COSMOLOGICAL ATTRACTORS AND

NO-HAIR THEOREMS

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Abstract

Interest in the attracting property of de Sitter space-time has grown during the 'inflationary era' of cosmology. In this dissertation we discuss the more important attempts to prove the so called 'cosmic no-hair conjecture' *ie* the proposition that all expanding universes with a positive cosmological constant asymptotically approach de Sitter space-time. After reviewing briefly the standard FRW cosmology and the success of the inflationary scenario in resolving most of the problems of standard cosmology, we carefully formulate the cosmic no-hair conjecture and discuss its limitations. We present a proof of the cosmic no-hair theorem for homogeneous space-times in the context of general relativity assuming a positive cosmological constant and discuss its generalisations. Since, in inflationary cosmology, the universe does not have a true cosmological constant but rather a vacuum energy density which behaves like a cosmological term, we take into account the dynamical role of the inflaton field in the no-hair hypothesis and examine the no-hair conjecture for the three main inflationary models, namely new inflation, chaotic inflation and power-law inflation. A generalisation of a well-known result of Collins and Hawking [21] in the presence of a scalar field matter source, regarding Bianchi models which can approach isotropy is given. In the context of higher order gravity theories, inflation emerges quite naturally without artificially imposing an inflaton field. The conformal equivalence theorem relating the solution space of these theories to that of general relativity is reviewed and the applicability of the no-hair theorems in the general framework of $f(R)$ theories is developed. We present our comments and conclusions about the present status of the cosmic no-hair theorem and suggest possible paths of future research in the field.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

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Acknowledgments

Contents

Chapter 1

Introduction

The theoretical model on which modern cosmology is based is the Friedmann-Robertson-Walker (FRW) cosmological model or, the hot big bang model. It assumes that the universe is perfectly homogeneous and isotropic. Einstein himself made the assumption of homogeneity and isotropy in order to simplify the mathematical analysis. Today the experimental observations strongly support this assumption at least for that part of the universe we can see. The FRW model predicts the expansion of the universe, the large-scale uniformity of the universe, the light-element abundances (with spectacular precision in the case of ^{4}He), and possibly the age of the universe. In view of these successes the FRW cosmology became known as the standard cosmology.

The most important cosmological discovery of the recent decades has been the detection of the cosmic background radiation (CBR). Its most striking feature is a temperature isotropy over a wide range of angular scales on the sky. The remarkable uniformity of the CBR indicates that at the end of the radiation-dominated period (some hundreds of thousands years after the big bang) the universe was almost completely isotropic. One then has a difficulty in explaining why there should be such an isotropy in the universe for the following reason. The finite velocity of light divides the universe into causally decoherent regions. Roughly speaking, if the age of the universe is T, then regions moving away because of the expansion of the universe and separated by a distance greater than

cT will not have enough time to communicate with each other. How did these separated regions come to be at the same temperature today to better than one part in ten thousand? There are two possible responses to this so-called horizon problem. The first is that the universe has always been isotropic which means that the initial conditions were such that the universe was and has remained homogeneous and isotropic. This seems to be statistically quite improbable. The second response is that the universe came about in a less symmetric state and evolved through some dynamical mechanisms towards a FRW state. Soon after the discovery of the cosmic background radiation isotropy, Misner and others suggested that the universe started off in a chaotic state with inhomogeneities and anisotropies of all kinds and that various dissipation processes damped out nearly all of these, leaving only the very small amounts that we see today. This programme was unable to show that the present state of the universe could be predicted independently of its initial conditions. In section 2.2 we discuss in more detail the difficulty in explaining the observed isotropy of the universe – unless one *assumes* that isotropy persists back to the big bang – and some other interrelated problems of the standard cosmology.

These problems led to the invention in 1981 of the inflationary scenario which is a modification of the standard hot big bang model. According to this scenario the very early universe underwent a short period of exponential expansion, or inflation, during which its radius increased by a factor of about 10^{50} times greater than in standard cosmology. This inflationary phase is also known as de Sitter phase since the de Sitter universe is a homogeneous and isotropic universe with radius growing exponentially with time. From times later than about 10^{-30} sec the history of the universe is described by the standard cosmology and all the successes of the later are maintained.

To see what this picture implies for our universe, consider a region which at time t as measured from the big bang has the size of the horizon distance at time t. The horizon distance at time t is approximately the distance travelled by light in time t *ie*, equals ct. This is evidently the greatest size of a causally coherent region possible. Thus at time $10^{-34} sec$ when inflation commences the size of this region is about $10^{-24} cm$.

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After inflation, at time about $10^{-30}sec$, its size has grown to approximately $10^{26}cm$. The observable universe at that epoch had a radius of about $10cm$, a minuscule part of the inflating region. Since the universe lay within a region which started as a causally coherent region, it would have had time to homogenise and isotropise. Thus inflation naturally explains the uniformity of the universe.

Today it is believed that inflation is the only way to solve most problems of the standard cosmology. It must be emphasised that the inflationary scenario is far from being a complete theory describing the very early universe. Several inflationary models have been developed during the last fifteen years, mainly because there exist different ways to generate the mechanism of inflation. These models have problems of their own. Inflation remains an area of active research.

Most models treat inflation in the context of a flat FRW cosmology. This seems paradoxical since one of the attractive features of the inflationary scenario is that it offers the possibility of explaining the present state of our universe without assuming special initial conditions. However, it is not obvious that cosmological models with non-FRW initial conditions will ever enter an inflationary epoch nor is it obvious that, if inflation occurs, initial inhomogeneities and anisotropies will be smoothed out. We now address this central issue of inflation: Does it proceed from very general initial conditions? To put it another way, does the universe forget its initial state during inflation and evolve exponentially fast towards a homogeneous and isotopic de Sitter space? With regard to this question it has been conjectured that *all expanding-universe models with positive cosmological constant asymptotically approach the de Sitter solution.* This is referred to as *the cosmic no-hair conjecture.* The term 'no-hair' denotes the loss of information regarding initial space-time geometry caused by evolution under the field equations.

In this dissertation we study the more important attempts to prove the cosmic no-hair conjecture (NHC). Our treatment will be purely classical, that is, we omit the relevant works based on quantum gravity. The organization of the material is as follows:

- Chapter Two: We begin by a brief review of the standard cosmological model

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and the problems of the standard cosmology. It follows a sketch of the inflationary scenario and its successes in the resolution of most of the problems of the standard model. We state the cosmic no-hair conjecture and discuss some of its limitations.

- Chapter Three: We discuss the cosmic NHC in the context of General Relativity assuming a positive cosmological constant.   The first – and probably the more important result – is a theorem due to Wald, who succeeded in proving that the cosmic NHC is true in the case of homogeneous space-times. We mention a slight generalisation of Jensen and Stein-Shabes as well as the totally different approach of Morrow-Jones and Witt.

- Chapter Four: In inflationary models the universe does not have a true cosmological constant.   Rather there is a vacuum energy density which during the slow evolution of the scalar field remains approximately constant and behaves like a cosmological term. In this chapter we take into account the dynamics of the scalar field in the no-hair hypothesis. We examine the NHC for three specific inflationary models, namely new inflation, chaotic inflation and power law inflation.

- Chapter Five: Interest in generalised theories of gravity has grown during the 'inflationary era' of cosmology, mainly because in the context of higher-order gravity theories inflation emerges quite naturally without the necessity of imposing an inflaton field. In this chapter we discuss the no-hair theorems in the general frame of $f(R)$ theories.

- Chapter Six: We present our comments and conclusions about the cosmic NHC. We also suggest some paths of research in this field.

Notation and conventions

In this dissertation we follow the sign conventions of Misner, Thorne and Wheeler (MTW [59]). In particular we use the metric signature $(-, +, +, +)$ and define the Riemann tensor by

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

so that

$$\nabla_c \nabla_d Z^a - \nabla_d \nabla_c Z^a = R^a{}_{bcd} Z^b.$$

The Ricci tensor is defined as the one-three contraction of the Riemann tensor so that

$$R_{bd} = R^a{}_{bad}.$$

The Einstein tensor is defined as

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R,$$

where g_{ab} is the metric tensor and R is the scalar curvature tensor. The Einstein field equations read therefore

$$G_{ab} = 8\pi T_{ab}.$$

Throughout most of this work, we use units where the gravitational constant, G, and the speed of light, c, are set equal to one. However, from section 4.3 to the end we use units such that $c = 8\pi G = 1$.

We employ the *abstract index notation* discussed in Wald [73]. Thus, latin indices on a tensor merely denote the type of the tensor (they are part of the notation for the tensor itself). Greek indices on a tensor represent its components in a given frame. In the cases where purely spatial tensor components occur, the range of the indices is explicitly denoted.

∇_a is the symbol for the covariant derivative operator. We occasionally use a semi-colon (;) to denote covariant differentiation. The symbol ∂_a stands for the ordinary derivative operator.

Here are some abbreviations most frequently used in the text.

RD	Radiation dominated.
MD	Matter dominated.
HE	Hawking and Ellis.
FRW	Friedmann-Robertson-Walker.
CBR	Cosmic background radiation.
GR	General relativity.
SEC	Strong energy condition.
WEC	Weak energy condition.
DEC	Dominant energy condition.
NHC	No-hair conjecture.
NHT	No-hair theorem.
HOG	Higher order gravity.
PPC	Positive pressure criterion.

The stress-energy-momentum tensor is usually written as stress tensor or energy-momentum tensor.

Chapter 2

The inflationary scenario and the cosmic no-hair conjecture

Modern theoretical cosmology is based on the investigation of the structure of our universe with the aid of general relativity. It is well known that the equations of general relativity cannot be solved for an arbitrary space-time and an arbitrary matter distribution. Hence, in order to simplify Einstein's equations we make the assumption that the universe is *homogeneous* and *isotropic*. Roughly speaking, homogeneous means that, if we were located in a different region of our universe, the basic characteristics of our surroundings would appear the same; and by isotropic we mean that there are no preferred directions in space. In the next section we begin by a brief review of the standard model, that is, the cosmological model which is constructed under the assumptions of homogeneity and isotropy of the universe.

2.1   The standard model

The homogeneous and isotropic expanding universe is described by the Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{2.1}$$

where $k = +1, -1$ or 0 for a closed, open or flat universe and $a(t)$ is the scale factor of the universe.

The evolution of the scale factor is governed by the Einstein equations

$$\ddot{a} = -\frac{4\pi}{3}(\rho + 3p)\,a \tag{2.2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho. \tag{2.3}$$

Here ρ is the density of the universe and p its pressure. From the last two equations one can find the conservation equation

$$\dot{\rho}\,a^3 + 3\left(\rho + p\right)a^2\,\dot{a} = 0. \tag{2.4}$$

Assuming an equation of state of the form $p = (\gamma - 1)\rho$ we deduce from (2.4) that

$$\rho \sim a^{-3\gamma}. \tag{2.5}$$

In particular for $p = 0$ (dust)

$$\rho \sim a^{-3} \quad \text{matter dominated} \tag{2.6}$$

and for $p = \frac{1}{3}\rho$ (radiation)

$$\rho \sim a^{-4} \quad \text{radiation dominated.} \tag{2.7}$$

In either case when a is small the curvature term k/a^2 in (2.3) is much smaller than the density term $(8\pi/3)\,\rho$ and the Friedmann equation (2.3) implies that

$$a \sim t^{\frac{2}{3\gamma}}. \qquad (2.8)$$

For a matter dominated universe

$$a \sim t^{\frac{2}{3}} \qquad (2.9)$$

while for a radiation dominated universe

$$a \sim t^{\frac{1}{2}} \ . \qquad (2.10)$$

Thus regardless of the spatial geometry ($k = \pm 1, 0$) the scale factor vanishes and the density becomes infinite as t goes to zero. It can be verified that the components of the curvature tensor $R^a{}_{bcd}$ also go to infinity as $t \to 0$. So the point $t = 0$ is referred to as the point of the initial cosmological singularity (Big Bang).

It is uncertain at present [43] what is the spatial geometry of the universe, ie what is the value of the scalar curvature k. It depends on the density ρ of the universe. In fact we see from the Friedmann equation (2.3) that the sign of k is determined by the ratio ρ/ρ_c of the actual density ρ to the critical density ρ_c, defined by

$$\rho_c = \frac{3H^2}{8\pi}, \qquad (2.11)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. If the quantity $\Omega \equiv \rho/\rho_c$ is less than one, the universe is open while, if it is larger than one, the universe is closed. Present day observations [43] give the values

$$40 \leq H \leq 100 \left(\frac{km}{sec.Mpc}\right) \qquad (2.12)$$

$$0.1 \leq \Omega \leq 2 \qquad (2.13)$$

so that the universe is not far from being flat.[1]

The existence of *horizons* is characteristic in a FRW space-time [35, 75]. The *particle horizon* delimits the causally connected part of the universe that an observer can see at a given time t. The null geodesic equation for the metric (2.1) gives

$$\frac{dr}{dt} = \frac{\sqrt{1 - kr^2}}{a(t)} \qquad (2.14)$$

and the physical distance travelled by light in time t (that is, the radius of the particle horizon) is

$$R_p = a(t) \int_0^{r(t)} \frac{dr}{\sqrt{1 - kr^2}} = a(t) \int_0^t \frac{dt'}{a(t')}. \qquad (2.15)$$

As an example consider a matter dominated universe, $a(t) \sim t^{\frac{2}{3}}$. Equation (2.15) gives $R_p = 3t = 2H^{-1}$ so that the radius of the observable part of the universe today (see (2.12)) is

$$0.9 \leq R_p \leq 2.25 \quad \left(10^{28} cm \right). \qquad (2.16)$$

The *event horizon* delimits that part of the universe from which we can (up to some time $t_{\max}$) receive information about events taking place now (at time t):

$$R_e(t) = a(t) \int_t^{t_{\max}} \frac{dt'}{a(t')}. \qquad (2.17)$$

As seen from (2.17), in a Minkowski space-time ($a(t) = const$) or in the Friedmann matter dominated flat universe ($a(t) \sim t^{\frac{2}{3}}$) there is no event horizon: $R_e \to \infty$ as

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[1]The controversy about the exact value of the Hubble parameter still holds. Recent observations have pushed up the lower bound of H with crucial implications for the standard cosmology. The present day value of H is probably (but not definitely) $80 \pm 17 km/ (sec . Mps)$.

$t_{\max} \to +\infty$. This is not the case for a de Sitter space-time as we shall see later.

2.2   Problems of the standard cosmology

The basic assumption of the Friedmann cosmology is that the universe is perfectly homogeneous and isotropic. Homogeneity (on the average) has been verified experimentally [43] and isotropy is the most striking consequence of the discovery of the cosmic background radiation (CBR) in 1964. In fact the CBR is uniform to about one part in 10^4 in different directions. It seems, however, very improbable that the initial state of the universe was exactly homogeneous and isotropic [21]. What is more reasonable is to assume that its initial state was less symmetric and through some mechanisms of isotropisation the initial nonuniformities became damped so that the universe could approach asymptotically its present state. However, as was shown by Collins and Hawking [21], the class of initial conditions for which the universe tends for large t to a Friedmann universe is of measure zero among all possible initial conditions.

Despite its simplicity the standard model is very successful in its predictions of the Hubble law, the cosmic background radiation and the abundances of the light elements [43, 75]. It provides a framework in which to discuss the history of the universe from at least as early as the time of nucleosynthesis ($t \simeq 10^{-2}$ to 10^2 sec and $T \simeq 10$ to $0.1 MeV$) until today ($t \simeq 15 Gyr$ and $T \simeq 2.75 K$). However, if it is extrapolated backward to times much earlier than one second after the big bang, several problems are raised. These problems can be stated in several different but equivalent ways. We review some of these problems below. For a more detailed exposition see [43, 49, 17].

The first problem is the *horizon problem*. If the universe were in causal contact at the end of the radiation dominated epoch, it might be imagined that microphysical processes smoothed out any temperature fluctuations thus explaining the above mentioned uniformity of the CBR. However, this is impossible because regions at, say, opposite directions in the sky were separated by many horizon distances [17] and they could not interact.

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The horizon problem is not an inconsistency in the standard model, but it rather represents a lack of predictive power. The problem is that the large-scale homogeneity of the universe is not explained or predicted by the model, but instead must be assumed.

The second problem is the *flatness problem.* As we can see from Friedmann equation (2.3)

$$\Omega(t) = \frac{1}{1 - x(t)} \tag{2.18}$$

$$x(t) = \frac{k}{a^2} \frac{3}{8\pi\rho} \propto \begin{cases} a^2(t) & \text{radiation dominated} \\ a(t) & \text{matter dominated} \end{cases} \tag{2.19}$$

so for the very early universe $x(t) \sim t$ and Ω was closer to unity. One can show that [49]

$$|\Omega(1\,\text{sec}) - 1| \leq \bigcirc\left(10^{-16}\right) \tag{2.20}$$

$$\left|\Omega(10^{-43}\,\text{sec}) - 1\right| \leq \bigcirc\left(10^{-60}\right). \tag{2.21}$$

As Linde [49] points out, if the density of the universe was initially (at Planck time) greater than ρ_c, say by $10^{-55}\rho_c$, the universe would be closed and its lifetime $t_c = \frac{4}{3}M$ (M is the 'total mass' of the universe [45]) would be so small that the universe would have collapsed long ago. If on the other hand the density near the Planck time was $10^{-55}\rho_c$ less than ρ_c, the present energy density in the universe would be vanishingly low and life could not exist. The difficulty in understanding why Ω was so extraordinarily close to one at these time scales is known as the flatness problem.

Like the horizon problem, the flatness problem is not an inconsistency in the standard model. The fact that the density of the early universe is almost equal to the critical density cannot be explained by the model but instead must be assumed in the initial conditions.

The third problem is the observed *small-scale inhomogeneity* of the universe. Al-

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though the universe is smooth on large scales, it contains important inhomogeneities such as stars, galaxies, clusters, and so on. In explaining such a structure, it is necessary to assume the existence of initial inhomogeneities. For a long time, the origin of these density perturbations remained obscure [17, 77].

The forth problem is that of the *unwanted relics* also known as the monopole problem. In the context of GUTs[2] a tremendous overproduction of magnetic monopoles occurs during the early stages of the universe. The expected monopole density is comparable to the baryon density giving an energy density in the universe about 15 orders of magnitude higher than the critical density ρ_c. The universe would have collapsed long ago. Besides monopoles, other structures such as domain walls can be produced following symmetry-breaking phase transitions in the early universe. For further discussion of these topological defects we refer to [43, 49].

There are some other problems [49] related to the four mentioned above which can be put under the general title 'Why is the Universe as it is?'. We do not continue this list. In the next section we shall discuss a proposal which claims to avoid most of them.

2.3 The Inflationary Scenario

Consider a scalar field φ described by the Lagrangian density

$$L = -\frac{1}{2}\partial_a\varphi\partial^a\varphi - V(\varphi).$$ (2.22)

Its energy-momentum tensor $T_{ab} = \partial_a\varphi\partial_b\varphi + g_{ab}L$ can be written as

$$T_{ab} = \partial_a\varphi\partial_b\varphi - \frac{1}{2}g_{ab}\left[\partial_c\varphi\partial^c\varphi + 2V(\varphi)\right].$$ (2.23)

If the field φ is homogeneous, its spatial derivatives vanish and in an appropriate coordinate system its energy-momentum tensor takes the form of the corresponding tensor

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[2]Grand unified theories

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of a perfect fluid

$$T_{\mu\nu} = diag\left(\rho, p, p, p\right),$$

(2.24)

where $\rho = \frac{1}{2}\,\dot{\varphi}^2 + V\left(\varphi\right)$ and $p = \frac{1}{2}\,\dot{\varphi}^2 - V\left(\varphi\right)$. (See also the introductory remarks in Chapter Four).

A constant scalar field φ over all space-time simply represents a restructuring of the vacuum in the sense that the vacuum energy density changes by a quantity proportional to $V\left(\varphi\right)$. If there were no gravitational effects, this change would be unobservable, but in general relativity it affects the properties of space-time. In fact $V\left(\varphi\right)$ enters into the Einstein equation in the following way:

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab} = 8\pi\left(T_{ab}^m - g_{ab}V\right),$$

(2.25)

where T_{ab} is the total energy-momentum tensor, T_{ab}^m is the energy-momentum tensor of ordinary matter and $-g_{ab}V$ is the energy-momentum tensor of the vacuum (the constant scalar field). Of course, $V\left(\varphi\right) = V$ is a constant. Note that (2.25) is just the Einstein equation with a cosmological constant Λ, viz.

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab} - \Lambda g_{ab},$$

(2.26)

where Λ is given by

$$\Lambda = 8\pi V\left(\varphi\right).$$

(2.27)

We may also view (2.25) or (2.26) in vacuum ($T_{ab}^m = 0$) as describing a perfect fluid with $p = -\rho = -V$ [35]. This large negative pressure has the effect that a homogeneous and isotropic universe expands exponentially. In fact equations (2.1), (2.2) and (2.3) imply that

$$a\left(t\right) = \begin{cases} H^{-1}\cosh Ht & \text{if } k = +1 \\ H^{-1}e^{Ht} & \text{if } k = 0 \\ H^{-1}\sinh Ht & \text{if } k = -1, \end{cases}$$

(2.28)

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where we have set

$$H \equiv \sqrt{\frac{\Lambda}{3}} = \sqrt{\frac{8\pi}{3}\rho}. \tag{2.29}$$

Note that, according to (2.4), the vacuum energy density does not change as the universe expands. The solution (2.1) and (2.28), obtained in 1917 by de Sitter, is referred to as *de Sitter space-time* and its interesting geometry is well described in HE [35]. As we shall see in the next section the de Sitter space-time plays a crucial role in inflationary cosmology. Notice that even the flat ($k = 0$) de Sitter space-time has an event horizon (cf. (2.17))

$$R_e(t) = H^{-1}. \tag{2.30}$$

An observer in an exponentially expanding universe can see only those events that take place no farther than H^{-1}.

The inflationary universe is a modification of the standard hot big bang model. The basic idea of inflation is that the early universe underwent a short period during which matter was in a metastable false vacuum state driving the evolution of the universe into exponential expansion. During this period the scale factor increased by a tremendous factor, perhaps 10^{50} times greater than in standard FRW cosmology.

Early constructions of inflationary models involved the notion of phase transitions present in all grand unified theories of elementary particles. It was assumed that the very early universe was in a hot ($T \geq 10^{14} GeV$), expanding state and the symmetry of the fundamental interaction was manifest so that the Higgs fields had zero values. During the expansion the temperature dropped and some small regions underwent a phase transition with at least one of the Higgs fields acquiring a non-zero value, resulting in a broken-symmetry state. However, for certain values of the parameters in most GUTs the rate of cooling is very fast compared with the rate of the phase transition. This causes the system to supercool to a negligible temperature with the Higgs field remaining at zero, resulting in a false vacuum. The false vacuum has a constant energy density $\rho = \rho_f$,

typically of the order of the fourth power of the characteristic mass scale of the theory, which for GUTs is

$$\rho_f \approx \left(10^{14} GeV\right)^4 \approx 10^{73} g\, cm^{-3}.$$ (2.31)

This tremendous energy density drives the universe – more precisely the region where the phase transition takes place – to exponential expansion with a minuscule time constant (see (2.29)). As already discussed, a homogeneous and isotropic universe with equation of state $p = -\rho$ (which is the case of the false vacuum) expands exponentially with time constant

$$H^{-1} = \left(\frac{8\pi}{3}\rho_f\right)^{-\frac{1}{2}} \approx \left(10^{10} GeV\right)^{-1} \approx 10^{-34}\, \text{sec}.$$ (2.32)

Meanwhile quantum or thermal fluctuations cause the Higgs field to deviate from zero. The field begins to increase with a rate similar to the speed of a ball rolling down the potential curve. In fact the equation of motion of a scalar field with Lagrangian density (2.22) is $\Box\varphi - V'(\varphi) = 0$. For a homogeneous scalar field, this equation in the de Sitter metric takes the form

$$\ddot{\varphi} + 3H\,\dot{\varphi} + V'(\varphi) = 0.$$ (2.33)

This is just the equation for a ball rolling down a hill with friction.[3] In the beginning the speed is very slow due to the 'flatness' of the potential – a feature common to all inflationary models. The damping term $3H\,\dot{\varphi}$ in (2.33) reflects the expansion of the universe and helps also to slow down the motion towards the steeper part of the potential. During this slow-rollover process the energy density remains very nearly equal to ρ_f and the exponential expansion continues. More precisely the magnitude of H (cf. (2.32))

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[3]Actually, this is the equation of particle sliding down a hill with friction; we retain however the commonly used expression 'slow-rollover process' when describing the evolution of the scalar field.

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Figure 2-1: Schematic illustration of an inflationary potential. The flatness of the potential is a common feature to all inflationary models. The qualitative phases of inflation are depicted: slow roll and coherent oscillations about the minimum of the potential.

changes very slowly so we may speak of a quasi-exponential expansion of a (flat) universe

$$a\left(t\right) = a_0 \exp \int_0^t H\left(t'\right) dt' \sim a_0 \exp\left(Ht\right) \tag{2.34}$$

rather than a strict de Sitter regime, of (2.28). Thus the universe has enough time to inflate.

Inflation comes to an end when H begins to decrease rapidly. This happens because the energy density falls to zero as the scalar field approaches its true-vacuum value. In fact, when the scalar field φ reaches the steep part of the potential, it falls quickly to the minimum and oscillates about it. These oscillations are damped by extra terms in the equation of motion which arise due to the coupling of φ with other fields in the theory [43]. The oscillations are interpreted in quantum field theory as a high density of scalar particles and the damping corresponds to the decay of these particles into lighter species. The decay products collide with each other and perhaps decay into still lighter species

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so that the energy rapidly thermalises. The system is reheated to a temperature about 1/3 times the phase transition temperature (as can be calculated by using conservation of energy). All the enormous false vacuum energy is therefore transformed in a hot gas of elementary particles in an equilibrium similar to the initial conditions assumed in the standard model. At this point the inflationary phase joins the standard FRW cosmology and accordingly the successes of the standard model are maintained.

To illustrate the modifications of the standard model resulting by including an inflationary transient stage we use the following parameters which are typical to most inflationary models[4]: $H^{-1} \sim 10^{-34}$ sec (cf. (2.32)), time required for φ to evolve to its equilibrium value $\Delta t \sim 10^{-32}$ sec $\approx 100 H^{-1}$. We assume that a smooth and *causally coherent* region of size less than $H^{-1} \sim 10^{-23} cm$ – the radius of the event horizon – undergoes inflation. During the (quasi-) de Sitter phase the scale factor grows by the factor $Z \equiv e^{H \Delta t} \approx e^{100} \approx 3 \times 10^{43}$.

We observe that the horizon problem is easily solved. Our observable universe was the result of inflating a very small region of space that was initially causally connected. After inflation the size of this region was $3 \times 10^{20} cm$, much larger than the radius of the observable universe (about $10cm$, as is calculated taking account that the temperature after reheating is approximately $10^{14} GeV$). The flatness problem is also solved: During inflation the energy density remained almost constant $\sim \rho_f$ while the scale factor grew by the factor $Z \approx e^{100}$ so that the ratio $x(t) = \frac{k}{a^2} \frac{3}{8\pi\rho}$ decreased by a factor $\sim e^{200}$. Thus the scale factor ('radius of curvature') of the universe today should still be much greater than the present observable radius (2.15), thereby explaining the flatness of the universe. We also see from (2.18) that Ω should be exponentially close to unity.[5] Further, the monopole

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[4]The values of these parameters depend on the model used. In some models the factor Z takes huge values of order $\sim 10^{10^5}$.

[5]The recent observational facts about the Hubble constant and Ω (see footnote on page 10) have put forward the construction of specific inflationary models that lead to $\Omega \neq 1$ today and solve all standard problems of classical cosmology (see for example [28]). Other modifications of the inflationary paradigm allow the possibility of a universe divided into infinitely many open universes with all possible values of Ω from 0 to 1. These results put theorists on the safe side because, if we find that $\Omega = 1$, it will prove inflationary cosmology since most inflationary models predict $\Omega = 1$ and no other theory makes

problem is solved in the following way: The number density of any unwanted relic is reduced during inflation by a factor of Z^3. That means, with the parameters used already, that the enormous monopole density predicted by GUTs in the standard cosmology (equal to the baryon density) is weakened by a factor of about 10^{130} in inflationary cosmology. Lastly, resolution of the fourth problem, namely the small scale structure of the universe is difficult to attain within the model described above. For more details see [43].

The version of inflation presented above is not the original model – usually referred to as *old inflation,* proposed by Guth in 1981 [33] and which proved to be unworkable. It is the variant proposed in 1982 by both Linde [50] and Albrecht and Steinhardt [1], often referred to as *new inflation.* However, this scenario is still far from being perfect. In order for the new inflationary universe to occur the underlying particle theory must contain a scalar field φ with the following properties:

i)The potential function $V(\varphi)$ must have a minimum at a non-zero value of φ.

ii)$V(\varphi)$ must be very flat in the vicinity of $\varphi = 0$. This guarantees the slow evolution of the scalar field to its equilibrium value thus permitting the universe to achieve enough inflation.

The early papers on the new inflationary scenario assumed that some of the Higgs fields responsible for breaking the grand unified symmetry would also play the role of the field φ which drives the inflation, but it was realised that this could not be the case [49]. In most newer models the φ field is an 'inflaton field' responsible for driving inflation. In 1983 Linde proposed a very simple and elegant model [51], the *chaotic inflation.* Although the scalar field is not part of any unified theory, the model successfully implements inflation and avoids most of the problems of the new inflation [49].

The inflationary scenario – although finality is not still achieved – is today the only way to solve the great puzzles of the standard model. Its implications for cosmology are manifold. For example inflation implies that the observable part of the universe is

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this prediction. If on the other hand we find that $\Omega \neq 1$, it will not disprove inflation, since we have models with $\Omega \neq 1$ and no other models of isotropic universe and $\Omega \neq 1$ are known so far. See [52] for a discussion.

many orders of magnitude smaller than the Universe as a whole and it is an impermissible extrapolation to draw any conclusions about the homogeneity of the later based on observations of such a tiny component. Outside the regions which underwent inflation large inhomogeneities could exist. While our immediate neighborhood and well beyond should be smooth and flat if our broader region inflated, it could be that the universe on the very largest scales is very irregular with regions inflating at different times and some regions never inflating at all. Moreover, the 'universes' that evolve in different inflationary regions could be quite different [49] due for example to a different breaking of the symmetry of the unified interaction. Furthermore all that we see today arose from 'nothing', in the form of false vacuum energy.

2.4   The cosmic no-hair conjecture

The observable universe today seems to be remarkably homogeneous and isotropic on a very large scale and the Friedmann cosmology is a successful cosmological model capable of describing its large-scale properties. In principle the inflationary scenario provides an explanation of the homogeneity and isotropy of the universe without assuming this symmetry as part of the initial conditions. However, most investigations of inflationary models incorporate homogeneity and isotropy from the outset. In the previous section we analysed inflation in the context of a FRW cosmology assuming that the inflating regions are smooth enough so that they can be regarded as de Sitter space-times.

It is not obvious that cosmological models with non-FRW initial conditions ever enter an inflationary epoch nor is it obvious that, if inflation occurs, initial inhomogeneities and anisotropies will be smoothed out eventually. Therefore the question of the naturalness of the inflationary scenario is posed in the sense that we have to ask: Does the inflationary phase in the evolution of the universe proceed from very general initial conditions?

With regard to the question of whether the universe evolves to a homogeneous and isotropic state during an inflationary epoch, Gibbons and Hawking [31] and Hawking and

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Moss [36] have put forward the following

Conjecture 1 *All expanding-universe models with positive cosmological constant asymptotically approach the de Sitter solution.*

This is referred to as *the cosmic no-hair conjecture.*

In general relativity solutions of the Einstein equations are believed to settle toward stationarity as the nonstationary parts dissipate in the form of gravitational radiation. Such a proposition is very difficult to prove, even for the simplest space-times. Even if we accept this principle, neither the final state of evolution, that is the stationary solution nor its uniqueness is at all obvious. The uniqueness assertions are known as 'no-hair conjectures', to denote the loss of information regarding initial space-time geometry, caused by evolution under the field equations. This information either radiates out to infinity or is hidden behind event horizons. The cosmic no-hair conjecture is an assertion of the uniqueness of the de Sitter metric as a stationary,[6] no-black-hole solution of the Einstein equation with positive cosmological constant.

A few comments about the no-hair conjecture (NHC) are necessary.

- A precise version of this conjecture is difficult to formulate, mainly because of the vagueness associated with the terms 'asymptotic approach' and 'expanding universe'.

- There is no general proof (or disproof) of this conjecture.

- Some counter-examples exist of the form 'initially expanding universe models recollapse to a singularity' without ever becoming de Sitter type universes, the most obvious one being the closed FRW space-time which collapses before it enters an inflationary phase (see [13, 9]).

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[6]The stationary nature of de Sitter space-time will be discussed in section 3.3.

- It should be possible for regions of the universe to collapse to black holes so that the universe approaches a de Sitter solution with black holes rather than a de Sitter solution. In addition other special behaviours should be possible such as an asymptotic approach to an Einstein static universe.

- Although the NHC is not generally valid as it stands, the number and diversity of the models that do obey this principle lead to the belief that perhaps a weaker version of the conjecture must be true.[7]

2.5 Summary

In this introductory chapter we reviewed the elements of the FRW cosmology and presented some of the problems of the standard model. These problems exhibit our inability to predict the present observable uniformity of the universe from general initial conditions. We outlined very briefly the basic ideas of the inflationary scenario and described the way it solves most of the problems of the standard cosmology. Finally we posed the question of how natural is inflation and discussed the cosmic no-hair conjecture. In the following chapters we review the most important attempts to prove the cosmic NHC. As we will see much progress has been done towards a no-hair theorem, in the special case of homogeneous cosmologies.

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[7]Alternatively it could be recast as something less than a principle against which models should be tested.

Chapter 3

The cosmic no-hair conjecture in general relativity

In this chapter we discuss the cosmic NHC in the context of general relativity. By this we mean that we do not take account of the dynamics of the inflaton field; rather we attribute the vacuum energy which is necessary for the inflation to a large cosmological constant. In other words we do not care about the origin of the cosmological term and we have no mechanism to finish inflation *ie* to drive the cosmological term to zero value.

Some evidence for the NHC has been discussed by Boucher and Gibbons [18], and Barrow [3]. They studied small perturbations of de Sitter space-time and found that they do not grow as the scale factor tends to infinity. Steigman and Turner [70] considered a perturbed FRW model dominated by shear or negative curvature when inflation begins in the context of new inflation. They found that the size of a causally coherent region after inflation is only slightly smaller than the usual one in a purely FRW model. Wald [74] was the first who succeeded in 1983 to prove that *'all expanding Bianchi cosmologies with positive cosmological constant Λ, except type-IX, evolve towards the de Sitter solution exponentially fast. The behavior of type-IX models is similar provided that Λ is greater than a certain bound'.* Wald's proof is the prototype for many subsequent works as we shall see in more detail in the next chapter.

3.1 Background from differential geometry

In this section we present the necessary geometric notions which are used in all subsequent chapters. We derive Raychaudhuri's equation which describes the behavior of a family of timelike geodesics in the presence of a gravitational field. Next we discuss the energy conditions on the matter content. As we shall see, the energy conditions, although not of geometric nature, are reasonable inequalities satisfied by the energy-momentum tensor. The extrinsic curvature of a hypersurface describes how the hypersurface is embedded in the space-time. Finally we present a very brief review of spatially homogeneous space-times.

3.1.1 Raychaudhuri's equation

Consider a smooth congruence of timelike geodesics in a space-time (M, g). The corresponding tangent vector field n is normalized to unit length, $\langle n, n \rangle = -1$. This means that the geodesics are parametrized by proper time t and $n = \partial/\partial t$. We define the *spatial metric* **h** by

$$h_{ab} = g_{ab} + n_a n_b. \tag{3.1}$$

Note that $h_a{}^b n_b = h^b{}_a n^a = 0$ so that $h^a{}_b = g^{ac} h_{cb}$ can be regarded as the projection operator onto the subspace of the tangent space perpendicular to n. In the following we are interested on the covariant derivative of n. Its geometric meaning will become clear immediately before section 3.1.4.

We define the *expansion, θ, shear, σ,* and *rotation, ω,* of the congruence by

$$\theta = h^{ab} \nabla_b n_a = \nabla_a n^a \tag{3.2}$$

$$\sigma_{ab} = \nabla_{(b} n_{a)} - \frac{1}{3} \theta h_{ab} \tag{3.3}$$

$$\omega_{ab} = \nabla_{[b} n_{a]}. \tag{3.4}$$

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The tensor fields σ_{ab} and ω_{ab} are purely spatial in the sense that $\sigma_{ab}n^b = \omega_{ab}n^b = 0$ and σ_{ab} is traceless. If the energy-momentum tensor of the matter fields is of the form of a fluid, then θ, σ, and ω, are *not* the expansion, shear and the rotation of the fluid unless the fluid happens to be moving along the geodesics.

The covariant derivative of n can be expressed as

$$\nabla_b n_a = n_{a;b} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}. \tag{3.5}$$

This can be verified by direct substitution from the previous defining equations (3.2) – (3.4).[1]

From the definition of the curvature tensor and the geodesic equation $n^a \nabla_a n^b = 0$ we have

$$\begin{aligned}
n^c \nabla_c \nabla_b n_a &= n^c \nabla_b \nabla_c n_a + R_{adcb} n^d n^c \\
&= \nabla_b \left(n^c \nabla_c n_a \right) - \left(\nabla_b n^c \right) \left(\nabla_c n_a \right) + R_{adcb} n^d n^c \\
&= - \left(\nabla_b n^c \right) \left(\nabla_c n_a \right) + R_{adcb} n^d n^c.
\end{aligned}$$

Taking the trace of the last equation we obtain

$$\frac{d\theta}{dt} \equiv n^c \nabla_c \left(\nabla_d n^d \right) = - \left(\nabla_d n^c \right) \left(\nabla_c n^d \right) - R_{cd} n^c n^d$$

and using (3.5) we get after some manipulation

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}n^a n^b. \tag{3.6}$$

This equation is known as the Raychaudhuri equation and plays an important role in the proof of the singularity theorems of general relativity.

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[1]In HE there is a projection $h_a{}^b$ for every index of the tensor field $n_{a;b}$ in the definitions of the shear and the rotation. Our definitions differ from those in HE because $n_{a;b}$ is purely spatial, $n_{a;b}n^a = n_{a;b}n^b = 0$. This is due to the fact that n is associated with a congruence of geodesics, not merely a congruence of timelike curves.

3.1.2 Energy conditions

The actual form of the energy-momentum tensor of the universe is very complicated since a large number of different matter fields contribute to form it. Therefore it is hopeless to try to describe the precise form of the energy-momentum tensor. However, there are some inequalities which it is physically reasonable to assume for the energy-momentum tensor. In many circumstances these inequalities are sufficient to prove via the field equations many important global results, for example the singularity theorems. In this section we discuss these inequalities, usually referred to as energy conditions.

At the end of the following section we shall see that, if the right-hand side of (3.6) is positive, the congruence expands while, if it is negative, geodesics in the congruence converge. We pay therefore attention to the sign in front of the last term on the right-hand side of the Raychaudhuri equation. By the Einstein equations this term can be written as

$$R_{ab}n^a n^b = 8\pi \left(T_{ab} - \frac{1}{2} T g_{ab} \right) n^a n^b = 8\pi \left(T_{ab} n^a n^b + \frac{1}{2} T \right). \tag{3.7}$$

The quantity $T_{ab}n^a n^b$ is the energy density as measured by an observer whose 4-velocity is n. For all known forms of matter this energy density is non-negative and therefore we impose that

$$T_{ab} u^a u^b \geq 0 \tag{3.8}$$

for all timelike vectors u. This condition is known as the *weak energy condition* (WEC). An other assumption usually accepted is the *strong energy condition* (SEC) which states that

$$T_{ab} n^a n^b \geq -\frac{1}{2} T \tag{3.9}$$

for all unit timelike vectors n. In fact it seems reasonable that the matter stresses will not be so large as to make the right-hand side of (3.7) negative. Finally, the *dominant energy condition* states that

$$T_{ab} u^a u^b \geq 0 \tag{3.10}$$

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and $T^a{}_b u^b$ is non-spacelike vector for all timelike vectors u. In particular the dominant energy condition implies that

$$|T_{\mu\nu}| \leq T_{00},\qquad(3.11)$$

where $T_{\mu\nu}$ are the components of T_{ab} in any orthonormal basis with n^a as the timelike element of this basis.

To be convinced that the above conditions are reasonable assumptions, let us see what they imply for a diagonalisable stress-energy tensor $T_{\mu\nu} = diag\,(\rho, p_1, p_2, p_3)$. This is for example the case of a perfect fluid with stress-energy tensor $T_{\mu\nu} = diag\,(\rho, p, p, p)$. It is easy to see that the energy conditions take the form

$$\rho \geq 0 \text{ and } \rho + p_i \geq 0 \quad (i = 1, 2, 3) \text{ (WEC)}$$

$$\rho + p_1 + p_2 + p_3 \geq 0 \text{ and } \rho + p_i \geq 0 \quad (i = 1, 2, 3) \text{ (SEC)}\qquad(3.12)$$

$$\rho \geq |p_i| \quad (i = 1, 2, 3) \text{ (DEC)}.$$

For further discussion see for example HE [35].

3.1.3 Extrinsic curvature

Consider now the case that the space-time (M, g) is globally hyperbolic and the congruence of the timelike geodesics is normal to a spacelike hypersurface Σ. In every point p of Σ the unit normal to Σ at p equals the (unit) tangent vector n of the geodesic passing through p. Then the *induced metric tensor* h on Σ coincides with the previously defined 'spatial metric' $h_{ab} = g_{ab} + n_a n_b$, (3.1). For the same reason the covariant derivative of n evaluated on Σ coincides with the *extrinsic curvature* K_{ab} of Σ, viz.

$$K_{ab} = \nabla_a n_b.\qquad(3.13)$$

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Of course the tensor field $\mathbf{K}$ is purely spatial (see footnote on page 25). Since the congruence is hypersurface orthogonal,[2] we have $\omega_{ab} = 0$ which implies that the extrinsic curvature tensor field is symmetric, ie $K_{ab} = K_{ba}$. Hence, taking the Lie derivative of the metric with respect to n we find

$$
\begin{aligned}
K_{ab} &= \tfrac{1}{2} \mathrm{L}_n g_{ab} \\
&= \tfrac{1}{2} \mathrm{L}_n \left(h_{ab} - n_a n_b \right) \\
&= \tfrac{1}{2} \mathrm{L}_n h_{ab},
\end{aligned}
\tag{3.14}
$$

where the geodesic equation was used. If a coordinate system adapted to n is used, then the components of the extrinsic curvature in these coordinates are

$$
K_{\mu\nu} = \frac{1}{2} \frac{\partial h_{\mu\nu}}{\partial t}.
\tag{3.15}
$$

The trace K of the extrinsic curvature is defined as

$$
K \equiv K^a{}_a = h^{ab} K_{ab} = \theta
\tag{3.16}
$$

so that K is equal to the mean expansion θ of the geodesic congruence orthogonal to Σ.

One has the following geometric interpretation of K [68]. Assume Σ to be a compact submanifold with boundary (otherwise, take a compact subset of Σ). For every p in Σ denote by γ_p the geodesic passing through p, ie, $\gamma_p : [0, \varepsilon] \to M$ is a future-directed geodesic orthogonal to Σ and satisfying $\gamma_p(0) = p$, with tangent vector field n. For all $t \in [0, \varepsilon]$, define $\Sigma_t \equiv \{ \gamma_p(t) : p \in \Sigma \}$, that is, Σ_t is the set of all points of Σ moved along each geodesic a parametric distance t. If $V(t)$ denotes the Riemannian volume of Σ_t then it can be shown

$$
V'(0) = \int_\Sigma K \Omega_\Sigma,
\tag{3.17}
$$

---

[2]A necessery and suficient condition that n be hypersurface orthogonal is $n_{[a} \nabla_b n_{c]} = 0$. See for example Wald [73], appendix B, p 436.

where Ω_Σ is the Riemannian volume element of Σ. Thus $K > 0$ roughly means that the future-directed geodesics orthogonal to Σ are, on the average, spreading out near Σ so as to increase the volume of Σ.

3.1.4 Homogeneous cosmologies

The space-times we deal with in this work are with a few exceptions spatially homogeneous. A space-time (M, g) is said to be *spatially homogeneous* if there exists a one-parameter family of spacelike hypersurfaces Σ_t foliating the space-time such that for each t and for each $p, q \in \Sigma_t$ there exists an isometry of M which takes p to q. Homogeneous cosmologies have been studied intensively over the past years (for reviews see Ryan and Shepley [66], MacCallum [54, 54]). One of the main reasons is that all possible geometries of the spacelike hypersurfaces fall into one of ten classes. Another equally important reason is that Einstein equations reduce to a system of ordinary differential equations. In fact because of the spatial symmetry only time variations are non-trivial.

The set of all isometries of a Riemannian manifold forms a Lie group G and the set of the Killing vector fields (that is, the set of infinitesimal generators of the isometries) constitutes the associated Lie algebra, with product the commutator. In our case $\dim G = \dim \Sigma_t = 3$ provided that G acts *simply transitively* on each Σ_t, that is, for every p, q in Σ_t there exists a *unique isometry* $\in G$ sending p to q. The algebraic structure of the group G can be described in terms of the Lie algebra since, if v, w are Killing vector fields, they satisfy

$$[v, w]^a = -C^a{}_{bc} v^b w^c, \tag{3.18}$$

where $C^a{}_{bc}$ are the *structure constants* of the Lie group. It follows immediately from the definition that

$$C^a{}_{bc} = -C^a{}_{cb} \tag{3.19}$$

and from the Jacobi identity for commutators that

$$C^e{}_{d[a}C^d{}_{bc]} = 0. \tag{3.20}$$

These two equations lead to all three-dimensional Lie groups, or equivalently, all possible sets of structure constants which satisfy (3.19) and (3.20). Bianchi was the first to classify all three-dimensional Lie groups into nine types. A slightly different version of this classification can be obtained in the following way (see Ellis and MacCallum [29]). The tensor field $C^c{}_{ab}$ can be decomposed as

$$C^c{}_{ab} = M^{cd}\epsilon_{dab} + \delta^c{}_{[a}A_{b]}, \tag{3.21}$$

where $\epsilon_{abc} = \epsilon_{[abc]}$ is a three form on the Lie algebra, $M^{cd} = M^{dc}$ and A_b is a 'dual' vector. We can solve for M^{ab} and A_b, taking $A_b = C^a{}_{ba}$ and $M^{ab} = \frac{1}{2}C^{(a}{}_{cd}\epsilon^{b)cd}$. Inserting (3.21) into the Jacobi identity (3.20) yields

$$M^{ab}A_b = 0. \tag{3.22}$$

Therefore the problem of finding all three-dimensional Lie groups is reduced to determine all dual vectors A_b and all symmetric tensors M^{ab} satisfying (3.22). If $A_b = 0$ (class A), there exist six Lie algebras determined by the rank and signature of M^{ab}. If $A_b \neq 0$ (class B), equation (3.22) implies that $rankM$ cannot be greater than two. Hence, in this case there exists four possibilities for the rank and signature of M^{ab}. These ten combinations are tabulated (see *eg* in Landau and Lifshitz [45], MacCallum [54]) and called the Bianchi models. For example, the Bianchi type-IX model is determined by $A_b = 0$, $rankM = 3$, $signatureM = (+ + +)$. One can find explicit formulas for $C^c{}_{ab}$ in different bases in Ryan and Shepley [66]. Useful formulas for the Ricci tensor and Einstein equations in terms of the structure constants and the spatial metric are also given in Ryan and Shepley [66].

The metric of a spatially homogeneous space-time is

$$g_{ab} = -n_a n_b + h_{ab}, \tag{3.23}$$

where $\mathbf{h}$ is the three-metric of the spatial slices and $n = \partial/\partial t$ is a unit timelike vector field, orthogonal to the homogeneous hypersurfaces. The vector field n defines the time coordinate of the space-time. There are many ways to put the metric in a useful form [53]. For example the spatial coordinates can be chosen as follows. We consider one homogeneous hypersurface Σ_0 and choose a basis of one-forms $\omega^1, \omega^2, \omega^3$ which are preserved under the isometries, that is, have zero Lie derivative with respect to the Killing vector fields. It follows that each one-form ω^i (the index i labels the one-form) satisfies (see for example Wald [73] section 7.2)

$$\nabla_{[a} \omega^i_{b]} = -\frac{1}{2} C^c{}_{ab} \omega^i_c, \tag{3.24}$$

with $C^c{}_{ab}$ the structure constants of the Lie group of isometries of the spacelike hypersurfaces. Then the (invariant) spatial metric can be written

$$h_{ab} = h_{ij} \omega^i_a \omega^j_b, \quad i, j = 1, 2, 3, \tag{3.25}$$

where the components h_{ij} are constant on Σ_0.

To construct the full metric we consider for $p \in \Sigma_0$ the unit normal vector n_p to Σ_0 at p (we used the symbol n to denote an arbitrary timelike vector field orthogonal to the homogeneous hypersurfaces for reasons that will soon become clear). Denote by γ_p the geodesic determined by (p, n_p). Then γ_p will be orthogonal to all the spatial hypersurfaces it intersects, because the tangent to γ_p remains orthogonal to all the spatial Killing vector fields (see for example O'Neill [63], ch.9, lemma 26). We label the other hypersurfaces by the proper time t of the intersection of γ_p with the hypersurface, hence, $t = const$ on each Σ_t. Then the vector field n defined by $n^a = -\nabla^a t$ will be everywhere orthogonal

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to each Σ_t and the integral curves of n are all geodesics since this is true along γ_p and hence is true everywhere on each Σ_t by spatial homogeneity. Now we 'Lie transport' the ω^i defined on Σ_0 throughout the space-time along n, ie,

$$L_n \omega^i = 0,$$

which implies that $\omega^i_a n^a = 0$ everywhere. We conclude that the metric (3.23) takes the form

$$g_{ab} = -\nabla_a t \nabla_b t + h_{ij}(t)\,\omega^i_a \omega^j_b, \tag{3.26}$$

or equivalently

$$ds^2 = -dt^2 + h_{ij}(t)\,\omega^i \omega^j, \quad i,j = 1,2,3. \tag{3.27}$$

It is now clear the property of homogeneous space-times mentioned at the beginning of this section, namely that the Einstein equations become ordinary differential equations with respect to time.

We note by $^{(3)}R$ the scalar curvature of the spacelike hypersurface which we think of as a Riemannian three-manifold with metric h. In what follows we make repeated use of a property of the scalar spatial curvature $^{(3)}R$, namely that $^{(3)}R$ is nonpositive in all Bianchi models except type-IX. To prove it, we write the scalar curvature $^{(3)}R$ in terms of the structure constants $C^a{}_{bc}$ of the Lie algebra of the symmetry group of the homogeneous hypersurface (see [45] or [66])

$$^{(3)}R = -C^a{}_{ab}C^c{}_c{}^b + \frac{1}{2}C^a{}_{bc}C^c{}_a{}^b - \frac{1}{4}C_{abc}C^{abc}. \tag{3.28}$$

All indices are lowered and raised with the spatial metric, h_{ab}, and its inverse h^{ab}. A rather lengthy calculation gives for $^{(3)}R$ (by substitution of (3.21) into (3.28) and using (3.22))

$$^{(3)}R = -\frac{3}{2}A_b A^b - h^{-1}\left(M^{ab}M_{ab} - \frac{1}{2}M^2\right), \tag{3.29}$$

where h is the determinant of h_{ab}, that is, $h^{-1} = \epsilon_{abc}\epsilon_{def}h^{ad}h^{be}h^{cf}$. From (3.29) it follows

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immediately that, if $^{(3)}R$ is positive then necessarily $M^{ab}M_{ab} < \frac{1}{2}M^2$, but then M^{ab} must be definite (positive or negative) as can be verified by considering a coordinate system where the tensor M^{ab} is diagonal. In this case (3.22) implies that $A_b = 0$. A look at the Bianchi classification shows that the combination $A_b = 0$ and $rankM = 3$ corresponds to the type-IX model. What we have shown is that in all Bianchi models except type-IX the three-scalar curvature is nonpositive

$$^{(3)}R \le 0 \ . \tag{3.30}$$

This ends the necessary geometric notions which will be used in the development of this chapter.

3.2   Proof of the cosmic no-hair conjecture for homogeneous cosmologies (Wald, 1983)

Our starting point is a (spatially) homogeneous space-time which, according to the results of the previous section, can be foliated by a one-parameter family of spacelike hypersurfaces Σ_t orthogonal to a congruence of timelike geodesics parametrized with proper time t. As usual, we denote by $n = \partial/\partial t$ the unit tangent vector field to the geodesics. When it happens that matter is moving along these geodesics, the expansion, shear and the rotation of the cosmic fluid coincide with the corresponding quantities of the geodesic congruence. However, we do not suppose that this be the case as the only property of the matter stress-energy tensor that we use is that it satisfy the strong and dominant energy conditions. We use the Einstein equations

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{3.31}$$

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to describe the evolution of Bianchi cosmologies. In what follows only two components of (3.31) are necessary: the time-time component

$$G_{ab}n^a n^b - \Lambda - 8\pi T_{ab} n^a n^b = 0 \qquad (3.32)$$

and the 'Raychaudhuri' equation

$$R_{ab}n^a n^b + \Lambda - 8\pi \left(T_{ab} - \frac{1}{2} g_{ab} T \right) n^a n^b = 0. \qquad (3.33)$$

The term $R_{ab}n^a n^b$ permits us to transform (3.33) to its more familiar form as follows. Firstly we decompose K_{ab} into its trace K and traceless part σ_{ab} (see (3.5) and (3.16)), viz.

$$K_{ab} = \frac{1}{3} K h_{ab} + \sigma_{ab}. \qquad (3.34)$$

We can then express $G_{ab}n^a n^b$ in terms of the three-geometry of the homogeneous hypersurface using the Gauss-Codacci equation (HE, [35])

$$\frac{1}{2} {}^{(3)}R = \frac{1}{2} R + R_{ab}n^a n^b - \frac{1}{2} \left(K^a{}_a \right)^2 + \frac{1}{2} K_{ab} K^{ab}. \qquad (3.35)$$

Observe that the sum of the first two terms on the right-hand side of the Gauss-Codacci equation equals $G_{ab}n^a n^b$, while the last term of this equation simplifies as $K_{ab}K^{ab} = \frac{1}{3}K^2 + \sigma_{ab}\sigma^{ab}$. Putting all these together in (3.32) we obtain

$$K^2 = 3\Lambda + \frac{3}{2} \sigma_{ab}\sigma^{ab} - \frac{3}{2} {}^{(3)}R + 24\pi T_{ab} n^a n^b. \qquad (3.36)$$

Eliminating $R_{ab}n^a n^b$ from (3.6) and (3.33) we obtain the Raychaudhuri equation with cosmological constant (compare to (3.6))

$$\dot{K} = \Lambda - \frac{1}{3} K^2 - \sigma_{ab}\sigma^{ab} - 8\pi \left(T_{ab} - \frac{1}{2} g_{ab} T \right) n^a n^b. \qquad (3.37)$$

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These last two equations are the basic tools in proving the NHT for homogeneous cosmologies. Our aim is to show that the metric h has the asymptotic form

$$h_{ab}(t) = \exp(Ht) h_{ab}(t_o)$$

by showing first that K is bounded – in fact it tends exponentially to a certain limit. This will be done by constructing certain inequalities starting from (3.36) and (3.37) and the energy conditions.

At the end of the previous section we proved that $^{(3)}R \leq 0$ in all Bianchi models except type-IX. We observe that, assuming a negative spatial scalar curvature, all terms in the right-hand side of (3.36) are positive and so we turn our attention to any Bianchi model which is not type-IX. Using the strong energy condition (3.9), the dominant energy condition (3.10) and (3.30), we obtain from (3.36) and (3.37)

$$\dot{K} \leq \Lambda - \frac{1}{3}K^2 \leq 0. \tag{3.38}$$

Assuming that the universe is initially expanding ie $K > 0$ at some arbitrary time $t = 0$, then the universe expands for ever ie $K > 0$ for all time since $K^2 \geq 3\Lambda$ implies that K cannot pass through zero. Therefore at all times we have

$$K \geq \sqrt{3\Lambda}. \tag{3.39}$$

The first inequality of (3.38) also implies

$$\frac{\dot{K}}{K^2 - 3\Lambda} \leq -\frac{1}{3} \tag{3.40}$$

which after integration yields

$$K \leq \frac{\sqrt{3\Lambda}}{\tanh(\alpha t)}, \qquad \text{where } \alpha = \sqrt{\frac{\Lambda}{3}}. \tag{3.41}$$

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For convenience we write together the inequalities (3.39) and (3.41)

$$\sqrt{3\Lambda} \le K \le \frac{\sqrt{3\Lambda}}{\tanh(\alpha t)}. \tag{3.42}$$

We see that the expansion rate K approaches $\sqrt{3\Lambda}$ exponentially fast. Equation (3.36) implies

$$0 \le \sigma_{ab}\sigma^{ab} \le \frac{2}{3}\left(K^2 - 3\Lambda\right) \le \frac{2\Lambda}{\sinh^2(\alpha t)}. \tag{3.43}$$

We deduce that the shear approaches almost exponentially to zero and the universe rapidly isotropises. From the same equations (3.36) and (3.41) we find that the energy density also approaches zero in a few time-constants α^{-1} since

$$T_{ab}n^a n^b \le \frac{\Lambda}{8\pi}\frac{1}{\sinh^2(\alpha t)}. \tag{3.44}$$

As a consequence of the dominant energy condition all components of the energy-momentum tensor rapidly approach zero (see (3.11)). From (3.14) and the fact that $K \to \sqrt{3\Lambda}$ as $\sigma_{ab} \to 0$ we see that the metric asymptotically has the form

$$h_{ab}(t) = \exp\left[2\alpha(t - t_0)\right]h_{ab}(t_0). \tag{3.45}$$

Finally (3.36) implies that the spatial curvature $^{(3)}R$ also goes to zero. The scaling property of the Ricci tensor,

$$^{(3)}R_{ab}(t) = \exp\left[-2\alpha(t - t_0)\right] {}^{(3)}R_{ab}(t_0),$$

(which can be verified by direct substitution in the formula giving the Ricci tensor in terms of the metric tensor) tells us that the universe becomes flat exponentially fast.

In conclusion, for $t \gg \alpha^{-1}$, any initially expanding Bianchi space-time not of type-IX becomes isotropic and flat (the shear and the spatial curvature vanish), expands at a constant rate $K = \sqrt{3\Lambda}$ and appears to be matter free ie, becomes an empty de Sitter

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space-time. A careful analysis of the asymptotic approach to de Sitter space-time can be found in Barrow and Götz [11].

What happens in the case of Bianchi type-IX models? In such models the scalar curvature $^{(3)}R$ can be positive, but the largest value it can take for fixed determinant h and given M^{ab} is (see Wald [74])

$$^{(3)}R_{\max} = \frac{3}{2} \frac{(\det M)^{2/3}}{h^{1/3}}.$$ (3.46)

From (3.36) and (3.37) we see that a large enough positive cosmological constant could still cause the same behavior to a Bianchi type-IX universe. Suppose that initially we have

$$\Lambda > \frac{1}{2} \, ^{(3)}R_{\max} = \frac{3}{4} \frac{(\det M)^{2/3}}{h^{1/3}}$$ (3.47)

and that the universe initially expands that is $K > 0$. From (3.14) it follows that $\dot{h}/h = 2K$. Hence h increases. We note that K remains positive by (3.36) provided that the above inequality for Λ is still satisfied. This inequality can be violated only if h becomes smaller than its initial value which is impossible. We conclude that K never passes through zero and h continues to increase. By the same type of arguments used in the other Bianchi models we can show that the universe approaches de Sitter space-time. Therefore Wald's theorem is proved.

We can show by a more physical argument why Wald's theorem is to be expected. For this purpose we need only the time-time component of Einstein equations (3.32) as our analysis will be quite qualitative.

Denoting the scale factors of the three principal axes of the universe by X_i, $i = 1, 2, 3$ and the mean scale factor by $a = V^{\frac{1}{3}}$, where $V = X_1 X_2 X_3$, (3.32) is written as

$$3H^2 \equiv 3 \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3} \left(\frac{\dot{V}}{V}\right)^2 = \Lambda + 8\pi T_{ab} n^a n^b + F(X_1, X_2, X_3).$$ (3.48)

The function F depends on the Bianchi model and contains all the information about

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the anisotropic expansion of the mean scale factor. The detailed form of F can be found in [29]. The only property of F that we will use is that in all Bianchi models F decreases at least as fast as a^{-2}. Equation (3.48) is the analog of the Friedmann equation for anisotropic cosmologies. For the FRW models, $X_1 = X_2 = X_3$ and $F = k/a^2$. The term $\dot{V}/V$ in (3.48) is the expansion K in Wald's theorem.

As the universe expands, the function F in (3.48) decreases at least as a^{-2} and the term $T_{ab}n^a n^b$ decreases as some power of a (for example as a^{-4} for a radiation dominated FRW universe). It is clear that the cosmological constant eventually dominates the terms $8\pi T_{ab}n^a n^b$ and F. This happens in about one Hubble time, $H_0^{-1} = (\Lambda/3)^{-\frac{1}{2}}$, and $a \sim \exp H_0 t$. We see that the anisotropic term F and the ordinary matter content of the universe decay exponentially with time and the space-time rapidly approaches de Sitter.

3.3 Other approaches to the cosmic no-hair conjecture in General Relativity

There are several attempts to generalise the no-hair theorem for homogeneous cosmologies proven by Wald to the cases of inhomogeneous or arbitrary space-times. We describe briefly two of these attempts. The first is due to Jensen and Stein-Shabes [41] and consists of a slight modification of Wald's theorem.

Theorem 1 *[Jensen & Stein-Shabes]. Let $(M, \mathbf{g})$ be an arbitrary synchronous space-time of any dimension with positive cosmological constant, the energy-momentum tensor satisfying the dominant and strong energy conditions and a scalar spatial curvature which is always positive. Then $(M, \mathbf{g})$ evolves towards a de Sitter space-time.*

A few comments about the assumptions of the theorem are necessary. Synchronous space-time is evidently a space-time that can be described by a synchronous reference system. This is always possible provided that the space-time is globally hyperbolic. Global

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hyperbolicity is also a necessary condition for the existence of spacelike hypersurfaces which foliate the space-time. Hence the notion of spatial curvature is meaningful.

The proof of the theorem goes exactly as in the homogeneous case and we do not repeat it here. In fact a careful reading of Wald's proof reveals that homogeneity is used only to show that the scalar three-curvature is nonpositive in all homogeneous cosmologies except for Bianchi type-IX models. In the present case the non-positivity of the spatial curvature is imposed from the beginning as an assumption.

Despite the fact that the proof of Jensen and Stein-Shabes is almost verbatim Wald's proof, the physical implications of their theorem are quite interesting. Consider all possible space-times which can be foliated by spacelike hypersurfaces. Foliation by spacelike hypersurfaces is not a severe restriction because as mentioned above, this is the case of globally hyperbolic space-times, a very large subset of all possible space-times. In fact Penrose [65] has conjectured that all physically reasonable space-times must be globally hyperbolic. Fix a hypersurface which will be considered as initial hypersurface Σ_0. It is clear that the concept of a scalar three-curvature of a given sign is meaningless as the curvature varies from point to point on Σ_0. However, it is possible that the scalar curvature has constant sign throughout Σ_0. It is reasonable to assume that about half of these special initial hypersurfaces have negative curvature and the corresponding space-times are good candidates for the validity of the NHT. It may be that the set of these special initial hypersurfaces is of measure zero in the set of all initial hypersurfaces. In an arbitrary initial hypersurface Σ_0 we can refine the argument by considering small regions on Σ_0, where the three-curvature has a given sign. This is always possible because $^{(3)}R$ is a continuous function of position on Σ_0. Then the regions of negative $^{(3)}R$ play the role of candidates for the validity of the NHT. However, this does not necessarily mean that these negatively curved regions will inflate. In order that the NHT be applicable, $^{(3)}R$ must be nonpositive for all subsequent time of interest, not only at the initial hypersurface. The detailed evolution of $^{(3)}R$ depends on the motion of the matter and the full set of Einstein equations must be used. A continuity argument of the time dependence

of $^{(3)}R$ does not guarantee that $^{(3)}R$ will remain negative for a few time constants $\sqrt{3/\Lambda}$. The above discussion though quite qualitative suggests that it is not unreasonable to believe that the NHC may be applicable to a wide class of space-times [64].

A quite different study of the cosmic NHC is a theorem of Morrow-Jones and Witt [58] which is formulated as follows.

Theorem 2 *[Morrow-Jones & Witt]. In the absence of black holes, the only locally static solutions to the Einstein equations in vacuum with positive cosmological constant are the de Sitter and Nariai solutions.*

An analogous result for $\Lambda < 0$ was obtained by Boucher, Gibbons and Horowitz [19] who proved that *anti-de Sitter* space-time is the unique solution to $R_{ab} = \Lambda g_{ab}$ which is strictly stationary and asymptotically anti-de Sitter. In an attempt to generalise this result for $\Lambda > 0$ the above authors noted that the main difficulty is the fact that de Sitter space-time has no spatial asymptotic regions.[3]

We recall that a space-time is said to be *stationary* if it admits a timelike Killing vector field, K. Any (smooth) vector field is the generator of diffeomorphisms. The timelike character of K guarantees that these diffeomorphisms are time translations while the Killing property of K means that the generated diffeomorphisms are isometries. Hence the space-time geometry does not change under the flow of the vector field. Therefore the definition agrees with the usual notion of stationarity as time translation invariance. Further a space-time is said to be *static* if it is stationary and the Killing field $K = \partial/\partial t$ is orthogonal to a family of (spacelike) hypersurfaces (see HE [35]).

The hypersurfaces can be thought of as surfaces of constant time labelled by the parameter t. It can be shown (see Wald [73]) that, if we choose arbitrary coordinates x^1, x^2, x^3 on one of the spacelike hypersurfaces, the metric components take the form

$$ds^2 = -V^2\left(x^1, x^2, x^3\right) dt^2 + h_{ij}\left(x^1, x^2, x^3\right) dx^i dx^j, \tag{3.49}$$

---

[3]The case $\Lambda = 0$ has been examined by Lichnerowicz (see for example [46]) who proved that the only strictly static, asymptotically flat solution is Minkowski space-time.

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where $V^2 = -K^a K_a$ and $i, j = 1, 2, 3$. We see immediately that the diffeomorphism defined by $t \rightarrow -t$ is an isometry. Thus the formal definition agrees with the usual notion of static space-time as being time-reflection invariant.

What does de Sitter space-time have to do with stationarity? How can we think of a stationary space-time which first contracts and then expands exponentially? The answer is that there is a coordinate system such that de Sitter space-time appears static. This is true because in general relativity there is no specially chosen time so that what appears dynamic with respect to one coordinate system may appear static with respect to another system of coordinates.

de Sitter space is more easily visualized as the hyperboloid

$$- \left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 = H^{-2} \qquad (3.50)$$

embedded in a five-dimensional Lorentz space. The radius of the hyperboloid is $H^{-1} = \sqrt{3/\Lambda}$ (cf. equation (2.29)). In coordinates (t, χ, θ, ϕ) defined on the hyperboloid by

$$
\begin{aligned}
x^0 &= H^{-1} \sinh Ht \\
x^1 &= H^{-1} \cosh Ht \cos \chi \\
x^2 &= H^{-1} \cosh Ht \sin \chi \cos \theta \\
x^3 &= H^{-1} \cosh Ht \sin \chi \cos \theta \\
x^4 &= H^{-1} \cosh Ht \sin \chi \sin \theta \sin \phi
\end{aligned}
\qquad (3.51)
$$

the de Sitter metric takes the usual form of a FRW universe

$$ds^2 = -dt^2 + H^{-2} \cosh^2 \left(Ht\right) \left[d\chi^2 + \sin^2 \chi \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right], \qquad (3.52)$$

with topology $\mathbf{R} \times S^3$. In coordinates (t, r, θ, ϕ), defined by

$$
\begin{aligned}
x^0 &= (H^{-2} - r^2)^{\frac{1}{2}} \sinh Ht \\
x^1 &= (H^{-2} - r^2)^{\frac{1}{2}} \cosh Ht \\
x^2 &= r \cos \theta \\
x^3 &= r \sin \theta \sin \phi \\
x^4 &= r \sin \theta \cos \phi
\end{aligned}
\tag{3.53}
$$

the metric takes the form of a static universe with event horizon H^{-1}

$$
ds^2 = - \left(1 - H^2 r^2\right) dt^2 + \left(1 - H^2 r^2\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right). \tag{3.54}
$$

The static Killing vector field is $K = \partial/\partial t$. Since there is a coordinate singularity at $r = H^{-1}$, de Sitter space is not strictly static by the definition given above. However, it is *locally static* in the sense that every point of the space-time has a neighborhood which admits a static Killing vector field. We give three examples to clarify the concept: Minkowski space-time is static, de Sitter space-time is locally static and Schwarzschild space-time is neither static nor locally static. In fact, in the interior of the Schwarzschild event horizon there is no timelike Killing vector field.

The Nariai metric [44] can be written as

$$
ds^2 = - \left(1 - \Lambda r^2\right) dt^2 + \left(1 - \Lambda r^2\right)^{-1} dr^2 + \Lambda^{-1} d\Omega^2, \tag{3.55}
$$

where $d\Omega^2$ is the metric of a two-sphere. The spatial geometry of this space-time is that of $\mathbf{R} \times S^2$, the radius of S^2 remaining constant. The Nariai solution is very unstable under perturbations of the S^2 part of the metric [67] and therefore it cannot represent a space-time toward which others may evolve. Therefore it seems plausible to regard de Sitter solution as the unique space-time toward which all other solutions evolve.

The proof of the theorem consists of the following steps: Locally static solutions must have at least one static Killing vector field which is timelike on a open region

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$U \subset M$. It follows that the solution on U is one of Petrov types I, D, O. Next it can be shown that there are no strictly static solutions. (This is easily proved in the case of compact spatial slices. If we assume that the spatial slices are non-compact, then the field equations and Meyer's theorem lead to a contradiction). Thus one can find two different static Killing vector fields K_1 and K_2 which are timelike on open sets U_1 and U_2 respectively. This excludes the possibility of a Petrov type I solution on $U_1 \cap U_2$. Standard results of the theory of elliptic nonlinear partial differential equations ensure that the metric is analytic on $U_1 \cap U_2$. On the other hand a solution is not Petrov type I iff a certain invariant Φ formed from combinations of the Riemann tensor vanishes. Since Φ is analytic and vanishes on $U_1 \cap U_2$, it vanishes everywhere on M. Thus the solutions are either everywhere on M of type D or everywhere on M of type O. If the solution is type O, the Weyl tensor is zero so the space-time has constant curvature and is therefore de Sitter space-time. If the solution is type D, the space-time is known to have the Nariai metric. For details of the proof see [58].

3.4 Summary

In this chapter we developed the necessary material from differential geometry which is needed for Wald's proof. It is also helpful in the following chapters. We presented Wald's proof in great detail for two reasons. Firstly it is the first attack to the cosmic NHC and secondly, as already mentioned, it constitutes the frame of all subsequent no-hair theorems for homogeneous space-times. We discussed a generalisation by Jensen and Stein-Shabes and closed this chapter with the totally different approach to the NHC of Morrow-Jones and Witt. In all of the above discussion we assumed the validity of Einstein equations with a positive cosmological constant and ordinary matter content, without having any inflationary model in mind. In the next chapter we examine whether the NHT remains true in the finite period of inflation.

Chapter 4

The cosmic no-hair conjecture in inflationary cosmology

In the previous chapter we discussed the NHC in the context of General Relativity, assuming a positive cosmological constant. However, in inflationary models the universe does not have a true cosmological constant. Rather there is a vacuum energy density, which during the slow evolution of the scalar field remains approximately constant and behaves like a cosmological term. Therefore we are faced with the question: Does the universe evolve towards a de Sitter type state before the potential energy of the scalar field reaches its minimum? The object of this chapter is to take into account the dynamics of the scalar field in the no-hair hypothesis.

In many inflationary models the particular form of the potential $V(\varphi)$ of the inflaton field is predicted by some particle theory. An alternative approach is to consider very simple forms of $V(\varphi)$ such as $m^2\varphi^2$ or $\lambda\varphi^4$, not directly related to any particular physical theory. This approach is reasonable since we do not really know which theory of particle physics best describes the very early universe. We shall examine the NHC for three specific inflationary models, namely new inflation, chaotic inflation and power-law inflation. The terminology derives mainly from the form of the potential function of the scalar field which drives the inflation.

In all models we assume that the scalar field φ is minimally coupled to gravity with a Lagrangian

$$L = -\frac{1}{2}\nabla_a\varphi\nabla^a\varphi - V(\varphi).$$

The energy-momentum tensor of the scalar field $T_{ab}^\varphi = \nabla_a\varphi\nabla_b\varphi + g_{ab}L$ is usually written as

$$T_{ab}^\varphi = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\left[\nabla_c\varphi\nabla^c\varphi + 2V(\varphi)\right]. \tag{4.1}$$

We suppose that the energy-momentum tensor T_{ab} of the remaining matter satisfies the strong and dominant energy conditions ie, $T_{ab}n^a n^b \geq -(1/2)T$ and $T_{ab}n^a n^b \geq 0$, where n is any unit, timelike vector field. In the previous chapter we chose n to be the tangent vector field to a congruence of timelike geodesics orthogonal to the homogeneous hypersurfaces. Although we do not assume that matter is moving along these geodesics, we can formally treat the scalar field as a perfect fluid with velocity vector field

$$u^a = \frac{\nabla^a\varphi}{\sqrt{-\nabla_a\varphi\nabla^a\varphi}}.$$

Furthermore, we construct the spacelike homogeneous hypersurfaces in such a way that u^a is hypersurface orthogonal. In other words we identify the vector field n with the velocity vector field u of the fluid representing the scalar field φ. In the following the field φ is supposed to be homogeneous so that its spatial derivatives vanish. With the above choice of n we have $n^a\nabla_a = \partial/\partial t$ hence,

$$T_{ab}^\varphi n^a n^b = \rho \quad \text{and} \quad \left(T_{ab}^\varphi - \frac{1}{2}g_{ab}T^\varphi\right)n^a n^b = \rho + 3p, \tag{4.2}$$

where

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad \text{and} \quad p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{4.3}$$

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4.1 Cosmic no-hair conjecture in new inflation

In this section we discuss the NHT in new inflation for homogeneous space-times. The particular form of the potential of the scalar field is of no importance provided that it has the general properties mentioned in Chapter Two. We begin with a qualitative discussion of a no-hair theorem in new inflation. For this purpose we need only the time-time component of the Einstein equations

$$G_{ab}n^a n^b = 8\pi\rho + 8\pi T_{ab}n^a n^b \qquad (4.4)$$

and the equation of motion of the scalar field

$$\Box\varphi - V'(\varphi) = 0.$$

Denoting the scale factors of the three principal axes of the universe by X_i, $i = 1, 2, 3$ and the mean scale factor by $a = V^{\frac{1}{3}}$, where $V = X_1 X_2 X_3$, (4.4) reads

$$\frac{1}{3}K^2 \equiv 3\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(\frac{\dot{V}}{V}\right)^2 = 8\pi\rho + 8\pi T_{ab}n^a n^b + F(X_1, X_2, X_3). \qquad (4.5)$$

(See also (3.48) and the remarks following that equation.) The only property of F that we use is that, for all Bianchi models, F decreases at least as fast as a^{-2}. The equation of motion of the scalar field $\Box\varphi - V'(\varphi) = 0$ becomes

$$\ddot{\varphi} + K\,\dot{\varphi} + V'(\varphi) = 0 \qquad (4.6)$$

because the spatial derivatives of φ vanish. As usual we assume that at the beginning of inflation the value φ_i of the field φ is far from the minimum of its potential. φ rolls slowly down its potential so the term $\ddot{\varphi}$ in (4.3) is negligible. Thus as long as φ is near φ_i the universe is endowed with an almost constant vacuum energy density $V_0 \equiv V(\varphi_i)$ or equivalently a cosmological term.

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It is now clear why Wald's result should be applicable in the present case. As the universe expands, the function F in (4.5) decreases at least as a^{-2}. Also the term $T_{ab}n^a n^b$ decreases with a (for example as a^{-4} for a radiation dominated FRW universe). However, the energy density of the scalar field, $8\pi\rho$, remains a constant ($\approx 8\pi V_0$) and eventually dominates the terms $8\pi T_{ab}n^a n^b$ and F. This happens in about one Hubble time, $H_0^{-1} = (8\pi V_0)^{-\frac{1}{2}}$, and $a \sim \exp H_0 t$. We see that the anisotropic term F and also the ordinary matter content of the universe decay exponentially with time and the space-time rapidly approaches that of de Sitter.

It remains to verify that in the time it takes the space-time to become nearly de Sitter, φ does not roll down to the minimum of its potential. The flatness of the potential in the vicinity of φ_i permits us to neglect the acceleration term $\ddot{\varphi}$ from (4.6) so that

$$\dot{\varphi} \approx -\frac{V'}{K}. \tag{4.7}$$

Integrating we find that

$$\int_{\varphi_i}^{\varphi} \frac{d\varphi}{V'(\varphi)} = -\int_0^{t_v} \frac{dt}{K}, \tag{4.8}$$

where t_v is the time when the universe becomes vacuum dominated. We wish to estimate the change $\delta\varphi$ of the scalar field before the universe becomes vacuum dominated. Assuming that $V'(\varphi) \approx V'(\varphi_i)$ and $a \sim t^n$ so that $K = 3nt^{-1}$ we find from (4.8) that

$$-\frac{\delta\varphi}{V'(\varphi_i)} \approx \frac{1}{6n}t_v^2.$$

The matching conditions of a and $\dot{a}$ at t_v (the time of change from power-law expansion to exponential expansion) yield $t_v = nH_0^{-1}$ and therefore $\delta\varphi$ can be written as

$$\delta\varphi \approx \frac{n}{2} \frac{-V'(\varphi_i)}{3H_0} H_0^{-1}. \tag{4.9}$$

Using (4.7) with $K = 3H_0$ we see that the change in φ during the pre-inflationary epoch is $n/2$ times the change in φ during the first Hubble time of the de Sitter epoch. In other

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words, during the preinflationary epoch φ does not change more than in the first e-fold of inflation. Thus the vacuum energy is not exhausted before the universe begins to inflate.

From the above qualitative analysis we conclude that the cosmic NHT must be true for homogeneous cosmological models in the context of new inflation. Moreover, the universe approaches de Sitter space-time before the inflaton field reaches the minimum of its potential. Of course the above results cannot be stated strictly in the form of a theorem as was the case of Wald's theorem. The reason is that we made use only of the general properties of the potential function $V(\varphi)$ and consequently we could not solve exactly (4.6) (see also the comments at the end of this section).

Another question related to isotropisation during inflation is the following. Since the period of exponential expansion is finite, it could be possible that curvature perturbations which decrease more slowly than the energy density of the universe, that is more slowly than a^{-4} (RD) or a^{-3} (MD), will eventually dominate over the energy density. Turner and Widrow [72] have shown that *although some homogeneous models will become again anisotropic, inflation postpones this event to an exponentially distant time in the future and models which inflate sufficiently to solve the horizon and flatness problems will today still be very isotropic.* Jensen and Stein-Shabes [40, 41] arrived at similar results, namely the anisotropies re-enter the horizon after a very long time.

Following Turner and Widrow [72] we use the spatial components of the Einstein equations

$$h^c{}_a h^d{}_b R_{cd} = 8\pi h^c{}_a h^d{}_b \left(T_{cd} - \frac{1}{2} g_{cd} T \right), \tag{4.10}$$

where the *total* (ordinary matter + scalar field) stress-energy tensor T_{ab} of the universe is that of a perfect, isotropic fluid with equation of state $p = (\gamma - 1)\rho$. In order that this fluid represent appropriately the matter content of the universe which evolves from an inflationary phase towards the matter dominated present epoch, the coefficient γ is not fixed, but varies according to the period of the evolution of the universe. Thus $\gamma = 0$ corresponds to vacuum energy, $\gamma = 4/3$ to a radiation dominated universe and $\gamma = 1$ to

a matter dominated universe. The conservation equation $\nabla_a T^{ab} = 0$ yields

$$\rho \sim V^{-\gamma} \tag{4.11}$$

(compare to (2.5)).

With notation as in (4.5) we denote the expansion rates in the principal directions of the universe by $b_i \equiv \dot{X}_i / X_i$. Hence, $K = b_1 + b_2 + b_3$. For diagonalisable Bianchi models K turns out to be the trace of the extrinsic curvature defined by (3.16). Recall that the diagonalisable Bianchi space-times are those for which the matrix h_{ij} of the three-metric of the homogeneous hypersurface, $h_{ab} = h_{ij}\omega_a^i\omega_b^j$, $i,j = 1,2,3$ can be put in diagonal form (cf. also (3.26) for notation). These are all Bianchi models except type -IV and -VII$_h$. In the diagonal case it is not difficult to show [66] that the Einstein tensor $G^a{}_b$ is diagonal with eigenvectors $(dt)^a$, $(\omega^1)^a$, $(\omega^2)^a$ and $(\omega^3)^a$. Since the energy-momentum tensor for a perfect fluid is diagonalisable ie, has eigenvalues $-\rho, p, p, p$, corresponding respectively to the eigenvectors $(dt)^a$, $(\omega^1)^a$, $(\omega^2)^a$,$(\omega^3)^a$, we conclude that, for all Bianchi models except type -IV and -VII$_h$, equations (4.10) can be written as [29]

$$\dot{b}_i + K b_i = F_i(X_1, X_2, X_3) + 4\pi(\rho - p) \quad i = 1,2,3. \tag{4.12}$$

The functions F_i decrease at least as $a^{-2} = V^{-\frac{2}{3}}$ (see Ellis and MacCallum [29] for the detailed form of the F_i). This is the only property of the F_i that we will use. For Bianchi types IV and VII$_h$ there is an additional term on the right-hand side of (4.12), due to the non-diagonalisability of these models. Since this term decreases as some power of a^{-1}, every argument used in this section is applicable to these models, but for the sake of simplicity we will not treat them separately. Taking the spatial trace of (4.10) we obtain

$$\frac{\ddot{V}}{V} = F_1 + F_2 + F_3 + 12\pi(\rho - p). \tag{4.13}$$

Equations (4.12) and (4.13) are reduced to the Friedmann equation in the case of isotropic

cosmologies, that is, $X_1 = X_2 = X_3 = a$ and $b_i = H \equiv \dot{a}/a$. If $F_i = 0$ and $\gamma \neq 1$, the expansion rates become asymptotically equal and we interpret the functions F_i as measuring the degree of anisotropy.

Turner and Widrow [72] assume that the universe inflates and that F_i can be treated as perturbations in (4.12) and (4.13). Denoting by F a typical F_i we write $F = \epsilon\rho$ where the function ϵ is supposed to have small initial value, say $\epsilon_i \approx 0.1$. With appropriate choices of γ it is possible to solve the above equations during the following periods: inflation, post-inflation phase, radiation dominated era and matter dominated period. For example, during inflation we have

$$
\begin{aligned}
\rho &\equiv \tfrac{H^2}{3} \approx V\left(\varphi\right) \\
\gamma &\approx 0 \\
\tfrac{\ddot{V}}{V} &\approx H^2 \Rightarrow V \sim \exp\left(Ht\right) \\
\dot{b_i} &+ K b_i = F + \tfrac{H^2}{3}.
\end{aligned}
\tag{4.14}
$$

Since $F \sim V^{-\frac{2}{3}} \sim \exp\left(-\tfrac{2}{3}Ht\right)$, the above equations can be easily solved giving, at the end of inflation,

$$
\epsilon \sim \epsilon_i \exp\left(-2N\right),
$$

where N is the number of e-folds the universe inflates. Similarly after inflation the corresponding equations yield

$$
\epsilon \sim \left(t^{2/\gamma}\right)^{\gamma - 2/3}.
$$

The last two equations allow the estimation of the present-day value of ϵ. We emphasise that γ depends on the period under consideration. For example $\gamma = 1$ during the matter dominated period (from $T \approx 10eV$ or $t \approx 10^{10}$ sec until today, $T = 2.7K$). Given that observations constrain ϵ_{today} to less than 10^{-4}, the necessary value of $N_{\min}$ can be computed. It happens that $N_{\min}$ has almost the same value [43] which is required to solve the horizon and flatness problems. Anisotropy will become important again when $\epsilon \approx 1$. This will happen in the exponentially distant future at time $t \sim \exp\left(3N - 3N_{\min}\right)10^{10}yr$

(see [72] for the details). As Turner and Widrow point out, a finite epoch of inflation does not smooth the universe globally. Rather it creates large smooth regions, sufficiently large to encompass our Hubble volume at this late moment of the history of the universe.

One may feel uncomfortable with two points in the above treatment of Turner and Widrow. The first point is that it is not proved that once the universe begins to inflate it will evolve towards de Sitter space-time *regardless* of the initial conditions *ie* the magnitude of the initial anisotropy. In the discussion above, the initial anisotropy is taken to be small enough so that it can be treated as a small perturbation of a FRW universe. The second point is that the scalar field φ, which is part of the dynamics, is treated phenomenologically. In fact, the device of a variable equation of state is used to bypass the study of the evolution of φ through its equation of motion (4.6). Of course we cannot solve exactly the equation (4.6) except for very special forms of the potential (see [39] for a simple example). Since we considered only the general properties of the potential as essential for our discussion we can only expect to find the asymptotic behavior of the solutions. In the following sections we shall see that a more specific type of the potential function endows a more active role to the equation of motion of the scalar field.

4.2 Cosmic no-hair conjecture in chaotic inflation: a quadratic model

Chaotic inflation is based on the evolution of a scalar field with a simple potential which usually has the form $V(\varphi) \sim \varphi^{2n}$, $n = 1, 2, 3, \ldots$ .The exact form of the potential is of no importance since the scalar field in this model is not part of any particle theory. In fact, as we have already mentioned at the end of the Chapter Two, the scalar field plays no other role than driving inflation.

Before examining the general case, we start with the simple example of a quadratic chaotic inflationary model for homogeneous space-times. We assume a homogeneous

scalar field with potential $V(\varphi) = \frac{1}{2}m^2\varphi^2$ which dominates the energy density of the usual matter of the universe. The result is exponential expansion provided that the scalar field is larger than (see [49])

$$\varphi_i = \sqrt{\frac{3}{4\pi}}.$$ (4.15)

We proceed in a way analogous to Wald's proof (see section 3.2). Consider the time-time component of Einstein's equation

$$G_{ab}n^a n^b = 8\pi\rho + 8\pi T_{ab}n^a n^b$$

and the 'Raychaudhuri' equation

$$R_{ab}n^a n^b = 8\pi\left(\frac{3}{2}\dot{\varphi}^2 - \rho\right) + 8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T\right)n^a n^b.$$

The last two equations can be expressed in terms of the three-geometry of the homogeneous hypersurface as follows:

$$K^2 = 24\pi\rho + \frac{3}{2}\sigma_{ab}\sigma^{ab} - \frac{3}{2}\,{}^{(3)}R + 24\pi T_{ab}n^a n^b$$ (4.16)

$$\dot{K} = -\frac{1}{3}K^2 - \sigma_{ab}\sigma^{ab} - 8\pi\left(\frac{3}{2}\dot{\varphi}^2 - \rho\right) - 8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T\right)n^a n^b.$$ (4.17)

All quantities appearing in these equations were defined in the preliminaries of the previous chapter. The term $\frac{3}{2}\dot{\varphi}^2 - \rho$ is just $\left(T^\varphi_{ab} - \frac{1}{2}g_{ab}T^\varphi\right)n^a n^b$.

The equation of motion of the scalar field $\Box\varphi - V'(\varphi) = 0$ becomes (see (4.6))

$$\dot{\rho} = -K\dot{\varphi}^2, \quad \text{or} \quad \ddot{\varphi} + K\dot{\varphi} + V'(\varphi) = 0.$$ (4.18)

Since we consider homogeneous cosmologies which are not of type-IX, the scalar curvature ${}^{(3)}R$ is nonpositive. Combining (4.16) and (4.17) with the energy conditions

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SEC (3.9) and DEC (3.10) we take

$$\dot{K} \leq 0. \tag{4.19}$$

Although K decreases monotonically, it cannot pass through zero since (4.16) implies that $K^2 - 24\pi\rho \geq 0$. Thus, if the universe is initially expanding, then it will expand for ever ie, $K > 0$ for all time.

Following Moss and Sahni [60], we consider the quantity

$$S = \frac{2}{3} \left(K^2 - 24\pi\rho \right) . \tag{4.20}$$

Taking the time derivative of S and using (4.17) and (4.18) we obtain

$$\dot{S} = -\frac{2}{3}KS - \frac{4}{3}K\sigma_{ab}\sigma^{ab} - 8\pi K \left(T_{ab} - \frac{1}{2}g_{ab}T \right) n^a n^b. \tag{4.21}$$

We see immediately that

$$\dot{S} \leq -\frac{2}{3}KS = -\frac{2}{3}S\sqrt{\frac{3}{2}S + 24\pi\rho}. \tag{4.22}$$

The above inequality cannot be integrated immediately because ρ is a function of time (although slowly-varying). However, during exponential expansion ρ is bounded below by the potential energy $V_1 = \frac{1}{2}m^2\varphi_1^2$, where φ_1 is given by (4.15), hence we have $24\pi\rho \geq 9m^2$. Integration of (4.22) yields

$$\sigma_{ab}\sigma^{ab} \leq S \leq \frac{6m^2}{\sinh^2(mt)}. \tag{4.23}$$

Here the first inequality follows directly from (4.16). Inequality (4.23) shows that the shear of the timelike congruence rapidly approaches zero. By the same type of reasoning as in Wald's proof we arrive at analogous results for the asymptotic behavior of the matter energy density and the geometry of the spacelike slices. (For a formal proof see the general case below). However, we have not yet answered the question of whether the universe isotropises ie, whether the shear vanishes before the end of the inflationary

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era. To this end we have to study the evolution of the scalar field via equation (4.18). Consider what happens during the time interval $0 < t < m^{-1}$. If K were zero, the solution would be $\varphi_0 \cos mt + m^{-1} \dot{\varphi}_0 \sin mt$, where φ_0 and $\dot{\varphi}_0$ are the initial values of φ and $\dot{\varphi}$. The presence of damping has the effect that φ does not decrease as in the case with zero damping factor that is, the solution of (4.18) satisfies

$$\varphi(t) > \varphi_0 \cos mt + m^{-1} \dot{\varphi}_0 \sin mt \quad \text{for } t < m^{-1}. \tag{4.24}$$

From (4.23) we see that the anisotropy dies away if exponential expansion continues up to time $t = m^{-1}$ that is, if $\varphi > \varphi_i$ at $t = m^{-1}$. Inequality (4.24) suggests that it suffices to take as initial conditions

$$\varphi_0 > 4\varphi_i \quad \text{and} \quad \left| \dot{\varphi}_0 \right| < \frac{1}{3} m\varphi_i. \tag{4.25}$$

Note that a larger anisotropy damps more efficiently the slow rolling of the scalar field thus producing more inflation.

The above analysis applies to all Bianchi cosmologies which are initially expanding except type-IX. It can be slightly generalised to the case of inhomogeneous cosmologies with negative spatial scalar curvature. In fact in the above proof we use no other property of homogeneous space-times which are non-type-IX than the nonpositivity of $^{(3)}R$. Therefore, assuming an inhomogeneous space-time with negative scalar three-curvature of the spacelike slices, we could proceed by repeating the above reasoning in the same way Jensen & Stein-Shabes claimed to generalise Wald's theorem (see theorem 1 in Chapter Three and the comments following).

In the case of Bianchi type-IX models the universe could recollapse before inflation begins. This is certainly true for some highly curved models. However, if initially $\Lambda_{effective} \equiv V(\varphi) > \frac{1}{2} \, {}^{(3)}R$ (compare to inequality (3.47) in Wald's proof), then it can be shown [42] that the universe avoids recollapse. Moreover, even in the case of a Bianchi type-IX cosmology the anisotropy is bounded as in (4.23), provided that the universe

expands.

There are two weak points of the above version of the cosmic no-hair theorem. Firstly the result depends on the initial conditions (4.25) for the scalar field. However, there is some evidence that even for large values of $\left|\dot{\varphi}\right|$, the shear disappears within a fixed interval of time. This was claimed numerically by Moss and Sahni [60] for a Bianchi type-I model. The second weakness is more serious. As we shall see shortly, in this model K tends to zero as $t \to \infty$. Therefore, if the shear decays more slowly than the rate of universal expansion, this would build up a large cumulative anisotropy with increasing time. Therefore, we must state more carefully what isotropisation of the universe means.

Collins and Hawking were the first to give a precise definition of isotropisation for a homogeneous universe [21]. According to their definition a model is said to approach isotropy if, as $t \to +\infty$,

    i) $h \to +\infty$, where h is the determinant of the spatial metric,

    ii) $T_{ab}n^a n^b > 0$ and $\dfrac{h_a{}^c T_{bc} n^b}{T_{ab}n^a n^b} \to 0$,

    iii) $\dfrac{\sigma}{K} \to 0$,   where $2\sigma^2 = \sigma_{ab}\sigma^{ab}$ and

    iv) $d_{ab} \equiv h^{-\frac{1}{3}} h_{ab}$ tend to some constants, d_{ab}^0, with $\det d_{ab} = 1$.

Condition (i) says that the universe expands indefinitely. It implies also that $K > 0$ and so it excludes a universe that could recollapse. Condition (ii) can be written more simply in a synchronous coordinate system: $T_{00} > 0$ and $T_{0i}/T_{00} \to 0$ as $t \to \infty$, $i = 1, 2, 3$. It requires the average velocity of the matter relative to the hypersurfaces of homogeneity to die away in time. Condition (iii) seeks to exclude those models in which the shear decays more slowly than the universe expands. Condition (iv) implies that the *cumulative anisotropy*, $\beta \equiv \int_0^t \sigma dt$, must approach a constant β_0 [12] as $t \to \infty$.

In order to examine which homogeneous cosmologies isotropise Collins and Hawking also made use of the dominant energy condition (DEC) and the *positive pressure criterion (PPC)* which requires

$$\sum_{k=1}^{3} T_{kk} \geq 0. \tag{4.26}$$

The first theorem of Collins and Hawking [21] states the following.

Theorem 3 *[Collins & Hawking 1973]. If the DEC and PPC are satisfied, the universe can approach isotropy only if it is one of the types I, V, VII$_0$ and VII$_h$.*

The proof is based on the Einstein field equations without a cosmological constant. We shall see that the theorem remains true in the context of the general chaotic inflation *ie* assuming a scalar field with a convex and positive potential having a local minimum equal to zero.

4.3 Cosmic no-hair conjecture in chaotic inflation: the general case

(For the rest of this dissertation, $c = 8\pi G = 1$). We are now moving on to the general case. This means that we are not concerned with a particular form of the scalar field potential. We consider an arbitrary convex, positive potential having a local minimum at $V(\varphi = 0) \equiv V_0$. As already mentioned, most of the potentials used in chaotic inflationary models belong to this class.

In terms of the three-geometry of the homogeneous hypersurfaces the time-time component of the Einstein tensor, the 'Raychaudhuri' equation and the spatial components of the Ricci tensor read

$$G_{ab}n^a n^b = \frac{1}{2}K^2 - \frac{1}{2}K_{ab}K^{ab} \tag{4.27}$$

$$R_{ab}n^a n^b = -\dot{K} - K_{ab}K^{ab} \tag{4.28}$$

$$h^c{}_a h^d{}_b R_{cd} = \dot{K}_{ab} + K K_{ab} - 2K_{ac}K^c{}_b - {}^{(3)}R_{ab}. \tag{4.29}$$

Equations (4.27) and (4.28) take the form of the corresponding ones found in Chapter Three provided that we decompose K_{ab} into its trace and traceless part

$$K_{ab}K^{ab} = \frac{1}{3}K^2 + \sigma_{ab}\sigma^{ab}. \tag{4.30}$$

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We use the abbreviations

$$T_s(n) \equiv \left(T_{ab} - \frac{1}{2}g_{ab}T\right)n^a n^b \quad \text{and} \quad T_w(n) \equiv T_{ab}n^a n^b \tag{4.31}$$

so that the SEC and DEC read $T_s(n) \geq 0$ and $T_w(n) \geq 0$. The Einstein equations corresponding to equations (4.27)–(4.29) become (recall that $8\pi G = 1$)

$$\frac{1}{2}K^2 - \frac{1}{2}K_{ab}K^{ab} = -\frac{1}{2}{}^{(3)}R + T_w(n) + \rho \tag{4.32}$$

$$-\dot{K} - K_{ab}K^{ab} = T_s(n) + \frac{1}{2}(\rho + 3p) \tag{4.33}$$

$$\begin{aligned}
&\dot{K}_{ab} + KK_{ab} - 2K_{ac}K^c{}_b - {}^{(3)}R_{ab} \\
&= h^c{}_a h^d{}_b \left(T_{cd} - \tfrac{1}{2}g_{cd}T\right) + \tfrac{1}{2}(\rho - p)h_{ab},
\end{aligned} \tag{4.34}$$

where (4.2) and (4.3) were used. The general asymptotic properties of the solutions to the above equations are summarized in the following

Lemma 1 *If we assume the following:*

The dominant and strong energy conditions hold

The scalar spatial curvature is nonpositive

The potential is convex and positive, hence $V(\varphi) \geq V_0$ for all φ

The universe is initially expanding, ie $K \geq 0$ at an (arbitrary chosen) initial time $t = 0$.

Then we obtain the following asymptotic forms for the solutions of the Einstein system (4.32)–(4.34):

i) $\dot{K} \leq 0$ and $K \geq 0$ for all $t \geq 0$ and

ii) ${}^{(3)}R$, $T_{ab}n^a n^b$, T, $\dot{\varphi}$ tend to zero and $K \to \sqrt{3V_0}$ as $t \to +\infty$.

Proof. i) If we had only a constant cosmological term, $\dot{K} \leq 0$ would follow directly from Raychaudhuri equation (4.33) and $K \geq 3\Lambda > 0$ from the time-time component of the Einstein equation (4.32) (see Wald's theorem). In the present case we can not proceed

in the same way because the energy-momentum tensor of the inflaton field φ violates the strong energy condition and the term $\rho + 3p$ in the right-hand side of (4.33) could be negative. However, addition of equations (4.32) and (4.33) yields

$$- \dot{K} = \frac{1}{2} \left(3 K_{ab} K^{ab} - K^2 \right) - \frac{1}{2} \, {}^{(3)}R + T_w \left(n \right) + T_s \left(n \right) + \frac{3}{2} \left(\rho + p \right). \tag{4.35}$$

Using the algebraic inequality

$$Tr \left(K^2 \right) \geq \frac{1}{3} \left(Tr K \right)^2, \tag{4.36}$$

(which can be easily verified by bringing the tensor K_{ab} to diagonal form) and the fact that $\rho + p = \dot{\varphi}^2 > 0$ we conclude from (4.35) that $\dot{K} \leq 0$. On the other hand, eliminating $K_{ab} K^{ab}$ from (4.32) and (4.33) we have

$$K^2 + \dot{K} = - \, {}^{(3)}R + \left(T_{ab} n^a n^b - \frac{1}{2} T \right) + \frac{3}{2} \left(\rho - p \right) \tag{4.37}$$

and, since $\rho - p = 2V \geq 0$, we conclude that

$$K^2 + \dot{K} \geq 0. \tag{4.38}$$

By a Raychaudhuri-type argument we find that $K \geq 0$ for $t \geq 0$.

ii) $K(t)$ is a decreasing function, bounded below by zero. Therefore $\lim_{t \to +\infty} K(t) \geq 0$ and $\lim_{t \to +\infty} \dot{K}(t) = 0$. Since the right-hand side of (4.35), is a sum of non-negative functions of time, we conclude that all these functions have zero limit as $t \to +\infty$. Finally, (4.37) implies that $K(t) \to \sqrt{3V_0}$ as $t \to +\infty$. ∎

A corollary of the above lemma is that the shear, σ^2, also tends to zero as $t \to +\infty$. In fact the term $3 K_{ab} K^{ab} - K^2$ in (4.35) equals $3 \sigma_{ab} \sigma^{ab}$.

We discuss now the possibility of isotropisation of a homogeneous universe with a scalar field in the spirit of the Collins and Hawking criterion on page 55. Equations (4.2) and (4.3) show that there exist time intervals where the scalar field behaves as

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ordinary matter, *ie*, like a perfect fluid with nonnegative pressure and also there exist time intervals where the PPC is violated. Therefore Theorem 3 is not directly applicable. It turns out to be convenient to consider a suitably defined average equation of state for the scalar field.

There are two quite different cases to consider depending on whether $V_0 > 0$ or $V_0 = 0$. In the first case $\lim_{t \to +\infty} K(t) = \sqrt{3V_0} \neq 0$ so that $\sigma/K \to 0$ as $t \to +\infty$ and the universe isotropises according to the Collins and Hawking criterion. The local minimum V_0 of the potential acts as a cosmological constant and the universe becomes de Sitter space with $H_0 = \frac{1}{3}K = \sqrt{\frac{1}{3}V_0}$. In short Wald's theorem is applicable and the conclusions at the end of its proof remain valid. In the second case the shear still vanishes asymptotically, but now K also tends to zero. It is not obvious whether $\lim_{t \to +\infty} \sigma/K$ exists and equals zero. This is the case of the quadratic model where as we saw vanishing of the shear does not guarantee that the universe isotropises. This is not surprising since, according to the theorems of Collins and Hawking [21], most homogeneous models (with ordinary matter) do *not* approach isotropy. A generalisation to the scalar field case of the first theorem of Collins and Hawking was given in 1991 by Heusler [37]. We discuss briefly the line of proof of this result.

To begin we define a time-dependent coefficient of state for the scalar field φ

$$\gamma(t) \equiv 2\left(1 - \frac{V}{\rho}\right), \tag{4.39}$$

so that we have an effective equation of state

$$p(t) = (\gamma(t) - 1)\rho(t). \tag{4.40}$$

Since the ratio V/ρ takes all values between 0 ($\varphi = 0$) and 1 ($\dot{\varphi} = 0$), $\gamma(t)$ takes all values between 0 (cosmological constant, $p = -\rho$) and 2 (stiff matter, $p = \rho$). Therefore there are time intervals when the scalar field does not behave like an effective cosmological term. We encountered a similar situation in the new inflation treatment of Turner and

Widrow [72] (see page 48) where the total matter content of the universe was modelled by a perfect fluid with a varying equation of state. Therefore, in order to investigate the (possible) asymptotic isotropisation of the homogeneous models, we need a kind of average equation of state. It turns out that it is convenient to define a new time coordinate by

$$\tau = \int K(t)\, dt. \tag{4.41}$$

Essentially we use the determinant of the spatial metric as new time parameter since $K = \left(\ln \sqrt{h}\right)^{\cdot}$. If we exclude models which could recollapse, condition (i) of the Collins and Hawking isotropisation criterion (see page 55) implies that τ is an increasing function of t and tends to $+\infty$ as $t \to +\infty$. Heusler [37] proves the following

Lemma 2 *Any homogeneous model of Bianchi type II, VI_0, VIII, IV or VI_h coupled to a scalar field, can only approach isotropy as $t \to +\infty$ if*

$$\frac{\rho}{K^2} \to \frac{1}{3}, \quad and \; \rho h^{\frac{1}{3}} \to +\infty.$$

The Bianchi models mentioned in the lemma are exactly those which do not contain an FRW solution. Under precisely the same assumptions we prove that

$$\left\langle \frac{V}{\rho} \right\rangle \geq \frac{2}{3},$$

where $\langle \rangle$ denotes the time average with respect to τ given by (4.41). From (4.18) we have

$$\dot{\rho} = -2K\left(\rho - V\right).$$

Using the definition of $\gamma(t)$ the above equation can be integrated to give

$$\rho(t) = \rho(0) \exp\left(-\int_0^t \gamma K\right).$$

Substitution of the above equation into the relation $\rho h^{\frac{1}{3}} \to \infty$ of the previous lemma

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together with the new time variable (4.41) yields

$$\lim_{\tau \to +\infty} \exp\left(\frac{2}{3}\tau - \int \gamma(\tau)\, d\tau\right) = +\infty.$$

We conclude that

$$\langle \gamma \rangle \equiv \lim_{\tau \to +\infty} \frac{1}{\tau} \int_{\tau_0}^{\tau} \gamma \leq \frac{2}{3}$$

and so, by (4.39), we finally obtain

$$\left\langle \frac{V}{\rho} \right\rangle \geq \frac{2}{3}. \tag{4.42}$$

The inequality (4.42) is inconsistent with isotropisation in the spirit of lemma 1. More precisely, we have the following

Lemma 3 *If $V(\varphi)$ is a convex, non-negative function with local minimum $V(0) = 0$ and $\dot{K} \leq 0 \leq K$, $K^2 + \dot{K} \geq 0$ for all $t \geq 0$, then every solution of (4.18) with $\rho(0) \geq 0$ and $\rho/K^2 \to c$ as $t \to +\infty$, where c is a strictly positive real number, satisfies*

$$\left\langle \frac{V}{\rho} \right\rangle \leq \frac{2}{3}. \tag{4.43}$$

The assumptions on K are reasonable in view of lemma 1, while $\rho/K^2 \to c$ is a necessary condition for isotropisation by Lemma 2. The proof of this assertion consists of taking the time average of (4.18) written in terms of the new time τ. The result is (for details we refer again to [37])

$$2\left\langle \frac{\rho}{K^2}\left(1 - \frac{V}{\rho}\right)\right\rangle = \left\langle \frac{\rho}{K^2}\frac{V'\varphi}{\rho}\right\rangle \tag{4.44}$$

and, since $\lim_{t \to +\infty} \rho/K^2 = c \neq 0$, this term can be eliminated in (4.44). On the other hand the convexity of $V(\varphi)$ implies that $\forall \varphi,\ \varphi V'(\varphi) \geq V(\varphi)$. Hence (4.44) yields

$$2 - 2\left\langle \frac{V}{\rho} \right\rangle = \left\langle \frac{V'\varphi}{\rho} \right\rangle \geq \left\langle \frac{V}{\rho} \right\rangle,$$

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that is,

$$\left\langle \frac{V}{\rho} \right\rangle \leq \frac{2}{3}. \quad \blacksquare$$

The preceding result, inequality (4.43), contradicts the necessary condition for isotropisation, inequality (4.42). This shows that the first theorem of Collins and Hawking (Theorem 3) remains true in the case of a scalar field. In conclusion one has the following result:[37]

Theorem 4 *[Heusler]. Any homogeneous cosmological model coupled to a scalar field with a convex, positive potential having a local minimum at $\varphi = 0$ with $V(0) = 0$ and ordinary matter satisfying the DEC and SEC can approach isotropy only if it is of the types I, V, VII_0 or VII_h.*

In [37] the Bianchi type-IX is also included in the statement of the above theorem. This is incorrect, since the nonpositivity of $^{(3)}R$ is a necessary condition of the byproduct $K^2 + \dot{K} \geq 0$ of lemma 1, which in turn is one of the assumptions of lemma 3. The spatial scalar curvature of Bianchi IX may be positive. In fact $^{(3)}R$ is positive if the three-dimensional Ricci tensor is isotropic (see Lemma 2 and Theorem 1 in Collins & Hawking [21]). Therefore a separate analysis is necessary if one wishes to include Bianchi IX in the above theorem.

4.3.1 Application: An exponential potential

Consider a model with a scalar field having a potential

$$V(\varphi) = V_0 \exp(-\lambda\varphi),$$

where V_0 and λ are constants. In the spirit of chaotic inflation such a field may not come from any particular particle theory. However, exponential potentials do arise in the effective four-dimensional theories induced by Kalusa-Klein theories. Moreover, many

higher order gravity theories naturally lead to inflation with an exponential potential as we will see in more detail in the next chapter.

An homogeneous and isotropic cosmological model driven by a scalar field with an exponential potential was studied by Halliwell [34]. He showed that there exists a solution with power-law inflation ($a(t) \sim t^p$, $p > 1$) and this solution is an attractor. Power-law inflation will be discussed in the following section. In this section we examine the question of isotropisation of homogeneous cosmologies based on such a scalar field. Following Ibanez, Hoogen and Coley [38] we write down the Einstein equations (see for example equations (4.16)–(4.18)) in the form

$$K^2 = 3\sigma^2 + \frac{3}{2}\dot{\varphi}^2 + 3V - \frac{3}{2}{}^{(3)}R$$

$$\dot{K} = -\frac{1}{3}K^2 - 2\sigma^2 - \dot{\varphi}^2 + V$$

$$\ddot{\varphi} + K\dot{\varphi} - \lambda V = 0,$$

where $2\sigma^2 = \sigma_{ab}\sigma^{ab}$. With the use of new expansion variables (see [38] for the details) and the time coordinate defined by (4.41) it can be shown that Lemma 2 implies that $\dot{\varphi}/K - \lambda/3 \to 0$ as $\tau \to +\infty$. As a consequence $\langle V/\rho \rangle = 1 - \lambda^2/6$. In order that the necessary condition for isotropisation, inequality (4.42), be satisfied λ^2 must be less than 2. Thus, if $\lambda^2 > 2$ and the model is not of Bianchi types I, V or VII, then it cannot isotropise. It is worth to note that the necessary condition $\lambda^2 \leq 2$ for isotropisation is compatible with the condition $\lambda^2 < 2$ which is necessary for inflation (see [34] and the discussion at the beginning of the following section).

The above treatment of the NHC is incomplete for two reasons. Firstly ordinary matter is not included in the model. Secondly, as the above authors themselves point out [38], their treatment does *not* imply that all models with $\lambda^2 \leq 2$ can approach isotropy. In the next section we will show precisely this, namely that *for $\lambda^2 < 2$ the isotropic power-law inflationary FRW solution is the unique attractor for any initially expanding*

Bianchi model (except probably type IX) with a scalar field having an exponential potential and ordinary matter satisfying the usual energy conditions.

4.4   Cosmic no-hair conjecture in power-law inflation

A common feature of all inflationary models is that the mean scale factor $a(t)$ of the universe accelerates with time

$$\ddot{a}(t) > 0 \tag{4.45}$$

or more generally the Lie derivative of the trace of the extrinsic curvature with respect to n is positive, $L_n K > 0$. So far we have considered models in which the scale factor expands exponentially $a(t) \propto \exp(Ht)$. There exists another class of models satisfying condition (4.45), the so called power-law inflationary models in which

$$a(t) \propto t^n, \quad n > 1. \tag{4.46}$$

Such a dependence on time of the scale factor can arise from a scalar field with an exponential potential, $V(\varphi) = V_0 \exp(-\lambda\varphi)$, where V_0 and λ are constants. In this case the Friedmann equation is

$$3\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{2}\dot{\varphi}^2 + V_0 \exp(-\lambda\varphi)$$

and the equation of motion of the scalar field is

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} - \lambda V_0 \exp(-\lambda\varphi) = 0.$$

Barrow [4] gives an exact solution of the Einstein equations with a scalar field of this type in a flat FRW metric. It is

$$a(t) \sim t^p, \ p = const, \ \varphi(t) = \sqrt{2p}\ln t, \tag{4.47}$$

provided that we take $V_0 = p(3p - 1)$ and $\lambda = \sqrt{2}/p$.

As we mentioned in the previous section, power-law inflation driven by a scalar field with exponential potential was also studied by Halliwell [34]. He showed that the solution (4.47) is an attractor for open $(k = -1)$ and flat $(k = 0)$ FRW models.

Most of the inflationary models of particle physics exhibit power-law rather than pure exponential expansion. As we shall see in more detail in the next chapter, many generalised theories of gravity which are conformally equivalent to the Einstein theory with a scalar field whose potential is of exponential type lead naturally to power-law inflation. Therefore, it seems natural to examine the applicability of the no-hair theorem in such inflationary models.

In the most general inhomogeneous case the problem is mathematically intractable. However, for homogeneous cosmologies Kitada and Maeda [42] proved a no-hair theorem in the spirit of Wald's theorem. Below we present the most interesting points of their work omitting the details of their proof which is similar to Wald's proof.

We assume that the inflaton field has a potential

$$V(\varphi) = V_0 \exp(-\lambda\varphi). \tag{4.48}$$

The solution (4.47) given by Barrow indicates that power-law inflation occurs whenever $\lambda < \sqrt{2}$. It is surprising that this is also the condition for isotropisation according to Heusler, as we saw in the previous section. A further restriction on λ of the form

$$0 \leq \lambda < \sqrt{\frac{2}{3}} \tag{4.49}$$

is necessary for the validity of inequality (4.59) below. Firstly we consider Bianchi cosmologies except type-IX. As usual the strong and dominant energy conditions for the energy-momentum tensor T_{ab} of the remaining matter are among our assumptions. Further, remembering the abbreviations (4.31) and that $\rho = \frac{1}{2}\dot{\varphi}^2 + V$, equations (4.16) and

(4.17) become

$$K^2 = 3\left(\frac{1}{2}\,\dot{\varphi}^2 + V\right) + \frac{3}{2}\sigma_{ab}\sigma^{ab} - \frac{3}{2}\,^{(3)}R + T_w(n) \qquad (4.50)$$

$$\dot{K} = -\dot{\varphi}^2 + V - \frac{1}{3}K^2 - \sigma_{ab}\sigma^{ab} - T_s(n). \qquad (4.51)$$

Following Kitada & Maeda [42] we define a new time coordinate τ by

$$\frac{d\tau}{dt} = \exp\left(-\frac{1}{2}\lambda\varphi\right). \qquad (4.52)$$

For simplicity we use again a dot to denote differentiation with respect to τ, that is, for a function f of time we have $df/dt = \dot{f}e^{-\frac{1}{2}\lambda\varphi}$. We also introduce new, tilded variables, $\tilde{A}$, where $\tilde{A} \equiv A\exp\left(\frac{1}{2}\lambda\varphi\right)$. The significance of these transformations will be discussed in the next chapter. In terms of these new variables equations (4.50) and (4.51) take the form

$$\widetilde{K}^2 = 3\left(\frac{1}{2}\,\dot{\varphi}^2 + V_0\right) + \frac{3}{2}\tilde{\sigma}_{ab}\tilde{\sigma}^{ab} - \frac{3}{2}\,^{(3)}\widetilde{R} + \widetilde{T}_w(n) \qquad (4.53)$$

$$\dot{\widetilde{K}} = -\dot{\varphi}^2 + V_0 - \frac{1}{3}\widetilde{K}^2 = \frac{\lambda}{2}\widetilde{K}\,\dot{\varphi} - \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} - \widetilde{T}_s(n) \qquad (4.54)$$

and the equation of motion of the scalar field, (4.18), becomes

$$\ddot{\varphi} - \frac{\lambda}{2}\,\dot{\varphi}^2 + \widetilde{K}\,\dot{\varphi} + \lambda V_0 = 0. \qquad (4.55)$$

We suppose that the universe is initially expanding ie, $K > 0 \iff \widetilde{K} > 0$, at some initial time and define the function $\widetilde{S}$ (compare with (4.20))

$$\widetilde{S} = \widetilde{K}^2 - 3\left(\frac{1}{2}\,\dot{\varphi}^2 + V_0\right). \qquad (4.56)$$

Using equations (4.54) and (4.55), and differentiating $\widetilde{S}$ with respect to τ (compare to (4.21)) we obtain the following inequality

$$\dot{\widetilde{S}} \leq -\frac{1}{3}\widetilde{S}\left(2\widetilde{K} - 3\lambda\,\dot{\varphi}\right). \qquad (4.57)$$

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Inequality (4.57) is similar to inequality (4.22). Further combining equations (4.53) and (4.56) we find

$$\widetilde{K}^2 - \frac{3}{2}\dot{\varphi}^2 \geq 3V_0 \qquad (4.58)$$

and from this inequality with $0 \leq \lambda < \sqrt{2/3}$ we conclude that

$$2\widetilde{K} - 3\lambda\dot{\varphi} \geq 3\alpha > 0, \qquad (4.59)$$

where

$$\alpha \equiv \left[\frac{4}{3}\left(1 - \frac{3}{2}\lambda^2\right)V_0\right]^{\frac{1}{2}}. \qquad (4.60)$$

With this bound, inequality (4.57) can be integrated yielding

$$0 \leq \widetilde{S}(\tau) \leq \widetilde{S}_0 \exp(-\alpha\tau), \qquad (4.61)$$

where $\widetilde{S}_0 = \widetilde{S}(0)$. Again, as in Wald's theorem, the time-time component of the Einstein equations, (4.53), implies that

$$0 \leq \quad \frac{1}{2}\widetilde{\sigma}_{ab}\widetilde{\sigma}^{ab}, \quad -\frac{1}{2}{}^{(3)}\widetilde{R}, \quad \widetilde{T}_w(n) \quad \leq \widetilde{S}(\tau). \qquad (4.62)$$

Inequalities (4.61) and (4.62) show that, during one e-folding time α^{-1}, the expansion rate $\widetilde{K}$ of the universe is dominated by the inflaton energy density; the shear and the three-curvature rapidly vanish *ie*, the spatial sections become nearly flat and isotropic and the universe appears to be matter-free.

We note that the expansion rate K does not vanish asymptotically as is the case for chaotic inflation studied by Moss & Sahni [60] (see the comment before the Collins & Hawking isotropisation criterion on page 55 and the analysis after the proof of lemma 1). Therefore this model of power-law inflation avoids the problems mentioned at the end of the discussion of the quadratic model and satisfies the third condition of the criterion of isotropisation of Collins and Hawking.

How fast does the universe isotropise with respect to the cosmic time t? We saw that isotropisation occurs rapidly in terms of the τ time. In order to return to time t we use the isotropic attracting solution (4.47). Inserting φ in (4.52) we find

$$e^{-\alpha\tau} \sim t^{-q},$$

where q is a positive constant depending on λ (refer to [42]). The dependence of the original variables on t is therefore

$$S, \quad \sigma_{ab}\sigma^{ab}, \quad {}^{(3)}R, \quad T_w(n) \quad \sim t^{-(2+q)},$$
$$K^2, \quad \rho \quad \sim t^{-2}. \tag{4.63}$$

We emphasise again that the anisotropy vanishes faster than the universe expands *ie*, no cumulative anisotropy develops for large t.

Wald's results for Bianchi type-IX cosmologies may be extended to cover the case of the power-law inflation model. Firstly we note that if, $\dot{\tilde{S}}_0 \geq 0$, equations (4.57)–(4.60) imply that $\dot{\tilde{S}} \leq -\alpha\tilde{S} \leq 0$ and so $\tilde{S}$ is bounded above, $\tilde{S}(\tau) \leq \max\left\{0, \tilde{S}_0 \exp(-\alpha\tau)\right\}$. Although the three-curvature of Bianchi type-IX models may be positive, it is bounded by ${}^{(3)}R_{\max} \sim h^{-\frac{1}{3}}$, where h is the determinant of the spatial metric (see (3.46)). At the end of Wald's proof we saw that in the case of a true cosmological constant, Λ the universe could still evolve towards de Sitter space-time provided that Λ is initially large enough, $\Lambda > \frac{1}{2} {}^{(3)}R_{\max}$ (cf. condition (3.47)). In the present situation we see that, if we define an effective cosmological 'constant' $\Lambda_{eff} \equiv V(\varphi)$, Wald's proof is extended to the case of the power-law inflation. In fact a careful analysis leads to the result that, if initially we have $\Lambda_{eff} > \frac{1}{2}{}^{(3)}R_{\max}$, then $-\frac{2}{3}\tilde{S} \leq^{(3)} \tilde{R} \leq^{(3)} \tilde{R}_{\max}(0) \exp(-3\alpha_{IX}\tau)$, that is, ${}^{(3)}R$ decays to zero (see [42] again for the technical details). The new time constant α_{IX}^{-1} is greater than α^{-1} which implies that the convergence time in type-IX is *longer* than that in the other Bianchi types.

So far we have seen that the power-law solution is the unique attractor for the open

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and flat FRW models as well as for almost all initially expanding Bianchi-type models, coupled with a scalar field having exponential potential. Is there enough reason to believe that the cosmic NHT is true in general for power-law inflation? A possible answer is given by Müller, Schmidt and Starobinsky [61], who constructed a generic inhomogeneous solution in the form of an asymptotic series that locally approaches the power-law solution (4.47) as $\to +\infty$. However, some inhomogeneous ever-expanding cosmological models undergoing power-law inflation do not approach homogeneity and isotropy as $t \to +\infty$. Barrow [4] gives such counter-examples using a class of metrics found by Wainwright and Goode (see [4] for references).

4.5 Summary

In this chapter we discussed several cosmic no-hair theorems in three large classes of inflationary models, namely new inflation, chaotic inflation and power-law inflation. We showed that the NHT remains true at least for homogeneous cosmologies, in the sense that the universe has enough time to isotropise before the inflaton field reaches the minimum of its potential. In the following chapter we shall discuss the attractor property of de Sitter space-time in the context of another mechanism of inflation which does not rely on a scalar field, but emerges naturally from curvature corrections to the Einstein Lagrangian.

Chapter 5

Cosmic no-hair conjecture in generalized theories of gravity

It is well known that the vacuum Einstein field equations can be derived from an action principle

$$S_E = -\frac{1}{2} \int L_E \, \epsilon, \tag{5.1}$$

where the Lagrangian density L_E is just the Ricci scalar R

$$L_E = R \tag{5.2}$$

and ϵ is the natural volume element $\epsilon = \sqrt{-g} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. If matter fields are included in the theory an appropriate L_{matter} term must be added to the Lagrangian density (5.2).

Einstein was the first to modify his original theory in an attempt to obtain a static cosmological model. This modified theory can be derived from a Lagrangian density

$$L = R - 2\Lambda, \tag{5.3}$$

where Λ is the cosmological constant. Since then there have been numerous attempts

to generalize the action (5.1) by considering action functionals that contain curvature invariants of higher than first order in (5.3). These Lagrangians generally involved linear combinations of all possible second order invariants that can be formed from the Riemann, Ricci and scalar curvatures, namely

$$R^2, \quad R_{ab}R^{ab}, \quad R_{abcd}R^{abcd}, \quad \varepsilon^{iklm}R_{ikst}R^{st}_{lm}.$$

The reasons for considering higher order generalizations of the action (5.1) are multiple. Firstly, there is no *a priori* physical reason, to restrict the gravitational Lagrangian to a linear function of R. Secondly it is hoped that higher order Lagrangians would create a first approximation to an as yet unknown theory of quantum gravity. For example a certain combination of the above second order invariants may have better renormalisation properties than general relativity [71]. Thirdly one expects that on approach to a space-time singularity, curvature invariants of all orders ought to play an important dynamical role. Far from the singularity, when higher order corrections become negligible, one should recover general relativity. It is also hoped that these generalised theories of gravity might exhibit better behavior near singularities.

In this chapter we consider higher order gravity (HOG) theories, wherein the Lagrangian density is an arbitrary analytic function of the scalar curvature R. This is obviously not the most general class of HOG theories, but the inclusion of curvature invariants other than R would greatly complicate matters [14]. There is another reason for this choice of the Langrangian closely related to the inflationary scenario as we shall see in the next section.

5.1 Field equations and the conformal equivalence theorem

We consider a Lagrangian density of the form

$$L = f(R),$$ (5.4)

where $f(R)$ is assumed to be an analytic function of the scalar curvature. By varying L with respect to the metric tensor $\mathbf{g}$, the action principle provides the vacuum field equations [22, 14]

$$f'R_{ab} - \frac{1}{2}fg_{ab} - \nabla_a\nabla_b f' + g_{ab}\Box f' = 0,$$ (5.5)

where $\Box = g^{ab}\nabla_a\nabla_b$ and a prime $(')$ denotes differentiation with respect to R. These are fourth order equations *ie* they contain fourth order derivatives of the metric. Therefore it is not surprising that very few solutions exist in the literature. For a discussion of cosmological solutions and stability issues see [14, 22, 25]. Among other difficulties related to the field equations (5.5) we mention the need for additional initial conditions in the formulation of the Cauchy problem, besides the usual ones in general relativity.

Fortunately there is a method to overcome most of these problems. This is the conformal equivalence theorem proved by Barrow and Cotsakis [8]: Under a suitable conformal transformation, equations (5.5) reduce to the Einstein field equations with a scalar field as a matter source. We can see this by considering the transformation[1]

$$\widetilde{\mathbf{g}} = \Omega^2\mathbf{g},$$ (5.6)

---

[1]Strictly speaking the metrics in the transformation (5.6) are said to be *conformally related*, while a conformal transformation is a diffeomorphism $f : (M, \mathbf{g}) \to (M, \widetilde{\mathbf{g}})$ such that $f^*\mathbf{g} = \Omega^2 g$; see *eg* [62] for a discussion.

where Ω^2 is a smooth strictly positive function chosen so that

$$\Omega^2 = f'(R).$$ (5.7)

Granted the relation between the tensors R_{ab} and R in the space-time $(M, \mathbf{g})$ to the corresponding ones $\tilde{R}_{ab}$ and $\tilde{R}$ in the space-time $(M, \tilde{\mathbf{g}})$ (see for example Wald [73], appendix D) we may transform the field equations (5.5) in the new space-time $(M, \tilde{\mathbf{g}})$. Moreover, on the introduction of the scalar field φ by

$$\varphi = \sqrt{\frac{3}{2}} \ln \left(f'(R) \right)$$ (5.8)

the conformally transformed field equations become

$$\tilde{R}_{ab} - \frac{1}{2} \tilde{g}_{ab} \tilde{R} = \nabla_a \varphi \nabla_b \varphi - \frac{1}{2} \tilde{g}_{ab} \left(\nabla_c \varphi \nabla^c \varphi \right) - \frac{1}{2} \tilde{g}_{ab} \left(f' \right)^{-2} \left(Rf' - f \right).$$ (5.9)

These are the Einstein equations for a scalar field source with potential

$$V = \frac{1}{2} \left(f' \right)^{-2} \left(Rf' - f \right).$$ (5.10)

The theorem of Barrow and Cotsakis allows one to study the dynamical properties of higher order gravity theories by analysing them in the conformal picture, that of Einstein's equations with a scalar field matter content. The authors state their result in D dimensions, but for our purposes a four-dimensional treatment is sufficient. If matter fields with energy-momentum tensor $T_{ab}^m(\mathbf{g})$ are present in the original space-time $(M, \mathbf{g})$, the field equations become

$$f' R_{ab} - \frac{1}{2} f g_{ab} - \nabla_a \nabla_b f' + g_{ab} \Box f' = T_{ab}^m(\mathbf{g}).$$ (5.11)

In the conformally related space-time, $(M, \tilde{\mathbf{g}})$, the corresponding Einstein equations be-

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come

$$\widetilde{R}_{ab} - \frac{1}{2}\widetilde{g}_{ab}\widetilde{R} = T_{ab}^m\left(\widetilde{\mathbf{g}}\right) + T_{ab}^{\varphi}\left(\mathbf{g},\widetilde{\mathbf{g}}\right), \tag{5.12}$$

where $T_{ab}^{\varphi}\left(\mathbf{g},\widetilde{\mathbf{g}}\right)$ is the right-hand side of (5.9). For a discussion on the question of the physical reality of the two metrics involved and other interpretation issues see [22, 23].

An interesting feature of HOG theories is that inflation emerges in these theories in a most direct way. In one of the first inflationary models, proposed in 1980 by Starobinsky [69], inflation is due to the R^2 correction term in a gravitational Lagrangian $L = R + \beta R^2$ where β is a constant. Instead of having to rely on the existence of a scalar field, inflation in the present context is driven by the higher order curvature terms present in the Langrangian without assuming a scalar field at all. Here the role of the scalar field is played by the scalar curvature of the space-time. The situation is not surprising under the light of the conformal transformation theorem stated above.

As a concrete example consider the quadratic Lagrangian theory

$$L = -2\Lambda + R + \beta R^2, \tag{5.13}$$

where Λ is the cosmological constant. With the use of equations (5.7) and (5.8) the conformal transformation (5.6) takes the form

$$\widetilde{\mathbf{g}} = \left(1 + 2\beta R\right)\mathbf{g}. \tag{5.14}$$

The potential of the scalar field (see (5.10)) in the conformally equivalent picture is

$$V = \frac{1}{8\beta}\left[1 - \exp\left(-\sqrt{\tfrac{2}{3}}\varphi\right)\right]^2. \tag{5.15}$$

As Maeda [55] has pointed out, this potential possesses a flat plateau $V\left(\varphi\right) \to 1/\left(8\beta\right)$ as $\varphi \to +\infty$ and leads to exponential inflation.

5.2 Stability of homogeneous and isotropic solutions

Before proceeding to discuss in detail the inflationary scenario and the applicability of the cosmic no-hair theorem in HOG theories we must first examine the existence of the de Sitter solution in the context of these theories. Moreover, if such a solution exists, we must study its stability properties.

In general relativity (GR) de Sitter space-time can be defined as the maximally symmetric vacuum solution of the Einstein equations (see HE [35]). To see this, recall that the space-times of constant curvature are locally characterized by the condition

$$R_{abcd} = \frac{1}{12} R \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) \tag{5.16}$$

which is equivalent to

$$R_{ab} = \frac{1}{4} R g_{ab}. \tag{5.17}$$

The second Bianchi identities, $R_{ab[cd;e]} = 0$, imply that R is covariantly constant ie

$$R = R_0 = const. \tag{5.18}$$

The Einstein tensor is therefore $R_{ab} - \frac{1}{2} R g_{ab} = -\frac{1}{4} R_0 g_{ab}$. Hence, one can regard these spaces as solutions of the vacuum Einstein equations with $\Lambda = \frac{1}{4} R_0$, ie the de Sitter solution. (If $R_0 = 0$, the solution is Minkowski space-time. The de Sitter space-time is the only maximally symmetric *curved* space-time).

To see what equations (5.16)–(5.18) imply for the HOG field equations we insert the above three constraints in (5.5) and take the following simple existence condition

$$R_0 f^{'} (R_0) = 2 f (R_0) . \tag{5.19}$$

Thus, given any $f(R)$ gravity theory, if there exists a solution R_0 of (5.19), then the theory contains the GR de Sitter solution with constant curvature R_0. For example for

the $f(R) = R - 2\Lambda$ theory (GR) the existence condition gives $R_0 = 4\Lambda$. For the purely quadratic theory $f(R) = R^2$, the condition (5.19) is identically satisfied that is, solutions exist for any value of R_0.

The general quadratic theory (5.13) admits the de Sitter space-time as a solution provided that $R_0 = 4\Lambda$. This solution is stable if $\beta < 0$, but unstable if $\beta > 0$ [14]. When Λ is zero, the exact de Sitter solution of GR corresponding to exponential inflation is not a solution of the quadratic theory. However, an exact solution of the $R + \beta R^2$ theory can be found (see [56, 14]), viz.

$$a_0(t) \sim \exp\left(Bt - At^2\right)$$
$$\text{with } A, B \text{ constants and } A = \tfrac{1}{72\beta} < 0.$$

$$(5.20)$$

In general $f(R)$ theories without a cosmological constant still admit a near-de Sitter solution corresponding to a quasi-exponential expansion of the universe. These solutions have the form $a(t) \sim \exp\left[H(t)t\right]$, where $H(t)$ is no longer constant but a slowly varying (decreasing) function of time. For example, in the quadratic case above, $H(t)$ is linearly decaying $H(t) = B - At$. Although mathematically more complicated, quasi-exponential expansion is more realistic as a model of inflation.

Since (5.20) is not solution of GR, the cosmic NHC in the framework of higher order gravity theories could be reformulated as follows [24]: *All solutions of the HOG theories derived from a Lagrangian L=f(R) with a metric which can be written in a synchronous form and a stress-energy tensor satisfying the strong and dominant energy conditions, asymptotically approach the quasi-de Sitter solution* (5.20). As Cotsakis and Flessas [24] point out, a stability analysis of the solution (5.20) is essentially equivalent to examining the validity of the above version of the cosmic NHC for solutions of the form

$$a(t) = e^{Bt - At^2}\left(1 + \epsilon(t)\right),$$
$$\epsilon(t) \ll 1.$$

$$(5.21)$$

If these solutions turn out to have a stable regime in HOG, then in that regime the solution

(5.20) will be an attractor for all solutions of the form (5.21). Thus, the cosmic NHT (in its HOG version) for homogeneous and isotropic space-times will follow immediately.

The stability analysis carried out by the above authors shows that conditions can always be found such that $\epsilon(t) \rightarrow 0$ as $t \rightarrow +\infty$ (see [24] for the details). This means that all solutions of HOG theories of the form (5.21) eventually settle down to the quasi-stationary state described by (5.20). Thus the above stated form of the cosmic NHT in HOG theories is true for homogeneous and isotropic cosmologies. The result does not rely on the corresponding NHT in general relativity and shows that small homogeneous and isotropic perturbations of the metric tensor tend to zero as $t \rightarrow +\infty$.

5.3 No-hair theorems for homogeneous space-times

A cosmic NHT for homogeneous cosmologies in a quadratic theory has been demonstrated by Maeda [55] (see also Mijic & Stein-Shabes [57]). The proof is based on the conformal equivalence theorem and thus relies on general relativity dynamics.

In the following we discuss the cosmic NHC for a quadratic Lagrangian (5.13) without cosmological constant. For a $f(R) = R + \beta R^2$ theory the field equations (5.5) take the form

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{\beta}{1 + 2\beta R}\left(2\nabla_a\nabla_b R - 2g_{ab}\Box R - \frac{1}{2}g_{ab}R^2\right). \quad (5.22)$$

With the corresponding conformal factor (5.7) we have the conformally related metric $\tilde{\mathbf{g}} = (1 + 2\beta R)\mathbf{g}$. Introducing the scalar field (see (5.8))

$$\varphi = \sqrt{\frac{3}{2}}\ln(1 + 2\beta R) \quad (5.23)$$

we can write the field equations in the conformal picture as

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}\tilde{g}_{ab}(\nabla_c\varphi\nabla^c\varphi) - \tilde{g}_{ab}V \quad (5.24)$$

Figure 5-1: The potential in the conformally equivalent system. The height of the plateau is $V_\infty = 1/8\beta$.

$$\Box\varphi - V'(\varphi) = 0,$$

where the potential V is (see (5.15))

$$V = \frac{1}{8\beta}\left[1 - \exp\left(-\sqrt{\tfrac{2}{3}}\varphi\right)\right]^2.$$

As already mentioned this potential has a long and flat plateau (see figure 5-1). When φ is far from the minimum of the potential, V is almost constant $V_\infty \equiv \lim_{\varphi\to+\infty} V(\varphi) = 1/(8\beta)$. Thus V has the general properties for inflation to commence and V_∞ behaves as a cosmological term.

Consider now homogeneous Bianchi-type space-times. According to the results of section 3.1.4 the metric can be written (we drop the tilde $\tilde{\ }$ for simplicity) as

$$ds^2 = -dt^2 + h_{ij}(t)\,\omega^i\omega^j \quad i,j = 1,2,3.$$

This fact greatly simplifies the matters because the scalar field defined by (5.23) can be treated as homogeneous. Moreover, as we already mentioned in section 3.1.4 the Einstein equations (5.24) become ordinary differential equations with respect to time.

It turns to be convenient to split the energy-momentum tensor of the scalar field (that is, the right-hand side of (5.24)) into a term without the potential $T_{ab} \equiv \nabla_a \varphi \nabla_b \varphi - \frac{1}{2} (\nabla \varphi)^2 g_{ab}$ plus the potential term $-V g_{ab}$. It is then easily seen that the T_{ab} part of the energy-momentum tensor satisfies the strong and dominant energy conditions. The time-time component of (5.24), the Raychaudhuri equation and the equation of motion of the scalar field are (compare to equations (3.36), (3.37) and (4.18))

$$K^2 = \frac{3}{2} \sigma_{ab} \sigma^{ab} - \frac{3}{2} \,^{(3)}R + \frac{3}{2} \dot{\varphi}^2 + 3V \tag{5.25}$$

$$\dot{K} = -\frac{1}{3} K^2 - \sigma_{ab} \sigma^{ab} - \dot{\varphi}^2 + V \tag{5.26}$$

$$\dot{\rho} = -K \dot{\varphi}^2, \quad \text{or} \quad \ddot{\varphi} + K \dot{\varphi} + V' = 0 . \tag{5.27}$$

We are now ready, to prove the cosmic NHC following Wald's method. It can be seen from figure (5-1) that, as long as inflation continues, V remains less than V_∞. Then, the above three equations imply

$$\sqrt{3V_\infty} \le K \le \frac{\sqrt{3V_\infty}}{\tanh \alpha t}$$

$$\sigma_{ab} \sigma^{ab} \le \frac{2V_\infty}{\sinh^2 \alpha t} \tag{5.28}$$

$$\dot{\varphi}^2 \le \frac{2V_\infty}{\sinh^2 \alpha t},$$

where $\alpha = \sqrt{V_\infty / 3} = \sqrt{24\beta}$. The proof is analogous to the way we arrived at equations (3.42)–(3.44) from equations (3.36) and (3.37).

Once the universe evolves on the plateau of the potential and is initially expanding, K is always positive by (5.25). This implies that $\dot{\rho} < 0$ and so, losing energy, the universe

evolves towards the minimum of the scalar field. Equations (5.28) show that anisotropy dies away in one Hubble time α^{-1} and the kinetic energy $\dot{\varphi}^2$ of the scalar field also disappears within α^{-1}. The universe reaches the potential minimum (when $\varphi = 0$) at which the cosmological term automatically vanishes. Using physical arguments Maeda claims that inflationary phase is a transient attractor. Hence inflation and consequent isotropisation is a quite natural phenomenon in R^2 gravity theory [55].

We discussed inflation in the equivalent space-time $(M, \tilde{\mathbf{g}})$, but it is not obvious that the above attractor property is maintained in the original space-time $(M, \mathbf{g})$. This is probably an unimportant question since there is much evidence that in most cases the rescaled metric $\tilde{\mathbf{g}}$ is the real physical metric [23]. However, since during inflation the scalar field φ changes very slowly and the two metrics are related by $\mathbf{g} = \exp\left(-\frac{2}{\sqrt{6}}\varphi\right)\tilde{\mathbf{g}}$, it is easily seen that inflation happens in the original picture also.

Berkin [16] arrived at similar results for Bianchi type-I and type-IX universes in $R + \beta R^2$ theory without relying on the conformal equivalence theorem. Using the field equations (5.5) of the theory he showed that Bianchi type-I model always isotropises. The same is true for Bianchi type-IX universes with the exception of some positive curvature models.

5.3.1   The D-dimensional case

It is interesting to note that in higher dimensions R^2 gravity theory exhibits power-law inflation. In a D-dimensional space-time (5.8) becomes

$$\varphi = \left(\frac{D-1}{D-2}\right)^{\frac{1}{2}} \ln f'(R)$$

(see for example [22] or [8]). Then in the case of the quadratic Lagrangian the potential (5.10) is

$$V = \frac{1}{8\beta}\left(1 - \exp\left(-\sqrt{\frac{D-2}{D-1}}\varphi\right)\right)^2 \exp\left(\frac{D-4}{\sqrt{(D-1)(D-2)}}\varphi\right).$$

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The potential has a very flat plateau in the four-dimensional case, but, when $D > 4$ it diverges exponentially as $\varphi \to +\infty$, ie,

$$V \sim \frac{1}{8\beta} \exp\left(\frac{D-4}{\sqrt{(D-1)(D-2)}}\varphi\right) \quad \text{for large } \varphi.$$

As discussed in Chapter Four, in the case of an exponential potential the expansion of the mean scale factor a is power-law. Assuming a flat FRW D-dimensional space-time the same analysis which led to the solution (4.47) now yields

$$\tilde{a}(t) \sim \tilde{t}^p, \quad p = \frac{4}{\lambda^2(D-2)}, \quad \varphi\left(\tilde{t}\right) = -\frac{2}{\lambda}\tilde{t}, \tag{5.29}$$

where $-\lambda$ is the coefficient in the exponent of the potential. Inflation occurs whenever $p > 1$; hence $|\lambda|$ must be less than $2/\sqrt{D-2}$. This restricts the dimension of the space-time to be smaller than ten. Therefore the dimension $D = 10$ is marginal for power-law inflation in the context of the $R + \beta R^2$ gravity theory.

Once again inflation in the rescaled system guarantees (power-law) inflation in the original system [55]. To see this we write (5.6) for a FRW metric as

$$ds^2 = \left(-d\tilde{t}^2 + \tilde{a}\left(\tilde{t}\right)^2 d\sigma^2\right) = \Omega^2\left(-dt^2 + a\left(t\right)^2 d\sigma^2\right). \tag{5.30}$$

By (5.7) we find from the solution (5.29)

$$\Omega = \exp\left(\frac{1}{\sqrt{(D-1)(D-2)}}\varphi\right) \propto \tilde{t}^{-m}, \quad m = \frac{2}{\lambda\sqrt{(D-1)(D-2)}} \tag{5.31}$$

as can be seen by inspection. Since $d\tilde{t} = \Omega dt$, we have $t \propto \tilde{t}^{1+m}$. The solution in the original system is therefore

$$a = \Omega^{-1}\tilde{a} \propto \tilde{t}^{p+m} \propto t^{\frac{p+m}{1+m}} \tag{5.32}$$

with the exponent of t larger than unity whenever $p > 1$.

A stability analysis in power-law inflationary solutions of HOG theories with La-

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grangian $L = R^n$ is given in Cotsakis and Saich [26]. We assume a homogeneous and isotropic D-dimensional space-time. To find out which theories exhibit power-law inflation we investigate under what conditions the potential (5.10) has the form $\exp(-\lambda\varphi)$. This leads to a Clairaut equation for $f(R)$ which has two solutions: $f(R) = R + 2\Lambda$, that is, general relativity with cosmological constant and $f(R) = R^n$, $n = const$. Using the conformal equivalence theorem a similar analysis with that given for the $R + \beta R^2$ case (cf. equations (5.30)–(5.32)) shows that power-law inflation occurs whenever $D \neq 2n$ and $a(t) \sim t^p$ with $p = (n-1)(2n-1)/(2-n)$. As the above authors show, the results do not approach general relativity in the limit $R \to 0$, but this can be rectified if the $f(R) = R^n$ theory is modified to be $f(R) = R + \beta R^n$. If we restrict ourselves to four dimensions and write down the analogue to the Raychaudhuri equation for the $f(R) = R^n$ theory in the FRW metric, we take a third order non-linear differential equation for the scale factor $a(t)$ [26]. We use the exact solution $a_0(t) \sim t^{\,p}$ to this equation as a background solution and consider (isotropic) perturbations of the form

$$a(t) = a_0(t)[1 + \epsilon(t)], \quad \epsilon(t) \ll 1. \tag{5.33}$$

This is similar to the perturbation analysis (see equation (5.21)) in the quasi-exponential inflation case. The result is $\epsilon(t) \sim t^\omega$, where ω depends on n. The stable solutions correspond (for $n > 0$) to $0 < n < 1/2$ and $5/4 < n < 2$. In particular, the power-law solution is unstable for any R^n theory with $n > 2$. The case $n = 2$ corresponds to a de Sitter solution which is stable against homogeneous and isotropic perturbations.

5.4 Summary

In this chapter we reviewed higher order gravity theories based on a Lagrangian $L = f(R)$. Apart from other reasons these theories are interesting because they provide a natural frame of inflation caused by higher order corrections to the Einstein–Hilbert Lagrangian. Thus there is no need for the deliberate introduction of a scalar field to drive

inflation. This fact becomes more transparent after one uses the conformal equivalence theorem discussed on page 72. We discussed the attractor property of inflating solutions in these theories for homogeneous and isotropic space-times. A NHT for homogeneous space-times in the $R + \beta R^2$ was proved making use of the conformal equivalence theorem. Finally, we discussed briefly D-dimensional space-times which seem to exhibit power-law inflation in both conformally equivalent frames.

Chapter 6

Concluding remarks

6.1 On the assumptions of the cosmic no-hair theorems

The property of the de Sitter space-time that it be an attractor for a large set of cosmological initial data has been exploited in various inflationary models. As already discussed, there exist initial data that do not allow inflation to produce a large, locally isotropic and homogeneous universe. One such set of initial data are those which evolve towards short-lived universes that recollapse before inflation can occur. This problem has been recognised implicitly in some proofs of no-hair theorems by restricting attention to models with nonpositive spatial curvature. In fact the general motif of the cosmic no-hair theorems is the following: Given an initially expanding space-time with nonpositive scalar spatial curvature and an energy-momentum tensor satisfying certain energy conditions, prove that the solution approaches asymptotically de Sitter. However, closed universes with positive spatial curvature need not recollapse. Consider for example the closed ($k = +1$) FRW universe (2.1)

$$ds^2 = -dt^2 + a^2\left(t\right)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

which has always positive spatial curvature, $^{(3)}R = 3/(2a^2)$. The Raychaudhuri equation and the time-time component of the Einstein equations become

$$\ddot{a} = -\frac{1}{6}(\rho + 3p)\,a \qquad (6.1)$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\rho}{3}. \qquad (6.2)$$

If the matter content is a perfect fluid with equation of state

$$p = (\gamma - 1)\,\rho,\ \gamma \in [0, 2]\,,$$

then this universe will recollapse only if

$$\rho > 0 \quad \text{and } \rho + 3p > 0. \qquad (6.3)$$

For instance, if we chose $\gamma = 2/3$, then $a(t) \sim t$ and, if $\gamma = 0$, then $a(t) \sim \cosh Ht$, with $H = const$. Thus, when $\rho + 3p < 0$, closed FRW models do not recollapse. As Barrow [4, 3], pointed out, the condition $\rho + 3p < 0$ is identical to that required to produce inflation. More generally inflation occurs whenever the energy-momentum tensor of the *total* matter content of the universe violates the strong energy condition

$$\left(T_{ab} - \frac{1}{2}g_{ab}T\right)n^a n^b > 0, \quad n^a \text{ timelike} \qquad \text{(SEC)}.$$

This conclusion holds even in the case of anisotropic and inhomogeneous models. Equation (6.1) generalises to (compare to (3.6))

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 + \nabla_a \dot{n}^a - \left(T_{ab} - \frac{1}{2}g_{ab}T\right)n^a n^b,$$

where $2\sigma^2 = \sigma_{ab}\sigma^{ab}$, $2\omega^2 = \omega_{ab}\omega^{ab}$ and $\nabla_a \dot{n}^a$ is the acceleration term. (Throughout this dissertation we always assume that the integral curves of the vector field n are timelike

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geodesics and so the acceleration term vanishes.) Defining the mean expansion scale factor $a(t)$ by $3\,\dot{a}\,/a \equiv \theta$, we can make the similarity of the above equation with the Friedmann equation (6.1) more transparent:

$$\ddot{a} = \left[-2\sigma^2 + 2\omega^2 + \nabla_a\,\dot{n}^a - \left(T_{ab} - \frac{1}{2}g_{ab}T \right) n^a n^b \right] a. \tag{6.4}$$

We see that the condition for inflation ($\ddot{a} > 0$) is also sufficient to prevent the recollapse of the universe. The sign of $\ddot{a}$ is determined not only by the matter term $\left(T_{ab} - \frac{1}{2}g_{ab}T \right) n^a n^b$ but also by the anisotropic terms on the right-hand side of (6.4).

We conclude that the conditions on the matter content of the universe necessary to achieve inflation are sufficient to prevent the recollapse of closed universes. When the SEC holds after inflation, a closed (isotropic) universe is free to recollapse. Therefore, in the presence of matter fields violating the SEC, the restriction $^{(3)}R \leq 0$ excludes a large class of ever expanding inflationary universes. Such models are for example space-times containing perfect fluid with hypersurface-orthogonal velocity fields ($\omega = 0$) which possess shear σ, mean expansion rate θ, and three-curvature $^{(3)}R$. We write again the equation of state

$$p = (\gamma - 1)\,\rho, \quad \gamma = const \in [0,2]\,, \tag{6.5}$$

the Raychaudhuri equation

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - 2\sigma^2 + \frac{1}{2}\left(\rho + 3p\right), \tag{6.6}$$

the time-time component of the Einstein equations

$$\theta^2 = 3\sigma^2 - \frac{3}{2}\,^{(3)}R + 3\rho \tag{6.7}$$

and the conservation equation

$$\dot{\rho} + \theta\left(\rho + p\right) = 0. \tag{6.8}$$

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A class of solutions to these equations was given by Collins [20]

$$\rho \propto \theta^2$$

$$\left(\frac{\sigma}{\theta}\right)^2 = \frac{1}{4}\left(3\gamma - 2\right)\left(1 - \frac{\rho}{\theta^2}\right) \tag{6.9}$$

$$\frac{^{(3)}R}{\theta^2} = -6\frac{2 - \gamma}{3\gamma - 2}\left(\frac{\sigma}{\theta}\right)^2.$$

We see from equations (6.5)–(6.9) that models violating the SEC ($3\gamma - 2 < 0$) which can drive inflation all possess $^{(3)}R > 0$. However, the cosmic NHT is not applicable in these inflationary models, because $\sigma/\theta \to$ constant as $t \to +\infty$. This means that anisotropy does not die away fast enough as the universe expands and the conditions of isotropisation of Collins & Hawking on page 55 do not hold.

Barrow [3] wrote that positive-spatial-curvature universes were excluded from the proofs of cosmic no-hair theorems because it was believed that they do not expand forever. This exclusion was based upon the incorrect equating of closed with recollapsing universes. We have seen that in most no-hair theorems Bianchi type-IX model occupies a special position among spatially homogeneous space-times. Lin and Wald have shown that the Bianchi type-IX model without cosmological constant will recollapse at some time provided that the matter satisfies the DEC and has nonnegative average pressure [47, 48]. This result could be generalised in the case of HOG theories, starting with quadratic Lagrangians. However, as we saw in Wald's theorem, any type-IX space-time with a positive cosmological constant approaches de Sitter space-time if initially (see condition (3.47))

$$\Lambda > \tfrac{1}{2}{}^{(3)}R_{\text{max}}. \tag{6.10}$$

Wald's criterion for isotropisation is unsatisfactory in discussing the generality of inflation in type-IX models for the following reasons. Firstly inequality (6.10) is a sufficient condition for inflation of the Bianchi type-IX space-time, hence there may be many in-

flationary solutions which do not satisfy the above criterion [27]. The second reason is that it is uncertain how probable are these universes initially satisfying inequality (6.10) among all possible initial conditions. It is known that the Bianchi type-IX space-time exhibits a chaotic behavior near the singularity with or without the cosmological term, like a particle oscillating in a triangle potential well [15]. The trajectory of the particle is inflationary if Λ dominates the spatial curvature, otherwise the universe recollapses. Den and Ishihara [27] investigated the chaotic trajectories near the singularity and found that there exists a strong mixing of inflationary and non-inflationary trajectories. In other words it is impossible to predict which trajectory will inflate if one sets the initial condition near the singularity. Since one expects that quantum gravitational effects play an important role near the singularity, an approach based on quantum cosmology seems to be more appropriate. Using a semiclassical analysis Yokoyama and Maeda [76] found that most of the classical trajectories are inflationary, even though they do not satisfy Wald's criterion at the beginning of the classical evolution.

The strong energy condition is assumed to hold in most no-hair theorems. In many respects SEC is an unrealistic energy condition in the context of inflation. Inflation occurs when a matter field violates the SEC, hence it is unreasonable to expect all the remaining matter fields to satisfy the SEC. Barrow [6] constructs simple counter-examples to the effect that if matter satisfies the weaker DEC (or the WEC) but violates the SEC, an initially de Sitter space-time may evolve towards the flat Friedmann universe $a(t) \sim t^{2/(3\gamma)}$ for large t. In conclusion, if the SEC is dropped, then the NHT is not true because the de Sitter solution may become unstable. The same is true in the $R + \beta R^2$ theory: de Sitter solution can become unstable if matter satisfies the DEC, but violates the SEC [7].

6.2 General review of the cosmic no-hair conjecture

How general are the cosmic no-hair theorems discussed so far? In the case of a true cosmological constant present for all time, homogeneous space-times except possibly the Bianchi type-IX do satisfy the cosmic NHT. From the discussion in Chapter Three one may have the impression that the cosmic NHT holds quite generally, if we exclude the Nariai metric as an attractor as this metric is unstable (see the theorem of Morrow-Jones and Witt in section 3.3). However, this theorem has a serious drawback: it is a statement about a certain asymptotic behavior of solutions to the Einstein equations in *vacuum*. We anticipate that if matter is included – possibly satisfying some energy conditions – then the theorem may not be true (cf comments immediately following Theorem 2 in section 3.3).

The no-hair theorems in the context of general relativity are quite satisfactory from a mathematical point of view. However, for inflation to be a natural phase in cosmological evolution, NHTs must be proved in the context of inflationary models where the cosmological term is *not* constant and has a *finite* duration. As we discussed in Chapters Four and Five, this problem can be considered as solved at least for homogeneous space-times in the following cases; in chaotic inflation up to certain constraints (see Heusler's theorem in section 4.3), in power-law inflation, in new inflation, although there is not a general proof taking account of the dynamical evolution of the inflaton field. This is mainly due to the fact that there does not exist a generally accepted particle model to provide an exact effective potential for the scalar field which drives inflation. The situation is similar in the case of inflation due to the vacuum polarization changing the gravitational Lagrangian at high values of the Ricci scalar. As we saw in Chapter Five, the attractor property of de Sitter space-time – as is usually referred the cosmic NHT in generalised gravity theories – has been established in many $f(R)$ theories. A generalization or reformulation of the cosmic NHC is possibly necessary in some theories (see the comments following the solution (5.20)). The applicability of the cosmic NHT in these theories is not surprising but is due to the conformal equivalence theorem [8] discussed in section

5.1. For a review of different inflation theories related by the conformal transformation theorem see Gottlöber *et al* [32].

For inhomogeneous space-times the situation is still rather obscure. Without the most general cosmological solution to the Einstein field equations no systematic research has been done in this direction to our knowledge. Apart from the above mentioned theorem (theorem 2 in section 3.3) a model with a scalar field with positive exponential potential as the only matter content has been studied and an inhomogeneous solution in the form of an asymptotic series approaching locally the power-law solution was found [61]. Counter-examples of inflating inhomogeneous space-times that do not approach homogeneity and isotropy as $t \to +\infty$ also exist [4].

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