First integrals for the Bianchi universes: Supplementation of the Noetherian integrals obtained by using Lie symmetries

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First integrals for the Bianchi universes

Supplementation of the Noetherian integrals with the first integrals obtained by using Lie symmetries

by

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Declaration

I, Hara Pantazi, affirm that the material contained in this dissertation has not (to my knowledge) been published elsewhere except where due reference has been made in the text, and that this dissertation is not being and has not been used for the award of any other degree or diploma in any university or other institution.

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Summary

In Chapter One we present the concept of symmetry which is the basis of this dissertation. In particular we are interested in symmetries of differential equations. We present the Lie method and we illustrate the procedure for finding first integrals. Noether's Theorem is presented in this chapter and also the procedure for the computation of Noetherian integrals. We describe the basic steps that we follow with the package Program `LIE` and we give in detail the example of Bianchi Type III.

Cappoziello *et al* have developed what they call the 'Noether symmetry approach' method and have applied this method to the Bianchi universes. In Chapter Two we describe their method and we pay attention to the application in the Bianchi universes. We are especially interested in the form of Noether's Theorem that they use and the results that they obtain with this form.

In Chapter Three we find the Lie symmetries of the Bianchi models in four cases, considering the existence of matter and potential: Case-1 where there is no matter and no potential. This is the simplest case since we obtain a system of three second order equations in three variables. The results are very helpful for the following cases. Case-2 where there is no potential: The system now contains four equations in four variables since matter is included. Case-3 where there is a constant potential. The system is the same as in the previous case plus the constant term of the potential in one of the equations. Case-4 where we have arbitrary potential. This is the most general case. The system now has the extra term of the function of the potential and its derivative with respect of ϕ. We also obtain the Noether symmetries for every case. We give the Lie algebras for the models.

In Chapter Four we completely solve the problem for the Bianchi types I, III and V. We apply the Lie and Noether methods as they have been developed in the first chapter. Especially for the Bianchi Type III we use a new reparametrisation in order to simplify the computations.

In Chapter Five we compare our results with the corresponding results of the 'Noether symmetry approach' for the Bianchi Types I and V in the second case. An interesting discussion arises from the comparison. In particular we point out the

important details that should be considered for the proper application of Noether's Theorem.

In three appendices we present the background material for Lie groups and Lie algebras, the Bianchi classification, the Bianchi Lagrangian and the forms of the potential.

Dedication

To the ones who supported me during this year

Prologue

We are interested in examining the 'Noether symmetry approach', a method which has been developed by Capozziello *et al* and recently has been applied to the Bianchi universes. There are particular points in the method which arouse our curiosity and are responsible for the creation of this dissertation. In particular we examine the problem of the Bianchi universes with two methods: The first one is to apply the Lie and Noether methods to find the symmetries for every type which is the main work of this dissertation. The second one is to examine the symmetries of the Bianchi models using the 'Noether symmetry approach'. We should note that both methods use Noether's Theorem. However, one should be careful to distinguish between the forms of the theorem which have been used in the two instances.

We find the Lie symmetries in the models in four cases: Case-1 where there is no matter and no potential. This case is the simplest one since we have a system of three equations with three variables λ, β_1 and β_2. The symmetries follow in almost all models easily. Considering the expression of the symmetries we continue in the second case where there is matter but no potential. The system has now the additional equation of ϕ. Moreover the derivative of ϕ appears in the λ equation. The symmetry $G = \partial/\partial\phi$ appears as a symmetry of the system since ϕ is an ignorable variable for the λ, β_1, β_2 and ϕ equations just as it is an ignorable coordinate in the Lagrangian. Hence the number of symmetries in the second case is always greater than what it is in the first case since there are symmetries which involve the ϕ term. Case-3, where we introduce a constant potential, does not present special difficulties since the system has almost the same form as in Case-2. The appearance of the constant term of the potential in the λ equation is responsible for 'losing' symmetries. An exception is the Bianchi Type I, where the Ricci scalar is zero. The constant potential appears in the expression of symmetries and gives new ones. The most general case is Case-4 in which we allow the potential to be an arbitrary function of ϕ. The function of the potential in the λ equation and its derivative in the ϕ equation are the terms responsible for the descrease the number of symmetries. The symmetry $G = \partial/\partial t$ appears in all models in all cases since the system is autonomous. We give the Lie algebras and we determine which of the Lie symmetries are Noether symmetries.

We completely solve the problem in Bianchi Types I, III and V. We note that the integrals that we obtain are separable integrals which yield the solution more easily since we need three integrals in the first case instead of six and four instead of eight in the next two cases. In the general case where the potential is an arbitrary function of ϕ we apply the general symmetry of the nonpotential case and find the conditions which should be satisfied for there to be a symmetry of the system. The potential should be an exponential function of ϕ. The condition that the energy function be zero was used by Capozziello *et al* to obtain the integrals. This condition is actually a first integral for the symmetry $\partial/\partial t$, a symmetry which is ignored by Capozziello *et al* since time is not allowed in the 'Noether symmetry approach'. If we impose this condition on the problem, we obtain the same results as Capozziello *et al* in the Bianchi Type I model, but this does not happen in the Bianchi Type V where the results are completely different. The Painlevé analysis supports our argument. The analyses of Capozziello *et al* are couched in the language of tangent bundles and differential geometry. Whilst that is fine as an exercise in this field, our results show that the problem under consideration, the integrability of the Bianchi models, is not given a comprehensive treatment by comparison with standard techniques. Unfortunately Capozziello *et al* do not mention the possible restrictive effect of their method. Although it seems that their method has all the advantages of an elegant method, the limitations of results is a serious matter and should be avoided. We have extended their results within the standard framework of symmetry analysis. We are aware that further results may lie within the ambit of a more generalised analysis.

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Contents

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We completely solve the problem in Bianchi Types I, III and V. We note that the integrals that we obtain are separable integrals which yield the solution more easily since we need three integrals in the first case instead of six and four instead of eight in the next two cases. In the general case where the potential is an arbitrary function of ϕ we apply the general symmetry of the nonpotential case and find the conditions which should be satisfied for there to be a symmetry of the system. The potential should be an exponential function of ϕ. The condition that the energy function be zero was used by Capozziello *et al* to obtain the integrals. This condition is actually a first integral for the symmetry $\partial/\partial t$, a symmetry which is ignored by Capozziello *et al* since time is not allowed in the 'Noether symmetry approach'. If we impose this condition on the problem, we obtain the same results as Capozziello *et al* in the Bianchi Type I model, but this does not happen in the Bianchi Type V where the results are completely different. The Painlevé analysis supports our argument. The analyses of Capozziello *et al* are couched in the language of tangent bundles and differential geometry. Whilst that is fine as an exercise in this field, our results show that the problem under consideration, the integrability of the Bianchi models, is not given a comprehensive treatment by comparison with standard techniques. Unfortunately Capozziello *et al* do not mention the possible restrictive effect of their method. Although it seems that their method has all the advantages of an elegant method, the limitations of results is a serious matter and should be avoided. We have extended their results within the standard framework of symmetry analysis. We are aware that further results may lie within the ambit of a more generalised analysis.

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Chapter 1

Symmetry Methods

1.1 Introduction

In this chapter we present the basic concepts of symmetry which we use in this dissertation. We start with a brief history and we continue to develop the Lie and Noether approaches. We determine the Lie symmetries for a simple example and give the procedure that we follow to find the first integrals. We present Noether's Theorem and we illustrate with an example the procedure of computing the Noether integrals. We also present the steps that we follow with Program LIE and the observations which leads us to the computation of the Lie symmetries.

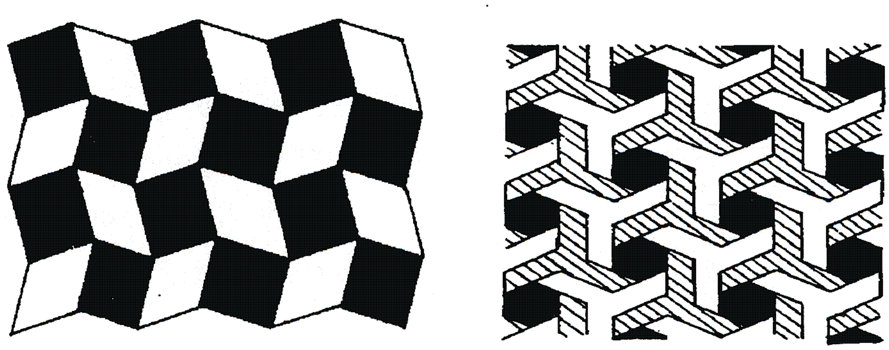

Table 1.1: Two examples of discretely symmetric patterns in the plane

1.2 Symmetry: a brief history

Symmetry is one of the ideas that have been used over the centuries in science and art and is closely connected with concepts of order, beauty and perfection. The concept of symmetry dates from the Paleolithic Period. In Mathematics we find symmetry being used in the Ionic School of Thales of Miletus and the Southern Italian School of the Pythagoreans [60]. Later mathematicians of the Athenian and Alexandrian periods rejected the reliance of the Ionian School on geometric symmetry. This attitude was a consequence of the paradoxes of Zeno of Elea [25]. Euclid's Elements contain virtually no mention of symmetry. The modern rise in the use of symmetry began at the turn of the nineteenth century.

The ancient Babylonians knew the formula for finding the roots of a quadratic equation. The cubic equation was apparently solved for the first time by Scipone del Ferro and the quartic by Ludovico Ferrari in the sixteenth century. The next problem was to solve the general quintic equation. Evariste Galois gave a proof of its impossibility based on the idea of symmetry. (The interested reader is referred to [53, 60]).

After Galois many mathematicians contributed to the fields of group theory. Two in particular were Felix Klein and Sophus Lie. Klein's main work is in discrete symmetry groups. In his famous lecture, The Erlangen Program, he shows the importance of the concept of symmetry with the folowing definition of geometry: *Geometry is the science which studies the properties of figures preserved under the transformations of a certain group of transformations*. Sophus Lie, on the other hand, devoted his entire career to the study of the theory of continuous groups (now known as Lie groups). There are two types of symmetries, the discrete and the continuous. A discrete symmetry is one which must be performed as a single operation. Translation in a planar lattice is an example of a discrete symmetry. A continuous symmetry is a continuous transformation which leaves invariant that upon which it operates and depends upon a continuous variable or parameter. Such a symmetry is the rotation of a circle about an axis through its centre normal to the plane of the circle. Lie considered not only continuous groups themselves, but went on to assign such groups to differential equations. The Lie theory of differential equations was highly valued by its creator and was for a time extremely popular. Over the years interest began to wane as mathematicians set their sights elsewhere. However, in branches of science such as Physics and Crystallography group theory found many applications. For the later the discrete groups are particularly important. In Physics Lie groups were associated with specific, usually conserved, quantities such as energy or angular momentum. The dimensional analysis so fondly used by engineers was the mere remnant of Lie's generalist approach to differential equations. In the 1950s, however, physicists (and later mathematicians) agreed that Lie theory describes the symmetries of the real objects modelled by differential equations. Hence finding the symmetries turned out to be of crucial importance.

1.3 Symmetries of differential equations

A transformation group is a mapping which depends upon one or more parameters. It is continuous if the parameter can vary continuously over a set of nonzero measure. If this mapping is continuously deformable from the identity, it can be written in the infinitesimal form

$$\bar{x}_i = x_i + \varepsilon\xi_i, \tag{1.1}$$

where ε is the parameter of smallness. The transformation (1.1) can be written in terms of the differential operator

$$G = \xi_i \frac{\partial}{\partial x_i} \tag{1.2}$$

as

$$\overline{x}_i = (1 + \varepsilon G)x_i. \tag{1.3}$$

In component form we have, in the case of two variables,

$$\overline{x} = x + \varepsilon\xi \tag{1.4}$$

$$\overline{y} = y + \varepsilon\eta, \tag{1.5}$$

where ξ and η are arbitrary functions and ε the infinitesimal.

Similarly we can express (1.4) and (1.5) in terms of a differential operator [22] acting on the variables x and y so that the infinitesimal transformation may be written as

$$\overline{x} = \left(1 + \varepsilon\xi\frac{\partial}{\partial x}\right)x \tag{1.6}$$

$$\overline{y} = \left(1 + \varepsilon\eta\frac{\partial}{\partial y}\right)y \tag{1.7}$$

and, if

$$G := \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial x}, \tag{1.8}$$

then

$$\overline{x} = (1 + \varepsilon G)\,x \tag{1.9}$$

$$\overline{y} = (1 + \varepsilon G)\,y. \tag{1.10}$$

The G is called the generator of the infinitesimal transformation. If one variable, y, is dependent upon the other variable, x, we must consider the effect of the transformation on derivatives. The first derivative transforms as (up to the first order in ε)

$$\begin{aligned} \frac{\mathrm{d}\overline{y}}{\mathrm{d}\overline{x}} &= \frac{\mathrm{d}(y + \varepsilon\eta)}{\mathrm{d}(x + \varepsilon\xi)} \\ &= \frac{y' + \varepsilon\eta'}{1 + \varepsilon\xi'} \\ &= y' + \varepsilon(n' - y'\xi'). \end{aligned} \tag{1.11}$$

Similarly

$$\frac{\mathrm{d}^2\overline{y}}{\mathrm{d}\overline{x}^2} = y'' + \varepsilon(n'' - 2y''\xi' - y'\xi''). \tag{1.12}$$

For the general case we can derive a differential operator by extension of the original operator (1.2). For an *nth* order differential equation the nth extension has the following form [44]

$$G^{[n]} = G + \sum_{i=1}^{n} \left\{ \eta^{(i)} - \sum_{j=1}^{i} \binom{i}{y} x^{(i+1-j)} \xi^{(j)} \right\} \frac{\partial}{\partial x^{(i)}}, \tag{1.13}$$

where $^{(i)}$ denotes $\mathrm{d}^i/\mathrm{d}x^i$.

If ξ and η are functions of x and y only, then we have a point transformation. A generalised transformation is one in which the coefficient functions depend upon derivatives of y as well. An nth order generalised transformation has the form

$$G = \xi\left(x, y, y', ..., y^{(n)}\right) \frac{\partial}{\partial x} + \eta\left(x, y, y', ..., y^{(n)}\right) \frac{\partial}{\partial y}. \tag{1.14}$$

We note that by the previous definition a point transformation is of zeroth order. We may define a symmetry as an operation which leaves invariant that upon which it operates [34]. When a symmetry corresponds to a point transformation, we call it a Lie point symmetry. If the transformation is generalised, then we obtain a generalised symmetry. Lie considered a special class of generalised symmetries, known as contact symmetries [16, 37, 38], for which the coefficient functions depend upon the first derivative of y in such way that the induced transformation in y' does not depend on y''.

If ξ and η contain an integral, we have a nonlocal symmetry [24]. The theory of nonlocal symmetries is quite a new theory which arose from interest in important problems. We work mainly with Lie point symmetries although the nonlocal approach could be useful in some points as we shall see in the following chapters.

A differential equation

$$E\left(x, y, y', ..., y^{(n)}\right) = 0 \tag{1.15}$$

possesses a symmetry

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \tag{1.16}$$

if

$$G^{[n]} E_{|_{E=0}} = 0. \tag{1.17}$$

Equation (1.17) means that the action of the nth extension of G on E is zero when the original equation is satisfied.

Since in Bianchi models time is the only independent variable, we confine our attention to the case of one independent variable, x, and four dependent variables, y_i, where $i = 1, 2, 3, 4$. We work with second order ordinary differential equations.

$$E(x, y_i, y_i', y_i'') = 0, \qquad i = 1, 2, 3, 4 \tag{1.18}$$

possesses a Lie point symmetry

$$G := \xi(x, y_i) \frac{\partial}{\partial x} + \eta_j(x, y_i) \frac{\partial}{\partial y_j} \tag{1.19}$$

if (1.18) is invariant under the infinitesimal transformation generated by the second extension of G, *viz.*

$$G^{[2]} = G + (\eta_j' - y_j'\xi') \frac{\partial}{\partial y_j'} + (\eta_j'' - 2y_j''\xi' - y_j'\xi'') \frac{\partial}{\partial y_j''} \tag{1.20}$$

whenever (1.18) holds. Note that (1.18) could be a scalar equation or a system of equations.

As a simple example we find the Lie point symmetries of the equation $y'' = 0$. The symmetry has the form

$$G = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \tag{1.21}$$

and from (1.13) the second extension takes the form

$$G^{[2]} = G + (\eta' - y'\xi') \frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'') \frac{\partial}{\partial y''}, \tag{1.22}$$

where

$$\xi' = \frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y} \tag{1.23}$$

$$\xi'' = \frac{\partial^2 \xi}{\partial x^2} + 2y'\frac{\partial^2 \xi}{\partial x \partial y} + y'^2\frac{\partial^2 \xi}{\partial y^2} + y''\frac{\partial \xi}{\partial y} \tag{1.24}$$

etc. So from (1.63)

$$\begin{aligned}
G^{[2]}y'' = 0 = \bigg[& \frac{\partial^2 \eta}{\partial x^2} + 2y'\frac{\partial^2 \eta}{\partial x \partial y} + y'^2\frac{\partial^2 \eta}{\partial y^2} + y''\frac{\partial \eta}{\partial y} \\
& - 2y''\left(\frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y}\right) \\
& - y'\left(\frac{\partial^2 \xi}{\partial x^2} + 2y'\frac{\partial^2 \xi}{\partial x \partial y} + y'^2\frac{\partial^2 \xi}{\partial y^2} + y''\frac{\partial \xi}{\partial y}\right)\bigg]\bigg|_{y''=0} \\
= & \frac{\partial^2 \eta}{\partial x^2} + 2y'\frac{\partial^2 \eta}{\partial x \partial y} + y'^2\frac{\partial^2 \eta}{\partial y^2} \\
& - y'\left(\frac{\partial^2 \xi}{\partial x^2} + 2y'\frac{\partial^2 \xi}{\partial x \partial y} + y'^2\frac{\partial^2 \xi}{\partial y^2}\right) \\
= & \; 0.
\end{aligned} \tag{1.25}$$

Since ξ and η are functions of x and y only, the y' dependence is explicit and we can separate by powers of y'. We have

$$\begin{aligned}
y'^3 &: \frac{\partial^2 \xi}{\partial y^2} = 0 \\
y'^2 &: \frac{\partial^2 \eta}{\partial y^2} - 2\frac{\partial^2 \xi}{\partial x \partial y} = 0 \\
y'^1 &: 2\frac{\partial^2 \eta}{\partial x \partial y} - \frac{\partial^2 \xi}{\partial x^2} = 0 \\
y'^0 &: \frac{\partial^2 \eta}{\partial x^2} = 0.
\end{aligned} \tag{1.26}$$

From (1.26a) we find that

$$\xi = a(x) + b(x)y$$

and (1.26b) gives

$$\eta = b'y^2 + c(x)y + d(x).$$

Equations (1.26c) and (1.26d) impose restrictions on the forms of ξ and η. Hence

$$0 = 2(2b''y + c') - a'' - b''y$$

$$0 = b'''y^2 + c''y + d''.$$

The functions a, b and c depend upon x only and so one can separate by independent powers of y. We find

$$d'' = 0$$

$$c'' = 0$$

$$b'' = 0$$

$$a'' = 2c'.$$

Hence the solution is

$$a = A_0 + A_1x + C_1x^2$$

$$b = B_0 + B_1x$$

$$c = C_0 + C_1x \tag{1.27}$$

$$d = D_0 + D_1x. \tag{1.28}$$

The symmetry has the form

$$G = \left(A_0 + A_1x + C_1x^2 + (B_0 + B_1x)y\right)\frac{\partial}{\partial x}$$

$$+ \left(B_1y^2 + (C_0 + C_1x)y + D_0 + D_1x\right)\frac{\partial}{\partial y}. \tag{1.29}$$

If in turn we set seven of the constants to be zero and the remaining one to be equal to unity, we obtain the following symmetries

$$G_1 = \frac{\partial}{\partial y}$$

$$G_2 = x\frac{\partial}{\partial y}$$

$$G_3 = \frac{\partial}{\partial x}$$

$$G_4 = x\frac{\partial}{\partial x}$$

$$G_5 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

$$G_6 = y\frac{\partial}{\partial y}$$

$$G_7 = y\frac{\partial}{\partial x}$$

$$G_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

The equation $y'' = 0$ possesses the maximal number of symmetries for a second order ordinary differential equation.

We use the symmetries to reduce the order of differential equations. From the generators we can find new variables which are invariants of the generator and functions of x, y and y'. Hence the order of the equations is reduced by one. Unfortunately there are cases that we are unable to find an expression of the solution even through the reduced equation.

1.4 First integrals

A first integral is a conserved quantity which is a function of the independent variable, the dependent variables and their derivatives up to the $(n-1)^{th}$ for an n^{th} order system. It is clear that the concept of a first integral is connected with the concept of invariance and so with the concept of symmetry. If we consider the equation of evolution of some system, then a first integral is a function of the dynamical variables that retains its initial value as the system evolves. From the geometric point of view a first integral can be interpreted as restricting the solution to a surface in the extended phase space. Hence, for a second order scalar equation, two independent first integrals define two surfaces, the intersection of which gives the solution of the equation as the trajectory in the three dimensional extended phase space [36].

If an equation

$$E\left(x, y, y', ..., y^{(n)}\right) = 0 \tag{1.30}$$

has a symmetry

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{1.31}$$

then a first integral $I\left(x, y, y', ..., y^{(n-1)}\right)$ of E associated with G is found from the solution of the equations

$$G^{[n-1]}I = 0 \tag{1.32}$$

and

$$\frac{\mathrm{d}I}{\mathrm{d}x}\bigg|_{E=0} = 0. \tag{1.33}$$

Equation (1.33) expresses the condition that I must be an integral. Equation (1.32) is an extra constraint which (hopefully) will make the process of solution easier. This reduces the number of first integrals to be found from (1.33) from n to $n-1$. Thus instead of finding all the independent first integrals of an equation we find a subclass of those first integrals which are invariant under the particular symmetry we choose. To solve an equation completely by symmetry methods we require more than one useful[1] symmetry. If the dimension of the space is n, then I in the context of a system of second order ordinary differential equations is a function of $2n+1$ variables x, y_i and y_i' which from (1.32) can be expressed in terms of $2n$ characteristics each of which is invariant under the infinitesimal transformation generated by $G^{[1]}$. If we consider (1.33), I becomes a function of $2n-1$ characteristics each of which is a first integral of the original equation (1.18).

For the second order equation (1.18) the function $I(x, y_j, y_j')$ is a first integral associated with the symmetry G if it possesses the properties

$$G^{[1]}I = 0 \tag{1.34}$$

$$\left.\frac{\mathrm{d}I}{\mathrm{d}x}\right|_{E=0} = 0$$

As a simple example we find the integrals for the symmetry $G_1 = \partial/\partial y$ of the equation $y'' = 0$. Since

$$\begin{aligned} G_1^{[1]} &= G_1 + (\eta' - y'\xi')\frac{\partial}{\partial y'} \\ &= 0\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 0\frac{\partial}{\partial y'}, \end{aligned} \tag{1.35}$$

the associated Lagrange's system is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dy'}{0} \tag{1.36}$$

and the characteristics are

$$u = y \tag{1.37}$$

$$v = y'. \tag{1.38}$$

[1]A useful symmetry is defined as a symmetry which may be used in the solution of differential equation either by reduction of order or the determination of first integrals.

The first integral is now

$$I(x, y, y') = J(u, v). \tag{1.39}$$

The requirement (1.33) gives

$$u'\frac{\partial J}{\partial u} + v'\frac{\partial J}{\partial v} = 0 \tag{1.40}$$

with the associated Lagrange's system

$$\frac{du}{v} = \frac{dv}{0} \tag{1.41}$$

for which v is a characteristic. Hence the integral is[2]

$$I_1 = y'. \tag{1.42}$$

The associated Lagrange's system for

$$G_5 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

is

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dy'}{y - xy'} \tag{1.43}$$

from which we obtain the two characteristics

$$u = \frac{y}{x} \qquad v = y - xy'. \tag{1.44}$$

The first integral has the form

$$I(x, y, y') = J(u, v). \tag{1.45}$$

The requirement $J' = 0$ gives

$$u'\frac{\partial J}{\partial u} + v'\frac{\partial J}{\partial v} = 0 \tag{1.46}$$

[2]Strictly $I_1 = f(y')$, but usually f is set as the identity.

with the associated Lagrange's system

$$\frac{du}{y'/x - y/x^2} = \frac{dv}{-xy''} \tag{1.47}$$

$$\Longleftrightarrow \frac{du}{-v} = \frac{dv}{0}. \tag{1.48}$$

We obtain the integral

$$I_2 = y - xy'. \tag{1.49}$$

One should note that the differential equation and the first integral have at least one symmetry in common, the symmetry that we used to obtain the first integral. There is always the possibility that the differential equation has a symmetry which is not necessarily a symmetry for the integral.

1.5 Noether's Theorem

Symmetries of differential equations are not the only symmetries which are used to provide integrals and solutions. The invariance of the Action Integral under infinitesimal transformation leads to Noether's theorem [47] which relates a first integral to each symmetry obtained from the Action Integral. We derive Noether's Theorem in the case of particulate motion in one degree of freedom. Under an infinitesimal transformation (1.1), where ξ and η are arbitrary differentiable functions,[3] the Action Integral

$$A = \int_{x_0}^{x^1} \mathcal{L}(x, y, y')dx \tag{1.50}$$

is invariant if

$$\overline{A} = \int_{\overline{x}_0}^{\overline{x}_1} \mathcal{L}\left(\overline{x}, \overline{y}, \overline{y}'\right) d\overline{x} \tag{1.51}$$

[3]One should be careful to be aware that, if the functions are not arbitrary, the generality of the treatment is restricted.

$$= A, \tag{1.52}$$

where $'$ denotes $d/d\overline{x}$ and $\mathcal{L}$ is an analytic function of x, y and y'. Hamilton's Principle makes the condition $\overline{A} = A$ slightly weaker: It is enough to consider the variation in the functional to be zero under a zero endpoint variation. We impose the restriction on ξ and η that they conform to the requirements of Hamilton's Principle, but allow for the freedom of a gauge function which contributes nothing to the variation. Thus we may write

$$\overline{A} = \int_{\overline{x}_0 = x_0}^{\overline{x}_1 = x_1} \mathcal{L}\left(\overline{x}, \overline{y}, \overline{y}'\right) d\overline{x} \tag{1.53}$$

$$= A + \int_{x_0}^{x_1} \frac{\mathrm{d}F}{\mathrm{d}x} dx. \tag{1.54}$$

Consider the transformations (1.1). We require that (1.54) be an identity for $\varepsilon = 0$ and we have

$$\int_{x_0}^{x_1} \left(\mathcal{L} + \varepsilon \left(\xi \frac{\partial \mathcal{L}}{\partial x} + \eta \frac{\partial \mathcal{L}}{\partial y} + \zeta \frac{\partial \mathcal{L}}{\partial y'} + \xi' \mathcal{L} \right) \right) dx = \int_{x_0}^{x_1} (\mathcal{L} + \varepsilon f') dx, \tag{1.55}$$

where the gauge function is such that $\overline{A}(\varepsilon = 0) = A$, and we have written $F = \varepsilon f$. The symmetry

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \tag{1.56}$$

is a Noether symmetry if

$$\xi \frac{\partial \mathcal{L}}{\partial x} + \eta \frac{\partial \mathcal{L}}{\partial y} + \zeta \frac{\partial \mathcal{L}}{\partial y'} + \xi' \mathcal{L} = f', \tag{1.57}$$

where $\zeta = \eta' - y'\xi'$. We call f a gauge function. In order to find the infinitesimal transformation under which the variation of the Action Integral is invariant we solve (1.57). We can rewrite (1.57) as

$$0 = \left\{ f - \left[\xi \mathcal{L} + (\eta - y'\xi) \frac{\partial \mathcal{L}}{\partial y'} \right] \right\}' - (\eta - y'\xi') \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \mathcal{L}}{\partial y'} \right) \right] \tag{1.58}$$

so that, when the variational principle is imposed on (1.54) for it to take a stationary value and consequently the Euler-Lagrange equation applies, (1.58) leads to the Noetherian integral

$$I = f - \left[\xi\mathcal{L} + (\eta - y'\xi)\frac{\partial \mathcal{L}}{\partial y'}\right]. \tag{1.59}$$

We note that the derivation is completely independent of the functional dependence of f, ξ and η. Hence the statement and the proof of Noether's Theorem can apply for more general symmetries such as generalised and nonlocal symmetries.

We give a simple example as an application of Noether's Theorem. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}y'^2. \tag{1.60}$$

We seek the Noether point symmetries for this Lagrangian. Hence we have

$$f' = (\eta' - y'\xi')y' + \frac{1}{2}\xi' y'^2 \tag{1.61}$$

$$\iff \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} = \left(\frac{\partial \eta}{\partial x} + y'\frac{\partial \eta}{\partial y} - y'\frac{\partial \xi}{\partial \xi} - y'^2\frac{\partial \xi}{\partial y}\right)y' + \frac{1}{2}\left(\frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y}\right)y'^2 \tag{1.62}$$

since ξ and η are functions of x and y. We have

$$\begin{aligned}
y'^3 &: \; -\frac{\partial \xi}{\partial y} + \frac{1}{2}\frac{\partial \xi}{\partial y} = 0 \\
y'^2 &: \; \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x} + \frac{1}{2}\frac{\partial \xi}{\partial x} = 0 \\
y'^1 &: \; \frac{\partial f}{\partial y} - \frac{\partial \eta}{\partial x} = 0 \\
y'^0 &: \; \frac{\partial f}{\partial x} = 0.
\end{aligned} \tag{1.63}$$

The system (1.63) has solution

$$\xi = A_0 + A_1 x + A_2 x^2 \tag{1.64}$$

$$\eta = \frac{1}{2}(A_1 + 2A_2 x)y + B_0 + B_1 x). \tag{1.65}$$

The five constants give the five Noether symmetries

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} \\ G_3 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \\ G_4 &= \frac{\partial}{\partial y} \\ G_5 &= x\frac{\partial}{\partial y} \end{aligned} \tag{1.66}$$

which are a subset of the eight Lie point symmetries of $y'' = 0$. From (1.59) we obtain the Noetherian integrals

$$\begin{aligned} I_1 &= -\frac{1}{2}y'^2 \\ I_2 &= -\frac{1}{2}y'(y - xy') \\ I_3 &= -\frac{1}{2}(y - xy')^2 \\ I_4 &= -y' \\ I_5 &= y - xy'. \end{aligned} \tag{1.67}$$

Once a Noether symmetry is found the first integral follows easily. For an nth order Lagrangian Noether's Theorem produces one integral per symmetry, but the corresponding Euler-Lagrange equation is of order $2n$ [55] and a Lie symmetry produces $(2n-1)$ integrals per symmetry. However, the calculation of these first integrals is nontrivial [20].

1.6 Computer methods

In order to obtain the Lie symmetries we use the package written by Alan Head, Program LIE [26]. Although there are many other packages [27] we prefer this particular one because we find it more sensible. Without the help of the package Program LIE we would not be able to investigate all the systems in the four cases for all the Bianchi models since the procedure by hand is too slow[4]. We give the basic steps of the procedure that we use for the computation of Lie symmetries. Since we obtain a system of second order equations with one independent variable and four dependent, for the Bianchi models we present the procedure for the case of four differential equations with one independent variable and four dependent. We enter the commands listed in Table 1.2. NIND# denotes the number of independent variables, NDEP# the number of dependent variables, DE#[n] represents the nth equation of the system and DV#[n] is the highest derivative of the nth equation[5]. Here one should be careful: The equation must be linear in that term. The next step is to define the variables. LIE interprets U(i) as dependent variables and X(i) as independent variables. In our case we have the dependent varibles u, v, w and ϕ so we enter them as U(1), U(2), U(3) and U(4). Time is the independent variable so we enter it as X(1). If one wishes to insert the second derivative of v, one types U(2,1,1), where the number 2 is the corresponding number which characterises the variable and the

[4]With but slight tongue in cheek one could say that the obtaining of the correct answer is more of a statistical matter!

[5]In principle the differential equations can be used to eliminate any term which occurs linearly in the equation. In practice the highest derivative is used. This is not the case when the symmetries of a function or first integral is to be determined.

ECHO: TRUE	\$
NIND#:1	\$
NDEP#:4	\$
DE#:DV#:	\$
DE#[1]:	\$
DE#[2]:	\$
DE#[3]:	\$
DE#[4]:	\$
DV#[1]:	\$
DV#[2]:	\$
DV#[3]:	\$
DV#[4]:	\$
ECHO:FALSE	\$
RDS()	\$

Table 1.2: *The commands for preparing a program to analyse a system of four equations using Program* LIE.

number 1 is the derivation with respect to the first independent variable. We enter the equation without the '=0' part. The next step is to type at the DOS prompt the command MULIE and continue with the commands

$$
\begin{array}{rl}
(i) & RDS(FILENAME, DAT); \\
(ii) & DOLIE(); \\
(iii) & DOSOLV(); \\
(iv) & DOCHECK(); \\
(v) & DOVEC(); \\
(vi) & DONZC();
\end{array}
$$

A common problem in this procedure arises when the machine gives us the comment 'exhaustion of memory'. If the problem arises in the process of DOLIE(); two options are available. The first one is to rearrange the equations so that the simpler ones are analysed first. However, the word simpler is relevant for the machine as the reader will note in the following example. The other option is to introduce new dependent variables to makes the system simpler. For this reason, as we see in Chapter Three, we use a new reparametrisation for the field equations. If the problem occurs in DOSOLV(); there are further options. In our case we overcame the problem in two ways: The first one is to enter the command DOINTCON(); after DOLIE();. Sometimes there are equations that are returned as unsolved in the route to determining the symmetries. If one can solve these, the solution can be inserted into LIE using EVSA#();.

We give an example of the procedure that we follow in order to obtain the Lie symmetries in Bianchi Type III. In the first case we have a system with three equations and three variables since there is no matter. After the use of Program LIE we obtain the four symmetries

$$G_1 = \frac{\partial}{\partial t}$$

$$G_2 = t\frac{\partial}{\partial t} + 4v\frac{\partial}{\partial v}$$

$$G_3 = u\frac{\partial}{\partial u} + 4w\frac{\partial}{\partial w}$$

$$G_4 = u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v}.$$

In the second case the system of equations is

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \frac{2}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} + \frac{1}{4}\dot{\phi}^2 = 0$$

$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} - \frac{4}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} = 0$$

(1.68)

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} + 4\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} = 0$$

$$\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0,$$

where ϕ represents the introduction of matter. We enter the system in the following form

$$ECHO : TRUE \ \$$$

$$DE\# : DV\# : \ \$$$

$$NIND\# : 1 \ \$$$

$$NDEP\# : 4 \ \$$$

$$DE\#[4] : U(1,1,1) * U(1) * U(2)^2 * U(3)^2 + 1/2 * U(1,1)^2 * U(2)^2 * U(3)^2$$

$$+ 3/8 * U(1)^2 * (U(2,1)^2 * U(3)^2 + 1/3 * U(3,1)^2 * U(2)^2)$$

$$+ 1/4 * U(4,1)^2 * U(1)^2 * U(2)^2 * U(3)^2 - 2/3 * U(2)^{3/2} * U(3)^{5/2} \ \$$$

$$DE\#[3] : U(2,1,1) * U(2) * U(1)^2 - U(2,1)^2 * U(1)^2$$

$$+ 3 * U(2,1) * U(1,1) * U(2) * U(1) - 4/3 * U(3)^{1/2} * U(2)^{3/2} \ \$$$

$$DE\#[2] : U(3,1,1) * U(3) * U(1)^2 * U(2) - U(3,1)^2 * U(1)^2 * U(2)$$

$$+ 3 * U(3,1) * U(1,1) * U(3) * U(1) * U(2) + 4 * U(3)^{5/2} * U(2)^{1/2} \ \$$$

$$DE\#[1] : U(4,1,1) * U(1) + 3 * U(4,1) * U(1,1) \ \$$$

$$DV\#[4] : U(1,1,1) \ \$$$

$DV\#[3] : U(2,1,1)$ \$

$DV\#[2] : U(3,1,1)$ \$

$DV\#[1] : U(4,1,1)$ \$

$ECHO : FALSE$ \$

$RDS()$ \$

If we apply the Program LIE to the system the machine gives us a message of memory exhausted. We note that the coefficient functions in the first case are first order polynomials. We expect that the symmetries in the second case will have almost the same expression as in the first case since the only difference is the extra term of ϕ in the first equation and the ϕ equation. Hence we expect that the symmetries in the second case will have coefficient functions which are first order polynomials. We should enter this information into the program in order to simplify the procedure of finding the symmetries. We achieve this by the use of the command $DOPOLYALL(1)$; after the $DOLIE()$; command. The program gives the symmetries

$$G_1 = \frac{\partial}{\partial t} \tag{1.69}$$

(1.70)

$$G_2 = t\frac{\partial}{\partial t} + 4v\frac{\partial}{\partial v} \tag{1.71}$$

(1.72)

$$G_3 = u\frac{\partial}{\partial u} + 4w\frac{\partial}{\partial w} \tag{1.73}$$

$$G_4 = u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v} \tag{1.74}$$

(1.75)

$$G_5 = \frac{\partial}{\partial \phi} \tag{1.76}$$

We note that the first four symmetries are the same as in the first case as they should be since they do not affect the extra term in ϕ of the u equation and the ϕ equation is homogeneous of degree zero in u. Also G_5 is a symmetry for the system since ϕ is an ignorable variable. Hence the general procedure that we follow is to find the symmetries in the first case where the system is simpler and after considering their expressions we impose the appropriate information on the program for the rest of the cases. Also in Chapter Three we shall see that the expression of the Ricci scalar is the term responsible for the different symmetries in the models. However, there are cases where the form of the Ricci scalar is not extremely different as in Bianchi Types VI and VII in class A where we obtain the same symmetries. Hence the expression of the Ricci scalar will give us an idea of the kind of the symmetries and so make the procedure easier.

Chapter 2

A Review of the Results from the Noether Symmetry Approach due to Capozziello et al

2.1 Introduction

Since our aim is to describe our universe we should be able to develop mathematical methods for the cosmological models which enable us to understand the evolution of the universe. Since the FRW model [54] is the model which describes perhaps most efficiently our universe, many methods have been applied to this model [12, 52]. One of them is the 'Noether symmetry approach' [10]. Recently [9] this method has been applied to the Bianchi models in order to obtain first integrals[1] in cases where the number of symmetries of the model permits us to describe it completely. In this chapter we describe the basic steps of this method and we give the application to the Bianchi universe as we find it in [9].

[1]The terminology of exact integration is used in Capozziello *et al* citeital.

2.2 The Noether symmetry approach

According to Capozziello *et al* the Noether symmetry approach is a method for finding symmetries of the Action Integral associated with the system of equations and the corresponding constants of the motion thereby reducing the dimension of the system. A constant of motion is a differentiable function on the tangent space which does not vary along the integral curves. Since the constants of motion are connected with the concept of invariance, the basis for this method is Noether's theorem[2]. Loosely speaking Noether's theorem can interpreted as 'If the Lagrangian is invariant under a one parameter group of transformations on the tangent space, then there is a constant of the motion which can be connected with this invariance' [45]. The one parameter group has an infinitesimal generator, X, on the tangent space. In order to obtain the constants of motion it is necessary to obtain the infinitesimal generator which is a point transformation. We present the basic steps of this method. The tangent space TQ which is spanned by $q_n, \dot{q}_n$ is tangent to the configuration space Q which is spanned by q_n. Let $\mathcal{L} = \mathcal{L}(q_n, \dot{q}_n)$ be the Lagrangian which describes our system, where $\dot{q}_n = \mathrm{d}q_n/dt$. Of course $\mathcal{L}$ contains also undefined terms of the scalar ϕ and the potential $V(\phi)$. The Lagrangian is taken to be independent of the time, which means that

$$\frac{\partial \mathcal{L}}{\partial t} = 0,$$

and nondegenerate[3], ie

$$\det\left[H\right]_{ij} = \det\left[\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}_j}\right] \neq 0.$$

The Euler-Lagrange equations arising from a zero endpoint variation of the Action

[2]It is actually a restricted form of Noether's theorem. In fact the infinitesimal transformation ignores the time and the gauge function is zero.

[3]To permit a one to one correspondence between symmetry and first integral [55].

Integral are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \frac{\partial \mathcal{L}}{\partial q_n} = 0, \quad i = 1, ..., n. \tag{2.1}$$

One identifies a symmetry by introducing a set of arbitrary, real, differentiable functions $\{X^n(q_n)\}$ which are functions of any q_n. These functions are actually the coefficients of the infinitesimal generator X. Contracting this with the field equations (2.1) we have

$$X^n \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \frac{\partial \mathcal{L}}{\partial q_n} \right) = 0 \tag{2.2}$$

so that

$$X^n \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} = X^n \frac{\partial L}{\partial q_n}. \tag{2.3}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(X^n \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \right) = \left(X^n \frac{\partial}{\partial q_n} + \frac{dX^n}{dt}\frac{\partial}{\partial \dot{q}_n} \right) \mathcal{L}. \tag{2.4}$$

The vector

$$X^n \frac{\partial}{\partial q_n} + \frac{dX^n}{dt}\frac{\partial}{\partial \dot{q}_n} \tag{2.5}$$

is the infinitesimal generator of the transformation and is the tangent lift of X onto the tangent space TQ. We denote this vector as X^T. If we apply this vector to the Lagrangian, then the quantity

$$\left(X^n \frac{\partial}{\partial q_n} + \frac{dX^n}{dt}\frac{\partial}{\partial \dot{q}_n} \right) \mathcal{L} \tag{2.6}$$

is the Lie derivative, $L_{X^T}\mathcal{L}$, which determines how the Lagrangian varies along the flow generated by X^T in TQ. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(X^n \frac{\partial \mathcal{L}}{\partial \dot{q}^n}\right) = L_X \mathcal{L}. \tag{2.7}$$

With this terminology Capozziello *et al* state Noether's theorem as:

Theorem: *If*

$$L_{X^T} \mathcal{L} = 0, \tag{2.8}$$

then the function

$$K = X^n \frac{\partial \mathcal{L}}{\partial \dot{q}^n} \tag{2.9}$$

is a constant of motion.

Equation (2.8) expresses that the Lagrangian density is constant along the integral curves of X^T and it is invariant under the transformation X^T.

We can express the (2.9) independently of coordinates. We introduce the Cartan one form $\theta_{\mathcal{L}}$ associated with $\mathcal{L}$ which is defined as

$$\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \mathrm{d}q_n. \tag{2.10}$$

We denote the contraction of the Cartan one form $\omega = \omega_i \mathrm{d}x^i$ and the vector $X = x^i \partial / \partial x^i$ as $i_X \omega = X^i \omega_i$. Hence we can rewrite the quantity $X^n \partial \mathcal{L} / \partial \dot{q}_n$ as

$$i_X \theta_{\mathcal{L}} = X^n \frac{\partial \mathcal{L}}{\partial \dot{q}_n}. \tag{2.11}$$

From (2.4) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(i_X \theta_{\mathcal{L}}) = L_X \mathcal{L}. \tag{2.12}$$

Marmo *et al* [45] define X to be a Noether symmetry[4] for the Lagrangian $\mathcal{L}$ if (2.8)

[4] We believe that this definition is a restriction as we discuss in Chapter Five.

holds. Hence for a Noether symmetry they obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(i_X\theta_{\mathcal{L}}\right)=0 \tag{2.13}$$

so that

$$i_X\theta_{\mathcal{L}}=K, \tag{2.14}$$

where K is a constant. The quantity $i_X\theta_{\mathcal{L}}$ is conserved and so is a constant of motion.

For the given Lagrangian and a point symmetry without a time component equation (2.8) reduces to an expression that is quadratic in $\dot{q}_n$ plus a term independent of $\dot{q}_n$. The coefficients of those terms are determined as functions of q_n and they must vanish (since $L_X\mathcal{L}=0$). This leads to a number of separate constraints that take the form of first order partial differential equations in the $X_n(q_n)$. A further constraint may arise from terms in the Lagrangian that are independent of $\dot{q}_n$. In theory a Noether symmetry is found. This gives a first integral of the field equation (2.1). Using K together with the energy function associated with $\mathcal{L}$[5]

$$E_{\mathcal{L}}:=\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i-\mathcal{L} \tag{2.15}$$

we can obtain a complete integration of equation (2.1) in the cases where the number of independent Noether symmetries is at least equal to the number of degrees of freedom of the system.[6] The equation

$$E_{\mathcal{L}}=0 \tag{2.16}$$

is the first order Einstein equation and represents a constraint [56].

The existence of the symmetry X gives us a constant of the motion via Noether's Theorem. A possible way to find it is to compute the Cartan one form associated

[5]This equation is actually the Noether integral for the symmetry $\partial/\partial t$. However, this symmetry is not considered there since the configuration space does not involve time.

[6]Recall that the existence of a Hamiltonian is implicit in the regular Hessian and so Liouville's theorem applies [8, 39, 40].

with $\mathcal{L}$, which, when contracted with X, gives the required constant of the motion. This constant, together with the 'energy function' associated with $\mathcal{L}$, provides the possibility of achieving complete integration of the models. The intrested reader will find more details about the method in [6, 45].

2.3 The Noether symmetry approach in the Bianchi models

In this section we present the application of the Noether symmetry approach to the Bianchi universes as described by Capozzello *et al* [9]. We find the form of the general symmetry by solving (2.8). Consider the restrictions which following from the solution of the (2.8). Capozziello *et al* obtain two cases for the potential. One is if it is a constant, the other if it is not. We give as an example the case of Bianchi Type I. We perform the procedure for finding the symmetries and the constants of motion in both cases of constant and variable potential. In Chapter Four we present the integrals for this model in the case where the potential is zero.

We consider the Lagrangian (3.10). The configuration space is spanned by the four variables $\lambda, \beta_1, \beta_2, \phi$ and so the vector field X has the form

$$X = L\frac{\partial}{\partial\lambda} + B_1\frac{\partial}{\partial\beta_1} + B_2\frac{\partial}{\partial\beta_2} + F\frac{\partial}{\partial\phi}, \tag{2.17}$$

where L, B_1, B_2, F are unknown functions of the four variables. To find the symmetry of the system it is sufficient to find the coefficients of X. The tangent lift X^T on TQ takes the form

$$X^T = X + \frac{dL}{dt}\frac{\partial}{\partial\dot{\lambda}} + \frac{dB_1}{dt}\frac{\partial}{\partial\dot{\beta}_1} + \frac{dB_2}{dt}\frac{\partial}{\partial\dot{\beta}_2} + \frac{dF}{dt}\frac{\partial}{\partial\dot{\phi}}, \tag{2.18}$$

where d/dt means the Lie derivative along the vector field X. Hence, for example,

$$\frac{dL}{dt} = \frac{\partial L}{\partial \lambda}\dot{\lambda} + \frac{\partial L}{\partial \beta_1}\dot{\beta}_1 + \frac{\partial L}{\partial \beta_2}\dot{\beta}_2 + \frac{\partial L}{\partial \phi}\dot{\phi}. \tag{2.19}$$

Equation (2.8) gives eleven partial differential equations since it is a homogeneous quadratic polynomial in $\dot{\lambda}, \dot{\beta}_1, \dot{\beta}_2, \dot{\phi}$ plus terms independent of the derivatives. Since the polyonimal has to be identically zero, each coefficient must be independently zero. The coefficients are

$$\begin{aligned} 3L + 2\tfrac{\partial L}{\partial \lambda} &= 0 \qquad & 3L + 2\tfrac{\partial B_1}{\partial \beta_1} &= 0 \\ 3L + 2\tfrac{\partial B_2}{\partial \beta_2} &= 0 \qquad & 3L + 2\tfrac{\partial F}{\partial \phi} &= 0 \end{aligned} \tag{2.20}$$

$$\begin{aligned} 4\tfrac{\partial L}{\partial \beta_1} - \tfrac{\partial B_1}{\partial \lambda} &= 0 \qquad & 4\tfrac{\partial L}{\partial \beta_2} - \tfrac{\partial B_2}{\partial \lambda} &= 0 \\ 6\tfrac{\partial L}{\partial \phi} - \tfrac{\partial F}{\partial \lambda} &= 0 \qquad & \tfrac{\partial B_1}{\partial \beta_2} + \tfrac{\partial B_2}{\partial \beta_1} &= 0 \\ 3\tfrac{\partial B_1}{\partial \phi} + 2\tfrac{\partial F}{\partial \beta_1} &= 0 \qquad & 3\tfrac{\partial B_2}{\partial \phi} + 2\tfrac{\partial F}{\partial \beta_2} &= 0 \end{aligned} \tag{2.21}$$

$$L\left(R^* + 6V(\phi) + N_1N_2N_3(1 + N_1N_2N_3)\right) + B_1\frac{\partial R^*}{\partial \beta_1} + B_2\frac{\partial R^*}{\partial \beta_2} + 2FV' = 0. \tag{2.22}$$

We note that (2.22) imposes restrictions on the class potentials in ϕ permitted. The solution of the first ten equations has the form

$$L = 0$$

$$B_1 = c\beta_2 + c_1\phi + c_0$$

$$B_2 = -cB_1 + c_2\phi + c_0'$$

$$F = F_0 - \frac{3}{2}(c_1\beta_1 + c_2\beta_2),$$

where c, c_0, c_0', c_1, c_2 and F_0 are arbitrary constants. At this point we should note that Capozzielo *et al* introduced the function $\bar{L}$ to satisfy the relation

$$L = e^{-3/2\lambda}\bar{L}(\beta_1, \beta_2, \phi) \tag{2.23}$$

and because of the structure of the system of the equation, $\bar{L}$ must satisfy the equations

$$\bar{L} = \bar{L}_1(\beta_1, \phi) + \bar{L}_2(\beta_2, \phi) \tag{2.24}$$

and

$$3e^{-3/2\lambda}\left(\bar{L}_1 + \bar{L}_2 - \frac{16}{9}\frac{\partial^2 \bar{L}_1}{\partial \beta_1^2}\right) + 2\frac{\partial \bar{B}_1}{\partial \beta_1} = 0, \tag{2.25}$$

where $\bar{B}_1$ is a function of β_1, β_2 and ϕ. From (2.25), since $\bar{B}_1$ does not depend on λ, we obtain

$$\frac{\partial \bar{B}_1}{\partial \beta_1} = 0. \tag{2.26}$$

Hence

$$\bar{L}_1 + \bar{L}_2 - \frac{16}{9}\frac{\partial^2 \bar{L}_1}{\partial \beta_1^2} = 0 \tag{2.27}$$

and on differentiation with respect ot β_2 we obtain

$$\frac{\partial \bar{L}}{\partial \beta_2} = 0 \tag{2.28}$$

so that $\bar{L}_2$ is a function of ϕ. On differentiation of (2.25) with respect to β_1 we obtain that

$$\bar{L}_1 = A_1(\phi)e^{\frac{3}{4}\beta_1} + A_2(\phi)e^{-\frac{3}{4}\beta_2} - \bar{L}_2(\phi) \tag{2.29}$$

and hence

$$\bar{L} = \bar{L}_1 + \bar{L}_2 = A_1(\phi)e^{\frac{3}{4}\beta_1} + A_2(\phi)e^{-\frac{3}{4}\beta_2}. \tag{2.30}$$

From (2.20) we obtain that

$$\frac{\partial \bar{B}_2(\beta_1, \beta_2, \phi)}{\partial \beta_2} = -\frac{3}{2}e^{-3\lambda}\bar{L}(\beta_1, \phi) \tag{2.31}$$

and with differentiation with respect to λ we obtain that $\bar{L} = 0$ and hence $L = 0$. Capozzielo *et al* have found that $\bar{L}_2$ and $\bar{L}_1$ are separately zero but there is no argument for this. The solutions must satisfy the restriction (2.22) which, for $L = 0$, takes the form

$$B_1\frac{\partial R^*}{\partial \beta_1} + B_2\frac{\partial R^*}{\partial \beta_2} = -2FV'. \tag{2.32}$$

We see that the left side is a function of all four variables and the right side one does not depend upon the variable λ. Since the terms which depend upon β_1, β_2 in the expression of R^* are exponential, the two sides after derivation with respect to λ should be independently zero which means that we obtain two subcases: when the potential V is constant and when $F = 0$. These constraints allow the symmetries to be found in every model. As a simple example we calculate the symmetries and the constants of motion for the Bianchi Type I. In the case where the potential V is constant the symmetry has the form

$$X = (c\beta_2 + c_1\phi + c_0)\frac{\partial}{\partial\beta_1} + (-c\beta_1 + c_2\phi + c_0')\frac{\partial}{\partial\beta_2}$$

$$+\left(F_0 - \frac{3}{2}(C_1\beta_1 + c_2\beta_2)\right)\frac{\partial}{\partial\phi} \tag{2.33}$$

with lift[7]

$$X^T = X + (c\dot{\beta}_2 + c_1\dot{\phi})\frac{\partial}{\partial\dot{\beta}_1} + (-c\dot{\beta}_1 + c_2\dot{\phi})\frac{\partial}{\partial\dot{\beta}_2} - \frac{3}{2}(c_1\dot{\beta}_1 + c_2\dot{\beta}_2)\frac{\partial}{\partial\dot{\phi}}. \tag{2.34}$$

Giving particular values to the constants as described in Chapter One we obtain the following symmetries:

$$X_1^T = \frac{\partial}{\partial\beta_1}$$

$$X_2^T = \frac{\partial}{\partial\beta_2}$$

$$X_3^T = \frac{\partial}{\partial\phi}$$

$$X_4^T = \phi\frac{\partial}{\partial\beta_1} - \frac{3}{2}\beta_1\frac{\partial}{\partial\phi} + \dot{\phi}\frac{\partial}{\partial\dot{\beta}_1} - \frac{3}{2}\dot{\beta}_1\frac{\partial}{\partial\dot{\phi}}$$

$$X_5^T = \phi\frac{\partial}{\partial\beta_2} - \frac{3}{2}\beta_2\frac{\partial}{\partial\phi} + \dot{\phi}\frac{\partial}{\partial\dot{\beta}_2} - \frac{3}{2}\dot{\beta}_2\frac{\partial}{\partial\dot{\phi}}$$

$$X_6^T = \beta_1\frac{\partial}{\partial\beta_2} - \beta_2\frac{\partial}{\partial\beta_1} + \dot{\beta}_1\frac{\partial}{\partial\dot{\beta}_2} - \dot{\beta}_2\frac{\partial}{\partial\dot{\beta}_1}.$$

The six symmetries are, by construction, independent. The nonzero Lie brackets are

[7]This coincides with the first extension which is the familiar term in the literature of differential equations.

$$
\begin{aligned}
\left[X_1^T, X_4^T\right] &= -\tfrac{3}{2}X_3^T & \left[X_2^T, X_5^T\right] &= -\tfrac{3}{2}X_3^T \\
\left[X_3^T, X_4^T\right] &= X_1^T & \left[X_3^T, X_5^T\right] &= X_2^T \\
\left[X_4^T, X_5^T\right] &= -\tfrac{3}{2}X_6^T & \left[X_1^T, X_6^T\right] &= X_2^T \\
\left[X_2^T, X_6^T\right] &= -X_1^T & \left[X_4^T, X_6^T\right] &= X_5^T \\
\left[X_5^T, X_6^T\right] &= X_4^T.
\end{aligned}
$$

Since we have found the symmetries, we are able to calculate the constants of motion. We use the form

$$K = i_X \partial_{\mathcal{L}} = X^n \frac{\partial \mathcal{L}}{\partial \dot{q}_n}. \tag{2.35}$$

Hence we have

$$K_1 = X_1^1 \frac{\partial \mathcal{L}}{\partial \dot{\beta}_1} = -3e^{3\lambda}\dot{\beta}_1$$

$$K_2 = X_2^2 \frac{\partial \mathcal{L}}{\partial \dot{\beta}_2} = -3e^{3\lambda}\dot{\beta}_2$$

$$K_3 = X_3^3 \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -2e^{3\lambda}\dot{\phi}$$

$$
\begin{aligned}
K_4 &= X_4^1 \frac{\partial \mathcal{L}}{\partial \dot{\beta}_1} + X_4^3 \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
&= -3e^{3\lambda}\phi\dot{\beta}_1 + 3e^{3\lambda}\beta_1\dot{\phi}
\end{aligned}
$$

$$K_5 = X_5^2 \frac{\partial \mathcal{L}}{\partial \dot{\beta}_2} + X_5^3 \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$= -3e^{3\lambda}\phi\dot{\beta}_2 + 3e^{3\lambda}\beta_2\dot{\phi}$$

$$K_6 = X_6^1 \frac{\partial \mathcal{L}}{\partial \dot{\beta}_1} + X_6^2 \frac{\partial \mathcal{L}}{\partial \dot{\beta}_2}$$

$$= 3e^{3\lambda}\beta_2\dot{\beta}_1 - 3e^{3\lambda}\dot{\beta}_2\beta_1,$$

where we denote by X_n^i the i-component of the n-vector. In practice we divide the first integrals by suitable numerical factors and we work with the simplified expressions for the first integrals. We observe that only five of the first integrals are independent.[8] In the case for which the potential V is not constant the general symmetry has the form

$$X = (c\beta_2 + c_0)\frac{\partial}{\partial \beta_1} + (-c\beta_1 + c_0')\frac{\partial}{\partial \beta_2} \tag{2.36}$$

with lift

$$X^T = X + c\dot{\beta}_2\frac{\partial}{\partial \dot{\beta}_1} - c\dot{\beta}_1\frac{\partial}{\partial \dot{\beta}_2}. \tag{2.37}$$

We obtain the symmetries

$$X_1^T = \frac{\partial}{\partial \beta_1}$$

$$X_2^T = \frac{\partial}{\partial \beta_2}$$

$$X_3^T = \beta_1\frac{\partial}{\partial \beta_2} - \beta_2\frac{\partial}{\partial \beta_1} + \dot{\beta}_1\frac{\partial}{\partial \dot{\beta}_2} - \dot{\beta}_2\frac{\partial}{\partial \dot{\beta}_1} \tag{2.38}$$

[8] We note that the constants are dependent since there is the relation $K_6 = (K_5K_1 - K_4K_2)/K_3$ and not the symmetries as is stated in [9]. Moreover we find forms for K_4 and K_5 different from the corresponding ones in that paper. Our expressions have the particular attraction of having a zero total derivative when the Euler-Lagrange equations are taken into account.

with Lie algebra

$$\left[X_1^T, X_3^T\right] = -X_2^T \qquad\qquad \left[X_2^T, X_3^T\right] = X_1^T.$$

The corresponding constants of motion, after the removal of constant multipliers, are

$$K_1 = e^{3\lambda}\dot{\beta}_1$$

$$K_2 = e^{3\lambda}\dot{\beta}_2$$

$$K_3 = e^{3\lambda}\left(\beta_2\dot{\beta}_1 - \beta_1\dot{\beta}_2\right).$$

Once one has found the first integrals one can substitute them into (2.16) and in some easy cases be able to achieve integration. The explicit performance of the procedure for finding exact solutions is not always possible. Most of the time it is very hard or impossible even in the cases where the potential is constant. For the Bianchi Type VII class A Capozziello *et al* obtained the symmetry

$$X = \frac{\partial}{\partial\beta_1} - \frac{3}{2}\frac{\partial}{\partial\beta_2} \tag{2.39}$$

which cannot be a Noether symmetry for $\mathcal{L}$ since $\frac{3}{2}\beta_1 + \beta_2$ is not an ignorable variable[9].

2.4 The exact integration.

In [9] Capozziello *et al* obtain the integrals for the Bianchi Types I and V in the case where the potential is zero.[10] They consider the restriction (2.16) which after the substitution of the constants of motion becomes

$$6\dot{\lambda}^2 - K^2e^{-6\lambda} - \alpha^2e^{-2\lambda} = 0, \tag{2.40}$$

[9]It is rarely appreciated that the standard integrals for Lagrangians which are autonomous and/or have ignorable coordinates are a consequence of Noether's Theorem in the case that the gauge function is taken as zero which is the situation in [9].

[10]Capozziello *et al* are referring in the case where the potential does not exist. We obtain this case as the second case in the following chapters.

where

$$K^2 = \frac{3}{2}\left(K_1^2 + K_2^2\right), \qquad \alpha^2 = 6a^2 \exp\left[\frac{2}{3a^2}\right]. \tag{2.41}$$

For Bianchi Type I $\alpha = 0$ and (2.40) gives the solution

$$\lambda = \frac{1}{3}\log\left[\sqrt{\frac{3}{2}}K(t - t_0)\right]. \tag{2.42}$$

Using the constants of motion we obtain the solution of the system. For Bianchi Type V after the reparametrisation $x = e^{2\lambda}$ equation (2.40) becomes[11]

$$\frac{3}{2}\dot{x}^2 - \frac{K^2}{x} - \alpha^2 = 0. \tag{2.43}$$

Whenever t is small, e^{λ} is proportional to $t^{1/2}$ and for large t it is proportional to $t^{1/3}$. The expression of K_3 gives

$$\phi = \sqrt{\frac{2}{3}}\frac{K_3}{K}\log(t - t_0), \tag{2.44}$$

so that for small t, ϕ is proportional to $t^{-1/2}$ and, for large t, ϕ is proportional to $\log t$.

2.5 The purpose of the current research

According to Marmo *et al* , the 'Noether approach' finds the Noether symmetries of the field equations for every model. In Chapter Four we present the integrable cases. With the assumption that there does not exist a potential, it is not hard to find the integrals as, for example, in Bianchi Types I and V [9]. The problem arises in cases where, although we have found sufficient number of Noether symmetries, the

[11] As we mentioned above, in this chapter we present the results as the reader would find them in the work of Capozziello *et al* . However, in Chapter Five we discuss the correct results.

exact integration is not straightforward even though we assume that the potential is constant as in Bianchi Type II [9]. Of course in cases for which the number of symmetries is fewer than the dimension of the system it is impossible to find the requisite number of integrals. This happens for the Bianchi Types $VIII$ and IX. An interesting idea which is the basis of this dissertation is to find the Lie and Noether symmetries of the field equations for every model. Our aim is to achieve completely integration in all these cases for which the number of symmetries is sufficient. As we observed in Chapter One, when one has found the Noether symmetries, it is an easy procedure to find an integral for every symmetry. Unfortunately in most cases the number of these integrals is insufficient to solve the problem even in the simple cases of models without matter. For this reason it is nessesary to obtain a new procedure which allows us to find more integrals. Hence we determine also the Lie symmetries of the system which allow us to find more integrals and, with the Noether ones, may give the solution of the problem.

Chapter 3

The Lie and Noether Symmetries of the Bianchi Models Possessing a Lagrangian

3.1 Introduction

In this chapter we work on the system of Euler-Lagrange equations which are derived from the Lagrangian (3.1). In the first section we give the transformations that we use in order to simplify the computations. We determine the Lie and the Noether symmetries for the Bianchi models by the method we have already described in Chapter One. We do not consider the cases of Bianchi Type IV and VII_h, the non-Lagrangian types since we examine those cases which are presented in [9]. We have four cases for every model: Case-1 in which there does not exist matter ϕ and potential V; Case-2 where there is matter but no potential; in Case-3 we have matter and a constant potential and Case-4, which is the most general since it involves matter and a general potential as a function of ϕ. We use Program LIE to find the Lie symmetries and we determine those which are Noetherian. In all cases we also find the algebra of the symmetries. We conclude with a discussion of the results that we have found. As the computations for each case are very similar, we simply summarise the results.

3.2 Transformations

The general Lagrangian has the form [9]

$$\mathcal{L} = e^{3\lambda}\left[R^* + 6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2 + 2V(\phi)\right]. \tag{3.1}$$

For Types IV and $VII_{h\neq 0}$ the Lagrangian is not known and so we do not consider them here. The Euler-Lagrange equations are

$$\begin{aligned}
&\ddot{\lambda} + \frac{3}{2}\dot{\lambda}^2 + \frac{3}{8}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) + \frac{1}{4}\dot{\phi}^2 - \frac{1}{12}e^{-3\lambda}\frac{\partial}{\partial\lambda}\left(e^{3\lambda}R^*\right) - \frac{1}{2}V(\phi) = 0 \\
&\ddot{\beta}_1 + 3\dot{\beta}_1\dot{\lambda} + \frac{1}{3}\frac{\partial R^*}{\partial \beta_1} = 0 \\
&\ddot{\beta}_2 + 3\dot{\beta}_2\dot{\lambda} + \frac{1}{3}\frac{\partial R^*}{\partial \beta_2} = 0 \\
&\ddot{\phi} + 3\dot{\phi}\frac{\dot{u}}{u} + V' = 0.
\end{aligned} \tag{3.2}$$

The Ricci scalar R^* has the form

CLASS A

$$\begin{aligned}
R^* = &-\frac{1}{2}e^{-2\lambda}\left[{N_1}^2 e^{4\beta_1} + e^{-2\beta_1}\left(N_2 e^{\sqrt{3}\beta_2} - N_3 e^{-\sqrt{3}\beta_2}\right)^2\right] \\
&-\frac{1}{2}e^{-2\lambda}\left[2N_1 e^{\beta_1}\left(N_2 e^{\sqrt{3}\beta_2} + N_3 e^{-\sqrt{3}\beta_2}\right)\right] + \frac{1}{2}N_1 N_2 N_3\left(1 + N_1 N_2 N_3\right).
\end{aligned} \tag{3.3}$$

$$R^{*}=2a^{2}e^{-2\lambda}\left(3-\frac{N_{2}N_{3}}{a^{2}}\right)e^{\beta} \tag{3.4}$$

with

$$\beta=\frac{2}{3a^{2}-N_{2}N_{3}}\left(N_{2}N_{3}\beta_{1}+\sqrt{-3a^{2}N_{2}N_{3}\beta_{2}}\right). \tag{3.5}$$

We set

$$u=e^{\lambda} \tag{3.6}$$

$$v=e^{\beta_{1}} \tag{3.7}$$

$$w=e^{\sqrt{3}\beta_{2}} \tag{3.8}$$

so that the system (3.2) is transformed to

$$\begin{aligned}
&\frac{\ddot{u}}{u}+\frac{1}{2}\frac{\dot{u}^{2}}{u}+\frac{3}{8}\left[\frac{\dot{v}^{2}}{v}+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^{2}\right]+\frac{1}{4}\dot{\phi}^{2}\\
&-\frac{1}{12}e^{-3\lambda}\frac{\partial}{\partial\lambda}\left(e^{3\lambda}R^{*}\right)-\frac{1}{2}V(\phi)=0\\
&\frac{\ddot{v}}{v}-\left(\frac{\dot{v}}{v}\right)^{2}+3\frac{\dot{u}}{u}\frac{\dot{v}}{v}+\frac{1}{3}\frac{\partial R^{*}}{\partial\beta_{1}}=0\\
&\frac{\ddot{w}}{w}-\left(\frac{\dot{w}}{w}\right)^{2}+3\frac{\dot{u}}{u}\frac{\dot{w}}{w}+\frac{\sqrt{3}}{3}\frac{\partial R^{*}}{\partial\beta_{2}}=0\\
&\ddot{\phi}+3\dot{\phi}\dot{\lambda}+V'=0
\end{aligned} \tag{3.9}$$

to be understood as being where R^{*} and its derivatives are expressed in terms of the new variables. We examine the Bianchi models in four cases: Case-1 where there is no matter and no potential. It is obvious that the terms of the potential and ϕ are missing in the u equation and also there is no ϕ equation. Hence we have a system of three equations with three dependent variables u, v and w. This is the easiest case of all the models and, once one has examined it, one is able to make inferences about the forms of the symmetries in the other cases. Case-2 no potential. In this case we have a system of four equations with four dependent variables u, v, w and ϕ. The term of the potential is missing in the u and ϕ equations. Case-3 with constant potential. The system has the same form as in the Case-2 plus the constant term of the potential in the u equation. Case-4 with arbitrary potential. It is the most general case. We have a system with four equations with four dependent variables. Moreover in the u equation there is the function of the potential and in the ϕ equation we have the derivative of the potential with respect to ϕ.

3.3 Bianchi Type I

$$\alpha = 0, \qquad N_1 = N_2 = N_3 = 0.$$

3.3.1 Case-1: no matter and no potential.

The Lagrangian is

$$\mathcal{L} = u^3 \left[6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]\right]. \tag{3.10}$$

From (3.10) we see that the Euler-Lagrange equations are

$$\begin{aligned} &\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] = 0 \\ &\frac{\ddot{v}}{v} + \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} = 0 \end{aligned} \tag{3.11}$$

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0.$$

If we use the command *DOINTCON* we obtain the symmetries

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t}$	
$G_3 = ut\frac{\partial}{\partial u} + \frac{3}{2}t^2\frac{\partial}{\partial t}$	
$G_4 = u\frac{\partial}{\partial u}$	
$G_5 = v\frac{\partial}{\partial v}$	$v\frac{\partial}{\partial v}$
$G_6 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_7 = v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$	$v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$

If we follow the usual procedure without the use of the command *DOINTCON*, then the machine gives us the symmetries in combination with the functions $F_{28}(v)$, $F_{45}(v)$, $F_{31}(w)$ and $F_{47}(w)$ which are determined from the unsolved system

$$-v\frac{\partial F_{28}(v)}{\partial v} + v\frac{\partial F_{45}(v)}{\partial v} + F_{28}(v) - F_{45}(v) = 0$$

$$-w\frac{\partial F_{31}(w)}{\partial w} + w\frac{\partial F_{47}(w)}{\partial w} + F_{31}(w) - F_{47}(w) = 0.$$

The solution of the system is

$$F_{28}(v) - F_{45}(v) = F_{95}v$$

$$F_{31}(w) - F_{41}(w) = F_{96}w,$$

where F_{95} and F_{96} are constants. We impose the solution on the program after the command $DOSOLV$ with the commands

$$EVSA\#(F\#(28, U2) - F(45, U2), \quad F\#(95) * U2)$$

$$EVSA\#(F\#(31, U3) - F(41, U3), \quad F\#(96) * U3)$$

where $F\#(95)$ and $F\#(96)$ are constants and we proceed to $DOCHECK$. The nonzero commutators are

$$\begin{aligned} [G_1, G_2'] &= G_1 & \qquad [G_2', G_3] &= 3G_3 \\ [G_1, G_3] &= 3G_2' & \qquad [G_5, G_7] &= -3G_6 \\ [G_6, G_7] &= G_5 & & \end{aligned}$$

with the algebra $sl(2,R) \oplus A_1 \oplus A_{3,6}$. We have set

$$G_2' = 3G_2 + G_4. \tag{3.12}$$

3.3.2 Case-2: No potential.

From the Lagrangian

$$\mathcal{L} = u^3 \left[6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 \right] \tag{3.13}$$

we have the following system

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{4}\dot{\phi}^2 = 0$$

$$\frac{\ddot{v}}{v} + \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} = 0$$

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0 \tag{3.14}$$

$$\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.$$

We obtain the symmetries with the command $DOINTCON$. We observe that, if we do not use the command $DOINTCON$, the machine returns the answer 'memory space exhausted'. Since in the first case the coefficient function of the $\partial/\partial t$ term is a polynomial at most of degree two in t, we impose this information with the command

$$EVSA\#(F\#(5, U1, U2, U3, U4, X1), \quad F\#(6) + F\#(7) * X_1 + F\#(8) * X_1^2),$$

where $F\#(6), F\#(7)$ and $F\#(8)$ are arbitrary constants. With this input we overcome the problem of exhaustion of memory and we obtain the system of unsolved equations

$$v\frac{\partial F_{53}(v)}{\partial v} - v\frac{\partial F_{55}(v)}{\partial v} + v\frac{\partial F_{60}(v)}{\partial v} - F_{53}(v) + F_{55}(v) - F_{60}(v) = 0$$

$$w\frac{\partial F_{68}(w)}{\partial w} - w\frac{\partial F_{70}(w)}{\partial w} + w\frac{\partial F_{72}(w)}{\partial w} - F_{68}(w) + F_{70}(w) - F_{72}(w) = 0.$$

We find the solution after the use of the commands

$$EVSA\#(F\#(53, U2) - F\#(55, U2) + F\#(60, U2), \quad F\#(98) * U2)$$

$$EVSA\#(F\#(68,U3) - F\#(70,U3) + F\#(72,U3), \quad F\#(99) * U3),$$

where $F\#(98)$ and $F\#(99)$ are arbitrary constants.

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t}$	
$G_3 = ut\frac{\partial}{\partial u} + \frac{3}{2}t^2\frac{\partial}{\partial t}$	
$G_4 = u\frac{\partial}{\partial u}$	
$G_5 = v\frac{\partial}{\partial v}$	$v\frac{\partial}{\partial v}$
$G_6 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_7 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$
$G_8 = v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$	$v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$
$G_9 = v\phi\frac{\partial}{\partial v} - \frac{3}{2}\log v\frac{\partial}{\partial \phi}$	$v\phi\frac{\partial}{\partial v} - \frac{3}{2}\log v\frac{\partial}{\partial \phi}$
$G_{10} = w\phi\frac{\partial}{\partial w} - \frac{1}{2}\log w\frac{\partial}{\partial \phi}$	$w\phi\frac{\partial}{\partial w} - \frac{1}{2}\log w\frac{\partial}{\partial \phi}.$

The nonzero brackets are

$$
\begin{aligned}
[G_1, G_2'] &= G_1 & [G_2', G_3] &= 3G_3 \\
[G_1, G_3] &= 3G_2' & [G_8, G_9] &= 3G_{10} \\
[G_8, G_{10}] &= -G_9 & [G_9, G_{10}] &= \tfrac{1}{2}G_8
\end{aligned}
$$

where G_2' is given above. The algebra is $4A_1 \oplus sl\,(2, R) \oplus SO(3)$.

3.3.3 Case-3: Constant potential.

The Lagrangian becomes

$$
\mathcal{L} = u^3 \left[6 \left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2} \left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3} \left(\frac{\dot{w}}{w}\right)^2 \right] - \dot{\phi}^2 + 2C \right] \tag{3.15}
$$

which gives the following system of Euler-Lagrange equations

$$
\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}C = 0 \\
&\frac{\ddot{v}}{v} + \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0 \\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.
\end{aligned} \tag{3.16}
$$

We use the command *DOINTCON* to obtain the symmetries. At this point one should be careful that, if one imposes on the program the information that the coefficient function of the $\partial/\partial t$ term is a polynomial of degree one and solves the new system, one looses the G_7[1], G_9 and G_{10} symmetries. The same happens if one sup-

[1]We note that G_7, even though it does not contain $\partial/\partial t$, is lost.

poses that the coefficient function of the $\partial/\partial u$ term is polynomial of second degree. Hence one should be very careful about which command one uses during the procedure of the computation of symmetries since there is the possibility to 'lose' some useful symmetries.

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = u\frac{\partial}{\partial u}$	
$G_3 = v\frac{\partial}{\partial v}$	$v\frac{\partial}{\partial v}$
$G_4 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_5 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$
$G_6 = v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$	$-v\log w\frac{\partial}{\partial v} + 3w\log v\frac{\partial}{\partial w}$
$G_7 = v\phi\frac{\partial}{\partial v} - \frac{3}{2}\log v\frac{\partial}{\partial \phi}$	$v\phi\frac{\partial}{\partial v} - \frac{3}{2}\log v\frac{\partial}{\partial \phi}$
$G_8 = w\phi\frac{\partial}{\partial w} - \frac{1}{2}\log w\frac{\partial}{\partial \phi}$	$w\phi\frac{\partial}{\partial w} - \frac{1}{2}\log w\frac{\partial}{\partial \phi}$
$G_9 = \exp[\sqrt{3}Ct]\frac{\partial}{\partial t} + \exp[\sqrt{3}Ct]u\frac{\partial}{\partial u}$	
$G_{10} = \frac{1}{\exp\left[\sqrt{3}Ct\right]}\frac{\partial}{\partial t} - \frac{u}{\exp\left[\sqrt{3}Ct\right]}\frac{\partial}{\partial u}$	

The nonzero brackets are

$$
\begin{aligned}
[G_6, G_7] &= 3G_8 & [G_7, G_8] &= 12G_6 \\
[G_6, G_8] &= -G_7 & [G_1, G_9] &= \sqrt{3}CG_9 \\
[G_9, G_{10}] &= -2\sqrt{3}G_1. & [G_1, G_{10}] &= -\sqrt{3}CG_{10}
\end{aligned}
$$

which is the algebra $SO(3) \oplus 4A_1 \oplus sl(2,R)$.

3.3.4 Case-4: Arbitrary potential.

In this case the Lagrangian is

$$
\mathcal{L} = u^3 \left[6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 + 2V(\phi) \right] \tag{3.17}
$$

which gives the system

$$
\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}V(\phi) = 0 \\
&\frac{\ddot{v}}{v} + \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0 \\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} + V' = 0.
\end{aligned} \tag{3.18}
$$

If we use the command $DOINTCON$, we obtain the symmetries

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = u\frac{\partial}{\partial u}$	
$G_3 = v\frac{\partial}{\partial v}$	$v\frac{\partial}{\partial v}$
$G_4 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_5 = v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$	$v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$

The nonzero brackets are

$$[G_4, G_5] = G_3, \qquad [G_3, G_5] = -3G_4$$

and the symmetries form the algebra $2A_1 \oplus A_{3,6}$.

3.4 Bianchi Type II

$$\alpha = 0, \qquad N_1 = 1, \qquad N_2 = 0, \qquad N_3 = 0.$$

3.4.1 Case-1: No matter and no potential.

The Lagrangian is

$$\mathcal{L} = u^3\left\{-\frac{1}{2}\frac{1}{u^2}v^4 + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]\right\}. \tag{3.19}$$

The Euler-Lagrange equations are

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{24}\frac{v^4}{u^2} = 0$$

$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} - \frac{2}{3}\frac{v^4}{u^2} = 0 \tag{3.20}$$

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0.$$

The symmetries of Bianchi Type II are

Lie symmetries	Noether Symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$	
$G_3 = -2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$	
$G_4 = -w\frac{\partial}{\partial w}$	$-w\frac{\partial}{\partial w}$

The non zero Lie brackets are

$$[G_1,\ G_3'] = G_1$$

where $G_3' = -G_2 - G_3$. The algebra is $2A_1 \oplus A_2$.

3.4.2 Case-2: No potential

The Lagrangian is

$$\mathcal{L} = u^3\left\{-\frac{1}{2}\frac{1}{u^2}v^4 + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2\right\}. \tag{3.21}$$

The Euler-Lagrange equations are

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{4}\dot{\phi}^2 + \frac{1}{24}\frac{v^4}{u^2} = 0$$

$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} - \frac{2}{3}\frac{v^4}{u^2} = 0$$

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0 \tag{3.22}$$

$$\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.$$

The symmetries are

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$	
$G_3 = -2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$	
$G_4 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_5 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$
$G_6 = w\phi\frac{\partial}{\partial w} + \frac{1}{2}\log w\frac{\partial}{\partial \phi}$	

The nonzero Lie brackets are

$$[G_5,\ G_6] = -G_4 \qquad [G_4,\ G_6] = \tfrac{1}{2}G_5$$

$$[G_1,\ G_2] = G_1 \qquad [G_1,\ G_3] = -2G_1.$$

which form the algebra $A_{3,6}\oplus\{A_2\oplus_s A_1\}$.

3.4.3 Case-3: Constant potential.

In this case the Lagrangian takes the form

$$\mathcal{L} = u^3\left\{-\frac{1}{2}\frac{1}{u^2}v^4 + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 + 2C\right\}. \tag{3.23}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}C + \frac{1}{24}\frac{v^4}{u^2} = 0\\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} - \frac{2}{3}\frac{v^4}{u^2} = 0\\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0\\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.
\end{aligned} \tag{3.24}$$

The command *DOINTCON* does not help in this case. Hence we use the command

$$EVSA\#(F\#(2, U1, U2, U3, U4, X1),\quad F\#(6) + F\#(7) * U2);$$

and we obtain the symmetries

Lie symmetries	Noether symmetries
$G_1 = \dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2 = w\dfrac{\partial}{\partial w}$	$w\dfrac{\partial}{\partial w}$
$G_3 = \dfrac{\partial}{\partial \phi}$	$\dfrac{\partial}{\partial \phi}$
$G_4 = w\phi\dfrac{\partial}{\partial w} + \frac{1}{2}\log w\dfrac{\partial}{\partial \phi}$	

The nonzero Lie brackets are

$$[G_2,\ G_4] = \frac{1}{2}G_3, \qquad [G_3,\ G_4] = G_2$$

which gives the algebra $A_{3,6} \oplus A_1$.

3.4.4 Case-4: Arbitrary potential.

The Lagrangian takes the form

$$\mathcal{L} = u^3\left\{c12\frac{1}{u^2}v^4 + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 + 2V(\phi)\right\} \tag{3.25}$$

The Euler-Lagrange equations are

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}V(\phi) + \frac{1}{24}\frac{v^4}{u^2} = 0$$

$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} - \frac{2}{3}\frac{v^4}{u^2} = 0 \tag{3.26}$$

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0$$

$$\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} + V' = 0.$$

The equations have the symmetries

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_3 = u\frac{\partial}{\partial u} + \frac{1}{2}v\frac{\partial}{\partial v}$	

We found the symmetries above after the use of the command

$$EVSA\#\,(F\#(5, U1, U2, U3, U4, X1), \quad F\#(6) + F\#(7) * X1)$$

$$EVSA\#\,(F\#(1, U1, U2, U3, U4, X1), \quad F\#(8) + F\#(9) * U1).$$

All the Lie brackets are zero. So the algebra is $3A_1$.

3.5 Bianchi Type III

$$\alpha = 1, \qquad N_1 = 0, \qquad N_2 = 1, \qquad N_3 = -1.$$

3.5.1 Case-1: No matter and no potential.

The Lagrangian takes the form

$$\mathcal{L} = u^3\left[8\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]\right]. \tag{3.27}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u}+\frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2+\frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]-\frac{2}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}}=0\\
&\frac{\ddot{v}}{v}-\left(\frac{\dot{v}}{v}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{v}}{v}-\frac{4}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}}=0\\
&\frac{\ddot{w}}{w}-\left(\frac{\dot{w}}{w}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{w}}{w}+4\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}}=0.
\end{aligned} \tag{3.28}$$

We use the *DOINTCON* command and obtain the symmetries

Lie symmetries	Noether Symmetries
$G_1=\frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2=t\frac{\partial}{\partial t}+4v\frac{\partial}{\partial v}$	
$G_3=u\frac{\partial}{\partial u}+4w\frac{\partial}{\partial w}$	
$G_4=u\frac{\partial}{\partial u}-4v\frac{\partial}{\partial v}$	

The only nonzero Lie bracket is

$$[G_1,G_2]=G_1$$

and the algebra is $A_2 \oplus 2A_1$.

3.5.2 Case-2: No potential

The Lagrangian

$$\mathcal{L} = u^3 \left[8\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 \right] \tag{3.29}$$

gives the equations

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \frac{2}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} + \frac{1}{4}\dot{\phi}^2 = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} - \frac{4}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} + 4\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} = 0 \\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.
\end{aligned} \tag{3.30}$$

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t} + 4v\frac{\partial}{\partial v}$	
$G_3 = u\frac{\partial}{\partial u} + 4w\frac{\partial}{\partial w}$	
$G_4 = u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v}$	
$G_5 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$
$G_6 = v\phi\frac{\partial}{\partial v} + w\phi\frac{\partial}{\partial w} - \frac{1}{2}(3\log v + \log w)\frac{\partial}{\partial \phi}$	

We found the symmetries with the command $DOINTCON$. The nonzero Lie brackets are

$$[G_1, G_2] = G_1 \qquad [X_3, G_6] = G_5 \qquad [X_4, G_6] = G_5$$

3.5.3 Case-3: Constant potential.

The Lagrangian

$$\mathcal{L} = u^3\left[8\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 + 2C\right] \quad . \tag{3.31}$$

gives the system

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \frac{2}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}C = 0$$

$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} - \frac{4}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} = 0$$

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} + 4\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} = 0 \tag{3.32}$$

$$\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.$$

If we use the command *DOINTCON*, we have the problem of exhaustion of memory. Since all of the coefficient functions of the previous case are linear, we overcome the problem with the use of the command *DOPOLYALL*(1) and obtain the symmetries

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = u\frac{\partial}{\partial u} + 4w\frac{\partial}{\partial w}$	
$G_3 = u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v}$	
$G_4 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$

All the brackets are zero and so the algebra is $4A_1$.

3.5.4 Case-4: Arbitrary potential.

In this case the Lagrangian is

$$\mathcal{L} = u^3\left[8\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 + 2V(\phi)\right]. \tag{3.33}$$

The Euler-Lagrange equations are

$$
\begin{aligned}
&\frac{\ddot{u}}{u}+\frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2+\frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]-\frac{2}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}}+\frac{1}{4}\dot{\phi}^2-\frac{1}{2}V(\phi)=0\\
&\frac{\ddot{v}}{v}-\left(\frac{\dot{v}}{v}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{v}}{v}+\frac{4}{3}\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}}=0\\
&\frac{\ddot{w}}{w}-\left(\frac{\dot{w}}{w}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{w}}{w}+4\frac{1}{u^2}\left(\frac{w}{v}\right)^{\frac{1}{2}}=0\\
&\ddot{\phi}+3\frac{\dot{u}}{u}\dot{\phi}+V'=0.
\end{aligned}
\tag{3.34}
$$

After the use of the same command as in the previous case we find that

Lie symmetries	Noether symmetries
$G_1=\frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2=u\frac{\partial}{\partial u}+4w\frac{\partial}{\partial w}$	
$G_3=u\frac{\partial}{\partial u}-4v\frac{\partial}{\partial v}$	

All the brackets are zero and so the algebra is $3A_1$.

3.6 Bianchi Type V

$$\alpha=1,\qquad N_1=0,\qquad N_2=0,\qquad N_3=0,\qquad b=0.$$

3.6.1 Case-1: No matter and no potential.

The Lagrangian takes the form

$$\mathcal{L} = u^3 \left\{ 6\frac{1}{u^2} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]\right\}. \tag{3.35}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 - \frac{1}{2}\frac{1}{u^2} + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0.
\end{aligned} \tag{3.36}$$

We use the *DOINTCON* command and obtain the symmetries

Lie symmetries	Noether Symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$	
$G_3 = v\frac{\partial}{\partial v}$	$v\frac{\partial}{\partial v}$
$G_4 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_5 = v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$	

The nonzero Lie brackets are

$$[G_1, G_2] = G_1 \qquad [G_3, G_5] = -3G_4 \qquad [G_4, G_5] = G_3$$

which form the algebra $A_2 \oplus A_{3,6}$.

3.6.2 Case-2: No potential

The Lagrangian is

$$\mathcal{L} = u^3 \left\{ 6\frac{1}{u^2} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 \right\}. \tag{3.37}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 - \frac{1}{2}\frac{1}{u^2} + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 + \frac{1}{4}\dot{\phi}^2\right] = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0 \\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.
\end{aligned} \tag{3.38}$$

We obtain the following symmetries

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$	
$G_3 = v\frac{\partial}{\partial v}$	$v\frac{\partial}{\partial v}$
$G_4 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_5 = v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$	
$G_6 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$
$G_7 = 2v\phi\frac{\partial}{\partial v} - 3\log v\frac{\partial}{\partial \phi}$	
$G_8 = 2w\phi\frac{\partial}{\partial w} - \log w\frac{\partial}{\partial \phi}$	

We have used the same command as in the previous case.

The nonzero brackets are

$$
\begin{aligned}
[G_3, G_5] &= -3G_4 & [G_4, G_5] &= G_3 \\
[G_3, G_7] &= -3G_6 & [G_4, G_8] &= -G_6 \\
[G_5, G_7] &= 3G_8 & [G_5, G_8] &= -G_7 \\
[G_6, G_7] &= 2G_3 & [G_6, G_8] &= 2G_4 \\
[G_1, G_2] &= G_1
\end{aligned}
$$

which form the algebra $A_2 \oplus \{A_1 \oplus_s \{2A_1 \oplus_s 3A_1\}\}$.

3.6.3 Case-3: Constant potential.

The Lagrangian is

$$
\mathcal{L} = u^3 \left\{ 6\frac{1}{u^2} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 + 2C \right\}. \tag{3.39}
$$

The Euler-Lagrange equations are

$$
\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 - \frac{1}{2}\frac{1}{u^2} + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}C\right] = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0 \\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.
\end{aligned} \tag{3.40}
$$

We use the command *DOINTCON* and obtain the symmetries

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = v\frac{\partial}{\partial v}$	$v\frac{\partial}{\partial v}$
$G_3 = w\frac{\partial}{\partial w}$	$w\frac{\partial}{\partial w}$
$G_4 = v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$	
$G_5 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$
$G_6 = v\phi\frac{\partial}{\partial v} - \frac{3}{2}\log v\frac{\partial}{\partial \phi}$	
$G_7 = w\phi\frac{\partial}{\partial w} - \frac{1}{2}\log w\frac{\partial}{\partial \phi}$	

We used the command *DOINTCON*.

The nonzero Lie brackets are

$$
\begin{array}{rclrcl}
[G_2, G_4] & = & -3G_3, & [G_3, G_4] & = & G_2 \\
[G_7, G_6] & = & -\frac{3}{2}G_5, & [G_3, G_7] & = & -\frac{1}{2}G_5 \\
[G_4, G_6] & = & 3G_7, & [G_4, G_7] & = & -G_6 \\
[G_4, G_7] & = & -G_6, & [G_5, G_6] & = & G_2 \\
[G_5, G_7] & = & G_3, & [G_6, G_7] & = & -\frac{1}{2}G_4.
\end{array}
$$

3.6.4 Case-4: Arbitrary potential.

The Lagrangian has the form

$$\mathcal{L} = u^3 \left\{ 6\frac{1}{u^2} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 + 2V(\phi) \right\}. \tag{3.41}$$

The system of equations is

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 - \frac{1}{2}\frac{1}{u^2} + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}V\right] = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} = 0 \\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} + V' = 0.
\end{aligned} \tag{3.42}$$

If we use the same command as in the previous cases, we have the problem of exhaustion of memory. We have used combinations of commands in order to find all the symmetries. Hence, if we impose on the program the information that the coefficient function of $\partial/\partial t$ is linear in t, then we lose the G_1 symmetry. If we impose the information that the coefficient function of the $\partial/\partial v$ term is linear in v, we lose the G_4 symmetry. We obtain the complete results by combining the partial results.

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = v\frac{\partial}{\partial v}$	
$G_3 = w\frac{\partial}{\partial w}$	
$G_4 = v\ln w\frac{\partial}{\partial v} - 3w\ln v\frac{\partial}{\partial w}$	

The nonzero Lie brackets are

$$[G_2, G_4] = -3G_3, \qquad [G_3, G_4] = G_2$$

and the algebra is $A_1 \oplus A_{3,6}$.

3.7 Bianchi Type VI: class A

$$\alpha = 0, \qquad N_1 = 1, \qquad N_2 = -1, \qquad N_3 = 0.$$

3.7.1 Case-1: No matter and no potential.

The Lagrangian takes the form

$$\mathcal{L} = u^3\left\{-\frac{1}{2u^2}\left[v^4 + \frac{w^2}{v^2} + 2vw\right] + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]\right\}. \tag{3.43}$$

The Euler-Lagrange equations are

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{24}\frac{1}{u^2}\left[v^4 + \left(\frac{w}{v}\right)^2 + 2vw\right] = 0$$

$$\frac{\ddot{v}}{v}-\left(\frac{\dot{v}}{v}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{v}}{v}+\frac{1}{3}\frac{1}{u^2}\left[-2v^4+\left(\frac{w}{v}\right)^2-vw\right]=0 \tag{3.44}$$

$$\frac{\ddot{w}}{w}-\left(\frac{\dot{w}}{w}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{w}}{w}-\frac{1}{u^2}\left[\left(\frac{w}{v}\right)^2+vw\right]=0.$$

We use the command *DOPOLYALL*(1) and obtain the symmetries

Lie symmetries	Noether Symmetries
$G_1=\frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2=t\frac{\partial}{\partial t}-\frac{1}{2}v\frac{\partial}{\partial v}-\frac{3}{2}w\frac{\partial}{\partial w}$	
$G_3=u\frac{\partial}{\partial u}+\frac{1}{2}v\frac{\partial}{\partial v}+\frac{3}{2}w\frac{\partial}{\partial w}$	

The only nonzero bracket is

$$[G_1,\ G_2]=G_1$$

and the algebra is $A_2 \oplus A_1$.

3.7.2 Case-2: No potential

The Lagrangian is

$$\mathcal{L}=u^3\left\{-\frac{1}{2u^2}\left[v^4+\frac{w^2}{v^2}+2vw\right]+6\left(\frac{\dot{u}}{u}\right)^2-\frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]-\dot{\phi}^2\right\}. \tag{3.45}$$

The Euler-Lagrange equations are

$$\frac{\ddot{u}}{u}+\frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2+\frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]+\frac{1}{24}\frac{1}{u^2}\left[v^4+\left(\frac{w}{v}\right)^2+2vw\right]+\frac{1}{4}\dot{\phi}^2=0$$

$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} + \frac{1}{3}\frac{1}{u^2}\left[-2v^4 + \left(\frac{w}{v}\right)^2 - vw\right] = 0$$

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} - \frac{1}{u^2}\left[\left(\frac{w}{v}\right)^2 + vw\right] = 0 \qquad (3.46)$$

$$\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.$$

We use the same command as in the previous case and obtain the symmetries

Lie symmetries	Noether symmetries
$G_1 = \dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2 = t\dfrac{\partial}{\partial t} - \frac{1}{2}v\dfrac{\partial}{\partial v} - \frac{3}{2}w\dfrac{\partial}{\partial w}$	
$G_3 = u\dfrac{\partial}{\partial u} + \frac{1}{2}v\dfrac{\partial}{\partial v} + \frac{3}{2}w\dfrac{\partial}{\partial w}$	
$G_4 = \dfrac{\partial}{\partial \phi}$	$\dfrac{\partial}{\partial \phi}$

We find the symmetries after the use of the command

$$EVSA\#(F\#(5,U1,U2,U3,U4,X1), \quad F\#(6) + F\#(7) * X1).$$

The nonzero Lie bracket is

$$[G_1, G_2] = G_1$$

and so the algebra is $A_2 \oplus 2A_1$.

3.7.3 Case-3 Constant potential.

In this case the Lagrangian is

$$\mathcal{L} = u^3 \left\{ \frac{1}{2u^2} \left[v^4 + \frac{w^2}{v^2} + 2vw \right] + 6 \left(\frac{\dot{u}}{u} \right)^2 - \frac{3}{2} \left[\left(\frac{\dot{v}}{v} \right)^2 + \frac{1}{3} \left(\frac{\dot{w}}{w} \right)^2 \right] - \dot{\phi}^2 + 2C \right\} \tag{3.47}$$

which gives the following equations

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2} \left(\frac{\dot{u}}{u} \right)^2 + \frac{3}{8} \left[\left(\frac{\dot{v}}{v} \right)^2 + \frac{1}{3} \left(\frac{\dot{w}}{w} \right)^2 \right] + \frac{1}{24} \frac{1}{u^2} \left[v^4 + \left(\frac{w}{v} \right)^2 + 2vw \right] + \frac{1}{4} \dot{\phi}^2 - \frac{1}{2} C = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v} \right)^2 + 3 \frac{\dot{u}}{u} \frac{\dot{v}}{v} + \frac{1}{3} \frac{1}{u^2} \left[-2v^4 + \left(\frac{w}{v} \right)^2 - vw \right] = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w} \right)^2 + 3 \frac{\dot{u}}{u} \frac{\dot{w}}{w} - \frac{1}{u^2} \left[\left(\frac{w}{v} \right)^2 + vw \right] = 0 \\
&\ddot{\phi} + 3 \frac{\dot{u}}{u} \dot{\phi} = 0.
\end{aligned} \tag{3.48}$$

After the use of the same commands in Program LIE as in the previous case we find the following symmetries

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = u\frac{\partial}{\partial u} + \frac{1}{2}v\frac{\partial}{\partial v} + \frac{3}{2}w\frac{\partial}{\partial w}$	
$G_3 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$

There is one nonzero Lie bracket

$$[G_1, G_2] = G_1$$

and the algebra is $A_2 \oplus A_1$.

3.7.4 Case-4: Arbitrary potential.

The Lagrangian is

$$\mathcal{L} = u^3 \left\{ -\frac{1}{2u^2} \left[v^4 + \frac{w^2}{v^2} + 2vw \right] + 6 \left(\frac{\dot{u}}{u} \right)^2 \right.$$

$$\left. -\frac{3}{2} \left[\left(\frac{\dot{v}}{v} \right)^2 + \frac{1}{3} \left(\frac{\dot{w}}{w} \right)^2 \right] - \dot{\phi}^2 + 2V(\phi) \right\}. \tag{3.49}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2} \left(\frac{\dot{u}}{u} \right)^2 + \frac{3}{8} \left[\left(\frac{\dot{v}}{v} \right)^2 + \frac{1}{3} \left(\frac{\dot{w}}{w} \right)^2 \right] + \frac{1}{24} \frac{1}{u^2} \left[v^4 + \left(\frac{w}{v} \right)^2 + 2vw \right] + \frac{1}{4} \dot{\phi}^2 - \frac{1}{2} V(\phi) = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v} \right)^2 + 3 \frac{\dot{u}}{u} \frac{\dot{v}}{v} + \frac{1}{3} \frac{1}{u^2} \left[-2v^4 + \left(\frac{w}{v} \right)^2 - vw \right] = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w} \right)^2 + 3 \frac{\dot{u}}{u} \frac{\dot{w}}{w} - \frac{1}{u^2} \left[\left(\frac{w}{v} \right)^2 + vw \right] = 0 \\
&\ddot{\phi} + 3 \frac{\dot{u}}{u} \dot{\phi} + V' = 0.
\end{aligned} \tag{3.50}$$

The system has the symmetries

Lie symmetries	Noether symmetries
$G_1 = \dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2 = u\dfrac{\partial}{\partial u} + \tfrac{1}{2}v\dfrac{\partial}{\partial v} + \tfrac{3}{2}w\dfrac{\partial}{\partial w}$	

We have used the command $DOPOLYALL(1)$. The algebra is $2A_1$ since the Lie bracket of G_1 and G_2 is zero.

3.8 Bianchi Type VI: class B

$$\alpha = a, \qquad N_1 = 0, \qquad N_2 = 1, \qquad N_3 = -1, \qquad b = 0.$$

3.8.1 Case-1: No matter and no potential.

The Lagrangian is

$$\mathcal{L} = u^3\left\{6\frac{1}{u^2} + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 - \frac{3a^2+1}{6u^2}\left(\frac{w^a}{v}\right)^{\frac{2}{3a^2+1}}\right]\right\}. \tag{3.51}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \frac{3a^2+1}{6u^2}\left(\frac{w^a}{v}\right)^{\frac{2}{3a^2+1}} = 0\\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}\dot{v}}{uv} - \frac{4}{3u^2}\left(\frac{w^a}{v}\right)^{\frac{2}{3a^2+1}} = 0\\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}\,\dot{w}}{u\,w} + \frac{4a}{u^2}\left(\frac{w^a}{v}\right)^{\frac{2}{3a^2+1}} = 0.
\end{aligned} \tag{3.52}$$

If we apply Program LIE to this system, a number of difficulties appears. The term $(w^a/v)^{\frac{2}{3a^2+1}}$ causes problems for the procedure to find the symmetries. For this reason we introduce a new reparametrisation in order to simplify the equations so that we can obtain results. We set

$$\jmath = w^{\frac{2a}{3a^2+1}}, \qquad\qquad n = v^{\frac{2}{3a^2+1}}. \tag{3.53}$$

The system of equations (3.52) is transformed to

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left(\frac{3a^2+1}{2}\right)^2 \left[\left(\frac{\dot{n}}{n}\right)^2 + \frac{1}{3a^2}\left(\frac{\dot{\jmath}}{\jmath}\right)^2\right] - \frac{(3a^2+1)}{6u^2}\frac{\jmath}{n} = 0 \\
&\frac{(3a^2+1)}{2}\frac{\ddot{n}}{n} - \left(\frac{\dot{n}}{n}\right)^2 + \frac{3(3a^2+1)}{2}\frac{\dot{u}\dot{n}}{un} - \frac{4}{3u^2}\frac{\jmath}{n} = 0 \\
&\frac{(3a^2+1)}{2}\frac{\ddot{\jmath}}{\jmath} - \left(\frac{\dot{\jmath}}{\jmath}\right)^2 + \frac{3(3a^2+1)}{2}\frac{\dot{u}\dot{\jmath}}{u\jmath} + \frac{4a^2}{u^2}\frac{\jmath}{n} = 0.
\end{aligned} \tag{3.54}$$

We can manipulate the last two equations and the system becomes

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left(\frac{3a^2+1}{2}\right)^2 \left[\left(\frac{\dot{n}}{n}\right)^2 + \frac{1}{3a^2}\left(\frac{\dot{\jmath}}{\jmath}\right)^2\right] - \frac{(3a^2+1)}{6u^2}\frac{\jmath}{n} = 0 \\
&\frac{\ddot{n}}{n} - \left(\frac{\dot{n}}{n}\right)^2 + 3\frac{\dot{u}\dot{n}}{un} - \frac{8}{3(3a^2+1)u^2}\frac{\jmath}{n} = 0 \\
&\frac{\ddot{\jmath}}{\jmath} - \left(\frac{\dot{\jmath}}{\jmath}\right)^2 + 3\frac{\dot{u}\dot{\jmath}}{u\jmath} + \frac{8a^2}{(3a^2+1)u^2}\frac{\jmath}{n} = 0.
\end{aligned} \tag{3.55}$$

We can further simplify the system by introducing the two constants

$$B = \frac{3a^2+1}{2}, \qquad\qquad C = \frac{1}{3a^2}. \tag{3.56}$$

Eventually the system takes the form

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}B^2\left[\left(\frac{\dot{n}}{n}\right)^2 + C\left(\frac{\dot{\jmath}}{\jmath}\right)^2\right] + \frac{1}{4}\dot{\phi}^2 - \frac{B}{3u^2}\frac{\jmath}{n} = 0$$

$$\frac{\ddot{n}}{n} - \left(\frac{\dot{n}}{n}\right)^2 + 3\frac{\dot{u}\dot{n}}{un} - \frac{4}{3Bu^2}\frac{\jmath}{n} = 0 \tag{3.57}$$

$$\frac{\ddot{\jmath}}{\jmath} - \left(\frac{\dot{\jmath}}{\jmath}\right)^2 + 3\frac{\dot{u}\dot{\jmath}}{u\jmath} + \frac{4}{3BCu^2}\frac{\jmath}{n} = 0.$$

We use the command *DOINTCON* and obtain the symmetries

Lie symmetries	Noether Symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$	
$G_3 = -\frac{1}{2}u\frac{\partial}{\partial u} + n\frac{\partial}{\partial n}$	
$G_4 = \frac{1}{2}u\frac{\partial}{\partial u} + \jmath\frac{\partial}{\partial \jmath}$	

The only nonzero bracket is $[G_1, G_2] = G_1$ and so the algebra is $A_2 \oplus 2A_1$.

3.8.2 Case-2: No potential

The Lagrangian becomes

$$\mathcal{L} = u^3\left\{-\frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 - \frac{3a^2+1}{6u^2}\left(\frac{w^a}{v}\right)^{\frac{2}{3a^2+1}}\right]\right.$$

$$\left. + 6\frac{1}{u^2} + 6\left(\frac{\dot{u}}{u}\right)^2 + \frac{1}{4}\dot{\phi}^2\right\}. \tag{3.58}$$

We obtain the following system of equations

$$
\begin{aligned}
&\frac{\ddot{u}}{u}+\frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2+\frac{3}{8}B^2\left[\left(\frac{\dot{n}}{n}\right)^2+C\left(\frac{\dot{\jmath}}{\jmath}\right)^2\right]-\frac{B}{3u^2}\frac{\jmath}{n}+\frac{1}{4}\dot{\phi}^2=0\\
&\frac{\ddot{n}}{n}-\left(\frac{\dot{n}}{n}\right)^2+3\frac{\dot{u}\dot{n}}{un}-\frac{4}{3Bu^2}\frac{\jmath}{n}=0\\
&\frac{\ddot{\jmath}}{\jmath}-\left(\frac{\dot{\jmath}}{\jmath}\right)^2+3\frac{\dot{u}\dot{\jmath}}{u\jmath}+\frac{4}{3BCu^2}\frac{\jmath}{n}=0\\
&\ddot{\phi}+3\frac{\dot{u}}{u}\dot{\phi}=0
\end{aligned}
\tag{3.59}
$$

which has the symmetries

Lie symmetries	Noether Symmetries
$G_1 = \dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2 = t\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial u}$	
$G_3 = -\frac{1}{2}u\dfrac{\partial}{\partial u} + n\dfrac{\partial}{\partial n}$	
$G_4 = \frac{1}{2}u\dfrac{\partial}{\partial u} + \mathrm{J}\frac{\partial}{\partial \mathrm{J}}$	
$G_5 = \dfrac{\partial}{\partial \phi}$	$\dfrac{\partial}{\partial \phi}$
$G_6 = n\phi\dfrac{\partial}{\partial n} + \mathrm{J}\phi\dfrac{\partial}{\partial \mathrm{J}}$ $-\frac{3}{2}B^2 C \log \mathrm{J}\dfrac{\partial}{\partial \phi} - \dfrac{3}{2}B^2 \log n\dfrac{\partial}{\partial \phi}$	

The command *DOINTCON* does not help us in this case. Hence we used the commands

$$EVSA\#(F\#(1,U1,U2,U3,U4,X1), \quad F\#(6)+F\#(7)*U1) \ .$$

$$EVSA\#(F\#(5,U1,U2,U3,U4,X1), \quad F\#(8)+F\#(9)*X1).$$

The nonzero brackets are

$$[G_1,G_2] = G_1 \qquad [G_3,G_6] = -\tfrac{3}{2}B^2G_5$$

$$[G_4,G_6] = -\tfrac{3}{2}B^2CG_5 \qquad [G_5,G_6] = G_3+G_4$$

which form the algebra $A_1 \oplus A_2 \oplus A_{3,6}$.

3.8.3 Case-3: Constant potential.

In this case the Lagrangian has the form

$$\mathcal{L} = u^3 \left\{ -\frac{3}{2}\left[\left(\frac{\dot v}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot w}{w}\right)^2 - \frac{3a^2+1}{6u^2}\left(\frac{w^a}{v}\right)^{\frac{2}{3a^2+1}} \right] \right.$$

$$\left. + 6\frac{1}{u^2} + 6\left(\frac{\dot u}{u}\right)^2 + \frac{1}{4}\dot\phi^2 + 2C \right\}. \qquad (3.60)$$

We obtain the system

$$\begin{aligned}
&\frac{\ddot u}{u} + \frac{1}{2}\left(\frac{\dot u}{u}\right)^2 + \frac{3}{8}B^2\left[\left(\frac{\dot n}{n}\right)^2 + C\left(\frac{\dot \jmath}{\jmath}\right)^2\right] - \frac{B}{3u^2}\frac{\jmath}{n} + \frac{1}{4}\dot\phi^2 - \frac{1}{2}C = 0 \\
&\frac{\ddot n}{n} - \left(\frac{\dot n}{n}\right)^2 + 3\frac{\dot u \dot n}{un} - \frac{4}{3Bu^2}\frac{\jmath}{n} = 0 \\
&\frac{\ddot \jmath}{\jmath} - \left(\frac{\dot \jmath}{\jmath}\right)^2 + 3\frac{\dot u \dot \jmath}{u\jmath} + \frac{4}{3BCu^2}\frac{\jmath}{n} = 0 \\
&\ddot\phi + 3\frac{\dot u}{u}\dot\phi = 0.
\end{aligned} \qquad (3.61)$$

We impose on Program LIE the information that the coefficient functions of the $\partial/\partial t$ and $\partial/\partial u$ terms are linear in t and u respectively and we obtain the results

Lie symmetries	Noether symmetries
$G_1 = \dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2 = -\frac{1}{2}u\dfrac{\partial}{\partial u} + n\dfrac{\partial}{\partial n}$	
$G_3 = \frac{1}{2}u\dfrac{\partial}{\partial u} + \jmath\dfrac{\partial}{\partial \jmath}$	
$G_4 = \partial\phi$	$\partial\phi$
$G_5 = n\phi\dfrac{\partial}{\partial n} + \jmath\phi\dfrac{\partial}{\partial \jmath} - \dfrac{3}{2}B^2C\log j\dfrac{\partial}{\partial\phi} - \dfrac{3}{2}B^2\log n\dfrac{\partial}{\partial\phi}$	

The nonzero brackets are

$$\begin{array}{rclcrcl} [G_2, G_4] & = & -G_2 - G_3 & & [G_2, G_5] & = & -\frac{3}{2}B^2G_4 \\ [G_4, G_5] & = & G_2 + G_3 & & [G_3, G_5] & = & -\frac{3}{2}B^2CG_4 \end{array}$$

so that the algebra is $2A_1 \oplus \{A_1 \oplus_s 2A_1\}$.

3.8.4 Case-4: Arbitrary potential.

The Lagrangian is

$$\mathcal{L} = u^3\left\{-\frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 - \frac{3a^2+1}{6u^2}\left(\frac{w^a}{v}\right)^{\frac{2}{3a^2+1}}\right]\right.$$

$$\left. + 6\frac{1}{u^2} + 6\left(\frac{\dot{u}}{u}\right)^2 + \frac{1}{4}\dot{\phi}^2 + 2V(\phi)\right\} \tag{3.62}$$

and gives the system

$$
\begin{aligned}
&\frac{\ddot{u}}{u}+\frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2+\frac{3}{8}B^2\left[\left(\frac{\dot{n}}{n}\right)^2+C\left(\frac{\dot{\jmath}}{\jmath}\right)^2\right]-\frac{B}{3u^2}\frac{\jmath}{n}+\frac{1}{4}\dot{\phi}^2-\frac{1}{2}V(\phi)=0\\
&\frac{\ddot{n}}{n}-\left(\frac{\dot{n}}{n}\right)^2+3\frac{\dot{u}\dot{n}}{un}-\frac{4}{3Bu^2}\frac{\jmath}{n}=0\\
&\frac{\ddot{\jmath}}{\jmath}-\left(\frac{\dot{\jmath}}{\jmath}\right)^2+3\frac{\dot{u}\dot{\jmath}}{u\jmath}+\frac{4}{3BCu^2}\frac{\jmath}{n}=0\\
&\ddot{\phi}+3\frac{\dot{u}}{u}\dot{\phi}+\frac{\partial V}{\partial\phi}=0.
\end{aligned}
\tag{3.63}
$$

We use the same command as in the previous case and obtain the symmetries

Lie symmetries	Noether symmetries
$G_1=\dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2=-\frac{1}{2}u\dfrac{\partial}{\partial u}+n\dfrac{\partial}{\partial n}$	
$G_3=\frac{1}{2}u\dfrac{\partial}{\partial u}+\jmath\dfrac{\partial}{\partial \jmath}$	

with the algebra $3A_1$

3.9 Bianchi VII

$$\alpha=0,\qquad N_1=1,\qquad N_2=1,\qquad N_3=0.$$

3.9.1 Case-1: No matter and no potential.

The Lagrangian takes the form

$$\mathcal{L} = u^3 \left\{ -\frac{1}{2u^2}\left[v^4 + \frac{w^2}{v^2} - 2vw\right] + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]\right\}. \tag{3.64}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{24}\frac{1}{u^2}\left[v^4 + \left(\frac{w}{v}\right)^2 - 2vw\right] = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} + \frac{1}{3}\frac{1}{u^2}\left[-2v^4 + \left(\frac{w}{v}\right)^2 + vw\right] = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} - \frac{1}{u^2}\left[\left(\frac{w}{v}\right)^2 - vw\right] = 0.
\end{aligned} \tag{3.65}$$

We use the $DOINTCON$ command and find the symmetries

Lie symmetries	Noether Symmetries
$G_1 = \dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2 = t\dfrac{\partial}{\partial t} - \frac{1}{2}v\dfrac{\partial}{\partial v} - \frac{3}{2}w\dfrac{\partial}{\partial w}$	
$G_3 = u\dfrac{\partial}{\partial u} + \frac{1}{2}v\dfrac{\partial}{\partial v} + \frac{3}{2}w\dfrac{\partial}{\partial w}$	

There is only one nonzero bracket,

$$[G_1, G_2] = G_1,$$

so that the algebra is $A_2 \oplus A_1$. If we impose on the program the information that the coefficient function of the $\partial/\partial t$ term or of the $\partial/\partial u$ term is linear, then we obtain the symmetries G_1 and $G_2 + G_3$. Hence we lose one symmetry. The same occurs in the subsequent cases.

3.9.2 Case-2: No potential

The Lagrangian is

$$\mathcal{L} = u^3\left\{-\frac{1}{2u^2}\left[v^4 + \frac{w^2}{v^2} - 2vw\right] + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 - \dot{\phi}^2\right]\right\}. \tag{3.66}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{24}\frac{1}{u^2}\left[v^4 + \left(\frac{w}{v}\right)^2 - 2vw\right] + \frac{1}{4}\dot{\phi}^2 = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{u}}{u} + \frac{1}{3}\frac{1}{u^2}\left[-2v^4 + \left(\frac{w^2}{v}\right)^2 + vw\right] = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} - \frac{1}{u^2}\left[\left(\frac{w}{v}\right)^2 - vw\right] = 0 \\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.
\end{aligned} \tag{3.67}$$

We use the command *DOINTCON* and obtain the symmetries

Lie symmetries	Noether symmetries
$G_1 = \frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2 = t\frac{\partial}{\partial t} - \frac{1}{2}v\frac{\partial}{\partial v} - \frac{3}{2}w\frac{\partial}{\partial w}$	
$G_3 = u\frac{\partial}{\partial u} + \frac{1}{2}v\frac{\partial}{\partial v} + \frac{3}{2}w\frac{\partial}{\partial w}$	
$G_4 = \frac{\partial}{\partial \phi}$	$\frac{\partial}{\partial \phi}$

In this case the algebra is $A_2 \oplus 2A_1$ as the only nonzero Lie bracket is $[G_1, G_2] = G_1$.

3.9.3 Case-3: Constant potential.

The Lagrangian

$$\mathcal{L} = u^3 \left\{ \frac{1}{2u^2}\left[v^4 + \frac{w^2}{v^2} - 2vw\right] + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 - \dot{\phi}^2\right] + 2C \right\} \tag{3.68}$$

gives the following Euler-Lagrange equations

$$\begin{aligned}
&\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{24}\frac{1}{u^2}\left[v^4 + \left(\frac{w}{v}\right)^2 - 2vw\right] + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}C = 0 \\
&\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{u}}{u} + \frac{1}{3}\frac{1}{u^2}\left[-2v^4 + \left(\frac{w}{v}\right)^2 + vw\right] = 0 \\
&\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} - \frac{1}{u^2}\left[\left(\frac{w}{v}\right)^2 - vw\right] = 0 \\
&\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0.
\end{aligned} \tag{3.69}$$

The symmetries of the system are

Lie symmetries	Noether symmetries
$G_1 = \dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2 = u\dfrac{\partial}{\partial u} + \frac{1}{2}v\dfrac{\partial}{\partial v} + \frac{3}{2}w\dfrac{\partial}{\partial w}$	
$G_3 = \dfrac{\partial}{\partial \phi}$	$\dfrac{\partial}{\partial \phi}$

We used the command $DOPOLYALL(1)$. There is only one nonzero Lie bracket

$$[G_1, G_2] = G_1.$$

Hence the algebra is $A_2 \oplus A_1$.

3.9.4 Case-4: Arbitrary potential.

In that case the Lagrangian takes the form

$$\mathcal{L} = u^3 \left\{ -\frac{1}{2u^2}\left[v^4 + \frac{w^2}{v^2} + 2vw\right] + 6\left(\frac{\dot{u}}{u}\right)^2 \right.$$
$$\left. - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2 - \dot{\phi}^2\right] + 2V(\phi) \right\}. \tag{3.70}$$

The Euler-Lagrange equations are

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] + \frac{1}{24}\frac{1}{u^2}\left[v^4 + \left(\frac{w}{v}\right)^2 - 2vw\right] + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}V(\phi) = 0$$

$$
\begin{aligned}
&\frac{\ddot v}{v}-\left(\frac{\dot v}{v}\right)^2+3\frac{\dot u}{u}\frac{\dot u}{u}+\frac{1}{3}\frac{1}{u^2}\left[-2v^4+\left(\frac{w}{v}\right)^2+vw\right]=0\\
&\frac{\ddot w}{w}-\left(\frac{\dot w}{w}\right)^2+3\frac{\dot u}{u}\frac{\dot w}{w}-\frac{1}{u^2}\left[\left(\frac{w}{v}\right)^2-vw\right]=0\\
&\ddot\phi+3\frac{\dot u}{u}\dot\phi+V'=0.
\end{aligned}
\tag{3.71}
$$

We use the same command as in previous case and we find the symmetries

Lie symmetries	Noether symmetries
$G_1=\frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2=u\frac{\partial}{\partial u}+\frac{1}{2}v\frac{\partial}{\partial v}+\frac{3}{2}w\frac{\partial}{\partial w}$	

The algebra is $2A_1$.

3.10 Bianchi Type $VIII$

$$\alpha=0,\qquad N_1=1,\qquad N_2=1,\qquad N_3=-1.$$

3.10.1 Case-1: No matter and no potential.

The Lagrangian is

$$
\begin{aligned}
\mathcal{L}=u^3\Bigg\{&-\frac{1}{2u^2}\left[v^4+\frac{1}{v^2}\left(w+\frac{1}{w}\right)^2-2v\left(w-\frac{1}{w}\right)\right]\\
&+6\left(\frac{\dot u}{u}\right)^2-\frac{3}{2}\left[\left(\frac{\dot v}{v}\right)^2+\frac{1}{3}\left(\frac{\dot w}{w}\right)^2\right]\Bigg\}.
\end{aligned}
\tag{3.72}
$$

The Euler-Lagrange equations are

$$
\begin{aligned}
&\frac{\ddot{u}}{u}+\frac{1}{2}\left(\frac{\dot{u}}{u}\right)^{2}+\frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^{2}+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^{2}\right]+\frac{1}{24}\frac{1}{u^{2}}\left[v^{4}+\frac{1}{v^{2}}\left(w+\frac{1}{w}\right)^{2}-2v\left(w-\frac{1}{w}\right)\right]=0\\
&\frac{\ddot{v}}{v}-\left(\frac{\dot{v}}{v}\right)^{2}+3\frac{\dot{u}}{u}\frac{\dot{v}}{v}+\frac{1}{3}\frac{1}{u^{2}}\left[-2v^{4}+\frac{1}{v^{2}}\left(w+\frac{1}{w}\right)^{2}+v\left(w-\frac{1}{w}\right)\right]=0\\
&\frac{\ddot{w}}{w}-\left(\frac{\dot{w}}{w}\right)^{2}+3\frac{\dot{u}}{u}\frac{\dot{w}}{w}-\frac{1}{u^{2}}\left[\frac{1}{v^{2}}\left(w^{2}-\frac{1}{w^{2}}\right)-v\left(w+\frac{1}{w}\right)\right]=0.
\end{aligned}
\tag{3.73}
$$

The symmetries of the system are

Lie symmetries	Noether Symmetries
$G_1=\frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$G_2=t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}$	

There is only one nonzero Lie bracket,

$$[G_1,G_2]=G_1,$$

and the algebra is A_2.

3.10.2 Case-2: No potential

The Lagrangian is

$$\mathcal{L}=u^{3}\left\{-\frac{1}{2u^{2}}\left[v^{4}+\frac{1}{v^{2}}\left(w+\frac{1}{w}\right)^{2}-2v\left(w-\frac{1}{w}\right)\right]\right.$$

$$+6\left(\frac{\dot{u}}{u}\right)^2-\frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]-\dot{\phi}^2\Big\}. \tag{3.74}$$

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\ddot{u}}{u}+\frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2+\frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]\\
&\quad+\frac{1}{24}\frac{1}{u^2}\left[v^4+\frac{1}{v^2}\left(w+\frac{1}{w}\right)^2-2v\left(w-\frac{1}{w}\right)\right]+\frac{1}{4}\dot{\phi}^2=0\\
&\frac{\ddot{v}}{v}-\left(\frac{\dot{v}}{v}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{v}}{v}+\frac{1}{3}\frac{1}{u^2}\left[-2v^4+\frac{1}{v^2}\left(w+\frac{1}{w}\right)^2+v\left(w-\frac{1}{w}\right)\right]=0\\
&\frac{\ddot{w}}{w}-\left(\frac{\dot{w}}{w}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{w}}{w}-\frac{1}{u^2}\left[\frac{1}{v^2}\left(w^2-\frac{1}{w^2}-v\left(w+\frac{1}{w}\right)\right)\right]=0\\
&\ddot{\phi}+3\frac{\dot{u}}{u}\dot{\phi}=0.
\end{aligned} \tag{3.75}$$

The symmetries that we find are

Lie symmetries	Noether symmetries
$G_1=\dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2=t\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial u}$	
$G_3=\dfrac{\partial}{\partial \phi}$	$\dfrac{\partial}{\partial \phi}$

There is one nonzero Lie bracket, *viz.*

$$[G_1,G_2]=G_1,$$

and the algebra is $A_2 \oplus A_1$.

3.10.3 Case-3: Constant potential.

In this case the Lagrangian becomes

$$\mathcal{L} = u^3 \left\{ -\frac{1}{2u^2} \left[v^4 + \frac{1}{v^2} \left(w + \frac{1}{w} \right)^2 - 2v \left(w - \frac{1}{w} \right) \right] \right.$$
$$\left. + 6 \left(\frac{\dot{u}}{u} \right)^2 - \frac{3}{2} \left[\left(\frac{\dot{v}}{v} \right)^2 + \frac{1}{3} \left(\frac{\dot{w}}{w} \right)^2 \right] - \dot{\phi}^2 + C \right\}. \tag{3.76}$$

The resulting Euler-Lagrange equations are

$$\frac{\ddot{u}}{u} + \frac{1}{2} \left(\frac{\dot{u}}{u} \right)^2 + \frac{3}{8} \left[\left(\frac{\dot{v}}{v} \right)^2 + \frac{1}{3} \left(\frac{\dot{w}}{w} \right)^2 \right]$$
$$+ \frac{1}{24} \frac{1}{u^2} \left[v^4 + \frac{1}{v^2} \left(w + \frac{1}{w} \right)^2 - 2v \left(w - \frac{1}{w} \right) \right] + \frac{1}{4} \dot{\phi}^2 - \frac{1}{2} C = 0$$
$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v} \right)^2 + 3 \frac{\dot{u}}{u} \frac{\dot{v}}{v} + \frac{1}{3} \frac{1}{u^2} \left[-2v^4 + \frac{1}{v^2} \left(w + \frac{1}{w} \right)^2 + v \left(w - \frac{1}{w} \right) \right] = 0$$
$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w} \right)^2 + 3 \frac{\dot{u}}{u} \frac{\dot{w}}{w} - \frac{1}{u^2} \left[\frac{1}{v^2} \left(w^2 - \frac{1}{w^2} - v \left(w + \frac{1}{w} \right) \right) \right] = 0$$
$$\ddot{\phi} + 3 \frac{\dot{u}}{u} \dot{\phi} = 0. \tag{3.77}$$

The system has only two symmetries

Lie symmetries	Noether symmetries
$G_1 = \dfrac{\partial}{\partial t}$	$\dfrac{\partial}{\partial t}$
$G_2 = \partial\phi$	$\dfrac{\partial}{\partial\phi}$

The Lie bracket is zero and so the algebra is $2A_1$.

3.10.4 Case-4: Arbitrary potential.

From the Lagrangian

$$\mathcal{L} = u^3\left\{-\frac{1}{2u^2}\left[v^4 + \frac{1}{v^2}\left(w + \frac{1}{w}\right)^2 - 2v\left(w - \frac{1}{w}\right)\right]\right.$$

$$\left. + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2 + 2V(\phi)\right\} \tag{3.78}$$

we obtain the Euler-Lagrange equations

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]$$

$$+ \frac{1}{24}\frac{1}{u^2}\left[v^4 + \frac{1}{v^2}\left(w + \frac{1}{w}\right)^2 - 2v\left(w - \frac{1}{w}\right)\right] + \frac{1}{4}\dot{\phi}^2 - \frac{1}{2}V = 0$$

$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} + \frac{1}{3}\frac{1}{u^2}\left[-2v^4 + \frac{1}{v^2}\left(w + \frac{1}{w}\right)^2 + v\left(w - \frac{1}{w}\right)\right] = 0 \tag{3.79}$$

$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} - \frac{1}{u^2}\left[\frac{1}{v^2}\left(w^2 - \frac{1}{w^2} - v\left(w + \frac{1}{w}\right)\right)\right] = 0$$

$$\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} + V' = 0. \tag{3.80}$$

The only symmetry is

$$G=\frac{\partial}{\partial t}$$

which is also a Noether symmetry. In all cases we used the command $DOPOLYALL(1)$.

3.11 Bianchi Type IX.

$$\alpha=0,\qquad N_1=1,\qquad N_2=1,\qquad N_3=1.$$

3.11.1 Case-1: No matter and no potential.

The Lagrangian has the form

$$\mathcal{L}=u^3\left\{-\frac{1}{2u^2}\left[v^4+\frac{1}{v^2}\left(w-\frac{1}{w}\right)^2-2v\left(w+\frac{1}{w}\right)\right]+1\right.$$
$$\left.+6\left(\frac{\dot u}{u}\right)^2-\frac{3}{2}\left[\left(\frac{\dot v}{v}\right)^2+\frac{1}{3}\left(\frac{\dot w}{w}\right)^2\right]\right\}. \tag{3.81}$$

The Euler-Lagrange equations are

$$\frac{\ddot u}{u}+\frac{1}{2}\left(\frac{\dot u}{u}\right)^2+\frac{3}{8}\left[\left(\frac{\dot v}{v}\right)^2+\frac{1}{3}\left(\frac{\dot w}{w}\right)^2\right]+\frac{1}{24}\frac{1}{u^2}\left[v^4+\frac{1}{v^2}\left(w-\frac{1}{w}\right)^2-2v\left(w+\frac{1}{w}\right)\right]=0$$

$$\frac{\ddot v}{v}-\left(\frac{\dot v}{v}\right)^2+3\frac{\dot u}{u}\frac{\dot v}{v}+\frac{1}{3}\frac{1}{u^2}\left[-2v^4+\frac{1}{v^2}\left(w-\frac{1}{w}\right)^2+v\left(w+\frac{1}{w}\right)\right]=0$$

$$\frac{\ddot w}{w}-\left(\frac{\dot w}{w}\right)^2+3\frac{\dot u}{u}\frac{\dot w}{w}-\frac{1}{u^2}\left[\frac{1}{v^2}\left(w^2-\frac{1}{w^2}\right)-v\left(w-\frac{1}{w}\right)\right]=0. \tag{3.82}$$

The system has the single symmetry

$$G=\frac{\partial}{\partial t}$$

which is also a Noether symmetry.

3.11.2 Case-2: No potential

The Lagrangian is

$$\mathcal{L} = u^3\left\{-\frac{1}{2u^2}\left[v^4 + \frac{1}{v^2}\left(w - \frac{1}{w}\right)^2 - 2v\left(w + \frac{1}{w}\right)\right] + 1\right.$$
$$\left. + 6\left(\frac{\dot{u}}{u}\right)^2 - \frac{3}{2}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right] - \dot{\phi}^2\right\}. \tag{3.83}$$

The Euler-Lagrange equations are

$$\frac{\ddot{u}}{u} + \frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2 + \frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2 + \frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]$$
$$+ \frac{1}{24}\frac{1}{u^2}\left[v^4 + \frac{1}{v^2}\left(w - \frac{1}{w}\right)^2 - 2v\left(w + \frac{1}{w}\right)\right] - \frac{1}{4}\dot{\phi}^2 = 0$$
$$\frac{\ddot{v}}{v} - \left(\frac{\dot{v}}{v}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{v}}{v} + \frac{1}{3}\frac{1}{u^2}\left[-2v^4 + \frac{1}{v^2}\left(w - \frac{1}{w}\right)^2 + v\left(w + \frac{1}{w}\right)\right] = 0$$
$$\frac{\ddot{w}}{w} - \left(\frac{\dot{w}}{w}\right)^2 + 3\frac{\dot{u}}{u}\frac{\dot{w}}{w} - \frac{1}{u^2}\left[\frac{1}{v^2}\left(w^2 - \frac{1}{w^2}\right) - v\left(w - \frac{1}{w}\right)\right] = 0$$
$$\ddot{\phi} + 3\frac{\dot{u}}{u}\dot{\phi} = 0. \tag{3.84}$$

In this case we find the additional symmetry

$$G = \frac{\partial}{\partial\phi}$$

which is also a Noether symmetry.

3.11.3 Case-3: Constant potential.

In this case the Lagrangian is

$$\mathcal{L} = u^3\left\{-\frac{1}{2u^2}\left[v^4 + \frac{1}{v^2}\left(w - \frac{1}{w}\right)^2 - 2v\left(w + \frac{1}{w}\right)\right] + 1\right.$$

$$+\,6\left(\frac{\dot u}{u}\right)^2-\frac{3}{2}\left[\left(\frac{\dot v}{v}\right)^2+\frac{1}{3}\left(\frac{\dot w}{w}\right)^2\right]-\dot\phi^2+2C\Bigg\}. \tag{3.85}$$

The Euler-Lagrange equations are

$$\frac{\ddot u}{u}+\frac{1}{2}\left(\frac{\dot u}{u}\right)^2+\frac{3}{8}\left[\left(\frac{\dot v}{v}\right)^2+\frac{1}{3}\left(\frac{\dot w}{w}\right)^2\right]$$

$$+\frac{1}{24}\frac{1}{u^2}\left[v^4+\frac{1}{v^2}\left(w-\frac{1}{w}\right)^2-2v\left(w+\frac{1}{w}\right)\right]-\frac{1}{4}\dot\phi^2-\frac{1}{2}C=0$$

$$\frac{\ddot v}{v}-\left(\frac{\dot v}{v}\right)^2+3\frac{\dot u}{u}\frac{\dot v}{v}+\frac{1}{3}\frac{1}{u^2}\left[-2v^4+\frac{1}{v^2}\left(w-\frac{1}{w}\right)^2+v\left(w+\frac{1}{w}\right)\right]=0$$

$$\frac{\ddot w}{w}-\left(\frac{\dot w}{w}\right)^2+3\frac{\dot u}{u}\frac{\dot w}{w}-\frac{1}{u^2}\left[\frac{1}{v^2}\left(w^2-\frac{1}{w^2}\right)-v\left(w-\frac{1}{w}\right)\right]=0$$

$$\ddot\phi+3\frac{\dot u}{u}\dot\phi=0. \tag{3.86}$$

The system has the same symmetry as above.

3.11.4 Case-4: Arbitrary potential.

From the Lagrangian

$$\mathcal{L}=u^3\Bigg\{-\frac{1}{2u^2}\left[v^4+\frac{1}{v^2}\left(w-\frac{1}{w}\right)^2-2v\left(w+\frac{1}{w}\right)\right]+1$$

$$+\,6\left(\frac{\dot u}{u}\right)^2-\frac{3}{2}\left[\left(\frac{\dot v}{v}\right)^2+\frac{1}{3}\left(\frac{\dot w}{w}\right)^2\right]-\dot\phi^2+2V(\phi)\Bigg\} \tag{3.87}$$

we obtain the following equations

$$\frac{\ddot{u}}{u}+\frac{1}{2}\left(\frac{\dot{u}}{u}\right)^2+\frac{3}{8}\left[\left(\frac{\dot{v}}{v}\right)^2+\frac{1}{3}\left(\frac{\dot{w}}{w}\right)^2\right]$$

$$+\frac{1}{24}\frac{1}{u^2}\left[v^4+\frac{1}{v^2}\left(w-\frac{1}{w}\right)^2-2v\left(w+\frac{1}{w}\right)\right]-\frac{1}{4}\dot{\phi}^2-\frac{1}{2}V=0$$

$$\frac{\ddot{v}}{v}-\left(\frac{\dot{v}}{v}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{v}}{v}+\frac{1}{3}\frac{1}{u^2}\left[-2v^4+\frac{1}{v^2}\left(w-\frac{1}{w}\right)^2+v\left(w+\frac{1}{w}\right)\right]=0$$

$$\frac{\ddot{w}}{w}-\left(\frac{\dot{w}}{w}\right)^2+3\frac{\dot{u}}{u}\frac{\dot{w}}{w}-\frac{1}{u^2}\left[\frac{1}{v^2}\left(w^2-\frac{1}{w^2}\right)-v\left(w-\frac{1}{w}\right)\right]=0$$

$$\ddot{\phi}+3\frac{\dot{u}}{u}\dot{\phi}+V'=0. \qquad (3.88)$$

In this case we obtain the single symmetry

$$G=\frac{\partial}{\partial t}$$

which is also a Noether symmetry. In all cases we used the command $DOPOLYALL(1)$.

3.12 Discussion

In this chapter we found the Lie and the Noether symmetries for the Bianchi universes. According to the presence of matter and a potential we had four cases for every model. In order to obtain the Lie symmetries we used Program LIE. In most of the cases we had to use specific commands for every case for every model. Case-1, where there is neither matter nor potential, is the easiest case since we have a system of three equations with three variables. The unsolved equations that we obtain during the application of Program LIE are the reason for the selection of the specific commands. The types of symmetries that we obtain in the first case give us the opportunity to 'guess' the type of the symmetries in the next cases. So we choose the commands in such a way so that the symmetries that we expect to find should agree with the symmetries in the first case.

All the models have in common the Noether symmetry, $\partial/\partial t$, since the system of equations is autonomous. Most of the models in the second case have in common the symmetry $\partial/\partial\phi$. The symmetries that we usually obtain in Cases-2 and -3 are combinations of the symmetries in the previous two cases. Only the Bianchi Type I in the case where the potential is constant gives a symmetry which involves the constant potential. This does not occur in the other models. The model VI in class B causes some problems when we use Program LIE. In order to avoid the difficulties we presented a new reparametrisation which allowed us to find the symmetries. For the models Bianchi Types $VIII$ and IX we obtained only the symmetries $\partial/\partial t$ and $\partial/\partial\phi$ as we expected.

Chapter 4

The Lie and Noether integrals of the Bianchi models

4.1 Introduction

In this chapter we give the integrals which we obtained from the symmetries for the Bianchi Types I, III and V. In particular for the Bianchi Type III we derive a new reparametrisation in order to simplify the calculation. We give the detailed procedure which leads us to the solution of the problem. We note that the expressions for the integrals are almost the same for the last two cases. We obtain the integrals for the three cases where either we do not have a potential or it is a constant. In the last section of this chapter we describe a method whereby, for an additional symmetry, we obtain the form for the potential. With the introduction of the potential, $V(\phi)$, we lose the $t\partial/\partial t$ symmetry. We seek to recover a related symmetry by the introduction of a term which counteracts the symmetry breaking effect of the potential. Hence we are able by the use of symmetry methods to obtain a subclass of potentials which satisfy the system of Euler-Lagrange equations. For this potential the symmetries must have a specific form.

4.2 Bianchi type I.

4.2.1 Case 1: No matter and no potential.

We use the symmetries in order to obtain the integrals using the method that we have already described in the first chapter. Hence we have

Symmetry	Integral
$G_4 = v\frac{\partial}{\partial v}$	$I_1 = u^3\frac{\dot{v}}{v}$
$G_5 = w\frac{\partial}{\partial w}$	$I_2 = u^3\frac{\dot{w}}{w}$
$G_7 = v\log w\frac{\partial}{\partial v} - 3w\log v\frac{\partial}{\partial w}$	$I_3 = \frac{1}{2}(\log u)^3\frac{\dot{u}^2}{u} - f_1(u)$

where

$$f_1(u) = \frac{1}{16}\left(I_1^2 + I_2^2\right)\left[(\log u)^3 + \frac{1}{2}(\log u)^2 + \frac{1}{6}\log u + \frac{1}{36}\right]u^{-6}. \tag{4.1}$$

Using the I_3, one of the integrals associated with G_7, we find

$$\dot{u} = \sqrt{g_1(u)}$$

$$\Longleftrightarrow \frac{du}{dt} = \sqrt{g_1(u)}$$

$$\Longleftrightarrow t - t_0 = \int\frac{du}{\sqrt{g_1(u)}}, \tag{4.2}$$

where

$$g_1(u) = \frac{2u^2 I_3}{(\log u)^3} + \frac{2u^2 f_1(u)}{(\log u)^3}. \tag{4.3}$$

Formally we invert (4.2) to find $u(t)$. The variables v and w follow from the integrals I_1 and I_2 respectively. Hence we have

$$v = \exp\left[\int \frac{I_1}{u^3} dt\right]$$

$$w = \exp\left[\int \frac{I_2}{u^3} dt\right].$$

We note that in this case we find the solution of the system using only three integrals since they are separable integrals.

4.2.2 Case-2: No potential.

We set

$$\alpha = \frac{2}{3K^{\frac{1}{2}}} \tag{4.4}$$

$$\beta = \left[\frac{2}{3}\frac{I_3^2}{K^2}\left(K - 6I_4^2\right)\right]^{\frac{1}{2}} \tag{4.5}$$

$$C = \frac{2I_3I_4}{K} \tag{4.6}$$

$$K = \frac{I_1^2 + I_2^2}{I_3^2} \tag{4.7}$$

and we obtain the following integrals

Symmetry	Integral
$G_5 = v\dfrac{\partial}{\partial v}$	$I_1 = u^3\dfrac{\dot{v}}{v}$
$G_6 = w\dfrac{\partial}{\partial w}$	$I_2 = u^3\dfrac{\dot{w}}{w}$
$G_7 = \dfrac{\partial}{\partial \phi}$	$I_3 = u^3\dot{\phi}$
$G_8 = v\log w\dfrac{\partial}{\partial v} - 3w\log v\dfrac{\partial}{\partial w}$	$I_4 = \dfrac{u^3}{I_3}\left[\dfrac{\dot{u}^2}{u} - \dfrac{1}{4}I_1 - \dfrac{1}{6}I_3^2u^{-6}\right]$
	$I_5 = t - \alpha\mathrm{arcsinh}\dfrac{u^3+C}{\beta}$

We note that the expression of the I_5 integral may have *arccosh* instead of *arcsinh* as we find after the calculations of the Lie integrals. We have

$$u = [\beta\sinh(t - I_5) - C]^{\frac{1}{3}} \tag{4.8}$$

$$\begin{aligned} v &= \exp\left[\int \frac{I_1}{u^3}dt\right] \\ &= \exp\left[I_1\int \frac{1}{[\beta\sinh(t-I_5) - C]}dt\right] \\ &= \exp\left[I_1\frac{1}{\sqrt{\beta^2+C^2}}\log\left(\frac{\beta\tanh\dfrac{t-I_5}{2} - \beta + \sqrt{\beta^2+C^2}}{-C\tanh\dfrac{t-I_5}{2} - \beta - \sqrt{\beta^2+C^2}}\right)\right] \end{aligned} \tag{4.9}$$

$$w = \exp\left[\int \frac{I_2}{u^3}dt\right]$$

$$= \exp\left[I_2 \int \frac{1}{[\beta \sinh(t - I_5) - C]} dt\right]$$

$$= \exp\left[I_2 \frac{1}{\sqrt{\beta^2 + C^2}} \log\left(\frac{\beta \tanh\frac{t - I_5}{2} - \beta + \sqrt{\beta^2 + C^2}}{-C \tanh\frac{t - I_5}{2} - \beta - \sqrt{\beta^2 + C^2}}\right)\right]. \quad (4.10)$$

4.2.3 Case 3: Constant potential.

The integrals have the same form as in previous case. The only difference is the expression for β_1 in which we have the extra term due to the potential

$$\beta = \left[\frac{2}{3} \frac{I_3^2}{K^2} \left(K - 6 I_4^2\right) - \frac{C}{2K}\right]^{\frac{1}{2}}. \quad (4.11)$$

4.3 Bianchi III.

For this model we use a new reparametrisation in order to simplify the computation to find the integrals. We obtain the results via a Riccati transformation. The integrals in Case-2 and Case-3 have essentially the same expressions.

4.3.1 Case 1: No matter and no potential.

Considering the form of the equations for this Type we set

$$\alpha = -2 \log u - \frac{1}{2} \log v + \frac{1}{2} \log w \quad (4.12)$$

$$\beta = \sqrt{3} \log v + \frac{1}{\sqrt{3}} \log w \quad (4.13)$$

$$\gamma = \log u. \quad (4.14)$$

Under the new reparametrisation the symmetry $G_3 = v\partial/\partial v + w\partial/\partial w$ transforms to $G = \partial/\partial\beta$ with associated Lagrange's system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}\alpha}{0} = \frac{\mathrm{d}\beta}{1} = \frac{\mathrm{d}\gamma}{0} = \frac{\mathrm{d}\dot{\alpha}}{0} = \frac{\mathrm{d}\dot{\beta}}{0} = \frac{\mathrm{d}\dot{\gamma}}{0}. \tag{4.15}$$

The characteristics of the system are

$$p = t, \quad u = \alpha, \quad w = \gamma, \quad x = \dot{\alpha}, \quad y = \dot{\beta}, \quad z = \dot{\gamma}. \tag{4.16}$$

Using these characteristics we obtain the system for the integrals

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}u}{x} = \frac{\mathrm{d}w}{z} = \frac{\mathrm{d}x}{\frac{3}{4}(x^2+y^2) - 4e^u} = \frac{\mathrm{d}y}{-3xy}$$

$$= \frac{\mathrm{d}z}{-\frac{3}{8}(x^2+y^2+4xz+8z^2)+\frac{2}{3}e^u}. \tag{4.17}$$

We have

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}y}{-3xy}$$

$$\Longleftrightarrow I_1 = e^{3\alpha}\dot{\beta} \tag{4.18}$$

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}x}{\frac{3}{4}(x^2+y^2) - 4e^u}$$

$$\Longleftrightarrow I_2 = \frac{\dot{\alpha}^2}{2\alpha^{\frac{3}{2}}} - f(\alpha) \tag{4.19}$$

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}u}{x}$$

$$I_3 = t - \int \frac{\mathrm{d}\alpha}{\dot{\alpha}}, \tag{4.20}$$

where

$$f(\alpha) = \frac{3}{4} I_1^2 \int \alpha^{-\frac{3}{2}} e^{-6\alpha} d\alpha + 4 \int \alpha^{-\frac{3}{2}} e^{\alpha} d\alpha. \tag{4.21}$$

We note that $\beta(t)$ follows from the quadrature of (4.18). The associated Lagrange's system for the symmetry $G_1 = \partial/\partial t$ is

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{0} = \frac{\mathrm{d}\gamma}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{0} \tag{4.22}$$

which gives the characteristics

$$u = \alpha, \quad v = \beta, \quad w = \gamma, \quad x = \dot{\alpha}, \quad y = \dot{\beta}, \quad z = \dot{\gamma}. \tag{4.23}$$

From the system

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}v}{y} = \frac{\mathrm{d}w}{z} = \frac{\mathrm{d}x}{\frac{3}{4}(x^2+y^2) - 4e^u} = \frac{\mathrm{d}y}{-3yx}$$

$$= \frac{\mathrm{d}z}{-\frac{3}{8}(x^2+y^2+4xz+8z^2) + \frac{2}{3}e^u} \tag{4.24}$$

we have

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}z}{-\frac{3}{8}(x^2+y^2+4xz+8z^2) + \frac{2}{3}e^u}$$

$$0 = \frac{\mathrm{d}z}{\mathrm{d}u} + \frac{3z^2}{x} + \frac{3}{2}z + \frac{3}{8}\frac{x^2+y^2}{x} - \frac{2}{3}\frac{e^u}{x} \tag{4.25}$$

in which we already have $x(u)$ and $y(u)$ from (4.19) and (4.18) respectively. We note that (4.25) is a Riccati equation. We use the generalised Riccati transformation [4]

$$z = f\frac{J'(u)}{J(u)}, \tag{4.26}$$

to obtain the second order equation

$$0 = f\frac{\jmath''}{\jmath} - fu\frac{\jmath'^2}{\jmath} + f'\frac{\jmath'}{\jmath} + \frac{3}{x}f^2\frac{\jmath'^2}{\jmath} + \frac{3}{8}\frac{x^2+y^2}{x} - \frac{2}{3}\frac{e^u}{x}. \tag{4.27}$$

We set

$$f = \frac{x}{3} \tag{4.28}$$

to remove the y'^2 terms and eventually obtain

$$0 = \jmath'' + \left(\frac{x'}{x} + \frac{3}{2}\right)\jmath' + \left(\frac{9}{8}\frac{x^2+y^2}{x^2} - 2\frac{e^u}{x}\right)\jmath \tag{4.29}$$

which is linear second order equation in $\jmath$ as a function of u. So we obtain $\dot{\gamma}$ as a function of α.

$$\frac{du}{x} = \frac{dw}{z}$$

$$\Longleftrightarrow \quad I_4 = \gamma - \int \frac{\dot{\gamma}}{\dot{\alpha}} da. \tag{4.30}$$

Hence we obtain

$$t - t_0 = \int \frac{da}{\sqrt{g(a)}}$$

$$\beta = \int \frac{I_2}{e^{3\alpha}} dt, \tag{4.31}$$

so that this case is formally reduced to quadrature and the solution of a linear second order differential equation[1].

[1]Although this is not quite us good as reduction to quadrature, there is no possibility of chaos as the second order ordinary differential equation is essentially that of an oscillator with time

4.3.2 Case 2: No potential.

In this case the integrals have the same form. The only difference is that γ depends on ϕ. From the symmetry $G_1 = \partial/\partial t$ we obtain the extra integral

$$I_4 = e^{3\gamma}\dot{\phi} \tag{4.32}$$

which we can solve to find

$$\phi = \int I_4 e^{-3\gamma} dt. \tag{4.33}$$

4.3.3 Case 3: Constant potential.

We obtain exactly the same integrals as in the previous case. We note that the second derivative of γ depends on the constant potential.

4.4 Bianchi V.

We find the Lie first integrals for the Bianch V model in the three cases. We note that for the first case the integrals have the same expression as the integrals in the Bianchi Type I with a different expression for the function f_1 and hence a different expression for the function g_1.

4.4.1 Case 1: No matter and no potential.

The integrals are

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dependent spring constant. As this is related to the autonomous oscillator by means of a time dependent rescaling transformation (Leach 1978), different trajectories are rescaled which does not represent the sensitive dependence upon the initial conditions required by chaos. The argument is due to Feix (private communication, Jan 1997) in response to the suggestions of the possibility of chaos in a time dependent linear oscillator by Lewis and Bouquet (private communication, also Jan 1997).

Symmetries	Integrals
$G_1 = \dfrac{\partial}{\partial t}$	$I_1 = u^3 \dfrac{\dot{v}}{v}$
	$I_2 = u^3 \dfrac{\dot{w}}{w}$
$G_7 = v\dfrac{\partial}{\partial v}$	$I_3 = \frac{1}{2}(\log u)^3 \dfrac{\dot{u}^2}{u} - f_1(u)$

where

$$f_1(u) = \frac{1}{4}\left[(\log u)^3 + \frac{3(\log u)^2}{2} + \frac{3\log u}{2} + \frac{3}{4}\right] u^{-2}$$

$$- \frac{1}{48}\left(3I_1^2 + I_2^2\right)\left[(\log u)^3 + \frac{1}{2}(\log u)^2 + \frac{1}{6}\log u + \frac{1}{36}\right] u^{-6}. \tag{4.34}$$

Using the integral I_3 we find

$$\dot{u} = \sqrt{g_1(u)}$$

$$\Longleftrightarrow \frac{du}{dt} = \sqrt{g_1(u)}$$

$$\Longleftrightarrow t - t_0 = \int \frac{du}{\sqrt{g_1(u)}}, \tag{4.35}$$

where

$$g_1(u) = \frac{2u^2 I_3}{(\log u)^3} + \frac{2u^2 f(u)}{(\log u)^3}. \tag{4.36}$$

From the inversion of (4.35) we find $u(t)$. The variables v and w follow from the integrals I_1 and I_2 respectively. Hence we have

$$v = \exp\left[\int \frac{I_1}{u^3} dt\right] \tag{4.37}$$

$$w = \exp\left[\int \frac{I_2}{u^3} dt\right]. \tag{4.38}$$

We note that we find the solution of the system using only three integrals since they are separable.

4.4.2 Case 2: No potential.

The integrals are

Symmetries	Integrals
$G_1 = \frac{\partial}{\partial t}$	$I_1 = u^3 \frac{\dot{v}}{v}$
	$I_2 = u^3 \frac{\dot{w}}{w}$
	$I_3 = u^3 \dot{\phi}$
$G_7 = v\frac{\partial}{\partial v}$	$I_4 = \frac{1}{2}(\log u)^3 \frac{\dot{u}^2}{u} - f_2(u)$

where

$$f_2(u) = \frac{1}{4}\left[(\log u)^3 + \frac{3(\log u)^2}{2} + \frac{3\log u}{2} + \frac{3}{4}\right] u^{-2}$$

$$-\frac{1}{48}\left(3I_1^2+I_2^2+2I_3^2\right)\left[(\log u)^3+\frac{1}{2}(\log u)^2+\frac{1}{6}\log u+\frac{1}{36}\right]u^{-6}. \quad (4.39)$$

Using the integral I_4 we obtain that

$$t-t_0=\int\frac{du}{\sqrt{g_2(u)}}, \quad (4.40)$$

where

$$g_2(u)=\frac{2u^2I_3}{(\log u)^3}+\frac{2u^2f_2(u)^3}{(\log u)}. \quad (4.41)$$

From the integrals I_1 and I_2 we obtain

$$v=\exp\left[\int\frac{I_1}{u^3}dt\right] \quad (4.42)$$

$$w=\exp\left[\int\frac{I_2}{u^3}dt\right]. \quad (4.43)$$

Furthermore using I_3 we obtain

$$\phi=\exp\left[\int\frac{I_3}{u^3}dt\right]. \quad (4.44)$$

Hence this case is reduced to quadratures.

4.4.3 Case 3: Constant potential.

The integrals have exactly the same form as in the previous case with a new function $f_3(u)$ instead of $f_2(u)$. We have

$$f_3(u)=\frac{1}{4}\left[(\log u)^3+\frac{3(\log u)^2}{2}+\frac{3\log u}{2}+\frac{3}{4}\right]u^{-2}+\frac{1}{8}C(\log u)^4$$

$$-\frac{1}{48}\left(3I_1^2+I_2^2+2I_3^2\right)\left[(\log u)^3+\frac{1}{2}(\log u)^2+\frac{1}{6}\log u+\frac{1}{36}\right]u^{-6}. \tag{4.45}$$

4.5 Counteracting the symmetry breaking potential.

In this section we present the Case 4 where the potential is a general function of ϕ for all the models in the both classes. In particular we describe the Bianchi Type I where the Ricci scalar is zero. We obtain the system of Euler-Lagrange equations

$$\ddot{\lambda}+\frac{3}{2}\dot{\lambda}^2+\frac{3}{8}\left(\dot{\beta}_1^2+\dot{\beta}_2^2\right)+\frac{1}{4}\dot{\phi}^2-\frac{1}{2}V(\phi)=0 \tag{4.46}$$

$$\ddot{\beta}_1+3\dot{\beta}_1\dot{\lambda}=0 \tag{4.47}$$

$$\ddot{\beta}_2+3\dot{\beta}_2\dot{\lambda}=0 \tag{4.48}$$

$$\ddot{\phi}+3\dot{\phi}\dot{\lambda}+V'=0. \tag{4.49}$$

The vector

$$G=t\frac{\partial}{\partial t} \tag{4.50}$$

is a symmetry for the first two cases but not for the last ones where we have the presence of the potential. The term of the potential is responsible for the loss of this symmetry. In order to 'fix' this problem we should add a term which could 'kill' the additional term of the potential in the λ and ϕ equations and should be also a symmetry for the other equations of the system. We assume the symmetry

$$G=t\frac{\partial}{\partial t}+a\frac{\partial}{\partial\phi}, \tag{4.51}$$

where a is a constant. We note that the new symmetry is a symmetry for the equations (4.47) and (4.48) since the additional term does not affect these two equations. If we apply the new symmetry in the (4.46) and (4.49), we obtain a specific form for the potential. Applying the second extension of (4.51) to the (4.46) and taking the equation into account we find

$$V'' + \frac{2}{a}V' = 0. \tag{4.52}$$

Similarly from (4.49) we obtain

$$V' + \frac{2}{a}V = 0. \tag{4.53}$$

Hence the potential has the form

$$V = Ke^{-2\phi/a}. \tag{4.54}$$

We note that, when one introduces a general form of symmetry for the system of the equations in the case where the potential is a general function of ϕ, one is able to find the specific form of the potential which satisfies the system of equations. Hence, we must posses the question. Have we found the most general form for the potential? In order to answer this question we should remember that the term of the potential is responsible for the symmetries that we lose from the Case 2. So we take the general form of the symmetry in Case 2 for which there is no potential,

$$G = (A + Bt)\frac{\partial}{\partial t} + (C + Dt)\frac{\partial}{\partial \lambda} + (E + F\phi + G\beta_2)\frac{\partial}{\partial \beta_1}$$

$$+(H + I\phi - G\beta_1)\frac{\partial}{\partial \beta_2} + (J - \frac{3}{2}F\beta_1 - \frac{3}{2}I\beta_2)\frac{\partial}{\partial \phi} \tag{4.55}$$

and apply the second extension to the system of the equations where the potential is general. Then we find that

- From (4.46) we obtain that $D = F = I = 0$. Hence from the ten initially possible symmetries we loose three, the ones which correspond to the three vanishing constants.

- From (4.49) we find that

$$V = Ke^{a\phi} \tag{4.56}$$

$$0 = Ja + 2B. \tag{4.57}$$

Hence we obtain six independent constants as we expect since in the Case 4 of this Type we found five independent symmetries.

- The conclusion is that, if we apply the general form of the symmetry, we find the same symmetries as in Case 4 plus another one and a specific form for the potential.

The other Bianchi models have extra terms in the Euler-Lagrange equations (3.2). Hence we should introduce a symmetry with an extra term. Considering the expression of the Ricci scalar in both classes and following the same thoughts as for the Bianchi Type I we are led to the symmetry

$$G = t\frac{\partial}{\partial t} + a\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \lambda}. \tag{4.58}$$

If we take the second extension of (4.58) and apply it to the system of the equations, we obtain that for both the classes A and B the potential is an exponential function of ϕ and has exactly the same form as in Bianchi Type I.

Chapter 5

Discussion

5.1 Introduction

In this chapter we present with details the strange points of the method which Capozziello *et al* have developed to apply Noether's Theorem in order to achieve integration of the problem of the Bianchi models. We compare our results with the ones that they have presented in [9] and we make clear the obscure points. We note their mistakes and correct the wrong results. In the last section we discuss what we could or could not do in order to avoid the traps of their method. Hence in this chapter we point out the details which one should consider whenever one wants to apply Noether's Theorem in the right way.

5.2 Observations

One of the things that aroused our interest is that the Noether symmetry approach as developed by Capozziello *et al* ignores the variable of time. As we have described in the second chapter, the infinitesimal generator X on the configuration space gives rise to the lift X^T on the tangent space which does not include the time variable. Hence the coefficient functions of the transformations have a special form since they are independent of time. Actually they do not consider time dependent transformations. Moreover the coefficient functions of the lift do not depend upon derivatives of the variables which means that they do not obtain velocity dependent tranformations. The symmetry $\partial/\partial t$ does not appear in their procedure although, as we have proved in Chapter Four, when we considered the whole space, it is a Noether symmetry as we expect since the Lagrangian is independent of time. A very interesting point is that in order to calculate the integrals Capozziello *et al* obtain the condition that the energy function is equal to zero. The energy function, as we have already commented, is actually the Noether integral of the symmetry $\partial/\partial t$. It seems that time cannot stay ignorable too long. This condition, that the energylike integral be zero, is actually a different problem from the one that we discuss. We obtain a system of four equations of the second order for which we obtain the integrals by applying symmetry methods. Capozziello *et al* , although they ignore time while they compute the symmetries, in order to find the integrals set the energylike integral to be equal to zero which is actually a constraint for the problem. If we impose this constraint in our case, then, as we will see in the next section, we obtain the same result for the Bianchi Type I but not for the Bianchi Type V.

Although in 1918 Noether [14, 15, 47] had already obtained the general form of Noether's Theorem, in the following years there have been developed forms [28, 41] of the theorem which actually restrict the original one. Hence one should pay attention to which form of Noether's Theorem is used. As we have already discussed in the first chapter, the presence of the gauge function is a consequence of the concept of the invariance of the Action Integral. The condition (2.8) does not allow the existence of the gauge function, something which is especially serious since it is a restriction which probably does not permit us to find the whole class of the solutions. The basic concept of Noether's Theorem is the concept of invariance, but what do we mean with the word invariant? and what should be invariant? Consider the original Noether's Theorem. The Action Integral should be constant under an infinitesimal transformation up to a gauge function. The transformation may be a velocity dependent transformation. We note that the condition $L_{X^T}\mathcal{L} = 0$ leads to the invariance of Action Integral under a point transformation without gauge function[1]. Hence Capozziello *et al* are referring to a restricted form of Noether's Theorem. The title of 'Noether symmetry approach' could give the wrong impression that this method uses the original Noether's Theorem. Of course the procedure that they follow after the restriction is a well defined procedure. As in all cases a restriction always causes problems [18]. A natural question arises. Are we sure that we obtain all Noether integrals and are they in the right form?

5.3 Cavete dona ferentes Græcos!

In order to answer this question we present our result in the original variables for the Bianchi Type I and V in the Case 2 where there does not exist a potential and we make a comparison with the corresponding ones in [9]. Hence for the Bianchi Type I we find that

$$u = [\beta \sinh(t - I_5) - C]^{1/3}. \tag{5.1}$$

Hence

$$e^{\lambda} = [\beta \sinh(t - I_5) - C]^{1/3} \tag{5.2}$$

and so

$$\lambda = \frac{1}{3}\log[\beta \sinh(t - I_5) - C]. \tag{5.3}$$

If we consider the condition that the energy function be zero, we have

$$0 = e^{3\lambda}\left[6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta_1}^2 + \dot{\beta_2}^2\right) - \dot{\phi}^2\right]$$

$$0 = 6\dot{\lambda}^2 e^{6\lambda} - \frac{3}{2}\left(I_1^2 + I_2^2\right) - I_3^2$$

[1]In the context of configuration space and tangent lifts the transformations may be independent of time. However, this is not necessary, since one can extend the space to include time as a dependent variable. However, this was not done by Capozziello *et al* [9].

$$A^2 = \left(\dot{\lambda} e^{3\lambda}\right)^2$$

$$\lambda = \frac{1}{3}\log(At + B), \tag{5.4}$$

where $A^2 = \frac{1}{4}\left(I_1^2 + I_2^2\right) + \frac{1}{6}I_3^2$ and B is an arbitrary constant, which has the same form as the one that Capozziello *et al* have found.

If we follow the same thoughts for the Bianchi Type V, when we consider the energy function to be equal to zero, we obtain

$$0 = e^{3\lambda}\left[6e^{-2\lambda} - 6\dot{\lambda}^2 + \frac{3}{2}\left(\dot{\beta_1}^2 + \dot{\beta_2}^2\right) + \dot{\phi}^2\right]$$

$$0 = 6e^{-2\lambda} - 6\dot{\lambda}^2 + \left[\frac{3}{2}\left(I_1^2 + I_2^2\right) + I_3^2\right] e^{-6\lambda}.$$

If we set $x = e^{2\lambda}$ in (5.5), we obtain

$$\frac{3}{2}\dot{x}^2 - \left[\frac{3}{2}\left(I_1^2 + I_2^2\right) + I_3^2\right]\frac{1}{x} - 6x = 0. \tag{5.5}$$

Hence

$$\dot{x}^2 = \left(I_1^2 + I_2^2 + \frac{2}{3}I_3^2\right)\frac{1}{x} + 4x$$

$$= \frac{D}{x} + 4x$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{D}{x} + 4x}$$

$$t - t_0 = \int \frac{\sqrt{x}}{\sqrt{4x^2 + D}}\mathrm{d}x, \tag{5.6}$$

where $D = I_1^2 + I_2^2 + \frac{2}{3}I_3^2$. For small x we obtain that

$$
\begin{aligned}
t - t_0 &= \int \sqrt{\frac{x}{D}} \left(1 + \frac{4x^2}{D}\right)^{-1/2} \mathrm{d}x \\
&= \frac{1}{\sqrt{D}} \int \sqrt{x} \left(1 - \frac{2x^2}{D} + ...\right) dx \\
&\simeq \frac{1}{\sqrt{D}} \int x^{1/2} \mathrm{d}x \\
&\simeq \frac{2}{3\sqrt{D}} x^{3/2}.
\end{aligned} \tag{5.7}
$$

Hence we have

$$
x \simeq \left[\frac{3\sqrt{D}}{2}(t - t_0)\right]^{2/3} \tag{5.8}
$$

$$
\lambda \simeq \frac{1}{3} \log \left[\frac{3\sqrt{D}}{2}(t - t_0)\right]. \tag{5.9}
$$

For large x we have

$$
\begin{aligned}
t - t_0 &= \int \frac{\sqrt{x}}{2x} \frac{\mathrm{d}x}{\sqrt{1 + \frac{D}{4x^2}}} \mathrm{d}x \\
&= \frac{1}{2} \int \frac{1}{\sqrt{x}} \left(1 - \frac{D}{8x^2} - ....\right) \mathrm{d}x \\
&\simeq \frac{1}{2} \int \frac{1}{\sqrt{x}} \mathrm{d}x \\
&\simeq x^{1/2}
\end{aligned}
$$

and so

$$
x \simeq (t - t_0)^2 \tag{5.10}
$$

$$\lambda \simeq \log(t - t_0). \tag{5.11}$$

We note that the results that we obtain differ from the corresponding ones which Capozziello *et al* have found. The explanation for this variation is because of their wrong substitution (2.43) of $x = e^{2\lambda}$ in the equation of energy. The right substitution yields us

$$\frac{3}{2}\dot{x}^2 - \frac{D^2}{x} - \alpha^2 x = 0. \tag{5.12}$$

In the original paper there is no x to multiply the α as we can observe from [9]. This error affects the behaviour of the other variables. Hence for the variable ϕ we find that for small x we have

$$\dot{\phi} = \frac{2I_3}{3\sqrt{D}(t - t_0)} \tag{5.13}$$

$$\phi = \frac{2I_3}{3\sqrt{D}} \log(t - t_0) \tag{5.14}$$

and for large x we obtain

$$\dot{\phi} = (t - t_0)^{-3} \tag{5.15}$$

$$\phi = -\frac{1}{2}(t - t_0)^{-2}. \tag{5.16}$$

The behaviour of ϕ differs from these which Capozziello *et al* have obtained. The same result we can be obtained using the Painlevé analysis [30, 31, 48, 49]. We set

$x = \alpha t^p$ in the equation

$$\dot{x}^2 = \frac{D}{x} + 4x \tag{5.17}$$

and we have

$$\alpha^2 p^2 t^{(2p-2)} = \frac{1}{\alpha} D t^{-p} + 4\alpha t^p. \tag{5.18}$$

For small x the dominant terms $\alpha^2 p^2 t^{2p-2}$ and $\frac{1}{\alpha} D t^{-p}$ give $p = \frac{2}{3}$ and $\alpha^3 = pD/4$.

For large x the dominant terms are $\alpha^2 p^2 t^{2p-2}$ and $4\alpha t^p$ and we obtain that $p = 2$ and $\alpha = 1$. As a check we set $x = t^2 + \beta t^{i+2}$ in the initial equation and we obtain $i = -1$, as required. Thus we have

$$x \simeq \frac{9D}{4}^{1/3} t^{2/3} \qquad t \simeq 0 \tag{5.19}$$

$$\tag{5.20}$$

$$x \simeq t^2 \qquad t \simeq \infty$$

which verifies the approximate evaluations of the integral made for small and large x.

5.4 Discussion

As we have already mentioned, the aim of this dissertation is to apply the Lie and Noether approaches to the Bianchi universe and to consider the results that Capozziello *et al* have obtained with their method. We have already developed the 'interesting' points of their method and results which caused our curiosity. The case that time is an ignored variable initially seems not to be a big misleading assumption since one could increase the dimension of the configuration space and consider the time as a dependent variable. However, this consideration leads to other considerations. If we consider time as a variable, then we ought to consider velocity dependent transformations which leads to a more general solution. Of course, if someone makes the calculation under the assumption of point transformations, the results will not be wrong, provided the computations are without errors, but they could be incomplete. The point is to obtain the biggest class of results and, if the problem is such that it necessarily imposes restrictions, then we should have the knowledge of the constraints that we have to use. Moreover the presence of time means that we have time dependent transformations and the coefficient functions should contain time. It seems that the absence of time is a serious matter when someone wants to obtain the most general result. Moreover a good problem for investigation is to use the energy equation instead of the λ equation and solve the new problem with the way which have developed in the first chapter of this dissertation.

Probably the most serious omission is the ignoring of the gauge function which actually makes for a very restricted form of Noether's theorem. In this case the integrals are correct, but there is always the possibility that integrals will be missed [18]. An interesting idea is to set the Lie derivative of the Lagrangian not to be zero but the derivative of a function which depends upon the time, and all the other variables, such that it is a total time derivative. That means

$$L_X\mathcal{L} = \left(X^n\frac{\partial}{\partial q_n} + \frac{dX^n}{dt}\frac{\partial}{\partial \dot{q}_n}\right)\mathcal{L} = \dot{f}. \tag{5.21}$$

Of course there are cases which the Lie and Noether approaches do not lead easily to a sufficient number of first integrals. An interesting idea is to consider the extension of the Lie symmetries to nonlocal ones [4, 5].

Appendix A

Spacetime symmetries, Lie groups and Lie algebras

A spacetime is called spatially homogeneous if there exists a one-parameter family of spacelike hypersurfaces foliating the spacetime such that for each t and for any two points p and q an isometry of the spacetime metric g_{ab} takes p into q. Hence every point lies in a homogeneous three dimensional section through spacetime which is everywhere spacelike, that is, the tangent vector of any curve lying on a homogeneous section is spacelike at every point [59]. Isotropy is the equivalence of directions. A spacetime is said to be spatially isotropic at each point if there exists a congruence of timelike curves with tangent vector field u^a filling the spacetime and satisfying the following property: For any point p and any two unit spatial tangent vectors s_1, s_2 orthogonal to u^a there exists an isometry which leaves p and u^a fixed but rotates s_1 into s_2. Thus in an isotropic universe it is impossible to construct a geometrically preferred tangent vector orthogonal to u_a [59]. We denote the Lie derivative of the tensor A along X by $L_X A$. At a point p in spacetime the field A is 'attached' to X and then X is able to transfer the vector $A(p)$ to a neighbouring point p'. The difference between the transported $A(p)$ and $A(p')$ in the limit as p' approaches p is $(L_X A)(p)$. In particular in the case where $A(p')$ is the same as the transported $A(p)$ for all the points p, $L_X A = 0$ and A is said to be an invariant for X.

A transformation which leaves the metric g invariant is called an isometry. An

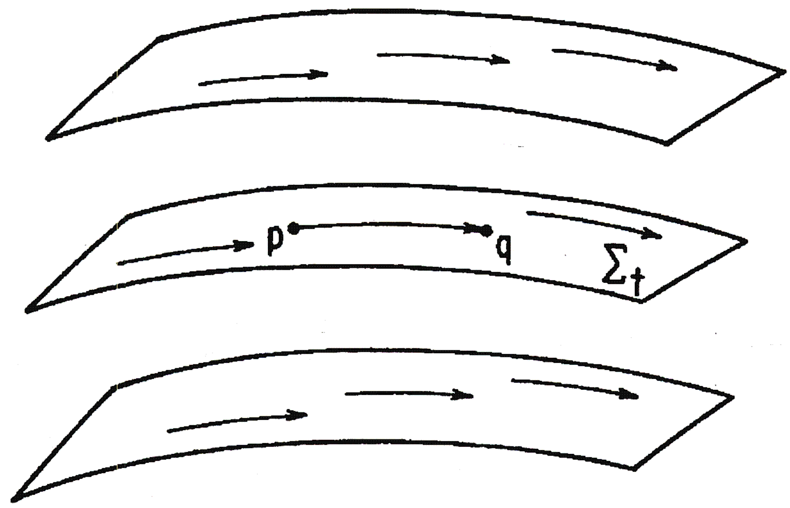


Figure A.1: The hypersurfaces of spatial homogeneity in spacetime.

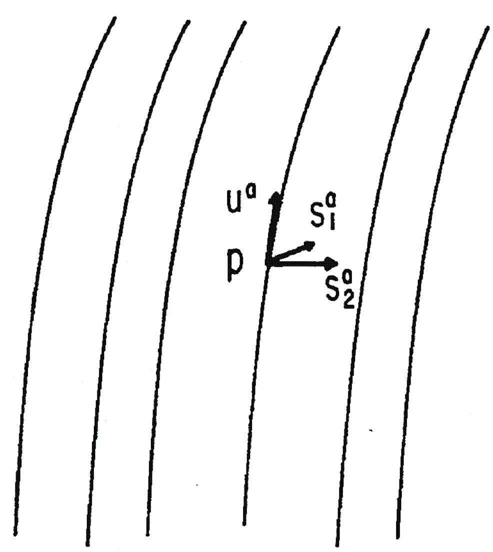


Figure A.2: The World lines of isotropic observers in spacetime. For any two vectors s_1^α, s_2^α at p which are orthogonal to u^α, there exists an isometry of the spacetime which leaves p fixed and rotates s_1^α into s_2^α.

infinitesimal isometry is described by a vector X called a Killing vector, which is said to generate isometries. A Killing vector thus satisfies

$$\mathcal{L}_X g = 0. \tag{A.1}$$

A Lie group G is a group which is also a differential manifold such that the map $G \times G \rightarrow G$ given by the algebraic product $(a, b) \rightarrow ab$ is differentiable [7].

Suppose that a group G acts as a group of transformations on another manifold M and consider the corresponding infinitesimal transformations. These define vector fields on M and these vector fields comprise a vector space dimension of which is equal to the dimension of G. We can define a skew-symmetric bilinear operation $[,]$ which satisfies the Jacobi identity

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0. \tag{A.2}$$

With this operation G becomes an algebra, called in this case a Lie algebra. It is known that every Lie algebra has associated with it a unique Lie group. Consequently, given the Lie algebra it is possible to reconstruct a Lie group from which it arises. The differential groups with the same Lie algebra differ only in their (global) topological properties. Thus the algebraic structure of G is defined entirely by that of its Lie algebra. We specify a Lie algebra by giving a basis $X_1, ..., X_n$ of vector fields and their commutators $[X_i, X_j]$. Since the commutators are again in the algebra, we must have

$$[X_\alpha, X_\beta] = C^\gamma_{\alpha\beta} X_\gamma \tag{A.3}$$

for some set of numbers $C^\gamma_{\alpha\beta}$ called the structure constants of algebra. From the properties of Lie algebra it follows that the structure constants must obey the following relations [7]

$$C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha} \tag{A.4}$$

$$C^\alpha_{\beta[\gamma} C^\beta_{\delta\epsilon]} = 0. \tag{A.5}$$

The structure of a group which is generated by r Killing vectors can be clearly recognised if one examines the commutativity of infinitesimal motions [57].

The structure constants are independent of the choice of coordinate system, but do depend upon the choice of basis of Killing vectors and can be simplified by suitable basis transformations.

Appendix B

Bianchi Classification

Our aim in this Appendix is to present the structure of the Bianchi models. We are interested in space-times which are foliated by homogeneous hypersurfaces of constant time. In our case homogeneity requires a three dimensional isometry group, which means that we must classify all three dimensional Lie algebras. Bianchi was the first to solve the problem of classifying three dimensional Lie algebras which are nonisomorphic, that is, groups whose structure constants cannot be converted into one another by linear transformations of the basis. The classification is determined by the dimension m of the algebra. The result which we present below is the so-called Bianchi classification. We obtain nine types of model according to m:

$$(a)\ m = 0 \quad Type\ I \tag{B.1}$$

$$(b)\ m = 1 \quad Type\ II,\ III \tag{B.2}$$

$$(c)\ m = 2 \quad Type\ IV,\ V,\ VI,\ VII \tag{B.3}$$

$$(d)\ m = 3 \quad Type\ VIII,\ IX. \tag{B.4}$$

Denote by X_α a basis of a three dimensional Lie algebra with structure constants $C^\gamma_{\alpha\beta}$ [29]. Let $\epsilon^{\beta\gamma\delta}$ be the totally skew tensor and define

$$t^{\alpha\delta} = \frac{1}{2} C^\alpha_{\beta\gamma} \epsilon^{\beta\gamma\delta}. \tag{B.5}$$

This contains all the information in $C^\gamma_{\alpha\beta}$. We may split it into symmetric and antisymmetric parts:

$$t^{\alpha\delta} = N^{\alpha\delta} + \epsilon^{\alpha\delta\eta} A_\eta. \tag{B.6}$$

This defines the vector A_η uniquely and also defines $N^{\alpha\delta}$. The Jacobi identity is equivalent to

$$N^{\alpha\beta} A_\beta = 0. \tag{B.7}$$

We distinguish the following cases. The first one is when $A_\beta = 0$. In this case we obtain the so called models of class A. Accordingly the Lie algebras are classified by the rank and signature of $N^{\alpha\beta}$. There exist six distinct Lie algebras [59]:

$$(I) \qquad N^{\alpha\beta} = 0 \tag{B.8}$$

$$(II)\ rank(N^{\alpha\beta}) = 1 \tag{B.9}$$

$$(VI_o)\ rank(N^{\alpha\beta}) = 2,\ signature\ -\ + \tag{B.10}$$

$$(VII_o)\ rank(N^{\alpha\beta}) = 2,\ signature\ +\ + \tag{B.11}$$

$$(VIII)\ rank(N^{\alpha\beta}) = 3,\ signature\ -\ +\ + \tag{B.12}$$

$$(IX)\ rank(N^{\alpha\beta}) = 3,\ signature\ +\ +\ +. \tag{B.13}$$

The other case is when $A_\beta \neq 0$. It is clear that the rank of the $N^{\alpha\beta}$ cannot be greater than two. So there are four possibilities for the rank and the signature.

$$(V)\ rank(N^{\alpha\beta}) = 0 \tag{B.14}$$

$$(IV)\ rank(N^{\alpha\beta}) = 1 \tag{B.15}$$

$$(III)\ rank(N^{\alpha\beta}) = 2,\ signature\ -\ + \tag{B.16}$$

$$(VI_h)\ rank(N^{\alpha\beta}) = 2,\ signature\ -\ + \tag{B.17}$$

$$(VII_h)\ rank(N^{\alpha\beta}) = 2,\ signature\ +\ + \tag{B.18}$$

We choose a base such that $A_\beta = (A,0,0)$ and $N^{\alpha\beta} = diag(N_1, N_2, N_3)$ [43]. By scaling the base we can set $N_{\alpha\beta}$ to be equal to 1 or -1. When the rank of the $N_{\alpha\beta}$ equals 2, we introduce a new parameter $h = A^2/(N_2 N_3)$ such that $A = \sqrt{|h|}$.
The subclassification of the Types VI and VII is as follows:

- h=0 : $A_\beta = 0$ and therefore we obtain the class A models VI_o, VII_o
- $h = 1$: We obtain the VII_h model
- $h = -1$: This corresponds to the type VI_h model.

We obtain the results in the table (2.20).
From the equation (B.6) we can calculate the structure constants. Their values for every type are given in Table (B.2)
We present analytically the basis X_α, the right-invariant basis W_α and its dual one form ω^α [43].

Table B.1: Bianchi classification

Class	A						B				
Type	I	II	VI_o	VII_o	VIII	IX	V	IV	III	VI_h	VII_h
Rank $N_{\alpha\beta}$	0	1	2	2	3	3	0	1	2	2	2
signature $N_{g\alpha\beta}$	0	1	0	2	1	3	0	1	0	0	2
A	0	0	0	0	0	0	1	1	1	$\sqrt{-h}$	$\sqrt{h}$
N_1	0	1	0	0	-1	1	0	0	0	0	0
N_2	0	0	-1	1	1	1	0	0	-1	-1	1
N_3	0	0	1	1	1	1	0	1	1	1	1

Table B.2: Structure Constants

Type	Structure constants
I	$C^{\alpha}_{\beta\gamma} = 0$
II	$C^1_{23} = 1$
III	$C^2_{23} = 1$
IV	$C^1_{13} = C^1_{23} = C^2_{23} = 1$
V	$C^1_{13} = C^2_{23} = 1$
VI	$C^1_{13} = 1,\ C^2_{23} = h,\ h \neq 0, 1$
VII	$C^1_{32} = C^2_{13} = 1,\ C^2_{23} = h,\ (h^2 < 4)$

VIII	$C^1_{23} = C^3_{12} = C^2_{13} = 1$
IX	$C^\alpha_{\beta\gamma} = \epsilon_{\alpha\beta\gamma}$

Table B.3: Basis vectors and forms in every Bianchi Type

Type	X_α	W_α	ω^α
I	∂x^1 ∂x^2 ∂x^3	∂x^1 ∂x^2 ∂x^3	$\mathrm{d}x^1$ $\mathrm{d}x^2$ $\mathrm{d}x^3$
II	∂x^1 ∂x^2 $\partial x^3 + x^2 \partial x^1$	∂x^2 $\partial x^2 + x^3 \partial x^1$ ∂x^3	$\mathrm{d}x^1 - x^3 \mathrm{d}x^2$ $\mathrm{d}x^2$ $\mathrm{d}x^3$
IV	$\partial x^1 - x^2 \partial x^2 - (x^2 + x^3)\, \partial x^3$ ∂x^2 ∂x^3	∂x^1 $e^{-x^1} (\partial x^2 - x^1 \partial x^3)$ $e^{-x^1} \partial x^3$	$\mathrm{d}x^1$ $e^{x_1} \mathrm{d}x^2$ $e^{x^1} (x^1 \mathrm{d}x^2 + \mathrm{d}x^3)$
V	$\partial x^1 - x^2 \partial x^2 - x^3 \partial x^3$ ∂x^2 ∂x^3	∂x^1 $e^{-x^1} \partial x^2$ $e^{-x^1} \partial x^3$	$\mathrm{d}x^1$ $e^{x_1} \mathrm{d}x^2$ $e^{x^1} \mathrm{d}x^3$

	VI, VII
X_α	$\partial x^1 + (x^3 - Ax^2)\,\partial x^2 + (x^2 - Ax^3)\,\partial x^3$
	∂x^2
	∂x^3
W_α	∂x^1
	$e^{-Ax^1}\cosh x^1 \partial x^2 + e^{-Ax^1}\sinh x^1 \partial x^3$
	$e^{-Ax^1}\sinh x^1 \partial x^2 + e^{-Ax^1}\cosh x^1 \partial x^3$
ω_α	$\mathrm{d}x^1$
	$e^{Ax_1}\cosh x^1 \mathrm{d}x^2 - e^{Ax^1}\sinh x^1 \mathrm{d}x^3$
	$e^{Ax^1}\sinh x^1 \mathrm{d}x^2 - e^{Ax^1}\cosh x^1 \mathrm{d}x^3$

	VII
X_α	$\partial x^1 + (x^3 - Ax^2)\,\partial x^2 + (x^2 + Ax^3)\,\partial x^3$
	∂x^2
	∂x^3
W_α	∂x^1
	$e^{-Ax^1}\cos x^1 \partial x^2 - e^{-Ax^1}\sin x^1 \partial x^3$
	$e^{-Ax^1}\sin x^1 \partial x^2 + e^{-Ax^1}\cos x^1 \partial x^3$
ω_α	$\mathrm{d}x^1$
	$e^{Ax_1}\cos x^1 \mathrm{d}x^2 - e^{Ax^1}\sin x^1 \mathrm{d}x^3$
	$e^{Ax^1}\sin x^1 \mathrm{d}x^2 + e^{Ax^1}\cos x^1 \mathrm{d}x^3$

	VIII
X_α	∂x^1
	$-\sinh x^1 \tanh x^2 \partial x^1 + \cosh x^1 \partial x^2 - \sinh x^1 \mathrm{sech} x^2 \partial x^3$
	$\cosh x^1 \tanh x^2 \partial x^1 - \sinh x^1 \partial x^2 + \cosh x^1 \mathrm{sech} x^2 \partial x^3$
W_α	$\sec x^2 \cos x^3 \partial x^1 - \sinh x^3 \partial x^2 - \tanh x^2 \cos x^3 \partial x^3$
	$\mathrm{sech} x^2 \sin x^3 \partial x^1 + \cos x^3 \partial x^2 - \tanh x^2 \sin x^3 \partial x^3$
	∂x^3
ω_α	$\cosh x^2 \cos x^3 \mathrm{d}x^1 - \sin x^3 \mathrm{d}x^2$
	$\cosh x^2 \sin x^3 \mathrm{d}x^1 + \sin x^3 \mathrm{d}x^2$
	$\sinh x^2 \mathrm{d}x^1 + \mathrm{d}x^3$

	IX
X_α	∂x^1
	$\sin x^1 \tan x^2 \partial x^1 + \cos x^1 \partial x^2 + \sin x^1 \sec x^2 \partial x^3$
	$\cos x^1 \tan x^2 \partial x^1 - \sin x^1 \partial x^2 + \cos x^1 \sec x^2 \partial x^3$
W_α	$\sec x^2 \cos x^3 \partial x^1 - \sin x^3 \partial x^2 + \tan x^2 \cos x^3 \partial x^3$
	$\sec x^2 \sin x^3 \partial x^1 + \cos x^3 \partial x^2 + \tan x^2 \sin x^3 \partial x^3$
	∂x^3
ω_α	$\cos x^2 \cos x^3 \mathrm{d}x^1 - \sin x^3 \mathrm{d}x^2$
	$\cos x^2 \sin x^3 \mathrm{d}x^1 + \cos x^3 \mathrm{d}x^2$
	$-\sin x^2 \mathrm{d}x^1 + \mathrm{d}x^3$

The canonical form for the structure constants does not fix the basis uniquely. In the table (B.4) we obtain the restrictions for the 3×3 matrix $N_{\alpha\beta}$. Hence types $VIII$ and IX have 6-dimensional sets of structure constants. The same occurs for types VI_h, VII_h except when h is fixed so that we obtain 5-dimensional sets. In Types VI_o and VII_o the matrix $N_{\alpha\beta}$ must be singular so that we have five independent values of constants. In Type II the first row of $N_{\alpha\beta}$ gives 3 independent values. In Type V the vector A_β gives the 3 dimensional set. In Type I there is no free choice of

Table B.4: Degrees of freedom

Type	I	II	VI_o	VII_o	VIII	IX	V	IV	VI_h	VII_h
d	0	3	5	5	6	6	3	5	5(6)	5(6)

constants. Since we have d degrees of freedom for the choice of the constants there are $9 - d$ degrees of freedom for the choice of the forms of the X_α.

Appendix C

The Bianchi Lagrangian and Potential Forms

We have already mentioned that in every hypersurface the metric is a function of time and using the dual forms ω^α can be expressed as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}(t)\omega^i\omega^j, \tag{C.1}$$

where the ω^i are three one forms obeying the relations

$$\mathrm{d}\omega^i = C^i_{jk}\omega^j\omega^k \tag{C.2}$$

with C^i_{jk} the structure constants of Table (B.2). Misner introduced the parametrisation $g_{ij}(t)e^{-2\Omega(t)}e^{2\beta_{ij}}$, where Ω is a scalar and β_{ij} a traceless 3×3 matrix. He then chose a coordinate condition $t \to \Omega$, that is, he chose Ω as his time coordinate. With this choice the β_{ij} become functions of Ω. This reduces the problem to one of finding the five independent components of β_{ij} as functions of Ω. There are special cases of β_{ij} which have fewer independent components. Diagonal βs have only two. This implies that the problem of Bianchi type universes reduces to that of the motion of a point, the 'universe point'. We can use λ as a new time variable and we make also a new reparametrisation of the β_-, β_+ to β_1, β_2. The general Lagrangian has the form [9]

$$\mathcal{L} = e^{3\lambda}\left[R^{*} + 6\dot{\lambda}^{2} - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2 + 2V(\phi)\right]. \tag{C.3}$$

The Ricci scalar R^{*} has the form

CLASS A

$$R^{*} = -\frac{1}{2}e^{-2\lambda}{N_1}^2 e^{4\beta_1} + e^{-2\beta_1}\left(N_2 e^{\sqrt{3}\beta_2} - N_3 e^{-\sqrt{3}\beta_2}\right)^2$$

$$-2N_1 e^{\beta_1}\left(N_2 e^{\sqrt{3}\beta_2} + N_3 e^{-\sqrt{3}\beta_2}\right) + \frac{1}{2}N_1 N_2 N_3 \left(1 + N_1 N_2 N_3\right). \tag{C.4}$$

CLASS B

$$R^{*} = 2a^2 e^{-2\lambda}\left(3 - \frac{N_2 N_3}{a^2}\right)e^{\beta} \tag{C.5}$$

with

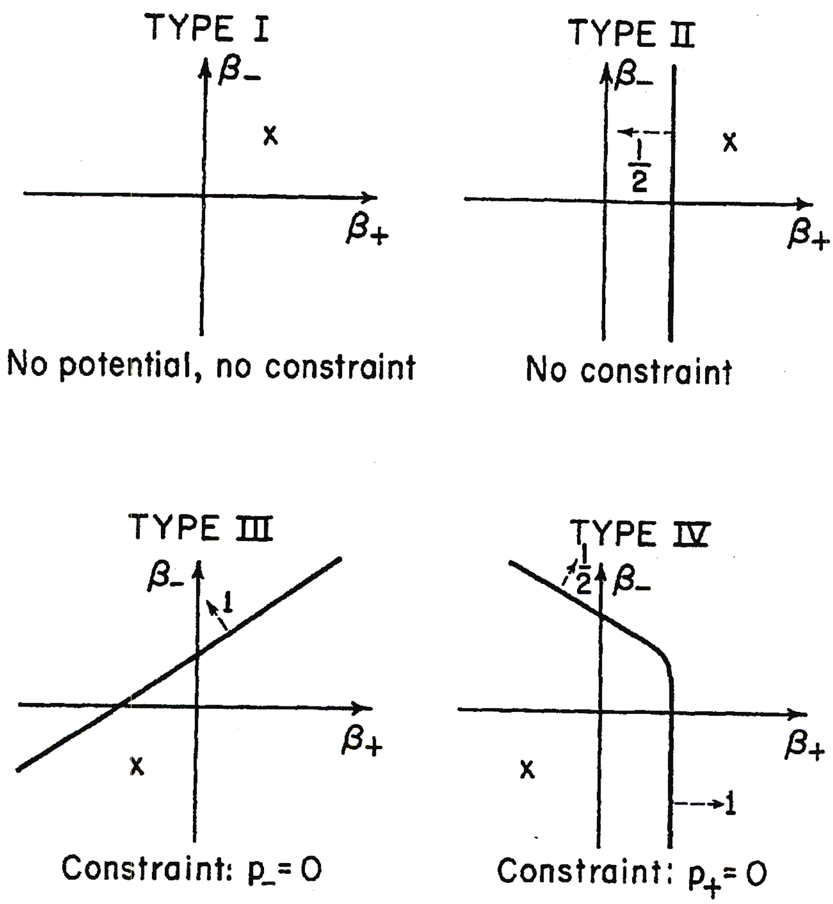
$$\beta = \frac{2}{3a^2 - N_2 N_3}\left(N_2 N_3 \beta_1 + \sqrt{-3}a^2 N_2 N_3 \beta_2\right). \tag{C.6}$$

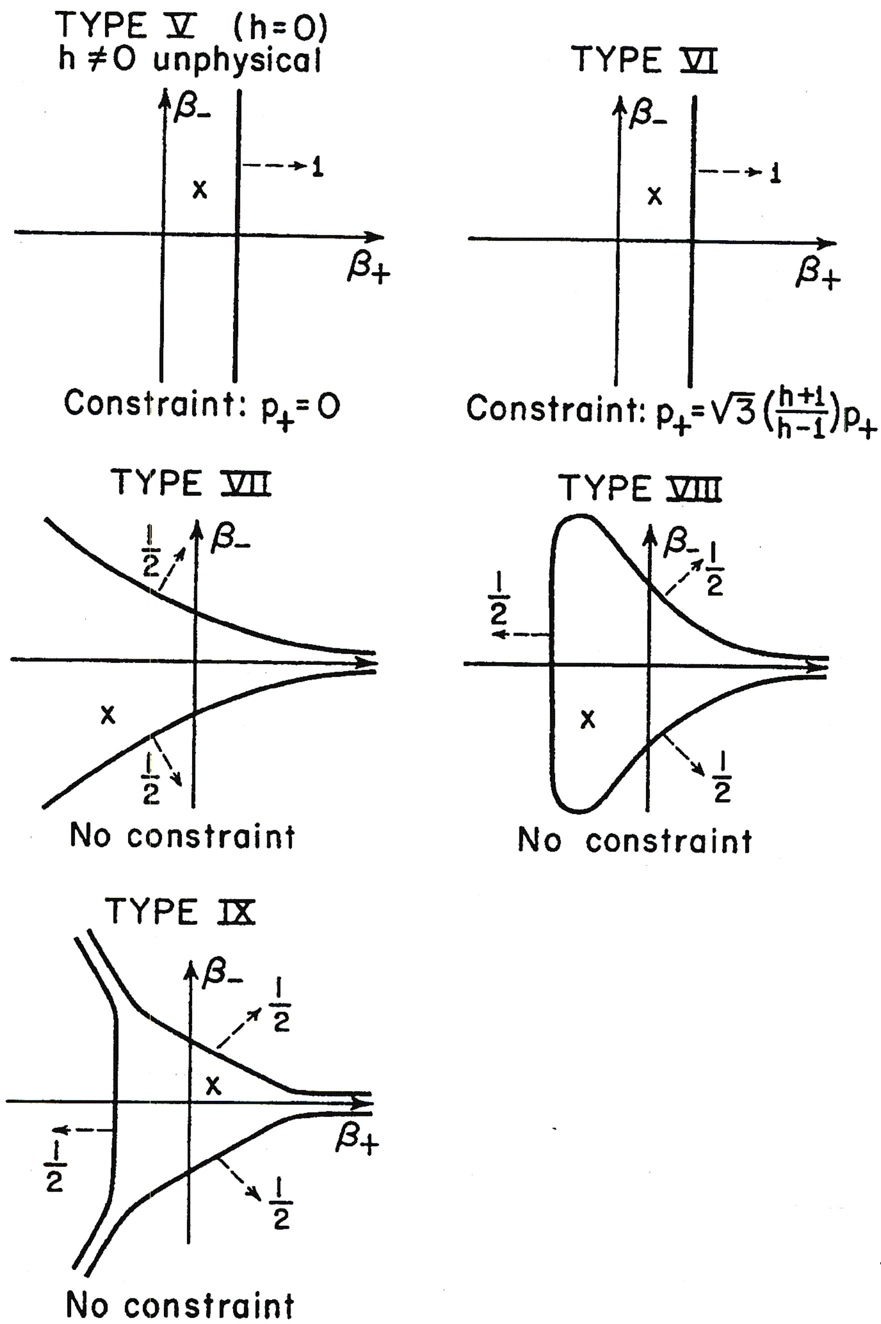
The forms of the potentials appearing in the Lagrangian (3.10) are given explicitly for each Bianchi type in the following Table. We note that the potentials are exponential in every case.

We present the potentials for Bianchi Type models in the diagonal case. The dashed arrows give the velocity of the wall associated with the potential. The symbol x marks the position of a generic universe point.

Type	Potential
I	0
II	$e^{-8\beta_+}$
III	$4e^{-(2\beta_+ - 2\sqrt{3}\beta_+)}$
IV	$e^{4\beta_+}(12 + e^{4\sqrt{3}\beta_-})$
V	$12e^{4\beta_+}$
$VI(h{\neq}0, 1)$	$4(1 + h + h^2)e^{4\beta_+}$
$VII(h^2 < 4)$	$2e^{4\beta_+}\left[\cosh(4\sqrt{3}\beta_-) + (2h^2 - 1)\right]$
VIII	$e^{-8\beta_+} + 2e^{4\beta_+}\left[\cosh(4\sqrt{3}\beta_- - 1)\right] + 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)$
IX	$e^{-8\beta_+} + 2e^{4\beta_+}\left[\cosh(4\sqrt{3}\beta_- 1)\right] - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)$

Figure C.1: The Potentials for Bianchi Type models. Types I to IV on this page and Types V to XI overleaf.





TYPE V (h=0)
h ≠0 unphysical
β_-
1
x
β_+
Constraint: $p_+=0$
TYPE VI
β_-
1
x
β_+
Constraint: $p_+=\sqrt{3}\left(\frac{h+1}{h-1}\right)p_+$
TYPE VII
$\frac{1}{2}$
β_-
x
$\frac{1}{2}$
No constraint
TYPE VIII
β_-
$\frac{1}{2}$
$\frac{1}{2}$
x
$\frac{1}{2}$
No constraint
TYPE IX
β_-
$\frac{1}{2}$
x
β_+
$\frac{1}{2}$
$\frac{1}{2}$
No constraint

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