + \vec{Z}_3 + \vec{W}_2, \quad \tilde{Z}^{[15]} = \vec{Z}_1 + \vec{Z}_3 - \vec{W}_2,$$

$$\tilde{Z}^{[16]} = \frac{1}{2}\vec{Z}_1 + \vec{Z}_3 + \vec{V}_1 - \frac{1}{2}\vec{V}_2,$$

$$\tilde{Z}^{[17]} = -\frac{1}{2}\vec{Z}_1 + (1+\beta)\vec{Z}_2 - \frac{1}{2}\vec{V}_2 + \vec{W}_1,$$

$$\tilde{Z}^{[18]} = (\frac{1}{2} + \alpha)\vec{Z}_1 + \vec{Z}_3 - \frac{1}{2}\vec{V}_2,$$

$$\tilde{Z}^{[19]} = \frac{n}{2}\vec{Z}_1 + (n-2)\vec{Z}_3 - \frac{n-2}{2}\vec{V}_2 + \mu\vec{W}_n, \quad n \geq 3,$$

$$\tilde{Z}^{[20]} = -\frac{1}{2}\vec{Z}_1 + \vec{Z}_3 - \frac{1}{2}\vec{V}_2 + \vec{W}_1.$$

To compact the generators we let the function $F(x)$ be as follows:

For vectors $\tilde{Z}^{[2]}$ and $\tilde{Z}^{[10]}$: (obtained from the vectors $\tilde{Z}^{(2)}$, $\tilde{Z}^{(4)}$ and $\tilde{Z}^{(11)}$, $\tilde{Z}^{(17)}_{\alpha=0}$ respectively) $F(x)$ is an analytic function with either:

$$\text{(i) } F'(0) = 0, \quad F''(0) = 0,$$
$$\text{or (ii) } F'(0) = 1, \quad F''(0) = 0,$$
$$\text{or (iii) } F'(0) = 0, \quad F''(0) = 1.$$

For vectors $\tilde{Z}^{[3]}$: (obtained from the vectors $\tilde{Z}^{(12)}$, $\tilde{Z}^{(15)}$ and $\tilde{Z}^{(16)}$) $F(x)$ is an analytic function with either:

$$\text{(i) } F'(0) = 0, \quad F''(0) = 0,$$
$$\text{or (ii) } F'(0) = 1, \quad F''(0) = 0,$$
$$\text{or (iii) } F'(0) = 0, \quad F''(0) = \pm 1.$$

4.4 Equations admitting an extension of algebra $L_{\mathcal{P}}$ by one

Following the procedure of in Section 3.5 we obtain equations of the form (1.1) such that they admit, together with three basis vectors (2.24) of the principal Lie algebra $L_{\mathcal{P}}$, also a fourth generator $\vec{X}_4$.

In Appendix B we give the result of the preliminary group classification of equation (1.1) with respect to a countable-dimensional subalgebra $L_{\#}$ of the equivalence algebra $L_{\mathcal{E}}$. For this particular classification we obtain 22 nonequivalent equations.

Chapter 5

Ovsiannikov's algorithm

In this chapter we demonstrate the application of the recently developed Ovsiannikov's algorithm to construct the optimal system of the subalgebras of a seven-dimensional solvable algebra of equation (1.1).

5.1 Preliminaries

In this section we will give some important definitions and notations that will be used in this chapter.

Definition 5.1 : Let L be an algebra. A subalgebra $J \subset L$ is called an *ideal* of L if for any $X \in J$, $Y \in L$, $[X, Y] \in J$.

Definition 5.2 : The ideal $L^{(1)} = [L, L]$ is called the *commutant* of the Lie algebra L. The commutant of the commutant $L^{(2)} = [L^{(1)}, L^{(1)}]$ is called the *second commutant* of the Lie algebra L. The $(k+1)th$ *commutant* is $L^{(k+1)} = [L^{(k)}, L^{(k)}]$.

L^q is a $q-$dimensional *solvable* Lie algebra if there exists a chain of subalgebras

$$L^{(1)} \subset L^{(2)} \subset \ldots \subset L^{(q-1)} \subset L^{(q)} = L^q$$

such that $L^{(k)}$ is a $k-$dimensional Lie algebra and $L^{(k-1)}$ is an ideal of $L^{(k)}$, $\quad k = 1, 2, \ldots, q$.

Definition 5.3 : The *Killing's polynomial* (or the *characteristic polynomial*) of the Lie algebra L^r for the variable $\vec{x}$ is

$$\chi(\vec{x}, \lambda) = \det(\lambda I_L - ad_{\vec{x}}) = \lambda^r - \tau_1(\vec{x})\lambda^{r-1} + \tau_2(\vec{x})\lambda^{r-2} - \ldots + (-1)^l \tau_{r-l}(\vec{x})\lambda^l, \tag{5.1}$$

where $\tau_{r-l} \neq 0$ and $l \geq 0$.

Definition 5.4 : The maximal value of the number l in equation (5.1), obtained when the vector $\vec{x}$ ranges over the whole space L^r, is called the *rank of the Lie algebra L^r*.

Definition 5.5 : Let G be a group acting on a set S and $x \in S$. The *stabilizer of x* is the set of elements $g \in G$ such that $gx = x$.

Definition 5.6 : The *normalizer* of a subalgebra K of a algebra L is defined by

$$Nor_L(K) = \{x \in L \mid [x, K] \subseteq K\}, \tag{5.2}$$

If $K = Nor_L(K)$, we call K *self-normalizing*.

5.2 Ovsiannikov's algorithm

In this section we will give a brief outline of Ovsiannikov's algorithm. A more detailed discussion of this method can be found in [9].

Let A be the group of inner automorphisms of a finite $n-$dimensional Lie algebra L. The calculation of the optimal system of subalgebras $\theta_A L$ begins by fixing the composition series of ideals

$$0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_S = L, \tag{5.3}$$

where each J_σ is the ideal in L, $J_\sigma \neq J_{\sigma+1}$ where it is impossible to condense the series (5.3) any further. For $s > 1$ there exists σ $(1 \leq \sigma \leq s)$ such that the factor algebra L/J_σ is isomorphic to some subalgebra $N \subset L$. This provides the algebra L with a *decomposition* into a semidirect sum of the proper ideal J (with $J = J_\sigma$) and the subalgebra N as follows:

$$L = J \oplus_s N. \tag{5.4}$$

In this case the group of inner automorphisms A is also decomposed into the semidirect product $A = A_J \otimes_s A_N$ of the proper invariant subgroup A_J and the subgroup A_N. The use of these decompositions allow the calculation of $\theta_A L$ in two steps:

Step 1 *The optimal system* $\theta_{A_N} N = \{N_p \mid p \in P\}$ *is calculated and the stabilizer* $A_p \subset A$ *of the subalgebra* N_p *is found for each* $p \in P$. N_P $(p = 1, 2, ..., P)$ *are representatives of* θN

Step 2 *The optimal system* $\theta_{A_p}(J + N_p) = \{K_{p,q} \mid q \in Q_p\}$ *is calculated.*

Then the set of all subalgebras $\{K_{p,q} \mid q \in Q_p\}$ is the optimal system $\theta_A L$. This two-step algorithm is performed as many times as the decompositions (5.4) admits the series (5.3).

An additional condition requires that the optimal system be a normalized one since for any $K \in \theta_A L$, $Nor_L K \in \theta_A L$. From [12] we see that the advantage of having normalized lists is that the problem of merging several sublists into a single overall list becomes greatly simplified.

5.3 The algebra L_7

In this chapter to simplify the calculations we choose the following basis for the algebra L_7:

$$\begin{aligned}
\vec{X}_1 &= \frac{\partial}{\partial u_x}, \\
\vec{X}_2 &= \frac{\partial}{\partial g}, \\
\vec{X}_3 &= \frac{\partial}{\partial x}, \\
\vec{X}_4 &= x\frac{\partial}{\partial u_x} - f\frac{\partial}{\partial g}, \\
\vec{X}_5 &= u_x\frac{\partial}{\partial u_x} - f\frac{\partial}{\partial f}, \\
\vec{X}_6 &= x\frac{\partial}{\partial x} + f\frac{\partial}{\partial f}, \\
\vec{X}_7 &= g\frac{\partial}{\partial g} + f\frac{\partial}{\partial f}.
\end{aligned} \tag{5.5}$$

The following relation exists between the basis (5.5) and the basis $\vec{Z}_1$, $\vec{Z}_2$, $\ldots$, $\vec{Z}_7$ from (3.4), namely,

$$\vec{X}_1 = \vec{Z}_2,\ \vec{X}_2 = \vec{Z}_5,\ \vec{X}_3 = \vec{Z}_1,\ \vec{X}_4 = \vec{Z}_7,$$

	$\vec{X}_1$	$\vec{X}_2$	$\vec{X}_3$	$\vec{X}_4$	$\vec{X}_5$	$\vec{X}_6$	$\vec{X}_7$
$\vec{X}_1$	0	0	0	0	$\vec{X}_1$	0	0
$\vec{X}_2$	0	0	0	0	0	0	$\vec{X}_2$
$\vec{X}_3$	0	0	0	$\vec{X}_1$	0	$\vec{X}_3$	0
$\vec{X}_4$	0	0	$-\vec{X}_1$	0	$\vec{X}_4$	$-\vec{X}_4$	0
$\vec{X}_5$	$-\vec{X}_1$	0	0	$-\vec{X}_4$	0	0	0
$\vec{X}_6$	0	0	$-\vec{X}_3$	$\vec{X}_4$	0	0	0
$\vec{X}_7$	0	$-\vec{X}_2$	0	0	0	0	0

Table 5.1: Commutators of L_7

$$\vec{X}_5 = \vec{Z}_6 - \vec{Z}_4,\ \vec{X}_6 = \vec{Z}_3 + \vec{Z}_4 - \vec{Z}_6,\ \vec{X}_7 = \vec{Z}_3.$$

The commutator relations for the algebra L_7 are given in Table 5.1.

The general vector $\mathbf{x} \in L_7$ is written in the form

$$\mathbf{x} = \sum_{\alpha=1}^{7} x^\alpha \vec{X}_\alpha$$

and hence every $\mathbf{x}$ is represented by the seven-dimensional vector $\vec{x} = (x^1,\ x^2, \ldots, x^7)$.

The inner derivation mapping $ad_{\mathbf{v}}$ for the general vector $\mathbf{v} = \sum_{\alpha=1}^{7} v^\alpha \vec{X}_\alpha$ is

$$\begin{aligned} ad_{\mathbf{v}}\mathbf{x} = [\mathbf{x}, \mathbf{v}] &= (v^5x^1 + v^4x^3 - v^3x^4 - v^1x^5)\vec{X}_1 \\ &+(v^7x^2 - v^2x^7)\vec{X}_2 + (v^6x^3 - v^3x^6)\vec{X}_3 \\ &+((v^5 - v^6)x^4 - v^4x^5 + v^4x^6)\vec{X}_4. \end{aligned}$$

The representation of the mapping $ad_{\mathbf{v}}$ in matrix form follows:

$$ad_{\mathbf{v}} = \begin{bmatrix} v^5 & 0 & v^4 & -v^3 & -v^1 & 0 & 0 \\ 0 & v^7 & 0 & 0 & 0 & 0 & -v^2 \\ 0 & 0 & v^6 & 0 & 0 & -v^3 & 0 \\ 0 & 0 & 0 & v^5 - v^6 & -v^4 & v^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Killing's polynomial of L_7 is

$$\chi(\vec{x},\lambda) = \det(\lambda I_L - ad_{\vec{x}}) = \begin{vmatrix} \lambda - x^5 & 0 & x^4 & -x^3 & -x^1 & 0 & 0 \\ 0 & \lambda - x^7 & 0 & 0 & 0 & 0 & -x^2 \\ 0 & 0 & \lambda - x^6 & 0 & 0 & -x^3 & 0 \\ 0 & 0 & 0 & \lambda - (x^5 - x^6) & -x^4 & x^4 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix}$$

$$= \lambda^3(\lambda - x^5)(\lambda - x^6)(\lambda - x^7)(\lambda - (x^5 - x^6)).$$

Thus the rank of the algebra L_7 is 3.

The commutants of the algebra L_7 have the form

$$\{0\} = L_7^{(3)} \subset \{\vec{X}_1\} = L_7^{(2)} \subset \{\vec{X}_1,\ \vec{X}_2,\ \vec{X}_3,\ \vec{X}_4\} = L_7^{(1)}.$$

Therefore the algebra L_7 is solvable.

The generators of the group of inner automorphisms of the algebra L_7 are:

$$
\begin{aligned}
\vec{A}_1 &= x^5\frac{\partial}{\partial x^1},\\
\vec{A}_2 &= x^7\frac{\partial}{\partial x^2},\\
\vec{A}_3 &= x^4\frac{\partial}{\partial x^1}+x^6\frac{\partial}{\partial x^3},\\
\vec{A}_4 &= -x^5\frac{\partial}{\partial x^1}+(x^5-x^{6)}\frac{\partial}{\partial x^4},\\
\vec{A}_5 &= -x^1\frac{\partial}{\partial x^1}-x^4\frac{\partial}{\partial x^4},\\
\vec{A}_6 &= -x^3\frac{\partial}{\partial x^3}+x^4\frac{\partial}{\partial x^4},\\
\vec{A}_7 &= -x^2\frac{\partial}{\partial x^2}.
\end{aligned}
\tag{5.6}
$$

These generators yield the following seven-dimensional group of linear transformations on the $(x^1, x^2, \ldots, x^7)$–space:

$$
\begin{aligned}
\vec{A}_1 &: \; x^{1\prime} = x^1 + a_1x^5,\\
\vec{A}_2 &: \; x^{2\prime} = x_2 + a_2x^7,\\
\vec{A}_3 &: \; x^{1\prime} = x^1 + a_3x^4, \quad x^{3\prime} = x^3 + a_3x^6,\\
\vec{A}_4 &: \; x^{1\prime} = x^1 - a_4x^5, \quad x^{4\prime} = x^4 + a_4(x^5 - x^6),\\
\vec{A}_5 &: \; x^{1\prime} = a_5x^1, \quad x^{4\prime} = a_5x^4,\\
\vec{A}_6 &: \; x^{1\prime} = a_6^{-1}x^3, \quad x^{4\prime} = a_6x^4,\\
\vec{A}_7 &: \; x^{1\prime} = a_7x^2,
\end{aligned}
\tag{5.7}
$$

where a_1, a_2, a_3, $a_4 \in \Re$ and a_5, a_6, $a_7 \in \Re^+$. The calculation for $\vec{A}_6$ is given in Appendix E.

The transformation (5.7) leaves the components x_5, x_6 and x_7 of the vector under consideration invariant.

The group of equivalence transformations includes the following reflections:

$$x^1 \longmapsto -x^1, \quad x^3 \longmapsto -x^3, \tag{5.8}$$

$$x^1 \longmapsto -x^1, \quad x^2 \longmapsto -x^2, \quad x^4 \longmapsto -x^4, \tag{5.9}$$

$$x^2 \longmapsto -x^2, \quad x^3 \longmapsto -x^3. \tag{5.10}$$

For our purposes we only use the transformations (5.8) and (5.9).

The algebra L_7 can be decomposed into a direct sum of the proper ideal $J = \{\vec{X}_1,\ \vec{X}_2,\ \vec{X}_3\}$ and the subalgebra $N = \{\vec{X}_4,\ \vec{X}_5,\ \vec{X}_6,\ \vec{X}_7\}$ as follows:

$$\begin{aligned} L_7 &= J \oplus N \\ &= \{\vec{X}_1,\ \vec{X}_2,\ \vec{X}_3\} \oplus \{\vec{X}_4,\ \vec{X}_5,\ \vec{X}_6,\ \vec{X}_7\}. \end{aligned} \tag{5.11}$$

The algebra L_7 is the factor algebra of L_{10} with respect to its ideal which is the three-dimensional principle Lie algebra $L_{\mathcal{P}}$ *i.e.* $L_7 = L_{10}/L_{\mathcal{P}}$.

5.4 Application of the algorithm

In this section we use the two-step algorithm to construct θL_7. In Step 1 we construct the optimal system of the algebra N. N_p $(p = 1, 2, \ldots, P)$ are representatives of θN. In Step 2 we complete every subalgebra N_p to the subalgebras $K_{p,q}$ $(q = 1, 2, \ldots, Q)$ which are representatives of the optimal system θL_7.

5.4.1 Step 1 : Construction of Optimal System θN

Every $s-$dimensional subalgebra $M_s \subset N$ $(s = 1, 2, 3)$ can be represented by matrix Q as follows:

$$Q = \begin{bmatrix} x^4 & x^5 & x^6 & x^7 \\ y^4 & y^5 & y^6 & y^7 \\ z^4 & z^5 & z^6 & z^7 \end{bmatrix}.$$

The approach we will use is to simplify this matrix Q by means of transformations of bases, inner automorphisms (5.6) and reflections (5.8) and (5.9). We will then divide the matrices we obtain into nonequivalent classes and in any class we select a representative having the simplest possible form.

First we assume that there is a nonzero element in the first column, say x^4. Then after $B-$transformations (linear combinations of rows) we obtain $y^4 = z^4 = 0$. We have $x^5 = x^6$ otherwise x^4 can be annulled by the automorphism $\vec{A}_4$.

Let the 2×3 submatrix Q_1 of the matrix Q have the form:

$$Q_1 = \begin{bmatrix} y^5 & y^6 & y^7 \\ z^5 & z^6 & z^7 \end{bmatrix}.$$

The $rank\ Q_1$ may be equal to $2, 1, 0$. When the $rank\ Q_1 = 2$, we have three-dimensional subalgebras of N and $rank\ Q_1 = 0, 1$ we have one- and two-dimensional subalgebras of N respectively. Therefore we have the following cases:

CASE 1 : $rank\ Q_1 = 2$

We reduce Q_1 by $B-$transformations preserving the first column of Q to one of the following forms :

$$Q_1^1 = \begin{bmatrix} 1 & 0 & y^{7\prime} \\ 0 & 1 & z^{7\prime} \end{bmatrix},$$

$$Q_1^2 = \begin{bmatrix} 1 & y^{6\prime} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q_1^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(i) For Q_1^1, matrix Q has the form:

$$Q = \begin{bmatrix} 1 & x^5 & x^6 & x^7 \\ 0 & 1 & 0 & y^{7\prime} \\ 0 & 0 & 1 & z^{7\prime} \end{bmatrix}, \; x^5 = x^6.$$

Using $B-$transformations we make $x^5 = x^6 = 0$ and as a result we obtain the following generators:

$$\begin{aligned} \vec{H}_1 &= \vec{X}_4 + x^7 \vec{X}_7, \\ \vec{H}_2 &= \vec{X}_5 + y^{7\prime} \vec{X}_7, \\ \vec{H}_3 &= \vec{X}_6 + z^{7\prime} \vec{X}_7. \end{aligned} \tag{5.12}$$

These generators have the following commutator relations:

$$[\vec{H}_1, \vec{H}_2] = \vec{X}_4, \quad [\vec{H}_1, \vec{H}_3] = -\vec{X}_4, \quad [\vec{H}_2, \vec{H}_3] = 0. \tag{5.13}$$

The vectors (5.12) generate a subalgebra if the commutators (5.13) are linear combinations of $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$. Therefore $x^7 = 0$. Thus the first subalgebra of N is

$$N_1 = \{\vec{X}_4, \vec{X}_5 + \alpha \vec{X}_7, \vec{X}_6 + \beta \vec{X}_7\}, \quad \forall \alpha, \beta \in \Re.$$

(ii) For Q_1^2, matrix Q has the form:

$$Q = \begin{bmatrix} 1 & x^5 & x^6 & x^7 \\ 0 & 1 & y^{6\prime} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x^5 = x^6.$$

Using $B-$transformations we make $x^5 = x^7 = 0$ .Thus $x^6 = 0$ and we obtain the following generators:

$$\begin{aligned} \vec{H}_1 &= \vec{X}_4, \\ \vec{H}_2 &= \vec{X}_5 + y^{6\prime}\vec{X}_6, \\ \vec{H}_3 &= \vec{X}_7. \end{aligned}$$

These generators have the following commutator relations:

$$[\vec{H}_1, \vec{H}_2] = (1 - y^{6\prime})\vec{X}_4, \quad [\vec{H}_1, \vec{H}_3] = 0, \quad [\vec{H}_2, \vec{H}_3] = 0.$$

The subalgebra of N is therefore

$$N_2 = \{\vec{X}_4, \vec{X}_5 + \alpha\vec{X}_6, \vec{X}_7\}, \quad \forall\alpha \in \Re.$$

(iii) For Q_1^3, we obtain the subalgebra

$$N_2 = \{\vec{X}_4, \vec{X}_6, \vec{X}_7\}, \quad \forall\alpha \in \Re.$$

CASE 2 : *rank* $Q_1 = 1$

We now consider two-dimensional subalgebras of N. The matrix Q then has the form:

$$Q = \begin{bmatrix} x^4 & x^5 & x^6 & x^7 \\ y^4 & y^5 & y^6 & y^7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad x^5 = x^6.$$

Again by $B-$transformations we reduce the submatrix Q_1 to one of the following possible forms:

$$Q_1^4 = \begin{bmatrix} 1 & y^{6\prime} & y^{7\prime} \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q_1^5 = \begin{bmatrix} 0 & 1 & y^{7\prime} \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q_1^6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(i) For Q_1^4, we obtain the following generators:

$$\begin{aligned} \vec{H}_1 &= \vec{X}_4 + x^7\vec{X}_7, \\ \vec{H}_2 &= \vec{X}_5 + y^{6\prime}\vec{X}_6 + y^{7\prime}\vec{X}_7, \end{aligned}$$

and therefore

$$[\vec{H}_1, \vec{H}_2] = (1 - y^{6\prime})\vec{X}_4.$$

These vectors generate subalgebras when either $x^7 = 0$ or $y^{6\prime} = 1$. For $x^7 = 0$ we obtain the subalgebra

$$N_4 = \{\vec{X}_4, \vec{X}_5 + \alpha\vec{X}_6 + \beta\vec{X}_7\}, \quad \forall\alpha, \beta \in \Re.$$

For $y^{6\prime} = 1$ and $x^7 \neq 0$, $\vec{H}_1$ becomes $\vec{X}_4 + \vec{X}_7$ as a result of $B-$transformations and automorphism $\vec{A}_5$ in (5.7) and hence

$$N_5 = \{\vec{X}_4 + \vec{X}_7, \vec{X}_5 + \vec{X}_6 + \beta\vec{X}_7\}, \quad \forall\beta \in \Re.$$

(ii) For Q_1^5, we have

$$N_6 = \{\vec{X}_4, \vec{X}_6 + \beta\vec{X}_7\}, \quad \forall\beta \in \Re.$$

(iii) For Q_1^6, we have

$$N_7 = \{\vec{X}_4 + \vec{X}_5 + \vec{X}_6, \vec{X}_7\}$$

and

$$N_8 = \{\vec{X}_4, \vec{X}_7\}.$$

CASE 3 : *rank* $Q_1 = 0$

In this case we obtain the following one-dimensional subalgebras of N:

$$N_9 = \{\vec{X}_4 + \vec{X}_5 + \vec{X}_6 + \alpha\vec{X}_7\}, \quad \forall\alpha \in \Re,$$

$$N_{10} = \{\vec{X}_4 + \vec{X}_7\},$$

$$N_{11} = \{\vec{X}_4\},$$

Suppose that $x^4 = y^4 = z^4 = 0$ in the first column of Q. The problem now simplifies greatly in order to compute all nonsimilar subalgebras of the algebra $\{\vec{X}_5, \vec{X}_6, \vec{X}_7\}$. The group of inner automorphisms acts trivially on this algebra (x^5, x^6, and x^7 are its invariants). We will only consider $B-$transformations. The *rank* of Q may equal $3, 2, 1$ or 0.

For *rank* $Q = 3$ we have

$$N_{12} = \{\vec{X}_5, \vec{X}_6, \vec{X}_7\}.$$

For the other cases we choose Q_1^i $(i = 1, \ldots, 6)$ as was discussed above. Hence the subalgebras of $\{\vec{X}_5, \vec{X}_6, \vec{X}_7\}$ are:

$$N_{13} = \{\vec{X}_5 + \alpha\vec{X}_7, \vec{X}_6 + \beta\vec{X}_7\},$$

$$N_{14} = \{\vec{X}_5 + \alpha\vec{X}_6, \vec{X}_7\},$$

$$N_{15} = \{\vec{X}_6, \vec{X}_7\},$$

$$N_{16} = \{\vec{X}_5, \alpha\vec{X}_6, \beta\vec{X}_7\},$$

$$N_{17} = \{\vec{X}_6 + \beta\vec{X}_7\},.$$

$$N_{18} = \{\vec{X}_7\},$$

$$N_{19} = \{0\}^1,$$

where $\alpha, \beta \in \Re$.

The subalgebras N_p $(p = 1, 2, \ldots, 19)$ obtained above are the entire representatives of the optimal system θN.

5.4.2 Step 2 : Construction of Optimal System θL_7

Here we illustrate Step 2 of the algorithm by constructing four- and five-dimensional subalgebras of L_7 corresponding to the subalgebra $N_7 = \{\vec{X}_4 + \vec{X}_5 + \vec{X}_6, \vec{X}_7\}$.

The four-dimensional subalgebras $L_4^7 \subset L_7$ are represented by the matrix:

[1] N_{19} corresponds to the ideal $\{\vec{X}_1, \vec{X}_2, \vec{X}_3\}$ in the decomposition (5.11) .

$$R = \left[\begin{array}{ccc|cccc} x^1 & x^2 & x^3 & 1 & 1 & 1 & 0 \\ y^1 & y^2 & y^3 & 0 & 0 & 0 & 1 \\ \hline z^1 & z^2 & z^3 & 0 & 0 & 0 & 0 \\ t^1 & t^2 & t^3 & 0 & 0 & 0 & 0 \end{array}\right].$$

Let R_1 be the 2×3 submatrix in the lower left corner of R. Since the *rank* $R = 4$, the *rank* $R_1 = 2$. The matrix R_1 can be reduced by $B-$transformations to one of the following forms:

$$R_1^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t^{3\prime} \end{bmatrix},$$

$$R_1^2 = \begin{bmatrix} 1 & z^{2\prime} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_1^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

CASE 1 : For R_1^1, we use $B-$transformations to bring $x^1 = x^2 = y^1 = y^2 = 0$ and the matrix R now has the form:

$$R = \left[\begin{array}{ccc|cccc} 0 & 0 & x^{3\prime} & 1 & 1 & 1 & 0 \\ 0 & 0 & y^{3\prime} & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t^{3\prime} & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore the four generators are

$$\begin{aligned}
\vec{H}_1 &= x^{3\prime}\vec{X}_3 + \vec{X}_4 + \vec{X}_5 + \vec{X}_3,\\
\vec{H}_2 &= y^{3\prime}\vec{X}_3 + \vec{X}_7,\\
\vec{H}_3 &= \vec{X}_1,\\
\vec{H}_4 &= \vec{X}_2 + t^{3\prime}\vec{X}_3.
\end{aligned}$$

The commutator relation of these generators are:

$$\begin{gathered}
[\vec{H}_1, \vec{H}_2] = (x^{3\prime} - y^{3\prime})\vec{X}_3 - y^{3\prime}\vec{X}_1,\\
[\vec{H}_1, \vec{H}_3] = -\vec{X}_1, \quad [\vec{H}_1, \vec{H}_4] = -t^{3\prime}\vec{X}_1 - t^{3\prime}\vec{X}_3, \qquad (5.14)\\
[\vec{H}_2, \vec{H}_3] = 0, \quad [\vec{H}_2, \vec{H}_4] = -\vec{X}_2, \quad [\vec{H}_3, \vec{H}_4] = 0.
\end{gathered}$$

The right hand side of each commutator in (5.14) must be linear combinations of $\vec{H}_1$, $\vec{H}_2$, $\vec{H}_3$, and $\vec{H}_4$. It therefore follows that $x^{3\prime} = y^{3\prime} = t^{3\prime} = 0$.

Therefore the first four-dimensional subalgebra of θL_7 is

$$L^7_{4,1} = \{\vec{X}_1, \vec{X}_2, \vec{X}_4 + \vec{X}_5 + \vec{X}_6, \vec{X}_7\}.$$

CASE 2 : For R^2_1, we proceed as in Case 1 to obtain the following four-dimensional subalgebra of θL_7 :

$$L^7_{4,2} = \{\vec{X}_1, \vec{X}_3, \vec{X}_4 + \vec{X}_5 + \vec{X}_6, \vec{X}_7\}.$$

CASE 3 : For R^3_1, after applying $B-$transformations and the automorphism $\vec{A}_3$

we obtain the following four generators :

$$\begin{aligned}\vec{H}_1 &= \vec{X}_4 + \vec{X}_5 + \vec{X}_6,\\ \vec{H}_2 &= y^{1\prime}\vec{X}_1 + \vec{X}_7,\\ \vec{H}_3 &= \vec{X}_2,\\ \vec{H}_4 &= \vec{X}_3.\end{aligned}$$

The commutator $[\vec{H}_1, \vec{H}_2] = -\vec{X}_1 - \vec{X}_3$. This is not a linear combination of the vectors $\vec{H}_1, \vec{H}_2, \vec{H}_3, \vec{H}_4$ and hence in this case we do not have a contribution to the optimal system θL_7.

Five-dimensional subalgebras $L_5^7 \subset L_7$ are represented by the matrix:

$$R = \left[\begin{array}{ccc|cccc} x^1 & x^2 & x^3 & 1 & 1 & 1 & 0 \\ y^1 & y^2 & y^3 & 0 & 0 & 0 & 1 \\ \hline z^1 & z^2 & z^3 & 0 & 0 & 0 & 0 \\ t^1 & t^2 & t^3 & 0 & 0 & 0 & 0 \\ u^1 & u^2 & u^3 & 0 & 0 & 0 & 0 \end{array}\right].$$

Let R_1 be the 3×3 submatrix in the lower left corner of R. Since the *rank* $R = 5$, the *rank* $R_1 = 3$. We reduce R_1 to the identity matrix by $B-$transformations. We then annul all x_k $(k = 1, 2, 3)$ by $B-$transformations which preserve the structure of N_7. We thus obtain only the following five-dimensional subalgebra:

$$L_{5,1}^7 = \{\vec{X}_1, \vec{X}_2, \vec{X}_3, \vec{X}_4 + \vec{X}_5 + \vec{X}_6, \vec{X}_7\}.$$

To summarize, the list of four- and five- dimensional subalgebras of θL_7 corresponding to the form N_7 consists of $L_{4,1}^7$, $L_{4,2}^7$ and $L_{5,1}^7$ respectively.

Proceeding analogously with the other elements of N_p we obtain all the possible subalgebras of θL_7. In [3], Chupakhin obtained the complete list of θL_7, which consists of 397 representatives.

We now need to normalize the optimal system θL_7. For example, we consider $L^7_{1,24} = \{\vec{X}_2 + \vec{X}_3 + \vec{X}_5\}$ and apply Definition 5.6 to obtain

$$Nor_{L_7} L_{1,24} = \{\vec{X}_2, \vec{X}_3, \vec{X}_5\}.$$

Also in [3], Chupakhin obtained 36 self-normalized subalgebras of θL_7.

The one-dimensional nonsimilar subalgebras $\theta_1(L_7)$ are presented in Appendix C. In [3], the list of the two-dimensional nonsimilar subalgebras $\theta_1(L_7)$ can be found. The complete list of the self-normalized subalgebras are presented in Appendix D.

Concluding Remarks

In this exercise a deeper understanding of the construction of principal Lie algebras and equivalence transformations and the construction of optimal systems of subalgebras using the methods of preliminary group classification and Ovsiannikov's algorithm has been gained.

Although not covered in this study, it would be interesting to extend this analysis to other classes of equations, for example, equations of the form:

$$u_{tt} = f(x, u_x, u_t)u_{xx} + g(x, u_x, u_t).$$

The resulting classification could then be compared with the results obtained in this research report.

Also, the problem of the preliminary group classification of equation (1.1) with respect to the two-dimensional extensions of the principal Lie algebra has still to be solved.

APPENDIX A

N	$\mathbf{Z}$	Invariant λ	Equation $u_{tt} =$	Additional generator $\vec{X}_4$
1	$\mathbf{Z}^{(1)}$	u_x	$\Phi u_{xx} + \Gamma$	$\frac{\partial}{\partial x}$
2	$\mathbf{Z}^{(2)}$	x	$\Phi u_{xx} + \Gamma$	$x\frac{\partial}{\partial u}$
3	$\mathbf{Z}^{(3)}$	u_x/x	$\Phi u_{xx} + \Gamma$	$t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$
4	$\mathbf{Z}^{(4)}_{\alpha\neq 0}$	u_x/x	$x^\sigma\{\Phi u_{xx} + \Gamma\}$	$(1 - \frac{\sigma}{2})t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$
5	$\mathbf{Z}^{(6)}_{\alpha=0}$	x	$u^\beta{}_x\{\Phi u_{xx} + \Gamma\}$	$\beta t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$
6	$\mathbf{Z}^{(6)}_{\alpha\neq 0}$	$u_x/x^{\sigma+1}$	$u^\gamma\{\Phi u_{xx} + x^\sigma\Gamma u_x\}$	$(2-\gamma)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2(\sigma+2)\frac{\partial}{\partial u}$
7	$\mathbf{Z}^{(7)}$	x	$\Phi u_{xx} - x^{-1}\Phi u_x + \Gamma$	$x^2\frac{\partial}{\partial u}$
8	$\mathbf{Z}^{(8)}$	u_x	$e^x\{\Phi u_{xx} + \Gamma\}$	$t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}$
9	$\mathbf{Z}^{(9)}$	u_x	$\Phi u_{xx} + \Gamma + x$	$2\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial u}$
10	$\mathbf{Z}^{(10)}$	$e^{-x}u_x$	$u^\beta{}_x\{\Phi u_{xx} + \Gamma u_x\}$	$\beta t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$
11	$\mathbf{Z}^{(11)}$	$x^2 - 2u_x$	$\Phi u_{xx} + \Gamma - x\Phi$	$2\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial u}$
12	$\mathbf{Z}^{(12)}$	x	$e^{u_x}\{\Phi u_{xx} + \Gamma\}$	$t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial y}$
13	$\mathbf{Z}^{(13)}$	x	$\Phi u_{xx} + \Gamma + u_x$	$(t^2 + 2x)\frac{\partial}{\partial y}$
14	$\mathbf{Z}^{(14)}$	u_x/x	$\Phi u_{xx} + \Gamma + \ln\lvert x\rvert$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (t^2 + 4u)\frac{\partial}{\partial y}$
15	$\mathbf{Z}^{(15)}$	$u_x - x\ln\lvert x\rvert$	$\Phi u_{xx} + \Gamma - \Phi\ln\lvert x\rvert$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (x^2 + 4u)\frac{\partial}{\partial u}$
16	$\mathbf{Z}^{(16)}$	x	$\Phi u_{xx} + (1-\Phi)x^{-1}u_x + \Gamma$	$(t^2 + x^2)\frac{\partial}{\partial u}$
17	$\mathbf{Z}^{(17)}$	x	$\Phi u_{xx} - (1+\Phi)x^{-1}u_x + \Gamma$	$(t^2 - x^2)\frac{\partial}{\partial u}$
18	$\mathbf{Z}^{(18)}$	$x^2 - 2u_x$	$e^x\{\Phi u_{xx} - x\Phi + \Gamma\}$	$t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial y}$
19	$\mathbf{Z}^{(19)}$	$x^2 - 2u_x$	$\Phi u_{xx} + (1-\Phi)x + \Gamma$	$2\frac{\partial}{\partial x} + (t^2 + x^2)\frac{\partial}{\partial u}$
20	$\mathbf{Z}^{(20)}$	$x^2 + 2u_x$	$\Phi u_{xx} + (1+\Phi)x + \Gamma$	$2\frac{\partial}{\partial x} + (t^2 - x^2)\frac{\partial}{\partial u}$
21	$\mathbf{Z}^{(21)}_{\alpha=0}$	x	$e^{u_x/x}\Phi\{u_{xx} - x^{-1}u_x - \ln\lvert\Phi\rvert + \Gamma\}$	$t\frac{\partial}{\partial t} - x^2\frac{\partial}{\partial x}$
22	$\mathbf{Z}^{(21)}_{\alpha\neq 0}$	$u_x - \sigma x\ln\lvert x\rvert$	$x^\sigma\Phi\{u_{xx} - \sigma\ln\lvert x\rvert - \ln\lvert\Phi\rvert + \Gamma\}$	$(2-\sigma)\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (\sigma x^2 + 4u)\frac{\partial}{\partial u}$
23	$\mathbf{Z}^{(22)}$	$u_x - x\ln\lvert x\rvert$	$\Phi u_{xx} + (1-\Phi)\ln\lvert x\rvert + \Gamma$	$(t^2 + x^2 + 4u)\frac{\partial}{\partial u}$
24	$\mathbf{Z}^{(23)}$	$u_x + x\ln\lvert x\rvert$	$\Phi u_{xx} + (1+\Phi)\ln\lvert x\rvert + \Gamma$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (t^2 - x + 4u)\frac{\partial}{\partial u}$
25	$\mathbf{Z}^{(24)}$	$e^{-x}u_x$	$e^{-x}\Phi u_{xx} + \Gamma + x$	$t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} + (t^2 + 2u)\frac{\partial}{\partial u}$
26	$\mathbf{Z}^{(25)}$	$u_x + \ln\lvert x\rvert$	$x^{-\beta}\{\Phi u_{xx} + x^{-1}\Gamma\}$	$(\beta+2)t\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial x} + 2(-x+u)\frac{\partial}{\partial u}$
27	$\mathbf{Z}^{(26)}_{\alpha=0}$	x	$\Phi u_x{}^{-1}u_{xx} + \Gamma + \ln\lvert u_x\rvert$	$t\frac{\partial}{\partial t} + (t^2 + 2u)\frac{\partial}{\partial u}$
28	$\mathbf{Z}^{(26)}_{\alpha\neq 0}$	$x^{-(1+\sigma)}u_x$	$x^{-\sigma}\Phi u_{xx} + \Gamma + \sigma\ln\lvert x\rvert$	$(2+\sigma)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + [\sigma t^2 + 2(2+\sigma)u]\frac{\partial}{\partial u}$
29	$\mathbf{Z}^{(27)}$	$u_x + x\ln\lvert x\rvert$	$x\Phi u_{xx} + \Gamma + u_x$	$t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (-t^2 - 2x + 2u)\frac{\partial}{\partial u}$

Table A : Result of the classification of Chapter 3 ($\sigma = 1/\alpha, \gamma = \beta/\alpha, \Phi$ and Γ are arbitary functions of λ).

APPENDIX B

N	$\tilde{Z}$	Invariant λ	Equation $u_{tt} =$	Additional generator $\vec{X}_4$
1	$\tilde{Z}^{[1]}$	u_x	$\Phi u_{xx} + \Gamma$	$\frac{\partial}{\partial x}$
2	$\tilde{Z}^{[2]}$	x	$\Phi u_{xx} - \frac{F''}{F'}\Phi u_x + \Gamma$	$F\frac{\partial}{\partial u}$
3	$\tilde{Z}^{[3]}$	u_x/x	$x^\alpha\{\Phi u_{xx} + \Gamma\}$	$(1-\frac{\alpha}{2})t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$
4	$\tilde{Z}^{[4]}$	x	$\Phi u_{xx} - \frac{1-F''\Phi}{F'}u_x + \Gamma$	$(t^2+2F)\frac{\partial}{\partial u}$
5	$\tilde{Z}^{[5]}_{\alpha=0}$	x	$u^\beta{}_x\{\Phi u_{xx} + \Gamma u_x\}$	$\beta t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$
6	$\tilde{Z}^{[5]}_{\alpha\neq0}$	$u_x/x^{\sigma+1}$	$u^\gamma\{\Phi u_{xx} + x^\sigma\Gamma\}$	$(2-\gamma)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2(\sigma+2)\frac{\partial}{\partial u}$
7	$\tilde{Z}^{[6]}$	u_x/x^{n+1} $-\mu\ln\|x\|$	$x^\beta\Phi\{\Phi u_{xx}$ $-x^{n-2}[\mu(n-1)\ln\|x\| + \Gamma]\}$	$(1-\frac{\beta}{2})t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ $+(nu + \frac{x^n}{n}\mu)\frac{\partial}{\partial u}$
8	$\tilde{Z}^{[7]}$	u_x	$e^x\{\Phi u_{xx} + \Gamma\}$	$t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}$
9	$\tilde{Z}^{[8]}$	u_x	$\Phi u_{xx} + \Gamma + x$	$2\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial u}$
10	$\tilde{Z}^{[9]}$	$e^{-x}u_x$	$u^\beta{}_x\{\Phi u_{xx} + \Gamma u_x\}$	$\beta t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$
11	$\tilde{Z}^{[10]}$	x	$e^{u_x/F'}\Phi\{u_{xx} - \frac{F''}{F'}u_x + \Gamma\}$	$t\frac{\partial}{\partial t} - 2F\frac{\partial}{\partial u}$
12	$\tilde{Z}^{[11]}$	u_x/x	$\Phi u_{xx} + \Gamma + \ln\|x\|$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (t^2+4u)\frac{\partial}{\partial u}$
13	$\tilde{Z}^{[12]}$	$u_x/x - \ln\|x\|$	$\Phi u_{xx} + \Gamma - \ln\|x\|$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (x^2+4u)\frac{\partial}{\partial u}$
14	$\tilde{Z}^{[13]}$	$u_x/x - \alpha\ln\|x\|$	$x^\alpha\Phi\{u_{xx} - \alpha\ln\|x\| + \Gamma\}$	$(2-\alpha)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (x^2+4u)\frac{\partial}{\partial u}$
15	$\tilde{Z}^{[14]}$	$u_x/x - x\ln\|x\|$	$\Phi u_{xx} + (1-\Phi)\ln\|x\| + \Gamma$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (t^2+x^2+4u)\frac{\partial}{\partial u}$
16	$\tilde{Z}^{[15]}$	$u_x/x + \ln\|x\|$	$\Phi u_{xx} + (1+\Phi)\ln\|x\| + \Gamma$	$2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (t^2-x^2+4u)\frac{\partial}{\partial u}$
17	$\tilde{Z}^{[16]}$	$e^{-x}u_x$	$e^{-x}\Phi u_{xx} + \Gamma + x$	$t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} + (t^2+2u)\frac{\partial}{\partial u}$
18	$\tilde{Z}^{[17]}$	$u_x + \ln\|x\|$	$x^{-\beta}\{\Phi u_{xx} + \frac{\Gamma}{x}\}$	$(\beta+2)t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (u-x)\frac{\partial}{\partial u}$
19	$\tilde{Z}^{[18]}_{\alpha=0}$	x	$\Phi {u_x}^{-1}u_{xx} + \Gamma + \ln\|u_x\|$	$t\frac{\partial}{\partial t} + (t^2+2u)\frac{\partial}{\partial u}$
20	$\tilde{Z}^{[18]}_{\alpha\neq0}$	$x^{-(1+\sigma)}u_x$	$x^{-\sigma}\Phi u_{xx} + \Gamma + \sigma\ln\|x\|$	$(1+\frac{\sigma}{2})t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ $+[\frac{\sigma}{2}t^2 + (2+\sigma)u]\frac{\partial}{\partial u}$
21	$\tilde{Z}^{[19]}$	u_x/x^{n+1} $-\mu\ln\|x\|$	$x^{2-n}\Phi u_x x + \Gamma$ $+[n-2-\mu(n-1)\Phi]\ln\|x\|$	$nt\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x}$ $+[2nu - (n-2)t^2 + 2\mu\frac{x^n}{n}]\frac{\partial}{\partial u}$
22	$\tilde{Z}^{[20]}$	$u_x + \ln\|x\|$	$x\Phi u_{xx} + \Gamma + u_x$	$t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (-t^2-2x+4u)\frac{\partial}{\partial u}$

Table B : Result of the classification of Chapter 4 ($\sigma = 1/\alpha, \gamma = \beta/\alpha$, Φ and Γ are arbitary functions of λ).

APPENDIX C

In Table C.1 and Table C.2 the one-dimensional nonsimilar subalgebras $\theta_1(L_7)$ are given. In the first column the number of the subalgebra, in the second column its basis and conditions for the parameters α, β and in the last column the basis of the normalizer of this subalgebra are given.

N	Basis of subalgebra	Normalizer of subalgebra
1	$\vec{X}_1$	L_7
2	$\vec{X}_2$	L_7
3	$\vec{X}_3$	$\vec{X}_1, \vec{X}_2, \vec{X}_3, \vec{X}_5, \vec{X}_6, \vec{X}_7$
4	$\vec{X}_4$	$\vec{X}_1, \vec{X}_2, \vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$
5	$\vec{X}_5$	$\vec{X}_2, \vec{X}_3, \vec{X}_5, \vec{X}_6, \vec{X}_7$
6	$\vec{X}_6$	$\vec{X}_1, \vec{X}_2, \vec{X}_5, \vec{X}_6, \vec{X}_7$
7	$\vec{X}_7$	$\vec{X}_1, \vec{X}_3, \vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$
8	$\vec{X}_4 + \vec{X}_5 + \vec{X}_6 + \alpha\vec{X}_7, \alpha \neq 0$	$\vec{X}_4, \vec{X}_5 + \vec{X}_6, \vec{X}_7$
9	$\vec{X}_4 + \vec{X}_5 + \vec{X}_6$	$\vec{X}_2, \vec{X}_4, \vec{X}_5 + \vec{X}_6, \vec{X}_7$
10	$\pm\vec{X}_2 + \vec{X}_4 + \vec{X}_5 + \vec{X}_6$	$\vec{X}_2, \vec{X}_4, \vec{X}_5 + \vec{X}_6$
11	$\vec{X}_3 + \vec{X}_4 + \vec{X}_7$	$\vec{X}_1, \vec{X}_3 + \vec{X}_4, \vec{X}_7$
12	$\vec{X}_4 + \vec{X}_7$	$\vec{X}_1, \vec{X}_4, \vec{X}_5 + \vec{X}_6$
13	$\pm\vec{X}_2 + \vec{X}_3 + \vec{X}_4$	$\vec{X}_1, \vec{X}_2, \vec{X}_3 + \vec{X}_4, \vec{X}_5 + \frac{1}{2}(\vec{X}_6 + \vec{X}_7)$
14	$\pm\vec{X}_2 + \vec{X}_4$	$\vec{X}_1, \vec{X}_2, \vec{X}_4, \vec{X}_5 + \vec{X}_7, \vec{X}_6 - \vec{X}_7$
15	$\vec{X}_3 + \vec{X}_4$	$\vec{X}_1, \vec{X}_2, \vec{X}_3 + \vec{X}_4, \vec{X}_5 + \frac{1}{2}\vec{X}_6, \vec{X}_7$

TABLE C.1 : Optimal system of one-dimensional subalgebras $\theta_1(L_7)$

N	Basis of subalgebra	Normalizer of subalgebra
16	$\vec{X}_5 + \alpha\vec{X}_6 + \beta\vec{X}_7, \alpha \neq 0, 1, \beta \neq 0$	$\vec{X}_5, \vec{X}_6, \vec{X}_7$
17	$\vec{X}_5 + \vec{X}_6 + \beta\vec{X}_7, \beta \neq 0$	$\vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$
18	$\vec{X}_2 + \vec{X}_5 + \alpha\vec{X}_6, \alpha \neq 0, 1$	$\vec{X}_2, \vec{X}_5, \vec{X}_6$
19	$\vec{X}_5 + \alpha\vec{X}_6, \alpha \neq 0, 1$	$\vec{X}_2, \vec{X}_5, \vec{X}_6, \vec{X}_7$
20	$\vec{X}_2 + \vec{X}_5 + \vec{X}_6$	$\vec{X}_2, \vec{X}_4, \vec{X}_5, \vec{X}_6$
21	$\vec{X}_5 + \vec{X}_6$	$\vec{X}_2, \vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$
22	$\vec{X}_3 + \vec{X}_5 + \beta\vec{X}_7, \beta \neq 0$	$\vec{X}_3, \vec{X}_5, \vec{X}_6, \vec{X}_7$
23	$\vec{X}_5 + \beta\vec{X}_7, \beta \neq 0$	$\vec{X}_3, \vec{X}_5, \vec{X}_6, \vec{X}_7$
24	$\vec{X}_2 + \vec{X}_3 + \vec{X}_5$	$\vec{X}_2, \vec{X}_3, \vec{X}_5$
25	$\vec{X}_2 + \vec{X}_3$	$\vec{X}_2, \vec{X}_3, \vec{X}_5, \vec{X}_6$
26	$\vec{X}_3 + \vec{X}_5$	$\vec{X}_2, \vec{X}_3, \vec{X}_5, \vec{X}_7$
27	$\vec{X}_1 + \vec{X}_6 + \alpha\vec{X}_7, \alpha \neq 0$	$\vec{X}_1, \vec{X}_6, \vec{X}_7$
28	$\vec{X}_6 + \beta\vec{X}_7, \beta \neq 0$	$\vec{X}_1, \vec{X}_5, \vec{X}_6, \vec{X}_7$
29	$\vec{X}_1 + \vec{X}_2 + \vec{X}_6$	$\vec{X}_1, \vec{X}_2, \vec{X}_6,$
30	$\vec{X}_1 + \vec{X}_6$	$\vec{X}_1, \vec{X}_2, \vec{X}_5, \vec{X}_6$
31	$\vec{X}_2 + \vec{X}_6$	$\vec{X}_1, \vec{X}_2, \vec{X}_5, \vec{X}_6$
32	$\vec{X}_3 + \vec{X}_7$	$\vec{X}_1, \vec{X}_3, \vec{X}_5, \vec{X}_7$
33	$\vec{X}_1 + \vec{X}_7$	$\vec{X}_1, \vec{X}_3, \vec{X}_4, \vec{X}_6, \vec{X}_7$
34	$\vec{X}_2 + \vec{X}_3$	$\vec{X}_1, \vec{X}_2, \vec{X}_3, \vec{X}_5, \vec{X}_6 + \vec{X}_7$
35	$\vec{X}_1 + \vec{X}_2$	$\vec{X}_1, \vec{X}_2, \vec{X}_3, \vec{X}_4, \vec{X}_5 + \vec{X}_6, \vec{X}_6$

TABLE C.2 : Optimal system of one-dimensional subalgebras $\theta_1(L_7)$

APPENDIX D

In Table D.1 and Table D.2 the self-normalized subalgebras of L_7 are given.

N	Basis of Subalgebra	Dimension
1	$\pm\vec{X}_2 + \vec{X}_3 + \vec{X}_4, \frac{1}{2}(\vec{X}_6 + \vec{X}_7)$	2
2	$\pm\vec{X}_2 + \vec{X}_4, \vec{X}_5 + \vec{X}_7, \vec{X}_6 - \vec{X}_7$	
3	$\vec{X}_3 + \vec{X}_4, \vec{X}_5 + \frac{1}{2}\vec{X}_6, \vec{X}_7$	
4	$\vec{X}_1 + \vec{X}_2, \vec{X}_3 + \vec{X}_4, \vec{X}_5 + \frac{1}{2}\vec{X}_6, \vec{X}_7$	
5	$\vec{X}_1, \pm\vec{X}_2 + \vec{X}_3 + \vec{X}_4, \vec{X}_5 + \frac{1}{2}(\vec{X}_6 + \vec{X}_7)$	3
6	$\vec{X}_5, \vec{X}_6, \vec{X}_7$	
7	$\vec{X}_1 + \vec{X}_2, \vec{X}_5 + \vec{X}_7, \vec{X}_6$	
8	$\vec{X}_2 + \vec{X}_3, \vec{X}_5, \vec{X}_7 + \vec{X}_6$	
9	$\vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
10	$\vec{X}_1, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
11	$\vec{X}_2, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
12	$\vec{X}_3, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
13	$\vec{X}_1, \pm\vec{X}_2 + \vec{X}_4, \vec{X}_5 + \vec{X}_7, \vec{X}_6 - \vec{X}_7$	
14	$\vec{X}_1 + \vec{X}_2, \vec{X}_4, \vec{X}_5 + \vec{X}_7, \vec{X}_6$	4
15	$\vec{X}_1, \vec{X}_3 + \vec{X}_4, \vec{X}_5 + \frac{1}{2}\vec{X}_6, \vec{X}_7$	
16	$\vec{X}_2, \vec{X}_3 + \vec{X}_4, \vec{X}_5 + \frac{1}{2}\vec{X}_6, \vec{X}_7$	
17	$\vec{X}_1, \vec{X}_2 + \vec{X}_3, \vec{X}_5, \vec{X}_6 + \vec{X}_7$	
18	$\vec{X}_1 + \vec{X}_2, \vec{X}_3, \vec{X}_5 + \vec{X}_7, \vec{X}_6$	
19	$\vec{X}_1, \vec{X}_2 + \vec{X}_3, \vec{X}_3 + \vec{X}_4, \frac{1}{2}(\vec{X}_6 + \vec{X}_7)$	

TABLE D.1 : Self-normalized subalgebras of L_7.

N	Basis of Subalgebra	Dimension
20	$\vec{X}_1, \vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$	5
21	$\vec{X}_2, \vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
22	$\vec{X}_1, \vec{X}_2, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
23	$\vec{X}_1, \vec{X}_3, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
24	$\vec{X}_2, \vec{X}_3, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
25	$\vec{X}_1, \vec{X}_2 + \vec{X}_3, \vec{X}_4, \vec{X}_5, \vec{X}_6 + \vec{X}_7$	
26	$\vec{X}_1, \pm\vec{X}_2 + \vec{X}_4, \vec{X}_3, \vec{X}_5 + \vec{X}_7, \vec{X}_6 - \vec{X}_7$	
27	$\vec{X}_1, \vec{X}_2, \vec{X}_3 + \vec{X}_4, \vec{X}_5 + \frac{1}{2}\vec{X}_6, \vec{X}_7$	
28	$\vec{X}_1, \vec{X}_2, \vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$	6
29	$\vec{X}_1, \vec{X}_3, \vec{X}_4, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
30	$\vec{X}_1, \vec{X}_2, \vec{X}_3, \vec{X}_5, \vec{X}_6, \vec{X}_7$	
31	L_7	7

TABLE D.2 : Self-normalized subalgebras of L_7.

APPENDIX E

Consider the vector

$$\vec{A}_6 = -x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4}.$$

Using the First Fundamental Theorem of Lie [7] we solve the following initial value problem to obtain the one-parameter Lie group of transformations:

$$\frac{dx^{3\prime}}{d\epsilon} = -x^3, \quad \frac{dx^{4\prime}}{d\epsilon} = x^4, \quad \frac{dx^{k\prime}}{d\epsilon} = 0, \quad k = 1, 2, 5, 6, 7$$

subject to the conditions $x^{j\prime} = x^j$, $j = 1, \ldots, 7$ when $\epsilon = 0$.

Considering only $\dfrac{dx^{3\prime}}{d\epsilon} = -x^3$ results in

$$\begin{aligned} x^{3\prime} &= -x^3\epsilon + x^3 \\ &= (1-\epsilon)x^3. \end{aligned}$$

From the definition of the group of transformation [7] we have $x^{3\prime\prime} = (1-\epsilon')x^{3\prime}$ and this leads to

$$x^{3\prime\prime} = (1-\epsilon')(1-\epsilon)x^3 = (1 - (\epsilon' + \epsilon - \epsilon'\epsilon))x^3.$$

Therefore the law of composition is $\phi(a, b) = a + b - ab$.

To find ϵ^{-1} we proceed as follows:

Let $a + b - ab = e$ or $b = \dfrac{a}{a-1}$ where e is the identity element. For $a = \epsilon$ and $b = \epsilon^{-1}$ we get

$$\begin{aligned} \Rightarrow \epsilon^{-1} &= \frac{\epsilon}{1-\epsilon} \\ \Rightarrow \frac{1}{\epsilon^{-1}} &= \frac{1-\epsilon}{\epsilon} \\ \Rightarrow \epsilon &= 1 - \epsilon^{-1}. \end{aligned}$$

Hence $\epsilon^{-1} = 1 - \epsilon$ implies $x^{3\prime} = \epsilon^{-1}x^3$.

Thus vector $\vec{A}_6$ yields the following one-parameter group of linear transformations:

$$\begin{aligned}
x^{1\prime} &= x^1, \\
x^{2\prime} &= x^2, \\
x^{3\prime} &= \epsilon^{-1}x^3, \\
x^{4\prime} &= \epsilon x^4, \\
x^{5\prime} &= x^5, \\
x^{6\prime} &= x^6, \\
x^{7\prime} &= x^7.
\end{aligned}$$

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