

*ALGEBRAIC PROPERTIES
OF
ORDINARY DIFFERENTIAL
EQUATIONS*

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*Algebraic Properties of Ordinary Differential
Equations*

by

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Volume II: The Papers

This thesis is submitted in fulfilment of the requirements for the degree of
Doctor of Science in the Faculty of Science, University of Natal

Durban

September 1995

Preface

Contained in this volume are copies of the papers by the author discussed in Volume I. Not included are papers mentioned in, for example, the Epilogue which refer to current trends and directions in this and related fields, but which do not constitute part of the literature discussed in the thesis.

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Quadratic Hamiltonians, quadratic invariants and the symmetry group $SU(n)$

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Journal of Mathematical Physics **19** (1978) 446–451

Quadratic Hamiltonians, quadratic invariants and the symmetry group $SU(n)$

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(Received 28 December 1976)

We show that any $2n$ -dimensional quadratic Hamiltonian may be transformed by a (usually time-dependent) linear canonical transformation into any other $2n$ -dimensional quadratic Hamiltonian, in particular that of the isotropic harmonic oscillator. This latter Hamiltonian possesses the symmetry group $SU(n)$ and $n^2 - 1$ linearly independent quadratic invariants which provide a basis for the generators of the group. Every other quadratic Hamiltonian is shown to have a quadratic invariant possessing $SU(n)$ symmetry. The free particle structure is given explicitly. The anisotropic oscillator is shown not to possess $SU(3)$ symmetry based on quadratic invariants. However, its wavefunctions and energy levels may be obtained directly from those of the isotropic oscillator whether the frequencies are commensurable or not.

1. INTRODUCTION

Since the pioneering work of Fock¹ on the Coulomb problem and Bargmann² on the isotropic time-independent oscillator, the study of dynamical symmetry groups in quantum mechanics has been one of the dominant features of the subject. Not the least of the reasons for this has been the connection between the existence of a particular symmetry and the solution of the Schrödinger equation.³

The existence of a symmetry group requires the existence of a set of constants of the motion such that all elements of the set commute (classically have zero Poisson bracket) with one particular element. The other elements or suitable linear combinations of them are then required to have commutation relations appropriate to the generators of the particular symmetry group sought. For problems with time-independent Hamiltonians, the Hamiltonian itself was taken as the central invariant and the search for a symmetry group became the search for sufficient other constants to constitute a basis. For both the Coulomb and oscillator problems, the angular momentum provided some of the required constants. The remaining constants were found in the Runge-Lenz vector⁴ for the former and in a symmetric matrix⁵ for the latter. Physically these constants describe the shape of the classical orbit.⁶ Each of these quantities has an unambiguous definition in terms of quantum mechanical operators and so the symmetry group was applicable to the quantum mechanical problem. This was not the case for the anisotropic oscillator with incommensurable frequencies.^{6,7} It has been pointed out frequently⁸ that the problem is in finding the appropriate operator expressions for classical expressions involving nonintegral powers.

The case of a time-dependent Hamiltonian produces the need to determine the basic constant of the motion, if any. A class of problems which has been of considerable interest is that which reduces to the time-dependent harmonic oscillator. This problem occurs in the motion of a charged particle in an electromagnetic field⁹⁻¹¹ or in the evolution of coherent states in lasers.¹² The existence of an exact invariant for the time-dependent oscillator was shown by Lewis¹⁰ and

Riesenfeld.¹¹ A simpler demonstration has been provided more recently.¹³ A discussion of the symmetry group of the three-dimensional time-dependent harmonic oscillator was given by Günther and Leach¹⁴ in which they showed that $SU(3)$ was the symmetry group of the invariant.

The use of canonical transformations in the solution of quantum mechanical problems has received considerable attention in recent years.¹⁵⁻¹⁹ The employment of time-dependent transformations^{13,20,21} has broadened the range of problems which can be successfully tackled. Such transformations have been particularly fruitful when applied to time-dependent oscillator Hamiltonians, in establishing both an interpretation for the invariant associated with the motion and that the motion is characterized by the symmetry group $SU(n)$ and also providing a relatively simple method for the solution of the Schrödinger equation.

In this paper we examine the general class of quadratic Hamiltonians describing some n -dimensional motion. We show that classically every such Hamiltonian has a quadratic constant of the motion which possesses $SU(n)$ as its symmetry group. In general, this is not the symmetry group of the Hamiltonian, but we may say that the Hamiltonian is characterized by the noninvariance group $SU(n)$. There is no problem in the transition to quantum mechanics. We give an explicit demonstration of the $SU(3)$ structure for a three-dimensional free particle and discuss the problem of the anisotropic oscillator with noncommensurable frequencies. In particular we show that its nondegenerate energy levels may be obtained by transformation methods from those of the degenerate isotropic oscillator.

2. LINEAR CANONICAL TRANSFORMATIONS

Writing the conjugate canonical coordinates (q, p) as

$$\begin{aligned} q_i &= \omega^\mu, \quad i = 1, n, \quad \mu = 1, n \\ p_i &= \omega^\mu, \quad i = 1, n, \quad \mu = n+1, 2n, \end{aligned} \quad (2.1)$$

Hamiltonian's equations of motion are

$$\dot{\omega} = \epsilon \frac{\partial H}{\partial \omega} \quad (2.2)$$

and the Poisson bracket of two scalars F and G is

$$[F, G]_{PB\omega} = \left(\frac{\partial F}{\partial \omega} \right)^T \epsilon \left(\frac{\partial G}{\partial \omega} \right), \quad (2.3)$$

where ϵ is the $2n \times 2n$ symplectic matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

The general linear transformation from coordinates ω to $\bar{\omega}$ is

$$\bar{\omega} = S\omega + r \Leftrightarrow \omega = \bar{S}\bar{\omega} + \bar{r}, \quad (2.4)$$

where S is a $2n \times 2n$ (real) matrix and r a $2n \times 1$ (real) column matrix. The condition that (2.4) be canonical is

$$S\epsilon S^T = \epsilon. \quad (2.5)$$

In terms of ω , the general quadratic Hamiltonian is

$$H(\omega) = \frac{1}{2} \omega^T A \omega + B^T \omega + C, \quad (2.6)$$

in which A is a $2n \times 2n$ symmetric matrix, B is a $2n \times 1$ column matrix, and C is a scalar. A , B , and C are coordinate free, but may be time-dependent. Under the transformation (2.4), $H(\omega)$ is transformed to $\bar{H}(\bar{\omega})$ where

$$\bar{H}(\bar{\omega}) = \frac{1}{2} \bar{\omega}^T \bar{A} \bar{\omega} + \bar{B}^T \bar{\omega} + \bar{C} \quad (2.7)$$

provided²⁰

$$\dot{S} = \epsilon \bar{A} S - S \epsilon A, \quad (2.8)$$

$$\dot{r} = \epsilon \bar{A} r + \epsilon \bar{B} - S \epsilon B. \quad (2.9)$$

Equations (2.8) and (2.9) are linear first order systems and so possess solutions²² with continuous first derivatives provided the elements of the matrices A, \bar{A}, B, \bar{B} are continuous functions of time over the interval of interest. In general S will contain $(2n)^2$ and r $(2n)$ arbitrary constants. The condition (2.5) imposes some constraint on the number of arbitrary constants of S , but does not determine them uniquely.

We note that C and \bar{C} do not appear in (2.8) and (2.9). In classical mechanics this reflects the invariance of Hamilton's equations of motion to the transition $H \rightarrow \bar{H}$: $\bar{H} = H - C$. Quantum mechanically, this invariance is expressed as an arbitrary time phase in the Schrödinger wave function.

3. THE FORM OF \bar{H}

In our previous applications of time-dependent linear canonical transformations,^{13,14,20,21} the signature of $\omega^T A \omega$ and $\bar{\omega}^T \bar{A} \bar{\omega}$ has been $2n$, i.e. the transformations have been between attractive oscillator Hamiltonians. This restriction is not implicit in any of (2.5), (2.8), or (2.9). That it is unnecessary may be illustrated by using the simple example of the one-dimensional oscillator with Hamiltonian

$$H = \frac{1}{2} (p^2 + \omega^2 q^2), \quad \omega \text{ constant}. \quad (3.1)$$

Some possible forms for \bar{H} and the required transformation matrices are

$$(i) \quad \bar{H} = \frac{1}{2} (P^2 - \omega^2 Q^2).$$

$$S = \begin{bmatrix} -(e^{\omega t} \dot{\alpha} + e^{-\omega t} \dot{\beta}), (e^{\omega t} \alpha + e^{-\omega t} \beta) \\ -\omega (e^{\omega t} \dot{\alpha} - e^{-\omega t} \dot{\beta}), \omega (e^{\omega t} \alpha - e^{-\omega t} \beta) \end{bmatrix}, \quad (3.2)$$

where

$$\alpha = A \sin \omega t + B \cos \omega t, \quad \beta = C \sin \omega t + D \cos \omega t, \quad (3.3)$$

$$2\omega^2 (AD - BC) = 1. \quad (3.4)$$

$$(ii) \quad \bar{H} = 0.$$

$$S = \begin{bmatrix} -\dot{\alpha}, & \alpha \\ -\dot{\beta}, & \beta \end{bmatrix}, \quad (3.5)$$

where

$$\alpha = A \cos \omega t + B \sin \omega t, \quad \beta = C \cos \omega t + D \sin \omega t, \quad (3.6)$$

$$\omega (AD - BC) = 1. \quad (3.7)$$

$$(iii) \quad \bar{H} = \omega P Q$$

$$S = \begin{bmatrix} -\dot{\alpha} e^{\omega t}, & \alpha e^{\omega t} \\ -\dot{\beta} e^{-\omega t}, & \beta e^{-\omega t} \end{bmatrix}, \quad (3.8)$$

where

$$\alpha = B \cos \omega t - A \sin \omega t, \quad \beta = D \cos \omega t - C \sin \omega t, \quad (3.9)$$

$$\omega (AD - BC) = 1. \quad (3.10)$$

$$(iv) \quad \bar{H} = P.$$

$$S = \begin{bmatrix} -\dot{\alpha}, & \alpha \\ -\dot{\beta}, & \beta \end{bmatrix}, \quad r = \begin{bmatrix} t+c \\ k \end{bmatrix}, \quad (3.11)$$

where α and β are the same as in (ii), c and k are arbitrary constants and use has been made of the equivalence of H and $(H - C)$.

The particular case $\bar{H} = 0$ is the one used in the solution of the Hamilton-Jacobi equation. However, the solution of Hamilton's equations for (3.1) may be obtained from the solution of the Hamilton's equations for any $\bar{H}(\bar{\omega})$ obtained from (3.1) via a canonical transformation. Generally H and \bar{H} are not numerically equal since

$$\bar{H} = H + \frac{\partial F}{\partial t}, \quad (3.12)$$

where F is the generating function of the transformation. For a linear transformation F is a quadratic form whose coefficients depend upon the elements of S (and r where applicable). If S is time-dependent, clearly $\bar{H} \neq H$.

4. THE ARCHTYPAL QUADRATIC HAMILTONIAN

From the foregoing, it is obvious that any quadratic Hamiltonian (2.6) is related to

$$H = \frac{1}{2} \omega^T \omega \quad (4.1)$$

by a linear canonical transformation. We term this the archtypal quadratic Hamiltonian because it possesses the dynamical symmetry group $SU(n)$ as an invariance symmetry group. The generators of the group may be written down in terms of constants of the motion described by (4.1) which are quadratic in ω .

Suppose \mathcal{C} is a time-independent quadratic form given by

$$\mathcal{C} = \frac{1}{2} \omega^T C \omega, \quad (4.2)$$

where C is a constant $2n \times 2n$ real symmetric matrix. \mathcal{C} is a constant of the motion provided

$$[\mathcal{C}, H]_{PB\omega} = \omega^T C^T \epsilon \omega = 0, \quad (4.3)$$

i.e., C has the form

$$C = \begin{bmatrix} U, & W \\ -W, & U \end{bmatrix}, \quad (4.4)$$

where U is a symmetric and W a skew-symmetric $n \times n$ matrix. We define the set of matrices $(U_{ij}, W_{ij}; i = 1, n; j = 1, n)$ as

$$[U_{ij}]_{mn} = \delta_{im} \delta_{nj} + \delta_{jm} \delta_{ni}, \quad (4.5)$$

$$[W_{ij}]_{mn} = \text{sgn}(j - i)(\delta_{im} \delta_{nj} - \delta_{jm} \delta_{ni}). \quad (4.6)$$

Any general C may be written as

$$C = \begin{bmatrix} \alpha_{ij} U_{ij}, & \beta_{ij} W_{ij} \\ -\beta_{ij} W_{ij}, & \alpha_{ij} U_{ij} \end{bmatrix}, \quad (4.7)$$

where the scalar coefficients α_{ij} and β_{ij} are symmetric in i and j .

Writing

$$U_{ij} = \frac{1}{2} \omega^T \begin{bmatrix} U_{ij}, & 0 \\ 0, & U_{ij} \end{bmatrix} \omega, \quad (4.8)$$

$$\mathbb{W}_{ij} = \frac{1}{2} \omega^T \begin{bmatrix} 0, & W_{ij} \\ -W_{ij}, & 0 \end{bmatrix} \omega, \quad (4.9)$$

we have a set of n^2 linearly independent constants of the motion which have zero Poisson bracket with H (4.1). Since

$$H = \frac{1}{2} \sum_i U_{ii}, \quad (4.10)$$

there are $n^2 - 1$ constants of the motion linearly independent of H . The U_{ij} are the components of the n -dimensional counterpart to the Fradkin tensor.⁵ The \mathbb{W}_{ij} are the components of the angular momentum tensor.

The above analysis applies equally well to quantum mechanics. From the $n^2 - 1$ constants of the motion we may obtain the standard generators of the Lie algebra $su(n)$ by suitable linear combinations. Thus for $n = 3$, the generators of $SU(3)$ for the quantum mechanical mechanical problem are

$$\begin{aligned} 2\sqrt{3} H_1 &= \mathbb{W}_{12}, \\ 12 H_2 &= U_{11} + U_{22} - 2 U_{33}, \\ 4\sqrt{3} E_3^\lambda &= \mathbb{W}_{23} + i\epsilon \mathbb{W}_{31} - \lambda(U_{13} + i\epsilon U_{23}), \\ 4\sqrt{6} E_{24} &= U_{11} - U_{22} + 2i\epsilon U_{12}, \end{aligned} \quad (4.11)$$

in which ϵ and λ take the values ± 1 independently. The H 's and E 's satisfy the usual commutation relations (c.f., Fradkin,⁵ Günther and Leach,¹⁴ and Sec. 6 for the discussion of the free particle).

5. BEHAVIOR UNDER TRANSFORMATION

Under the linear canonical transformation

$$\bar{\omega} = S\omega + r \iff \omega = \bar{S}\bar{\omega} + \bar{r}, \quad (2.4)$$

the Hamiltonian $\bar{H}(\bar{\omega})$ which gives a description of the motion in $\bar{\omega}$ equivalent to that of $H(\omega)$, (4.1) in ω is given by

$$\bar{H}(\bar{\omega}) = H(\omega) + \partial F / \partial t. \quad (3.12)$$

In $\bar{\omega}$ we also have an expression for H which we write as $H(\bar{\omega})$. Since

$$\dot{H}(\bar{\omega}) = 0, \quad (5.1)$$

$H(\bar{\omega})$ is a nontrivial constant of the motion described by $\bar{H}(\bar{\omega})$ because $H(\omega)$ is nonzero. In general $[H(\bar{\omega}), \bar{H}(\bar{\omega})]_{PB\bar{\omega}}$ is nonzero unless $\partial F / \partial t$ has zero Poisson bracket with $H(\bar{\omega})$.

As the transformation from ω to $\bar{\omega}$ is both canonical and nondegenerate, the symmetry group of $H(\omega)$ is preserved for $H(\bar{\omega})$. Suppose the generators of $SU(n)$ for $H(\omega)$ are $\mathbb{C}_N(\omega)$, $N = 1, n^2 - 1$ with $\mathbb{C}_N(\omega)$ having the form

$$\mathbb{C}_N = \alpha_{ij}^N U_{ij} + \beta_{ij}^N \mathbb{W}_{ij}. \quad (5.2)$$

Then

$$\begin{aligned} \text{(i)} \quad & \mathbb{C}_N(\omega) \neq 0, \\ \text{(ii)} \quad & \mathbb{C}_M(\omega) \neq \mathbb{C}_N(\omega), \quad M \neq N, \\ \text{(iii)} \quad & [\mathbb{C}_N(\omega), H(\omega)]_{PB\omega} = 0, \end{aligned} \quad (5.3)$$

$$\text{(iv)} \quad [\mathbb{C}_M(\omega), \mathbb{C}_N(\omega)]_{PB\omega} = f_{MN}^K \mathbb{C}_K(\omega),$$

where K, M and N range over the values 1 to $n^2 - 1$ and the f_{MN}^K are the structure constants. These properties are invariant under a linear canonical transformation, (5.4) (i) and (ii) due to the nondegeneracy and (5.4) (iii) and (iv) due to the canonicity of the transformation. For example,

$$\begin{aligned} [\mathbb{C}_M(\bar{\omega}), \mathbb{C}_N(\bar{\omega})]_{PB\bar{\omega}} &= \left\{ \frac{\partial}{\partial \bar{\omega}} \mathbb{C}_M(\bar{\omega}) \right\}^T \epsilon \left\{ \frac{\partial}{\partial \bar{\omega}} \mathbb{C}_N(\bar{\omega}) \right\} \\ &= \left\{ \frac{\partial}{\partial \omega} \mathbb{C}_M(\omega) \right\}^T S \epsilon S^T \left\{ \frac{\partial}{\partial \omega} \mathbb{C}_N(\omega) \right\} \\ &= f_{MN}^K \mathbb{C}_K(\bar{\omega}) \end{aligned}$$

since $S \epsilon S^T = \epsilon$.

For every \bar{H} obtainable from H (4.1) under a linear canonical transformation, there exists a constant of the motion $H(\bar{\omega})$ possessing the symmetry group $SU(n)$ which is thereby a (usually noninvariance) symmetry group characterizing \bar{H} . Alternatively, any given quadratic Hamiltonian may be transformed to the archtypal form (4.1) which possesses $n^2 - 1$ associated constants. In the original coordinates, (4.1) and the associated constants provide the (noninvariance) symmetry group for the Hamiltonian.

We emphasize that these results apply equally well to quantum mechanics when the usual conventions are observed. We note that the definitions of H and the \mathbb{C}_N are already symmetric in the products of ω^μ and ω^ν .

6. THE FREE PARTICLE AND $SU(3)$

In three dimensions, a free particle has the Hamiltonian

$$H = \frac{1}{2} p^2, \quad (6.1)$$

where p is a three-vector. The archtypal form

$$\bar{H} = \frac{1}{2} (P^2 + Q^2) \quad (6.2)$$

is obtained by the transformation

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \alpha, & -\alpha t + \beta \\ \dot{\alpha}, & -\dot{\alpha} t + \dot{\beta} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}, \quad (6.3)$$

in which

$$\alpha = A \cos t + B \sin t, \quad \beta = C \cos t + D \sin t. \quad (6.4)$$

The transformation (6.3) is canonical provided the constant 3×3 matrices A , B , C , and D satisfy

$$AC^T = CA^T, \quad BD^T = DB^T, \quad AD^T - CB^T = I. \quad (6.5)$$

In particular we may set

$$A = I, \quad D = I, \quad C = 0, \quad B = 0, \quad (6.6)$$

so that the transformation matrix is

$$S = \begin{bmatrix} I \cos t & -It \cos t + I \sin t \\ -I \sin t & Its \sin t + I \cos t \end{bmatrix}. \quad (6.7)$$

Applying the transformation (6.3) to the generators of $SU(3)$ given in (4.9), in the (q, p) coordinates we obtain the quantum mechanical generators

$$\begin{aligned} 2\sqrt{3} H_1 &= q_1 p_2 - q_2 p_1, \\ 12 H_2 &= q_1^2 + q_2^2 - 2q_3^2 + (1+t^2)(p_1^2 + p_2^2 - 2p_3^2) \\ &\quad - 2t(q_1 p_1 + q_2 p_2 - 2q_3 p_3), \\ 4\sqrt{3} E_4^\lambda &= q_2 p_3 - q_3 p_2 + i\epsilon(q_3 p_1 - q_1 p_3) \\ &\quad - \lambda\{q_1 q_3 + (1+t^2)p_1 p_3 - t(q_1 p_3 + p_1 q_3)\} \\ &\quad - i\lambda\epsilon\{q_2 q_3 + (1+t^2)p_2 p_3 - t(q_2 p_3 + p_2 q_3)\}, \\ 4\sqrt{6} E_{2\epsilon} &= q_1^2 - q_2^2 + (1+t^2)(p_1^2 - p_2^2) - 2t(q_1 p_1 - q_2 p_2) \\ &\quad + 2i\epsilon\{q_1 q_2 + (1+t^2)p_1 p_2 - t(q_1 p_2 + p_1 q_2)\}. \end{aligned} \quad (6.8)$$

(Note that there is no necessity to write H_2 or $E_{2\epsilon}$ in symmetric form since $i\hbar$ terms cancel.)

Using these generators we directly confirm the $SU(3)$

commutation relations, viz.

$$\begin{aligned} [H_1, H_2] &= 0, \\ [H_1, E_4^\lambda] &= \epsilon \hbar (2\sqrt{3})^{-1} E_4^\lambda, \quad [H_2, E_4^\lambda] = \epsilon \lambda \hbar (2)^{-1} E_4^\lambda, \\ [H_1, E_{2\epsilon}] &= \epsilon \hbar (\sqrt{3})^{-1} E_{2\epsilon}, \quad [H_2, E_{2\epsilon}] = 0, \\ [E_4^\lambda, E_{2\epsilon}] &= 0, \quad [E_4^\lambda, E_{2\epsilon}] = 0, \\ [E_4^\lambda, E_{-4}^\lambda] &= \epsilon \hbar (2\sqrt{3})^{-1} H_1 + \epsilon \lambda \hbar (2)^{-1} H_2, \\ [E_4^\lambda, E_{-4}^\lambda] &= -\epsilon \lambda \hbar (\sqrt{6})^{-1} E_{2\epsilon}, \\ [E_4^\lambda, E_{-2\epsilon}] &= \epsilon \lambda \hbar (\sqrt{6})^{-1} E_{-4}^\lambda, \quad [E_{2\epsilon}, E_{-2\epsilon}] = \epsilon \hbar (\sqrt{3})^{-1} H_1. \end{aligned} \quad (6.9)$$

The invariant for the motion is

$$I = \frac{1}{2} [q_1^2 + q_2^2 + q_3^2 + (1+t^2)(p_1^2 + p_2^2 + p_3^2) - t(q_1 p_1 + p_1 q_1 + q_2 p_2 + p_2 q_2 + q_3 p_3 + p_3 q_3)] \quad (6.10)$$

and, as has been directly demonstrated, I possesses the dynamical symmetry of $SU(3)$. As $\partial I / \partial t$ is nonzero, I does not commute with H and so $SU(3)$ is a noninvariance symmetry group for H (6.1).

7. THE ANISOTROPIC OSCILLATOR: NONINVARIANCE GROUP

The three-dimensional time-independent anisotropic oscillator has the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^3 (p_i^2 + \omega_i^2 q_i^2). \quad (7.1)$$

It is transformed to the archtypal oscillator Hamiltonian by the linear canonical transformation with coefficient matrix,

$$S = \begin{bmatrix} \omega_1^{1/2} \cos(\omega_1 - 1)t & 0 & 0 & -\omega_1^{-1/2} \sin(\omega_1 - 1)t & 0 & 0 \\ 0 & \omega_2^{1/2} \cos(\omega_2 - 1)t & 0 & 0 & -\omega_2^{-1/2} \sin(\omega_2 - 1)t & 0 \\ 0 & 0 & \omega_3^{1/2} \cos(\omega_3 - 1)t & 0 & 0 & -\omega_3^{-1/2} \sin(\omega_3 - 1)t \\ \hline \omega_1^{1/2} \sin(\omega_1 - 1)t & 0 & 0 & \omega_1^{-1/2} \cos(\omega_1 - 1)t & 0 & 0 \\ 0 & \omega_2^{1/2} \sin(\omega_2 - 1)t & 0 & 0 & \omega_2^{-1/2} \cos(\omega_2 - 1)t & 0 \\ 0 & 0 & \omega_3^{1/2} \sin(\omega_3 - 1)t & 0 & 0 & \omega_3^{-1/2} \cos(\omega_3 - 1)t \end{bmatrix} \quad (7.2)$$

The transformed Hamiltonian

$$\bar{H} = \frac{1}{2} \sum_{i=1}^3 (P_i^2 + Q_i^2) \quad (7.3)$$

is isotropic and exhibits the full degeneracy associated with $SU(3)$. This degeneracy is associated with the expression for \bar{H} in the (q, p) coordinate system which is the quadratic invariant

$$I = \frac{1}{2} \sum_{i=1}^3 (\omega_i q_i^2 + \omega_i^{-1} p_i^2). \quad (7.4)$$

I (7.4) commutes with H (7.1). The constants of the motion which provide a basis for the generators of the $SU(3)$ group of I are

$$I_{ii} = \frac{1}{2} (\omega_i q_i^2 + \omega_i^{-1} p_i^2), \quad (7.5)$$

$$\begin{aligned} I_{ij} &= (\omega_i^{1/2} \omega_j^{1/2} q_i q_j + \omega_i^{-1/2} \omega_j^{-1/2} p_i p_j) \cos(\omega_i - \omega_j)t \\ &\quad + (\omega_i^{1/2} \omega_j^{-1/2} q_i p_j - \omega_i^{-1/2} \omega_j^{1/2} q_j p_i) \sin(\omega_i - \omega_j)t, \end{aligned} \quad (7.6)$$

$$\begin{aligned} L_{ij} &= -(\omega_i^{1/2} \omega_j^{1/2} q_i q_j + \omega_i^{-1/2} \omega_j^{-1/2} p_i p_j) \sin(\omega_i - \omega_j)t \\ &\quad + (\omega_i^{1/2} \omega_j^{-1/2} q_i p_j - \omega_i^{-1/2} \omega_j^{1/2} q_j p_i) \cos(\omega_i - \omega_j)t \end{aligned} \quad (7.7)$$

(no summation on either i or j in (7.5-7). Although $[I_{ii}, H]$ is zero,

$$[I_{ij}, H] = -i\hbar (\omega_i - \omega_j) L_{ij}, \quad (7.8)$$

$$[L_{ij}, H] = i\hbar (\omega_i - \omega_j) I_{ij}. \quad (7.9)$$

Thus the $SU(3)$ basis for I does not provide a like basis

for H .

We point out that this does not exclude $SU(3)$ from being the symmetry group for the anisotropic oscillator, but it does exclude the possibility of a quadratic basis. When the ratios of frequencies are rational, the symmetry group exists both classically and quantum mechanically.^{7,17} When the frequencies are incommensurable the classical generators involve irrational powers.⁶ Although considerable progress has been made in constructing consistent quantum mechanical operators when the powers are rational,¹⁸ irrational powers as yet defy description. In this case it is possible that a consistent quantum mechanical discussion of the operators is not possible.

8. THE ANISOTROPIC OSCILLATOR: ENERGY STATES

The effect of a linear canonical transformation on the Schrödinger wavefunction is well established.^{19,23} Under the transformation

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \quad (8.1)$$

we have

$$\psi(q, t) = \int_{-\infty}^{\infty} dQ K_1(q, Q, t) \bar{\psi}(Q, t), \quad (8.2)$$

where the kernel $K_1(q, Q, t)$ is given by

$$K_1(q, Q, t) = (2\pi)^{-N/2} |\det S_2|^{-1/2} \exp\{i F_1(q, Q, t)\}, \quad (8.3)$$

$$2F_1(q, Q, t) = -q^T S_2^T S_4^T q - q^T S_1^T S_2^T q + 2Q^T S_2^T q, \quad (8.4)$$

provided S_2 is nonsingular. If S_2 is singular, the expression for K_1 may be written in an alternate form. For our discussion this is not the case and (8.3) with (8.4) suffices.

The energy levels of the motion described by $\psi(q, t)$ may be calculated without the form of $\psi(q, t)$ being known since

$$\begin{aligned} \langle \psi_n | H | \psi_n \rangle &= i\hbar \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dQ' \bar{\psi}_n^*(Q, t) K_1^*(q, Q, t) \\ &\times \left\{ \bar{\psi}_n(Q', t) \frac{\partial}{\partial t} K_1(q, Q', t) \right. \\ &\left. + K_1(q, Q', t) \frac{\partial}{\partial t} \bar{\psi}_n(Q', t) \right\}. \quad (8.5) \end{aligned}$$

For the anisotropic oscillator, $K_1(q, Q, t)$ may be constructed from (7.2) and $\bar{\psi}_n(Q, t)$ is the Schrödinger wavefunction for the isotropic oscillator. It is merely a matter of persistent calculation to show that

$$\langle \psi_n | H | \psi_n \rangle = \hbar \omega_1 (n_1 + \frac{1}{2}) + \hbar \omega_2 (n_2 + \frac{1}{2}) + \hbar \omega_3 (n_3 + \frac{1}{2}) \quad (8.6)$$

for

$$\langle \bar{\psi}_n | \bar{H} | \bar{\psi}_n \rangle = \hbar (n_1 + n_2 + n_3 + \frac{3}{2}). \quad (8.7)$$

Thus, when ω_1 , ω_2 , and ω_3 are noncommensurable, non-degenerate energy levels are obtained even though the energy levels of $\bar{\psi}$ are degenerate.

It may have been observed that we wrote the kernel of (8.2) as $K_1(q, Q, t)$ which was expressed in terms of an $F_1(q, Q, t)$. This is not usually done in the literature (cf. Boon and Seligman²³). We do this to emphasize

that classically the function $F_1(q, Q, t)$ is the generating function of the first type for the canonical transformation from \bar{H} to H . Should we so desire, the other generating functions may be used to construct a kernel. Then we may transform wavefunctions from momentum to coordinate, coordinate to momentum, and momentum to momentum representations in (Q, P) and (q, p) , respectively. Thus

$$\psi(q, t) = \int_{-\infty}^{\infty} dP K_3(q, P, t) \bar{\phi}(P, t), \quad (8.8)$$

$$\phi(p, t) = \int_{-\infty}^{\infty} dQ K_2(p, Q, t) \bar{\psi}(Q, t), \quad (8.9)$$

$$\phi(p, t) = \int_{-\infty}^{\infty} dP K_4(p, P, t) \bar{\phi}(P, t). \quad (8.10)$$

9. COMMENT

We have restricted the present discussion to quadratic Hamiltonians, quadratic invariants, and linear transformations for two reasons. Firstly the theory has a particularly elegant form and secondly there is no difficulty in the transition to quantum mechanics. For every quantum mechanical problem described by a quadratic Hamiltonian, the group $SU(n)$ may be associated in at least a noninvariant way.

It would appear that the wavefunction and energy levels for any such Hamiltonian may be obtained via (8.2) and (8.5) from the archtypal quadratic Hamiltonian. This is certainly the case when the signature of the quadratic part of the Hamiltonian is $2n$. However, when (8.5) is applied in the case of a free particle Hamiltonian a quadratic function of time is obtained. Evidently the transition from classical to quantum mechanics imposes some constraints on the validity of the application of (8.2) and (8.5), a feature which has been noted in a different context by Kennedy and Kerner.²⁴ The nature of these constraints will be the subject of further investigation.

Another worthy area of investigation is the applicability of the ideas used here to general Hamiltonian systems. It has been pointed out¹⁶ that classically every $2n$ -dimensional Hamiltonian may be transformed to any other $2n$ -dimensional Hamiltonian by a suitable canonical transformation. If the canonical transformation is a point transformation, the transition to quantum mechanics is always possible.²⁵ However, in the more general case, the extent to which such results may be applied in quantum mechanics will be determined by whether or not the quantum mechanical operators in the different coordinate systems are uniquely related.

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The Lie and Lie-admissible symmetries of dynamical systems

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Hadronic Journal **3** (1979) 390–439

The Lie and Lie-admissible symmetries of dynamical systems

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Abstract

In this paper we recall Lie's method for the construction of the generators of Lie symmetry groups from given second-order equations of motion, and provide an illustrative example. The method is then adapted to the equations of motion in their equivalent first-order (vector field) form, and an example is discussed in terms of Hamilton's equations. The first-order version of Lie's method is then studied for the construction of Lie-admissible symmetry groups, that is, connected Lie groups realized in such a way to admit a non-Lie, but Lie-admissible algebra in the neighborhood of the identity, as a nonconservative extension of the conventional Lie description of conservative mechanics. Some problems of using Lie's method for the construction of a Lie-admissible symmetry when the transformation includes a coordinate dependent time change are discussed.

*Supported by the U.S. Department of Energy under Contract Number AS02-78ER04742

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CHAPTER 1. INTRODUCTORY REMARKS

1.1. Exact and broken Lie symmetries. The history of Lie's symmetries¹ in physics is rather remarkable. After several decades of lack of general interest following their original proposal of 1891, Lie's symmetries were brought to the attention of the physics community by rather forceful, pioneering studies at the beginning of this century. Since that time, Lie's symmetries have progressively evolved, up to the current level of diversified physical applications. Almost needless to recall, a rather primary role, for the achievement of this position in physics, has been the application of Lie's symmetries for the conventional relativities of current use in particle physics, such as Galilei's and Einstein's special relativity.

Nevertheless, with the passing of time, it is becoming more and more clear that Lie's symmetries are generally broken in nature. The broken character of the unitary SU(3) symmetry for the classification of the strongly interacting particles was identified at the very inception of this line of study. Subsequently, the broken character of internal symmetries has been subjected to studies in rather diversified fields.

The situation for the space-time symmetries is, at this moment, quite intriguing.

Discrete space-time symmetries. Leo-Yang discovery of the parity violation in weak interactions brought to light, rather forcefully, the broken character of the discrete space-time symmetries. This character was subsequently confirmed by the discovery of the PC and T violations in the weak decays of light hadrons.

Classical continuous space-time symmetries. The breaking of the conventional Galilei symmetry in Newtonian mechanics is today established in rather incontrovertible terms. In fact, the assumption of the

symmetry considered literally implies the assumption of the validity of the perpetual motion in our environment. In general, the Newtonian forces are nonconservative and Galilei-form-noninvariant. As a result, Galilei's relativity is broken in Newtonian mechanics not only at the level of the conservation laws (physical profile), but also at the level of the form-invariance of the equations of motion (group theoretical profile). A classification of the breakings of Galilei's relativity has been provided by one of us (see ref.², Section 2.14; for a review, see in Part A of these Proceedings-ref.³, Section 2.2).

The situation for the Poincaré symmetry is predictably more controversial. According to a number of authors, the Poincaré symmetry is an exact symmetry for the classical motion of charged particles. According to Santilli (loc. cit.) the Poincaré symmetry is generally broken in nature, with the exception of the conditions originally conceived by Einstein, that is, motion of charged particles in vacuum under long range electromagnetic interactions. The breaking of the rotational part of the Poincaré symmetry is, in any case, established by the lack of conservation of the angular momentum for the motion of charged particles in a resistive medium. The breaking of the full Poincaré symmetry is (independently) inferred from conditions of compatibility with the breaking of the full Galilei symmetry, as established by the systems of our everyday experience.

This situation confirms the approximative character of physical theories. Galilei's symmetries provides a first approximation of physical reality (of Newtonian character) under the condition that the particles can be well approximated as being point-like (e.g., the sun and the planets of our solar system in Newtonian approximation). Whenever the point-like approximation becomes excessive (e.g., for the motion of an extended object in a resistive medium), Galilei's symmetry becomes excessively approximated

for any physically meaningful description, that is, it is broken. According to Santilli (loc. cit.), exactly the same situation occurs for Einstein's special relativity in classical mechanics, including gravitational extensions. In this latter case the geometry can be consistently considered as being of local Lorentz character for the motion of test particle in vacuum in the exterior of astrophysical bodies (again, when the test particle can be effectively approximated as being point-like), but the geometry must necessarily be non-Lorentz in local character for the interior case, to avoid perpetual-motion-type of approximations, e.g., for the motion of a satellite in Earth's atmosphere, or for the motion of a proton in the core of a star (for details, see ref.³, Sections 2.1 and 2.4).

Quantum mechanical continuous space-time symmetries. This is an open problem at this time, understandably, because of the current lack of sufficient experimental information. There is a rather general agreement that the Galilei and Poincaré symmetries are exact for the quantum mechanical description of charged particles moving in vacuum under large mutual distances and electromagnetic interactions only (e.g., the structure of atoms). On the contrary, there is a rather sharp disagreement at this moment on the problem of the validity or invalidity of the same symmetries for the strong interactions. According to one group of researchers, Einstein special relativity is still exactly valid for particles under strong interactions. According to another group of researchers, the relativity must be necessarily violated to allow a technical characterization of the extended character of the hadrons. In particular, according to this second view, conventional space-time symmetries are broken beginning at the nuclear level (e.g., dissipative nuclear processes), and this breaking persists at the hadronic level (e.g., models of hadronic constituents with wave packets in complete mutual penetration), as well as

at the level of the interior problem of gravitation (e.g., for structure models of stars with non-point-like constituents). Predictably, this situation calls for an experimental resolution (see ref.³, Section 2.4 for the state of the art on available proposals for experiments).

1.2. Lie-admissible treatment of broken Lie symmetries. The situation indicated in paragraph 1.1 was sufficiently well identified at the beginning of this decade. Santilli therefore initiated a study aiming at the identification of methods for the treatment of broken Lie symmetries in classical mechanics and quantum mechanics²⁻⁵. The guiding idea is essentially the following. One of the reasons of the physical effectiveness of Lie's symmetries is that they are equipped with an array of mathematical structures, such as (a) analytic formulations (Hamilton's equations, variational principles, etc.); (b) algebraic formulations (Lie's theorems, the Poincaré-Birkhoff-Witt theorem, etc.); and (c) geometrical formulations (symplectic geometry, Lie's derivative, etc.). In order to achieve a treatment of broken Lie symmetries which is sufficiently diversified to allow physical applications, it appears advisable to (1) identify the forces responsible for the breaking of the original symmetry; (2) identify the mathematical structure of the broken symmetry; and (3) achieve for the broken symmetry a diversification of mathematical methods as close as possible to that for the original exact symmetry.

Aspect (1) was readily resolved. All Lie symmetries considered in paragraph 1.1 are broken in classical mechanics because of "non-Hamiltonian forces", that is, forces nonderivable from a potential. It is possible that the same situation occurs also at the quantum mechanical level, in the sense that these nonpotential forces can be interpreted as an approxi-

mation of nonlocal forces (e.g., via polynomial expansions in the velocities). The studies under consideration suggested the generalization of Hamilton's equations into a form capable of accommodating unrestricted forces. The brackets of the time evolution law of these equations violate the Lie algebra laws, and verify instead those of the covering Lie-admissible algebras.

The studies of aspect (2) resulted in the proposal that a "broken Lie symmetry" be identified with a Lie-admissible symmetry, that is, a symmetry of the equations of motion possessing a Lie-admissible (non-Lie) algebra in the neighborhood of the identity. On more technical grounds, the original symmetry is considered broken at the level of the enveloping, associative, Lie-admissible, algebra. The replacement of this associative envelope with a covering, nonassociative, Lie-admissible, form ensures the capability of admitting a Lie-admissible algebras in the neighborhood of the identity (covering algebraic profile), as well as the exponentiation to a connected group (covering group-theoretical profile). Initial applications of these ideas resulted to be encouraging. For instance, the application of the treatment for the broken SU(3) symmetries in hadron physics reproduced the Gell-Mann-Okubo mass formula identically, while achieving identification of the broken context. Similarly, the treatment of the breaking of the symmetry under translations in time provided the capability to reach an algebraic representation of the time rate of variation of the energy under nonconservative forces.

Owing to the encouraging results for aspects (1) and (2), the studies for aspects (3) are now in full expansions, and they are generally referred to the "Lie-admissible formulations", that is, to the generalization of Lie's theory of Lie-admissible character at the analytic, algebraic and geometrical level. For a review of the state of the art, see ref.³

For the case of the connected space-time symmetries in Newtonian mechanics, these studies can be essentially applied as follows. Consider a Newtonian system possessing an exact symmetry (ES) under a Lie group G (e.g., the complete Galilei group or one of its subgroups)

$$(\mathfrak{m} \ddot{\vec{r}} - \vec{f})^{ES} = 0; \quad \vec{r} \in \mathbb{R}^n. \quad (1.1)$$

Suppose that the G-symmetry is broken (BS) because of the presence of additive forces (generally) nonderivable from a potential

$$[(\mathfrak{m} \ddot{\vec{r}} - \vec{f})^{ES} - \vec{F}]^{BS} = 0 \quad (1.2)$$

The Lie-admissible treatment of the broken G-symmetry essentially consists of the replacement of the Lie symmetry group G of eqs. (1.1) with a Lie-admissible symmetry group \hat{G} of eqs. (1.2). \hat{G} is constructed in such a way that the original Lie formulations for the exact symmetry are recovered identically at all levels (analytic, algebraic and geometrical) at the value of null symmetry breaking forces \vec{F} .

When G is the time component of Galilei's relativity (in canonical realization), the Lie-admissible covering \hat{G} has the structure

$$\hat{G} : \quad e^{t S^{\mu\nu} \frac{\partial H}{\partial a^\nu \partial a^\mu}}; \quad a = (r, p) \quad (1.3)$$

where H is the generator of G, that is, the nonconserved total energy, and S is the Lie-admissible (rather than Lie) tensor. In the neighborhood of the identity, the Lie-admissible group provides the characterization of the time rate of variation of physical quantities X_i , $i = 1, 2, \dots$ (say, energy, angular momentum, etc.) according to the rule

$$\dot{X}_i = \frac{dX_i}{dt} = (X_i - e^{dt S^{\mu\nu} \frac{\partial H}{\partial a^\nu \partial a^\mu}} X_i) / dt = (X_i, H) \neq 0 \quad (1.4)$$

where (X_i, H) is the Lie-admissible product characterized by the tensor S. At the limit of null F-forces: (i) the product (X_i, H) becomes the conventional Poisson product; (ii) the conservation laws of the physical quantities X_i are recovered identically; and (iii) the group \hat{G} recovers the conventional group G identically. This is, in essence, the covering of the time component of Galilei's relativity in Newtonian mechanics proposed by Santilli in ref.², with a conjectured extension to the entire 10-parameter Galilei group (see ref.³ for the state of the art).

Notice that the Lie-admissible group (1.3) admits an operator-type reformulation. It is therefore expected that the Lie-admissible treatment of broken Lie symmetries admits a quantum mechanical image.

Finally, it should be indicated here that the theorems of direct universality of the Lie-admissible algebras in Newtonian and quantum mechanics (ref.³) establish that all broken Lie symmetries can be treated via the Lie-admissible techniques, although the use of other techniques, of course, is not excluded.

1.3. Statement of the problem. The state of the art in the construction of the Lie-admissible treatment of broken Lie symmetries in Newtonian mechanics can be summarized as follows. We know how to construct the Lie-admissible generalization of Hamilton's equations for all Newtonian systems (including nonlocal forces). We also know how to generalize a Lie algebra

into a covering Lie-admissible algebra. In turn, this allows the construction of the symplectic-admissible generalization of the underlying symplectic two-forms. Finally, we know how to construct the covering Lie-admissible group (1.3) for all local, autonomous, nonconservative systems.⁴

Nevertheless, a central problem which is open at this time consists of general methods for the construction of the covering Lie-admissible symmetry \hat{G} from given equations of motion (1.2). This paper is devoted to initial studies along these lines. Notice that our problem is at least two-fold. First, we need general methods for the construction of symmetries from given equations of motion. Second, these methods should be such to allow a redefinition of the symmetry as a Lie-admissible symmetry.

The last problem is made technically identifiable by the property that all Lie-admissible groups admit a reformulation in terms of conventional Lie groups (although in different generators without direct physical meaning). This is easily seen by rewriting group (1.3) in the vector field form

$$\hat{G} : e^{tX(a)} \quad (1.5)$$

As a result of this property, the first part of our study will be devoted to the review of Lie's method for the construction of symmetries from given differential equations. This method (which has been regrettably forgotten to a considerable extent) has been brought again to the attention of our community by some of us. For a recent review of the state of the art see the article by Eliezer⁶.

We shall subsequently study the formulation of Lie's method for first-order systems of differential equations, that is, for the reduction of equations (1.2) to an equivalent first-order form. This reduction is recommendable for all symmetries (whether of exact or broken Lie character), in order to have the proper grounds for analytic, algebraic, and geometrical formulations.

We shall then identify a number of aspects of the Lie-admissible reformulation of Lie's method, and consider in detail some of them, e.g., the Lie-admissible reformulation (1.3) of a given Lie symmetry (1.5).

No quantum mechanical aspect will be considered in this paper. Also, this paper is written for the physics audience without any use at all of familiar mathematical languages (such as vector fields, cotangent bundle, Lie's derivative, etc.). The interested mathematicians can however easily reformulate our analysis in the conventional language of differential geometry.

CHAPTER 2. LIE'S METHOD

2.1. Invariance of form.

There has been a recent revival of interest in a method which S. Lie¹ used about a century ago to analyse the symmetry properties of a differential equation by considering the groups of continuous transformations which preserve the form of the differential equation. The notion of "invariance of form" has in the intervening years become a dominant theme in physical theory, with the advent of the theories of relativity and quantum mechanics. For a given dynamical system we describe as Lie symmetries those which arise from transformations which leave the dynamical equations of motion invariant. The family of such transformations are found to satisfy the group axioms and thus we obtain the symmetry group associated with the system.

In the context of ordinary differential equations, Lie's aim was to construct a general integration method whereby for a first order equation a solution may be obtained by quadrature and, for a higher degree equation, a reduction in order could be effected and the solution obtained thereafter by a requisite number of quadratures.

The approach had much success but also showed some limitation. For example, for a given first order first degree equation, it was found that a knowledge of the group does in fact lead to a solution by quadrature. However there is no systematic method available for determining the uncountably infinite number of groups. For a first order equation which is not of first degree, a group need not necessarily exist⁷.

In the case of second order equations of the first degree the situation is markedly different. When there is only one dependent variable, there exist at most eight independent groups of point transformations which leave the equation invariant. These are obtained by solving a master equation. The method has been applied to some second order equations of motion of dynamical systems, and significant results have been obtained for the free particle, for various oscillator systems and for the Kepler problem⁸⁻¹³.

It is also of interest to note that in the case of second order equations, if we widen the transformations from point-transformations to velocity-dependent transformations, the situation bears a similarity to the first order case and the number of groups becomes infinite⁸.

2.2. Local and Global transformations.

We consider some aspects of a one-parameter group of transformations of the x, y plane into itself such that

$$\bar{x} = f(x, y, \alpha), \quad \bar{y} = g(x, y, \alpha) \tag{2.1}$$

where $\alpha = \alpha_0$ corresponds to the identity transformation and the group properties are satisfied. Lie's method emphasises the importance of the local structure of the group. The global structure may be derived from the local structure. Accordingly the method determines, in the first instance, the group of infinitesimal transformations which leave a given differential equation invariant.

With $\alpha = \alpha_0 + \delta\alpha$, we take as transformation

$$\bar{x} = x + \xi(x, y)\delta\alpha, \quad \bar{y} = y + \eta(x, y)\delta\alpha. \tag{2.2}$$

The functions ξ and η define the transformation locally. The finite form of the transformation may then be found by integrating the differential system

$$\frac{d\bar{x}}{\xi(\bar{x}, \bar{y})} = \frac{d\bar{y}}{\eta(\bar{x}, \bar{y})} = d\tau \quad (2.3)$$

with the initial conditions $\bar{x} = x, \bar{y} = y$ at $\tau = 0$ where e.g. $\tau = \alpha - \alpha_0$.

At each point (\bar{x}, \bar{y}) , ξ and η determine a direction of gradient η/ξ . As τ varies, the point (\bar{x}, \bar{y}) describes a curve which we call a path curve. Since the path curve as a whole goes into itself under the transformation, the path curve is also called an invariant curve.

The infinitesimal transformation (2.2) may be expressed concisely by the symbol U where

$$U = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y} \quad (2.4)$$

Then

$$Ux = \xi, Uy = \eta, \bar{x} = (1 + \delta\tau U)x, \bar{y} = (1 + \delta\tau U)y.$$

The finite form of the transformation may be written formally as

$$\bar{x} = [\exp \tau U]x, \bar{y} = [\exp \tau U]y.$$

For a function $\phi(x,y)$, we may write

$$\phi(\bar{x}, \bar{y}) = \phi(x,y) + \delta\tau U\phi$$

where $U\phi = \xi \frac{\partial \phi}{\partial x} + \eta \frac{\partial \phi}{\partial y}$ is the directional derivative of ϕ along the path curve at (x, y) .

$\phi(x,y)$ is said to be an invariant when $\phi(\bar{x}, \bar{y}) = \phi(x,y)$, for which a necessary and sufficient condition is $U\phi = 0$.

A family of curves

$$\psi(x, y) = \text{constant}$$

may be said to be an invariant in the sense $\psi(\bar{x}, \bar{y}) = F(\psi(x,y))$, so that geometrically a member of the family would transform into another member or into itself under a transformation of the group. In terms of the operator U , we may write the condition for this:

$U\psi = G(\psi)$ where $G(\psi)$ is a function of ψ . Except for path curves for which $U\psi = 0$, we may without loss of generality write the equation of family as $F(\psi) = \text{constant}$ such that $UF = 1$.

2.3. First order differential equations

Let us now consider a first order differential equation of the form

$$\phi(x, y, y') = y' - f(x, y) = 0 \quad (2.5)$$

where $y' = \frac{dy}{dx}$. The solution of (2.5) in the form $K(x, y) = \text{constant}$ will represent a singly infinite family of curves, which will be called integral curves of the system.

We consider now the effect of the transformation (2.2) on this differential equation. The point transformation induces changes in y', y'' and in all higher derivatives. Let $\delta\tau \eta^{(k)}$ represent the change in the k th derivative $\frac{d^k y}{dx^k} = y^{(k)}$. It can be shown that

$$\eta^{(k)} = \frac{d}{d\bar{x}} \eta^{(k-1)} - y^{(k)} \frac{d\xi}{dx} \quad (2.6)$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots$$

The change in any function $\phi(x, y, y', y'', \dots)$ may be written $\delta_{\tau} U^{(n)} \phi$ where

$$U^{(n)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^{(1)} \frac{\partial}{\partial y'} + \dots + \eta^{(k)} \frac{\partial}{\partial y^{(k)}} + \dots \quad (2.7)$$

Let $U^{(k)}$ denote that part of the series for $U^{(n)}$ which terminates at $\eta^{(k)} \frac{\partial}{\partial y^{(k)}}$. The condition that the equation (2.5) retains its form under the transformation is equivalent to $U^{(n)} \phi = 0$, which reduces to $U^{(1)} \phi = 0$, as ϕ involves only x, y and y' .

Hence
$$-\xi \frac{\partial f}{\partial x} - \eta \frac{\partial f}{\partial y} + \eta^{(1)} = 0. \quad (2.8)$$

Substituting for $\eta^{(1)}$ and replacing y' by f yields

$$\frac{\partial \eta}{\partial x} + f \left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x} \right) - f^2 \frac{\partial \xi}{\partial y} - \xi \frac{\partial f}{\partial x} - \eta \frac{\partial f}{\partial y} = 0 \quad (2.9)$$

as the condition that ξ, η satisfy for equation (2.5) to be invariant under the transformation. Geometrically this property is that an integral curve transforms either into itself or into another integral curve.

Defining ρ by the equation

$$\eta = \xi f + \rho, \quad (2.10)$$

and substituting in (2.9) then yields

$$\frac{\partial \rho}{\partial x} + f \frac{\partial \rho}{\partial y} = \rho \frac{\partial f}{\partial y}, \quad (2.11)$$

showing that the solution of (2.9) may be written: ξ is arbitrary and η is given by (2.10) and (2.11).

Solution of (2.11) is based on that of

$$\frac{dx}{1} = \frac{dy}{f} = \frac{d\rho}{\rho \frac{\partial f}{\partial y}} \quad (2.12)$$

which incorporates equation (2.5).

We may also view (2.10) as arising from a condition $U\psi = 1$ where $\psi = \text{constant}$ is the family of integral curves of (2.5).

It may also be shown that whenever the condition (2.9) is satisfied, an integrating factor may be found which provides a solution of (2.5) by quadrature.

We now consider the invariant functions $v(x, y, y')$ of $U^{(1)}$. Such a function is given by solutions of

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = \frac{dy'}{\eta^{(1)}(x,y,y')} \quad (2.13)$$

The first pair of (2.13) yield the invariant function $u(x,y) = \text{constant}$ of U , that is $Uu = 0$. Any other pair of (2.13) is then integrated eliminating the appropriate variable using $u = \text{constant}$. The solution is $v(x, y, y') = \text{constant}$. This is the so called first differential invariant of U and we have $U^{(1)}v = 0$.

It is clear that the most general first order differential equation invariant under a given group with generator U is

$$v(x, y, y') = H(u(x, y)), \quad (2.14)$$

for any function H .

2.4 Second order equations

The equations we would be concerned with are dynamical equations involving time t as the independent variable, and therefore a change of notation is desirable. We consider an equation of motion of the form

$$\ddot{x} + g(x, \dot{x}, t) = 0 \tag{2.15}$$

The point transformation is

$$\bar{t} = t + \delta t \xi(t, x), \quad \bar{x} = x + \delta x \eta(t, x) \tag{2.16}$$

with associated quantities

$$\left. \begin{aligned} U^{(k)} &= \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \dots + \eta^{(k)} \frac{\partial}{\partial x^{(k)}} \\ \eta^{(k)} &= \frac{d\eta^{(k-1)}}{dt} - x^{(k)} \frac{d\xi}{dt} \\ \frac{d}{dt} &= \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \ddot{x} \frac{\partial}{\partial \dot{x}} + \dots \end{aligned} \right\} \tag{2.17}$$

Invariance of (2.15) leads to the condition

$$U^{(2)}(\ddot{x} + g) = 0 \tag{2.18}$$

where after differentiation we substitute $\ddot{x} = -g$, and obtain

$$\eta^{(2)} + \xi \frac{\partial g}{\partial t} + \eta \frac{\partial g}{\partial x} + \eta^{(1)} \frac{\partial g}{\partial \dot{x}} = 0 \tag{2.19}$$

Substitution for $\eta^{(1)}$ and $\eta^{(2)}$ yields the master equation

$$\begin{aligned} & \frac{\partial^2 \eta}{\partial t^2} + \left(2 \frac{\partial^2 \eta}{\partial x \partial t} - \frac{\partial^2 \xi}{\partial t^2} \right) \dot{x} + \left(\frac{\partial^2 \eta}{\partial x^2} - 2 \frac{\partial^2 \xi}{\partial x \partial t} \right) \dot{x}^2 - \frac{\partial^2 \xi}{\partial x^2} \dot{x}^3 \\ & - \left(\frac{\partial \eta}{\partial t} - 2 \frac{\partial \xi}{\partial t} - 3 \dot{x} \frac{\partial \xi}{\partial x} \right) g + \xi \frac{\partial g}{\partial t} + \eta \frac{\partial g}{\partial x} + \left\{ \frac{\partial \eta}{\partial t} + \left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial t} \right) \dot{x} - \frac{\partial \xi}{\partial x} \dot{x}^2 \right\} \frac{\partial g}{\partial \dot{x}} = 0. \end{aligned} \tag{2.20}$$

When we consider the invariance of the whole family of integral curves and not of any one member, the equation (2.20) holds for all values of \dot{x} . Hence we equate to zero the coefficients of linearly independent functions of \dot{x} and obtain a set of partial differential equations for ξ and η . On solving these we obtain the required group of transformations. We shall find that there are at most eight one-parameter groups.

2.5 An example

To illustrate the method we consider the equation of the damped free particle:

$$\ddot{x} + \gamma \dot{x} = 0. \tag{2.21}$$

Substituting $g = \gamma \dot{x}$ in (2.20), and equating to zero terms of the same power in \dot{x} , we have

$$\left. \begin{aligned} \frac{\partial^2 \xi}{\partial x^2} &= 0 \\ \frac{\partial^2 \eta}{\partial x^2} - 2 \frac{\partial^2 \xi}{\partial x \partial t} + 2\gamma \frac{\partial \xi}{\partial x} &= 0 \\ 2 \frac{\partial^2 \eta}{\partial x \partial t} - \frac{\partial^2 \xi}{\partial t^2} + \gamma \frac{\partial \xi}{\partial t} &= 0 \\ \frac{\partial^2 \eta}{\partial t^2} + \gamma \frac{\partial \eta}{\partial t} &= 0 \end{aligned} \right\} \tag{2.22}$$

Solving these we have

$$\left. \begin{aligned} \xi &= A + Bx + Ce^{-\gamma t} + (D + Ex)e^{\gamma t} \\ \eta &= F + Gx - \gamma Dx^2 + (H - C\gamma x)e^{-\gamma t} \end{aligned} \right\} \quad (2.23)$$

where A, B, ..., H are arbitrary constants. We note that there are eight possible independent solutions for ξ, η . The explicit solutions are given in a table in the next section.

2.6 First Integrals

When a first integral $I(x, \dot{x}, t)$ exists, the equation $I = c$ would represent a surface in (x, \dot{x}, t) space. The solution curves will be on this surface, and if there exists a transformation which permutes solution curves within the surface, then

$$U^{(1)} I(x, \dot{x}, t) = 0. \quad (2.24)$$

This equation together with the constancy of I may be used to determine I . For given ξ, η we solve

$$\frac{dt}{\xi} = \frac{dx}{\eta} = \frac{d\dot{x}}{\eta^{(1)}} \quad (2.25)$$

to obtain invariant $u(x, t)$ and differential invariant $v(x, \dot{x}, t)$.

Then any function of the form $\phi(u, v)$ is invariant under the operation of $U^{(1)}$. It is also a first integral if its total time derivative is zero, giving

$$\frac{d\phi}{dt} = -\frac{\partial \phi}{\partial u} / \frac{\partial \phi}{\partial v} = \psi(u, v) \quad (2.26)$$

where ψ is calculated after substituting for \ddot{x} . The first integral is obtained where possible from the first-order equation (2.26) by quadrature. We observe that the existence of the group has enabled a reduction in order from second to first. If equation (2.26) proves intractable, the existence of a second group may provide a way to determine an integrating factor. This matter is developed in detail elsewhere by Prince¹⁰.

We illustrate by applying this method to the example of Section 2.5 for the particular transformation $\xi = x, \eta = -\gamma x^2$ which belongs to the group. Equation (2.25) yields

$$\frac{dt}{x} = \frac{dx}{-\gamma x^2} = \frac{d\dot{x}}{-2\gamma x\dot{x} - \dot{x}^2} \quad (2.27)$$

from which we obtain

$$u = xe^{\gamma t}, \quad v = \dot{x}/x(\dot{x} + \gamma x). \quad (2.28)$$

Then $\frac{dv}{du} = \frac{dv}{dt} / \frac{du}{dt}$ which on evaluation after substituting for \ddot{x}

is found to be $-v/u$. Integration gives uv as a constant of motion.

Hence we take

$$I = \dot{x}e^{\gamma t}/(\dot{x} + \gamma x). \quad (2.29)$$

We give below a table of the values of ξ and η for the eight one-parameter transformations, and the associated first integrals for the damped free particle:

ξ	η	I
1	0	$\dot{x} + \gamma x$
$e^{\gamma t}$	0	$\dot{x} e^{\gamma t}$
$-\frac{1}{\gamma} e^{-\gamma t}$	$x e^{-\gamma t}$	$\dot{x} + \gamma x$
x	$-\gamma x^2$	$(\dot{x} + \gamma x) / \dot{x} e^{\gamma t}$
$x e^{\gamma t}$	0	$\dot{x} e^{\gamma t} / (\dot{x} + \gamma x)$
0	x	$\dot{x} e^{\gamma t} / (\dot{x} + \gamma x)$
0	1	$\dot{x} e^{\gamma t}$
0	$e^{-\gamma t}$	$\dot{x} + \gamma$

The group of these eight one-parameter transformations is the Lie symmetry group for the damped free particle.

The Noether method, which is applicable for studying symmetries when the system has a Lagrangian, provides only a sub-group of the Lie group. Noether's method is equivalent to finding the transformation functions $\xi(x, t)$ and $\eta(x, t)$ and a gauge function $f(x, t)$ such that

$$U^{(1)}L = -\dot{\xi}L + f \tag{2.30}$$

This leads to a set of partial differential equations for ξ , η and f . With each transformation there is an associated first integral

$$J = (\xi\dot{x} - \eta) \frac{\partial L}{\partial \dot{x}} - \xi L + f. \tag{2.31}$$

Elements of the Lie group which are not in the Noether group also have physical significance. Such an element in the Kepler problem leads to the symmetry described by the Runge-Lenz vector

CHAPTER 3. LIE METHOD FOR FIRST ORDER SYSTEMS.

3.1. Introductory Remarks

It is of some importance to see how the preceding results may be obtained if a given second order equation (Newtonian equation of motion) is reduced to a pair of first order equations. The ultimate aim is to obtain an equivalent set of results when working in a formulation of mechanics which is based on first order equations such as those of Hamilton or their generalizations.

Firstly we consider the invariance of a general pair of first order equations of the first degree in (x, y, z) space under point transformations of that space. The restriction is then made, that the pair be equivalent to a second order equation of first degree in (x, y) space ($z = G(x, y, y')$). To obtain useful results it is found to be necessary that the z transformation be induced by the x, y transformations in the treatment of the pair of first order equations.

By way of example we apply the approach to a dynamical problem in Hamiltonian mechanics.

3.2. The Lie method for simultaneous first order equations

Consider the pair of first degree first order ordinary differential equations

$$y' = f(x, y, z) \tag{3.1}$$

$$z' = g(x, y, z)$$

in (x, y, z) space. Two functionally independent solutions

$$\begin{aligned} \omega(x, y, z) &= c = \text{constant} \\ \lambda(x, y, z) &= k = \text{constant} \end{aligned} \tag{3.2}$$

of (3.1) provide two families of surfaces in the 3-space and any other solution of (3.1) will be a function of ω and λ . The form invariance of (3.1) under a point transformation in the 3-space will come about when the transformation permutes solutions (3.2). The converse is also true: any point transformation in 3-space permuting (3.2) will leave (3.1) form invariant.

Suppose such a point transformation has the generator

$$\begin{aligned} V &= \bar{\xi}(x, y, z) \frac{\partial}{\partial x} + \bar{\eta}(x, y, z) \frac{\partial}{\partial y} \\ &+ \bar{\zeta}(x, y, z) \frac{\partial}{\partial z} . \end{aligned} \tag{3.3}$$

The above invariance requirements are expressed as

$$V^{(1)} \left\{ \begin{array}{l} y' - f(x, y, z) \\ z' - g(x, y, z) \end{array} \right\} = 0 \tag{3.4}$$

or

$$\begin{aligned} \bar{\eta}^{(1)} - V f &= 0 \\ \bar{\zeta}^{(1)} - V g &= 0 \end{aligned}$$

when (3.1) holds (form invariance of (3.1)) or, equivalently,

$$\begin{aligned} V\omega &= \Omega(\omega, \lambda) \\ V\lambda &= \Lambda(\omega, \lambda) \end{aligned} \tag{3.5}$$

when (3.2) holds (permutation of solutions (3.2)). Note that

$$\begin{aligned} V^{(1)} &= \bar{\xi} \frac{\partial}{\partial x} + \bar{\eta} \frac{\partial}{\partial y} + \bar{\zeta} \frac{\partial}{\partial z} \\ &+ \bar{\eta}^{(1)} \frac{\partial}{\partial y'} + \bar{\zeta}^{(1)} \frac{\partial}{\partial z'} \end{aligned} \tag{3.6}$$

with for example $\bar{\eta}^{(1)} = \frac{d\bar{\eta}}{dx} - y' \frac{d\bar{\xi}}{dx}$.

It is also worth noting that without loss of generality ω and λ may be chosen such that

$$\begin{aligned} V\omega &= 0 \\ V\lambda &= 1 , \end{aligned} \tag{3.7}$$

(compare with §2.3)¹⁴. By considering (3.7) or the partial differential equations (3.4) it may be shown that

$$\begin{aligned} \bar{\eta} &= \rho(x, y, z) + f\bar{\xi} \\ \bar{\zeta} &= \sigma(x, y, z) + g\bar{\xi} , \end{aligned} \tag{3.8}$$

where ρ and σ are related to ω and λ .

It is clear from (3.8) that as in the simple first order case there is an uncountable infinity of triples $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ available for a given pair of first order equations of first degree (3.1).

3.3. Equivalence to a second order equation

Suppose the pair (3.1) is regarded as the decomposition of a second order equation of first degree, say

$$y'' = F(x, y, y') \quad (3.9)$$

that is, we introduce a dummy variable

$$z = G(x, y, y') \quad (3.10)$$

in (3.9) to produce (3.1) as

$$y' = f(x, y, z) \quad (3.11)$$

$$z' = g(x, y, z)$$

(3.9) is reclaimed from (3.11) as

$$y'' = f_x + f f_y + g f_z \quad (3.12)$$

where z is replaced in (3.12) through (3.10).

Since equations (3.11) have identical solutions to (3.12) we look for a restricted class of point transformations of (x, y, z) space with (3.11) invariant in the sense of §3.2, which is equivalent to the set of point transformations in (x, y) space leaving (3.12) invariant in the sense of §2.4 .

Now we are really dealing with a second order equation in (x, y) space so we should only consider transformations in (x, y, z) space of the form

$$\begin{aligned} \bar{x} &= f(x, y, z) \\ \bar{y} &= g(x, y, z) \end{aligned} \quad (3.13)$$

with the z transformation being induced by (3.13) through the induced transformation in y' and (3.10). Explicitly, the appropriate infinitesimal operator in (x, y, z) space is the $U^{(1)}$ operator of (x, y) space written in (x, y, z) variables rather than in (x, y, y') variables

$$\begin{aligned} v &= U^{(1)} = (Ux) \frac{\partial}{\partial x} + (Uy) \frac{\partial}{\partial y} + (U^{(1)}z) \frac{\partial}{\partial z} \\ &= \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \zeta(x, y, z) \frac{\partial}{\partial z} \end{aligned} \quad (3)$$

where (3.10) and (3.11) have been used to eliminate y' from $U^{(1)}z$. To treat the invariance of (3.11) using (3.14), v is extended in the usual way,

$$v^{(1)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z} + \eta^{(1)} \frac{\partial}{\partial y'} + \zeta^{(1)} \frac{\partial}{\partial z'} \quad (3)$$

with

$$\begin{aligned} \eta^{(1)} &= \frac{d\eta}{dx} - y' \frac{d\xi}{dx} \\ \zeta^{(1)} &= \frac{d\zeta}{dx} - z' \frac{d\xi}{dx} \end{aligned} \quad (3)$$

The variables x, y, z are now regarded as independent quantities and the first equation of (3.11) is to be considered as an equation of motion rather than as a definition.

It may be shown that the invariance of (3.11) under (3.14)

$$\begin{aligned} v^{(1)}(y' - f(x, y, z)) &= 0 \\ v^{(1)}(z' - g(x, y, z)) &= 0 \end{aligned} \quad (3)$$

when (3.11) hold, ensures the invariance of (3.12) considered as a

second order equation in (x, y, z) space, that is, (3.17) implies that

$$v^{(2)} (y'' - f_x - ff_y - gf_z) = 0, \quad (3.18)$$

when (3.12) holds, is identically satisfied. This is to be expected as (3.12) and (3.11) have identical solutions in (x, y, z) space. Moreover, the invariance requirement (3.18) is identical with the invariance requirement under U of (3.12), considered as a second order equation in (x, y) space, which is that

$$U^{(2)} (y'' - f_x - ff_y - gf_z) = 0. \quad (3.19)$$

This is seen by expanding (3.18) as

$$\eta^{(2)} = V(f_x + ff_y + gf_z) \quad (3.20)$$

and (3.19) as

$$\eta^{(2)} = U^{(1)} (f_x + ff_y + gf_z) \quad (3.21)$$

where in (3.20) y'' is everywhere replaced through (3.12) and z is everywhere retained and in (3.21) y'' is replaced everywhere through (3.12) and z is not retained.

In summary, if a pair of first order equations of first degree is to be treated rather than an equivalent second order equation of first degree, the results of the treatment of the second order problem are regained exactly if the point transformations involved in the first order theory are restricted to those generated by operators of the form

$$V = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \zeta(x, y, z) \frac{\partial}{\partial z} \quad (3.22)$$

The one-to-one correspondence of the results of the two approaches is through the identification

$$V(x, y, z) = U^{(1)} (x, y, y'(x, y, z)). \quad (3.23)$$

Note firstly that a relation (3.10) must exist, that is at some stage z must be introduced as a dummy variable and secondly that we have achieved equivalence of the two methods through sacrificing the z -dependence of the x and y transformations in (x, y, z) space.

Finally we note that there exist at most eight generators of the type (3.22) leaving invariant a given pair of first order equations (3.11). The generators form a Lie algebra identical to the algebra for the corresponding second order problem.

3.4. Construction of constants of motion

Any functionally independent pair of solutions $\omega(x, y, z)$ and $\lambda(x, y, z)$ of (3.11) are first integrals $I(x, y, y')$ and $J(x, y, y')$ of the equivalent second order equation (3.12). In the first of equations (3.7) we saw that without loss of generality one of these, say $\omega(x, y, z)$ may be chosen such that

$$V\omega(x, y, z) = 0. \quad (3.24)$$

This, coupled with the property of first integrals, viz.

That

$$\frac{d}{dx} \omega(x, y, z) = 0 \quad (3.25)$$

enables the construction of ω as follows. Equation (3.24) indicates

that $w(x, y, z)$ must be a function of $u_1(x, y, z)$ and $u_2(x, y, z)$, the invariants of the operator V which are found by quadrature (c.f. §2.3). It is written as

$$w(x, y, z) = \phi(u_1, u_2) . \quad (3.26)$$

Substituting this into (3.24) we have

$$\begin{aligned} \frac{\partial \phi}{\partial u_1} \frac{du_1}{dx} + \frac{\partial \phi}{\partial u_2} \frac{du_2}{dx} &= 0 \\ \Leftrightarrow \frac{du_2}{du_1} &= \psi(u_1, u_2) . \end{aligned} \quad (3.27)$$

Equation (3.27) indicates how the pair of equations (3.11) is reduced to a single first order equation when a group is available (c.f. §2.6). The function $\psi(u_1, u_2)$ is determined by calculation

$$\frac{du_2}{du_1} = \frac{du_2}{dx} / \frac{du_1}{dx} \quad (3.28)$$

using the functional forms of u_1 and u_2 in terms of x, y and z and the equations of motion (3.11). Equation (3.27) is then solved to obtain the invariant

$$w = \phi(u_1, u_2) = \text{constant} .$$

When a simple quadrature is not possible, the existence of another group of point transformations leaving (3.11) invariant may be used to provide an integrating factor for (3.27) ¹⁴.

3.5 Application to a problem in Hamiltonian mechanics

From the foregoing discussion it is obvious that the Lie method may be applied to particular types of pairs of first order equations. As an example we look at Hamilton's equations and as indicated above we shall use point transformations only. We may consider the method applied in the canonical framework without reference to Newtonian or Lagrangian formulations. It should be emphasized that, if a problem has a parallel description in the other formulation, then the results are identical and the equivalence is established through (3.23).

In keeping with the usual Hamiltonian methodology we write the generator of the group as

$$Y(q, p, t) = \xi(q, t) \frac{\partial}{\partial t} + \eta(q, t) \frac{\partial}{\partial q} + \zeta(q, p, t) \frac{\partial}{\partial p} . \quad (3.29)$$

Hamilton's equations

$$\dot{q} - \frac{\partial H}{\partial p} = 0 \quad \dot{p} + \frac{\partial H}{\partial q} = 0 \quad (3.30)$$

are invariant under the group when

$$Y^{(1)} \left\{ \begin{array}{l} \dot{q} - \frac{\partial H}{\partial p} \\ \dot{p} + \frac{\partial H}{\partial q} \end{array} \right\} = 0 \quad (3.31)$$

with (3.30) in force. The operator $Y^{(1)}$ is the first extension of Y . From (3.31) we have

$$\eta^{(1)} - Y \left(\frac{\partial H}{\partial p} \right) = 0, \quad \zeta^{(1)} + Y \left(\frac{\partial H}{\partial q} \right) = 0 \quad (3.32)$$

and the first of these may be rearranged to give an expression for ζ in terms of $\xi, \eta, \dot{\xi}, \dot{\eta}, q, p$ and t . This is substituted in the second of (3.32) using the definition of $\zeta^{(1)}$. In the resulting equation since ξ and η independent of p , we may equate coefficients of linearly independent functions of p to zero thereby obtaining ξ and η . These are then substituted in the expression for ζ . The invariants are calculated as in §3.4.

We illustrate the method with a simple example, the damped free particle. It has Hamiltonian¹⁵

$$H = \frac{1}{2} p^2 e^{-\gamma t} \quad (3.33)$$

whence

$$\zeta = e^{\gamma t} \eta^{(1)} + \dot{\gamma} p \xi. \quad (3.34)$$

When (3.34) is substituted into the second of (3.32) and the algebra worked through we obtain the generators and invariants listed in Table 1.

If p is replaced by $\dot{q} \exp(\gamma t)$ the results for ξ, η and I are identical with those given in Section 2.6.

TABLE 1.

$\xi(q, t)$	$\eta(q, t)$	$\xi(q, p, t)$	$I(q, p, t)$
1	0	γp	$\gamma q + p e^{-\gamma t}$
$e^{\gamma t}$	0	0	p
$\frac{1}{\gamma} e^{-\gamma t}$	$q e^{-\gamma t}$	$-\gamma q - p e^{-\gamma t}$	$\gamma q + p e^{-\gamma t}$
0	q	p	$p / (\gamma q + p e^{-\gamma t})$
0	1	0	p
0	$e^{-\gamma t}$	$-\gamma$	$\gamma q + p e^{-\gamma t}$
q	$-\gamma q^2$	$-\gamma p (q + p e^{-\gamma t})$	$(\gamma q + p e^{-\gamma t}) / p$
$q e^{\gamma t}$	0	$-p^2$	$p / (\gamma q + p e^{-\gamma t})$

Table 1. Generators and invariants for the damped free particle with Hamiltonian $H = \frac{1}{2} p^2 \exp(-\gamma t)$.

CHAPTER 4. APPLICATION OF THE LIE METHOD TO THE
LIE ADMISSIBLE GENERALIZATION OF MECHANICS

4.1 Introduction

In this chapter we investigate the relevance of the previously outlined Lie method for finding point symmetries to the Lie admissible generalization of mechanics proposed by one of us in a number of papers²⁻⁵. Since within both this Lie admissible mechanics and Lie's theory of differential equations one is concerned with transformation groups which leave invariant a given system of equations, it appears that there is a strong motivation for carrying out such an investigation and a reasonable hope that a common ground will exist between the two approaches of mutual benefit to both.

We emphasise, however, right at the outset that the work we report here is still very much at the exploratory stage and instead of being in a position to formulate concretely established results we can at this point only outline the problems which appear relevant and indicate possible ways in which they may be resolved. Nevertheless since we believe strongly in the motivations, we feel confident that a difficulties can eventually be overcome.

The essence of the Lie admissible generalization resides in the extension of the canonical Poisson Brackets

$$(A, B) = \omega^{\mu\nu} \frac{\partial A}{\partial a^\mu} \frac{\partial B}{\partial a^\nu}, \quad a^\mu = (q, p), \quad \omega^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4.1)$$

to the generalised Lie admissible brackets

$$(A, B) = S^{\mu\nu} \frac{\partial A}{\partial a^\mu} \frac{\partial B}{\partial a^\nu}; \quad S^{\mu\nu} = \omega^{\mu\nu} + T^{\mu\nu}, \quad (4.2)$$

with the replacement $[A, B] \rightarrow (A, B)$ being carried out in the transition from a (variationally) self-adjoint(SA) to a non-self-adjoint (NSA) system of equations via the addition of NSA forces¹⁵. The tensor $T^{\mu\nu}$ which is some prescribed function of the additional forces and is required to vanish as the forces go to zero, essentially carries the generalization. This generalization in fact opens the way to a natural procedure for changing the symmetry group of the SA system into a corresponding one appropriate to the NSA counterpart. Thus, in the case of a SA system where a canonical formalism is necessarily available, one realizes the symmetry group (taken here for concreteness to be the Galilei Group) via the infinitesimal canonical transformations

$$a^\mu + \bar{a}^\mu = a^\mu + \delta\theta^i [a^\mu, G_i] \quad (4.3)$$

with the ten generating functions G_i being the appropriate ones associated with the Galilei group. In the transition to the NSA equation the conventional canonical formalism is lost, but nevertheless one is strongly motivated to write the analogue of the above symmetry group with the aid of the generalised bracket, as the Lie admissible group¹

$$a^\mu + \bar{a}^\mu = a^\mu + \delta\theta^i (a^\mu, G_i) \quad (4.4)$$

The finite group elements are obtained from this by exponentiation

$$\bar{a}^\mu = \left\{ e^{\delta\theta^i S^{\alpha\beta}} \frac{\partial G_i}{\partial a^\beta} \frac{\partial}{\partial a^\alpha} \right\} a^\mu \quad (4.5)$$

It has been found⁴ that in the case where consideration is restricted to the generator of time translation (i.e. all G_i are absent but one which is then identified with the (SA) Hamiltonian) the Lie-admissible group (4.5) can always be constructed and furthermore the resulting transformations $a^\mu \rightarrow \bar{a}^\mu$ do indeed

keep the NSA equations form invariant. Unfortunately a general proof of invariance under the complete Lie admissible group is not yet available.

4.2. Some open problems

It is at this stage that we feel a useful point of contact may result by a parallel investigation with the Lie method. As outlined in the previous chapters, this method concerns itself with the direct construction of point transformation groups of the form

$$\begin{aligned} t + \bar{t} &= t + \delta\theta^1 \xi_1(t, q) \\ q + \bar{q} &= q + \delta\theta^1 \eta_1(t, q) \end{aligned} \tag{4.6}$$

which keep a given second order equation form invariant and thereby map one of its solutions into another solution. Whether or not such a solution also satisfy the homogeneous Euler equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

is accidental and, indeed, the construction of symmetries is essentially the same for both SA and NSA equations of motion. Thus it appears that the point symmetries, extracted from a NSA equation by the direct Lie method, should have a considerable bearing on the Lie admissible group. Because this latter group is arrived at in a rather formal and indirect way, a simple straightforward comparison of the two approaches does not seem to be possible. It is precisely in this search for a 'common ground' that in the following we list via numbered paragraphs (not necessarily in any order of priority) what appear to us relevant points of discussion concerning such a

link-up.

(i) It is immediately apparent that in order to come closer to a comparison we must, in the case of the Lie methods, pass from the second order (in time) Lagrangian formulation of the SA problem to an equivalent first order Hamiltonian canonical formalism. Such an 'induced' Lie method of point transformation within the Hamiltonian theory has been discussed in the previous chapter where, in addition, the general procedure outlined for reducing any second order system to first order, establishes a well defined way of reducing also the NSA equations to pairs of first-order ones. Thus, if L is the Lagrangian of the SA problem, one defines for both the SA and NSA problems the co-ordinate $p = \frac{\partial L}{\partial \dot{q}}$. In the former case p will be the canonical momentum while in the latter case it will merely play the rôle of a fixed construct in terms of q, \dot{q} and t. The Lie method now applied directly in the co-ordinates t, q, p leads for both problems to infinitesimal transformations of the form

$$\begin{aligned} t + \bar{t} &= t + \delta\theta^1 \xi_1(t, q) \\ q + \bar{q} &= q + \delta\theta^1 \eta_1(t, q) \\ p + \bar{p} &= p + \delta\theta^1 \zeta_1(t, q, p) \end{aligned} \tag{4.7}$$

or

$$\begin{aligned} t + \bar{t} &= t + \delta\theta^1 \xi_1(t, q) \\ a^\mu + \bar{a}^\mu &= a^\mu + \delta\theta^1 \sigma_1^\mu(t, q, p), \quad \sigma_1^\mu = (\eta_1, \zeta_1) \end{aligned}$$

(ii) The appearance of co-ordinate dependent time transformations, which arise in general in the Lie theory and are indicated in (4.6) and (4.7), causes difficulties. Thus if Hamilton's equations are to

exist subsequent to a transformation of the type (4.7) (in which for the present we even generalise $\eta_1(t, q)$ to $\eta_1(t, q, p)$), that is if the given Hamiltonian equations

$$\frac{da^\mu}{dt} = [a^\mu, H(a, t)]_{(a)} \quad a^\mu = (q, p) \quad (4.8)$$

are to lead to the new ones

$$\frac{d\bar{a}^\mu}{dt} = [\bar{a}^\mu, \bar{H}(\bar{a}, \bar{t})]_{(\bar{a})} \quad (4.9)$$

then the integrability conditions for the existence of a new Hamiltonian function \bar{H} rule out a^μ -dependent time transformations i.e., one can have only $\xi_1 = \xi_1(t)$ (notice, however, that NSA systems are generally treated via the Lie-admissible generalization of Hamilton's equations³).

A complete resolution of the difficulty can be achieved in the case of the SA problem by restricting oneself to the so called 'Noether subgroup' of point symmetries. This subgroup is mathematically distinguished from the full point symmetry group of Lie by being that subset of point transformations which keep invariant the action functional $S[c] = \int_c L dt$ along arbitrary curves c and change the functional form of the Lagrangian $L(q, \dot{q}, t)$ at most by a total time derivative $\frac{d}{dt} F(q, t)$. It can be generally shown for systems with natural Lagrangians that the transformations of this sub-group have the property that the time part of the transformation does not involve q i.e. the ξ_1 of the Noether subgroup are at most functions of t . (An additional well known and well used property of this subgroup is that it immediately supplies closed form integral invariants equal in number to the number of its essential parameters when c is an extremal.)

Clearly it is exactly the Noether subgroup, or one of its subgroups in turn, which is identified with the space-time group such as the Galilei Group, when in the SA problem one stipulates Galilean invariance.

Now although by restricting oneself to the Noether subgroup, appropriately translated into the Hamiltonian language of the SA problem, one is left with meaningful transformations of the type (4.7) with $\xi_1 = \xi_1(t)$, nevertheless one cannot do the corresponding thing for the NSA counterpart, for the obvious reason that, in terms of the given co-ordinates, no Lagrangian and hence no Noether subgroup exist for it. Thus any attempt at identifying in general the Lie admissible group with some subgroup of the NSA Lie point group, with this latter subgroup arising in some natural way as an 'image' or 'shadow' of the Noether subgroup, appears to be fraught with considerable difficulty since it is not clear how one is to define such a 'shadow'.

Furthermore, the Lie admissible covering group and the group from which it arises by generalisation of the Poisson brackets, are both postulated to have the *same* number of parameters and this leads to the requirement that any 'shadow' subgroup, if it is to be identifiable with the Lie admissible covering groups, must have the same dimension as the 'object' group (= Noether group or its sub-group) of which it is a shadow. This requirement may perhaps turn out to be too severe if one allows only point groups to be candidates for the Lie admissible group, since it is conceivable that by introducing a 'bad enough' NSA force, one might reduce the complete NSA point transformation group to the trivial identity, in which case any concept of a 'shadow' evaporates.

(iii) The discussion in the previous paragraph seems to motivate an extension of the Lie methods to a wider class of transformation than just point transformations, if one is to have a useful confluence with the Lie admissible theory. Although this may not be an invariable conclusion, it is nevertheless prompted also by an examination of the generalised brackets of the form

$$(q, G_1) = S^{11} \frac{\partial G_1}{\partial q} + S^{12} \frac{\partial G_1}{\partial p} \quad (4.10)$$

a quantity which is eventually to be identified with the function η_1 arising in the Lie symmetry group of the NSA equation. If a point group is to fit the bill, then (q, G_1) cannot turn out to be a function involving p , whereas a priori there does not seem to be any strong reason why this should be the case in view of the fact that the generators G_1 and indeed the NSA forces themselves appearing in the Lie-admissible tensor $S^{\mu\nu} = \omega^{\mu\nu} + T^{\mu\nu}$, may be strongly p -dependent. Consequently, one may need to widen the point groups to dynamical groups by including functional or path-dependent transformations. This, however, carries with it the problems already mentioned in the previous sections, that present techniques do not provide a systematic way of arriving at such dynamical groups, in sharp contrast with the point transformations.

(iv) If one eventually succeeds in constructing a satisfactory 'shadow' symmetry group of the NSA equation, then one must be careful in the process of identification with the Lie admissible group, to recognise that the parameters θ^1 in the shadow group are merely mathematical parameters and do not yet have any physical connotations in contrast with the θ^1 appearing in the Lie admissible group. Thus

a problem will arise as to what linear combinations of the form

$$\eta_1' = L_{1j} \eta_j$$

$$G_1' = L_{1j} G_j$$

are to be associated respectively with the generalised brackets (q, G_1) and (p, G_1) . We expect the resolution of such ambiguities to be effected by purely algebraic methods.

(v) In the attempt to identify a suitable 'shadow' subgroup within the NSA symmetry group it may perhaps be useful to resort to the 'inverse problem' which involves constructing a Lagrangian also for the NSA equation.¹⁵ The co-ordinates in which this is possible will in general not be physical, but one may thereby establish a Noether type subgroup of symmetries in the NSA case which when mapped back to the original (physical) co-ordinates establishes some invariant notion of a 'shadow'.

It is interesting to try this for the simple case of a free particle in one dimension upon which one superimposes a NSA linear velocity damping force. Writing

$$q = x$$

to indicate an inertial (physical) co-ordinate, we have

$$SA: \quad \ddot{x} = 0 \quad \leftrightarrow \quad L_{SA} = \frac{1}{2} \dot{x}^2$$

$$NSA: \quad \ddot{x} + \gamma \dot{x} = 0 \quad \leftrightarrow \quad L_{NSA} = \frac{1}{2} \dot{x}^2 e^{\gamma t}$$

We define

$$p = \frac{\partial L_{SA}}{\partial \dot{x}} = \dot{x}$$

then (x, p) are the canonical co-ordinates of the SA problem and only they are to appear in both the SA & NSA problems.

The Lie method applied separately to the two equations leads to 8 infinitesimal operators for both as follows

<u>SA</u>	<u>NSA</u>
$X_1 = \frac{\partial}{\partial t}$	$X_1^1 = e^{\gamma t} \frac{\partial}{\partial t}$
$X_2 = t^2 \frac{\partial}{\partial t} + xt \frac{\partial}{\partial x}$	$X_2^1 = e^{-\gamma t} \frac{\partial}{\partial t} - \gamma x e^{-\gamma t} \frac{\partial}{\partial x}$
$X_3 = t \frac{\partial}{\partial t} + \frac{1}{2} x \frac{\partial}{\partial x}$	$X_3^1 = \frac{\partial}{\partial t} - \frac{1}{2} \gamma x \frac{\partial}{\partial x}$
$X_4 = t \frac{\partial}{\partial x}$	$X_4^1 = e^{-\gamma t} \frac{\partial}{\partial x}$
$X_5 = \frac{\partial}{\partial x}$	$X_5^1 = \frac{\partial}{\partial x}$
$X_6 = x \frac{\partial}{\partial t}$	$X_6^1 = x e^{\gamma t} \frac{\partial}{\partial t}$
$X_7 = tx \frac{\partial}{\partial t} + x^2 \frac{\partial}{\partial x}$	$X_7^1 = x \frac{\partial}{\partial t} - \gamma x^2 \frac{\partial}{\partial x}$
$X_8 = x \frac{\partial}{\partial x}$	$X_8^1 = x \frac{\partial}{\partial x}$

(4.11)

In both cases, the first five operators X_i $i=1,2,\dots,5$ generate the Noether subgroups of the corresponding Lagrangian, allowing us to identify the shadow subgroup with the NSA Noether sub group.

To attempt a further identification with the Lie admissible group, we need to calculate what infinitesimal change is induced

in the co-ordinate $p = \dot{x}$ by

- (a) the Noether sub group of the SA problem
- (b) the Noether sub group of the NSA problem .

These are readily calculated by using the appropriate first extensions $X_i^{(1)}$ $i = 1,2,\dots, 5$ for the SA & NSA problems. Thus we get the function $\zeta_i(t, x, p)$ for the Noether subgroups

SA	NSA
$\zeta_1 = 0$	$\zeta_1 = -\gamma p e^{\gamma t}$
$\zeta_2 = x - tp$	$\zeta_2 = \gamma^2 x e^{-\gamma t}$
$\zeta_3 = -\frac{1}{2} p$	$\zeta_3 = -\frac{1}{2} \gamma p$
$\zeta_4 = 1$	$\zeta_4 = -\gamma e^{-\gamma t}$
$\zeta_5 = 0$	$\zeta_5 = 0$

At this point a complication arises in so far as the 1-dimensional Galilei group $G(1, 1)$ has three and not five parameters (its infinitesimal operators are X_1, X_4, X_5 in the SA set). Thus the space-time symmetry group we wish to cover by a Lie admissible group is a proper subgroup of the SA Noether group and we must look for its 'shadow' in turn in the NSA Noether counterpart. Since this is somewhat problematic we shall not pursue the example further but instead draw the general conclusion that rather than covering only a given space-time symmetry group (such as the Galilei group) it may be profitable to look for generalizations which cover always the Noether symmetry group containing this space-time group as a proper or improper subgroup.

CHAPTER 5. CONCLUDING REMARKS

Our studies indicate that Lie's method must be subjected to a number of additional developments before it can be effectively applied to the construction of the Lie-admissible symmetries of Newtonian systems. The most visible and important one is the extension to arbitrary transformations in (t,q,p)-space, rather than point transformations in (t,q)-space. In fact, such an extension is needed both classically as well as quantum mechanically (see ref.³, Section 2.3).

Nevertheless, our studies indicate also that the approach is quite promising, particularly in regards to the problem of the relativity of Newtonian mechanics (no relativistic, field theoretical, and quantum mechanical extensions!). The case of the linearly damped particle worked out in the preceding chapters points out this possibility quite clearly. Recall that, in this case, we have the transition from the free SA particle in one dimension to a NSA system

$$(\ddot{x})_{SA} = 0 \longrightarrow [(\ddot{x})_{SA} + \gamma \dot{x}]_{NSA} = 0. \quad (5.1)$$

The relativity for the SA subsystem is the conventional Galilei relativity in one-space dimension. By ignoring scalar extensions, the relativity group G(1.1) is 3-dimensional and it is given by translations in time $T_1(t)$, translations in space $T_1(x)$, and Galilei boosts $G_1(x)$. The meaning of the relativity in which we are interested here is that the group G(1.1) constitutes a symmetry group of the equation of motion in such a way to represent the conservation laws of the total energy, linear momentum, and uniform motion of the center of mass, that is, the first integrals

$$H = \frac{1}{2} p^2, \quad p = \dot{x}, \quad G = x \quad (5.2)$$

The abstract Lie group G(1.1) is unique. Nevertheless, it can be realized in different ways when applied specifically to the system considered. A most significant difference which has been indicated earlier is that the group G(1.1) can be realized either for configuration space treatments along generators of type (4.11), or for phase space treatments, in which case the generators are those of eqs. (5.2) (see, for instance, reference¹⁶, p. 390). As indicated in the introductory remarks, we prefer the phase space treatment because it is more suitable for the direct application of analytic/algebraic/geometrical techniques. With the symbol G(1.1) we shall therefore refer to the canonical realization of the Galilei group in one-space dimension with generators (5.2) and with unit mass.

When the free particle enters a dissipative medium and experiences a linearly-dependent velocity-drag force, all conservation laws (5.2) are lost. The covering Lie-admissible relativity proposed by Santilli^{2,3} is intended to achieve

- (A) the form-invariance of the extended, NSA, system; in such a way that
- (B) the behaviour of the symmetry in the neighborhood of the identity characterizes, this time, the time rate of variation of the physical quantities (5.2); while
- (C) the conventional Galilei relativity is recovered identically at all levels at the limit of null NSA force.

As familiar (see, for a review, ref.³, Section 1.2), the use of Lie's groups prohibits the achievement of all three conditions A, B and C. In fact, condition B calls for the realization of a symmetry in such a way to admit a non-Lie algebra in the neighborhood of the identity because the anticommutativity of the Lie product allows only the

representation of the particular case $\dot{H} = [H, H] = 0$, and prohibits the representation of the broader physical law $\dot{H} = f(t) \neq 0$.

The use of a Lie-admissible group does indeed allow the achievement of the three conditions A, B, and C. We are therefore interested in a Lie-admissible covering $\hat{G}(1.1)$ of the Lie group $G(1.1)$, and, more specifically, in Lie-admissible coverings $\hat{T}_1(t)$, $\hat{T}_1(x)$, and $\hat{G}_1(x)$ of the individual components $T_1(t)$, $T_1(x)$, and $G_1(x)$ of $G(1.1)$ which are capable of providing the algebraic representation of the time rate of variations of H , p and G

$$\dot{X}_i = (X_i, H) \neq 0; \quad i = 1, 2, 3; \quad X_1 = H, \quad X_2 = p, \quad X_3 = G \quad (5.3)$$

in such a way that the conventional Galilei setting is recovered identically all all levels (including the conservation laws) at the limit of null value of the NSA force.

The first condition to achieve these goals is to realize $\hat{G}(1.1)$ in terms of the generators of $G(1.1)$. This is clearly essential on physical grounds to preserve the direct physical significance of the generators. In fact, the force which breaks the $G(1.1)$ symmetry is not derivable from a potential. Therefore, the physical quantities of the original SA systems and of its NSA extension coincide. For more details, see the locally quoted references.

The second condition for the desired objectives is to exponentiate the time evolution law into a Lie-admissible group. For the NSA system considered here this task has been worked out in details in ref. 4, p.1304 and foll., resulting in the first, desired, Lie-admissible covering

$$\hat{T}_1(t) : (a') = \begin{pmatrix} x' \\ p' \end{pmatrix} = e^{t S^{\mu\nu} \frac{\partial H}{\partial a^\nu} \frac{\partial}{\partial a^\mu}} \begin{pmatrix} x \\ p \end{pmatrix}$$

$$= \begin{pmatrix} x + \frac{t}{\Gamma}(x, H) + \dots \\ p + \frac{t}{\Gamma}(p, H) + \dots \end{pmatrix} = \begin{pmatrix} x - p(e^{-\gamma t} - 1)/\gamma \\ p \exp(-\gamma t) \end{pmatrix} \quad (5.4)$$

$$(S) = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}$$

$$(A, B) = \frac{\partial A}{\partial a^\mu} S^{\mu\nu} \frac{\partial B}{\partial a^\nu}$$

It is an instructive exercise for the interested reader to verify that the Lie-admissible group (5.4) verifies the three conditions under consideration, that is, (1) it leaves the equations of motion form invariant, i.e.,

$$\begin{pmatrix} \dot{x} - p \\ \dot{p} + \gamma p \end{pmatrix} = 0 \rightarrow \begin{pmatrix} \dot{x}' - p' \\ \dot{p}' + \gamma p' \end{pmatrix} = 0 \quad (5.5)$$

(2) it represents the time rate of variations (5.3) via rule (1.4); and (3) it recovers the conventional Galilei group $T_1(t)$ identically at the limit $F_{NSA} = -\gamma p = 0$, i.e., at the limit $\gamma = 0$. As a final remark, notice that the group $\hat{T}_1(t)$ can be constructed via the use of the Lie-admissible techniques, not only for the system considered here, but for all autonomous systems, without any need of Lie's method.

When passing to the construction of the Lie-admissible coverings $\hat{T}_1(x)$ and $\hat{G}_1(x)$ the situation is different. It is at this point that

Lie's method becomes particularly significant. In fact, one may, in principle:

- (a) construct the symmetries $\hat{T}_1(x)$ and $\hat{G}_1(x)$ as ordinary Lie symmetries, via the methods considered in the preceding chapters, in which case the generators do not have a direct physical meaning; and then
- (b) reformulate the symmetries in terms of the generators of $T_1(x)$ and $G_1(x)$, which yields automatically Lie-admissible groups, by making sure that the groups $\hat{T}_1(x)$ and $\hat{G}_1(x)$ recover the original groups $T_1(x)$ and $G_1(x)$ identically at the limit $F_{NSA} = 0$.

The inspection of Table 1 of Chapter 3 and of eqs. (4.11) indicates that these objectives have realistic possibilities of actuation for the equation of motion considered, as well as of potential extension to equations of motion in more than one dimension and with more complex forces. These problems will be considered in a separate paper.

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The complete symmetry group of the one-dimensional time-dependent harmonic oscillator

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Journal of Mathematical Physics **21** (1980) 300-304

The complete symmetry group of the one-dimensional time-dependent harmonic oscillator

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(Received 12 January 1979)

The five invariants for the time-dependent one-dimensional harmonic oscillator Hamiltonian are constructed. Using the linear transformation to the time-dependent oscillator Hamiltonian, the five invariants for the latter are obtained. The differential operators which generate the dynamical symmetry of this Hamiltonian have the same commutator relations as these of the time-independent problem. An additional three operators are obtained using the method of extended Lie groups and have the same properties as those for the time-independent problem. Thus the complete dynamical symmetry of the time-dependent problem is the eight-parameter Lie group $SL(3, R)$.

1. INTRODUCTION

In recent years there has been considerable interest in harmonic oscillator systems, both time independent and time dependent. This interest has expressed itself in several ways, especially in the construction of invariants and the determination of symmetry groups for such systems. Different approaches are found in the literature. To some extent these differences follow from different concepts of what is the basic dynamical expression for the oscillator system—Newton's equation(s) of motion, the Lagrangian, or the Hamiltonian. There are differences in the results obtained. To be more precise consider the one-dimensional time-independent harmonic oscillator with equation of motion, Lagrangian, and Hamiltonian, respectively,

$$\ddot{q} + q = 0, \quad (1.1)$$

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2, \quad (1.2)$$

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 \quad (1.3)$$

(there is no essential loss of generality in taking the customary ω^2 as unity). The application of the Lie theory of extended groups to Eq. (1.1)¹ showed that the complete symmetry group was the eight-parameter Lie group $SL(3, R)$. Applying Noether's theorem to Eqs. (1.2), Lutzky² obtained a five-parameter subgroup of $SL(3, R)$ corresponding to two linear and three quadratic constants of the motion. To obtain the additional three-parameter subgroup, Lutzky used the Langrange's equation of motion. The members of this subgroup do not preserve the invariance of the action integral as does a Noether-derived operator, but they do preserve the invariance of the equation of motion since solutions are transformed into solutions. There does not appear to have been a similar treatment for the Hamiltonian (1.3) although for this problem the extension of the results for the Lagrangian (1.2) is particularly obvious.

In considering the time-dependent one-dimensional harmonic oscillator (we restrict ourselves to one dimension to keep the discussion as simple as possible; the extension to higher dimensions is more a matter of algebraic rather than conceptual difficulty), again there have been different approaches. Defining the problem by

$$\ddot{q} + \omega^2(t)q = 0, \quad (1.4)$$

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2(t)q^2, \quad (1.5)$$

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)q^2, \quad (1.6)$$

Lewis,³ applying Kruskal's method⁴ in closed form, constructed an exact invariant which is equally valid for the Lagrangian (1.5) or the Hamiltonian (1.6). This was

$$I_L = \frac{1}{2}\{q^2/\rho^2 - (\rho\dot{q} - \dot{\rho}q)^2\}, \quad (1.7)$$

$$I_H = \frac{1}{2}\{q^2/\rho^2 - (\rho p - \dot{\rho}q)^2\}, \quad (1.8)$$

where the suffixes L and H refer to Lagrangian and Hamiltonian formulation, respectively, and $\rho(t)$ is any solution of

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}. \quad (1.9)$$

Leach⁵ used the method of time-dependent linear canonical transformations to obtain a form similar to Eq. (1.8) by transforming Eq. (1.6) to (1.3). Lutzky⁶ applied Noether's theorem to the Lagrangian (1.5) to obtain Eq. (1.7).

None of the writers mentioned above has provided a discussion of the symmetry group and invariants of the one-dimensional time-dependent harmonic oscillator. In this note we provide such discussion in the context of the Hamiltonian formalism. We start with a simple method for obtaining the five invariants for the Hamiltonian (1.3). Using the linear canonical transformation between Eqs. (1.3) and (1.6) we construct the five invariants of Eq. (1.6) from those of Eq. (1.3). The five corresponding group generators are given. The remaining three operators which leave Newton's equation of motion invariant are also given. It is demonstrated that the operators have the same commutator properties as those for the time-independent problem, hence showing that the time-dependent oscillator also possesses the dynamical symmetry of $SL(3, R)$.

2. THE INVARIANTS OF $H(1.3)$

All manner of polynomial invariants for $H(1.3)$ may be constructed by postulating a general form for the invariant with undertermined coefficients and then using the requirement that, if I is an invariant of H ,

$$\frac{dI}{dt} = [I, H]_{PB} + \frac{\partial I}{\partial t} = 0. \quad (2.1)$$

Writing $H(1.3)$ as

$$H = \frac{1}{2} \mathbf{z}^T \mathbf{z}, \quad (2.2)$$

where

$$\mathbf{z} = \begin{bmatrix} q \\ p \end{bmatrix}, \quad (2.3)$$

a linear invariant, denoted by I_1 , has the form

$$I_1 = \mathbf{r}^T \mathbf{z}, \quad (2.4)$$

in which \mathbf{r} is a coordinate free 2-vector.

Since the Poisson bracket of two scalars F and G is given by

$$[F, G]_{\text{PB}} = \left(\frac{\partial F}{\partial \mathbf{z}} \right)^T J \left(\frac{\partial G}{\partial \mathbf{z}} \right), \quad (2.5)$$

where J is the 2×2 symplectic matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (2.6)$$

Eq. (2.1) is simply

$$\mathbf{r}^T \mathbf{z} + \dot{\mathbf{r}}^T J \mathbf{z} = 0. \quad (2.7)$$

Making use of the linear independence of the elements of \mathbf{z} , Eq. (2.7) reduces to

$$\dot{r}_1 = r_2, \quad \dot{r}_2 = -r_1, \quad (2.8)$$

where r_1 and r_2 are the elements of \mathbf{r} . This Hamiltonian system of first order linear differential equations has a singular point at

$$r_1 = 0, \quad r_2 = 0. \quad (2.9)$$

This particular solution is trivial in the present context. The general solution of Eq. (2.8) is

$$r_1 = a \cos t + b \sin t, \quad (2.10a)$$

$$r_2 = -a \sin t + b \cos t, \quad (2.10b)$$

in which a and b are arbitrary constants. The general linear invariant of $H(1.3)$ is

$$I_1 = (a, b) \mathbf{z} \cos t + (b, -a) \mathbf{z} \sin t. \quad (2.11)$$

If we write the row vector (a, b) as \mathbf{c}^T , the invariant is

$$I_1 = \mathbf{c}^T (I \cos t - J \sin t) \mathbf{z}, \quad (2.12)$$

e., to within a factor of the magnitude of \mathbf{c} , I_1 is the magnitude of the projection of the vector

$$\mathbf{I}_1 = (I \cos t - J \sin t) \mathbf{z} \quad (2.13)$$

in the direction of some arbitrary constant vector \mathbf{c} . Clearly the basic invariant is I_1 , whose two linearly independent elements are

$$I_{11} = q \cos t - p \sin t, \quad (2.14a)$$

$$I_{12} = q \sin t + p \cos t. \quad (2.14b)$$

We observe that the invariant vector \mathbf{I}_1 is simply the position vector of the point on the phase plane occupied by a particle at time $t = 0$. The time development of \mathbf{z} is given

$$\mathbf{z} = (I \cos t + J \sin t) \mathbf{I}_1 \quad (2.15a)$$

$$= (I \cos t + J \sin t) \mathbf{z}(0) \quad (2.15b)$$

which describes a circle of radius $z(0)$ in the phase plane. We

note that

$$\frac{\partial I_{12}}{\partial t} = I_{11}, \quad (2.16a)$$

$$\frac{\partial I_{11}}{\partial t} = -I_{12}. \quad (2.16b)$$

This indicates that there are only two symmetry mappings associated with the first order invariants, a result which follows from the discussion given by Katzin and Levine.⁷

We could proceed to construct the quadratic invariants of $H(1.3)$ by postulating the form

$$I_2 = \frac{1}{2} \mathbf{z}^T M \mathbf{z}, \quad (2.17)$$

in which M is a 2×2 coordinate free symmetric matrix, and solving the equation corresponding to Eq. (2.17), viz.,

$$\mathbf{z}^T (JM + \frac{1}{2} \dot{M}) \mathbf{z} = 0. \quad (2.18)$$

For the one-dimensional system being considered here it eventuates that there are only three types of quadratic invariant which are given by the following products of the linear invariants:

$$2I_{21} = I_{11}^2 + I_{12}^2 = p^2 + q^2, \quad (2.19a)$$

$$2I_{22} = 2I_{11}I_{12} = (q^2 - p^2) \sin 2t + 2qp \cos 2t, \quad (2.19b)$$

$$2I_{23} = I_{11}^2 - I_{12}^2 = (q^2 - p^2) \cos 2t - 2qp \sin 2t. \quad (2.19c)$$

The result is not surprising. Any quadratic form in two variables is a linear combination of the three linearly independent expressions given above and no more than three linearly independent quadratic expression can be formed from two linearly independent variables.

The five invariants which have been obtained here correspond to the five invariants derived by Lutzky in Ref. 2 where he used Noether's theorem to obtain the five Lie group operators and hence the invariants. The slight differences are due to the present writer wishing to keep the form of the invariants related to their physical interpretations. Thus, I_{11} and I_{12} are the initial positions (at time $t = 0$) in the phase while I_{21} is the Hamiltonian which in this problem is the conserved energy.

We point out that the method outlined here generalizes easily to multidimensional oscillators and that suitable operators may be obtained for the corresponding quantum mechanical problem.⁸ In the latter problem the products are symmetrized, a process which follows naturally if a matrix formulation is used. For the quantum mechanical problem it is convenient to define new operators. They are

$$A^\pm = 2^{-1/2} \{ I_{11} \pm i I_{12} \}, \quad (2.20a)$$

$$B^\pm = I_{23} \pm i I_{22}, \quad (2.20b)$$

$$C = I_{21}. \quad (2.20c)$$

The A 's are the time-dependent creation and annihilation operators. We note that in higher dimensional problems there is an additional class of invariants whose elements constitute the angular momentum tensor.

3. TRANSFORMATION TO THE TIME-DEPENDENT OSCILLATOR

A canonical transformation of the Hamiltonian

$$H = \frac{1}{2} \mathbf{z}^T A \mathbf{z} \quad (3.1)$$

to the Hamiltonian

$$\bar{H} = \frac{1}{2} \bar{\mathbf{z}}^T \bar{A} \bar{\mathbf{z}} \quad (3.2)$$

may be accomplished by the linear transformation

$$\bar{\mathbf{z}} = S \mathbf{z}, \quad (3.3)$$

where (cf. Ref. 5)

$$\dot{S} = J \bar{A} \dot{S} - S J A, \quad (3.4a)$$

$$S J S^T = J. \quad (3.4b)$$

The equation (3.4a) arises from the requirement that the description of the time development of the system be equivalent in either the \mathbf{z} or $\bar{\mathbf{z}}$ coordinate system, i.e.,

$$\dot{\bar{\mathbf{z}}} = J \frac{\partial \bar{H}}{\partial \bar{\mathbf{z}}} = J \bar{A} S \mathbf{z} \quad (3.5a)$$

and

$$\dot{\mathbf{z}} = \frac{d(S\mathbf{z})}{dt} = \dot{S}\mathbf{z} + S J A \mathbf{z}. \quad (3.5b)$$

The number of arbitrary constants in the general solution of Eq. (3.4a) is reduced by the requirement (3.4b), which is the condition that the transformation be canonical.

For the particular case of the transformation from $H(1.3)$ to $H(1.6)$ we take $H(1.3)$ as \bar{H} . The transformation has been shown to be³

$$\mathbf{z} \rightarrow \bar{\mathbf{z}}: \mathbf{z} = R S \bar{\mathbf{z}}, \quad (3.6)$$

where

$$R = \begin{bmatrix} C_1 \cos W_1 + C_2 \cos W_2, & -C_1 \sin W_1 - C_2 \sin W_2 \\ C_1 \sin W_1 - C_2 \sin W_2, & C_1 \cos W_1 - C_2 \cos W_2 \end{bmatrix}, \quad (3.7a)$$

$$S = \begin{bmatrix} \rho^{-1} & 0 \\ -\dot{\rho} & \rho \end{bmatrix}, \quad (3.7b)$$

and

$$W_1 = \int_{t_0}^t (\rho^{-2} - 1) dt', \quad W_2 = \int_{t_0}^t (\rho^{-2} + 1) dt', \quad (3.8a)$$

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}, \quad (3.8b)$$

$$C_1^2 - C_2^2 = 1. \quad (3.8c)$$

4. INVARIANTS FOR THE TIME-DEPENDENT OSCILLATOR HAMILTONIAN

As invariance is independent of the coordinate representation, there are five invariants for $H(1.6)$ which are obtained by expressing the invariant derived in section two in terms of the new coordinates. Using the transformation (3.6) with the specific expressions for R and S given by Eqs. (3.7a) and (3.7b), respectively, we have

$$I_{11} = (\cosh C + \sinh C) \{ \rho^{-1} q \cos W - (\rho p - \dot{\rho} q) \sin W \}, \quad (4.1a)$$

$$I_{12} = (\cosh C - \sinh C) \{ \rho^{-1} q \sin W + (\rho p - \dot{\rho} q) \cos W \}, \quad (4.1b)$$

$$2I_{21} = \cosh 2C \{ \rho^{-2} q^2 + (\rho p - \dot{\rho} q)^2 \} + \sinh 2C \{ \rho^{-2} q^2 - (\rho p - \dot{\rho} q)^2 \cos 2W \},$$

$$- 2\rho^{-1} q (\rho p - \dot{\rho} q) \sin 2W \}, \quad (4.1c)$$

$$2I_{22} = \rho^{-2} q^2 \sin 2W - (\rho p - \dot{\rho} q)^2 \times \sin 2W + 2\rho^{-1} q (\rho p - \dot{\rho} q) \cos 2W, \quad (4.1d)$$

$$2I_{23} = \cosh 2C \{ \rho^{-2} q^2 \cos 2W - (\rho p - \dot{\rho} q)^2 \cos 2W - 2\rho^{-1} q (\rho p - \dot{\rho} q) \sin 2C \} + \sinh 2C \{ \rho^{-2} q^2 + (\rho p - \dot{\rho} q)^2 \}, \quad (4.1e)$$

in which we have written

$$\cosh C = C_1, \quad \sinh C = C_2, \quad (4.2a)$$

$$W = \int_{t_0}^t \rho^{-2} dt'. \quad (4.2b)$$

If C_2 is set at zero, I_{21} is the form of the invariant reported by Lewis.³ The general form given above was reported by Leach.^{5b}

We may define a set of invariants independent of C by writing

$$J_{11} = (\cosh C - \sinh C) I_{11} = \rho^{-1} q \cos W - (\rho p - \dot{\rho} q) \sin W, \quad (4.3a)$$

$$J_{12} = (\cosh C + \sinh C) I_{12} = \rho^{-1} q \sin W + (\rho p - \dot{\rho} q) \cos W, \quad (4.3b)$$

$$2J_{21} = 2(I_{21} \cosh 2C - I_{23} \sinh 2C) = \rho^{-2} q^2 (\rho p - \dot{\rho} q)^2, \quad (4.3c)$$

$$2J_{22} = 2I_{22} = \{ \rho^{-2} q^2 - (\rho p - \dot{\rho} q)^2 \} \sin 2W + 2\rho^{-1} q (\rho p - \dot{\rho} q) \cos 2W, \quad (4.3d)$$

$$2J_{23} = 2(-I_{21} \sinh 2C + I_{23} \cosh 2C) = \{ \rho^{-2} q^2 - (\rho p - \dot{\rho} q)^2 \} \cos 2W - 2\rho^{-1} q (\rho p - \dot{\rho} q) \sin 2W. \quad (4.3e)$$

We note that the Lewis invariant now occurs without a parameter in Eq. (4.3c).

The physical interpretation of some of these invariants is facilitated if we make use of the intermediate Hamiltonian

$$H' = \frac{1}{2} \rho^{-2} (p'^2 + q'^2), \quad (4.4)$$

which is related to $H(1.6)$ by the canonical transformation

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} \rho^{-1} & 0 \\ -\dot{\rho} & \rho \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}. \quad (4.5)$$

Then

$$J_{11} = q' \cos W - p' \sin W, \quad (4.6a)$$

$$J_{12} = q' \sin W + p' \cos W, \quad (4.6b)$$

$$J_{21} = \frac{1}{2} (q'^2 + p'^2). \quad (4.6c)$$

In the (q', p') phase plane, taking $t_0 = 0$,

$$J_{11} = q'(0), \quad J_{12} = p'(0). \quad (4.7)$$

The motion of the particle along the phase space trajectory is given by

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} \cos W & \sin W \\ -\sin W & \cos W \end{bmatrix} \mathbf{J}_1, \quad (4.8)$$

where \mathbf{J}_1 is defined similarly to \mathbf{I}_1 . This is a circular motion and J_{21} simply represents the constancy of the radius, being half the square of the radius.

Reverting to Eq. (4.3) we may obtain the formal solution for q by eliminating $\rho\dot{p} - \dot{\rho}q$ from (4.3a) and (4.3b). This is

$$q = \rho(J_{11} \cos W + J_{12} \sin W), \quad (4.9)$$

which is of the same form as that given in Ref. 6, Eq. (10).

Finally, we note that only two of the five invariants listed are functionally independent. It seems to us to be natural to select J_{11} and J_{12} as the independent quantities. Then

$$2J_{21} = (J_{11})^2 + (J_{12})^2, \quad (4.9a)$$

$$2J_{23} = (J_{11})^2 - (J_{12})^2, \quad (4.9b)$$

$$2J_{22} = 2J_{11}J_{12}. \quad (4.9c)$$

These relations are the same as those which are found in the time-independent case (cf. Ref. 8).

5. OPERATORS OF THE FIVE-PARAMETER SUBGROUP

Now that we have the five invariants for the Hamiltonian of the time-dependent harmonic oscillator it is a simple task to obtain the corresponding differential operators which are the generators of transformations. To facilitate comparison with the time-independent harmonic oscillator as discussed by Lutzky,² we adopt the nomenclature used in that paper. To summarize this, a generator G is given by

$$G(q,t) = \xi(q,t) \frac{\partial}{\partial t} + \eta(q,t) \frac{\partial}{\partial q} \quad (5.1)$$

and the corresponding invariant, in the Lagrangian formulation, is

$$\Phi(q,\dot{q},t) = (\xi\dot{q} - \eta) \frac{\partial L}{\partial \dot{q}} - \xi L + f(q,t). \quad (5.2)$$

In this instance

$$L(q,\dot{q},t) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2(t)q^2. \quad (5.3)$$

We may determine ξ , η and f from the invariants given by

Eqs. (4.3a)–(4.3c) and we have, replacing p by \dot{q} ,

$$\Phi_1(q,\dot{q},t) = -2J_{22}(q,\dot{q},t), \quad (5.4a)$$

$$\Phi_2(q,\dot{q},t) = -2J_{23}(q,\dot{q},t), \quad (5.4b)$$

$$\Phi_3(q,\dot{q},t) = -2J_{12}(q,\dot{q},t), \quad (5.4c)$$

$$\Phi_4(q,\dot{q},t) = 2J_{11}(q,\dot{q},t), \quad (5.4d)$$

$$\Phi_5(q,\dot{q},t) = 2J_{21}(q,\dot{q},t). \quad (5.4e)$$

The corresponding operators are

$$G_1 = \sin 2W \frac{\partial}{\partial W} + (\rho\dot{\rho} \sin 2W + \cos 2W)q \frac{\partial}{\partial q}, \quad (5.5a)$$

$$G_2 = \cos 2W \frac{\partial}{\partial W} + (\rho\dot{\rho} \cos 2W - \sin 2W)q \frac{\partial}{\partial q}, \quad (5.5b)$$

$$G_3 = \rho \cos W \frac{\partial}{\partial q}, \quad (5.5c)$$

$$G_4 = \rho \sin W \frac{\partial}{\partial q}, \quad (5.5d)$$

$$G_5 = \frac{\partial}{\partial W} + \rho\dot{\rho}q \frac{\partial}{\partial q}. \quad (5.5e)$$

We note that if ω is constant and equal to one, $\rho^{-2} = \omega = 1$, $W = t$, and the expressions for the generators are the same as those given by Lutzky Ref. 2, Eqs. (24a)–(24e) for the time-independent problem. These operators generate a five-parameter Lie group and have the following commutative relations:

$$[G_1, G_2] = -2G_5, \quad [G_5, G_1] = 2G_2, \quad [G_2, G_5] = 2G_1, \quad (5.6a)$$

$$[G_3, G_4] = 0, \quad (5.6b)$$

$$[G_3, G_1] = [G_2, G_4] = [G_5, G_4] = G_3, \quad (5.6c)$$

$$[G_1, G_4] = [G_3, G_5] = [G_2, G_3] = G_4, \quad (5.6d)$$

which are exactly the same as those for the time-independent problem.

We conclude this section by listing the expressions for ξ , η , and f . The order corresponds to the one to five ordering of the Φ 's and G 's:

$\xi(q,t)$	$\eta(q,t)$	$f(q,t)$
$\rho^2 \sin 2W$	$\frac{\partial}{\partial q} \{ \frac{1}{2}q^2(\rho\dot{\rho} \sin 2W + \cos 2W) \}$	$\frac{\partial}{\partial t} \{ \frac{1}{2}q^2(\rho\dot{\rho} \sin 2W + \cos 2W) \}$
$\rho^2 \cos 2W$	$\frac{\partial}{\partial q} \{ \frac{1}{2}q^2(\rho\dot{\rho} \cos 2W - \sin 2W) \}$	$\frac{\partial}{\partial t} \{ \frac{1}{2}q^2(\rho\dot{\rho} \cos 2W - \sin 2W) \}$
0	$\frac{\partial}{\partial q} \{ q\rho \cos W \}$	$\frac{\partial}{\partial t} \{ q\rho \cos W \}$
0	$\frac{\partial}{\partial q} \{ q\rho \sin W \}$	$\frac{\partial}{\partial t} \{ q\rho \sin W \}$
ρ^2	$\frac{\partial}{\partial q} \{ \frac{1}{2}q^2 \rho\dot{\rho} \}$	$\frac{\partial}{\partial t} \{ \frac{1}{2}q^2 \rho\dot{\rho} \}$

It should be noted that the ξ 's listed in Eqs. (5.7) are as coefficients of $\partial/\partial t$. In Eq. (5.6), $\partial/\partial W$ is used, the two being related by

$$\frac{\partial}{\partial W} = \rho^2 \frac{\partial}{\partial t}. \quad (5.8)$$

The appropriate variable to use in the operators in W and not t because W is now the effective time variable. This is most readily seen from the Hamiltonian H' introduced in Eq. (4.4). Under the time scale transformation from t to W , Hamilton's equations for H' are

$$\frac{dq'}{dW} = p', \quad \frac{dp'}{dW} = -q'. \quad (5.9)$$

6. OPERATORS OF THE THREE-PARAMETER SUBGROUP

The five invariants obtained in Sec. 4 relate to the Hamiltonian. The complete dynamical symmetry of the time-dependent oscillator is made up not only of the five corresponding operators, but also any operators which leave the Newtonian equation of motion invariant. These operators, as well as those given in Sec. 5, may be obtained by using the Lie theory of extended groups. This method is quite adequately described in Wulfman and Wybourne (1.b) and it is not proposed to repeat their working. We shall simply summarize the relevant results.

Suppose that the generator G defined by

$$G = \xi(q,t) \frac{\partial}{\partial t} + \eta(q,t) \frac{\partial}{\partial q} \quad (6.1)$$

is a member of the complete dynamical symmetry group. Its second extension is

$$\begin{aligned} G'' = & \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \{ \eta_t + (\eta_q - \xi_t) \dot{q} - \xi_q \dot{q}^2 \xi \} \frac{\partial}{\partial \dot{q}} \\ & + \{ \eta_{tt} + (2\eta_{qt} - \xi_{tt}) \dot{q} \\ & + (\eta_{qq} - 2\xi_{qt}) \dot{q}^2 - \xi_{qq} \dot{q}^3 \\ & + (\eta_q - 2\xi_t - 3\dot{q}\xi_q) \dot{q} \} \frac{\partial}{\partial \ddot{q}}. \end{aligned} \quad (6.2)$$

If the Newtonian equation of motion is

$$N(\ddot{q}, \dot{q}, q, t) = 0, \quad (6.3)$$

$$G''N = 0 \quad (6.4)$$

since G is a generator of the symmetry group. The requirement that Eq. (6.4) be true whenever Eq. (6.3) is true leads to a set of partial differential equations for ξ and η .

In the case of the time-dependent harmonic oscillator, Newton's equation is

$$\ddot{q} + \omega^2(t)q = 0. \quad (6.5)$$

When G'' acts on this equation and the resulting differential equations are solved, in addition to the five operators already given, we obtain

$$G_6 = q \frac{\partial}{\partial q}, \quad (6.6a)$$

$$G_7 = \rho^{-1} q \sin W \frac{\partial}{\partial W} + (\rho \sin W + \rho^{-1} \cos W) q^2 \frac{\partial}{\partial q}, \quad (6.6b)$$

$$G_8 = \rho^{-1} q \cos W \frac{\partial}{\partial W} + (\rho \cos W - \rho^{-1} \sin W) q^2 \frac{\partial}{\partial q}. \quad (6.6c)$$

These three operators form a subgroup with the commutation relations

$$[G_6, G_7] = G_7, \quad [G_6, G_8] = G_8, \quad [G_7, G_8] = 0, \quad (6.7)$$

which are the same as for the time-independent problem.

These three operators also have the same commutator relations with the other five operators as in the time-independent case. They are

$$[G_6, G_1] = [G_6, G_2] = [G_6, G_5] = 0, \quad (6.8a)$$

$$[G_6, G_3] = -G_3, \quad [G_6, G_4] = -G_4, \quad (6.8b)$$

$$[G_7, G_1] = -G_7, \quad [G_7, G_2] = -G_8, \quad (6.8c)$$

$$[G_8, G_1] = G_8, \quad [G_8, G_2] = -G_7, \quad (6.8d)$$

$$[G_7, G_3] = -\frac{1}{2}(G_1 + 3G_6), \quad (6.8e)$$

$$[G_8, G_4] = \frac{1}{2}(-G_1 + 3G_6),$$

$$[G_7, G_4] = \frac{1}{2}(G_2 - G_5),$$

$$[G_8, G_3] = -\frac{1}{2}(G_2 + G_5), \quad (6.8f)$$

$$[G_7, G_5] = -G_8 \quad [G_8, G_5] = G_7. \quad (6.8g)$$

Thus we have the result that the complete dynamical symmetry of the time-dependent one-dimensional harmonic oscillator is $SL(3, R)$.

ACKNOWLEDGMENTS

We thank C.J. Eliezer and G. Prince for useful discussions.

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The complete symmetry group of a forced harmonic oscillator

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Journal of the Australian Mathematical Society **22** (Series B) (1980) 12–21

THE COMPLETE SYMMETRY GROUP OF A FORCED HARMONIC OSCILLATOR

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(Received 14 June 1979)

(Revised 10 July 1979)

Abstract

The complete symmetry group of a forced harmonic oscillator is shown to be $S(3, R)$ in the one-dimensional case. Approaching the problem through the Hamiltonian invariants and the method of extended Lie groups, the method used is that of time-dependent point transformations. The result applies equally well to the forced repulsive oscillator and a particle moving under the influence of a coordinate-free force. The generalization to n -dimensional systems is discussed.

1. Introduction

Over the last few years there has been a resurgence of interest in the symmetries of dynamical systems. The original impetus for the construction of dynamical symmetry groups came from the description of unexpected degeneracies in spectra. For the basic integrable classical systems, the Kepler problem and the harmonic oscillator, the symmetry groups are respectively $SO(4)$ and $SU(3)$ (for three dimensions). Similarities between time-independent and time-dependent oscillator systems led to the study of the invariants, and consequently the symmetries, of time-dependent quadratic Hamiltonians. It was seen that time-dependent and time-independent quadratic Hamiltonians were related by linear canonical transformations [5]. Previously known invariants [4] for the time-independent oscillator were shown to have their counterparts for time-dependent oscillators [3, 6]. In particular, the Lewis invariant [12] was given a simple explanation. The existence of these invariants enabled the construction of symmetry groups for quadratic Hamiltonians [7]. As the invariants usually had non-zero Poisson brackets with the Hamiltonian, they were called non-invariance symmetry groups.

Further work [9] on quadratic Hamiltonians showed that there are more linear

and quadratic invariants than is necessary to form the operators of the special unitary group. The significance of these invariants became obvious in light of the work of Anderson and Davison [1], Wulfman and Wybourne [17], and Lutzky [13]. Anderson and Davison showed that the complete symmetry group for the one-dimensional time-independent harmonic oscillator and the free particle was $Sl(3, R)$. The term complete is used to indicate that the group is the largest admitted by the problem. Wulfman and Wybourne used the method of extended Lie groups to deduce the same result for the oscillator. Lutzky used Noether's theorem and a modification of the theory of extended Lie groups to repeat the result.

The extension of these results to other systems has been undertaken recently. Leach has shown that $Sl(3, R)$ applies to the one-dimensional time-dependent harmonic oscillator [10] and to the repulsive oscillator [11]. Of the eight generators required, five came from the two linear and three quadratic invariants of the Hamiltonian and the remaining three from the use of the method of extended Lie groups. Prince and Eliezer [15] extended the work on the time-dependent oscillator to n -dimensions, showing that the group is $Sl(n+2, R)$. They followed Lutzky's method.

In this note, the complete symmetry group of a forced one-dimensional harmonic oscillator is shown to be $Sl(3, R)$. The method used is based on the theory of linear canonical transformations. Lutzky's approach has not been attempted, but it is expected that the results would be equally easy to obtain. Some work was done using the method of Wulfman and Wybourne. The resulting partial differential equations were not easy to solve and it is believed that the method adopted here is simpler. It also has the advantage of showing the essential similarity of linear Hamiltonian systems.

2. Lie symmetry groups and invariants

The main ideas of the theory of Lie symmetry groups may be found in Bluman and Cole [2]. Some of the relevant results are summarized here for the benefit of the reader. A one-parameter infinitesimal point transformation from coordinates (q, t) to coordinates (\bar{q}, \bar{t}) , where

$$\bar{t} = t + \xi(q, t) \delta\alpha, \quad \bar{q} = q + \eta(q, t) \delta\alpha, \quad (2.1)$$

is generated by the operator

$$G(q, t) \equiv \xi(q, t) \frac{\partial}{\partial t} + \eta(q, t) \frac{\partial}{\partial q}. \quad (2.2)$$

To find the variations induced on the derivatives it is necessary to use the extended

group operator. For a function involving the first derivative, the operator is

$$G^{(1)} \equiv \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \eta^{(1)} \frac{\partial}{\partial \dot{q}} \quad (2.3)$$

and, for one involving the second derivatives, it is

$$G^{(2)} \equiv \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \eta^{(1)} \frac{\partial}{\partial \dot{q}} + \eta^{(2)} \frac{\partial}{\partial \ddot{q}}, \quad (2.4)$$

where

$$\left. \begin{aligned} \eta^{(1)} &\equiv \frac{d\eta}{dt} - \dot{q} \frac{d\xi}{dt}, \\ \eta^{(2)} &\equiv \frac{d\eta^{(1)}}{dt} - \ddot{q} \frac{d\xi}{dt}, \\ \frac{d}{dt} &\equiv \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q}. \end{aligned} \right\} \quad (2.5)$$

In particular, if it is desired to study the symmetry group of a differential equation of the second order, such as

$$N(q, \dot{q}, \ddot{q}, t) = 0, \quad (2.6)$$

the second extended operator must be used. An operator G is said to be the generator of a one-parameter symmetry group for (2.6) if, whenever (2.6) is satisfied,

$$G^{(2)} N(q, \dot{q}, \ddot{q}, t) = 0 \quad (2.7)$$

$$\Leftrightarrow \xi \frac{\partial N}{\partial t} + \eta \frac{\partial N}{\partial q} + \eta^{(1)} \frac{\partial N}{\partial \dot{q}} + \eta^{(2)} \frac{\partial N}{\partial \ddot{q}} = 0. \quad (2.8)$$

It is possible to obtain the generator of a one-parameter symmetry group if the motion corresponding to the differential equation (2.6) has an integral invariant. In terms of the Lagrangian formulation, if there exists an invariant, it may be written in the form

$$\phi(q, \dot{q}, t) = (\xi \dot{q} - \eta) \frac{\partial L}{\partial \dot{q}} - \xi L + f(q, t). \quad (2.9)$$

The operator G is obtained by the use of the ξ and η which satisfies (2.9) for a particular $\phi(q, \dot{q}, t)$. In the Hamiltonian formulation the corresponding relation is

$$\phi(q, p, t) = \xi H - \eta p + f(q, t). \quad (2.10)$$

The function $f(q, t)$ performs the task of soaking up remaining terms. For each invariant ϕ there corresponds only one generator. It has been noted by Lutzky [14] that, were f allowed to depend upon q, \dot{q}, t (or q, p, t), this uniqueness property

is lost. From (2.10) it is obvious that, for those Hamiltonians which are at most quadratic in the momentum, invariants are also at most quadratic in the momentum if they are to give rise to a symmetry group generator.

It should be emphasized that, although to each invariant (subject to the limitations mentioned above) there exists a generator of a one-parameter Lie group, there may be one-parameter groups which do not correspond to the Lagrangian (or Hamiltonian) invariants. The generators of such groups are formed from the solutions of (2.8). They have the property of leaving the equation of motion invariant, but not the action integral. Only the Lagrangian (Hamiltonian) derived generators have this property. There has been some work done on finding constants which correspond to these additional generators, but so far the results have not been reported in the literature [16].

3. Invariants for the forced oscillator

The one-dimensional, time-independent, forced harmonic oscillator has Newtonian equation of motion

$$+\ddot{q}q+f=0, \quad (3.1)$$

where f is taken to be a continuous function of time over the interval of time which is of interest. The corresponding Hamiltonian is

$$H = \frac{1}{2}(p^2 + q^2) + fq, \quad (3.2)$$

the conjugate momentum, p , being \dot{q} . This may be transformed to the unforced oscillator

$$\bar{H} = \frac{1}{2}(P^2 + Q^2) \quad (3.3)$$

by means of a linear canonical transformation. Rewriting (3.2) and (3.3) as

$$H = \frac{1}{2}z^T I z + b^T z, \quad (3.4)$$

$$\bar{H} = \frac{1}{2}\bar{z}^T I \bar{z}, \quad (3.5)$$

where

$$z = \begin{bmatrix} q \\ p \end{bmatrix}, \quad \bar{z} = \begin{bmatrix} Q \\ P \end{bmatrix}, \quad b = \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad (3.6)$$

the transformation is

$$z = S\bar{z} + r, \quad (3.7)$$

where [5]

$$\hat{S} = JIS - SJI, \quad (3.8)$$

$$\hat{t} = JIr + Jb. \quad (3.9)$$

The matrix J is the 2×2 symplectic matrix. There is no loss of generality in taking S to be the identity. The solution of (3.9) is given by

$$\mathbf{r}(t) = R(t)\mathbf{r}(0) + \int_0^t R(t)R^{-1}(\tau)J\mathbf{b}(\tau) d\tau, \quad (3.10)$$

where $R(t)$ is the 2×2 matrix formed from the solution set of the homogeneous equation corresponding to (3.9). Explicitly,

$$R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \quad (3.11)$$

Again there is no loss of generality if $\mathbf{r}(0)$ is set at zero. The transformation relating the two Hamiltonians is then

$$\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} Q \\ P \end{bmatrix} + \begin{bmatrix} g(t) \\ -h(t) \end{bmatrix}, \quad (3.12)$$

where

$$g(t) = \int_0^t \sin(\tau-t)f(\tau) d\tau, \quad (3.13)$$

$$h(t) = \int_0^t \cos(\tau-t)f(\tau) d\tau. \quad (3.14)$$

The two linear and three quadratic invariants for (3.2) are (compare with [8]):

$$\left. \begin{aligned} 2\phi_1(q, p, t) &= -\{(q-g)^2 - (p+h)^2\} \sin 2t - 2(q-g)(p+h) \cos 2t, \\ 2\phi_2(q, p, t) &= -\{(q-g)^2 - (p+h)^2\} \cos 2t + 2(q-g)(p+h) \sin 2t, \\ \phi_3(q, p, t) &= -(q-g) \sin t - (p+h) \cos t, \\ \phi_4(q, p, t) &= (q-g) \cos t - (p+h) \sin t, \\ 2\phi_5(q, p, t) &= (p+h)^2 + (q-g)^2. \end{aligned} \right\} \quad (3.15)$$

The ordering adopted is that which has been used in recent papers [10, 13, 15]. From (2.2) and (2.10) it follows that the corresponding generators are

$$\left. \begin{aligned} G_1 &= \sin 2t \frac{\partial}{\partial t} + \{(q-g) \cos 2t - h \sin 2t\} \frac{\partial}{\partial q}, \\ G_2 &= \cos 2t \frac{\partial}{\partial t} - \{(q-g) \sin 2t + h \cos 2t\} \frac{\partial}{\partial q}, \\ G_3 &= \cos t \frac{\partial}{\partial q}, \\ G_4 &= \sin t \frac{\partial}{\partial q}, \\ G_5 &= \frac{\partial}{\partial t} - h \frac{\partial}{\partial q}. \end{aligned} \right\} \quad (3.16)$$

These five generators have commutation relations which are appropriate for the five-parameter proper subgroup of $Sl(3, R)$. They are

$$\left. \begin{aligned} [G_1, G_2] &= -2G_5, & [G_5, G_1] &= 2G_2, & [G_2, G_5] &= 2G_1, \\ [G_3, G_4] &= 0, \\ [G_3, G_1] &= [G_2, G_4] = [G_5, G_4] = G_3, \\ [G_1, G_4] &= [G_3, G_5] = [G_2, G_3] = G_4. \end{aligned} \right\} \quad (3.17)$$

4. Transformation of the generators

The remaining three generators, which also constitute a proper subgroup of $Sl(3, R)$, may be obtained by solving the partial differential equation (2.8) for $\xi(q, t)$ and $\eta(q, t)$. This has the disadvantage of including the derivation of the generators already obtained. Further, for this problem, the ordinary equations which emerge are coupled, which complicates the algebra. An alternative method, which has been used elsewhere [11], is to compare the expressions for G_1 to G_5 for the harmonic oscillator with those obtained in this case, for the forced oscillator. By noting the variations, an educated guess can be made as to the expressions for G_6 to G_7 . The guess can be checked by substitution of ξ and η in (2.8). There is another method which may be applied in this problem. Before outlining it, the generators for the harmonic oscillator described by (3.3) are listed. They are

$$\left. \begin{aligned} G_1(Q, t') &= \sin 2t' \frac{\partial}{\partial t'} + Q \cos 2t' \frac{\partial}{\partial Q}, \\ G_2(Q, t') &= \cos 2t' \frac{\partial}{\partial t'} - Q \sin 2t' \frac{\partial}{\partial Q}, \\ G_3(Q, t') &= \cos t' \frac{\partial}{\partial Q}, \\ G_4(Q, t') &= \sin t' \frac{\partial}{\partial Q}, \\ G_5(Q, t') &= \frac{\partial}{\partial t'}, \\ G_6(Q, t') &= Q \frac{\partial}{\partial Q}, \\ G_7(Q, t') &= Q \sin t' \frac{\partial}{\partial t'} + Q^2 \cos t' \frac{\partial}{\partial Q}, \\ G_8(Q, t') &= Q \cos t' \frac{\partial}{\partial t'} - Q^2 \sin t' \frac{\partial}{\partial Q}. \end{aligned} \right\} \quad (4.1)$$

The transformation from \bar{H} to H may be written as

$$\left. \begin{aligned} q &= Q + g(t'), \\ p &= P - h(t'), \\ t &= t', \end{aligned} \right\} \quad (4.2)$$

where the distinction is made between t and t' for reasons which become apparent shortly. The transformation (4.2) is a point transformation and the second equation adds no further information to that contained in the other two. (Note that $dg(t')/dt' = -h(t')$.) The differential operators in (4.1), namely $\partial/\partial t'$ and $\partial/\partial Q$, take the following expression in the new coordinates,

$$\left. \begin{aligned} \frac{\partial}{\partial t'} &= \frac{\partial t}{\partial t'} \frac{\partial}{\partial t} + \frac{\partial q}{\partial t'} \frac{\partial}{\partial q} = \frac{\partial}{\partial t} - h \frac{\partial}{\partial q}, \\ \frac{\partial}{\partial Q} &= \frac{\partial t}{\partial Q} \frac{\partial}{\partial t} + \frac{\partial q}{\partial Q} \frac{\partial}{\partial q} = \frac{\partial}{\partial q}. \end{aligned} \right\} \quad (4.3)$$

If, in the first five equations in (4.1), Q is replaced by $q-g$, t' by t , $\partial/\partial t'$ by $\partial/\partial t - L\partial/\partial q$ and $\partial/\partial Q$ by $\partial/\partial q$, the operators are now in the form given in (3.16). The remaining generators for the forced oscillator are

$$\left. \begin{aligned} G_6 &= (q-g) \frac{\partial}{\partial q}, \\ G_7 &= (q-g) \sin t \frac{\partial}{\partial t} + \{(q-g)^2 \cos t - (q-g)h \sin t\} \frac{\partial}{\partial q}, \\ G_8 &= (q-g) \cos t \frac{\partial}{\partial t} - \{(q-g)^2 \sin t + (q-g)h \cos t\} \frac{\partial}{\partial q}. \end{aligned} \right\} \quad (4.4)$$

It is simply a matter of algebra to verify that G_6 , G_7 and G_8 do form a proper subgroup of $SL(3, R)$. The commutation relations are

$$[G_6, G_7] = G_7, \quad [G_6, G_8] = G_8, \quad [G_7, G_8] = 0. \quad (4.5)$$

The commutation relations between the generators of the two subgroups are also the standard ones, being

$$\left. \begin{aligned} [G_6, G_1] &= [G_6, G_2] = [G_6, G_5] = 0, \\ [G_7, G_1] &= -G_7, \quad [G_7, G_2] = -G_8, \quad [G_7, G_5] = -G_8, \\ [G_8, G_1] &= G_8, \quad [G_8, G_2] = -G_7, \quad [G_8, G_5] = G_7, \\ [G_6, G_3] &= -G_3, \quad [G_6, G_4] = -G_4, \\ [G_7, G_3] &= -\frac{1}{2}(G_1 + 3G_6), \quad [G_8, G_4] = \frac{1}{2}(-G_1 + 3G_6), \\ [G_7, G_4] &= \frac{1}{2}(G_2 - G_5), \quad [G_8, G_3] = -\frac{1}{2}(G_2 + G_5). \end{aligned} \right\} \quad (4.6)$$

Thus the complete symmetry group of the one-dimensional forced harmonic oscillator is $SI(3, R)$.

5. Discussion

The result obtained here for the forced oscillator applies equally well to an oscillator with a moving source of potential for it is essentially the same problem, having the same equation of motion. The transformation between the two Hamiltonians was of the form of identity plus a time-dependent displacement. Thus the symmetry group of any forced linear system will be the same as the corresponding unforced linear system. As the symmetry group $SI(3, R)$ is applicable to the free particle, attractive and repulsive (both time-independent and time-dependent) oscillators, it is applicable to the case of a particle moving under the influence of a coordinate-free force and to the forced attractive and repulsive oscillator. It is also applicable to motion relative to frames of reference in rectilinear motion (the equivalent for the oscillators to a moving source of potential). It might be noted that this is independent, in the case of oscillators, of whether the forcing term (or source movement) is resonant or not.

For n -dimensional systems without coupling, damping or rotating axes, the corresponding result is that the symmetry group is $SI(n+2, R)$. The excluded cases await further investigation.

The use of point transformations may make the investigation of the symmetries of other systems easier. To take the example of the time-dependent harmonic oscillator which has been studied in other ways [10, 15], the Hamiltonians

$$H = \frac{1}{2}\{p^2 + \omega^2(t)q^2\} \quad (5.1)$$

and

$$H = \rho^{-2}\{P^2 + Q^2\} \quad (5.2)$$

are related by the point transformation

$$\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 1/\rho & 0 \\ -\dot{\rho} & \rho \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix}, \quad (5.3)$$

where $\rho(t)$ is any solution of

$$\ddot{\rho} + \omega^2(t)\rho = 1/(\rho^3). \quad (5.4)$$

Hence the symmetries of the systems corresponding to H and H are the same. The change of time variable from t to W , where

$$W = \int_{t_0}^t \rho^{-2} dt', \quad (5.5)$$

makes (5.2) the equivalent to the time-independent harmonic oscillator which possesses $SI(3, R)$ symmetry. The problem of determining whether a given dynamical system possesses this symmetry is reduced to finding a point transformation relating it to a system which does.

Acknowledgements

The writer wishes to thank Professor C. J. Eliezer, who introduced him to the subject of dynamical symmetries some years ago, and G. E. Prince for stimulating discussions on the subject of Lie groups.

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$sl(2, R)$ and the repulsive oscillator

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Journal of Physics A: Mathematical and General **13** (1980) 1991–2000

SI(3, R) and the repulsive oscillator

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Received 31 May 1979, in final form 24 October 1979

Abstract. The complete symmetry group of a particle moving in one dimension under the influence of a negative quadratic potential ('the repulsive oscillator') is shown to be $SI(3, R)$. The generators of the five-parameter subgroup are obtained from the two linear and three quadratic invariants of the Hamiltonian. The additional generators required for the three-parameter subgroup are obtained from the method of extended Lie groups. It is inferred that an n -dimensional, uncoupled, undamped and unforced linear system has the complete symmetry group $SI(n+2, R)$.

1. Introduction

Discussion of the symmetry groups of dynamical systems, especially classical, has widened in recent years. Initially, discussions were of purely geometrical symmetries, for example rotational invariance. The concept of dynamical symmetry as contrasted to geometrical symmetry arose from the necessity to explain the existence of degeneracies in spectra which were over and above those expected on purely geometric grounds.

For the non-relativistic Kepler problem the conserved Runge–Lenz vector provided additional generators which showed that $SO(4)$ was the appropriate symmetry group. In the case of the non-relativistic isotropic harmonic oscillator, the conserved Jauch–Hill–Fradkin tensor performed a similar task. In this case the symmetry group was $SU(3)$. Each constant of the motion associated with these symmetry groups has zero Poisson bracket with the (appropriate) Hamiltonian. Subsequent development has been in the construction of non-invariance symmetry groups. These have taken the form of non-invariance super-groups as studied by Mukunda *et al* (1965) and of non-invariance groups for time-dependent systems such as those studied by Günther and Leach (1977) and Leach (1978a).

More recently, attention has been given to the complete symmetry groups of dynamical systems. The basic systems studied have been one-dimensional and linear. Linear systems are important physically and have the advantage of being amenable to mathematical treatment. It would appear that the concentration on one-dimensional systems has been from a desire to highlight the symmetry rather than to engage in a demonstration of algebraic dexterity. However, there is a more serious difficulty in the treatment of multi-dimensional linear systems which is related to the diagonalisation of symmetric matrices by symplectic transformations (cf Williamson 1937).

Anderson and Davison (1974) showed that the one-dimensional, time-independent, harmonic oscillator and the free particle both possessed the complete symmetry group $SI(3, R)$. The result for the oscillator was obtained also by Wulfman and Wybourne (1976) who employed the method of extended Lie groups. In an elegant

paper, Lutzky (1978) combined Noether's Theorem with a modification of the extended theory to obtain the same result. Leach (1979) showed that the one-dimensional, time-dependent, harmonic oscillator also has $Sl(3, R)$ as its complete symmetry group. The method adopted was based on a combination of linear canonical transformations of the Hamiltonian and the method of extended Lie groups. The complete symmetry group for an n -dimensional time-dependent harmonic oscillator (uncoupled) was shown to be $Sl(n+2, R)$ by Prince and Eliezer (1980). They followed Lutzky's method. By implication the corresponding time-independent problem also possesses $Sl(n+2, R)$ symmetry.

In this paper, the complete symmetry of a particle moving in one dimension under the influence of a negative, time-independent, quadratic potential is shown to be $Sl(3, R)$. With this result established, it may be inferred that the complete symmetry group of an n -dimensional linear system, without damping, coupling or forcing terms, is $Sl(n+2, R)$. This is the case whether the potential terms are time-independent or time-dependent. (The generators for the one-dimensional, time-dependent negative quadratic potential are listed in the Appendix.) It remains to be seen whether the result extends to the three categories of systems which have been excluded.

The main decision to be made when embarking on the determination of a complete symmetry group is which method is to be adopted. In this paper the method used is that which combines the Hamiltonian invariants and the extended Lie theory. The other two methods are believed to be equally suitable in this instance. However, the present writer is not convinced that this will be the case in all instances, especially when applying the method of extended Lie groups. The main motivation for the choice in this problem is that it lies within the author's programme of demonstrating the essential sameness of all classical quadratic Hamiltonians.

The development of the paper reflects that purpose. The canonical transformation from attractive to repulsive oscillator is derived and from this the two linear and three quadratic invariants obtained. Using standard theory the associated generators may then be written down. The form of the remaining three generators is suggested. They are shown to satisfy the partial differential equations arising from the method of extended Lie groups. The eight generators are shown to have the commutation relations appropriate to the symmetry group $Sl(3, R)$, which establishes the result. A comparison of these generators with those of the time-independent and time-dependent harmonic oscillator suggests the generators for the time-dependent repulsive oscillator and they are listed in the Appendix. On a matter of terminology, the system described here is called the 'repulsive oscillator'. On a point of semantics this is, in a sense, nonsensical, but it is believed to be a suitable description.

2. Canonical transformation from attractive to repulsive oscillator

The repulsive oscillator has Newtonian equation of motion

$$\ddot{q} - q = 0 \quad (2.1)$$

and associated Hamiltonian

$$H = \frac{1}{2}(p^2 - q^2) \quad (2.2)$$

in which

$$p = \dot{q}. \quad (2.3)$$

In terms of the two-vector $z^T = (q, p)$, the Hamiltonian is

$$H = \frac{1}{2} z^T A z \tag{2.4}$$

where the 2×2 real symmetric matrix A is given by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.5}$$

The attractive oscillator has Hamiltonian

$$\bar{H} = \frac{1}{2} \bar{z}^T I \bar{z} \tag{2.6}$$

where I is the 2×2 identity. A linear canonical transformation from \bar{H} to H is accomplished by

$$z = S \bar{z} \tag{2.7}$$

where the 2×2 real matrix S satisfies the system of equations (Leach 1977)

$$\dot{S} = JAS - SJI, \tag{2.8}$$

J being the 2×2 symplectic matrix. The requirement that the transformation be canonical imposes the constraint that

$$SJS^T = J. \tag{2.9}$$

The system of equations (2.8) may be rewritten as

$$\dot{u} = Mu \tag{2.10}$$

where

$$u^T = (S_{11}, S_{12}, S_{21}, S_{22}) \tag{2.11}$$

$$M = \begin{pmatrix} J & I \\ I & J \end{pmatrix}. \tag{2.12}$$

Setting $t_0 = 0$, the solution of (2.10) is

$$\begin{aligned} u(t) &= \exp(tM)u(0) \\ &= \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \cosh t \cos t + \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \cosh t \sin t \right. \\ &\quad \left. + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \sinh t \cos t + \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \sinh t \sin t \right\} u(0). \end{aligned} \tag{2.13}$$

Writing

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad u(0) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{2.14}$$

the constraint (2.9) becomes

$$u_1^T J u_2 = 1. \tag{2.15}$$

As it is desired to express the invariants of \bar{H} in terms of z , the inverse transformation is required. Since

$$\begin{aligned} S^{-1} &= -JS^T J \\ &= -J[u_1(t), u_2(t)]J, \end{aligned} \tag{2.16}$$

when $u_1(t)$ and $u_2(t)$ are substituted from (2.13), the inverse is

$$S^{-1} = (I \sin t - J \cos t)(-u_1 \sinh t - u_2 \cosh t, u_1 \cosh t + u_2 \sinh t). \quad (2.17)$$

3. The invariants

The Hamiltonian (2.6) has five invariants (see Leach (1979) *The Complete Symmetry Group of a One-dimensional Forced Harmonic Oscillator* (unpublished)), two linear and three quadratic in the canonical variables. (There are, of course, invariants of higher degree, but their Poisson bracket relations, which generate more invariants, do not constitute a closed set and so are not suitable to provide the finite number of generators which is a feature of second-order equations.) The linear invariants are given by the elements of the vector

$$C_1 = (I \cos t - J \sin t)\bar{z}. \quad (3.1)$$

The quadratic invariants are given by the elements of the matrix

$$C_2 = C_1 C_1^T \\ = (I \cos t - J \sin t)\bar{z}\bar{z}^T(I \cos t + J \sin t). \quad (3.2)$$

There are three linearly independent elements of C_2 . The usual forms of the invariants are given by

$$[C_2]_{12} = [C_2]_{21}, \quad \frac{1}{2}([C_2]_{11} + [C_2]_{22}), \quad \frac{1}{2}([C_2]_{11} - [C_2]_{22}).$$

However, the expression in (3.2) is suitable for the present formalism. At the appropriate stage in the development, the invariants will be regrouped.

Applying the transformation

$$\bar{z} = S^{-1}z \quad (3.3)$$

with S^{-1} as given in (2.7), the linear invariants become

$$C_1 = -J(u_1 \cosh t + u_2 \sinh t, u_1 \sinh t + u_2 \cosh t)Jz. \quad (3.4)$$

This may be written in a form free of the vectors u_1 and u_2 by premultiplying by $S(0)$. Then

$$C'_1 = S(0)C_1 \\ = \begin{pmatrix} u_1^T \\ u_2^T \end{pmatrix} C_1 \\ = (I \cosh t - K \sinh t)z \quad (3.5)$$

where the matrix K is

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.6)$$

Under the transformation the matrix of quadratic invariants becomes

$$C_2 = J(u_1 \cosh t + u_2 \sinh t, u_1 \sinh t + u_2 \cosh t)Jz z^T J^T \\ \times (u_1 \cosh t + u_2 \sinh t, u_1 \sinh t + u_2 \cosh t)^T J^T. \quad (3.7)$$

This may be converted to an invariant matrix free of u_1 and u_2 by premultiplying by $S(0)$ and postmultiplying by $S(0)^T$. Thus

$$\begin{aligned} C'_2 &= S(0)C_2S(0)^T \\ &= (I \cosh t - K \sinh t)zz^T(I \cosh t - K \sinh t), \end{aligned} \tag{3.8}$$

a result which could have been anticipated by the expression for C'_1 in (3.5). It should be emphasised that the rearrangement of C_1 and C_2 to obtain C'_1 and C'_2 has nothing to do with the canonical transformation from \bar{H} to H . The point is that, for all possible choices of the parameters of the transformation, the same set of invariants suffices. Hereafter the prime is dropped and C_1 and C_2 refer to the parameter-free forms given by (3.5) and (3.8).

4. The generators of the five-parameter subgroup

Corresponding to the two linear and three quadratic invariants there is a five-parameter group which is a subgroup of the complete group. Before proceeding to obtain the generators of the subgroup, the five invariants are written down in the order and form which corresponds to the usage of Lutzky (1978) and Leach (1979). This will facilitate comparison. (See also the discussion at the end of § 6.) Thus

$$\begin{aligned} \phi_1(q, p, t) &= -[C_2]_{12} \\ &= \frac{1}{2}[(q^2 + p^2) \sinh 2t - 2qp \cosh 2t] \end{aligned} \tag{4.1a}$$

$$\begin{aligned} \phi_2(q, p, t) &= -\frac{1}{2}([C_2]_{11} - [C_2]_{22}) \\ &= -\frac{1}{2}(q^2 - p^2) \end{aligned} \tag{4.1b}$$

$$\begin{aligned} \phi_3(q, p, t) &= -C_{12} \\ &= -(-q \sinh t + p \cosh t) \end{aligned} \tag{4.1c}$$

$$\begin{aligned} \phi_4(q, p, t) &= C_{11} \\ &= q \cosh t - p \sinh t \end{aligned} \tag{4.1d}$$

$$\begin{aligned} \phi_5(q, p, t) &= \frac{1}{2}([C_2]_{11} + [C_2]_{22}) \\ &= -\frac{1}{2}[(q^2 + p^2) \cosh 2t + 2qp \sinh 2t]. \end{aligned} \tag{4.1e}$$

The generator of a one-parameter group is given by

$$G(q, t) = \xi(q, t) \partial/\partial t + \eta(q, t) \partial/\partial q \tag{4.2}$$

and the corresponding invariant, in the Lagrangian formulation, is

$$\phi(q, \dot{q}, t) = (\xi\dot{q} - \eta) \partial L/\partial \dot{q} - \xi L + f(q, t). \tag{4.3}$$

In the Hamiltonian formulation, making use of

$$p = \partial L/\partial \dot{q}, \quad L = p\dot{q} - H, \tag{4.4}$$

the invariant is

$$\phi(q, p, t) = \xi H - \eta p + f(q, t). \tag{4.5}$$

With H as given by (2.2) and in the order corresponding to the listing of the invariants in (4.1a-e), ξ , η and $f(q, t)$ are given by

$\xi(q, t)$	$\eta(q, t)$	$f(q, t)$	
$\sinh 2t$	$q \cosh 2t$	$q^2 \sinh 2t$	
1	0	0	
0	$\cosh t$	$q \sinh t$	
0	$\sinh t$	$q \cosh t$	
$\cosh 2t$	$q \sinh 2t$	$q^2 \cosh 2t$	(4.6)

and the generators are

$$G_1 = \sinh 2t \partial/\partial t + q \cosh 2t \partial/\partial q \quad (4.7a)$$

$$G_2 = \partial/\partial t \quad (4.7b)$$

$$G_3 = \cosh t \partial/\partial q \quad (4.7c)$$

$$G_4 = \sinh t \partial/\partial q \quad (4.7d)$$

$$G_5 = \cosh 2t \partial/\partial t + q \sinh 2t \partial/\partial q. \quad (4.7e)$$

It is noted that for the first, second and fifth,

$$\xi = \frac{\partial^2}{\partial q^2}[2F(q, t)], \quad \eta = \frac{\partial^2}{\partial q \partial t}[F(q, t)], \quad f = \frac{\partial^2}{\partial t^2}[F(q, t)] \quad (4.8)$$

with $F(q, t)$ being $\frac{1}{4}q^2 \sinh 2t$, $\frac{1}{4}q^2$ and $\frac{1}{4}q^2 \cosh 2t$ respectively. For the third and fourth,

$$\xi = \frac{\partial^2}{\partial q^2}[F(q, t)], \quad \eta = \frac{\partial^2}{\partial q \partial t}[F(q, t)], \quad f = \frac{\partial^2}{\partial t^2}[F(q, t)] \quad (4.9)$$

with $F(q, t)$ being $q \sinh t$ and $q \cosh t$ respectively.

5. The generators of the three-parameter subgroup

The maximum number of one-parameter groups for a second-order differential equation is eight (Bluman and Cole 1974). Of these, five have been obtained using the Hamiltonian formulation and transformation theory. The three remaining are obtained using the method of extended Lie groups by applying the second extension of the generator to the Newtonian equation of motion. This equation of motion is

$$\ddot{q} - q = 0. \quad (5.1)$$

The second extension of the operator

$$G(q, t) = \xi(q, t) \partial/\partial t + \eta(q, t) \partial/\partial q \quad (5.2)$$

is (Prince and Eliezer 1980)

$$G^{(2)}(q, t) = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \eta^{(1)} \frac{\partial}{\partial \dot{q}} + \eta^{(2)} \frac{\partial}{\partial \ddot{q}} \quad (5.3)$$

where

$$\eta^{(1)} = d\eta/dt - \dot{q} d\xi/dt \tag{5.4}$$

$$\eta^{(2)} = d\eta^{(1)}/dt - \ddot{q} d\xi/dt \tag{5.5}$$

$$d/dt = \partial/\partial t + \dot{q} \partial/\partial q. \tag{5.6}$$

G is the generator of a one-parameter group provided

$$G^{(2)}(\ddot{q} - q) = 0 \tag{5.7}$$

whenever equation (5.1) is satisfied. This gives rise to the set of partial differential equations

$$\xi_{qq} = 0 \tag{5.8a}$$

$$\eta_{aq} - 2\xi_{qt} = 0 \tag{5.8b}$$

$$2\eta_{qt} - \xi_{tt} - 3q\xi_a = 0 \tag{5.8c}$$

$$\eta_{tt} - \eta + q\eta_a - 2q\xi_t = 0. \tag{5.8d}$$

The solution of these equations is not particularly difficult in this case. However, for some other problems it is not a trivial task. One way to avoid having to solve these equations is to compare the known solutions for G_1 to G_5 already obtained with those of a similar problem. In this case the attractive oscillator is suitable for comparison. It is observed that for G_1 to G_5 , in the coefficients of the $\partial/\partial t$ terms, \cos goes to \cosh and \sin to \sinh while for the $\partial/\partial q$ terms \cos goes to \cosh and \sin to $-\sinh$. For the attractive oscillator (cf Lutzky (1978)),

$$G_6 = q \partial/\partial q \tag{5.9a}$$

$$G_7 = q \sin t \partial/\partial t + q^2 \cos t \partial/\partial q \tag{5.9b}$$

$$G_8 = q \cos t \partial/\partial t - q^2 \sin t \partial/\partial q. \tag{5.9c}$$

This suggests that for the repulsive oscillator

$$\xi_6 = 0, \quad \eta_6 = q \tag{5.10a}$$

$$\xi_7 = q \sinh t, \quad \eta_7 = q^2 \cosh t \tag{5.10b}$$

$$\xi_8 = q \cosh t, \quad \eta_8 = q^2 \sinh t. \tag{5.10c}$$

Each of these pairs in turn satisfies the equations (5.8a-d). Therefore it is proposed that for the repulsive oscillator

$$G_6 = q \partial/\partial q \tag{5.11a}$$

$$G_7 = q \sinh t \partial/\partial t + q^2 \cosh t \partial/\partial q \tag{5.11b}$$

$$G_8 = q \cosh t \partial/\partial t + q^2 \sinh t \partial/\partial q. \tag{5.11c}$$

6. The complete symmetry group

The final test for the generators is whether they have commutation relations appropriate to one of the established groups. One reason for this is to identify the group. Another reason is to check the accuracy of the expressions for the generators, especially

those of G_6 , G_7 and G_8 in view of the method used to obtain them. The commutation relations are given in table 1. These are the standard relations, as reported in recent literature, for the symmetry group $Sl(3, \mathcal{R})$ which is therefore the complete symmetry group for the repulsive oscillator.

Table 1. The commutation relations. The (i, j) th entry is the bracket $[G_i, G_j]$.

$G_i \backslash G_j$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	$-2G_5$	$-G_3$	G_4	$-2G_2$	0	G_7	$-G_8$
G_2	$2G_5$	0	G_4	G_3	$2G_1$	0	G_8	G_7
G_3	G_3	$-G_4$	0	0	G_4	G_3	$\frac{1}{2}(G_1 + 3G_6)$	$\frac{1}{2}(G_2 + G_5)$
G_4	$-G_4$	$-G_3$	0	0	$-G_3$	G_4	$-\frac{1}{2}(G_2 - G_5)$	$\frac{1}{2}(G_1 - 3G_6)$
G_5	$2G_2$	$-2G_1$	G_4	G_3	0	0	G_8	$-G_7$
G_6	0	0	$-G_3$	$-G_4$	0	0	G_7	G_8
G_7	$-G_7$	$-G_8$	$-\frac{1}{2}(G_1 + 3G_6)$	$\frac{1}{2}(G_2 - G_5)$	$-G_8$	$-G_7$	0	0
G_8	G_8	$-G_7$	$-\frac{1}{2}(G_2 + G_5)$	$-\frac{1}{2}(G_1 - 3G_6)$	G_7	$-G_8$	0	0

Those who are familiar with the form of the generators for the attractive oscillator will have observed that the operators G_2 and G_5 appear to have been interchanged for the repulsive oscillator. This is not the case. The operators G_2 and G_5 for the repulsive oscillator are respectively the counterparts of G_2 and G_5 for the attractive oscillator. This is most easily seen from the corresponding quadratic invariants (see Leach 1978b). Under the transformation from attractive to repulsive oscillator, the quadratic invariants transform as

$$(q^2 - p^2) \sin 2t + 2qp \cos 2t \rightarrow -(q^2 + p^2) \sinh 2t + 2qp \cosh 2t \quad (6.1a)$$

$$(q^2 - p^2) \cos 2t - 2qp \sin 2t \rightarrow q^2 - p^2 \quad (6.1b)$$

$$q^2 + p^2 \rightarrow (q^2 + p^2) \cosh 2t - 2qp \sinh 2t. \quad (6.1c)$$

For the attractive oscillator G_5 corresponds to the Hamiltonian, whereas for the repulsive oscillator it is G_2 which corresponds to the Hamiltonian. The Hamiltonian (in a time-independent context) is the generator of time translations and so it is appropriate that the generator corresponding to the Hamiltonian in each case is $\partial/\partial t$. In order to preserve the pattern of the commutation relations of the generators for $Sl(3, \mathcal{R})$ in the form adopted by recent writers, the generator of pure time translations may vary from problem to problem. For the attractive oscillator it is G_5 and for the repulsive oscillator discussed here it is G_2 . In the case of the alternative form for the Hamiltonian of the repulsive oscillator, namely

$$H = pq, \quad (6.2)$$

G_1 becomes $\partial/\partial t$.

7. Comment

The complete symmetry group for the one-dimensional attractive oscillator, the free

particle and the repulsive oscillator is $Sl(3, R)$ in each instance. For the one-dimensional time-dependent attractive oscillator it has been demonstrated by Leach (1979) that the complete symmetry group is also $Sl(3, R)$. This is also the case for the one-dimensional time-dependent repulsive oscillator for which the generators are given in the Appendix. For an n -dimensional uncoupled time-dependent attractive oscillator system, Prince and Eliezer (1980) have shown that the complete symmetry group is $Sl(n + 2, R)$. From the results summarised above, it may be inferred that the complete symmetry group of an n -dimensional linear system is $Sl(n + 2, R)$ provided that the system is uncoupled, undamped and unforced.

The question now arises as to whether the complete symmetry group of any n -dimensional linear system is also $Sl(n + 2, R)$. It is to be reported elsewhere (Leach (1979) *The Complete Symmetry Group of a One-dimensional Forced Harmonic Oscillator* (unpublished)) that the result holds if forcing is present. The situation with respect to damping is the subject of current investigation. For a coupled system the position is as yet obscure, but it is hoped that some indication will be forthcoming in the near future.

Appendix

The one-dimensional time-dependent repulsive oscillator is described by the Newtonian equation of motion

$$\ddot{q} - \omega^2(t)q = 0 \tag{A1}$$

and has the Hamiltonian

$$H = \frac{1}{2}(p^2 - \omega^2(t)q^2). \tag{A2}$$

The Hamiltonian (A2) is related to the corresponding time-dependent Hamiltonian

$$\bar{H} = \frac{1}{2}(P^2 - Q^2) \tag{A3}$$

by the linear canonical transform

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} 1/\rho, & 0 \\ -\dot{\rho}, & \rho \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{A4}$$

where $\rho(t)$ is any solution of the 'auxiliary equation' (cf Eliezer and Gray 1976)

$$\ddot{\rho} - \omega^2(t)\rho = 1/\rho^3. \tag{A5}$$

The generators of the complete symmetry group $Sl(3, R)$ are

$$G_1 = \sinh 2W \partial/\partial W + (-\rho\dot{\rho} \sinh 2W + \cosh 2W)q \partial/\partial q \tag{A6a}$$

$$G_2 = \partial/\partial W + \rho\dot{\rho}q \partial/\partial q \tag{A6b}$$

$$G_3 = \rho \cosh W \partial/\partial q \tag{A6c}$$

$$G_4 = \rho \sinh W \partial/\partial q \tag{A6d}$$

$$G_5 = \cosh 2W \partial/\partial W + (\rho\dot{\rho} \cosh 2W + \sinh 2W)q \partial/\partial q \tag{A6e}$$

$$G_6 = q \partial/\partial q \tag{A6f}$$

$$G_7 = \rho^{-1}q \sinh W \partial/\partial W + (-\dot{\rho} \sinh W + \rho^{-1} \cosh W)q^2 \partial/\partial q \tag{A6g}$$

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$$G_8 = \rho^{-1} q \cosh W \partial/\partial W + (\dot{\rho} \cosh W + \rho^{-1} \sinh W) q^2 \partial/\partial q \quad (\text{A6h})$$

where

$$W = \int_{t_0}^t \rho^{-2} dt' \quad (\text{A7})$$

is the effective time variable.

Acknowledgment

The writer thanks Professor C J Eliezer and G E Prince for many stimulating discussions on the subject of complete symmetry groups.

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Journal of the Australian Mathematical Society (Series B) **23** (1981) 173-186

**APPLICATIONS OF THE LIE THEORY OF
EXTENDED GROUPS IN HAMILTONIAN MECHANICS:
THE OSCILLATOR AND THE KEPLER PROBLEM**

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(Received 24 September 1980)

(Revised 22 December 1980)

Abstract

The method of the Lie theory of extended groups has recently been formulated for Hamiltonian mechanics in a manner which is consistent with the results obtained using the Newtonian equation of motion. Here the method is applied to the three-dimensional time-independent harmonic oscillator and to the classical Kepler problem. The expected constants of motion are obtained. Previously unobserved relations between generators and invariants are also noticed.

1. Introduction

There is, it would seem, a never-ending search for methods which provide a way of determining symmetries and invariants for dynamical systems. Two useful methods are Noether's theorem (in its various forms) and the method of the Lie theory of extended groups. The present application of both methods is to test systems for the presence of exact symmetries and, if they exist, to determine the associated constants of the motion. It would not greatly surprise us if in the future they were, in some sense, used to determine approximate symmetries, for example, of an adiabatic type.

The two methods are based on the concept of an invariance under an infinitesimal transformation of the dynamical variables. For Noether's theorem, the object which is left invariant is the Action Integral and, for the Lie method, it is the equation(s) of motion. The latter method is less restrictive than the

former and provides a greater number of invariants and/or allows a more general class of problem to be treated.

Before proceeding further it is proper that we define the type of infinitesimal transformation about which we speak. We are dealing with point transformations only. As an example of the greater generality of the Lie method we cite a one-dimensional linear system. Allowing point transformations only, Noether's theorem yields five generators of symmetry whereas the Lie method provides eight. However, the more serious failings of Noether's theorem occur when multi-dimensional systems are studied. It does not provide the Jauch-Hill-Fradkin tensor for the harmonic oscillator nor the Runge-Lenz vector for the classical Kepler problem. The Lie method does [5], [6].

To repair this deficiency in Noether's theorem, the use of velocity-dependent transformations has been proposed [1], [4]. Certainly the constants mentioned above satisfy the equations obtained with the more general type of transformation. Unfortunately, the wider class of admissible transformations results in an infinite number of symmetries for which no systematic method of determination exists. The same fault applies to the Lie method if the inclusion of velocity-dependent transformations is allowed. However, the inclusion is not necessary for the Lie method since all the useful invariants may be found with coordinate-dependent transformations only. As far as we are aware, invariants of value in describing the motion are either linear or quadratic in the velocities (momenta). It is these invariants which we term useful here. The difficulties associated with the use of velocity-dependent transformations in Noether's theorem are delineated more fully in the Appendix.

Until recently [7], the application of the Lie method in mechanics has been to the Newtonian equation of motion whereas Noether's theorem may be applied in either a Lagrangian or Hamiltonian context. Once the Newtonian results were obtained, the results could be translated into Hamiltonian form [2], but it seemed to us to be a messy approach. As many problems occur in a Hamiltonian framework, we judged it better to formulate the Lie method in the Hamiltonian framework in such a way that the results obtained would be consistent with the results for the corresponding Newtonian system when it exists. The formulation turned out to be very straightforward and we simply quote the results here. The operator

$$Y(\mathbf{q}, \mathbf{p}, t) = \xi(\mathbf{q}, t)\partial/\partial t + \eta_i(\mathbf{q}, t)\partial/\partial q_i + \zeta_i(\mathbf{q}, \mathbf{p}, t)\partial/\partial p_i \quad (1.1)$$

is a generator of a one-parameter Lie group for a Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ with equations of motion

$$\dot{q}_i - \partial H/\partial p_i = 0 \quad \text{and} \quad \dot{p}_i + \partial H/\partial q_i = 0, \quad (1.2)$$

provided the action of the first extension of Y on equations (1.2) gives zero whenever equations (1.2) are satisfied. The first extension is

$$Y^{(1)}(\dot{\mathbf{q}}, \dot{\mathbf{p}}, \mathbf{q}, \mathbf{p}, t) = Y(\mathbf{q}, \mathbf{p}, t) + \eta_i^{(1)}(\mathbf{q}, \mathbf{p}, t) \partial / \partial \dot{q}_i + \zeta_i^{(1)}(\mathbf{q}, \mathbf{p}, t) \partial / \partial \dot{p}_i, \quad (1.3)$$

where we have specifically written $\eta_i^{(1)}$ and $\zeta_i^{(1)}$ as functions of \mathbf{q} , \mathbf{p} and t to indicate that whenever \dot{q}_j appears it is replaced by $\partial H / \partial p_j$ and \dot{p}_j by $-\partial H / \partial q_j$. Thus

$$\eta_i^{(1)} = \frac{\partial \eta_i}{\partial t} + \frac{\partial H}{\partial q_j} \frac{\partial \eta_i}{\partial q_j} - \frac{\partial H}{\partial p_i} \left(\frac{\partial \xi}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial \xi}{\partial q_j} \right) \quad (1.4)$$

and

$$\zeta_i^{(1)} = \frac{\partial \zeta_i}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial \zeta_i}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial \zeta_j}{\partial p_j} + \frac{\partial H}{\partial q_i} \left(\frac{\partial \xi}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial \xi}{\partial q_j} \right). \quad (1.5)$$

When the set of generators $\{Y_i\}$ is determined, the associated invariants are obtained as solutions to the pair of equations

$$Y_i(\mathbf{q}, \mathbf{p}, t) I(\mathbf{q}, \mathbf{p}, t) = 0 \quad (1.6)$$

and

$$\frac{d}{dt} \{I(\mathbf{q}, \mathbf{p}, t)\} = 0. \quad (1.7)$$

This is the main drawback with the Lie method in comparison with Noether's theorem as the latter gives the invariant immediately. There are some instances in which the integration of equations (1.6) and (1.7), or their equivalents in the Newtonian picture, is not necessary. The time-dependent one-dimensional harmonic oscillator [3] may be taken as an example. The generators for that problem may be converted to the generators for the corresponding time-independent oscillator by a common transformation. Applying the inverse transformation to the oscillator invariants, the invariants for the time-dependent problem are obtained immediately.

The plan of this paper is as follows. In Section 2 the generators for the class of Hamiltonian systems to which the oscillator and the Kepler problem belong are shown to have a particular form. This simplifies the algebra when the two problems are considered in Sections 3 and 4.

2. Possible form of the generator

Suppose we have a Hamiltonian for which the equations of motion have the form

$$\dot{q}_i - f_{ij}(t) p_j = 0 \quad (2.1)$$

and

$$\dot{p}_i + g_i(\mathbf{q}, t) = 0, \quad (2.2)$$

which is commonly the case when the basis is cartesian. The matrix $[f_{ij}]$ is regular and may, without loss of generality, be taken as symmetric, although this is not necessary. Applying the first extension of the operator Y given in equation (1.3) to equations (2.1) and (2.2) we have

$$\eta_i^{(1)} - Y(\mathbf{q}, \mathbf{p}, t)f_{ij}(t)p_j = 0 \quad (2.3)$$

and

$$\zeta_i^{(1)} + Y(\mathbf{q}, \mathbf{p}, t)g_i(\mathbf{q}, t) = 0. \quad (2.4)$$

On rearranging equation (2.3) we obtain

$$\zeta_i = f^{ik}\eta_k^{(1)} - \xi f^{ik}(\partial f_{kj}/\partial t)p_j, \quad (2.5)$$

where $[f^{ij}]$ is the inverse of f_{ij} , that is,

$$f^{ik}f_{kj} = \delta_j^i. \quad (2.6)$$

This expression for ζ_i is substituted into equation (2.4). The terms of third order in the momenta yield

$$\frac{\partial^2 \xi}{\partial q_i \partial q_j} = 0 \quad (2.7)$$

and so

$$\xi(\mathbf{q}, t) = a(t) + b_i(t)q_i. \quad (2.8)$$

Collecting and rearranging the terms of second order in the momenta,

$$\left\{ f^{ik}f_{mn}f_{jl} \frac{\partial^2 \eta_k}{\partial q_j \partial q_n} - 2\delta^i m f_{nl} \frac{\partial^2 \xi}{\partial q_n \partial t} - \delta_m^i \frac{\partial f_{jl}}{\partial t} \frac{\partial \xi}{\partial q_j} - f_{jl} f^{ik} \frac{\partial f_{km}}{\partial t} \frac{\partial \xi}{\partial q_j} \right\} \\ \times \{ \delta_{ul} \delta_{om} + \delta_{ol} \delta_{um} \} = 0, \quad (2.9)$$

where equation (2.6) has also been used. For equation (2.9), it is obvious that η is at most quadratic in \mathbf{q} . From the particular case $f_{ij} = \delta_{ij}$ we get

$$\eta_i(\mathbf{q}, t) = b_j q_j q_i + c_{ij}(t)q_j + d_i(t). \quad (2.10)$$

There is no point, at this stage, in considering the coefficients of first and zero-th order powers of p since these contain the functions $g_i(\mathbf{q}, t)$. Had we allowed the $f_{ij}(t)$ to be functions of \mathbf{q} as well, equation (2.10) would have been much more complex. Such dependence would occur when curvilinear coordinates are used and may, as a separate topic, be worthy of discussion. However, the cartesian form which we are using here is sufficient for the present purpose. To round off this section we complement equations (2.8) and (2.10)

with the expression for $\zeta_i(\mathbf{q}, \mathbf{p}, t)$ for the case when f_{ij} is δ_{ij} . It is

$$\zeta_i(\mathbf{q}, \mathbf{p}, t) = -\dot{a}p_i + \ddot{b}_j q_j q_i + \dot{b}_j p_j q_i - b_j p_j p_i + \dot{c}_{ij} q_j + c_{ij} p_j + \dot{d}_i. \quad (2.11)$$

3. Harmonic oscillator

The three-dimensional time-independent harmonic oscillator has the Hamiltonian

$$H = \frac{1}{2}(\mathbf{p}^2 + \mathbf{q}^2) = \frac{1}{2}(p_i p_i + q_i q_i), \quad i = 1, 3. \quad (3.1)$$

Equations (2.2) and (2.3) become

$$\eta_i^{(1)} - \zeta_i = 0 \quad \text{and} \quad \zeta_i^{(1)} + \eta_i = 0. \quad (3.2)$$

Corresponding to the development in the previous section, the first of (3.2) gives ζ_i in terms of η_i , ξ and their derivatives. Using the second of (3.2), the general formulae obtained for ζ_i , η_i and ξ , and separating by coefficients of powers of p and q in turn, we obtain the following set of ordinary differential equations

$$\left\{ \begin{array}{l} 2\ddot{a}\delta_{ik} + \ddot{c}_{ik} = 0, \quad \ddot{a}\delta_{ik} - 2\dot{c}_{ik} = 0, \\ \ddot{b}_i + b_i = 0, \quad \text{and} \quad \ddot{d}_i + d_i = 0. \end{array} \right\} \quad (3.3)$$

The solution of this set of equations is trivial and results in the following expressions for ξ , η and ζ :

$$\xi(\mathbf{q}, t) = A + B \sin 2t + C \cos 2t + (D_j \sin t + E_j \cos t)q_j, \quad (3.4)$$

$$\eta_i(\mathbf{q}, t) = (D_j \cos t - E_j \sin t)q_j q_i + F_{ik} q_k + (B \cos 2t - C \sin 2t)q_i + G_i \sin t + H_i \cos t, \quad (3.5)$$

$$\begin{aligned} \zeta_i(\mathbf{q}, \mathbf{p}, t) = & -(B \cos 2t - C \sin 2t)p_i - (D_j \sin t + E_j \cos t)q_j q_i \\ & + (D_j \cos t - E_j \sin t)p_j q_i - (D_j \sin t + E_j \cos t)p_j p_i \\ & - 2(B \sin 2t - C \cos 2t)q_i + F_{ij} p_j + G_i \cos t - H_i \sin t. \end{aligned} \quad (3.6)$$

In all, for the three-dimensional problem, the generator contains twenty-four constants of integration giving twenty-four linearly independent generators. In passing, we note that for the corresponding n -dimensional system there are $n^2 + 4n + 3$ constants and so an equal number of generators (compare with [1]). From the expressions given in equations (3.4) to (3.6) we may write down the individual generators. These are

$$Y_{1i} = \sin t \partial / \partial q_i + \cos t \partial / \partial p_i, \quad i = 1, 3, \quad (3.7)$$

$$Y_{2i} = \cos t \partial / \partial q_i - \sin t \partial / \partial p_i, \quad i = 1, 3, \quad (3.8)$$

$$Y_3 = \partial / \partial t, \quad (3.9)$$

$$Y_4 = \sin 2t \partial / \partial t + q_j \cos 2t \partial / \partial q_j - (2q_j \sin 2t + p_j \cos 2t) \partial / \partial p_j, \quad (3.10)$$

$$Y_5 = \cos 2t \partial/\partial t - q_j \sin 2t \partial/\partial q_j - (2q_j \cos 2t - p_j \sin 2t) \partial/\partial p_j, \quad (3.11)$$

$$Y_{6i} = q_i \sin t \partial/\partial t + q_i q_j \cos t \partial/\partial q_j - (q_i q_j \sin t - p_i q_j \cos t - p_i p_j \sin t) \partial/\partial p_j, \quad i = 1, 3, \quad (3.12)$$

$$Y_{7i} = q_i \cos t \partial/\partial t - q_i q_j \sin t \partial/\partial q_j - (q_i q_j \cos t + p_i q_j \sin t + p_i p_j \cos t) \partial/\partial p_j, \quad i = 1, 3, \quad (3.13)$$

$$Y_{8ij} = q_j \partial/\partial q_i + q_i \partial/\partial q_j + p_j \partial/\partial p_i + q p_i \partial/\partial p_j, \quad i, j = 1, 3, \quad (3.14)$$

$$Y_{9ij} = q_j \partial/\partial q_i - q_i \partial/\partial q_j + p_j \partial/\partial p_i - p_i \partial/\partial p_j, \quad i \neq j = 1, 3. \quad (3.15)$$

Note that the generators for Y_{8ij} and Y_{9ij} are obtained from the symmetric and antisymmetric parts of F_{ij} , respectively.

We do not intend to discuss the group structure in detail here as it corresponds with the results given by Prince and Eliezer [5] for the time-dependent system, that is, the group is $Sl(3+2, R)$. Furthermore, we shall not explore all the invariants in detail. Of the generators with time-dependent coefficients, Y_{1i} and Y_{2i} are associated with the invariants which represent the initial conditions. The others are not so informative about the motion.

However, we do wish to make some comment about the generators Y_3 , Y_{8ij} and Y_{9ij} . The antisymmetric generator Y_{9ij} produces the invariants

$$I_{9ij1} = q_i p_j - q_j p_i, \quad i \neq j, \quad (3.16)$$

and

$$I_{9ij2} = q_i^2 + q_j^2 + p_i^2 + p_j^2, \quad i \neq j, \quad (3.17)$$

there being three of each type. The first type consists of the components of angular momentum. To each component of the angular momentum there is one invariant of the second type which represents a hypersphere in the four-dimensional subspace of phase space given by the coordinates (q_i, q_j, p_i, p_j) . This hypersphere is invariant under rotations. The symmetric generator Y_{8ij} gives rise to what might be described as the hyperbolic counterpart of the invariant hypersphere. The two constants may be considered to be the orthogonal projections of a pair of rotating (hyper-) hyperboloids specified respectively by

$$f_1(q_i, q_j, p_i, p_j) = (q_i^2 - q_j^2) - (p_i^2 - p_j^2) \quad (3.18)$$

and

$$f_2(q_i, q_j, p_i, p_j) = q_i p_i - q_j p_j. \quad (3.19)$$

So we see that the symmetric generator is not of use to describe the motion whereas the antisymmetric operator gives rise to the angular momentum and, when summed, the energy, both of which are useful quantities.

We have purposely left the simplest generator, that of time-translations, to the last as it displays a richness of detail which is generally overlooked. We suspect the reason for this lies in the consequence of the use of Noether's theorem as in that context this generator gives rise to the conserved energy alone. However, we are working within the scheme of the Lie method and shall display a far richer result. We recall that an invariant associated with a generator satisfies the dual conditions

$$Y(\mathbf{q}, \mathbf{p}, t)I(\mathbf{q}, \mathbf{p}, t) = 0 \quad \text{and} \quad \dot{I}(\mathbf{q}, \mathbf{p}, t) = 0. \quad (3.20)$$

For Y_3 the associated Lagrange's system for the first of (3.20) is

$$dt/1 = dq_i/0 = dp_i/0, \quad i = 1, 3, \quad (3.21)$$

whence, trivially,

$$u_i = q_i \quad \text{and} \quad v_i = p_i. \quad (3.22)$$

Rewriting the second of (3.20) in terms of the u_i and v_i , we have

$$du_i/du_j = v_i/v_j, \quad dv_i/dv_j = u_i/u_j \quad \text{and} \quad dv_i/du_j = -u_i/v_j. \quad (3.23)$$

Taking the third of (3.23),

$$u_i du_j + dv_i v_j = 0, \quad (3.24)$$

and adding to it the same with i and j interchanged, we obtain

$$d(u_i u_j + v_i v_j) = 0. \quad (3.25)$$

Integration of equation (3.25) gives the Jauch-Hill-Fradkin tensor

$$A_{ij} = q_i q_j + p_i p_j. \quad (3.26)$$

Half the trace of A_{ij} gives the energy. The first two of equation (3.23) may be rewritten as

$$du_i v_j - du_j v_i = 0 \quad \text{and} \quad u_i dv_j - u_j dv_i = 0. \quad (3.27)$$

When these are added and integrated we have the angular momentum tensor

$$L_{ij} = q_i p_j - q_j p_i. \quad (3.28)$$

The Jauch-Hill-Fradkin tensor does not arrive from the application of Noether's theorem. The reason for this stems from two facts. The first is that only one invariant is found per generator and the $\partial/\partial t$ generator, by not containing the coordinates specifically, can be regarded as a diagonal operator which, in combination with a diagonal Lagrangian (Hamiltonian), gives a diagonal invariant, the trace of the tensor. The second is that the only other possible source could be one of the time independent generators Y_{8ij} and Y_{9ij} , actually their Noether equivalents, and neither of these provides the tensor when Noether's theorem is used.

It must be emphasized that we are not promoting the use of the generator, $\partial/\partial t$, to obtain all of the time-independent first integrals. This might require considerable ingenuity, rather more, in fact, than is required when the generator directly associated with the particular first integral is used. What we do wish to demonstrate is that, in the Lie method, such first integrals are implied in the generator of time translations.

4. Classical Kepler problem

The Hamiltonian for the classical Kepler problem is

$$H = \frac{1}{2} \mathbf{p}^2 - \mu/r \quad (4.1)$$

and, in light of the development in Section 2, we take cartesian coordinates. Equations (2.2) and (2.3) become

$$\eta_i^{(1)} - \zeta_i = 0 \quad \text{and} \quad \zeta_i^{(1)} + \mu \eta_j (\delta_{ij}/r^3 - 3q_i q_j/r^5) = 0. \quad (4.2)$$

Following the procedure used in Section 3 we find that

$$\left. \begin{aligned} a(t) &= At + B, & b_i(t) &= 0, \\ C_{ij}(t) &= \frac{2}{3} A \delta_{ij} + C_{ij} & \text{and} & \quad d_i(t) = 0, \end{aligned} \right\} \quad (4.3)$$

where the constant matrix $[C_{ij}]$ is skew-symmetric. The generators of one-parameter Lie groups are

$$Y_1 = \partial/\partial t, \quad (4.4)$$

$$Y_2 = t \partial/\partial t + \frac{2}{3} q_j \partial/\partial q_j - \frac{1}{3} p_j \partial/\partial p_j, \quad (4.5)$$

$$Y_{3ij} = q_i \partial/\partial q_j - q_j \partial/\partial q_i + p_i \partial/\partial p_j - p_j \partial/\partial p_i, \quad i \neq j = 1, 3, \quad (4.6)$$

a total of five in all since Y_3 is skew-symmetric.

Starting with Y_{3ij} , the associated Lagrangian system is

$$dt/0 = dq_i/q_j = dq_j/(-q_i) = dp_i/p_j = dp_j/(-p_i), \quad i \neq j, \quad (4.7)$$

for which

$$\left. \begin{aligned} u_1 &= t, & u_{2ij} &= q_i^2 + q_j^2, \\ v_{1ij} &= p_i^2 + p_j^2 & \text{and} & \quad v_{2ij} = q_i p_j - q_j p_i. \end{aligned} \right\} \quad (4.8)$$

From (4.8) and the equations of motion we have

$$\begin{aligned} du_{2ij}/du_1 &= 2(q_i p_i + q_j p_j), & dv_{1ij}/du_1 &= -2(q_i p_i + q_j p_j) \mu/r^3 \\ & & \text{and} & \quad dv_{2ij}/du_1 = 0. \end{aligned} \quad (4.9)$$

From the third of equation (4.9) it follows immediately that v_{2ij} is a first integral, in fact the angular momentum,

$$L_{ij} = q_i p_j - q_j p_i. \quad (4.10)$$

Combining the first two of (4.9),

$$dv_{1ij} + (\mu/r^3) du_{2ij} = 0. \quad (4.11)$$

Summing this over the three permissible values of i and j and returning to the canonical coordinates,

$$d(2p^2) + 4\mu dr/r^2 = 0, \quad (4.12)$$

which, on integration, gives the conserved energy,

$$E = H = \frac{1}{2}p^2 - \mu/r. \quad (4.13)$$

For Y_2 we have

$$dt/t = (dq_j)/(\frac{2}{3}q_j) = (dp_j)/(-\frac{1}{3}p_j), \quad j = 1, 3, \quad (4.14)$$

and so

$$u_i = q_i t^{-2/3} \quad \text{and} \quad v_i = p_i t^{1/3}, \quad i = 1, 3. \quad (4.15)$$

From (4.15) and the equations of motion we have

$$\frac{du_i}{du_j} = \frac{3v_i - 2u_i}{3v_j - 2u_j} \quad \text{and} \quad \frac{dv_i}{du_j} = \frac{v_i - 3\mu_i P^{-3}}{3v_j - 2u_j}, \quad (4.16)$$

where

$$P^2 = u_k u_k. \quad (4.17)$$

There are three possible solutions to equations (4.16), the components of the Runge-Lenz vector,

$$R_i = u_i v_j v_j - u_j v_i v_j - \mu u_i / P = q_i p_j p_j - q_j p_i p_j - \mu q_i / r. \quad (4.18)$$

That the energy and angular momentum cannot be solutions may be verified by allowing Y_2 to act on each and noting that the result is non-zero.

The associated Lagrange system for Y_1 is

$$dt/1 = dq_i/0 = dp_i/0, \quad i = 1, 3, \quad (4.19)$$

and so

$$u_i = q_i \quad \text{and} \quad v_i = p_i, \quad i = 1, 3. \quad (4.20)$$

From (4.2) and the equations of motion we have the relations

$$du_j/du_i = v_j/v_i, \quad dv_j/dv_i = u_j/u_i \quad \text{and} \quad dv_j/du_i = -(\mu u_j)/(r^3 v_i). \quad (4.21)$$

The first and second of these may be combined to give

$$d(u_i v_j - u_j v_i) = 0, \quad (4.22)$$

which yields the conserved angular momentum tensor

$$L_{ij} = q_i p_j - q_j p_i. \quad (4.23)$$

Using the third of (4.21) in diagonal form and summing we have

$$v_i dv_i - \mu d(1/r) = 0, \quad (4.24)$$

which integrates to give the conserved energy

$$E = H = \frac{1}{2} \mathbf{p}^2 - \mu/r. \quad (4.25)$$

Now we take the third of (4.21) in its general form, multiply by $u_{(i/j)}$ and sum on the repeated index to obtain

$$u_i v_i dv_j - \mu u_j d(1/r) = 0 \quad \text{and} \quad u_j v_i dv_i - \mu u_i d(1/r) = 0, \quad (4.26)$$

which may be combined to give

$$u_i d(v_j v_j) - u_j v_j dv_i - \mu u_i d(1/r) = 0. \quad (4.27)$$

Employing the first and third of equation (4.21), this may be rewritten as

$$d(u_i v_i v_j - u_j v_i v_i - \mu u_i/r) = 0, \quad (4.28)$$

which integrates immediately to the components of the Runge-Lenz vector (4.18).

5. Comment

In this short note we have illustrated the use of the method of the Lie theory of extended groups in the context of Hamilton's equations of motion. We have demonstrated that the method may be applied in a self-contained way and yield results consistent with those obtained by the more usual application of the method to the Newtonian equations of motion. This consistency is achieved by restricting the choice of infinitesimal transformations to point transformations. The restriction is not unreasonable since it is well-known from the Lie theory of differential equations that unrestricted transformations will lead to an infinite number of generators which cannot be determined systematically.

The two problems which were chosen for this demonstration are well-known for their properties and their usefulness in illustrating theory in mechanics. The oscillator is the paradigm of linear systems and the Kepler problem the simplest of the important non-linear systems. In using these well-worked problems we have, even still, been able to make a point which, when made, is obvious, but has not, to our knowledge, been made. The classical invariants which are used to

determine the equations of the orbits, energy, angular momentum and Jauch-Hill-Fradkin tensor/Runge-Lenz vector, may be found, in the Lie approach, from the generator of time translations. Since, in Hamiltonian mechanics, the Hamiltonian is the generator of time translations for time-independent quantities, any such quantity which has zero Poisson bracket with the Hamiltonian is an invariant under time translation. That the same invariant arises from more than one generator is not exceptional as, in the case of the free particle in one-dimension, eight generators give rise to only three linearly independent invariants when the Lie method is used.

The Lie method yields more than Noether's theorem, under the same type of permitted transformations, for two reasons. The first is simply the possibility of obtaining more generators. The second is that, for a given generator, the equations of motion enter into the determination of possible invariants. We saw, especially in dealing with the $\partial/\partial t$ generator, that this permits manipulations which are not possible with Noether's theorem since, for the latter, the relationship between generator and invariant is predetermined.

In examining a particular problem with a view to establishing its orbit characteristics, we are not interested in every possible invariant, but only those which are of use in describing the orbit. Of the many invariants associated with the harmonic oscillator only the energy, angular momentum and Jauch-Hill-Fradkin tensor are used for this purpose. For the Kepler problem it happens that all the invariants found are of this type and this may well be a feature of non-linear systems. It has been noted elsewhere that linear systems are special [9]. The Lie method does involve some further manipulation which could, in some problems, depend upon the ingenuity of the investigator to determine the invariants. The beauty of Noether's theorem is that the theory does not require further ingenuity once the generators are known. The failure of Noether's theorem is that it does not produce all the known orbit-determining invariants. If Noether's theorem could be suitably generalized so that all of these invariants are available, its utility would be greatly enhanced. For reasons already mentioned, velocity-dependent transformations are not suitable. The question is this, is there some other way to generalize the theorem to produce the desired results?

Appendix

In the Introduction, the use of Noether's theorem with point transformations was criticized on the grounds that it did not give rise to all of the known first integrals. In particular, for the problems under consideration here, Noether's theorem does not yield the Jauch-Hill-Fradkin tensor for the oscillator on the

Runge–Lenz vector for the Kepler problem. Those first integrals may be obtained if velocity-dependent transformations are admitted, the latter being given as an example by Levy–Le Blond [4] and also in the text by Saletan and Cromer [8]. However, the reader will recall that the introduction of velocity-dependent transformations was also criticized in the Introduction as it gave rise to an infinite number of generators of symmetry transformations for which there is no systematic method of determination. The apparent contradiction in the last two sentences is readily resolved. In the paper and text cited, the authors do not derive their generators, but simply quote them and demonstrate that the Runge–Lenz vector follows. The concern of the present paper is the determination of generators. The problems of finding velocity-dependent transformations are demonstrated explicitly below.

Suppose that the Action Integral

$$J = \int_{t_0}^{t_1} \{ \mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t) \} dt \quad (\text{A1})$$

is invariant under the infinitesimal transformation generated by

$$Y(\mathbf{q}, \mathbf{p}, t) = \xi(\mathbf{q}, \mathbf{p}, t) \partial/\partial t + \eta(\mathbf{q}, \mathbf{p}, t) \cdot \partial/\partial \mathbf{q} + \zeta(\mathbf{q}, \mathbf{p}, t) \cdot \partial/\partial \mathbf{p}. \quad (\text{A2})$$

Then, from Noether's theorem, there exists a first integral

$$I(\mathbf{q}, \mathbf{p}, t) = \mathbf{p} \cdot \eta(\mathbf{q}, \mathbf{p}, t) - H(\mathbf{q}, \mathbf{p}, t)\xi(\mathbf{q}, \mathbf{p}, t) - f(\mathbf{q}, t), \quad (\text{A3})$$

where $f(\mathbf{q}, t)$ is the gauge-variant contribution. Note that the exclusion of \mathbf{p} from f does not indicate any loss of generality. To determine the generator, (A3) is differentiated with respect to time and set equal to zero. This gives rise to the equation

$$\begin{aligned} \mathbf{p} \cdot \left(\frac{\partial \eta}{\partial t} + [\eta, H]_{PB} \right) - \frac{\partial H}{\partial \mathbf{q}} \cdot \eta - \frac{\partial H}{\partial t} \xi - H \left(\frac{\partial \xi}{\partial t} + [\xi, H]_{PB} \right) \\ - \frac{\partial f}{\partial t} - [f, H]_{PB} = 0. \end{aligned} \quad (\text{A4})$$

In view of the fact that ξ and η have been assumed to contain \mathbf{p} , (A4) cannot be separated by coefficients of powers of p_i as before. This is the basis for the remark made above that the generators cannot be determined systematically when velocity (momentum)-transformations are admitted.

It is, of course, possible to postulate forms for $\xi(\mathbf{q}, \mathbf{p}, t)$ and $\eta(\mathbf{q}, \mathbf{p}, t)$ as polynomials in \mathbf{p} with undetermined coefficients, substitute these and determine the coefficients. However, this procedure provides no guarantee of completeness in the set of operators. In the case of the Kepler problem, for example, it may be verified by direct substitution in (A4) that an operator is the generator of a symmetry transformation, which is the case of the example given by Saletan and Cromer ([8], page 82).

It will be recalled that the use of velocity dependent transformations was also criticized on the grounds that there exist in infinite number of them. This is a well-known consequence of Lie's theory of differential equations, but may be demonstrated in a very trivial fashion. Suppose that it has been shown that

$$Y(\mathbf{q}, \mathbf{p}, t) = \xi(\mathbf{q}, \mathbf{p}, t) \frac{\partial}{\partial t} + \eta(\mathbf{q}, \mathbf{p}, t) \cdot \frac{\partial}{\partial \mathbf{q}} + \zeta(\mathbf{q}, \mathbf{p}, t) \cdot \frac{\partial}{\partial \mathbf{p}} \quad (\text{A5})$$

is a generator of a symmetry transformation. Then, if $G(I)$ is an arbitrary function of invariants (denoted generically by I),

$$\tilde{Y}(\mathbf{q}, \mathbf{p}, t) = G(I) Y(\mathbf{q}, \mathbf{p}, t) \quad (\text{A6})$$

is also a generator. This follows from direct substitution in (A4). As an infinite number of such functions $G(I)$ exists, there is an infinite number of velocity-dependent generators. For, even if the original Y were the generator of a point transformation, its derivatives \tilde{Y} must be velocity-dependent since first integrals are functions of \mathbf{p} as well as of \mathbf{q} (and t).

In this Appendix it has been shown that Noether's theorem does have disadvantages when velocity-dependent transformations are admitted. The basic cause is very practical. How are generators to be found if (A4) cannot be solved? This difficulty with Noether's theorem is compounded by the fact that, without velocity-dependent transformations, it provides an incomplete set. For those two reasons it is suggested that the method of the Lie theory of extended groups, based on point transformations, is more satisfactory.

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An exact invariant for a class of time-dependent anharmonic oscillators with cubic anharmonicity

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Journal of Mathematical Physics **22** (1981) 465–470

An exact invariant for a class of time-dependent anharmonic oscillators with cubic anharmonicity

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(Received 23 October 1979; accepted for publication 29 February 1980)

An exact invariant is constructed for a class of time-dependent anharmonic oscillators using the method of the Lie theory of extended groups. The presence of the anharmonic term imposes a constraint on the nature of the time dependence. For a sub-class it is possible to obtain an energy-like integral and a condition under which the motion is bounded.

PACS numbers: 03.20. + i

1. INTRODUCTION

In the investigation of the behavior of plasma one of the models which was early adopted was that of the motion of a charged particle in an axially symmetric field. An advantage of such a model was that, under suitable approximations, the radial equation reduced to that of the simple harmonic oscillator. One could not ask for a better model as far as the resulting mathematics was concerned. However, the Zeta machine and its like did not work and a more refined model was needed. This led to the equation of the time-dependent harmonic oscillator to take into account time-varying fields. This oscillator system had attracted earlier attention, primarily as an approximation to the lengthening pendulum. Unfortunately, unlike the time-independent oscillator, there was no known exact invariant for the time-dependent oscillator.

That deficiency was overcome by Lewis¹ using a method based on Kruskal's scheme.² The Lewis invariant has attracted a considerable amount of attention from a variety of viewpoints.³⁻⁵ We were able to offer a simple derivation and interpretation of the invariant⁶ and to provide invariants for similar linear systems.⁷ Indeed each member of the whole class of quadratic Hamiltonians was shown to be equivalent to any other member to within a (time-dependent) linear canonical transformation. The application of these results to quantum mechanics was begun by Lewis and Riesenfeld⁸ and extended by us.⁹ Naturally for quantum mechanics the class of permissible linear transformations is restricted, but, for a time-dependent quadratic Hamiltonian of constant signature, there would appear to be no difficulty. In fact the quantum mechanical results for time-dependent linear transformations closely parallel those for the time-independent case as reported by Wolf and others.¹⁰

We have not heard of the Zeta machine for many years, but it appears as if its successors require an even better model than that of the time-dependent harmonic oscillator. It was suggested¹¹ that a time-dependent anharmonic oscillator with cubic anharmonicity in the Hamiltonian would be suitable as a starting point and that an invariant for such a system could be informative. Assuming that a nonlinear system required nonlinear transformations, we proposed a scheme for constructing an invariant related to the Hamiltonian by means of such a transformation.¹² In general it was anticipated that the transformation would be an infinite series and probably divergent as well. However, in celestial mechanics

truncation of similar series has been used with some success, for instance in the work of Gustavson¹³ on the notorious Hénon-Heiles problem.

To provide the solution to a problem in the form of an infinite series is sometimes acceptable (as in the case of the ordinary oscillator), but generally speaking it may be regarded as less than satisfactory. The whole advantage of the Lewis invariant is that it is concise, easy to work and has a precise "physical" interpretation. We became convinced that useful transformations must nearly always be linear although the opposite viewpoint has recently been advanced by Maharatna, Dutt, and Chattarji.¹⁴ It is our opinion that the results obtained here support our viewpoint.

It is evident that in general a time-dependent problem will not possess an invariant. This is not surprising. The point is to be developed elsewhere. It will be seen in the case of the problem discussed here that the determination and interpretation of such invariants rely on point transformations of the type $t \rightarrow T, q \rightarrow Q$. In the Hamiltonian context this means a linear canonical transformation coupled with a change of time scale. That such should be the case is fortunate for the results may readily be extended to quantum mechanics. Indeed the linear transformation belongs to the class of transformations for which the Schrödinger wavefunctions are related by means of a geometric transform rather than the more general integral transform.^{10a}

Apart from the context of Hamilton-Jacobi theory, time-dependent transformations have not much been used until recently. Nevertheless we do express some surprise that the methods employed here have not been adopted generally. Specifically, the theories of Lie and Noether have been around for a considerable time and yet, to our knowledge, have only been applied, in a context similar to the present one, in recent years. Many of the ideas employed in this paper have been developed in earlier papers to which reference is made when appropriate.

2. THE PROBLEM

As reported above, it has been suggested that a better model for the motion of a charged particle in an axial field should include allowance for anharmonicity. Let us be more generous and allow velocity dependent damping and a coordinate free forcing term. Whether the addition of such terms is helpful to the model is unknown to us and we leave that matter to the physicists. All we wish to do here is to

provide the maximum possible flexibility for the model.

The Newtonian equation of motion for such a particle may be written as

$$\ddot{q} + a(t)\dot{q} + b(t)q + c(t)q^2 + d(t) = 0. \quad (2.1)$$

The time-dependent parameters a , b , c , and d are not specified as to properties, but are assumed to be as good as the occasion requires. Following Caldriola and Kanai,¹⁵ we use an integrating factor to construct a Lagrangian which is

$$L = \frac{1}{2}\dot{q}^2 A - (\frac{1}{2}bq^2 + \frac{1}{3}cq^3 + dq)A, \quad (2.2)$$

where

$$A = A(t) = \exp \int_{t_0}^t a(t') dt'. \quad (2.3)$$

The conjugate momentum is

$$p = qA \quad (2.4)$$

and the Hamiltonian is

$$H = \frac{1}{2}p^2 A^{-1} + (\frac{1}{2}bq^2 + \frac{1}{3}cq^3 + dq)A. \quad (2.5)$$

Under the change of time scale given by

$$T = \int_{t_0}^t A^{-1}(t') dt', \quad (2.6)$$

the problem may be discussed in terms of the equivalent system with Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}bq^2 + \frac{1}{3}cq^3 + dq. \quad (2.7)$$

Note that under the succession of transformations used here, it is the form of the expression for the Hamiltonian rather than the precise relationship of one symbol to the corresponding preceding one which is of interest. The alternative normal procedure would require a new set of symbols for each equation and naturally would be followed in a practical application. Under the translation

$$q \rightarrow q + r(t), \quad p \rightarrow p + s(t), \quad (2.8)$$

we obtain a new Hamiltonian of form

$$H = \frac{1}{2}p^2 + \frac{1}{2}bq^2 + \frac{1}{3}cq^3 + g(t), \quad (2.9)$$

provided, in terms of the coefficients of Eq. (2.7),

$$\ddot{r} + br - cr^2 - d = 0, \quad s = \dot{r}. \quad (2.10)$$

As far as Eq. (2.9) is concerned, we may ignore the garbage term $g(t)$ as it plays no role in the equations of motion. It should be noted that in quantum mechanics $g(t)$ would appear in the phase.

Apart from the cubic term, Eq. (2.9) is now in time-dependent oscillator form and so the appropriate transformation is (cf. Ref. 6)

$$Q = \rho^{-1}q, \quad P = \rho p - \dot{\rho}q, \quad (2.11)$$

where $\rho(t)$ satisfies the auxiliary equation

$$\ddot{\rho} + b\rho = \rho^{-3}. \quad (2.12)$$

The Hamiltonian now takes the form

$$H = \rho^{-2} \{ \frac{1}{2}P^2 + \frac{1}{2}Q^2 + \frac{1}{3}cQ^3 \}, \quad (2.13)$$

which, under the change of time scale

$$T = \int_{t_0}^t \rho^{-2}(t') dt', \quad (2.14)$$

becomes

$$H = \frac{1}{2}P^2 + \frac{1}{2}Q^2 + \frac{1}{3}BQ^3. \quad (2.15)$$

Thus we see that all Newtonian equations of motion of the form of Eq. (2.1) may be discussed in terms of the Newtonian equation

$$\ddot{q} + q + B(t)q^2 = 0, \quad (2.16)$$

provided the original linear coefficient is sufficient to warrant the final positive sign. In terms of the physical model this restriction is reasonable. We are looking at a physical situation which does involve attraction to the first order. Otherwise kinking is inevitable.

3. CHOICE OF APPROACH

The initial problem has been reduced to a discussion of the three (equivalent) alternative forms, respectively, the Hamiltonian, the Lagrangian, and the Newtonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{1}{3}Bq^3, \quad (3.1)$$

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2 - \frac{1}{3}Bq^3, \quad (3.2)$$

$$N = \ddot{q} + q + Bq^2 = 0. \quad (3.3)$$

Considering that we have possible quantum mechanical applications in mind, we must pose the question: Upon which of Eqs. (3.1), (3.2), or (3.3) do we base our analysis? The Hamiltonian is most closely related to quantum mechanics. However, our experience of applying transformations to nonlinear Hamiltonians has not been happy.^{12,16} The Lagrangian is not far removed from the Hamiltonian and is susceptible to treatment by the generalized Noether's theorem. Furthermore any invariant found in this way has a corresponding Hamiltonian invariant.¹⁷ Also the invariants are easily found using Noether's theorem as there is an explicit formula. On the other hand, the Newtonian equation of motion is the most generous when it comes to providing invariants derived by the method of the Lie theory of extended groups.¹⁸

The approach chosen here is that of the Lie theory of extended groups. Those who are familiar with the method will know that the price of generality is a more awkward determination of invariants *vis à vis* the Noether method. This drawback may be alleviated by the use of point transformations which we have discussed elsewhere.¹⁹ This procedure is particularly effective when there is only one possible invariant. Essentially a new time and coordinate system is found in which the invariant is a function of coordinate and velocity only. The transformation is one of time scale and linear in the coordinate. This relates very well to the use of change of time scale and linear canonical transformation in the Hamiltonian formalism. It will be particularly attractive if the invariant (in the new coordinates) is energy-like since this will have useful quantum mechanical applications. If it is not, then at least we will have an invariant.

4. THE METHOD OF THE LIE THEORY OF EXTENDED GROUPS

Although the Lie theory of extended groups has enjoyed a rightful resurgence of attention recently,²⁰ it may not be familiar to all readers. Accordingly we provide a brief

resumé relevant to Newtonian equations of motion. Suppose there exists a transformation with generator

$$G = \xi(q,t) \frac{\partial}{\partial t} + \eta(q,t) \frac{\partial}{\partial q}. \quad (4.1)$$

(We restrict ourselves to the one-dimensional problem.) If under the transformation generated by this operator a Newtonian equation of motion is to remain invariant, then, writing the equation as

$$N(\ddot{q}, \dot{q}, q, t) = 0, \quad (4.2)$$

we require

$$G^{(2)}N(\ddot{q}, \dot{q}, q, t) = 0, \quad (4.3)$$

whenever Eq. (4.2) is satisfied. The second extension of G , denoted by $G^{(2)}$ is given by

$$G^{(2)} = G + \eta^{(1)} \frac{\partial}{\partial \dot{q}} + \eta^{(2)} \frac{\partial}{\partial \ddot{q}}, \quad (4.4)$$

$$\eta^{(1)} = \dot{\eta} - \xi \dot{q}, \quad \eta^{(2)} = \ddot{\eta} - \xi \ddot{q} - 2\dot{\xi} \dot{q}. \quad (4.5)$$

Presuming that Eq. (4.3) has non-trivial solutions for ξ and η , a constant of the motion may be found by imposing the double requirement that

$$G^{(1)}\Phi(\dot{q}, q, t) = 0, \quad (4.6)$$

$$D\Phi(\dot{q}, q, t) = 0, \quad (4.7)$$

where $D \equiv d/dt$.

In the analysis of certain linear systems²¹ it became obvious that the task of finding the G 's, let alone solving Eqs. (4.6) and (4.7), was complicated. It had been observed that linear systems of the same dimension had the same symmetry group, $Sl(n+2, R)$.²² There was also the Hamiltonian result about the equivalence of linear systems under linear canonical transformations. This suggested that a point transformation of the type $t \rightarrow T, q \rightarrow Q$ would simplify matters. In quantum mechanics a time-independent energy-type invariant is desirable²³ and in classical mechanics it is most useful. The generator for such a constant is

$$G = \frac{\partial}{\partial t}, \quad (4.8)$$

provided of course that the appropriate space-time frame of reference is used. Supposed that in solving Eq. (4.3) a generator of the form

$$G = f(t) \frac{\partial}{\partial t} + \{g(t)q + h(t)\} \frac{\partial}{\partial q} \quad (4.9)$$

is obtained. All one needs to do is to change to new space and time variables, Q and T , by means of transformation linear in Q such that now

$$G = \partial/\partial T \quad (4.10)$$

and an invariant independent of the new time may be obtained. As a function of Q and \dot{Q} it may be an energy type integral, in which case it has suitable quantum mechanical features. If the invariant is not of that form, the quantum mechanical applications are not so obvious, but at least an invariant does exist. We note that it has been found that, in the case for which there are several generators, a common transformation reduces them to a commonly simpler form.

This is the case with linear systems.²⁴ For nonlinear systems the existence of several generators does not usually occur. Indeed it should be mentioned in passing that even in the instance of integrable nonlinear systems there may only be one generator.²⁵ This seems strange as an integrable system has two constants of integration for a one-dimensional system. It is hoped to discuss this point at another time.

To conclude this section we make a brief summary of the generalized Noether's theorem. Suppose a transformation with generator G leaves the action integral invariant, with G as defined as in Eq. (4.1). Then there exists an invariant given by

$$\Phi(\dot{q}, q, t) = -\{\xi L + (\eta - \xi \dot{q})(\partial L / \partial \dot{q}) + f(q, t)\}, \quad (4.11)$$

where $f(q, t)$ is determined along with ξ and η .

5. A USEFUL RESULT

As a result, which is useful in this work and which does not appear to have been stated before, concerns the form of the generator G for the class of Newtonian equations of motion

$$N(\ddot{q}, q, t) \equiv \ddot{q} + g(q, t) = 0. \quad (5.1)$$

Under the requirement that

$$G^{(2)}N(\ddot{q}, q, t) = 0, \quad (5.2)$$

we have

$$\xi \frac{\partial g}{\partial t} + \eta \frac{\partial g}{\partial q} + (\ddot{\eta} - \xi \ddot{q} - 2\dot{\xi} \dot{q}) = 0. \quad (5.3)$$

Remembering that both ξ and η are functions of q and t only, we separate coefficients of powers of \dot{q} to obtain

$$\frac{\partial^2 \xi}{\partial q^2} = 0, \quad (5.4)$$

$$\frac{\partial^2 \eta}{\partial q^2} - 2 \frac{\partial^2 \xi}{\partial q \partial t} = 0, \quad (5.5)$$

$$2 \frac{\partial^2 \eta}{\partial q \partial t} - \frac{\partial^2 \xi}{\partial t^2} + 3g \frac{\partial \xi}{\partial q} = 0, \quad (5.6)$$

$$\frac{\partial^2 \eta}{\partial t^2} - g \frac{\partial \eta}{\partial q} + 2g \frac{\partial \xi}{\partial t} + \xi \frac{\partial g}{\partial t} + \eta \frac{\partial g}{\partial q} = 0. \quad (5.7)$$

From (5.4)

$$\xi = a(t) + b(t)q. \quad (5.8)$$

From (5.5)

$$\eta = b(t)q^2 + c(t)q + d(t). \quad (5.9)$$

Thus ξ is at most linear in q and η at most quadratic in q . We shall use these forms in the development below.

A similar result may be obtained for Noether's theorem. The Lagrangian corresponding to the Newtonian equation (5.1) is

$$L = \frac{1}{2}\dot{q}^2 - F(q, t), \quad F(q, t) = \int^q g(q', t) dq'. \quad (5.10)$$

The Noether invariant (4.11) has the form

$$\Phi(\dot{q}, q, t) = \xi \left\{ \frac{1}{2}\dot{q}^2 + F(q, t) \right\} - \eta \dot{q} - f(q, t). \quad (5.11)$$

Taking the total time derivative of (5.11) and equating pow-

ers of \dot{q} to zero we obtain

$$\frac{\partial \xi}{\partial q} = 0, \quad (5.12)$$

$$\frac{1}{2} \frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial q} = 0, \quad (5.13)$$

$$F \frac{\partial \xi}{\partial q} - \frac{\partial \eta}{\partial t} - \frac{\partial f}{\partial q} = 0, \quad (5.14)$$

$$F \frac{\partial \xi}{\partial t} + \xi \frac{\partial F}{\partial t} + \eta g - \frac{\partial f}{\partial t} = 0. \quad (5.15)$$

From Eq. (5.12) we have

$$\xi(q, t) = a(t), \quad (5.16)$$

and using this in Eq. (5.13),

$$\eta(q, t) = \frac{1}{2} \dot{a}(t) q + b(t). \quad (5.17)$$

From Eq. (5.14) we find that $f(q, t)$ has the form

$$f(q, t) = -\frac{1}{4} \ddot{a}(t) q^2 - \dot{b}(t) q. \quad (5.18)$$

The arbitrary function of time from the integration of Eq. (5.14) is neglected. The possible expressions for $a(t)$ and $b(t)$ are then found from Eq. (5.15). At this point we simply note that the Noether's theorem generators are simpler in general form to those obtained using the Lie theory. A comparison of the expressions for ξ and η might tempt some to identify $c(t)$ in (5.9) with $\frac{1}{2} \dot{a}(t)$. As will be seen below, this is not the case.

6. APPLICATION OF THE LIE THEORY TO THE ANHARMONIC OSCILLATOR

The reduced problem is the existence of the invariants for the Newtonian equation of motion

$$\ddot{q} + q + B(t)q^2 = 0. \quad (6.1)$$

From Sec. 5 we know that the generator is of the form

$$G(q, t) = (a + bq)\partial/\partial t + (bq^2 + cq + d)\partial/\partial q, \quad (6.2)$$

only $\eta^{(2)}$ is required and it is

$$\begin{aligned} \eta^{(2)} &= \ddot{\eta} - \dot{\xi}\dot{q} - 2\xi\ddot{q} \\ &= \ddot{b}q^2 + 3\dot{b}q\dot{q} - 3bq\ddot{q} \\ &\quad + \ddot{c}q + 2\dot{c}\dot{q} + c\ddot{q} + \ddot{d} - \dot{q}\ddot{a} - 2\dot{q}\dot{a}. \end{aligned} \quad (6.3)$$

Substituting into

$$G^{(2)}(\ddot{q} + q + Bq^2) = 0, \quad (6.4)$$

and equating powers of \dot{q} to zero, we find that

$$b = 0, \quad (6.5)$$

$$\ddot{d} + d = 0, \quad (6.6)$$

$$\ddot{c} + 2\dot{a} + 2Bd = 0, \quad (6.7)$$

$$a\dot{B} + Bc + 2\dot{a}B = 0, \quad (6.8)$$

$$2\dot{c} - \dot{a} = 0. \quad (6.9)$$

From Eqs. (6.6-9) it follows that

$$d = D \sin t + E \cos t, \quad (6.10)$$

$$2c = \dot{a} + \alpha, \quad (6.11)$$

$$B = Ka^{-5/2} \exp\left[-\frac{1}{2}\alpha \int dt'/a(t')\right], \quad (6.12)$$

where α , K , D , and E are constants and $a(t)$ satisfies the third

order equation

$$\ddot{a} + 4\dot{a} + 2Bd = 0. \quad (6.13)$$

Thus the generator of a one parameter symmetry group for Eq. (6.1) is

$$G(q, t) = a\partial/\partial t + \left\{\frac{1}{2}(\dot{a} + \alpha)q + d\right\}\partial/\partial q, \quad (6.14)$$

provided $B(t)$ takes the form specified in Eq. (6.12). In view of the result for Noether's theorem, a generator for a transformation which preserves the action is

$$G(q, t) = a\partial/\partial t + (\frac{1}{2}\dot{a}q + d)\partial/\partial q, \quad (6.15)$$

i.e., α must be taken as zero.

From Eq. (6.13) we see that for $d \neq 0$, there is a considerable increase in the complexity of the differential equation defining a . From Eq. (6.12) the range of functions $B(t)$ is increased for $\alpha \neq 0$. However, this is at the price of removing the resulting invariant from the Noether class. As far as ease of manipulation is concerned, clearly the case $d = 0$, $\alpha = 0$ is the simplest. As to whether the non-Noetherian case $d = 0$, $\alpha \neq 0$ is simpler than $d \neq 0$, $\alpha = 0$, it is not so easy to judge except when it comes to obtaining explicit expressions. The general case $d \neq 0$, $\alpha \neq 0$ is clearly the most complex. Because the case $\alpha \neq 0$ is non-Noetherian we would expect some qualitative differences in the invariant which will probably have direct bearing on possible quantum mechanical applications. For the moment we shall examine the general case and then discuss the particular cases in turn.

7. DEFINING EQUATION FOR THE INVARIANT

In Sec. 4 we suggested that the process of finding the invariant would be simplified by a transformation to a new space-time coordinate system. We define the transformation as

$$T = f(t), \quad Q = g(t)q + h(t). \quad (7.1)$$

By imposing the requirement that $G(q, t)$ as given in Eq. (6.15) take the form

$$G(Q, T) = \partial/\partial T, \quad (7.2)$$

we see that

$$af' = 1, \quad ag' + \frac{1}{2}(\dot{a} + \alpha)g = 0, \quad ah' + dg = 0, \quad (7.3)$$

so that the parameters of the transformation are given by

$$f(t) = \int a^{-1}(t') dt', \quad (7.4)$$

$$g(t) = a^{-1/2}(t) \exp(-\frac{1}{2}\alpha f(t)), \quad (7.5)$$

$$h(t) = -\int d(t') a^{-3/2}(t') \exp(-\frac{1}{2}\alpha f(t')) dt'. \quad (7.6)$$

Applying the transformation to the Newtonian equation of motion (6.1), it now takes the time-independent form

$$Q'' + \alpha Q' + KQ^2 + MQ + N = 0, \quad (7.7)$$

where the constants M and N are given by

$$M = \frac{1}{2}\{a\ddot{a} - \frac{1}{2}(\dot{a} + \alpha)^2 + \alpha(\dot{a} + \alpha) + 2a^2 - 4Kh\}, \quad (7.8)$$

$$N = h(M + Kh) + g\{\frac{1}{2}(\dot{a} - \alpha)d - ad\}, \quad (7.9)$$

and the prime represents differentiation with respect to T . The constancy of the expressions on the right-hand sides of

Eqs. (7.9) and (7.10) may be confirmed by direct differentiation and the use of the differential equations for the various parameters.

In the normal way of finding the invariant corresponding to an operator G , we would solve the Eqs. (4.6) and (4.7). However, as Eq. (7.8) is independent of T and the invariant is a function of Q and Q' , we rewrite it as

$$Q' \frac{dQ'}{dQ} + \alpha Q' + KQ^2 + MQ + N = 0. \quad (7.10)$$

The invariant is then obtained by quadrature.

For the case $\alpha = 0$, we immediately obtain the energy-like integral

$$\Phi(Q', Q) = \frac{1}{2}Q'^2 + \frac{1}{3}KQ^3 + \frac{1}{2}MQ^2 + NQ. \quad (7.11)$$

This is a Noetherian invariant and so has a corresponding Hamiltonian form. We now examine the problem from the Hamiltonian viewpoint.

8. HAMILTONIAN VIEWPOINT

The Hamiltonian corresponding to the original Newtonian is

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{1}{3}Bq^3. \quad (8.1)$$

The transformation which reduces the Newtonian equation to a time-independent form involved both a change of time scale and a linear transformation of the coordinate. In the Hamiltonian context such a transformation is accomplished in two stages. The first involves a linear canonical transformation which removes the time dependence to a multiplicative factor. The second is a change of time scale so that the Hamiltonian is now the invariant. Thus under the transformations we expect the Hamiltonian to become

$$\bar{H}(Q, P, T) = \frac{1}{2}P^2 + \frac{1}{3}KQ^3 + \frac{1}{2}MQ^2 + NQ, \quad (8.2)$$

which is the Hamiltonian version of Eq. (7.12). We now verify this result.

The form of the linear transformation may be inferred from Eqs. (7.1) and (7.3) (with $\alpha = 0$) and the fact, implicit in Eq. (8.2), that $Q' = P$. Thus

$$Q = qg + h, \quad P = agq + agp - dg, \quad (8.3)$$

where

$$g = a^{-1/2}, \quad h = \int dg/a. \quad (8.4)$$

The type two generating function is

$$F_2(q, P, t) = gPq + \dot{a}q^2/4a + dq/a + hP. \quad (8.5)$$

Then

$$H'(Q, P, t) = H(q, p, t) + \partial F_2(q, P, t) / \partial t \\ = a^{-1} \{ \frac{1}{2}P^2 + \frac{1}{3}KQ^3 + \frac{1}{2}MQ^2 + NQ \}. \quad (8.6)$$

The change of time scale, $T = \int a^{-1}(t') dt'$, yields the Hamiltonian (8.2).

We emphasize that an invariant of the form given in Eq. (8.2) exists only in the case $\alpha = 0$, i.e., for functions $B(t)$ in Eq. (8.1) given by

$$B(t) = Ka^{-5/2}, \quad (8.7)$$

where a is a solution of the nonlinear equation

$$\ddot{a} + 4\dot{a} + 2Ka^{-5/2}(D \sin t + E \cos t) = 0. \quad (8.8)$$

In view of the form of Eq. (8.8) it is rather fortunate that the foregoing analysis may be performed without an explicit knowledge of $a(t)$. In terms of the original coordinates the invariant is

$$I(q, \dot{q}, t) \\ = \frac{1}{2}a\dot{q}^2 - \frac{1}{2}a\dot{q}\dot{q} + \frac{1}{3}Bq^3/a + (\frac{1}{2}a + \ddot{a}/a)q^2 - d\dot{q} \\ + \{ gh(a\ddot{a} - \frac{1}{2}\dot{a}^2 + 2a^2 - 2Kh) + (a\dot{d} - \dot{a}d)/a \} q \\ + d^2/2a + 4Kh^3/3 + 3Mh^2/2 + gh(\dot{a}d/2 - a\dot{d}). \quad (8.9)$$

For the simpler case when d is zero, this reduces to

$$I(q, \dot{q}, t) = a \{ \frac{1}{2}\dot{q}^2 + \frac{1}{3}q^2 + \frac{1}{3}Bq^3 \} - \frac{1}{2}a\dot{q}\dot{q} + \ddot{a}q^2/a, \quad (8.10)$$

which is not overly complicated.

9. DISCUSSION

We have seen that an invariant may be obtained for certain time-dependent anharmonic systems. From the invariant we may obtain information regarding the boundedness of the motion of the particle. To illustrate this, we consider the simplest case $\alpha \equiv 0, d \equiv 0$. Then

$$a(t) = A + B \sin 2t + C \cos 2t, \quad (9.1)$$

$$B(t) = K(A + B \sin 2t + C \cos 2t)^{-5/2}, \quad (9.2)$$

$$M = A^2 - B^2 - C^2. \quad (9.3)$$

By requiring that $B(t)$ be finite and real we have

$$A > (B^2 + C^2)^{1/2}, \quad M > 0.$$

In this particular instance, the transformation from (q, p) to (Q, P) is

$$Q = gq, \quad P = agq + agp, \quad (9.4)$$

so that, when $p = 0$, the invariant is

$$I = \frac{1}{3}KQ^3 + \frac{1}{2}(M + \frac{1}{4}\dot{a}^2)Q^2. \quad (9.5)$$

The values of I for which this cubic has three real distinct roots may be obtained from the discriminant of Cardan's formula.²⁶ We find that

$$I < (M + \frac{1}{4}\dot{a}^2)^3/6K^2 \quad (9.6)$$

and, since $\frac{1}{2}\dot{a} (= B \cos 2t - C \sin 2t)$ may be zero, the motion will be bounded for

$$I < M^3/6K^2. \quad (9.7)$$

A similar analysis may be performed for the case $d \neq 0, \alpha \equiv 0$, but the result is more complicated due to the extra terms. When $\alpha \neq 0$, the stability of the motion is more difficult to determine because of the nature of the differential equation (7.11) determining the invariant. However, for particular cases it would be amenable to numerical treatment.

The value of the results obtained here depends upon the type of time variation of the field found in the experimental situation. If this time variation is one of the types allowed by the theory for an exact invariant, it would be most gratifying. If it is not, it may be possible to use the results obtained here as an approximation to find bounds within which the motion will remain.

ACKNOWLEDGMENTS

We wish to thank Ralph Lewis of Los Alamos for originally suggesting this problem, Willy Sarlet of Ghent for a preprint of a paper on invariants for certain nonlinear systems,²⁷ Geoffrey Prince and Professor C.J. Eliezer of La Trobe for stimulating conversations on the Lie method.

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A further note on the Hénon-Heiles problem

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Journal of Mathematical Physics **22** (1981) 679-682

A further note on the Hénon–Heiles problem

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(Received 12 November 1979; accepted for publication 14 March 1980)

The method of the Lie theory of extended groups is applied to the Hénon–Heiles problem. Only one generator for a one-parameter group is found. The corresponding first integral is the energy. It is inferred that no other exact integral exists.

PACS numbers: 03.20 + i

I. INTRODUCTION

The Hénon–Heiles problem¹ concerns the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3 \quad (1.1)$$

($p_1 = \dot{q}_1, p_2 = \dot{q}_2$). It has been posed as a model for the motion of a galactic cluster. Computer analysis of the problem has suggested that for sufficiently small values of the energy, there exists a first integral independent of the energy. This has been termed the third integral (the first being the energy, the second being the angular momentum of the total system of which the Hamiltonian above is part). The tantalizing suggestiveness of the numerical results has led to much effort to find the third integral. It would be fair to say that progress has not been great and that the problem remains intractable. Indeed the term “notorious” has been applied to the problem,² a comment which doubtless stems from the sense of frustration produced.

In this note another method of attack is employed, that of the Lie theory of extended groups. Recently it has been used with considerable success on linear systems³ and with some success on a nonlinear time-dependent system.⁴ In the latter instance one of the chief results is that the possible existence of an invariant is determined by the nature of the time dependence.⁵

Before commencing the analysis, we give a brief summary of the method. Given a system of Newtonian equations of motion

$$\mathbf{N}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, t) = \mathbf{0}, \quad (1.2)$$

the system admits a one-parameter Lie group with generator

$$G(\mathbf{q}, t) = \xi(\mathbf{q}, t) \partial / \partial t + \boldsymbol{\eta}(\mathbf{q}, t) \cdot \nabla_{\mathbf{q}} \quad (1.3)$$

provided

$$G^{(2)}\mathbf{N} = \mathbf{0} \quad (1.4)$$

whenever Eq. (1.2) is satisfied. $G^{(2)}$ is the second extension of G and is given by

$$G^{(2)} = G + \boldsymbol{\eta}^{(1)} \cdot \nabla_{\dot{\mathbf{q}}} + \boldsymbol{\eta}^{(2)} \cdot \nabla_{\ddot{\mathbf{q}}}, \quad (1.5)$$

where

$$\boldsymbol{\eta}^{(1)} = \dot{\boldsymbol{\eta}} - \dot{\xi} \dot{\mathbf{q}}, \quad \boldsymbol{\eta}^{(2)} = \ddot{\boldsymbol{\eta}} - \ddot{\xi} \dot{\mathbf{q}} - 2\dot{\xi} \ddot{\mathbf{q}}. \quad (1.6)$$

If such a generator exists, there exists a corresponding first integral, $I(\mathbf{q}, \dot{\mathbf{q}}, t)$, which is constructed by applying the double requirement that

$$G^{(1)}I = 0, \quad DI = 0, \quad (1.7)$$

where D represents the total time derivative. If more than

one generator of a one-parameter Lie group exists, there may be more than one first integral, although it does not follow automatically. As a trivial counterexample, the one-dimensional free particle has eight linearly independent generators; yet only three linearly independent first integrals are obtained.

We mention that the Lie theory is more general than Noether's theorem⁶ in its conception. The generators for the latter constitute a subset of those of the former. This distinction applies to nonlinear as well as to linear systems. An excellent example of this distinction is seen in the treatment of the classical Kepler problem by Prince and Eliezer.⁷

2. THE FORM OF THE GENERATORS

Applying the Lie method leads to sufficient complexity in the case of one-dimensional systems, let alone in systems of higher order, to warrant the determination of the permissible form of the generators for a given type of Newtonian equation before considering a particular problem. Suppose the system has Newtonian equations of the form

$$\ddot{q}_l + f_l(q, t) = 0, \quad l = 1, n. \quad (2.1)$$

Adopting the usual convention of summation on repeated indices, the twice-extended generator is

$$G^{(2)} = \xi \partial / \partial t + \eta_i \partial / \partial q_i + (\dot{\eta}_i - \dot{\xi} \dot{q}_i) \partial / \partial \dot{q}_i + (\ddot{\eta}_i - \ddot{\xi} \dot{q}_i - 2\dot{\xi} \ddot{q}_i) \partial / \partial \ddot{q}_i. \quad (2.2)$$

Applying this to Eq. (2.1) and separating out the terms which are of second and third order in the velocities, we have

$$\dot{q}_i \dot{q}_j \partial^2 \xi / \partial q_i \partial q_j = 0 \quad (2.3)$$

$$\dot{q}_i \dot{q}_j \partial^2 \eta_i / \partial q_i \partial q_j - 2\dot{q}_i \dot{q}_j \partial^2 \xi / \partial q_i \partial t = 0. \quad (2.4)$$

From Eq. (2.3) it is apparent that

$$\xi(\mathbf{q}, t) = a(t) + b_i(t) q_i. \quad (2.5)$$

Substituting this into Eq. (2.4), we have

$$\dot{q}_i \dot{q}_j \partial^2 \eta_i / \partial q_m \partial q_n = 2\dot{q}_i \dot{q}_j \dot{b}_i. \quad (2.6)$$

Differentiating with respect to \dot{q}_m and \dot{q}_n in turn and assuming that η_i is sufficiently regular for the order of differentiation to be immaterial, we have

$$\partial^2 \eta_i / \partial q_m \partial q_n = \delta_{im} \dot{b}_n + \delta_{in} \dot{b}_m. \quad (2.7)$$

It then follows that

$$\eta_i(\mathbf{q}, t) = \dot{b}_k q_k q_i + c_{ik}(t) q_k + d_i(t). \quad (2.8)$$

Thus for a system of Newtonian equations of the type given

by Eq. (2.1), a generator of one-parameter Lie group has the form

$$G(q,t) = (a + b_k q_k) \partial / \partial t + (b_k q_k q_l + c_{lk} q_k + d_l) \partial / \partial q_l, \quad (2.9)$$

where $a, b, c,$ and d are functions of time to be determined according to the particular functions $f_l(q,t)$.

3. EQUATIONS DETERMINING THE TIME-DEPENDENT FUNCTIONS

As the algebra involved in the determination of the functions $a, b, c,$ and d tends to be messy no matter what the system may be, it is just as easy to consider a general two-dimensional system of which the Hénon–Heiles problem is a particular case. We take the Hamiltonian to be

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + Aq_1^3 + Bq_1^2 q_2 + Cq_1 q_2^2 + Dq_2^3, \quad (3.1)$$

the Hénon–Heiles case being given by $A = 0 = C, B = 1, D = -\frac{1}{3}$. The two Newtonian equations corresponding to Eq. (3.1) may be written as

$$\ddot{q}_l + q_l + f_{mn}{}^l q_m q_n = 0, \quad l = 1, 2, \quad (3.2)$$

where $f_{mn}{}^l$ is symmetric in the indices m and n and the repeated indices are summed now over 1 and 2 only. Applying the second extension of the operator given by Eq. (2.9) to Eq. (3.2), the coefficients of the terms of second and third order in the velocities vanish identically. The terms linear in the velocities are

$$3\ddot{b}_k(\dot{q}_k q_l + q_k \dot{q}_l) + 2\dot{c}_{lk} \dot{q}_k - \ddot{a} \dot{q}_l - b_k \dot{q}_l (q_k + f_{mn}{}^k q_m q_n) - 2b_k \dot{q}_k (q_l + f_{mn}{}^l q_m q_n) = 0. \quad (3.3)$$

From the coefficients of the second order terms in the displacements, it is obvious that

$$b_1 \equiv 0, \quad b_2 \equiv 0. \quad (3.4)$$

From the terms now remaining, it follows that

$$2c_{ij} = \dot{a} \delta_{ij} + \alpha_{ij} \quad (3.5)$$

where the four α_{ij} are as yet arbitrary constants.

Turning now to be velocity independent terms, those independent of the coordinates yield

$$\dot{d}_l + d_l = 0, \quad (3.6)$$

those linear in the coordinates give

$$2d_m q_n f_{mn}{}^l + \dot{c}_{lm} q_m + 2\dot{a} q_l = 0, \quad (3.7)$$

and the second order terms are

$$2\dot{a} q_m q_n f_{mn}{}^l - c_{lk} q_m q_n f_{mn}{}^k + c_{mk} q_k q_n f_{mn}{}^l + c_{nk} q_k q_m f_{mn}{}^l = 0. \quad (3.8)$$

Differentiating Eq. (3.8) with respect to q_i and q_j in succession and making use of Eq. (3.5),

$$5\dot{a} f_{ij}{}^l + \alpha_{kl} f_{kj}{}^l + \alpha_{kl} f_{ki}{}^l - \alpha_{lk} f_{ij}{}^k = 0. \quad (3.9)$$

4. SOME POSSIBLE GENERATORS

From Eq. (3.6) it is evident that

$$d_l = E_l \sin t + F_l \cos t, \quad l = 1, 2. \quad (4.1)$$

Substituting for $c(t)$ in Eq. (3.7), four equations result, viz.

$$\ddot{a} + 4\dot{a} + 4d_m f_{m1}{}^1 = 0, \quad (4.2)$$

$$\ddot{a} + 4\dot{a} + 4d_m f_{m2}{}^2 = 0, \quad (4.3)$$

$$d_m f_{m2}{}^1 = 0, \quad d_m f_{m1}{}^2 = 0. \quad (4.4)$$

The pair of equations in (4.4) are identical and, in terms of the coefficients in Eq. (3.1), are

$$Bd_1 + Cd_2 = 0. \quad (4.5)$$

A consistency condition between Eqs. (4.2) and (4.3) (in the case $\dot{a}(t) \neq 0$) requires

$$(3A - C)d_1 + (B - 3D)d_2 = 0. \quad (4.6)$$

If $\dot{a}(t) \equiv 0$, Eqs. (4.2) and (4.3) require that

$$3Ad_1 + Bd_2 = 0, \quad Cd_1 + 3Dd_2 = 0. \quad (4.7)$$

For the moment let us confine our attention to the case for which $a(t)$ is a constant. There exist three relations, Eqs. (4.5) and (4.7), between d_1 and d_2 . For these to be consistent $A, B, C,$ and D are related by

$$B^2 = 3AC, \quad C^2 = 3BD, \quad BC = 9AD. \quad (4.8)$$

To within a scaling constant, possible values which the coefficients may take are

$$A \quad B \quad C \quad D \quad (4.9a)$$

$$1 \quad 0 \quad 0 \quad 0 \quad (4.9b)$$

$$0 \quad 0 \quad 0 \quad 1 \quad (4.9c)$$

$$1 \quad \pm 3x \quad 3x^2 \quad \pm x^3 \quad (4.9d)$$

$$-1 \quad \pm 3x \quad -3x^2 \quad \pm x^3 \quad (4.9e)$$

where x is a positive constant. The generators for such systems are

$$G_1 = \partial / \partial t, \quad (4.10)$$

corresponding to $a(t)$ constant and

$$G_2 = \sin t \partial / \partial q_2, \quad G_3 = \cos t \partial / \partial q_2$$

$$G_2 = \sin t \partial / \partial q_1, \quad G_3 = \cos t \partial / \partial q_1$$

$$G_2 = \sin t (\pm x \partial / \partial q_1 - \partial / \partial q_2),$$

$$G_3 = \cos t (\pm x \partial / \partial q_1 - \partial / \partial q_2)$$

$$G_2 = \sin t (\pm x \partial / \partial q_1 + \partial / \partial q_2),$$

$$G_3 = \cos t (\pm x \partial / \partial q_1 + \partial / \partial q_2), \quad (4.11)$$

corresponding to (4.9a) to (4.9d), respectively.

First integrals for the systems above are easily constructed by using the result that if I is a first integral, then so also is $G^{(1)}I$. Taking the first of (4.9), the energy is

$$E = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + q_1^2 + q_2^2) + Aq_1^3, \quad (4.12)$$

$$I_1 = G_2^{(1)}E = q_2 \sin t + \dot{q}_2 \cos t, \quad (4.13)$$

$$I_2 = G_3^{(1)}E = q_2 \cos t - \dot{q}_2 \sin t. \quad (4.14)$$

In the case of the third of (4.9),

$$E = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + q_1^2 + q_2^2) + A(q_1 \pm xq_2)^3, \quad (4.15)$$

$$I_1 = \sin t (\pm xq_1 - q_2) + \cos t (\pm x\dot{q}_1 - \dot{q}_2), \quad (4.16)$$

$$I_2 = \cos t (\pm xq_1 - q_2) - \sin t (\pm x\dot{q}_1 - \dot{q}_2), \quad (4.17)$$

The resemblance of Eqs. (4.16) and (4.17) to Eqs. (4.13) and

(4.14) is not accidental. Consider the transformation

$$\bar{t} = t \quad Q_1 = q_1 \pm xq_2, \quad Q_2 = \pm xq_1 + q_2. \quad (4.18)$$

Under this transformation the generators become

$$G_1 = \partial/\partial t, \quad (4.19)$$

$$G_2 = -(1+x^2)\sin t \partial/\partial Q_2,$$

$$G_3 = -(1+x^2)\cos t \partial/\partial Q_2, \quad (4.20)$$

and the energy is

$$E = \frac{1}{2}(1+x^2)(\dot{Q}_1^2 + \dot{Q}_2^2 + Q_1^2 + Q_2^2) + AQ_1^3. \quad (4.21)$$

The transformation has separated the system into two uncoupled parts so that four linearly independent first integrals exist. The integrals linear in the coordinate and velocity correspond to the harmonic oscillator part of Eq. (4.21). They represent $Q_2(0)$ and $\dot{Q}_2(0)$. That similar integrals do not exist for Q_1 is not surprising in view of the quadrature required to express Q_1 as a function of time.

The generators discussed above for the case when $a(t)$ is a constant do not apply to the Hénon-Heiles problem since the coefficients for that problem do not fit in with the scheme in (4.9). Allowing $a(t)$ to be not constant is of no use for the Hénon-Heiles problem as the consistency condition of Eq. (4.6) is not satisfied. Moreover, returning to Eq. (3.9), the admissible time-dependent forms of $a(t)$, which involve sines and cosines [cf. Eq. (4.2)] can only occur when all the f_{mn}^l are zero. The last remaining source of a generator is to be found in the α part of $c(t)$. From Eq. (3.9) we see that the α 's must satisfy

$$\alpha_{kl}f_{kj}^l + \alpha_{kj}f_{ki}^l - \alpha_{ik}f_{ij}^k = 0. \quad (4.22)$$

In terms of the coefficients in Eq. (3.1) the conditions in equation (4.22) constitute the system of equations

$$\begin{bmatrix} 3A & -B & 2B & 0 \\ 0 & 3A-C & C & B \\ 2B & 0 & 2C-3A & -B \\ -C & 2B-3D & 0 & 2C \\ C & B & 3D-B & 0 \\ 0 & 2C & -C & 3D \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \end{bmatrix} = 0. \quad (4.23)$$

Unfortunately, when the Hénon-Heiles values $A = 0 = C$, $B = 1$, $D = -\frac{1}{3}$ are substituted, all of the α 's must be zero and so no integral can arise from this source.

For the possible cases listed in (4.9), α 's do exist. In corresponding order they are

$$\begin{array}{cccc} \alpha_{11} & \alpha_{12} & \alpha_{21} & \alpha_{22} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & \pm x & \pm x & 1 \\ 1 & \pm x & \pm x & 1 \end{array} \quad (4.24)$$

For example, the integral corresponding to the first case is

$$I_3 = t - \arctan q_2/\dot{q}_2. \quad (4.25)$$

Finally we note for other values of the coefficients A , B , C , and D , α -based generators do not exist.

5. DISCUSSION

We have seen that the Hénon-Heiles problem gives rise to only one one-parameter symmetry group. Consequently, any first integral of the motion must satisfy the requirement that

$$G^{(1)}(\mathbf{q}, \dot{\mathbf{q}}, t) I(\mathbf{q}, \dot{\mathbf{q}}, t) = 0, \quad (5.1)$$

where $G^{(1)}$ is the first extension of the generator G . In this case G is simply the generator of time translations and so

$$G^{(1)} = G = \partial/\partial t. \quad (5.2)$$

Thus the invariant satisfies the equation

$$\partial I(\mathbf{q}, \dot{\mathbf{q}}, t)/\partial t = 0. \quad (5.3)$$

In the formulation of this problem the velocity and momentum are identical, and we may rewrite (5.3) as

$$\partial I(\mathbf{q}, \mathbf{p}, t)/\partial t = 0, \quad (5.4)$$

from which it follows that I is a function of the canonical variables only. Since the invariant, now $I(\mathbf{q}, \mathbf{p})$, has zero total time-derivative,

$$[I, H]_{PB} = 0, \quad (5.5)$$

i.e., it has zero Poisson bracket with the Hamiltonian. We have already seen in an earlier paper⁸ that the only time-independent invariant with this property is the Hamiltonian itself.

Those who are familiar with the application of the Lie theory to linear systems will know that there exist generators for which the corresponding invariants contain time explicitly. In particular there are invariants corresponding to the initial conditions of the motion. Such invariants do not arise in the case of the Hénon-Heiles problem, nor is this lack of occurrence peculiar to it. A similar situation applied to the one-dimensional anharmonic oscillator⁴ and to the Kepler problem.⁹ We surmise that the apparent absence of initial condition type invariants for nonlinear systems may be due to nonlinear operations being required to invert expressions for variables to expressions for constants (of integration).

6. CONCLUSION

The Lie method has given a negative answer to the question of the existence of an integral other than the energy for the Hénon-Heiles problem. It is known that a formal integral exists for which various expansion techniques are available.¹⁰ The result of our investigations suggests that such series are indeed formal. The Lie method, being based on the Newtonian equations of motion, provides the largest possible set of generators of dynamical symmetries. For the Hénon-Heiles problem only one generator exists, and hence there is only one exact first integral, the energy (\equiv the Hamiltonian).

What then is to become of the Hénon-Heiles problem? In spite of the contrary evidence presented here, there is still the fact that the system does possess remarkable regularity at low energies. This suggests that the formal integral is a reasonable approximation for small enough values of the variables. It may be possible to use the idea of an approximate symmetry to construct a corresponding approximate integral. This is a matter for future investigation.

ACKNOWLEDGMENTS

I am indebted to G. E. Prince of La Trobe for stimulating discussions on the subject of the Lie method and for providing preprints of his work and to Professor A. Giorgilli of Milan, whose query regarding my earlier work on this problem⁸ prompted me to try again.

¹The literature is extensive. The original paper was M. Hénon and C. Heiles, *Astron. J.* **69**, 73–9 (1964). For a recent survey see R. C. Churchill, G. Pecelli, and D. L. Rod, "A survey of the Hénon–Heiles Hamiltonian with applications to related examples," in *Como Conference Proceedings on Stochastic Behaviour in Classical and Quantum Mechanical Systems* (Springer, New York, to appear). This contains a comprehensive set of references.

²T. Bountis, "Non-linear models in dynamics and statistical mechanics," Ph.D. thesis (University of Rochester, 1978).

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⁴P. G. L. Leach, "An exact invariant for a class of time-dependent anharmonic oscillators with cubic anharmonicity," Research Report AM79:10 (Department of Applied Mathematics, La Trobe University, 1979).

⁵The same point is made in a different context in W. Sarlet and L. Y. Bahar, "A direct construction of first integrals for certain non-linear dynamical systems," preprint (Department of Mechanical Engineering and Mechanics, Drexel University).

⁶The particular version referred to here is the one used for example by M. Lutzky, *J. Phys. A* **11**, 249–58 (1978) and not the one involving velocity-dependent transformations found for example in D. S. Djukic, *Arch. Mech. Stosow.* **26**, 243–9 (1974).

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⁹P. G. L. Leach, "Applications of the Lie theory of extended groups in Hamiltonian mechanics: the oscillator and the Kepler problem," Research Report AM80:01 (Department of Applied Mathematics, La Trobe University, 1980).

¹⁰A. Giorgilli and L. Galgani, *Celest. Mech.* **17**, 267–80 (1978) and the references cited therein. Also A. Giorgilli (private communication 18 June 1979).

First integrals versus configurational invariants and a weak form of complete integrability

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Physica 17D (1985) 87-98

FIRST INTEGRALS VERSUS CONFIGURATIONAL INVARIANTS AND A WEAK FORM OF COMPLETE INTEGRABILITY

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Received 18 October 1984

Revised 23 April 1985

A confusion over the concept of first integrals, which has been created in a recent paper by Hall [13] is clarified. The clear distinction between first integrals and functions which are first integrals only on a specific, fixed hypersurface is discussed. Hall's terminology of configurational invariants is adopted for the latter case. The possible relevance of knowing configurational invariants for a Hamiltonian system is illustrated by results concerning a weak form of the theory on complete integrability.

1. Introduction

First integrals, constants of the motion, conservation laws, exact invariants, configurational invariants; these are a number of different names for what is essentially the same thing. Mathematicians studying ordinary differential equations will always use the first name; physicists are more inclined to talk about constants of the motion or conservation laws, while people who have used one of the last two names wanted, in the first place, to distinguish from approximate, asymptotic or adiabatic invariants. The number of recent publications devoted to the search for, say, first integrals is enormous. Yet, do we always know what we are talking about?

Many of the recent contributions to the field consider Hamiltonian systems of the standard type $H = \frac{1}{2}\sum_i p_i^2 + V(q, t)$ and are directed towards

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identifying suitable potentials $V(q, t)$, for which it is possible to achieve an explicit construction of a first integral, usually of a prescribed polynomial nature in the p 's. A few references in that respect, which will enable the reader to track down many more, are: Lewis and Leach [1], Leach et al. [2], Grammaticos and Dorizzi [3], Ray and Reid [4], Katzin and Levine [5, 6], Leach [7]. When we slightly narrow the issue by considering potentials $V(q)$ which do not depend explicitly on time, we of course know a first integral from the outset, namely the energy integral. So the question becomes: what are the potentials for which a second first integral can be obtained? For two-degrees-of-freedom systems, this question was already discussed by Whittaker [8] and in fact goes back to Darboux [9], who first derived the partial differential equation which V must satisfy in order that a second integral, quadratic in the momenta, exists. The complete class of solutions of this partial differential equation has only recently been ob-

tained independently by Thompson [10], by Dorizzi et al. [11] and by Ankiewicz and Pask [12]. Hall [13] has also paid a great deal of attention to Whittaker's analysis. Unfortunately, he has confused the issue by claiming that Whittaker's discussion suffers from a conceptual error, which e.g. would imply that the results of refs. 10–12 are still incomplete. Hall introduced the terminology "configurational invariants" to distinguish them, as he said, from microscopic (adiabatic) invariants such as the magnetic moment for the motion of a charged particle. It is clear that he has no distinction in mind between configurational invariants and first integrals. His claim essentially is that in searching for a second integral of a conservative system, the momenta or velocities can no longer be treated as independent, constrained as they are by the already known energy integral. A similar confusion can be seen in papers by Xanthopoulos [14] and Fordy [15].

The first aim of the present paper is to clarify this issue. As we will argue in detail, there is certainly no error in Whittaker's approach. On the other hand, Hall's paper contains many interesting features. However, one simply must realize that his "configurational invariants" are generally not genuine first integrals of the given system. Instead, they are first integrals of the reduced system along a specific fixed energy surface. Once a precise mathematical meaning is given to this notion of configurational invariants, it remains valuable to study systems for which they exist. After all, systems for which more than one genuine first integral is known are not common, so it will often be useful to have some other tools which may provide qualitative insight into at least a subclass of possible motions. To enhance this point, we will develop in the second part of the paper a new theory concerning configurational invariants, which in fact is a weak form of the well-known complete integrability property for n degrees-of-freedom systems having n integrals in involution. Our approach will be classical and analytical in nature and it seems only just that we will follow as closely as possible Whittaker's proof of the standard com-

plete integrability theorem. A geometrical picture of the results will be presented in the last section.

2. First integrals and configurational invariants

To fix the ideas, let us consider a general system of second-order differential equations in \mathbb{R}^2 , written in the form

$$\begin{aligned}\ddot{x} &= f_1(x, y, \dot{x}, \dot{y}), \\ \ddot{y} &= f_2(x, y, \dot{x}, \dot{y}).\end{aligned}\tag{1}$$

All functions will tacitly be assumed to be sufficiently smooth in an open neighbourhood of a point of their domain. Suppose that

$$F(x, y, \dot{x}, \dot{y}) = E\tag{2}$$

is a first integral of (1) and recall that, by definition, this means

$$\dot{x}\partial F/\partial x + \dot{y}\partial F/\partial y + f_1\partial F/\partial \dot{x} + f_2\partial F/\partial \dot{y} \equiv 0\tag{3}$$

for all x, y, \dot{x} and \dot{y} . By means of such a first integral, the order of the given system can be reduced by one. Explicitly, if we assume that the relation (2) for example can be solved (locally) for \dot{y} , say

$$\dot{y} = \phi(x, y, \dot{x}, E),\tag{4}$$

then the 4th-order system (1) is reduced to the 3rd-order system

$$\begin{aligned}\ddot{x} &= f_1(x, y, \dot{x}, \phi(x, y, \dot{x}, E)) \\ &\triangleq \bar{f}_1(x, y, \dot{x}, E), \\ \dot{y} &= \phi(x, y, \dot{x}, E),\end{aligned}\tag{5}$$

which depends upon the parameter E .

Instead of looking for a second integral of (1), we could search for a first integral of the reduced system (5). Such an integral will generally depend on the parameter E and the defining relation, an

identity of the type (3), will be required to hold for all values of E in some open interval. With this correct interpretation, there is no distinction whatsoever between first integrals of (1) and first integrals of (5): the two concepts are in one-to-one correspondence. To be very specific, if

$$I(x, y, \dot{x}, E) = C \tag{6}$$

represents a first integral of (5), we have

$$\dot{x} \partial I / \partial x + \phi \partial I / \partial y + \dot{f}_1 \partial I / \partial \dot{x} \equiv 0, \tag{7}$$

for all x, y, \dot{x} and E . Defining a function $J(x, y, \dot{x}, \dot{y})$ by

$$J(x, y, \dot{x}, \dot{y}) = I(x, y, \dot{x}, F(x, y, \dot{x}, \dot{y})), \tag{8}$$

we obtain a first integral of the original system (1). Indeed, computing \dot{J} we obtain

$$\begin{aligned} \dot{J} = & \dot{x} \frac{\partial I}{\partial x}(x, y, \dot{x}, F(x, y, \dot{x}, \dot{y})) \\ & + \dot{y} \frac{\partial I}{\partial y}(-) + f_1 \frac{\partial I}{\partial \dot{x}}(-) + \dot{F} \frac{\partial I}{\partial E}(-), \end{aligned}$$

where the notation $(-)$ is used to avoid repetition of lengthy arguments. The last term is zero because F is a first integral and since (7) holds for all E , we may replace E by $F(x, y, \dot{x}, \dot{y})$; the resulting identity precisely says that the remaining terms in the expression for \dot{J} vanish. We leave it to the reader to verify explicitly that if $J(x, y, \dot{x}, \dot{y}) = K$ is a first integral of (1), then the function

$$I(x, y, \dot{x}, E) \triangleq J(x, y, \dot{x}, \phi(x, y, \dot{x}, E)) \tag{9}$$

yields a first integral of the reduced system (5).

The purpose of this rather elementary exercise is to exhibit unequivocally that in looking for genuine first integrals of (1) different from an a priori known integral like (2), it is not necessary to take account somehow of a constraint imposed by (2). Thus, contradicting Hall [13], there is absolutely no error in Whittaker's reasoning. This being said, one can very well contemplate studying the reduced system (5) along a specific fixed level surface

$E = E_0$. The point to realize then is that first integrals of the system

$$\begin{aligned} \ddot{x} &= \bar{f}_1(x, y, \dot{x}, E_0), \\ \dot{y} &= \phi(x, y, \dot{x}, E_0), \end{aligned} \tag{10}$$

where E_0 this time is a given number, are generally not first integrals of the original system (1). Even if we start off with a function depending on all phase space variables of (1) (and possibly also depending on time), whose restriction to a particular level surface $E = E_0$ is a first integral of (10), the given function need not be related to any first integral of (1). Yet, it may be a valuable instrument for the qualitative study of the dynamics associated with (1). We propose to adopt Hall's terminology for such functions, this time of course making them appear as being properly distinguished from first integrals of (1).

Definition. A function $J(x, y, \dot{x}, \dot{y}, t)$ is said to be a configurational invariant of (1) with respect to the first integral (2), if the restriction of J to some fixed surface $E = E_0$, i.e.

$$\bar{J}(x, y, \dot{x}, t, E_0) = J(x, y, \dot{x}, \phi(x, y, \dot{x}, E_0), t),$$

is a first integral of the corresponding reduced system (10).

We will frequently talk about configurational invariants without specifying the first integral with respect to which they are defined. It is evident that first integrals of (1) are configurational invariants but not vice versa.

A simple application is in order to clarify what preceeds, so let us consider, with Hall [13], the system

$$\begin{aligned} \ddot{x} &= \Omega \dot{y} - V_x, \\ \ddot{y} &= -\Omega \dot{x} - V_y, \end{aligned} \tag{11}$$

where Ω and V are functions of x and y , and subscripts are used here to denote partial derivatives. We first investigate the existence of genuine

first integrals of (11), which are linear in the velocities. Putting

$$J(x, y, \dot{x}, \dot{y}) = S(x, y)\dot{x} + T(x, y)\dot{y} + K(x, y), \quad (12)$$

we obtain

$$\dot{J} = S_x \dot{x}^2 + (S_y + T_x) \dot{x} \dot{y} + T_y \dot{y}^2 + (K_x - T\Omega) \dot{x} + (K_y + S\Omega) \dot{y} - (SV_x + TV_y).$$

In accordance with the definition, J will be a first integral provided we have

$$\begin{aligned} S_x = 0, \quad T_y = 0, \quad S_y + T_x = 0, \\ K_x - T\Omega = 0, \quad K_y + S\Omega = 0, \quad SV_x + TV_y = 0. \end{aligned} \quad (13)$$

These conditions are easily solved and imply that V and Ω must be arbitrary functions of the argument $u = c(x^2 + y^2) + ay + bx$, a , b and c being arbitrary constants.

Next, we seek conditions under which the same expression (12) for J will be a configurational invariant of (11) with respect to the energy integral. Following our definition, this means that the function

$$\begin{aligned} \bar{J}(x, y, \dot{x}, E_0) = S(x, y)\dot{x} + T(x, y) \\ \times \sqrt{2(E_0 - V) - \dot{x}^2} + K(x, y) \end{aligned} \quad (14)$$

should be a first integral of the corresponding reduced system for some fixed value of E_0 . Computing $\dot{\bar{J}}$ we find

$$\begin{aligned} \dot{\bar{J}} = \dot{x}(-\Omega T + K_x) + \dot{x} \sqrt{2(E_0 - V) - \dot{x}^2} (S_y + T_x) \\ + \dot{x}^2 (-T_y + S_x) + \sqrt{2(E_0 - V) - \dot{x}^2} (S\Omega + K_y) \\ + [-SV_x + 2(E_0 - V)T_y - TV_y]. \end{aligned}$$

If this is to be zero for all x , y and \dot{x} , one easily

deduces the following requirements:

$$\begin{aligned} S_y + T_x = 0, \quad S_x - T_y = 0, \\ K_x - \Omega T = 0, \quad K_y + S\Omega = 0, \\ -SV_x + 2(E_0 - V)T_y - TV_y = 0. \end{aligned} \quad (15)$$

These are precisely the conditions derived (in a slightly different way) by Hall. It is clear that they allow for a larger class of potentials. Hall has solved the conditions (15) quite skilfully. He further indicated how to investigate the existence of N th-degree polynomial (in the velocities) configurational invariants and pointed out that the highest degree coefficients will be the real and imaginary parts of an arbitrary complex analytic function. Similar observations can be seen in [14].

Note that if the conditions (15) would be required to hold for all values of E_0 , the last equation would imply $T_y = 0$, whereby the set of conditions (15) would reduce to the conditions (13) for a first integral. This is in agreement with our general observations at the beginning of this section.

As a final comment, Hall claimed that Whittaker's analysis failed to produce necessary and sufficient conditions for the existence of a second integral whereas this would be true for his results. He even appealed to the celebrated KAM theorem to make his point. However, since both (13) and (15) directly follow from interpreting the definition of the concept in question, there is no doubt that each set of conditions, within its own proper context, represents necessary and sufficient conditions. If the conditions (15) are used in looking for genuine first integrals of (11), then they are necessary, but not sufficient.

3. Some further illustrative examples

Consider the equations of motion for a general central force problem in polar coordinates,

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) = F(r), \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \end{aligned} \quad (16)$$

Through the angular momentum integrál

$$mr^2\dot{\theta} = l, \tag{17}$$

we get a reduction to the third-order system

$$\begin{aligned} \ddot{r} &= \frac{l^2}{m^2r^3} + \frac{F}{m}, \\ \dot{\theta} &= \frac{l}{mr^2}. \end{aligned} \tag{18}$$

Let us look at a fixed level surface $l = l_0$ and try to construct a configurational invariant of (16) with respect to the first integral (17). This is of course particularly simple here, since the decoupled r -equation for the reduced system (18) at $l = l_0$ produces its own energy integral

$$I = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l_0^2}{mr^2} + V(r).$$

Not surprisingly, moreover, the expression I happens to be a first integral of (18) for any value of l_0 , so that we are fortunate enough to recover an additional first integral of the original system (16), which in agreement with (8) reads

$$J = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r).$$

For a less trivial example, we go back to the application discussed in the preceding section and construct a particular solution of the conditions (15), which does not satisfy the requirements (13). Taking, for simplicity, $\Omega = 0$ and choosing $S = x$, $T = y$, the conditions (15) will be satisfied if we take $K = 0$ and let $W = V - E_0$ be any solution of the partial differential equation

$$xW_x + yW_y + 2W = 0.$$

The general solution of this equation is

$$W = x^{-2}f(y/x),$$

where the function f is arbitrary. To fix the ideas, we further take $E_0 = 0$. The system (11) we are

talking about thus reads

$$\begin{aligned} \ddot{x} &= x^{-3}[2f + yx^{-1}f'], \\ \ddot{y} &= -x^{-3}f', \end{aligned} \tag{19}$$

where the prime denotes differentiation of f with respect to its argument. The energy integral for (19) is given by

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + x^{-2}f, \tag{20}$$

and from (12), taking account of our choice for S and T , it is clear that a configurational invariant with respect to E is obtained as

$$J = x\dot{x} + y\dot{y}. \tag{21}$$

It is straightforward to verify that

$$\dot{J} = 2E, \tag{22}$$

which confirms that J is a first integral of the reduced system on the fixed energy surface $E = 0$. Needless to say, we are assuming here that the function f takes negative values in a suitable domain. Using J , we can reduce the system on $E = 0$ one step further to obtain the following second-order system:

$$\begin{aligned} \dot{x} &= (x^2 + y^2)^{-1}[Jx \pm yg(yx^{-1}, J)], \\ \dot{y} &= (x^2 + y^2)^{-1}[Jy \mp xg(yx^{-1}, J)], \end{aligned} \tag{23}$$

where $g(u, J) = [-2f(u)(1 + u^2) - J^2]^{1/2}$. At this point, we note that (23) actually is a gradient system, i.e. we have

$$\dot{x} = G_x, \quad \dot{y} = G_y, \tag{24}$$

where the function $G(x, y, J)$ is defined by

$$\begin{aligned} G(x, y, J) &= \frac{1}{2}J \ln(x^2 + y^2) \\ &\mp \int^{y/x} (1 + u^2)^{-1}g(u, J) du. \end{aligned} \tag{25}$$

Moreover, defining a new function $I(x, y, J)$ by

$$I(x, y, J) = \partial G / \partial J, \quad (26)$$

it is straightforward to check that I is a first integral of (23). The integration of the system (19), constrained to the fixed energy surface $E = 0$, thus is reduced to solving a single ordinary differential equation. For the example under consideration this is perhaps not surprising since (21) in fact produces the remaining constant through a further integration. The key to the further reduction in a more general context, however, turns out to be the fact that the Poisson bracket of the functions E and J has the following property:

$$[E, J] = -2E, \quad (27)$$

which we would like to call a weak involution property. Recall that we are dealing with a two degrees-of-freedom system (19) for which we originally dispose of two functions E and J , one being a first integral, the other one being a first integral on $E = 0$ only and that by (27) E and J appear to be in involution on $E = 0$. This is clearly reminiscent of the well-known situation of a completely integrable Hamiltonian system. Inspired by this example and following closely Whittaker's treatment of complete integrability [8, paragraph 148], we will develop in the next section a general theory concerning the above illustrated weak form of complete integrability, which will amply underscore the relevance of knowing configurational invariants.

4. A weak form of complete integrability

Consider a general Hamiltonian system with n degrees of freedom, governed by the Hamiltonian $H(q, p, t)$. Let $\Phi_n(q, p, t)$ be a first integral of our system, i.e. we have $\dot{\Phi}_n = 0$. As before, the dot will always stand for total time differentiation along the given equations of motion. We further assume that we know $n - 1$ configurational invariants with respect to Φ_n , denoted by $\Phi_\alpha(q, p, t)$, $\alpha = 1, \dots, n$

$- 1$, such that the n functions Φ_i , $i = 1, \dots, n$ are in weak involution. By this we mean

$$\dot{\Phi}_\alpha|_{\Phi_n=0} = 0, \quad \alpha = 1, \dots, n-1, \quad (28)$$

$$[\Phi_i, \Phi_j]|_{\Phi_n=0} = 0, \quad i, j = 1, \dots, n. \quad (29)$$

The fact that we are picking out here the level surface $\Phi_n = 0$ is just a matter of simplicity of notations and does not involve any loss of generality with respect to taking any other fixed surface $\Phi_n = \text{constant}$. Greek indices will always run from 1 to $n - 1$, latin indices are running from 1 to n and we will be using the summation convention over repeated indices. We now introduce n functions F_i by

$$F_\alpha = \dot{\Phi}_\alpha - a_\alpha, \quad F_n = \dot{\Phi}_n, \quad (30)$$

where the a_α are arbitrary constants. Assuming that the Jacobian $(\partial\Phi_i/\partial p_j) = (\partial F_i/\partial p_j)$ is non-singular in a suitable neighbourhood, we can locally solve the relations $F_i = 0$ for the momenta p_j , yielding

$$p_i = f_i(q, t, a_\alpha). \quad (31)$$

We finally put

$$G_i(q, p, t, a_\alpha) = p_i - f_i(q, t, a_\alpha). \quad (32)$$

As a result of our assumptions, we obviously have

$$\dot{F}_i|_{F_n=0} = 0,$$

and therefore a fortiori

$$\dot{F}_i|_{F=0} = 0, \quad (33)$$

where the notation $F = 0$ refers to $F_j = 0$ for all $j = 1, \dots, n$.

In a similar fashion, since $[F_i, F_j] = [\Phi_i, \Phi_j]$, we equally know that $[F_i, F_j]|_{F_n=0} = 0$ and hence

$$[F_i, F_j]|_{F=0} = 0, \quad i, j = 1, \dots, n. \quad (34)$$

Whittaker's treatment of the complete integrability theorem relies heavily on a kind of lemma, formulated in his paragraph 147, which in a free interpretation would state: if n functions, say F_i ,

are in involution then the corresponding functions G_j , constructed from the relations $F_i = 0$ as above, are also in involution. The point to realize now is that the starting assumption in this statement can be considerably weakened, without altering the conclusion. To be precise, if the F_i satisfy the weak involution property (34), then the G_i still are in (strong) involution. In fact, even the converse is true.

Lemma.

$$[F_i, F_j]|_{F=0} = 0 \Leftrightarrow [G_i, G_j] = 0. \quad (35)$$

Proof. On the one hand, a simple calculation shows that

$$[G_i, G_j] = 0 \Leftrightarrow \frac{\partial f_i}{\partial q_j} - \frac{\partial f_j}{\partial q_i} = 0. \quad (36)$$

On the other hand, we have

$$[F_i, F_j]|_{F=0} = \frac{\partial F_i}{\partial q_l}(q, f(q, t), t) \frac{\partial F_j}{\partial p_l}(q, f(q, t), t) - \frac{\partial F_i}{\partial p_l}(-) \frac{\partial F_j}{\partial q_l}(-)$$

(the possible presence of extra parameters like a_α in the functions f_i is of course totally irrelevant for these considerations). Computing the partial derivative with respect to q_l of the identities $F_i(q, f(q, t), t) \equiv 0$, we obtain

$$\frac{\partial F_i}{\partial q_l}(q, f(q, t), t) + \frac{\partial F_i}{\partial p_k}(-) \frac{\partial f_k}{\partial q_l} \equiv 0. \quad (37)$$

Using these to replace the $\partial F/\partial q$ -terms, the above expression straightforwardly reduces to

$$[F_i, F_j]|_{F=0} = \frac{\partial F_i}{\partial p_k}(q, f(q, t), t) \times \frac{\partial F_j}{\partial p_l}(-) \left(\frac{\partial f_l}{\partial q_k} - \frac{\partial f_k}{\partial q_l} \right). \quad (38)$$

In view of (36) and the non-singularity of the matrix $(\partial F_i/\partial p_k)$, the equivalence (35) now is evident.

A perhaps more obvious and somewhat similar equivalence is the following:

$$\dot{F}_i|_{F=0} = 0 \Leftrightarrow \dot{G}_i|_{G=0} = 0. \quad (39)$$

Indeed, we have

$$\dot{F}_i|_{F=0} = \left(\frac{\partial F_i}{\partial t} + \frac{\partial F_i}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial F_i}{\partial p_j} \frac{\partial H}{\partial q_j} \right) \Big|_{p=f(q,t)}$$

Using (37) and the analogous identity

$$\frac{\partial F_i}{\partial t}(q, f(q, t), t) + \frac{\partial F_i}{\partial p_j}(-) \frac{\partial f_j}{\partial t} \equiv 0, \quad (40)$$

this can be rewritten in the form

$$\dot{F}_i|_{F=0} = - \left[\frac{\partial F_i}{\partial p_j} \left(\frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial q_k} \frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial q_j} \right) \right] \Big|_{p=f(q,t)}$$

or

$$\dot{F}_i|_{F=0} = \frac{\partial F_i}{\partial p_j}(q, f(q, t), t) \cdot (\dot{G}_j|_{G=0}),$$

from which (39) follows.

We now return to our general setting in which the functions F_i are constructed out of one first integral and $n-1$ related configurational invariants (see (30)). Our functions f_i , as defined in (31) thus depend on $n-1$ constant parameters a_α . From (34), using our lemma, it follows that the $f_i(q, t, a)$ satisfy (36). In addition, from (33), in view of the equivalence (39), we have

$$\frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial q_k} \frac{\partial H}{\partial p_k}(q, f(q, t, a), t) + \frac{\partial H}{\partial q_j}(q, f(q, t, a), t) = 0.$$

Using (36) again, this can be rewritten as

$$\frac{\partial f_j}{\partial t} + \frac{\partial H_1}{\partial q_j} = 0, \quad (41)$$

where the function $H_1(q, t, a)$ is defined by

$$H_1(q, t, a) = H(q, f(q, t, a), t). \quad (42)$$

The relations (36) and (41) imply the existence of a function $W(q, t, a)$ such that

$$f_j = \frac{\partial W}{\partial q_j}, \quad H_1 = -\frac{\partial W}{\partial t}. \quad (43)$$

Finally, we introduce $n - 1$ new functions $I_\alpha(q, t, a)$, defined by

$$I_\alpha(q, t, a) = \frac{\partial W}{\partial a_\alpha}. \quad (44)$$

Proposition.

$$\dot{I}_\alpha|_{F=0} = 0. \quad (45)$$

Proof.

$$\begin{aligned} \dot{I}_\alpha|_{F=0} &= \frac{\partial I_\alpha}{\partial t} + \frac{\partial I_\alpha}{\partial q_k} \frac{\partial H}{\partial p_k}(q, f(q, t, a), t) \\ &= \frac{\partial^2 W}{\partial t \partial a_\alpha} + \frac{\partial^2 W}{\partial q_k \partial a_\alpha} \frac{\partial H}{\partial p_k}\left(q, \frac{\partial W}{\partial q}, t\right) \\ &= \frac{\partial}{\partial a_\alpha} \left(\frac{\partial W}{\partial t} + H_1 \right), \end{aligned}$$

which is zero in view of (43).

The above proposition already covers all properties we observed about the example (19) of the previous section. It is, however, of interest to relate all results back to functions defined on the original phase space. For that purpose it is useful that we enter into greater detail concerning the way the functions $f_i(q, t, a)$ can be obtained from the relations $F_j = 0$. Let us, for example, assume that the relation $\Phi_n = 0$ can be solved for one of the momenta which we denote by p_n , yielding

$$p_n = \phi_n(q, p_\beta, t). \quad (46)$$

Define next

$$\bar{\Phi}_\alpha(q, p_\beta, t) = \Phi_\alpha(q, p_\beta, \phi_n(q, p_\beta, t), t). \quad (47)$$

Solving the relations $\bar{\Phi}_\alpha - a_\alpha = 0$ for the variables p_β , we of course obtain

$$p_\beta = f_\beta(q, t, a_\alpha), \quad (48)$$

while the remaining function f_n finally is defined as

$$f_n(q, t, a_\alpha) = \phi_n(q, f_\beta(q, t, a_\alpha), t). \quad (49)$$

The considerations which will follow do not really depend on this particular way of constructing the f_i , but doing it this way seems most suitable to visualize what is happening.

As we have learned already in section 1, in calculations which involve a lot of substitutions of part of the variables in terms of new ones, it is often rather dangerous to be sloppy about the precise arguments appearing in each function. Therefore, we would like to be very explicit here concerning the following quite elementary observations. First of all, let $g(q, p, t)$ be any function in which we first substitute the relations (31) for the p 's, thus defining

$$\bar{g}(q, t, a) = g(q, f(q, t, a), t).$$

If we next define a function $\bar{\bar{g}}(q, p_\beta, t)$ by

$$\bar{\bar{g}}(q, p_\beta, t) = \bar{g}(q, t, \bar{\Phi}_\alpha(q, p_\beta, t)),$$

it is straightforward to verify explicitly that

$$\bar{\bar{g}}(q, p_\beta, t) = g(q, p, t)|_{p_n = \phi_n(q, p_\beta, t)} = g|_{\Phi_n = 0}. \quad (50)$$

Secondly, Φ_n being a first integral, we know that $\dot{\Phi}_n = 0$. The corresponding statement about the function $p_n - \phi_n(q, p_\beta, t)$ is

$$(\dot{p}_n - \dot{\phi}_n)|_{\Phi_n = 0} = 0. \quad (51)$$

Finally, using (51), it is easy to check that

$$\Phi_\alpha|_{\phi_n=0} = 0 \Leftrightarrow \bar{\Phi}_\alpha|_{\phi_n=0} = 0. \quad (52)$$

We now return to our proposition and proceed as follows. We define functions $\bar{\Psi}_\alpha$ by

$$\bar{\Psi}_\alpha(q, p_\beta, t) = I_\alpha(q, t, \bar{\Phi}_\gamma(q, p_\beta, t)). \quad (53)$$

The property (45) then implies

$$\dot{\bar{\Psi}}_\alpha|_{\phi_n=0} = 0. \quad (54)$$

Indeed, (45) means that

$$\frac{\partial I_\alpha}{\partial t} + \frac{\partial I_\alpha}{\partial q_k} \frac{\partial H}{\partial p_k}(q, f(q, t, a), t) \equiv 0,$$

for all q, t and a . In particular, we preserve an identity if we replace the a_α by $\bar{\Phi}_\alpha(q, p_\beta, t)$, yielding

$$\begin{aligned} & \frac{\partial I_\alpha}{\partial t}(q, t, \bar{\Phi}_\alpha(q, p_\beta, t)) \\ & + \frac{\partial I_\alpha}{\partial q_k}(-) \left(\frac{\partial H}{\partial p_k}(q, p, t) \right) \Big|_{p_n=\phi_n} \equiv 0, \end{aligned} \quad (55)$$

whereby the observation (50) about an arbitrary function f here has been applied to $\partial H/\partial p_k$. On the other hand, we have

$$\begin{aligned} \dot{\bar{\Psi}}_\alpha|_{\phi_n=0} &= \frac{\partial I_\alpha}{\partial t}(q, t, \bar{\Phi}_\alpha(q, p_\beta, t)) \\ & + \frac{\partial I_\alpha}{\partial q_k}(-) \frac{\partial H}{\partial p_k} \Big|_{p_n=\phi_n} + \frac{\partial I_\alpha}{\partial a_\beta}(-) \dot{\bar{\Phi}}_\beta \Big|_{\phi_n=0}. \end{aligned}$$

The statement (54) now follows from (55) and (52). Finally, to obtain functions which are defined at the same level as the $\Phi_\alpha(q, p, t)$, we set

$$\Psi_\alpha(q, p, t) = I_\alpha(q, t, \Phi_\gamma(q, p, t)). \quad (56)$$

It is obvious then that $\bar{\Psi}_\alpha$ is related to Ψ_α in exactly the same way as $\bar{\Phi}_\alpha$ is related to Φ_α . Hence, from (54) and making again use of the

equivalence (52), we conclude that also

$$\dot{\Psi}_\alpha|_{\phi_n=0} = 0, \quad (57)$$

in other words, the functions $\Psi_\alpha(q, p, t)$ provide us with $n-1$ additional configurational invariants. Summarizing what precedes, we have proved the following "weak complete integrability" theorem:

Theorem. Let $H(q, p, t)$ determine a Hamiltonian system with n degrees of freedom. Assume we know a first integral Φ_n and $n-1$ related configurational invariants Φ_α (in the sense of (28)), such that the functions Φ_α, Φ_n are in weak involution in the sense of (29). Then there exist $n-1$ further configurational invariants Ψ_β so that the integration of the system along the hypersurface $\Phi_n=0$ is reduced to solving a single first-order ordinary differential equation.

We end this section by giving an interpretation of these results in terms of a canonical transformation, which in fact will produce a simple, alternative proof of the above theorem and provide some additional insights. From (43) it is clear that $W(q, t, a_\alpha)$ is a solution of the Hamilton-Jacobi equation. It is, however, not a complete solution because there are only $n-1$ constants involved. Yet, we can use W to build up a canonical transformation to new variables (Q, P) , which is generated by the following generating function of F_2 -type:

$$F_2(q, P, t) = W(q, t, P_\alpha) + q_n P_n. \quad (58)$$

The transformation formulae then read

$$p_\beta = \frac{\partial F_2}{\partial q_\beta} = \frac{\partial W}{\partial q_\beta}(q, t, P_\alpha), \quad (59a)$$

$$p_n = \frac{\partial F_2}{\partial q_n} = \frac{\partial W}{\partial q_n}(q, t, P_\alpha) + P_n, \quad (59b)$$

$$Q_\beta = \frac{\partial F_2}{\partial P_\beta} = \frac{\partial W}{\partial P_\beta}(q, t, P_\alpha), \quad (59c)$$

$$Q_n = \frac{\partial F_2}{\partial P_n} = q_n. \quad (59d)$$

Assuming (as before) that the matrix $(\partial^2 W / \partial q_\beta \partial P_\alpha)$ is non-singular, we can express the old q -coordinates in terms of the new variables, from (59c,d). It is important to realize thereby that these expressions $q = q(Q, P_\alpha, t)$ will not depend on P_n . The new Hamiltonian $K(Q, P, t)$ explicitly reads

$$K = \left[H \left(q, \frac{\partial W}{\partial q_\beta}, \frac{\partial W}{\partial q_n} + P_n, t \right) + \frac{\partial W}{\partial t} \right] \Big|_{q=q(Q, P_\alpha, t)} \quad (60)$$

From the way we explicitly constructed (48) it is clear that, if we solve the relations (59a) for the P_α as functions of the old variables, we exactly recover the functions $\bar{\Phi}_\alpha(q, p_\beta, t)$. Substituting these functions into the right-hand side of (59c), it is equally obvious that the Q_α , in terms of the old variables, precisely reproduce the functions $\bar{\Psi}_\alpha$ (see (44) and (53)). Finally, substituting for P_α into (59b) and looking at (49), we learn that the hypersurface $\Phi_n = 0$ in the new coordinates is transformed into $P_n = 0$. We therefore are interested primarily in the transformed differential equations along $P_n = 0$. Now, the variable P_n only enters in one argument in the expression (60) for $K(Q, P, t)$ and by (43) it then is clear that

$$K(Q, P, t)|_{P_n=0} \equiv 0. \quad (61)$$

This implies

$$\dot{Q}_\alpha \Big|_{P_n=0} = \frac{\partial K}{\partial P_\alpha} \Big|_{P_n=0} = \frac{\partial}{\partial P_\alpha} (K|_{P_n=0}) = 0. \quad (62a)$$

and similarly

$$\dot{P}_\alpha \Big|_{P_n=0} = 0, \quad (62b)$$

$$\dot{P}_n \Big|_{P_n=0} = 0. \quad (62c)$$

Eqs. (62a), (62b) confirm that the functions $\bar{\Phi}_\alpha, \bar{\Psi}_\alpha$, or equivalently Φ_α, Ψ_α are configurational invariants with respect to Φ_n . Eq. (62c), in view of the observation made with (51), is in agreement with the fact that Φ_n is a first integral. The remain-

ing equation will precisely produce the reduced single differential equation mentioned in the theorem, and explicitly reads

$$\dot{Q}_n \Big|_{P_n=0} = \frac{\partial K}{\partial P_n} \Big|_{P_n=0} = \left[\frac{\partial H}{\partial p_n} \left(q, \frac{\partial W}{\partial q}, t \right) \right] \Big|_{q=q(Q, P_\alpha, t)}, \quad (62d)$$

whereby the variables (Q_α, P_α) in the right-hand side of (62d) are treated as constants.

5. Geometrical interpretation and discussion

It seems to us that most treatments of the reduction and integrability of Hamiltonian systems, also in the modern literature, are primarily concerned with the complete dynamics and deal with global symmetries and first integrals (see e.g. [16] or [17]). Our analysis of the preceding section is related to a reduction of only part of the dynamics, namely the restriction to an invariant submanifold $\Phi_n = 0$. Yet, the tools for sketching a geometrical picture of these results are certainly available in the literature, so let us briefly go into such an interpretation here.

We consider a time-dependent Hamiltonian system, described by a smooth function H on an extended phase space $T^*M \times \mathbb{R}$ with natural coordinates (q_i, p_i, t) . We further have n independent functions Φ_i at our disposal, satisfying the requirements formulated at the beginning of section 4. Passing to the so-called homogeneous formulation of the problem, we consider the manifold $T^*(M \times \mathbb{R}) \cong T^*M \times \mathbb{R} \times \mathbb{R}$, with coordinates (q_i, t, p_i, e) and whose canonical symplectic form reads $\Omega = dp_i \wedge dq_i + de \wedge dt$. In what follows, we will make no notational distinction between a function f , defined on $T^*M \times \mathbb{R}$, and the same f , regarded as a function on $T^*(M \times \mathbb{R})$. The relation $H + e = 0$ defines a $(2n+1)$ -dimensional co-isotropic submanifold N of $T^*(M \times \mathbb{R})$. The triplet $(T^*(M \times \mathbb{R}), \Omega, N)$ is called a homogeneous system (see e.g. [18]). The dynamics of the time-dependent Ham-

iltonian system can be identified with the characteristic distribution of $\Omega|_N$, more precisely, the integral curves of this distribution project onto the integral curves of the Hamiltonian system on $T^*M \times \mathbb{R}$. Using $\{, \}$ to denote Poisson brackets on $T^*(M \times \mathbb{R})$, the requirements on the functions Φ_i now read,

$$\begin{aligned} \{\Phi_n, H + e\} &= 0, \quad \{\Phi_\alpha, H + e\}|_{\Phi_n=0} = 0, \\ \{\Phi_i, \Phi_j\}|_{\Phi_n=0} &= 0. \end{aligned} \tag{63}$$

The interpretation of the regularity requirements, which were inherent in section 4, is as follows. The point $(a, 0, 0) \in \mathbb{R}^{n+1}$ is supposed to be a regular value of the mapping

$$\begin{aligned} \phi: T^*(M \times \mathbb{R}) &\rightarrow \mathbb{R}^{n+1}, \\ (q, t, p, e) &\mapsto (\Phi_i(q, p, t), H(q, p, t) + e), \end{aligned}$$

for each a in some open subset V of \mathbb{R}^{n-1} . It then follows from (63) that for each $a \in V$, $P_a = \phi^{-1}(a, 0, 0)$ is a Lagrangian submanifold of $(T^*(M \times \mathbb{R}), \Omega)$ (P_a is $n + 1$ -dimensional and $\Omega|_{P_a} = 0$), which is clearly embedded in N . Stated otherwise, P_a is a solution of the Hamilton–Jacobi problem associated with the homogeneous system $(T^*(M \times \mathbb{R}), \Omega, N)$ (see [18]). Locally, P_a has a generating function W_a and since $\text{Im}(dW_a)$ is contained in the submanifold N defined by $H + e = 0$, it follows that

$$H\left(q, \frac{\partial W_a}{\partial q}, t\right) + \frac{\partial W_a}{\partial t} = 0.$$

In this way, we recover the conclusion that the existence of functions Φ_i satisfying (63) implies the existence of a $(n - 1)$ -parameter solution of the Hamilton–Jacobi equation. The derivation of $n - 1$ further invariants I_α then proceeds as before. Note in passing that the same conclusions remain true if the first of conditions (63) is weakened to hold along $\Phi_n = 0$ only, i.e. Φ_n , instead of being a first integral, defines an invariant relation.

For another aspect of interest, let us go back to the example (19) of section 3. Whereas this exam-

ple was constructed in looking for potentials allowing for a configurational invariant linear in the velocities, the same example actually also belongs to the class of potentials for which a second integral, quadratic in the velocities exists (see e.g. [10]) and therefore is completely integrable. This is most easily illustrated as follows. The Hamiltonian for (19), in polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ reads

$$H = \frac{1}{2}(p_r^2 + r^{-2}p_\theta^2) + (r \cos \theta)^{-2}f(\tan \theta). \tag{64}$$

The Hamilton–Jacobi equation corresponding to (64) can be solved by separation of variables. Explicitly, the separation constant λ , which will provide the second integral, is given by

$$(p_r^2 - 2E)r^2 = \lambda = -p_\theta^2 - 2 \cos^{-2}\theta f(\tan \theta), \tag{65}$$

where E of course is the energy. Looking at the first of relations (65), we now recognize a simple relation between the two integrals E and λ and our configurational invariant J . Indeed, $J = x\dot{x} + y\dot{y} = rp_r$, and hence

$$\lambda = J^2 - 2Er^2. \tag{66}$$

We observe that the first integral λ reduces to the configurational invariant J^2 along $E = 0$. Moreover, whereas E and J^2 are in weak involution, E and λ are in (strong) involution. It may be an interesting question to study under what circumstances such an “extension” of configurational invariants is generally possible. The hard part in this question is the strong involution requirement on E and λ . As a matter of fact, going back to the general setting of section 4, one can show that it is locally always possible to find first integrals μ_α , which reduce to the configurational invariants Φ_α along $\Phi_n = 0$, but these μ_α will only satisfy the weak involution property. For the above example, $\mu = J - 2tE$ is a first integral of (19), reducing to the configurational invariant J along $E = 0$, but unlike E and λ , E and μ merely are in weak involution.

As a final remark, recall that our study grew out of the concern to clarify the confusion created by

the mixture of different concepts in [13]. We have insisted on being very explicit about the arguments appearing in each function throughout our analysis and this has everything to do with knowing at each step whether one is dealing with an equation or with an identity. That this is not a luxury may appear from another example of a confusing statement concerning first integrals. Recall that the function $W(q, t, a_\alpha)$, defined by (43) in fact provides an (incomplete) solution of the Hamilton–Jacobi equation. Gelfand and Fomin [19, pp. 90–91] state a theorem saying that the partial derivatives of W with respect to the available constants then provide first integrals of the system. Our proposition (45) in fact gives a more correct interpretation of this situation.

Acknowledgements

Most of this work was conceived while one of us (PGLL) was a visitor at the Instituut voor Theoretische Mechanica, Gent. The generous hospitality of the Instituut is gratefully acknowledged. The visit was made possible by a travel grant from the University of the Witwatersrand, Johannesburg and the Council for Scientific and Industrial Research of South Africa. The authors are indebted to Prof. R. Mertens for his continuing support.

Note added in proof

It has been brought to our attention that the distinction between first integrals and configurational invariants has also been made in a recent

paper by Sitaram and Varma [20], in which configurational invariants, which are there called “quasi-invariants”, are studied in relation with generalised killing vector fields.

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Quaestiones Mathematica 8 (1985) 241-274

Quaestiones Mathematicae

8(1985), 241-274

THE LINEAR SYMMETRIES OF A NONLINEAR DIFFERENTIAL EQUATION

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ABSTRACT

The Lie point symmetries of the non-linear differential equation $\ddot{q} + 3c\dot{q} + q^3 = 0$ which arises in the study of the modified Emden equation are shown to have an unexpectedly rich algebra associated with them, a feature which enables the above non-linear differential equation to be transformed into the simplest linear second order ordinary differential equation.

AMS Subject Classification: 34A34, 22E70

1. INTRODUCTION

The pioneering works of S. Lie [18] and E. Noether [22] have generated a considerable amount of interest over the years, more so recently. Although a vast amount of energy has been expended over the last several years in search of new methods to determine the symmetries and associated first integrals of both linear and non-linear dynamical systems (especially classical), the methods of the Lie theory of extended groups and Noether's theorem or a combination of these [20] are still often used.

The method of the Lie theory of extended groups is based on the invariance of the Newtonian equation of motion under infinitesimal transformation. More recently, the Lie method has been modified to apply to the Hamilton's equations of motion [23].

Noether's theorem (at least the version usually encountered) is based on the invariance of the Action functional (see for e.g. [19]) under infinitesimal transformation. Still other versions of this powerful and well established theorem appear in the literature. An excellent comparative review of these versions may be found in the paper by Sarlet and Catrijn [27] in which many additional references are quoted. It is worthwhile mentioning here, that Noether's theorem may also be applied in a Hamiltonian context [4].

If the class of admissible infinitesimal transformations is restricted to point transformations then for one-dimensional linear classical dynamical systems, Noether's theorem yields five generators of symmetry

whereas the Lie method provides eight. This is a serious limitation which is reflected in the solution of the one-dimensional simple harmonic oscillator in which some of the excluded generators are associated with the periodicity of the motion [28]. Moreover, for multi-dimensional systems, Noether's theorem does not provide the Jauch-Hill-Fradkin [7,6] tensor for the time-independent isotropic harmonic oscillator nor the Runge-Lenz [26,15] vector for the classical Kepler problem. The Lie method does [24,25]. Another distinct advantage of the Lie method is that it allows a more general class of problems to be treated since it does not require the existence of a Lagrangian. However, in spite of the above limitations, Noether's theorem does possess the beauty of having an explicit formula for the first integrals once the symmetry generators are known. The Lie method, on the other hand, requires further manipulation which could, in some problems (such as the one we are investigating), depend upon the ingenuity of the investigator to determine the first integrals.

When velocity-dependent transformations are admitted, the first integrals mentioned above do satisfy the equations obtained from Noether's theorem [16], but unfortunately, this velocity dependence results in an infinite number of symmetry generators which cannot be determined explicitly. No general method of determination exists although classes of first integrals classified by the nature of their velocity dependence may be systematically determined [8,13,27]. However the admittance of velocity-dependent transformations is not necessary for the Lie method since all the useful information may be obtained with point transformations only. Noether's theorem with point transformations has been criticized by Leach [11].

amongst others, precisely because of its inadequacy in determining the complete (finite) algebra of symmetries. This, to our mind, is not a serious deficiency if the principal aim of the researcher is in the determination of the first integrals.

One-dimensional linear systems, especially classical, have received a tremendous amount of attention over the years. Lie himself first obtained the generators for the free particle. Anderson and Davison [1] showed that the free particle and the simple harmonic oscillator both possessed the complete symmetry group $SL(3, R)$. The most widely discussed system viz. the harmonic oscillator and the simplest system viz. the free particle can both be considered to be paradigms of dynamical systems for obvious reasons.

One-dimensional non-linear systems have received little attention thus far. In the work presented here, we treat an interesting class of non-linear systems. Curiously the free particle is used as a vehicle for our discussion. This is done in an unconventional manner as we shall see later and the dichotomy of free particle system and nonlinear system elucidated.

We are interested in the following autonomous, non-linear system,

$$\ddot{q} + \frac{4n}{n+3} q \dot{q} + \frac{2(n^2-1)}{(n+3)^2} q^3 = 0 \quad , \quad n \neq -3 \quad , \quad (1.1)$$

which arises in the study of the modified Emden equation,

$$\ddot{x} + q(t)\dot{x} + x^n = 0 \quad (1.2)$$

by considering its first integrals. Of the various Emden-type equations, preference was given to equation (1.2) by Leach [14] and Moreira [21]. The former applied Noether's theorem with velocity-dependent transformation to equation (1.2) to obtain first integrals. Furthermore, it was shown [14,21] that equation (1.1) is of paramount importance in obtaining these first integrals. However, in this work we focus attention on some of the remarkable aspects of equation (1.1) which we believe have not been exploited before.

We treat the more general system

$$\ddot{q} + \alpha q \dot{q} + \beta q^3 = 0 \quad (1.3)$$

where α and β are as yet arbitrary. In doing so, we achieve a measure of elegance in our presentation, as well as in our calculation.

It is necessary to decide upon which method (Lie method or Noether's theorem) is to be used in the treatment of equation (1.3). Our choice is the Lie method. We could put (1.3) into a Lagrangian formulation and use Noether's theorem. However, as the reader will see below, it is the Lie algebraic features of (1.3) which are of paramount importance to our discussion.

2. CALCULATION OF THE GENERATORS USING THE METHOD OF THE LIE THEORY
OF EXTENDED GROUPS

In the study of the Lie theory of extended groups, relevant to Newtonian equations of motion, the one-parameter infinitesimal point transformations act on solution curves in (q, t) space, transforming solution curves into solution curves. This is equivalent to the requirement that the Newtonian equation of motion be form invariant under these transformations. For such a transformation, the group operator is

$$G(q, t) = \xi(q, t) \frac{\partial}{\partial t} + \eta(q, t) \frac{\partial}{\partial q}. \quad (2.1)$$

The extended group operator is also used. For a function involving the first derivative, the operator is

$$G^{(1)} = G + \eta^{(1)} \frac{\partial}{\partial \dot{q}} \quad (2.2)$$

and, for one involving the second derivative, it is

$$G^{(2)} = G^{(1)} + \eta^{(2)} \frac{\partial}{\partial \ddot{q}}, \quad (2.3)$$

where

$$\eta^{(1)} = \frac{d\eta}{dt} - \dot{q} \frac{d\xi}{dt}$$

$$\eta^{(2)} = \frac{d\eta^{(1)}}{dt} - \ddot{q} \frac{d\xi}{dt}$$

and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} + \ddot{q} \frac{\partial}{\partial \dot{q}}. \quad (2.4)$$

In particular, for the Newtonian equation of motion

$$N(\ddot{q}, \dot{q}, q, t) = 0, \quad (2.5)$$

the second extended operator (2.3) must be used. An operator G is said to be the generator of a one-parameter symmetry group for (2.5) if, whenever (2.5) is satisfied,

$$G^{(2)} N(\ddot{q}, \dot{q}, q, t) = 0 \quad (2.6)$$

equivalently

$$\xi \frac{\partial N}{\partial t} + \eta \frac{\partial N}{\partial q} + \eta^{(1)} \frac{\partial N}{\partial \dot{q}} + \eta^{(2)} \frac{\partial N}{\partial \ddot{q}} = 0. \quad (2.7)$$

More details on the Lie theory of extended groups may be found in Bluman and Cole [3].

For our purpose

$$N(\ddot{q}, \dot{q}, q) = \ddot{q} + \alpha q \dot{q} + \beta q^3 = 0. \quad (2.8)$$

Under the dual requirements (2.5) and (2.6) we have

$$\eta^{(2)} + \alpha \dot{q} \eta + 3\beta q^2 \dot{\eta} + \alpha q \dot{\eta} - \alpha q \dot{q} \dot{\xi} = 0 \quad (2.9)$$

which, when different powers of \dot{q} are treated as linearly independent (as they are, since this is a second order equation) yields, on equating the coefficients of separate powers of \dot{q} to zero (bearing in mind that both ξ and η are functions of q and t only), for the third and second powers

$$\dot{q}^3 : \xi = a(t)q + b(t) \quad (2.10)$$

and

$$\dot{q}^2 : \eta = \dot{a}q^2 - \frac{\alpha}{3} aq^3 + c(t)q + d(t). \quad (2.11)$$

The coefficient of \dot{q} gives rise to two cases:

$$(i) \quad \alpha^2 = 9\beta$$

$$(ii) \quad \alpha^2 = 9\beta.$$

In the first case we find that

$$a = 0$$

$$b = At + B$$

$$c = -A$$

$$d = 0$$

(2.12)

where A and B are constants. There are only two generators for this case.

These are

$$G_1 = \frac{\partial}{\partial t} \quad (2.13)$$

$$G_2 = t \frac{\partial}{\partial t} - q \frac{\partial}{\partial q}. \quad (2.14)$$

We discover that this case is of not much importance to our central theme. Besides, it has already been treated elsewhere (Leach [14]).

We do, however, quote results which are of relevance to our future discussion. We find that

$$[G_1, G_2] = G_1,$$

where

$$G_2 = \rho(q, t)G_1. \quad (2.15)$$

Therefore G_1 and G_2 are generators of type III [2], and the canonical form of these generators is

$$\bar{G}_1 = \frac{\partial}{\partial Q} \quad (2.16)$$

$$\bar{G}_2 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}, \quad (2.17)$$

where T is the independent variable and Q the dependent variable.

The case (i) belongs to one of the classes of differential equations which have two generators of symmetry and it is convenient, at this stage, to

provide a table for the canonical forms of two-parameter groups which seldom appears in the literature. This table will be of value for future calculations.

Table 1

Canonical Forms of Two-Parameter Groups

Group Type	$[G_1, G_2]$	Connectedness of G_1 and G_2	Canonical Forms of G_1 and G_2
I	0	$G_2 = \rho(q, t)G_1$ (unconnected)	$\bar{G}_1 = \frac{\partial}{\partial T}$, $\bar{G}_2 = \frac{\partial}{\partial Q}$
II	0	$G_2 = \rho(q, t)G_1$ (connected)	$\bar{G}_1 = \frac{\partial}{\partial Q}$, $\bar{G}_2 = T \frac{\partial}{\partial Q}$
III	G_1	$G_2 = \rho(q, t)G_1$ (unconnected)	$\bar{G}_1 = \frac{\partial}{\partial Q}$, $\bar{G}_2 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}$
IV	G_1	$G_2 = \rho(q, t)G_1$ (connected)	$\bar{G}_1 = \frac{\partial}{\partial Q}$, $\bar{G}_2 = Q \frac{\partial}{\partial Q}$

However, it is the second case which provides the results of great interest to us and we find that

$$\begin{aligned}
 d &= -At + B \\
 c &= \frac{\alpha}{2} At^2 + Ct - D \\
 b &= \frac{\alpha}{6} At^3 + (C + \frac{\alpha}{2} B)t^2 + Et - F \\
 a &= \frac{-\alpha^2}{36} At^4 - \frac{\alpha}{6} (C + \frac{\alpha}{3} B)t^3 - \frac{\alpha}{6} (E - D)t^2 + Gt + H \quad (2.18)
 \end{aligned}$$

where A through H are constants. Thus for the case $\alpha^2 = 9B$, the differential equation (1.3) has eight generators of symmetry. Hence we can write (1.3) as

$$\ddot{q} + \alpha q \dot{q} + \left(\frac{\alpha}{3}\right)^2 q^3 = 0. \quad (2.19)$$

Under the transformation

$$q = \frac{3}{\alpha} \theta \quad (2.20)$$

(2.19) becomes

$$\ddot{\theta} + 3\theta \dot{\theta} + \theta^3 = 0. \quad (2.21)$$

So we might as well, from the start, set $\alpha = 3$ in (2.19) since (2.19) is equivalent to (2.21) under the transformation (2.20). Rewriting equation (2.21) in terms of q we obtain

$$\ddot{q} + 3q \dot{q} + q^3 = 0. \quad (2.22)$$

One may enquire as to how this equation is related to the original equation (1.1).

In (1.1) set

$$\begin{aligned}\alpha &= \frac{4n}{n+3} \\ \beta &= \frac{2(n^2-1)}{(n+3)^2}.\end{aligned}\tag{2.23}$$

For the case $\alpha^2 = 9\beta$ we must have

$$n = 3.\tag{2.24}$$

Thus (1.1) reduces to

$$\ddot{q} + 2q\dot{q} + \left(\frac{2}{3}\right)^2 q^3 = 0.\tag{2.25}$$

Clearly (2.25) is equivalent to (2.22). Therefore (2.22) forms the basis of our discussion. Equation (2.22) has eight generators of symmetry. It is well known that the maximum number of symmetry generators for a second order ordinary differential equation is eight (c.f. Bluman and Cole [3]). However, we point out that this result has been shown to be peculiar to second order linear systems as a proven fact. For a non-linear system, this is unexpected and indeed remarkable!

It may generally be stated that all non-linear autonomous second order systems have at least one symmetry generator whereas most non-linear non-autonomous second order systems have no symmetry generator at all [12].

Before we proceed to list the eight generators, it is worthwhile mentioning that this was not the order in which we previously obtained them, i.e., in terms of putting the constants A through H equal to one with all others zero in turn.

The eight generators are

$$G_1 = \frac{t^2}{2} q \frac{\partial}{\partial t} + (tq^2 - \frac{1}{2}t^2q^3 - q) \frac{\partial}{\partial q}$$

$$G_2 = q \frac{\partial}{\partial t} - q^3 \frac{\partial}{\partial q}$$

$$G_3 = tq \frac{\partial}{\partial t} + (q^2 - tq^3) \frac{\partial}{\partial q}$$

$$G_4 = (-\frac{1}{2}t^2q + t) \frac{\partial}{\partial t} + (\frac{1}{2}t^2q^3 - tq^2) \frac{\partial}{\partial q}$$

$$G_5 = (\frac{1}{2}t^3 - \frac{1}{2}t^4q) \frac{\partial}{\partial t} + (-t + \frac{3}{2}t^2q + \frac{1}{2}t^4q^3 - t^3q^2) \frac{\partial}{\partial q}$$

$$G_6 = (-\frac{1}{2}t^3q + t^2) \frac{\partial}{\partial t} + (tq + \frac{1}{2}t^3q^3 - \frac{3}{2}t^2q^2) \frac{\partial}{\partial q}$$

$$G_7 = (-\frac{1}{2}t^3q + \frac{3}{2}t^2) \frac{\partial}{\partial t} + (1 + \frac{1}{2}t^3q^3 - \frac{3}{2}t^2q^2) \frac{\partial}{\partial q}$$

$$G_8 = -\frac{\partial}{\partial t}$$

We observe that most of the generators are rather complicated. It is also observed that

$$G_4 = \rho(q, t)G_2 \quad (2.26)$$

where

$$\rho(q, t) = t - \frac{1}{q}. \quad (2.27)$$

This fact will be used later.

3. THE COMMUTATION RELATIONS

The Lie algebra of the symmetries of a system is represented by the commutation relations between generators

$$[G_i, G_j] = G_i G_j - G_j G_i. \quad (3.1)$$

In this way the underlying symmetry group may be obtained - up to isomorphism.

All one-dimensional linear systems have common properties (Leach [9]). This follows from the equivalence of all linear systems under transformation and the preservation of algebraic properties under transformation [5]. Moreover, these systems possess the complete symmetry group $SL(3, R)$.

We wish to identify the complete symmetry group for the non-linear system (2.22). The generators of (2.22) given in the previous section should have commutation relations appropriate to one of the eight parameter groups $SL(3, \mathbb{R})$, $SU(3)$ or $GL(2, \mathbb{C})$. The latter two being complex.

The commutation relations are given in Table 2. These relations are appropriate to the symmetry group $SL(3, \mathbb{R})$. Therefore the complete symmetry group for (2.22) is $SL(3, \mathbb{R})$. This will become apparent later.

Table 2

The (i, j) th entry is the Lie bracket $[G_i, G_j]$

$G_i \backslash G_j$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	$-G_2$	$-G_3$	0	G_5	0	$(G_7 - 2G_6)$	G_3
G_2		0	0	G_2	$(G_1 + G_4)$	G_3	$(3G_3 + G_8)$	0
G_3			0	0	G_6	0	$(2G_1 - G_4)$	G_2
G_4				0	G_5	G_6	$2G_6$	$-(G_3 + G_8)$
G_5					0	0	0	$(3G_6 - G_7)$
G_6						0	G_5	$(2G_4 - G_1)$
G_7							0	$3G_4$
G_8								0

The entries below the main diagonal have been omitted because of the skewness of the Lie bracket.

It is observed from the table that the triplets $\{G_1, G_2, G_3\}$ and $\{G_4, G_5, G_6\}$ constitute three-parameter subgroups. A feature of great interest here is that the triplets have identical algebras. This will be discussed more fully in the next section.

4. GENERATOR FROM FIRST INTEGRAL

Conventionally the symmetry generators of a one-dimensional linear system are used to determine the invariants or first integrals associated with the system. Here the reverse procedure is adopted for the free particle. This has been done generally for one-dimensional linear systems by Leach [9]. He used the simple harmonic oscillator as a vehicle for his discussion. We employ his technique for the free particle.

The free particle has the equation of motion

$$\ddot{q} = 0 \quad (4.1)$$

with Hamiltonian

$$H = \frac{1}{2}p^2, \quad (4.2)$$

where

$$p = \dot{q}.$$

The two first integrals are easily seen to be

$$\begin{aligned} I_1 &= p \\ I_2 &= q - tp \end{aligned} \quad (4.3)$$

and we include their quotient

$$I_3 = \frac{q - tp}{p} \quad (4.4)$$

since this has been shown by Leach [9] to be relevant from the oscillator.

We seek the set of one-parameter symmetry groups with which each of I_1, I_2, I_3 is associated.

ξ and η of the symmetry generator

$$G = \xi(q, t) \frac{\partial}{\partial t} + \eta(q, t) \frac{\partial}{\partial q} + \zeta(q, p, t) \frac{\partial}{\partial p} \quad (4.5)$$

is determined by the following equations:

$$\zeta \frac{\partial I}{\partial p} + \eta \frac{\partial I}{\partial q} + \xi \frac{\partial I}{\partial t} = 0 \quad (4.6)$$

$$\eta^{(1)} - \zeta \frac{\partial^2 H}{\partial p^2} - \eta \frac{\partial^2 H}{\partial q \partial p} - \xi \frac{\partial^2 H}{\partial t \partial p} = 0 \quad (4.7)$$

where

$$\eta^{(1)} = \dot{\eta} - \dot{\xi} \frac{\partial H}{\partial p}, \quad (4.8)$$

by eliminating ζ between (4.6) and (4.7) and insisting that ξ and η be independent of p [9].

Thus for the invariant $I_1 = p$, we obtain

$$\frac{d\eta}{dt} - p \frac{d\xi}{dt} = 0 \quad (4.9)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + p \frac{\partial}{\partial q} \quad (4.10)$$

(4.9) yields a partial differential equation in which the terms are grouped together in powers of p . By equating coefficients of separate powers of p to zero we obtain

$$p^2 : \xi = a(t) \quad (4.11)$$

$$p : \eta = \dot{a}q + b(t) \quad (4.12)$$

and

$$p^0 : a = At + B \quad (4.13)$$

$$b = C,$$

where a and b are determined using (4.12).

Therefore the triplet of generators with which I_1 is associated is

$$\begin{aligned}G_1 &= t \frac{\partial}{\partial t} + q \frac{\partial}{\partial q} \\G_2 &= \frac{\partial}{\partial t} \\G_3 &= \frac{\partial}{\partial q} .\end{aligned}\tag{4.14}$$

In like manner, for I_2 , we obtain

$$\begin{aligned}G_4 &= t \frac{\partial}{\partial t} \\G_5 &= t^2 \frac{\partial}{\partial t} + tq \frac{\partial}{\partial q} \\G_6 &= t \frac{\partial}{\partial q} .\end{aligned}\tag{4.15}$$

Similarly for I_3

$$\begin{aligned}G_7 &= tq \frac{\partial}{\partial t} + q^2 \frac{\partial}{\partial q} \\G_8 &= q \frac{\partial}{\partial t} \\G_9 &= q \frac{\partial}{\partial q} .\end{aligned}\tag{4.16}$$

By comparing the usual forms for the generators of the free particle with those obtained here using the reverse procedure for the free particle we observe that G_1 is linearly dependent. This can easily be verified by noting that

$$G_1 = G_4 + G_9 . \quad (4.17)$$

On the other hand, it may be possible to include G_1 in the formation of a linearly independent set of generators for the free particle. We defer discussion of this point to a later stage.

Our prime concern here, however, is the commutation properties of the nine generators which are simpler than those for the usual eight as we shall see below.

We calculate the commutators for each of the triplets of generators associated with I_1 , I_2 and I_3 respectively.

For $I_1 = p$:

$$[G_1, G_2] = -G_2$$

$$[G_1, G_3] = -G_3$$

$$[G_2, G_3] = 0 .$$

(4.18)

For $I_2 = q - pt$

$$[G_4, G_5] = G_5$$

$$[G_4, G_6] = G_6$$

$$[G_5, G_6] = 0. \quad (4.19)$$

For $I_3 = (q - pt)/P$:

$$[G_7, G_8] = 0$$

$$[G_7, G_9] = -G_7$$

$$[G_8, G_9] = -G_8. \quad (4.20)$$

Each of the triplets of generators $\{G_1, G_2, G_3\}$, $\{G_4, G_5, G_6\}$ and $\{G_7, G_8, G_9\}$ form a closed subgroup. This follows immediately from (4.18), (4.19) and (4.20).

It is worthwhile noting that each of the three sets of commutation relations given above may be written in the form

$$[X_1, X_2] = X_2$$

$$[X_1, X_3] = X_3$$

$$[X_2, X_3] = 0 \quad (4.21)$$

This can be accomplished by a mere change in sign of G_7 and G_9 . Thus it follows from (4.21) that the algebraic properties of the triplets are identical.

5. TRANSFORMATION OF THE GENERATORS

It has been found that, in the case for which there are several generators, a common transformation reduces them to a commonly simpler form. This is the case with linear systems [10]. Moreover, this should likewise be the case for non-linear systems such as (2.22). That indeed it is so, will be seen shortly.

It can easily be verified from sections 3 and 4 that the commutation relations of the triplets $\{G_1, G_2, G_3\}$ and $\{G_4, G_5, G_6\}$ for (2.22) are identical to those for the free particle. Thus we can use the free particle as a vehicle for our discussion.

By means of the change in space and time variables $(q, t) \rightarrow (Q, T)$, the generators transform as

$$G(q, t) \rightarrow \bar{G}(Q, T)$$

where

$$\bar{G}(Q, T) = \left(\epsilon \frac{\partial T}{\partial t} + \eta \frac{\partial T}{\partial q} \right) \frac{\partial}{\partial T} + \left(\epsilon \frac{\partial Q}{\partial t} + \eta \frac{\partial Q}{\partial q} \right) \frac{\partial}{\partial Q}. \quad (5.1)$$

We observe (from Table 2) that

$$[G_2, G_3] = 0,$$

where

$$G_3 = \rho(q, t)G_2. \quad (5.2)$$

Therefore, G_2 and G_3 are generators of type I. The canonical form is (see Table 1)

$$\bar{G}_2 = \frac{\partial}{\partial T}$$

$$\bar{G}_3 = \frac{\partial}{\partial Q}. \quad (5.2)$$

We seek the transformation from (q, t) to (Q, T) which casts G_2 and G_3 into canonical form.

G_2 assumes canonical form provided

$$\begin{aligned} \xi \frac{\partial Q}{\partial t} + \eta \frac{\partial Q}{\partial q} &= 0 \\ \xi \frac{\partial T}{\partial t} + \eta \frac{\partial T}{\partial q} &= 1 \end{aligned} \quad (5.3)$$

where $\xi = q$ and $\eta = -q^3$.

By applying the method of characteristics for first order partial differential equations to (5.3) we obtain

$$\frac{dt}{q} = \frac{dq}{-q^3}$$

$$\frac{dt}{q} = \frac{dq}{-q^3} = \frac{dT}{T} \quad (5.4)$$

for which the solutions are

$$Q = t - \frac{1}{q},$$

$$T = -\frac{t^2}{2} + \frac{t}{q}. \quad (5.5)$$

Under this transformation the other generators are reduced to a simpler form.

We list the eight generators. They are

$$\bar{G}_1 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}$$

$$\bar{G}_2 = \frac{\partial}{\partial T}$$

$$\bar{G}_3 = \frac{\partial}{\partial Q}$$

$$\bar{G}_4 = T \frac{\partial}{\partial T}$$

$$\bar{G}_5 = T^2 \frac{\partial}{\partial T} + QT \frac{\partial}{\partial Q}$$

$$\bar{G}_6 = T \frac{\partial}{\partial Q}$$

$$\bar{G}_7 = TQ \frac{\partial}{\partial T} + (Q^2 + 2T) \frac{\partial}{\partial Q}$$

$$\bar{G}_8 = Q \frac{\partial}{\partial T} - \frac{\partial}{\partial Q}.$$

By comparing the generators obtained here with those of the free particle, we observe that \bar{G}_7 to \bar{G}_8 have the same structure and form as G_1 to G_6 of the free particle. This was to be expected in view of our remarks in paragraph two of this section. Furthermore, it is observed that a G_1 -type generator (viz. \bar{G}_7) is included in the above complete set of generators. The possibility of this inclusion was hinted at in the previous section.

In section two it was observed that

$$G_4 = \rho(q, t)G_2, \quad (5.6)$$

where

$$\rho(q, t) = t - \frac{1}{q}. \quad (5.7)$$

From Table 2 we notice that

$$[G_2, G_4] = G_2. \quad (5.8)$$

It follows from Table 1 that G_2 and G_4 are generators of type IV and that the canonical form is

$$\bar{G}_2 = \frac{\partial}{\partial Q}$$

$$\bar{G}_4 = Q \frac{\partial}{\partial Q} . \quad (5.9)$$

By comparing (5.9) with the corresponding generators given in the previous page, we notice that an interchange of the Q and T variables have taken place. Thus the canonical transformation which results in (5.9) is

$$Q = -\frac{t^2}{2} + \frac{t}{q} ,$$

$$T = t - \frac{1}{q} . \quad (5.10)$$

This is obtained by merely interchanging Q and T in (5.5). The other generators are also reduced to a simpler form under transformation (5.10).

We list the eight generators:

$$\bar{G}_1 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}$$

$$\bar{G} = \frac{\partial}{\partial Q}$$

$$\bar{G}_3 = \frac{\partial}{\partial T}$$

$$\bar{G}_4 = Q \frac{\partial}{\partial Q}$$

$$\bar{G}_5 = QT \frac{\partial}{\partial T} + Q^2 \frac{\partial}{\partial Q}$$

$$\bar{G}_6 = Q \frac{\partial}{\partial T}$$

$$\bar{G}_7 = (T^2 + 2Q) \frac{\partial}{\partial T} + TQ \frac{\partial}{\partial Q}$$

$$\bar{G}_8 = -\frac{\partial}{\partial T} + T \frac{\partial}{\partial Q}.$$

Once again we have a G_I -type generator appearing in our list. Just as before, six of the generators given above have the same structure and form as six of the generators of the free particle.

6. THE SOLUTION

Under the transformation (5.5) or (5.10), equation (2.22) takes the canonical form

$$\frac{d^2 Q}{dT^2} = 0. \quad (6.1)$$

We can simply write this as

$$Q'' = 0. \quad (6.2)$$

Where the prime denotes differentiation with respect to T . This is the equation of a free particle in the transformed coordinates. In view of the linear symmetries obtained in the previous chapter, it is not in the least surprising that the transformed equation assumes such a simple form. However, in isolation it is quite remarkable!

In section four it was noted that the functionally independent invariants I_1 and I_2 were associated with two three-parameter subgroups of the symmetry group $SL(3, R)$. Moreover, these two invariants were sufficient to determine the solution of the free particle. In view of the preceding discussions, this should also be the case for (6.2).

Therefore we have

$$\bar{I}_1 = Q'$$

$$\bar{I}_2 = Q - TQ' . \quad (6.3)$$

The solution is immediate:

$$Q = \bar{I}_2 + \bar{I}_1 T . \quad (6.4)$$

In terms of the original variables we have (using (5.5))

$$q = \frac{2(1 + \bar{I}_1 t)}{\bar{I}_1 t^2 + 2t - 2\bar{I}_2} . \quad (6.5)$$

That (6.5) is the general solution of (2.22) can easily be checked by direct substitution.

By using the transformation (5.10), we obtain the following:

7. CONCLUSION

In this work we have shown that the non-linear differential equation

$$\ddot{q} + \alpha q \dot{q} + \left(\frac{\alpha}{3}\right)^2 q^3 = 0, \quad (7.1)$$

where α is an arbitrary constant, is transformable to the Newtonian equation of motion of a free particle. In the process we used the free particle as a vehicle for our discussion. In doing so, we observed that the symmetry properties of the free particle were injected into (7.1) under a canonical transformation which was obtained via commutator considerations. The triplets associated with the respective systems provided the solution of (7.1).

Furthermore, it was shown that the complete symmetry group for (7.1) is $SL(3, \mathbb{R})$. From the results established above, it may be inferred that the complete symmetry group for any non-linear system is $SL(3, \mathbb{R})$ provided the system has eight symmetries. However, the classification of such systems is a difficult matter and requires further investigation.

ACKNOWLEDGEMENTS: One of the authors (F.M.) wishes to thank Prof. D.P. Mason and Dr. R. Maartens for useful comments. This work was supported in part by a grant from the Council for Scientific and Industrial Research (CSIR) of South Africa.

$$\begin{aligned}
 q &= \frac{2(I_1 + t)}{t^2 + 2I_1 t + 2I_2} \\
 &= \frac{2(1 + \frac{1}{I_1}t)}{\frac{1}{I_1}t^2 + 2t + 2\frac{I_2}{I_1}}.
 \end{aligned}
 \tag{6.6}$$

(6.6) is the same as (6.5) if

$$\bar{I}_1 = \frac{1}{I_1}$$

and

$$\bar{I}_2 = -\frac{I_2}{I_1}. \tag{6.7}$$

Writing out the expressions for I_1 and I_2 in terms of q , \dot{q} and t we obtain

$$I_1 = -t + \frac{q}{q^2 + \dot{q}} \tag{6.8}$$

and

$$I_2 = \frac{t^2}{2} + \frac{1 - tq}{q^2 + \dot{q}} \tag{6.9}$$

which have been called resonance integrals [17].

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Received 1985.04.03

First integrals for the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$

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Journal of Mathematical Physics **26** (1985) 2510-2514

First Integrals for the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$

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(Received 18 April 1984; accepted for publication 1 March 1985)

It is shown that the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$ possesses first integrals for functions $\alpha(t)$ other than kt^{-1} . The function $\alpha(t)$ is obtained explicitly in the case $n = 3$ and parametrically for other $n(\neq 2)$. The case $n = 2$ is seen to be particularly difficult to solve.

I. INTRODUCTION

The Emden equation of index n ,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n, \quad \theta(0) = 1, \quad \theta'(0) = 0, \quad \xi > 0, \quad (1.1)$$

arises in the study of equilibrium configurations of a spherical gas cloud acting under the mutual attractions of its molecules and subject to the laws of thermodynamics. The index n is given by

$$n = (c_v - c)/(c_p - c_v), \quad (1.2)$$

where c_v is the specific heat at constant volume, c_p is the specific heat at constant pressure, and c is the assumed constant in the relationship between heat input dQ and temperature change dT , i.e.,

$$dQ = c dT. \quad (1.3)$$

Chandrasekhar¹ discusses the equation extensively and reports complete solutions in the cases $n = 0, 1, 5$.

The Emden equation may also be considered as an equation in dynamics, viz.,

$$\ddot{q} + (2/t)\dot{q} + q^n = 0, \quad (1.4)$$

which represents in general an anharmonic oscillator subject to damping dependent upon the velocity. In the case $n = 5$, Logan² illustrates the use of Noether's theorem with point transformations by applying the theorem to the variational integral

$$J = \int_{t_0}^{t_1} t^2 \left(\frac{1}{2} \dot{q}^2 - \frac{1}{6} q^6 \right) dt, \quad (1.5)$$

the integrand of which is a Lagrangian from which (1.4) may be obtained. He obtained the first integral

$$I = \frac{1}{2} t^3 \dot{q}^6 + \frac{1}{2} t^3 \dot{q}^2 + \frac{1}{2} t^2 q \dot{q}. \quad (1.6)$$

This same first integral was obtained by Sarlet and Bahar³ by introducing a time-dependent integrating factor. They extended their treatment to the more general equation

$$\ddot{q} + \beta(t)\dot{q} + \alpha(t)q^m = 0, \quad m \neq -1 \quad (1.7)$$

and found that the first integral

$$I = \left(\dot{q}^2 + \frac{2\alpha}{m+1} q^{m+1} \right) \exp \left(2 \int \beta(t') dt' \right) \times \left[C + C_4 \int \exp \left(- \int \beta(t'') dt'' \right) dt' \right]$$

$$- C_4 q \dot{q} \exp \left(\int \beta(t') dt' \right) \quad (1.8)$$

exists when $\alpha(t)$ and $\beta(t)$ satisfy the relation

$$\alpha^{-2/(m+3)} \exp \left(- \frac{4}{m+3} \int \beta(t') dt' \right) - C_4 \int \exp \left(- \int \beta(t'') dt'' \right) dt' = C \quad (1.9)$$

and C and C_4 are constants. Furthermore, they showed that the result was equivalent to using the gauge-invariant Noether approach. The extension to the gauge-variant Noether approach was mentioned, but the calculations and integral were not given.

Moreira,⁴ as an example of the application of the Lewis-Leach⁵ direct method to Newtonian equations of motion, studied the equation

$$\ddot{q} + \alpha(t)\dot{q} + q^n = 0. \quad (1.10)$$

He found the first integral

$$I = \exp \left(m \int \alpha(t') dt' \right) \left\{ \dot{q}^2 + \alpha q \dot{q} (2 - m) + [2/(n+1)] q^{n+1} + \frac{1}{2} q^2 (m-2) [(m-1)\alpha^2 + \dot{\alpha}] \right\}, \quad (1.11)$$

where $m = 2(n+1)/(n+3)$, provided that $\alpha(t)$ satisfied the differential equation

$$\ddot{\alpha} + (3m-2)\alpha\dot{\alpha} + \alpha^3 m(m-1) = 0. \quad (1.12)$$

He did not provide the general solution of (1.12), but gave two particular solutions, viz.,

$$\alpha(t) = 2/mt, \quad \alpha(t) = [(m-1)t]^{-1}. \quad (1.13)$$

Feix and Lewis,⁶ in their study of first integrals for dissipative nonlinear systems using rescaling (the equivalent in Newtonian mechanics of generalized canonical transformations in Hamiltonian mechanics⁷), examined the equation

$$\ddot{x} + \beta(t)\dot{x} + \frac{\partial \phi(x,t)}{\partial x} = 0. \quad (1.14)$$

As an example they treat the case in which

$$\phi(x,t) = \gamma(t) [x^{m+1}/(m+1)], \quad m \neq -1. \quad (1.15)$$

However, they specified $\beta(t)$ to be proportional to t^{-1} .

In this note we intend to apply Noether's theorem to an equation of Emden type. It is necessary to decide which of the various forms of the equation given above is to be used.

Our choice is that of Moreira (1.10). The reason for this is as follows. Consider the equation

$$\ddot{x} + \beta(t)\dot{x} + \gamma(t)x^n = 0, \quad (1.16)$$

which was discussed in Refs. 3 and 6. Under the transformation

$$(x, t) \rightarrow (X, T: X = x, T = f(t)), \quad (1.17)$$

(1.17) becomes

$$X'' + (\dot{f}f^{-2} + \beta f^{-1})X' + \gamma f^{-2}X^n = 0, \quad (1.18)$$

where prime denotes d/dT . If we set

$$\gamma = (\dot{f})^2, \quad \alpha(T(t)) = \dot{f}f^{-2} + \beta f^{-1}, \quad (1.19)$$

we have an equation of type (1.10). Should it happen that $\gamma(t)$ is negative we could replace the first of (1.19) by $\gamma = -\dot{f}^{-2}$ to keep \dot{f} real. If $\gamma(t)$ has zeros, the transformation (1.17) would strictly apply to the interval between successive zeros and matching of the results for successive intervals would have to be undertaken.

One may wonder what relevance this has to the original Emden equation (1.1). This equation comes from the equation of equilibrium¹

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho, \quad (1.20)$$

where r is the radial variable in the gaseous sphere, ρ is the density of the gas, P is its pressure, and G is the universal gravitational constant, by means of the substitutions

$$\rho = \lambda \theta^n, \quad P = K\lambda^{(n+1)/n} \theta^{n+1} \quad (1.21)$$

and suitable rescaling. The density and pressure are related by

$$P = K\rho^{(n+1)/n}. \quad (1.22)$$

If we imagine that the constant K is replaced by a function of r , we obtain an equation of type (1.10) in terms of a variable σ , where

$$P = K_1 \sigma^{(n+1)/n}, \quad \sigma = f(r)\rho. \quad (1.23)$$

II. APPLICATION OF NOETHER'S THEOREM

Following Sarlet and Cantrijn,⁸ if the functional

$$J = \int_{t_0}^{t_1} L(q, \dot{q}, t) dt, \quad (2.1)$$

where L is regular in \dot{q} , admits a gauge-variant symmetry generated by

$$G = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial q}, \quad (2.2)$$

where $\tau = \tau(q, \dot{q}, t)$ and $\xi = \xi(q, \dot{q}, t)$, there exists a first integral

$$I = \left[L\tau + \frac{\partial L}{\partial \dot{q}} (\xi - \dot{q}\tau) \right] - f(q, \dot{q}, t), \quad (2.3)$$

where f , τ , and ξ are determined by the equations

$$\begin{aligned} L \frac{\partial \tau}{\partial \dot{q}} + \frac{\partial L}{\partial \dot{q}} \left(\frac{\partial \xi}{\partial \dot{q}} - \dot{q} \frac{\partial \tau}{\partial \dot{q}} \right) &= \frac{\partial f}{\partial \dot{q}}, \\ \tau \frac{\partial L}{\partial t} + \xi \frac{\partial L}{\partial q} + L \left(\frac{\partial \tau}{\partial t} + \dot{q} \frac{\partial \tau}{\partial q} \right) &= 0, \end{aligned} \quad (2.4)$$

$$\begin{aligned} &+ \frac{\partial L}{\partial \dot{q}} \left\{ \frac{\partial \xi}{\partial t} + \dot{q} \frac{\partial \xi}{\partial q} - \dot{q} \left(\frac{\partial \tau}{\partial t} + \dot{q} \frac{\partial \tau}{\partial q} \right) \right\} \\ &= \frac{\partial f}{\partial t} + \dot{q} \frac{\partial f}{\partial q}. \end{aligned} \quad (2.5)$$

Since, further,

$$\xi - \dot{q}\tau = -g \frac{\partial I}{\partial \dot{q}}, \quad (2.6)$$

where

$$\frac{\partial^2 L}{\partial \dot{q}^2} g = 1, \quad (2.7)$$

we may without loss of generality set $\tau = 0$ and do so.

The equation of motion we study is

$$\ddot{q} + \alpha(t)\dot{q} + q^n = 0, \quad n \neq 1, 0, 1. \quad (2.8)$$

(The cases $n = 0, 1$ are excluded since they are linear systems and have already been treated elsewhere.⁹ For the case $n = -1$ see the Appendix.) A Lagrangian for this equation is

$$L(q, \dot{q}, t) = \frac{1}{2} A(t) \dot{q}^2 - [A(t)/(n+1)] q^{n+1}, \quad (2.9)$$

where

$$A(t) = \exp \left(\int^t \alpha(t') dt' \right). \quad (2.10)$$

As the first integrals obtained in Refs. 2, 3, 4, and 6 were quadratic in \dot{q} , we look for one of the same type. From (2.6) it is evident that

$$\xi = a(q, t)\dot{q} + b(q, t). \quad (2.11)$$

From (2.3) we see that the first integral is

$$I = A\dot{q}(a\dot{q} + b) - f. \quad (2.12)$$

Substituting for L , ξ , and τ into (2.4) and integrating with respect to \dot{q} we see that

$$f = \frac{1}{2} aA\dot{q}^2 + c(q, t) \quad (2.13)$$

and (2.12) becomes

$$I = \frac{1}{2} aA\dot{q}^2 + bA\dot{q} - c. \quad (2.14)$$

Substituting for L , ξ , τ , and f into (2.5) and equating the coefficients of like powers of \dot{q} to zero, we have

$$(\dot{q}^3) \quad A \frac{\partial a}{\partial \dot{q}} = \frac{1}{2} \frac{\partial}{\partial \dot{q}} (aA), \quad (2.15)$$

$$(\dot{q}^2) \quad A \frac{\partial a}{\partial t} + A \frac{\partial b}{\partial \dot{q}} = \frac{1}{2} \frac{\partial}{\partial t} (aA), \quad (2.16)$$

$$(\dot{q}^1) \quad -aAq^n + A \frac{\partial b}{\partial t} = \frac{\partial c}{\partial \dot{q}}, \quad (2.17)$$

$$(\dot{q}^0) \quad -Abq^n = \frac{\partial c}{\partial t}. \quad (2.18)$$

From (2.15), (2.16), and (2.17) in turn we find that

$$a = \sigma(t), \quad (2.19)$$

$$b = -\frac{1}{2A} (\dot{\sigma}A - \sigma\dot{A})\dot{q} - \gamma(t), \quad (2.20)$$

$$\begin{aligned} c &= -\frac{\sigma A q^{n+1}}{n+1} - \frac{1}{4} A \frac{d}{dt} \left[\frac{1}{A} (\dot{\sigma}A - \sigma\dot{A}) \right] q^2 \\ &\quad - A\gamma q - \delta(t). \end{aligned} \quad (2.21)$$

Substitution for b and c into (2.18) gives the consistency condition

$$\frac{q^{n+1}}{n+1} \frac{d}{dt}(\sigma A) + \frac{1}{4} \frac{d}{dt} \left\{ A \frac{d}{dt} \left[\frac{1}{a} (\dot{\sigma} A - \sigma \dot{A}) \right] \right\} q^2 + \frac{d}{dt} (A \dot{\gamma}) q + \frac{1}{2} (\dot{\sigma} A - \sigma \dot{A}) q^{n+1} + A \gamma q^n + \delta = 0. \quad (2.22)$$

Since $n \neq -1, 0, 1$, we distinguish two cases.

(1) $n = 2$: Equating the coefficients of like powers of q to zero, we have

$$(q^3) \quad \frac{1}{3} \frac{d}{dt}(\sigma A) + \frac{1}{2} (\dot{\sigma} A - \dot{A}) = 0, \quad (2.23)$$

$$(q^2) \quad \frac{1}{4} \frac{d}{dt} \left\{ A \frac{d}{dt} \left[\frac{1}{A} (\dot{\sigma} A - \sigma \dot{A}) \right] \right\} + A \gamma = 0, \quad (2.24)$$

$$(q^1) \quad \frac{d}{dt} (A \dot{\gamma}) = 0, \quad (2.25)$$

$$(q^0) \quad \delta = 0. \quad (2.26)$$

(2) $n \neq 2$: Equating the coefficient of like powers of q to zero, we have

$$(q^{n+1}) \quad \frac{1}{n+1} \frac{d}{dt}(\sigma A) + \frac{1}{2} (\dot{\sigma} A - \sigma \dot{A}) = 0, \quad (2.27)$$

$$(q^n) \quad A \gamma = 0, \quad (2.28)$$

$$(q^2) \quad \frac{d}{dt} \left\{ A \frac{d}{dt} \left[\frac{1}{A} (\dot{\sigma} A - \sigma \dot{A}) \right] \right\} = 0, \quad (2.29)$$

$$(q^1) \quad \frac{d}{dt} (A \dot{\gamma}) = 0, \quad (2.30)$$

$$(q^0) \quad \delta = 0. \quad (2.31)$$

III. THE CASE $n \neq 2$

It is evident that $\gamma = 0$ and δ is an ignorable constant. From (2.27) we obtain

$$(n+3)(\dot{\sigma}/\sigma) = (n-1)(\dot{A}/A), \quad n \neq -3, \quad (3.1)$$

$$\sigma \dot{A} = 0, \quad n = -3. \quad (3.2)$$

From (3.2) it follows that either $\sigma = 0$, in which case the first integral is trivially a constant, or $\dot{A} = 0$, in which case $\alpha = 0$. As neither is of interest to us we take $n \neq -3$ henceforth. Integrating (3.1), we have

$$\sigma^{n+3} = K_1 A^{n-1}, \quad (3.3)$$

where K_1 is some constant. If we make the substitution

$$\sigma = \rho^{n-1}, \quad A = K_2 \rho^{n+3}, \quad (3.4)$$

(2.29) becomes

$$\rho^2 \ddot{\rho} + (4n-3) \rho \dot{\rho} \dot{\rho} + 2n(n-2) \dot{\rho}^3 = 0. \quad (3.5)$$

Since

$$A(t) = \exp \left(\int \alpha(t') dt' \right),$$

we have

$$\alpha = (n+3) \dot{\rho} \rho^{-1}, \quad (3.6)$$

and the function $\alpha(t)$, for which a first integral exists, satisfies

$$\ddot{\alpha} + [4n/(n+3)] \alpha \dot{\alpha} + [2(n^2-1)/(n+3)^2] \alpha^3 = 0, \quad (3.7)$$

which comes from (3.5). In terms of ρ , the first integral (2.14) is

$$I = \frac{1}{2} \rho^{2n+2} \dot{q}^2 + 2 \dot{\rho} \rho^{2n+1} \dot{q} q + \rho^{2n+2} q^{n+1} / (n+1) - \{ (n-2) \rho^{2n} \dot{\rho}^2 + \rho^{2n+1} \ddot{\rho} \} q^2. \quad (3.8)$$

The invariance of I is easily checked by direct differentiation. An explicit form for the integral requires a knowledge of the functional expression for ρ . As we also require α , we look for the solution of (3.7) and then obtain ρ by quadrature of (3.6).

It is relatively straightforward to perform the first integration of (3.7). We obtain

$$C_1 \left(\dot{\alpha} + \frac{n+1}{n+3} \alpha^2 \right)^{n+1} = C_2 \left(\dot{\alpha} + \frac{n-1}{n+3} \alpha^2 \right)^{n-1}, \quad (3.9)$$

$$|C_1| + |C_2| \neq 0.$$

(This form is suggested by the similar result found in Kamke,¹⁰ p. 329, No. 1.204.) Before considering the next integration of (3.9) in general, we note two particular cases. If $C_1 = 0$,

$$\dot{\alpha} + [(n-1)/(n+3)] \alpha^2 = 0, \quad (3.10)$$

which may be integrated to give

$$\alpha(t) = (K_1 + [(n-1)/(n+3)]t)^{-1}, \quad (3.11)$$

and from (3.6)

$$\rho(t) = K_2 (K_1 + [(n-1)/(n+3)]t)^{1/(n-1)}. \quad (3.12)$$

Likewise, if $C_2 = 0$,

$$\alpha(t) = (K_1 + [(n+1)/(n+3)]t)^{-1}, \quad (3.13)$$

$$\rho(t) = K_2 (K_1 + [(n+1)/(n+3)]t)^{1/(n+1)}, \quad (3.14)$$

where K_1 and K_2 are arbitrary constants and K_2 may be set at unity since it has only a scaling effect on the first integral. These results are in accordance with those of Moreira⁴ if K_1 is set at zero. For ρ as given in (3.12) the first integral is

$$I_1 = \frac{1}{2} \left(K + \frac{n-1}{n+3} t \right)^{(2n+2)/(n-1)} q^2 + \frac{2}{n+3} \left(K + \frac{n-1}{n+3} t \right)^{(n+3)/(n-1)} q \dot{q} + \left(K + \frac{n-1}{n+3} t \right)^{(2n+2)/(n-1)} \frac{q^{n+1}}{n+1}, \quad (3.15)$$

and for ρ as given in (3.14) the first integral is

$$I_2 = \frac{1}{2} \left(K + \frac{n+1}{n+3} t \right)^2 \dot{q}^2 + \frac{2}{n+3} \left(K + \frac{n+1}{n+3} t \right) q \dot{q} + \left(K + \frac{n+1}{n+3} t \right)^2 \frac{q^{n+1}}{n+1} + \frac{2}{(n+3)^2} q^2. \quad (3.16)$$

The integration of (3.9) is not simple when both C_1 and C_2 are nonzero. If we make the substitution

$$u(t) = -\alpha^{-1}(t), \quad (3.17)$$

(3.9) becomes, after suitable rearrangement,

$$u = k \left[\frac{(\dot{u} + (n+1)/(n+3))^{n+1}}{(\dot{u} + (n-1)/(n+3))^{n-1}} \right]^{1/4}. \quad (3.18)$$

The solution of (3.18) may be written in parametric form (Kamke,¹⁰ p. 30, 4.17) as

$$u = k \left[\frac{(\eta + (n+1)/(n+3))^{n+1}}{(\eta + (n-1)/(n+3))^{n-1}} \right]^{1/4}, \quad (3.19)$$

$$t = \frac{1}{2} k \int^{\eta} \left[\frac{(\eta' + (n+1)/(n+3))^{n-3}}{(\eta' + (n-1)/(n+3))^{n+3}} \right]^{1/4} d\eta'. \quad (3.20)$$

In the particular case $n = 3$, we obtain

$$t = \frac{1}{2} k \int^{\eta} \left(s + \frac{1}{3} \right)^{-3/2} ds \quad (3.21)$$

and so

$$\alpha(t) = 3(t - K_1)/[K_2 + (t - K_1)^2], \quad (3.22)$$

$$\rho(t) = \{K_2 + (t - K_1)^2\}^{1/4}. \quad (3.23)$$

Hence the differential equation

$$\ddot{q} + 3(t - K_1)\dot{q}/[K_2 + (t - K_1)^2] + q^3 = 0 \quad (3.24)$$

possesses the first integral

$$I = [K_2 + (t - K_1)^2]^2 \dot{q}^2 + 2(t - K_1)[K_2 + (t - K_1)^2] q \dot{q} + \frac{1}{2}[K_2 + (t - K_1)^2]^2 q^4 - K_2 q^2. \quad (3.25)$$

For general n , we make the substitution

$$\eta + \frac{n-1}{n+2} = \frac{2}{n+3} \frac{1}{x^2}, \quad x \neq 0, \quad (3.26)$$

so that (3.20) is now

$$t = k \left(\frac{n+3}{2} \right)^{1/2} \int^x (1 + x'^2)^{n-3/4} dx'. \quad (3.27)$$

This integral may be evaluated in the case where n is an odd integer, but, with the exception of $n = 3$, it is not possible to invert the result to obtain x and so u as an explicit function of t .

Note that, if in (3.22) K_1 and K_2 are set equal to zero, $\alpha(t) = 3/t$ and, if $K_2 = -K_1^2$ and the limit $K_1 \rightarrow \infty$ is taken, $\alpha(t) = 3/2t$. These are the results obtained by Moreira for the case $n = 3$.

IV. THE CASE $n = 2$

We recall the equations for this case [(2.23) - (2.25)], viz.,

$$\frac{1}{3} \frac{d}{dt}(\sigma A) + \frac{1}{2}(\dot{\sigma} A - \sigma \dot{A}) = 0, \quad (4.1)$$

$$\frac{1}{4} \frac{d}{dt} \left\{ A \frac{d}{dt} \left[\frac{1}{A}(\dot{\sigma} A - \sigma \dot{A}) \right] \right\} + A\gamma = 0, \quad (4.2)$$

$$\frac{d}{dt}(A\dot{\gamma}) = 0. \quad (4.3)$$

We observe that, when $\gamma \equiv 0$, we obtain the same parametric forms, (4.3), (3.19), and (3.20), defining $\alpha(t)$ as for the general case. However, we proceed to consider the possibilities of richer results in the case $\gamma \neq 0$.

From (4.1),

$$\sigma^2 = KA, \quad (4.4)$$

so that (4.2) and (4.3) become

$$\frac{d}{dt}(\sigma^5 \ddot{\sigma}) - \sigma^5 \gamma = 0, \quad (4.5)$$

$$\frac{d}{dt}(\sigma^5 \dot{\gamma}) = 0. \quad (4.6)$$

Substituting for γ from (4.5) into (4.6) and integrating once, we have

$$\frac{d^2}{dt^2}(\sigma^5 \ddot{\sigma}) - 5 \frac{\dot{\sigma}}{\sigma} \frac{d}{dt}(\sigma^5 \ddot{\sigma}) = M. \quad (4.7)$$

It has not been possible to make any progress on the general solution of (4.7). However, a number of particular solutions are available. These are, for $M = 0$,

$$\sigma(t) = K_1 t + K_2, \quad (4.8)$$

$$\sigma(t) = (K_1 t + K_2)^3, \quad (4.9)$$

$$\sigma(t) = (K_1 t + K_2)^{1/3}, \quad (4.10)$$

and for $M \neq 0$,

$$\sigma(t) = (K_1 t + K_2)^{2/3}, \quad (4.11)$$

for which the value of M is given by

$$M = \frac{2}{3} K_1^4. \quad (4.12)$$

The corresponding solutions for α and γ are

$$\alpha(t) = 5K_1/(K_1 t + K_2), \quad \gamma(t) = 0, \quad (4.13)$$

$$\alpha(t) = 15K_1/(K_1 t + K_2), \quad \gamma(t) = 96K_1^3, \quad (4.14)$$

$$\alpha(t) = 5K_1/3(K_1 t + K_2), \quad \gamma(t) = 0, \quad (4.15)$$

$$\alpha(t) = \frac{10K_1}{3(K_1 t + K_2)}, \quad \gamma(t) = -\frac{4K_1^3}{9(K_1 t + K_2)^{7/3}}. \quad (4.16)$$

The solutions of (4.13) and (4.15) for $\alpha(t)$ are the special solutions obtained from the integration of (3.9) with C_1 and C_2 zero, respectively. The corresponding first integrals are

$$I = \frac{1}{2}(t + K)^6 \dot{q}^2 + 2(t + K)^5 q \dot{q} + \frac{1}{2}(t + K)^6 q^3, \quad (4.17)$$

for

$$\ddot{q} + 5\dot{q}/(t + K) + q^2 = 0, \quad (4.18)$$

$$I = \frac{1}{2}(t + K)^{18} \dot{q}^2 + 6(t + K)^{17} q \dot{q} + \frac{1}{2}(t + K)^{18} q^3 - 6(t + K)^{16} q^2 - 96(t + K)^{15} \dot{q}, \quad (4.19)$$

for

$$\ddot{q} + 15\dot{q}/(t + K) + q^2 = 0, \quad (4.20)$$

$$I = \frac{1}{2}(t + K)^2 \dot{q}^2 + \frac{1}{2}(t + K) q \dot{q} + \frac{1}{2}(t + K)^2 q^3 + \frac{1}{2} q^2, \quad (4.21)$$

for

$$\ddot{q} + 5\dot{q}/3(t + K) + q^2 = 0 \quad (4.22)$$

and

$$I = \frac{1}{2}(t + K)^4 \dot{q}^2 + \frac{1}{2}(t + K)^3 q \dot{q} + \frac{1}{2}(t + K)^4 q^3 + \frac{1}{2}(t + K)^2 q^2 + \frac{1}{2}(t + K) \dot{q} + \frac{1}{2} q, \quad (4.23)$$

for

$$\ddot{q} + 10\dot{q}/3(t + K) + q^2 = 0. \quad (4.24)$$

The results given in (4.17) and (4.22) are the same as those obtained by Moreira⁴ when n is replaced by 2 in his general formula.

V. CONCLUSION

We have seen that the differential equation

$$\ddot{q} + \alpha(t)\dot{q} + q^n = 0$$

possesses first integrals for more general $\alpha(t)$ than has previously been reported. However, it is only in the case $n = 3$ that it is possible to write the most general $\alpha(t)$ as an explicit function of time. For other $n (\neq 2)$ it is possible only to define $\alpha(t)$ parametrically. The case $n = 2$ is especially difficult and only four particular functions $\alpha(t)$ have been recognized.

In an attempt to avoid the impasse covered by the general nonintegrability of (3.27) we examined the differential equation for $\alpha(t)$, (3.7), for Lie symmetries. For $n \neq 3$ two generators of symmetry were found. Unfortunately they yielded the same information as is already contained in (3.9) and (3.20) and so added nothing to what was already known. The integrable case, $n = 3$, was found to have eight generators of symmetry. Inasmuch as (3.7) is rather nonlinear and the maximum number of generators for a second-order ordinary differential equation is eight, this result was unexpected, although a similar result has been observed before.¹¹

ACKNOWLEDGMENTS

We express our appreciation to I. de C. Moreira and H. R. Lewis for supplying the preprints referenced as 4 and 6, respectively. It is a pleasure to thank the referee for making constructive suggestions and for supplying the contents of the Appendix.

APPENDIX: THE CASE $n = 1$

The referee has kindly supplied the results for the case $n = -1$, i.e., for the equation

$$\ddot{q} + \alpha(t)\dot{q} + 1/q = 0. \quad (\text{A1})$$

These cases emerge. For

$$\alpha(t) = K, \quad (\text{A2})$$

where K is constant,

$$I = \frac{1}{2} \dot{q}^2 + Kq\dot{q} + \log q + \frac{1}{2} K^2 q^2 + Kt. \quad (\text{A3})$$

For

$$\alpha(t) = K \tan(Kt + M), \quad (\text{A4})$$

where K and M are constants (either both real or both purely imaginary),

$$I = \frac{1}{2} \dot{q}^2 + K \tan(Kt + M)q\dot{q} - \frac{1}{2} K^2 q^2 + \log q - \log \cos(Kt + M). \quad (\text{A5})$$

Finally, for

$$\alpha(t) = -1/(t + M), \quad (\text{A6})$$

where, again, M is a constant,

$$I = \frac{1}{2} \dot{q}^2 - q\dot{q}/(t + M) + \log q - \log(t + M). \quad (\text{A7})$$

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¹⁰E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen* (Chelsea, New York, 1971).

¹¹J. L. Reid, on the symmetries of the differential equation

$$y'' + 4y' + 2Ky^{-3/2} \exp\left[-\frac{1}{2}k \int y^{-1}(x') dx'\right]$$

$$\times (M \sin x + N \cos x) = 0$$

(private communication dated 20 August 1981).

Symmetries of nonlinear differential equations and linearisation

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Journal of Physics A: Mathematical and General **20** (1987) 277-292

Symmetries of non-linear differential equations and linearisation

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Received 10 December 1985, in final form 2 May 1986

Abstract. A non-linear ordinary differential equation is linearisable if it possesses $SL(3, \mathcal{R})$ symmetry. The conditions under which the Abelian two-dimensional subalgebras of $sl(3, \mathcal{R})$ are sufficient for linearisation are established.

1. Introduction

The Lie theory of extended groups has regained a lot of attention in recent studies of ordinary and partial differential equations. Knowing point symmetries of ordinary differential equations generally allows one to reduce the order of the equations (see, e.g., Bluman and Cole 1974, Ovsiannikov 1978). Alternatively, one can exploit the symmetries to construct first integrals for the given equations. When the equations are of Lagrangian type, this is most easily done in the context of Noether's theorem, but it has been pointed out by many authors that the point symmetries of Noether type do not exhaust the point symmetries of the differential equations (see, e.g., Lutzky 1978, Prince and Eliezer 1981, Wulfman and Wybourne 1976, Leach 1981). Concerning a single second-order equation, we know from Lie's counting theorem that there can be at most eight point symmetries (Ovsiannikov 1978, Anderson and Davison 1974). It is further known that all linear equations do have a full eight-parameter group of symmetries which is $SL(3, \mathcal{R})$. Mahomed and Leach (1985) recently added a new element to the discussion in investigating a non-linear equation of the type

$$\ddot{q} + \alpha q \dot{q} + \beta q^3 = 0 \quad (1)$$

which itself arose in a study of the generalised Emden equation (Leach 1985). They found that such an equation has only two point symmetries, unless $\alpha^2 = 9\beta$, in which case there are eight symmetries, whose generators again exhibit the $sl(3, \mathcal{R})$ commutation relations. The latter result means that there exists a *point transformation*, reducing the equation to any linear equation (such as the free particle equation) and that the non-linear equation accordingly can be readily solved. They further inferred from this example that any non-linear equation may have the $SL(3, \mathcal{R})$ group, provided it has eight symmetries. Certainly, in view of the invariance of the commutation relations of symmetry generators under arbitrary point transformations, having $SL(3, \mathcal{R})$ symmetry is a necessary and sufficient condition for a second-order differential equation

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to be linearisable (by which we mean linearisable through a point transformation). The question then arises whether we need to know the full eight symmetry generators of an equation before we can conclude that the linearisation will exist. The following property (cf Lie 1891) will be the starting point for the investigation of this paper.

Proposition. In order that a second-order ordinary differential equation has the $\mathfrak{sl}(3, \mathcal{R})$ algebra, it is necessary and sufficient that it has the nilpotent algebra

$$[G_1, G_2] = 0 \quad [G_2, G_3] = 0 \quad [G_1, G_3] = G_2 \quad (2)$$

henceforth referred to as \mathfrak{N} .

Proof. For a general second-order equation $\ddot{q} = f(\dot{q}, q, t)$, we write the generator of a point symmetry in the form

$$G = \tau(t, q)\partial/\partial t + \xi(t, q)\partial/\partial q. \quad (3)$$

Assuming that the equation has the algebra \mathfrak{N} then means that we have three symmetry generators G_1, G_2, G_3 , satisfying the commutation relations (2).

If this is the case, we cannot have that both $G_1 = \rho(q, t)G_2$ and $G_3 = \psi(q, t)G_2$ (for suitable functions ρ and ψ) as the first two commutations then would imply $G_2(\rho) = 0$, $G_2(\psi) = 0$, which in turn would lead to $[G_1, G_3] = 0$, contradicting the last relation in (2). Without loss of generality, we may therefore assume that no function ρ exists such that $G_1 = \rho G_2$. As a result, there exists a regular point transformation $Q = Q(q, t)$, $T = T(q, t)$, transforming the generators G_1 and G_2 to the standard form

$$\bar{G}_1 = \partial/\partial T \quad \bar{G}_2 = \partial/\partial Q. \quad (4)$$

Representing, in these coordinates, \bar{G}_3 in a general form like (3), it follows from the commutation relations involving G_3 that one must have

$$\partial\tau/\partial Q = 0 \quad \partial\xi/\partial Q = 0 \quad \partial\tau/\partial T = 0 \quad \partial\xi/\partial T = 1.$$

Therefore (disregarding constant multiples of \bar{G}_1 and \bar{G}_2), the transformed equation must have the symmetry

$$\bar{G}_3 = T \partial/\partial Q. \quad (5)$$

From (4) it is obvious that the right-hand side of the differential equation for Q cannot depend on T and Q . Expressing the invariance with respect to the generator (5) then straightforwardly shows that it cannot depend on $Q' = dQ/dT$ as well. The transformed equation will thus be of the form $Q'' = \text{constant}$ (Lie 1891), which is an obviously linear differential equation and consequently has $SL(3, \mathcal{R})$ symmetry. The original equation in q accordingly has the same symmetry, which completes the proof of the sufficiency. The necessity is trivial because \mathfrak{N} is a subalgebra of $\mathfrak{sl}(3, \mathcal{R})$.

Remark. Whenever a non-linear differential equation is found to have eight symmetry generators, the above result also contains a hint for actually trying to construct a coordinate transformation which does the linearisation. Indeed, it will be advantageous to search for three combinations of the obtained generators which constitute \mathfrak{N} . If this can be done, a linearisation will follow from transforming two of the commuting symmetries to the standard form (4).

One can push the question which has been dealt with in the above proposition a bit further and ask whether linearisability will not even follow from assuming less than the three-dimensional algebra \mathfrak{K} . This is indeed the case. As a matter of fact, it follows from our argumentation that to within removing from G_3 a constant multiple of G_1 , G_2 and G_3 commute and further satisfy $G_3 = \psi(t, q)G_2$ for some function ψ . This implies the existence of a coordinate transformation (already established by Lie (1891)) transforming the generators to

$$\bar{G}_2 = \partial/\partial Q \quad \bar{G}_3 = T \partial/\partial Q \quad (6)$$

as a result of which the differential equation appears in the form $Q'' = F(T)$. Once again this is a linear equation having $SL(3, \mathcal{R})$ symmetry.

The other subalgebra of (2) constitutes a more appealing subject for further study, if only because it is not trivial. Having two commuting symmetries which are independent means that the differential equation transforms to the general form (Lie 1891)

$$Q'' = F(Q') \quad (7)$$

which in itself is already a significant reduction, since equations of type (7) may be solved by two consecutive quadratures. In the following sections, we wish to investigate the possible existence of more symmetries for (7), thereby keeping an eye on the identification of further criteria which will ensure linearisability.

2. The symmetry conditions for a class of non-linear equations

Starting from the assumption that the equations under investigation have two independent commuting symmetries, we take the preliminary transformation to the standard form (4) of these symmetries for granted and rewrite the resulting equation in lower case variables as

$$\ddot{q} = f(\dot{q}). \quad (8)$$

A generator of the form (3) will be a symmetry for (8) if and only if τ and ξ satisfy the following requirement (invariance of (8) under the second extension of G):

$$\xi_{tt} + \dot{q}(2\xi_{tq} - \tau_{tt}) + \dot{q}^2(\xi_{qq} - 2\tau_{tq}) - \dot{q}^3\tau_{qq} + (\xi_q - 2\tau_t - 3\dot{q}\tau_q)f = [\xi_t + \dot{q}(\xi_q - \tau_t) - \dot{q}^2\tau_q]f_q \quad (9)$$

where the suffices refer to partial derivatives.

It is impossible to proceed further with such a complicated equation without making further assumptions about the nature of the function $f(\dot{q})$. The polynomial character in \dot{q} of all the coefficients in (9) strongly suggests looking at the case where f itself is a polynomial. Obviously we are not interested in the case that f is linear in \dot{q} . Suppose then that the leading order term in f is of degree n with $n \geq 3$. We write

$$f = k\dot{q}^n + l\dot{q}^{n-1} + \dots$$

Looking at the coefficients of \dot{q}^{n+1} and \dot{q}^n in (9), we then obtain the following conditions:

$$(n-3)k\tau_q = 0 \quad (10)$$

$$(n-4)l\tau_q + k[(n-2)\tau_t - (n-1)\xi_q] = \tau_{qq}\delta_{n3} \quad (11)$$

where δ_{n3} is the Kronecker delta.

For any n greater than 3, it is clear that (10) and (11) allow τ at most to be a function of t and ξ to be a linear function in q with possible time-dependent coefficients. One can subsequently look at the terms of degree 0, 1 and 2 in (9) and infer from a simple analysis that in the most favourable case there can be at most three symmetry generators, provided the coefficients in f satisfy a number of algebraic relations. The bigger n is, the larger the number of such restrictions on the coefficients will be so that the chances of obtaining more than the two given symmetries become smaller with increasing n . In conclusion, linearisability is certainly not possible for a polynomial f of degree greater than three and the case $n=3$ appears to be of a peculiar nature, because it is the only case for which the coefficients of the highest-order terms in (9) automatically cancel out, leaving the possibility for a non-trivial q dependence in τ . For all these reasons, we further restrict ourselves to polynomials of degree 3. For completeness, we study the quadratic polynomial case in an appendix. The reader may find some similarities between our analysis and a recent paper by Aguirre and Krause (1985). These authors also treat the case of a polynomial of degree 3 (with coefficients possibly depending on q and t). However they merely compute the commutation relations of symmetry generators, without investigating the conditions under which such generators exist and without analysing the nature of the resulting algebra, nor the question of linearisability.

Considering a differential equation of the form

$$\ddot{q} = g\dot{q}^3 + a\dot{q}^2 + b\dot{q} + c \quad g \neq 0 \quad (12)$$

where all coefficients are constant, we can make a preliminary rescaling of time to make the coefficient of the high-order term equal to one. Having done so, a transformation of the form

$$q = Q - \frac{1}{3}aT \quad t = T \quad (13)$$

will eliminate the quadratic term on the right-hand side. So, without loss of generality, we may assume that $g = 1$, $a = 0$ in (12) and therefore restrict our attention to equations of the form

$$\ddot{q} = \dot{q}^3 + b\dot{q} + c. \quad (14)$$

Let us repeat first of all that an equation like (14) can certainly be solved by two quadratures. Being able to solve a differential equation does not necessarily mean, however, that the same equation can be linearised by a point transformation. Among other things, we wish to find out under what circumstances a linearisation for (14) exists, thereby keeping in mind that we regard (14) as a representative of a whole class of differential equations which can be transformed to it once two commuting symmetries are known. It will appear soon that the study of the full symmetry group of (14) is quite interesting in its own right.

With an $f(\dot{q})$ as in (14), equating the coefficients of like powers of \dot{q} in the symmetry condition (9) give rise to the following set of partial differential equations:

$$\tau_{qq} + 2\xi_q - \tau_t = 0 \quad (15)$$

$$2b\tau_q + 3\xi_t - \xi_{qq} + 2\tau_{tq} = 0 \quad (16)$$

$$b\tau_t + 3c\tau_q - 2\xi_{tq} + \tau_{tt} = 0 \quad (17)$$

$$b\xi_t - c\xi_q + 2c\tau_t - \xi_{tt} = 0. \quad (18)$$

We can solve (15) for ξ_q and use the result to solve (16) for ξ_t , thus obtaining

$$\xi_q = \frac{1}{3}(\tau_t - \tau_{qq}) \quad (19)$$

$$\xi_t = -\frac{1}{6}(\tau_{qqq} + 3\tau_{tq} + 4b\tau_q). \quad (20)$$

Substituting these expressions into (17) and (18) we obtain two partial differential equations involving τ alone. Expressing in addition that we want the system (19) and (20) to be completely integrable for ξ we end up with the following requirements on τ (because of the order of the equations, it is preferable to abandon the index notation for partial derivatives at this stage):

$$\frac{\partial^3 \tau}{\partial t \partial q^2} + 3c \frac{\partial \tau}{\partial q} + b \frac{\partial \tau}{\partial t} = 0 \quad (21)$$

$$\frac{\partial^4 \tau}{\partial t \partial q^3} - b \frac{\partial^3 \tau}{\partial q^3} + 3 \frac{\partial^3 \tau}{\partial t^2 \partial q} + 3c \frac{\partial^2 \tau}{\partial q^2} + b \frac{\partial^2 \tau}{\partial t \partial q} - 4b^2 \frac{\partial \tau}{\partial q} + 9c \frac{\partial \tau}{\partial t} = 0 \quad (22)$$

$$\frac{\partial^4 \tau}{\partial q^4} + 4b \frac{\partial^2 \tau}{\partial q^2} + 3 \frac{\partial^2 \tau}{\partial t^2} = 0. \quad (23)$$

The strategy now is clear: we have to solve the above equations for τ first, after which (19) and (20) will determine the corresponding components ξ of symmetry generators. Before embarking upon this task, let us try to simplify the equations for τ . In the first place, using $\partial/\partial q$ of condition (21), (22) readily simplifies to

$$-b \frac{\partial^3 \tau}{\partial q^3} + 3 \frac{\partial^3 \tau}{\partial t^2 \partial q} - 4b^2 \frac{\partial \tau}{\partial q} + 9c \frac{\partial \tau}{\partial t} = 0. \quad (24)$$

Next, observe that the combination $(21)_{qq} - (23)_t$, results in a homogeneous third-order partial differential equation for τ , namely

$$\frac{\partial^3 \tau}{\partial t^3} + b \frac{\partial^3 \tau}{\partial t \partial q^2} - c \frac{\partial^3 \tau}{\partial q^3} = 0. \quad (25)$$

In order to see whether (25) can actually replace the fourth-order equation (23), suppose conversely that τ is a solution of the set of equations (21), (24) and (25). Then, we can arrive at an equation of type (23) in two different ways. Indeed, it is straightforward to verify that

$$(24)_q - 3(21)_t \sim b(23)$$

$$3(25)_q - (24)_t - 4b(21)_q \sim c(23).$$

Therefore we distinguish two cases.

Case I. b and c not both zero. It then follows from the above observations that the original conditions on τ can equivalently be replaced by the set of third-order conditions (21), (24) and (25).

Case II. $b = c = 0$. The requirements for τ readily simplify to

$$\frac{\partial^3 \tau}{\partial t \partial q^2} = 0 \quad \frac{\partial^3 \tau}{\partial t^2 \partial q} = 0 \quad \frac{\partial^4 \tau}{\partial q^4} + 3 \frac{\partial^2 \tau}{\partial t^2} = 0. \quad (26)$$

Remark. There is still a certain amount of redundancy hidden in the conditions for case I. For example, when $b = 0$ ($c \neq 0$) one can show that, apart from the homogeneous equation (25), the other requirements integrate to the single condition $\tau_{,q} + 3c\tau = \text{constant}$. On the other hand, for $b \neq 0$ we have seen above that (21) and (24) actually imply (23) and therefore also (25). It is, however, more convenient to study first the solutions of the homogeneous equation (25) anyway and in doing so it will turn out that a discussion about b being zero or not is rather irrelevant.

All partial differential equations for τ we have been dealing with have certain remarkable properties which can be of great help in constructing particular solutions. Consider for example equation (21) which we rewrite in the form

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \tau}{\partial q^2} + b\tau \right) + \frac{\partial}{\partial q} (3c\tau) = 0.$$

This form implies that there must exist a function ϕ such that

$$\frac{\partial^2 \tau}{\partial q^2} + b\tau = \frac{\partial \phi}{\partial q} \quad -3c\tau = \frac{\partial \phi}{\partial t}. \quad (27)$$

One can easily verify that if τ is a particular solution of (21), then the function ϕ generated by (27) is another particular solution of the same equation. In other words, (27) is a so-called auto-Bäcklund transformation for (21) (Rogers and Shadwick 1982). The general property behind this observation is as follows. Consider two relations of the form

$$a \frac{\partial^{k+l} u}{\partial x^k \partial y^l} + cu = \frac{\partial v}{\partial y} \quad b \frac{\partial^{i+j} u}{\partial x^i \partial y^j} + eu = \frac{\partial v}{\partial x} \quad (28)$$

where a , b , c and e are constants. Expressing the integrability condition $v_{xy} = v_{yx}$ results in a partial differential equation to be satisfied by u . Eliminating, on the other hand, the derivatives of u from (28) by calculating the combination

$$b \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} \left(\frac{\partial v}{\partial y} \right) - a \frac{\partial^k}{\partial x^k} \frac{\partial^l}{\partial y^l} \left(\frac{\partial v}{\partial x} \right)$$

and making use of (28) again to express everything in terms of v , results in exactly the same partial differential equation to be satisfied by v .

Returning to our equations for τ one can observe that there are actually different ways of rewriting some of them in a form like (28) and there is always one particular solution at hand, namely $\tau = \text{constant}$. Through the process of (27), for example, this solution will generate an infinite number of polynomial in q and t solutions of equation (21), which of course will be drastically limited by the restrictions coming from the other equations. We will not use this technique here, because the general solution of our equations (21), (24), and (25) can be obtained in a straightforward manner.

3. Obtaining the full symmetry group

3.1. Triple root case

We start with the simple case II, for which we have to solve equations (26). The first of these implies that τ must be of the form

$$\tau = f(q) + h(t)q + g(t) \quad (29)$$

of equation (25) in the form

$$\left(\frac{\partial}{\partial t} + 2\alpha \frac{\partial}{\partial q}\right) \left(\frac{\partial}{\partial t} - \alpha \frac{\partial}{\partial q}\right)^2 \tau = 0 \quad (31)$$

with $\alpha \neq 0$. We then have

$$b = -3\alpha^2 \quad c = -2\alpha^3. \quad (32)$$

The general solution of (31) is given by (compare with (29))

$$\tau = f(q - 2\alpha t) + g(q + \alpha t) + (q - 2\alpha t)h(q + \alpha t) \quad (33)$$

where f , g and h are as yet arbitrary functions of their arguments. According to the results of the previous section, we next have to turn to equations (21) and (24). Equation (21) becomes

$$-2\alpha f''' + \alpha(g''' - 9\alpha^2 g') + \alpha(q - 2\alpha t)(h''' - 9\alpha^2 h') = 0$$

where the prime denotes differentiation with respect to the appropriate argument. In view of the different arguments involved and the fact that $\alpha \neq 0$, it follows that we must have

$$f''' = 0 \quad g''' - 9\alpha^2 g' = 0 \quad h''' - 9\alpha^2 h' = 0. \quad (34)$$

The remaining condition (24) then turns out to be identically satisfied. Solving equations (34) is a straightforward matter. The resulting solution for τ takes the form

$$\begin{aligned} \tau = & c_1 + c_2(q - 2\alpha t) + c_3(q - 2\alpha t)^2 + c_4 \sinh 3\alpha(q + \alpha t) \\ & + c_5 \cosh 3\alpha(q + \alpha t) + c_6(q - 2\alpha t) \sinh 3\alpha(q + \alpha t) \\ & + c_7(q - 2\alpha t) \cosh 3\alpha(q + \alpha t). \end{aligned} \quad (35)$$

Just as in the previous case, this means that we again have a full eight-parameter group of symmetries, the generators of which are given in table 3. For shorthand, we introduce the new variables

$$T = q + \alpha t \quad Q = q - 2\alpha t. \quad (36)$$

We do not list the commutation relations of these generators. They turn out to be rather messy and it would be a major task to try and reorganise the set of generators in such a way that some standard $sl(3, \mathcal{R})$ pattern would emerge. The hints which were explored in the introduction appear to be of great help here for convincing

Table 3.

	τ	ξ
G_1	0	1
G_2	1	0
G_3	Q	$-\alpha Q$
G_4	Q^2	$-\alpha Q^2 - Q$
G_5	$\sinh 3\alpha T$	$\frac{1}{2}\alpha \sinh 3\alpha T - \frac{3}{2}\alpha \cosh 3\alpha T$
G_6	$\cosh 3\alpha T$	$\frac{1}{2}\alpha \cosh 3\alpha T - \frac{3}{2}\alpha \sinh 3\alpha T$
G_7	$Q \sinh 3\alpha T$	$\frac{1}{2}(\alpha Q - 1) \sinh 3\alpha T - \frac{1}{2}(3\alpha Q + 1) \cosh 3\alpha T$
G_8	$Q \cosh 3\alpha T$	$-\frac{1}{2}(3\alpha Q + 1) \sinh 3\alpha T + \frac{1}{2}(\alpha Q - 1) \cosh 3\alpha T$

ourselves that we are indeed dealing again with $SL(3, \mathcal{R})$ symmetry. To see this, observe that we have

$$G_7 - G_8 = Q(G_5 - G_6) \quad (37)$$

and

$$(G_5 - G_6)(Q) = 0 \quad (38)$$

from which it follows that

$$[G_7 - G_8, G_5 - G_6] = 0. \quad (39)$$

The symmetries $G_7 - G_8$ and $G_5 - G_6$ therefore can play the role of G_2 and G_3 in the introductory discussion leading to equation (6) and their transformation to the standard form (6) will linearise the equation, meaning that the algebra of symmetries must be a representation of $sl(3, \mathcal{R})$.

3.3. Three distinct factors for equation (25)

In this case, the factorisation of (25) is

$$\left(\frac{\partial}{\partial t} + 2\alpha\frac{\partial}{\partial q}\right)\left(\frac{\partial}{\partial t} - (\alpha + \gamma)\frac{\partial}{\partial q}\right)\left(\frac{\partial}{\partial t} - (\alpha - \gamma)\frac{\partial}{\partial q}\right)\tau = 0. \quad (40)$$

The constant α thereby is real, whereas γ can be real or purely imaginary, depending on whether our related cubic equation has three distinct real roots or one real and two complex conjugate roots. In both cases we certainly have

$$\gamma \neq \pm 3\alpha \quad \gamma \neq 0 \quad (41)$$

and the expression for the coefficients b and c in terms of α and γ is given by

$$b = -(3\alpha^2 + \gamma^2) \quad c = -2\alpha(\alpha^2 - \gamma^2). \quad (42)$$

The general solution of (40) is

$$\tau = f(g - 2\alpha t) + g[q + (\alpha + \gamma)t] + h[q + (\alpha - \gamma)t]. \quad (43)$$

Inserting this expression into the remaining conditions (21) and (24) again gives rise to simple differential equations for f , g and h and it follows as before that there are eight symmetry generators. For writing them down in a real form, one has to make a distinction between the case γ real and the case γ purely imaginary. Since the full expressions become rather complicated, we prefer to continue the probe into the nature of the algebra after a simplifying transformation.

3.3.1. Three real roots. The factorisation (40) of the partial differential equation (25) means that we are actually looking at the case for which our differential equation (14) can be written as

$$\ddot{q} = (\dot{q} - 2\alpha)[\dot{q} + (\alpha + \gamma)][\dot{q} + (\alpha - \gamma)]. \quad (44)$$

Under the coordinate transformation

$$\bar{q} = q + (\alpha + \gamma)t \quad \bar{t} = q + (\alpha - \gamma)t \quad (45)$$

equation (44) reduces to the form

$$\bar{q}'' = (3\alpha - \gamma)\bar{q}'^2 - (3\alpha + \gamma)\bar{q}'. \quad (46)$$

So we are down to an equation of the type discussed in the appendix. We can further reduce it to one of the standard types treated there by the subsequent transformation

$$\bar{q} = (3\alpha - \gamma)\bar{q} - \frac{1}{2}(3\alpha - \gamma)\bar{t} \quad \bar{t} = \bar{t}. \quad (47)$$

This brings us to the equation

$$\bar{q}'' = \bar{q}'^2 - \frac{1}{4}(3\alpha + \gamma)^2 \quad (48)$$

which clearly is of type (A2). Since we found $SL(3, \mathcal{R})$ symmetry for (A2), we can draw the same conclusion about (44) and the symmetry generators, if wanted, can be obtained from those in table A1, using the transformations (45) and (47).

3.3.2. One real and two complex conjugate roots. The factorisation (40), with $\gamma = i\omega$, now relates to a differential equation of the form

$$\ddot{q} = (\dot{q} - 2\alpha)(\dot{q}^2 + 2\alpha\dot{q} + \alpha^2 + \omega^2). \quad (49)$$

The transformation

$$\bar{q} = -(9\alpha^2 + \omega^2)t \quad \bar{t} = q - 2\alpha t \quad (50)$$

is found to reduce the cubic right-hand side (49) again to a quadratic one, explicitly

$$\bar{q}'' = \bar{q}'^2 - 6\alpha\bar{q}' + 9\alpha^2 + \omega^2. \quad (51)$$

Putting

$$\tilde{q} = \bar{q} - 3\alpha\bar{t} \quad \tilde{t} = \bar{t} \quad (52)$$

eventually brings us to the standard type

$$\tilde{q}'' = \tilde{q}'^2 + \omega^2 \quad (53)$$

for which the symmetry generators are listed in table A1. Again, we have $SL(3, \mathcal{R})$ symmetry.

Let us briefly discuss the meaning of the results of this section. When one performs an arbitrary coordinate transformation

$$T = F(q, t) \quad Q = G(q, t) \quad (54)$$

on the free particle equation $Q'' = 0$, there results a differential equation which is at most cubic in \dot{q} . Explicitly it becomes

$$[FG]_{t,q}\ddot{q} + [F, G]_{q,q^2}\dot{q}^3 + ([F, G]_{t,q^2} + 2[F, G]_{q,tq})\dot{q}^2 + ([F, G]_{q,t^2} + 2[F, G]_{t,tq})\dot{q} + [F, G]_{t,t^2} = 0 \quad (55)$$

where, for example, $[F, G]_{q,q^2}$ is a shorthand notation for

$$[F, G]_{q,q^2} = \frac{\partial F}{\partial q} \frac{\partial^2 G}{\partial q^2} - \frac{\partial^2 F}{\partial q^2} \frac{\partial G}{\partial q}.$$

Equation (55) therefore represents the most general type of second-order differential equation which can be linearised by a point transformation. It is not at all obvious from this observation that all constant coefficient equations like (12) are linearisable, because this involves imposing four partial differential equations on the two functions F and G . Our results, however, show that indeed all equations of type (12) are linearisable. Moreover, we started looking at such equations as representing a whole

class of equations which have two commuting symmetries. In this respect, we are now able to state the following conclusion: a second-order differential equation which has two commuting point symmetries G_1, G_2 (G_2 not being of the form $\rho(q, t)G_1$ for some function ρ) is linearisable, if and only if the transformation which brings G_1, G_2 into standard form (4) reduces the differential equation to one which is at most cubic in \dot{q} .

To complete this study, we construct in the next section a linearising transformation for all cases we have been led to distinguish.

4. Linearising transformations

4.1. The triple root case

For the equation

$$\ddot{q} = \dot{q}^3 \quad (56)$$

we observed in § 3 that the algebra $\{G_1, G_2, G_3\}$ is \mathfrak{N} . Now, even though the commuting generators G_1 and G_2 are in standard form, our equation (56) is not linear! This can only mean that we have, so to speak, the wrong coordinate playing the role of t . The transformation

$$Q = t \quad T = q \quad (57)$$

indeed reduces the equation to $Q'' = -1$, so that the relation

$$Q = -\frac{1}{2}T^2 + c_1T + c_2 \quad (58)$$

through (57) implicitly defines the solution of (56).

4.2. The double root case

For a differential equation of the form

$$\ddot{q} = (\dot{q} - 2\alpha)(\dot{q} + \alpha)^2 \quad (59)$$

we know from the discussion in the previous section that a linearisation can be obtained if we transform the symmetry generators $G_5 - G_6$ and $G_7 - G_8$ to the form (6). It turns out, however, that the coordinates (36), which were merely introduced to simplify the notation, do the job as well. They reduce equation (59) to the linear equation

$$Q'' = 3\alpha Q' \quad (60)$$

whose solution is given by

$$Q = c_1 + c_2 e^{3\alpha T}. \quad (61)$$

The solution of (59) thus is implicitly defined by (61) and the transformation formulae (36).

4.3. The case of three distinct real roots

In view of the reduction which was achieved in the preceding section, we merely have to linearise equation (A2). From the observations in the appendix, it follows that a

linearisation will be obtained if we transform for example the symmetries G_7 and G_3 to the standard form $\partial/\partial T, \partial/\partial Q$. Such a transformation must satisfy the requirements

$$\begin{aligned} e^{q-\omega t} \left(\frac{\partial T}{\partial q} \frac{\partial}{\partial T} + \frac{\partial Q}{\partial q} \frac{\partial}{\partial Q} \right) &= \frac{\partial}{\partial T} \\ e^{-2\omega t} \left(\frac{\partial T}{\partial t} \frac{\partial}{\partial T} + \frac{\partial Q}{\partial t} \frac{\partial}{\partial Q} \right) + \omega e^{-2\omega t} \left(\frac{\partial T}{\partial q} \frac{\partial}{\partial T} + \frac{\partial Q}{\partial q} \frac{\partial}{\partial Q} \right) &= \frac{\partial}{\partial Q} \end{aligned} \quad (62)$$

or

$$\begin{aligned} \frac{\partial Q}{\partial q} &= 0 & e^{-2\omega t} \frac{\partial Q}{\partial t} &= 1 \\ e^{q-\omega t} \frac{\partial T}{\partial q} &= 1 & \frac{\partial T}{\partial t} + \omega \frac{\partial T}{\partial q} &= 0. \end{aligned}$$

We straightforwardly obtain

$$Q = \frac{1}{2\omega} e^{2\omega t} \quad T = -e^{-q+\omega t} \quad (63)$$

and the reduced differential equation becomes $Q'' = 0$. Hence, the relation

$$c_1 e^{-q-\omega t} + c_2 e^{-2\omega t} = 1 \quad (64)$$

implicitly defines the solution of (A2). For obtaining the solution of (44), we just have to replace ω by $\frac{1}{2}(3\alpha + \gamma)$ and take account of the two linear transformations (45) and (47).

4.4. The case of one real and two complex conjugate roots

For linearising equation (A4), we can, for example, exploit the fact that the generators G_7 and G_8 commute and are proportional to each other. Therefore, a transformation which reduces G_7 to $\partial/\partial Q$ and G_8 to $T \partial/\partial Q$ will do the job. Such a transformation is easily obtained as

$$T = \tan \omega t \quad Q = -e^{-q}/\cos \omega t \quad (65)$$

and the reduced equation happens to be the free particle equation. As a result, the relation

$$(c_1 \sin \omega t + c_2 \cos \omega t) e^q = 1 \quad (66)$$

defines the solution of (A4) and obtaining the solution of (49) is just a matter of taking further account of the linear transformations (50) and (52).

5. Discussion

The general problem of equivalence for second-order equations $\ddot{q} = H(\dot{q}, q, t)$ under point transformation was considered by Tresse (1896). He constructed all semi-invariants for such equations. For our discussion, however, it suffices to mention only the functionally independent order-four semi-invariants of which there are two. The vanishing of one of them $\partial^4 H / \partial \dot{q}^4$ is a necessary condition for linearisation (i.e.

reduction to any linear equation such as, for example the free particle equation). This, one may recall, also follows from equation (55). Furthermore, for an equation in normal form (Arnold 1983)

$$\ddot{q} = A(t)q^2 + O(|q|^3 + |\dot{q}|^3)$$

the other semi-invariant is the scalar invariant

$$I = 5A\ddot{A} - 12\dot{A}^2$$

which is constructed from the differential form of order $\frac{5}{2}$, namely

$$\omega = A(t)|dt|^{5/2}.$$

It turns out that the vanishing of the form ω is a necessary condition for linearisation. The two conditions $\omega = 0$ and $\partial^4 H / \partial \dot{q}^4 = 0$, however, are sufficient for linearisation. The geometric theory of second-order equations, therefore, also provides criteria for linearisation. In this regard we further cite the work of Cartan (1924) who in particular investigated second-order equations cubic in the first derivative. The question of linearisability of differential equations has recently been discussed by other authors as well. We can mention, for example, a paper by Berkovich (1979) on ordinary differential equations, quadratic in \dot{q} and a contribution by Kumei and Bluman (1982), dealing primarily with systems of partial differential equations. The ideas put forward in our paper bear some resemblance to the latter reference, because we discuss linearisability in terms of the existence of certain point symmetries of the differential equation.

We have shown that every equation which has two commuting (non-proportional) symmetries will be linearisable, provided that the transformation which brings these symmetries in their standard form reduces the differential equation to one which is at most cubic in \dot{q} . An interesting topic for further study would be to find practical criteria for testing given second-order differential equations with respect to the existence of two such commuting point symmetries.

We have also observed that, when the equation has two proportional commuting symmetries, linearisation is immediate. In this way, we have in fact treated two of the four possible cases in the classification of two-dimensional algebras of symmetry generators of a single second-order differential equation (Lie 1891). The other two cases are characterised by

$$[G_1, G_2] = G_1 \quad \text{with} \quad G_2 \neq \rho(q, t)G_1 \quad (67)$$

or

$$[G_1, G_2] = G_1 \quad \text{with} \quad G_2 = \rho(g, t)G_1. \quad (68)$$

In the latter case, the standard form for the generators is given by $\bar{G}_1 = \partial / \partial Q$, $\bar{G}_2 = Q\partial / \partial Q$ and the transformed differential equation is linear in Q' (Lie 1891). Whenever an equation has two point symmetries (68), we can therefore again conclude that it will have $SL(3, \mathcal{R})$ symmetry. Just as it was the case with two commuting symmetries, the situation is less trivial when G_1 and G_2 are not proportional, as in (67). The standard form of such symmetries then is given by

$$\bar{G}_1 = \frac{\partial}{\partial Q} \quad \bar{G}_2 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}$$

and the transformed differential equation is $Q'' = T^{-1}F(Q')$ (Lie 1891). The study, along the lines of the present paper, of the possible existence of more than two point symmetries for the differential equation corresponding to this case is currently under investigation.

Finally, we would like to point out that one can think of generalising this study to systems with more than one degree of freedom, whereby the generalisation of the case treated in detail here would be the most appealing one. We would then be talking about systems

$$\ddot{q}^i = f^i(t, q, \dot{q}) \quad i = 1, \dots, n$$

having up to $n + 1$ commuting point symmetries, which accordingly can be transformed to the standard form $\partial/\partial T, \partial/\partial Q^1, \dots, \partial/\partial Q^n$.

Acknowledgments

This work was accomplished while one of us (WS) was visiting the University of the Witwatersrand. He gratefully acknowledges the hospitality of the Department of Applied Mathematics of that institution. FM wishes to thank the CSIR of South Africa for financial assistance.

Appendix. Quadratic in \dot{q} equations

Considering a general equation of the form

$$\ddot{q} = a\dot{q}^2 + b\dot{q} + c \quad (\text{A1})$$

we can perform a preliminary rescaling of q to make $a = 1$. Having done so, a transformation of the form $Q = q + \frac{1}{2}bt$ will further eliminate the \dot{q} term. Henceforth, we can restrict our attention to the case where $a = 1$ and $b = 0$ in (A1) and we have to distinguish two different situations.

A1. The right-hand side has real roots

The equation under investigation has the form

$$\ddot{q} = \dot{q}^2 - \omega^2. \quad (\text{A2})$$

The general symmetry requirement (9) for this case gives rise to the following set of partial differential equations:

$$\begin{aligned} \tau_q + \tau_{qq} &= 0 \\ \xi_q - \xi_{qq} + 2\tau_{tq} &= 0 \\ 2\xi_t - 3\omega^2\tau_q - 2\xi_{tq} + \tau_{tt} &= 0 \\ \omega^2\xi_a - 2\omega^2\tau_t - \xi_{tt} &= 0. \end{aligned} \quad (\text{A3})$$

They can easily be solved and are found to produce the eight symmetry generators given in table A1.

The partial differential equations for obtaining all point symmetries of (A4) of course differ only slightly from (A3). We again find eight generators which are listed in table A3. The commutator algebra is given in table A4. The \mathfrak{N} subalgebra is found with the generators $\{G_8, G_7, \frac{1}{2}(G_2 + G_3)\}$ or $\{G_5, G_6, \frac{1}{2}(G_2 + G_3)\}$. So it is by now no longer a surprise that we again have $SL(3, \mathcal{R})$ symmetry. The linearisation of (A4) via point transformation is given by (65). Once more the transformed equation is the free particle equation.

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Equivalence classes of differential equations

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Nonlinear Evolutions J P Lions and F Calogero edd (World Scientific,
Singapore, 1987) 781-790

EQUIVALENCE CLASSES OF DIFFERENTIAL EQUATIONS

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ABSTRACT

We show the necessity to consider general pseudo-group of point transformations $Q = F(t, q)$, $T = G(t, q)$ in order to determine completely all equivalence classes of second-order ordinary differential equations, written in normal form $\ddot{q} = f(t, q, \dot{q})$, admitting Lie algebras of point symmetries.

Over the last few decades there has been a number of publications devoted to the study of ordinary differential equations (o.d.e.'s) utilising symmetry techniques. In particular, one-dimensional systems of second-order o.d.e.'s have received a fair amount of attention, especially linear systems.

It is well known that all linear single second-order o.d.e.'s possess the eight-dimensional Lie algebra $sl(3, \mathbb{R})$ as their symmetry algebra.^{1,2} Moreover, recent advances have shown that any nonlinear second-order o.d.e. also has the Lie algebra $sl(3, \mathbb{R})$ provided it has eight generators of symmetry.^{2,3} This follows from the invariance of the symmetry algebra under a point transformation, a feature which enables the nonlinear equation to be transformed to any linear second-order equation (such as the free particle equation). Thus a necessary and sufficient condition for a second-order o.d.e. to be linearizable via a point transformation is that it admits the Lie algebra $sl(3, \mathbb{R})$.

The question then arises as to whether we need to know the full eight symmetries of an equation before we can conclude that a linearization will exist. If not, one may enquire as to what are the subalgebras of $sl(3, \mathbb{R})$ which will result in the linearization of an equation?

It is not necessary to answer the above question in its entire generality since we do not require all the linearizing subalgebras of $sl(3, \mathbb{R})$. For our purpose it suffices to

consider only the minimal linearizing algebras L (see §2 for a formal definition) which properly contain all point symmetry subalgebras associated with equivalence classes of nonlinearizable second-order o.d.e.'s. Thus, if we were to use a method to determine (the number of) point symmetries of a second-order o.d.e., this method should be based within a framework in which the maximum symmetry algebra is at least of dimension $s = \dim \max \{l \in L\}$.

In this work we show that $s = 8$. The implication here is that the maximum (unique) symmetry algebra for second-order o.d.e.'s of dimension one cannot be anything less than $sl(3, \mathbb{R})$ if all equivalence classes of equations admitting point symmetry algebras are to be included. This does not, however, rule out the possibility of having maximal symmetry algebras of dimensions less than eight providing they do exist. They do, indeed, exist. The following question immediately suggests itself. What is the least dimension of these algebras so that they do not omit any equivalence class of equations admitting point symmetry algebras? We show that it is three.

This brings us to some of the well known approaches to tackling second-order equations. Amongst the most frequently used is the Painlevé-Gambier classification of second-order o.d.e.'s (see e.g. Ince⁴) which produces fifty equations. In this context we further cite the Cartan equivalence method as applied to second-order equations^{5,6}. These studies reveal a maximum symmetry algebra of dimension six since they are based on pseudo-groups of transformations of the restricted type $Q = F(t, q)$, $T = G(t)$. Consequently, these methods fail to take into account certain equivalence classes of equations on which we are able to report as we, in general, allow for mixing of the independent and dependent variables (i.e. $Q = F(t, q)$, $T = G(t, q)$).

Another study, one which motivated our investigations, is the classical Noether theorem (Noether's theorem with point transformations⁷). This theorem establishes a precise correspondence between equivalence classes of symmetries (invariance transformations of the action functional up to gauge terms) and first integrals. It should be pointed out that such a link is strictly with respect to (*w.r.t.*) a fixed Lagrangian description, bearing in mind that there are ambiguities in the possible Lagrangian description of a given second-order equation. The inverse problem for Lagrangian mechanics guarantees the existence of a Lagrangian for every one-dimensional system although the construction of such a Lagrangian can be highly nontrivial despite there being infinitely many of them. Thus, in principle, the range of applicability of Noether's theorem extends to all one-dimensional systems. This theorem has a framework in which the maximal symmetry algebras are of dimension five. Hence we expect Noether's theorem to provide sufficient information as to the point symmetry structure of any given second-order o.d.e. Of course, this should be *w.r.t.* the natural Lagrangian. For instance, the Noether theorem applied to the free particle equation $\ddot{q} = 0$, *w.r.t.* the natural (in this case the usual) Lagrangian $L = \frac{1}{2} \dot{q}^2$,

yields five symmetries whereas *w.r.t.* the 'unusual' Lagrangian $L = -\frac{1}{2} t\dot{q}^2 + q\dot{q} \ln \dot{q} - q\dot{q}$ it gives rise to only one symmetry!

1. THE SYMMETRY APPROACH

Consider a general second-order o.d.e., written in normal form

$$\ddot{q} = f(t, q, \dot{q}), \quad (1.1)$$

where dot denotes differentiation *w.r.t.* t . A vector field

$$Y = \xi(t, q, \dot{q}) \frac{\partial}{\partial t} + \eta(t, q, \dot{q}) \frac{\partial}{\partial q} + \zeta(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}}$$

is said to be a dynamical symmetry of the second-order equation field

$$\Gamma = \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} + f(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}},$$

if (see⁷)

$$\mathcal{L}_Y \Gamma \equiv [Y, \Gamma] = g\Gamma \quad (1.2)$$

for a suitable function g .

In this work we concern ourselves with point transformations only. Hence we require the co-ordinate functions ξ and η of the vector field Y to be independent of \dot{q} . To denote this we write

$$G^{(1)} = \xi(t, q) \frac{\partial}{\partial t} + \eta(t, q) \frac{\partial}{\partial q} + \zeta(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}}.$$

The operator

$$G = \xi(t, q) \frac{\partial}{\partial t} + \eta(t, q) \frac{\partial}{\partial q} \quad (1.3)$$

is referred to as the generator of a Lie point symmetry of Γ . Furthermore, $G^{(1)}$ is called the first extension of G .

The set of all operators $\{G\}$ admitted by a given o.d.e. of the form (1.1) (i.e. (1.2) holds) generates a finite-dimensional Lie algebra of dimension at most eight.

Alternatively, given a finite-dimensional algebra realization in terms of vector fields in co-ordinates t and q , one can seek a second-order o.d.e. associated with this realization by invoking (1.2). This, however, is not always possible as there are realizations which are not symmetries of a second-order o.d.e.⁸.

2. SUBALGEBRAS OF $sl(3, \mathbb{R})$ AND LINEARIZATION.

We use the Mubarakzyanov classification of real low dimension Lie algebras given in refs.^{9,10}.

Suppose an equation of the type (1.1) admits the one-dimensional algebra A_1 . Then it has at least one symmetry of the form (1.3). We can reduce the equation to a co-ordinate independent form by means of a point transformation which brings the symmetry to a generator of co-ordinate translation. Thus an equation with a single symmetry belongs to the equivalence class of

$$\ddot{q} = f(t, \dot{q}),$$

where $\partial/\partial q$ is the canonical realization of the algebra A_1 and f is a definite function of t and \dot{q} . We now treat the case of equations admitting a two-dimensional algebra (abelian $2A$, or solvable A_2). Second-order o.d.e's possessing two generators of symmetry have four canonical forms for their generators^{1,2}. They are, with their associated differential equations,

Table I

TYPE	$[G_1, G_2]$	CONNECTEDNESS OF G_1 AND G_2	CANONICAL FORMS OF G_1 AND G_2	FORM OF EQUATION
I	0	$\forall \rho G_2 \neq \rho(t, q)G_1$ (unconnected)	$\bar{G}_1 = \frac{\partial}{\partial T}$ $\bar{G}_2 = \frac{\partial}{\partial Q}$	$Q'' = F(Q')$
II	0	$\exists \rho G_2 = \rho(t, q)G_1$ (connected)	$\bar{G}_1 = \frac{\partial}{\partial Q}$ $\bar{G}_2 = T \frac{\partial}{\partial Q}$	$Q'' = F(T)$
III	G_1	$\forall \rho G_2 \neq \rho(t, q)G_1$ (unconnected)	$\bar{G}_1 = \frac{\partial}{\partial Q}$ $\bar{G}_2 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}$	$TQ'' = F(Q')$
IV	G_1	$\exists \rho G_2 = \rho(t, q)G_1$ (connected)	$\bar{G}_1 = \frac{\partial}{\partial Q}$ $\bar{G}_2 = Q \frac{\partial}{\partial Q}$	$Q'' = Q'F(T)$

If an equation admits $2A_1$ or A_2 with G_1, G_2 satisfying $G_2 = \rho(t, q)G_1$ for some function (i.e. type II or IV), then it is linearizable via a point transformation (it has $SL(3, \mathbb{R})$ symmetry). Consider e.g. the type II representative equation. Under the

transformation

$$\tilde{Q} = Q - \int^T \int^R F(s) ds dR \quad \tilde{T} = T$$

the type II equation is equivalent to $\tilde{Q}'' = 0$. The six additional operators are easily obtained from the symmetries of $\tilde{Q}'' = 0$, using the above transformation.

Hereafter we shall refer to the realizations belonging to the equivalence classes of symmetries of type I to IV as $2A_1^I$, $2A_1^{II}$, A_2^{III} and A_2^{IV} respectively.

Let us introduce the following definition:

DEFINITION: Given a realization \mathcal{A} of a subalgebra of $sl(3, \mathbb{R})$, a realization \mathcal{L} of a subalgebra of $sl(3, \mathbb{R})$ is said to be a *minimal subalgebra for linearization w.r.t. \mathcal{A}* (MSL w.r.t. \mathcal{A}) if

- (i) $\mathcal{A} \subseteq \mathcal{L}$
- (ii) any second-order o.d.e. admitting \mathcal{L} is linearizable
- (iii) no proper subalgebra of \mathcal{L} has the properties (i) and (ii)

Hence $2A_1^{II}$ is a MSL w.r.t. $2A_1^I$ or $2A_1^I$ is a MSL w.r.t. $A_1: (\partial/\partial q)$

Likewise for A_2^{IV} we have A_1^{IV} is a MSL w.r.t. A_2^{IV} or A_2^{IV} is a MSL w.r.t. A_1 .

In what follows we restrict $\mathcal{A} \subset \mathcal{L}$. This we can do as $\mathcal{L} \geq 2$ from the above. Our choice of \mathcal{A} will be as follows: we require \mathcal{A} to be associated with representatives of equivalence classes of non-linearizable second-order o.d.e.'s. In this case \mathcal{A} can be at most of dimension three⁸. Accordingly we have

$$2A_1^{II}/A_2^{IV} \text{ is a MSL w.r.t. } A_1.$$

We focus attention on three dimensional algebras. There are eleven Lie algebras of dimension three¹⁰ (decomposable and indecomposable). For convenience and easy reference we list them

Table II

Algebra	Nonzero commutation relations
$3A_1$	
$A_1 \oplus A_2$	$[G_1, G_3] = G_1$
$A_{3,1}$ (Weyl)	$[G_2, G_3] = G_1$
$A_{3,2}$	$[G_1, G_3] = G_1, [G_2, G_3] = G_1 + G_2$
$A_{3,3}$ ($D \otimes T_2$)	$[G_1, G_3] = G_1, [G_2, G_3] = G_2$

$A_{3,4} (E(1,1))$	$[G_1, G_3] = G_1, [G_2, G_3] = -G_2$
$A_{3,5}^a (0 < a < 1)$	$[G_1, G_3] = G_1, [G_2, G_3] = aG_2$
$A_{3,6} (E(2))$	$[G_1, G_3] = -G_2, [G_2, G_3] = G_1$
$A_{3,7}^b (b > 0)$	$[G_1, G_3] = bG_1 - G_2, [G_2, G_3] = G_1 + bG_2$
$A_{3,8} (SL(2, \mathcal{R}))$	$[G_1, G_2] = G_1, [G_2, G_3] = G_3, [G_3, G_1] = -2G_2$
$A_{3,9} (SO(3))$	$[G_1, G_2] = G_3, [G_2, G_3] = G_1, [G_3, G_1] = G_2$

The realizations of these algebras in terms of vector fields in two co-ordinates and their associated equations were considered in ref.⁸⁾ We only list the relevant realizations and corresponding equations

Table III

Let $\partial/\partial t = p$ and $\partial/\partial q = r$

$A_{3,1}$	r	p	tr	$\ddot{q} = A$
$A_{3,2}$	r	$p - (\ln t)r$	$tp + qr$	$t\ddot{q} = -1 + A \exp(-\dot{q})$
	r	$-(\ln t)r$	$tp + qr$	$t\ddot{q} = -\dot{q} + B$
or	r	p	$tp + (t+q)r$	$\ddot{q} = A \exp(-\dot{q})$
	$-r$	tr	$p + qr$	$\ddot{q} = B \exp t$
$A_{3,3}$	p	r	$tp + qr$	$\ddot{q} = 0$
	r	tr	qr	$\ddot{q} = 0$
$A_1 \oplus A_2, A_{3,4}, A_{3,5}^a$	$(0, -1, a)$			
	p	r	$tp + aqr$	$\ddot{q} = A\dot{q}^{\frac{a-2}{a-1}}$
	r	tr	$(1-a)tp + qr$	$\ddot{q} = Bt^{\frac{1-2a}{a-1}}$
or	r	$t^{1-a}r$	$tp + qr$	$t\ddot{q} = -a\dot{q} + A$
	r	$t^{1-a}p$	$tp + qr$	$t\ddot{q} = (a-1)\dot{q} + B\dot{q}^{\frac{2a-1}{a-1}}$
$A_{3,6}, A_{3,7}^b$	$(0, b)$			
	p	r	$(bt+q)p$	$\ddot{q} = A(1+\dot{q})^{\frac{1}{2}}$
			$+(bq-t)r$	$\times \exp(b \tan^{-1} \dot{q})$
	tr	r	$(1+t^2)p$	$\ddot{q} = B(1+t^2)^{-\frac{1}{2}}$

			$+(tq + bq)r$	$\times \exp(b \tan^{-1} t)$
$A_{3,8}$	r	$tp + qr$	$2tqp$	$t\ddot{q} = \dot{q}^3 + \dot{q}$
			$+(q^2 - t^2)r$	$+A(1 + \dot{q}^2)^{\frac{3}{2}}$
	r	$tp + qr$	$2tqp + q^2r$	$t\ddot{q} = A\dot{q}^3 - \frac{1}{2}\dot{q}$
$A_{3,9}$	$(1 + t^2)p + tqr$	$tqp + (1 + q^2)r$	$qp - tr$	$\ddot{q} = 0$

Of the above algebras most have two realizations. We shall once again denote them as A^I, A^{II} . It follows from the above (and ref.⁸) that there are five representatives of equivalence classes of equations having three symmetries (in each case $A \neq 0$ and $a \in \mathbb{R}$). They are

$$\begin{aligned}
 t\ddot{q} &= \dot{q}^3 + \dot{q} + A(1 + \dot{q}^2)^{3/2} \\
 t\ddot{q} &= A\dot{q}^3 - \frac{1}{2}\dot{q} \\
 t\ddot{q} &= (a-1)\dot{q} + A\dot{q}^{\frac{2a-1}{a-1}} \quad \text{or} \quad \ddot{q} = A\dot{q}^{\frac{2a-2}{a-1}} \quad a \neq 0, 1/2, 1, 2 \\
 t\ddot{q} &= -1 + A \exp(-\dot{q}) \quad \text{or} \quad \ddot{q} = A \exp(-\dot{q}) \\
 \ddot{q} &= A(1 + \dot{q}^2)^{3/2} \exp(a \tan^{-1} \dot{q}).
 \end{aligned} \tag{2.1}$$

The remaining equations of Table III are linear including equation (2.1c) when $a = 0, 1/2, 1$ or 2 . It is now a straightforward matter to obtain the MSL *w.r.t.* $2A_1^I$ or A_2^{III} . However, in order to investigate the MSL *w.r.t.* the realizations associated with equations (2.1) we need to look at least at four dimensional algebras. Below, we list the four-dimensional algebras of relevance⁸ together with their three-dimensional subalgebras¹⁰

Table IV

Algebra	Nonzero Commutators	Dimension 3
$2A_2$	$[G_1, G_2] = G_2, [G_3, G_4] = G_4$	$A_1 \oplus A_2 : (G_1, G_3; G_2) (G_1, G_4; G_2)$ $(G_1, G_3; G_4) (G_2, G_3; G_4)$ $A_{3,3} : (G_1 + G_3; G_2, G_4)$ $A_{3,4} : (G_1 - G_3; G_2, G_4)$ $A_{3,5}^a : (G_1 + xG_3; G_2, G_4),$ $a = \begin{cases} x, & 0 < x < 1 \\ 1/x, & 1 < x < \infty \end{cases}$
$A_{3,8} \oplus A_1$	$[G_1, G_3] = 2G_2, [G_1, G_2] = G_1,$ $[G_2, G_3] = G_3$	$A_2 \oplus A_1 : (G_2, G_4; G_1)$ $A_{3,8} : (; G_1, G_2, G_3)$

$A_{4,7}$	$[G_1, G_4] = 2G_1, [G_2, G_4] = G_2,$ $[G_3, G_4] = G_2 + G_3,$ $[G_2, G_3] = G_1$	$A_{3,1} : (G_2, G_3; G_1)$ $A_{3,5}^{\frac{1}{2}} : (G_4; G_1, G_2)$
$A_{4,8}$	$[G_2, G_3] = G_1, [G_2, G_4] = G_2,$ $[G_3, G_4] = -G_3$	$A_{3,1} : (G_2, G_3; G_1)$ $A_2 \oplus A_1 : (G_4, G_1; G_2), (G_4, G_1; G_3)$
$A_{4,9}^b$ ($0 < b < 1$)	$[G_2, G_3] = G_1,$ $[G_1, G_4] = (1+b)G_1,$ $[G_2, G_4] = G_2,$ $[G_3, G_4] = bG_3$	$A_{3,1} : (G_2, G_3; G_1)$ $A_{3,5}^v : (G_4; G_1, G_2),$ $v = \begin{cases} 1+b, & 1+b < 1 \\ 1/(1+b), & 1+b > 1 \end{cases}$ $A_{3,4} : (G_4; G_1, G_3), b = -1/2$ $A_{3,5}^w : (G_4; G_1, G_3),$ $w = \begin{cases} b/(1+b), & b/(1+b) < 1 \\ (1+b)/b, & b/(1+b) > 1 \end{cases}$
$A_{4,9}^1$	$[G_2, G_3] = G_1, [G_1, G_4] = 2G_1,$ $[G_2, G_4] = G_2, [G_3, G_4] = G_3$	$A_{3,1} : (G_2, G_3; G_1)$ $A_{3,5}^{1/2} : (G_4; G_1, \cos \phi G_2 + \sin \phi G_3)$ $0 \leq \phi < \pi$
$A_{4,9}^0$	$[G_2, G_3] = G_1, [G_1, G_4] = G_1,$ $[G_2, G_4] = G_2$	$A_{3,1} : (G_2, G_3; G_1)$ $A_2 \oplus A_1 : (G_3, G_4; G_1)$ $A_{3,3} : (G_4; G_1, G_2)$ $A_{3,2} : (G_4 + xG_3; G_1, G_2) (x \neq 0)$
$A_{4,12}$	$[G_1, G_3] = G_1, [G_2, G_3] = G_2,$ $[G_1, G_4] = -G_2, [G_2, G_4] = G_1$	$A_{3,3} : (G_3; G_1, G_2)$ $A_{3,6} : (G_4; G_1, G_2)$ $A_{3,7}^{ x } : (G_4 + xG_3; G_1, G_2) (x \neq 0)$

We give the MSL *w.r.t.* $2A_1^I, A_2^{III}$ and the realizations associated with (2.1b) to (2.1e). For the MSL *w.r.t.* the realization associated with (2.1a) we must go beyond dimension four. A discussion of this case will be given after Table V.

Table V

\mathcal{L}	\mathcal{A}	\mathcal{L}	\mathcal{A}
$A_{3,1}$	$2A_1^I$	$A_{3,9}$	A_1
$A_{3,2}^{II}$	A_2^{III}	$2A_2$	$A_{3,5}^{aI}$
$A_{3,3}^I$	$2A_1^I / A_2^{III}$	$A_{3,8} \oplus A_1$	$A_{3,8}^{II}$
$A_{3,5}^{aII}$	A_2^{III}	$A_{4,9}^b$	$A_{3,5}^{aI}$
$A_{3,5}^{\frac{1}{2}I}$	$2A_1^I$	$A_{4,9}^a$	$A_{3,2}^I$
$A_1 \oplus A_2^I$	$2A_1^I / A_2^{III}$	$A_{4,12}$	$A_{3,6}^I / A_{3,7}^{bI}$

Let us now examine equation (2.1a) and its associated realization $A_{3,8}^I$ (see Table III). For $A = 0$, (2.1a) is reducible to the free particle equation via the point transformation²

$$T = q \quad Q = \frac{t^2}{2} + \frac{q^2}{2}. \quad (2.2)$$

Under this transformation, the standard $A_{3,8}^I$ realization becomes

$$\begin{aligned} \bar{G}_1 &= \partial/\partial T + T\partial/\partial Q & \bar{G}_2 &= T\partial/\partial T + 2Q\partial/\partial Q \\ \bar{G}_3 &= 2(T^2 - Q)\partial/\partial T + 2TQ\partial/\partial Q \quad (\text{still } A_{3,8}^I) \end{aligned} \quad (2.3)$$

Expressing equation (2.1a) in these co-ordinates (T, Q) , we have

$$Q'' = A\Theta(T, Q, Q'), \quad (2.4)$$

where Θ is a definite function of T, Q, Q' governed by (2.2). The latter equation (2.4) is linearizable only if $A = 0$. The reason being that (2.1a) which point transforms to it is linearizable only if $A = 0$. To obtain the MSL *w.r.t.* $A_{3,8}^I$ we use the realization (2.3) which are free particle operators. Our eventual MSL can only contain free particle operators. If we append (2.3) with any free particle operator, it does not close under commutation until we have generated the entire $sl(3, \mathbb{R})$ algebra. Hence the MSL *w.r.t.* $A_{3,8}^I$ is $sl(3, \mathbb{R})$.

We have shown that the highest dimension for \mathcal{L} corresponds to $sl(3, \mathbb{R})$, the maximum symmetry algebra for second-order o.d.e.'s. This is only realised when $\mathcal{A} = A_{3,8}^I$. Consequently, we can state the following theorem:

THEOREM: If \mathcal{A} is any realization except $A_{3,8}^I$ and $\dim \mathcal{A} \leq 3$ then $2 \leq \dim \mathcal{L} \leq 4$.

Thus, if the maximum symmetry algebra is of dimension less than eight then it will exclude the equivalence class of equations associated with the realization $A_{3,8}^I$. In this case the maximum Lie algebra will be at least of dimension four. An example of this occurs in the study of the local equivalence problem^[5,6] for (1.1) under the restricted pseudo-group of point transformations $Q = F(t, q)$, $T = G(t)$. The maximum Lie algebra in this case is six dimensional. Further equivalence classes omitted by this method are associated with the realizations $A_{3,6}^I$ and $A_{3,7}^I$. The Painlevé-Gambier classification also fails to take into account the above mentioned equivalence classes of equations as it is based on a subgroup of the restricted pseudo-group.

Finally, the least dimension of the maximal algebras for a symmetry method is three. This follows from Tables III and V. The underlying pseudo-group of point transformations is of the general type. For example, Noether's theorem with general point transformations has a framework in which the maximal symmetry algebras are of dimension five.

Acknowledgement: F.M. thanks Drs. W. Sarlet, F. Cantrijn and Prof. P. Winternitz for useful discussions.

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Lie point symmetries for systems of second order linear ordinary differential equations

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Quaestiones Mathematica 11 (1988) 95-117

**Lie point symmetries for systems of second order
linear ordinary differential equations**

BY

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Abstract

The equations for the Lie point symmetries of autonomous systems of second order linear ordinary differential equations are derived. The results for a two dimensional system are treated in detail and some consideration is extended to higher dimensional systems. The effect of the introduction of time-dependent elements into the coefficient matrix is briefly discussed.

AMS Subject Classification: 34A30, 22E60, 17B15

1. INTRODUCTION.

It is now roughly a century since Lie [4] introduced the concept of the invariance of a differential equation under an infinitesimal transformation. In the case of a point transformation, the transformation is generated by the operator, or generator,

$$G = \tau(x, t) \frac{\partial}{\partial t} + \eta(x, t) \frac{\partial}{\partial x} \quad (1.1)$$

Transformations of derivatives of x are induced by the extensions of G where the k times extended operator is defined by

$$G^{(k)} = G^{(k-1)} + \left(\eta^k - \sum_{i=1}^k \binom{k}{i} x^{k+1-i} \tau^i \right) \frac{\partial}{\partial x^k} \quad (1.2)$$

in which x^j , τ^j and η^j denotes the j th total time derivative of x , τ and η respectively. The definition is completed by noting that $G^{(0)} = G$. A k th order differential equation $N^{(k)} = 0$ possesses the symmetry G if

$$G^{(k)} N^{(k)} \Big|_{N^{(k)}=0} = 0. \quad (1.3)$$

The set of all generators of a differential equation constitutes an algebra under the operation of the Lie bracket. The number of generators depends on the differential equation. Lie showed that the maximal number was eight for second order linear equations and $n + 4$ for higher order linear equations.

Interest in the general question of symmetries of differential equations faded until recent years. This is not to say that there was no interest in symmetries and symmetry groups. The close relationship between group properties and physics led to considerable interest in certain groups such as $SO(3)$, $SO(4)$ and $SU(3)$. As the importance of groups became more manifest, it was not surprising that the symmetries of differential equations again came under scrutiny. Initially the studies were based in physical problems such as the harmonic oscillator and variants. In retrospect the progress appears incredibly slow, but in the works of Anderson and Davison [1], Wulfman and Wybourne [15], Lutzky [5], Prince and Eliezer [10,11], Prince and Leach [12] and Leach [2], there was much learning. Most of these works discussed second order linear equations which, as eventually became obvious, all have the symmetry algebra $sl(3, R)$. However, the n -dimensional harmonic oscillator and the Kepler problem were also studied. Recently Mahomed and Leach [6] found that the nonlinear equation

$$\ddot{x} + 3x\dot{x} + x^3 = 0 \quad (1.4)$$

also has $sl(3, R)$ symmetry and so, since symmetries are preserved under point transformations, (1.4) could be transformed to a linear equation. This observation led to a flurry of activity. Sarlet, Mahomed and Leach [13] and Mahomed and Leach [7] found the criteria whereby nonlinear second order equations could be linearized by a point transformation. Recently Mahomed and Leach [8] showed that the number of symmetries which a second order equation may possess is 0, 1, 2, 3 or 8.

The possibility of linearizing a nonlinear second order equation has great practical importance in the area of computation as it is much less expensive to solve numerically a linear equation than it is to solve a nonlinear equation. This is the more so when it comes to systems of equations. One is interested in determining criteria whereby a system of nonlinear equations can be linearized by a point transformation. However, before one starts it does help if one knows something of the algebraic properties of systems of linear equations.

What is known is that all one-dimensional linear equations have $sl(3, R)$ symmetry and the n -dimensional free particle and harmonic oscillator (autonomous and time-dependent) both have $sl(n + 2, R)$ symmetry.

It is the purpose of this paper to go some way to filling in the gaps in our knowledge. In §2 we look at a system of two linear equations with constant coefficients so that we may gain some insight. The restriction to constant coefficients does not, in our experience, constitute a serious restriction as will be seen in a subsequent section, but it enables us to keep a clear view of the important results.

2. SYSTEM OF TWO LINEAR EQUATIONS.

The general two dimensional system of linear ordinary differential equations with constant coefficients is

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}\tag{2.1}$$

with a, b, c and d constants and denoting differentiation with respect to the independent variable t . By elementary similarity transformations (see G.F. Wilkinson's monograph [14], p46), which do not effect the number of symmetries, this may be written in the upper triangular form

$$\begin{aligned}\bar{x} &= ax + cy \\ \bar{y} &= dy\end{aligned}\tag{2.2}$$

and it is this system which we analyze. If the generator is

$$G = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\tag{2.3}$$

where τ, ξ and η are functions of x, y and t , the second extension of G is

$$G^{(2)} = G + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial}{\partial \dot{x}} + (\dot{\eta} - \dot{y}\dot{\tau}) \frac{\partial}{\partial \dot{y}} + (\ddot{\xi} - 2\dot{x}\ddot{\tau} - \dot{x}\dot{\tau}) \frac{\partial}{\partial \ddot{x}} + (\ddot{\eta} - 2\dot{y}\ddot{\tau} - \dot{y}\dot{\tau}) \frac{\partial}{\partial \ddot{y}}\tag{2.4}$$

The action of $G^{(2)}$ on the system (2.2) with the sense of the natural generalization of (1.3) to two coupled equations produces a pair of complicated partial differential equations. We do not write them down here, but indicate what is done. The two equations contain terms with powers of \dot{x} and \dot{y} up to the third degree. As all functions are functions of x, y and t only, we equate coefficients of separate powers $\dot{x}^i \dot{y}^j$ to zero using a REDUCE procedure. The resulting equations are

$$\frac{\partial^2 \tau}{\partial x^2} = \frac{\partial^2 \tau}{\partial x \partial y} = \frac{\partial^2 \tau}{\partial y^2} = 0\tag{2.5}$$

$$\frac{\partial^2 \xi}{\partial x^2} = 2 \frac{\partial^2 \tau}{\partial x \partial t}, \quad \frac{\partial^2 \xi}{\partial x \partial y} = \frac{\partial^2 \tau}{\partial y \partial t}, \quad \frac{\partial^2 \xi}{\partial y^2} = 0\tag{2.6}$$

$$\frac{\partial^2 \eta}{\partial x^2} = 0, \quad \frac{\partial^2 \eta}{\partial x \partial y} = \frac{\partial^2 \tau}{\partial x \partial t}, \quad \frac{\partial^2 \eta}{\partial y^2} = 2 \frac{\partial^2 \tau}{\partial y \partial t}\tag{2.7}$$

$$2 \frac{\partial^2 \xi}{\partial x \partial t} - \frac{\partial^2 r}{\partial t^2} - 3(ax + by) \frac{\partial r}{\partial x} - dy \frac{\partial r}{\partial y} = 0$$

$$\frac{\partial^2 \xi}{\partial y \partial t} - (ax + by) \frac{\partial r}{\partial y} = 0$$

$$\frac{\partial^2 \xi}{\partial t^2} + (ax + by) \frac{\partial \xi}{\partial x} + dy \frac{\partial \xi}{\partial y} - 2(ax + by) \frac{\partial r}{\partial t} - a\xi - b\eta = 0$$

$$\frac{\partial^2 \eta}{\partial x \partial t} - dy \frac{\partial r}{\partial x} = 0$$

$$2 \frac{\partial^2 \eta}{\partial y \partial t} - \frac{\partial^2 r}{\partial t^2} - (ax + by) \frac{\partial r}{\partial x^2} - 3dy \frac{\partial r}{\partial y} = 0$$

$$\frac{\partial^2 \eta}{\partial t^2} + (ax + by) \frac{\partial \eta}{\partial x} + dy \frac{\partial \eta}{\partial y} - 2dy \frac{\partial r}{\partial t} - d\eta = 0 \quad (2.8)$$

From equations (2.5-7) we see that

$$r = b_1(t)x + b_2(t)y + c(t)$$

$$\xi = (\dot{b}_1 x + \dot{b}_2 y)x + d_{11}(t)x + d_{12}(t)y + e_1(t)$$

$$\eta = (\dot{b}_1 x + \dot{b}_2 y)y + d_{21}(t)x + d_{22}(t)y + e_2(t) \quad (2.9)$$

It is a straightforward, albeit tedious and lengthy, task to substitute equations (2.9) into equations (2.8) and equate coefficients of distinct powers of x and y to zero so that the equations which the time-dependent functions satisfy are determined. We omit the detail and quote the results. In the analysis it becomes evident that there are distinct cases.

Case (i) The coefficients of (2.2), a , b and d , take the form $d = a$, $b = 0$.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
G_1	0	0	0	0	0	αG_6	$-\alpha G_7$	αG_8	$-\alpha G_9$
G_2		0	$-G_3$	G_4	0	$-G_6$	$-G_7$	0	0
G_3			0	$G_5 - G_2$	$-G_3$	0	0	$-G_6$	G_7
G_4				0	G_4	$-G_8$	$-G_9$	0	0
G_5					0	0	0	$-G_8$	G_9
G_6						0	0	0	0
G_7							0	0	0
G_8								0	0
G_9									0
G_{10}									
G_{11}									
G_{12}									
G_{13}									
G_{14}									
G_{15}									

Table I(a)

G_{10}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}
$2\alpha G_{10}$	$-2\alpha G_{11}$	αG_{12}	$-\alpha G_{13}$	αG_{14}	$-\alpha G_{15}$
0	0	G_{12}	G_{13}	0	0
0	0	G_{14}	G_{15}	0	0
0	0	0	0	G_{12}	G_{13}
0	0	0	0	G_{14}	G_{15}
0	$-2\alpha G_7$	G_{10}	$G_1 - \alpha(3G_2 + G_3)$	0	$-2\alpha G_3$
$2\alpha G_6$	0	$G_1 + \alpha(3G_2 + G_3)$	G_{11}	$2\alpha G_3$	0
0	$-2\alpha G_9$	0	$-2\alpha G_4$	G_{10}	$G_1 - \alpha(G_2 + 3G_3)$
$2\alpha G_8$	0	$2\alpha G_4$	0	$G_1 + \alpha(G_2 + 3G_3)$	G_{11}
0	$-4\alpha G_1$	0	$-2\alpha G_{12}$	0	$-2\alpha G_{14}$
	0	$2\alpha G_{13}$	0	$2\alpha G_{15}$	0
		0	0	0	0
			0	0	0
				0	0
					0

Table I(b)

We find the 15 generators (\sqrt{a} is written as α)

$$\begin{aligned}
 G_1 &= \frac{\partial}{\partial t} & G_9 &= e^{-\alpha t} \frac{\partial}{\partial y} \\
 G_2 &= x \frac{\partial}{\partial x} & G_{10} &= e^{2\alpha t} \left(\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} \right) \\
 G_3 &= y \frac{\partial}{\partial x} & G_{11} &= e^{-2\alpha t} \left(\frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} - \alpha y \frac{\partial}{\partial y} \right) \\
 G_4 &= x \frac{\partial}{\partial y} & G_{12} &= x e^{\alpha t} \left(\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} \right) \\
 G_5 &= y \frac{\partial}{\partial y} & G_{13} &= x e^{-\alpha t} \left(\frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} - \alpha y \frac{\partial}{\partial y} \right) \\
 G_6 &= e^{\alpha t} \frac{\partial}{\partial x} & G_{14} &= y e^{\alpha t} \left(\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} \right) \\
 G_7 &= e^{-\alpha t} \frac{\partial}{\partial x} & G_{15} &= y e^{-\alpha t} \left(\frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} - \alpha y \frac{\partial}{\partial y} \right) \\
 G_8 &= e^{\alpha t} \frac{\partial}{\partial y} & &
 \end{aligned} \tag{2.10}$$

The Lie bracket operation between these generators gives the commutation table shown in Tables I(a) and I(b). That the Lie algebra is $sl(4, R)$ follows from the results of Prince and Eliezer [10]

Case(ii) The coefficients of (2.2), a , b and d , take the form $d \neq a, b = 0$

We find the 7 generators (\sqrt{a} is written as α and \sqrt{d} as β)

$$\begin{aligned}
 G_1 &= \frac{\partial}{\partial t} & G_5 &= e^{-\alpha t} \frac{\partial}{\partial x} \\
 G_2 &= x \frac{\partial}{\partial x} & G_6 &= e^{\beta t} \frac{\partial}{\partial y} \\
 G_3 &= y \frac{\partial}{\partial y} & G_7 &= e^{-\beta t} \frac{\partial}{\partial y} \\
 G_4 &= e^{\alpha t} \frac{\partial}{\partial x} & &
 \end{aligned} \tag{2.11}$$

Comparing this with the results for Case (i) we see that we have lost the interchange operators, G_3 and G_4 , plus $G_{10} - G_{15}$. The commutation relations are given in

Table II.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_1	0	0	0	αG_4	$-\alpha G_5$	βG_6	$-\beta G_7$
G_2		0	0	$-G_4$	$-G_5$	0	0
G_3			0	0	0	$-G_6$	$-G_7$
G_4				0	0	0	0
G_5					0	0	0
G_6						0	0
G_7							0

Table II Commutation relations for the 7 generators of Case (ii).

Case (iii) The coefficients of (2.2), a , b and d , take the form $a = 0 = d$, $b \neq 0$.

In this case there are 8 generators and they are

$$\begin{aligned}
 G_1 &= \frac{\partial}{\partial t} & G_5 &= \frac{\partial}{\partial x} \\
 G_2 &= 2x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} & G_6 &= t \frac{\partial}{\partial x} \\
 G_3 &= 2y \frac{\partial}{\partial y} - t \frac{\partial}{\partial t} & G_7 &= bt^2 \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial y} \\
 G_4 &= y \frac{\partial}{\partial x} & G_8 &= bt^3 \frac{\partial}{\partial x} + 6t \frac{\partial}{\partial y}.
 \end{aligned} \tag{2.12}$$

The commutation relations are given in Table III.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	G_1	$-G_1$	0	0	G_5	$2bG_6$	$3G_7$
G_2		0	0	$-2G_4$	$-2G_5$	$-G_6$	0	G_8
G_3			0	$2G_4$	0	$-G_6$	$-2G_7$	$-3G_8$
G_4				0	0	0	0	$-6G_6$
G_5					0	0	0	0
G_6						0	0	0
G_7							0	0
G_8								0

Table III Commutation relations for the 8 generators of Case (iii)

Case (iv) $b \neq 0$, a , d not both zero.

In this case 7 generators are again found. The precise form of the generators depends upon the relationship between a and d . We list the generators for the case in which $a \neq d$ and both are non-zero. They are (with \sqrt{a} written as α , \sqrt{d} as β and $b/(d-a)$ as k)

$$\begin{aligned}
 G_1 &= \frac{\partial}{\partial t} & G_5 &= e^{-\alpha t} \frac{\partial}{\partial x} \\
 G_2 &= (x - ky) \frac{\partial}{\partial x} & G_6 &= e^{\beta t} \left(k \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \\
 G_3 &= y \left(ky \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) & G_7 &= e^{-\beta t} \left(k \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \\
 G_4 &= e^{\alpha t} \frac{\partial}{\partial x} & &
 \end{aligned} \tag{2.13}$$

The commutation relations are listed in Table IV which we see is just the same as for case (ii)

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_1	0	0	0	αG_4	$-\alpha G_5$	βG_6	$-\beta G_7$
G_2		0	0	$-G_4$	$-G_5$	0	0
G_3			0	0	0	$-G_6$	$-G_7$
G_4				0	0	0	0
G_5					0	0	0
G_6						0	0
G_7							0

Table IV Commutation table for the 7 generators of Case (iv)

In concluding this section we observe that, for the two dimensional system (2.1) which we have written in the standard form (2.2):

- (i) There exist three possible numbers of generators, viz. 15, 8 and 7.
- (ii) The maximal number of generators is obtained only if the coefficient matrix is a scalar multiple of the identity. This scalar may be zero. The Lie algebra is $sl(4, R)$ which, of course, was already known from other studies. We note that, even if the scalar were time-dependent, the result still holds.
- (iii) When the coefficient matrix is diagonal with unequal entries (one of which may be zero), the number of generators is reduced to 7. This is equally so when the coefficient matrix is fully triangular. However, we have treated them separately with the benefit of hindsight after making the calculations of §3.

(iv) In the case for which the coefficient matrix is strictly upper triangular an extra symmetry is introduced to make 8 in all.

In our investigation of two-dimensional systems we have found that different cases occur. Apart from preparing us for the treatment of n dimensional systems we have already found a distinction between one-dimensional linear systems and greater-than-one dimensional systems. For the former the symmetry algebra is always the same, $sl(3, R)$, which means that a point transformation exists which will transform one equation to any other. That is not the case when the dimensionality exceeds one. This will be particularly relevant when the study of symmetries of systems of nonlinear differential equations is undertaken.

3. A SYSTEM OF n LINEAR EQUATIONS.

Having seen how the procedure works in practice we turn now to an autonomous system of n second order linear differential equations with constant coefficients

$$\ddot{x} = Ax \Leftrightarrow \ddot{x}_i = a_{ij}x_j, \quad i = 1, n \quad (3.1)$$

The summation convention is used throughout unless otherwise specifically stated.

Writing the generator as

$$G = \tau \frac{\partial}{\partial t} + \eta_k \frac{\partial}{\partial x_k} \quad (3.2)$$

where, again, τ and η_k are functions of x_i and t . The second extension of G is

$$G^{(2)} = \tau \frac{\partial}{\partial t} + \eta_k \frac{\partial}{\partial x_k} + (\dot{\eta}_k - \dot{x}_k \dot{\tau}) \frac{\partial}{\partial \dot{x}_k} + (\ddot{\eta}_k - 2\ddot{x}_k \dot{\tau} - \dot{x}_k \ddot{\tau}) \frac{\partial}{\partial \ddot{x}_k}, \quad k = 1, n \quad (3.3)$$

The invariance of (3.1) under the action of $G^{(2)}$ with (3.1) applying leads to the system of equations

$$\begin{aligned} & \frac{\partial^2 \eta_k}{\partial t^2} + 2\dot{x}_j \frac{\partial^2 \eta_k}{\partial x_j \partial t} + \dot{x}_i \dot{x}_j \frac{\partial^2 \eta_k}{\partial x_i \partial x_j} + a_{ij} x_j \frac{\partial \eta_k}{\partial x_j} - 2a_{ki} x_i \left(\frac{\partial r}{\partial t} + \dot{x}_j \frac{\partial r}{\partial x_j} \right) \\ & - \dot{x}_k \left(\frac{\partial^2 r}{\partial t^2} + 2\dot{x}_j \frac{\partial^2 r}{\partial x_j \partial t} + \dot{x}_i \dot{x}_j \frac{\partial^2 r}{\partial x_i \partial x_j} + a_{ij} x_j \frac{\partial r}{\partial x_i} \right) = a_{kj} \eta_j, \quad k = 1, n. \end{aligned} \quad (3.4)$$

The terms of third degree in the \dot{x} 's give

$$\dot{x}_k \left(\dot{x}_i \dot{x}_j \frac{\partial^2 r}{\partial x_i \partial x_j} \right) = 0, \quad k = 1, n,$$

i.e., the Hessian of r with respect to x is zero. Thus

$$r = b_i(t)x_i + c(t). \quad (3.5)$$

The terms of second degree in the \dot{x} 's give

$$\dot{x}_i \dot{x}_j \frac{\partial^2 \eta_k}{\partial x_i \partial x_j} - 2\dot{x}_k \dot{x}_j \frac{\partial^2 r}{\partial x_j \partial t} = 0, \quad k = 1, n$$

from which it follows that

$$\eta_k = \dot{b}_i x_i x_k + d_{kj}(t)x_j + e_k(t). \quad (3.6)$$

Substituting for r and η_k into terms of first degree in the \dot{x} 's we have

$$\begin{aligned} & 2\dot{x}_j (\dot{b}_j x_k + \dot{b}_i x_i \delta_{kj} + \dot{d}_{kj}) - 2a_{ki} x_i \dot{x}_j b_j \\ & - \dot{x}_k (\dot{b}_i x_i + \dot{c} + a_{ij} x_j b_i) = 0, \quad k = 1, n. \end{aligned}$$

The terms independent of x give

$$2\dot{d}_{kj} = \delta_{kj} \dot{c}, \quad k, j = 1, n$$

from which it follows that

$$d_{kj} = d_{kj}^{\circ} + \frac{1}{2} \delta_{kj} \dot{c} \Leftrightarrow D = D^{\circ} + \frac{1}{2} \dot{c} I \quad (3.7)$$

where d_{kj}° are constants and $D = [d_{kj}]$, $D^{\circ} = [d_{kj}^{\circ}]$. The terms dependent on \dot{x} and x may be collected by differentiating with respect to \dot{x}_m and x_{ℓ} . This yields

$$2(\bar{b}_m \delta_{k\ell} - b_m a_{k\ell}) + (\bar{b}_{\ell} - a_{i\ell} b_i) \delta_{km} = 0, \quad k, \ell, m = 1, n \quad (3.8)$$

After substituting for τ and η_k the terms left in (3.4) are

$$\begin{aligned} & \ddot{b}_i x_i x_k + \bar{d}_{kj} x_j + \bar{e}_k + a_{ij} x_j (\dot{b}_i x_k + \dot{b}_{\ell} x_{\ell} \delta_{ik} + d_{ki}) \\ & - 2a_{ki} x_i (\dot{b}_i x_j + \dot{c}) - a_{kj} (\dot{b}_j x_i x_j + d_{ji} x_i + e_j) = 0 \quad k = 1, n. \end{aligned} \quad (3.9)$$

The terms independent of x give

$$\bar{e}_k - a_{kj} e_j = 0 \quad (3.10)$$

and those linearly dependent on x are

$$\bar{d}_{ki} x_i + (d_{kj} a_{ji} - a_{kj} d_{ji}) x_i - 2\dot{c} a_{ki} x_i = 0 \quad k = 1, n$$

from which, with (3.7), it follows that

$$\ddot{c} I - 4\dot{c} A + 2[D^{\circ}, A] = 0. \quad (3.11)$$

Differentiating the quadratic terms in (3.9) with respect to x_m and then x_{ℓ} we obtain

$$(\ddot{b}_m + \dot{b}_i a_{im}) \delta_{k\ell} - 2\dot{b}_m a_{k\ell} + (\ddot{b}_{\ell} + \dot{b}_i a_{i\ell}) \delta_{km} - 2\dot{b}_{\ell} a_{km} = 0 \quad k, \ell, m = 1, n. \quad (3.12)$$

The generators are now obtained by solving equations (3.8, 10, 11 and 12). In principle (3.10) is straightforward as it is just the original system (3.1) and will have

$2n$ arbitrary constants in the solution. Hence all linear systems have a generator of the form

$$G_k = e_k \frac{\partial}{\partial x_k} \quad k = 1, 2n, \quad (3.13)$$

where e_k is a solution of (3.1) (Note that there is no summation in (3.13).) Inspired by the two dimensional example we firstly consider the two special cases $A = \alpha I$ and $A = D$ where I is the unit matrix and D a diagonal matrix with not all entries equal.

Case (i) $A = \alpha I$

Equation (3.8) becomes

$$2(\bar{b}_m - \alpha b_m)\delta_{k\ell} + (\bar{b}_\ell - \alpha b_\ell)\delta_{km} = 0 \quad k, \ell, m = 1, n$$

from which it follows that

$$\bar{b}_m - \alpha b_m = 0 \quad m = 1, n \quad (3.14)$$

Substituting (3.14) in (3.12) leaves (3.12) satisfied identically. The solution of (3.14) will have $2n$ arbitrary constants and so there are $2n$ generators of the form

$$G_k = b_i x_i \frac{\partial}{\partial t} + \dot{b}_i x_i x_k \frac{\partial}{\partial x_k} \quad k = 1, 2n \quad (3.15)$$

where the b 's are the solutions of (3.14) and there is summation on i but not on k . Turning now to (3.11) it becomes

$$\ddot{c} - 4\alpha\dot{c} = 0 \quad (3.16)$$

which has three solutions and gives three generators of the form

$$G_k = c_k(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{c}_k x_k \frac{\partial}{\partial x_k} \quad k = 1, 3 \quad (3.17)$$

where $c_k(t)$ is one of the solutions of (3.16) and there is no summation on k . Finally there are the n^2 elements of D° which give generators of the form

$$G_k = x_i \frac{\partial}{\partial x_k} \quad i, k = 1, n. \quad (3.18)$$

In all there are $(n+2)^2 - 1$ generators and one can identify the algebra of the generators with respect to the Lie bracket as $sl(n+2, R)$.

Case (ii) $A = D$

Equation (3.8) is now

$$2(\bar{b}_m - b_m D_{kk})\delta_{k\ell} + (\bar{b}_\ell - b_\ell D_{\ell\ell})\delta_{km} = 0 \quad k, \ell, m = 1, n.$$

Setting $k = \ell \neq m$ and $k = \ell = m$ we have

$$\begin{aligned} \bar{b}_m - b_m D_{kk} &= 0 \\ \bar{b}_m - b_m D_{mm} &= 0 \end{aligned} \quad k, m = 1, n$$

respectively and, since not all the diagonal elements of D are equal, it follows that all of the b 's must be zero. This leaves (3.12) identically satisfied.

Equation (3.11) has diagonal terms of the form

$$\ddot{c} - 4\dot{c}D_{ii} = 0$$

because the diagonal terms of the commutator are zero. It follows that c is a constant, C_0 and the d_{ii}° arbitrary. The off diagonal terms are

$$d_{ik}^\circ (D_{kk} - D_{ii}) = 0.$$

Here d_{ik}° will be zero whenever D_{kk} and D_{ii} differ. Since at least one of the D_{ii} must differ from the rest, there are at most $(n-1)^2 + 1$ d_{ij}° 's. If all elements of D differ, there are just n elements of D° and these are the diagonal elements. Hence the minimum number of generators is $3n + 1$ and the maximum is $n^2 + n + 2$. Whenever two of the diagonal elements of D are equal there are two extra off diagonal d_{ij}° 's. In general, if k of the diagonal elements of D are equal, there are an additional $k(k-1)$ d_{ij}° 's.

Recognizing already from the considerations of Case (ii) that the situation becomes more complex the greater the value of n , we consider a general matrix A which has been transformed to upper triangular form. It follows that all b 's are zero and again it is the c 's and d 's which have to be considered. From (3.11) it is evident that \dot{c} is at most a constant which depends on the elements of D° and A and so c contributes only one generator additional to those from the d 's. The number of independent elements of D° depends upon the multiplicities of eigenvalues of A . Since A is in upper triangular form, the multiplicities are given by the number of repeated elements on the leading diagonal. To be concise let us consider (3.11) in the case $n = 3$. It may be rewritten as

$$\begin{bmatrix} a_{11}I - A^T & a_{12}I & a_{13}I \\ 0 & a_{22}I - A^T & a_{23}I \\ 0 & 0 & a_{33}I - A^T \end{bmatrix} \begin{bmatrix} d_1^{\circ T} \\ d_2^{\circ T} \\ d_3^{\circ T} \end{bmatrix} = -4 \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} c \quad (3.19)$$

in which $d_i^{\circ T}$ is the transpose of the i th row of D° and a_i^T is the transpose of the i th row of A . We may solve this iteratively by starting at the bottom block. The number of arbitrary d° 's depends upon the multiplicity of the eigenvalues of A .

If it is one, there are three arbitrary d° 's, d_{ii}° and $\dot{c} = 0$. If it is greater than one the situation is rather more complex as relations between d° 's occur which depend upon the off diagonal elements of A . For example, if we take $a_{22} = a_{33}$ and a_{12}, a_{13} ,

$a_{23} \neq 0$, we have as arbitrary d 's, d_{11} , d_{23} and d_{33} with $\dot{c} = 0$, and all other d 's apart from d_{12} and d_{13} automatically zero. Likewise with $a_{11} = a_{22} = a_{33}$ and a_{12}, a_{13}, a_{23} nonzero we again find three arbitrary d 's. However, if we put a_{12} and a_{23} equal to zero, the number of arbitrary d 's increases to six. To sum up this case we may consider (3.19) as three systems of equations with the coefficient matrix in each case having at most rank 2. The augmented matrix in the case of maximal rank can always be made of the same rank as the coefficient matrix by appropriate choices of higher d 's and \dot{c} so that there are always three arbitrary d 's. However, this is not necessarily the case if the rank is not maximal. We can conclude that for an n -dimensional system there will be n arbitrary d 's. However, there may be more if the rank is not maximal and some of the off diagonal terms are zero.

In Case (iii) of §2 we saw that, when A was strictly upper triangular, there was one extra generator. The equation for the d 's is just (3.19) with $a_{ii} = 0$ which is the equivalent of putting $a_{11} = a_{22} = a_{33}$. We saw above that the presence of non-zero a_{12}, a_{13} and a_{23} had no effect on the number of arbitrary d 's, but that, with a_{12} and a_{23} zero, there was an increase from three to six. This effect can be seen more clearly if we move to a four-dimensional system. In the case that the only non-zero elements of A are a_{13}, a_{14} and a_{24} , the arbitrary d 's are d_{i3}^o and d_{i4}^o , $i = 1, 4$. Also \dot{c}, d_{11}, d_{12} and d_{21} are specified in terms of the others. The result is unchanged if $a_{14} = 0$. If the only non-zero element is a_{14} , d_{11} and d_{ij} , $i = 1, 4, j = 2, 4$ are non-zero and $\dot{c} = d_{11}/4$.

4. DISCUSSION.

In the Introduction we stated that we would confine our attention to autonomous systems for the sake of clarity in the exposition. We wish now to note the effects of time dependence in the elements of the coefficient matrix. The effect of this is to add the term $\tau \dot{a}_{kj} x_j$ to the right hand side of (3.4). As this term is independent of the velocities the structures of τ and the η_k are unaffected. The only changes to the

equations to be solved, viz. (3.8, 10, 11 and 12) are to (3.11) and (3.12) which now become

$$\ddot{c}I - 4\dot{c}A - 2c\dot{A} + 2[D^\circ, A] = 0 \quad (4.1)$$

$$\begin{aligned} & (\ddot{b}_m + \dot{b}_i a_{im}) \delta_{k\ell} - 2\dot{b}_m a_{k\ell} - b_m \dot{a}_{k\ell} \\ & + (\ddot{b}_\ell + \dot{b}_i a_{i\ell}) \delta_{km} - 2\dot{b}_\ell a_{km} - b_\ell \dot{a}_{km} = 0 \end{aligned} \quad (4.2)$$

$k, \ell, m = 1, n.$

We note that equation (3.8) is unchanged and so the only case for which a nontrivial solution for b exists is in the case $A = \alpha(t)I$ for which

$$\ddot{b}_m - \alpha \dot{b}_m = 0. \quad (3.14)$$

It is readily verified that (3.14) satisfies (4.1) identically and so no b 's are lost. It is just that now the solution of (3.14) is not so trivial.

We do not wish to repeat the long analysis of §3 for the equation (4.1). Let us rather see the flavour of the results for the case of a two-dimensional system. We look at (4.1) for the various cases considered in §2.

Case (i) $a_{22} = a_{11}, a_{12} = 0$

Equation (4.1) reduces to just

$$\ddot{c} - 4\dot{c}\dot{a}_{11} - 2c\ddot{a}_{11} = 0$$

for which there are three solutions. There are no constraints on the d° 's and so the Lie algebra is still $sl(4, R)$ (as expected from the work of Prince and Eliezer [10]).

Case (ii) $a_{22} \neq a_{11}, a_{12} = 0$

Now (4.1) reduces to

$$\ddot{c} - 4\dot{c}a_{11} - 2ca_{11} = 0$$

$$\ddot{c} - 4\dot{c}a_{22} - 2ca_{22} = 0$$

$$d_{21}^{\circ} = 0$$

$$d_{21}^{\circ} = 0$$

From the first two of these, it is evident that in general only the trivial solution $c = 0$ exists and in this case the number of symmetries is reduced from 7 to 6. However, if $a_{11}(t) \propto a_{22}(t) \forall t$, we see that $c(t) = C(a_{11})^{-\frac{1}{2}}$ and so the number of symmetries stays at 7.

Case (iii) $a_{11} = 0 = a_{22}, a_{12} \neq 0$

Equation (4.1) is now

$$\ddot{c} - 2a_{12}d_{21}^{\circ} = 0$$

$$\ddot{c} + 2a_{12}d_{21}^{\circ} = 0$$

$$2\dot{c}a_{12} + c\dot{a}_{12} + a_{12}(d_{11}^{\circ} - d_{22}^{\circ}) = 0$$

From the first two it is evident that $\ddot{c} = 0$ and $d_{21}^{\circ} = 0$. The third equation is best viewed as defining permissible functions $a_{12}(t)$. From this point of view the number of generators is 10-4e's, 3d's and 3c's. If one has a given a_{12} , the number of generators will usually be reduced to seven - 4e's and 3d's unless the solution

$$c(t) = c_0 a_{12}^{-\frac{1}{2}} - a_{12}^{-\frac{1}{2}} (d_{11} - d_{22}) \int a_{12}^{\frac{1}{2}} dt$$

satisfies $\ddot{c} = 0$ in which case there will be eight.

Case (iv) $a_{12} \neq 0$, a_{11}, a_{22} not both zero

The relevant equations are

$$\ddot{c} - 4\dot{c}a_{11} - 2c\dot{a}_{11} - 2a_{12}d_{21} = 0$$

$$\ddot{c} - 4\dot{c}a_{22} - 2c\dot{a}_{22} + 2a_{12}d_{21} = 0$$

$$d_{21}(a_{11} - a_{22}) = 0$$

$$2\dot{c}a_{12} + c\dot{a}_{12} + a_{12}(d_{11} - d_{22}) + 2d_{12}(a_{22} - a_{11}) = 0$$

It is evident that $d_{21} = 0$ whether $a_{11} = a_{22}$ or not. In general there will not exist a c which satisfies the three equations. Hence $d_{11} = d_{22}$, $d_{12} = 0$ and there are only five symmetries in all.

In summary we see that in the two-dimensional case introduction of time-dependent elements into the coefficient matrix A can affect the number of symmetries, usually downwards. We would expect the same to be the case in higher dimensions. Interestingly the system of highest symmetry is not affected by the introduction of the single permissible function of time. This indicates that the system

$$\dot{\bar{x}} = \alpha(t)x$$

can always be transformed to the equivalent time-independent system by a point transformation. Indeed the transformation is well-known being

$$x_i = \bar{x}_i/\rho(t) \quad T = \int^t \rho^{-2}(s)ds$$

where $\rho(t)$ is a solution of the Lewis-Pinney equation [3,9]

$$\ddot{\rho} - \alpha\rho = \rho^{-3}.$$

The general reduction in symmetry means that point transformations are not available to render the system time-independent.

CONCLUSION.

Part of the motivation for this work was the idea to extend recent work [7,8,13] on the linearization of single nonlinear differential equations to systems of equations. We believed that the identification of the symmetries of linear equations would help when nonlinear systems were analysed. Certainly we knew that, if the nonlinear system had $(n+2)^2 - 1$ symmetries, then it could be linearized by a point transformation. We now see that systems with rather fewer symmetries can still be linearized in the same way. That a nonlinear system could be linearized would mean a great reduction in computational effort.

ACKNOWLEDGEMENTS.

We thank F.M. Mahomed for valuable discussions on the identification of Lie algebras. One of us (VMG) thanks the Foundation for Research and Development of the CSIR for financial support.

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Received 1987.06.04. and in revised form 1987.10.16

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Proceedings of the Workshop on Finite Dimensional Integrable Nonlinear Dynamical Systems P G L Leach and W-H Steeb edd (World Scientific, Singapore, 1988) 178–189

NORMAL FORMS FOR THIRD ORDER EQUATIONS

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ABSTRACT

We show that a linear third order ordinary differential equation has either five or seven point symmetries. Then it is shown that a necessary and sufficient condition for a third order equation to be linearizable is that it admits the three dimensional Abelian algebra $3A_1$. Furthermore, we establish representatives for all third order equations possessing three symmetries.

1. INTRODUCTION

The study of the algebraic properties of ordinary differential equations (*o.d.e.'s*) was initiated by Lie¹⁾ in the previous century. He had in mind the integrability of equations (linear or nonlinear) which admitted a one or multi-parameter group of invariant point transformations. These invariance or symmetry transformations generate a Lie algebra.²⁾

Over the years first and second order *o.d.e.'s* have received much attention using the Lie symmetry method. In particular Lie obtained the two canonical

forms of generators for each of the two real two-dimensional Lie algebras (see e.g. ref.³) and their associated representative equations.

In this work we investigate the analogous problem for third order *o.d.e.'s*. In other words we find the representatives for third order equations which possess three generators of symmetry.

Another study which has generated much interest is the Painlevé analysis for *o.d.e.'s*.⁴ The question whether or not the Painlevé property (this property is linked to the integrability of the equation⁵) holds has been investigated for first and second order equations in detail. The only first order equations with the Painlevé property are generalized Riccati equations. For second order *o.d.e.'s* there are fifty canonical equations (hereafter referred to as the Painlevé equations), six of which define new transcendents. The determination of third order equations was carried out by Bureau⁶. This work, however, is incomplete.

In previous papers^{7,8} we mentioned the existence of second order equations with no point symmetries (such as PI, the first of the Painlevé equations which defines a new transcendent, see ref.⁴) that do possess the Painlevé property. We also noted that there exist equations with two or more symmetries (such as (2.1e) of ref.⁷) devoid of the Painlevé property. These remarks also hold true for third order equations. For example,

$$\ddot{q} = \bar{q}^2 \exp \dot{q}$$

admits the three dimensional algebra $A_{3,3}$, but does not have the Painlevé property since it contains a transcendental function of \dot{q} . One cannot reduce the above equation to a Painlevé equation via a point transformation. Moreover, it is not clear how one would reduce an equation to a Painlevé equation using a contact transformation.

In view of the above remarks the third order representative equations obtained here, using symmetry techniques, supplement the existing Painlevé equations of Bureau.

2. LINEARITY AND ABELIAN STRUCTURE

Consider the general third order linear equation

$$\ddot{q} + B_2 \dot{q} + B_1 q + B_0 = g(t) \quad (2.1)$$

where the B_i 's are functions of t and the dot denotes d/dt . By means of the transformation $q = f\bar{q} + q_p$, where q_p is a particular solution of (2.1) and $f = \exp[\frac{1}{3} \int^t B_2(s) ds]$, equation (2.1) is reducible to a homogeneous equation independent of the \bar{q} term. Hence, without loss of generality we may treat

$$\ddot{\bar{q}} + B_1\dot{\bar{q}} + B_0\bar{q} = 0. \quad (2.2)$$

We analyze (2.2) for its point symmetries. If the generator of point symmetry of (2.2) is

$$G = \xi(t, q) \frac{\partial}{\partial t} + \eta(t, q) \frac{\partial}{\partial q}, \quad (2.3)$$

the third extension of G is

$$G^{(3)} = G + \sum_{j=1}^3 \eta^{(j)} \frac{\partial}{\partial q^{(j)}}, \quad (2.4)$$

where

$$\eta^{(k)} = \frac{d}{dt} \eta^{(k-1)} - q^{(k)} \frac{d\xi}{dt}, \quad k = 1, 3 \quad \eta^{(0)} = \eta$$

and $q^{(j)}$ denotes the j th total time derivative of q . The symmetry requirement for (2.2), viz. $G^{(3)}(\ddot{\bar{q}} + B_1\dot{\bar{q}} + B_0\bar{q}) = 0$ whenever (2.2) holds gives rise to a partial differential equation. Equating coefficients of separate powers of $\bar{q}^i \bar{q}^j$ to zero then results in a set of five determining equations

$$\begin{aligned} \xi_q &= 0 & \eta_{qq} &= 0 \\ \eta_{tq} &= \xi_{tt} & 3\eta_{ttq} - \xi_{ttt} + 2B_1\xi_t + \xi\dot{B}_1 &= 0 \\ \eta_{ttt} - B_0q\eta_q + 3B_0q\xi_t + B_1\eta_t + B_0\eta + \xi\dot{B}_0 &= 0. \end{aligned}$$

Solving the first pair for ξ and η yields

$$\xi = a(t) \quad \eta = b(t)q + c(t). \quad (2.5)$$

The functions a, b and c are constrained by the remaining determining equations to satisfy

$$\begin{aligned} \dot{b} &= \dot{a} \\ 3\ddot{b} - \ddot{a} + 2B_1\dot{a} + a\dot{B}_1 &= 0 \\ \ddot{b} + 3\dot{a}B_0 + a\dot{B}_0 + B_1\dot{b} &= 0 \\ \ddot{c} + B_1\dot{c} + B_0c &= 0. \end{aligned} \tag{2.6}$$

It is immediately observed that (2.6a) and (2.6d) give rise to four point symmetries for equation (2.2). We examine conditions (2.6b) and (2.6c) more closely. Substituting (2.6a) into (2.6b) and (2.6c), we obtain

$$\begin{aligned} 2\ddot{a} + 2B_1\dot{a} + a\dot{B}_1 &= 0 \\ \frac{d}{dt} \ddot{a} + 3\dot{a}B_0 + a\dot{B}_0 + B_1\ddot{a} &= 0. \end{aligned} \tag{2.7}$$

We have

$$a = A(\dot{B}_1 - 2B_0)^{-1/3}, \quad A \text{ constant} \tag{2.8}$$

provided $\dot{B}_1 \neq 2B_0$. However, if $B_0 = \dot{B}_1/2$, (2.7b) turns out to be precisely $d(2.7a)/dt$ implying that the two conditions (2.7) can now be replaced by the single condition (2.7a). Consequently equation (2.2) has seven point symmetries whenever $B_0 = \dot{B}_1/2$ and five otherwise.

As an aside it is interesting to note that (2.7a) is related to the well-known Lewis-Pinney (cf. ref.⁹) equation which commonly is found in the study of time dependent second order systems.

Equation (2.2) has a seven dimensional Lie algebra when $B_0 = \dot{B}_1/2$, the maximum algebra that one can expect for third order *o.d.e.*'s. The Lie algebra is isomorphic to the Lie algebra of $\ddot{q} = 0$ which is (see ref.¹⁰) $3A_1 \otimes_s (sl(2, \mathbb{R}) \otimes A_1)$ (where \otimes and \otimes_s denote direct and semidirect product respectively).

When $\dot{B}_1 \neq 2B_0$, equation (2.2) possesses a five dimensional algebra. In this case the Lie algebra is $3A_1 \otimes_s (A_1 \otimes A_1)$. We again have a three dimensional

Abelian algebra $3A_1$. The $sl(2, \mathbb{R})$ algebra of the previous case now *reduces* to the one dimensional Abelian algebra A_1 .

We have in fact shown that if a third order *o.d.e.* is linearizable via a point transformation then it admits the algebra $3A_1$. This follows from the invariance of the symmetry algebra under a point transformation as well as by noting that the three linearly independent solutions of (2.6d) generate $3A_1$, which turns out to be a subalgebra for both the five and seven dimensional algebras either of which can be admitted by a linear equation. The question then arises as to whether the converse holds. Indeed it does hold. If a third order *o.d.e.* of the form

$$\ddot{q} = H(\bar{q}, \dot{q}, q, t) \quad (2.9)$$

admits the algebra $3A_1$, then its generators of symmetry satisfy $3A_1$. In a previous paper⁸⁾ we showed that there exists only one realization of the algebra $3A_1$ in terms of vector fields in two co-ordinates. This realization, in canonical form, is given by

$$G_1 = \frac{\partial}{\partial q} \quad G_2 = t \frac{\partial}{\partial q} \quad G_3 = h(t) \frac{\partial}{\partial q}, \quad (2.10)$$

where $h(t)$ is not linear in t . It now remains to show that the third order equation which is left invariant under the operators (2.10) is linear. Expressing the invariance of a third order equation of the form (2.9) with respect to each of the first two operators of (2.10) in turn, we find that H cannot be a function of q and \dot{q} . The remaining operator then implies that the equation be

$$\ddot{q} + \alpha(t)\dot{q} + \beta(t) = 0, \quad (2.11)$$

which is linear. The function α is in terms of h and β is arbitrary. Equation (2.11) admits the maximum algebra only if $\alpha = 0$, otherwise it has the five dimensional algebra.

Formally, we have proved the following result:

Theorem 1. A necessary and sufficient condition for a third order equation of the form (2.9) to be linearizable via a point transformation is that it admits the three dimensional Abelian algebra $3A_1$. ◊

Remark: We have seen that a general third order linear *o.d.e.* (2.2) is only reducible to $\ddot{x} = 0$ via a point transformation provided $B_0 = \dot{B}_1/2$ (cf. ref.¹⁰).

To conclude this section we give an application from electrodynamics. The equation of motion of a radiating charged particle embedded in a radiation field, the Langevin equation, is¹¹

$$m\ddot{x} = eE(t) + F(t, x, \dot{x}) + m\tau\ddot{x}, \quad \tau = 2e^2/3mc^3 \quad (2.12)$$

where m is the mass and e the charge of the particle, F the given external force and E the electric field. For a force defined by (cf.¹²)

$$F(t, x, \dot{x}) = F_1(t)x + F_2(t)\dot{x} + F_3(t),$$

(2.12) can be written in the form (2.1) and so its invariance properties are just those deduced above.

3. EQUATIONS WITH THREE SYMMETRIES

In this section we establish representatives for all third order equations, of the form (2.9), which admit three dimensional point symmetry algebras. Thus, if an equation possesses three point symmetries, it can be reduced via a point transformation to one of the canonical equations derived here.

The realizations of real three dimensional Lie algebras in terms of vector fields in two co-ordinates were obtained in ref.⁸) We need only associate third order equations to each of the realizations which are generators of symmetry. This we do by invoking the symmetry requirement

$$G^{(3)}(\ddot{q} - H(t, q, \dot{q}, \ddot{q})) = 0 \quad \text{whenever (2.9) holds} \quad (3.1)$$

for each G belonging to every three dimensional algebra (subalgebra of $3A_1 \otimes, (sl(2, \mathbb{R}) \otimes A_1)$) realization. As in the case of second order *o.d.e.*'s, it may happen that a realization does not leave invariant any third order equation. This occurs for the $so(3)$ ($A_{3,0}$) realizations (see Table II).

There are eleven real Lie algebras of dimension three.¹³) They are

Table I

Algebra	Nonzero commutation relations
$3A_1$	
$A_1 \oplus A_2$	$[G_1, G_3] = G_1$
$A_{3,1}$ (Weyl)	$[G_2, G_3] = G_1$
$A_{3,2}$	$[G_1, G_3] = G_1, [G_2, G_3] = G_1 + G_2$
$A_{3,3}$ ($D \otimes T_2$)	$[G_1, G_3] = G_1, [G_2, G_3] = G_2$
$A_{3,4}$ ($E(1,1)$)	$[G_1, G_3] = G_1, [G_2, G_3] = -G_2$
$A_{3,5}^a$ ($0 < a < 1$)	$[G_1, G_3] = G_1, [G_2, G_3] = aG_2$
$A_{3,6}$ ($E(2)$)	$[G_1, G_3] = -G_2, [G_2, G_3] = G_1$
$A_{3,7}^b$ ($b > 0$)	$[G_1, G_3] = bG_1 - G_2, [G_2, G_3] = G_1 + bG_2$
$A_{3,8}$ ($SL(2, \mathcal{R})$)	$[G_1, G_2] = G_1, [G_2, G_3] = G_3, [G_3, G_1] = -2G_2$
$A_{3,9}$ ($SO(3)$)	$[G_1, G_2] = G_3, [G_2, G_3] = G_1, [G_3, G_1] = G_2$

The realizations of these algebras in terms of vector fields in two coordinates are listed below

Table II

Let $\partial/\partial t = p$ and $\partial/\partial q = r$

$3A_1$	r	tr	$h(t)r$
$A_{3,1}$	r	p	tr
$A_{3,2}$	r	p	$tp + (t+q)r$
	r	$-(\ln t)r$	$tp + qr$
$A_1 \oplus A_2$ ($a = 0$),	$A_{3,3}$ ($a = 1$),	$A_{3,4}$ ($a = -1$),	$A_{3,5}^a$ ($0 < a < 1$)

	p	r	$tp + aqr$
	r	tr	$(1 - a)tp + qr$
$A_{3,6} (b = 0), A_{3,7}^b (b > 0)$	p	r	$(bt + q)p + (bq - t)r$
	tr	r	$(1 + t^2)p + (tq + bq)r$
$A_{3,8}$	r	qr	q^2r
	r	$tp + qr$	$2tqp + q^2r$
	r	$tp + qr$	$2tqp + (q^2 - t^2)r$
$A_{3,9}$	$i(\sin q)r$	r	$-i(\cos q)r \quad i = \sqrt{-1}$
	$(1 + t^2)p + tqr$	$tqp + (1 + q^2)r$	$qp - tr$

Of the above algebras most have two realizations. The algebra $A_{3,8}$ has three realizations. We shall denote the above realizations as A^I, A^{II} and A^{III} whenever there are more than one.

We associate third order *o.d.e.*'s to each of the realizations except that of $A_{3,9}$ which is not a subalgebra of $3A_1 \otimes_s (sl(2, \mathbb{R}) \otimes A_1)$. This involves the solution of a system of three linear first order partial differential equations which arise from using the symmetry requirement for each of the three operators of a given realization. One can then use the theory of complete systems to obtain the solution of the system. However, if the operators are simple as they usually are, then one can use the procedure adopted in §2 whereby we obtained the equation associated with $3A_1$. The results are summarized in Table III.

Table III

Realization	Canonical Equation
$3A_1$	$\ddot{q} + \alpha(t)\dot{q} + \Upsilon(t) = 0$
$A_{3,1}$	$\ddot{q} = \Upsilon(\dot{q})$
$A_{3,2}^I$	$\ddot{q} = \dot{q}^2 \Upsilon(\dot{q} \exp \dot{q})$

$A_{3,2}^{II}$	$t^2 \ddot{q} = \Upsilon(t\bar{q} + \dot{q}) + 2\dot{q}$
$A_{3,3}^I$	$\ddot{q} = \bar{q}^2 \Upsilon(\dot{q})$
$A_{3,3}^{II}$	$\ddot{q} = \bar{q} \Upsilon(t)$
$A_1 \oplus A_2^I, A_{3,4}^I, A_{3,5}^{\alpha I}$	$\ddot{q} = \bar{q}^{\frac{2-a}{2}} \Upsilon(\bar{q} \dot{q}^{\frac{2-a}{2}}) \quad a \neq 1, 2$
$A_{3,5}^{\frac{1}{2} I}$	$\dot{q} \ddot{q} = \Upsilon(\bar{q}) \quad a = 2$
$A_1 \oplus A_2^{II}, A_{3,4}^{II}, A_{3,5}^{\alpha II}$	$\ddot{q} = \bar{q}^{\frac{2-a}{2}} \Upsilon(t^{\frac{2-a}{2}} \bar{q}) \quad a \neq 1, 1/2$
$A_{3,5}^{\frac{1}{2} II}$	$t \ddot{q} = \Upsilon(\bar{q}) \quad a = 1/2$
$A_{3,6}^I, A_{3,7}^{bI}$	$\ddot{q} = 3\bar{q}\bar{q}^2(1 + \dot{q}^2)^{-1} + (1 + \dot{q}^2)^2 \exp(2b \tan^{-1} \dot{q})$ $\times \Upsilon(\bar{q}(1 + \dot{q}^2)^{-3/2} \exp(-b \tan^{-1} \dot{q}))$
$A_{3,6}^{II}, A_{3,7}^{bII}$	$\ddot{q} = 3t\bar{q}(1 + t^2)^{-1} + (1 + t^2)^{-5/2} \exp(b \tan^{-1} t)$ $\times \Upsilon(\bar{q}(1 + t^2)^{3/2} \exp(-b \tan^{-1} t))$
$A_{3,8}^I$	$\dot{q} \ddot{q} = \frac{3}{2} \bar{q}^2 + \dot{q}^2 \Upsilon(t)$
$A_{3,8}^{II}$	$t^2 \dot{q} \ddot{q} = \dot{q}^5 \Upsilon((t\bar{q} + \frac{1}{2}\dot{q})/\dot{q}^3) + 3t^2 \bar{q}^2$
$A_{3,8}^{III}$	$t^2(1 + \dot{q}^2) \ddot{q} = (1 + \dot{q}^2)^3 \Upsilon((t\bar{q} - \dot{q} - \dot{q}^3)/(1 + \dot{q}^2)^{3/2})$ $+ 3t^2 \bar{q}^2 \dot{q}$

In the above table Υ is an arbitrary function of its argument. We observe that there are fifteen classes of equations each of which is reducible to a second order equation (some of the equations are even reducible to first order). Four of them are parameter (a or b) dependent.

Note that the representation of the generators in canonical form does not necessarily lead to the simplest expression for the differential equation. For example the equation associated with $A_{3,8}^{II}$ becomes, under the transformation $Q = t, T = q,$

$$Q^2 Q''' + \Gamma\left(\frac{1}{2} Q'^2 - Q Q''\right) = 0$$

in which $'$ denotes d/dT .

We also notice that two of the classes in the above table are linear. Hence we can state the following result.

Theorem 2. If a third order *o.d.e.* admits operators, G_i ; $i = 1, 3,$ of the

form (2.3) such that $G_1 = \rho(t, q)G_2$ and $G_3 = \psi(t, q)G_2$ for some functions ρ and ψ with $\rho\psi \neq 1$, then it is linearizable via a point transformation.

The proof of this theorem follows upon observing that the only representative equations which admit proportional operators are those associated with $3A_1$, $A_{3,3}^{II}$ and $A_{3,8}^I$. Among these only the last mentioned violates the condition $\rho\psi \neq 1$ and this equation has exactly three symmetries for Υ arbitrary. \circ

In conclusion we prove

Theorem 3. There does not exist any third order *o.d.e.* having exactly six point symmetries.

Proof. A third order *o.d.e.* cannot possess exactly a six dimensional algebra of point symmetries containing the Abelian algebra $3A_1$ as this would, by Theorem 1, imply linearization and consequently a seven dimensional point symmetry algebra for the equation. Moreover, the only six dimensional algebra (subalgebra of $3A_1 \otimes (sl(2, \mathbb{R}) \otimes A_1)$) which does not contain $3A_1$ is $2A_1 \otimes (sl(2, \mathbb{R}) \otimes A_1)$. We show that an equation cannot have exactly six point symmetries which generate this algebra. Let us consider the $sl(2, \mathbb{R})$ (equivalently $A_{3,8}$) structure. There are three realizations of $A_{3,8}$ (see Table II). Writing G_4 as (2.3) and invoking $sl(2, \mathbb{R}) \otimes A_1$ with the operators $A_{3,8}^I$ representing G_i , $i = 1, 3$ we obtain $G_4 = a(t)\partial/\partial t$ with $a(t)$ arbitrary. A transformation of the form $T = \int^t 1/a(s) ds$, $Q = q$ is then used to eliminate $a(t)$. Hence without loss of generality we have

$$G_1 = \frac{\partial}{\partial q} \quad G_2 = q \frac{\partial}{\partial q} \quad G_3 = q^2 \frac{\partial}{\partial q} \quad G_4 = \frac{\partial}{\partial t}. \quad (3.2)$$

Representing the realization of $2A_1$ as $\{G_5, G_6\}$ and acting with A_1 (G_4) on $2A_1$ by dilation we find that

$$G_j = \alpha_j(q) \exp(-t) \frac{\partial}{\partial t} + \beta_j(q) \exp(-t) \frac{\partial}{\partial q}, \quad j = 5, 6$$

with α_j and β_j functions of q . The equation invariant under (3.2) is (see Table III)

$$\dot{q} \ddot{q} = \frac{3}{2} \bar{q}^2 + \dot{q}^2 B, \quad B \text{ constant.}$$

This equation is not invariant under operators of the form G_5 or G_6 . In a similar manner one can obtain the operators

$$\begin{aligned} G_1 &= \frac{\partial}{\partial q} & G_2 &= t \frac{\partial}{\partial t} + q \frac{\partial}{\partial q} \\ G_3 &= 2tq \frac{\partial}{\partial t} + q^2 \frac{\partial}{\partial q} & G_4 &= t \frac{\partial}{\partial t}. \end{aligned} \quad (3.3)$$

where $\{G_i; i = 1, 3\}$ is the realization $A_{3,8}^{II}$. Invariance with respect to (3.3) implies (see Table III)

$$\dot{q} \ddot{q} = 3\bar{q}^2,$$

which by means of $T = q$, $Q = t$ is simply $Q''' = 0$ whence we have a seven dimensional algebra of point symmetries. Lastly we find that $A_{3,8}^{III}$ is not a subalgebra of $sl(2, \mathbb{R}) \otimes A_1$. This completes the proof. \diamond

Acknowledgements. We thank Professor Louis Michel for sending us the preprint referenced 9). F.M. also wishes to thank the CSIR of South Africa and the University of the Witwatersrand for financial support.

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Maximal subalgebra associated with a first integral of a system possessing $sl(3, R)$ algebra

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Journal of Mathematical Physics **29** (1988) 1807–1813

Maximal subalgebra associated with a first integral of a system possessing $sl(3, \mathbb{R})$ algebra

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(Received 26 August 1987; accepted for publication 6 April 1988)

Conventionally the symmetries of a dynamical system are used to determine the first integrals associated with the system. In this paper the reverse procedure is adopted for one-dimensional systems possessing $SL(3, \mathbb{R})$ symmetry. The equivalence of such systems enables the free particle equation to be used as a vehicle. It is observed that for three particular first integrals, each gives rise to a triplet of generators having isomorphic algebras. It is then shown how the knowledge of a first integral and its associated triplet enables one to obtain the remaining integrals and triplets.

I. INTRODUCTION

In this paper we shall use the Hamiltonian version of the Lie method.^{1,2} Suppose a one-parameter symmetry group has the generator

$$Y(q, p, t) = \xi(q, t) \frac{\partial}{\partial t} + \eta(q, t) \frac{\partial}{\partial q} + \zeta(q, p, t) \frac{\partial}{\partial p}. \quad (1.1)$$

As a consequence of its property of transforming solution curves in (q, p, t) space into solution curves, the first extension of $Y(q, p, t)$,

$$Y^{(1)}(q, p, t) = Y(q, p, t) + \eta^{(1)}(q, p, t) \frac{\partial}{\partial \dot{q}} + \zeta^{(1)}(q, p, t) \frac{\partial}{\partial \dot{p}}, \quad (1.2)$$

where

$$\eta^{(1)} = \dot{\eta} - \dot{\xi} \frac{\partial H}{\partial p}, \quad \zeta^{(1)} = \dot{\zeta} + \dot{\xi} \frac{\partial H}{\partial q}, \quad (1.3)$$

preserves the form of Hamilton's equations

$$\dot{q} - \frac{\partial H}{\partial p} = 0, \quad \dot{p} + \frac{\partial H}{\partial q} = 0. \quad (1.4)$$

The action of $Y^{(1)}(q, p, t)$ on (1.4) produces the equations

$$\eta^{(1)} - Y \frac{\partial H}{\partial p} = 0, \quad \zeta^{(1)} + Y \frac{\partial H}{\partial q} = 0, \quad (1.5)$$

which may be manipulated to obtain Y (this being possible due to ξ and η being free of the variable p).

Assuming the existence of solutions for Y , these constitute the complete symmetry group for the dynamical system. Once the generators are found the associated first integrals may also be found.

Here we work from a different viewpoint. Given a constant of the motion we find the set of one-parameter symmetry groups with which it is associated. It is known already that in some cases the same constant arises from different generators. As an example, the simple harmonic oscillator with

$$H = \frac{1}{2}(p^2 + q^2) \quad (1.6)$$

has three generators,

$$\begin{aligned} G_6 &= q \frac{\partial}{\partial q}, \\ G_7 &= q \sin t \frac{\partial}{\partial t} + q^2 \cos t \frac{\partial}{\partial q}, \\ G_8 &= q \cos t \frac{\partial}{\partial t} - q^2 \sin t \frac{\partial}{\partial q} \end{aligned} \quad (1.7)$$

(in the usual notation^{3,4}), all of which give rise to the same first integral,

$$I(q, p, t) = (q \cos t - p \sin t) / (q \sin t + p \cos t). \quad (1.8)$$

At this stage we should note that the functional form given to a first integral is a matter of choice. Another form for I (1.8) (Ref. 5) is

$$I'(q, p, t) = t - \arctan(q/p). \quad (1.9)$$

What we shall do is show that for a one-dimensional system possessing $SL(3, \mathbb{R})$ symmetry, all of the generators of the complete symmetry group may be found from three constants only. We look at the relationship between combinations of generators and various different constants of the motion. The common features of the action of the set of generators on the constants mentioned above are noted. Finally we take a new look at the commutation properties of the Lie algebra which may be written in a simplified form. The vehicle for this discussion is the free particle. The results are true for all one-dimensional systems having $sl(3, \mathbb{R})$ algebra.

II. GENERATOR FROM FIRST INTEGRAL

In the usual discussion of Lie symmetry groups the transformations act on solution curves in (q, t) space, transforming solution curves into solution curves. In the Hamiltonian context, which we adopt here, we have trajectories in (q, p, t) space. The advantage of considering (q, p, t) space is that the motion, being of the form $I(q, p, t) = C$, represents invariant surfaces in this space, and we may consider infinitesimal transformations that transform solutions into solutions on the same invariant surface.

Let $Y(q, p, t)$ be the generator of such an infinitesimal transformation (i.e., $YI = 0$). Referring to Fig. 1, suppose that Y transforms the trajectory (1) into trajectory (2) and, in particular, that P is transformed to P' . Let the vectors τ_1

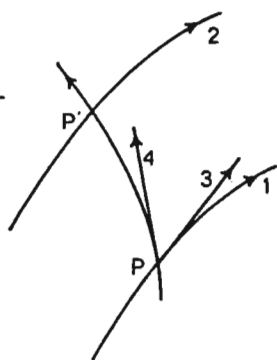


FIG. 1. Curves (1) and (2) are solution curves in (q,p,t) space lying on an invariant surface. At P, (3) is tangent to the solution curve (1) and is represented by τ_1 . The transformation path in the invariant surface from P to P' is (4) and is represented by the vector τ_2 .

and τ_2 be tangential to the solution curve and the transformation curve at P, respectively. The vectors have the following properties:

$$\tau_1 \times \tau_2 \propto \nabla I (\neq 0), \quad \tau_1 \cdot \nabla I = 0, \quad \tau_2 \cdot \nabla I = 0, \quad (2.1)$$

where ∇ is the gradient operator in (q,p,t) space. The three properties listed in (2.1) are not independent, and we consider the first, which essentially is the rule for permutation of solutions, and the third, which is that the infinitesimal transformation is in the invariant surface.

Writing

$$\tau_1 = \hat{t} + \frac{\partial H}{\partial p} \hat{q} - \frac{\partial H}{\partial q} \hat{p},$$

$$\tau_1 = \xi \hat{t} + \eta \hat{q} + \zeta \hat{p}, \quad (2.2)$$

$$\nabla = \hat{t} \frac{\partial}{\partial t} + \hat{q} \frac{\partial}{\partial q} + \hat{p} \frac{\partial}{\partial p},$$

where \hat{q} , \hat{p} , and \hat{t} are unit vectors in the direction of q , p , and t increasing, respectively, and letting the proportionality in the first of (2.1) be $\gamma(q,p,t)$, we have

$$\begin{bmatrix} 0 & \partial I / \partial p & \partial I / \partial q & \partial I / \partial t \\ -\partial I / \partial p & 0 & 1 & -\partial H / \partial p \\ -\partial I / \partial q & -1 & 0 & -\partial H / \partial q \\ -\partial I / \partial t & \partial H / \partial p & \partial H / \partial q & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ \xi \\ \eta \\ \zeta \end{bmatrix} = 0. \quad (2.3)$$

The condition that the transformation be canonical, viz.,

$$[\bar{q}, \bar{p}]_{PB_{q,p}} = 1, \quad (2.4)$$

reduces in the infinitesimal case to

$$\frac{\partial \eta}{\partial q} + \frac{\partial \zeta}{\partial p} = 0. \quad (2.5)$$

From (2.3) we see that this is equivalent to

$$[\gamma, I]_{PB_{q,p}} + [\xi, H]_{PB_{q,p}} = 0. \quad (2.6)$$

In addition to the first integral satisfying (2.3), it also satisfies (1.5). In particular, taking the first of each set we have

$$\xi \frac{\partial I}{\partial p} + \eta \frac{\partial I}{\partial q} + \zeta \frac{\partial I}{\partial t} = 0, \quad (2.7)$$

$$\eta^{(1)} - \xi \frac{\partial^2 H}{\partial p^2} - \eta \frac{\partial^2 H}{\partial q \partial p} - \xi \frac{\partial^2 H}{\partial t \partial p} = 0. \quad (2.8)$$

Eliminating $\zeta(q,p,t)$ between (2.7) and (2.8) and insisting that ξ and η be independent of p , we may determine those ξ and η permitted by the particular invariant I . The expression for ξ is obtained from (2.7) or (2.8) and that for η from one of (2.3). We observe that the rank of the coefficient matrix in (2.3) is always 2.

III. SOME EXAMPLES

To illustrate the method, we consider a few examples. The examples appended are linear as well as nonlinear.

Example (a): As is customary nowadays, we begin by considering the simple harmonic oscillator that has Hamiltonian (1.6). There exist six standard constants of the motion, which are listed in Table I together with the generators for which they are invariants. The order of generators follows the usage of Lutzky.³ For simplicity the generators are written in unextended form. Starting from the six linearly independent constants listed in Table I we may work backwards to find the generator(s) of the transformation(s) under which each remains invariant. We illustrate the method with J_3 . Substituting into (2.7) and (2.8) we have

$$\xi \cos t + \eta \sin t + (q \cos t - p \sin t) \xi = 0, \quad (3.1)$$

$$\eta^{(1)} - \xi = 0. \quad (3.2)$$

We eliminate ξ between (3.1) and (3.2) and then impose the additional constraint that η and ξ be independent of p to obtain

TABLE I. Symmetry generators and associated invariants for the one-dimensional harmonic oscillator. The generators are given in unextended form for the purpose of simplicity.

Generator	Invariant
$G_1 = \sin 2t \frac{\partial}{\partial t} + q \cos 2t \frac{\partial}{\partial q}$	$J_1 = \frac{1}{2}(p^2 - q^2) \sin 2t - qp \cos 2t$
$G_2 = \cos 2t \frac{\partial}{\partial t} - q \sin 2t \frac{\partial}{\partial q}$	$J_2 = \frac{1}{2}(p^2 - q^2) \cos 2t + qp \sin 2t$
$G_3 = \cos t \frac{\partial}{\partial q}$	$J_3 = q \sin t + p \cos t$
$G_4 = \sin t \frac{\partial}{\partial q}$	$J_4 = q \cos t - p \sin t$
$G_5 = \frac{\partial}{\partial t}$	$J_5 = \frac{1}{2}(p^2 + q^2)$
$G_6 = q \frac{\partial}{\partial q}$	$J_6 = (q \cos t - p \sin t) / (q \sin t + p \cos t)$
$G_7 = q \sin t \frac{\partial}{\partial t} + q^2 \cos t \frac{\partial}{\partial q}$	$J_7 = (q \cos t - p \sin t) / (q \sin t + p \cos t)$
$G_8 = q \cos t \frac{\partial}{\partial t} - q^2 \sin t \frac{\partial}{\partial q}$	$J_8 = (q \cos t - p \sin t) / (q \sin t + p \cos t)$

$$\frac{\partial \eta}{\partial t} + p \frac{\partial \eta}{\partial q} - p \left(\frac{\partial \xi}{\partial t} + p \frac{\partial \xi}{\partial q} \right) + \eta \tan t + (q - p \tan t) \xi = 0. \quad (3.3)$$

Equating coefficients of independent powers of p to zero we see that

$$\xi(q, t) = a(t), \quad \eta(q, t) = (\dot{a} + a \tan t)q + b(t), \quad (3.4)$$

where $a(t)$ and $b(t)$ are solutions of

$$d^2(a/\cos t)/dt^2 + a/\cos t = 0, \quad \dot{b} + b \tan t = 0. \quad (3.5)$$

Thus we have

$$a(t) = A \cos^2 t + C \cos t \sin t, \quad b(t) = B \cos t, \quad (3.6)$$

from which it follows that

$Y(q, p, t)$

$$\begin{aligned} &= (A \cos^2 t + C \sin t \cos t) \frac{\partial}{\partial t} \\ &+ \{(-A \sin t \cos t + C \cos^2 t)q + B \cos t\} \frac{\partial}{\partial q} \\ &+ \{(A \sin t \cos t + C \sin^2 t)p - [A(\cos^2 t - \sin^2 t) \\ &+ 2C \sin t \cos t]q - B \cos t\} \frac{\partial}{\partial p}. \end{aligned} \quad (3.7)$$

Similar calculations may be performed for the other constants of the motion. It eventuates that, for J_1, J_2 , and J_5 , only one generator is obtained in each case and it is the one listed in Table I. For J_4 a three-parameter solution is again obtained and, not surprisingly, J_6, J_7 , and J_8 give rise to a common three-parameter solution consisting of a linear combination of $G_6^{(1)}(q, p, t)$, $G_7^{(1)}(q, p, t)$, and $G_8^{(1)}(q, p, t)$.

We emphasize that the first extensions are written in terms of the momentum p and not the velocity \dot{q} . It makes no

difference to the functional form of the operator in this case, but does, say, for the damped free particle where $p = p(\dot{q}, t)$. The three triplets of generators are of interest to us and we list them in unextended form in Table II. To mark the departure from the usual expressions used we now employ the symbol $X(q, t)$ to refer to an operator in (q, t) space (i.e., the "unextended" operator). The corresponding operator in (q, p, t) space (i.e., the "extended" operator) is written as $Y(q, p, t)$. In Table II the generators are listed against their invariants (now relabeled I_1, I_2, I_3) and the corresponding γ 's are also given.

We defer discussion of the algebraic properties of the generators listed in Table II to the next section. In this section we compare the usual forms for the generators and first integrals given in Table I with those adopted here, show how higher-order integrals are constructed from linear combinations of the generators, and observe the pattern of the action of all the generators on all the integrals in the Table II format. In terms of the X 's, the G 's, which are usually used to determine the commutator relations that identify the algebra of the $SL(3, \mathbb{R})$ group, are given by

$$\begin{aligned} G_1 &= X_{22} - X_{33}, & G_2 &= -X_{23} - X_{32}, & G_3 &= X_{21}, \\ G_4 &= X_{31}, & G_5 &= -X_{23} + X_{32}, & & \\ G_6 &= -X_{11} = X_{22} + X_{33}, & G_7 &= -X_{12}, & G_8 &= X_{13}. \end{aligned} \quad (3.8)$$

The linear dependence of the X 's is seen in the expressions for G_6 , which are equivalent to the linear relation

$$X_{11} + X_{22} + X_{33} = 0. \quad (3.9)$$

We further observe that the X 's are related among each other as follows:

$$qX_{i1} = \cos tX_{i2} + \sin tX_{i3}, \quad i = 1, 3, \quad (3.10)$$

$$X_{1i} + q \cos tX_{2i} + q \sin tX_{3i} = 0, \quad i = 1, 3. \quad (3.11)$$

TABLE II. The three invariants with associated generators and constants of proportionality. Again, X 's instead of Y 's are listed for reasons of clarity.

Invariant	Generator	Constant of proportionality
$I_1 = \frac{q \cos t - p \sin t}{q \sin t + p \cos t}$	$X_{11} = -q \frac{\partial}{\partial q}$	$\gamma_{11} = (q \sin t + p \cos t)^2$
	$X_{12} = -q \sin t \frac{\partial}{\partial t} - q^2 \cos t \frac{\partial}{\partial q}$	$\gamma_{12} = (q \sin t + p \cos t)^2$ $\times (q \cos t - p \sin t)$
	$X_{13} = q \cos t \frac{\partial}{\partial t} - q^2 \sin t \frac{\partial}{\partial q}$	$\gamma_{13} = (q \sin t + p \cos t)^3$
$I_2 = q \sin t + p \cos t$	$X_{21} = \cos t \frac{\partial}{\partial q}$	$\gamma_{21} = 1$
	$X_{22} = \sin t \cos t \frac{\partial}{\partial t} + q \cos^2 t \frac{\partial}{\partial q}$	$\gamma_{22} = q \cos t - p \sin t$
	$X_{23} = -\cos^2 t \frac{\partial}{\partial t} + q \sin t \cos t \frac{\partial}{\partial q}$	$\gamma_{23} = q \sin t + p \cos t$
$I_3 = q \cos t - p \sin t$	$X_{31} = \sin t \frac{\partial}{\partial q}$	$\gamma_{31} = -1$
	$X_{32} = \sin^2 t \frac{\partial}{\partial t} + q \sin t \cos t \frac{\partial}{\partial q}$	$\gamma_{32} = -q \cos t + p \sin t$
	$X_{33} = -\sin t \cos t \frac{\partial}{\partial t} + q \sin^2 t \frac{\partial}{\partial q}$	$\gamma_{33} = -q \sin t + p \cos t$

The γ 's, which are also listed in Table II, are constants, which perhaps is not unexpected. We note that in each case,

$$\gamma_{12} = I_3 \gamma_{11}, \quad \gamma_{13} = I_2 \gamma_{11}, \quad i = 1, 3. \quad (3.12)$$

By inspection, the relations between the constants listed in Tables I and II are

$$\begin{aligned} J_1 &= -\frac{1}{2} I_2 I_3, \quad J_2 = \frac{1}{2} (I_3^2 - I_2^2), \\ J_3 &= I_2, \quad J_4 = I_3, \\ J_5 &= \frac{1}{2} (I_2^2 + I_3^2), \quad J_6 = J_1 (= J_7 = J_8). \end{aligned} \quad (3.13)$$

The members of each of the three classes of first integrals—linear, quadratic, and quotient—constitute a complete set of each class. In the case of the quotient integral, I_1 , it will be appreciated that the theory involved in determining the integral does not distinguish between I_1 and its reciprocal.

Considering the relations between the I 's and J 's and the X 's and G 's, it is instructive to calculate the integral that results from a linear combination of the X 's. As an example of the calculation we take

$$\tilde{X}_1 = mX_{21} + nX_{31}. \quad (3.14)$$

The associated integral is a solution of

$$\tilde{Y}_1 \tilde{I}_1 = 0, \quad \frac{d(\tilde{I}_1)}{dt} = 0. \quad (3.15)$$

The first of (3.15) is

$$\frac{dt}{0} = \frac{dq}{m \cos t + n \sin t} = \frac{dp}{-m \sin t + n \cos t}, \quad (3.16)$$

from which it follows that \tilde{I}_1 is a function of u_1 and u_2 , where $u_1 = t$, $u_2 = m(q \sin t + p \cos t) - n(q \cos t - p \sin t)$.

As a matter of terminology we point out that here we use u_1 and u_2 to emphasize the traditional equivalence of the canonical coordinates in Hamiltonian mechanics. Were we using q , \dot{q} , and t as variables, the usual u and v where v represents the differential invariant would be appropriate. Using the second of (3.15) in the standard fashion,

$$\frac{du_2}{du_1} = \frac{du_2}{dt} \left(\frac{du_1}{dt} \right)^{-1} = 0. \quad (3.18)$$

Hence the invariant is

$$\tilde{I}_1 = mI_2 - nI_3, \quad (3.19)$$

i.e., simply a linear combination.

Combining X_{23} and X_{32} in the same manner we obtain

$$\tilde{I}_2 = (m + n)J_2 + (m - n)J_5, \quad (3.20)$$

i.e., a linear combination of two of the quadratic integrals. The choices $m = -1, n = 1$ and $m = 1, n = 1$ yield J_2 and J_5 , respectively (up to a constant multiplicative factor there is an arbitrariness in how an integral is written). These two choices coincide with the combinations given in (3.8). Similarly the combination of X_{22} and X_{33} gives

$$\tilde{I}_3 = (p \sin t - q \cos t)^m (p \cos t + q \sin t)^{-n}, \quad (3.21)$$

yielding J_1 and J_6 for the choices $m = 1, n = -1$ and $m = 1, n = 1$, respectively.

We note that in the three examples of integrals considered, the first gave simply a linear combination of the two linear integrals, the second a linear combination of the qua-

dratic integrals, whereas the third provided an integral of arbitrary order depending only upon our choice of m and n . We do not wish to labor the matter of combinations of generators, but we point out that the relationship between the combination of generators and the combination of the associated integrals is not simple. The reason for the diversity of the results in (3.19), (3.20), and (3.21) possibly lies in the commutation relations between generators. The effect of the commutation relations would become obvious if exponentiation were used due to the Campbell–Baker–Hausdorff formula.

To conclude this example we observe the action of each of the Y 's on the three integrals I_1, I_2 , and I_3 . These are given in Table III. The table may be summarized as follows. If

$$Y_{ij} I_j = A_{ij}, \quad (3.22)$$

where A_{ij} is some combination of I_1, I_2 , and I_3 , then

$$Y_{12} I_j = I_3 A_{ij}, \quad Y_{13} I_j = I_2 A_{ij}. \quad (3.23)$$

The connection with (3.12) is obvious.

We have treated the simple harmonic oscillator in great detail. For the remaining examples, we merely give a sketch.

Example (b): Let us now consider the damped free particle, which has the Hamiltonian

$$H = \frac{1}{2} p^2 e^{-kt}, \quad p = \dot{q} e^{kt}. \quad (3.24)$$

The appropriate first integrals are

$$I_1 = q/p - (1 - e^{-kt})/k, \quad (3.25)$$

$$I_2 = p, \quad I_3 = q - p(1 - e^{-kt})/k.$$

The triplets of generators corresponding to each first integral are

$$\begin{aligned} X_{11} &= -q \frac{\partial}{\partial q}, \\ X_{12} &= \frac{q(1 - e^{kt})}{k} \frac{\partial}{\partial t} - q^2 \frac{\partial}{\partial q}, \\ X_{13} &= q e^{kt} \frac{\partial}{\partial t}; \end{aligned} \quad (3.26)$$

$$\begin{aligned} X_{21} &= \frac{\partial}{\partial q}, \\ X_{22} &= -\frac{(1 - e^{kt})}{k} \frac{\partial}{\partial t} + q \frac{\partial}{\partial q}, \end{aligned} \quad (3.27)$$

TABLE III. The effect of each Y acting on each invariant. The expressions in parentheses illustrate the effect more explicitly. A Y_{2i} operator has the effect of dividing by I_3 and a Y_{3i} operator by I_2 , in contrast to the multiplying effect shown by Y_{12} and Y_{13} .

	I_1	I_2	I_3
Y_{11}	0	$-I_2$	$-I_3$
Y_{12}	0	$-I_3 I_2$	$-I_3^2$
Y_{13}	0	$-I_2^2$	$-I_2 I_3$
Y_{21}	$1/I_2 (= I_1/I_3)$	0	$1 (= I_3/I_3)$
Y_{22}	$I_1 (= I_3 I_1/I_3)$	0	$I_3 (= I_3 I_3/I_3)$
Y_{23}	$1 (= I_2 I_1/I_3)$	0	$I_2 (= I_2 I_3/I_3)$
Y_{31}	$-I_1/I_2$	$1 (= I_2/I_2)$	0
Y_{32}	$-I_1^2 (= -I_3 I_1/I_2)$	$I_3 (= I_3 I_2/I_2)$	0
Y_{33}	$-I_1 (= -I_2 I_1/I_2)$	$I_2 (= I_2 I_2/I_2)$	0

$$\begin{aligned}
X_{23} &= -e^{kt} \frac{\partial}{\partial t}; \\
X_{31} &= \frac{(1 - e^{-kt})}{k} \frac{\partial}{\partial q}, \\
X_{32} &= \frac{(e^{kt} + e^{-kt} - 2)}{k^2} \frac{\partial}{\partial t} + \frac{q(1 - e^{-kt})}{k} \frac{\partial}{\partial q}, \\
X_{33} &= \frac{(1 - e^{kt})}{k} \frac{\partial}{\partial t}.
\end{aligned} \tag{3.28}$$

The linear dependence relation is again (3.9) and the following relations amongst the generators are observed:

$$\begin{aligned}
qX_{i1} &= e^{kt}X_{i2} + [(e^{kt} - 1)/k]X_{i3}, \quad i = 1, 3, \\
X_{1i} + qe^{kt}X_{2i} - qke^{kt}X_{3i} &= 0.
\end{aligned} \tag{3.29}$$

It may be verified by direct calculation that the first integrals and generators have the properties summarized in Tables III and IV.

We next treat two familiar examples of nonlinear systems.

Example (c): The differential equation

$$\dot{q} + 3q\dot{q} + q^3 = 0 \tag{3.30}$$

occurs in the investigation of univalued functions defined by second-order differential equations and in the study of the modified Emden equation.^{6,7} Recently it has also been treated in Ref. 8. Equation (3.30) has the Hamiltonian

$$\begin{aligned}
H &= t/q - pq^2 - \frac{1}{2}(t^2 - 2pq^2)^{1/2}, \\
p &= \frac{1}{2} \frac{t^2}{q^2} - \frac{1}{2} \frac{1}{(q^2 + \dot{q})^2}.
\end{aligned} \tag{3.31}$$

The first integrals are (cf. Ref. 7)

$$\begin{aligned}
I_1 &= -t + (t^2 - 2pq^2)^{1/2}, \\
I_2 &= t^2/2 + (1/q)(1 - tq)(t^2 - 2pq^2)^{1/2}, \\
I_3 &= \frac{t^2q + 2(1 - tq)(t^2 - 2pq^2)^{1/2}}{-2tq + 2q(t^2 - 2pq^2)^{1/2}}.
\end{aligned} \tag{3.32}$$

The triplets of generators corresponding to each first integral are

$$\begin{aligned}
X_{11} &= \frac{1}{2} t^2 q \frac{\partial}{\partial t} + \left(tq^2 - \frac{1}{2} t^2 q^3 - q \right) \frac{\partial}{\partial q}, \\
X_{12} &= tq \frac{\partial}{\partial t} + (q^2 - tq^3) \frac{\partial}{\partial q},
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
X_{13} &= q \frac{\partial}{\partial t} - q^3 \frac{\partial}{\partial q}; \\
X_{21} &= \left(\frac{1}{2} t^3 q - \frac{1}{2} t^2 \right) \frac{\partial}{\partial t} \\
&\quad + \left(1 - 2tq - \frac{1}{2} t^3 q^3 + \frac{3}{2} t^2 q^2 \right) \frac{\partial}{\partial q}, \\
X_{22} &= (t^2 q - t) \frac{\partial}{\partial t} + (2tq^2 - t^2 q^3 - q) \frac{\partial}{\partial q}, \\
X_{23} &= (tq - 1) \frac{\partial}{\partial t} + (q^2 - tq^3) \frac{\partial}{\partial q}; \\
X_{31} &= \left(\frac{1}{2} t^3 - \frac{1}{4} tq^4 \right) \frac{\partial}{\partial t} \\
&\quad + \left(-t + \frac{3}{2} t^2 q + \frac{1}{4} t^4 q^3 - t^3 q^2 \right) \frac{\partial}{\partial q}, \\
X_{32} &= \left(-\frac{1}{2} t^3 q + t^2 \right) \frac{\partial}{\partial t} \\
&\quad + \left(tq + \frac{1}{2} t^3 q^3 - \frac{3}{2} t^2 q^2 \right) \frac{\partial}{\partial q}, \\
X_{33} &= \left(-\frac{1}{2} t^2 q + t \right) \frac{\partial}{\partial t} + \left(\frac{1}{2} t^2 q^3 - tq^2 \right) \frac{\partial}{\partial q}.
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
X_{21} &= \left(\frac{1}{2} t^3 q - \frac{1}{2} t^2 \right) \frac{\partial}{\partial t} \\
&\quad + \left(1 - 2tq - \frac{1}{2} t^3 q^3 + \frac{3}{2} t^2 q^2 \right) \frac{\partial}{\partial q}, \\
X_{22} &= (t^2 q - t) \frac{\partial}{\partial t} + (2tq^2 - t^2 q^3 - q) \frac{\partial}{\partial q}, \\
X_{23} &= (tq - 1) \frac{\partial}{\partial t} + (q^2 - tq^3) \frac{\partial}{\partial q}; \\
X_{31} &= \left(\frac{1}{2} t^3 - \frac{1}{4} tq^4 \right) \frac{\partial}{\partial t} \\
&\quad + \left(-t + \frac{3}{2} t^2 q + \frac{1}{4} t^4 q^3 - t^3 q^2 \right) \frac{\partial}{\partial q}, \\
X_{32} &= \left(-\frac{1}{2} t^3 q + t^2 \right) \frac{\partial}{\partial t} \\
&\quad + \left(tq + \frac{1}{2} t^3 q^3 - \frac{3}{2} t^2 q^2 \right) \frac{\partial}{\partial q}, \\
X_{33} &= \left(-\frac{1}{2} t^2 q + t \right) \frac{\partial}{\partial t} + \left(\frac{1}{2} t^2 q^3 - tq^2 \right) \frac{\partial}{\partial q}.
\end{aligned} \tag{3.35}$$

The linear dependence relation is given by

$$X_{11} = X_{22} + X_{33}, \tag{3.36}$$

and the relations among the operators are

$$\begin{aligned}
(t - 1/q)X_{1i} &= X_{2i}, \quad i = 1, 3, \\
(t/q) - t^2/2 X_{1i} &= X_{3i}.
\end{aligned} \tag{3.37}$$

Example (d): The equation

$$t\ddot{q} = \dot{q}^3 + \dot{q} \tag{3.38}$$

was considered in Ref. 9. It has the Hamiltonian

$$H = -4t(q^3 p/2)^{1/2} - 4tq, \quad p = 2q^2 - 2t^2/\dot{q}^2, \tag{3.39}$$

and first integrals

$$\begin{aligned}
I_1 &= 2(q^2 - p/2)^{1/2} + 2q, \\
I_2 &= t^2 - q^2 - 2q(q^2 - p/2)^{1/2}, \\
I_3 &= \frac{t^2 - q^2 - 2q(q^2 - p/2)^{1/2}}{2q + 2(q^2 - p/2)^{1/2}}.
\end{aligned} \tag{3.40}$$

The triplets of generators associated with each first integral are

TABLE IV. Commutation relations between the X 's. The entry is [X column, X row]. The relations for the Y 's are, naturally, the same.

	X_{11}	X_{12}	X_{13}	X_{21}	X_{22}	X_{23}	X_{31}	X_{32}	X_{33}
X_{11}	0	$-X_{12}$	$-X_{13}$	X_{21}	0	0	X_{31}	0	0
X_{12}	X_{12}	0	0	$X_{22} - X_{11}$	$-X_{12}$	$-X_{13}$	X_{32}	0	0
X_{13}	X_{13}	0	0	X_{23}	0	0	$X_{33} - X_{11}$	$-X_{12}$	$-X_{13}$
X_{21}	$-X_{21}$	$X_{11} - X_{22}$	$-X_{23}$	0	X_{21}	0	0	X_{31}	0
X_{22}	0	X_{12}	0	$-X_{21}$	0	$-X_{23}$	0	X_{32}	0
X_{23}	0	X_{13}	0	0	X_{23}	0	$-X_{21}$	$X_{33} - X_{22}$	$-X_{23}$
X_{31}	$-X_{31}$	$-X_{32}$	$X_{11} - X_{33}$	0	0	X_{21}	0	0	X_{31}
X_{32}	0	0	X_{12}	$-X_{31}$	$-X_{32}$	$X_{22} - X_{33}$	0	0	X_{32}
X_{33}	0	0	X_{13}	0	0	X_{23}	$-X_{31}$	$-X_{32}$	0

$$\begin{aligned}
 X_{11} &= \left(\frac{t}{2} - \frac{1}{2} t^{-1} q^2 \right) \frac{\partial}{\partial t} + q \frac{\partial}{\partial q}, \\
 X_{12} &= -t^{-1} q \frac{\partial}{\partial t} + \frac{\partial}{\partial q},
 \end{aligned}
 \tag{3.41}$$

$$\begin{aligned}
 X_{13} &= \frac{1}{2} t^{-1} \frac{\partial}{\partial t}; \\
 X_{21} &= \left(\frac{1}{2} tq - \frac{1}{2} t^{-1} q^3 \right) \frac{\partial}{\partial t} + q^2 \frac{\partial}{\partial q}, \\
 X_{22} &= -t^{-1} q^2 \frac{\partial}{\partial t} + q \frac{\partial}{\partial q},
 \end{aligned}
 \tag{3.42}$$

$$\begin{aligned}
 X_{23} &= \frac{1}{2} t^{-1} q \frac{\partial}{\partial t}; \\
 X_{31} &= \left(\frac{t^3}{2} - \frac{1}{2} t^{-1} q^4 \right) \frac{\partial}{\partial t} + (t^2 q + q^3) \frac{\partial}{\partial q}, \\
 X_{32} &= -(tq + t^{-1} q^3) \frac{\partial}{\partial t} + (t^2 + q^2) \frac{\partial}{\partial q}, \\
 X_{33} &= \left(\frac{1}{2} t + \frac{1}{2} t^{-1} q^2 \right) \frac{\partial}{\partial t}.
 \end{aligned}
 \tag{3.43}$$

The linear dependence relation is again (3.36) and the relations among the generators are

$$\begin{aligned}
 qX_{1i} &= X_{2i}, \quad i = 1, 3, \\
 (t^2 + q^2)X_{1i} &= X_{3i}.
 \end{aligned}
 \tag{3.44}$$

IV. COMMUTATION RELATIONS

Each of the triplets $\{X_{1i}\}$, $\{X_{2i}\}$, and $\{X_{3i}\}$ constitutes a Lie subalgebra. The commutation relations are

$$\begin{aligned}
 [X_{11}, X_{12}] &= -X_{12}, \quad [X_{11}, X_{13}] = -X_{13}, \\
 [X_{12}, X_{13}] &= 0,
 \end{aligned}
 \tag{4.1}$$

$$\begin{aligned}
 [X_{21}, X_{22}] &= \pm X_{21}, \quad [X_{21}, X_{23}] = 0, \\
 [X_{22}, X_{23}] &= \mp X_{23},
 \end{aligned}
 \tag{4.2}$$

$$\begin{aligned}
 [X_{31}, X_{32}] &= 0, \quad [X_{31}, X_{33}] = \pm X_{31}, \\
 [X_{32}, X_{33}] &= \pm X_{32}.
 \end{aligned}
 \tag{4.3}$$

Clearly each of the three sets of commutation relations given above can be written in the form

$$[Z_1, Z_2] = 0, \quad [Z_1, Z_3] = Z_1, \quad [Z_2, Z_3] = Z_2, \tag{4.4}$$

and so the algebraic properties of the triplets are identical.

The Lie algebra represented by the commutation relations (4.4) is denoted by $A_{3,3}$ (see Refs. 10 and 11). The question now arises as to whether the three triplets of generators, having isomorphic algebras (4.1), (4.2), and (4.3), associated with each of the examples treated in the previous section, can be transformed into a canonical triplet of generators by a point transformation. To answer this we prove the following result.

Proposition: In order that a differential equation $\ddot{q} = N(\dot{q}, q, t)$ possess $\mathfrak{sl}(3, \mathbb{R})$ algebra it is necessary and sufficient that it have the algebra $A_{3,3}$.

Proof: To prove sufficiency, suppose that $\ddot{q} = N(\dot{q}, q, t)$ has the algebra $A_{3,3}$. The generators of symmetry G_i ($i = 1, 3$) then satisfy the $A_{3,3}$ commutation relations. Assume first that $G_2 = \rho(t, q)G_1$ and $G_3 = \psi(t, q)G_1$ for suitable nonconstant functions ρ and ψ . This certainly is a strong

condition. There is, however, no less generality in assuming the conjunction than there is in assuming the disjunctive statement, since $G_2 = \rho G_1$ implies $G_3 = \psi G_1$ and vice versa. Invoking $A_{3,3}$ we obtain $G_1 \rho = 0$ and $G_1 \psi = 1$. Transforming G_1 to $\bar{G}_1 = \partial / \partial Q$ and taking the simplest solutions of the aforementioned equations we arrive at the generators

$$\bar{G}_1 = \frac{\partial}{\partial Q}, \quad \bar{G}_2 = T \frac{\partial}{\partial Q}, \quad \bar{G}_3 = Q \frac{\partial}{\partial Q}. \tag{4.5}$$

Expressing the invariance of the differential equation for Q with respect to (4.5) results in the free particle equation $Q'' = 0$.

By assuming that $G_2 \neq \phi(t, q)G_1$ for any function ϕ , we straightforwardly obtain

$$\bar{G}_1 = \frac{\partial}{\partial T}, \quad \bar{G}_2 = \frac{\partial}{\partial Q}, \quad \bar{G}_3 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}, \tag{4.6}$$

the differential equation once more being the free particle equation. Proof of the case for which the generators G_1 and G_3 are unconnected is omitted as it also yields (4.6).

Thus it follows that the equation for q has the $\mathfrak{sl}(3, \mathbb{R})$ algebra. Proof of the necessity is trivial since $A_{3,3}$ is a subalgebra of $\mathfrak{sl}(3, \mathbb{R})$. \diamond

We have two realizations for the algebra $A_{3,3}$, viz., (4.5) and (4.6). On examining each of the triplets of generators obtained in the examples we notice that they are unconnected. For example, $X_{12} \neq \rho X_{11}$, $X_{13} \neq \psi X_{11}$, and $X_{12} \neq \phi X_{13}$ for any functions ρ , ψ , and ϕ . Therefore each of the triplets of the examples can be reduced to the standard form (4.6) by means of a point transformation that transforms the original equation to the free particle equation $Q'' = 0$. It is thus instructive to look at the free particle equation. The Hamiltonian is

$$H = \frac{1}{2} P^2, \quad P = Q',$$

and the first integrals are

$$I_1 = P, \quad I_2 = Q - TP, \quad I_3 = (Q - TP)/P. \tag{4.7}$$

Here I_1 has associated the triplet [cf. (4.6)]

$$\bar{X}_{11} = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}, \quad \bar{X}_{12} = \frac{\partial}{\partial T}, \quad \bar{X}_{13} = \frac{\partial}{\partial Q}.$$

For I_2 , we have

$$\bar{X}_{12} = T^2 \frac{\partial}{\partial T} + TQ \frac{\partial}{\partial Q}, \quad \bar{X}_{22} = T \frac{\partial}{\partial T}, \quad \bar{X}_{23} = T \frac{\partial}{\partial Q},$$

and for I_3 ,

$$\bar{X}_{31} = TQ \frac{\partial}{\partial T} + Q^2 \frac{\partial}{\partial Q}, \quad \bar{X}_{32} = Q \frac{\partial}{\partial T}, \quad \bar{X}_{33} = Q \frac{\partial}{\partial Q}.$$

We immediately observe these relations among the generators [cf. Examples (c) and (d)]:

$$\begin{aligned}
 T\bar{X}_{1i} &= \bar{X}_{2i}, \quad i = 1, 3, \\
 Q\bar{X}_{1i} &= \bar{X}_{3i}.
 \end{aligned}
 \tag{4.8}$$

By inspection we observe that the I_2 generators are equivalent to the I_3 generators under the interchange transformation

$$\bar{T} = Q, \quad \bar{Q} = T. \tag{4.9}$$

The I_1 generators are form-invariant under (4.9). It is easily

verified that the I_2 generators transform into the I_1 generators via the transformation

$$\bar{T} = -1/T, \quad \bar{Q} = Q/T. \quad (4.10)$$

Hence using (4.9) together with (4.10) we deduce that

$$\bar{T} = -1/Q, \quad \bar{Q} = T/Q, \quad (4.11)$$

reduces the I_3 generators to the I_1 generators. Clearly the above transformations leave the free particle equation invariant.

Hence the knowledge of a first integral J together with its associated triplet $\{G_i: i = 1,3\}$ enables an equation to be reduced to the free particle equation by a point transformation that transforms the G_i 's to the standard form (4.6). Accordingly one can obtain the remaining first integrals from (4.7) and the associated triplets from (4.8). The foregoing examples can easily be shown to illustrate this. However, we consider the time-dependent oscillator as a further example. The equation is

$$\ddot{q} + \omega^2(t)q = 0, \quad (4.12)$$

with the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2, \quad p = \dot{q}.$$

A first integral is given by

$$I = 1/(q\dot{C} - pC),$$

where C is a particular solution of (4.12). The triplet of generators associated with I are obtained by using (2.7) and (2.8). They are

$$\begin{aligned} X_{11} &= C^2 \int C^{-2} dt \frac{\partial}{\partial t} + q \left(1 + C\dot{C} \int C^{-2} dt \right) \frac{\partial}{\partial q}, \\ X_{12} &= C^2 \frac{\partial}{\partial t} + C\dot{C}q \frac{\partial}{\partial q}, \\ X_{13} &= C \frac{\partial}{\partial q}. \end{aligned} \quad (4.13)$$

The transformation which reduces (4.13) to (4.6) is given by

$$T = \int C^{-2} dt, \quad Q = q/C.$$

It is now a simple matter to obtain the remaining first integrals and triplets by using (4.7) and (4.8) together with the above transformation. The first integrals are

$$\begin{aligned} J &= \frac{q}{C} - (pC - q\dot{C}) \int C^{-2} dt, \\ K &= \frac{q}{C(pC - q\dot{C})} - \int C^{-2} dt, \end{aligned} \quad (4.14)$$

and the triplets are given by

$$\begin{aligned} \int C^{-2} dt X_{1i} &= X_{2i}, \quad i = 1,3, \\ (q/C)X_{1i} &= X_{3i}, \quad i = 1,3. \end{aligned} \quad (4.15)$$

ACKNOWLEDGMENT

F.M.M. thanks the Council of Scientific and Industrial Research of South Africa and the University of the Witwatersrand for financial support.

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Journal of Mathematical Physics **29** (1988) 2563–2569

Analysis and solution of a nonlinear second-order differential equation through rescaling and through a dynamical point of view

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(Received 3 May 1988; accepted for publication 3 August 1988)

The solutions of the equation $\ddot{y} + y\dot{y} + \beta y^3 = 0$, where β is a free parameter, are investigated. For $\beta = \frac{1}{3}$ the equation is linearizable through an eight-parameter symmetry group and is completely integrable. For $\beta \neq \frac{1}{3}$ only two symmetries subsist, but through a dynamical description the analytical asymptotic solutions and their behavior are given according to the value of β and according to the initial conditions.

1. INTRODUCTION

The differential equation

$$y'' + yy' + y^3/9 = 0 \quad (1)$$

arises in the study of the modified Emden equation¹⁻³

$$\ddot{q} + \alpha(t)\dot{q} + \gamma(t)q^m = 0$$

in the case $m = 3$. It is also found in the study of univalued functions defined by second-order differential equations. Equation (1) is a member of the Riccati hierarchy and can be transformed into a linear third-order equation by means of the standard transformation $y(x) = 3u'(x)/u(x)$. It is also a member of the class of equations represented by

$$y'' + 3a(x)yy' + b(x)y' + a^2(x)y^3 + c(x)y^2 + d(x)y + e(x) = 0.$$

This is the general form of a second-order ordinary differential equation linear in the first derivative that can be transformed into a linear second-order equation by means of a point transformation.⁴⁻⁶ As a consequence Eq. (1) possesses the algebra $sl(3, R)$.

By virtue of its interest in both mathematical and physical contexts, we here make further investigations into Eq. (1). To provide interpretations based on an understanding of physics, we recast the problem as the classical mechanical problem of a particle moving in a one-dimensional potential. Further, we modify (1) so that the Newtonian equation of motion is taken to be

$$\ddot{q} + q\dot{q} + \beta q^3 = 0. \quad (2)$$

We are interested in the behavior of the solution of (2), in particular for varying values of β . One question to be addressed is the following. For $\beta = \frac{1}{3}$, (2) is linearizable, possesses eight symmetries, and is completely integrable. Consequently, we could expect that this remarkable mathematical property corresponds to an important physical one appearing (or disappearing) for this value which consequently would appear as a critical one. By setting the problem in the context of classical mechanics, we shall see

that there are other values of β for which something can be deduced about (2) analytically and that the critical value of β is not $\frac{1}{3}$. This is something that was not detected in the symmetry analysis referred to above since, for all values of $\beta \neq \frac{1}{3}$, there exist only two symmetries, i.e., the Lie point symmetry analysis distinguishes between $\beta = \frac{1}{3}$ and $\beta \neq \frac{1}{3}$ only. We note that Ince⁷ includes (2) in his detailed discussion of second-order nonlinear equations. However, our treatment is differently based.

In the case $\beta = \frac{1}{3}$, application of the Riccati transformation $q = 3u/u$ to (2) gives the third-order equation

$$\ddot{u} = 0$$

with solution

$$u(t) = A_0 + A_1 t + A_2 t^2,$$

whence

$$q(t) = \frac{3(A_1 + 2A_2 t)}{A_0 + A_1 t + A_2 t^2}, \quad q(0) = \frac{3A_1}{A_0}, \quad (3)$$

$$\dot{q}(0) = \frac{3(2A_0 A_2 - A_1^2)}{A_0^2}.$$

From (3) we see that only A_1/A_0 and A_2/A_0 matter and consequently we take $A_0 = 1$.

The asymptotic behavior of $q(t)$ depends not only on the value of A_2 but also on the existence and sign of the roots of $1 + A_1 t + A_2 t^2 = 0$. If there is no positive root for this equation, the asymptotic behavior ($t \rightarrow \infty$) of $q(t)$ is

$$q(t) = 6/t, \quad A_2 \neq 0, \quad (3')$$

$$q(t) = 3/t, \quad A_2 = 0.$$

On the other hand, if a positive root exists, then the solution exhibits an explosive character [i.e., $q(t)$ goes to infinity in a finite time]. The problem is to obtain the boundary curves for the initial conditions.

If $A_2 < 0$, the equation $1 + A_1 t + A_2 t^2$ has real roots of opposite sign and consequently one is positive leading to an explosive solution. If $A_2 > 0$, two cases occur. For $A_1 > 0$, either there is no root or the two roots are negative and consequently no explosive solution can take place. For $A_1 < 0$, if

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the roots exist, they are both positive and the explosive solution takes place.

Consequently, the boundary curves are

$$A_2 = 0, \quad \text{for } A_1 > 0, \quad (3'')$$

$$A_1^2 - 4A_2 = 0, \quad \text{for } A_1 < 0. \quad (3''')$$

Taking into account (3), (3'') and (3''') can be written, respectively, as

$$A_2 = 0 \Rightarrow \dot{q}_0 = -3A_1^2.$$

This relation together with

$$q_0 = 3A_1 \Rightarrow \dot{q}_0 = -\frac{1}{3}q_0^2.$$

Taking into account the two last relations of (3),

$$A_1^2 - 4A_2 = 0 \Rightarrow \dot{q}_0 = -\frac{1}{4}q_0^2.$$

Figure 1 gives the sign of A_1, A_2 , and $\Delta = A_1^2 - 4A_2$ and shows that the boundary curve for initial conditions leading to an explosive solution are given by the two half-parabolas. In fact, we are going to show the generalization of this result for $0 < \beta < \frac{1}{4}$.

II. SELF-SIMILAR ANALYSIS

For β taking on general values, we may use rescaling² to determine the asymptotic behavior of $q(t)$. The transformation

$$(t, q) \rightarrow (\bar{t}, \bar{q}): \quad \bar{t} = a^r t, \quad \bar{q} = a^p q$$

is self-similar if

$$B - 2A = 2B - A = 3B.$$

This system has the consistent solution $B = -A$ with A arbitrary. We note that this transformation corresponds to the second of the Lie point symmetries associated with (2),⁴ viz.,

$$G_2 = t \frac{\partial}{\partial t} - q \frac{\partial}{\partial q}$$

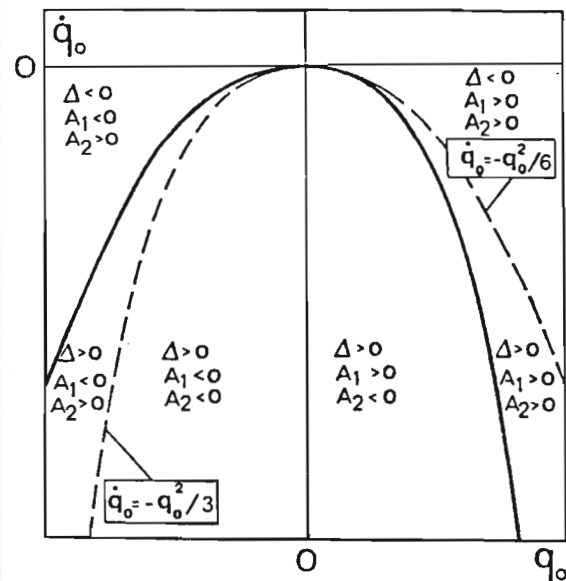


FIG. 1. The boundary curve between explosive and nonexplosive solutions is the half-parabola $\dot{q}_0 = -q_0^2/6$ for $q_0 < 0$ and the half-parabola $\dot{q}_0 = -q_0^2/3$ for $q_0 > 0$ in the case $\beta = \frac{1}{4}$.

as can be seen by setting $a = (1 + \epsilon)^{-1}$. As (2) is autonomous, the other symmetry is obviously

$$G_1 = \frac{\partial}{\partial t}.$$

We will see, later on, that the combination of these two symmetries will give both the asymptotic behavior and the classification of the initial conditions. The elementary invariants of the similarity transformation are⁸

$$\xi = qt, \quad \eta = \dot{q}t^2. \quad (4)$$

Noting that

$$\frac{d\xi}{dt} = \frac{1}{t}(\dot{q}t^2 + qt) = \frac{1}{t}(\eta + \xi),$$

we take as new variables the invariants ξ (for position) and $\omega = \eta + \xi$ (for velocity), so that Eq. (2) becomes

$$\frac{d\xi}{d\theta} = \omega, \quad (5a)$$

$$\frac{d\omega}{d\theta} = (3 - \xi)\omega + (\xi^2 - 2\xi - \beta\xi^3), \quad (5b)$$

where $\theta = \log t$ is the new time.

Equations (5) are interpreted as a system of first-order equations of motion in the new phase space-time (ξ, ω, θ) of a particle moving under the influence of a velocity-dependent drag force $(3 - \xi)\omega$ and a force derivable from the potential

$$V(\xi) = \frac{1}{4}\beta\xi^4 + \xi^2 - \frac{1}{2}\xi^3. \quad (6)$$

We note that the velocity-dependent drag force is damping for $\xi > 3$ and accelerating for $\xi < 3$.

III. BEHAVIOR OF THE POTENTIAL WITH VARYING β

For the convenience of later reference, we categorize the behavior of the potential as β varies downwards. This is presented in Fig. 2. We first note that $V(\xi)$ is always zero at the origin $\xi = 0$.

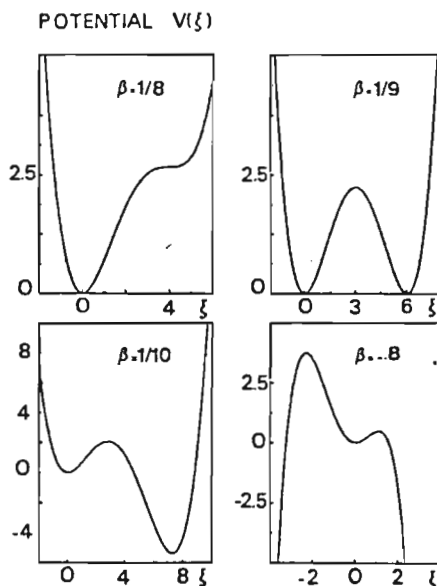


FIG. 2. Plots of the potential $V(\xi) = \frac{1}{4}\beta\xi^4 - \frac{1}{2}\xi^3 + \xi^2$ for different values of β .

$\beta > \frac{1}{2}$: $V(\xi)$ has a single minimum at $\xi = 0$. It is strictly monotonic increasing for $|\xi| \rightarrow \infty$.

$\beta = \frac{1}{2}$: $V(\xi)$ has a single minimum at $\xi = 0$, a stationary point of inflection ($dV/d\xi = d^2V/d\xi^2 = 0$) at $\xi = 4$, and now it is monotonic increasing for $|\xi| \rightarrow \infty$.

$\frac{1}{2} > \beta > \frac{1}{4}$: The stationary point of inflection of the potential splits into a maximum at ξ_M in [3,4] and a minimum at ξ_m in [4,6].

$\beta = \frac{1}{4}$: $V(\xi)$ has minima at $\xi = 0$ and $\xi_m = 6$ and a maximum at $\xi_M = 3$. Note that the potential reads $V(\xi) = (\xi - 6)^2 \xi^2 / 36$ in this peculiar case and that it is symmetric about the maximum $\xi_M = 3$. Moreover, the extremum ξ_M is the limit abscissa between a damping and an accelerating velocity-dependent drag force.

$0 < \beta < \frac{1}{4}$: The maximum ξ_M is in [2,3] and the minimum ξ_m is in $[6, +\infty[$. As $\beta \rightarrow 0$, ξ_M moves towards the limit value 2 and the minimum ξ_m behaves as $1/\beta$. The potential is strictly monotonic increasing for $|\xi| \rightarrow \infty$.

$\beta = 0$: The maximum is at $\xi_M = 2$. The potential is strictly monotonic increasing in $[0, 2]$ and strictly monotonic decreasing in $] -\infty, 0]$ and in $[2, +\infty[$.

$\beta < 0$: $V(\xi)$ has a minimum at $\xi = 0$ and two maxima on either side of the origin. The maximum on the left of the origin moves in from $\xi = -\infty$ for β increasingly negative and away from $\xi = 2$ to the right. From this rather detailed analysis, it appears that the critical values of β are $\beta = 0$ and $\beta = \frac{1}{4}$.

IV. SOME SPECIAL SOLUTIONS

Before we discuss the qualitative behavior of the motion for varying β , we consider some special solutions. Elimination of θ from Eqs. (5a) and (5b) gives

$$\omega \frac{d\omega}{d\xi} = (3 - \xi)\omega + \xi^2 - 2\xi - \beta\xi^3. \quad (7)$$

An ansatz of the form

$$\omega = a_0 + a_1\xi + a_2\xi^2 \quad (8)$$

for the solution of (7) yields the following possible solutions:

$$(i) \quad \omega = 2\xi - \frac{1}{2}\xi^2, \quad \beta = \frac{1}{2}, \quad (9)$$

$$(ii) \quad \omega = -6 + 3\xi - \frac{1}{2}\xi^2, \quad \beta = \frac{1}{4}, \quad (10)$$

$$(iii) \quad \omega = \xi + a_2\xi^2, \quad \beta = -a_2(2a_2 + 1). \quad (11)$$

[It is a trivial matter to show that a finite polynomial solution to (7) can only have the form of (8).]

We may then solve (5a) for each of these solutions. We also list here the corresponding solution of the original equation (2).

Case (i),

$$\xi = 6e^{2\theta}/(K + e^{2\theta}), \quad q = 6t/(K + t^2); \quad (12)$$

case (ii),

$$\xi = \frac{3K + 6e^\theta}{K + e^\theta}, \quad q = \frac{3K + 6t}{Kt + t^2} = \frac{6(\frac{1}{2}K + t)}{-K^2/4 + (\frac{1}{2}K + t)^2}; \quad (13)$$

case (iii),

$$\xi = -e^\theta/(K + a_2e^\theta), \quad q = -1/(K + a_2t); \quad (14)$$

where in all cases K is a constant of integration. We see that

solutions (i) and (ii) are almost identical. As expected, they are particular forms of the general solution (3) for the $\beta = \frac{1}{2}$ case.

We consider case (iii) and the condition $\beta = -a_2(2a_2 + 1)$ in more detail. The expression for the potential (6) is now

$$V(\xi) = -(a_2/4)(2a_2 + 1)\xi^4 + \xi^2 - \frac{1}{2}\xi^3. \quad (15)$$

If we are interested in real valued solutions to (2), the interpretation in terms of a_2 is valid for $\beta < \frac{1}{2}$ since for $\beta > \frac{1}{2}$, a_2 as given by (11) becomes complex. Hence the special solution (iii) is only to be considered for $\beta < \frac{1}{2}$. We recall that it is this value of β that separates two distinct regions of behavior of the potential. From Fig. 2 and Sec. III we see that the potential $V(\xi)$ has three extrema, $\xi_0 = 0$, $\xi_1 = \xi_m$, and $\xi_2 = \xi_M$, for $0 < \beta < \frac{1}{2}$. These extrema are obtained from Eq. (6) and they are given by

$$(\beta\xi_i^2 - \xi_i + 2)\xi_i = 0 \quad (i = 0, 1, 2). \quad (16)$$

In fact, Eq. (11) is transformed into Eq. (16) if we take $-1/a_2$ as the new variable. Consequently, introducing the solution ξ_m and ξ_M of Eq. (16) for a given β , we can write the special solution (iii) [Eq. (11)] as

$$\omega_M(\xi) = \xi(1 - \xi/\xi_M), \quad (17a)$$

$$\omega_m(\xi) = \xi(1 - \xi/\xi_m). \quad (17b)$$

Equations (17a) and (17b) show clearly the important role that will be played by the special solution (iii). The curves $\omega_M(\xi)$ and $\omega_m(\xi)$, interpreted as initial conditions in the phase space (ξ, ω) , are frontier curves for different types of time evolution. Indeed, the solution $\omega_M(\xi)$ describes a particle arriving with a zero velocity on the top of the potential hill (with possible subsequent bifurcations) while, for the solution $\omega_m(\xi)$, the particle falls in the potential bottom and has a zero velocity at $\xi = \xi_m$ (consequently playing the role of an attractor solution). These two curves are exactly the ones we found in the case $\beta = \frac{1}{2}$ (cf. the discussions of the end of Sec. I where we found that the boundary curve for bifurcating initial conditions is indeed the self-similar solution going through the point $\xi = \xi_M = 3$).

It is interesting to understand why such simple special solutions can be obtained and to see the roles of the two symmetries G_1 and G_2 (see Sec. II) which for our equation exists for all values of β .

In fact, the introduction of the new phase space (ξ, ω) makes the system autonomous (i.e., invariant under the symmetry $\partial/\partial\theta$). To compute the boundaries, we must consequently solve Eq. (7). In our case, the solutions of this equation can be obtained by using the symmetry G_1 [i.e., the invariance of the equation in (q, \dot{q}) space]. Let us consider at the initial time $t = 1$ the conditions $q = \xi_m$, $\dot{q} = -\xi_m$, which correspond in the (ξ, ω) space to $\xi = \xi_m$, $\omega = 0$. In this (ξ, ω) space, nothing happens and the particle is motionless. Of course, in the (q, \dot{q}) space we have an evolution with

$$q = \xi_m/t, \quad \dot{q} = -\xi_m/t^2.$$

Let us consider the system at $t = T$. Since the system is invariant under time translation, we can reintroduce the values ξ_m/T and $-\xi_m/T^2$ as new initial position and velocity and

bring back the clock to $t = 1$. The new ξ and ω are consequently

$$\xi = \xi_m/T, \quad \omega = -\xi_m/T^2 + \xi_m/T.$$

Eliminating T , we indeed obtain

$$\omega = \xi(1 - \xi/\xi_m) \quad (18)$$

and we have obtained the parabola going through the equilibrium point $(\xi_m, 0)$. In the same way, we obtain the parabola going through the other equilibrium point $(\xi_M, 0)$. These boundary curves appear consequently as prolongation of the two equilibrium points obtained by going into the (q, \dot{q}) space (where evolution takes place) and coming back to the (ξ, ω) space.

V. PAINLEVÉ ANALYSIS OF EQ. (2)

We perform the Painlevé analysis of (2) as follows. First, we determine the dominant behavior by substituting

$$q(t) \propto \alpha_0(t - t_1)^n \quad (19)$$

into (2). We find that $n = -1$ and all terms are dominant. This reflects the fact that (2) is invariant under the scale change $t \rightarrow at, q \rightarrow a^{-1}q$ as we saw already in the self-similar analysis of Sec. II. The value of α_0 is determined from the solution of

$$2\alpha_0 - \alpha_0^2 + \beta\alpha_0^3 = 0, \quad (20)$$

i.e.,

$$\alpha_0 = (1 \pm \sqrt{1 - 8\beta})/2\beta \Leftrightarrow \beta = (\alpha_0 - 2)/\alpha_0^2 \quad (21)$$

since $\alpha_0 \neq 0$. We have two particular solutions corresponding to the two roots. They are $\alpha_0^+(t - t_1)^{-1}$ and $\alpha_0^-(t - t_1)^{-1}$. These solutions fail to be real for $\beta > \frac{1}{8}$. In the second step of the Painlevé analysis we determine the resonances (Kowaleski exponents). Writing

$$q = \alpha_0\tau^{-1} + \alpha_1\tau^{-1+r}, \quad \tau = t - t_1, \quad (22)$$

and substituting into (2) we find that

$$r^2 + (\alpha_0 - 3)r + (\alpha_0 - 4) = 0, \quad (23)$$

so that r takes the values $r_1 = -1$, which always occurs in such analyses, and $r_2 = 4 - \alpha_0$. Since the resonances are required to be integral and the leading behavior is determined by $(t - t_1)^{-1}$, we have that $4 - \alpha_0$ must be a non-negative integer, i.e., $\alpha_0 \in \{3, 2, 1, 0, -1, \dots\}$. Consequently this specifies the value of β . For α_0^+ we have the sequence of permissible (α_0^+, β) to be $\{-n, -(n+2)/n^2, n \in \mathbb{Z}^+\}$ and, for α_0^- , $\{3, \frac{1}{2}, 2, 0, 1, -1\}$, where in the case of $(2, 0)$, l'Hôpital's rule must be used.

By way of example, the solution for $\alpha_0^- = 3$ and $\beta = \frac{1}{8}$ is

$$q(t) = 3(t - t_1)^{-1} + a_1 \left[1 - \frac{a_1}{3}(t - t_1) + \frac{a_1^2}{3^2}(t - t_1)^2 - \frac{a_1^3}{3^3}(t - t_1)^3 + \frac{a_1^4}{3^4}(t - t_1)^4 - \frac{a_1^5}{3^5}(t - t_1)^5 \dots \right].$$

Performing the summation, we obtain

$$q(t) = 3(t - t_1)^{-1} + 3a_1/[3 + a_1(t - t_1)], \quad (24)$$

where a_1 is arbitrary. We recover a particular case of the general solution given by (3).

For $\alpha_0^- = 1$ and $\beta = -1$, it is

$$q(t) = (t - t_1)^{-1} + a_3(t - t_1)^2 \left[1 + \frac{a_3}{21}(t - t_1)^3 + \frac{a_3^2}{3.21}(t - t_1)^6 + \frac{11a_3^3}{39.21.21}(t - t_1)^9 + \frac{4a_3^4}{39.21.21}(t - t_1)^{12} \dots \right], \quad (25)$$

where a_3 is arbitrary.

VI. QUALITATIVE BEHAVIOR

Now we present qualitative behavior obtained through numerical simulations of the system (5). The discussion is made in the phase space (ξ, ω) according to the values of initial conditions ξ_i and ω_i (ξ_i : initial position; ω_i : initial velocity), and according to the range in which the parameter β lies.

A. The case $0 < \beta < \frac{1}{8}$

We first study the range bounded by the critical values $\beta = 0$ and $\beta = \frac{1}{8}$. In this interval, we have the two special solutions $\omega_M(\xi) = \xi(1 - \xi/\xi_M)$ [Eq. (17a)] and $\omega_m(\xi) = \xi(1 - \xi/\xi_m)$ [Eq. (17b)], where $\xi_M = (1 - \sqrt{1 - 8\beta})/2\beta$ and $\xi_m = (1 + \sqrt{1 - 8\beta})/2\beta$.

As shown in Fig. 3, these solutions are two parabolic trajectories in the phase space. These curves pass through the point $(0, 0)$ where they have a common tangent line, $(d\omega/d\xi) = 1$.

The "large" parabola corresponds to $\omega = \omega_m(\xi)$ and the intersection with the ξ axis is at $\xi = \xi_m$. The small parabola represents $\omega = \omega_M(\xi)$ and $\xi = \xi_M$ is the intersection with the ξ axis.

In order to structure the discussion, we divide the phase space into four strips defined by

$$S_1 = \{\xi/\xi > \xi_m\},$$

$$S_2 = \{\xi/\xi_M < \xi < \xi_m\},$$

$$S_3 = \{\xi/0 < \xi < \xi_M\},$$

$$S_4 = \{\xi/\xi < 0\}.$$

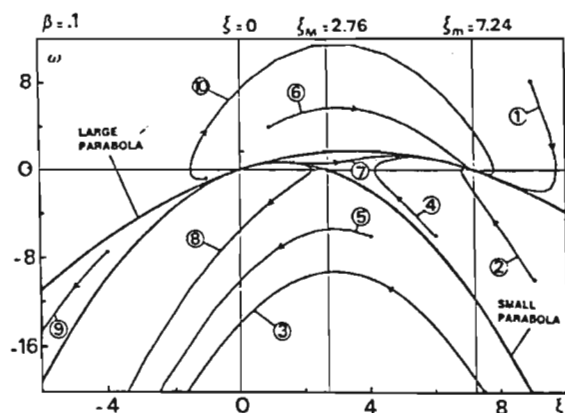


FIG. 3. Evolution in the phase space (ξ, ω) for $\beta = \frac{1}{8}$ ($0 < \beta < \frac{1}{8}$). The "large" and "small" parabola represent, respectively, $\omega = \omega_m(\xi)$ and $\omega = \omega_M(\xi)$. The curves 1, 2, 4, 6, 7, 10 lead to the attractor point $\xi = \xi_m$ and $\omega = 0$. The curves 3, 5, 8, 9 describe explosive solutions.

We now describe the behavior for different initial conditions (ξ_i, ω_i) . We denote the asymptotic solution for the time-evolution of the particle by $(\xi_\infty, \omega_\infty)$.

1. $\xi_i \in S_1$

We have the following results:

- (i) $\omega_i > \omega_M(\xi)$, then $(\xi_\infty, \omega_\infty) = (\xi_m, 0)$, (26)
- (ii) $\omega_i < \omega_M(\xi)$, then $\omega_\infty(\xi) \sim -\xi^2/\xi_M$, for $\xi \rightarrow -\infty$. (27)

In case (i), the particle always asymptotically falls into the bottom of the potential with a zero velocity. Two features must be pointed out. First, if the particle moves initially towards the positive ξ , it comes back after a finite time and during the way back, the work done by the friction force $(3 - \xi)\omega$ exactly balances the potential energy in such a way that the particle reaches ξ_m with a zero velocity (curve ① in Fig. 3). As a matter of fact, also for negative initial velocity [but greater than $\omega_M(\xi_i)$], the particle cannot pass over the top ($\xi = \xi_M$) of the potential and it is finally trapped at $\xi = \xi_m$ (curve ②). For the second case [case (ii)], the initial velocity is highly negative. The particle climbs up the top of the potential and for $\xi < 0$, the ω -dependent force exceeds the potential force. As a consequence, an explosive solution arises and the particle moves outwards ($\xi \rightarrow -\infty$) with an increasing negative velocity (curve ③). The behavior is given by Eq. (27) which is obtained by seeking a solution of the form $A\xi^n$ in Eq. (7). We may now solve q as a function of t . Combining on one hand (4) and (26), we obtain the evolution corresponding to case (i),

$$q(t) \sim \frac{1 + \sqrt{1 - 8\beta}}{2\beta} \cdot \frac{1}{t} = \frac{\xi_m}{t}, \quad t \rightarrow \infty. \quad (28)$$

This is an extension of solution (12) to $\beta \neq \frac{1}{2}$ for large t and $K > 0$. On the other hand, combining (4) and (27) we obtain [case (ii)]

$$q(t) \sim \frac{1}{t_0 \log(t/t_0)} \sim \frac{1}{t - t_0}, \quad t \rightarrow t_0. \quad (29)$$

The particle moves to infinity in a finite time and we recover the results of the Painlevé analysis as given in Sec. V. This solution is again an extension of (12) to $\beta \neq \frac{1}{2}$, where the constant K is negative.

2. $\xi_i \in S_2$

The behavior is similar to the one obtained for the case $\xi_i \in S_1$. The curve $\omega = \omega_M(\xi)$ separates the initial conditions in two classes. The first class [$\omega > \omega_M(\xi)$] leads to evolution (28) (curve ④), corresponding to the attractor, while the second class [$\omega < \omega_M(\xi)$] leads to solution (29) (curve ⑤) corresponding to the explosive solution). A blowup of Fig. 3 in the region $[0, \xi_m]$ is shown in Fig. 4.

3. $\xi_i \in S_3$

The asymptotic behavior is again the same as the case $\xi_i \in S_1$. Nevertheless (see Fig. 4), the transient stage is quite different. In fact, for $\omega_i > \omega_M(\xi_i)$, the particle climbs up the top of the potential before falling to $\xi = \xi_m$ with a zero velocity (curves ⑥ and ⑦). For $\omega_i < \omega_M(\xi_i)$, the particle cannot

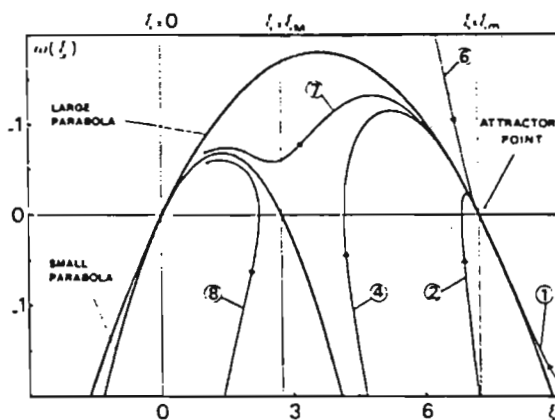


FIG. 4. Blowup of the region $[0, \xi_m]$ shown in Fig. 3.

not jump over the top and, if ω_i is positive, the particle falls back and for large negative ξ , $\omega_\infty(\xi)$ is given by (27) (curve ⑧).

4. $\xi_i \in S_4$

We have the following results:

- (i) $\omega_i > \omega_m(\xi_i)$, then $(\xi_\infty, \omega_\infty) = (\xi_m, 0)$; (30)
- (ii) $\omega_i < \omega_m(\xi_i)$, then $\omega_\infty(\xi) \sim -\xi^2/\xi_M$, for $\xi \rightarrow -\infty$. (31)

For high negative velocity [$\omega_i < \omega_m(\xi_i)$] the particle climbs up the potential (the velocity dependent force exceeds the potential force) and experiences an explosive instability (curve ⑨). By contrast for $\omega_i > \omega_m(\xi_i)$ the particle is always trapped at $\xi = \xi_m$ with a zero velocity (curve ⑩).

Let us point out that this result is again a generalization of the result obtained for $\beta = \frac{1}{2}$. In that case, the explosive instability appears at time τ for which $1 + A_1\tau + A_2\tau^2 = 0$. Taking (3) into account we find that, for $t \rightarrow \tau$, we have

$$q(t) = \frac{3(A_1 + A_2\tau)}{A_2(\tau - \tau')} \frac{1}{t - \tau}. \quad (32)$$

In the above formula, τ' is the second root of $1 + A_1t + A_2t^2 = 0$. Since $\xi = qt$ and $\omega = qt + \dot{q}t^2$, we have, for $t \rightarrow \tau$,

$$\begin{aligned} \xi &\approx \frac{3\tau(A_1 + 2A_2\tau)}{A_2(\tau - \tau')} \frac{1}{t - \tau}, \\ \omega &\approx \frac{3\tau^2(A_1 + 2A_2\tau)}{A_2(\tau - \tau')} \frac{(-1)}{(t - \tau)^2}. \end{aligned} \quad (33a)$$

For the computation of ω , we can neglect qt when $t \rightarrow \tau$. We have consequently, for $t \rightarrow \tau$,

$$\frac{\omega}{\xi^2} = \left(-\frac{1}{3} \right) \frac{A_2(\tau - \tau')}{A_1 + 2A_2\tau}. \quad (33b)$$

But since τ and τ' are the two roots of $A_2t^2 + A_1t + 1 = 0$, the second member of (33b) is just equal to $(-\frac{1}{3})$, i.e., to $-1/\xi_M$ in agreement with (31).

As a conclusion of this section, we have two kinds of asymptotic behavior. The particle falls to $\xi_\infty = \xi_m$ with $\omega_\infty = 0$ for $\xi_i > 0$ and $\omega_i > \omega_M(\xi_i)$ and for $\xi_i < 0$ and $\omega_i > \omega_m(\xi_i)$.

If $\omega_i < \omega_M(\xi_i)$ for $\xi_i > 0$ and if $\omega_i < \omega_m(\xi_i)$ for $\xi_i < 0$, the particle moves towards the region of negative ξ with an increasing negative velocity and $\omega(\xi) \sim -\xi^2/\xi_M$ for

$\xi \rightarrow -\infty$ (explosive solution). This behavior is described in terms of q and t by Eqs. (28), (29), and (31). Note that (28), (29), and (31) also display the asymptotic behavior for the case $\beta = \frac{1}{2}$. Consequently, asymptotic solutions obtained analytically for the case $\beta = \frac{1}{2}$ may be generalized to the whole range $0 < \beta < \frac{1}{2}$.

B. The case $\beta > \frac{1}{2}$

For $\beta > \frac{1}{2}$, the situation is completely modified. The potential has only a minimum at $\xi = 0$. The two pivot points $\xi = \xi_m$ and $\xi = \xi_M$ and the corresponding "small" and "large" parabolas do not exist any more. Since we have shown that for $0 < \beta < \frac{1}{2}$, these elements give both bifurcation boundaries and the attracting solutions, we can wonder if, for $\beta > \frac{1}{2}$, there is any interest in working in the new space. Consequently, the study in terms of q , \dot{q} , and t appears more relevant. In the following, we consider therefore the basic equation (2) and we will present simulations performed in the phase space (q, \dot{q}) .

Equation (2) holds symmetry properties: it is invariant under the transformation $t \rightarrow -t$ and $q \rightarrow -q$. The trajectories in the phase space are therefore expected to be symmetric with respect to the \dot{q} axis. Starting initially from $-q_0 < 0$ with a zero velocity ($\dot{q} = 0$), the particle moves towards the region $q > 0$ and reaches the point q_0 with zero velocity. For the same reason, the evolution from this point q_0 with this initial zero velocity brings back the particle at $-q_0$. In other words, the trajectory in the (q, \dot{q}) space is a closed curve. The particle oscillates in the range $[-q_0, +q_0]$.

Figure 5 exhibits the evolution for $\beta = 0.15$, $\beta = 0.3$, and $\beta = 0.6$. The initial position is $-q_0 = -1$.

We note that the larger the value of β , the larger is the amplitude of \dot{q} . This behavior comes out from the fact that the potential $\beta q^4/4$ stiffens for increasing β . Consequently, the particle is more accelerated and the amplitude of the velocity increases.

The lower part ($\dot{q} < 0$) of the trajectory presents a little bump. This is a slowing down of the particle, when going from $q > 0$ to $q < 0$, due to the velocity-dependent drag force.

As one brings β nearer the critical value $\frac{1}{2}$, the particle passes through $q = 0$ at a slower pace and in the limit $\beta = \frac{1}{2}$, the particle arrives at $q = 0$ with no velocity while the time to reach that point goes to infinity. It is this approach to the

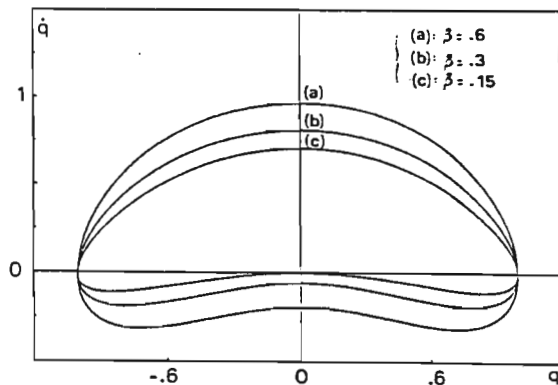


FIG. 5. Solution in the phase space (q, \dot{q}) for different values of β ($\beta > \frac{1}{2}$).

point $q = \dot{q} = 0$ that is subsequently described and solved in the new space. For $\beta > \frac{1}{2}$ we, of course, do not need this new space.

C. The case $\beta < 0$

Since we now recover the self-similar solutions, we come back to the (ξ, ω) space. Figure 6 exhibits the two parabolic trajectories in the phase space (ξ, ω) . These trajectories correspond to the special solutions (17a) and (17b) for $\beta < 0$. But in contradistinction to Sec. VI A, the curve $\omega_m(\xi)$ is now concave. Note that ξ_m is always negative while ξ_M always lies in the range $[0, 2]$.

The new feature brought in this subsection is that no attractor point can exist in the potential $V(\xi)$.

This can be easily seen in terms of the variables q and t . Indeed, coming back to Eq. (2), the force $-\beta q^3$ derives from the potential $V(q) = (\beta/4)q^4$, which is always negative and monotonic decreasing for $q \rightarrow +\infty$. Consequently, the point $(q, \dot{q}) = (0, 0)$ is unstable and the particle always falls to the right or to the left.

The question is now the following: Starting from $q < 0$ (resp. $q > 0$) with a positive velocity \dot{q} (resp. $\dot{q} < 0$) can a particle reach the top ($q = 0$) of the potential and fall towards the zone $q > 0$ (resp. $q < 0$)?

The answer is yes. It is obtained from the study in the (ξ, ω) phase space given in Fig. 6. Keeping in mind that the curves $\omega_m(\xi)$ and $\omega_M(\xi)$ are trajectories of the particle and noting that two distinct trajectories cannot cross, the phase space (ξ, ω) separates in two regions. The frontier between these two regions is given by the curve $\omega = \omega_m(\xi)$ for $\xi < 0$ and by $\omega = \omega_M(\xi)$ for $\xi > 0$. We see, therefore, that this property, already obtained in the range $0 < \beta < \frac{1}{2}$, is preserved for $\beta < 0$.

The first region contains every initial point (ξ_i, ω_i) whose evolution brings asymptotically the particle towards $\xi \rightarrow +\infty$ with positive velocity. For any initial condition (ξ_i, ω_i) taken in the second region, the particle always moves towards $\xi \rightarrow -\infty$ with negative velocity. The initial points labeled ① to ④ (respectively, ⑤ to ⑧) give the evolution towards $\xi \rightarrow +\infty$ (resp. $-\infty$) with an increasing positive (respectively, negative) velocity [$\omega(\xi) \sim \xi^2$].

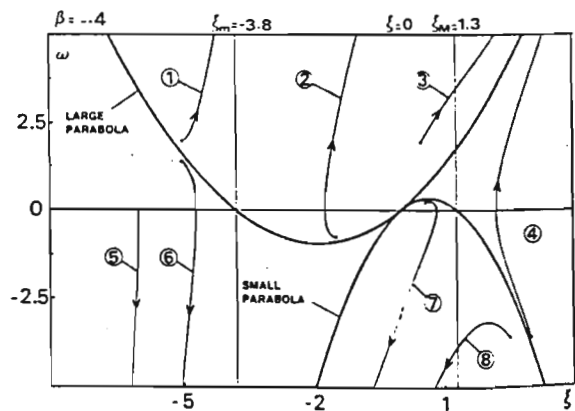


FIG. 6. Evolution in the phase space (ξ, ω) for $\beta = -0.4$ ($\beta < 0$). The curves ① to ④ (resp. ⑤ to ⑧) describe a particle moving towards $\xi > 0$ (resp. $\xi < 0$).

VII. CONCLUSION

The purpose of this paper was, first, to exhibit the importance and role of the different symmetries obtained by the usual Lie analysis, and, second, to show that rescaling and casting the problem in a new phase space allows one to use qualitative arguments about the sign of the drag force, the form of the new potential, and the role of the self-similar solutions.

The first intriguing result is the critical position from a purely "mathematical" point of view of the case $\beta = \frac{1}{2}$ where everything can be analytically expressed while from a "physical" point of view it plays no role at all. This somewhat arbitrary distinction between mathematical and physical points of view just means that the bifurcation values (for the parameter) or the bifurcation boundaries (for the initial conditions) will be labeled physical results while the fact that an analytical expression can be given is a mathematical one.

The second result was the "pivot" role played by the self-similar solutions $q = \xi_m/t$ and ξ_M/t . Moreover, since the equation is also invariant under time translation, these solutions can be extended (curves $\omega = \xi - \xi^2/\xi_m$ and $\omega = \xi - \xi^2/\xi_M$) providing all the interesting results about the boundaries for the initial conditions leading to bifurcations and the nature of the asymptotic solutions. When these self-similar solutions disappear, the nature of the general solution is totally modified.

The third result is the interesting concept of rescaling where, introducing the new dynamical variables, we can find easily the equilibrium points and their nature not only in

their neighborhood but, sometimes, far away in a strongly nonlinear region. Of course, in some situations ($\beta > \frac{1}{2}$), the new phase space does not present any interest and the problem is simpler in the original (q, \dot{q}, t) space.

Now we point out that these methods and concepts can be generalized to the case of the second-order differential equation with self-similar solutions [either time invariant as (3) or not]. Of course, the results will depend on the structure of the different terms of the equation, but the methodology will essentially remain the same. We will present later work on this generalization.

ACKNOWLEDGMENT

One of us (P. G. L. Leach) thanks the CNRS at Orléans for its hospitality in July 1987 during which time part of this work was undertaken.

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Lie algebras associated with scalar second-order ordinary differential equations

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Journal of Mathematical Physics **30** (1989) 2770-2775

Lie algebras associated with scalar second-order ordinary differential equations

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(Received 18 November 1988; accepted for publication 28 June 1989)

Second-order ordinary differential equations are classified according to their Lie algebra of point symmetries. The existence of these symmetries provides a way to solve the equations or to transform them to simpler forms. Canonical forms of generators for equations with three-point symmetries are established. It is further shown that an equation cannot have exactly $r \in \{4, 5, 6, 7\}$ point symmetries. Representative(s) of equivalence class(es) of equations possessing $s \in \{1, 2, 3, 8\}$ point symmetry generator(s) are then obtained.

I. INTRODUCTION

It is by now well established that when one realizes a real low-dimensional Lie algebra in terms of vector fields in two coordinates more than one canonical form may occur. A familiar example of this is found in the two canonical forms of generators obtained for each of the two real two-dimensional Lie algebras $2A_1$ and A_2 given in Lie.¹

Lie showed that if a second-order ordinary differential equation $\ddot{q} = E(t, q, \dot{q})$ admits a two-dimensional algebra (Abelian algebra $2A_1$ or the solvable algebra A_2) of point symmetries, then its point symmetries G_1 and G_2 can be either connected [i.e., there exists a function $\rho(t, q)$ such that $G_2 = \rho(t, q)G_1$] or unconnected [i.e., for any function $\psi(t, q)G_2 \neq \psi(t, q)G_1$]. Thus he was led to distinguish between connected and unconnected operators for each of the algebras $2A_1$ and A_2 , respectively. Consequently he had four cases (see Ref. 2) to consider (Types I–IV, see Ref. 3). Lie deduced that, if a second-order equation admits a two-dimensional algebra with operators G_1 and G_2 satisfying $G_2 = \rho(t, q)G_1$ for a suitable function ρ (i.e., Type II or IV of Ref. 3, then it is linearizable via a point transformation, i.e., it has $SL(3, \mathbb{R})$ symmetry (see Ref. 2).

In Ref. 2 second-order equations having two commuting unconnected point symmetries (Type I) were investigated for linearizability. It was shown that such an equation is linearizable provided the regular point transformation that brings these symmetries into their canonical form reduces the equation to one which is at most cubic in the first derivative. Accordingly, for equations of this type, the complete symmetry group $SL(3, \mathbb{R})$ and the corresponding linearizing point transformations were obtained.

The study of second-order equations admitting two noncommuting unconnected point symmetries (Type III), with a view to linearization, was undertaken in Ref. 3. It was found that such an equation has the $SL(3, \mathbb{R})$ symmetry group provided the point transformation that casts these symmetries to their canonical form reduces the equation to the form

$$t\ddot{q} = a\dot{q}^3 + b\dot{q}^2 + (1 + b^2/3a)\dot{q} + b/3a + b^3/27a^2, \quad (i)$$

where $a (\neq 0)$ and b are arbitrary constants.

A necessary condition for a second-order equation to

admit the $sl(3, \mathbb{R})$ algebra (see Refs. 2 and 3) is that it be of the form

$$\ddot{q} = \mathcal{A}(t, q)\dot{q}^3 + \mathcal{B}(t, q)\dot{q}^2 + \mathcal{C}(t, q)\dot{q} + \mathcal{D}(t, q), \quad (ii)$$

where the functions \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} are analytic. Sufficient conditions for a second-order equation to admit the $sl(3, \mathbb{R})$ algebra are given by (see Ref. 3)

$$3\mathcal{A}_{tt} + 3\mathcal{A}_t\mathcal{C} - 3\mathcal{A}_q\mathcal{D} + 3\mathcal{A}\mathcal{C}_t + \mathcal{C}_{qq} - 6\mathcal{A}\mathcal{D}_q + \mathcal{B}\mathcal{C}_q - 2\mathcal{B}\mathcal{B}_t - 2\mathcal{B}_{tq} = 0, \quad (iii)$$

$$6\mathcal{A}_t\mathcal{D} - 3\mathcal{B}_q\mathcal{D} + 3\mathcal{A}\mathcal{D}_t + \mathcal{B}_{tt} - 2\mathcal{C}_{tq} - 3\mathcal{B}\mathcal{D}_q + 3\mathcal{D}_{qq} + 2\mathcal{C}\mathcal{C}_q - \mathcal{C}\mathcal{B}_t = 0, \quad (iv)$$

where the suffices refer to partial derivatives.

In Ref. 2 and 3 we have treated all four cases (Types I–IV) of second-order equations admitting two-dimensional algebras of point symmetries which are linearizable via a point transformation. Thus, if an equation of the form (ii) passes the linearization test, i.e., conditions (iii) and (iv) hold, then one requires only two-point symmetries of the equation to obtain a linearizing point transformation for the equation.

In this paper we thoroughly investigate second-order equations admitting real Lie algebras of dimension two and higher (and at most eight which is the maximum dimension for such equations). We give a complete and rigorous treatment of three-dimensional Lie algebra realizations in terms of vector fields defined on the plane (see Sec. II) since in this respect there have been omissions in the works of Lie. Lie did not take into account the realizations of the family of algebras $A_{3,7}$ (b is a real parameter) and the algebras $A_{3,6}$ and $A_{3,9}$ (see Table I for their commutation relations). The canonical realizations in two coordinates obtained for the low-dimensional Lie algebras (see Tables I–III) are utilized systematically to derive all equivalence classes of second-order equations (see Sec. III) which admit point symmetry algebras. More precisely, we obtain five representatives [see (45)] of equivalence classes of second-order equations having exactly three-point symmetries. It is also deduced that second-order equations possessing exactly two-point symmetries can belong to one of two equivalence classes [see (46)]. Moreover, we prove, in Sec. II, that a second-order equation cannot admit exactly $r \in \{4, 5, 6, 7\}$ point symme-

TABLE I. Lie algebras of dimension 3.

Algebra	Nonzero commutation relations
$3A_1$	
$A_1 \oplus A_2$	$[G_1, G_3] = G_1$
$A_{3,1}$ (Weyl)	$[G_2, G_3] = G_1$
$A_{3,2}$	$[G_1, G_3] = G_1, [G_2, G_3] = G_1 + G_2$
$A_{3,3}$ ($D \otimes T_2$)	$[G_1, G_3] = G_1, [G_2, G_3] = G_2$
$A_{3,4}$ ($E(1,1)$)	$[G_1, G_3] = G_1, [G_2, G_3] = -G_2$
$A_{3,5}$ ($0 < a < 1$)	$[G_1, G_3] = G_1, [G_2, G_3] = aG_2$
$A_{3,6}$ ($E(2)$)	$[G_1, G_3] = -G_2, [G_2, G_3] = G_1$
$A_{3,7}$ ($b > 0$)	$[G_1, G_3] = bG_1 - G_2, [G_2, G_3] = G_1 + bG_2$
$A_{3,8}$ ($SL(2, \mathbb{R})$)	$[G_1, G_2] = G_1, [G_2, G_3] = G_3, [G_3, G_1] = -2G_2$
$A_{3,9}$ ($SO(3)$)	$[G_1, G_2] = G_3, [G_2, G_3] = G_1, [G_3, G_1] = G_2$

tries. This in effect means that a second-order equation can admit exactly one of 0, 1, 2, 3, or 8 point symmetries.

II. LIE ALGEBRA REALIZATIONS IN TWO COORDINATES

The Lie algebra classification used is the Mubarakzhanov classification given in Patera *et al.*⁴ For the three- and four-dimensional algebras our list also includes the decomposable algebras. The notation is the same as given in Ref. 4. Thus when referring to A^r_j , we simply mean the j th algebra of dimension r . The superscript(s), if any, indicate(s) the parameter(s) on which the algebra depends. The range of parameters is restricted to avoid double counting and algebraic sums of lower algebras. Assignment of specific values to a parameter singles out special algebras, within a family, which are well-known and which in some cases result in the linearization of the associated second-order equation. There are 11 Lie algebras of dimension three⁴ (decomposable and indecomposable) two of which depend on parameters. For convenience and easy reference we present their commutation relations in a convenient basis $\{G_i; i = 1, 3\}$ in Table I.

The decomposable Lie algebras are the Abelian algebra $3A_1$ and the non-Abelian one $A_1 \oplus A_2$. In many cases we have indicated the corresponding Lie groups in parentheses. These are the Weyl group, the semidirect product of dilations and translations $D \otimes T_2$, the Euclidean group $E(2)$, the pseudo-Euclidean group $E(1, 1)$, the special linear group $SL(2, \mathbb{R})$, and the special orthogonal group $SO(3)$.

The following easily verified identities will be used in the proof of theorems. Let G_1, G_2, G_3 be operators of the form

$$G = \xi(t, q) \frac{\partial}{\partial t} + \eta(t, q) \frac{\partial}{\partial q}.$$

Then (a) $[G_1, G_2] = (-G_2\rho)G_2$, if $G_1 = \rho(t, q)G_2$ for a suitable function ρ , (b) $[G_1, G_3] = (G_1\psi)G_2 + \psi[G_1, G_2]$, if $G_3 = \psi(t, q)G_2$ for a suitable function ψ .

Theorem 1: A second-order ordinary differential equation does not admit the Abelian Lie algebra, $3A_1$.

Proof: Suppose a general second-order equation of the form $\ddot{q} = H(\dot{q}, q, t)$ has the Abelian algebra. The generators of symmetry G_i ($i = 1, 3$) then satisfy the Abelian commutation relations $3A_1$. This being the case, we firstly show that

one cannot have $G_1 = \rho(t, q)G_2$ or $G_3 = \psi(t, q)G_2$ (for any nonconstant functions ρ and ψ). Assume the contrary is true. Then for the case $G_1 = \rho G_2$, the first and last commutators of $3A_1$ imply

$$G_2\rho = \xi_2 \frac{\partial \rho}{\partial t} + \eta_2 \frac{\partial \rho}{\partial q} = 0,$$

$$G_3\rho = \xi_3 \frac{\partial \rho}{\partial t} + \eta_3 \frac{\partial \rho}{\partial q} = 0.$$

As ρ cannot be a constant, at least one of $\partial\rho/\partial t$ or $\partial\rho/\partial q$ is nonzero. This in turn leads to $\xi_2\eta_3 - \xi_3\eta_2 = 0$ and consequently

$$G_3 = (\xi_3/\xi_2)G_2.$$

Invoking the second commutator, $[G_2, G_3] = 0$, we obtain

$$G_2(\xi_3/\xi_2) = 0.$$

Therefore we must have $\xi_3/\xi_2 = f(\rho)$ for some function f (since also $G_2\rho = 0$). Thus eventually we have

$$G_1 = \rho G_2, \quad G_3 = f(\rho)G_2 \text{ with } G_2\rho = 0.$$

Transforming G_2 to $\bar{G}_2 = \partial/\partial Q$ and solving $\bar{G}_2\bar{\rho} = 0$ [$\bar{\rho} = \bar{\rho}(Q, T)$ where $Q = Q(t, q)$ and $T = T(t, q)$ are the transformed coordinates] we obtain $\bar{\rho} = g(T)$ for some function g . Without loss of generality we may set $g(T) = T$ making the generators appear as (cf. Lie¹)

$$\bar{G}_1 = T \frac{\partial}{\partial Q}, \quad \bar{G}_2 = \frac{\partial}{\partial Q}, \quad \bar{G}_3 = f(T) \frac{\partial}{\partial Q}. \tag{1}$$

Expressing the invariance of the differential equation for Q with respect to (1) results in f being linear in T contradicting the linear independence of the G_i 's. The preceding argument applies equally to the case $G_3 = \psi G_2$ since G_1 and G_3 play interchangeable roles. It follows therefore that $G_1 \neq \rho G_2$ and $G_3 \neq \psi G_2$ for any functions ρ and ψ . This, however, leads to (upon reducing G_1, G_2 to canonical form)

$$\bar{G}_1 = \frac{\partial}{\partial T}, \quad \bar{G}_2 = \frac{\partial}{\partial Q}, \quad \bar{G}_3 = \alpha \frac{\partial}{\partial T} + \beta \frac{\partial}{\partial Q}$$

(α, β constants and $\alpha \neq 0$) which obviously are linearly dependent. \diamond

An immediate consequence of the above theorem is that a differential equation does not admit a Lie algebra which contains the Abelian three-dimensional algebra $3A_1$, as a subalgebra. As a result we need not consider all the real Lie algebras of dimension four (24 if we include, as we have, the decomposable ones). Indeed we have tabulated only nine (see Table II below). Of the five-dimensional algebras, Theorem 1 eliminates all except three of the indecomposable algebras $A_{5,36}, A_{5,37}$, and $A_{5,40}$ (see Table III) and two of the decomposable algebras $A_{3,8} \oplus A_2$ and $A_{3,9} \oplus A_2$. We list the four-dimensional algebras (in convenient basis $\{G_i; i = 1, 4\}$) of relevance together with their three-dimensional subalgebras (Patera and Winternitz⁵) in Table II. The subalgebras listed in Table II are enclosed in parentheses (rather than in braces), e.g., $(G_1, G_3; G_2)$. This is to indicate that they are maximal subalgebras of the Lie algebras considered. The generators of the derived algebra are written to the right of the semicolon.

Concerning the five-dimensional real Lie algebras (indecomposable), all but three of the algebras do not contain

TABLE II. Relevant algebras of dimension 4 and their three-dimensional subalgebras.

Algebra	Nonzero commutators	Dimension 3
$2A_2$	$[G_1, G_2] = G_2,$ $[G_3, G_4] = G_4$	$A_1 \oplus A_2: (G_1, G_3; G_2)(G_1, G_4; G_2)$ $(G_1, G_3; G_4)(G_2, G_3; G_4)$ $A_{3,3}: (G_1 + G_3; G_2, G_4)$ $A_{3,4}: (G_1 - G_3; G_2, G_4)$ $A_{3,5}^a: (G_1 + xG_3; G_2, G_4),$ $a = \begin{cases} x, & 0 < x < 1, \\ 1/x, & 1 < x < \infty, \end{cases}$
$A_{3,8} \oplus A_1$	$[G_1, G_3] = 2G_2,$ $[G_1, G_2] = 1,$ $[G_2, G_3] = G_3$	$A_2 \oplus A_1: (G_2, G_4; G_1)$
$A_{3,9} \oplus A_1$	$[G_1, G_2] = G_3,$ $[G_2, G_3] = G_1,$ $[G_3, G_4] = G_2$	$A_{3,8}: (G_1, G_2, G_3)$ $A_{3,9}: (G_1, G_2, G_3)$
$A_{4,7}$	$[G_1, G_4] = 2G_1,$ $[G_2, G_4] = G_2,$ $[G_3, G_4] = G_2 + G_3,$ $[G_2, G_3] = G_1$	$A_{3,1}: (G_2, G_3; G_1)$ $A_{3,5}^{1/2}: (G_4; G_1, G_2)$
$A_{4,x}$	$[G_2, G_3] = G_1,$ $[G_2, G_4] = G_2,$ $[G_3, G_4] = -G_1$	$A_{3,1}: (G_2, G_3; G_1)$
$A_{4,9}^b$ ($0 < b < 1$)	$[G_2, G_3] = G_1,$ $[G_1, G_4] = (1+b)G_1,$ $[G_2, G_4] = G_2,$ $[G_3, G_4] = bG_3$	$A_2 \oplus A_1: (G_4, G_1; G_2), (G_4, G_1; G_3)$ $A_{3,1}: (G_2, G_3; G_1)$ $A_{3,5}^c: (G_4; G_1, G_2),$ $v = \begin{cases} 1+b, & 1+b < 1, \\ 1/(1+b), & 1+b > 1, \end{cases}$ $A_{3,4}: (G_4; G_1, G_1), \quad b = -\frac{1}{2}$ $A_{3,5}^d: (G_4; G_1, G_1),$ $w = \begin{cases} b/(1+b), & b/(1+b) < 1, \\ (1+b)/b, & b/(1+b) > 1. \end{cases}$
$A_{4,9}^e$	$[G_2, G_3] = G_1,$ $[G_1, G_4] = 2G_1,$ $[G_2, G_4] = G_2,$ $[G_3, G_4] = G_3$	$A_{3,1}: (G_2, G_3; G_1)$ $A_{3,5}^{1/2}: (G_4; G_1, \cos \phi G_2 + \sin \phi G_3)$ $0 < \phi < \pi$
$A_{4,9}^f$	$[G_2, G_3] = G_1,$ $[G_1, G_4] = G_1,$ $[G_2, G_4] = G_2$	$A_{3,1}: (G_2, G_3; G_1)$ $A_2 \oplus A_1: (G_3, G_4; G_1)$ $A_{3,3}: (G_4; G_1, G_2)$ $A_{3,2}: (G_4 + xG_1; G_1, G_2) \quad (x \neq 0)$ $A_{3,1}: (G_2, G_3; G_1)$
$A_{4,10}$	$[G_2, G_3] = G_1,$ $[G_2, G_4] = -G_1,$ $[G_3, G_4] = G_2$	$A_{3,1}: (G_2, G_3; G_1)$
$A_{4,11}^g$ ($a > 0$)	$[G_2, G_3] = G_1,$ $[G_1, G_4] = 2aG_1,$ $[G_2, G_4] = aG_2 - G_3,$ $[G_3, G_4] = G_2 + aG_3$	$A_{3,1}: (G_2, G_3; G_1)$
$A_{4,12}$	$[G_1, G_3] = G_1,$ $[G_2, G_3] = G_2,$ $[G_1, G_4] = -G_2,$ $[G_2, G_4] = G_1$	$A_{3,1}: (G_3; G_1, G_2)$ $A_{3,6}: (G_4; G_1, G_2)$ $A_{3,7}^h: (G_4 + xG_1; G_1, G_2) \quad (x \neq 0)$

$3A_1$ as a subalgebra. Their commutation relations in a convenient bases $\{G_i; i = 1, 5\}$ are given in Table III.

We shall discuss the Lie algebras listed in Tables II and III after investigating the three-dimensional Lie algebra realizations of Table I.

Proposition 2: If a second-order equation admits the Lie algebra $sl(2, \mathbb{R})$ ($A_{3,8}$), then it has either three or eight generators of symmetry.

Proof: Suppose an equation admits the Lie algebra $sl(2, \mathbb{R})$, i.e., the generators of symmetry satisfy the $sl(2, \mathbb{R})$ commutation relations. Proceeding as in Theorem 1 we can-

TABLE III. Relevant algebras of dimension 5.

Algebra	Nonzero commutation relations
$A_{3,36}$	$[G_2, G_3] = G_1, [G_1, G_4] = G_1, [G_2, G_4] = G_2$ $[G_2, G_5] = -G_2, [G_3, G_5] = G_3$
$A_{3,37}$	$[G_2, G_3] = G_1, [G_1, G_4] = 2G_1, [G_2, G_4] = G_2$ $[G_3, G_4] = G_3, [G_2, G_5] = -G_3, [G_3, G_5] = G_2$
$A_{3,40}$	$[G_1, G_2] = 2G_1, [G_1, G_3] = -G_2, [G_2, G_3] = 2G_3$ $[G_1, G_4] = G_3, [G_2, G_4] = G_4, [G_2, G_5] = -G_5,$ $[G_3, G_5] = G_4$

not have $G_1 = \rho(t, q)G_2$ or $G_3 = \psi(t, q)G_2$ (for any nonconstant functions ρ and ψ) as, in the case $G_1 = \rho G_2$, the first and last commutators yield

$$G_2 \rho + \rho = 0, \quad G_3 = (1/\rho)G_2, \quad (2)$$

with the remaining one giving rise to

$$G_2(1/\rho) = 1/\rho, \quad (3)$$

these in turn leading to the generators [on transforming G_2 to $\bar{G}_2 = Q(\partial/\partial Q)$ and choosing the simplest solution of (2a)]

$$\bar{G}_1 = \frac{\partial}{\partial Q}, \quad \bar{G}_2 = Q \frac{\partial}{\partial Q}, \quad \bar{G}_3 = Q^2 \frac{\partial}{\partial Q}, \quad (4)$$

which as is seen by elementary calculation cannot be symmetries of a second-order equation! It follows therefore that $G_1 \neq \rho G_2$ for any function ρ . Similarly it can be shown that no function ψ exists such that $G_3 = \psi G_2$. Hence $G_1 \neq \rho G_2$ and $G_3 \neq \psi G_2$ for any functions ρ and ψ . Consequently, there exists a regular point transformation $Q = Q(t, q)$, $T = T(t, q)$, transforming G_1, G_2 to the canonical form

$$\bar{G}_1 = \frac{\partial}{\partial Q}, \quad \bar{G}_2 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}. \quad (5a)$$

The commutators involving \bar{G}_3 then give us

$$\bar{G}_3 = (2TQ + \alpha T^2) \frac{\partial}{\partial T} + (Q^2 + \beta T^2) \frac{\partial}{\partial Q}, \quad (5b)$$

where α, β are constants.

It is now a simple matter to obtain the differential equations for Q having at least the three symmetries (5). They are

$$TQ'' = \frac{-1}{(\alpha^2 + 4\beta)} \{ (2Q' + \alpha)(2Q'^2 + 2\alpha Q' - 2\beta) + A(2Q'^2 + 2\alpha Q' - 2\beta)^{3/2} \}, \quad (6)$$

with $\alpha^2 + 4\beta \neq 0, A$ constant and

$$TQ'' = -\frac{1}{2}(Q' + \alpha/2) + B(Q' + \alpha/2)^3, \quad (7)$$

with $\alpha^2 + 4\beta = 0, B$ constant. The emergence of two differential equations (6) and (7) should imply the existence of two canonical forms for the generators. This in fact is the case. We find that the linear transformations

$$\bar{T} = (i/2)(\alpha^2 + 4\beta)^{1/2} T, \quad i = \sqrt{-1},$$

$$\bar{Q} = Q + (\alpha/2)T, \quad \alpha^2 + 4\beta \neq 0,$$

and

$$\bar{T} = B^{-1/2}T, \quad B \neq 0,$$

$$\bar{Q} = Q + (\alpha/2)T, \quad \alpha^2 + 4\beta = 0,$$

do provide reductions to the two canonical forms

$$\bar{G}_1 = \frac{\partial}{\partial \bar{Q}}, \quad \bar{G}_2 = \bar{T} \frac{\partial}{\partial \bar{T}} + \bar{Q} \frac{\partial}{\partial \bar{Q}}, \quad (8)$$

$$\bar{G}_3 = 2\bar{T}\bar{Q} \frac{\partial}{\partial \bar{T}} + (\bar{Q}^2 - \bar{T}^2) \frac{\partial}{\partial \bar{Q}}$$

and

$$\bar{G}_1 = \frac{\partial}{\partial \bar{Q}}, \quad \bar{G}_2 = \bar{T} \frac{\partial}{\partial \bar{T}} + \bar{Q} \frac{\partial}{\partial \bar{Q}}, \quad (9)$$

$$\bar{G}_3 = 2\bar{T}\bar{Q} \frac{\partial}{\partial \bar{T}} + \bar{Q}^2 \frac{\partial}{\partial \bar{Q}}.$$

The associated differential equations are

$$\bar{T}\bar{Q}'' = \bar{Q}'^3 + \bar{Q}' + \mathcal{A}(1 + \bar{Q}'^2)^{3/2}, \quad (10)$$

$$\bar{T}\bar{Q}'' = \bar{Q}'^3 - \frac{1}{2}\bar{Q}', \quad (11)$$

where \mathcal{A} is a constant. It now becomes clear that Eq. (10) has more than the three given symmetries (8) whenever $\mathcal{A} = 0$. This follows straightforwardly from the results of Ref. 3 [see Sec. I, (i)]. It is also evident that Eq. (7) is linear when $B = 0$. For $B \neq 0$ we have the form (11). Equation (10) has exactly the three symmetries (8) whenever \mathcal{A} is nonzero.

It should also be mentioned that the transformations

$$\bar{T} = Q + aT, \quad \bar{Q} = Q + bT, \quad a, (\neq b) \text{ constants,}$$

with $\alpha = a + b, \beta = -ab$ and

$$\bar{T} = Q + aT, \quad \bar{Q} = T^{1/2},$$

with $\alpha = 2a, \beta = -a^2$ do provide reductions to the two Lie canonical forms

$$\bar{G}'_1 = \frac{\partial}{\partial \bar{T}} + \frac{\partial}{\partial \bar{Q}}, \quad \bar{G}'_2 = \bar{T} \frac{\partial}{\partial \bar{T}} + \bar{Q} \frac{\partial}{\partial \bar{Q}}, \quad (12a)$$

$$\bar{G}'_3 = \bar{T}^2 \frac{\partial}{\partial \bar{T}} + \bar{Q}^2 \frac{\partial}{\partial \bar{Q}},$$

and

$$\bar{G}'_1 = \frac{\partial}{\partial \bar{T}}, \quad \bar{G}'_2 = \bar{T} \frac{\partial}{\partial \bar{T}} + \frac{1}{2}\bar{Q} \frac{\partial}{\partial \bar{Q}}, \quad (12b)$$

$$\bar{G}'_3 = \bar{T}^2 \frac{\partial}{\partial \bar{T}} + \bar{T}\bar{Q} \frac{\partial}{\partial \bar{Q}}.$$

The corresponding differential equations are

$$(\bar{T} - \bar{Q})\bar{Q}'' + 2(\bar{Q}' + \bar{A}\bar{Q}'^{3/2} + \bar{Q}'^2) = 0, \quad (13a)$$

$$\bar{Q}^3\bar{Q}'' + \bar{B} = 0. \quad (13b)$$

where \bar{A} and \bar{B} are constants.

Proposition 3: If a second-order equation admits the Lie algebra $A_{3,7}^b$ ($b > 0$) $A_{3,6}$, then it has either three or eight generators of symmetry.

Proof: Suppose an equation admits the algebra $A_{3,7}^b$ ($b > 0$), $A_{3,6}$. Then we cannot have $G_2 = \rho(t, q)G_1$ and $G_3 = \psi(t, q)G_1$ for any nonconstant functions ρ and ψ as this

would lead to $\rho^2 + 1 = 0$. We thus assume $G_2 \neq \rho G_1$ for any ρ . This produces the following generators:

$$\bar{G}_1 = \frac{\partial}{\partial T}, \quad \bar{G}_2 = \frac{\partial}{\partial Q}, \quad (14)$$

$$\bar{G}_3 = (bT + Q) \frac{\partial}{\partial T} + (bQ - T) \frac{\partial}{\partial Q}.$$

The differential equation for Q is

$$Q'' = A(1 + Q'^2)^{3/2} \exp(b \arctan Q'), \quad (15)$$

where A is a constant. It is fairly simple to observe that the rhs of Eq. (15) cannot be a polynomial which is at most cubic in Q' unless $A = 0$. Hence it follows from the conclusion of Ref. 2 (see Sec. I) that (15) is not linearizable via a point transformation if $A \neq 0$. It has three symmetries. For $A = 0$ Eq. (15) clearly has five more symmetries.

We next assume that $G_3 \neq \psi G_1$ for any function ψ . Without loss of generality we may further assume that $G_1 = \phi(t, q)G_2$ for some function ϕ . Since we have $[G_1, G_2] = 0$, the reduction to canonical form of the generators G_1 and G_2 is given by

$$\bar{G}_1 = T \frac{\partial}{\partial Q}, \quad \bar{G}_2 = \frac{\partial}{\partial Q}, \quad (16a)$$

and

$$\bar{G}_3 = (1 + T^2) \frac{\partial}{\partial T} + (TQ + bQ) \frac{\partial}{\partial Q}, \quad (16b)$$

following easily from the commutators involving G_3 . The associated differential equation is

$$Q'' = B(1 + T^2)^{-3/2} \exp(b \arctan T), \quad (17)$$

where B is a constant. This equation clearly is linear. \diamond

The canonical forms (14) and (16) are presented for the first time. The realizations of the family of Lie algebras $A_{3,7}^b$ ($b > 0$) and $A_{3,6}$ and the corresponding second-order equations were not considered by Lie.

Remark: Suppose we have an equation of the precise form (15) or (17) but with $b < 0$. Then we can introduce the basis $\{V_1 = \bar{G}_2, V_2 = \bar{G}_1, V_3 = -\bar{G}_3\}$ so that the resulting Lie algebra is of the desired form $A_{3,7}^{-b}$ ($-b > 0$).

Proposition 4: If an equation admits the Lie algebra $A_{3,2}$, then it has either three or eight generators of symmetry.

Proof: Suppose an equation admits the algebra $A_{3,2}$. We find that the symmetry generators G_i cannot be proportional to each other. Therefore we resort to proof by cases. In the first case we assume that no function ψ exists such that $G_3 = \psi(t, q)G_1$. This, proceeding as before, gives rise to the following generators:

$$\bar{G}_1 = \frac{\partial}{\partial Q}, \quad \bar{G}_2 = \beta \frac{\partial}{\partial T} - \ln T \frac{\partial}{\partial Q}, \quad \beta \text{ constant,} \quad (18)$$

$$\bar{G}_3 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}$$

(disregarding an additive constant multiple of G_1). Expressing the invariance of a general differential equation for Q with respect to (18) yields

$$TQ'' = -1/\beta + A \exp(-\beta Q'), \quad \beta \neq 0 \quad (19)$$

and

$$TQ'' = -Q' + B, \quad \beta = 0, \quad (20)$$

where A and B are constants. Equation (20) is linear. It is immediate from the results of Ref. 3 [see (i)] that (19) is not linearizable (via a point transformation) if $A \neq 0$. In this case (19) has exactly three symmetries.

The transformations

$$\bar{Q} = Q + \frac{T}{\beta} \ln T - \frac{T}{\beta}, \quad \bar{T} = \frac{T}{\beta}, \quad \beta \neq 0,$$

and

$$\bar{Q} = -Q, \quad \bar{T} = -\ln T, \quad \beta = 0,$$

give the Lie canonical forms

$$\bar{G}'_1 = \frac{\partial}{\partial \bar{Q}}, \quad \bar{G}'_2 = \frac{\partial}{\partial \bar{T}}, \quad (21)$$

$$\bar{G}'_3 = \bar{T} \frac{\partial}{\partial \bar{T}} + (\bar{T} + \bar{Q}) \frac{\partial}{\partial \bar{Q}},$$

and

$$\bar{G}'_1 = -\frac{\partial}{\partial \bar{Q}}, \quad \bar{G}'_2 = \bar{T} \frac{\partial}{\partial \bar{Q}}, \quad (22)$$

$$\bar{G}'_3 = \frac{\partial}{\partial \bar{T}} + \bar{Q} \frac{\partial}{\partial \bar{Q}}.$$

The associated differential equations are

$$\bar{Q}'' = \bar{A} \exp(-\bar{Q}'), \quad (23)$$

$$\bar{Q}'' = \bar{B} \exp(\bar{T}), \quad (24)$$

where \bar{A} and \bar{B} are constants.

The case $G_2 \neq \rho G_1$ for ρ arbitrary gives the generators (21) above. \diamond

Proposition 5: If an equation admits the Lie algebra $A_{3,5}^a$ ($0 < |a| < 1$), $A_{3,4}$, $A_1 \oplus A_2$, then it has either three or eight generators of symmetry.

Proof: Suppose an equation admits the algebra $A_{3,5}^a$ ($0 < |a| < 1$), $A_{3,4}$, $A_1 \oplus A_2$.

Then we certainly must have $G_2 \neq \rho(t, q)G_1$ or $G_3 \neq \psi(t, q)G_1$ for any functions ρ and ψ . Otherwise the commutators would imply $G_1\psi = a (\neq 1)$ and $G_1\rho = 1$. Thus we first assume that no function ρ exists such that $G_2 = \rho G_1$. As a result the reduction to canonical form of the generators G_1 and G_2 is immediate and we finally obtain

$$\bar{G}'_1 = \frac{\partial}{\partial \bar{T}}, \quad \bar{G}'_2 = \frac{\partial}{\partial \bar{Q}}, \quad (25)$$

$$\bar{G}'_3 = T \frac{\partial}{\partial \bar{T}} + aQ \frac{\partial}{\partial \bar{Q}}.$$

The differential equation for Q invariant under at least the three symmetry generators (25) is

$$Q'' = AQ''^{(a-2)/(a-1)}, \quad (26)$$

where A is a constant. It follows from Ref. 2 (see Sec. I) that Eq. (26) is not linearizable (via a point transformation) if $A \neq 0$ and $a \neq 0, \frac{1}{2}, 2$. In this case (26) has three symmetries. Note that for $|a| > 1$ we can introduce the basis $\{V_1 = \bar{G}'_2, V_2 = \bar{G}'_1, V_3 = (1/a)\bar{G}'_3\}$ so that the resulting Lie algebra is $A_{3,5}^{1/a}$ ($0 < 1/|a| < 1$).

For the second part of the proof we assume $G_3 \neq \psi G_1$ for any function ψ . This gives rise to the symmetry generators

$$\bar{G}'_1 = \frac{\partial}{\partial Q}, \quad \bar{G}'_2 = \alpha T^{1-a} \frac{\partial}{\partial T} + \beta T^{1-a} \frac{\partial}{\partial Q}, \quad (27)$$

$$\bar{G}'_3 = T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q},$$

where α and β are constants. The associated differential equations having at least these three symmetries are then given by

$$TQ'' = (1/\alpha)(a-1)(\alpha Q' - \beta) + B(\alpha Q' - \beta)^{(2a-1)/(a-1)}, \quad \alpha \neq 0, \quad (28)$$

$$TQ'' = -aQ' + C, \quad \alpha = 0, \quad (29)$$

where B and C are constants. Equation (29) being linear has eight symmetries, three of them given by (27) (with $\alpha = 0$). By means of a linear transformation we can reduce (28) to the form [may as well set $\alpha = 1, \beta = 0$ in (28)]

$$TQ'' = (a-1)Q' + \mathcal{B}Q''^{(2a-1)/(a-1)}, \quad (30)$$

where \mathcal{B} is a constant. Evidently this equation is not linearizable if $\mathcal{B} \neq 0$ and $a \neq 0, \frac{1}{2}$. For $a = 2$ and $\mathcal{B} \neq 0$, (30) takes the form

$$TQ'' = \mathcal{B}Q'^3 + Q'. \quad (31)$$

By a rescaling of time we can make the constant \mathcal{B} unity, whence we have the familiar equation of Ref. 3 [see (i)] for which we obtained eight-point symmetries. \diamond

Under the transformation

$$Q = t, \quad T = a^{1/a}q^{1/a}, \quad a \neq 0, \quad (32)$$

(25) (in lower case variables and with $a \neq 0$) is equivalent to (27) ($\alpha = 1, \beta = 0$). When $a = 0$ ($\alpha = 1, \beta = 0$) the generators (27) are reducible to the simple form

$$\bar{G}'_1 = \frac{\partial}{\partial Q}, \quad \bar{G}'_2 = T \frac{\partial}{\partial T}, \quad \bar{G}'_3 = Q \frac{\partial}{\partial Q}, \quad (33)$$

where $\bar{G}'_3 = \bar{G}'_3 - \bar{G}'_2$. The generators (25) (in lower case variables and with $a = 0$) are now equivalent to (33) by means of the transformation

$$T = \exp q, \quad Q = t.$$

We also point out that the transformation

$$Q = q, \quad T = \beta t^{1-a}, \quad \beta \neq 0,$$

maps (27) (in lower case variables with $\alpha = 0$) to

$$\bar{G}'_1 = \frac{\partial}{\partial Q}, \quad \bar{G}'_2 = T \frac{\partial}{\partial Q}, \quad (34)$$

$$\bar{G}'_3 = (1-a)T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}.$$

The related equation is

$$Q'' = DT^{(1-2a)/(a-1)}, \quad (35)$$

where D is a constant.

In view of the above discussion we have two canonical forms for the generators. They are either the Lie canonical forms given by (25) and (34) or those given by (27) upon setting $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 0$, respectively.

We now prove theorems relating to the linearizability (via a point transformation) of a second-order equation.

Theorem 6: In order that a differential equation $\ddot{q} = N(\dot{q}, q, t)$ possesses $\mathfrak{sl}(3, \mathbb{R})$ algebra it is necessary and sufficient that it has the algebra (A) $A_1 \oplus A_2$, $A_{3,5}^{1/2}$, (B) $A_{3,3}$, (C) $A_{3,1}$, (D) $A_{3,9}$ [so(3)].

Proof (A): To prove sufficiency we need only refer to Proposition 5 and the discussion following it. We restrict our attention to (27) with $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 0$, respectively, bearing in mind that a can take on the values 0, $\frac{1}{2}$, or 2. When $a = 2$ we always can introduce a basis so that the Lie algebra is $A_{3,5}^{1/2}$. This much is apparent from the previous theorem. The relevant equations (29) and (30) each have eight generators of symmetry whenever $a = 0, \frac{1}{2}, 2$. Thus they admit the $\mathfrak{sl}(3, \mathbb{R})$ algebra. The original equation $\ddot{q} = N(\dot{q}, q, t)$ accordingly has the same algebra. Proof of the necessity is trivial because $A_1 \oplus A_2$ and $A_{3,5}^{1/2}$ are subalgebras of $\mathfrak{sl}(3, \mathbb{R})$. \diamond

We proved part (B) of the above theorem in Ref. 6 which deals with the reverse procedure to find the point symmetry algebra from three given first integrals of a second-order equation that possesses $\mathfrak{sl}(3, \mathbb{R})$ algebra. The algebra $A_{3,3}$ arises in connection with each of the three triplets of symmetry generators of the second-order equation.

The proof of part (C) is given in Ref. 2. The algebra $A_{3,1}$ plays an important role in the determination of linearizing transformations for equations of Type I considered in Ref. 2. In Ref. 2 the Lie algebra $A_{3,1}$ was referred to as the \mathfrak{K} algebra. Let us also mention that some of the structure of the proofs in this paper are closely patterned along the lines of the proofs of the propositions of Refs. 2 and 6. We now prove part (D).

Proof (D): Suppose an equation has the Lie algebra so(3) ($A_{3,9}$). Then we cannot have $G_1 = \rho(t, q)G_2$ or $G_3 = \psi(t, q)G_2$ (for any nonconstant functions ρ and ψ) as, in the case $G_1 = \rho G_2$, we deduce

$$(G_2\rho)^2 + \rho^2 = -1, \quad G_3 = -(G_2\rho)G_2, \quad (36)$$

which in turn lead to the generators [on transforming G_2 to $\bar{G}_2 = \partial/\partial Q$ and choosing the simplest solution of (36a)]

$$\bar{G}_1 = i \sin Q \frac{\partial}{\partial Q}, \quad \bar{G}_2 = \frac{\partial}{\partial Q}, \quad (37)$$

$$\bar{G}_3 = i \cos Q \frac{\partial}{\partial Q}, \quad i = \sqrt{-1},$$

which are not symmetries of a second-order equation. Therefore $G_1 \neq \rho G_2$ for any function ρ . Likewise we can show that no function ψ exists such that $G_3 = \psi G_2$. Hence $G_1 \neq \rho G_2$ and $G_3 \neq \psi G_2$. This being the case we always can choose coordinates in which one of the generators appears as a generator of time translation. Thus the only admissible transformation from now on will be of the form

$$Q = \alpha(q), \quad T = t + \beta(q), \quad (38)$$

where α and β are as yet arbitrary. We write the three generators as

$$G_1 = \frac{\partial}{\partial t}, \quad G_2 = \xi_2 \frac{\partial}{\partial t} + \eta_2 \frac{\partial}{\partial q}, \quad (39)$$

$$G_3 = \xi_3 \frac{\partial}{\partial t} + \eta_3 \frac{\partial}{\partial q},$$

where the ξ 's and η 's are functions of t and q . Invoking the commutators $[G_1, G_2] = G_3$ and $[G_1, G_3] = -G_2$ of the Lie algebra so(3), we obtain

$$\frac{\partial \xi_2}{\partial t} = \xi_3, \quad \frac{\partial \xi_3}{\partial t} = -\xi_2, \quad (40)$$

$$\frac{\partial \eta_2}{\partial t} = \eta_3, \quad \frac{\partial \eta_3}{\partial t} = -\eta_2,$$

which in turn imply

$$\frac{\partial^2 \xi_2}{\partial t^2} + \xi_2 = 0, \quad \frac{\partial^2 \eta_2}{\partial t^2} + \eta_2 = 0.$$

Consequently ξ_2 and η_2 are

$$\xi_2 = a \cos t + b \sin t, \quad \eta_2 = c \cos t + d \sin t,$$

where a, b, c , and d are functions of q . The coordinate functions ξ_3 and η_3 are immediately given by

$$\xi_3 = -a \sin t + b \cos t, \quad \eta_3 = -c \sin t + d \cos t.$$

The remaining commutator $[G_2, G_3] = G_1$ then yields the conditions

$$cb' - da' = 1 + a^2 + b^2, \quad cd' - dc' = bd + ac. \quad (41)$$

Since the generators G_1, G_2 , and G_3 are unconnected we must have $c \neq 0$ or $d \neq 0$. We firstly assume that $c \neq 0$. Without loss of generality we may further take $d = 0$. The reason for this is straightforward. For under the transformation (38) G_2 transforms to

$$\bar{G}_2 = \bar{\xi}_2 \frac{\partial}{\partial T} + \bar{\eta}_2 \frac{\partial}{\partial Q},$$

where

$$\bar{\xi}_2 = \bar{a} \cos T + \bar{b} \sin T, \quad \bar{\eta}_2 = \bar{c} \cos T + \bar{d} \sin T,$$

with

$$\bar{d} = \alpha'(c \sin \beta + d \cos \beta).$$

Similar expressions hold for \bar{a}, \bar{b} , and \bar{c} . Hence, if $d \neq 0$ at the outset, we can make $\bar{d} = 0$ by the requirement that β be determined from the relation $\cot \beta = -c/d$. This relation is not violated when G_3 transforms to \bar{G}_3 .

Henceforth we work in coordinates in which $d = 0$. In order to preserve this property we further restrict the transformations (38) to be of the form

$$Q = \alpha(q), \quad T = t. \quad (42)$$

The conditions (41) with $d = 0$ imply

$$a = 0, \quad cb' = 1 + b^2,$$

making the generators appear in simplified form. By means of the transformation [set $\alpha = b$ in (42)]

$$Q = b(q), \quad T = t,$$

the generators G_i acquire the form

$$\bar{G}_1 = \frac{\partial}{\partial T}, \quad \bar{G}_2 = Q \sin T \frac{\partial}{\partial T} + (1 + Q^2) \cos T \frac{\partial}{\partial Q}, \quad (43)$$

$$\bar{G}_3 = Q \cos T \frac{\partial}{\partial T} - (1 + Q^2) \sin T \frac{\partial}{\partial Q}.$$

The generators (43) are reduced even further via the transformation

$$\bar{T} = \tan T, \quad \bar{Q} = Q / \cos T.$$

They transform to

$$\begin{aligned} \bar{G}_1 &= (1 + \bar{T}^2) \frac{\partial}{\partial \bar{T}} + \bar{T} \bar{Q} \frac{\partial}{\partial \bar{Q}}, \\ \bar{G}_2 &= \bar{T} \bar{Q} \frac{\partial}{\partial \bar{T}} + (1 + \bar{Q}^2) \frac{\partial}{\partial \bar{Q}}, \\ \bar{G}_3 &= \bar{Q} \frac{\partial}{\partial \bar{T}} - \bar{T} \frac{\partial}{\partial \bar{Q}}. \end{aligned} \quad (44)$$

The differential equation for \bar{Q} is the free particle equation. A similar argument holds for the case $d \neq 0$, once again giving rise to the above generators (43). It therefore follows that the equation for q has the $\mathfrak{sl}(3, \mathbb{R})$ algebra, concluding sufficiency. The necessity is trivial since $\mathfrak{so}(3)$ is a subalgebra of $\mathfrak{sl}(3, \mathbb{R})$. \diamond

We point out that the generators (43) were previously obtained by Wulfman and Wybourne⁷ in their treatment of the simple harmonic oscillator.

The $\mathfrak{so}(3)$ algebra when realized in two coordinates has a unique canonical form, viz., (44) which are generators of symmetry of a second-order equation. The $\mathfrak{so}(3)$ realizations derived above were omitted by Lie.

We now concentrate on four- and higher-dimensional Lie algebras. The four-dimensional algebras of interest are listed in Table II. Each one of them contains a three-dimensional subalgebra which implies linearization by Theorem 6. Hence, if a second-order equation admits a four-dimensional algebra, then it is linearizable. However, there are four-dimensional algebras which are not admitted by any second-order equation. In fact one can prove the following.

Proposition 7: A second-order equation does not admit the Lie algebra (A) $A_{3,9} \oplus A_1$, (B) $A_{4,7}$, (C) $A_{4,10}$, and (D) $A_{4,11}$ ($a > 0$), respectively.

Remark: We only prove the result for (A). The proofs for (B), (C), and (D) are similar and are therefore omitted.

Proof: Suppose a second-order equation admits the algebra $A_{3,9} \oplus A_1$. Then we can use the $A_{3,9}$ realization given in (43) (now using lower case) since (43) is the only canonical realization of $A_{3,9}$ that generates symmetries of a second-order equation.

Writing G_4 in the general form $G_4 = \xi(\partial/\partial t) + \eta(\partial/\partial q)$ and applying $[G_1, G_4] = 0$, we find that ξ and η are independent of t . The remaining commutators then require that both ξ and η be zero, contradicting the assumption. \diamond

A second-order equation does not admit the five-dimensional algebra $A_{3,9} \oplus A_2$. This is a consequence of Proposition 7 by noting that $A_{3,9} \oplus A_1$ is a subalgebra of $A_{3,9} \oplus A_2$. Moreover, it is not difficult to verify $A_{3,8} \oplus A_2$ is not admitted by a second-order equation. Each of the Lie algebras $A_{5,36}$, $A_{5,37}$, and $A_{5,40}$ (see Table III) has a four-dimensional sub-

algebra and hence imply linearization for a second-order equation that admits it.

We now investigate the case of a second-order equation which admits a six- or seven-dimensional real Lie algebra. The only simple algebra of dimension less than eight (greater than five) is $\mathfrak{so}(3,1)$ (see, e.g., Ref. 8). If a second-order equation admits $\mathfrak{so}(3,1)$ then it is linearizable since $\mathfrak{so}(3,1)$ contains $\mathfrak{so}(3)$ as a subalgebra which implies linearization by Theorem 6. Every six- or seven-dimensional nonsemisimple real Lie algebra is Levi decomposable or can be written as a direct sum of lower dimensional Lie algebras. A six- or seven-dimensional nonsemisimple real Lie algebra has a four-dimensional subalgebra (see Ref. 8). This in turn implies linearization for a second-order equation that admits a six- or seven-dimensional nonsemisimple Lie algebra.

An immediate consequence of the preceding propositions, theorems, and related discussions is the following interesting result.

Theorem 8: A second-order equation does not admit exactly an $\mathfrak{r} \in \{4,5,6,7\}$ dimensional point symmetry algebra.

III. EQUIVALENCE CLASSES OF EQUATIONS

A second-order ordinary differential equation has either 0, 1, 2, 3, or 8 point symmetries. We exclude the case of equations possessing no point symmetry as we cannot in general write representatives of equivalence classes for such equations. If an equation has one-point symmetry, it can be reduced to an autonomous form by means of a point transformation which brings the symmetry to a generator of time translation. Thus an equation with a single point symmetry belongs to the equivalence class of

$$\ddot{q} = f(q, \dot{q}),$$

where $\partial/\partial t$ (realizes A_1) is the standard form for the symmetry and f is a definite function of q and \dot{q} . We treat the case of equations possessing two-point symmetries after investigating the three-point symmetry case. For the equivalence classes of equations having three-point symmetries we need only recall the results of the previous section. It follows that there are five representatives of equivalence classes (in each case $A \neq 0$ and $a \in \mathbb{R}$). They are

$$t\ddot{q} = \dot{q}^3 + \dot{q} + A(1 + \dot{q}^2)^{3/2}, \quad (45a)$$

$$t\ddot{q} = A\dot{q}^3 - \frac{1}{2}\dot{q}, \quad (45b)$$

$$\begin{aligned} t\ddot{q} &= (a-1)\dot{q} + A\dot{q}^{(2a-1)/(a-1)} \text{ or } \dot{q} \\ &= A\dot{q}^{(a-2)/(a-1)}, \quad a \neq 0, \frac{1}{2}, 1, 2, \end{aligned} \quad (45c)$$

$$t\ddot{q} = -1 + A \exp(-\dot{q}) \text{ or } \dot{q} = A \exp(-\dot{q}), \quad (45d)$$

$$\ddot{q} = A(1 + \dot{q}^2)^{3/2} \exp(a \arctan \dot{q}). \quad (45e)$$

It is now simple to deduce that equations possessing two point symmetries belong to either of the equivalence classes

$$\ddot{q} = f(\dot{q}), \quad t\ddot{q} = g(\dot{q}), \quad (46)$$

where f is not a polynomial which is at most cubic in \dot{q} and i is not of the form given in (45) and g is not linear in \dot{q} and neither is g of the form given in (i) nor of the form given in (45).

Equations admitting the richest number of point sym-

metrics belong to the equivalence class of the free particle equation, $\ddot{q} = 0$.

IV. CONCLUSION

In this work we have shown how second-order ordinary differential equations can be classified by investigating the realizations of real low-dimensional Lie algebras in terms of vector fields in two coordinates. In this way we were able to associate differential equations to those realizations that are generators of symmetry of a second-order equation. Our results show that an equation admitting exactly an r -dimensional ($r \in \{1, 2, 3, 8\}$) Lie algebra of point symmetry generator(s) possess(es) a canonical representative for the corresponding differential equation.

For the case $r = 1$ the differential equation can be transformed to an autonomous form. The case $r = 2$ resulted in two classes of representative equations. For equations having three-point symmetries ($r = 3$ case), we obtained five representatives of equivalence classes. The subject of linearization ($r = 8$ case) was addressed and a number of linearizability results were proved.

We have in fact obtained a general (local) structure theory for second-order equations which admit point symmetry algebras.

Finally, let us note that many further questions remain open. We, however, content ourselves with just a few of them. It would certainly be of great interest to compare the Lie classification of equations given here with the Painlevé classification which produces 50 equations (see Refs. 9–11 and references therein). The transformation up to which the Painlevé classification was done is given by

$$T = \alpha(t), \quad Q = \frac{\beta(t)q + \gamma(t)}{\delta(t)q + \tau(t)}.$$

The question which arises is this. Is there an overlap between the Lie classification and the Painlevé classification? The first step in answering this question would be to determine the symmetries of the Painlevé equations. Alternatively, one may attempt to reduce some of the Lie equations to the Painlevé ones. This certainly is not a trivial task. For example, how would one go about reducing Eq. (45e) to a Painlevé equation since (45e) contains a transcendental function. If one cannot perform such a reduction, then this would mean that the Painlevé classification is inexhaustive and needs supplementation. Moreover, the Painlevé classification was achieved under the restrictive point transformation given

above so one would expect incompleteness. However, the Lie classification also requires supplementation in the case of equations possessing no symmetry as we cannot write representatives of such equations. Even in the case of equations having one symmetry the Lie classification is found to be too general. To remedy this requires further investigation. Nevertheless, it should be pointed out that a preliminary investigation shows that the Painlevé classification does provide representatives for equations having zero or one symmetry.

Recent investigations (Kamran *et al.*¹², Kamran and Shadwick¹³) using Cartan's equivalence method have studied the equivalence of differential equations of the form $\ddot{q} = F(\dot{q}, q, t)$ under the restricted point transformation

$$T = \phi(t), \quad Q = \psi(t, q).$$

This was motivated by the Painlevé classification and as such should be viewed against the Painlevé background discussed above.

ACKNOWLEDGMENTS

We thank Professor W. Sarlet and Dr. F. Cantrijn for fruitful discussions from which we benefitted. F. M. also wishes to thank Professor P. Winternitz, Professor R. T. Sharp and Dr. J. van der Jeugt for illuminating discussions on Lie algebras.

F. M. thanks the CSIR and the University of the Witwatersrand for financial support. We thank the referee for constructive suggestions which undoubtedly have enhanced the presentation of the paper.

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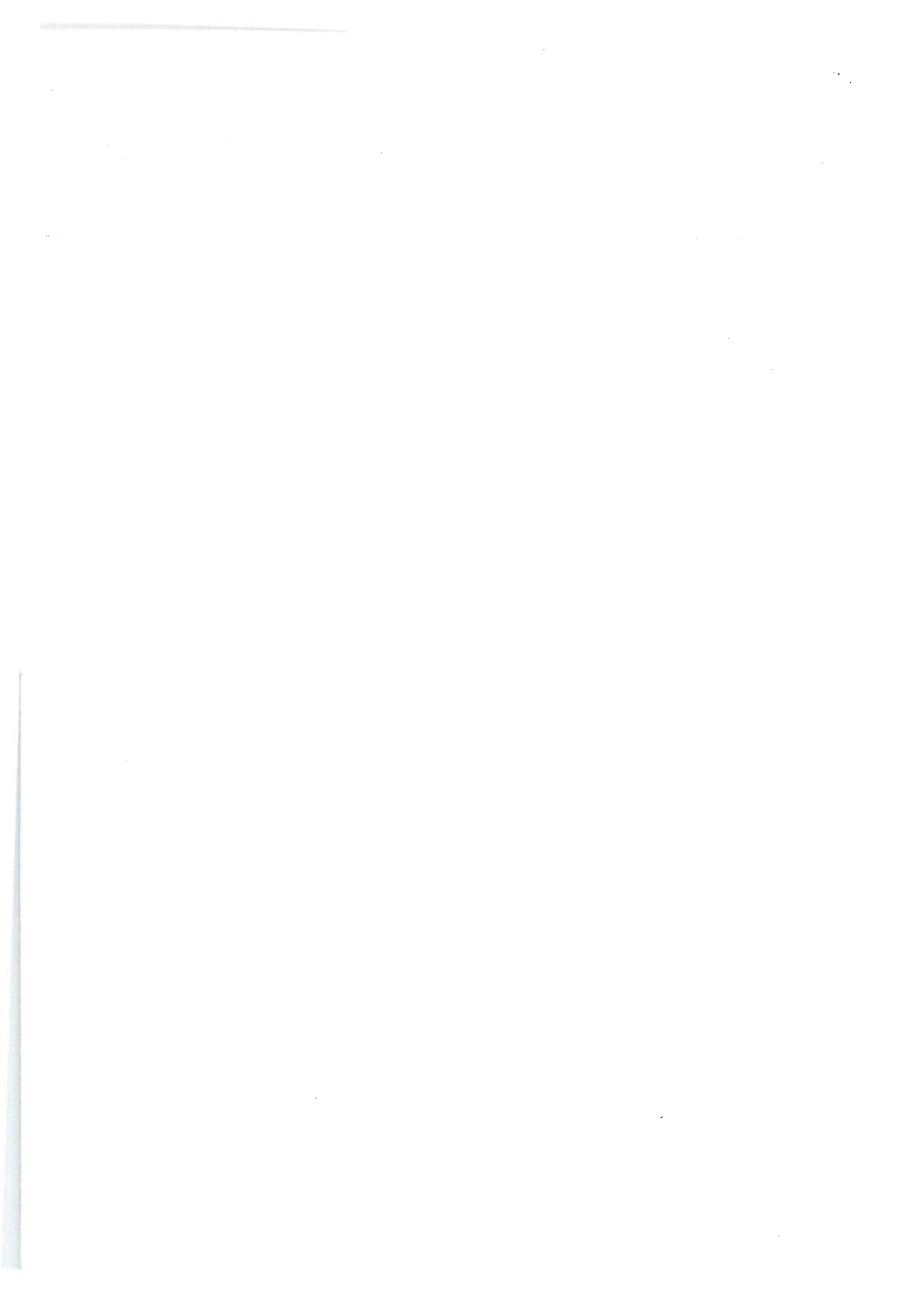
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Symmetry Lie algebras of n th order ordinary differential equations

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Journal of Mathematical Analysis and Applications **151** (1990) 80-107

Symmetry Lie Algebras of n th Order Ordinary Differential Equations

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Submitted by S. P. Diliberto

Received August 24, 1988

We show that an n th ($n \geq 3$) order linear ordinary differential equation has exactly one of $n+1$, $n+2$, or $n+4$ (the maximum) point symmetries. The Lie algebras corresponding to the respective numbers of point symmetries are obtained. Then it is shown that a necessary and sufficient condition for an n th ($n \geq 3$) order equation to be linearizable via a point transformation is that it must admit the n dimensional Abelian algebra $nA_1 = A_1 \oplus A_1 \oplus \dots \oplus A_1$. We discuss in detail the symmetry realizations of $(n-1)A_1 \oplus A_1$. Finally, we prove that an n th ($n \geq 3$) order equation $q^{(n)} = H(t, q, \dots, q^{(n-1)})$ cannot admit exactly an $n+3$ dimensional algebra of point symmetries which is a subalgebra of $nA_1 \oplus gl(2, \mathbb{R})$. © 1990 Academic Press, Inc.

1. INTRODUCTION

The study of point symmetries of ordinary differential equations has enjoyed a renewal of interest over the last few decades. Initial investigations were motivated by physical problems such as the one dimensional harmonic oscillator (see, e.g., [21, 1, 7]). Indeed, most of the earlier works dealt with second order linear equations which, as is now well known [14, 1, 13], have the symmetry algebra $sl(3, \mathbb{R})$. This in effect means that any second order linear equation is, by means of a diffeomorphism of the plane, equivalent to the equation of the free particle, i.e., a linear second order equation belongs to the equivalence class of the free particle equation.

In a recent paper [11] we have shown that the number of point symmetries which a second order equation can possess is exactly one of 0, 1, 2, 3, or 8. If a nonlinear second order equation has eight symmetries, which can only mean $sl(3, \mathbb{R})$ algebra, then it is linearizable via a point transformation. Thus a second order equation with eight point symmetries also belongs to the equivalence class of the free particle equation [20].

The subject of the present paper is the investigation of the Lie algebraic properties of n th order ($n \geq 3$) equations. In studying this, one should note that the $n = 2$ case differs substantially from the $n \geq 3$ case in three respects. First, the maximum number of point symmetries for second order ($n = 2$) equations is $2 + 6$ (Lie [9]) whereas for higher order ($n \geq 3$) equations it is $n + 4$ (Lie [10]). Second, all linear second order equations are equivalent to the free particle equation whereas a linear higher order ($n \geq 3$) equation, as we shall show (see Sections 2, 3, and 4), can belong to one of three equivalence classes depending upon whether it has $n + 1$, $n + 2$, or the maximum number ($n + 4$) of point symmetries. Third, the full Lie algebra of a second order equation is a subalgebra of the maximum Lie algebra $sl(3, \mathbb{R})$, whereas the full Lie algebra of a higher order ($n \geq 3$) equation, as we shall see in Section 6, need not be a subalgebra of its maximum algebra.

The outline of this work is as follows. In Sections 2, 3, and 4 we investigate the conditions under which the general linear higher order equation can have exactly one of $n + 1$, $n + 2$, or $n + 4$ point symmetries. The Lie algebras corresponding to the respective numbers of point symmetries are then obtained. In Section 5 we discuss linearizability of n th ($n \geq 3$) order equations using transformation properties. This is dictated by the underlying algebraic structure. In this section we also discuss in detail the symmetry realizations of $(n - 1)A_1 \oplus A_1$ and their associated equations. Finally in Section 6 we show that an n th order ($n \geq 3$) equation cannot admit exactly an $n + 3$ dimensional point symmetry algebra which is a subalgebra of $nA_1 \oplus gl(2, \mathbb{R})$.

To place this work in its proper perspective, we would like to refer the reader to recent papers on this subject by Krause and Michel [3, 4]. These authors treat equations belonging to the equivalence class $y^{(n)} = 0$, and thus only discuss the $n + 4$ point symmetries which are the single case that occurs for this problem. Let us also mention the survey paper by Neuman [16] which contains several of the recent as well as the classical references on the topic of linear n th order equations.

2. SYMMETRY CONDITION FOR LINEAR EQUATIONS

Consider the general n th order linear equation

$$q^{(n)} + \sum_{i=0}^{n-1} B_i(t) q^{(i)} = g(t), \quad (2.1)$$

where the B_i 's and g are functions of the time t and $q^{(j)}$ denotes the j th

total time derivative of q . The implication of making infinitesimal transformations is that these functions are analytic. By means of the transformation

$$q = f\bar{q} + q_p, \quad (2.2)$$

where q_p is a particular solution of the nonhomogeneous equation (2.1) and $f = \exp[(1/n) \int^t B_{n-1}(s) ds]$, the original equation (2.1) is reducible to a homogeneous equation independent of the $q^{(n-1)}$ term. Hence without loss of generality we may treat

$$q^{(n)} + \sum_{i=0}^{n-2} B_i(t) q^{(i)} = 0, \quad n \geq 3. \quad (2.3)$$

We shall refer to (2.3) as the canonical form of the general linear equation (2.1). We analyse Eq. (2.3) for its point symmetries. Let us note that the symmetry algebras of (2.1) and (2.3) are isomorphic since the symmetry algebra of a differential equation is invariant under changes of coordinate (cf. Section 5). (The reader is referred to Olver [17] for an up to date and thorough introduction to the application of the Lie theory to differential equations.)

If the generator of point symmetry of (2.3) is

$$G = \xi(t, q) \frac{\partial}{\partial t} + \eta(t, q) \frac{\partial}{\partial q}, \quad (2.4)$$

the n th extension (or prolongation in the terminology of [17]) of G (which induces transformations of the derivatives of q up to the n th) is

$$G^{[n]} = G + \sum_{j=1}^n \eta^{[j]} \frac{\partial}{\partial q^{(j)}}, \quad (2.5)$$

where

$$\eta^{[k]} = \frac{d}{dt} \eta^{[k-1]} - q^{(k)} \frac{d\xi}{dt}, \quad k = 1, n \quad \eta^{[0]} = \eta.$$

Equivalently we can write $\eta^{[k]}$, using binomial coefficients, as

$$\begin{aligned} \eta^{[k]} &= \eta^{(k)} - \sum_{i=1}^k \binom{k}{i} q^{(k+1-i)} \xi^{(i)} \\ &= \eta^{(k)} - \sum_{j=0}^{k-1} \binom{k}{j} q^{(j+1)} \xi^{(k-j)} \end{aligned} \quad (2.6)$$

in which $\xi^{(l)}$ and $\eta^{(l)}$ denote the l th total time derivative. Using induction

on the recursion relations for the $\eta^{[k]}$'s given above, it is not difficult to verify the relations (2.6). An n th order ordinary differential equation $E(t, q, q^{(1)}, \dots, q^{(n)}) = 0$ possesses the point symmetry G if

$$G^{(n)}E|_{E=0} = 0. \tag{2.7}$$

Applying the symmetry requirement (2.7) (and using (2.6)) to (2.3) we obtain

$$\begin{aligned} \eta^{(n)} - \sum_{j=1}^n \binom{n}{j} q^{(n+1-j)} \xi^{(j)} + \xi \sum_{i=0}^{n-2} B_i^{(1)}(t) q^{(i)} \\ + \sum_{i=0}^{n-2} B_i(t) \left[\eta^{(i)} - \sum_{j=1}^i \binom{i}{j} q^{(i+1-j)} \xi^{(j)} \right] = 0, \quad n \geq 3. \end{aligned} \tag{2.8}$$

Guided by the expressions for $\eta^{(3)}$ and $\eta^{(4)}$ (and perhaps $\eta^{(5)}, \eta^{(6)}, \dots$, bearing in mind that $n \geq 3$) (note that $\dot{q} = q^{(1)}$ and $\ddot{q} = q^{(2)}$),

$$\begin{aligned} \eta^{(3)} &= \eta_{iii} + 3\dot{q}\eta_{iiq} + 3\dot{q}^2\eta_{iqq} + \dot{q}^3\eta_{qqq} + 3\ddot{q}\eta_{iq} + 3\dot{q}\ddot{q}\eta_{qq} + q^{(3)}\eta_q \\ \eta^{(4)} &= \eta_{iiii} + 4\dot{q}\eta_{iiiq} + 6\dot{q}^2\eta_{iiqq} + 4\dot{q}^3\eta_{iqqq} + \dot{q}^4\eta_{qqqq} + 6\ddot{q}\eta_{iiq} \\ &\quad + 12\dot{q}\ddot{q}\eta_{iqq} + 4q^{(3)}\eta_{i/q} + 4\dot{q}q^{(3)}\eta_{qq} + 3\ddot{q}^2\eta_{qq} + q^{(4)}\eta_q + 6\dot{q}^2\ddot{q}\eta_{qqq} \\ &\quad \vdots \end{aligned} \tag{2.9}$$

we write down the relation for the terms containing $q^{(n-1)}$ in (2.8). We observe that only the expression $[\eta^{(n)} - \sum_{j=1}^n \binom{n}{j} q^{(n+1-j)} \xi^{(j)}]$ of (2.8) has terms involving $q^{(n-1)}$:

$$\begin{aligned} nq^{(n-1)}\eta_{iq} + n\dot{q}q^{(n-1)}\eta_{qq} - \binom{n}{2} q^{(n-1)}(\xi_{ii} + 2\dot{q}\xi_{iq} + \dot{q}^2\xi_{qq} + \ddot{q}\xi_q) \\ - \binom{n}{n} \dot{q}(nq^{(n-1)}\xi_{iq} + n\dot{q}q^{(n-1)}\xi_{qq}) - \binom{n}{n-1} \ddot{q}q^{(n-1)}\xi_q = 0. \end{aligned} \tag{2.10}$$

Since ξ and η are functions of q and t only, we may equate coefficients of separate powers of $\ddot{q}q^{(n-1)}$, $\dot{q}q^{(n-1)}$, and $q^{(n-1)}$ to zero, which results in a subset of three determining equations:

$$\xi_q = 0, \quad \eta_{qq} = 0, \quad \eta_{iq} = \frac{n-1}{2} \xi_{ii}.$$

This implies that ξ and η are of the form

$$\xi = a(t), \quad \eta = \left(\frac{n-1}{2} a^{(1)} + \alpha \right) q + b(t), \tag{2.11}$$

where a, b are as yet arbitrary functions of t and α is a constant.

Using Leibnitz's rule for differentiating a product we compute $\eta^{(k)}$ from (2.11):

$$\eta^{(k)} = \left(\frac{n-1}{2} a^{(1)} + \alpha \right) q^{(k)} + b^{(k)} + \frac{n-1}{2} \sum_{j=0}^{k-1} \binom{k}{j} q^{(j)} a^{(k+1-j)}, \quad k = 1, n.$$

Thus $\eta^{[k]}$ (using (2.6)) becomes

$$\begin{aligned} \eta^{[k]} = & \sum_{j=0}^{k-1} \binom{k}{j} \left\{ \frac{n-1}{2} q^{(j)} a^{(k+1-j)} - q^{(j+1)} a^{(k-j)} \right\} \\ & + \left(\frac{n-1}{2} a^{(1)} + \alpha \right) q^{(k)} + b^{(k)}. \end{aligned} \quad (2.12)$$

Recall that (2.8) is

$$\eta^{[n]} + \xi \sum_{i=0}^{n-2} B_i^{(1)}(t) q^{(i)} + \sum_{i=0}^{n-2} B_i(t) \eta^{[i]} = 0, \quad n \geq 3. \quad (2.13)$$

Substitution of (2.12) and (2.11) into (2.13) results in the functions a and b satisfying the condition

$$\begin{aligned} & \sum_{j=0}^{n-1} \binom{n}{j} \left\{ \frac{n-1}{2} q^{(j)} a^{(n+1-j)} - q^{(j+1)} a^{(n-j)} \right\} + b^{(n)} \\ & + \left(\frac{n-1}{2} a^{(1)} + \alpha \right) q^{(n)} + \sum_{i=0}^{n-2} B_i(t) \left\{ \left(\frac{n-1}{2} a^{(1)} + \alpha \right) q^{(i)} + b^{(i)} \right. \\ & \left. + \sum_{j=0}^{i-1} \binom{i}{j} \left[\frac{n-1}{2} q^{(j)} a^{(i+1-j)} - q^{(j+1)} \right] \right\} + a \sum_{i=0}^{n-2} B_i^{(1)} q^{(i)} = 0. \end{aligned} \quad (2.14)$$

The coefficient of $\left(\frac{n-1}{2} a^{(1)} + \alpha \right)$ clearly is identically zero. The $q^{(k)}$, $k = 0, n-1$ independent terms give

$$b^{(n)} + \sum_{i=0}^{n-2} B_i(t) b^{(i)} = 0, \quad (2.15)$$

whence the n linearly independent solutions of the original equation (2.3) generate the Abelian point symmetry algebra $nA_1 = A_1 \oplus A_1 \oplus \dots \oplus A_1$ (where \oplus denotes n -fold direct sum) admitted by (2.3). The generator of the Abelian algebra (nA_1) is sometimes referred to a superposition operator (see [5]). A further point symmetry for (2.3) is evident from (2.11) since η contains the constant α . This symmetry acts on nA_1 by dilatation.

Condition (2.14) now reduces to

$$\begin{aligned} & \sum_{j=0}^{n-1} \binom{n}{j} \frac{n-1}{2} q^{(j)} a^{(n+1-j)} - \sum_{j=0}^{n-2} \binom{n}{j} q^{(j+1)} a^{(n-j)} \\ & + \sum_{i=1}^{n-2} B_i(t) \cdot \sum_{j=0}^{i-1} \left\{ \frac{n-1}{2} \binom{i}{j} q^{(j)} a^{(i+1-j)} - \binom{i}{j} q^{(j+1)} a^{(i-j)} \right\} \\ & + na^{(1)} \sum_{i=0}^{n-2} B_i(t) q^{(i)} + a \sum_{i=0}^{n-2} B_i^{(1)}(t) q^{(i)} = 0. \end{aligned} \quad (2.16)$$

Note that to obtain finally (2.16) we have replaced $q^{(n)}$ by $-\sum_{i=0}^{n-2} B_i(t) q^{(i)}$ in the first sum of (2.14). Separating out the $q^{(n-i)}$ ($1 \leq i \leq n$) terms we obtain the differential equations for a . They are

$$\begin{aligned} & \left\{ \binom{n}{n-i} \frac{n-1}{2} - \binom{n}{n-i-1} \right\} a^{(i+1)} \\ & + \sum_{j=2}^{i-1} B_{n-j} a^{(i-j+1)} \left\{ \binom{n-j}{n-i} \frac{n-1}{2} - \binom{n-j}{n-i-1} \right\} \\ & - B_{n-i} \binom{n-i}{n-i-1} a^{(1)} + na^{(1)} B_{n-i} + a B_{n-i}^{(1)} = 0, \quad i = 1, n. \end{aligned}$$

We can write this more compactly as

$$\begin{aligned} & \frac{(n+1)! (i-1)}{(n-i)! (i+1)! 2} a^{(i+1)} + ia^{(1)} B_{n-i} + a B_{n-i}^{(1)} \\ & + \sum_{j=2}^{i-1} B_{n-j} \frac{(n-j)! [n(i-j-1) + i + j - 1]}{(n-i)! (i-j+1)! 2} a^{(i-j+1)} = 0, \quad i = 1, n. \end{aligned} \quad (2.17)$$

Clearly, Eq. (2.17) vanishes identically for $i = 1$ ($B_{n-1} = 0$). This is to be expected since the relation $\xi = a(t)$ is an outcome of equating the terms containing $q^{(n-1)}$ to zero (see (2.10)).

3. THE $n + 4$ DIMENSIONAL SYMMETRY ALGEBRA FOR LINEAR EQUATIONS

It is instructive at this stage to determine the point symmetries of the simplest equation, which is the one treated in Ref. [3]. It is of course the simplest subset of our general approach.

$$q^{(n)} = 0, \quad n \geq 3. \quad (3.1)$$

We already have $n+1$ point symmetries for (3.1). Indeed, (2.15) ($B_i=0$) implies

$$b = \sum_{i=0}^{n-1} \mathcal{B}_i t^i, \quad \mathcal{B}_i \text{ constants}$$

and (2.11) gives $q\partial/\partial q$. Moreover, invoking (2.17) we have, since $B_i=0$,

$$a = \mathcal{A}_0 + \mathcal{A}_1 t + \mathcal{A}_2 t^2, \quad \mathcal{A}_i \text{ constants.}$$

Hence Eq. (3.1) admits $n+4$ point symmetries which (see [3, 4]) constitute the Lie algebra $nA_1 \oplus, (sl(2, \mathbb{R}) \oplus A_1) \simeq nA_1 \oplus, gl(2, \mathbb{R})$ (where \oplus and \oplus_s denote direct and semidirect sums, respectively).

Let us now consider the case when the B_i 's are not all zero. For $i=2$, (2.17) reduces to

$$\frac{(n+1)!}{(n-2)! 4!} a^{(3)} + a^{(1)} B_{n-2} + \frac{1}{2} a B_{n-2}^{(1)} = 0. \quad (3.2)$$

As an aside it is interesting to note that (3.2) is related to the well-known Lewis-Pinney [8, 19] equation which commonly is found in the study of time-dependent second order equations.

It is well known that an n th ($n \geq 3$) order ordinary differential equation can have at most $n+4$ point symmetries. Lie [10] provided a geometric proof (see also Ref. [2] for a recent contribution). Hence it is most appropriate that we first investigate (2.3) for the maximum number of point symmetries. To this end we already have $n+1$ symmetries with the B_i 's still arbitrary. A further three symmetries will arise from (3.2) provided that (2.17) vanishes identically for all $i \geq 3$. We discuss the conditions under which this will occur. It turns out that the B_i 's can be determined recursively in terms of B_{n-2} . We indicate the procedure whereby this can be achieved. Setting $i=3$ in (2.17) we find

$$\frac{(n+1)!}{(n-2)! 4!} a^{(4)} + \frac{3}{n-2} a^{(1)} B_{n-3} + \frac{1}{n-2} a B_{n-3}^{(1)} + a^{(2)} B_{n-2} = 0. \quad (3.3)$$

Substituting $d(3.2)/dt$ into (3.3) then results in

$$3a^{(1)}\Gamma_3 + a\Gamma_3^{(1)} = 0, \quad (3.4)$$

where

$$\Gamma_3 = \frac{1}{2} B_{n-2}^{(1)} - \frac{1}{n-2} B_{n-3}. \quad (3.5)$$

In order for (3.4) to vanish identically we must have $\Gamma_3 = 0$, i.e.,

$$B_{n-3} = \frac{n-2}{2} B_{n-2}^{(1)}. \quad (3.6)$$

It follows that if (3.6) is satisfied then (3.3) turns out to be precisely $d(3.2)/dt$, implying that there is essentially only one condition, viz., (3.2).

We apply the above reasoning for $i=4$ (in (2.17)). Starting with

$$\begin{aligned} & \frac{(n+1)!}{(n-2)! 4!} a^{(5)} + a^{(3)} B_{n-2} \frac{5(n+5)}{18} + a^{(2)} \frac{5}{n-2} B_{n-3} \\ & + \frac{40}{3(n-2)(n-3)} a^{(1)} B_{n-4} + \frac{10}{3(n-2)(n-3)} a B_{n-4}^{(1)} = 0 \end{aligned} \quad (3.7)$$

and substituting $d^2(3.2)/dt^2$ and (3.6) into (3.7) we end up with

$$4a^{(1)}\Gamma_4 + a\Gamma_4^{(1)} = 0, \quad (3.8)$$

where

$$\Gamma_4 = \frac{1}{2} B_{n-2}^{(2)} - \frac{10}{3(n-2)(n-3)} B_{n-4} + B_{n-2}^2 \frac{(n-2)! (5n+7)}{(n+1)! 3}. \quad (3.9)$$

For the single third order condition (3.2) to remain valid we require $\Gamma_4 = 0$, which gives

$$B_{n-4} = \frac{3(n-2)(n-3)}{20} B_{n-2}^{(2)} + B_{n-2}^2 \frac{(n-2)! (5n+7)(n-2)(n-3)}{(n+1)! 10}. \quad (3.10)$$

Here we find that, if (3.6) and (3.10) hold, (3.7) is identical to $d^2(3.2)/dt^2$.

Proceeding in the above manner we obtain a few more coefficients in terms of B_{n-2} . This is rather straightforward albeit lengthy and tedious. We have

$$\begin{aligned} B_{n-5} &= (n-2)(n-3)(n-4) \left\{ \frac{B_{n-2}^{(3)}}{30} + B_{n-2} B_{n-2}^{(1)} \frac{(n-2)! (5n+7)}{(n+1)! 10} \right\}, \\ B_{n-6} &= (n-2) \cdots (n-5) \left\{ \frac{B_{n-2}^{(4)}}{168} + B_{n-2}^{(2)} B_{n-2} \frac{(n-2)! (21n+29)}{(n+1)! 140} \right. \\ & \quad \left. + (B_{n-2}^{(1)})^2 \frac{(n-2)! (7n+10)}{(n+1)! 56} + B_{n-2}^3 \frac{(n-2)!^2 (35n^2 + 112n + 93)}{(n+1)!^2 210} \right\}, \end{aligned}$$

$$\begin{aligned}
B_{n-7} &= (n-2) \cdots (n-6) \left\{ \frac{B_{n-2}^{(5)}}{1120} + B_{n-2}^2 B_{n-2}^{(1)} \frac{(n-2)!^2 (35n^2 + 112n + 93)}{(n+1)!^2 140} \right. \\
&\quad \left. + B_{n-2}^{(1)} B_{n-2}^{(2)} \frac{(n-2)! 3(7n+10)}{(n+1)! 280} + B_{n-2} B_{n-2}^{(3)} \frac{(n-2)! (14n+19)}{(n+1)! 420} \right\}, \\
B_{n-3} &= (n-2) \cdots (n-7) \left\{ \frac{B_{n-2}^{(6)}}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 4} + B_{n-2} B_{n-2}^{(4)} \frac{(n-2)! (3n+4)}{(n+1)! 9 \cdot 8 \cdot 7} \right. \\
&\quad + (B_{n-2}^{(2)})^2 \frac{(n-2)! (9n+13)}{(n+1)! 8 \cdot 5^2 \cdot 4} + B_{n-2}^{(1)} B_{n-2}^{(3)} \frac{(n-2)! (12n+17)}{(n+1)! 8 \cdot 6 \cdot 5 \cdot 3} \\
&\quad + B_{n-4}^4 \frac{(n-2)!^3 (175n^3 + 945n^2 + 1769n + 1143)}{(n+1)!^3 7 \cdot 6 \cdot 5^2 \cdot 4} \\
&\quad + B_{n-2}^2 B_{n-2}^{(2)} \frac{(n-2)!^2 (315n^2 + 996n + 817)}{(n+1)!^2 7 \cdot 6 \cdot 5^2 \cdot 4} \\
&\quad \left. + B_{n-2} (B_{n-2}^{(1)})^2 \frac{(n-2)!^2 (210n^2 + 678n + 568)}{(n+1)!^2 8 \cdot 7 \cdot 6 \cdot 5} \right\} \\
&\quad \vdots \tag{3.11}
\end{aligned}$$

Each of the above relations is a consequence of

$$ia^{(1)}\Gamma_i + a\Gamma_i^{(1)} = 0 \tag{3.12}$$

when $\Gamma_i = 0$ for $i = 5, 6, 7$, and 8 , respectively, taking into account that the condition $\Gamma_i = 0$ yields B_{n-i} in terms of B_{n-2} .

The following general discussion is now in order. We note that once the third order condition (3.2) holds, the $n-1$ functions B_{n-k} ($k = 2, n$) satisfy $n-2$ equations which arise from (2.17). By the procedure outlined above one can obtain the coefficients B_{n-i} ($i > 2$) in terms of B_{n-2} . We have in fact done this for $i = 3, 8$. It therefore follows that if we repeat the same argument a further $n-8$ times, i.e., for the remaining values of $i \in \{9, 10, \dots, n\}$, by substituting for each $i \in \{9, 10, \dots, n\}$ $d^{i-2}(3.2)/dt^{i-2}$ into (2.17) together with the relation for B_{n-i+1} in terms of B_{n-2} , one will arrive at Eq. (3.12). Setting $\Gamma_i = 0$ will then yield B_{n-i} in terms of B_{n-2} .

An attractive endeavour worth pursuing would be to determine the general relationship between the B_{n-i} 's and the arbitrary coefficient B_{n-2} .

We have seen that the maximum number of point symmetries $n+4$ applies only to linear equations (2.3) whenever their coefficients satisfy relations of the form (3.11). The implication here is that each coefficient of the equation can only be written in terms of one arbitrary function of time. This certainly is highly exceptional. In general, when the $n-1$ coefficients

are arbitrary, Eq. (2.3) has $n + 1$ symmetries which generate the Lie algebra $nA_1 \oplus_s A_1$ (see discussion following Eq. (2.15)).

We identify the $n + 4$ dimensional Lie algebra of (2.3) when it has the maximum number of point symmetries. We already have at our disposal the $n + 1$ dimensional subalgebra $nA_1 \oplus_s A_1$ of the desired algebra. It now remains only to look at the commutation properties of three additional generators of symmetry. These are readily available on invoking (3.2). Suppose the three linearly independent solutions of (3.2) are a_1, a_2 , and a_3 . Then by (2.11) the additional symmetry generators are

$$G_i = a_i \frac{\partial}{\partial t} + \frac{n-1}{2} a_i^{(1)} q \frac{\partial}{\partial q}, \quad i = 1, 3. \quad (3.13)$$

We obtain the commutation relations of the generators (3.13). Writing $[G_i, G_j] = \sum_{k=1}^3 \varepsilon_{ijk} F_k$, where ε_{ijk} is the permutation symbol, we find that

$$F_k = w_k \frac{\partial}{\partial t} + \frac{n-1}{2} w_k^{(1)} q \frac{\partial}{\partial q}, \quad k = 1, 3, \quad (3.14)$$

where

$$\sum_{k=1}^3 \varepsilon_{ijk} w_k = a_i a_j^{(1)} - a_j a_i^{(1)}, \quad i, j = 1, 3. \quad (3.15)$$

The w_k 's are nonzero since the a_i 's are linearly independent and one can easily deduce that the w_k 's are functionally related, i.e.,

$$\sum_{i=1}^3 a_i w_i = 0. \quad (3.16)$$

Moreover, it is a simple matter to verify that Eq. (3.2) has seven point symmetries. This means that (3.2) is reducible to $Q''' = 0$ via a point transformation ([12], see also [3, 4, 6]). The transformation that does the reduction is given by

$$Q = q/a_1, \quad T = a_2/a_1, \quad (3.17)$$

provided $a_3 = a_2^2/a_1$. This is always the case if a_1, a_2 , and (so) a_3 are properly chosen, i.e., there is no special property (such as a subcase) involved. We shall discuss transformations of type (3.17) in greater detail after proving Theorem 1. At present we concern ourselves with the Lie algebra of the G_i 's. The requirement that the operators G_i ($i = 1, 3$) close under commutation, taking into consideration the generators of $nA_1 \oplus_s A_1$, gives rise to

$$\begin{aligned}
 w_3 &= \gamma_1 a_1 + \gamma_2 a_2 \\
 w_1 &= \gamma_3 a_3 + \gamma_4 a_2 \\
 w_2 &= -\gamma_4 a_1 - (\gamma_1 + \gamma_3) a_2 - \gamma_2 a_3,
 \end{aligned}
 \tag{3.18}$$

where the γ_i 's are constants. To obtain the above form we have used the functional relation $a_3 = a_2^2/a_1$ as well as (3.16). It is now apparent from (3.18) that the operators G_i generate a three dimensional Lie algebra. In fact, (3.18) defines a class of Lie algebras. For example, if we set $\gamma_1 = \gamma_3 = 0$ and $\gamma_2 = \gamma_4 = 1$, we arrive at the algebra $[G_1, G_2] = G_2$, $[G_2, G_3] = G_2$, $[G_1, G_3] = G_1 + G_3$ which by a change of basis is isomorphic to the classically well-known algebra $[G_1, G_2] = 0$, $[G_2, G_3] = G_2$, $[G_1, G_3] = G_1$ (denoted as $A_{3,3}$ in Ref. [10]). However, since $\{w_1, w_2, w_3\}$ is linearly independent, (3.18) gives rise to a simple algebra isomorphic to $sl(2, \mathbb{R})$ (and not to $so(3)$). Hence, without loss of generality, we may set $\gamma_2 = \gamma_4 = 0$ and $\gamma_1 = \gamma_3 = 1$ in (3.18). The generators G_1, G_2, G_3 , and $q\partial/\partial q$ generate the algebra $sl(2, \mathbb{R}) \oplus A_1$. It then emerges that the $n+4$ dimensional Lie algebra of (2.3), provided (3.11) holds, is $nA_1 \oplus (sl(2, \mathbb{R}) \oplus A_1)$, where nA_1 is an Abelian ideal of the algebra.

We list the linear time-dependent equations (2.3) (up to order eight) which have $n+4$ ($n=3, 8$) point symmetries. As a consequence of this we will be in a position to characterise all constant coefficient linear equations (2.3) which admit $n+4$ point symmetries. Straightforward manipulations utilising relations (3.6), (3.10), and (3.11) yield (in each case the B_i 's are functions of t)

$$\begin{aligned}
 \text{(a)} \quad & q^{(3)} + B_1 q^{(1)} + \frac{1}{2} B_1^{(1)} q = 0, \\
 \text{(b)} \quad & q^{(4)} + B_2 q^{(2)} + B_2^{(1)} q^{(1)} + \left(\frac{3}{10} B_2^{(2)} + \frac{9}{100} B_2^2 \right) q = 0, \\
 \text{(c)} \quad & q^{(5)} + B_3 q^{(3)} + \frac{3}{2} B_3^{(1)} q^{(2)} + \left(\frac{9}{10} B_3^{(2)} + \frac{16}{100} B_3^2 \right) q^{(1)} \\
 & + \left(\frac{1}{5} B_3^{(3)} + \frac{16}{100} B_3 B_3^{(1)} \right) q = 0, \\
 \text{(d)} \quad & q^{(6)} + B_4 q^{(4)} + 2B_4^{(1)} q^{(3)} + \left(\frac{63}{35} B_4^{(2)} + \frac{259}{35^2} B_4^2 \right) q^{(2)} \\
 & + \left(\frac{28}{35} B_4^{(3)} + \frac{518}{35^2} B_4 B_4^{(1)} \right) q^{(1)} \\
 & + \left(\frac{5}{35} B_4^{(4)} + \frac{130}{35^2} B_4^{(1)^2} + \frac{155}{35^2} B_4 B_4^{(2)} + \frac{225}{35^3} B_4^3 \right) q = 0,
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & q^{(7)} + B_5 q^{(5)} + \frac{5}{2} B_5^{(1)} q^{(4)} + \left(3B_5^{(2)} + \frac{1}{4} B_5^2 \right) q^{(3)} + \left(2B_5^{(3)} + \frac{3}{4} B_5^{(1)} B_5 \right) q^{(2)} \\
 & + \left(\frac{5}{7} B_5^{(4)} + \frac{295}{4 \cdot 14^2} B_5^{(1)2} + \frac{88}{14^2} B_5^{(2)} B_5 + \frac{36}{14^3} B_5^3 \right) q^{(1)} \\
 & + \left(\frac{3}{2 \cdot 14} B_5^{(5)} + \frac{177}{4 \cdot 14^2} B_5^{(1)} B_5^{(2)} + \frac{39}{2 \cdot 14^2} B_5 B_5^{(3)} + \frac{54}{14^3} B_5^2 B_5^{(1)} \right) q = 0,
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & q^{(8)} + B_6 q^{(6)} + 3B_6^{(1)} q^{(5)} + \left(\frac{9}{2} B_6^{(2)} + \frac{47}{8 \cdot 21} B_6^2 \right) q^{(4)} \\
 & + \left(4B_6^{(3)} + \frac{47}{2 \cdot 21} B_6 B_6^{(1)} \right) q^{(3)} + \left(\frac{15}{7} B_6^{(4)} + \frac{1773}{4 \cdot 21^2} B_6^{(2)} B_6 + \frac{1485}{4 \cdot 21^2} B_6^{(1)2} \right. \\
 & + \left. \frac{3229}{16 \cdot 21^3} B_6^3 \right) q^{(2)} + \left(\frac{9}{14} B_6^{(5)} + \frac{393}{2 \cdot 21^2} B_6 B_6^{(3)} + \frac{891}{2 \cdot 21^2} B_6^{(1)} B_6^{(2)} \right. \\
 & + \left. \frac{9687}{16 \cdot 21^3} B_6^2 B_6^{(1)} \right) q^{(1)} + \left(\frac{1}{12} B_6^{(6)} + \frac{40}{21^2} B_6^{(4)} B_6 \right. \\
 & + \left. \frac{51}{80 \cdot 21} B_6^{(2)2} + \frac{113}{24 \cdot 21} B_6^{(1)} B_6^{(3)} \right. \\
 & + \left. \frac{25}{256 \cdot 21^2} B_6^4 + \frac{4135}{96 \cdot 21^2} B_6^{(2)} B_6^2 + \frac{347}{24 \cdot 21^2} B_6 B_6^{(1)2} \right) q = 0,
 \end{aligned}$$

(3.19)

Assuming now that the coefficients B_i are constant ($B_i^{(j)} = 0$, for $j = 1, 2, \dots$), we observe by inspection that the equations of odd order contain only odd derivatives of q and likewise equations of even order contain only even derivatives of q . In both these cases one can write the above equations in their factored form. It is then a small step to deduce that in the general case one would obtain the following. For n odd, $n + 4$ point symmetries occur only if the equation has the form

$$\frac{d}{dt} \left\{ \prod_{i=1}^{(n-1)/2} \left(\frac{d^2}{dt^2} + \frac{(2i)^2}{(n+1)} B_{n-2} \right) \right\} q = 0. \tag{3.20}$$

For n even we have

$$\left\{ \prod_{i=1}^{n/2} \left(\frac{d^2}{dt^2} + \frac{(2i-1)^2}{(n+1)} B_{n-2} \right) \right\} q = 0, \tag{3.21}$$

which has $n + 4$ point symmetries. Both series (odd and even) are derivable from a second order equation.

4. THE $n+1$ AND $n+2$ DIMENSIONAL SYMMETRY ALGEBRA
FOR LINEAR EQUATIONS

Having discussed at length the $n+4$ symmetry case for linear equations (2.3), it is natural to enquire under what circumstances (2.3) would possess $n+1$, $n+2$, or $n+3$ point symmetries.

We choose first to look at the situation when (2.3) possesses exactly $n+2$ point symmetries. No doubt the discerning reader may already have perceived that exactly $n+2$ symmetries do exist for (2.3). Indeed it is immediate that for fixed i ($i \geq 3$), (3.12) is a first order equation in a provided $\Gamma_i(B_{n-i}, B_{n-2}) \neq 0$. Therefore for fixed i , say $i=k$, we can solve for a in terms of B_{n-2} and B_{n-k} , i.e.,

$$a = \mathcal{C} \Gamma_k^{-1/k}(B_{n-k}, B_{n-2}), \quad \mathcal{C} (\neq 0) \text{ constant.} \quad (4.1)$$

Once (4.1) holds, the $n-1$ coefficient functions $B_{n-j}(t)$ ($j=2, n$) are constrained to satisfy $n-2$ equations which arise from (2.17) upon substitution of the relation (4.1). Thus the linear equation (2.3) admits exactly $n+2$ point symmetries only if each of its time-dependent coefficients can, in principle, be written in terms of one arbitrary function of time.

It is an easy task to identify the Lie algebra of (2.3) when it has exactly $n+2$ point symmetries. The $n+2$ dimensional Lie algebra is $nA_1 \oplus (2A_1)$. The $sl(2, \mathbb{R})$ subalgebra of the maximum algebra now *reduces* to the one dimensional Abelian subalgebra A_1 which clearly is an outcome of (4.1).

Let us now consider the possible existence of exactly $n+3$ point symmetries for (2.3). In order for (2.3) to possess exactly $n+3$ symmetries, Eq. (2.17), which is a condition on a , must be implied by a second order equation in a in the sense that we have seen (2.17) implied by the third order equation (3.2) for the $n+4$ symmetry case. But this cannot occur for the simple reason that each stage ($i \geq 3$ in (2.17)) one would have to contend with either the first order equation (3.12) ($\Gamma_i(B_{n-i}, B_{n-2}) \neq 0$) or (3.12) vanishing identically, i.e., $\Gamma_i = 0, \forall i \geq 3$, meaning that one would be left with the third order equation (3.2).

By implication it now becomes obvious that in general the linear equation (2.3) has exactly $n+1$ point symmetries whenever at least one of its time-dependent coefficients can be expressed in terms of at least two arbitrary functions.

It is opportune to once again look at the constant coefficient linear equations (2.3). This time, however, we take the coefficients to be arbitrary constants. We observe that $a(t) = a_0$ (a_0 constant) is always a solution of (2.17). So all constant coefficient linear equations (2.3) have at least $n+2$ point symmetries. In particular, if an equation (2.3) has constant coefficients which are not of the form contained in (3.20) or (3.21), then it possesses exactly $n+2$ point symmetries.

To illustrate some aspects of the foregoing discussion on linear equations (2.3) with time-dependent coefficients, we consider examples of linear third and fourth order equations of the form (2.3). We already know the form for the maximum number of point symmetries (see (3.19a, b)). Let us then investigate the conditions for exactly $n+2$ and $n+1$ point symmetries for $n=3$ and $n=4$, respectively.

For third order linear equations we assume that (3.4) is a first order equation in a where Γ_3 is given by (3.5)| $_{n=3}$. Then (cf. (4.1))

$$a = \mathcal{C}(B_1^{(1)} - 2B_0)^{-1/3}, \quad \mathcal{C} (\neq 0) \text{ constant.} \quad (4.2)$$

Thus a third order equation (2.3) would have exactly $n+2=3+2$ point symmetries provided (in contract to (3.19a))

$$B_0 \neq \frac{1}{2}B_1^{(1)} \quad (4.3)$$

and the solution a given in (4.2) satisfies (3.2)| $_{n=3}$. This in turn implies that there is a functional dependency relation between B_0 and B_1 , i.e., each coefficient can be written in terms of one function. Let us now look at linear fourth order equations. Assume first that (3.4) is a first order equation in a for $n=4$ where Γ_3 is given by (3.5)| $_{n=4}$. Then we have (cf. (4.1))

$$a = \mathcal{C}(B_2^{(1)} - B_1)^{-1/3}, \quad \mathcal{C} (\neq 0) \text{ constant.} \quad (4.4)$$

Substituting $d^2(3.2)/dt^2|_{n=4}$ into (3.7)| $_{n=4}$, we obtain (cf. (3.8))

$$\frac{5}{2}a^{(2)}(B_2^{(1)} - B_1) + 4a^{(1)}\Gamma_4 + a\Gamma_4^{(1)} = 0, \quad (4.5)$$

where Γ_4 is given by (3.9)| $_{n=4}$. From insertion of the relation (4.4) into (4.5), there results

$$(B_2^{(1)} - B_1) A^{(1)} - \frac{4}{3}(B_2^{(2)} - B_1^{(1)})A = 0 \quad (4.6)$$

with

$$A = 100B_0 - 50B_1^{(1)} - 9B_2^2 + 20B_2^{(2)}. \quad (4.7)$$

Thus B_0 , B_1 , and B_2 are constrained to satisfy (4.6). We can therefore obtain B_0 in terms of B_1 and B_2 . There are two possibilities for B_0 . They are

$$B_0 = \begin{cases} \frac{1}{100}(50B_1^{(1)} + 9B_2^2 - 20B_2^{(2)}) \\ \frac{1}{100}(50B_1^{(1)} + 9B_2^2 - 20B_2^{(2)}) + \gamma(B_2^{(1)} - B_1)^{4/3}, \end{cases} \quad \gamma \neq 0. \quad (4.8)$$

The equation (4.8) is deduced from (4.6) by either setting $A=0$ or treating (4.6) as a first order equation in $(B_2^{(1)} - B_1)$. So a fourth order equation

(2.3) would have exactly six point symmetries if B_0 is given by (4.8), with the additional requirement that a given by (4.4) must satisfy (3.2)| $_{n=4}$. Thus, B_0 of (4.8) (and B_1 and B_2) can be expressed in terms of one function.

We next assume that $\Gamma_3 = 0$ in (3.4). We then have (see (3.6)| $_{n=4}$)

$$B_1 = B_2^{(1)}. \quad (4.9)$$

Solving (3.8) (with Γ_4 as in (3.9)| $_{n=4}$) for a gives

$$a = \mathcal{A}(30B_2^{(2)} + 9B_2^2 - 100B_0)^{-1/4}, \quad \mathcal{A} \neq 0. \quad (4.10)$$

In this case a fourth order equation (2.3) would have exactly six point symmetries provided (in contrast to (3.19b))

$$B_0 \neq \frac{3}{10} B_2^{(2)} + \frac{9}{100} B_2^2, \quad (4.11)$$

a given by (4.10) satisfies (3.2)| $_{n=4}$, and (4.9) holds. We again deduce that each coefficient of the equation is expressible in terms of one function. It is now simple to characterise those third and fourth order equations (2.3) that have exactly $n+1$ ($n=3, 4$) point symmetries. For example, one could choose any two coefficients as arbitrary functions.

5. LINEARITY AND ABELIAN STRUCTURE

At this juncture it is important to remember that all linear equations (2.3) have at least $n+1$ point symmetries which generate the Lie algebra $nA_1 \oplus_s A_1$. This implies that all linear equations, written in canonical form (2.3), have a generic Abelian structure in the form of the Abelian n dimensional algebra nA_1 .

Let us go one step further and make the assertion that if an n th order equation

$$q^{(n)} = H(t, q, \dots, q^{(n-1)}) \quad (5.1)$$

is linearizable via a point transformation then it admits the Abelian algebra nA_1 . In order to prove this assertion we need only show that *the induced symmetry algebra of (5.1) is invariant under a point transformation*. (We have assumed this simple statement to be true in going from (2.1) to (2.3)). To this end suppose (5.1) is linearizable to $Q^{(n)} = \bar{H}$ (of form (2.3)) by $Q = \bar{F}(t, q)$, $T = \bar{G}(t, q)$ (or equivalently in inverted form by $q = F(T, Q)$, $t = G(T, Q)$). Forming the commutator of any two, say $G_i =$

$\xi_i(T, Q) \partial/\partial T + \eta_i(T, Q) \partial/\partial Q$, $i = 1, 2$, of the $n + 1$ symmetry generators of $Q^{(n)} = \bar{H}$, we have

$$[G_1(T, Q), G_2(T, Q)] = (G_1 \xi_2 - G_2 \xi_1) \partial/\partial T + (G_1 \eta_2 - G_2 \eta_1) \partial/\partial Q. \quad (5.2)$$

The commutator of these same operators in coordinates (t, q) is given by

$$[\bar{G}_1(t, q), \bar{G}_2(t, q)] = \{\bar{G}_1(G_2 t) - \bar{G}_2(G_1 t)\} \partial/\partial T + \{\bar{G}_1(G_2 q) - \bar{G}_2(G_1 q)\} \partial/\partial Q, \quad (5.3)$$

where, for example, $G_2 t = \xi_2 \partial t/\partial T + \eta_2 \partial t/\partial Q$. It is not difficult to show that the right-hand sides of (5.2) and (5.3) are identical. This completes the proof.

The question now arises as to whether the converse of the above assertion holds. Indeed it does hold. If an n th order equation of the form (5.1) admits the algebra nA_1 , then its generators of symmetry

$$G_i = \xi_i(t, q) \partial/\partial t + \eta_i(t, q) \partial/\partial q, \quad i = 1, n \quad (5.4)$$

satisfy the Abelian commutation relations. Let us consider the case when the G_i 's are unconnected, i.e., $G_j \neq \rho(t, q) G_k$ for any two indices j and k such that $j \neq k$ and for any function ρ . A point transformation can then be performed to reduce G_j and G_k to

$$\bar{G}_j = \partial/\partial Q, \quad \bar{G}_k = \partial/\partial T. \quad (5.5)$$

Writing $G_l (j \neq l \neq k)$ in these coordinates and invoking the commutators involving \bar{G}_j and \bar{G}_k , we find that $\{\bar{G}_j, \bar{G}_k, \bar{G}_l\}$ is linearly dependent. Thus we cannot have a linearly independent set of unconnected operators which generate the Lie algebra nA_1 . We therefore assume connectedness for the operators, i.e., for any two indices j and k ($j < k$) there exists a non-constant function $\psi_{jk}(t, q)$ such that $G_j = \psi_{jk}(t, q) G_k$. We have assumed $j < k$ because of the skew symmetry of the Lie bracket. Moreover, since $G_i = \psi_{ij} G_j (i < j)$ and $G_j = \psi_{jk} G_k (j < k)$, we require $\psi_{ij} \psi_{jk} = \psi_{ik} (i < j < k)$. Hence we can write $n - 1$ of the operators, say G_i , $i = 1, n - 1$ in terms of G_n . A point transformation will then reduce G_j and G_k to

$$\bar{G}_j = \partial/\partial Q, \quad \bar{G}_k = T \partial/\partial Q.$$

Representing \bar{G}_l for any l such that $j \neq l \neq k$ in (T, Q) coordinates, we obtain

$$\bar{G}_l = f_l(T) \partial/\partial Q,$$

where the set of n functions $\{1, T, f_l(T)\}$ is linearly independent. This

means that a realization for the Abelian Lie algebra nA_1 is given by (using lower case variables)

$$G_1 = \partial/\partial q, \quad G_2 = t\partial/\partial q, \quad G_i = f_i(t) \partial/\partial q, \quad i = 3, n, \quad (5.6)$$

with $\{1, t, f_i|_{i=3}^n\}$ linearly independent. We shall refer to (5.6) as the canonical realization of nA_1 .

Expressing the invariance of $q^{(n)} = H$ with respect to G_1 and G_2 results in

$$q^{(n)} = H_2(t, \bar{q}, \dots, q^{(n-1)}), \quad H_2 \text{ arbitrary.}$$

Invariance under G_3 gives rise to

$$q^{(n)} = \frac{f_3^{(n)}}{f_3^{(n-1)}} q^{(n-1)} + H_3(t, u^1, \dots, u^{n-3}), \quad (5.7)$$

where

$$u^k = q^{(n-k)} - \frac{f_3^{(n-k)}}{f_3^{(n-k-1)}} q^{(n-k-1)}, \quad k = 1, n-3$$

and H_3 is an arbitrary function of its $n-2$ arguments. Further invariance with respect to G_4 of (5.7) results in H_3 being of the form

$$H_3 = \frac{x^0}{x^1} u^1 + H_4(t, v^1, v^2, \dots, v^{n-4}),$$

where

$$x^k = f_4^{(n-k-1)} \frac{f_3^{(n-k)}}{f_3^{(n-k-1)}} - f_4^{(n-k)},$$

$$v^k = u^k - \frac{x^k}{x^{k+1}} u^{k+1},$$

and H_4 is an arbitrary function of its $n-3$ arguments. It follows that subsequent invariances under G_i for all $i \geq 5$ will result in H_i being an arbitrary function of $n-i+1$ arguments. Hence we can write down the most general equation that will remain invariant under (5.6) as

$$q^{(n)} = \sum_{i=2}^{n-1} b_i(t) q^{(i)} + H_n(t), \quad (5.8)$$

where the b_i 's are $n - 2$ functions of the $n - 2$ coordinate functions $f_j(t)$ and are given by (since (5.8) is invariant under the G_i 's of (5.6))

$$f_i^{(n)} = \sum_{k=2}^{n-1} f_i^{(k)} b_k, \quad i = 3, n.$$

The b_i 's are now obviously obtained from

$$b_p = \frac{\sum_{k=3}^n f_k^{(n)} F_{kp}}{\det(f)}, \quad p = 2, n - 1, \tag{5.9}$$

where F_{kp} is the cofactor of $f_k^{(p)}$. This completes the proof.

Formally, we have proved the following result:

THEOREM 1. *A necessary and sufficient condition for an n th ($n \geq 3$) order equation of the form (5.1) to be linearizable via a point transformation is that it admit the n dimensional Abelian algebra nA_1 .*

Remark. A striking yet simple fact is that the above theorem even holds for $n = 1$, i.e., for first order equations. It follows that all first order equations that admit a symmetry are (locally) equivalent to each other. However, the above theorem does not in general hold for $n = 2$. The reason for this is that there are two realizations for $2A_1$, viz., $\{\partial/\partial q, t\partial/\partial q\}$ and $\{\partial/\partial q, \partial/\partial t\}$. Theorem 1 applies to the former realization whereas the latter realization does not imply linearization for the associated second order equation (see [11]).

We note from the above (see (5.9)) that in the simplest case one would have $f_k = t^{k-1}$ ($k = 3, n$) and (5.8) would reduce to ($b_p = 0$ for all p)

$$q^{(n)} = H_n(t). \tag{5.10}$$

Clearly, (5.10) is the most general equation which will remain invariant under (5.6) whenever $f_k = t^{k-1}$ ($k = 3, n$). Moreover, it is immediate that the translation $\bar{q} = q - q_p$, where

$$q_p = \int^t \left[\int^{S_n} \cdots \left[\int^{S_{n-k}} b(S_{n-k-1}) dS_{n-k-1} \right] \cdots dS_{n-1} \right] dS_n, \quad k = 0, n - 2$$

is a solution of (5.10), transforms the solutions of (5.10) into solutions of $\bar{q}^{(n)} = 0$. The operators $t^{k-1} \partial/\partial q$ ($k = 1, n$) (see (5.6)) are invariant under this translation. It is apparent that Eq. (5.10) admits the maximum algebra.

Let us go back to (2.15). Assuming that Eq. (2.15) (equivalently (2.3))

has the solution set $\{b_i: i=1, n\}$, the symmetries of (2.3) which generate nA_1 are given by

$$G_k = b_k \partial/\partial q, \quad k = i, n. \quad (5.11)$$

It follows from the preceding discussion that the point transformation that reduces (5.11) to $T^{k-1} \partial/\partial Q$ ($k=1, n$) will transform (2.3) to a form that admits the maximum algebra. The transformation

$$Q = q/b_1, \quad T = b_2/b_1, \quad (5.12)$$

which we used previously (see (3.17)), will reduce (2.3) to $Q^{(n)} = 0$ provided that $b_k = b_2^{k-1}/b_1^{k-2}$, $k=2, n$. Of course only that special class of equations (2.3) whose coefficients satisfy relations of the type (3.11) would have the property that their entire solution set could be spanned by just two solutions $\{b_1, b_2\}$, i.e., are derivable from a second order equation. The resultant linear equation is referred to as an iterative linear equation (see Neuman [16] and Krause and Michel [4]).

We now consider the n ($n > 3$) dimensional subalgebra (of $nA_1 \oplus, (sl(2, \mathbb{R}) \oplus A_1)$) structure $(n-1)A_1 \oplus, A_1$. The $n=3$ case has been treated in Ref. [12]. Hence we mainly discuss the $n > 3$ case here.

THEOREM 2. *If an n th ($n > 3$) order ordinary differential equation admits $(n-1)A_1 \oplus, A_1$, then it has generators of symmetry equivalent to exactly one of forms*

- (a) $G_1 = \partial/\partial q, G_2 = t\partial/\partial q, G_j = f_j(t) \partial/\partial q, j=3, n$
- (b) $G_1 = \partial/\partial q, G_2 = t\partial/\partial q, G_j = f_j(t) \partial/\partial q (j=3, n-1),$
 $G_n = q \partial/\partial q$ (5.13)
- (c) $G_1 = \partial/\partial q, G_2 = t\partial/\partial q, G_j = f_j(t) \partial/\partial q (j=3, n-1),$
 $G_n = \partial/\partial t + \alpha q \partial/\partial q$
- (d) $G_1 = \partial/\partial q, G_2 = t\partial/\partial q, G_j = t^{j-1} \partial/\partial q (j=3, n-1),$
 $G_n = (\beta_1 t^2 + \beta_2 t + \beta_3) \partial/\partial t + (\beta_1(n-2)t + \beta_4) q \partial/\partial q,$

where α and β_i are real parameters such that β_1 and β_2 are not both zero.

Proof. Assume that an equation of the general form (5.1) admits the Lie algebra $(n-1)A_1 \oplus, A_1$ ($n > 3$). Then, the symmetry generators of the form (5.4) admitted by (5.1) will satisfy the commutation relations of $(n-1)A_1 \oplus, A_1$. We consider first the Abelian subalgebra $(n-1)A_1$. The

canonical realization of this subalgebra in terms of vector fields in two coordinates is (cf. (5.6))

$$G_1 = \partial/\partial q, \quad G_2 = t\partial/\partial q, \quad G_m = f_m(t) \partial/\partial q, \quad m = 3, n-1. \quad (5.14)$$

Writing the n th vector field G_n (realization of A_1) in the general form (2.4) and invoking the commutators $[G_1, G_n]$ and $[G_2, G_n]$ of $(n-1)A_1 \oplus_s A_1$, we find that $G_n = a(t) \partial/\partial t + (b(t)q + c(t)) \partial/\partial q$, where c is an as yet arbitrary function of t and the functions $a(t)$ and $b(t)$ are given linearly in terms of the elements of $\{1, t, f_i |_{i=3}^{n-1}\}$.

We first assume that $a(t) = 0$. Then G_n can be brought to the form

$$\bar{G}_n = b(T) Q \partial/\partial Q \quad (5.15)$$

by means of the point transformation $T = t, Q = q + c(t)/b(t)$, provided that $b \neq 0$. The generators (5.14) remain invariant under this transformation. For $b = 0$, G_n reduces to $G_n = c(t) \partial/\partial q$, where $\{1, t, f_m(t) |_{m=3}^{n-1}, c(t)\}$ is linearly independent. In this case we have the Abelian algebra nA_1 (see Theorem 1). Thus we have the form (5.13a).

We discuss the case $b \neq 0$. Invariance of an equation of the form (5.1) under (5.14) yields, by the argument preceding Theorem 1 (note that the penultimate step in the proof of Theorem 1 will yield the following equation, bearing in mind that the leading term after t will always be $q^{(n-1)}$ for each $H_i, 3 \leq i \leq n-1$),

$$q^{(n)} = \sum_{k=3}^{n-1} c_k(t) q^{(k)} + H_{n-1} \left(t, q^{(n-1)} + \sum_{k=2}^{n-2} d_k(t) q^{(k)} \right). \quad (5.16)$$

The functions $c_k(t)$ and $d_k(t)$ are in terms of the coordinate functions $f_j(t)$ of the generators (5.14). More precisely we have, since (5.16) is invariant under (5.14) and H_{n-1} is arbitrary in its arguments, the relations

$$f_i^{(n)} = \sum_{k=3}^{n-1} f_i^{(k)} c_k, \quad \sum_{k=2}^{n-2} f_i^{(k)} d_k = -f_i^{(n-1)}, \quad i = 3, n-1. \quad (5.17)$$

Thus the c_i 's and the d_i 's are given by

$$c_p = \frac{\sum_{k=3}^{n-1} f_k^{(n)} F_{kp}}{\det(f_k^{(p)})}, \quad p = 3, n-1, \quad d_s = \frac{\sum_{k=3}^{n-1} -f_k^{(n-1)} F'_{ks}}{\det(f_k^{(s)})}, \quad s = 2, n-2, \quad (5.18)$$

where F_{kp} and F'_{ks} are the cofactors of the elements $f_k^{(p)}$ ($p = 3, n-1$) and $f_k^{(s)}$ ($s = 2, n-2$), respectively.

We impose the additional requirement that G_n (dropping the bar and using lower case) of (5.15) must be a symmetry vector field of (5.16) subject

to (5.17). We obtain, recalling that the n th prolongation of G_n is given by $G_n^{[n]} = \sum_{j=0}^n \sum_{i=0}^j \binom{j}{i} q^{(i)} b^{(j-i)} \partial/\partial q^{(j)}$, the condition

$$H_{n-1} - u \frac{\partial H_{n-1}}{\partial u} = 0, \quad u = q^{(n-1)} + \sum_{k=2}^{n-2} d_k(t) q^{(k)},$$

provided that $b^{(1)} = 0$. Therefore in order for G_n of (5.15) to be a symmetry of (5.16), we must have $b = \mathcal{B} \neq 0$, a constant. The resultant differential equation is linear,

$$q^{(n)} = \sum_{k=3}^{n-1} c_k(t) q^{(k)} + g(t) q^{(n-1)} + g(t) \sum_{k=2}^{n-2} d_k(t) q^{(k)}, \quad (5.19)$$

where g is a function of t . As a consequence we have derived the form (5.13b).

We discuss the situation $a \neq 0$. The transformation

$$T = t, \quad Q = q + \left[\int^t -\frac{c(s)}{a(s)} \left(\exp \int^s -\frac{b(r)}{a(r)} dr \right) ds \right] \exp \int^t \frac{b(s)}{a(s)} ds$$

reduces G_n to

$$\bar{G}_n = a(T) \partial/\partial T + b(T) Q \partial/\partial Q. \quad (5.20)$$

The generators (5.14) remain invariant under this space translation. We require that (5.20) (using lower case) to be a symmetry of (5.16) subject to (5.17). The n th prolongation of G_n is (using (2.6))

$$G_n^{[n]} = G_n + \sum_{k=1}^n b q^{(k)} \partial/\partial q^{(k)} + \sum_{k=1}^n \sum_{i=0}^{k-1} \binom{k}{i} [q^{(i)} b^{(k-i)} - q^{(i+1)} a^{(k-i)}] \partial/\partial q^{(k)}.$$

Acting with $G_n^{[n]}$ on (5.16), whenever (5.16) holds, we find that G_n is a symmetry of (5.16) provided that either $a^{(1)} = 0$ and $b^{(1)} = 0$ or $a^{(3)} = 0$ and $b^{(1)} = a^{(2)}(n-2)/2$ with $d_k = c_k = 0$. In the first case we have $a = \mathcal{A} \neq 0$ and $b = \mathcal{B}$ (both constants). Thus we can write G_n as

$$G_n = \partial/\partial t + \alpha q \partial/\partial q, \quad \alpha = \mathcal{B}/\mathcal{A} \text{ constant}, \quad (5.21)$$

with the condition for invariance given by

$$b H_{n-1} - b u \frac{\partial H_{n-1}}{\partial u} - a \frac{\partial H_{n-1}}{\partial t} = 0, \quad u = q^{(n-1)} + \sum_{k=2}^{n-2} d_k q^{(k)},$$

provided that $a^{(1)} = 0$ and $b^{(1)} = 0$. The associated differential equation is

$$q^{(n)} = \sum_{k=3}^{n-1} c_k q^{(k)} + J \left(q^{(n-1)} \exp(-\alpha t) + \exp(-\alpha t) \sum_{k=2}^{n-2} d_k q^{(k)} \right) \exp(\alpha t), \tag{5.22}$$

where J is arbitrary in its argument and c_k, d_k are independent of t . Hence we have derived the form (5.13c).

In the second case ($d_k = c_k = 0$) we have the symmetry requirement

$$(b - na^{(1)}) H_{n-1} - a^{(2)} u n / 2 + [(n-1)a^{(1)} - b] u \frac{\partial H_{n-1}}{\partial u} - a \frac{\partial H_{n-1}}{\partial t} = 0, \quad u = q^{(n-1)}, \tag{5.23}$$

provided that $a^{(3)} = 0$ and $b^{(1)} = a^{(2)}(n-2)/2$. Thus we have the symmetry G_n

$$G_n = (\beta_1 t^2 + \beta_2 t + \beta_3) \partial / \partial t + (\beta_1(n-2)t + \beta_4) q \partial / \partial q, \tag{5.24}$$

where β_i are constants such that β_1 and β_2 are not both zero. If $\beta_1 = \beta_2 = 0$, then we would have the previous case, which is more general. To determine the most general equation invariant under (5.24), we distinguish two cases. First we assume $\beta_1 = 0$ ($\beta_2 \neq 0$). Then we can, without loss of generality, take $\beta_2 = 1$. Now solving for H_{n-1} from (5.23) we obtain the equation

$$q^{(n)} = (t + \beta_3)^{\beta_4 - n} J(q^{(n-1)}(t + \beta_3)^{n - \beta_4 - 1}), \tag{5.25}$$

where J is arbitrary in its argument. Next, we suppose that $\beta_1 \neq 0$. Then we can set $\beta_1 = 1$. Solving (5.23) under these circumstances we arrive at the equation

$$(t^2 + \beta_2 t + \beta_3) q^{(n)} + n t q^{(n-1)} = q^{(n-1)} J(q^{(n-1)}(t^2 + \beta_2 t + \beta_3)^{n/2} \exp[(n\beta_2/2 - \beta_2 - \beta_4) s(t)]), \tag{5.26}$$

where J is arbitrary in its argument and $s(t)$ is given by (set $\Delta = \beta_2^2 - 4\beta_3$)

$$s(t) = \begin{cases} \frac{-2}{\sqrt{\Delta}} \tanh^{-1} \frac{\beta_2 + 2t}{\sqrt{\Delta}}, & \Delta > 0 \\ \frac{-2}{\beta_2 + 2t}, & \Delta = 0 \\ \frac{2}{\sqrt{-\Delta}} \tan^{-1} \frac{\beta_2 + 2t}{\sqrt{-\Delta}}, & \Delta < 0. \end{cases}$$

Since $d_k = c_k = 0$ we can set (in (5.14)) $f_m(t) = t^{m-1}$ ($m = 3, n-1$). This, together with (5.24), yields the form (5.13d). This completes the proof. ■

It is immediate from the preceding theorem that linearization is only implied by the first two symmetry realizations of (5.13). As a matter of fact the first realization (Abelian nA_1) was treated in Theorem 1. We now focus our attention on the second realization. We observe that G_n of (5.13b) acts as a dilatation operator. It is a simple matter to verify that only the symmetry generators (5.13b) possess the commutation relations $[G_i, G_n] = G_i$, $[G_i, G_j] = 0$, $i, j = 1, n-1$. Hence it is now opportune to state the following linearizability result.

THEOREM 3. *An n th order ($n > 3$) equation of the form (5.1) is linearizable via a point transformation if and only if it admits the n dimensional Lie algebra*

$$[G_i, G_n] = G_i, \quad [G_i, G_j] = 0, \quad i, j = 1, n-1. \quad (5.27)$$

Proof. The proof for sufficiency follows from Theorem 2 and the preceding discussion. The necessity is straightforward since if an n th order equation is linearizable and put in the canonical form (2.3), then it admits the $n+1$ dimensional Lie algebra $nA_1 \oplus_s A_1$, which contains $(n-1)A_1 \oplus_s A_1$ as a subalgebra. ■

Remark. Theorem 3 does not apply to the cases $n=2$ and $n=3$. For $n=2$ there are two realizations of the algebra $A_1 \oplus_s A_1 \cong A_2$, viz., $A_2^I: \{\partial/\partial q, t\partial/\partial t + q\partial/\partial q\}$ and $A_2^{II}: \{\partial/\partial q, q\partial/\partial q\}$. Theorem 3 does not apply to the former realization (see [11]). For $n=3$ there are two realizations of the algebra (5.27). They are $A_{3,3}^I: \{\partial/\partial t, \partial/\partial q, t\partial/\partial t + q\partial/\partial q\}$ and $A_{3,3}^{II}: \{\partial/\partial q, t\partial/\partial q, q\partial/\partial q\}$. The above theorem applies to the latter realization whereas the former realization does not in general yield linearization for a third order equation (see [12]). The Lie group corresponding to the algebra (5.27) is a semidirect product of space translations ($q \rightarrow q + \varepsilon\tau(t)$, where $\tau \in \{1, t, f_m(t)|_{m=3}^{n-1}\}$), and space dilatations ($q \rightarrow q \exp \varepsilon$). We denote this group by $T_{n-1}^q \oplus_s D^q$. The superscript q is used to distinguish this case from the general case of translations and dilatations in both t and q . (For example, (5.13d) with $\beta_1 = \beta_3 = 0$ does not yield linearization. The same is true for A_2^I and $A_{3,3}^I$.)

6. NONEXISTENCE OF EXACTLY AN $n+3$ DIMENSIONAL SYMMETRY ALGEBRA WHICH IS A SUBALGEBRA OF $nA_1 \oplus_s gl(2, \mathbb{R})$

We are now in a position to prove the following theorem.

THEOREM 4. *There does not exist any n th ($n \geq 3$) order ordinary differential equation having exactly an $n+3$ dimensional point symmetry algebra which is a subalgebra of $nA_1 \oplus_s gl(2, \mathbb{R})$.*

Proof. An n th ($n \geq 3$) order ordinary differential equation cannot possess exactly an $n + 3$ dimensional Lie algebra of point symmetries containing the Abelian algebra nA_1 , the algebra (5.27) for $n > 3$, or $A''_{3,3}$ for (see [12]) $n = 3$, as these would, by Theorems 1 and 3 and the fact that a linear n th ($n \geq 3$) order equation does not admit exactly an $n + 3$ dimensional algebra of point symmetries, imply linearization and consequently an $(n + 4)$ th dimensional point symmetry algebra for the equation. This in effect means that the only $n + 3$ dimensional algebra structure (subalgebra of $nA_1 \oplus_s (sl(2, \mathbb{R}) \oplus A_1)$) that we need to investigate is $(n - 1)A_1 \oplus_s (sl(2, \mathbb{R}) \oplus A_1)$. We show that an n th ($n \geq 3$) order equation cannot have this algebra as its maximum symmetry algebra. Let us consider the subalgebra $(n - 1)A_1 \oplus_s A_1$. In the foregoing we have discussed the possibility of linearizing an equation admitting this algebraic structure. Let us then investigate the case where linearization need not follow. For $n > 3$, the symmetry realizations of the algebra $(n - 1)A_1 \oplus_s A_1$ were obtained in Theorem 2. Recall that in the case where linearization is not in general implied by the operators (5.13) (i.e., (5.13c, d)), we can write, using (5.14) and (5.20),

$$G_1 = \partial/\partial q, \quad G_2 = t\partial/\partial q, \quad G_i = f_i(t) \partial/\partial q \quad (i = 3, n - 1), \quad (6.1)$$

$$G_n = a(t) \partial/\partial t + b(t) q\partial/\partial q,$$

where $a(t) \neq 0$ and $b(t)$ are given linearly in terms of the elements of $\{1, t, f_i |_{i=3}^{n-1}\}$. Of course in the case of linearization we had $a(t) = 0$. Writing $G_{n+i} = \xi_i \partial/\partial t + \eta_i \partial/\partial q$, $i = 1, 3$, realization of $sl(2, \mathbb{R})$, in their general form and invoking the commutators $[G_1, G_{n+i}]$ of $(n - 1)A_1 \oplus_s sl(2, \mathbb{R})$ (subalgebra of $(n - 1)A_1 \oplus_s (sl(2, \mathbb{R}) \oplus A_1)$), we arrive at

$$\frac{\partial \xi_i}{\partial q} = 0, \quad \eta_i = g_i(t)q + h_i(t), \quad i = 1, 3, \quad (6.2)$$

where g_i is a linear combination of the elements of $\{1, t, f_i |_{i=3}^{n-1}\}$ and h_i is a function of t . The commutation relations of the algebra $sl(2, \mathbb{R})$ then imply (since $\partial \xi_j/\partial q = 0$) ($i, j = 1, 3$)

$$\xi_i = 0, \quad \eta_i \frac{\partial \eta_j}{\partial q} - \eta_j \frac{\partial \eta_i}{\partial q} = \epsilon_{ij1} \eta_3 - 2\epsilon_{ij2} \eta_2 + \epsilon_{ij3} \eta_1. \quad (6.3)$$

Substituting η_i of (6.2) into (6.3) yields $g_i = h_i = 0$, $i = 1, 3$, whence we cannot have exactly an $n + 3$ dimensional algebra of symmetries, which is a subalgebra of $nA_1 \oplus_s gl(2, \mathbb{R})$, for any n th ($n > 3$) order equation.

For $n = 3$ the algebra of interest is $2A_1 \oplus_s (sl(2, \mathbb{R}) \oplus A_1)$. We consider the Abelian $2A_1$ subalgebra. There are two realizations of $2A_1$ in terms of vector fields on the plane. They are $2A'_1: \{\partial/\partial q, \partial/\partial t\}$ and

$2A_1'' : \{\partial/\partial q, t\partial/\partial q\}$. First we focus on the $2A_1''$ realization. The realization of $2A_1 \oplus A_1$ in this case (excluding the linear case $A_{3,3}''$) is (cf. (6.1))

$$G_1 = \partial/\partial q, \quad G_2 = t\partial/\partial q, \quad G_3 = a(t)\partial/\partial t + b(t)q\partial/\partial q, \quad a \neq 0. \quad (6.4)$$

Utilising the same argumentation as for $n > 3$, it is quite straightforward from (6.2) and (6.3) that we cannot have exactly a 3 + 3 dimensional symmetry algebra (subalgebra of $3A_1 \oplus gl(2, \mathbb{R})$) for a third order equation admitting two generators that are reducible to $2A_1''$.

Next we look at the $2A_1'$ realization. Writing G_3 in its general form and invoking $2A_1 \oplus A_1$ with the operators $G_1 = \partial/\partial q$ and $G_2 = \partial/\partial t$ representing $2A_1'$, we obtain (disregarding constant multiples of G_1 and G_2) G_3 given by (a_3, b_3, c_3 , and d_3 are real constants)

$$G_3 = (a_3q + b_3t)\partial/\partial t + (c_3q + d_3t)\partial/\partial q. \quad (6.5)$$

As a result of the classification of three dimensional Lie algebras [18], G_3 can assume one of the standard forms (see [12])

$$G_3 = \begin{cases} t\partial/\partial t + aq\partial/\partial q, & |a| \leq 1 \\ t\partial/\partial q \\ t\partial/\partial t + (t+q)\partial/\partial q \\ (bt+q)\partial/\partial t + (bq-t)\partial/\partial q, & b \geq 0. \end{cases} \quad (6.6)$$

Let $G_i, i=4, 6$ realize $sl(2, \mathbb{R})$. Then the algebra $2A_1 \oplus sl(2, \mathbb{R})$ implies (a_i, b_i, c_i , and d_i are real constants)

$$G_i = (a_iq + b_it)\partial/\partial t + (c_iq + d_it)\partial/\partial q, \quad i=4, 6. \quad (6.7)$$

Moreover, since $G_i, i=4, 6$ constitute $sl(2, \mathbb{R})$ and commute with G_3 , the constants a_i, b_i, c_i , and $d_i, i=4, 6$ are constrained to satisfy ($i, j=4, 6$)

$$\begin{aligned} \text{(a)} \quad & c_i = -b_i \\ \text{(b)} \quad & \varepsilon_{ij4}c_6 - 2\varepsilon_{ij5}c_5 + \varepsilon_{ij6}c_4 = a_id_j - a_jd_i \\ \text{(c)} \quad & -\frac{1}{2}\varepsilon_{ij4}a_6 + \varepsilon_{ij5}a_5 - \frac{1}{2}\varepsilon_{ij6}a_4 = a_ic_j - a_jc_i \\ \text{(d)} \quad & -\frac{1}{2}\varepsilon_{ij4}d_6 + \varepsilon_{ij5}d_5 - \frac{1}{2}\varepsilon_{ij6}d_4 = c_id_j - c_jd_i \\ \text{(e)} \quad & a_id_3 - a_3d_i = 0 \\ \text{(f)} \quad & 2a_3b_i + (c_3 - b_3)a_i = 0 \\ \text{(g)} \quad & 2d_3b_i + (c_3 - b_3)d_i = 0. \end{aligned} \quad (6.8)$$

We show that the realizations $2A_1', G_3$ (any one of the forms given in (6.6)) and $G_i, i=4, 6$ are not symmetries of any third order equation. To that end

let us first assume that $G_3 = t\hat{c}/\partial t + aq\partial/\partial q$ ($a_3 = 0$, $b_3 = 1$, $c_3 = a$, and $d_3 = 0$). We solve (6.8) to obtain the precise form of the generators (6.7). It is immediate that (6.8e) is identically satisfied. The last two sets of conditions (6.8f, g) give rise to ($i = 4, 6$)

$$(a-1)a_i = 0, \quad (a-1)d_i = 0.$$

If $a \neq 1$, then clearly $a_i = d_i = 0$, $i = 4, 6$. The remaining sets of equations of (6.8) imply that $c_i = b_i = 0$, $i = 4, 6$. Thus we must have $a = 1$. This means that we need only concentrate on (6.8a-d). Let us now suppose that $a_k = 0$ for fixed $k \in \{4, 5, 6\}$. We take $a_5 = 0$. Further, we cannot have $b_5 = 0$ ($c_5 = 0$), for if we do, then $a_4 = b_4 = c_4 = d_4 = 0$. For $c_5 \neq 0$, a simple analysis of (6.8a-d) shows that only in the cases $c_5 = \frac{1}{2}$ and $c_5 = -\frac{1}{2}$ do we have linearly independent operators G_i , $i = 3, 6$ occurring. In the case $c_5 = -\frac{1}{2}$ we obtain (taking a linear combination of the original operators)

$$G_4 = -q\partial/\partial t, \quad G_5 = \frac{1}{2}t\partial/\partial t - \frac{1}{2}q\partial/\partial q, \quad G_6 = t\partial/\partial q. \quad (6.9)$$

For $c_5 = \frac{1}{2}$ we obtain a linear combination of (6.9). The most general third order equation invariant under G_1 , G_2 , and $G_3 = t\partial/\partial t + q\partial/\partial q$ is

$$q^{(3)} = \bar{q}^2 Y(\bar{q}), \quad Y \text{ arbitrary.} \quad (6.10)$$

Invariance under G_6 of (6.10) results in Y being a constant. Equation (6.10) with Y constant is now easily seen not to be invariant with respect to G_4 and G_5 of (6.9). In a similar manner one can show that by assuming $a_4 = 0$ or $a_6 = 0$, the resultant operators G_i , $i = 4, 6$ do not leave Eq. (6.10) invariant. Hence we cannot have $a_j = 0$ for any $j \in \{4, 5, 6\}$. Without loss of generality we can set $a_i = 1$, $i = 4, 6$. Thereafter invoking the first four sets of Eqs. (6.8), we obtain a linear combination of the operators $G_3 = t\partial/\partial t + q\partial/\partial q$ and (6.9), thereby implying the nonexistence of any third order equation which admits these operators together with $2A'_1$ as symmetry generators.

In an analogous manner one can easily show, by assuming G_3 to be any one of the other three forms (6.6b-d) and solving (6.8), that there are no additional generators of the form (6.7). In fact the system (6.8) has the trivial solution. This completes the proof. ■

It is of great interest to note that there do exist n th order ($n \geq 3$) (non-linearizable) equations admitting exactly an $n+3$ dimensional algebra of point symmetries. We consider an example originally due to Michel [15]. The third order equation

$$3\bar{q}^2 - 2\dot{\bar{q}}q^{(3)} = 0 \quad (6.11)$$

possesses the full Lie algebra $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ of dimension $n + 3 = 3 + 3$. The generators of symmetry are given by

$$\begin{aligned} \text{(a)} \quad G_1 &= \frac{\partial}{\partial q}, & G_2 &= q \frac{\partial}{\partial q}, & G_3 &= q^2 \frac{\partial}{\partial q}, \\ \text{(b)} \quad G_1 &= \frac{\partial}{\partial t}, & G_2 &= t \frac{\partial}{\partial t}, & G_3 &= t^2 \frac{\partial}{\partial t}. \end{aligned} \quad (6.12)$$

Both sets (6.12a, 6.12b) of vector fields are equivalent to the $A'_{3,8}$ realization [12]. It is easy to derive Eq. (6.11). We only have to seek invariance of the equation associated with $A'_{3,8}$, viz., $\dot{q}q^{(3)} = \frac{3}{2}\dot{q}^2 + \dot{q}^2 Y(t)$ (Y arbitrary function of t) with respect to the operators (6.12b). We note that $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ is not a subalgebra of $3A_1 \oplus gl(2, \mathbb{R})$.

ACKNOWLEDGMENTS

We thank Professor Louis Michel of the IHES (Bures-sur-Yvette) for sending us Refs. [3, 4] and for pointing out an error in an earlier version of Theorem 4. F. M. also thanks the Foundation for Research Development of the Republic of South Africa and the University of the Witwatersrand for financial support.

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The relationship between the symmetries of and the existence of conserved vectors for the equation $\ddot{\mathbf{r}} + f(r)\mathbf{L} + g(r)\hat{\mathbf{r}} = \mathbf{0}$

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Journal of Physics A: Mathematical and General **23** (1990) 2765–2774

The relationship between the symmetries of and the existence of conserved vectors for the equation $\ddot{\mathbf{r}} + f(r)\mathbf{L} + g(r)\mathbf{r} = \mathbf{0}$

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Received 17 May 1989, in final form 27 February 1990

Abstract. The existence of explicit expressions for conserved vectors for the charge-monopole problem and the Kepler problem is well known. The Lie algebras of the point transformations under which the equations of motion are invariant have been found more recently. The recent discovery of an explicit expression for a Laplace-Runge-Lenz-like vector for the equation of motion $\ddot{\mathbf{r}} + h'\mathbf{L}/r + (hh' + kr^{-2})\hat{\mathbf{r}} = \mathbf{0}$, $h = h(r)$, from which one of the equations for the orbit is easily obtained, has prompted the question of what is the Lie algebra of the point symmetries of the equation $\ddot{\mathbf{r}} + f(r)\mathbf{L} + g(r)\hat{\mathbf{r}} = \mathbf{0}$ of which each of the above problems is a member. In the case that $f(r) \equiv 0$ it is well known that Laplace-Runge-Lenz-like vectors exist. The existence of such conserved vectors does not imply a particular algebraic structure of the Lie point symmetries of the equation of motion. However, the existence of such symmetries provides a systematic method for constructing the vectors.

1. Introduction

Some years ago Fradkin (1967) demonstrated that all central potential problems possessed the dynamical symmetries O_4 and SU_3 . The former followed from the Lie algebra $so(4)$ (for bounded motion, $so(3, 1)$ for unbounded motion) of the components of two conserved vectors under the operation of taking the Poisson bracket. From the nature of the potential it is evident that the angular momentum $\mathbf{L} (= \mathbf{r} \times \dot{\mathbf{r}}$, throughout this paper the mass has been scaled to 1) is a conserved vector. The other conserved vector was a conserved vector in the plane of the motion which could be written as a linear combination of generalisations of the Laplace-Runge-Lenz vector (Laplace 1799, Runge 1923, Lenz 1924) and Hamilton's vector (Hamilton 1847) of the Kepler problem. The latter group was obtained by using the angular momentum and by constructing an appropriate generalisation of the Jauch-Hill-Fradkin tensor (Jauch and Hill 1940, Fradkin 1965) of the simple harmonic oscillator problem.

In this paper we are concerned with the connection of the symmetries of the differential equation governing the motion and the existence of conserved vectors rather than the algebraic structure of the first integrals. To this purpose the central result of Fradkin (1967) is that every central potential problem has a vector of Laplace-Runge-Lenz type in addition to the angular momentum. Peres (1979) was led to a restricted form of this result since he assumed a particular structure for the Laplace-Runge-Lenz-like vector. Yoshida (1987) demonstrated the relationship of Peres' result to that of Fradkin and later extended the existence of such a vector for all motions in the plane (Yoshida 1989).

Generally speaking, multidimensional nonlinear dynamical systems are not integrable, are lacking in symmetry and are given to chaotic evolution. However, there are some completely integrable nonlinear systems which have become well worn paradigms. The most noteworthy of these is the Kepler problem with equation of motion (in reduced co-ordinates)

$$\ddot{\mathbf{r}} = -\frac{\mu \hat{\mathbf{r}}}{r^2}. \quad (1.1)$$

(Throughout this paper overdot means differentiation with respect to time and caret denotes a unit vector.)

In addition to the conserved angular momentum vector \mathbf{L} and energy, the Kepler problem has two additional related conserved vectors, Hamilton's vector (Hamilton 1847)

$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu \hat{\boldsymbol{\theta}}}{L} \quad (1.2)$$

and the better known Laplace-Runge-Lenz vector (Laplace 1799, Runge 1923, Lenz 1924)

$$\mathbf{J} = \mathbf{L} \times \mathbf{K} = \mathbf{L} \times \dot{\mathbf{r}} + \mu \hat{\mathbf{r}}. \quad (1.3)$$

The latter vector may be used to obtain the orbit equation since, if the polar angle θ is measured from \mathbf{J} , the scalar product $\mathbf{J} \cdot \mathbf{r}$ leads to

$$r = \frac{L^2}{\mu - J \cos \theta}. \quad (1.4)$$

(Hamilton's vector could equally be used to obtain the orbit equation, but the derivation is not as elegant.)

The Laplace-Runge-Lenz vector has been termed inessential (Kaplan 1986) since the conservation of angular momentum and energy implies integrability by Liouville's theorem (Whittaker 1944). This is certainly a valid and correct viewpoint in so far as integrability is concerned. However, the Laplace-Runge-Lenz vector classically leads to the orbit equation (1.4) in a trivial way and quantally provides the source of the hidden degeneracy in the spectrum of the hydrogen atom. Considerations of this nature cause us to be rather more excited than Kaplan about the existence of explicit expressions for vectors such as the Laplace-Runge-Lenz vector.

Another system which has been known for a long time to possess a conserved vector is the charge-monopole problem with equation of motion

$$\ddot{\mathbf{r}} = -\frac{\lambda \mathbf{r} \times \dot{\mathbf{r}}}{r^3} = -\frac{\lambda \mathbf{L}}{r^3} \quad (1.5)$$

which has the conserved vector (Poincaré 1896)

$$\mathbf{Q} = \mathbf{L} - \lambda \hat{\mathbf{r}}. \quad (1.6)$$

It is usual to regard (1.6) as a generalised angular momentum (Moreira *et al* 1985) since the surface on which the motion takes place can be obtained from (1.6) by taking the scalar product of \mathbf{Q} with \mathbf{r} . As the motion is three dimensional, this is one of the two equations required to specify the orbit. The angle θ between \mathbf{Q} and \mathbf{r} is constant, so that the orbit lies on a right circular cone with its vertex at the origin and the direction of its axis of symmetry is $\hat{\mathbf{P}}$.

Although Fradkin (1965) and Yoshida (1989) have demonstrated the formal existence of conserved vectors of Laplace-Runge-Lenz type for any planar motion, the number of examples for which explicit expressions were known was limited. However, in the last decade the number of dynamical systems possessing explicit conserved vectors from which the orbit equation may be derived has been increased. Katzin and Levine (1983) and Leach (1985) derived the Laplace-Runge-Lenz-type vector

$$\mathbf{J} = \mathbf{L} \times (\mathbf{u}\dot{\mathbf{r}} - \dot{\mathbf{u}}\mathbf{r}) + \mu\hat{\mathbf{r}} \quad (1.7)$$

for the time-dependent Kepler problem with equation of motion

$$\ddot{\mathbf{r}} = -\ddot{\mu}\mathbf{r}/\mu - \mu\hat{\mathbf{r}}/(r^2u) \quad (1.8)$$

where $u(t)$ is an arbitrary function of time. Then Jezewski and Mittleman (Mittleman and Jezewski 1982, Jezewski and Mittleman 1983) obtained the vector

$$\mathbf{J} = \frac{\mathbf{L} \times \dot{\mathbf{r}}}{L^2} + \mu z'(\theta)\hat{\boldsymbol{\theta}} + \mu z(\theta)\hat{\mathbf{r}} \quad (1.9)$$

where $z(\theta) = \int_{\theta_0}^{\theta} \sin(\theta - \eta)(h - \alpha\eta)^{-2} d\eta$ and h and α are constants, for the Kepler problem with a drag law proposed by Danby (1962) which has the equation of motion

$$\ddot{\mathbf{r}} + \frac{\alpha\dot{\mathbf{r}}}{r^2} + \frac{\mu\mathbf{r}}{r^3} = \mathbf{O}. \quad (1.10)$$

(The notation used follows the usage of Gorrington and Leach (1988a) rather than that of Jezewski and Mittleman.) Recently Thompson (1987), in an investigation of the Kepler-charge monopole problem with equation of motion

$$\ddot{\mathbf{r}} - \frac{\mathbf{L}}{r^3} + \left(-\frac{1}{r^3} + \frac{k}{r^2} \right) \hat{\mathbf{r}} = \mathbf{O} \quad (1.11)$$

discovered the conserved vector

$$\mathbf{J} = \mathbf{L} \times \dot{\mathbf{r}} + \frac{\mathbf{L}}{r} + k\hat{\mathbf{r}}. \quad (1.12)$$

(One should note that this motion is not planar and so does not fall within the ambit of the results of Fradkin (1965) and Yoshida (1989).) All of these studies were computationally rather complicated. However, adapting a particularly simple and elegant method promoted by Collinson (1973), Leach (1987), Leach and Gorrington (1987) and Gorrington and Leach (1987, 1988a, b, 1989a, b) were not only able to recover the examples cited above but also to provide whole new classes of problems for which a Laplace-Runge-Lenz-type vector existed in explicit form. Thereby they opened up the way for a direct calculation of the orbit for these problems. We mention just the one of these which is relevant to the purpose of this paper. The equation of motion

$$\ddot{\mathbf{r}} + \frac{h'(r)}{r} \mathbf{L} + \left(hh' + \frac{k}{r^2} \right) \hat{\mathbf{r}} = \mathbf{O} \quad (1.13)$$

has the conserved Laplace-Runge-Lenz-type vector

$$\mathbf{J} = \mathbf{L} \times \dot{\mathbf{r}} + h\mathbf{L} + k\hat{\mathbf{r}} \quad (1.14)$$

where $h(r)$ is an arbitrary differentiable function of r (Leach and Gorrington 1988).

The richness of the Kepler problem and of the charge-monopole problem in terms of explicit expressions for conserved quantities attracted attention to the Lie point symmetries of their equations of motion. For the Kepler problem with equation of motion (1.1) the following symmetry generators were found (Leach 1981, Prince and Eliezer 1981):

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t} & G_2 &= t \frac{\partial}{\partial t} + \frac{2}{3} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \\ G_3 &= z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} & G_4 &= x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} & G_5 &= y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \end{aligned} \quad (1.15)$$

The non-zero commutation relations are

$$\begin{aligned} [G_1, G_2] &= G_1 & [G_3, G_4] &= G_5 \\ [G_4, G_5] &= G_3 & [G_5, G_3] &= G_4 \end{aligned} \quad (1.16)$$

from which it is evident that the Lie algebra is the direct sum $\mathfrak{a}_2 \oplus \mathfrak{so}(3)$. In passing we note that G_3 , G_4 and G_5 follow from the invariance under rotation of (1.1), G_1 from its invariance under time translation and G_2 , the generator specifically related to the Laplace-Runge-Lenz vector, indicates invariance under the similarity transformation $(t, \mathbf{r}) \rightarrow (\bar{t}, \bar{\mathbf{r}}: t = \alpha \bar{t}, \mathbf{r} = \alpha^{2/3} \bar{\mathbf{r}})$. The two-dimensional algebra \mathfrak{a}_2 of G_1 and G_2 has recently found application in the analysis of one-dimensional nonlinear second-order differential equations (Leach *et al* 1988).

The algebraic structure of the Kepler problem extends to the time-dependent Kepler problem (1.8) since they are related by a point transformation (Leach 1985).

For the charge-monopole problem (1.5) Moreira *et al* (1985) found the generators

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t} & G_2 &= t \frac{\partial}{\partial t} + \frac{1}{2} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \\ G_3 &= t \left(t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) & G_4 &= z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \\ G_5 &= x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} & G_6 &= y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \end{aligned} \quad (1.17)$$

with non-zero commutation relations

$$\begin{aligned} [G_1, G_2] &= G_1 & [G_2, G_3] &= G_3 & [G_3, G_1] &= -2G_2 \\ [G_4, G_5] &= G_6 & [G_5, G_6] &= G_4 & [G_6, G_4] &= G_5. \end{aligned} \quad (1.18)$$

This algebra is the direct sum of the subalgebras of G_1 , G_2 and G_3 and G_4 , G_5 and G_6 and is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(3)$. (Moreira *et al* (1985) use $\mathfrak{so}(2, 1)$ rather than $\mathfrak{sl}(2, \mathbb{R})$.)

We observe that the charge-monopole problem has a richer algebraic structure than the Kepler problem. This means that, as Thompson (1987) observed, they cannot be related by a point transformation.

Equation (1.13) with h constant and k non-zero is just the Kepler problem (1.1). With k zero and h non-constant it cannot be reduced to the charge-monopole problem (1.5). Nevertheless (1.13) does possess a Laplace-Runge-Lenz-type vector just as (1.1) and (1.5) do and it is of interest to determine the Lie algebra associated with it. However, to make the results of this paper of greater generality we consider the equation

$$\ddot{\mathbf{r}} + f(r)\mathbf{L} + g(r)\mathbf{r} = \mathbf{O} \quad (1.19)$$

of which (1.1), (1.5) and (1.13) are particular instances.

2. The determining equations

A second-order differential equation is invariant under a Lie point transformation generated by a symmetry generator, G , if the action of the twice extended generator, $G^{(2)}$, on the differential equation is zero whenever the differential equation holds. We shall perform the analysis in a cartesian basis as it is easier to detect computational errors due to the structure of the equations obtained. We define G to be

$$G = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z} \quad (2.1)$$

where τ , ξ , η and ζ are functions of t , x , y and z . The twice extended operator is

$$\begin{aligned} G^{(2)} = G + (\xi - \dot{x}\tau) \frac{\partial}{\partial \dot{x}} + (\eta - \dot{y}\tau) \frac{\partial}{\partial \dot{y}} + (\zeta - \dot{z}\tau) \frac{\partial}{\partial \dot{z}} \\ + (\ddot{\xi} - 2\dot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial}{\partial \ddot{x}} + (\ddot{\eta} - 2\dot{y}\dot{\tau} - \dot{y}\ddot{\tau}) \frac{\partial}{\partial \ddot{y}} + (\ddot{\zeta} - 2\dot{z}\dot{\tau} - \dot{z}\ddot{\tau}) \frac{\partial}{\partial \ddot{z}} \end{aligned} \quad (2.2)$$

where

$$\dot{\tau} = \frac{\partial \tau}{\partial t} + \dot{x} \frac{\partial \tau}{\partial x} + \dot{y} \frac{\partial \tau}{\partial y} + \dot{z} \frac{\partial \tau}{\partial z} \quad (2.3)$$

$$\begin{aligned} \ddot{\tau} = \frac{\partial^2 \tau}{\partial t^2} + 2\dot{x} \frac{\partial^2 \tau}{\partial x \partial t} + 2\dot{y} \frac{\partial^2 \tau}{\partial y \partial t} + 2\dot{z} \frac{\partial^2 \tau}{\partial z \partial t} + \dot{x}^2 \frac{\partial^2 \tau}{\partial x^2} + \dot{y}^2 \frac{\partial^2 \tau}{\partial y^2} + \dot{z}^2 \frac{\partial^2 \tau}{\partial z^2} + 2\dot{x}\dot{y} \frac{\partial^2 \tau}{\partial x \partial y} \\ + 2\dot{y}\dot{z} \frac{\partial^2 \tau}{\partial y \partial z} + 2\dot{z}\dot{x} \frac{\partial^2 \tau}{\partial z \partial x} + \ddot{x} \frac{\partial \tau}{\partial x} + \ddot{y} \frac{\partial \tau}{\partial y} + \ddot{z} \frac{\partial \tau}{\partial z} \end{aligned} \quad (2.4)$$

etc. The three components of (1.19) are

$$\begin{aligned} \ddot{x} + f(y\dot{z} - z\dot{y}) + gx = 0 \quad \ddot{y} + f(z\dot{x} - x\dot{z}) + gy = 0 \\ \ddot{z} + f(x\dot{y} - y\dot{x}) + gz = 0. \end{aligned} \quad (2.5)$$

The partial differential equations which determine τ , ξ , η and ζ are obtained thus. $G^{(2)}$, (2.2), is applied to each of (2.5) with the derivatives expanded as in (2.3) and (2.4). All second derivatives are removed using (2.5). Since we have assumed a point transformation, the terms can be collected as coefficients of various combinations of powers of the total time derivatives \dot{x} , \dot{y} and \dot{z} and coefficients of linearly independent combinations then set to zero. This process is somewhat tedious to do by hand, but it was readily handled by REDUCE. Forty-eight partial differential equations were obtained, twelve of which were superfluous being a twofold repetition of the six second-order equations for τ (equation (2.6) below). We list the equations in groups according to the order in which they were subsequently analysed:

$$\begin{aligned} \frac{\partial^2 \tau}{\partial x^2} = 0 \quad \frac{\partial^2 \tau}{\partial x \partial y} = 0 \quad \frac{\partial^2 \tau}{\partial y^2} = 0 \\ \frac{\partial^2 \tau}{\partial y \partial z} = 0 \quad \frac{\partial^2 \tau}{\partial z^2} = 0 \quad \frac{\partial^2 \tau}{\partial z \partial x} = 0 \\ \frac{\partial^2 \xi}{\partial x^2} = 2 \frac{\partial^2 \tau}{\partial x \partial t} + f \left(y \frac{\partial \tau}{\partial z} - z \frac{\partial \tau}{\partial y} \right) \quad \frac{\partial^2 \xi}{\partial y^2} = fz \frac{\partial \tau}{\partial y} \end{aligned} \quad (2.6)$$

$$\frac{\partial^2 \xi}{\partial z^2} = -fy \frac{\partial \tau}{\partial z} \quad 2 \frac{\partial^2 \xi}{\partial x \partial y} = 2 \frac{\partial^2 \tau}{\partial y \partial t} + f \left(2z \frac{\partial \tau}{\partial x} - x \frac{\partial \tau}{\partial z} \right) \quad (2.7)$$

$$2 \frac{\partial^2 \xi}{\partial y \partial z} = f \left(z \frac{\partial \tau}{\partial z} - y \frac{\partial \tau}{\partial y} \right) \quad 2 \frac{\partial^2 \xi}{\partial z \partial x} = 2 \frac{\partial^2 \tau}{\partial z \partial t} + f \left(x \frac{\partial \tau}{\partial y} - 2y \frac{\partial \tau}{\partial x} \right)$$

$$\frac{\partial^2 \eta}{\partial x^2} = -fz \frac{\partial \tau}{\partial x} \quad \frac{\partial^2 \eta}{\partial y^2} = 2 \frac{\partial^2 \tau}{\partial y \partial t} + f \left(z \frac{\partial \tau}{\partial x} - x \frac{\partial \tau}{\partial z} \right)$$

$$\frac{\partial^2 \eta}{\partial z^2} = fx \frac{\partial \tau}{\partial z} \quad 2 \frac{\partial^2 \eta}{\partial x \partial y} = 2 \frac{\partial^2 \tau}{\partial x \partial t} + f \left(y \frac{\partial \tau}{\partial z} - 2z \frac{\partial \tau}{\partial y} \right) \quad (2.8)$$

$$2 \frac{\partial^2 \eta}{\partial y \partial z} = 2 \frac{\partial^2 \tau}{\partial z \partial t} + f \left(2x \frac{\partial \tau}{\partial y} - y \frac{\partial \tau}{\partial x} \right) \quad 2 \frac{\partial^2 \eta}{\partial z \partial x} = f \left(x \frac{\partial \tau}{\partial x} - z \frac{\partial \tau}{\partial z} \right)$$

$$\frac{\partial^2 \zeta}{\partial x^2} = fy \frac{\partial \tau}{\partial x} \quad \frac{\partial^2 \zeta}{\partial y^2} = -fx \frac{\partial \tau}{\partial y}$$

$$\frac{\partial^2 \zeta}{\partial z^2} = 2 \frac{\partial^2 \tau}{\partial z \partial t} + f \left(x \frac{\partial \tau}{\partial y} - y \frac{\partial \tau}{\partial x} \right) \quad 2 \frac{\partial^2 \zeta}{\partial x \partial y} = f \left(y \frac{\partial \tau}{\partial y} - x \frac{\partial \tau}{\partial x} \right) \quad (2.9)$$

$$2 \frac{\partial^2 \zeta}{\partial y \partial z} = 2 \frac{\partial^2 \tau}{\partial y \partial t} + f \left(z \frac{\partial \tau}{\partial x} - 2x \frac{\partial \tau}{\partial z} \right) \quad 2 \frac{\partial^2 \zeta}{\partial z \partial x} = 2 \frac{\partial^2 \tau}{\partial x \partial t} + f \left(2y \frac{\partial \tau}{\partial z} - z \frac{\partial \tau}{\partial y} \right)$$

$$2 \frac{\partial^2 \xi}{\partial x \partial t} - \frac{\partial^2 \tau}{\partial t^2} + f \left(y \frac{\partial \xi}{\partial x} + y \frac{\partial \xi}{\partial z} - z \frac{\partial \xi}{\partial y} - z \frac{\partial \eta}{\partial x} \right) + g \left(3x \frac{\partial \tau}{\partial x} + y \frac{\partial \tau}{\partial y} + z \frac{\partial \tau}{\partial z} \right) = 0$$

$$2 \frac{\partial^2 \eta}{\partial y \partial t} - \frac{\partial^2 \tau}{\partial t^2} + f \left(z \frac{\partial \xi}{\partial y} + z \frac{\partial \eta}{\partial x} - x \frac{\partial \eta}{\partial z} - x \frac{\partial \xi}{\partial y} \right) + g \left(x \frac{\partial \tau}{\partial x} + 3y \frac{\partial \tau}{\partial y} + z \frac{\partial \tau}{\partial z} \right) = 0 \quad (2.10)$$

$$2 \frac{\partial^2 \zeta}{\partial z \partial t} - \frac{\partial^2 \tau}{\partial t^2} + f \left(x \frac{\partial \eta}{\partial z} + x \frac{\partial \zeta}{\partial y} - y \frac{\partial \zeta}{\partial x} - y \frac{\partial \xi}{\partial z} \right) + g \left(x \frac{\partial \tau}{\partial x} + y \frac{\partial \tau}{\partial y} + 3z \frac{\partial \tau}{\partial z} \right) = 0$$

$$2 \frac{\partial^2 \xi}{\partial y \partial t} + f \left(y \frac{\partial \zeta}{\partial y} + z \frac{\partial \xi}{\partial x} - x \frac{\partial \xi}{\partial z} - z \frac{\partial \eta}{\partial y} - \zeta - z \frac{\partial \tau}{\partial t} \right) - \frac{zf'}{r} (x\xi + y\eta + z\zeta) + 2xg \frac{\partial \tau}{\partial y} = 0$$

$$2 \frac{\partial^2 \xi}{\partial z \partial t} + f \left(y \frac{\partial \zeta}{\partial z} + x \frac{\partial \xi}{\partial y} - y \frac{\partial \xi}{\partial x} - z \frac{\partial \eta}{\partial z} + \eta + y \frac{\partial \tau}{\partial t} \right) + \frac{yf'}{r} (x\xi + y\eta + z\zeta) + 2xg \frac{\partial \tau}{\partial z} = 0$$

$$2 \frac{\partial^2 \eta}{\partial x \partial t} + f \left(z \frac{\partial \xi}{\partial x} + y \frac{\partial \eta}{\partial z} - z \frac{\partial \eta}{\partial y} - x \frac{\partial \zeta}{\partial x} + \zeta + z \frac{\partial \tau}{\partial t} \right) + \frac{zf'}{r} (x\xi + y\eta + z\zeta) + 2yg \frac{\partial \tau}{\partial x} = 0$$

$$2 \frac{\partial^2 \eta}{\partial z \partial t} + f \left(z \frac{\partial \xi}{\partial z} + x \frac{\partial \eta}{\partial y} - y \frac{\partial \eta}{\partial x} - x \frac{\partial \zeta}{\partial z} - \xi - x \frac{\partial \tau}{\partial t} \right) - \frac{xf'}{r} (x\xi + y\eta + z\zeta) + 2yg \frac{\partial \tau}{\partial z} = 0$$

$$2 \frac{\partial^2 \zeta}{\partial x \partial t} + f \left(x \frac{\partial \eta}{\partial x} + y \frac{\partial \zeta}{\partial z} - z \frac{\partial \zeta}{\partial y} - y \frac{\partial \xi}{\partial x} - \eta - y \frac{\partial \tau}{\partial t} \right) - \frac{yf'}{r} (x\xi + y\eta + z\zeta) + 2zg \frac{\partial \tau}{\partial x} = 0$$

$$2 \frac{\partial^2 \zeta}{\partial y \partial t} + f \left(x \frac{\partial \eta}{\partial y} + z \frac{\partial \zeta}{\partial x} - x \frac{\partial \zeta}{\partial z} - y \frac{\partial \xi}{\partial y} + \xi + x \frac{\partial \tau}{\partial t} \right) + \frac{xf'}{r} (x\xi + y\eta + z\zeta) + 2zg \frac{\partial \tau}{\partial y} = 0$$

$$\frac{\partial^2 \xi}{\partial t^2} + f \left(y \frac{\partial \xi}{\partial t} - z \frac{\partial \eta}{\partial t} \right) + g \left(\xi + 2x \frac{\partial \tau}{\partial t} - x \frac{\partial \xi}{\partial x} - y \frac{\partial \xi}{\partial y} - z \frac{\partial \xi}{\partial z} \right) + \frac{xg'}{r} (x\xi + y\eta + z\zeta) = 0$$

(2.11)

$$\frac{\partial^2 \eta}{\partial t^2} + f \left(z \frac{\partial \xi}{\partial t} - x \frac{\partial \zeta}{\partial t} \right) + g \left(\eta + 2y \frac{\partial \tau}{\partial t} - x \frac{\partial \eta}{\partial x} - y \frac{\partial \eta}{\partial y} - z \frac{\partial \eta}{\partial z} \right) + \frac{y g'}{r} (x\xi + y\eta + z\zeta) = 0 \quad (2.12)$$

$$\frac{\partial^2 \zeta}{\partial t^2} + f \left(x \frac{\partial \eta}{\partial t} - y \frac{\partial \xi}{\partial t} \right) + g \left(\zeta + 2z \frac{\partial \tau}{\partial t} - x \frac{\partial \zeta}{\partial x} - y \frac{\partial \zeta}{\partial y} - z \frac{\partial \zeta}{\partial z} \right) + \frac{z g'}{r} (x\xi + y\eta + z\zeta) = 0$$

where ' denotes differentiation with respect to r .

The solution of equations (2.6)–(2.12) was performed block by block. Although the calculations are lengthy, they are routine and we merely quote the results. For general f and g the equation

$$\dot{r} + f(r)L + g(r)r = 0 \quad (2.13)$$

obviously has the algebra $\mathfrak{a}_1 \oplus \mathfrak{so}(3)$ with \mathfrak{a}_1 representing invariance under time translation and $\mathfrak{so}(3)$ the usual rotational invariance. Specifically, the generators of $\mathfrak{so}(3)$ are

$$G_1 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \quad G_2 = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \quad G_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \quad (2.14)$$

and the generator of \mathfrak{a}_1 is

$$G_4 = \frac{\partial}{\partial t} \quad (2.15)$$

Special cases of (2.13) with additional symmetry are

$$\ddot{r} + \frac{\mu r}{r^{4A/(K+A)}} = 0 \quad (2.16)$$

which has the algebra $\mathfrak{a}_2 \oplus \mathfrak{so}(3)$, the additional symmetry being

$$G_5 = 2At \frac{\partial}{\partial t} + (A+K) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \quad (2.17)$$

$$\ddot{r} + \frac{\lambda L}{r^3} + \frac{\mu r}{r^4} = 0 \quad (2.18)$$

which has the algebra $\mathfrak{sl}(2, R) \oplus \mathfrak{so}(3)$ with

$$G_6 = t^2 \frac{\partial}{\partial t} + t \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \quad (2.19)$$

and

$$\ddot{r} + \frac{\lambda L}{r^3} + \left(\frac{\mu}{r^4} - \varepsilon \right) r = 0 \quad (2.20)$$

which has the same algebra, but now

$$G_5 = e^{2t\sqrt{\varepsilon}} \left(\frac{1}{\sqrt{\varepsilon}} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \quad (2.21)$$

$$G_6 = e^{-2t\sqrt{\varepsilon}} \left(\frac{1}{\sqrt{\varepsilon}} \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \right). \quad (2.22)$$

(For $\varepsilon < 0$, G_5 and G_6 can be replaced by \bar{G}_5 and \bar{G}_6 which would be expressed in terms of sine and cosine functions.)

The additional symmetry (2.17) for the power-law central force represents invariance under the self-similar transformation

$$t = \alpha^{2A} \bar{t} \quad r = \alpha^{K+A} \bar{r} \quad (2.23)$$

and is lost if the central force is not a power law.

The equations of motion (2.18) and (2.20) both possess the algebra $\mathfrak{sl}(2, \mathcal{R}) \oplus \mathfrak{so}(3)$ regardless of the values of the parameters. They may be regarded as direct extensions of the results of Moreira *et al* (1985) for the algebra of the charge monopole problem.

The term $\mu r r^{-4}$ can be interpreted as a centripetal force and the term ϵr represents a harmonic repulsor or oscillator depending upon the sign of ϵ or a 'free' particle if $\epsilon = 0$. The latter term does not affect the algebra nor the integrability of the equation. Indeed the same conserved vector exists as for (1.5), namely

$$\mathbf{P} = \mathbf{L} - \frac{\lambda \mathbf{r}}{r} \quad (2.24)$$

and L is also constant in both cases. We note that the motion continues to be on the surface of a cone. Another scalar integral

$$I = \frac{1}{2}(\dot{r}^2 - \mu/r^2 - \epsilon r^2) \quad (2.25)$$

also exists. Equation (2.20) does not belong to the class of problems (1.13) treated by Leach and Gorringe (1988). However, (2.18) does in the particular case that $\mu = -\lambda^2$. Then, in addition to the three integrals above, there is also the conserved vector

$$\mathbf{J} = \mathbf{L} \times \dot{\mathbf{r}} - \frac{\lambda \mathbf{L}}{r} = \mathbf{P} \times \dot{\mathbf{r}} \quad (2.26)$$

from which the orbit equation is just as easily obtained by taking the scalar product with \mathbf{r} as was the case with (2.24).

3. Connection between Laplace-Runge-Lenz-like vectors and symmetries

We have seen that, when in (2.13) $f(r) \equiv 0$ and $g(r)$ is a power law, the algebra of the symmetries is $\mathfrak{a}_2 \oplus \mathfrak{so}(3)$ no matter the degree of the power law. However, when $g(r)$ is not a power law, G_5 is lost. Now, in the case of the Kepler problem G_5 is usually associated with the existence of the Laplace-Runge-Lenz vector of that problem (Lévy-Leblond 1971). However, Leach (1981) demonstrated that the vector could be obtained from G_4 , which represents invariance under time translation, a property of any autonomous first integral. As Fradkin (1965) has demonstrated, any central force law will have a Laplace-Runge-Lenz-like vector and so it is evident that the presence of G_5 is not essential for its existence although it may well help in the determination of an explicit expression for the vector.

In this paper, because we were interested in equations of the types (1.1), (1.5) and (1.13), we have not considered the general planar problem for which Yoshida (1989) demonstrated the existence of a Laplace-Runge-Lenz-like vector. However, elsewhere (Gorringe and Leach 1987) we showed that an explicit expression for a Laplace-Runge-Lenz-like vector for the autonomous equation of motion

$$\ddot{\mathbf{r}} + g\hat{\mathbf{r}} + h\hat{\boldsymbol{\theta}} = \mathbf{0} \quad (3.1)$$

could be found by Collinson's method (Collinson 1973) only when

$$g(r, \theta) = \frac{U''(\theta) + U(\theta)}{r^2} + \frac{2V'(\theta)}{r^{3/2}} \quad (3.2)$$

$$h(r, \theta) = \frac{V(\theta)}{r^{3/2}} \quad (3.3)$$

where U and V are arbitrary functions of θ . Excluding the cases for which the force is constant or $h \equiv 0$ and g a power-law potential, one finds that (3.1) possesses one symmetry in addition to the obvious one of the generator of time translations only when

$$g = (K \sin 2\theta + L \cos 2\theta) W(w_1) G + X(w_1) G \quad (3.4)$$

$$h(r, \theta) = G^{-1} W(w_1) \quad (3.5)$$

where

$$G = (F + K \cos 2\theta - L \sin 2\theta)^{-1/2} \exp\left((E - 2A_1) \int \frac{d\theta}{F + K \cos 2\theta - L \sin 2\theta}\right) \quad (3.6)$$

$$w_1 = r^2 (F + K \cos 2\theta - L \sin 2\theta) \exp\left(-2E \int \frac{d\theta}{F + K \cos 2\theta - L \sin 2\theta}\right) \quad (3.7)$$

and W and X are arbitrary functions of w_1 and A_1 , F , K and L are arbitrary constants. Even when the r dependence in (3.4) is put in the form of (3.2) and (3.3), which is possible, the freedom of choice of $U(\theta)$ and $V(\theta)$ is restricted. (The method of calculation follows that outlined in section 2.)

The point which may be inferred from the two previous paragraphs is that the number of symmetries for a planar motion is not related to the existence of a conserved vector of Laplace-Runge-Lenz type at all. However, in terms of the construction of an explicit expression for the vector, the existence of symmetries provides a method. A function $I(r, \theta, \dot{r}, \dot{\theta}, t)$ is invariant under the action of a symmetry G if $G^{[1]}I = 0$. In principle we can find I as a function of four independent characteristics. The requirement that I be a first integral, i.e. $\dot{I} = 0$, reduces this number to three independent characteristics each of which is a first integral. It is in this sense that the existence of symmetries is of value. It is not in the demonstration of existence but in the process of construction of explicit expressions for the first integrals.

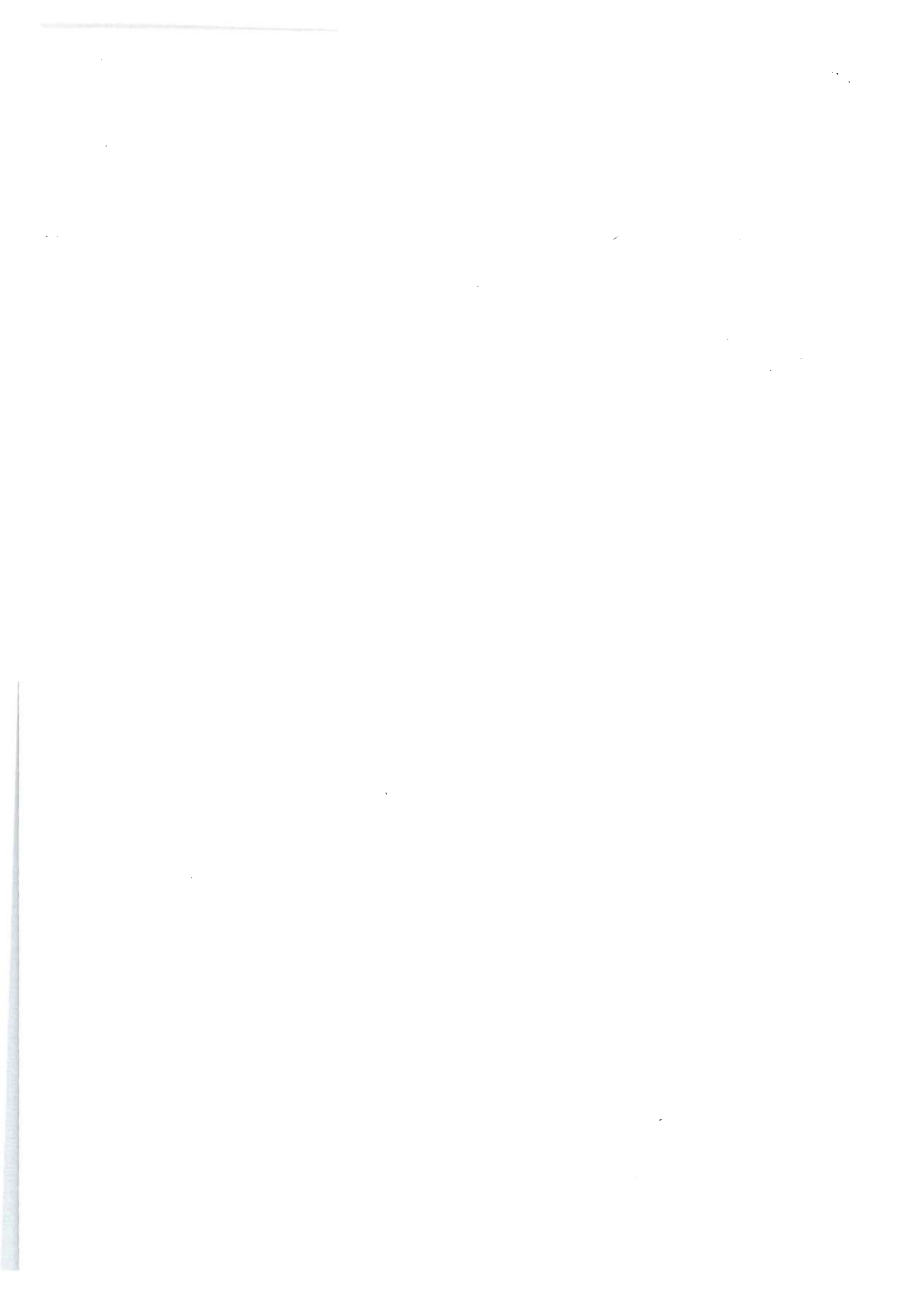
Finally we remark that one of the main results of this paper has been the determination of the Lie algebras of the Lie point symmetries admitted by (1.19). We have seen that the Kepler problem shares the same algebra with any power-law potential. Moving away from planar motions, the algebra of the symmetries of (2.18) and (2.19) is independent of the values of the parameters λ , μ and ε (excepting that λ and μ may not both be zero). By way of contrast, McIntosh and Cisneros (1970) in their study of the monopole-Kepler and monopole-oscillator problems take $\mu = -\lambda^2$ to ensure closed orbits. This choice had already been made for the former problem by Zwanziger (1968) so that a Laplace-Runge-Lenz-like vector could be explicitly constructed. (The reason for the case of construction of the vector becomes obvious from the more general work on the construction of Laplace-Runge-Lenz-like vectors for (1.19) by Leach and Goringe (1988).) It would be interesting to see whether the relaxation of this condition, which is not required for the symmetries of the differential equation (the source of a construction base), leads to significant results for the monopole-oscillator problem.

Acknowledgments

We thank Barbara Abraham Shrauner for drawing our attention to the paper of Moreira *et al* (1985). VMG is funded by the Foundation for Research Development of South Africa and the University of the Witwatersrand, Johannesburg. It is a pleasure to thank the referees and adjudicator for many helpful comments and suggestions.

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Generalized Ermakov systems

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Physics Letters A **158** (1991) 102-106

Generalized Ermakov systems

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Received 9 April 1991; revised manuscript received 6 June 1991; accepted for publication 2 July 1991

Communicated by A.P. Fordy

The nature of generalized Ermakov systems is explained in terms of the symmetry algebra $sl(2, \mathbb{R})$. The concept of weak generalized Ermakov systems is introduced.

In 1880 Ermakov [1] obtained a first integral for the time-dependent harmonic oscillator,

$$\ddot{q} + \omega^2(t)q = 0, \quad (1)$$

by introducing an auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}, \quad (2)$$

eliminating the ω^2 between (1) and (2) and multiplying by the integrating factor $\rho\dot{q} - \dot{\rho}q$. It is

$$I = \frac{1}{2} [(\rho\dot{q} - \dot{\rho}q)^2 + (q/\rho)^2], \quad (3)$$

which is usually called the Lewis invariant, following Lewis' rediscovery of it in 1966 [2] using the asymptotic method of Kruskal [3].

Ray and Reid [4] popularized Ermakov systems in the western literature and introduced generalized Ermakov systems of the form

$$\ddot{x} + \omega^2(t)x = \frac{1}{x^3} f(y/x), \quad (4)$$

$$\ddot{y} + \omega^2(t)y = \frac{1}{y^3} g(y/x), \quad (5)$$

where f and g are arbitrary functions of their arguments. A first integral for the system (4) and (5) is obtained by eliminating ω^2 between the two and multiplying by the integrating factor $x\dot{y} - \dot{x}y$. It is

$$I = \frac{1}{2} (x\dot{y} - \dot{x}y)^2 + \int^{y/x} [uf(u) - u^{-3}g(u)] du. \quad (6)$$

In fact the presence of the ω^2 terms in (4) and (5) is a bit misleading in two respects. In the first place the transformation

$$T = \cot\left(\int \rho^{-2} dt\right),$$

$$X = \rho^{-1}x \csc T, \quad Y = \rho^{-1}y \csc T, \quad (7)$$

where

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}, \quad (8)$$

transforms (4) and (5) to

$$\ddot{X} = \frac{1}{X^3} f(Y/X), \quad (4')$$

$$\ddot{Y} = \frac{1}{Y^3} g(Y/X). \quad (5')$$

(Eq. (8), called the Pinney equation [5], is endemic in studies of time-dependent systems.) In the second place the ω^2 can be replaced by *anything* (a point already noted by Ray [6], Ray and Reid [7] and Goedert [8]) and the terms can still be eliminated to give

$$x\dot{y} - \dot{x}y = \frac{x}{y^3} g(y/x) - \frac{y}{x^3} f(y/x). \quad (9)$$

There have been many studies of Ermakov invariants in two dimensions. In particular we mention

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the generally quantum mechanically oriented works of Wollenberg [9] and Ray [10], extensions to velocity-dependent potentials [11] and time-dependent potentials [12]. More recent contributions are by Athorne et al. [13] and Athorne [14] who in the latter paper was concerned with particular functional forms of $f(y/z)$ and $g(y/x)$ in (4) and (5).

In the context of the usual usage of the phrase, generalized Ermakov system, (9) is really the equation of interest. We examine (9) for the existence of Lie point symmetries. To our knowledge this is the first time that the Lie point symmetries of the differential equation (9) have been studied. However, we note that Korsch employed the dynamical group $SO(2,1)$ in a study of the Hamiltonian form of (1) [15]. Recall that a system of ordinary differential equations

$$N(x, \dot{x}, \ddot{x}, t) = 0 \tag{10}$$

possesses a symmetry

$$G = \zeta(x, t) \frac{\partial}{\partial t} + \eta_i(x, t) \frac{\partial}{\partial x_i}, \tag{11}$$

if

$$G^{[2]}N|_{N=0} = 0, \tag{12}$$

where

$$G^{[2]} = G + (\dot{\eta}_i - \dot{x}_i \dot{\zeta}) \frac{\partial}{\partial \dot{x}_i} + (\ddot{\eta}_i - 2\dot{x}_i \ddot{\zeta} - \dot{x}_i \dot{\zeta}) \frac{\partial}{\partial \ddot{x}_i} \tag{13}$$

is the second extension of G . The usual routine calculation leads to there being three symmetries for (9), viz.

$$G_1 = \frac{\partial}{\partial t}, \quad G_2 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

$$G_3 = t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} + ty \frac{\partial}{\partial y}, \tag{14}$$

with commutation relations

$$[G_1, G_2] = 2G_1, \quad [G_1, G_3] = G_2, \quad [G_2, G_3] = 2G_3, \tag{15}$$

which is the algebra $A_{3,8}$, better known as $sl(2, \mathbb{R})$.

These same symmetries apply to (4') and (5') separately and we confine our attention to systems

of this type. Since the generalized Ermakov system (4') and (5') has the first integral (6), which is of angular momentum type, it makes sense to go over to plane polar coordinates (r, θ) . The system (4') and (5') is

$$\ddot{r} - r\dot{\theta}^2 = \frac{1}{r^3} [\sec^2 \theta f(\tan \theta) + \csc^2 \theta g(\tan \theta)], \tag{4''}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r^3} [-\sec^2 \theta \tan \theta f(\tan \theta) + \csc^2 \theta \cot \theta g(\tan \theta)]. \tag{5''}$$

The force is derivable from a potential if

$$\tan^2 \theta f'(\tan \theta) + g'(\tan \theta) = 0 \tag{16}$$

and the potential is

$$V = \frac{1}{2r^2} [\sec^2 \theta f(\tan \theta) + \csc^2 \theta g(\tan \theta)]. \tag{17}$$

The first integral (6) is

$$I = \frac{1}{2} (r^2 \dot{\theta})^2 + \int_{\tan \theta}^{\tan \theta} [uf(u) - u^{-3}g(u)] du$$

$$= \frac{1}{2} (r^2 \dot{\theta})^2 + \frac{1}{2} [u^2 f(u) + u^{-2} g(u)]|_{u=\tan \theta}$$

$$- \frac{1}{2} \int_{\tan \theta}^{\tan \theta} [u^2 f'(u) + u^{-2} g'(u)] du \tag{18}$$

and, when the force is derivable from a potential, we can use (16) to obtain

$$I = \frac{1}{2} (r^2 \dot{\theta})^2 + \frac{1}{2} [\sec^2 \theta f(\tan \theta) + \csc^2 \theta g(\tan \theta)]. \tag{18'}$$

If we take (4'') and (5'') to represent the equations of motion of a particle of mass one and (16) applies, the Hamiltonian of the system is

$$H = \frac{1}{2} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + V = \frac{1}{2} \left(p_r^2 + \frac{I}{r^2} \right), \tag{19}$$

so that the system is immediately integrable (cf. ref. [16]). It is evident that generalized Ermakov systems are really Cartesian forms of a system of equations which have the polar form

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(\theta)}{r^3}, \tag{20}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{G(\theta)}{r^3}, \quad (21)$$

with the symmetries

$$G_1 = \frac{\partial}{\partial t}, \quad G_2 = 2t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r},$$

$$G_3 = t^2 \frac{\partial}{\partial t} + tr \frac{\partial}{\partial r}, \quad (22)$$

which represent time-translation, self-similar and conformal transformations respectively. There is a Hamiltonian if

$$G = -\frac{1}{2}F'. \quad (23)$$

The Ermakov invariant comes from the angular component of the equation of motion (21) and is

$$I = \frac{1}{2}(r^2\dot{\theta})^2 - \int G(\theta) d\theta \quad (24)$$

$$= \frac{1}{2}[p_\theta^2 + F(\theta)], \quad (24')$$

if the system is Hamiltonian.

As we saw above, the Ermakov invariant in the two-dimensional case comes from the angular component of the equation of motion. This provides the clue for generalization to higher dimensions which is straightforward if polar coordinates are used. This is not the only way to look for generalization to higher dimensions. Lutzky [17] increased the number of Cartesian equations, but kept the single auxiliary equation (2). (See also ref. [18].) The criterion is invariance under the second extension of the symmetries in (22). In three dimensions we have that the system

$$\ddot{r} - r\dot{\theta}^2 - r \sin^2\theta \dot{\phi}^2 = \frac{F(\theta, \phi)}{r^3}, \quad (25)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta \cos\theta = \frac{G(\theta, \phi)}{r^3}, \quad (26)$$

$$r \sin\theta \ddot{\phi} + 2\dot{r}\dot{\phi} \sin\theta + 2r\dot{\theta}\dot{\phi} \cos\theta = \frac{H(\theta, \phi)}{r^3 \sin\theta}, \quad (27)$$

possesses the $sl(2, \mathbb{R})$ symmetry of (22). The system has a Hamiltonian provided

$$G(\theta, \phi) = -\frac{1}{2} \frac{\partial F}{\partial \theta}, \quad H(\theta, \phi) = -\frac{1}{2} \frac{\partial F}{\partial \phi}, \quad (28)$$

with a potential given by

$$V = \frac{1}{2r^2} F(\theta, \phi). \quad (29)$$

Multiplication of (27) by $r^3 \sin^3\theta \dot{\phi}$ gives

$$\frac{d}{dt} \frac{1}{2} (r^2 \sin^2\theta \dot{\phi})^2 = H(\theta, \phi) \sin^2\theta \dot{\phi}, \quad (30)$$

so that, if

$$H(\theta, \phi) = -\frac{\frac{1}{2}U'(\phi)}{\sin^2\theta}, \quad (31)$$

(27) has the first integral

$$J = \frac{1}{2} (r^2 \sin^2\theta \dot{\phi})^2 + \frac{1}{2} U(\phi), \quad (32)$$

which is really just the type of Ermakov invariant met in the two-dimensional case. Multiplication of (26) by $r^3\dot{\theta}$ gives, when (30) is taken into account,

$$\frac{1}{2} \frac{d}{dt} \left((r^2\dot{\theta})^2 + \frac{(r^2 \sin^2\theta \dot{\phi})^2}{\sin^2\theta} \right) = \dot{\theta}G + \dot{\phi}H. \quad (33)$$

A first integral

$$I = \frac{1}{2} \left((r^2\dot{\theta})^2 + \frac{(r^2 \sin^2\theta \dot{\phi})^2}{\sin^2\theta} + W(\theta, \phi) \right) \quad (34)$$

exists provided

$$G = -\frac{1}{2} \frac{\partial W}{\partial \theta}, \quad H = -\frac{1}{2} \frac{\partial W}{\partial \phi}, \quad (35)$$

and this is the Ermakov invariant appropriate to three dimensions. Although we have used manipulation of the equations of motion (25)–(27) to obtain I and J , they can be recovered by using the standard method based on symmetries. This is given in the appendix. Note that, if $W=F$, the system is Hamiltonian. The requirement that both I and J exist leads to the form of the Hamiltonian given by Landau and Lifschitz [16], but this is not the case if only I is required to exist as then (35) is a less stringent condition. The generalization to higher dimensions is obvious.

The existence of an Ermakov invariant has been based on the invariance of (9) under the representation (14) of the algebra $sl(2, \mathbb{R})$. It is appropriate to introduce the concepts of generalized Ermakov systems and of weak generalized Ermakov systems. We define a generalized Ermakov system as one in which the original equations of motion (such as (4'))

and (5')) are invariant under action of the $sl(2, \mathbb{R})$ representation (14) and a weak generalized Ermakov system as one in which (9) holds, but the original equations of motion are not invariant under the action of $sl(2, \mathbb{R})$. The notion is easily extended to higher dimensions. Thus the three-dimensional system

$$\ddot{r} - r\dot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2 = S(t, r, \theta, \phi), \quad (36a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta = -\frac{1}{2r^3} \frac{\partial W(\theta, \phi)}{\partial \theta}, \quad (36b)$$

$$r \sin \theta \ddot{\phi} + 2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\theta}\dot{\phi} \cos \theta = -\frac{1}{2r^3 \sin \theta} \frac{\partial W(\theta, \phi)}{\partial \phi}, \quad (36c)$$

where S can be any function of r, θ, ϕ and t , need have no symmetries at all, but it does possess an Ermakov invariant (both if (31) holds) and so it is of the weak generalized Ermakov class. The reason for this is that eqs. (36b,c) do have $sl(2, \mathbb{R})$ symmetry and it is from these equations that the invariant(s) is (are) found.

Finally we note that for dimensions greater than two it is possible for a generalized Ermakov system to have higher symmetry. If in (36) $r^3 S(t, r, \theta, \phi)$ and W depend only on θ , there is the additional symmetry $so(2)$ and the full symmetry algebra is $so(2) \oplus sl(2, \mathbb{R})$.

Appendix. Symmetry-based derivation of I and J

We note that the Ermakov invariant (24') is invariant under the three symmetries (22). (This is not as unusual as it may seem. See ref. [19].) This suggests that the simplest symmetry $G_1 = \partial/\partial t$ be used in the derivation of the invariants for the three-dimensional generalization. We recall the standard symmetry-based method for determining first integrals. We use the symmetry G to determine the characteristics invariant under the first extension of G and write the invariant as a function of these characteristics. The requirement that the total time derivative be zero leads to a second linear partial differential equation the characteristics of which are first

integrals of the original system of differential equations.

Given that the variables are $t, r, \theta, \phi, \dot{r}, \dot{\theta}$ and $\dot{\phi}$, invariance under $G_1 = \partial/\partial t$ gives the last six as characteristics and the integral is $F(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi})$. The associated Lagrange system [20] is

$$\begin{aligned} \frac{dr}{\dot{r}} &= \frac{d\theta}{\dot{\theta}} = \frac{d\phi}{\dot{\phi}} = \frac{d\dot{r}}{F/r^3 + r\dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2} \\ &= \frac{d\dot{\theta}}{G/r^4 - 2\dot{r}\dot{\theta}/r + \dot{\phi}^2 \sin \theta \cos \theta} \\ &= \frac{d\dot{\phi}}{H/(r^4 \sin^2 \theta) - 2\dot{r}\dot{\phi}/r - 2\dot{\theta}\dot{\phi} \cot \theta}. \end{aligned} \quad (A.1)$$

The general idea is to manipulate the equations in (A.1) until a denominator is zero and the numerator is an exact differential of a function. This function is then a first integral. To avoid writing down lengthy equations we refer to the members of (A.1) as (A.1_i)–(A.1_{vi}) respectively and just state the manipulations involved.

To obtain J (31) we proceed as follows. In the denominator of (A.1_{vi}) the term $-2\dot{r}\dot{\phi}/r$ is eliminated by the addition of $(2\dot{\phi}/r)$ (A.1_i) and $-2\dot{\theta}\dot{\phi} \cot \theta$ by the addition of $(2\dot{\theta} \cot \theta)$ (A.1_{ii}). This just leaves the H term. If (A.1_{vi}) is multiplied numerator and denominator by $\dot{\phi}$, the H term is removed by the subtraction of $[H/(r^4 \sin^2 \theta)]$ (A.1_{iii}). The denominator of (A.1_{vi}) is now zero. The numerator is

$$\dot{\phi} \left(d\dot{\phi} + \frac{2\dot{\phi}}{r} dr + 2\dot{\theta} \cot \theta d\theta \right) - \frac{H}{r^4 \sin^2 \theta} d\phi. \quad (A.2)$$

Multiplication of this by $r^4 \sin^4 \theta$ (numerator and zero denominator) gives

$$d\left(\frac{1}{2} r^4 \sin^4 \theta \dot{\phi}^2\right) - H \sin^2 \theta d\phi, \quad (A.3)$$

which is an exact differential if (31) is satisfied and so we have J .

To find I we manipulate (A.1_{vi}) to obtain (A.2). To remove $-2\dot{r}\dot{\theta}/r$ and $\dot{\phi}^2 \sin \theta \cos \theta$ from (A.1_v) we add $(2\dot{\theta}/r)$ (A.1_i) and subtract $(\dot{\phi} \sin \theta \cos \theta)$ (A.1_{iii}). The G term is removed by multiplying numerator and denominator by $\dot{\theta}$ and subtracting $(A.1_{ii})/r^4$ which makes the denominator zero and the numerator

$$\dot{\theta} \left(d\dot{\theta} + \frac{2\dot{\theta}}{r} dr - \dot{\phi} \sin \theta \cos \theta d\phi \right) - \frac{G}{r^4} d\theta. \quad (A.4)$$

The combination $(r^4 \sin^2 \theta)$ (A.2) + (r^4) (A.4) yields

$$d\left(\frac{1}{2}r^4\dot{\theta}^2 + \frac{1}{2}r^4 \sin^2\theta \dot{\phi}^2\right) - G d\theta - H d\phi, \quad (\text{A.5})$$

which is an exact differential if (35) is satisfied and so we have I .

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First integrals associated with the additional symmetry of central force problems with power law potentials

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Quæstiones Mathematicæ 14 (1991) 277–289

FIRST INTEGRALS ASSOCIATED WITH THE
ADDITIONAL SYMMETRY OF CENTRAL FORCE
PROBLEMS WITH POWER LAW POTENTIALS

V.M. GORRINGE AND P.G.L. LEACH

ABSTRACT. It has recently been shown by Leach and Gorrynge that the equation of motion $\ddot{r} + \mu r^{-\frac{2}{\alpha}} = 0$ is invariant under the action of the symmetry $G = t \partial/\partial t + \alpha r \partial/\partial r$ in addition to the usual time and rotation symmetries in contrast to the occurrence of two extra symmetries for the charge monopole problem (Moreira et al, J Phys, 18 (1985) L427). The integrals associated with G are calculated and an interesting connection with Hamilton-Jacobi theory demonstrated.

1. Introduction. A second order ordinary differential equation

$$(1.1) \quad \ddot{r} + \mu r^{-\frac{2}{\alpha}} = 0,$$

where overdot denotes differentiation with respect to time, is said to possess a Lie point symmetry

$$(1.2) \quad G := \tau(t, x_i) \frac{\partial}{\partial t} + \eta_j(t, x_i) \frac{\partial}{\partial x_j}$$

Both of us acknowledge the support of the Foundation for Research Development of South Africa and one of us (V M G) the support of the University of the Witwatersrand, Johannesburg through a postgraduate bursary .

if (1.1) is invariant under the infinitesimal transformation generated by the second extension of G

$$(1.3) \quad G^{[2]} := G + (\eta_j - \dot{x}_j \dot{\tau}) \frac{\partial}{\partial \dot{x}_j} + (\ddot{\eta}_j - 2\ddot{x}_j \dot{\tau} - \dot{x}_j \ddot{\tau}) \frac{\partial}{\partial \ddot{x}_j}$$

whenever (1.1) holds.

A function $I(t, x_i, \dot{x}_i)$ is a first integral of (1.1) associated with the symmetry G if it possesses the properties

$$(1.4) \quad G^{[1]} I = 0$$

$$(1.5) \quad \left. \frac{dI}{dt} \right|_{E=0} = 0,$$

where $G^{[1]}$ the first extension of G is given in (1.3) by truncation after the terms involving the differential operators $\partial/\partial \dot{x}_j$. If the dimension of the space is N , I is a function of $2N+1$ variables t, x_i, \dot{x}_i which by virtue of (1.4) are expressible in terms of $2N$ characteristics each of which is invariant under the infinitesimal transformation generated by $G^{[1]}$. By virtue of (1.5) I becomes now a function of $2N-1$ characteristics each of which is a first integral of the original differential equation (1.1).

The relationship between the first integral I and the Lie point symmetry G is one of the primary reasons for the investigation of these symmetries of differential equations. The determination of first integrals is the major part of determining integrable systems. Of course the method of extended Lie groups is not the only way to determine the first integrals of a system. One can use the "direct" method, Noether's theorem or Hamilton-Jacobi theory. One's experience with all methods is that in a particular application to a difficult problem any method other than the one being used is more appropriate.

The purpose of this paper is to investigate the first integrals associated with the "additional" symmetry of the equation of motion

$$(1.6) \quad \ddot{r} + \mu r^{-\frac{2}{\alpha}} r = 0.$$

We adopt the usual convention $\tilde{r} = (x_1, x_2, x_3)^T = (x, y, z)^T$ and $r = \|\tilde{r}\|_2$. Since, for appropriate values of the constants μ and α , (1.6) includes the free particle and the isotropic simple harmonic oscillator, we insist that μ be non-zero and α finite. Although both problems fit into the structure of (1.6), the number of their symmetries is much higher and their general properties are sufficiently well-known in the literature to warrant any further discussion. In the case $\alpha \neq 1$, (1.6) corresponds to the power law potential Hamiltonian

$$(1.7) \quad H = \frac{1}{2} \tilde{p} \cdot \tilde{p} + \frac{\alpha\mu}{2\alpha - 2} r^{2-\frac{2}{\alpha}}$$

and, when $\alpha = 1$, to the logarithmic potential Hamiltonian

$$(1.8) \quad H = \frac{1}{2} \tilde{p} \cdot \tilde{p} + \mu \log r,$$

where the momentum canonically conjugate to \tilde{r} is $\tilde{p} = \dot{\tilde{r}}$. (This can be interpreted either as taking the mass of the particle to be unity or as rescaling to make it unity.) Although (1.7) and (1.8) have different structures, there is no essential difference in the results obtained and we will consider (1.7) as the generic case and simply quote the relevant results for (1.8) at the appropriate sections in the text below.

At the beginning of the previous paragraph we mentioned the "additional" symmetry of (1.6). Let us explain what we mean. Recently it has been shown [Leach and Goringe 1988] that (1.6) has the five symmetries

$$\begin{aligned}
 G_1 &= \frac{\partial}{\partial t} \\
 G_2 &= y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \\
 G_3 &= x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \\
 G_4 &= z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \\
 (1.9) \quad G_5 &= t \frac{\partial}{\partial t} + \alpha \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)
 \end{aligned}$$

with the non-zero commutation relations

$$\begin{aligned}
 (1.10) \quad [G_2, G_3] &= G_4, [G_3, G_4] = G_2, [G_4, G_2] = G_3 \\
 [G_1, G_5] &= G_1
 \end{aligned}$$

These commutation relations show that the Lie algebra of the symmetries is the direct sum $a_2 \oplus so(3)$. The additional symmetry is G_5 since for an autonomous general central force law only G_1 through G_4 are found and the algebra is $a_1 \oplus so(3)$.

It is usual to identify G_1 with the conservation of energy and G_2 through G_4 with the separate conservation of the three components of angular momentum. However, in the case of the Kepler problem (for which $\alpha = \frac{2}{3}$) it was shown [Leach 1981] that the conserved energy, angular momentum and Laplace-Runge-Lenz vector could all be found from G_1 although the last is normally associated with G_5 [Lévy-Leblond 1971, Prince and Eliezer 1981, Leach 1981]. The question naturally arises: What other integrals are associated with G_5 for the Kepler problem, $\alpha = \frac{2}{3}$, and for general α , what integrals?

2. First integrals associated with G_5 . As we know that the motion described by (1.6) is planar, there is no loss of generality in reducing the number of spatial dimensions to two. From the discussion following eqns(1.4,5) it is evident that we lose two integrals, but, as the reduction of dimension from three to two reduces the number of non-trivially

conserved components of angular momentum from three to one, this is no real loss. We can write G_5 in two ways as

$$(2.1) \quad G_5 = t \frac{\partial}{\partial t} + \alpha \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)$$

and

$$(2.2) \quad G_5 = t \frac{\partial}{\partial t} + \alpha r \frac{\partial}{\partial r}.$$

For the time being we shall use the second form although the first does have a useful function to which we shall refer below. The first extension of (2.2) is

$$(2.3) \quad G_5^{[1]} = t \frac{\partial}{\partial t} + \alpha r \frac{\partial}{\partial r} + 0 \frac{\partial}{\partial \theta} + (\alpha - 1) \dot{r} \frac{\partial}{\partial \dot{r}} - \dot{\theta} \frac{\partial}{\partial \dot{\theta}}$$

in which we include $\partial/\partial\theta$ with a zero coefficient to remind us that all variables must be present. The characteristics of (1.4) are found from the associated Lagrange's system

$$(2.4) \quad \frac{dt}{t} = \frac{dr}{\alpha r} = \frac{d\theta}{0} = \frac{d\dot{r}}{(\alpha - 1)\dot{r}} = \frac{d\dot{\theta}}{-\dot{\theta}}$$

and are

$$(2.5) \quad \begin{array}{ll} u_1 = rt^{-\alpha} & v_1 = \dot{r}t^{1-\alpha} \\ u_2 = \theta & v_2 = \dot{\theta}t. \end{array}$$

The radial and angular components of the equation of motion (1.6) are

$$(2.6) \quad \ddot{r} = r\dot{\theta}^2 - \mu r^{1-\frac{1}{\alpha}} \quad \ddot{\theta} = -2\dot{r}\dot{\theta}/r$$

so that

$$(2.7) \quad \begin{aligned} \dot{u}_1 &= t^{-1}(v_1 - \alpha u_1) & \dot{v}_1 &= t^{-1} \left[(1 - \alpha)v_1 + u_1 v_2^2 - \mu u_1^{1-\frac{2}{\alpha}} \right] \\ \dot{u}_2 &= t^{-1} v_2 & \dot{v}_2 &= t^{-1} [v_2 - 2v_1 v_2 / u_1]. \end{aligned}$$

The characteristics of (1.5) are found from the associated Lagrange's system

$$(2.8) \quad \frac{du_1}{v_1 - \alpha u_1} = \frac{du_2}{v_2} = \frac{dv_1}{(1 - \alpha)v_1 + u_1 v_2^2 - \mu u_1^{1-\frac{2}{\alpha}}} = \frac{dv_2}{v_2 - 2v_1 v_2 / u_1} = \frac{dt}{t}$$

to which dt/t has been added in the sense of an auxiliary variable [Ince 1956]. To determine the characteristics of (2.8) we must take rather more complex combinations than in the case of (2.4). To facilitate description of the combinations we label the i th member of (2.8) as (2.8.i, $i = 1, 5$).

After some simplification the combination $\left(u_1 v_2^2 + \mu u_1^{1-\frac{2}{\alpha}} \right)$ (2.8.1) $+ v_1$ (2.8.3) $+ u_1^2 v_2$ (2.8.4) gives

$$(2.9) \quad \frac{d \left[\frac{1}{2} v_1^2 + \frac{1}{2} u_1^2 v_2^2 + \frac{\alpha \mu}{2(\alpha-1)} u_1^{2-\frac{2}{\alpha}} \right]}{2(1-\alpha) \left[\frac{1}{2} v_1^2 + \frac{1}{2} u_1^2 v_2^2 + \frac{\alpha}{2(\alpha-1)} \mu u_1^{2-\frac{2}{\alpha}} \right]} = \frac{dt}{t}$$

which can be immediately integrated to give the characteristic and first integral

$$(2.10a) \quad w_1 = \left[\frac{1}{2} v_1^2 + \frac{1}{2} u_1^2 v_2^2 + \frac{\alpha \mu}{2(\alpha-1)} u_1^{2-\frac{2}{\alpha}} \right] t^{2(\alpha-1)}.$$

In terms of the original variables this is

$$(2.10b) \quad w_1 = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 + \frac{\alpha \mu}{2(\alpha-1)} r^{2-\frac{2}{\alpha}}$$

which is the conserved energy in a more recognisable form. In the case $\alpha = 1$ the ultimate term in (2.10b) is replaced by $\mu \log r$.

The combination $2u_1v_2(2.8.1) + u_1^2(2.8.4)$ reduces to

$$(2.11) \quad \frac{d(u_1^2v_2)}{(1-2\alpha)u_1^2v_2} = \frac{dt}{t}$$

so that the second characteristic is

$$(2.12a) \quad w_2 = u_1^2v_2t^{2\alpha-1}$$

or, in terms of the original variables,

$$(2.12b) \quad w_2 = r^2\dot{\theta}$$

the one component of the conserved angular momentum which we did not make zero when going from three dimensions to two dimensions. (It is easier to use the cartesian form of G_5 to find the components of angular momentum in three dimensions.)

In establishing the conservation of energy and angular momentum using the similarity variables of G_5 (i.e., u_i and v_i) it has been necessary to use the auxiliary variable t in (2.8.5). This is because the denominators in the left hand members of (2.9) and (2.11) have been non-zero. One way to obtain a zero denominator is to take the combination $t(2.8.1) - (v_1 - \alpha u_1)(2.8.5)$ which gives

$$(2.13) \quad \frac{tdu_1 + \alpha u_1 dt - v_1 dt}{0}$$

As the denominator of (2.13) is zero, the numerator is the differential of a characteristic although in its present form it is not obvious what the characteristic is. However, if we multiply (2.13) by $t^{\alpha-1}[v_1 t^{\alpha-1}]^{-1}$, it becomes

$$(2.14) \quad \frac{-dt + [v_1 t^{\alpha-1}]^{-1} d(u_1 t^\alpha)}{0}$$

If we substitute for v_2 in (2.10a) from (2.12a) and rearrange the resulting expression, we have

$$(2.15) \quad (v_1 t^{\alpha-1})^2 = 2w_1 - \frac{w_2^2}{(u_1 t^\alpha)^2} - \frac{\alpha\mu}{\alpha-1} (u_1 t^\alpha)^{2-\frac{2}{\alpha}}$$

so that the characteristic derived from (2.14) is

$$(2.16a) \quad w_3 = -t + \int^{u_1 t^\alpha} \left[2w_1 - \frac{w_2^2}{\eta^2} - \frac{\alpha\mu}{\alpha-1} \eta^{2-\frac{2}{\alpha}} \right]^{-\frac{1}{2}} d\eta$$

or, in terms of the original co-ordinates

$$(2.16b) \quad w_3 = -t + \int^r \left[2w_1 - \frac{w_2^2}{\eta^2} - \frac{\alpha\mu}{\alpha-1} \eta^{2-\frac{2}{\alpha}} \right]^{-\frac{1}{2}} d\eta.$$

(For the logarithmic potential the last term in crochets is replaced by $-2\mu \log \eta$.)

In principle with the determination of the first integral (2.16b) the problem of solving the equation of motion (1.6) is reduced to the final quadrature

$$(2.17) \quad \theta = \int^t \frac{w_2}{r^2} dt'$$

in which integral r is replaced by $r(t)$ from (2.16b). For general α the integral in (2.16b) cannot be evaluated in closed form and in fact is known in terms of elliptic or circular functions for only fourteen values of α [Whittaker 1944; Taff 1985].

3. Discussion. The reader will have observed the connection of (2.16b) with Hamilton's characteristic function of Hamilton-Jacobi theory. For the Hamiltonian (1.7) (the usual substitution will give the result for that of (1.8)) the solution to the Hamilton-Jacobi equation

$$(3.1) \quad \frac{1}{2} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{\alpha \mu}{\alpha - 1} r^{2-\frac{2}{\alpha}} \right] + \frac{\partial S}{\partial t} = 0$$

is, in the notation of this paper,

$$(3.2) \quad S(r, \theta, t) = -w_1 t + \int^r \left[2w_1 - \frac{w_2^2}{\eta^2} - \frac{\alpha \mu}{\alpha - 1} \eta^{2-\frac{2}{\alpha}} \right]^{\frac{1}{2}} d\eta + w_2 \theta$$

so that

$$(3.3) \quad w_3 = \frac{\partial S}{\partial w_1}$$

is a constant as predicted by the theory. However in obtaining (2.16b) using the symmetry of the equation of motion (1.6) we have in fact used a rather less sophisticated argument. It goes like this. It is self evident that w_3 defined by

$$(3.4) \quad w_3 = -t + \int^t dt$$

is a constant. Writing $dt = dr(dt/dr) = dr/\dot{r}$, (3.4) becomes

$$(3.5) \quad w_3 = -t + \int^r \frac{dr}{\dot{r}}$$

and (2.16b) follows since we can express \dot{r} in terms of w_1, w_2 and r .

So far we have not mentioned analogues to the Laplace-Runge-Lenz vector. Consider G_5 in cartesian co-ordinates when $\alpha = \frac{2}{3}$. In two-dimensions the characteristics arising from (1.4) are

$$(3.6) \quad \begin{aligned} u_1 &= xt^{-\frac{2}{3}} & v_1 &= \dot{x}t^{\frac{1}{3}} \\ u_2 &= yt^{-\frac{2}{3}} & v_2 &= \dot{y}t^{\frac{1}{3}} \end{aligned}$$

and the associated Lagrange's system arising from (1.5) is

$$(3.7) \quad \frac{du_1}{v_1 - \frac{2}{3}u_1} = \frac{du_2}{v_2 - \frac{2}{3}u_2} = \frac{dv_1}{\frac{1}{3}v_1 - \mu u_1 P^{-3}} = \frac{dv_2}{\frac{1}{3}v_2 - \mu u_1 P^{-3}} = \frac{dt}{t}$$

where $P^2 = u_1^2 + u_2^2$. Using the same convention as for (2.8) the combination

$$(v^2 - \mu u_2^2 P^{-3})(3.7.1) - (v_1 v_2 - \mu u_1 u_2 P^{-3})(3.7.2) - u_2 v_2 (3.7.3) + (2u_1 v_2 - u_2 v_1)(3.7.4)$$

gives

$$(3.8) \quad \frac{d(u_1 v_2^2 - u_2 v_1 v_2 - \mu u_1 P^{-1})}{0}$$

so that the characteristic is

$$(3.9) \quad \begin{aligned} w_4 &= (u_1 v_2 - u_2 v_1)v_2 - \frac{\mu u_1}{P} \\ &= (x\dot{y} - y\dot{x})\dot{y} - \frac{\mu x}{r} \end{aligned}$$

which is readily recognized as one of the components of the Laplace-Runge-Lenz vector in two dimensional space. The combination

$$\begin{aligned} & - (v_1 v_2 - \mu u_1 u_2 P^{-3})(3.7.1) + (v_1^2 - \mu u_1^2 P^{-3})(3.7.2) \\ & - (u_1 v_2 - 2u_2 v_1)(3.7.3) - u_1 v_1 (3.7.4) \end{aligned}$$

gives the other component

$$(3.10) \quad \begin{aligned} w_5 &= -(u_1 v_2 - u_2 v_1) v_2 - \frac{\mu u_2}{P} \\ &= -(x\dot{y} - y\dot{x})\dot{x} - \frac{\mu y}{r} . \end{aligned}$$

To the best of our knowledge this is the first explicit demonstration of the derivation of the Laplace-Runge-Lenz vector from G_5 although it has been implied by Lévy-Leblond [11] and Prince and Eliezer [14].

Although the Laplace-Runge-Lenz vector is very convenient for obtaining the orbit equation [1, 9] it is not essential [6] for the integration of (1.6). In principle, as Fradkin [2] has shown, a Laplace-Runge-Lenz vector exists for all central force problems and so for all α . It will certainly not be quadratic in the velocity. The reason for this is simple. Ignoring the free particle and isotropic harmonic oscillator the only central force problem apart from the Kepler problem which admits a second integral quadratic in the velocity is in the case $\alpha = \frac{1}{2}$ as we shall show. The Hamiltonian [8]

$$(3.11) \quad H = \frac{1}{2} \underset{\sim}{p} \cdot \underset{\sim}{p} - \frac{1}{2} \frac{\ddot{\rho}}{\rho} r^2 + \frac{1}{\rho^2} U \left(\frac{r}{\rho} \right) ,$$

where $\rho = \rho(t)$ is an arbitrary function of time and U an arbitrary function of its argument has a first integral quadratic in the velocity given by

$$(3.12) \quad I = \frac{1}{2} (\rho p_r - \dot{\rho} r)^2 + \frac{1}{2} r^{-2} p_\theta^2 \rho^2 + U \left(\frac{r}{\rho} \right) .$$

In the case $\alpha = \frac{1}{2}$, (1.7) is

$$(3.13) \quad H = \frac{1}{2} \underset{\sim}{p} \cdot \underset{\sim}{p} - \frac{1}{2} \mu r^{-2} .$$

The potential in (3.13) can be obtained from that in (3.11) in two ways: (i) set $\rho = 1$, $U = -\frac{1}{2} \mu r^{-2}$ and (ii) let $U = -\frac{1}{2} \mu (r/\rho)^{-2} - \frac{1}{8} (r/\rho)^2$ and

set $\ddot{\rho}/\rho + \frac{1}{4\rho^4} = 0$, i.e., $\rho = t^{\frac{1}{2}}$. Corresponding to (i) there is the usual energy integral and to (ii) we have

$$\begin{aligned} w_6 &= tw_1 - \frac{1}{2} r\dot{r} \\ (3.14) \qquad &= -w_1w_3 \end{aligned}$$

as can be verified by direct calculation from (2.16.b) with $\alpha = \frac{1}{2}$. (The reader will have noted that the choice of $U(r/\rho)$ and the solution $\rho = r^{\frac{1}{2}}$ have been selected to give a result in accordance with (2.16.b)). The case $\alpha = \frac{1}{2}$ happens to belong to the class of monopole problems for which the algebra is $sl(2, R) \oplus so(3)$ [Moreira et al 1985, Leach & Gorringer 1988] and the additional symmetry gives rise to an integral which is quadratic in t , but, as one would expect, it is not algebraically independent of w_1, w_2 and w_3 .

4. Conclusion. We have seen that the additional symmetry which the equation of motion for a particle moving under the influence of a central power law force possesses leads naturally to the conservation laws of energy and angular momentum and to a time-dependent first integral which arises from the Hamilton-Jacobi equation when the first two conservation laws are present. In general we do not expect a fourth independent conservation law/first integral for then, with (2.17), the system would be overdetermined. The explicit expression for the Laplace-Runge-Lenz vector for the Kepler problem ($\alpha = 2/3$) appears almost anomalous in the general power law central force context although explicit expressions for similar vectors are known for other types of force laws (Mittleman and Jezewski 1982, Mittleman and Jezewski 1983, Thompson 1987, Leach and Gorringer 1987, Gorringer and Leach 1988). We have been unable to determine any for general α , but we do know that, if such a conserved quantity exists, it will not be quadratic in the velocities and we doubt that it will be autonomous.

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Received 9 March 1990 .

Contact symmetry algebras of scalar second-order ordinary differential equations

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Journal of Mathematical Physics **32** (1991) 2051-2055

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(Received 22 March 1990; accepted for publication 3 April 1991)

There is a natural link between contact symmetries (symmetries that leave invariant the first-order contact form) and first integrals of a given second-order ordinary differential equation. It will be shown that the contact symmetry algebra of a general second-order ordinary differential equation is infinite dimensional and it is generated by the functionally independent first integrals of the equation. Moreover, the contact symmetry algebras of second-order equations admitting one, two, three, and eight point symmetries are obtained.

I. INTRODUCTION

Whereas there have been many publications devoted to the study of the point symmetry structure of ordinary differential equations, the same is not true for the contact symmetry structure. Since the initial works of Lie¹ on this subject there have been few contributions to this area. Notable among these are the works of Campbell² and Eisenhart,³ who in the main consider the geometry of contact transformations. Recently, there have been books by Anderson and Ibragimov⁴ and Ovsiannikov⁵ devoted to Lie's first-order contact transformation groups and their generalization. We also mention a paper by Schwarz⁶ dealing with the contact symmetries (not to be confused with generalized symmetries) of the one-dimensional simple harmonic oscillator. The Lie algebra of contact symmetries of the simple harmonic oscillator is shown to be infinite dimensional. Moreover, the Lie algebra of point symmetries, viz., $sl(3, \mathbb{R})$ is recovered as a subalgebra of the infinite-dimensional algebra. In Sec. III we present a generalization of this result. More precisely, we determine the contact symmetries of the general linear second-order equation. We also obtain the contact symmetry algebras of nonlinear second-order equations admitting one, two, three, and eight point symmetries, respectively. These results are natural extensions of the results presented in our previous work on point symmetry algebras (see Ref. 7) and provide a direct relationship between (contact) symmetries and first integrals of a second-order ordinary differential equations.

II. CONTACT SYMMETRY CONDITION

We begin with an operator

$$Y = \xi(t, q, \dot{q}) \frac{\partial}{\partial t} + \eta(t, q, \dot{q}) \frac{\partial}{\partial q}, \quad (1)$$

where ξ and η are analytic functions. The j th prolongation (see, e.g., Ref. 8) of Y is the operator (note $q^{(1)} = \dot{q}$ and $q^{(2)} = \ddot{q}$)

$$Y^{[j]} = Y + \sum_{k=1}^j \eta^{(k)} \frac{\partial}{\partial q^{(k)}}, \quad j = 1, 2, \quad (2)$$

where $\eta^{(k)}$ is given by the recursion relation ($\eta^{[0]} \equiv \eta$)

$$\eta^{(k)} = \frac{d}{dt} \eta^{(k-1)} - q^{(k)} \frac{d}{dt} \xi, \quad k = 1, 2, \quad (3)$$

with the total differentiation operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} + \ddot{q} \frac{\partial}{\partial \dot{q}}. \quad (4)$$

Then it is said that Y , as in (1), is a generator of a generalized symmetry (or simply a generalized symmetry) of the second-order equation written in normal form $\ddot{q} = E(t, q, \dot{q})$, E analytic, if

$$Y^{[2]}[\ddot{q} - E(t, q, \dot{q})] = 0,$$

whenever $\ddot{q} = E$. Once E is given, the above symmetry condition, after using (3) and (4), can be written as a partial differential equation in ξ and η . There exists, in general, an infinite number of generalized symmetries for any second-order equation. However, there is no systematic method of determining these generalized symmetries except if one makes a suitable ansatz for the coefficient functions (see, e.g., Ref. 9). Even then the calculation can be unwieldy.

We specialize to generators of first-order tangent or contact transformations. We discuss the notion of contact transformations as applied to second-order equations. Consider the first prolongation of the operator (1) of the form

$$X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \eta^{(1)} \frac{\partial}{\partial \dot{q}}, \quad (5)$$

in which the coefficient functions ξ , η , and $\eta^{(1)}$ depend only on t , q , and \dot{q} . Using the recursion relation (3) we can write $\eta^{(1)}$ in its expanded form as

$$\eta^{(1)} = \eta_t + \dot{q} \eta_q + \ddot{q} \eta_{\dot{q}} - \dot{q}(\xi_t + \dot{q} \xi_q + \ddot{q} \xi_{\dot{q}}),$$

where the suffices refer to partial derivatives. Since ξ , η , and $\eta^{(1)}$ do not depend on \ddot{q} , we have the following determining equations of the one-parameter group of transformation generated by X

$$\eta_q = \dot{q} \xi_{\dot{q}}, \quad (6)$$

$$\eta^{(1)} = \eta_t + \dot{q}(\eta_q - \xi_t) - \dot{q}^2 \xi_{\dot{q}}.$$

By a contact symmetry we shall mean a symmetry vector field X defined in \mathbb{R}^3 , as in (5), such that (6) is satisfied. It follows, by virtue of our definition, that a contact symmetry is a restricted type of generalized symmetry. We introduce the auxiliary analytic function $W(t, q, \dot{q})$ defined by (cf. Refs. 4-6)

$$W(t, q, \dot{q}) = \eta(t, q, \dot{q}) - \dot{q} \xi(t, q, \dot{q}). \quad (7)$$

Consequently, we can write ξ , η , and $\eta^{(1)}$ in terms of W as

$$\begin{aligned} \xi &= -W_{\dot{q}}, \quad \eta = W - \dot{q}W_{\dot{q}}, \\ \eta^{(1)} &= W_t + \dot{q}W_{\dot{q}} = W^{(1)} - \dot{q}W_{\dot{q}}. \end{aligned} \quad (8)$$

Conversely, for an arbitrary function $W(t, q, \dot{q})$, Eqs. (8) give a solution of the determining Eqs. (6). We can also write relations for $\eta^{(2)}$ in terms of W . We have

$$\eta^{(2)} = W^{(2)} - q^{(3)}W_{\dot{q}}. \quad (9)$$

The contact symmetry condition for a second-order equation is

$$X^{(1)}[\ddot{q} - E(t, q, \dot{q})] = 0, \quad (10)$$

whenever $\ddot{q} = E$. Here, we have identified $X^{(1)}$ with $Y^{(2)}$ of (2). Equation (10) upon using (8) and (9) can be written solely in terms of W and its partial derivatives.

If W is linear in \dot{q} , the corresponding contact symmetry is a prolonged point symmetry. A point symmetry is often referred to as a zeroth-order contact symmetry.^{4,5} One can then speak of a contact symmetry (in the way we have defined it) as being a first-order contact symmetry.

III. CONTACT SYMMETRY ALGEBRA

Consider a second-order equation written in normal form, viz.,

$$\ddot{q} = E(t, q, \dot{q} = p), \quad (11)$$

where E is analytic. The contact symmetry condition for (11) is (10) whenever (11) holds. In terms of W , we can write (10) as

$$\begin{aligned} W^{(2)} - p^{(2)}W_p - (W^{(1)} - \dot{p}W_p)E_p \\ - (W - pW_p)E_q + W_pE_t = 0. \end{aligned} \quad (12)$$

Expanding (12), subject to (11) holding, we obtain a linear second-order partial differential equation which after some suitable rearrangement can be written as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial q} + E\frac{\partial}{\partial p}\right)^2 W \\ - E_p\left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial q} + E\frac{\partial}{\partial p}\right)W - E_qW = 0. \end{aligned} \quad (13)$$

We recognize that $\partial/\partial t + p\partial/\partial q + E\partial/\partial p$ is the operator of total differentiation along $\ddot{q} = E$ which we denote by Γ . Hence, we can express (13) more compactly as

$$\Gamma^2(W) - E_p\Gamma(W) - E_qW = 0. \quad (14)$$

For a given E , the solution of (14), via (8), will generate the contact symmetry algebra of (11).

It is instructive to first consider the example of the free particle equation, $\ddot{q} = 0$. This will provide a cue for a more general treatment. Invoking (14) with $E = 0$, upon introducing the function $S(t, q, p)$, leads to

$$\Gamma(S) = 0 \quad \text{and} \quad \Gamma(W) = S. \quad (15)$$

The solution of (15) is straightforward and we have

$$W(t, q, p) = tU(I, J) + V(I, J), \quad (16)$$

where

$$I = p, \quad J = q - pt, \quad (17)$$

and U, V are arbitrary functions of their arguments. Thus the

free particle equation admits an infinite-dimensional Lie algebra of contact symmetries. Its generators of contact symmetries are defined by (8) and (16). As for the simple harmonic oscillator (see Table 1 of Schwarz⁶), one can also obtain the eight point symmetries of the free particle equation by an appropriate choice of the functions U and V . If we choose U and V as (U and V are obtained by setting $W_{pp} = 0$)

$$\begin{aligned} U &= (c_0 + c_1p)(q - tp) + c_2 + c_3p + c_5p^2, \\ V &= c_1(q - tp)^2 + (c_4 + c_5p)(q - tp) + c_6 + c_7p, \end{aligned} \quad (18)$$

where the c_i 's are constants, then W is linear in p

$$\begin{aligned} W &= (c_0tq + c_1q^2 + c_2t + c_4q + c_6) \\ &\quad - p(c_0t^2 + c_1tq - c_3t - c_4t - c_5q - c_7). \end{aligned} \quad (19)$$

Consequently there are eight point symmetries that generate the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$.

What are the salient features that arise from treating the free particle equation? First, we observe that the contact symmetry algebra is infinite dimensional. Second, we notice that the arguments of the two functions U and V given in (17) are functionally independent first integrals of the equation itself. That is, the contact symmetries are generated by the first integrals of $\ddot{q} = 0$. In a sense there is a link between the contact symmetries and the first integrals of the free particle equation. The properties outlined are also satisfied by the one-dimensional simple harmonic oscillator.⁶ Hence, they are satisfied by two of the paradigms of one-dimensional systems.

Are the above features peculiar to general linear second-order equations? More generally, do they hold even for nonlinear second-order equations?

For a general linear second-order equation we require E of (11) to be of the form (note $p = \dot{q}$)

$$E = -\beta(t)p - \gamma(t)q - \delta(t), \quad (20)$$

where $\beta(t)$, $\gamma(t)$, and $\delta(t)$ are arbitrary analytic functions. Substituting (20) into (14) we obtain

$$\Gamma^2(W) + \beta(t)\Gamma(W) + \gamma(t)W = 0, \quad (21)$$

in which the coefficients β , γ depend only on t [cf. the general case (14)].

We implement a procedure whereby (21) can be reduced to the simple form [cf. (15)]

$$\bar{\Gamma}^2(\bar{W}(T, Q, P)) = 0, \quad (22)$$

where T, Q (note $P = Q'$), and \bar{W} are the transformed variables to be defined and $\bar{\Gamma} = \partial/\partial T + P\partial/\partial Q$. To perform the reduction of (21) to (22), we at first treat the differential equation (21) as a linear homogeneous second-order ordinary differential equation in W . Then one can look upon (22) as the transformed "free particle" equation in \bar{W} with $\bar{\Gamma}$ the operator of total differentiation d/dT along $Q'' = 0$ ($' = d/dT$). Notwithstanding this, we simultaneously have to effect a transformation of the general linear nonhomogeneous equation, i.e., (11) with E given by (20) to the free particle equation $Q'' = 0$.

Let us indicate how one obtains the point transformation that reduces the general linear equation

$$\ddot{q} + \beta(t)\dot{q} + \gamma(t)q + \delta(t) = 0 \quad (23)$$

to the free particle equation $Q'' = 0$. In Ref. 10 it was shown that

$$[F,G]_{q,t}\ddot{q} + [F,G]_{q^2,q}\dot{q}^3 + ([F,G]_{q^2,t} + 2[F,G]_{t,q,q})\dot{q}^2 + ([F,G]_{t^2,q} + 2[F,G]_{t,q,t})\dot{q} + [F,G]_{t^2,t} = 0, \quad (24)$$

where, for example, $[F,G]_{t^2,t}$ is a shorthand for $F_{tt}G_t - F_tG_{tt}$, is the most general type of second-order ordinary differential equation equivalent to the free particle equation $Q'' = 0$ by means of the diffeomorphism $Q = F(t,q)$, $T = G(t,q)$, with Jacobian $[F,G]_{q,t}$ not identically zero at least locally. By setting the coefficients of \dot{q}^3 and \dot{q}^2 in (24) to zero and solving for F and G one can obtain the point transformation that will reduce (23) to the free particle equation (see Refs. 11 and 12). It turns out that (23) is reducible to $Q'' = 0$ if

$$\begin{aligned} \beta(t) &= (\ln AC^2)', \quad C \neq 0, \\ \gamma(t) &= \left(\frac{\ddot{A}}{A}\right)' + \frac{\ddot{A}}{A} \frac{\dot{C}}{C} + \frac{\ddot{C}}{C}, \\ \delta(t) &= \frac{1}{C} \left[\left(\frac{\ddot{A}}{A}\right)' D + \frac{\ddot{A}}{A} \dot{D} + \ddot{D} \right], \end{aligned} \quad (25)$$

and

$$Q = \dot{A}(Cq + D), \quad T = A. \quad (26)$$

Accordingly the point symmetry algebra of (23) is the same as that for the free particle equation which is $\mathfrak{sl}(3, \mathbb{R})$ (see Refs. 13–15). We show that (23) admits an infinite-dimensional algebra of contact symmetries that contains $\mathfrak{sl}(3, \mathbb{R})$ as a subalgebra.

We require the following transformation scheme

$$(t, q, p = \dot{q}, W) \mapsto (T, Q, P = Q', \bar{W}),$$

such that the transformation in t is the same for both $(t, W) \mapsto (T, \bar{W})$ and $(t, q) \mapsto (T, Q)$, i.e. $T = \bar{T}$. This can be done by using the result of the preceding discussion. It follows that the transformation

$$\begin{aligned} \bar{W} &= \dot{A}CW, \quad T = A, \quad Q = \dot{A}(Cq + D), \\ P &= \ddot{A}(Cq + D)/\dot{A} + (\dot{C}q + \ddot{D}) + Cp, \end{aligned} \quad (27)$$

where A , C , and D are functions of t related to β , γ , and δ by (25), will reduce (21) to (22). The solution of (22) in T , Q , P , and \bar{W} variables clearly is of the form (16). This, in turn, implies that the solution of (21) is [using (27)] of the form

$$W(t, q, p) = (A/\dot{A}C)U(I, J) + (1/\dot{A}C)V(I, J), \quad (28)$$

with I given by (27b) and J by

$$J = \dot{A}(Cq + D) - ACp - A(\dot{C}q + \ddot{D}) - A\ddot{A}(Cq + D)/\dot{A}. \quad (29)$$

Thus for linear equations we have the following conclusion.

The contact symmetry algebra of the general linear second-order equation is infinite dimensional and is generated by the functionally independent first integrals of the equation.

Let us now consider an example of a nonlinear equation that has the $\mathfrak{sl}(3, \mathbb{R})$ algebra. We investigate

$$\ddot{q} + \dot{q}^2 = 0, \quad (30)$$

for contact symmetries. Invoking (14) with $E = -\dot{q}^2 = -p^2$ we have

$$\Gamma^2(W) + 2p\Gamma(W) = 0. \quad (31)$$

Unlike (21), (31) contains p as a coefficient of $\Gamma(W)$. To write (31) in the form (21), we require p to be given explicitly in terms of t . Accordingly, we need at least a first integral of (30). For this problem it is clear that (31) can be written as

$$\Gamma^2(W) + [2/(t+A)]\Gamma(W) = 0, \quad (32)$$

involving a constant of integration A of (30). Now we can reduce (32) to the form [cf. (22)]

$$\bar{\Gamma}^2(\bar{W}(T, q, p)) = 0 \quad (33)$$

by means of the transformation $(t, W) \mapsto (T, \bar{W})$, where

$$\bar{\Gamma} = \frac{\partial}{\partial T} + p \frac{dt}{dT} \frac{\partial}{\partial q} + \frac{dt}{dT} E \frac{\partial}{\partial p}.$$

The change of variables that brings (32) to the form (33) is

$$\bar{W} = W, \quad T = 1/(t+A). \quad (34)$$

The solution of (33) is straightforward and we eventually have that W of (31) takes the form

$$W = pU(I, J) + V(I, J), \quad (35)$$

where $I (= A)$, J are

$$I = 1/p - t, \quad J = p \exp q, \quad (36)$$

which are two functionally independent first integrals of (30).

Admittedly the nonlinear example, admitting $\mathfrak{sl}(3, \mathbb{R})$, just treated is a simple one. However, the above procedure equally well applies to any nonlinear equation that admits the $\mathfrak{sl}(3, \mathbb{R})$ algebra. The most general second-order equation that admits the $\mathfrak{sl}(3, \mathbb{R})$ algebra is given by (24). Equation (24) is reducible to the simplest equation $Q'' = 0$ by means of the regular point transformation $Q = F(t, q)$, $T = G(t, q)$. The solution of (24) is $F(t, q) = G(t, q)I_1 + I_2$ which can be (locally) solved for q in terms of t . This in turn implies that (14) can be cast in the form (21) from which one can deduce (33) and hence the contact symmetry algebra of (24). Alternatively, one can resort to the implicit form (16) where W is now a function of $T = G$, $Q = F$, and $P = dQ/dT$ with I and J given by (17) in terms of $T = G$, $Q = F$, and $P = dQ/dT$.

We have in fact proved the following theorem.

Theorem: The contact symmetry Lie algebra of a linear or linearizable (via a point transformation) second-order equation is the infinite-dimensional algebra $\mathfrak{a}_3 \oplus \mathfrak{sl}(3, \mathbb{R})$ (\oplus denotes semidirect sum and \mathfrak{a}_3 denotes the Lie algebra of purely contact symmetries) and it is generated by the functionally independent first integrals of the equation.

In the general case we cannot always express E_q and E_p of (14) explicitly in terms of t . This in general involves *a priori* knowledge of the solution (or, at worst, the first integrals) of the nonlinear equation (11). However, one can locally solve (11) for q as a function of t with respect to some given initial conditions. Then, in principle, one can obtain the contact symmetry algebra of the nonlinear equation

(11). Thus the contact symmetry algebra of a second-order equation possessing no point symmetry is infinite dimensional and we shall denote it as a_0 .

Let us now consider the equivalence class of equations that admit one point symmetry. Then we can write (11) in the form $\ddot{q} = E(t,p)$. This equation admits the point symmetry algebra A_1 (see Ref. 7) realized by the canonical vector field $\partial/\partial q$. We investigate the contact symmetry algebra of this equation. Invoking (14) with $E = E(t,p)$, and introducing the function $S(t,q,p)$, gives rise to

$$\Gamma(S) = E_p \text{ and } \ln \Gamma(W) = S. \quad (37)$$

The solution of (37) is

$$W(t,q,p) = U(I,J) \left[\int^p \frac{1}{E} \exp\left(\int^{\bar{p}} \frac{E_{p'}}{E} dp'\right) d\bar{p} \right] + V(I,J), \quad (38)$$

where U and V are arbitrary functions of the first integrals of $\ddot{q} = E(t,p)$.

The contact symmetry algebra of an equation admitting exactly one point symmetry is infinite dimensional and is given by $a_1 \oplus A_1$.

We obtain the contact symmetry algebra of equations having two point symmetries by invoking (38). First, we look at equations admitting the $2A_1$ (Abelian) algebra of point symmetries (see Ref. 7). This means that (11) must admit $\partial/\partial q$ and $\partial/\partial t$ (in standard form). Hence (11) can be brought to the form $\ddot{q} = E(p)$. Utilizing (38) with $E = E(p)$ we immediately get [cf. (35)]

$$W(t,q,p) = U(I,J)p + V(I,J), \quad (39)$$

where I and J are the first integrals

$$I = \int^p \frac{dr}{E(r)} - t, \quad (40)$$

$$J = p \int^p \frac{dr}{E(r)} - \int^p \int^r \frac{ds}{E(s)} dr - q.$$

The contact symmetry algebra here is infinite dimensional and given by $a_{21} \oplus 2A_1$.

The canonical equation admitting the A_2 (see Ref. 7) algebra of point symmetries is $\ddot{q} = t^{-1}H(p)$ [set $E = t^{-1}H(p)$ in (11)] with generators $\partial/\partial q$ and

$t\partial/\partial t + q\partial/\partial q$. The first integrals are

$$I = t^{-1} \exp\left(\int^p \frac{dr}{H(r)}\right),$$

$$J = p \exp\left(\int^p \frac{dr}{H(r)}\right) - t^{-1}q \exp\left(\int^p \frac{dr}{H(r)}\right) - \int^p \exp\left(\int^r \frac{ds}{H(s)}\right) dr. \quad (41)$$

We substitute $E = t^{-1}H(p)$ in (38). This, upon using (41a), leads to

$$W(t,q,p) = U(I,J) \int^p \exp\left(\int^s \frac{dr}{H(r)}\right) ds + V(I,J). \quad (42)$$

I and J are, as usual, the first integrals given in (41). The contact symmetry algebra is infinite dimensional, given by $a_{22} \oplus A_2$.

It remains now to discuss the contact symmetry algebra of second-order equations admitting exactly three point symmetries. We use the following notation. When referring to a_3 , we mean the j th ($j = 1, 2, \dots, 7$) algebra of purely contact symmetries which forms a semidirect sum with a three-dimensional algebra. There are five representative classes of second-order equations possessing exactly three point symmetries (see Ref. 7). We list the equations (in each case $B \neq 0, b \in \mathbb{R}$ and U, V are arbitrary functions of the functionally independent first integrals of the respective equations) together with the form of W and the contact symmetry algebra in Table I.

Note that although $Up + V$ is repeated thrice in the table, the associated contact Lie algebra is different in each case since U and V for each case have different integral dependencies.

Thus we have obtained a complete classification scheme for the contact symmetry algebras of scalar second-order ordinary differential equations. For a given differential equation, the contact Lie algebra is a semidirect sum of the point symmetry (including, trivially, the zero point symmetry) and the purely contact symmetry algebras. Moreover, the contact Lie algebra is generated by the first integrals of the equation itself and this provides a direct relationship between the symmetries and the first integrals.

TABLE I. Equations with the form of W and the contact symmetry algebra.

Equation	$W(t,q,p)$	Contact algebra
$\ddot{q} = B \exp(-p)$ or $\ddot{q} = -1 + B \exp(-p)$	$Up + V$ $U \ln(1 - B \exp(1 - p)) + V$	$a_{31} \oplus A_{3,2}$
$\ddot{q} = (b-1)p + Bp^{(2b-1)/(b-1)}$ or $\ddot{q} = Bp^{(b-2)/(b-1)} b \neq 0, 1/2, 1, 2$	$Up^{b/(b-1)} + V$ $Up + V$	$a_{32} \oplus A_{3,4}$ (if $b = -1$) $a_{33} \oplus A_{3,5}$ (if $b \neq -1$)
$\ddot{q} = B(1 + p^2)^{3/2} \exp(b \tan^{-1} p)$	$Up + V$	$a_{34} \oplus A_{3,6}$ (if $b = 0$) $a_{35} \oplus A_{3,7}$ (if $b \neq 0$)
$\ddot{q} = p^3 + p + B(1 + p^2)^{3/2}$ $\ddot{q} = Bp^3 - p/2$	$U(Bp + (1 + p^2)^{1/2}) + V$ $U(Bp + 1/2p) + V$	$a_{36} \oplus A_{3,8}$ $a_{37} \oplus A_{3,8}$

ACKNOWLEDGMENT

We thank the Foundation for Research Development of South Africa for support through a research grant.

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Shear-free spherically symmetric solutions with conformal symmetry

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General Relativity and Gravitation **23** (1991) 261–267

Shear-free Spherically Symmetric Solutions with Conformal Symmetry

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Received January 24th, 1990

Dyer, McVittie and Oattes (1987) presented the field equations for shear-free perfect fluid spacetimes which are spherically symmetric and admit a conformal symmetry. Two special solutions of these equations are found. Furthermore, in the general case, one field equation is solved in terms of a Painlevé transcendent, while the remaining equation is reduced to an Emden-Fowler equation.

1. INTRODUCTION

In trying to solve the highly nonlinear field equations of general relativity it is usually assumed that the spacetime admits symmetries. Recently a number of solutions have been found in spherically symmetric models with the assumption that the spacetime is invariant under a conformal Killing vector (see e.g. Ref. 1 for the static case and Ref. 2 for the nonstatic case). Dyer, McVittie and Oattes [3] assume a particular form of a conformal Killing vector in the t - r plane in spherically symmetric shear-free perfect fluid spacetimes. They derive the field equations with these restrictions, but are unable to find any closed form solutions of these equations.

In this paper we study the two field equations of [3] in greater detail. The appropriate results that we need from [3] are given in Section 2. In

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Section 3 we find an exact expanding solution for the field equations with the specified conformal Killing vector by demanding that an arbitrary function of integration vanishes. We show that this solution is a member of a wider class of known solutions that in general do not admit a conformal symmetry. In Section 4 we express the field equations in terms of canonical coordinates. We find a special non-expanding solution which is non-static. Furthermore, we show that one field equation has a solution in terms of a Painlevé transcendent, while the other is an Emden-Fowler equation.

We follow the notation and conventions of [3] throughout.

2. FIELD EQUATIONS WITH CONFORMAL SYMMETRY

In this section we give the relevant results from Dyer et al. [3] necessary for our solutions. For a shear-free perfect fluid source the spherically symmetric metric can be written in comoving coordinates as [3,4]

$$ds^2 = A^2(t, r)dt^2 - B^2(t, r)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \quad (1)$$

where the fluid four-velocity is $u^\alpha = A^{-1}\delta^\alpha_t$. If there is no heat flow then

$$\frac{\dot{B}}{B} = AH(t) \quad (2)$$

where $3H = \Theta =$ the fluid expansion rate [4]. The condition for pressure isotropy

$$4\frac{B'}{B} + \frac{1}{r} = \frac{\dot{B}''}{\dot{B}'} + 2\left(\frac{\dot{B}}{\dot{B}'}\right)\left(\frac{B'}{B}\right)^2 \quad (3)$$

yields the first integral

$$\left(\frac{B'}{B}\right)' - \left(\frac{B'}{B}\right)^2 - \left(\frac{B'}{B}\right)\frac{1}{r} = \frac{G(r)}{B} \quad (4)$$

where $G(r)$ is an arbitrary function of integration.

If the spacetime is invariant under a conformal Killing vector ξ , then

$$\mathcal{L}_\xi g_{\alpha\beta} = 2\phi g_{\alpha\beta} \quad (5)$$

where ϕ is the conformal factor. The particular ξ chosen in [3] is, after a coordinate transformation,

$$\xi^\gamma = t\delta^\gamma_t + r\delta^\gamma_r \quad (6)$$

The motivation for the choice (6) is that subclasses of McVittie's solutions [5] admit a conformal Killing vector in the $t-r$ plane [3]. The converse question is addressed by [3]: to find solutions of the field equations with ξ specified by (6). Note that (6) maps fluid flow lines into fluid flow lines (since $\mathcal{L}_\xi u^\alpha = -\phi u^\alpha$), and is invariant under the Killing vectors $X_{(a)}$ ($a = 1, 2, 3$) of spherical symmetry (i.e. $[X_{(a)}, \xi] = 0$). Note also that in flat spacetime (6) generates a self-similarity. This is reflected in the occurrence of the r/t dependence of A/B [see (7) below].

With (1) and (6) the conformal Killing equations (5) becomes

$$\begin{aligned} r \frac{B'}{B} + t \frac{\dot{B}}{B} &= \phi - 1 \\ r \frac{A'}{A} + t \frac{\dot{A}}{A} &= \phi - 1 \end{aligned}$$

which imply

$$\frac{A}{B} = \sigma(r/t) \equiv \sigma(\mu) \tag{7}$$

where σ is an arbitrary function. From (2) and (7)

$$\dot{B} = B^2 \sigma(\mu) H. \tag{8}$$

Using (3) and (8) gives

$$\left(\frac{B'}{B}\right)' - \left(\frac{B'}{B}\right)^2 - \left(\frac{B'}{B}\right) \frac{1}{r} = \frac{T(\mu)}{r^2} \tag{9}$$

where

$$T(\mu) = (1/2\sigma)(\mu\sigma_\mu - \mu^2\sigma_{\mu\mu}). \tag{10}$$

At this stage it is convenient to distinguish between the two cases of vanishing/nonvanishing function of integration $G(r)$: $G = 0$ and $G \neq 0$. We observe from (4) and (9) that $G = 0 \Leftrightarrow T = 0$.

Case 1 ($G = 0$):

With $G = 0$ the field equations (9) and (10) become

$$\left(\frac{B'}{B}\right)' - \left(\frac{B'}{B}\right)^2 - \left(\frac{B'}{B}\right) \frac{1}{r} = 0 \tag{11}$$

$$\sigma_\mu - \mu\sigma_{\mu\mu} = 0. \tag{12}$$

We present the general solution to (11) and (12) in Section 3. Note that this case was not considered by Dyer et al. [3].

Case 2 ($G \neq 0$):

In this case we have from eqs. (4) and (9), and then (8) that

$$B = \frac{r^2 G}{T} \Rightarrow T_\mu = rt^2 \sigma GH \quad (13)$$

which yields

$$G = r^{-1-m} \quad (14)$$

$$H = t^{m-2} \quad (15)$$

$$\sigma = \mu^m T_\mu \quad (16)$$

where m is an arbitrary constant. With the form of σ in (16), (10) becomes

$$\mu^2 T_{\mu\mu\mu} + \mu(2m-1)T_{\mu\mu} + (m^2 - 2m + 2T)T_\mu = 0. \quad (17)$$

Clearly Dyer et al. [3] obtained (13)-(17) by assuming that $G \neq 0$ and $T \neq 0$. Equations (4) and (17) need to be integrated to obtain a solution to the field equations with $G \neq 0$. We take up this problem in Section 4.

3. A PARTICULAR EXPANDING SOLUTION

Here we consider the Case 1 ($G = 0$) which was omitted in [3]. For this special case we can integrate the field equations and obtain a consistent solution. We assume that $H \neq 0$. If $H = 0$ then $B = 0$ by (2), thereby forcing the expansion to vanish. The resulting solution will then be static or can be generated by a static solution (since the shear also vanishes) (Ref. 4, p.166).

On integration (11) gives the metric function

$$B = [r^2 K(t) + L(t)]^{-1} \quad (18)$$

where $K(t)$ and $L(t)$ are arbitrary functions of integration. The differential equation (12) is of the Euler-Cauchy form and has the solution

$$\sigma = a\mu^2 + b = ar^2/t^2 + b$$

where a and b are constants of integration. Then (7) gives the remaining metric function

$$A = (ar^2/t^2 + b)[r^2 K(t) + L(t)]^{-1}. \quad (19)$$

Substituting the metric functions (18) and (19) into (2) gives

$$K(t) = -a \int t^{-2} H(t) dt$$

$$L(t) = -b \int H(t) dt.$$

Thus (18) and (19), with the above restrictions on $K(t)$ and $L(t)$, are a solution to the field equations for Case 1.

Solutions with the metric function $B(t, r)$ of the form (18) have been reported previously (Ref. 4, p.168). In these solutions, the other metric function $A(t, r)$ is obtained from (2) by specifying $H(t)$, and the functions $K(t)$ and $L(t)$ are arbitrary. Here the functions of integration $K(t)$ and $L(t)$ are determined by the function $H(t)$. This restriction is a consequence of the conformal symmetry (6). The conformal factor is [with the help of (18) and (19)]

$$\phi = \frac{L - t\dot{L} - r^2 K - r^2 t \dot{K}}{L + r^2 K}$$

so that in general the solution admits a proper (i.e. non-homothetic) conformal symmetry.

4. SOLUTIONS WITH $G(r) \neq 0$

In Case 2 ($G \neq 0$) the function of integration $G(r)$ assumes the form (14). By (13) we observe that $B = B(r, \mu)$ is a separable function of the variables r and $\mu = r/t$. This severe restriction did not arise in Section 3 when $G = 0$. With G given by (14), we can write (4) as

$$r^2 \frac{B_{rr}}{B} - 2r^2 \left(\frac{B_r}{B}\right)^2 - r \frac{B_r}{B} + 2r\mu \frac{B_{r\mu}}{B} - 4r\mu \frac{B_r B_\mu}{B^2} + \mu^2 \frac{B_{\mu\mu}}{B} - 2\mu^2 \left(\frac{B_\mu}{B}\right)^2 - \mu \frac{B_\mu}{B} = \frac{r^{1-m}}{B}. \quad (20)$$

If we let $\eta = \ln r$, $\eta + \zeta = \ln \mu$ and $y^{-1} = B$, then (20) transforms to

$$\frac{\partial^2 y}{\partial \eta^2} - 2 \frac{\partial y}{\partial \eta} = -e^{(1-m)\eta} y^2 \quad (21)$$

which has the advantage of not containing an explicit ζ dependence. We have been unable to exhibit the general solution of the nonlinear, second

order partial differential equation (21). However (21) is readily seen to admit a special solution $y = (m - 1)(3 - m) \exp(m - 1)\eta$, i.e.

$$B = (m - 1)^{-1}(3 - m)^{-1}r^{1-m}. \quad (22)$$

The solution (22) is non-expanding since $\dot{B} = 0$, which implies $H = 0$ by (8). Then (13) shows that $T_\mu = 0$, and by (22)

$$T = (m - 1)(3 - m).$$

Then eq. (10) may be solved for σ . After the change of variable $\rho = \ln \mu$, (10) becomes

$$\frac{d^2\sigma}{d\rho^2} - 2\frac{d\sigma}{d\rho} + 2(m - 1)(3 - m)\sigma = 0$$

which has the advantage of being a linear equation with constant coefficients. Thus $\sigma(\rho)$ is readily found in closed form once m is specified. Then A is determined by (7) and (22). The time-derivative of the fluid pressure p is found from the general expression of p (Ref. 4, p.166):

$$\dot{p} = 2B^{-3}B'(A'/A)^*.$$

This is readily seen to be non-zero. Thus the special non-expanding solution given by (22) is not static.

We now show that in the general (expanding) case, the second field equation (17) has a solution in terms of a Painlevé transcendent. First, we observe that (17) has a first integral

$$\mu^2 T_{\mu\mu} + (2m - 3)\mu T_\mu + (m - 1)(m - 3)T + T^2 = k \quad (23)$$

where k is a constant. We make the change of variables

$$T(\mu) = \gamma y(x) + T_0, \quad x = \ln(\mu/\beta)$$

where T_0 is the constant solution of (23), given by

$$T_0^2 + (m - 1)(m - 3)T_0 - k = 0. \quad (24)$$

Then (23) becomes

$$\frac{d^2y}{dx^2} + 2(m - 2)\beta \frac{dy}{dx} + k\beta^2 y + \beta^2 \gamma y^2 = 0. \quad (25)$$

Equation (25) is of Painlevé form (Kamke: Ref. 6, eq. 6.23, p.547) if we let

$$2(m-2)\beta = 5a, \quad k\beta^2 = 6a^2, \quad \beta^2\gamma = -6.$$

These equations show that β and γ (and hence a) are arbitrary non-zero constants, while k is determined by m :

$$k = 24(m-2)^2/25. \quad (26)$$

Then T_0 is determined by (24), which always has real roots, by (26). Thus for k given by (26), the second field equation (17) has a solution given by the solution of (25):

$$y = a^2 C_1^2 e^{-2ax} \mathcal{P}(C_1 e^{-ax} + C_2, 0, 1) \quad (27)$$

where \mathcal{P} is a Painlevé transcendent (Ref. 6, eq. 6.23, p.547) and C_1, C_2 are arbitrary constants.

Finally, we show that the remaining field equation (4) in the general case ($G \neq 0 \neq H$) can be transformed to a particular case of the Emden-Fowler equation. With the transformation $4w(t, x) = 1/B(t, r)$ and $x = r^{-2}$, (4) becomes [using (14)]:

$$xw'' + 2w' + x^{(m-3)/2}w^2 = 0 \quad (28)$$

which is an Emden-Fowler equation (Ref. 6, eq. 6.74, p.560). Thus, given the Painlevé solution (27), we have reduced the general problem to the solution of the Emden-Fowler equation (28).

ACKNOWLEDGEMENT

SDM thanks the Centre for Nonlinear Studies of the University of the Witwatersrand for hospitality and support during a visit when this work was done.

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Self-similar solutions of the generalized Emden-Fowler equation

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International Journal for Nonlinear Mechanics **27** (1992) 575-582

SELF-SIMILAR SOLUTIONS OF THE GENERALIZED EMDEN-FOWLER EQUATION

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(Received 25 April 1991)

Abstract—When, in the generalized Emden-Fowler equation $y'' + f(x)y^n = 0$, the function $f(x)$ takes certain forms, the equation possesses one or two Lie point symmetries and in some cases closed-form solutions can be obtained.

1. INTRODUCTION

The literature devoted to the generalized Emden-Fowler equation

$$y'' + f(x)y^n = 0 \quad (1.1)$$

and its various special forms is vast. Wong [1] in his review paper of 1975 lists 144 references and is rather selective. For example in Vol. 91 of the *Monthly Notices of the Royal Astronomical Society*, in which Fowler [2] provides an astrophysically motivated discussion, there are also papers by Milne [3], Fairclough [4], Hopf [5] and Russell [6] on the Emden-Fowler equation which doubtless reflects the strong interest in stellar structure at the time and the value of Emden's model. Equations of the form (1.1) (and more general ones) occur as reductions of partial differential equations and there is continuing work being done, in particular by Atkinson and Peletier [7-9].

Even in the case of the Emden-Fowler equation ($f(x) = x^m$) closed-form solutions are scarce although we should mention that of Kustaanheimo and Qvist [10] which arises as a solution to the Einstein field equations. Indeed from the very beginning [11-13] studies were concerned rather more with the existence and behaviour of solutions and this is still very much the case [14].

It is the objective of this paper to search for (and find) explicit solutions of equation (1.1) using the methods of Lie point symmetry analysis. The motivation comes from general relativity in which the Einstein field equations in the spherically symmetric shear-free model [10] reduce after transformation to equation (1.1) in the case $n = 2$. Recently [15, 16] some new solutions have been found within this class of models. (It is amusing to note that Fowler [13, p. 261] remarks that "the chief interest lies . . . in n odd rather than n even".)

The paper is structured as follows. In Section 2 we briefly recall the principles of Lie symmetry analysis and apply it to equation (1.1). We see that there is a natural splitting into consideration of $n \neq 2$ and $n = 2$ and the two cases are treated in Sections 3 and 4, respectively. (Note that we do not consider the trivial cases $n = 0, 1$ for which the symmetry structure and integrability are well known.)

2. GOVERNING EQUATIONS FOR THE POINT SYMMETRIES

In general an ordinary differential equation

$$N(x, y, y', \dots, y^{(n)}) = 0 \quad (2.1)$$

where ' denotes d/dx and $(^{(n)}) d^n/dx^n$, admits a Lie point symmetry

$$G = \xi(x, y)\partial/\partial x + \eta(x, y)\partial/\partial y \quad (2.2)$$

if

$$G^{(n)}N|_{N=0} = 0 \quad (2.3)$$

where $G^{(n)}$ is the n th extension of G . When equation (2.1) is a second-order equation,

$$G^{(2)} = G + (\eta' - \xi'y')\frac{\partial}{\partial y'} + (\eta'' - 2\xi'y'' - \xi''y')\frac{\partial}{\partial y''} \quad (2.4)$$

where

$$\eta' = \frac{\partial\eta}{\partial x} + y'\frac{\partial\eta}{\partial y} \quad (2.5)$$

etc. The functions $\xi(x, y)$ and $\eta(x, y)$ are found by equating coefficients of independent powers of y' in equation (2.3) separately to zero. In the case of the generalized Emden-Fowler equation (1.1), ξ and η are required to satisfy the four coupled partial differential equations

$$\frac{\partial^2\xi}{\partial y^2} = 0 \quad (2.6)$$

$$\frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\xi}{\partial x\partial y} = 0 \quad (2.7)$$

$$2\frac{\partial^2\eta}{\partial x\partial y} + 3fy^n\frac{\partial\xi}{\partial y} - \frac{\partial^2\xi}{\partial x^2} = 0 \quad (2.8)$$

$$\frac{\partial^2\eta}{\partial x^2} - fy^n\frac{\partial\eta}{\partial y} + 2fy^n\frac{\partial\xi}{\partial x} + \xi f'y^n + n\eta fy^{n-1} = 0. \quad (2.9)$$

From equations (2.6)–(2.8) we find that

$$\xi = a(x) \quad \eta = c(x)y + d(x) \quad (2.10)$$

and a and c are related via

$$a'' = 2c'. \quad (2.11)$$

In the case of equation (2.9) the last term on the left-hand side signals the need to distinguish between $n \neq 2$ and $n = 2$. In the case $n \neq 2$ we obtain in addition to equation (2.11)

$$d = 0 \quad c'' = 0 \quad (2.12)$$

$$(n-1)fc + 2fa' + f'a = 0. \quad (2.13)$$

When $n = 2$, the equations are

$$d'' = 0 \quad c'' = -2fd \quad (2.14)$$

$$cf + af' + 2a'f = 0. \quad (2.15)$$

We shall now treat the two cases in turn.

3. CASE $n \neq 2$

From equations (2.11)–(2.13) we have

$$a = A_0 + A_1x + A_2x^2$$

$$c = \frac{1}{2}a' + \frac{\alpha}{n-1}$$

$$f = Ka^{-(n+2)/2} \exp\left[-\alpha \int^x \frac{ds}{a(s)}\right]. \quad (3.1)$$

We note that for $\alpha \neq 0$, $f(x)$ is an algebraic function only in the case that the discriminant of the coefficients of $a(x)$ is positive.

Before we proceed with the general implications of the general result in equation (3.1) we consider the restriction of equation (1.1) to the standard Emden-Fowler equation in which $f(x) = x^m$. Now equation (2.13) is a constraint on $a(x)$ and $c(x)$ and we find that

$$(m + n + 3)A_2 = 0 \quad (3.2)$$

$$\alpha = -\frac{1}{2}(2m + n + 3)A_1 \quad (3.3)$$

$$mA_0 = 0. \quad (3.4)$$

In the case $m = 0$, we can have three generators of symmetry if $n = -3$ and $\alpha = 0$, namely

$$G_1 = \frac{\partial}{\partial x} \quad (3.5)$$

$$G_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad (3.6)$$

$$G_3 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \quad (3.7)$$

with the commutation relations

$$[G_1, G_2] = 2G_1, \quad [G_1, G_3] = G_2, \quad [G_2, G_3] = 2G_3$$

and so the algebra is evidently $sl(2, R)$. However, the case $m = 0$ is not really of interest.

In the general case $m \neq 0$, if $m + n + 3 \neq 0$ there is just the single symmetry

$$G_1 = x \frac{\partial}{\partial x} - \frac{m+2}{n-1} y \frac{\partial}{\partial y} \quad (3.8)$$

which is the generator of a self-similar transformation. The Emden-Fowler equation is transformed to autonomous form by the rescaling transformation [17]

$$Y = yx^{(m+2)/(n-1)} \quad X = \log x \quad (3.9)$$

and becomes

$$Y'' - \frac{2m+n+3}{n-1} Y' + \frac{m+2}{n-1} \frac{m+n+1}{n-1} Y + Y^n = 0. \quad (3.10)$$

For general m and n equation (3.10) can only be reduced to a first-order equation.

When $m + n + 3 = 0$, in addition to the symmetry in equation (3.8), we also have

$$G_2 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \quad (3.11)$$

which represents a conformal transformation. Since $[G_1, G_2] = G_2$, we may use G_2 to reduce

$$y'' + x^{-(n+3)} y^n = 0 \quad (3.12)$$

to quadratures [18]. The invariants of G_2 are found from

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dy'}{y - xy'} \quad (3.13)$$

and are

$$u = yx^{-1} \quad v = y'x - y. \quad (3.14)$$

This leads to the first-order equation

$$\frac{dv}{du} = -\frac{u^n}{v}$$

which can be immediately integrated to give the first integral

$$I = \frac{1}{2}v^2 + \frac{1}{n+1}u^{n+1}. \quad (3.15)$$

As $u' = vx^{-2}$, equation (3.15) can be rewritten as the quadrature

$$-\frac{1}{x} + \frac{1}{x_0} = \int^{yx^{-1}} \frac{du}{\left(2I - \frac{2u^{n+1}}{n+1}\right)^{1/2}}. \quad (3.16)$$

(In the case $n = -1$, $u^{n+1}/n + 1$ is replaced by $\log u$.) It is possible to perform the quadrature in closed form whenever n takes the form $-(i-1)/(i+1)$ where i is an integer, but inversion to obtain y as a global function of x is possible only in the case $n = -3$. (Strictly speaking when n is not an integer one should write $x^m y^n = x^{-n-3} y^n$ as $xy|x|^{-n-4}|y|^{n-1}$.)

We now revert to the discussion of the generalized Emden-Fowler equation with f as given by equation (3.1). Under the transformation

$$X = \int^x \frac{ds}{a(s)} \quad Y = ya^{-1/2} \exp\left(-\frac{\alpha X}{n-1}\right) \quad (3.17)$$

(which transforms the single generator to $\partial/\partial X$) equation (1.1) with f as in equation (3.1) becomes

$$Y'' + \frac{2\alpha}{n-1} Y' + MY + KY^n = 0 \quad (3.18)$$

where

$$M = \left(\frac{\alpha}{n-1}\right)^2 - \frac{\Delta}{4} \quad (3.19)$$

and Δ is the discriminant $A_1^2 - 4A_0A_2$. If $\alpha = 0$, equation (3.18) is readily reduced to quadratures. In the case $\Delta > 0$ and $\alpha = \pm 2\sqrt{\Delta}/(n-1)$ equation (3.18) can be reduced to

$$\frac{d^2v}{du^2} + \operatorname{sgn}(K)v^n = 0 \quad (3.20)$$

by the transformation

$$Y = v \exp\left[\left(-\frac{\alpha}{n-1} \pm \frac{1}{2}\sqrt{\Delta}\right)X\right] \\ u = (|K|/\Delta)^{1/2} \exp[\pm\sqrt{\Delta}X]. \quad (3.21)$$

The reduction of equation (3.20) to quadrature is immediate. In this case f is algebraic and is given by

$$f(x) = K(2A_2)^{n+3}(2A_2x + A_1 + \sqrt{\Delta})^{-(n+3)/2 \pm 2/(n-1)} \\ \times (2A_2x + A_1 - \sqrt{\Delta})^{-(n+3)/2 \mp 2/(n-1)} \quad (3.22)$$

since

$$A_0 + A_1x + A_2x^2 = \left[x - \left(-\frac{A_1 + \sqrt{\Delta}}{2A_2}\right)\right] \left[x - \left(-\frac{A_1 - \sqrt{\Delta}}{2A_2}\right)\right].$$

4. THE CASE $n = 2$

In the case of the Emden-Fowler equation with $f = x^m$, equations (2.11) and (2.14) yield

$$d = D_0 + D_1x \\ a = A_0 + A_1x + A_2x^2 - \frac{4D_0x^{m+3}}{(m+1)(m+2)(m+2)} - \frac{4D_1x^{m+4}}{(m+2)(m+3)(m+4)} \\ c = \frac{1}{2}a' + \alpha = \frac{1}{2}A_1 + \alpha + A_2x - \frac{2D_0x^{m+2}}{(m+1)(m+2)} - \frac{2D_1x^{m+3}}{(m+2)(m+3)} \quad (4.1)$$

provided $m \neq -1, -2, -3, -4$. (In fact explicit calculations for the excluded cases show that there is nothing special about them. For $m = -1$, the equation $y'' + x^{-1}y^2 = 0$ has the single symmetry $x\partial/\partial x - y\partial/\partial y$; for $m = -2$, $y'' + x^{-2}y^2 = 0$ has $x\partial/\partial x$; for $m = -3$, $y'' + x^{-3}y^2 = 0$ has $x\partial/\partial x + y\partial/\partial y$ and, for $m = -4$, $y'' + x^{-4}y^2 = 0$ has $x\partial/\partial x + 2y\partial/\partial y$. In other words, the symmetry is of the self-similar kind found for all other values of m . Note also that under the transformation $x \rightarrow x^{-1}$, $y \rightarrow y/x$, $m = -1$ is mapped to $m = 4$ and

$m = -2$ is mapped to $m = -3$.) The consistency condition (2.15) leads to

$$A_0 = 0 \quad (4.2a)$$

$$\alpha + \frac{1}{2}(2m + 5)A_1 = 0 \quad (4.2b)$$

$$(m + 5)A_2 = 0 \quad (4.2c)$$

$$(7m + 15)D_0 = 0 \quad (4.2d)$$

$$(7m + 20)D_1 = 0. \quad (4.2e)$$

Condition (4.2b) is just the self-similar transformation and condition (4.2c) is the $m + n + 3 = 0$ case which represents a conformal transformation. The treatment of each case is the same as for the general n case and will not be repeated here. It is conditions (4.2d) and (4.2e) which introduce something new for the case $n = 2$.

When $m = -20/7$, the two symmetries are

$$G_1 = 7x \frac{\partial}{\partial x} + 6y \frac{\partial}{\partial y}$$

$$G_2 = 343x^{8/7} \frac{\partial}{\partial x} + (196x^{1/7}y + 12x) \frac{\partial}{\partial y}$$

with $[G_1, G_2] = G_2$, which means that the solution of

$$y'' + x^{-20/7}y^2 = 0 \quad (4.3)$$

can be reduced to quadratures with G_2 [18].

The integral invariants of G_2 are

$$u = x^{-4/7}y - \frac{6}{49}x^{2/7}$$

$$v = x^{4/7}y' - \frac{4}{7}x^{-3/7}y - \frac{12}{343}x^{3/7}$$

the reduced equation is

$$\frac{dv}{du} = -\frac{u^2}{v}$$

and, since $v = x^{8/7}u'$, the solution of equation (4.3) is reduced to the quadrature

$$7(x_0^{-1/7} - x^{-1/7}) = \int_{u_0}^u \frac{d\eta}{(2I - \frac{2}{3}\eta^3)^{1/2}} \quad (4.4)$$

where I is a constant and $u_0 = u(x_0, y_0)$.

When $m = -15/7$, the two symmetries are

$$G_1 = -7x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$$

$$G_2 = 343x^{6/7} \frac{\partial}{\partial x} + (147x^{-1/7}y - 12) \frac{\partial}{\partial y}$$

and $[G_1, G_2] = G_2$. The integral invariants of G_2 are

$$u = x^{-3/7}y - \frac{6}{49}x^{-2/7}$$

$$v = x^{3/7}y' - \frac{3}{7}x^{-4/7}y + \frac{12}{343}x^{-3/7}$$

and the result corresponding to equation (4.4) is

$$7(x^{1/7} - x_0^{1/7}) = \int_{u_0}^u \frac{d\eta}{(2I - \frac{2}{3}\eta^3)^{1/2}}$$

(Note that the transformation $x \rightarrow x^{-1}$, $y \rightarrow y/x$, maps the $m = 14/7$ equation to the $m = -20/7$ equation.)

We note that, for those cases in which the solution is reducible to quadratures due to the possession of two symmetries with the property $[G_1, G_2] = G_2$, G_1 is a self-similar symmetry whereas G_2 appears to be of a conformal type. Bouquet *et al.* [20] studied equations invariant under time translation and self-similarity the generators of which also have the algebra $[G_1, G_2] = G_2$ and in that case it was the time translation symmetry which led to the reduction to quadratures.

We turn now to the generalized Emden-Fowler equation for which equations (2.11), (2.14) and (2.15) give

$$\begin{aligned} d &= D_0 + D_1 x \\ c &= \frac{1}{2} a' + \alpha \\ f &= K a^{-5/2} \exp\left(-\alpha \int \frac{dx}{a}\right) \end{aligned} \quad (4.5)$$

where $a(x)$ is the solution of the non-linear equation

$$a''' = -4Kda^{-5/2} \exp\left(-\alpha \int \frac{dx}{a}\right). \quad (4.6)$$

(It is interesting to note that the expression for f in equation (4.5) is the same as that found by Leach [20] in his study of a time-dependent oscillator with a quadratic anharmonic term. The equation for $a(x)$ differed from equation (4.6).) Without solving the equation for $a(x)$ we can transform the generalized Emden-Fowler equation with f as given in equation (4.5) as follows. The symmetry

$$G = a \frac{\partial}{\partial x} + \left[\left(\frac{1}{2} a' + \alpha \right) y + d \right] \frac{\partial}{\partial y} \quad (4.7)$$

is transformed to

$$G = \frac{\partial}{\partial X} \quad (4.8)$$

by means of the transformation

$$X = \int \frac{dx}{a}, \quad Y = ya^{-1/2} \exp\left(-\alpha \int \frac{dx}{a}\right) - \int \left[da^{3/2} \exp\left(-\alpha \int \frac{dx}{a}\right) \right] dx. \quad (4.9)$$

Under this transformation the generalized Emden-Fowler equation becomes

$$Y'' + 2\alpha Y' + KY^2 + \left(\frac{1}{2}M + \alpha^2\right)Y + N = 0 \quad (4.10)$$

where K is the constant in f and

$$\begin{aligned} M &= aa'' - \frac{1}{2}a'^2 + 4K \int \left[da^{-3/2} \exp\left(-\alpha \int \frac{dx}{a}\right) \right] dx \\ N &= \frac{1}{4}I(4\alpha^2 + 2M - 4KI) + a^{-1}(a' + 2\alpha)I' + a^{-2}I'' \end{aligned} \quad (4.11)$$

where

$$I = \int \left[da^{-3/2} \exp\left(-\alpha \int \frac{dx}{a}\right) \right] dx$$

and M and N are constants. (In fact M comes from the formal integration of equation (4.6).)

The constant term in equation (4.10) can be removed by a translation, $Y = Z + J$, where J satisfies

$$KJ^2 + \left(\frac{1}{2}M + \alpha^2\right)J + N = 0 \quad (4.12)$$

and equation (4.10) becomes

$$Z'' + 2\alpha Z' + KZ^2 + \left(\frac{1}{2}M + \alpha^2 + 2KJ\right)Z = 0. \quad (4.13)$$

This is only possible if the discriminant

$$\Delta_1 = \left(\frac{1}{2}M + \alpha^2\right)^2 - 4KN \geq 0.$$

Under the transformation

$$u = \exp(-\frac{2}{3}\alpha X) \quad Z = \frac{4\alpha^2}{25K} vu^2 \quad (4.14)$$

equation (4.13) is transformed to

$$v'' + v^2 = 0 \quad (4.15)$$

which is easily reduced to a quadrature. For this to be possible, α must take the value

$$\alpha^2 = \frac{25}{24} \sqrt{\Delta_1}. \quad (4.16)$$

In view of the expression for Δ_1 , equation (4.16) leads to the requirement that

$$KN \geq m^2/4.$$

Thus we see that without actually solving equation (4.6) we have been able to delineate the conditions under which the generalized Emden-Fowler equation can be reduced to quadratures.

Although we have been unable to solve equation (4.6) in general, it is easy to obtain some particular solutions in the case $\alpha = 0$. If one assumes that a has the form

$$a = (A_0 + A_1 x)^\beta$$

it is straightforward to see that consistency of equation (4.6) demands that $\beta = 6/7$ and d be constant. This recovers the $m = -20/7$ case with the obvious generalization to $A_0 + A_1 x$ instead of x .

The trial function

$$a = (A_0 + A_1 x + A_2 x^2)^\beta$$

yields $\beta = 4/7$, d a monomial and $\Delta = 0$ and so the $m = -15/7$ case is recovered. (Once again, these two cases are limited by a transformation, namely $x \rightarrow x^{-1}$, $y \rightarrow y/x$.)

5. CONCLUSION

In this paper we have sought similarity solutions to the Emden-Fowler equation

$$y'' + x^m y^n = 0$$

and the generalized Emden-Fowler equation

$$y'' + f(x)y^n = 0.$$

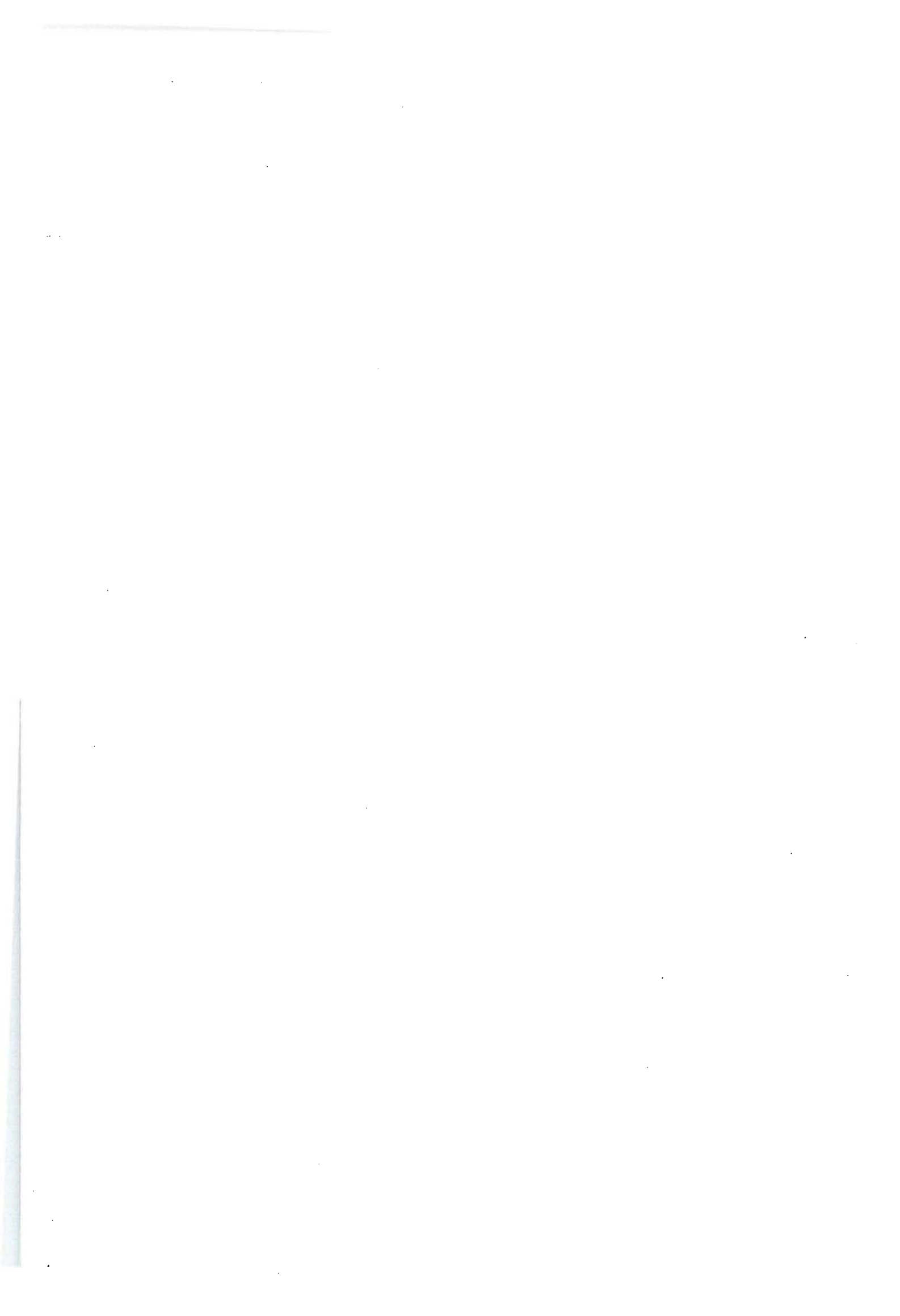
We have found a variety of solutions for various values of m and n in the former case and various functional forms of $f(x)$ and values of n in the second case. It is interesting to note that it was the order-two equations arising in solutions to the Einstein field equations which provided the richest structure.

Acknowledgements—One of us (SDM) thanks the Centre for Non-linear Studies of the University of the Witwatersrand, Johannesburg for hospitality whilst this work was undertaken and another (PGLL) M. Fairman and F. M. Mahomed for useful discussions.

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On the elementary Schrödinger bound states and their multiplets

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Journal of Mathematical Physics **33** (1992) 2785–2794

On the elementary Schrödinger bound states and their multiplets

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(Received 31 January 1992; accepted for publication 15 April 1992)

The problem of the existence of elementary bound states is discussed. A—trivial—observation that every elementary wave function $\psi^{[l]}(r)$ is an exact bound state for an appropriate potential, $V(r) = V^{[l]}[\psi(r), r]$, is shown to lead to a very transparent form of the “quasiexact” (QE) solvability condition $V^{[l]} = V^{[l]}$ for doublets and multiplets of the ψ 's. In this sense, the particular class of elementary ansätze, $\psi^{[l]}(r) = r^l \text{polynomial}(r^2) \times \exp[r^\mu \text{polynomial}(r^2)]$, also defines the particular class of QE-solvable potentials. They have an elementary nonpolynomial (rational) form, possibly also with a strongly singular—repulsive—core at the origin. The properties of these forces are discussed in detail.

I. INTRODUCTION

It is well known that the Schrödinger equation for harmonic oscillators is exactly solvable. Their potential $V = V^{[0]}(r) = Ar^2$ with $A > 0$ is a rather exceptional force. In accord with Newton,¹ we may only list three other completely solvable radial Schrödinger equations,

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V^{[l]}(r) \right] = E\psi(r), \quad l=0,1,\dots \quad (1)$$

The first one, with Coulombic $V^{[1]}(r) = B/r$ and $B < 0$, proves obtainable simply by the change of variables $r \rightarrow r' = r^2$ in the harmonic oscillator, Eq. (1).² In the second case the square-well potential function, $V^{[2]}(r) = Ar^\infty$, remains discontinuous. Finally Newton's last case, the solvable—repulsive—spike with $V^{[3]}(r) = Ar^{-2}$ does not possess any bound states at all.

Weaker definitions of the exact solvability may be—and are currently being—accepted. Even the mere restriction of Eq. (1) to a single partial wave ($l = l_{\text{fixed}}$) enables one to solve this equation already for a fairly broad class of the so-called Bargmann potentials.¹

Recently, a strengthening tendency toward further weakening of the solvability concept may be noticed. Following the pioneering work of Singh *et al.*³ devoted to the potential

$$V(r) = Ar^2 + Br^4 + Cr^6, \quad C > 0, \quad (2)$$

many people started feeling happy even when a *single* solution of Eq. (1) acquired an elementary form. The construction of these partial or “quasiexact” (QE) solutions was extended by Magyari⁴ to arbitrary polynomials.

During the later development the interest in polynomial QE forces was complemented by studies of the non-polynomial interaction

$$V(r) = Ar^2 + \frac{Br^2}{1 + Cr^2}, \quad C > 0, \quad A > 0 \quad (3)$$

(Refs. 5–11), of the strongly singular potential

$$V(r) = Ar^2 + Br^{-4} + Cr^{-6}, \quad C > 0, \quad A > 0 \quad (4)$$

(Ref. 12), and of the various more complicated descendants of these examples. Here we intend to propose a further extension or “natural closure” of this QE class of $V(r)$'s.

We shall start from a fixed, postulated structure of the ground-state $\psi(r)$'s. In the present paper we shall only use exponentials, powers, and polynomials as elementary functions. Thus we take the wave functions to be of the form

$$\psi(r) = r^l \text{polynomial}(r^2) \times \exp[r^\mu \text{polynomial}(r^2)]. \quad (5)$$

Arbitrary superpositions of the above examples (2)–(4) will be shown to enter this QE-solvability scheme in a natural way (Sec. II).

In Sec. III, the formal challenge of constructing and/or proving the simultaneous existence of several elementary bound states $\psi = \psi^{[j]}(r)$, $j=1,2,\dots,t$ for the same potential $V(r)$ will be analyzed. Next, Sec. IV will be devoted to the underlying algebraic considerations. Section V will complement it by a short discussion of the possible moves beyond the imminent QE-solvability limitations. In Sec. VI we provide a summary.

II. THE CONCEPT OF QE SOLVABILITY

In the light of the various possible (e.g., perturbative^{13,14}) applications of the QE solutions and/or of their multiplets, we believe that their completion and general analysis remains a topical problem. An indirect indication of its importance may be documented by the multiple (independent) rediscoveries of, say, the QE results of Singh *et al.*³ (cf. e.g., Refs. 15–19).

The initial encouragement of our present new QE study stemmed from a rather puzzling mixture of trivial and nontrivial features in virtually any QE-solvable problem. In particular, a strong motivation came from constructions of multiplets, as described, e.g., in Ref. 12 on the singular equation (4) or in Refs. 9 and 20–22, devoted to the nonpolynomial anharmonicity (3).

A. The reconstruction of a single potential

An explicit construction of a single elementary bound state becomes entirely trivial once we proceed in an “inverse” manner²³ and postulate some (i.e., arbitrary, not necessarily elementary) form of the exact bound state $\psi(r)$.^{7,24} As a consequence our partially solvable forces will be easily obtainable from Eq. (1) once we reread it as an explicit formula for, or definition of, $V(r)$ itself, viz.,

$$V(r) = E - \frac{l(l+1)}{r^2} + \frac{\psi''(r)}{\psi(r)}. \quad (6)$$

Obviously any imposed ground-state wave function $\psi = \psi_{l,0}^{(1)}(r)$ generates the corresponding potential $V = V_{l,0}^{(1)}(r)$ explicitly, via the simple rearrangement (6) of Eq. (1). In a phenomenological context the trivial inverse procedure is often used, e.g., in spin systems²³ or laser theory.²⁵ Of course, most papers on partial or “quasiexact” solvability of Eq. (6) deal with its less trivial aspects. Thus let us start with their short—and by far incomplete—review.

B. The possible applications of QE solvability

For the sake of brevity people often select polynomial potentials as the simplest models that may be partially solvable.^{4,26} Several formal aspects appeared as a motivation for these studies. We may recall, e.g., the presence of continued fractions that terminate³ and/or the above-mentioned change of variables, giving the Coulomb + linear + harmonic Fivel modification of Eq. (2).²⁷

Results of this type may be generalized in certain very natural ways.²⁸ Again a broad variability of the QE forces themselves (e.g., after the same change of variables or after a less straightforward transition to the noncentral²⁹ or many-body¹⁹ cases) is worth mentioning.

QE considerations are usually assisted by detailed analyses of some separate examples. Indeed, starting from the simplest sextic anharmonic example of Singh *et al.*,³

and ending up, e.g., with quasi-Schrödinger equations in some curved spaces,¹⁸ the examples always clarify our understanding of the QE phenomenon. In this setting studies of the algebraic properties of the QE models¹⁷ or of their supersymmetric background³⁰ or recurrent features³¹ should also be recalled. QE examples play an important role in the analyses of $1/N$ expansions,³² in the approximative extrapolations,³³ etc.

A methodically important extension of the QE horizon was provided by transition to nonpolynomial forces of the form $V = \text{polynomial}/\text{polynomial}$,^{5–7,10} possibly also with singularities in $V(r)$ at finite values of r .¹¹ The well-behaved forces of the latter type became very popular as a testing ground. A particular emphasis has been laid upon the simplest nontrivial nonpolynomial anharmonicity, $x^2/(1+gx^2)$ ^{8–11} and/or its screened-Coulombic equivalent.³⁴ We remind the reader of the pertinent interesting algebraic results and observations of Whitehead *et al.*,³⁵ who proposed to treat certain sets of solutions as multiplets of Sturmians. Adhikari *et al.*³⁶ reemphasized the relevance of rational examples in supersymmetric quantum mechanics.

In a purely computational context new features of the nonpolynomial case were noticed in Refs. 37 and 38. Whenever these forces become singular, we immediately lose the firm background of standard theory.³⁹ This is the reason why the singular case has been studied separately by several authors.^{12,40} Vice versa, we may only apply the relevant and well-established Floquet mathematical theory³⁹ to the simplest (e.g., inverse-sextic) singular anharmonicities.⁴¹

In a way an ultimate methodical QE extreme is represented by the forces that are nonpolynomial (rational) and strongly singular (repulsive) in the origin at the same time. Particular attention has to be paid to the extremely natural character of this “last-step” generalization, which seems to have been, rather undeservedly, neglected in the literature up to now. It preserves all the merits of feasibility of the simpler QE potentials.

In this paper we shall study, within these extended QE limits, as specified by Eq. (5) above, the general problem of existence (and also of the construction) of multiplets—mostly doublets—of bound states. We shall put emphasis upon the specification and representations of some underlying algebra and symmetries. Marginal attention will also be paid to the role that QE solutions may have in the various numerical algorithms and standard as well as new approximation schemes.

III. THE QE CONSTRUCTIONS

If one solves the radial Schrödinger equation (1) numerically, one may encounter unexpected difficulties indeed. For example, one of the standard methods for numerical computation based on the Prüfer transform⁴² in the NAG Library⁴³ might fail for singular potentials (cf.

Ramatlhape *et al.*,⁴⁴ who reported that the more robust algorithm of Gorringer and Leach⁴⁵ was to be preferred). Thus, precisely the above-mentioned exact solutions of the singular and fractional type are needed there for testing. Of course, the complexity of their explicit construction may vary.

A. Trivial case—The ground-state-generated QE potentials

Let us postulate each exceptional eigenstate $\psi^{[i]}(r) = \psi(r)$ of a potential $V^{[i]}(r)$ in the form (5) or, better,

$$\psi(r) = D(r)\exp[-F(r)], \tag{7}$$

with $D(r)$ merely polynomial,

$$D(r) = \sum_{n=0}^N d_n r^{2n}, \tag{8}$$

and a slightly more complicated exponent

$$F(r) = -\lambda \log r + \sum_{m=0}^M f_m r^{2m+\mu}. \tag{9}$$

The simplest illustration may be taken over from Singh *et al.*³ With trivial $D=1$ and the simplest nontrivial $F(r) = \alpha r^2 + \beta r^4$ inserted in Eq. (6), we recover their sextic polynomial potential (2) immediately. Similarly, in accord with Flessas,⁵ we may choose trivial $F(r) = \alpha r^2$ and nontrivial $D(r) = 1 + gr^2$ and get the second “next-to-solvable” potential of Eq. (3). We also obtain the strongly singular force (4) from the trivial $D=1$ and nontrivial $F(r) = \alpha r^2 + \beta r^{-2}$. Thus we may expect that the next-to-the-simplest examples will result from a combined choice of nontrivial D 's and F 's.

In general, we may rewrite Eq. (6) simply as an explicit formula or prescription,

$$V(r) - E + \frac{l(l+1)}{r^2} = \frac{\psi''(r)}{\psi(r)} \Leftrightarrow W(r) - E = F'^2 - F'' + \frac{(D'' - 2D'F')}{D}, \tag{10}$$

for the “unknown” potential. Thus let us summarize the straightforward overall strategy that applies in the case of a single ground-state input ψ . We postulate an elementary formula (7)–(9) for ψ and get the explicit definition equation (10) of the corresponding “partially solvable” potential.

By way of further illustration let us fix nontrivial $N=1$ and, say, $M=2$ in Eq. (9). Then, with the simplest negative $\mu=-2$, we have

$$D(r) = 1 + gr^2, \quad F(r) = \alpha r^2 + b \log r + cr^{-2}, \quad a, c, g > 0, \tag{11}$$

after an inessential change in notation. By immediate evaluation, we again arrive at the specification of the underlying QE solvable potential

$$W(r) - E = 4a^2 r^2 + (4ab - 2a) + \frac{(b + b^2 - 8ac)}{r^2} - (4b + 6) \frac{c}{r^4} + 4 \frac{c^2}{r^6} + \frac{8cgr^{-2} + (2g - 4bg) - 8agr^2}{1 + gr^2}. \tag{12}$$

Various potentials discussed in the older literature may be obtained at the special values of the couplings from Eq. (12), e.g., the rational equation (3) results from the choice of $c=0$, the multinomial singular equation (4) is obtained at $g=0$, and the polynomial—harmonic oscillator—comes out for trivial $g=c=b=0$. Again, we may emphasize that the requirement of regularity $c=0$, as is usually used in the literature, does not imply any simplification.

B. The QE multiplets

Now that we have guaranteed, in the trivial manner, the existence of “the simplest” ground state of the form (7) and (11), let us consider a search for the second bound state solution of the same Schrödinger equation (1) with the potential (12). In general, the assumption of elementarity of this second bound state $\hat{\psi}$ becomes much less trivial.

Obviously we must require that the new, overcared elementary factors are of the form

$$\hat{D} = DP, \quad \hat{F} = F, \tag{13}$$

with a polynomial P . The shift of energy and the coincidence of potentials,

$$\hat{E} = E + \epsilon, \quad \hat{W}(r) = W(r), \tag{14}$$

form the further conditions of existence of our doublet or, in general, of the several overcared components of a multiplet.

Insertion into Eq. (10) gives immediately a simpler formulation,

$$\epsilon = \frac{D'' - 2D'F'}{D} - \frac{\hat{D}'' - 2\hat{D}'F'}{\hat{D}}, \tag{15}$$

of our algebraic problem. We may also rewrite it as our final formula,

$$\epsilon = -\frac{2D'P'}{DP} + \frac{2P'F' - P''}{P}, \tag{16}$$

which must be satisfied by the new “unknown” wave function or, in the present notation, by the polynomial $P=P(r)$.

C. The simplest doublet—The constructive proof of existence

We take $D(r)$ and $F(r)$, as given in (11), to denote

$$P=P(r)=1+fr^2+hr^4+zr^6, \tag{17}$$

and try to satisfy Eq. (16). In the first step we notice that the right-hand side of Eq. (16) becomes singular in the limit $r \rightarrow 0$. Thus, as long as the left-hand side—the energy difference ϵ —must remain r independent and finite, we have to fix the value of the coupling f by the condition

$$f=0. \tag{18}$$

As an immediate counterpart to the latter postulate, let us also imagine that the limiting transition $r \rightarrow \infty$ simplifies Eq. (16) considerably. The second formula,

$$\epsilon=24a, \tag{19}$$

then follows simply from the leading-order part of this limit in Eq. (16).

In the third step we notice that the right-hand side function of Eq. (16) also grows indefinitely near the (pair of) roots of $D(r)$. The disappearance of such a spurious pole necessitates that the polynomial D divides the product of polynomials $D'P'$, so that we have to fix

$$z=\frac{2}{3}gh. \tag{20}$$

We are left with the sufficiently simple form of Eq. (16). It already admits a comparatively quick reduction to the almost elementary set of equations,

$$a=-\frac{2}{3}hc, \quad 2b=3+4cg, \quad 2hc+6cg^2=9g, \tag{21}$$

from which we have, in conjunction with the requirements of normalizability and regularity, the final specification of the (unique) solution,

$$h=\frac{-3a}{2c} < 0, \quad z=\frac{-ga}{c} < 0, \tag{22}$$

$$b=2gc+\frac{2}{3} > 0, \quad g=\frac{1}{4c} [3+(9+8ac)^{1/2}] > 0.$$

An important observation concerns the nodal zeros of our new state ψ . As long as both h and z are negative, our second wave function may and must have just one nodal zero and, therefore, it has to be interpreted as the first excited state of our system.

In summary, our construction specifies the couplings compatible with the current normalizability and regular-

ity requirements. At least one QE doublet of bound states exists. We notice that this doublet degenerates to the example (3) for $c=0$.

IV. THE ALGEBRAIC BACKGROUND OF QE SOLVABILITY

Let us assume the existence of the second QE solution with \bar{D} replacing D and $E+\epsilon$ replacing E . Then, Eq. (15) can be written in the form

$$(D\bar{D}'-D'\bar{D})'-2F'(D\bar{D}'-D'\bar{D})+\epsilon D\bar{D}=0. \tag{23}$$

We may also choose any triplet of integers I, J , and K as the respective highest (finite) powers of r in F, D , and \bar{D} ,

$$F=\sum f r^i, \quad D=\sum d r^j, \quad \bar{D}=\sum \bar{d} r^k. \tag{24}$$

In this—slightly modified—notation, Eq. (23) becomes

$$\sum_j \sum_k [(j+k-1)(k-j)d\bar{d}_k r^{j+k-2}] - 2 \sum_i \sum_j \sum_k [if_i(k-j)d\bar{d}_k r^{j+k-2}] + \epsilon \sum_j \sum_k d\bar{d}_k r^{j+k} = 0, \tag{25}$$

in which the highest powers of r are $J+K-2, I+J+K-2$, and $J+K$ in the first, second, and third sums, respectively.

A. The nonlinear coupled algebraic equations

In the case that $I > 2$ the terms in the second sum in Eq. (25) must self-cancel for powers greater than $J+K$. A simple calculation shows that $J=K$ and that, if D and \bar{D} are normalized to $d_j=1=\bar{d}_k$, the coefficients $d_{j-1}=\bar{d}_{k-1}$, etc. until the other terms in (25) are to be considered. If I is sufficiently large, all the coefficients $d_j=\bar{d}_k$ and so $\epsilon=0$, i.e., there is only one state for the potential given by Eq. (10).

If I is not too large a number, there still remains some freedom in Eq. (25), which is now reduced to a system of $2K+1$ equations in the $K+3-I$ undetermined coefficients \bar{d}_k and the parameter ϵ . As such, it is a constrained eigenvalue problem. Although this feature has not explicitly been noted before, we do not labor it, but rather consider it an alternative approach to determining the set of polynomials D that satisfy Eq. (10).

As an elementary example we take

$$F=ar^4+br^2, \quad D=c+r^2, \quad \bar{D}=d+r^2 \tag{26}$$

(by implication W is even and so \bar{D} is chosen as even), which, when substituted into (25), leads to the three non-trivial equations (the other two are zero equals zero):

$$\epsilon-16a(c-d)=0,$$

$$\epsilon(c+d) - 8b(c-d) = 0, \tag{27}$$

$$\epsilon cd + 2(c-d) = 0.$$

The solutions of (27) are

$$c = [b \pm \sqrt{b^2 + 2a}] / 4a,$$

$$d = [b \pm \sqrt{b^2 + 2a}] / 4a,$$

$$\epsilon = \pm 8 \sqrt{b^2 + 2a},$$

which are consistent with the well-known results for the even sextic potential.

The case $I=2$ is typical of the case when a harmonic oscillator is perturbed by a potential that contains negative powers or is a rational function, the $\lambda x^2 / (1 + gx^2)$ perturbation being the best studied example. In (25), the highest powers of r are $J+K-2$, $J+K$, and $J+K$ in the first, second, and third sums, respectively. Equating coefficients of separate powers of r to zero starting from the $(J+K)$ th, we find that

$$\epsilon = 4f_2(K-J), \tag{28}$$

which means that multiplets can only occur when $K \neq J$, which is very opposite to the case $I > 2$. The next power gives

$$\frac{\bar{d}_{K-1}}{d_K} = \frac{d_{J-1}}{d_J} + \frac{f_1}{f_2}(K-J), \tag{29}$$

and expressions for \bar{d}_{K-n} , $n=2, \dots$, are readily found by routine manipulation.

In the context of $I=2$ the ansatz (13) may profitably be used, as it has been observed^{7,9} that the denominator of a rational potential occurs in the wavefunction. The formula (16) becomes, in an obvious notation,

$$\sum_j \sum_l l(l+2j-1) d_p r^{j+l-2} - 2 \sum_i i f_i \left[\sum_j \sum_l l d p_l \right] \times r^{j+l-2} + \epsilon \sum_j \sum_l d p_l r^{j+l} = 0, \tag{30}$$

with the highest power in P being r^L . The expression for ϵ is as in (28), with $K-J$ replaced by L . The next highest power of r yields

$$p_{L-1} = \frac{f_1 L}{2f_2} p_L \tag{31}$$

and further coefficients follow by means of routine arithmetic.

By way of example we take the well-known interaction

$$W = x^2 + \frac{\lambda x^2}{1 + gx^2}, \quad x \in (-\infty, \infty) \tag{32}$$

in one dimension. The state

$$\psi = (1 + gx^2) \exp(-\frac{1}{2}x^2) \tag{33}$$

exists, provided the coupling constant λ is related to g via

$$\lambda = -2g(g+2). \tag{34}$$

The energy is given by

$$E = 1 - 2g. \tag{35}$$

The nature of the potential in (32) is such that the wave functions are either even or odd. We write

$$P = \sum p_i x^{2i+\delta}, \tag{36}$$

where $\delta=0$ or 1 , depending upon whether the state is even or odd. With this, $D=1+gx^2$ and $F=\frac{1}{2}x^2$, Eq. (30) becomes

$$\begin{aligned} & \sum (2i+\delta)(2i+\delta-1) p_i x^{2i+\delta-2} + \sum [\epsilon + g(2i+\delta) \\ & \times (2i+\delta+3) - 2(2i+\delta)] p_i x^{2i+\delta} \\ & + g \sum [\epsilon - 2(2i+\delta)] p_i x^{2i+\delta+2} = 0. \end{aligned} \tag{37}$$

The coefficient matrix for the vector of p_i 's is tridiagonal. The last term permits the separation of a finite part of the eigenvalue problem by setting $\epsilon = 2(2I+\delta)$ and $p_i = 0$, $i=I+1, \dots$. The truncated part of (37) is now an eigenvalue problem on the parameter g .

We give some elementary results. When $\delta=0$, for $I=2$, $\epsilon=8$,

$$P = p_0(1 - 4x^2 + 4x^4/7), \quad g = -\frac{1}{28}, \tag{38}$$

and for $I=3$, $\epsilon=12$,

$$\begin{aligned} P &= p_0(1 - 6x^2 + 4(g+1)x^4 - 8(g+1)x^6/27), \\ g &= \frac{-20 \pm \sqrt{148}}{63}. \end{aligned} \tag{39}$$

It is noted that in both cases $g < 0$ so that we are treating the singular potential discussed in Refs. 10 and 36. The original state is the second excited state and the subsequent states are the sixth and fourth or eighth (depending on the sign in g), respectively.

Let us now put $\delta=1$. When $I=1$, $\epsilon=6$, $g=-\frac{2}{3}$ we get $P = p_0 x(1 - 2x^2/9)$, which is the fifth excited state. When

$I=2$, $\epsilon=10$, $g = (-13 \pm \sqrt{133})/18$, and we have $P=p_0x(1-(2g+4)/3x^2+(g+4)/15x^4)$. We may also note that, again, in both cases we are in the singular, $g < 0$, regime.

In full analogy with the above analysis of $r \rightarrow \infty$, we might parallel it by the similar discussion concerning the lowest powers of r and the equation (25) in the vicinity of zero $r=0$. This is quite straightforward and will not be done here.

B. The underlying "quasidynamical" Lie algebras

It is well known that many quasiexactly solvable problems may be related to a finite-dimensional representation of a Lie algebra. Turbiner¹⁷ was the first to treat the problem of the sextic potential via the representation

$$J^+ = z^2 \frac{d}{dz} - 2jz, \quad J^0 = z \frac{d}{dz} - j, \quad J^- = \frac{d}{dz}, \quad (40)$$

of $sl(2, R)$, where $z=x^2$ and $j(j+1)$ is the eigenvalue of the Casimir operator. Indeed, the commutators of these operators give the standard $sl(2, R)$ relations

$$[J^+, J^-] = -2J^0, \quad [J^0, J^\pm] = \pm J^\pm. \quad (41)$$

One of the difficulties with this representation of the elements of $sl(2, R)$ is that one is limited to polynomial potentials of degree six. Recently, de Souza Dutra and Boschi Filho⁴⁶ used a generalization of this representation to attain polynomial potentials of higher degree via the algebra $so(2, 1)$ [isomorphic to $sl(2, R)$]. The representation used was

$$T_1 = \alpha_2 x^{2-j} \left(\frac{d}{dx}\right)^2 + \alpha_1 x^{1-j} \left(\frac{d}{dx}\right) + \alpha_0 x^{-j},$$

$$T_2 = -\frac{i}{j} x \left(\frac{d}{dx}\right) - i\sigma, \quad T_3 = \alpha x^j, \quad (42)$$

which satisfies the $so(2, 1)$ algebra

$$[T_1, T_2] = -iT_1; \quad [T_2, T_3] = -iT_3; \quad [T_1, T_3] = -iT_2, \quad (43)$$

provided only that

$$\sigma = \frac{1}{2j} \left[\frac{\alpha_1}{\alpha_2} + j - 1 \right], \quad \lambda = -(2\alpha_2 j^2)^{-1}. \quad (44)$$

The constants α_1 , α_2 , and j are selected to match the different elements of the Hamiltonian.

The obvious question is whether the most general Hamiltonian may be given a representation in terms of the generators of the $sl(2, R)$ or $so(2, 1)$ algebras. The work of de Souza Dutra and Boschi Filho provides a clue in this regard.

Their approach was to write

$$H = \sum a_k (T_3)^k T_1 + \sum b_l (T_3)^l T_2 + \sum c_m (T_3)^m, \quad (45)$$

and, since T_3 was independent of $D := d/dz$, the third term permitted the introduction of an arbitrary polynomial potential. The ansätze of (42) and (45) can be naturally generalized to

$$T_1 = f_0 + f_1 D + f_2 D^2,$$

$$T_2 = g_0 + g_1 D, \quad (46)$$

$$T_3 = h_0,$$

and

$$H = a(T_3) T_1 + b(T_3) T_2 + c(T_3), \quad (47)$$

where the coefficient functions are to be determined.

The primary constraint on the generators in (46) is that they satisfy (43) and hence the system of coupled equations,

$$-if_2 = 2f_2 g'_1,$$

$$-if_1 = f_1 g'_1 - f'_1 g_1 + 2f_0 g'_0 + f_2 g'_1,$$

$$-if_0 = -f'_0 g_1 + f_1 g'_0 + f_2 g'_0, \quad (48)$$

$$-ig_0 = f_0 h'_0 + f_2 h'_0,$$

$$-ig_1 = 2f_2 h'_0,$$

$$-ih_0 = g_1 h'_0,$$

must be solved. This leads to the solutions

$$\begin{aligned}
 T_1 &= \frac{c+g_0^2}{h_0} + i\left(\frac{g_0}{h_0} - \frac{g_0'}{h_0'}\right) - \frac{1}{2}i\left(\frac{g_0}{h_0'} + i\frac{h_0 h_0'}{h_0'^2}\right)D - \frac{h_0}{h_0'^2}D^2, \\
 T_2 &= g_0 - \frac{ih_0}{h_0'}D, \\
 T_3 &= h_0,
 \end{aligned}
 \tag{49}$$

where g_0 and h_0 are arbitrary functions and c is an arbitrary constant.

It might be wondered whether a more general representation would be possible. Suppose that we take the generators for the $sl(2, R)$ algebra to be

$$\begin{aligned}
 J^+ &= a + bD + cD^2, \\
 J^- &= d + eD + fD^2, \\
 J^0 &= g + hD + iD^2,
 \end{aligned}
 \tag{50}$$

where the functions $a-i$ are to be determined through the relations

$$[J^+, J^-] = -2J^0, \quad [J^0, J^\pm] = \pm J^\pm.$$

After a calculation of tedious length we find that

$$\begin{aligned}
 J^+ &= G_1, \\
 J^- &= L^2G_1 + 2LG_2 + G_3, \\
 J^0 &= LG_1 + G_2,
 \end{aligned}
 \tag{51}$$

where

$$\begin{aligned}
 G_1 &= a + ruD + r^2D^2, \\
 G_2 &= \frac{1}{4}\left\{(r'-u)\left(2LQ + \int \frac{1}{r}\right) - 1\right\} \\
 &\quad - \frac{1}{2}\left(2LQ + \int \frac{1}{r}\right)D, \\
 G_3 &= \frac{1}{4}\left(2LQ + \int \frac{1}{r}\right)^2.
 \end{aligned}
 \tag{52}$$

Here L and Q are arbitrary constants, r and u are arbitrary functions, and the function a is the solution of the first-order equation:

$$\begin{aligned}
 0 &= \frac{1}{4}r^2\left\{(r'-u)\left(2LQ + \int \frac{1}{r}\right)\right\}' + \frac{1}{4}ru\left\{(r'-u)\right. \\
 &\quad \times \left. \left(2LQ + \int \frac{1}{r}\right)\right\}' + \frac{1}{2}r\left(2LQ + \int \frac{1}{r}\right)a' + a.
 \end{aligned}
 \tag{53}$$

This can be integrated to give

$$\begin{aligned}
 a &= K\left(2LQ + \int \frac{1}{r}\right)^{-2} + \frac{1}{4}(r'-u)^2 \\
 &\quad - \frac{1}{4}(r'-u)\left(2LQ + \int \frac{1}{r}\right)^{-1} - \frac{1}{2}r(r'-u)',
 \end{aligned}$$

where K is an arbitrary constant. However, there is no generalization, as $G_1, G_2,$ and G_3 are essentially the same as $T_1, T_2,$ and T_3 . In fact, in terms of writing down a Hamiltonian operator as a combination of J 's there is less freedom than with the T 's.

In summary, the analysis of the rational QE forces with a strong singularity in the origin remains very similar to its preceding simpler special cases. Now an important feature of the generalization may be seen in the increase of singularities in the underlying differential equation. Only some very general methodical considerations may be applied now to the construction of the non-QE solutions. This is very much of a challenge for applications.

V. A FEW ANALYTIC ASPECTS OF QE SOLVABILITY

One of the most straightforward generalizations of the QE solutions may be based on a replacement of the polynomial [say, $D(r)$ (8)] by its $N \rightarrow \infty$ Taylor-series limit. In accord with the underlying theory of ordinary differential equations,³⁹ every (i.e., normalizable as well as non-normalizable) solution $\psi(r)$, which pertains to the force $V(r)$ regular in the origin may then be expressed in the form of Eq. (7). Thus, the new, $N \rightarrow \infty$ extended class of $D(r)$'s becomes already "too rich"—its restriction must be guaranteed by an independent imposition of the physical boundary conditions.⁴⁷ Nevertheless, terminating solutions remain useful here as a non-numerical source of information.

A. QE solutions as points of degeneracy of the power-series method

Of course, a limiting transition $N \rightarrow \infty$ in $D(r)$ (8) may (and often does⁴⁷) violate the physical asymptotic boundary conditions. Hence any QE-like formal solution,

$$\psi(r) = \text{Taylor series} \times \text{an auxiliary exponential}, \tag{54}$$

need not represent a bound state, in general. Vice versa, there exists a number of examples^{3,47} where it does specify the exact solution at the correct energy.

In the QE context, it is interesting to notice that the standard Frobenius power-series ansatz may happen to coincide with an exact (i.e., QE) solution if and only if its

auxiliary exponential factor remains asymptotically equivalent to the physical WKB wave function,

$$\psi(r) = \psi_{\text{WKB}}(r) \sum_{n=0}^N h_n r^{n+l+1}. \tag{55}$$

In such a case the QE solutions may be obtained simply from the standard $N \rightarrow \infty$ Frobenius recurrences for the h 's,³⁹ complemented by the obvious conditions of termination at a finite dimension $M < \infty$,

$$\begin{pmatrix} Q_{0,0} & Q_{0,1} & 0 & 0 & \cdots & 0 \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & 0 & \cdots & 0 \\ & & \cdots & & & \\ Q_{M-1,0} & Q_{M-1,1} & & & & Q_{M-1,M} \\ Q_{M,0} & Q_{M,1} & & & & Q_{M,M} \\ Q_{M+1,0} & Q_{M+1,1} & & & & Q_{M+1,M} \\ & & \cdots & & & \end{pmatrix} \begin{pmatrix} h_0^{[q]} \\ h_1^{[q]} \\ \vdots \\ h_M^{[q]} \end{pmatrix} = 0. \tag{56}$$

Here $M = M^{[q]}$ and the superscript q is introduced to run over all the members of the multiplet, $q = 1, 2, \dots, t$.

Often the overcomplete set of linear algebraic QE equations (56) still remains finite and may be solved non-numerically (cf. e.g., Refs. 4, 7, or 12). In the infinite-dimensional limit, $M \rightarrow \infty$, these equations may (but they need not) remain solvable and their solutions may (though they need not) remain physical. For more details on a rigorous analysis of this limiting transition (and on the related computational, so-called "Hill-determinant," method), we refer, e.g., to Hautot's review.⁴⁷

B. Forces singular in the origin and doubly infinite Hill determinants

In the standard mathematical textbooks (e.g., Ref. 39), the usage of the name of Hill determinants differs from Ref. 47. The name remains reserved to situations where the force $V(r)$ is strongly repulsive in the origin. In such a context the above limiting transition $N \rightarrow \infty$ must be complemented by the decrease of $\lambda \rightarrow -\infty$ in Eq. (8). *Mutatis mutandis* most of the other statements of the preceding section remain valid again, and only the matrix Q in Eq. (56) has to be replaced by its doubly infinite generalization

$$\begin{pmatrix} & & \cdots & & \\ Q_{-L,0} & Q_{-L,1} & \cdots & Q_{-L,M} & \\ & & \cdots & & \\ Q_{-1,0} & Q_{-1,1} & \cdots & Q_{-1,M} & \\ Q_{0,0} & Q_{0,1} & \cdots & Q_{0,M} & \\ & & \cdots & & \\ Q_{M,0} & Q_{M,1} & \cdots & Q_{M,M} & \\ Q_{M+1,0} & Q_{M+1,1} & \cdots & Q_{M+1,M} & \\ & & \cdots & & \end{pmatrix} \begin{pmatrix} h_0^{[q]} \\ h_1^{[q]} \\ \vdots \\ h_M^{[q]} \end{pmatrix} = 0. \tag{57}$$

The new situation is, of course, not entirely identical. Let us only mention that the new generalized ansatz,

$$\psi(r) = \text{Laurent series} \times \text{an auxiliary exponential}, \tag{58}$$

need not suffice for representing bound states anymore. In fact, we have to replace l in Eq. (55) by (a pair of) free parameters so that only a pair of the QE-like ansätze (58) with $l = l_1$ and $l = l_2$ may—and must—be employed, in general. More details may be found, e.g., in Ref. 41.

C. QE solutions as starting points of perturbative considerations

In the standard (say, Rayleigh-Schrödinger) perturbation theory, people often combine numerical algorithms with perturbative considerations and the computations may start from a QE solution $\psi^{[0]}(r)$ and potential $V^{[0]}(r)$ in principle. Thus, whenever the latter force lies sufficiently close to the one under investigation, the use of QE states may be of importance. Now let us briefly mention a few other possible strategies.

In the simplest examples of type (2), several new perturbationlike uses of the QE solutions may be found in the recent literature (Refs. 22 and 33). Here let us just briefly mention the most straightforward one. Whenever we use the Hill-determinant method [i.e., basically an infinite-dimensional generalization of Eq. (56),⁴⁷] we may use an appropriate q th QE state as an approximation.¹⁴

Whenever we admit a (complex) pole in $V(r)$, we cannot use the above simple power-series methods at all—the rational functions $V(r)$ and their QE solutions $\psi(r)$ are better understood in a variational-like setting.³⁵ Then we may assume that a bound state in question is written as a finite sum,

$$|\psi\rangle = \sum_n |n\rangle h_n, \tag{59}$$

in some complete—not necessarily orthonormalized—set of states $|n\rangle$. In the spirit of the above-mentioned—now inapplicable—Hill-determinant constructions, we may even use a WKB-inspired choice of the type

$$\langle r|n\rangle = D(r)r^n \exp[-F(r)]. \quad (60)$$

More details may be found in Ref. 48.

VI. SUMMARY

An appeal of our new QE-solvable examples lies in their methodical aspects. They may combine a strong singularity in the origin with the presence of poles of $V(r)$ in the complex plane, so that any other non-numerical way of constructing bound states may represent a real challenge formally.

For polynomial potentials we may treat QE solvability as a certain degeneracy of a more universal Hill-determinant method. Polynomial plus singular interactions permit an application of Floquet theory and Laurent (=doubly infinite) power-series expansions in the similar spirit.⁴¹ Unfortunately, the more complicated forces (and especially those with singularities in the complex plane) cannot be treated in such a spirit, so that a common background of the QE phenomenon is not clear and unique.

Here we proposed a possible QE-classification scheme based on the ψ -initiated reconstruction of the potential. This approach proves efficient, even in the case of QE multiplets. An unexpected simplicity of the related algebra has been demonstrated on a number of examples here. Several particular bound-state solutions with elementary structure seem tractable by the common and transparent formalism.

The presence of features not excluded in the present QE examples (and, especially, poles lying near the real axis) usually restricts the use and/or convergence of the variational methods. Thus all the possible QE-solvable forces may find applications in the computational (as well as perturbative, etc.) approximation techniques.

Originally our present study was initiated by an optimistic expectation that an applicability of standard techniques and/or reinterpretations of the elementary QE bound states will adhere to some suitable Lie algebra or SUSY representations. In this context our present results remain unsatisfactory and incomplete. Nevertheless, we hope that we shall be able to cover more of these algebraic aspects of the QE phenomenon in the not too distant future.

ACKNOWLEDGMENTS

One of us (P.G.L.L.) thanks the Nuclear Physics Institute of the Czechoslovak Academy of Sciences for its

kind hospitality while this work was carried out. He also thanks the Foundation for Research Development of South Africa for its continuing support. We thank K. S. Govinder and C. S. Schei for assistance in the preparation of the manuscript.

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$sl(2, R)$, Ermakov systems and the magnetic monopole

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*Modern Group Analysis: Advanced analytical and computational methods in
mathematical physics* N H Ibragimov, M Torrisi and A Valent edd (Kluwer,
Dordrecht, 1993) 255–264

$sl(2, R)$, ERMAKOV SYSTEMS AND THE MAGNETIC MONOPOLE

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Abstract. A representation of the group $sl(2, R)$ is the symmetry algebra of the Lewis-Ermakov invariant. Under certain circumstances it is also the symmetry algebra of the associated pair of second order differential equations. The most general pair of sodes possessing the same $sl(2, R)$ symmetry is called a generalized Ermakov system. In three dimensions the system of sodes invariant under $sl(2, R)$ symmetry contains the magnetic monopole as a special case. The general system in three dimensions always admits a Poincaré vector and in fact three such vectors always exist. A special case of weak generalized Ermakov systems in three dimensions is the monopole-Kepler problem.

1. Ermakov systems

In 1880 V P Ermakov [1] put forward a method to find a first integral of

$$\ddot{q} + \omega^2(t)q = 0 \quad (1)$$

by introducing the auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \frac{A}{\rho^3}. \quad (2)$$

Elimination of $\omega^2(t)$ between (1) and (2) by the combination $\rho(1) - (2)q$, multiplication by the integrating factor $(\rho\dot{q} - \dot{\rho}q)$ and integration give the first integral

$$I = \frac{1}{2}(\rho\dot{q} - \dot{\rho}q)^2 + \frac{1}{2}A \left(\frac{q}{\rho}\right)^2 \quad (3)$$

known as the Ermakov invariant. It is also known as the Lewis invariant after Ralph Lewis who found it in 1966 by an independent method [2]. An elementary generalisation was provided by Ermakov who replaced the A/ρ^3 term by $\rho^{-3}f(q/\rho)$.

In 1979 Ray and Reid [3] treated the system

$$\begin{aligned} \ddot{x} + \omega^2(t)x &= x^{-3}f(y/x) \\ \ddot{y} + \omega^2(t)y &= y^{-3}g(y/x) \end{aligned} \quad (4)$$

and used the same procedure as Ermakov to determine the first integral

$$I = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + \int^{y/x} [uf(u) - u^{-3}g(u)] du. \quad (5)$$

In fact I exists for any system of the form [4]

$$\begin{aligned}\ddot{x} + \square x &= x^{-3}f(y/x) \\ \ddot{y} + \square y &= y^{-3}g(y/x).\end{aligned}\quad (6)$$

The structure of I suggests angular momentum. In plane polar coordinates the Ermakov system is

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 + \square r &= r^{-3}F(\theta) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= r^{-3}G(\theta)\end{aligned}\quad (7)$$

and the Ermakov invariant is

$$I = \frac{1}{2}(r^2\dot{\theta})^2 - \int G(\theta)d\theta \quad (8)$$

and is manifestly the autonomous first integral of the angular equation in system (7).

2. Algebraic properties of the Ermakov invariant

If a function $I(t, x, \dot{x})$ is invariant under the action of the first extension of a Lie point symmetry $G = \tau(t, x)\partial/\partial t + \eta_i(t, x)\partial/\partial x_i$,

$$G^{[1]}I := \tau \frac{\partial I}{\partial t} + \eta_i \frac{\partial I}{\partial x_i} + (\dot{\eta}_i - \dot{x}_i \dot{\tau}) \frac{\partial I}{\partial \dot{x}_i} = 0. \quad (9)$$

In the case of

$$I = \frac{1}{2}(r^2\dot{\theta})^2 - \int G(\theta)\dot{\theta} \quad (10)$$

Leach [5] showed that there were three point symmetries, viz.

$$\begin{aligned}G_1 &= \frac{\partial}{\partial t} \\ G_2 &= 2t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \\ G_3 &= t^2 \frac{\partial}{\partial t} + tr \frac{\partial}{\partial r}\end{aligned}\quad (11)$$

with $[G_1, G_2] = 2G_1$, $[G_1, G_3] = G_2$, $[G_2, G_3] = 2G_3$, which is a representation of the Lie algebra $sl(2, R)$. The system of differential equations

$$\mathbf{E}(t, x, \dot{x}, \ddot{x}) = 0 \quad (12)$$

is invariant under the Lie point symmetry $G = \tau\partial/\partial t + \eta_i\partial/\partial x_i$ if

$$G^{[2]}\mathbf{E}(t, x, \dot{x}, \ddot{x})|_{\mathbf{E}=0} = 0, \quad (13)$$

where

$$G^{[2]} = \tau \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial x_i} + (\dot{\eta}_i - \dot{x}_i \dot{\tau}) \frac{\partial}{\partial \dot{x}_i} + (\ddot{\eta}_i - 2\dot{x}_i \ddot{\tau} - \dot{x}_i \dot{\tau}) \frac{\partial}{\partial \ddot{x}_i}. \quad (14)$$

The system

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= r^{-3}F(\theta) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= r^{-3}G(\theta) \end{aligned} \tag{15}$$

has the same symmetries as I . This is not the case for the general system (7). However, if the additional term is $\omega^2 r$, another representation of $sl(2, R)$ leaves (7) invariant.

3. Generalized Ermakov systems from the $sl(2, R)$ viewpoint: two dimensions

An inverse problem is created [6] if one takes the $sl(2, R)$ representation, (11), and seeks the form of $f(t, r, \theta, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}) = 0$ invariant under all three symmetries. The action of G_1 removes t from f . G_2 leads to the five characteristics $\theta, u = r\dot{r}, v = r^2\dot{\theta}, w = r^3\ddot{r}$ and $x = r^4\ddot{\theta}$. G_3 reduces the number of characteristics to four θ, u, v, w and $y = x + 2uv$ to give the general form

$$f(\theta, r^2\dot{\theta}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^3\ddot{r}) = 0. \tag{16}$$

How is this done? Consider G_2 .

$$G_2^{[2]} = 2t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} + 0 \frac{\partial}{\partial \theta} - \dot{r} \frac{\partial}{\partial \dot{r}} - 2\dot{\theta} \frac{\partial}{\partial \dot{\theta}} - 3\ddot{r} \frac{\partial}{\partial \ddot{r}} - 4\ddot{\theta} \frac{\partial}{\partial \ddot{\theta}} \tag{17}$$

and

$$G_2^{[2]} f|_{f=0} = 0 \tag{18}$$

leads to the associated Lagrange's system for the characteristics

$$\frac{dr}{r} = \frac{d\theta}{0} = \frac{d\dot{r}}{-\dot{r}} = \frac{d\dot{\theta}}{-2\dot{\theta}} = \frac{d\ddot{r}}{-3\ddot{r}} = \frac{d\ddot{\theta}}{-4\ddot{\theta}} \tag{19}$$

(recall $\partial f/\partial t = 0$). These are easily solved.

The general two-dimensional system invariant under $sl(2, R)$ is

$$\begin{aligned} F(\theta, r^2\dot{\theta}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}) &= 0 \\ G(\theta, r^2\dot{\theta}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}) &= 0. \end{aligned} \tag{20}$$

Attention is confined to the forms

$$\begin{aligned} r^3\ddot{r} &= f_1(\theta, r^2\dot{\theta}) \\ r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} &= g(\theta, r^2\dot{\theta}) \end{aligned} \tag{21}$$

which can be written in the more physical forms

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= r^{-3}f(\theta, r^2\dot{\theta}) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= r^{-3}g(\theta, r^2\dot{\theta}) \end{aligned} \tag{22}$$

($f_1 = f + (r^2\dot{\theta})^2$). Note that (22) differs from (15) by the inclusion of the magnitude of the angular momentum in the arguments of f and g . If new time, $\tau = \int r^{-2} dt$, and the inverse radial distance $\chi = r^{-1}$, are introduced the system (22) becomes

$$\chi'' + [\theta'^2 + f(\theta, \theta')] \chi = 0 \quad (23)$$

$$\theta'' = g(\theta, \theta'). \quad (24)$$

Under 'appropriate' conditions (24) may be integrated to give

$$I = M(\theta, \theta') \quad (25)$$

which is the generalisation of the Ermakov invariant (5). Assuming that (25) can be inverted (which is always possible locally) as

$$\theta' = N(\theta, I), \quad (26)$$

this can be integrated to give

$$\tau - \tau_0 = \int \frac{d\theta}{N(\theta, I)}. \quad (27)$$

Finally, with the inversion of (27) to give

$$\theta = J(\tau, I, \tau_0), \quad (28)$$

it is evident that (23) reduces to the time-dependent harmonic oscillator.

4. Generalised Ermakov systems from the $sl(2, R)$ viewpoint: three dimensions

The extension of the results of § 3 to three dimensions is straight forward [7]. For an equation of the form

$$f(t, r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}, \ddot{r}, \ddot{\theta}, \ddot{\phi}) = 0 \quad (29)$$

to be invariant under the action of the second extensions of the generators of $sl(2, R)$ given in (11) the characteristics are $\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}$. Consideration is restricted to the system of equations

$$\begin{aligned} r^3\ddot{r} &= f(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \\ r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} &= g(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \\ r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi} &= h(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}). \end{aligned} \quad (30)$$

In terms of the new time T and the inverse radial distance χ these are

$$\begin{aligned} \chi'' &= f(\theta, \phi, \theta', \phi') \chi \\ \theta'' &= g(\theta, \phi, \theta', \phi') \\ \phi'' &= h(\theta, \phi, \theta', \phi') \end{aligned} \quad (31)$$

which are too general as they stand. This generality is reduced by the imposition of an additional symmetry requirement. A natural one is invariance under the rotation group $so(3)$. Invariance under the algebra $so(3)$ represented by

$$\begin{aligned} G_4 &= \frac{\partial}{\partial \phi} \\ G_5 &= \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \\ G_6 &= \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \end{aligned} \tag{32}$$

requires that the system take the form

$$\begin{aligned} r^3 \ddot{r} &= A_1(L) \\ r^4 \ddot{\theta} + 2r^3 \dot{r} \dot{\theta} &= r^4 \dot{\phi}^2 \sin \theta \cos \theta + B(L)r^2 \dot{\theta} - C(L)r^2 \dot{\phi} \sin \theta \\ r^4 \ddot{\phi} + 2r^3 \dot{r} \dot{\phi} &= -2r^4 \dot{\theta} \dot{\phi} \cot \theta + \frac{1}{\sin \theta} [B(L)r^2 \dot{\phi} \sin \theta + C(L)r^2 \dot{\theta}] \end{aligned} \tag{33}$$

where $L^2 := r^4(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)$ is the square of the magnitude of the angular momentum. In vector form (33) takes the compact form

$$\ddot{\mathbf{r}} = \frac{1}{r^3} \{ A(L)\hat{\mathbf{r}} + B(L)\hat{\boldsymbol{\omega}} + C(L)\hat{\mathbf{L}} \}, \tag{34}$$

where

$$\hat{\boldsymbol{\omega}} := \hat{\mathbf{L}} \times \hat{\mathbf{r}} \quad \text{and} \quad A = A_1 + L^2.$$

The full symmetry, $sl(2, R) \oplus so(3)$, of (34) can be broken by the addition of an additional radial term without affecting the properties inherent in the $\hat{\boldsymbol{\omega}}$ and $\hat{\mathbf{L}}$ components just as was the case of the two-dimensional system described by (7). Some particular cases are

- (i) $B = C = 0$. L is conserved and so $A(L)$ is constant. This represents a Newton-Cotes spiral with an excess/deficit of angular momentum.
- (ii) $A = B = 0$. $C(L) = \lambda L$. This describes a particle moving in the field of a magnetic monopole. There is the conserved Poincaré vector [8]

$$\mathbf{P} = \mathbf{L} + \lambda \hat{\mathbf{r}}. \tag{35}$$

The motion is on the surface of a cone with semivertex angle $\arccos(\lambda/P)$. To determine the Poincaré vector for (34) assume a vector of the form

$$\mathbf{P} := I\hat{\mathbf{r}} + J\hat{\boldsymbol{\omega}} + K\hat{\mathbf{L}}.$$

The requirement that $\dot{\mathbf{P}}$ be zero leads to

$$\frac{d}{dt} \begin{pmatrix} I \\ J \\ K \end{pmatrix} = r^{-2} \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} \begin{pmatrix} I \\ J \\ K \end{pmatrix}. \tag{36}$$

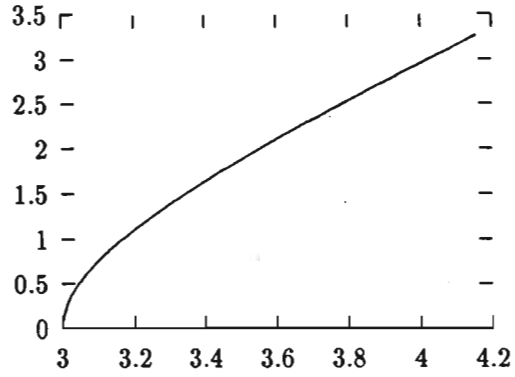


Fig. 1. $B = C = 0$; $A(L)$ is constant. $A(L) > 0$. Newton-Cotes spiral with an excess of angular momentum.

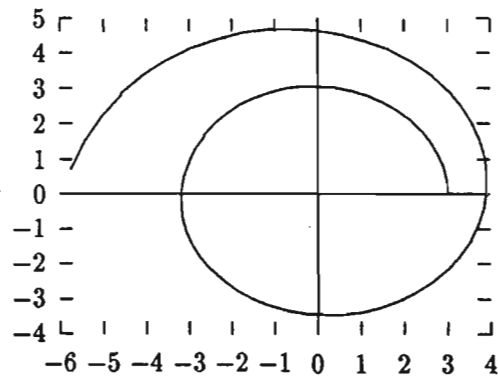


Fig. 2. $B = C = 0$; $A(L)$ is constant. $A(L) < 0$. Newton-Cotes spiral with deficit of angular momentum.

In terms of new time, $T := \int r^{-2} dt$, (36) is

$$\frac{d}{dT} \begin{pmatrix} I \\ J \\ K \end{pmatrix} = \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} \begin{pmatrix} I \\ J \\ K \end{pmatrix}. \quad (37)$$

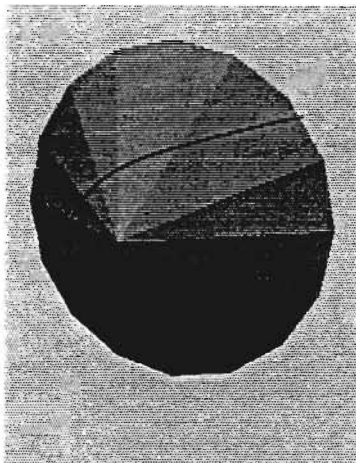


Fig. 3. $A = B = 0$; $C(L) = \lambda L$. Particle moving in the field of a magnetic monopole. Motion is on the surface of a cone with semivertex angle $\arccos(\lambda/P)$.

Returning to eq(34) the vector product with r is

$$\dot{L} = r^{-2}(B\hat{L} - C\hat{\omega}) \tag{38}$$

and the scalar product of (38) with L gives

$$\begin{aligned} L\dot{L} &= r^{-2}BL \\ \Leftrightarrow L' &= B(L). \end{aligned} \tag{39}$$

Integration of (39) yields $L(T)$ so that (37) has the form

$$I' = M(T)I$$

and leads to three vectors of Poincaré type.

Eq (37) can be solved in principle. Normalization of the conserved vector, P , means that there are only two independent variables. Under the Weierstrass transformation [9]

$$\xi = \frac{I + iJ}{1 - K} \quad \eta = \frac{I + iJ}{1 + K} \tag{40}$$

both ξ and η satisfy

$$\omega' + iL\omega + \frac{iL}{2C}(1 - \omega^2) = 0. \tag{41}$$

Under the transformation

$$\omega = C + \frac{2iCu'}{Lu} + i\left(\frac{C}{L}\right)^2\left(\frac{L}{C}\right)' \tag{42}$$

(41) becomes

$$u'' \left\{ \frac{1}{4} \left(\frac{C'}{C} - \frac{L'}{L} + iL \right)^{1/2} - \frac{1}{2} \left(\frac{C'}{C} - \frac{L'}{L} + iL \right)' + \frac{L^2}{4C^2} \right\} u = 0 \quad (43)$$

which is just the equation for the time-dependent oscillator. The radial component of (34) is

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) = r^{-3} A(L). \quad (44)$$

In terms of χ and T (44) is

$$\chi'' + [L^2 + A(L)]\chi = 0 \quad (45)$$

and, since $L = L(T)$, this also is the time-dependent harmonic oscillator.

5. The Pinney equation and time-dependent systems

The time-dependent oscillator

$$\ddot{q} + \omega^2(t)q = 0 \quad (46)$$

is transformed to the time-independent oscillator

$$Q'' + \Omega^2 Q = 0 \quad (47)$$

by the transformation

$$Q = \frac{q}{\rho} \quad T = \int \rho^{-2}(t) dt, \quad (48)$$

where $\rho(t)$ satisfies the equation

$$\ddot{\rho} + \omega^2 \rho = \frac{\Omega^2}{\rho^3}, \quad (49)$$

which is known as the Pinney equation [10].

The transformation

$$Q = \frac{q - \alpha}{\rho} \quad T = \int \rho^{-2}(t) dt \quad (50)$$

with

$$\ddot{\rho} + \omega^2 \rho = \frac{\Omega^2}{\rho^3},$$

where Ω^2 is often taken as unity, and its Hamiltonian version

$$Q = \frac{q - \alpha}{\rho}, \quad P = \rho(q - \dot{\alpha}) - \dot{\rho}(q - \alpha), \quad T = \int \rho^{-2}(t) dt \quad (51)$$

are used in both classical and quantum mechanics.

The Hamiltonian [11]

$$H = \frac{1}{2}\rho^2 + F(t)q + \frac{1}{2}\omega^2(t)q^2 + \frac{1}{\rho^2}U\left(\frac{q - \alpha}{\rho}\right) \quad (52)$$

is transformed to the autonomous form

$$\bar{H} = \frac{1}{2}P^2 + U(Q) \tag{53}$$

provided ρ satisfies (49) and α is a solution of

$$\ddot{\alpha} + \omega^2(t)\alpha = F(t). \tag{54}$$

6. The Pinney equation and $sl(2, R)$

The symmetries of eq(49) are

$$\begin{aligned} G_1 &= u_1(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{u}_1(t) \rho \frac{\partial}{\partial \rho} \\ G_2 &= u_2(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{u}_2(t) \rho \frac{\partial}{\partial \rho} \\ G_3 &= u_3(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{u}_3(t) \rho \frac{\partial}{\partial \rho}, \end{aligned} \tag{55}$$

where u_1, u_2 and u_3 are the three linearly independent solutions of

$$\ddot{u} + 4\omega^2 u + 4\omega \dot{\omega} u = 0 \tag{56}$$

and the Lie algebra of the symmetries is $sl(2, R)$. When (56) is multiplied by u , the resultant equation is easily integrated to give

$$u\ddot{u} - \frac{1}{2} \dot{u}^2 + 2\omega^2 u^2 = cst. \tag{57}$$

Let $u = \rho^2, cst = 2\Omega^2$. Then (57) becomes

$$\rho^3 \ddot{\rho} + \omega^2 \rho^4 = \Omega^2 \tag{58}$$

or, on division by ρ^3 ,

$$\ddot{\rho} + \omega^2 \rho = \frac{\Omega^2}{\rho^3}.$$

The Pinney equation is, in a sense, the second integral of (56).

Note that (56) is a tode which has the maximal number of symmetries, seven (3 + 4), with the algebra $A_1 \oplus 3A_1 \oplus sl(2, R)$. The Pinney equation is the integral associated with the $sl(2, R)$ part of the algebra. The solution of the Pinney equation (49) can be found as follows. Eq(49) has the symmetry

$$G = u(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{u}(t) \rho \frac{\partial}{\partial \rho}. \tag{59}$$

A new independent variable, $T = \int u^{-1} dt$, and a new dependent variable, $V = \rho u^{-1/2}$, are introduced. Eq(49) is transformed to

$$V'' + \Omega^2 V = \frac{\Omega^2}{V^3} \tag{60}$$

which is readily solved in terms of the solutions of [10]

$$Y'' + \Omega^2 Y = 0 \quad (61)$$

and is

$$V^2 = A \sin^2 \Omega t + 2B \sin \Omega t \cos \Omega t + C \cos^2 \Omega t, \quad (62)$$

where

$$2AC - B^2 = 1. \quad (63)$$

7. Higher order equations and the Pinney connection

Equation (56) has maximal symmetry and is also self-adjoint since

$$\left(-\frac{d}{dt}\right)^3 u + \left(-\frac{d}{dt}\right)^1 (4\omega^2 u) + \left(-\frac{d}{dt}\right)^0 (4\omega\dot{u}) = 0$$

gives

$$-[\ddot{u} + 4\omega^2 \dot{u} + 4\omega\dot{u}] = 0.$$

Eq(56) is the first of the hierarchy of higher order equations which possesses the maximal symmetry $A_1 \oplus nA_1 \oplus sl(2, R)$. They are all self-adjoint and can be transformed to autonomous form by the transformation

$$T = \int u^{-1} dt \quad V = \frac{v}{u^{(n-1)/2}}. \quad (64)$$

There should be an equation of order $(n-1)$ which has the essential feature of the Pinney equation, viz. the Lie algebra $sl(2, R)$. It is an open question whether other conditions should be imposed.

Acknowledgements

The Foundation for Research Development of South Africa is thanked for its support.

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International Journal for Nonlinear Mechanics **28** (1993) 705-712

SYMMETRIES OF HAMILTONIAN ONE-DIMENSIONAL SYSTEMS

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(Received 1 December 1992)

Abstract—The integrability of one-dimensional time-dependent particle Lagrangians is investigated via the techniques of Noether's theorem and the method of Lie symmetries.

1. INTRODUCTION

The theory of autonomous Hamiltonian systems built by Liouville and, more recently, by Arnold is one of the great achievements of the theory of non-linear differential equations. The one-dimensional case which describes the motion of a particle in Newtonian mechanics

$$\begin{aligned}\dot{q} &= p, \\ \dot{p} &= f(q)\end{aligned}\tag{1.1}$$

or

$$\ddot{q} = f(q)\tag{1.2}$$

(where the dot stands for the total time derivative) can be solved by elementary means. A first integral is the Hamiltonian

$$H(q, p) = \frac{p^2}{2} + V(q),\tag{1.3}$$

where the potential $V(q)$ is connected to $f(q)$ through the classical relation $f(q) = -dV/dq$. Physically, the Hamiltonian is the conserved energy E . In addition to (1.3), one can obtain a second first integral by the quadrature

$$J = t - \int \frac{dq}{\sqrt{2[E - V(q)]}}.\tag{1.4}$$

Recently, it was shown in [1] through the formalism of canonical transformations that the non-autonomous case

$$\ddot{q} = f(q, t)\tag{1.5}$$

leads to first integrals, i.e. provided one first integral, I , explicitly exists, the independent second one, J , is immediately derived.

Using the integrating factor technique, this result was recovered in [2] and an explicit form for J was given in terms of quadratures. As a consequence, if one first integral can be found, the second one is easily obtained and equation (1.5) is completely integrated.

However, since the Hamiltonian

$$H(q, p, t) = \frac{1}{2}p^2 + V(q, t)\tag{1.6}$$

is not a first integral, this result is difficult to use because a first integral is required initially.

The proof of the extension of Liouville's theorem is very close to the theory of Lie symmetries which can also be used to find a first integral. The aim of this article is to describe all the geometric (i.e. not generalized) symmetries of equation (1.5) and its natural Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - V(q, t) \quad (1.7)$$

and the forms of function $f(q, t)$ (resp. $V(q, t)$) for which symmetries occur. In Section 2 we determine the Noetherian symmetry and integral of (1.7). In Section 3 we use this first integral to determine the second first integral following the procedures described in [2]. The Lie symmetries of (1.5) are determined in Section 4. For generally admissible $f(q, t)$, one Lie symmetry differs from the corresponding Noetherian symmetry by a parameter-dependent self-similar term. Although the order of the differential equation can always be reduced by one, in general closed-form integrability occurs only when the parameter is zero, i.e. in the Noetherian case. However, some exceptions are found and they are integrated in Section 5. We finish with some concluding remarks in Section 6.

2. THE SEARCH FOR NOETHERIAN SYMMETRIES

Equation (1.5) has the Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - V(q, t), \quad (2.1)$$

where the potential $V(q, t)$ is given by

$$-\frac{\partial}{\partial q} V(q, t) = f(q, t). \quad (2.2)$$

The Lagrangian (2.1) has a Noetherian symmetry

$$X = \xi(q, t) \frac{\partial}{\partial t} + \eta(q, t) \frac{\partial}{\partial q} \quad (2.3)$$

and an associated first integral $I(q, p, t)$ if [3]

$$X^{(1)}(L) + \dot{\xi}L = \dot{\varphi}, \quad (2.4)$$

where $\varphi(q, t)$ is a gauge function and $X^{(1)}$ is the once extended operator

$$X^{(1)} = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + (\dot{\eta} - p\dot{\xi}) \frac{\partial}{\partial p}, \quad (2.5)$$

with p given by $p = \dot{q}$. If condition (2.4) is satisfied, the first integral is

$$I = \varphi - \left[\xi L + (\eta - p\xi) \frac{\partial L}{\partial p} \right]. \quad (2.6)$$

After substitution of (2.1) into (2.4) and expansion of the total derivatives in terms of partial derivatives, we separate the coefficients of independent powers of p and integrate the first three equations to obtain the coefficients of the symmetry together with the equation for the potential:

$$\xi = a(t), \quad (2.7)$$

$$\eta = \frac{1}{2}a'(t)q + b(t), \quad (2.8)$$

$$\varphi = \frac{1}{4}a''(t)q^2 + b'(t)q + c(t), \quad (2.9)$$

$$a \frac{\partial V}{\partial t} + (\frac{1}{2}a'q + b) \frac{\partial V}{\partial q} = -\frac{1}{4}a'''q^2 - b''q - c' - a'V, \quad (2.10)$$

where the prime indicates the derivative of a function with respect to its argument. Equation (2.10) is a linear partial differential equation for the potential $V(q, t)$, the characteristics of which are found from the solution of the associated Lagrange's system:

$$\frac{dt}{a} = \frac{dq}{\frac{1}{2}a'q + b} = \frac{-dV}{\frac{1}{4}a'''q^2 + b''q + c' + a'V}. \quad (2.11)$$

A combination of the first and second terms of (2.11) gives the first characteristic u to be

$$u = qa^{-1/2} - \int ba^{-3/2} dt. \quad (2.12)$$

In the literature this characteristic is usually written as $(q - \alpha)/\rho$ (cf. [4]) and we shall make use of the redefinition

$$a = \rho^2 \quad \text{and} \quad b = \rho^3 \left(\frac{\alpha}{\rho} \right)'. \quad (2.13)$$

The second characteristic v is found from the first and third terms of (2.11) with (2.12).

Using (2.13) to replace a and b after integration, by parts when necessary, we obtain

$$v = -\rho^2 V - \frac{1}{2} \rho \ddot{\rho} q^2 + (\alpha \rho \ddot{\rho} - \rho^2 \ddot{\alpha}) q + \frac{1}{2} \rho^2 \dot{\alpha}^2 - \rho \dot{\rho} \dot{\alpha} + \frac{1}{2} \dot{\rho}^2 \alpha^2 - c. \quad (2.14)$$

Hence, we obtain the form of the potential V :

$$V = -\frac{1}{2} \frac{\ddot{\rho}}{\rho} q^2 + (\ddot{\rho} \alpha - \rho \ddot{\alpha}) \frac{q}{\rho} + \frac{1}{2 \rho^2} (\dot{\rho} \alpha - \rho \dot{\alpha})^2 - \frac{c}{\rho^2} + \frac{1}{\rho^2} \chi \left(\frac{q - \alpha}{\rho} \right), \quad (2.15)$$

where χ is an arbitrary function of its argument.

Introducing (2.7)–(2.9) into (2.6) and using the redefinition (2.13) of a and b , the first integral I is easily derived:

$$I = \frac{1}{2} [\rho(p - \dot{\alpha}) - \dot{\rho}(q - \alpha)]^2 + \chi \left(\frac{q - \alpha}{\rho} \right). \quad (2.16)$$

We have already obtained the result by different methods (e.g. in [4, 5]).

3. THE PROLONGATION OF NOETHER'S CASE WITH LIOUVILLE'S THEOREM

In this section we begin with a result originally presented in [2]. Starting with a given function $I(q, p, t)$, the Hamiltonian $H(q, p, t)$ having I as the first integral is derived together with the canonically conjugate function to $I(q, p, t)$, $J(q, p, t)$. The derivation of J is obtained through a generating function $F_n(q, I, t)$, which is also connected to the expression of $H(q, p, t)$. (Note that it makes no essential difference if one uses a generating function of the form $F_n(p, I, t)$; see ref. [2].) As a consequence, a knowledge of one first integral provides a knowledge of the second one. This result is expressed as follows. Denoting by G the inverse function of $I(q, p, t)$ with respect to p ,

$$p = G(q, I, t), \quad (3.1)$$

the second first integral reads

$$J(q, I, t) = \int_0^q G_1(x, I, t) dx - \int_0^t (GG_1)(0, I, y) dy, \quad (3.2)$$

where the subscript I represents the partial derivative. In this formulation the generating function does not appear and this explicit derivation of J (equation (3.2)) is given in [6].

We apply this result to the first integral defined by (2.16) which is easily inverted with respect to p . After some calculation we obtain

$$J_1 = \int_0^q \frac{1}{\rho} \left\{ 2 \left[I - \chi \left(\frac{x - \alpha}{\rho} \right) \right] \right\}^{-1/2} dx - \int_0^t \frac{dy}{\rho^2(y)} + \int_0^t \left\{ 2 \left[I - \chi \left(-\frac{\alpha}{\rho}(y) \right) \right] \right\}^{-1/2} \left(\frac{-\alpha' \rho + \rho' \alpha}{\rho^2} \right) (y) dy. \quad (3.3)$$

Then, if we introduce the auxiliary function K defined by

$$K(x, y) = \int \frac{dx}{\sqrt{2[y - \chi(x)]}}, \quad (3.4)$$

we get explicitly J_1 :

$$J_1 = \frac{1}{2} \left[K \left(\frac{q-x}{\rho}, I \right) - K \left(-\frac{x}{\rho}, I \right) + K \left(-\frac{x}{\rho}, I \right) - K \left(-\frac{x}{\rho}(0), I \right) - \int \frac{dt}{\rho^2} \right]. \quad (3.5)$$

By eliminating the constant term $K \left(-\frac{x}{\rho}(0), I \right)$, the final expression

$$J = K \left(\frac{q-x}{\rho}, I \right) - \int \frac{dt}{\rho^2} \quad (3.6)$$

is obtained. We may replace I with $I(q, p, t)$ in order to have J in the standard variables (q, p, t) , but we can also consider that this formula provides the expression of q as a function of t and two arbitrary constants I and J , i.e. the solution of the original equation.

Another way to solve completely equation (1.5) is to use a change of variables. This is the equivalent of a generalized canonical transformation in Hamiltonian mechanics [7]. The change of variables is natural by considering (2.16):

$$T = \int \frac{dt}{\rho^2}, \quad Q = \frac{q-x}{\rho}. \quad (3.7)$$

The symmetry found in Section 2 becomes

$$X = \frac{\partial}{\partial T}. \quad (3.8)$$

The first integral (2.16) is now

$$I = \frac{1}{2} \dot{Q}^2 + \chi(Q), \quad (3.9)$$

where the dot now stands for the derivative d/dT .

In addition, the equation of motion (1.15) with $V(q, t)$ given by (2.15) is

$$\ddot{Q} = -\chi'(Q). \quad (3.10)$$

This equation is autonomous and can be reduced to a quadrature. The second first integral is obviously

$$J = \int \frac{dQ}{\sqrt{2[I - \chi(Q)]}} - T. \quad (3.11)$$

This is exactly the first integral (3.6) found by the first method.

Finally, it is interesting to note that J as given by (3.6) or (3.11) is similar to J as given by (1.4). As a consequence, for time-dependent Hamiltonians, $I(q, p, t)$ (equation (2.16)) plays the same role as E (or H) when H is time-independent. This result is not only valid for the family of potentials given by (2.15) but also holds for any time-dependent Hamiltonian [1].

4. THE SEARCH FOR LIE SYMMETRIES

We apply the Lie analysis to (1.5). We consider the symmetry

$$X = \xi(q, t) \frac{\partial}{\partial t} + \eta(q, t) \frac{\partial}{\partial q}. \quad (4.1)$$

After extension of the operator (4.1) to the second order in the derivatives and computation of the determining equation, four equations are obtained:

$$\xi_{qq} = 0, \quad (4.2)$$

$$\eta_{qq} - 2\xi_{qt} = 0, \quad (4.3)$$

$$3f\xi_q + \xi_{tt} - 2\eta_{qt} = 0, \quad (4.4)$$

$$\xi f_t + \eta f_q = \eta_{tt} + (\eta_q - 2\xi_t) f. \quad (4.5)$$

Equations (4.2) and (4.3) give

$$\xi = a(t) + d(t)q, \quad (4.6)$$

$$\eta = b(t) + c(t)q + d'(t)q^2, \quad (4.7)$$

where a, b, c and d are four arbitrary functions. To solve (4.4), there are two possibilities. If ξ_q is not zero, then f is linear in q and we obtain the case of the oscillator (already considered in [6]). In the following we suppose that f is not linear in q and we get, from (4.4),

$$\zeta = a(t), \quad (4.8)$$

$$\eta = \left[\frac{1}{2}a'(t) + \beta\right]q + b(t), \quad (4.9)$$

where β is a constant. By comparing equations (2.8) and (4.9) it is clear that the Lie symmetry contains the Noetherian case and that for $\beta = 0$ we recover the Noetherian result. This will be illustrated later. Equation (4.5) can be solved by the method of characteristics already used in Section 2. We obtain the two characteristics

$$u = e^{-\beta T} \left(\frac{q}{\rho} - e^{\beta T} \int b\rho^{-3} e^{-\beta T} dT \right) \quad (4.10)$$

and

$$v = e^{-\beta T} (f\rho^3 - \rho^2 \dot{\rho}q) + \int [(\ddot{\rho}b - \rho\ddot{b})e^{-\beta T}] dT, \quad (4.11)$$

where we have introduced $\rho(t)$ and $T(t)$ by the definitions

$$a = \rho^2 \quad \text{and} \quad T = \int \frac{dt}{a(t)}. \quad (4.12)$$

Hence, the general solution of (4.5) is

$$f = e^{\beta T} \rho^{-3} \left\{ -\phi(u) + \int [(\rho\ddot{b} - \ddot{\rho}b)e^{-\beta T}] dT \right\} + \frac{\dot{\rho}}{\rho} q, \quad (4.13)$$

where ϕ is an arbitrary function of its argument. The symmetry is now

$$X = \rho^2 \frac{\partial}{\partial t} + [(\rho\dot{\rho} + \beta)q + b] \frac{\partial}{\partial q}. \quad (4.14)$$

We introduce a transformation for t and q in the same spirit as $[q - \alpha(t)]/\rho$ for q in Section 3. We consider the variables T and Q defined by

$$T = \int \frac{dt}{\rho^2} \quad \text{and} \quad Q = \rho^{-1}q - e^{\beta T} \int b\rho^{-3} e^{-\beta T} dT, \quad (4.15)$$

under which (1.5) with f given in (4.13) is now

$$\ddot{Q} = -e^{\beta T} \phi(e^{-\beta T} Q), \quad (4.16)$$

where the overdot is d/dT . In terms of Q and T the symmetry (4.14) is

$$X = \frac{\partial}{\partial T} + \beta Q \frac{\partial}{\partial Q}. \quad (4.17)$$

The conventional approach would be to introduce a transformation which would convert (4.14) to $\partial/\partial T$. However, this would introduce a velocity-dependent term into (1.5) (cf. [8]). Consequently, we must be content to have a non-autonomous equation (equation (4.16)), but it will still be of the form (1.5).

The symmetry (4.17) allows the reduction of equation (4.16) to a first-order equation. If we choose the new variables x and y defined by

$$x = Qe^{-\beta T} \quad \text{and} \quad y = \dot{Q}e^{-\beta T} - \beta x, \quad (4.18)$$

we obtain

$$y \frac{dy}{dx} + 2\beta y + \phi(x) + \beta^2 x = 0, \quad (4.19)$$

which, being an Abel's equation of the second kind, is not integrable in the general case. There are two exceptions. The first one is obtained for $\beta = 0$. This case corresponds to the

existence of a Noetherian symmetry and from (4.16) it is clear that the evolution is completely integrable and the two first integrals are given by (2.16) and (3.11). The second integrable case arises for ϕ linear in x . This special case makes (4.19) homogeneous and it can be integrated by elementary means.

At this stage we may place this result in the framework of the approach described in [6]. To use it we must verify conditions (3.3) of [6]:

$$\frac{\partial}{\partial q}(\eta - p\xi) + \frac{\partial}{\partial p}(\dot{\eta} - f\xi) = 0. \quad (4.20)$$

We deduce that β must be zero. This case is therefore especially fruitful and the different techniques (Noether's theorem, direct Lie symmetries or Lie symmetries in the Hamiltonian context) lead to the same conclusion. On the contrary, none of the methods gives a general solution in the case $\beta \neq 0$.

The existence of the constant β in the symmetry has been found in [8] in studies of particular forms of the generalized Emden-Fowler equation although forms (4.16) and (4.17) were not given there. In [9] it was observed that a further symmetry of a particular case of the generalized Emden-Euler equation existed for a special value of β ($\neq 0$) and this enabled the equation to be reduced to a quadrature. In the next part of this paper we explore the possibility of the same property being true for the general equation (4.16).

5. THE CASE OF LIE SYMMETRY WITH $\beta \neq 0$

Since for $\beta = 0$ we have the same situation as for Noether's theorem, it is clear that the case $\beta \neq 0$ departs from the results of the previous sections. We have proved that the existence of a Lie symmetry for (1.5) allows us to reduce the problem to the canonical form (in (4.16) Q , T and $-\phi$ are, respectively, replaced with q , t and F)

$$\ddot{q} = e^{\beta t} F(e^{-\beta t} q), \quad (5.1)$$

with the corresponding symmetry

$$X = \frac{\partial}{\partial t} + \beta q \frac{\partial}{\partial q}. \quad (5.2)$$

Equation (5.1) is an extension of the family of equations [10] $\ddot{q} = t^{k-2} F(q/t^k)$ for which the symmetry reads $X = t \frac{\partial}{\partial t} + kq \frac{\partial}{\partial q}$. For $k = 1/2$ it can be shown that the equation is always completely integrable for any function F (this case corresponds therefore to the case $\beta = 0$, with the existence of a Noetherian symmetry), whereas for $k \neq 1/2$ only some forms of F are admissible to make the equation completely integrable.

In order to be able to solve completely (5.1), a second symmetry X' is required. We need X' to be of the same form as in (4.1), where ξ and η are given by (4.8) and (4.9):

$$X' = a(t) \frac{\partial}{\partial t} + \left\{ \left[\frac{1}{2} a'(t) + \gamma \right] q + b(t) \right\} \frac{\partial}{\partial q}. \quad (5.3)$$

Notice that the constant appearing in (4.9) is now written as γ since the parameter β is already present in (5.1).

The determining equation is

$$\left[\left(\frac{1}{2} a' + \gamma - \beta a \right) q e^{-\beta t} + b e^{-\beta t} \right] F' + \left(\frac{1}{2} a' - \gamma + \beta a \right) F = \frac{1}{2} a''' q e^{-\beta t} + b'' e^{-\beta t}. \quad (5.4)$$

For (5.4) to be integrable, each term must be a function of $q e^{-\beta t}$ up to a multiplicative function of time. Alternatively, each term must be zero which places no constraint on F but the symmetry found is the X given by (5.2). We consider the ratio of the coefficients of F and $q e^{-\beta t} F'$ written in the form

$$\frac{\frac{1}{2} a' - \gamma + \beta a}{\frac{1}{2} a' + \gamma - \beta a} = M - 1, \quad (5.5)$$

where M is an arbitrary constant. This equation is considered as a differential equation for

$a(t)$, the solution of which is given by

$$a = \frac{\gamma}{\beta} + A \exp\left(\frac{2M}{M+4} \beta t\right), \quad (5.6)$$

where A is a constant. Since the coefficient of F' must be the product of a function of $qe^{-\beta t}$ and a function of t , we set

$$b = -K \frac{4A\beta}{M+4} \exp\left(\frac{3M+4}{M+4} \beta t\right), \quad (5.7)$$

where K is an arbitrary constant. Equation (5.4) is then rewritten as

$$(qe^{-\beta t} + K)F' - (M+1)F = \left(\frac{\beta}{M+4}\right)^2 [-M^3qe^{-\beta t} + K(3M+4)^2], \quad (5.8)$$

which is easily integrated to give

$$F = B(qe^{-\beta t} + K)^{M+1} + \left(\frac{\beta M}{M+4}\right)^2 (qe^{-\beta t} + K) - K\beta^2, \quad (5.9)$$

where B is an arbitrary constant.

The second symmetry X' is

$$X' = \exp\left(\frac{2M\beta}{M+4} t\right) \left[(M+4) \frac{\partial}{\partial t} + (M\beta q - 4K\beta e^{\beta t}) \frac{\partial}{\partial q} \right] + \frac{\gamma}{\beta} \left(\frac{\partial}{\partial t} + \beta q \frac{\partial}{\partial q} \right). \quad (5.10)$$

The last term on the right-hand side is clearly proportional to X and therefore it can be omitted without loss of generality. Consequently, X' becomes

$$X' = \exp\left(\frac{2M\beta}{M+4} t\right) \left[(M+4) \frac{\partial}{\partial t} + (M\beta q - 4K\beta e^{\beta t}) \frac{\partial}{\partial q} \right]. \quad (5.11)$$

The two symmetries are now used to reduce equation (5.1). We take variables which convert X' to $\partial/\partial T$:

$$T = -\frac{M+4}{2M\beta} \exp\left(-\frac{2M\beta}{M+4} t\right), \quad (5.12)$$

$$Q = \exp\left(\frac{4\beta}{M+4} t\right) (qe^{-\beta t} + K), \quad (5.13)$$

and equation (5.1) becomes

$$\ddot{Q} = BQ^{M+1}. \quad (5.14)$$

Since it is autonomous, the Hamiltonian is a first integral:

$$H = \frac{1}{2} \dot{Q}^2 - \frac{B}{M+2} Q^{M+2}. \quad (5.15)$$

The second one is given by (1.4):

$$J = t - \int \frac{dQ}{\sqrt{2\left(E + \frac{B}{M+2} Q^{M+2}\right)}}. \quad (5.16)$$

6. CONCLUSION

In this paper we compared two approaches (Noetherian and Lie symmetries) leading to the integration of the Hamiltonian equation $\ddot{q} = f(q, t)$ (equation (1.5)). The existence of a Noetherian symmetry always leads to an explicit first integral, from which a second one can be deduced by Liouville's theorem. In that case (1.5) is therefore always completely integrable.

In contrast, one Lie symmetry is not enough, in general, to get total integration. This result is explained as follows. The class of the Lie symmetries is larger than the Noetherian

one. They differ by the introduction of an additional constant β present in the Lie symmetries. As a consequence, for $\beta \neq 0$, the Noetherian property is not satisfied and complete integrability is not preserved in general.

In this paper we exhibited a family of equations of the type (1.5) having $\beta \neq 0$ but possessing a second Lie symmetry. Combining these two symmetries we deduced therefore that (1.5) became completely integrable. As a consequence, this constitutes an extension of the Noether's result which is valid only for $\beta = 0$.

We illustrate these results by the case of dilatations. If the symmetry of equation (1.5) is

$$X = t \frac{\partial}{\partial t} + kq \frac{\partial}{\partial q}, \quad (6.1)$$

the Lagrangian satisfies

$$X^{(1)}(L) + D(\xi)L = (2k - 1)L. \quad (6.2)$$

The case for which Noether's theorem applies is $k = 1/2$. The reduced equation is

$$y \frac{dy}{dx} = \chi(x) - k(k - 1)x - (2k - 1)y, \quad (6.3)$$

where

$$x = qt^{-k}, \quad y = pt^{-k-1} - kqt^{-k}. \quad (6.4)$$

Hence, as in the general case, only the case for which the right-hand side of (6.2) is zero—here, $k = 1/2$ —is completely solvable.

We have confined our attention to geometrical symmetries because it is this case which must be studied in detail. As is known (see, for example, the results of [6]), the existence of a generalized symmetry usually enables the integration of the equation to be carried to completion (see [11]).

Acknowledgement—P. G. L. Leach thanks the Foundation for Research Development of South Africa for its continuing support.

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Hidden symmetries of nonlinear ordinary differential equations

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Exploiting Symmetry in Applied and Numerical Analysis E Allgower, K George and R Miranda edd (Lecture Notes in Applied Mathematics **29** American Mathematical Society, Providence, 1993) 1–10

Hidden Symmetries of Nonlinear Ordinary Differential Equations

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ABSTRACT. Hidden symmetries of ordinary differential equations of type I and type II are analyzed for three equations of physical interest: the modified Painlevé-Ince equation, the Pinney equation and semiconductor transport equations.

1. Introduction

The analytical solution of nonlinear, ordinary differential equations (NLODEs) was put on a systematic footing by Sophus Lie¹ and is discussed in many monographs.²⁻¹⁰ Many ad hoc approaches known before his work were shown to be feasible because of the underlying symmetries of the NLODEs. The symmetries of the ODEs were identified by the condition that the ODEs be invariant under Lie point (continuous) groups. The ODEs invariant under a one-parameter group were shown to be reducible in order by one; those invariant under a two-parameter group were shown to be reducible by two orders if for nonabelian two-parameter groups the group invariants of the normal subgroup were chosen to reduce the original ODE first by one order. Investigations of the invariance of ODEs under multi-parameter groups by Lie showed that the NLODEs invariant under Lie point groups could be reduced to quadratures if the associated Lie algebra was solvable and had dimension equal to or greater than the order of the ODE.^{8,10}

Two systematic methods were developed by Lie, one of which is called the classical Lie method, in the use of symmetries to solve ordinary differential equations of both linear and nonlinear varieties. The classical Lie method determines the Lie point groups symmetries of a set of ODEs by an invariance condition. The method also applies to PDEs but we discuss only ODEs here. From the resultant group variable transformations to ODEs invariant under translations in one variable can be

1991 Mathematics Subject Classification. Primary 34A34, 22E70.

Supported by the Southwestern Bell Foundation Grant.

This paper is in final form and no version of it will be submitted for publication elsewhere.

found. Such transformed equations can be reduced in order by one if the new variables are the differential invariants of the group. The order of the ODE is reduced by one for each group used. The inverse method, also developed by Sophus Lie, starts with a group as represented by the group generator and calculates the most general form of the NLODE. Tables of the the general forms NLODEs invariant under common Lie groups have been compiled.^{3,6}

Neither of these methods finds all the symmetries. In recent years nontrivial applications of contact symmetries, also developed by Sophus Lie, have occurred in the transformation of nonlinear PDEs to linear PDEs.^{9,11} Generalized (Lie-Bäcklund) symmetries have been introduced to describe the symmetries of evolution NLPDEs which support solitons and used to find additional solutions.^{8,12} Contact and generalized symmetries depend on variable transformations which are functions of the first derivatives and all orders of derivatives respectively. However, there are variable transformations that depend on integrals of the variables. Potential symmetries are examples where the NLPDEs must be put in conservation form.^{9,13} Alternatively, inverse differential operators have been developed to treat these nonlocal transformations in a direct fashion but assumptions are made about the form of the inverse operators.^{14,15} Nonlocal group generators have been studied briefly.⁸ These nonlocal symmetries are evidence that the direct methods for finding Lie point symmetries, contact symmetries or generalized symmetries may be missing some symmetries.

All the present direct methods for calculating nonlocal symmetries have limitations. The introduction of inverse operators is restrictive since the nonlocal variables are chosen to be integrals of the dependent variables integrated over one of the independent variables. From an analysis of nonlocal symmetries of NLODEs we have found that the nonlocal variables may be integrated over an intricate combination of independent and dependent variables. The potential method requires that the PDEs be put in conservation form which is not always possible and make the special assumption that the integration constant is zero. The Direct Method depends on the proper choice of ansatz for the solution. The nonclassical method can be difficult to apply as the right additional symmetry relation must be appended to the original PDEs. However, one may apply the inverse method as has been done for Lie point symmetries and determine general forms of NLDEs. This approach has the advantage that one does not need an a priori assumption about the form of the differential equations or the inverse differential operators. It is well suited to the exploration of the different forms of NLDEs with hidden symmetries, the determination of the types of variable transformations between different reduced NLDEs and the possible forms of the nonlocal group generators. The inverse method for the hidden symmetries of which two types have been identified so far^{16,17} possesses some unique aspects not found in the inverse method for Lie point symmetries. For type I hidden symmetries first-order ODEs were investigated first. They were chosen because there is no direct method for finding the symmetries of first-order ODEs that is easily applicable yet the equations for many practical problems reduce to first-order ODEs. The direct method is not useful since the determining equation for first-order ODEs, although a linear PDE, is so complicated that solving it has yielded few solutions. Instead ad hoc guesses and the inverse method for Lie point transformations have been applied to find the symmetries. For type II hidden symmetries the hidden symmetries of second-order ODEs were investigated first by choosing a ubiquitous first-order ODE which was then increased

in order by a suitable variable transformation. As the first-order ODE is constructed to be invariant under a one-parameter group, it can be reduced by the direct method to quadratures. The type I hidden symmetries of ordinary differential equations are the symmetries of normal subgroups that vanish when a higher-order ODE is reduced in order by one by the non-normal subgroup differential invariants. The resultant reduced, lower-order ODE possesses the hidden symmetry and is invariant under an extended nonlocal group generator. No general direct method of finding these nonlocal group generators from a given ODE has yet been devised. The type I hidden symmetry was noted in an example by Olver⁸ and has been investigated recently. A systematic study of the type I hidden symmetries of first-order ODEs associated with nonabelian, two-parameter projective subgroups has been reported.¹⁶ The solved form, that is the highest-order derivative appears to the first power, of the general second-order ODE invariant under the two-parameter group was first found for each subgroup. These second-order ODEs were then reduced to first-order by the variables which are the differential invariants of the non-normal subgroup. These first-order ODEs lost the point symmetry of the normal subgroup. Alternatively, the second-order ODEs were reduced to separable first-order ODEs by the normal subgroup differential invariants. These first-order ODEs can be reduced to quadratures by the direct method as they are invariant under a known one-parameter group. A nonlocal variable transformation between the variables of these two first-order ODEs enabled a solution to be given for the ODE with the hidden symmetry. The inverse method is more indirect than that used for Lie point symmetries in that the general forms of the second-order ODEs invariant under nonabelian two-parameter groups were first calculated and then reductions made to the first-order ODEs which had the hidden symmetries. Applications of type I hidden symmetries to the Vlasov characteristic equation and a reaction-diffusion equation have been made.¹⁶

Type II hidden symmetries are those which appear when the order of an ODE is reduced by one and the reduced ODE is determined to be invariant under a new group. An example of these hidden symmetries was given by Olver and they have been studied for energy conserving equations since these equations are so widespread in science and engineering.¹⁷ In this hidden symmetry the procedure for determining general forms of the energy-conserving equations with hidden symmetries was to choose the first-order ODE invariant under a group and then to increase the order of the ODE by using the differential invariants of the projective subgroups. The group under which the first-order ODE was invariant was chosen such that it was not a symmetry of the resultant second-order ODE. The direct classical Lie method was applied to the second-order ODE to ascertain that the symmetry of the first-order ODE was lost as a point symmetry in the transformation to the second-order ODE. Again an inverse method was used since the hidden symmetry of the second-order ODE can not be determined directly. To avoid hidden symmetries one reduces by invariants of normal subgroups.

The two inverse methods for determining the hidden symmetries of ODEs differ from each other and from the inverse method for point symmetries. For the type I hidden symmetry the general form of an ODE of some order that is invariant under a multi-parameter group is calculated. The ODE is then reduced in order by variables which are the differential invariants of a non-normal subgroup; the resultant ODE has one or more hidden symmetries corresponding to the lost symmetries of the associated normal subgroups. On the other hand for the type II hidden symmetries the general form of an ODE invariant under one or more groups is calculated and then the order of that ODE increased where at least one of the symmetries of the lower

order ODE is lost. For the point symmetries the inverse method is simpler; one calculates the ODE in any given order for a group by means of the differential invariants.

The application of the inverse methods to finding hidden symmetries of three ODEs is presented here. In section 2 the Painlevé-Ince equation is discussed. In section 3 the Pinney equation is discussed. In section 4 a semiconductor transport equation is discussed. The conclusions are presented in section 5. Since we deal with specific differential equations, we use a modified inverse method. The modified procedure is to increase the order of the ODE once and then to reduce the higher-order ODE, which for our three cases is a third-order ODE, by the differential invariants of the various groups under which the third-order ODE is invariant. The delicacy in the procedure is in the choice of the group by which differential invariants one increases the order of the original ODE. In the choice one is guided by the tables compiled with projective subgroups of type I and type II hidden symmetries.^{16,17} No attempt was made to increase the order of the three second-order ODEs to higher orders than three but in principle there is no reason not to try increasing the order of the ODEs more.

2. Modified Painlevé-Ince Equation

We begin with the modified Painlevé-Ince equation to illustrate the technique for particular NLODEs.¹⁸ The NLODE where ' denotes differentiation with respect to x is

$$(1) \quad y'' + \sigma yy' + \beta y^3 = 0.$$

The modified Painlevé-Ince equation arose in the solution of the modified Emden equation. The original Emden equation modeled the thermodynamic behavior of a spherical cloud of gas molecules interacting by gravitational attraction.¹⁹ For $\beta \neq 1/9$ Eq.(1) is invariant under a two-parameter group and for $\beta = 1/9$ Eq. (1) is invariant an eight-parameter group. This was determined by Leach and collaborators.^{20,21} Consequently, even for $\beta \neq 1/9$ this equation can be reduced to quadratures. However, the solutions reported in the dictionary edited by Kamke²² or found by Leach²⁰ involve several integrations.

The technique here is to transform Eq.(1) to a third-order ODE which can then be reduced to another second-order ODE of simpler form. As has been already stated there is at present no systematic method for choosing the group for increasing the order of Eq.(1) here. We can look at the groups under which Eq.(1) is invariant. We find that

$$(2) \quad U_{1a} = \frac{\partial}{\partial x}, \quad U_{2a} = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

The scaling transformation represented by U_{2a} suggests a Riccati transformation. The Riccati transformation, which increases the order of an ODE, uses the differential invariants of the scaling transformation in the dependent variables.¹⁶ Calculations with different scaling symmetries of third-order ODEs show that if one of two scaling symmetries is used to reduce the order of the third-order ODE, the reduced second-order ODE will retain a scaling symmetry in the new variables. Furthermore

scaling groups are common non-normal subgroups. If we let

$$(3) \quad y = \frac{\Gamma u_z}{u}, \quad x = z,$$

we find

$$(4) \quad u^2 u_{zzz} + M u u_z u_{zz} + N u_z^3 = 0$$

for $M = \sigma\Gamma - 3$ and $N = 2 + \beta\Gamma^2 - \sigma\Gamma$.

Eq.(4) is invariant under three groups represented by the group generators

$$(5) \quad U_{1b} = \frac{\partial}{\partial z}, \quad U_{2b} = z \frac{\partial}{\partial z}, \quad U_{3b} = u \frac{\partial}{\partial u}.$$

The differential invariants of all three groups lead to a general form for the third-order ODE

$$(6) \quad F\left(\frac{u^2 u_{zzz}}{u_z^3}, \frac{u u_{zz}}{u_z^2}\right) = 0.$$

Comparing Eqs.(4) and (6), we see that Eq.(4) is linear in the invariants together with an additive constant. Reduction of Eq.(4) by the differential invariants of U_{1b} leads to a linear ODE where the transformation is

$$(7) \quad \bar{y} = \left(\frac{\Gamma u_z}{u}\right)^2, \quad \bar{x} = \Gamma \ln u$$

The resulting second-order ODE is

$$(8) \quad \frac{d^2 \bar{y}}{d\bar{x}^2} + \frac{\sigma d\bar{y}}{d\bar{x}} + 2\beta \bar{y} = 0$$

which has well-known solutions for RLC circuits or for damped harmonic oscillators among others. The third-order Eq.(4) has a type II hidden symmetry since Eq.(8) reduced from it is invariant under an eight-parameter group, an increase over the two-parameter group expected.

Two other reductions of order of Eq.(4) are possible. Reducing Eq.(4) by the differential invariants of U_{2b} results in an ODE with a type I hidden symmetry. Besides the symmetry represented by the group generator U_{2b} the symmetry represented by U_{1b} has been lost. If we reduce Eq.(4) by the differential invariants of U_{3b} , we recover the original ODE, Eq.(1).

The advantage in transforming to Eq.(8) over reducing the original Eq.(1) to quadratures is that the parametric solutions are simpler.¹⁸ The transformation from Eq.(1) to Eq.(8) was originally guessed by experience with the projective subgroups. The direct variable transformation between the two Eqs.(1) and (8) is nonlocal and is

$$(9) \quad \bar{y} = y^2, \quad \bar{x} = \int y \, dx$$

3. Pinney Equation

The Pinney equation has wide applications in accelerator physics as well as in the solutions of the one-dimensional Vlasov-Maxwell equations.^{23,24} It is

$$(10) \quad y'' + \omega^2(x)y = \frac{M}{y^3}$$

for M a constant and the prime ' denotes differentiation with respect to x . It is the first integral of the following equation as has been shown by Leach.²⁵ He derived the differential equations and evaluated their Lie point symmetries for which the hidden symmetries are here analyzed. The equation is

$$(11) \quad u''' + 4\omega^2 u' + 4\omega\omega' u = 0$$

where $u = y^2$ and again the prime ' denotes differentiation with respect to x . The Pinney equation is invariant under a three-parameter group. On the other hand the third-order ODE, Eq.(11), is invariant under a seven-parameter group which is the maximal dimension of a group under which a third-order ODE is invariant.

The group structure of the third-order ODE, Eq.(11), is of interest since it possesses a type II hidden symmetry and a second-order ODE reduced from it has a type I hidden symmetry. The group generators under which Eq.(11) is invariant are

$$(12) \quad U_{1c} = u \frac{\partial}{\partial u}, \quad U_{2cj} = f_j(x) \frac{\partial}{\partial u}, \quad U_{3cj} = f_j(x) \frac{\partial}{\partial x} + u f_j'(x) \frac{\partial}{\partial u}$$

for $j = 1, 2, 3$ and $f_j(x)$ are linearly independent solutions of Eq.(11). If the third-order ODE (11) is reduced in order by the differential invariants associated with U_{1c} , then the second-order ODE has lost the three point symmetries associated with U_{2cj} . We note that

$$(13) \quad [U_{2cj}, U_{1c}] = U_{2cj}$$

where the groups associated with U_{2cj} are the normal subgroups and that associated with U_{1c} is the non-normal subgroup. Reducing Eq.(11) by the non-normal subgroup variables of U_{1c} loses the symmetries of the normal subgroups of U_{2cj} . The inherited point group associated with the second-order ODE reduced from Eq.(11) by the differential invariants of U_{1c} is a three-parameter group represented by U_{3cj} . This is expected since the group of U_{1c} is used in the reduction and the loss of the three symmetries of U_{2cj} leave three of the symmetries of Eq.(11). However, when the reduction is made the actual group under which the reduced ODE is invariant is an eight-parameter group! The variable transformation in the reduction of order is

$$(14) \quad v = \frac{u_z}{u}, \quad z = x.$$

The reduced equation is

$$(15) \quad v_{zz} + 3vv_z + v^3 + 4\omega^2 v + 4\omega\omega_z = 0.$$

Three of the group generators are of the form of the first extension of U_{3cj} in the (z, v) variables but the function $f_j(x)$ is now replaced by $p_j(z)$. They are

$$(16) \quad U_{3cj}^e = p_j(z) \frac{\partial}{\partial z} + (u p_j'(z) - p_j''(z)) \frac{\partial}{\partial u}$$

with $j = 1, 2, 3$. The equation satisfied by $p_j(z)$ has a more general form than that for $f_j(x)$. The five other group generators are associated with groups that are not symmetries of the third-order ODE (11).

The type I hidden symmetries associated with Eq.(15) are the three symmetries of the normal subgroup of U_{2cj} . The type II hidden symmetries associated with Eq.(11) are the new symmetries found for Eq.(15). Not only are five new groups found but the three groups which have a group generator of the form of U_{3cj} are more general. The type II hidden symmetry is of a sort that not only adds new symmetries but modifies those already present. This last property was not been seen before in ODEs with a simpler group structure.

4. Semiconductor Transport Equation

The semiconductor transport equations for a steady-state, (no time dependence) one-dimensional system with a constant collision frequency are the differential form of Gauss's law, the momentum conservation equation and the vanishing of the divergence of the current density.²⁶ These can be combined and dimensionless coordinates defined to give single second-order ODE with ' denoting differentiation with respect to x

$$(17) \quad 2\bar{v}^2 \bar{v}'' + 2\bar{v}\bar{v}' + 2\beta\bar{v}\bar{v}' + \bar{v} - 1 = 0.$$

where $-\bar{v} = v/v_E$ for $v_E = J/q_0 n_0$, the parameter $\beta = 1/\sqrt{2} \omega_p \tau$ is not the same β as in section 2 and $x = X/x_0$ for $x_0 = J/(\sqrt{2} q_0 n_0 \omega_p)$. Here v is the carrier speed, J is the current density, $q_0 n_0$ is the background charge density, ω_p is the carrier plasma frequency and τ is the collision time (inverse collision frequency) and X is the position of the carrier with respect to the emitting electrode.

Eq.(17) is invariant under translations in x . To discuss the nonlocal transformation we use the differential invariants, $w_z = dw/dz$ and $z = x$. For ease of calculation we let

$$(18) \quad \bar{v} = \frac{1}{w_z}.$$

With this transformation Eq.(17) becomes a nonlinear third-order ODE.

$$(19) \quad 2w_2 w_{zzz} - 6 w_{zz}^2 + 2\beta w_z^2 w_{zz} - w_z^4 + w_z^5 = 0.$$

It is invariant under translations in z and in w . The group generators are

$$(20) \quad U_{1d} = \frac{\partial}{\partial z}, \quad U_{2d} = \frac{\partial}{\partial w}.$$

The differential invariants of U_{2d} were used to increase the order of Eq.(17) to the third-order ODE. On the other hand the differential invariants of U_{1d} can be used as the variables to reduce the second-order ODE to a another second-order ODE. We let the new variables be

$$(21) \quad \bar{v} = 1/w_z \text{ and } T = w/\sqrt{2}.$$

The likely procedure would be to use w_z as the new dependent variable and then transform the results in the second-order ODE. Here we do two transformations of the variables in one step. We find

$$(22) \quad \frac{d^2 \bar{v}}{dT^2} + \sqrt{2\beta} \frac{d\bar{v}}{dT} + \bar{v} = 1$$

$$\text{with } T = \omega_p \int_0^x \frac{dX'}{\bar{v}}.$$

The new second-order ODE, Eq.(22), is invariant under an eight-parameter group since it is a linear ODE; Eq.(19) is invariant under a two-parameter group. Hence, the third-order ODE, Eq.(19), has a type II hidden symmetry. Interestingly, Eq.(22) is of the same form as Eq.(8). Both for the modified Painlevé-Ince equation and the set of semiconductor transport equations a nonlocal transformation to the well-known, linear harmonic oscillator equation has been shown.

5. Conclusion

Hidden symmetries of ordinary differential equations originating from three second-order ODEs of considerable physical and mathematical interest have been demonstrated. This has been done by increasing the order of the original ODEs and then reducing the order of the third-order ODEs by the differential invariants of the various subgroups. Both type I and type II hidden symmetries have been identified. The type I hidden symmetries occur when the order of the ODE is reduced by the non-normal subgroup variables and the resultant lower-order ODE loses a symmetry. The type II hidden symmetries occur when the order of an ODE is reduced and new symmetries not present in the higher order ODE occur. The occurrence of these hidden symmetries is associated with nonlocal transformations between ODEs of the same order. The transformations are not local since the order of the associated Lie algebra differs in the two ODEs whereas a point transformation preserves the order of the Lie algebra.

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Kepler's third law and the oscillator's isochronism

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American Journal of Physics **61** (1993) 991–995

sending it, in anagrammatic form, to Oldenburg, who was then the secretary of the Royal Society. The solution of the anagrams were given in Huygens' book *Horologium Oscillatorium* which has been published in Paris in 1673, 14 yr before publication of Newton's *Principia*.

¹³The Bible, Gen. 1. 1 and Gen. 2. 20.

¹⁴One of us (Marek Abramowicz) has published a short account on Newtonian inward centrifugal force in an International Center for Theoretical Physics preprint (September, 1990) and presented it at several lectures, including one for a general public at the Smithsonian Institution Resident Associate Program (October, 1991).

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¹⁷The optical reference geometry was introduced by M. A. Abramowicz, B. Carter, and J.-P. Lasota, "Optical reference geometry for stationary and static dynamics," *Gen. Relat. Grav.* 20, 1173-1183 (1988). In the time independent (static) optical reference geometry *light trajectories agree with geodesic lines in space*. This follows from the Fermat principle, according to which in any static spacetime light moves between two points in space on a trajectory for which the light travel time is minimal.

Kepler's third law and the oscillator's isochronism

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(Received 15 October 1992; accepted 30 March 1993)

Two classes of differential equations which have Kepler-like and oscillatorlike elliptical orbits are shown to have generalizations of the conserved angular momentum, energy, and Laplace-Runge-Lenz vector (Jauch-Hill-Fradkin tensor for the oscillator case). Both possess a generator of self-similar transformations and the related period-semimajor axis relation is a generalization of Kepler's third law in which the constant of proportionality depends explicitly on the eccentricity of the orbit.

I. INTRODUCTION

It is well known that the Kepler problem, described in reduced coordinates by the equation of motion

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = 0, \quad (1.1)$$

where μ is a positive constant, possesses the constants of the motion, energy

$$E = \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{\mu}{r}, \quad (1.2)$$

angular momentum

$$\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}, \quad (1.3)$$

Hamilton's vector¹

$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu}{L} \hat{\theta} \quad (1.4)$$

and Laplace-Runge-Lenz vector²⁻⁴

$$\mathbf{J} = \mathbf{K} \times \mathbf{L} = \dot{\mathbf{r}} \times \mathbf{L} - \mu \hat{\mathbf{r}}, \quad (1.5)$$

where $\hat{\mathbf{r}}$ and $\hat{\theta}$ are the unit vectors in plane polar coordinates in the plane of the motion. If θ is measured from \mathbf{J} ,

the scalar product of \mathbf{r} with Eq (1.5) leads to the equation of the orbit

$$r = \frac{L^2}{\mu + J \cos \theta} \quad (1.6)$$

which is an ellipse, parabola, or hyperbola according to whether $\mu > J$, $\mu = J$ or $\mu < J$.

The equation of motion (1.1) is invariant under the actions of the second extensions of the three elements of the rotation group SO(3), time translation $G = \partial/\partial t$, and self-similarity

$$G = t \frac{\partial}{\partial t} + \frac{2}{3} r \frac{\partial}{\partial r}. \quad (1.7)$$

(In the case of the generator of self-similar transformations, for example, the second extension is

$$G^{[2]} = t \frac{\partial}{\partial t} + \frac{2}{3} r \frac{\partial}{\partial r} - \frac{1}{3} \dot{r} \frac{\partial}{\partial \dot{r}} - \frac{4}{3} \ddot{r} \frac{\partial}{\partial \ddot{r}}.)$$

This last symmetry is generally associated with the Laplace-Runge-Lenz vector and Kepler's third law of planetary motion

$$TR^{-3/2} = \frac{2\pi}{\mu^{1/2}},$$

where T is the period of revolution and R is the semi-major axis of the elliptical orbit. The transformations induced by this symmetry transform orbits into orbits of the same eccentricity.

It is also well known that the simple harmonic oscillator, described by the equation of motion

$$\ddot{\mathbf{r}} + \omega^2 \mathbf{r} = 0, \quad (1.8)$$

where ω^2 is a positive constant, possesses, as constants of the motion, energy

$$E = \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{1}{2} \omega^2 r^2, \quad (1.9)$$

angular momentum usually written as the rank two tensor

$$L_{ij} = \epsilon_{ijk} L_k, \quad (1.10)$$

where the L_k are the components of the vector \mathbf{L} in Eq. (1.3), and Jauch-Hill-Fradkin tensor^{5,6}

$$A_{ij} = \dot{x}_i \dot{x}_j + \omega^2 x_i x_j. \quad (1.11)$$

(There are also explicitly time-dependent constants of the motion, but they are not relevant to the purpose of this paper.) Note that E is not an independent integral since $E = \frac{1}{2} \text{Tr}(A)$. The equation of the orbit is given by the quadratic form

$$\mathbf{r}^T [\text{Tr}(A)I - A] \mathbf{r} = L^2 \quad (1.12)$$

or, in terms of plane polar coordinates

$$r^2 = \frac{L^2}{E - (E^2 - \omega^2 L^2)^{1/2} \cos 2\theta}. \quad (1.13)$$

Of the many symmetries of Eq. (1.8) there is the self-similar one

$$G = r \frac{\partial}{\partial r} \quad (1.14)$$

which is associated with the isochronism of the oscillator orbits, viz.

$$T = \frac{2\pi}{\omega}. \quad (1.15)$$

In their studies of Danby's model⁷ for the motion of a low altitude satellite subject to atmospheric drag described by the equation of motion

$$\ddot{\mathbf{r}} + \frac{\alpha \dot{\mathbf{r}}}{r} + \frac{\mu \mathbf{r}}{r^3} = 0. \quad (1.16)$$

Jezewski and Mittleman obtained an analytic solution of the equation of motion⁸ and demonstrated that there existed first integrals which were the direct analogues of the angular momentum, the energy and the Laplace-Runge-Lenz vector of the classical Kepler problem.⁹ Leach¹⁰ re-derived their results by vectorial manipulation of the equation of motion following Collinson's treatment of the Kepler problem.¹¹ (The method had previously been used by Bleuler and Kustaanheimo,¹² but that paper was not known to the present authors at the time.) Subsequently Gorrington and Leach¹³⁻¹⁵ showed that orbit equations could be found for a wide variety of problems described by the equation of motion

$$\ddot{\mathbf{r}} + f \dot{\mathbf{r}} + g \mathbf{r} = 0 \quad (1.17)$$

under certain constraints on the functions f and g . In particular it was noted that for suitable f and g the orbits were

conic sections of the form of Eq. (1.6). The existence of closed orbits for these velocity-dependent equations of motion is in marked contrast to the classic result of Bertrand,¹⁶ who showed that for general initial conditions there are no classical orbits for position-dependent potentials vanishing at infinity except for the Kepler problem.

In this paper we study the generalization of Kepler's third law for conic section orbits of equations of motion of the type of Eq. (1.17). We are not specifically concerned with the inverse problem of given an orbit find the most general equation of motion which can provide it. This problem has a long history (see Whittaker)¹⁷ and has more recently been treated by Broucke.¹⁸ Rather we are interested in those equations of the type of Eq. (1.17) which possess a self-similar symmetry akin to Eq. (1.7).

We determine formulas for the periods of the motions for elliptic orbits of both Kepler and oscillator type and find the appropriate generalizations of Kepler's third law. In doing so we highlight the peculiar positions of Kepler's third law for the Kepler problem and the isochronism of the simple harmonic oscillator. As a final observation we find that there exists a drag law for the Kepler problem giving rise to oscillatorlike orbits which may be of interest in satellite studies and one for the oscillator giving rise to Kepler-like orbits.

As the expressions for some of the constants of motion are not standard, we provide brief derivations of them to make the paper self-contained.

II. KEPLER-TYPE ORBITS

The equation of motion of the type in Eq. (1.17) which we study here is

$$\ddot{\mathbf{r}} - \frac{1}{2}(\alpha + 3) \frac{\dot{\mathbf{r}}}{r} + \mu r^\alpha \mathbf{r} = 0 \quad (2.1)$$

which reduces to the Kepler problem when $\alpha = -3$. It is evident that Eq. (2.1) is invariant under the second extension of the three generators of the rotation group $SO(3)$, time translation $G = \partial/\partial t$, and self-similarity, the last being

$$G = t \frac{\partial}{\partial t} - \frac{2}{\alpha} r \frac{\partial}{\partial r}. \quad (2.2)$$

(The derivation for a power law force can be found in Leach and Gorrington¹⁹ and none of the symmetry is lost by the addition of the "drag" force.)

The vector product of \mathbf{r} with Eq. (2.1) gives

$$\dot{\mathbf{L}} - \frac{1}{2}(\alpha + 3) \frac{\dot{\mathbf{r}}}{r} \mathbf{L} = 0, \quad (2.3)$$

where the angular momentum vector is defined in Eq. (1.3). Although Eq. (2.3) indicates that L is not conserved, $\dot{\mathbf{L}}$ is and the motion is in a plane. The scalar part of Eq. (2.3) gives

$$L = k r^{(\alpha+3)/2} \quad (2.4)$$

and, since $L = r^2 \dot{\theta}$, we have that

$$k = \dot{\theta} r^{-(\alpha-1)/2} \quad (2.5)$$

is a constant of the motion and $k = k \hat{\mathbf{L}}$ can be thought of as a generalized angular momentum. Division of Eq. (2.1) by L and use of Eq. (2.5) yield an exact derivative which integrated to give the Hamilton-like vector

$$E = \frac{1}{2} \text{Tr}(A_{ij}) = \frac{1}{2} \frac{\dot{r} \cdot \dot{r}}{r^2} + \frac{1}{2} \mu r^2. \quad (3.8)$$

A straightforward calculation shows that

$$r^T [\text{Tr}(A)I - A] r = k^2 \quad (3.9)$$

and this is the equation for the orbit. (This is just the same as for the standard oscillator orbit except that L^2 is replaced by k^2 which reinforces the interpretation of k as a generalized angular momentum.) The standard form of the orbit equation is obtained when the matrix is in diagonal form. Due to the constancy of \hat{L} the motion is in a plane and we take x_3 to be the variable in the direction of \hat{L} . The eigenvalues of the 2×2 matrix in Eq. (3.9) which determines the orbit in the plane are

$$\lambda = E \pm (E^2 - \mu k^2)^{1/2} \quad (3.10)$$

and, if we take x and y to be in the direction of the principal axes, the standard form of the elliptical orbit in Cartesian coordinates is

$$\frac{x^2}{[E + (E^2 - \mu k^2)^{1/2}]/\mu} + \frac{y^2}{[E - (E^2 - \mu k^2)^{1/2}]/\mu} = 1. \quad (3.11)$$

The polar form of Eq. (3.10) ($x = r \cos \theta$, $y = r \sin \theta$) is

$$r^2 = \frac{k^2}{E - (E^2 - \mu k^2)^{1/2} \cos 2\theta}. \quad (3.12)$$

The period, T , is found from Eqs. (3.4) and (3.12) and is

$$T = \frac{4}{k} \int_0^{\pi/2} \left(\frac{k^2}{E - (E^2 - \mu k^2)^{1/2} \cos 2\theta} \right)^{(4-\alpha)/4} d\theta. \quad (3.13)$$

Since

$$\begin{aligned} E - (E^2 - \mu k^2)^{1/2} \cos 2\theta \\ = (\mu k^2)^{1/2} \left[\frac{E}{(\mu k^2)^{1/2}} - \left[\left(\frac{E}{(\mu k^2)^{1/2}} \right)^2 - 1 \right]^{1/2} \cos 2\theta \right], \end{aligned} \quad (3.14)$$

Eq. (3.13) becomes

$$T = 2\pi \mu^{(\alpha-4)/8} k^{-\alpha/4} P_{(\alpha-4)/4}(z), \quad (3.15)$$

where

$$z = \frac{E}{(\mu k^2)^{1/2}} \quad (3.16)$$

and the interval of integration is changed from 0 to $\pi/2$ to 0 to π by the substitution $\eta = 2\theta$. The argument of the Legendre function is found to be a function of the eccentricity as follows. From Eq. (3.11) the semimajor and semiminor axes have lengths given by

$$a^2 = [E + (E^2 - \mu k^2)^{1/2}]/\mu \quad (3.17)$$

and

$$b^2 = [E - (E^2 - \mu k^2)^{1/2}]/\mu, \quad (3.18)$$

respectively, so that

$$a^2 + b^2 = \frac{2E}{\mu}, \quad (3.19)$$

$$a^2 b^2 = \frac{k^2}{\mu}. \quad (3.20)$$

With Eqs. (3.19), (3.20) and $b^2 = a^2(1 - e^2)$ Eq. (3.16) becomes

$$z = \left[\frac{1}{4} \left(\frac{2E}{\mu} \right)^2 \left(\frac{k^2}{\mu} \right)^{-1} \right]^{1/2} = \frac{2 - e^2}{2(1 - e^2)^{1/2}}. \quad (3.21)$$

The μ , k part of Eq. (3.15) is

$$\mu^{-1/2} \left(\frac{k^2}{\mu} \right)^{-\alpha/8} = \mu^{-1/2} a^{-\alpha/2} (1 - e^2)^{-\alpha/8} \quad (3.22)$$

and Eq. (3.15) can be written in the form of a generalized Kepler's third law as (with R in place of a)

$$TR^{\alpha/2} = \frac{2\pi}{\mu^{1/2}} (1 - e^2)^{-\alpha/8} P_{(\alpha-4)/4} \left(\frac{2 - e^2}{2(1 - e^2)^{1/2}} \right) \quad (3.23)$$

which is consistent with the symmetry of Eq. (2.2).

IV. DISCUSSION

To have all of the results readily accessible for the discussion we summarize them here. The equation

$$\ddot{r} - \frac{1}{2}(\alpha + 3) \frac{\dot{r}}{r} \dot{r} + \mu r^\alpha = 0 \quad (4.1)$$

describes a motion in the plane which has the following properties:

- (i) the orbit is a conic section with the origin at a focus (Kepler-like),
- (ii) the areal velocity is $\frac{1}{2} k r^{(\alpha+3)/2}$, and
- (iii) in the case of an elliptical orbit the period and semimajor axis are related by

$$TR^{\alpha/2} = \frac{2\pi}{\mu^{1/2}} (1 - e^2)^{-(\alpha+1)/4} P_{(\alpha-1)/2} [(1 - e^2)^{-1/2}]. \quad (4.2)$$

The equation

$$\ddot{r} - \frac{1}{2} \alpha \frac{\dot{r}}{r} \dot{r} + \mu r^\alpha = 0 \quad (4.3)$$

describes a motion in the plane which has the following properties:

- (i) the orbit is an ellipse with the origin at the geometric center (oscillatorlike),
- (ii) the areal velocity is $1/2 k r^{\alpha/2}$ and
- (iii) the period and semimajor axis are related by

$$TR^{\alpha/2} = \frac{2\pi}{\mu^{1/2}} (1 - e^2)^{-\alpha/8} P_{(\alpha-4)/4} \left(\frac{2 - e^2}{2(1 - e^2)^{1/2}} \right). \quad (4.4)$$

Both Eqs. (4.1) and (4.3) have the symmetry

$$G = t \frac{\partial}{\partial t} - \frac{2}{\alpha} r \frac{\partial}{\partial r}. \quad (4.5)$$

In the case of Eq. (4.1) with $\alpha = -3$ we recover the usual Kepler's laws of planetary motion. In particular we have in Eq. (4.2), that $T^2 \propto R^3$ irrespective of the value of the eccentricity of the ellipse. [Recall that $P_{-2}(x) = P_1(x) = x$.] In the case of Eq. (4.3) with $\alpha = 0$ we have, in

$$\mathbf{K} = \frac{\dot{\mathbf{r}}}{L} - \frac{\mu}{k^2} \hat{\theta}. \quad (2.6)$$

A Laplace–Runge–Lenz-like vector follows from

$$\mathbf{J} = \mathbf{K} \times \hat{\mathbf{L}} = \frac{\dot{\mathbf{r}} \times \mathbf{L}}{L^2} - \frac{\mu}{k^2} \hat{\mathbf{r}}. \quad (2.7)$$

If θ is measured from \mathbf{J} , the scalar product of \mathbf{J} and \mathbf{r} leads to the equation for the orbit

$$r = \frac{1}{\mu/k^2 + J \cos \theta}. \quad (2.8)$$

The scalar product of Eq. (2.1) with $\dot{\mathbf{r}} r^{-(\alpha+3)}$ leads to the energylike integral

$$E = \frac{1}{2} \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{r^{\alpha+3}} - \frac{\mu}{r}. \quad (2.9)$$

The scalars K , k , and E are related by the formula

$$K^2 = \frac{2E}{k^2} + \frac{\mu^2}{k^4}. \quad (2.10)$$

The period of the motion, T , is found from Eqs. (2.5) and (2.8) (recalling that $J=K$) to be

$$T = \frac{2}{k} \int_0^\pi \left(\frac{\mu}{k^2} + K \cos \theta \right)^{(\alpha-1)/2} d\theta. \quad (2.11)$$

From Eq. (2.10)

$$\begin{aligned} \frac{\mu}{k^2} + K \cos \theta = & \left(\frac{-2E}{k^2} \right)^{1/2} \left[\frac{\mu}{k(-2E)^{1/2}} \right. \\ & \left. + \left[\left(\frac{\mu}{k(-2E)^{1/2}} \right)^2 - 1 \right]^{1/2} \cos \theta \right] \end{aligned} \quad (2.12)$$

($E < 0$ for an elliptical orbit) and Eq. (2.11) can be written as²⁰

$$T = \frac{2\pi}{k} \left(\frac{-2E}{k^2} \right)^{(\alpha-1)/4} P_{(\alpha-1)/2}(z), \quad (2.13)$$

where

$$z = \frac{\mu}{k(-2E)^{1/2}} \quad (2.14)$$

and $P_\nu(z)$ is the Legendre function of the first kind. The semimajor axis length R is found from Eq. (2.8) as

$$R = \frac{1}{2} [r(0) + r(\pi)] = \frac{\mu}{k^2} \left(\frac{\mu^2}{k^4} - K^2 \right)^{-1}$$

which with Eq. (2.10) becomes

$$R = \frac{\mu}{(-2E)}. \quad (2.15)$$

We recall that for an ellipse the length of the semilatus rectum is

$$l = \frac{b^2}{R} = R(1 - e^2), \quad (2.16)$$

where R is the semimajor axis length, b the semiminor axis length, and e is the eccentricity. From Eq. (2.8) with $\theta = \pi/2$ we have

$$l = \frac{k^2}{\mu} \quad (2.17)$$

so that Eq. (2.14) becomes

$$z = (1 - e^2)^{-1/2}$$

using Eqs. (2.15)–(2.17). Hence Eq. (2.13) is

$$TR^{\alpha/2} = \frac{2\pi}{\mu^{1/2}} (1 - e^2)^{-(\alpha+1)/4} P_{(\alpha-1)/2}[(1 - e^2)^{-1/2}] \quad (2.18)$$

which is the appropriate form of the generalized Kepler's third law for Eq. (2.1) and is consistent with the symmetry of Eq. (2.2) since z is invariant under the action of G . (That this is so follows from the effect of the first extension of G , G^1 on the right-hand side of Eq. (2.14) when k and E are expressed in terms of r , \dot{r} , and θ . An alternative approach is to introduce a broader concept of the first extension of a symmetry operator which we shall call $G[\text{ext}]$. We include in $G[\text{ext}]$ similarity terms which account for dynamical quantities of interest. Thus for k the extra term is $-\alpha^{-1}k\partial/\partial k$ and for E it is $2\alpha^{-1}E\partial/\partial E$. The invariance of z under $G[\text{ext}]$ follows immediately.)

III. OSCILLATOR-TYPE ORBITS

This time the form of Eq. (1.17) to be studied is

$$\ddot{r} - \frac{1}{2} \alpha \frac{\dot{r}}{r} + \mu r^\alpha = 0 \quad (3.1)$$

which reduces to the standard oscillator equation when $\alpha=0$. It is evident that Eq. (2.2) is a symmetry of Eq. (3.1). [For $\alpha=0$, the appropriate form of Eq. (2.2) is $G=r\partial/\partial r$.]

The vector product of \mathbf{r} with Eq. (3.1) gives

$$\dot{\mathbf{L}} - \frac{1}{2} \alpha \frac{\dot{r}}{r} \mathbf{L} = 0 \quad (3.2)$$

from which it follows that $\hat{\mathbf{L}}$ is constant and

$$L = kr^{\alpha/2} \quad (3.3)$$

so that

$$k = \dot{\theta} r^{(4-\alpha)/2} \quad (3.4)$$

is a constant of the motion and $\mathbf{k} = k\hat{\mathbf{L}}$ can be regarded as a generalized angular momentum.

The analogue of the Jauch–Hill–Fradkin tensor is obtained as follows. The Cartesian form of Eq. (3.1) is

$$\ddot{x}_i - \frac{1}{2} \alpha \frac{\dot{r}}{r} \dot{x}_i + \mu r^\alpha x_i = 0, \quad i=1,3. \quad (3.5)$$

The combination (3.5) $\dot{x}_j + \dot{x}_j(3.5)_j$ gives

$$\ddot{x}_i \dot{x}_j + \dot{x}_i \ddot{x}_j - \alpha \frac{\dot{r}}{r} \dot{x}_i \dot{x}_j + \mu r^\alpha (x_i \dot{x}_j + \dot{x}_i x_j) = 0 \quad (3.6)$$

which, on division by r^α , can be integrated to give the conserved tensor

$$A_{ij} = \frac{\dot{x}_i \dot{x}_j}{r^\alpha} + \mu x_i x_j. \quad (3.7)$$

The energylike integral is

(4.4), the well-known isochronism of the standard oscillator.

We have constructed the equations of motion to have elliptical orbits [for $E < 0$ in the case of Eq. (4.1); in the case of Eq. (4.3) E is always positive] and so further comment on (i) is unnecessary. The conservation of areal velocity is lost as angular momentum is not conserved. Angular momentum has been replaced by the conservation of a generalized angular momentum which is related to the geometry of the orbit in the same way as angular momentum was when it was conserved [replace k^2 by L^2 in Eqs. (2.17) and (3.20)]. However, in Eqs. (4.2) and (4.4) we have a direct link to Kepler's third law if allowance is made for the different value of the exponent of R . The feature of this generalized Kepler's third law is that its constant of proportionality depends explicitly on the eccentricity of the orbit.

When $\alpha = -3$, it is well known that for the Kepler problem [i.e., Eq. (4.1) with $\alpha = -3$] the transformations induced by Eq. (4.5) map solutions onto solutions of the same eccentricity.²¹ The same applies to the oscillator when $\alpha = 0$ [i.e., Eq. (4.3) with $\alpha = 0$] [Eq. (4.5) is replaced by $r\partial/\partial r$]. This is not reflected in the general period-semimajor axis relations for both Eqs. (4.1) and (4.3). For general α not only does Eq. (4.5) map solutions onto solutions of the same eccentricity, but the period-semimajor axis relations are specific to orbits of the same eccentricity. However, we should note some specific instances in which the constant of proportionality does not depend on the eccentricity. In the case of Eq. (4.1) this is so when $\alpha = -1$. In the cases $\alpha = 1$ and 3 the constant of proportionality can be made independent of the eccentricity if the semilatus rectum, $l = R(1 - e^2)$, is used instead of the semimajor axis length. When $\alpha = 1$,

$$Tl^{1/2} = \frac{2\pi}{\mu^{1/2}} \quad (4.6)$$

and, when $\alpha = 3$

$$Tl^{3/2} = \frac{2\pi}{\mu^{1/2}} \quad (4.7)$$

It does not appear possible to obtain similar results for Eq. (4.3) due to the complicated argument of the Legendre function.

If, in Eq. (4.1), we set $\alpha = 0$, we obtain an oscillator with an additional velocity-dependent force. The effect of the additional force is to change the usual oscillator orbit into a Kepler-type orbit. From Eq. (4.2) we see that isochronism is preserved only for orbits of the same eccentricity. On the other hand, if, in Eq. (4.3), we set $\alpha = -3$, we have the Kepler problem with an additional velocity-dependent force (opposite in sign to the one mentioned above). The

usual Kepler orbit is replaced by an oscillator orbit and Kepler's third law is replaced by a similar law in which the constant of proportionality depends upon the eccentricity of the orbit.

ACKNOWLEDGMENT

This work was supported in part by the Foundation for Research Development of South Africa.

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NATIVE LANGUAGES

Given their extensive training in mathematical techniques, the preponderance of mathematics in particle physicists' accounts of reality is no more hard to explain than the fondness of ethnic groups for their native language.

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The algebraic structure of generalised Ermakov systems in three dimensions

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Journal of Physics A: Mathematical and General **26** (1993) 4035–4046

The algebraic structure of generalized Ermakov systems in three dimensions

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Received 9 February 1993

Abstract. The characteristic algebra of generalized Ermakov systems is $sl(2, \mathbb{R})$. The structure of these systems in three dimensions is obtained. A subset in the form of an equation of motion with the additional requirement of $so(3)$ symmetry is studied. It includes the classical equation of the magnetic monopole. The existence of three vectors of Poincaré type is established. Consideration is given to weak generalized Ermakov systems in which the symmetry breaking occurs in the radial equation.

1. Introduction

Since Ray and Reid (1979) revealed them to the Western literature, Ermakov systems and their generalizations have attracted wide attention. The Ermakov system (Ermakov 1880) is the combination of the equation for a time-dependent oscillator

$$\ddot{q} + \omega^2(t)q = 0 \quad (1.1)$$

and an auxiliary equation, generally known as the Pinney equation (Pinney 1950),

$$\ddot{\rho}^2 + \omega^2(t)\rho = \rho^{-3}. \quad (1.2)$$

Elimination of $\omega^2(t)$ between (1.1) and (1.2), the introduction of an integrating factor, $\rho\dot{q} - \dot{\rho}q$, and integration produces the first integral

$$I = \frac{1}{2}[(\rho\dot{q} - \dot{\rho}q)^2 + (q/\rho)^2] \quad (1.3)$$

which is usually called the Lewis invariant after H Ralph Lewis who 'rediscovered' it in 1966 (Lewis 1967, 1968) in an application of Kruskal's asymptotic method (Kruskal 1962).

The generalized Ermakov system of Ray and Reid (1979)

$$\ddot{x} + \omega^2(t)x = x^{-3}f(y/x) \quad (1.4)$$

$$\ddot{y} + \omega^2(t)y = y^{-3}g(y/x) \quad (1.5)$$

where f and g are arbitrary functions of their arguments, has a first integral obtained in the same way as that for (1.1) and (1.2) which is

$$I = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + \int^{y/x} [uf(u) - u^{-3}g(u)] du. \quad (1.6)$$

The Ermakov invariant (1.6) persists if $\omega^2(t)$ is replaced by *anything* (Ray 1980, Ray and Reid 1980, Goedert 1990, Leach 1991).

Generalized Ermakov systems have been widely treated and we refer the reader to Leach (1991) for references. Our particular interest in this paper is the relationship between the structure of the differential equations of a three-dimensional generalized Ermakov system and its underlying Lie algebraic structure. In part this has been motivated by the observation of Leach (1991) that the nature of generalized Ermakov systems is explained in terms of the Lie symmetry algebra $sl(2, \mathbb{R})$ and is more obvious if plane polar coordinates (r, θ) are used. Then (1.4), (1.5) and (1.6) become

$$\ddot{r} - r\dot{\theta}^2 + \omega^2(t)r = r^{-3}F(\theta) \quad (1.7)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = r^{-3}G(\theta) \quad (1.8)$$

$$I = \frac{1}{2}(r^2\dot{\theta})^2 - \int G(\theta) d\theta. \quad (1.9)$$

The Ermakov–Lewis invariant (1.9) possesses the three Lie point symmetries

$$G_1 = \partial/\partial t \quad (1.10)$$

$$G_2 = 2t\partial/\partial t + r\partial/\partial r \quad (1.11)$$

$$G_3 = t^2\partial/\partial t + tr\partial/\partial r \quad (1.12)$$

with the Lie brackets

$$[G_1, G_2] = 2G_1 \quad [G_1, G_3] = G_2 \quad [G_2, G_3] = 2G_3 \quad (1.13)$$

which are the relations for the Lie algebra $sl(2, \mathbb{R})$. The invariant (1.9) is obtained by integration of the angular equation (1.8) which also possesses $sl(2, \mathbb{R})$ symmetry. With the representation (1.10)–(1.12) of $sl(2, \mathbb{R})$ used here this is not the case for (1.7) unless the $\omega^2(t)r$ term (or anything else) is absent. However, we do note that the $\omega^2(t)r$ term can be accommodated by a different set of operators which are related to (1.10)–(1.12) by means of a point transformation (Leach and Gorrynge 1990). In Leach (1991) it was proposed that the expression *generalized Ermakov* system be restricted to systems for which the equations of motion possessed $sl(2, \mathbb{R})$ symmetry and that systems which had only the invariant be termed *weak generalized Ermakov* systems.

The observations reported by Leach (1991) were based on a Lie algebraic analysis of (1.4), (1.5) and (1.6). It was natural to ask what systems possessed $sl(2, \mathbb{R})$ symmetry. Govinder and Leach (1992) showed that the most general system of second-order ordinary differential equations in two dimensions invariant under the action of the $sl(2, \mathbb{R})$ representation (1.10)–(1.12) consisted of equations of the form

$$F(\theta, r^2\dot{\theta}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}) = 0. \quad (1.14)$$

(The extension to higher dimensions requires just the addition of the arguments $\phi, r^2\dot{\phi}, r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}, \dots$) A subset of equations of the type (1.14) can be written in terms of the form of the Newtonian equation of motion of a particle as

$$\ddot{r} - r\dot{\theta}^2 = r^{-3}f(\theta, r^2\dot{\theta}) \tag{1.15}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = r^{-3}g(\theta, r^2\dot{\theta}). \tag{1.16}$$

Equations (1.15) and (1.16) take a simpler form if we introduce new time

$$T := \int r^{-2} dt \tag{1.17}$$

and the inverse radial distance $\chi = r^{-1}$. They become

$$\chi'' + [\theta'^2 + f(\theta, \theta')]\chi = 0 \tag{1.18}$$

$$\theta'' = g(\theta, \theta') \tag{1.19}$$

where $' := (d/dT)$. There is a local solution of (1.19), $\theta = \theta(T, \theta_0, \theta'_0)$, and, when this is substituted into (1.18) together with $\theta' = \theta'(T, \theta_0, \theta'_0)$, (1.18) is reduced to the equation for a two-parameter family of time-dependent harmonic oscillators, the family of general solutions of which will contain four parameters. Under suitable conditions on $g(\theta, \theta')$ (1.19) can be integrated to give the first integral

$$I = M(\theta, \theta') \tag{1.20}$$

which is the generalization of the Ermakov–Lewis invariant (1.9). Again under suitable conditions (1.20) can be integrated to give a second first integral

$$J = T - \int \frac{d\theta}{N(\theta, I)} \tag{1.21}$$

where $N(\theta, I)$ is obtained by the inversion of (1.20). In general this will not be possible as $M(\theta, \theta')$ will be only locally defined (for example on open neighbourhoods of analytic points of g in (θ, θ') space) and typically $N(\theta, I)$ will be infinitely branched. In the case that f and g in (1.18) and (1.19) are free of θ' we obtain from (1.19) the usual expression for the Ermakov–Lewis invariant

$$I = \frac{1}{2}\theta'^2 - \int g(\theta) d\theta \tag{1.22}$$

and (1.21) becomes

$$J = T - \int d\theta / \left\{ 2 \left[I + \int^\theta g(s) ds \right] \right\}^{1/2}. \tag{1.23}$$

We note that, if we impose the additional requirement that equation (1.14) (and so (1.18) and (1.19)) be invariant under the action of the rotation group in two dimensions with generator $\partial/\partial\theta$ so that the algebra is $sl(2, \mathbb{R}) \oplus so(2)$, (1.19) is now

$$\theta'' = g(\theta'). \tag{1.24}$$

The Ermakov–Lewis invariant is given by

$$I = \int \frac{\theta' d\theta'}{g(\theta')} - \theta \quad (1.25)$$

and an explicitly time-dependent integral is given by

$$K = T - \int \frac{d\theta'}{g(\theta')}. \quad (1.26)$$

The integral (1.23) comes from the elimination of θ' between (1.25) and (1.26).

In this paper we extend the consideration of systems of differential equations invariant under $sl(2, \mathbb{R})$ to three dimensions. We have already noted that the generalization of (1.14) to higher dimensions is trivial. However, we find that the imposition of rotational invariance by making the invariance algebra $sl(2, \mathbb{R}) \oplus so(3)$ yields an interesting class of differential equations which includes the classical equation for the magnetic monopole. The invariance of this equation under the elements of the algebra $sl(2, \mathbb{R}) \oplus so(3)$ has already been reported by Moreira *et al* (1985) (although they preferred to use the isomorphic algebra $so(2, 1) \oplus so(3)$). The monopole is also known to possess a conserved vector called Poincaré's vector (Poincaré 1896). We shall see that the general system to be discussed here possesses three such vectors and that the solution of the system of equations reduces to the determination of the three Poincaré vectors and the solution of the radial equation corresponding to (1.18). We should point out that, in the case of the monopole, the vector usually referred to as Poincaré's vector is obtained by elementary vectorial manipulation of the equation of motion. The derivation of the two other vectors which, because of their nature, we also term Poincaré vectors is by no means transparent even in this simple case. We also consider weak generalized Ermakov systems in three dimensions.

2. Equations invariant under $sl(2, \mathbb{R}) \oplus so(3)$

In spherical polar coordinates ($x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$) the equation corresponding to (1.14) invariant under the representation (1.10)–(1.12) of $sl(2, \mathbb{R})$ is

$$F(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}) = 0. \quad (2.1)$$

To make reasonable sense as a system of second-order differential equations in three dependent variables we need a system of three equations of the form

$$r^3\ddot{r} = f(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \quad (2.2)$$

$$r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = g(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \quad (2.3)$$

$$r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi} = h(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}). \quad (2.4)$$

(One could conceive of variations on this. By way of example—not definitive nor intended to be exclusive—(2.4) could be replaced by

$$H(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}, I) = 0 \quad (2.5)$$

where I is a parameter which may be taken to be the value of a first integral. If I has a particular value, I_0 , in which case it could just as well be omitted from (2.5), we are in the

realm of configuration invariants. To keep the discussion concise we do not digress into this specialized area. The reader is referred to Sarlet *et al* (1985) for a discussion of the relationship between systems of second-order equations, first integrals and configurational invariants.)

In terms of the new time T and inverse radial distance χ equations (2.2)–(2.4) are

$$\chi'' = -f(\theta, \phi, \theta', \phi')\chi \tag{2.6}$$

$$\theta'' = g(\theta, \phi, \theta', \phi') \tag{2.7}$$

$$\psi'' = h(\theta, \phi, \theta', \phi'). \tag{2.8}$$

In contrast to the pair of equations (1.18) and (1.19) for which (1.19) was ‘in principle’ integrable and so (1.18) reduced to the time-dependent oscillator, the situation with the system (2.6)–(2.8) is much more complex. Given θ and ϕ as functions of T , (2.6) is straightforward enough as it is linear in χ .

We confine our attention to systems for which, in addition to invariance under $sl(2, \mathbb{R})$ there is also rotational invariance, i.e. the system of equations is also invariant under the action of the generators of $so(3)$, namely

$$G_4 = \partial/\partial\phi \tag{2.9}$$

$$G_5 = \sin\phi\partial/\partial\theta + \cot\theta\cos\phi\partial/\partial\phi \tag{2.10}$$

$$G_6 = \cos\phi\partial/\partial\theta - \cot\theta\sin\phi\partial/\partial\phi. \tag{2.11}$$

We add this constraint from considerations of possible physical applications. The applications of $(\partial/\partial\phi)$ to (2.2)–(2.4) (alternatively (2.6)–(2.8), but the sequel suggests that the former should be used) is simple enough, f , g and h must be ϕ free. As the second extension of G_5 ($\propto G_6$) mixes $\dot{\theta}$ and $\dot{\phi}$ terms, (2.3) and (2.4) must be treated as a coupled system whereas (2.2) can be treated by itself. The actual analysis involved makes for a pleasant and straightforward exercise. The action of $G_6^{[2]}$ makes no difference to the result which is to be expected since $G_6 = [G_4, G_5]$. (Given a symmetry $G = \tau\partial/\partial t + \eta_i\partial/\partial x_i$, the second extension is $G^{[2]} = G + (\dot{\eta}_i - \dot{x}_i\dot{\tau})\partial/\partial\dot{x}_i + (\dot{\eta}_i - 2\dot{x}_i\dot{\tau} - \dot{x}_i\ddot{\tau})\partial/\partial\ddot{x}_i$.) We find that the most general system of the form (2.2)–(2.4) invariant under $sl(2, \mathbb{R}) \oplus so(3)$ is

$$r^3\ddot{r} = A_1(L) \tag{2.12}$$

$$r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = r^4\dot{\phi}^2\sin\theta\cos\theta + B(L)r^2\dot{\theta} - C(L)r^2\dot{\phi}\sin\theta \tag{2.13}$$

$$r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi} = -2r^4\dot{\theta}\dot{\phi}\cot\theta + \frac{1}{\sin\theta}[B(L)r^2\dot{\phi}\sin\theta + C(L)r^2\dot{\theta}] \tag{2.14}$$

where A_1 , B and C are arbitrary functions of their argument L , where

$$L^2 := r^4(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \tag{2.15}$$

is the square of the magnitude of the angular momentum. The three equations (2.12)–(2.14) may be written in the compact vectorial form

$$\ddot{\mathbf{r}} = \frac{1}{r^3}\{A(L)\hat{\mathbf{r}} + B(L)\hat{\boldsymbol{\omega}} + C(L)\hat{\mathbf{L}}\} \tag{2.16}$$

where we have replaced $A_1(L)$ by $A(L) + L^2$. In an obvious notation \hat{r} and \hat{L} are the unit vectors in the direction of the radius vector and the angular momentum vector $L(= r \times \dot{r})$. The unit vector $\hat{\omega} := \hat{L} \times \hat{r}$ is in the direction of the rate of change of \hat{r} and is the natural generalization of $\hat{\theta}$ in plane polar coordinates.

In terms of the definition of generalized and weak generalized Ermakov systems (2.2)–(2.4) represents the three-dimensional form of the generalized Ermakov system. The addition of some extra term to (2.2) would be in the spirit of the meaning of weak generalized Ermakov system as given by Leach (1991). However, two points should be made. The first is that under suitable (for example analyticity) conditions (2.2)–(2.4) have integrals, i.e. constants of integration, defined over some local neighbourhood. The existence of one or more global first integrals for (2.3), (2.4) or a combination of (2.3) and (2.4) would require some constraints on the functions g and h . The second is that we have chosen the radial equation to be the one which leads to the symmetry breaking. It made sense in two dimensions as we were guaranteed the ‘in principle’ existence of an Ermakov–Lewis invariant provided that the system maintained $sl(2, \mathbb{R})$ symmetry in the radial equation. This is lost in the general three-dimensional case and further thought needs to be given to a correct terminology.

To conclude this section we make some observations about (2.16). For B and C zero and $A(L)$ a constant (L is conserved) we have the equation for a Newton–Cotes spiral (Whittaker 1944) which, in essence, is the free particle in the plane with an excess or deficit of angular momentum. For A and B zero and $C(L)$ proportional to L ($= \lambda L$) a constant (L is again conserved) we have the classic equation of a particle moving in the field of a magnetic monopole. In this case it is well known that there exists the first integral

$$P = L + \lambda \hat{r} \quad (2.17)$$

and the motion is on the surface of a cone of semi-vertex angle given by $\cos^{-1}(C/PL)$ (Poincaré 1896). It is only more recently that Moreira *et al* (1985) demonstrated that the algebra was $so(2, 1) \oplus so(3)$ (isomorphic to $sl(2, \mathbb{R}) \oplus so(3)$). We note that the classical monopole is a Hamiltonian system and the components of the Poincaré vector possess the algebra $so(3)$ under the operation of taking the Poisson bracket (Mladenov 1988).

3. Poincaré vector for (2.16)

The combination of the existence of the Poincaré vector (2.17) and the symmetry algebra $sl(2, \mathbb{R}) \oplus so(3)$ for the classical monopole equation

$$\ddot{r} = C(L)\hat{L}/r^3 \quad (3.1)$$

suggests that it may be fruitful to look for a similar vector for the general equation (2.16). Elementary manipulation of (3.1) produces (2.17) more or less without trying. This is not the case with (2.16). However, an equally simple-minded approach does yield interesting results. We assume the existence of a vector of Poincaré type given by

$$P := I\hat{r} + J\hat{\omega} + K\hat{L} \quad (3.2)$$

where I , J and K are functions to be determined. Requiring that \dot{P} be zero when (2.16) is satisfied leads to the system of equations

$$\frac{d}{dt} \begin{pmatrix} I \\ J \\ K \end{pmatrix} = r^{-2} \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} \begin{pmatrix} I \\ J \\ K \end{pmatrix} \quad (3.3)$$

which in terms of new time T are

$$\begin{pmatrix} I \\ J \\ K \end{pmatrix}' = r^{-2} \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} \begin{pmatrix} I \\ J \\ K \end{pmatrix}. \quad (3.4)$$

Equations (3.4) have a geometrical interpretation. They are the Serret–Frenet formulae associated with a curve of curvature L and torsion $C(L)/L$, parametrized by T . An orthonormal triad of solution vectors represents the principal triad of the curve, consisting of tangent, normal and binormal vectors.

As an aside we note that this approach is not feasible for the two-dimensional system of equations since \hat{r} and $\hat{\theta}$ are then multiples of $\dot{\theta}$ and each multiple is a property of the geometry of the plane and is independent of the mechanics. The only way to make progress would be to specify the \dot{r} and θ dependence in P . This has not been necessary in the present case because the dynamics is introduced via $\hat{\omega}$.

The scalar product of (2.16) with \hat{r} is

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) = r^{-3} A(L) \quad (3.5)$$

which in terms of χ and T is

$$\chi'' + \{L^2 + A(L)\}\chi = 0. \quad (3.6)$$

The vector product of \mathbf{r} with (2.16) gives

$$\dot{L} = r^{-2} \{B\dot{L} - C\dot{\omega}\} \quad (3.7)$$

so that

$$L\dot{L} = r^{-2} BL \quad (3.8)$$

or

$$L' = B(L) \quad (3.9)$$

which gives the first integral

$$M = T - \int \frac{dL}{B(L)}. \quad (3.10)$$

This can be interpreted as an equation defining T in terms of L or L in terms of T . Naturally, if B is zero, the magnitude of the angular momentum is constant.

By virtue of (3.10), (3.6) becomes the by now familiar time-dependent oscillator which characterizes the radial equation for generalized Ermakov systems expressed in the appropriate coordinates.

In like fashion (3.4) is now a three-dimensional non-autonomous first-order system of differential equations. Its structure is suggestive of a time-dependent oscillator written as a system of first-order equations. However, the analogy only helps for a constant L . Before going into the details of the method of solution of (3.4) one comment is appropriate. As a three-dimensional first-order linear system it has three linearly independent solutions. This means that there are in fact three 'Poincaré' vectors.

In view of the geometric interpretation of equations (3.4) some natural examples to consider would be ones which have standard properties of curves. The simplest one is a curve of constant curvature which means that B is zero. The solution of (3.4) is

$$\begin{aligned} \begin{pmatrix} I \\ J \\ K \end{pmatrix} &= \exp \left\{ \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} T \right\} \begin{pmatrix} I_0 \\ J_0 \\ K_0 \end{pmatrix} \\ &= \begin{bmatrix} C \\ L\Omega \\ 0 \\ L \\ \Omega \end{bmatrix} I_0 + \begin{bmatrix} \frac{L}{\Omega} \sin \Omega T \\ \cos \Omega T \\ -\frac{C}{L\Omega} \sin \Omega T \end{bmatrix} J_0 + \begin{bmatrix} \frac{L}{\Omega} \cos \Omega T \\ -\sin \Omega T \\ -\frac{C}{L\Omega} \cos \Omega T \end{bmatrix} K_0 \end{aligned} \quad (3.11)$$

where $\Omega^2 := L^2 + C^2/L^2$ and the scaling has been chosen so that the norm of each vector is one. The standard magnetic monopole has $C = \lambda L$ and is associated with a curve of constant torsion. The usual Poincaré vector of the literature has $I_0 = \Omega$, $J_0 = 0$ and $K_0 = 0$ and is

$$P = L + \lambda \hat{r} \quad (3.12)$$

but we emphasize that there are in fact three vectors.

The solution (3.11) also applies in the case that there is a constant ratio of torsion to curvature, i.e. $C = \lambda L^2$. There is a Poincaré vector $\hat{L} + \lambda \hat{r}$ (regardless of A and B) which is time-independent. Then the general solution of (3.4) is written using $\int^T L(\tau) d\tau$.

Since L is constant the solution of the radial equation (3.6) is simply

$$\chi(T) = E \sin \omega T + F \cos \omega T \quad (3.13)$$

where $\omega^2 := L^2 + A(L)$ and E and F are constants of integration. (We consider only ω real and non-zero. The other two possibilities can be treated in a similar fashion.) From (3.13) and the definition of T we have

$$t = \int \frac{dT}{(E \sin \omega T + F \cos \omega T)^2} \quad (3.14)$$

which is easily evaluated and inverted to give T in terms of t and, through (3.13), we have

$$r(t) = \left[\frac{1}{E^2 + F^2} + \frac{2\omega^2}{F^2} (t - t_0)^2 \right]^{1/2}. \quad (3.15)$$

With the general solution of (3.2) inverted to give \hat{r} , multiplication by r gives $\mathbf{r}(t)$ with six constants of integration and hence the general solution. This solution applies to all problems associated with constant curvature.

We would expect to find three conserved vectors as the form posited for P spans the space (except for exceptional points where degeneracy occurs). One is reminded of the work of Fradkin (1965, 1967) and Yoshida (1987, 1989) on the existence of Laplace–Runge–Lenz vectors for central force and other three-dimensional problems.

The procedure described in detail above for the constant curvature case applies *mutatis mutandis* for the general equation (2.16). Usually the equations become non-autonomous with a consequent increase in the degree of difficulty of solution. This is particularly the

case with (3.4) which in the autonomous case is solved by a straightforward exponentiation of the coefficient matrix by time. Nevertheless the general equation can be treated.

By construction P is a constant vector and I , J and K are not independent when the magnitude of P is specified. (In the case of the single vector there is not much point to it, but, when there are three vectors spanning the space, there is no small appeal in specifying unit vectors.) Only two dependent variables are needed and we introduce the transformation (the so-called Weierstrass transformation of Forsyth (1904); see also Kamke (1971)),

$$\xi = \frac{I + iJ}{1 - K} \quad \eta = -\frac{I + iJ}{1 + K}. \quad (3.16)$$

Together with the normalization of P , (3.16) leads to a common differential equation for ξ and η which is of Riccati form, namely

$$w' + iLw + \frac{iL}{2C}(1 - w^2) = 0 \quad (3.17)$$

where w stands for ξ and η in turn. The transformation

$$w = 2iCy'/Ly \quad (3.18)$$

yields the linear second-order equation

$$y'' + \left(\frac{C'}{C} - \frac{L'}{L} + iL \right) y' + \frac{L^2}{4C^2} y = 0 \quad (3.19)$$

which is trivially related via

$$y = \left(\frac{L}{C} \right)^{1/2} u e^{-i/2 \int L dt} \quad (3.20)$$

to the standard time-dependent harmonic oscillator (TDHO)

$$u'' + \left\{ \frac{1}{4} \left(\frac{C'}{C} - \frac{L'}{L} + iL \right)^2 - \frac{1}{2} \left(\frac{C'}{C} - \frac{L'}{L} + iL \right)' + \frac{L^2}{4C^2} \right\} u = 0. \quad (3.21)$$

Given the solution for u , ξ and η follow through (3.18) and (3.20). The components of P are given by

$$I = \frac{1 - \xi\eta}{\xi - \eta} \quad J = \frac{i(1 + \xi\eta)}{\xi - \eta} \quad K = \frac{\xi + \eta}{\xi - \eta}. \quad (3.22)$$

Needless to remark the tricky business is always the solution of the TDHO equation (3.21). We illustrate this with what appears to be a fairly innocuous set of functions B and C being proportional to L , i.e.

$$B = \alpha L \quad C = \beta L. \quad (3.23)$$

Then

$$L = L_0 e^{\alpha T} \quad (3.24)$$

and (3.21) becomes

$$u'' - \frac{1}{4}[L_0^2 e^{2\alpha T} + 2i\alpha L_0 e^{\alpha T} - \beta^{-2}]u = 0 \quad (3.25)$$

which is Whittaker's differential equation in slightly disguised form. With the solution to (3.25) the route back to ξ and η via (3.18) and (3.20) is straightforward. To keep things simple we take $A(L)$ to be zero. The solution of the radial equation (3.6) is

$$\chi = EJ_0(L_0 e^{\alpha T}) + FY_0(L_0 e^{\alpha T}) \quad (3.26)$$

where J_0 and Y_0 are Bessel's functions and, as before, E and F are constants. However, the determination of t is via

$$t - t_0 = \int \frac{dT}{\{EJ_0(L_0 e^{\alpha T}) + FY_0(L_0 e^{\alpha T})\}^2} \quad (3.27)$$

for which a closed expression is not known.

4. Some 'weak' considerations

Leach (1991) proposed that systems with Ermakov invariants which did not possess $sl(2, \mathbb{R})$ symmetry should be termed 'weak'. Athorne (1991), although not disagreeing with the distinction, noted that other classifications—such as Hamiltonian and non-Hamiltonian—were also important. Indeed, the point of that letter was that those (non-Hamiltonian) systems described, and which had only *one* global invariant, could be understood as 'linear extensions' of an underlying Hamiltonian system with appropriate choice of time variable. Here we wish to consider a few examples of systems for which only the angular equations possess $sl(2, \mathbb{R})$ symmetry. We maintain $so(3)$ symmetry overall so that the radial equation has the form

$$\ddot{r} - \frac{L^2}{r^3} = \frac{1}{r^3}A(L) + f(r, L) \quad (4.1)$$

where $f(r, L)$ is the symmetry-breaking term. The analysis of the angular equations is the same which means that, in principle, we have $L = L(T)$ and the three Poincaré vectors. In terms of the inverse radial variable χ and new time (4.1) is

$$\chi'' + [A(L) + L^2]\chi + \frac{1}{\chi^2}f\left(\frac{1}{\chi}, L\right) = 0. \quad (4.2)$$

When f is zero, (4.2), as the equation for the TDHO, is transformed to autonomous form by the transformation

$$J = \frac{\chi}{\rho} \quad \tau = \int \rho(T)^{-2} dT \quad (4.3)$$

where ρ is a solution of the Pinney equation (Pinney 1950)

$$\frac{d^2\rho}{dT^2} + [A(L) + L^2]\rho = \rho^{-3}$$

and $L = L(T)$ through (3.10). One could hope that for some functions f that the transformation (4.3) would render it autonomous. For this to happen it is necessary for $\rho = g(L)$ and the argument of f to be $\chi^{-1}g(L)$, where g is a solution to a Pinney-type equation with L as independent variable containing $A(L)$ and $B(L)$.

Such constraints are not required in a few cases. If the additional force is due to a Newton-Cotes potential, (4.2) is as if f were zero and $A(L)$ changed. For a Kepler-type potential

$$f\left(\frac{1}{\chi}, L\right) = \mu(L)\chi^2 \quad (4.5)$$

so that (4.3) is just the non-homogeneous time-dependent oscillator and is solved in standard fashion. For an oscillator type potential

$$f\left(\frac{1}{\chi}, L\right) = \mu(L)\chi^{-1} \quad (4.6)$$

which makes (4.2) a Pinney equation with time-dependent coefficients. If μ is independent of L or $B = 0$, this can be treated as if it were the standard TDHO problem. If such is not the case, the best that one can do is to introduce a time-dependent transformation which converts (4.2) to a generalized Emden-Fowler equation of order -3 .

5. Conclusion

In the case that (2.16) has a Hamiltonian representation the Poincaré vectors will have the Lie algebra $so(4)$ under the operation of taking the Poisson bracket. The question is under what circumstances does it have a Hamiltonian? (One would not expect the usual Poisson bracket relation $[z_\mu, z_\nu]_{PB} = J_{\mu\nu}$, ($z_i = q_i, z_{n+1} = p_i$ and J is the $2n \times 2n$ symplectic matrix) but more the monopole type of relation, i.e. seek $H : \dot{q} = [q, H]_{PB}$ and $\dot{p} = [p, H]_{PB}$ lead to the equation of motion.) There are two cases of (2.16) to consider: (i) when (2.16) is itself Hamiltonian; and (ii) when (2.16) possesses a global invariant which is not, however, a Hamiltonian function for the system. In the latter case the possibility arises that this invariant is a Hamiltonian function for a subsystem on an appropriate phase space, as in Athorne (1991).

Acknowledgments

KSG thanks the Foundation for Research Development of South Africa for its support. PGLL expresses his appreciation to Sisters M MacFarlane and S Hucks and their staff of the Orthopaedic Ward of Addington Hospital, Durban, for their tender care while part of the manuscript was written and to an unnameable motorist who made his stay possible.

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Generalized Ermakov systems in terms of $sl(2, R)$ invariants

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Quaestiones Mathematica 16 (1993) 405–412

GENERALIZED ERMAKOV SYSTEMS IN TERMS OF $sl(2, R)$ INVARIANTS

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ABSTRACT. Conventional generalized Ermakov systems are shown to be a subset of the class of second order ordinary differential equations invariant under $sl(2, R)$ symmetry. When the system is two-dimensional, it can be reduced to a one-dimensional time-dependent simple harmonic oscillator by a suitable choice of new time and distance variables.

1991 *Mathematics Subject Classification.* 58F07, 22E70.

In a recent letter [5] it was shown that the Ermakov invariant [4] and its generalization [10] are invariant under the action of the generators of the Lie algebra $sl(2, R)$. (For a more comprehensive bibliography on generalized Ermakov systems see the references cited in ref [5].) It was suggested in ref [5] that the name generalized Ermakov system be restricted to the case in which the original system of differential equations also possesses $sl(2, R)$ symmetry and that otherwise the system be termed weak.

In [5] the observation that the Ermakov invariant had the nature of a generalized angular momentum led to the treatment of generalized Ermakov systems in polar co-ordinates (plane polar for two-dimensional

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Work supported in part by the Foundation for Research Development of South Africa.

systems, spherical and hyperspherical for three- and n -dimensional systems).

The natural question to ask is: "What is the general form of a differential equation invariant under the action of the generators of $sl(2, R)$?" In view of the comment above the representation of $sl(2, R)$ used is expressed in terms of polar co-ordinates, ie,

$$(1) \quad G_1 = \frac{\partial}{\partial t}$$

$$(2) \quad G_2 = 2t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}$$

$$(3) \quad G_3 = t^2 \frac{\partial}{\partial t} + tr \frac{\partial}{\partial r}.$$

We seek the structure of the general second order ordinary differential equation (sode)

$$(4) \quad f(t, r, \theta, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}) = 0$$

invariant under the second extension of the symmetries (1), (2) and (3). G_1 is easily disposed as it is its own second extension and invariance under it requires that f in (4) be free of t . The second extension of G_2 is

$$(5) \quad G_2^{[2]} = 2t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} + 0 \frac{\partial}{\partial \theta} - \dot{r} \frac{\partial}{\partial \dot{r}} - 2\dot{\theta} \frac{\partial}{\partial \dot{\theta}} - 3\ddot{r} \frac{\partial}{\partial \ddot{r}} - 4\ddot{\theta} \frac{\partial}{\partial \ddot{\theta}},$$

where we include the $0(\partial/\partial\theta)$ term to remind ourselves that θ is an argument of f , and its application to (4) gives a linear partial differential equation for f , viz.

$$(6) \quad r \frac{\partial f}{\partial r} + 0 \frac{\partial f}{\partial \theta} - \dot{r} \frac{\partial f}{\partial \dot{r}} - 2\dot{\theta} \frac{\partial f}{\partial \dot{\theta}} - 3\ddot{r} \frac{\partial f}{\partial \ddot{r}} - 4\ddot{\theta} \frac{\partial f}{\partial \ddot{\theta}} = 0$$

(recall that f is t -free due to G_1) the characteristics of which are found by the solution of the associated Lagrange's system

$$(7) \quad \frac{dr}{r} = \frac{d\theta}{0} = -\frac{d\dot{r}}{\dot{r}} = -\frac{d\dot{\theta}}{2\dot{\theta}} = -\frac{d\ddot{r}}{3\ddot{r}} = -\frac{d\ddot{\theta}}{4\ddot{\theta}}$$

and are $\theta, u = r\dot{r}, v = r^2\dot{\theta}, w = r^3\ddot{r}$ and $x = r^4\ddot{\theta}$ so that (4) is now

$$(8) \quad f(\theta, u, v, w, x) = 0.$$

The invariance of (8) under the action of $G_3^{[2]}$ reduces the number of characteristics to four, viz. θ, v, w and $y = x + 2uv$ so that the general structure of a second order ordinary differential equation invariant under $sl(2, R)$ is

$$(9) \quad f(\theta, r^2\dot{\theta}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^3\ddot{r}) = 0.$$

We note that in spherical polar co-ordinates the characteristics $\phi, r^4\dot{\phi}$ and $r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}$ would also be functions and the general form of the equation is

$$(10) \quad f(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}^2) = 0.$$

The generalization to hyperspherical co-ordinates is obvious.

The relationship between Ermakov systems and classical mechanics is fairly obvious (Ermakov *did* start off with the equation for the classical harmonic oscillator.). We therefore continue the analysis in two (spatial) dimensions. In two dimensions we require two equations which, in general, would be

$$(11) \quad F(\theta, r^2\dot{\theta}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}) = 0$$

$$(12) \quad G(\theta, r^2\dot{\theta}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}) = 0.$$

We note that a particular case of the general result represented by eqq (11) and (12) was reported by Ray and Reid [9] who used an interesting combination of Noether's theorem and an *ad hoc* method. We confine our attention to the subset of (11) and (12) which can be written in the normal forms

$$(13) \quad r^3\ddot{r} = f_1(\theta, r^2\dot{\theta})$$

$$(14) \quad r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = g(\theta, r^2\dot{\theta}).$$

The implicit function theorem [2, p171] does impose some restrictions on F and G for this inversion to be possible. In particular (13) and (14) may

only have local validity. In such a case the development below would be confined to a collection of neighbourhoods. However, in terms of possible physical applications based on the ideas of Newtonian mechanics, we assume that (13) and (14) have more than local validity. Equations (13) and (14) can be recast in the form of the equation of motion of a classical particle if we take $f_1 = f + (r^2\dot{\theta})^2$ so that (13) and (14) become

$$(15) \quad \ddot{r} - r\dot{\theta}^2 = \frac{1}{r^3}f(\theta, r^2\dot{\theta})$$

$$(16) \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r^3}g(\theta, r^2\dot{\theta}).$$

The usual generalized Ermakov systems have f and g free of $r^2\dot{\theta}$.

In order to make the structure of these equations more transparent we introduce new time τ defined by

$$(17) \quad \tau = \int r^{-2} dt$$

and the inverse radial distance $\chi = 1/r$. If derivatives with respect to τ are denoted by ', '' etc, (15) and (16) become

$$(18) \quad \chi'' + [\theta'^2 + f(\theta, \theta')]\chi = 0$$

$$(19) \quad \theta'' = g(\theta, \theta').$$

Eq (19) is effectively a first order equation for θ' with θ as dependent variable. In terms of the Lie theory for the integration of a first order ode an integrating factor can be found such that

$$(20) \quad g(\theta, \theta') = -\theta' \frac{\partial M(\theta, \theta')}{\partial \theta} \bigg/ \frac{\partial M(\theta, \theta')}{\partial \theta'}$$

so that (19) integrates to

$$(21) \quad M(\theta, \theta') = h,$$

where h is a constant. (Note: We are here concerned with principle. In practice there may be technical difficulties!) Given the structure assumed for g the implicit function theorem [2, p165] guarantees inversion of (21) to

$$\theta' = N(\theta, h)$$

(at least locally) so that

$$(22) \quad \tau - \tau_0 = \int \frac{d\theta}{N(\theta, h)}.$$

This can also be inverted (locally) to give

$$(23) \quad \theta = J(\tau, h, \tau_0).$$

Now that θ is known, (18) becomes the differential equation in (χ, τ) space of the classical time-dependent linear oscillator if the coefficient of χ is positive, the free particle if zero and the linear repulsor if negative. We remark that the new time defined in (17) is almost familiar except that $r(t)$ is used instead of $\rho(t)$ [6,7]. Another way to look at the definition of new time is as

$$(24) \quad \tau = \int r^{-2} dt = \int r^{-2} \frac{dt}{d\theta} d\theta = \int (r^2 \dot{\theta})^{-1} d\theta$$

so that τ is the measure of time in which the time rate of change of angle is the angular momentum. This angular momentum interpretation and resulting oscillator equation (18) remind one of the interpretation of the Pinney equation [8] by Eliezer and Gray [3].

Eqq (15) and (16) reduce to the equations for a Newton-Cotes spiral in the case $f = \text{constant}$ and $g = 0$ [10, p83]. The qualitative features of a spiral are maintained for the generalized Ermakov system in the cases that (i) $\theta'^2 + f(\theta, \theta') < 0$ since $\chi(\tau)$ is unbounded and so $r \rightarrow 0$ (see fig 1) and (ii) $\theta'^2 + f(\theta, \theta') > 0$ since $\chi(\tau)$ passes through zero and so $r \rightarrow \infty$ (see fig 2). However, it is possible to obtain closed orbits. One such is depicted in fig 3.

Some general comments are in order. The reduction of the nonlinear equation (15) to that of the linear time-dependent oscillator combines the method of Whittaker [11, p78] and the introduction of the 'new time' τ . In the case that the angular momentum $L(= r^2 \dot{\theta})$ is conserved, the new time is just $L\theta$ which Whittaker uses. In the general case the procedure adopted here is very similar to that found in Athorne et al [1]. In the two-dimensional case the connection between the angular variable

and new time is obvious. Consider, however, the set of equations in spherical polars corresponding to (13) and (14). It is

$$(25) \quad r^3 \ddot{r} = f_1(\theta, \phi, r^2 \dot{\theta}, r^2 \dot{\phi})$$

$$(26) \quad r^4 \ddot{\theta} + 2r^3 \dot{r} \dot{\theta} = g_1(\theta, \phi, r^2 \dot{\theta}, r^2 \dot{\phi})$$

$$(27) \quad r^4 \ddot{\phi} + 2r^3 \dot{r} \dot{\phi} = h_1(\theta, \phi, r^2 \dot{\theta}, r^2 \dot{\phi}).$$

Although this set can be recast in a form which has the left hand sides as the components of the acceleration in spherical polar co-ordinates and the new time (17) introduced, there is no connection with an angle because there are two to choose from. A possible approach is to impose further symmetry constraints on equation (25-27). Work on this approach is underway.

In this letter we have concentrated on generalized Ermakov systems (as defined in ref[1]). It is apparent that the weak systems can be included if in (15) we allow f to include, say, additional radial forces. The integration of (16) to obtain an Ermakov invariant is unaffected. In general (18) would become nonlinear in χ and the possibility of closed form integration much reduced, except in such special cases as the Kepler inverse-square force.

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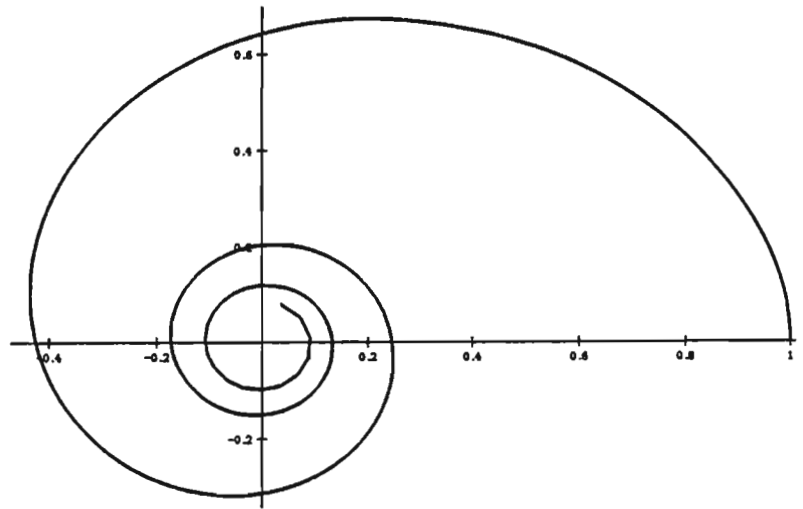


Figure 1: Inwardly spiralling orbit for $f = -\theta'^2 - 1$ and $g = \theta'$ with initial conditions $r(0) = 1$, $\dot{r}(0) = 0$, $\theta(0) = 0$ and $\dot{\theta}(0) = 0.9$

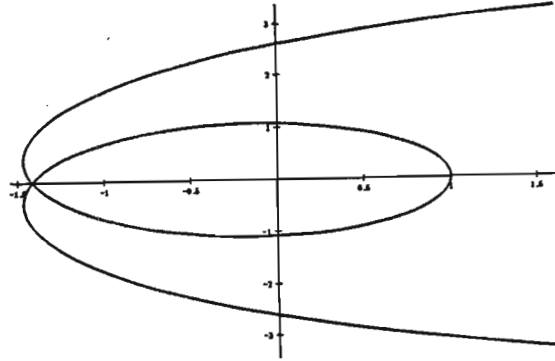


Figure 2: Outwardly spiralling orbit for $f = -0.9375\theta^2$ and $g = 0$ with initial conditions $r(0) = 3.6$, $\dot{r}(0) = -0.025$, $\theta(0) = -5\pi/3$ and $\dot{\theta}(0) = 0.0075$

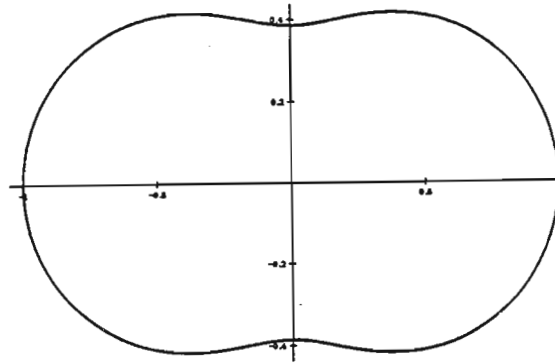


Figure 3: Closed orbit for $f = \theta^2(a - 2\cos 2\theta)$ and $g = 0$ with initial conditions $r(0) = 1$, $\dot{r}(0) = 0$, $\theta(0) = 0$ and $\dot{\theta}(0) = 0.1$ ($a \approx -1.4551$ is related to the eigenvalue of the Mathieu function $ce(\theta, 1)$.)

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Received 10 October 1992.

The Painlevé test, hidden
symmetries and the equation
$$y'' + yy' + ky^3 = 0$$

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Journal of Physics A: Mathematical and General **26** (1993) 5017–5024

The Painlevé test, hidden symmetries and the equation $y'' + yy' + ky^3 = 0$

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Received 24 February 1993, in final form 26 May 1993

Abstract. For general values of the parameter, k , the equation $y'' + yy' + ky^3 = 0$ can be reduced to quadrature via a Lie algebraic approach, either direct or through hidden symmetries. For specific values of k , mostly in $(\frac{1}{9}, \frac{1}{8})$, the solution can be expressed in parametric form. For these values of k the equation passes the weak Painlevé test. For some other values of k the equation passes the Painlevé test, but the solution cannot in general be expressed parametrically.

1. Introduction

The differential equation

$$y'' + yy' + ky^3 = 0 \quad (1.1)$$

has occurred in studies in a variety of fields such as univalued functions defined by second order differential equations (Golubev 1950), the generalized Emden equation (Moreira 1984, Leach 1985), the Riccati equation (Chisholm and Common 1987) and in the modelling of the fusion of pellets (Erwin *et al* 1984), has itself been the object of study by Mahomed and Leach (1985), Leach *et al* (1988) and Abraham-Shrauner (1992) and as one of a class of equations by Bouquet *et al* (1991). These studies were mainly concerned with the Lie point symmetries of (1.1) and numerical properties of the solution for various values of the parameter k .

For general values of the parameter, k , equation (1.1) is invariant under the actions of the second extensions of the Lie point symmetries

$$G_1 = \frac{\partial}{\partial x} \quad (1.2)$$

associated with invariance under translation in the independent variable and

$$G_2 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \quad (1.3)$$

which is the generator of self-similar transformations. In the particular case that $k = \frac{1}{9}$ (1.1) possesses eight Lie point symmetries (Mahomed and Leach, 1985) with the algebra $s(3, R)$ which implies that the equation is linearizable. The transformation is

$$Y = \frac{1}{2}x^2 - \frac{x}{y} \quad X = x - \frac{1}{y}. \quad (1.4)$$

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Although the value of $k = \frac{1}{9}$ is significant in terms of the Lie algebraic properties of (1.1), this is not a critical value of k as far the behaviour of the solution is concerned. That value is $k = \frac{1}{8}$ (Leach *et al* 1988).

An interesting feature of (1.1) is that it has hidden symmetries associated with it no matter the value of k (Abraham-Shrauner 1992) in that it is related to a linear second-order equation by means of a non-local transformation.

The focal point of the study of any differential equation is the question of its integrability. Three approaches to this question are Lie analysis, Painlevé analysis and numerical studies. The purpose of this paper is to provide a unified treatment of (1.1) so that already known features are connected to some which have yet to be explored. In particular we compare two methods of reducing (1.1) to quadratures with the results of the Painlevé analysis of the equation. From this comparison we will see that the passing of the Painlevé or weak Painlevé test and the reduction to a parametric solution via the symmetries are closely related.

The parameter, k , which appears in (1.1) is written in that form for the convenience of the presentation of the equation. In the analysis which follows in the following sections it will be seen that another parameter, related to k , emerges naturally. As we wish to relate the reduction to quadrature to the results of the Painlevé analysis, we initiate that analysis first.

2. Painlevé analysis: initial considerations

Following the procedure outlined in Ramani *et al* (1989) we determine the leading-order behaviour by setting

$$y = \alpha \chi^p \quad (2.1)$$

where $\chi = x - x_0$, and find that there is a simple pole ($p - 1$), that all terms are dominant and that the coefficient α satisfies

$$k\alpha^2 - \alpha + 2 = 0. \quad (2.2)$$

The resonances are found by substituting

$$y = \alpha \chi^{-1} + \beta \chi^{r-1} \quad (2.3)$$

into the dominant terms (all in this case) and requiring that the coefficient of β be zero. When (2.2) is used, the resonances are found to be $r = -1$ (as required) and $r = 4 - \alpha$.

Equation (1.1) passes the Painlevé test if r is a positive integer which fixes α to one of the sequence of integers 3, 2, 1, -1, ... (The value $\alpha = 0$ is omitted due to (2.2).) One could contemplate the weak Painlevé test with α a rational number. However, for the moment we leave this analysis with two remarks. For $k > \frac{1}{8}$, α is complex. It has already been found (Leach *et al* 1988) that the critical value of k is $\frac{1}{8}$ and this is supported by the results so far of the Painlevé analysis. We have identified a parameter, α , related to k which plays a critical role in the analysis.

3. Reduction of order

Since (1.1) has two symmetries G_1 , (1.2), and G_2 , (1.3), with $[G_1, G_2] = G_1$, the usual reduction of order is through the normal subgroup, G_1 , which yields an Abel equation of the second kind with the symmetry G_2 preserved in the new co-ordinates. This symmetry

is used to transform the Abel equation into one of variables separable form. Under the transformation

$$u = \frac{1}{y} \quad v = -\frac{y'}{y^2} \quad (3.1)$$

equation (1.1) becomes

$$uvv' = 2v^2 - v + k \quad (3.2)$$

which is immediately reduced to the quadrature

$$\int \frac{du}{u} = \frac{1}{2} \int \frac{v dv}{(v - \frac{1}{2})(v + (2 - \alpha)/2\alpha)} \quad (3.3)$$

when (2.2) is taken into account. On integration (3.3) becomes

$$Ku^{2(4-\alpha)} = \left(v - \frac{1}{\alpha}\right)^{2-\alpha} \left(v + \frac{2-\alpha}{2\alpha}\right)^2 \quad (3.4)$$

where K is the constant of integration, except in the special cases $\alpha = 2(k = 0)$ and $\alpha = 4(k = \frac{1}{8})$ which are

$$Ku^2 = v - \frac{1}{2} \quad (3.5)$$

and

$$Ku^2 = (v - \frac{1}{4}) \exp[-\frac{1}{4}(v - \frac{1}{4})] \quad (3.6)$$

respectively. The reduction above is for $k \leq \frac{1}{8}$. For $k > \frac{1}{8}$ the quadrature of (3.2) gives

$$Ku^4 = (2v^2 - v + k) \exp\left\{\frac{1}{\sqrt{(8k-1)}} \tan^{-1}\left[\frac{2v-1}{\sqrt{(8k-1)}}\right]\right\}. \quad (3.7)$$

There still remains the further quadrature because of the transformation (3.1). Even if the immediate step of $v = u'$ is used, non-local inversion of (3.4) is not possible for general α .

4. Hidden symmetry reduction

The basic idea behind the hidden symmetry approach (Abraham-Shrauner 1993) is to *increase* the order of the given differential equation by one via a non-local transformation and then to reduce the order using one of the other symmetries to an equation of the same order as the original equation but with (one hopes) more symmetries. Equation (1.1) belongs to the class (Bouquet *et al* 1991)

$$y'' + y^m y' + ky^{2m+1} = 0 \quad (4.1)$$

for which the non-local transformation is a slight generalization of the Riccati transformation. It is

$$y = \left(1 + \frac{2}{m}\right)^{1/m} (u'/u)^{1/m}. \quad (4.2)$$

In the case of (1.1) $m = 1$ and the resultant third-order equation is

$$u^2 u''' + (9k - 1)u'^3 = 0. \quad (4.3)$$

It is immediately obvious why (1.1) is easily integrated in the case of $k = \frac{1}{9}$.

Equation (4.3) has the three symmetries

$$G_1 = \frac{\partial}{\partial x} \quad (4.4)$$

$$G_2 = x \frac{\partial}{\partial x} \quad (4.5)$$

$$G_3 = u \frac{\partial}{\partial u} \quad (4.6)$$

with $[G_1, G_2] = G_1$, $[G_1, G_3] = 0$ and $[G_2, G_3] = 0$, i.e., the algebra is $A_1 \oplus A_2$ (compared with A_2 of (1.1)). The symmetry G_3 is a natural consequence of the Riccati transformation. The reduction of the self-similar symmetry (1.3) which contains both x and y terms to one only in x may have some connection with the fact that all terms in (1.1) are dominant in the Painlevé analysis.

When (4.4) is used to reduce the order of (4.3), the resulting equation is an Euler equation in the square of the new independent variable. A further transformation brings this into a linear constant coefficient form. Combining the two transformations we have

$$v = \log u \quad w = u^2 \quad (4.7)$$

and (4.3) becomes

$$w'' - w' + 2(9k - 1)w = 0. \quad (4.8)$$

Since (4.8) is a linear second-order equation, it is invariant under the actions of the generators of the eight-element algebra $sl(3, R)$.

The solution of (4.8) is

$$w = Ae^{\lambda_1 v} + Be^{\lambda_2 v} \quad (4.9)$$

where

$$\lambda_{1,2} = \frac{1}{2} \pm \delta \quad \delta = \frac{3}{2} \sqrt{1 - 8k} \quad (4.10)$$

(except in the case $k = \frac{1}{8}$). The solution $u(x)$ of (4.3) is obtained from the inversion of the quadrature

$$x - x_0 = \int \frac{du}{u^{1/4}(Au^\delta + Bu^{-\delta})^{1/2}}. \quad (4.11)$$

A full discussion of the integration in (4.11) would necessitate the treatment of a number of cases resulting in a certain amount of repetition. To avoid this we consider only the case A and B both positive. The substitution

$$u^\delta = (B/A)^{1/2} \sinh \eta \quad (4.12)$$

brings (4.11) to the form

$$x - x_0 = \frac{1}{\delta A^{1/2}} \left(\frac{B}{A} \right)^{3/8\delta - 1/4} \int \sinh^{3/4\delta - 1/2} \eta \, d\eta \quad (4.13)$$

which can be evaluated when the exponent on the sinh is an integer. The form of the integral varies with the oddness or evenness of the integer.

For an integer, $2n$, (4.10) gives

$$k = \frac{n(2n + 1)}{(4n + 1)^2} \tag{4.14}$$

and (Gradshteyn and Ryzhik (1980), equation 2.412.2)

$$x - x_0 = \frac{(4n + 1)B^n}{3 \cdot 2^{2n-1} A^{n+1/2}} \left\{ (-1)^n \binom{2n}{n} \eta + 2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sinh(2n - 2k)\eta}{2n - 2k} \right\} \tag{4.15}$$

For an odd integer, $2n + 1$, (4.10) gives

$$k = \frac{(n + 1)(2n + 1)}{(4n + 3)^2} \tag{4.16}$$

and (Gradshteyn and Ryzhik (1980), equation 2.412.3)

$$x - x_0 = \frac{2(-1)^n (4n + 3)B^{n+1/2}}{3A^n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\cosh^{2k+1} \eta}{2k + 1} \tag{4.17}$$

In terms of η

$$y = \frac{3A^{(n+1)/2}}{B^{n/2}} \frac{\cosh \eta}{\sinh^{n+1} \eta} \tag{4.18}$$

when

$$\delta = \frac{3}{2(2n + 1)} \tag{4.19}$$

In the case that n is an even integer, (4.18) with (4.13) provides a parametric solution and, when n is an odd integer, (4.18) with (4.17) does the same.

It is possible to eliminate η only in the cases $n = 0 (k = 0)$, $n = 1 (k = \frac{1}{9})$ and $n = 3 (k = 6/49)$. The even values of n suffer from the mixture of η and $\sinh \eta$ terms. Nevertheless the solution is well-defined for all integer n . We note that, except for the trivial case $k = 0$, the parameter k belongs to the open interval $(\frac{1}{9}, \frac{1}{8})$, i.e., between the value for which (1.1) is linearizable and the critical value.

If $k > \frac{1}{8}$, the solution of (4.8) is

$$w = e^{v/2} (A \sin \delta v + B \cos \delta v) \tag{4.20}$$

where

$$\delta = \frac{3}{2} \sqrt{8k - 1} \tag{4.21}$$

The integral corresponding to (4.11) cannot be evaluated in terms of a finite combination of elementary functions.

5. Weak Painlevé property

In section 4 we saw that (1.1) could be solved in parametric form when

$$k = \frac{n(n + 1)}{2(2n + 1)^2} \quad n = 0, 1, \dots \tag{5.1}$$

for which, since $k < \frac{1}{8}$, the constant α in the Painlevé analysis takes the values

$$\alpha = \frac{2(2n+1)}{n+1}, \frac{2(2n+1)}{n} \quad (5.2)$$

(with $n \neq 0$ in the latter case) and the Kowalevskaya exponents are

$$r = \frac{2}{n+1}, -\frac{2}{n} \quad (5.3)$$

respectively.

For general n , r is non-integer and so (1.1) does not have the Painlevé property for these values of k . However, it is possible that (1.1) does have the weak Painlevé property since r is rational. In the case of the first resonance calculation shows that the expansion should be in powers $\chi^{2/(n+1)}$ rather than $\chi^{1/(n+1)}$. Without going into the details of what is a routine calculation we find that

$$y = \frac{2(2n+1)}{(n+1)} \frac{1}{\chi} \left\{ 1 + \left(\frac{\chi}{b}\right)^{2/(n+1)} + \frac{n^2+4n+1}{n+5} \left(\frac{\chi}{b}\right)^{4/(n+1)} + \frac{2n^4+19n^3+48n^2+23n+4}{2(n+5)(n+7)} \left(\frac{\chi}{b}\right)^{6/(n+1)} + \dots \right\} \quad (5.4)$$

where b is the arbitrary constant which enters at the resonance. The weak Painlevé property is satisfied. As we have already seen in section 4, for these values of k the solution of (1.1) can be written in parametric form.

It has been claimed (Ablowitz *et al* 1980) that a negative resonance (apart from $r = -1$ which represents the arbitrariness of x_0 (Grammaticos *et al* 1982)) is purely formal. This is not always the case. When all terms in an equation are dominant, it is obvious that the leading term behaviour does not indicate whether the leading term is the lowest or highest power in the series. The existence of a negative resonance (apart from $r = -1$) indicates that we have the *highest* power and not the *lowest* power. Hence it is appropriate to insert the ansatz

$$y(\chi) = \chi^{-1} \sum_{k=0}^{\infty} a_k \chi^{-2k/n} \quad (5.5)$$

into (1.1). We emphasise that this is the correct interpretation of a negative resonance (apart from $r = -1$) only in the case that all terms are dominant. We find that

$$y = \frac{2(2n+1)}{n} \chi^{-1} \left\{ 1 + \left(\frac{b}{\chi}\right)^{2/n} + \frac{n^2-2n-2}{n-4} \left(\frac{b}{\chi}\right)^{4/n} + \frac{4n^4-17n^3-5n^2+24n+12}{2(n-4)(n-6)} \left(\frac{b}{\chi}\right)^{6/n} + \dots \right\} \quad (5.6)$$

where again b is the arbitrary constant introduced at the resonance. As with all series solutions obtained using the Painlevé method, (5.6) is a formal solution. We conjecture that it represents an essential singularity at $\chi = 0$. This result, (5.6), is invalid for $n = 4, 6, \dots$ at which values the arbitrary constant, b , must be zero. These values give particular solutions to (1.1). It is an easy calculation to show for n an even integer, $2m$, that

$$y = \frac{4m+1}{m\chi} \quad (5.7)$$

is a particular solution of (1.1).

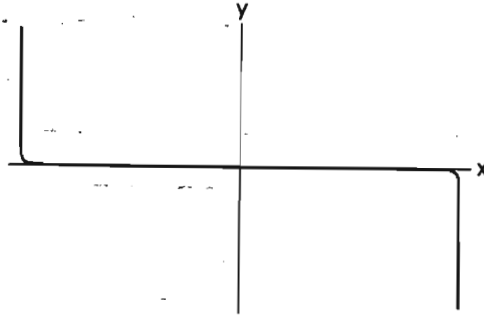


Figure 1. $k = \frac{1}{8}$; $y(0) = 0$, $y'(0) = -1$. Divergent solution.

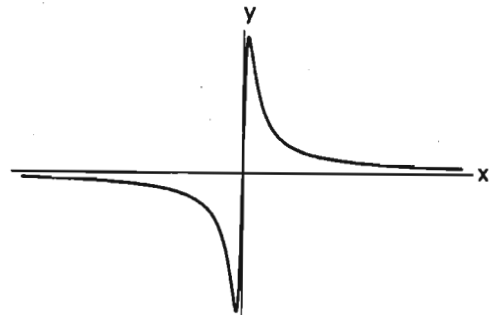


Figure 2. $k = \frac{1}{8}$; $y(0) = 0$, $y'(0) = 1$. Decaying solution.

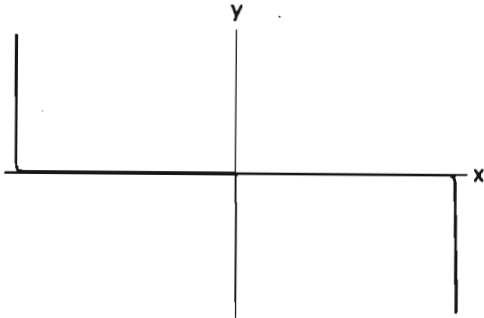


Figure 3. $k = \frac{1}{8}$; $y(0) = 0$, $y'(0) = -1$. Divergent solution indicates that $k = \frac{1}{8}$ is the limiting case of the $k < \frac{1}{8}$ behaviour.

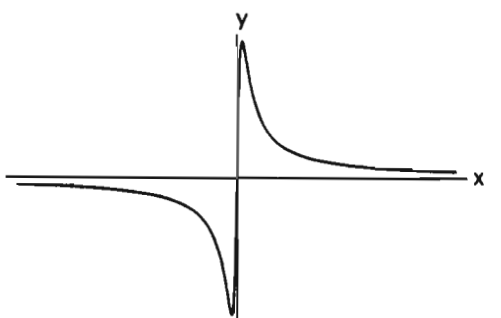


Figure 4. $k = \frac{1}{8}$; $y(0) = 0$, $y'(0) = 1$. Decaying solution indicates that $k = \frac{1}{8}$ is the limiting case of the $k < \frac{1}{8}$ behaviour.

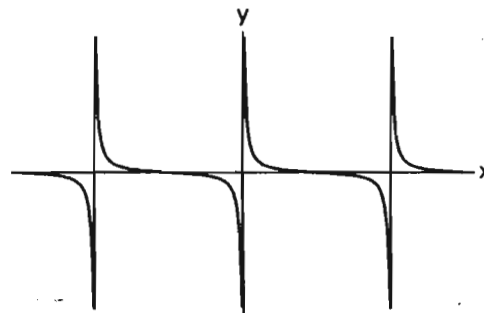


Figure 5. $k = \frac{1}{7}$; $y(0) = 0$, $y'(0) = 1$. The solution is oscillatory for all initial conditions.

6. Discussion

An interesting feature of (1.1) is the connection between the passing of the weak Painlevé test and the reduction of the solution to parametric form when k takes the values given in (5.1). Equation (1.1) can be reduced to quadratures via the symmetry approach—either direct or hidden—for any value of the parameter, k . However, the weak Painlevé property

is found only for the particular values of k given by (5.1). We observe the curious property that, apart from the trivial case $k = 0$, this feature is restricted to values of k in $(\frac{1}{9}, \frac{1}{8})$. (Recall that, when $k = \frac{1}{9}$, the parameter is easily eliminated.) In terms of the behaviour of the solution, $k = \frac{1}{8}$ is the critical value in that the solution passes from non-oscillatory to oscillatory (see figures 1 to 5). In Leach *et al* (1988) it was stressed that it was this value rather than $k = \frac{1}{9}$ which was the critical value. However, the present analysis strongly suggests that $k = \frac{1}{9}$ is also a critical value although in a sense different to normal usage.

In the initial treatment of the Painlevé analysis we noted that the Kowalevskaya exponent was $4 - \alpha$. Clearly the Painlevé test is satisfied for $\alpha = 3, 2, 1, -1, \dots$ (α cannot be zero). By way of example in the particular case $\alpha = 1$ we find that

$$y = \frac{1}{x} \left[1 + (b\chi)^3 + \frac{4}{27}(b\chi)^6 + \frac{53}{27}(b\chi)^9 + \dots \right] \quad (6.1)$$

i.e. y is essentially a function in the variable $[b(x - x_0)]^3$. It appears that for general integer α , the expansion is in powers of $[b(x - x_0)]^{4-\alpha}$.

If one performs the reduction of order via the hidden symmetry method with the substitution $y = 4u'/u$, the resulting quadrature is

$$x - x_0 = \int \frac{du}{(Au^q + Bu^{-q})^{1/2}} \quad (6.2)$$

where $q = 2(4 - \alpha)/\alpha$. This can be evaluated in closed form for $\alpha = 3, 2, 1$ which correspond to $k = \frac{1}{9}, 0$ and -1 respectively. In contrast to the situation when the weak Painlevé test is satisfied, the passing of the Painlevé test does not coincide with expression of the solution in parametric form (except for the special cases noted above).

Acknowledgments

This work was supported by the University of Natal Research Fund and the Foundation for Research Development of South Africa.

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International Journal of Nonlinear Mechanics **29** (1994) 177-185



THE LIE ANALYSIS AND SOLUTIONS FOR A CLASS OF SECOND-ORDER NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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(Received 25 March 1993)

Abstract—A class of differential equations discussed by Ranganathan [*Int. J. Non-Linear Mech.* **23**, 421 (1988) and **24**, 19 (1989)] and Kara and Mahomed [*Int. J. Non-Linear Mech.* **27**, 919 (1992)] is subjected to the Lie analysis to determine those cases for which a solution can be found.

1. INTRODUCTION

Ranganathan [1, 2] has investigated the existence of first integrals and solutions of second-order non-linear ordinary differential equations of the general form

$$\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2q^{-1} + f(t)q^n, \quad (1.1)$$

where $p(t)$, $r(t)$ and $f(t)$ are arbitrary functions of time and μ and n are arbitrary constants with neither being unity. Ranganathan generalised the method of Reid [3, 4] which is based on homogeneous combinations of two linearly independent solutions, $w_1(t)$ and $w_2(t)$, of the associated linear equation

$$\ddot{w} + p(t)\dot{w} + r(t)w = 0. \quad (1.2)$$

The work of Reid was a generalisation of that of Pinney [5] which is closely related to the celebrated paper of Ermakov [6].

More recently Kara and Mahomed [7] approached (1.1) from the point of view of equivalent Lagrangians. In particular they treated the subsets of (1.1) which are related to the canonical representations of second-order equations possessing the Lie algebras $sl(3, R)$ and $sl(2, R)$.

In this paper we consider (1.1) as a representation of an equation of simpler structure from which it can be obtained by means of a point transformation of Cartan form. This simpler equation is studied by the standard methods of Lie analysis.

2. SIMPLIFICATION OF (1.1)

When the first term on the right-hand side of (1.1) is taken to the left-hand side and the equation multiplied by $q^{-\mu}$, it is evident that it has the form

$$\frac{1}{1-\mu}(q^{1-\mu})'' + \frac{p}{1-\mu}(q^{1-\mu})' + rq^{1-\mu} = fq^{n-\mu} \quad (2.1)$$

or, after the substitution

$$s = \frac{q^{1-\mu}}{1-\mu} \quad (2.2)$$

is made,

$$\ddot{s} + p\dot{s} + (1-\mu)rs = (1-\mu)^{(n+1)/(1-\mu)}fs^{(n-\mu)/(1-\mu)}. \quad (2.3)$$

In the case that $n = \mu$, (2.3) becomes the linear equation

$$\ddot{s} + p\dot{s} + (1-\mu)rs = f \quad (2.4)$$

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Contributed by W. F. Ames.

(recall that $n = 1$ has already been excluded). Being linear (2.6) possesses the eight-element Lie algebra $sl(3, R)$ and corresponds to one of the types treated by Kara and Mahomed via the technique of equivalent Lagrangians. We do not pursue this linear equation further.

Under the transformation

$$s(t) = y(x)v(t), \quad x = u(t), \quad (2.5)$$

(2.3) becomes

$$y''\dot{u}^2v + y'(\ddot{u}v + 2\dot{u}\dot{v} + p\dot{u}v) + y(\ddot{v} + p\dot{v} + (1 - \mu)rv) = (1 - \mu)^{\sigma+1}fy^\sigma v^\sigma, \quad (2.6)$$

where

$$\sigma = \frac{n - \mu}{1 - \mu}. \quad (2.7)$$

The standard treatment of (2.6) is to select the functions u and v to remove the y' term, set the coefficient of y constant (Leach [8] chose 1 for the case $\sigma = 2$ and Leach *et al.* [9] 0 for the general case) and concentrate the variable coefficient in the non-linear term y^σ . Note that Feix and Lewis [10] treated a subset of this class by requiring that the coefficient of y^σ also be a constant. However, Moreira [11] and Leach [12] elected to place the variable coefficient at y' , make the coefficient of y zero and that of y^σ one. All of these are different representations of the generalized Emden-Fowler equation

$$\psi'' + \phi(x)\psi^n = 0, \quad (2.8)$$

on which there is a vast literature (see [9] and the references cited therein).

One of the problems encountered in the group theoretic treatment of (2.6) when the unspecified variable coefficient is placed at the non-linear term (cf. [8, 9]) is that, when $\sigma = 2$, a function which arises in the analysis is required to satisfy a very non-linear equation. This equation is not obviously solvable. A similar problem arose when the unspecified variable coefficient was placed at y' in [11, 12]. The method of [11] was based on the "direct" method of Lewis and Leach [13] (see also Gascon *et al.* [14]) and that of [12] on Noether's theorem. The final equation to be solved was the same. The general solution cannot be obtained in analytic form [15]. On the basis of these experiences, we examine (2.6) with the variable coefficient placed with the y term, the coefficient of y' is set at zero and that of y^σ at one. Thus, we set

$$\ddot{u}v + 2\dot{u}\dot{v} + p\dot{u}v = 0 \quad (2.9)$$

so that

$$\dot{u} = v^{-2} \exp \left[\int p(x) dt \right] \quad (2.10)$$

and, from the coefficient of y^σ , $v(t)$ is such that

$$(1 - \mu)^{\sigma+1}fv^{\sigma+3} \exp \left[2 \int p(t) dt \right] = 1 \quad (2.11)$$

except in the case $\sigma = -3$ which we exclude from the general discussion and treat separately. With u and v now specified we assume that the coefficient of y can be written as a function of the new independent variable, $F(x)$.

3. THE CASE $\sigma = -3$

Before we proceed to the general case, we consider the case $\sigma = -3$ which corresponds to $n = 4\mu - 3$. We take the coefficient of y to be zero, i.e. v is a solution of the linear equation

$$\ddot{v} + p\dot{v} + (1 - \mu)rv = 0. \quad (3.1)$$

The equation to be considered is

$$y'' = G(x)y^{-3}. \quad (3.2)$$

We recall that a second-order ordinary differential equation

$$\mu(x, y, y', y'') = 0 \quad (3.3)$$

possesses a Lie point symmetry

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (3.4)$$

provided

$$X^{[2]}N(x, y, y', y'')|_{N=0} = 0, \quad (3.5)$$

where

$$X^{[2]} = X + (\eta' - y'\xi') \frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'') \frac{\partial}{\partial y''} \quad (3.6)$$

is the second extension of X .

A routine calculation shows that

$$X = a(x) \frac{\partial}{\partial x} + c(x)y \frac{\partial}{\partial y} \quad (3.7)$$

subjected to the requirements

$$c'' = 0, \quad (3.8)$$

$$a'' = 2c', \quad (3.9)$$

$$(4c - 2a')G = aG', \quad (3.10)$$

i.e.

$$c = C_0 + C_1x, \quad (3.11)$$

$$a = A_0 + A_1x + C_1x^2 \quad (3.12)$$

and

$$(4C_0 - 2A_1)G = (A_0 + A_1x + C_1x^2)G'. \quad (3.13)$$

For general G , (3.13) requires that all constants be zero, i.e. no symmetry exists. If G is a constant, K , there are three symmetries, namely

$$X_1 = \frac{\partial}{\partial x}, \quad (3.14)$$

$$X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad (3.15)$$

$$X_3 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad (3.16)$$

which have the Lie algebra $sl(2, R)$ and so belong to the second class considered by Kara and Mahomed [7]. The solution of (3.2) is trivial being

$$y = (A + 2Bx + Cx^2)^{1/2}, \quad (3.17)$$

with

$$AC - B^2 = K. \quad (3.18)$$

If (3.13) is regarded as the defining differential equation for G , we have

$$G = K \exp \left[\int \frac{(4C_0 - 2A_1) dx}{(A_0 + A_1x + C_1x^2)} \right], \quad (3.19)$$

which has three forms depending on the sign of the discriminant of the denominator of the integrand. There is just the one symmetry, namely

$$X = (A_0 + A_1x + C_1x^2) \frac{\partial}{\partial x} + (C_0 + C_1x)y \frac{\partial}{\partial y}. \quad (3.20)$$

In this case (3.2) is transformed to the autonomous form

$$Y'' + (2C_0 - A_1)Y' + C_0^2Y = KY^{-3} \quad (3.21)$$

under the transformation

$$X = \int \frac{dx}{A_0 + A_1x + C_1x^2}, \quad (3.22)$$

$$Y = y \exp \left(- \int \frac{C_0 + C_1x}{A_0 + A_1x + C_1x^2} dx \right). \quad (3.23)$$

Equation (3.21) does not fall into the class of equations to which Reid's method [4] applies and does not satisfy the Painlevé test. (Put $u^2 = y$. There is a pole at $p = 1$ and resonances at $r = -1, 1$, but the Y' term ruins the further analysis.) It can be reduced to an Abel's equation of the second kind, the solution of which is by no means obvious. We conclude that the only useful case for $\sigma = -3$ is when there are three symmetries.

4. THE CASE $\sigma \neq -3, 2$

Examination of the existence of a symmetry for the equation

$$y'' + F(x)y = y^\sigma \quad (4.1)$$

soon reveals that the case $\sigma = 2$ is distinct. We consider the case $\sigma \neq 2$ first. The only possible structure of a symmetry of (4.1) is

$$X = a(x) \frac{\partial}{\partial x} + c(x)y \frac{\partial}{\partial y} \quad (4.2)$$

and this is provided $a(x)$ and $c(x)$ satisfy the requirements:

$$2c' = a'', \quad (4.3)$$

$$(\sigma - 1)c = -2a', \quad (4.4)$$

$$c'' + aF' + 2a'F = 0. \quad (4.5)$$

From (4.3) and (4.4) it follows that

$$a = A_0 + A_1x, \quad (4.6)$$

$$c = -\frac{2A_1}{\sigma - 1}, \quad (4.7)$$

so that F is determined by

$$(A_0 + A_1x)F' + 2A_1F = 0. \quad (4.8)$$

For general $F(x)$ there is no symmetry. If $F(x)$ is a constant, K , there is one symmetry and the solution (1.1) is reduced to the quadrature

$$x - x_0 = \int \frac{dy}{\left[C - Ky^2 + 2\frac{y^{\sigma+1}}{\sigma+1} \right]^{1/2}}, \quad (4.9)$$

where C is a constant of integration. Recalling that $\sigma \neq -3, 2, 1$ the solution can be expressed in terms of elliptic functions for certain values of σ . (See Section 6.)

The general solution of (4.8) is

$$F(x) = \frac{K}{(A_0 + A_1x)^2} \quad (4.10)$$

and for $K \neq 0$ there is just one symmetry, namely

$$X = (A_0 + A_1x) \frac{\partial}{\partial x} - \frac{2A_1}{\sigma - 1} y \frac{\partial}{\partial y}. \quad (4.11)$$

With $F(x)$ as given in (4.10) the autonomous form of the equation is

$$Y'' - A_1 \left(\frac{\sigma + 3}{\sigma - 1} \right) Y' + \left(K + \frac{2A_1^2(\sigma + 1)}{(\sigma - 1)^2} \right) Y = Y^\sigma, \quad (4.12)$$

where

$$X = \frac{1}{A_1} \log(A_0 + A_1 x), \quad (4.13)$$

$$Y = y(A_0 + A_1 x)^{2/(\sigma-1)}. \quad (4.14)$$

As was the case for $\sigma = -3$, (4.12) is not amenable to further treatment unless M is put equal to zero which brings us back to the $F(x) = K$ case mentioned above.

In the special case that $F(x)$ is zero, i.e. the transformation which satisfies (2.11) also makes the coefficient of y in (2.6) zero, there are two symmetries,

$$X_1 = \frac{\partial}{\partial x}, \quad (4.15)$$

$$X_2 = x \frac{\partial}{\partial x} + \frac{2}{1-\sigma} y \frac{\partial}{\partial y}, \quad (4.16)$$

for

$$y'' = y^\sigma. \quad (4.17)$$

The solution is reduced to the quadrature (cf. (4.9))

$$x - x_0 = \int \left(\frac{dy}{C + \frac{y^{\sigma+1}}{\sigma+1}} \right)^{1/2}. \quad (4.18)$$

General solutions can be obtained in terms of elliptic integrals for certain values of σ . (See Section 6.)

Single-parameter solutions can be obtained by putting $C = 0$. Then

$$y = \left[\frac{(1-\sigma)}{2(1+\sigma)^{1/2}} (x - x_0) \right]^{-2/(\sigma-1)}. \quad (4.19)$$

5. THE CASE $\sigma = 2$

The equation

$$y'' + F(x)y = y^2 \quad (5.1)$$

has a symmetry of the form

$$X = a(x) \frac{\partial}{\partial x} + [c(x)y + d(x)] \frac{\partial}{\partial y} \quad (5.2)$$

provided

$$2c' - a'' = 0, \quad (5.3)$$

$$c + 2a' = 0, \quad (5.4)$$

$$c'' + aF' + 2a'F = 2d, \quad (5.5)$$

$$d'' + dF = 0. \quad (5.6)$$

From (5.3) and (5.4) it is evident that

$$a = A_0 + A_1 x, \quad (5.7)$$

$$c = -2A_1. \quad (5.8)$$

Equation (5.5) may be regarded as a definition of $d(x)$, namely

$$d = A_1 F + \frac{1}{2}(A_0 + A_1 x)F' \quad (5.9)$$

so that (5.6) becomes the defining equation for $F(x)$

$$(A_0 + A_1 x)F''' + 4A_1 F'' + (A_0 + A_1 x)FF' + 2A_1 F^2 = 0. \quad (5.10)$$

The change of variables

$$X = A_0 + A_1x, \quad W(x) = F(x)/A_1^2 \quad (5.11)$$

reduces (5.10) to its essential form

$$XW''' + 4W'' + XWW' + 2W^2 = 0. \quad (5.12)$$

It has not been possible to obtain the general solution of (5.12), but in a very real sense it is not necessary. The change of variables

$$X = \int \frac{dx}{a}, \quad (5.13)$$

$$Y = a^2y - \int adx \quad (5.14)$$

reduces (5.1) to the autonomous form

$$Y'' - 5A_1Y' + (6A_1^2 - M)Y - N = Y^2, \quad (5.15)$$

when (5.5)–(5.7) hold. The constants M and N are given by

$$M = 2 \int adx - a^2F, \quad (5.16)$$

$$N = \left(\int adx \right)^2 - Fa^2 \int adx - 6a'^2ad + 3a^2a'd - a^3d'. \quad (5.17)$$

Equation (5.15) can be reduced in order to an Abel's equation of the second kind for which a closed-form solution is not obvious. This observation is independent of any solution of (5.12), apart from the trivial one $W(X) = 0$ for which (5.1) can be solved in terms of elliptic integrals.

In the case that $A_1 = 0$ (5.15) can be reduced to an elliptic integral. Equation (5.10) reduces to

$$F''' + FF' = 0, \quad (5.18)$$

which also can be reduced to an elliptic integral.

6. DISCUSSION

The possible solutions of

$$\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2q^{-1} + f(t)q^n \quad (6.1)$$

can be divided into a number of cases.

(a) $n = \mu$.

If
$$s(t) = As_1(t) + Bs_2(t) \quad (6.2)$$

is the general solution of the linear equation

$$\ddot{s} + p(t)\dot{s} + (1 - n)r(t)s = f(t), \quad (6.3)$$

the solution of (6.1) is, trivially,

$$q = [(1 - n)(As_1(t) + Bs_2(t))]^{1/(1-n)}. \quad (6.4)$$

This is the $sl(3, R)$ case of Kara and Mahomed [7].

(b) $n = 4\mu - 3$.

The standard form of (6.1) is

$$y'' = Ky^{-3}, \quad (6.5)$$

which has the solution

$$y = (A + 2Bx + Cx^2)^{1/2}, \quad (6.6)$$

where

$$AC - B^2 = K. \quad (6.7)$$

This is the $sl(2, R)$ case of Kara and Mahomed [7] and (6.5) is the elementary form of the Pinney equation [5]. The relationships between the variables (q, t) and (x, y) are

$$q(t) = \left[\frac{4}{1-n} y(x)v(t) \right]^{4/(1-n)}, \quad (6.8)$$

$$x = u(t), \quad (6.9)$$

where

$$\ddot{u}v + 2\dot{u}\dot{v} + p\dot{u}v = 0, \quad (6.10)$$

$$\ddot{v} + p\dot{v} + \left(\frac{1-n}{4} \right) rv = 0 \quad (6.11)$$

and also we have required that

$$\left(\frac{1-n}{4} \right)^{3(n-1)/4} f(t) \exp \left[2 \int p(t) dt \right] = K. \quad (6.12)$$

The easiest way to interpret this case is to select arbitrary functions $u(t)$ and $v(t)$. From (6.10) and (6.11) this choice establishes

$$p(t) = -\frac{\ddot{u}}{\dot{u}} - \frac{2\dot{v}}{v}, \quad (6.13)$$

$$r(t) = \frac{4}{n-1} \left[\frac{1}{3} \frac{(v^3)''}{v^3} + \frac{\dot{v}\ddot{u}}{v\dot{u}} \right]. \quad (6.14)$$

The third function $f(t)$ is obtained from (6.12) as

$$f(t) = K_1 \dot{u}v^4 \quad (6.15)$$

and the general solution is

$$q(t) = \{v(t)[A + 2Bu(t) + Cu^2(t)]\}^{2/(1-n)}, \quad (6.16)$$

where

$$AC - B^2 = K_1 \left(\frac{1-n}{4} \right)^{(3n-11)/4}. \quad (6.17)$$

(c) $n = 2 - \mu$.

In the case that $A_1 = 0$ (5.15) can be reduced to an incomplete elliptic integral of the first type and so $Y(X)$ is expressed in terms of a Jacobi elliptic function. With $A_0 = 1$, (5.5) gives

$$d = \frac{1}{2} F' \quad (6.18)$$

so that the transformation from (x, y) to (X, Y) , (5.13) and (5.14), becomes

$$x = X, \quad y = Y + \frac{1}{2} F. \quad (6.19)$$

From (5.10) the function $F(x)$ is a solution of

$$F'' + \frac{1}{2} FF' = 0 \quad (6.20)$$

and $F(x)$ is also given by a Jacobi elliptic function. (We note that for special cases of the values of constants of integration simpler relations will occur.) For arbitrary functions $u(t)$ and $v(t)$, the coefficient functions in (1.1) are

$$p(t) = -\frac{\ddot{u}}{\dot{u}} - \frac{2\dot{v}}{v}, \quad (6.21)$$

$$r(t) = \frac{1}{n-1} \left[\dot{u}^2 F(u(t)) + \frac{1}{3} \frac{(v^3)''}{v^3} + \frac{\dot{v}\ddot{u}}{v\dot{u}} \right], \quad (6.22)$$

$$f(t) = \frac{1}{(n-1)^{2(n-1)}} v^{-2} \quad (6.23)$$

and the solution is

$$q(t) = \{(n-1)v(t)\{Y[u(t)] + \frac{1}{2}F[u(t)]\}\}^{1/(n-1)}. \quad (6.24)$$

The two arbitrary constants of $q(t)$ are found in the argument of the elliptic function in $Y(x)$. The constants in $F(x)$ are parameters in (1.1) due to (6.22).

(d) Other values of n .

In Section 4 we saw that the solution of

$$y'' + F(x)y = y^\sigma \quad (6.25)$$

can be reduced to quadrature in the cases $F(x) = K$ and $F(x) = 0$. Consider the case $F(x) = K$. Equation (6.25) can be integrated to give

$$y'^2 = C - Ky^2 + \frac{2}{\sigma+1}y^{\sigma+1}. \quad (6.26)$$

Following Broucke [16], we put

$$y = w^\alpha \quad (6.27)$$

so that

$$w' = \frac{1}{\alpha} \left[Cw^{2-2\alpha} - Kw^2 + \frac{2}{\sigma+1}w^{\alpha(\sigma-1)+2} \right]^{1/2}. \quad (6.28)$$

The quadrature of (6.28) can be expressed in terms of elliptic integrals [17] if for suitable α it takes the form

$$w' = \frac{1}{\alpha} wk [P(w)]^{1/2}, \quad (6.29)$$

where k is an integer and $P(w)$ is at most a fourth degree polynomial. (Note that some cases can reduce to simpler integrals.) We find that the possible (α, σ) pairs are $[-1/2, (l-2m)/l]$, $1, m \in (0, 1, 2, 3, 4)$, $(1, m-1), m \in (0, 1, 2, 3, 4)$, and $[-(m-2)/2, (m+2)/(m-2)]$, $m \in (0, 1, 3, 4)$, excluding parameter values which lead to $\sigma = -3, -1, 0, 1$ and 2 .

Whichever integrals can be evaluated and converted to give $y(x)$ provide solutions for (1.1) in the form

$$q(t) = (1-\mu)[y(u(t))v(t)]^{1/(1-\mu)}, \quad (6.30)$$

where u and v are arbitrary functions, provided $p(t)$ and $f(t)$ are given by (6.21) and (6.23), respectively, and

$$r(t) = \frac{1}{1-\mu} \left[K\dot{u}^2 - \frac{\ddot{v}}{v} + \frac{2\dot{v}^2}{v^2} - \frac{\ddot{u}\dot{v}}{\dot{u}v} \right]. \quad (6.31)$$

When $K = 0$, (6.28) becomes

$$w' = \frac{1}{\alpha} \left[Cw^{2-2\alpha} + \frac{2}{\sigma+1}w^{\alpha(\sigma-1)+2} \right]^{1/2}, \quad (6.32)$$

which can be expressed in terms of elliptic integrals if α and σ belong to the number pairs

$$\left(1-k, \frac{m+k-1}{1-k} \right) \quad \text{or} \quad \left(-\frac{1}{2}(m+2k-2), \frac{m-2k+2}{m+2k-2} \right),$$

where k is an integer not equal to one and $m \in \{0, 1, 2, 3, 4\}$. Values of m and k which make $\sigma = -3, -1, 0, 1$ or 2 are excluded. The solution for (1.1) is given by (6.30) with $p(t), f(t)$ and $r(t)$ given by (6.21), (6.23) and (6.31) (with $K = 0$).

Acknowledgements—This work was supported in part by the University of Natal Research Office and the Foundation for Research Development of South Africa.

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Solution of generalized Emden-Fowler equations with two symmetries

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International Journal of Nonlinear Mechanics **29** (1994) 529–538



0020-7462(93)E0021-M

SOLUTION OF GENERALIZED EMDEN-FOWLER EQUATIONS WITH TWO SYMMETRIES

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Abstract—The generalized Emden–Fowler equation $y'' + p(x)y' + r(x)y = f(x)y^n$ has a single point symmetry under a certain constraint on $f(x)$. Although the order of the equation can be reduced by one, integration of the resulting Abel's equation of the second kind in closed form is not generally possible. Under a stronger constraint there exist two symmetries G_1 and G_2 , such that $[G_1, G_2] = (\text{cst})G_2$ and reduction to quadratures becomes trivial. The special cases $n = 2$ and $n = -3$ are treated in detail.

1. INTRODUCTION

The generalized Emden–Fowler equation

$$y'' + p(x)y' + r(x)y = f(x)y^n \quad (1.1)$$

is the simplest second-order ordinary differential equation which contains a single nonlinear term. It arises frequently in the modelling of problems in one dimension and as the radial equation in spherically symmetric problems. (See Wong [1] for a review which, even then, was very selective in its list of references (cf. Leach *et al.* [2].) The origin of the name is found in the works on stellar structure by Lane [3] and Emden [4] and the more mathematical analyses initiated by Fowler [5, 6]. The more generalized form of (1.1) can be found in the papers of Feix and Lewis [7], Leach [8] and Basu and Ray [9]. Variations on (1.1) have been considered by Ranganathan [10, 11] and further discussed by Kara and Mahomed [12], but they do not constitute any real generalization. (See Lemmer and Leach [13].)

Indeed the degree of generalization is a real question. It is a well-known fact that the second and third terms of (1.1) can be removed by a Kummer–Liouville transformation:

$$y(x) = u(x)v(t) \quad t = t(x) \quad (1.2)$$

and this approach is found in, e.g., the papers of Leach [8] (for the particular case $n = 2$) and Leach *et al.* [2]. In this paper we do not remove these terms by means of a preliminary transformation for a very specific reason. By keeping these terms we find that the analysis gives rise to a particular type of third-order linear differential equation which provides significant insight into the properties of the mathematical problem under consideration. This is by no means critical to the analysis, but it is a nice point to be appreciated by those who enjoy the study of the structure of differential equations.

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The purpose of this paper is to solve (1.1) in the sense of reduction to quadratures. To this end we examine the equation for Lie point symmetries. It is well-known that the possession of a point symmetry enables one to reduce the order of an ordinary differential equation by one. It is equally well-known that this does not guarantee integrability and, indeed, in the case of second-order equations, the reduction usually leads to an Abel's equation of the second kind, from which little joy is to be expected. However, if a second-order equation possesses two symmetries, G_1 and G_2 , with the property that $[G_1, G_2] = (\text{cst})G_2$, reduction of order using G_2 leads to a first-order equation which inherits a symmetry from G_1 and so is reducible to zeroth order, i.e. the solution is given by a quadrature. (In general two symmetries with this algebra reduce an n th order equation to one of $(n - 2)$ th order.) Consequently, we look for the subset of equations of type (1.1) which have two symmetries.

The requirement that (1.1) possess one symmetry imposes a relationship amongst the functions $p(x), r(x)$ and $f(x)$, which we shall regard as a constraint on the last, i.e. we require the nonlinearity to fit in with the linear structure. (This is simply a matter of arbitrary choice. One could equally demand that the damping term be consistent with the "potential" terms.) Under this constraint, (1.1) can be transformed to autonomous form by requiring that the symmetry take the form $\partial/\partial X$ in the new coordinates. This is the approach found in several papers on the subject (cf. Leach [8], Feix and Lewis [7] and Leach *et al.* [2]). In this paper we investigate the conditions under which the autonomous form of (1.1) possesses a second point symmetry. When this does occur, it is this second symmetry which plays the role of G_2 referred to above. The reduction to quadratures becomes very simple.

The price of integrability is the imposition of a further constraint on the freedom of choice of the function, $f(x)$ (in the sense noted above).

We find that the analysis naturally separates into two cases, $n \neq 2$ and $n = 2$. (We exclude $n = 0, 1$ for obvious reasons.) The latter gives a richer result compared with a general value of n . However, when $n = -3$, the results are distinct from all other values of n . Two symmetries simply cannot occur. Either there is one and the problem of dealing with an intractable Abel's equation of the second kind remains, or there are three. In the latter case the equation is of Ermakov-Pinney type (Ermakov [14], Pinney [15]) and the quadrature is reduced to an explicit formula.

We should note that recently Berkovich [16, 17] has also mentioned the existence of a second symmetry, but as an aside and in a context of intent different from that of the present paper.

2. THE BASIC EQUATIONS

A second-order differential equation

$$N(x, y, y', y'') = 0 \quad (2.1)$$

possesses a Lie point symmetry

$$G = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (2.2)$$

if

$$G^{[2]}N|_{N=0} = 0, \quad (2.3)$$

where

$$G^{[2]} := G + (\eta' - y'\xi') \frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'') \frac{\partial}{\partial y''} \quad (2.4)$$

is the second extension of G , which is necessary to deal with the infinitesimal transformations in y' and y'' induced by the action of G . It is a standard procedure to show that the application of (2.4) to (1.1) requires that G necessarily take the form

$$G = a(x) \frac{\partial}{\partial x} + (c(x)y + d(x)) \frac{\partial}{\partial y}. \quad (2.5)$$

Two possible cases emerge immediately. They are

(i) $n \neq 2$, for which

$$\begin{aligned} 2c' - a'' + ap' + a'p &= 0, \\ c'' + c'p + ar' + 2a'r &= 0, \\ af' + [2a' + (n - 1)c]f &= 0, \\ d'' + d'p + dr &= 0, \\ dnf &= 0 \end{aligned} \tag{2.6}$$

and

(ii) $n = 2$, for which

$$\begin{aligned} 2c' - a'' + ap' + a'p &= 0, \\ c'' + c'p + ar' + 2a'r &= 2df, \\ af' + (2a' + c)f &= 0, \\ d'' + d'p + dr &= 0. \end{aligned} \tag{2.7}$$

We consider each in turn.

3. CASE $n \neq 2$

From the last of (2.6) it is immediately evident that $d = 0$. The functions a and c satisfy the system of equations

$$2c' - a'' + ap' + a'p = 0, \tag{3.1}$$

$$c'' + c'p + ar' + 2a'r = 0 \tag{3.2}$$

and f is given by

$$\frac{f'}{f} = -\frac{2a' + (n - 1)c}{a}. \tag{3.3}$$

From (3.1)

$$c' = \frac{1}{2}a'' - \frac{1}{2}(ap' + a'p), \tag{3.4}$$

so that (3.2) is

$$\frac{1}{2}a''' - (p' + \frac{1}{2}p^2 - 2r)a' - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)'a = 0. \tag{3.5}$$

Equation (3.5) is a linear equation of the form

$$y''' + By' + \frac{1}{2}B'y = 0, \tag{3.6}$$

which is self-adjoint and has the maximal symmetry for a third-order ordinary differential equation, viz. $A_1 \oplus 3A_1 \oplus sl(2, R)$ (Mahomed and Leach [18]). Equation (3.6) has an integrating factor, y , and the integrated equation reduces to the Ermakov-Pinney equation (Ermakov [14], Pinney [15]) on the substitution $y = \rho^2$.

Equation (3.4) is readily integrated to give

$$c = C_0 + \frac{1}{2}a' - \frac{1}{2}ap \tag{3.7}$$

and (3.3) becomes

$$\frac{f'}{f} = -\left\{ \frac{n + 3a'}{2} \frac{1}{a} + (n - 1) \frac{C_0}{a} - \frac{n - 1}{2} p \right\}. \tag{3.8}$$

4. THE SPECIAL CASE $n = -3, C_0 = 0$

From (3.8) it is evident that the case $n = -3$ and $C_0 = 0$ is special, since then

$$\frac{f'}{f} = -2p, \tag{4.1}$$

i.e. f is independent of a and there are the three symmetries

$$G_i = a_i \frac{\partial}{\partial x} + \frac{1}{2}(a_i' - a_i p) \frac{\partial}{\partial y}, \quad (4.2)$$

where the three functions $a_i(x)$ are the linearly independent solutions of the third-order equation (3.5). The algebra of the symmetries is $sl(2, R)$.

Under the standard representation of $sl(2, R)$ the three symmetries have the form

$$\begin{aligned} G_1 &= \frac{\partial}{\partial X}, \\ G_2 &= 2X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y}, \\ G_3 &= X^2 \frac{\partial}{\partial X} + XY \frac{\partial}{\partial Y} \end{aligned}$$

and the equation is

$$Y'' = KY^{-3}. \quad (4.3)$$

The solution of (4.3) is

$$Y = [A + 2BX + CX^2]^{1/2}, \quad (4.4)$$

where $AC - B^2 = K$. This follows from the well-known result of Pinney [15] that, if $u(x)$ and $v(x)$ are linearly independent solutions of

$$y'' = \omega^2(x)y = 0, \quad (4.5)$$

the solution of

$$y'' + \omega^2(x)y = \frac{K}{y^3} \quad (4.6)$$

is

$$y = [Au^2 + 2Buv + Cv^2]^{1/2} \quad (4.7)$$

with $AC - B^2 = K/W^2$, where W is the value of the Wronskian of the solutions $u(x)$ and $v(x)$.

From (4.1)

$$f(x) = K \exp \left[-2 \int p \, dx \right]. \quad (4.8)$$

The transformation from

$$y'' + py' + ry = K \exp \left[-2 \int p \, dx \right] y^{-3} \quad (4.9)$$

to (4.3), viz.

$$Y'' = KY^{-3},$$

is

$$X = \frac{1}{2\sqrt{-M}} \exp \left[2\sqrt{-M} \int \frac{dx}{a} \right] \quad (4.10)$$

$$Y = \exp \left[\sqrt{-MX} \right] ya^{-1/2} \exp \left[\frac{1}{2} \int p \, dx \right], \quad (4.11)$$

where M is a parameter, and the solution follows. (For $M > 0$ the relevant exponentials are replaced by trigonometric functions.)

The value of M is found from the integration of (3.5), viz.

$$\frac{1}{2} a''' - (p' + \frac{1}{2} p^2 - 2r)a' - \frac{1}{2} (p' + \frac{1}{2} p^2 - 2r)' a = 0.$$

When (3.5) is multiplied by the integrating factor a , it is trivially integrated to give

$$\frac{1}{2} aa'' - \frac{1}{2} a'^2 - \frac{1}{2} (p' + \frac{1}{2} p^2 - 2r)a^2 = M. \quad (4.12)$$

5. CASE $n = -3$ AND $C_0 \neq 0$

In this case (3.8) yields

$$f = K \exp \left[-2 \int \left(p - \frac{2C_0}{a} \right) dx \right] \tag{5.1}$$

and there is the single symmetry

$$G = a \frac{\partial}{\partial x} + (C_0 + \frac{1}{2}(a' - ap))y \frac{\partial}{\partial y}. \tag{5.2}$$

However, recall that there are three a 's from the solution of (3.5) and so three independent f 's for a given C_0 .

With f as in (5.1), the autonomous form of (1.1) is

$$Y'' + 2C_0 Y' + (M + C_0^2)Y = KY^{-3} \tag{5.3}$$

and the transformation is

$$X = \int \frac{dx}{a}$$

$$Y = ya^{-1/2} \exp \left[\frac{1}{2} \int \left(p - \frac{2C_0}{a} \right) dx \right]. \tag{5.4}$$

Equation (5.3) has only the one symmetry given by $\partial/\partial X$ and the standard reduction of order via $\eta = Y$ and $\zeta = Y'$ gives

$$\zeta \zeta' + 2C_0 \zeta + (M + C_0^2)\eta = K\eta^{-3}, \tag{5.5}$$

which is an Abel's equation of the second kind and no closed-form solution is apparent.

6. CASE $n \neq -3, 2$

We return to the consideration of the case for general index $n (\neq 0, 1, 2, -3)$. We recall that we have the equation

$$y'' + py' + ry = fy^n \tag{6.1}$$

with the symmetry

$$G = a \frac{\partial}{\partial x} + cy \frac{\partial}{\partial y} \tag{6.2}$$

and that

$$c = C_0 + \frac{1}{2}(a' - ap), \tag{6.3}$$

$$f = Ka^{-(n+3)/2} \exp \left[\frac{n-1}{2} \int \left(p - \frac{2C_0}{a} \right) dx \right], \tag{6.4}$$

where $a(x)$ is a solution of

$$\frac{1}{2}a''' - (p' + \frac{1}{2}p^2 - 2r)a' - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)'a = 0 \tag{6.5}$$

or, equivalently, the integrated form

$$\frac{1}{2}aa'' - \frac{1}{4}a'^2 - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)a^2 = M. \tag{6.6}$$

The autonomous form of (6.1), with f as given in (6.4), is

$$Y'' + 2C_0 Y' + (M + C_0^2)Y = KY^n, \tag{6.7}$$

where the transformation from (6.1) to (6.7) is

$$X = \int \frac{dx}{a}$$

$$Y = ya^{-1/2} \exp \left[\frac{1}{2} \int \left(p - \frac{2C_0}{a} \right) dx \right]. \tag{6.8}$$

The symmetry (6.2) now has the form

$$G_1 = \frac{\partial}{\partial X}. \quad (6.9)$$

The standard reduction of (6.7) under (6.9) leads to an Abel's equation of the second kind, from which very little joy can be expected. However, it is valid to pose the question: "Are there any circumstances under which (6.7) has two symmetries?" Any symmetry of (6.7), apart from (6.9), must have the form

$$G_2 = a \frac{\partial}{\partial X} + cY \frac{\partial}{\partial Y}, \quad (6.10)$$

where $a(X)$ and $c(X)$ have to be determined. Such a symmetry exists if there is a nontrivial solution to the system

$$2c' - a'' + 2C_0 a' = 0, \quad (6.11)$$

$$(n-1)c = -2a', \quad (6.12)$$

$$c'' + 2C_0 c' + 2(M + C_0^2)a' = 0. \quad (6.13)$$

Equations (6.11) and (6.12) combine to give

$$\frac{n+3}{n-1} a'' - 2C_0 a' = 0, \quad (6.14)$$

which explains the peculiarity of the $n = -3$ case. Either C_0 is zero, which leads to the three symmetries given in (4.2), or a' is zero, which leads to only one symmetry.

In the general case ($n \neq -3$), we solve (6.14) for a and c follows from (6.12). We find

$$a = A_0 + A_1 \exp \left[2C_0 \left(\frac{n-1}{n+3} \right) X \right], \quad (6.15)$$

$$c = -\frac{4C_0 A_1}{n+3} \exp \left[2C_0 \left(\frac{n-1}{n+3} \right) X \right], \quad (6.16)$$

but we also require consistency with (6.13) and this imposes the constraint

$$M = -C_0^2 \left(\frac{n-1}{n+3} \right)^2. \quad (6.17)$$

Hence

$$Y'' + 2C_0 Y' + \frac{8(n+1)}{(n+3)^2} C_0^2 Y = KY^n \quad (6.18)$$

has the two symmetries

$$G_1 = \frac{\partial}{\partial X}, \quad (6.19)$$

$$G_2 = \exp \left[2C_0 \left(\frac{n-1}{n+3} \right) X \right] \left(\frac{\partial}{\partial X} - \frac{4C_0}{n+3} Y \frac{\partial}{\partial Y} \right). \quad (6.20)$$

Since $[G_1, G_2] = (\text{cst})G_2$, reduction should be done using G_2 (Olver [19]). This is made easier by the change of variables

$$\mathcal{X} = \frac{1}{2C_0} \left(\frac{n+3}{n-1} \right) \exp \left[-2C_0 \left(\frac{n+1}{n-3} \right) X \right], \quad (6.21)$$

$$\mathcal{Y} = Y \exp \left[\frac{2C_0 X}{n+3} \right], \quad (6.22)$$

which gives

$$G_2 = \frac{\partial}{\partial \mathcal{X}} \quad (6.23)$$

and

$$\mathcal{Y}'' = K\mathcal{Y}^n. \quad (6.24)$$

The reduction to quadratures is obvious.

For the existence of one symmetry, $a(x)$ is a solution of

$$\frac{1}{2}a''' - (p' + \frac{1}{2}p^2 - 2r)a' - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)'a = 0 \tag{6.25}$$

or its integrated version

$$\frac{1}{2}aa'' - \frac{1}{4}a'^2 - (p' + \frac{1}{2}p^2 - 2r)a^2 = M \tag{6.26}$$

and there are three such functions, $a(x)$. For the existence of two symmetries, $a(x)$ is a solution of

$$\frac{1}{2}aa'' - \frac{1}{4}a'^2 - (p' + \frac{1}{2}p^2 - 2r)a^2 = -C_0^2 \left(\frac{n-1}{n+3}\right)^2, \tag{6.27}$$

which, with $a = \rho^2$, has the Ermakov-Pinney form

$$\rho'' - \left(p' + \frac{1}{2}p^2 - 2r\right)\rho = -\frac{C_0^2}{\rho^3} \left(\frac{n-1}{n+3}\right)^2, \tag{6.28}$$

i.e. $a(x)$ is now a two-parameter function (excluding C_0). Note that the two-parameter function, $a(x)$, which is the solution of (6.27) satisfies (6.25), but it is not a general solution.

7. THE CASE $n = 2$

Equation (1.1) is now

$$y'' + py' + ry = fy^2 \tag{7.1}$$

and it has a symmetry of the form

$$G = a(x)\frac{\partial}{\partial x} + [c(x)y + d(x)]\frac{\partial}{\partial y}, \tag{7.2}$$

provided

$$2c' - a'' + ap' + a'p = 0, \tag{7.3}$$

$$c'' + c'p + ar' + 2a'r = 2df, \tag{7.4}$$

$$af' + (2a' + c)f = 0, \tag{7.5}$$

$$d'' + d'p + dr = 0. \tag{7.6}$$

Note that the fifth equation for the $n \neq 2$ case is now not separate, but coalesces with the second. The solution of (7.6) gives d . From (7.3) we have

$$c = C_0 + \frac{1}{2}(a' - ap) \tag{7.7}$$

and from (7.5)

$$f = Ka^{-5/2} \exp\left[\frac{1}{2}\int\left(p - \frac{C_0}{a}\right)\right]. \tag{7.8}$$

The remaining equation, (7.4), is now

$$\frac{1}{2}a''' - (p' + \frac{1}{2}p^2 - 2r)a' - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)'a = 2Kda^{-5/2} \exp\left[\frac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]. \tag{7.9}$$

Multiplication of (7.9) by a and integration gives

$$M = \frac{1}{2}aa'' - \frac{1}{4}a'^2 - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)a^2 - 2K\int\frac{d}{a^{3/2}} \exp\left[\frac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]. \tag{7.10}$$

Multiplication of (7.9) by $a \int d/a^{3/2}$ and integration gives

$$N = K \left\{ \int \frac{d}{a^{3/2}} \exp\left[\frac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right] \right\}^2 - \int \left\{ \left[\frac{1}{2}aa''' - (p' + \frac{1}{2}p^2 - 2r)aa' - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)'a^2 \right] \left[\int \frac{d}{a^{3/2}} \exp\left[\frac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right] \right] \right\}. \tag{7.11}$$

The autonomous form of (7.1) with f , as given in (7.8), viz.

$$Y'' + 2C_0 Y' + (M + C_0^2)Y + N = KY^2, \quad (7.12)$$

is obtained by the transformation

$$X = \int \frac{dx}{a}$$

$$Y = y \exp\left(-\int \frac{c}{a}\right) - \int \left[\frac{d}{a} \exp\left(-\int \frac{c}{a}\right)\right], \quad (7.13)$$

where, as usual,

$$c = C_0 + \frac{1}{2}(a' - ap). \quad (7.14)$$

The standard analysis of (7.12) shows that it has two symmetries if the constraint

$$\left(M + \frac{C_0^2}{25}\right)\left(M + \frac{49C_0^2}{25}\right) + 4KN = 0 \quad (7.15)$$

applies. The symmetries of

$$Y'' + 2C_0 Y' + (M + C_0^2)Y - \frac{1}{4K}\left(M + \frac{C_0^2}{25}\right)\left(M + \frac{49C_0^2}{25}\right) = KY^2 \quad (7.16)$$

are

$$G_1 = \frac{\partial}{\partial X}, \quad (7.17)$$

$$G_2 = \exp\left[\frac{2C_0 X}{5}\right] \left\{ \frac{\partial}{\partial X} - \left[\frac{4C_0}{5} Y - \frac{2C_0}{5K} \left(M + \frac{C_0^2}{25}\right) \right] \frac{\partial}{\partial Y} \right\}. \quad (7.18)$$

The transformation (X, Y) to $(\mathcal{X}, \mathcal{Y})$, which gives

$$G_2 = \frac{\partial}{\partial \mathcal{X}}, \quad (7.19)$$

transforms (7.16) to the standard form

$$\mathcal{Y}'' = K\mathcal{Y}^2, \quad (7.20)$$

the solution of which is straightforward.

8. CONCLUSION

In this investigation we have seen that for $n \neq -3, 2$ the equation

$$y'' + py' + ry = fy^n \quad (8.1)$$

has a Lie point symmetry, provided that

$$f = Ka^{-(n+3)/2} \exp\left[\frac{n-1}{2} \int \left(p - \frac{2C_0}{a}\right)\right], \quad (8.2)$$

where

$$\frac{1}{2}a''' - (p' + \frac{1}{2}p^2 - 2r)a' - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)a = 0 \quad (8.3)$$

or

$$\frac{1}{2}aa'' - \frac{1}{4}a'^2 - \frac{1}{2}(p' + \frac{1}{2}p^2 - 2r)a^2 = M. \quad (8.4)$$

Equation (8.1) has two symmetries, provided the constraint

$$M = -C_0^2 \left(\frac{n-1}{n+3}\right)^2 \quad (8.5)$$

is satisfied. The solution of equation (8.1) is then trivially reduced to a quadrature. In general there are five parameters, viz. C_0, K, A_1, A_2 and A_3 . Two symmetries exist on the family of hypersurfaces in the five-dimensional parameter space defined by (8.5).

For $n = 2$ the equation

$$y'' + py' + ry = fy^2 \quad (8.6)$$

has

$$f = Ka^{-5/2} \exp \left[\frac{1}{2} \int \left(p - \frac{2C_0}{a} \right) \right], \quad (8.7)$$

where

$$\frac{1}{2} a''' - \left(p' + \frac{1}{2} p^2 - 2r \right) a' - \frac{1}{2} \left(p' + \frac{1}{2} p^2 - 2r \right)' a = 2Kda^{-5/2} \exp \left[\frac{1}{2} \int \left(p - \frac{2C_0}{a} \right) \right] \quad (8.8)$$

and

$$d'' + pd' + rd = 0. \quad (8.9)$$

Equation (8.6) has two symmetries, provided the constraint

$$\left(M + \frac{C_0^2}{25} \right) \left(M + \frac{49C_0^2}{25} \right) + 4KN = 0, \quad (8.10)$$

where M and N are the values of the two integrals of (8.8), is satisfied. Since (8.8) can be rewritten as a fourth-order ordinary differential equation, the two symmetries exist on the family of hypersurfaces defined by (8.10) in an eight-dimensional parameter space.

The case $n = -3$ has either one symmetry ($C_0 \neq 0$) or three symmetries ($C_0 = 0$). In the former case, solution in closed form via the reduction of order to an Abel's equation of the second kind is not at all obvious. In the latter case, the solution is trivial as the equation is now an Ermakov–Pinney equation.

One can consider the question of the integrability of (1.1) from the viewpoint of Lie point symmetries of the differential equation as a matter of levels of constraints. If there is no constraint imposed on the relationship among $p(x)$, $r(x)$ and $f(x)$, there is no inference provided by considerations of symmetry. When $f(x)$ is constrained by (6.4) and (6.5) (for $n \neq 2$; for $n = 2$ the relevant equations are (8.7) and (8.8)), a single symmetry exists. When $n \neq 2$, the parameter space is five dimensional and, when $n = 2$, it is eight dimensional (there are two parameters in $d(x)$). In both cases integrability is guaranteed on a hypersurface in each of the parameter spaces. In this respect the additional constraint plays a role similar to that of a first integral of prescribed value in the case of configurational invariants. (Sarlet *et al.* [20]).

As a final remark we note that $n = 2$ marks a transitional case from the integrable linear equation to general values of n . This is indicated by integrability (in the sense of reduction to quadratures) on a hypersurface in an eight dimensional parameter space, whereas for general values of n the space is only five dimensional. The exception to this is when $n = -3$ and $C_0 = 0$. The equation is then the Ermakov–Pinney equation, which can be interpreted as arising from the integration of a third order linear equation of maximal symmetry or, as was proposed by Eliezer and Gray [21], the radial equation of a higher order system with rotational symmetry.

Acknowledgements—The draft of this manuscript was written while one of us (PGLL) was enjoying the hospitality of the Department of Mathematics at the University of the Aegean. The typescript was prepared by K. S. Govinder. The Foundation for Research Development of South Africa is thanked for its continuing support.

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The algebraic structure of the first integrals of third order linear equations

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Journal of Mathematical Analysis and Applications (1995)

The algebraic structure of the first integrals of third order linear equations

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Abstract

We use the Lie theory of extended groups to analyse the first integrals of scalar third order linear ordinary differential equations. The analysis reveals three natural classes for which the equation has four, five or seven symmetries. We show that each class has first integrals with a specific number of symmetries. Illustrative examples of each class are given. Comparison is made with the integrals of nonlinear equations.

1 Introduction

The search for first integrals of differential equations is an active area of research as this is the first step in the path to finding their solutions. Leach and Mahomed [1] carried out a detailed analysis of first integrals of linear second order equations. The importance of that work lies in the fact that mechanics is inherently a second order system. However, in some cases, second order equations depend upon some parameter. In such cases further progress can be made by treating the expression as a first integral of some third order equation. Also, it is becoming increasingly evident that the study of third order equations (and hence their first integrals) is useful as they arise in many physical situations such as in relativistic astrophysics [2, 3]. It is therefore important to study the behaviour of first integrals of third order equations.

As is well-known, a second order ordinary differential equation

$$N(x, y, y', y'') = 0 \quad (1.1)$$

is invariant under the infinitesimal generator

$$G = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (1.2)$$

iff

$$G^{[2]}N|_{N=0} = 0, \quad (1.3)$$

where

$$G^{[2]} = G + (\eta' - y'\xi') \frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'') \frac{\partial}{\partial y''} \quad (1.4)$$

is called the second extension of G . We say that (1.1) has the point symmetry given by (1.2). In the case of first integrals the situation is slightly different. Consider the second order first integral

$$I = f(x, y, y', y''). \quad (1.5)$$

I has the symmetry (1.2) iff

$$G^{[2]}I = 0, \quad (1.6)$$

ie, G annihilates I and does not leave it invariant as in the case of symmetries of differential equations. It is this difference in treatment of expressions that sometimes

allows us further progress. Note that the procedure for finding symmetries of equations is a special case (that of $I = 0$) of the procedure for finding symmetries of first integrals of the same order.

In this paper we consider the general first integral of a linear third order equation. We show how the three cases for which the equation has four, five and seven symmetries arise naturally from the analysis of the first integrals. We subsequently provide the explicit forms of the symmetries of these first integrals. We give examples to illustrate each case. The analysis is confined to linear equations as there is not yet a general form for third order nonlinear equations possessing n ($n < 7$) symmetries. However, we conclude with two examples of third order nonlinear equations and show that a formulation of a general theory as in the case of first integrals of linear equations is not obvious.

2 The general case

Consider

$$\widehat{I} = a(x)y'' + b(x)y' + c(x)y + d, \quad (2.1)$$

which is a first integral of

$$y''' + \frac{a' + b}{a}y'' + \frac{b' + c}{a}y' + \frac{c'}{a}y + \frac{d'}{a} = 0. \quad (2.2)$$

To simplify the subsequent calculations, we assume that (2.2) is written in normal form. This implies that a translation and a factor have been used to remove the d'/a and y'' terms respectively, ie, effectively we put

$$b = -a' \quad d' = d = 0. \quad (2.3)$$

We therefore confine our analysis to the first integral

$$I = ay'' - a'y' + cy \quad (2.4)$$

and its differential equation

$$y''' + \frac{c - a''}{a}y' + \frac{c'}{a}y = 0. \quad (2.5)$$

For a general third order linear equation in normal form

$$y''' + f(x)y' + g(x)y = 0 \quad (2.6)$$

it follows that $a(x)$ and $c(x)$ are determined from the third order system

$$a'' + fa = c \quad (2.7a)$$

$$c' = ag, \quad (2.7b)$$

ie the solution set contains three arbitrary functions.

Applying the procedure explained in §1, we find that the first integral (2.4) has a symmetry of the form (1.2) iff

$$\begin{aligned} & \xi a' y'' - \xi a'' y' + \xi c' y + a \left\{ \frac{\partial^2 \eta}{\partial x \partial y} + 2y' \frac{\partial^2 \eta}{\partial x \partial y} + y'^2 \frac{\partial^2 \eta}{\partial y^2} + y'' \frac{\partial \eta}{\partial y} - 2y'' \left(\frac{\partial \xi}{\partial x} + y' \frac{\partial \xi}{\partial y} \right) \right. \\ & \left. - y' \left(\frac{\partial^2 \xi}{\partial x^2} + 2y' \frac{\partial^2 \xi}{\partial x \partial y} + y'^2 \frac{\partial^2 \xi}{\partial y^2} + y'' \frac{\partial \xi}{\partial y} \right) \right\} - a' \left(\frac{\partial \eta}{\partial x} + y' \frac{\partial \eta}{\partial y} - y' \frac{\partial \xi}{\partial x} - y'^2 \frac{\partial \xi}{\partial y} \right) + c\eta = 0. \end{aligned} \quad (2.8)$$

The coefficient of y'' in (2.8) gives the functional forms of ξ and η as

$$\xi = p(x) \quad (2.9)$$

$$\eta = \left(2p' - \frac{pa'}{a} \right) y + q(x) \quad (2.10)$$

which imply that the coefficient of y'^2 in (2.8) is identically zero. The terms in (2.8) that do not involve y or any of its derivatives now give

$$aq'' - a'q' + cq = 0, \quad (2.11)$$

which always has two solutions. Thus (2.4) will always have at least two symmetries of the form

$$G_1 = q_1(x) \frac{\partial}{\partial y} \quad (2.12a)$$

$$G_2 = q_2(x) \frac{\partial}{\partial y} = q_1(x) \int \frac{a(x)}{q_1(x)^2}, \quad (2.12b)$$

where $q_1(x)$ and $q_2(x)$ are the solutions of (2.11) and $a(x)$ is a solution of (2.7). Note that (2.11) can be differentiated to give an equation of the form (2.5). Thus $q_1(x)$ and

$q_2(x)$ are also solutions of (2.5). In fact all three solutions of (2.5) are applicable. We choose them pairwise for a given integral. Hence the three independent first integrals of (2.5) will each have a pair of solutions of (2.5) in their symmetries.

The coefficient of y' in (2.8) gives a second order equation for p with solution

$$p = Aa + Ba \int \frac{1}{a}, \quad (2.13)$$

where A and B are constants of integration. We will in general, then, obtain two symmetries from p of the form

$$G_3 = a \frac{\partial}{\partial x} + a'y \frac{\partial}{\partial y} \quad (2.14a)$$

$$G_4 = a \int \frac{1}{a} \frac{\partial}{\partial x} + \left(a' \int \frac{1}{a} + 2 \right) y \frac{\partial}{\partial y}. \quad (2.14b)$$

However, the solutions of (2.13) must be consistent with

$$pc' + a \left(2p' - \frac{pa'}{a} \right)'' - a' \left(2p' - \frac{pa'}{a} \right)' - c \left(2p' - \frac{pa'}{a} \right) = 0, \quad (2.15)$$

the coefficient of y in (2.8). Substituting (2.13) into (2.15) gives

$$A[a'c + ac' + aa''' - a'a''] + B \left\{ [a'c + ac' + aa''' - a'a''] \int \frac{1}{a} + 2 \left[a'' - \frac{a'^2}{a} + c \right] \right\} = 0. \quad (2.16)$$

For both solutions of p to persist (and hence two symmetries) the coefficients of both A and B must vanish, ie,

$$a'c + ac' + aa''' - a'a'' = 0 \quad (2.17)$$

$$a'' - \frac{a'^2}{a} + c = 0. \quad (2.18)$$

From (2.18)

$$c = -a \left(\frac{a'}{a} \right)' \quad (2.19)$$

and (2.17) is satisfied identically. Thus if (2.19) holds, p gives rise to two symmetries. Note that (2.18) is a second order equation and so only two of the three solutions for $a(x)$ in (2.7) apply. This implies that we obtain two first integrals with four symmetries. Note further, that (2.19) implies that

$$\left(\frac{c - a''}{a} \right)' = 2 \left(\frac{c'}{a} \right) \quad (2.20)$$

in (2.5) which is now

$$y''' - \left[\left(\frac{a'}{a} \right)' + \frac{a''}{a} \right] y' - \frac{1}{a} \left[a \left(\frac{a'}{a} \right)' \right]' y = 0. \quad (2.21)$$

The relationship (2.20) implies that (2.21) has maximal (seven) symmetry [4].

Suppose, now, that (2.17) is satisfied giving

$$c = -a \left(\frac{a'}{a} \right)' + \frac{k}{a}, \quad (2.22)$$

where k is a constant. Clearly, for $k \neq 0$, (2.18) is not satisfied and p has only one solution. Note that (2.22) still gives (2.20). Hence a third order equation with maximal symmetry has two first integrals that have four symmetries and one with three. The first integral with three symmetries will involve the remaining solution for $a(x)$ (in (2.7)) that does not satisfy (2.18). The differential equation now becomes

$$y''' - \left[\left(\frac{a'}{a} \right)' + \frac{a''}{a} \right] y' - \frac{1}{a} \left[a \left(\frac{a'}{a} \right)' \right]' y + k \left[\frac{y'}{a} - \frac{a'y}{a^2} \right] = 0. \quad (2.23)$$

The three independent first integrals of (2.23) have the symmetries (for $k = 0$)

$$G'_1 = q_1(x) \frac{\partial}{\partial y} \quad (2.24a)$$

$$G'_2 = a_1(x) \frac{\partial}{\partial y} \quad (2.24b)$$

$$G'_3 = a_1(x) \frac{\partial}{\partial x} + a'_1 y \frac{\partial}{\partial y} \quad (2.24c)$$

$$G'_4 = a_1 \int \frac{1}{a_1} \frac{\partial}{\partial x} + \left(a'_1 \int \frac{1}{a_1} + 2 \right) y \frac{\partial}{\partial y}, \quad (2.24d)$$

$$G''_1 = q_2(x) \frac{\partial}{\partial y} \quad (2.25a)$$

$$G''_2 = a_2(x) \frac{\partial}{\partial y} \quad (2.25b)$$

$$G''_3 = a_2(x) \frac{\partial}{\partial x} + a'_2 y \frac{\partial}{\partial y} \quad (2.25c)$$

$$G''_4 = a_2 \int \frac{1}{a_2} \frac{\partial}{\partial x} + \left(a'_2 \int \frac{1}{a_2} + 2 \right) y \frac{\partial}{\partial y} \quad (2.25d)$$

and (for $k \neq 0$)

$$\tilde{G}_1 = q_1(x) \frac{\partial}{\partial y} \quad (2.26a)$$

$$\tilde{G}_2 = q_2(x) \frac{\partial}{\partial y} \quad (2.26b)$$

$$\tilde{G}_3 = a_3(x) \frac{\partial}{\partial x} + a'_3 y \frac{\partial}{\partial y}, \quad (2.26c)$$

where $q_1(x)$ and $q_2(x)$ are solutions of (2.11) and hence (2.23) and the a_i 's ($i = 1, 2, 3$) are solutions of (2.7).

In a constructive approach one firstly solves (2.7) for $a(x)$ and $c(x)$ and then substitutes in (2.18) to find the relationship which the constants of integration must satisfy for $p(x)$ and hence yield four symmetries. There is a doubly infinite family of integrals with four symmetries, but only two linearly independent ones. The third linearly independent integral, obtained by a selection of constants of integration which does not satisfy (2.18), has only three symmetries.

Suppose, now, that the coefficient of B in (2.16) is zero. This gives

$$c = -a \left(\frac{a'}{a} \right)' + \frac{k}{a(\int a^{-1})^2}. \quad (2.27)$$

When $k \neq 0$, the coefficient of A in (2.16) is not zero. This implies that p produces only one symmetry of the form

$$G_3 = a \int \frac{1}{a} \frac{\partial}{\partial x} + \left(a' \int \frac{1}{a} + 2 \right) y \frac{\partial}{\partial y}. \quad (2.28)$$

Note that while c and a are related via (2.27), (2.20) does not hold in (2.5). This is the case for which (2.5) has five symmetries [4]. The three independent first integrals of a third order linear equation with five symmetries have the symmetries

$$G'_1 = q_1(x) \frac{\partial}{\partial y} \quad (2.29a)$$

$$G'_2 = q_2(x) \frac{\partial}{\partial y} \quad (2.29b)$$

$$G'_3 = a_1 \int \frac{1}{a_1} \frac{\partial}{\partial x} + \left(a'_1 \int \frac{1}{a_1} + 2 \right) y \frac{\partial}{\partial y}, \quad (2.29c)$$

$$G_1'' = q_1(x) \frac{\partial}{\partial y} \quad (2.30a)$$

$$G_2'' = q_3(x) \frac{\partial}{\partial y} \quad (2.30b)$$

$$G_3'' = a_2 \int \frac{1}{a_2} \frac{\partial}{\partial x} + \left(a_2' \int \frac{1}{a_2} + 2 \right) y \frac{\partial}{\partial y} \quad (2.30c)$$

and

$$\tilde{G}_1 = q_2(x) \frac{\partial}{\partial y} \quad (2.31a)$$

$$\tilde{G}_2 = q_3(x) \frac{\partial}{\partial y} \quad (2.31b)$$

$$\tilde{G}_3 = a_3 \int \frac{1}{a_3} \frac{\partial}{\partial x} + \left(a_3' \int \frac{1}{a_3} + 2 \right) y \frac{\partial}{\partial y}, \quad (2.31c)$$

where the q_i 's and a_i 's ($i = 1, 2, 3$) are solutions of (2.11) and (2.7) respectively.

Finally, when both the coefficients of A and B in (2.16) are nonzero, $p = 0$ and contributes no symmetries. This implies that the coefficients of y' and y in (2.5) are unrelated. This is the case for which (2.5) has four symmetries [4]. The three independent first integrals of (2.5) then have the pairs of symmetries (2.29a–2.29b), (2.30a–2.30b) and (2.31a–2.31b) respectively.

3 Example I: $n = 7$

We firstly consider the generic third order equation with maximal (seven) symmetry, viz.

$$y''' = 0 \quad (3.1)$$

with symmetries [5]

$$G_1 = \frac{\partial}{\partial y} \quad (3.2a)$$

$$G_2 = x \frac{\partial}{\partial y} \quad (3.2b)$$

$$G_3 = x^2 \frac{\partial}{\partial y} \quad (3.2c)$$

$$G_4 = y \frac{\partial}{\partial y} \quad (3.2d)$$

$$G_5 = \frac{\partial}{\partial x} \quad (3.2e)$$

$$G_6 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad (3.2f)$$

$$G_7 = x^2 \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \quad (3.2g)$$

and thus has the Lie algebra $3A_1 \oplus_s (sl(2, R) \oplus A_1)$ [4]. It is easily verified that (3.1) has the three independent first integrals

$$I_1 = y'' \quad (3.3a)$$

$$I_2 = xy'' - y' \quad (3.3b)$$

$$I_3 = \frac{1}{2}x^2y'' - xy' + y. \quad (3.3c)$$

We call the first integrals (3.3) initial condition first integrals as, for $x = 0$, we have

$$I_1 = y_0'' \quad (3.4a)$$

$$I_2 = -y_0' \quad (3.4b)$$

$$I_3 = y_0. \quad (3.4c)$$

In (3.3) I_1 has the symmetries

$$G_1 = \frac{\partial}{\partial y} \quad (3.5a)$$

$$G_2 = x \frac{\partial}{\partial y} \quad (3.5b)$$

$$G_3 = \frac{\partial}{\partial x} \quad (3.5c)$$

$$G_4 = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \quad (3.5d)$$

I_2 ,

$$X_1 = \frac{\partial}{\partial y} \quad (3.6a)$$

$$X_2 = x^2 \frac{\partial}{\partial y} \quad (3.6b)$$

$$X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad (3.6c)$$

and I_3 ,

$$Y_1 = x \frac{\partial}{\partial y} \quad (3.7a)$$

$$Y_2 = x^2 \frac{\partial}{\partial y} \quad (3.7b)$$

$$Y_3 = x^2 \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \quad (3.7c)$$

$$Y_4 = x \frac{\partial}{\partial x}. \quad (3.7d)$$

The Z_1 and the Z_2 symmetries (where Z refers to G , X and Y in turn) contain the solutions of the equations (3.1) and correspond to (2.12a) and (2.12b). Note that the linear solution implies that $k = 0$ in (2.22) and hence does not appear in the three symmetry case (3.6). Note further that the Z_3 and Z_4 symmetries also contain solutions of the original equation. This is because the functions $a_i(x)$ are solutions of the adjoint equation of (2.5) and in the case of maximal symmetry this equation is self-adjoint.

The Lie brackets of the symmetries of I_1 are

$$\begin{aligned} [G_1, G_2] &= 0 & [G_2, G_3] &= -G_1 & [G_3, G_4] &= G_3 \\ [G_1, G_3] &= 0 & [G_2, G_4] &= G_2 \\ [G_1, G_4] &= 2G_1 \end{aligned} \quad (3.8)$$

(which is the Lie algebra $A_{4,9}^1$ [6]) and of I_3 are

$$\begin{aligned} [Y_1, Y_2] &= 0 & [Y_2, Y_3] &= 0 & [Y_3, Y_4] &= Y_4 \\ [Y_1, Y_3] &= Y_2 & [Y_2, Y_4] &= -2Y_2 \\ [Y_1, Y_4] &= -Y_1. \end{aligned} \quad (3.9)$$

Clearly the correspondence is

$$\begin{aligned} Y_1 &\longrightarrow G_2 \\ Y_2 &\longrightarrow -G_1 \\ Y_3 &\longrightarrow G_3 \\ Y_4 &\longrightarrow -G_4 \end{aligned}$$

and (3.9) is also the Lie algebra $A_{4,9}^1$. If we let the co-ordinates in I_3 be (x, y) and in I_1 , $(\mathcal{X}, \mathcal{Y})$, the transformation that takes I_3 to I_1 (and vice versa) is

$$\mathcal{X} = -\frac{1}{x} \quad \mathcal{Y} = -\frac{y}{x^2}. \quad (3.10)$$

This transformation just maps I_2 to itself and

$$X_1 \longrightarrow -X_1$$

$$X_2 \longrightarrow -X_2$$

$$X_3 \longrightarrow -X_3.$$

Note that (3.6) forms the Lie algebra $A_{3,4}$ which is better known as the algebra of the pseudo-Euclidean group $E(1,1)$. Equation (3.1) is obviously invariant under (3.10) and, in \mathcal{Y} new variables, is

$$\mathcal{Y}''' = 0. \quad (3.11)$$

It is instructive to consider a slightly less trivial example of an equation with seven symmetries to see how the constructive approach works in practice. Consider

$$y''' + y' = 0 \quad (3.12)$$

for which (cf (2.6)) $f = 1$ and $g = 0$. Then

$$a = A_1 \sin x + A_2 \cos x + C_0 \quad (3.13a)$$

$$c = C_0. \quad (3.13b)$$

Four symmetries exist when (2.18) is satisfied, ie

$$A_1^2 + A_2^2 = C_0^2. \quad (3.14)$$

Two choices are $C_0 = 1$ and $A_2 = \pm 1$ which give the four symmetry integrals

$$I_1 = (1 + \cos x)y'' + \sin xy' + y \quad (3.15a)$$

$$I_2 = (1 - \cos x)y'' - \sin xy' + y. \quad (3.15b)$$

A three symmetry integral can be obtained by the choice, say, of $A_1 = 1$ and is

$$I_3 = \sin xy'' - \cos xy'. \quad (3.15c)$$

The symmetries follow from (2.24), (2.25) and (2.26) respectively.

4 Example II: $n = 5$

Consider now the linear equation

$$y''' - y = 0 \quad (4.1)$$

with the five symmetries

$$G_1 = e^x \frac{\partial}{\partial y} \quad (4.2a)$$

$$G_2 = e^{\omega x} \frac{\partial}{\partial y} \quad (4.2b)$$

$$G_3 = e^{\omega^2 x} \frac{\partial}{\partial y} \quad (4.2c)$$

$$G_4 = y \frac{\partial}{\partial y} \quad (4.2d)$$

$$G_5 = \frac{\partial}{\partial x}, \quad (4.2e)$$

where $1 + \omega + \omega^2 = 0$. The Lie brackets of (4.2) form the Lie algebra $3A_1 \oplus_s (2A_1)$.

We choose the three independent first integrals of (4.1) to be

$$I_1 = e^{-x} (y + y' + y'') \quad (4.3a)$$

$$I_2 = e^{-\omega x} (\omega y + y' + \omega^2 y'') \quad (4.3b)$$

$$I_3 = e^{-\omega^2 x} (\omega y + \omega^2 y' + y''). \quad (4.3c)$$

I_1 has the symmetries

$$G_1 = e^{\omega x} \frac{\partial}{\partial y} \quad (4.4a)$$

$$G_2 = e^{\omega^2 x} \frac{\partial}{\partial y} \quad (4.4b)$$

$$G_3 = \frac{1}{\omega} \left(\frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad (4.4c)$$

I_2 ,

$$X_1 = e^x \frac{\partial}{\partial y} \quad (4.5a)$$

$$X_2 = e^{\omega^2 x} \frac{\partial}{\partial y} \quad (4.5b)$$

$$X_3 = \frac{1}{\omega^2} \left(\frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \quad (4.5c)$$

and I_3 ,

$$Y_1 = e^{\omega x} \frac{\partial}{\partial y} \quad (4.6a)$$

$$Y_2 = e^x \frac{\partial}{\partial y} \quad (4.6b)$$

$$Y_3 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \quad (4.6c)$$

The Lie brackets of (4.4), (4.5) and (4.6) each form the Lie algebra $A_{3,3}$ which is the algebra of the group comprising the semi-direct product of dilations and translations $D \otimes_s T_2$. We can transform the first integrals (and hence their symmetries) in a cyclic manner by setting

$$x \longrightarrow \omega X.$$

This transformation leaves (4.1) invariant.

5 Example III: $n = 4$

As the final linear example we consider [7]

$$y''' + f(x)y'' + y' + f(x)y = 0, \quad (5.1)$$

where $f(x)$ is an arbitrary function of x , with symmetries

$$G_1 = \sin x \frac{\partial}{\partial y} \quad (5.2a)$$

$$G_2 = \cos x \frac{\partial}{\partial y} \quad (5.2b)$$

$$G_3 = z(x) \frac{\partial}{\partial y} \quad (5.2c)$$

$$G_4 = y \frac{\partial}{\partial y}, \quad (5.2d)$$

where

$$z(x) = \int_0^x \exp\left(-\int f(u)du\right) \sin(x-u)du, \quad (5.3)$$

which form the Lie algebra $3A_1 \oplus A_1$. The three independent first integrals of (5.1) are

$$I_1 = y' \sin x - y \cos x - z(x)(y'' + y) \exp \left(\int f(u) du \right) \quad (5.4a)$$

$$I_2 = y' \cos x + y \sin x - z(x)(y'' + y) \exp \left(\int f(u) du \right) \quad (5.4b)$$

$$I_3 = (y'' + y) \exp \left(\int f(u) du \right) \quad (5.4c)$$

with the symmetries

$$G_1 = \sin x \frac{\partial}{\partial y} \quad (5.5a)$$

$$G_2 = z(x) \frac{\partial}{\partial y}, \quad (5.5b)$$

$$X_1 = \cos x \frac{\partial}{\partial y} \quad (5.6a)$$

$$X_2 = z(x) \frac{\partial}{\partial y} \quad (5.6b)$$

and

$$Y_1 = \sin x \frac{\partial}{\partial y} \quad (5.7a)$$

$$Y_2 = \cos x \frac{\partial}{\partial y} \quad (5.7b)$$

respectively. Each pair forms the Lie algebra $2A_1$ and contains the solutions of the original equation. The search for the transformation to cycle through the first integrals (5.4) requires an Abel's formula for third order equations which, to our knowledge, has yet to be discovered.

6 Nonlinear examples

We now consider a version of the Kummer-Schwarz equation [8], viz.

$$2y'y''' - 3y''^2 = 0 \quad (6.1)$$

with the symmetries

$$G_1 = \frac{\partial}{\partial x} \quad (6.2a)$$

$$G_2 = x \frac{\partial}{\partial x} \quad (6.2b)$$

$$G_3 = x^2 \frac{\partial}{\partial x} \quad (6.2c)$$

$$G_4 = \frac{\partial}{\partial y} \quad (6.2d)$$

$$G_5 = y \frac{\partial}{\partial y} \quad (6.2e)$$

$$G_6 = y^2 \frac{\partial}{\partial y} \quad (6.2f)$$

which form the Lie algebra $sl(2, R) \oplus_s sl(2, R)$. Three independent first integrals are

$$I_1 = \frac{y''^2}{y'^3} \quad (6.3a)$$

$$I_2 = y - \frac{2y'^2}{y''} \quad (6.3b)$$

$$I_3 = x + \frac{2y''(yy'' - y'^2)}{y''(yy'' - 2y'^2)} \quad (6.3c)$$

with the symmetries

$$G_1 = \frac{\partial}{\partial y} \quad (6.4a)$$

$$G_2 = \frac{\partial}{\partial x} \quad (6.4b)$$

$$G_3 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad (6.4c)$$

$$X_1 = \frac{\partial}{\partial x} \quad (6.5a)$$

$$X_2 = x \frac{\partial}{\partial x} \quad (6.5b)$$

and

$$Y_1 = y \frac{\partial}{\partial y} \quad (6.6)$$

respectively.

Consider also a nonlinear third order equation with four symmetries [7], viz.

$$(1 + y'^2)y''' = (3y' + 1)y''^2. \quad (6.7)$$

The symmetries of (6.7) are

$$G_1 = \frac{\partial}{\partial x} \quad (6.8a)$$

$$G_2 = \frac{\partial}{\partial y} \quad (6.8b)$$

$$G_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad (6.8c)$$

$$G_4 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \quad (6.8d)$$

Three independent first integrals are

$$I_1 = \frac{y''}{(1 + y'^2)^{3/2}} \exp(-\arctan y') \quad (6.9a)$$

$$I_2 = 2y + \frac{(1 + y')(1 + y'^2)}{y''} \quad (6.9b)$$

$$I_3 = 2x + \frac{(1 + y'^2)(1 - y')}{y''} \quad (6.9c)$$

with the symmetries

$$G_1 = \frac{\partial}{\partial x} \quad (6.10a)$$

$$G_2 = \frac{\partial}{\partial y} \quad (6.10b)$$

$$G_3 = (x + y) \frac{\partial}{\partial x} + (y - x) \frac{\partial}{\partial y}, \quad (6.10c)$$

$$X_1 = \frac{\partial}{\partial x} \quad (6.11)$$

and

$$Y_1 = \frac{\partial}{\partial y} \quad (6.12)$$

respectively. It is apparent from the above that the relationship between the first integrals (6.3) and (6.9) and their respective symmetries (6.4–6.6) and (6.10–6.12) is

not at all obvious. Indeed there is some question as to where the beginning of the resolution of this problem lies. Similar to the linear case, each nonlinear equation will have a different normal form that depends on its Lie algebra. However, while there is just one Lie algebra for a particular dimension (any one of four, five or seven) that relates to third order linear equations, in the case of nonlinear equations the number is as yet undetermined. This needs to be resolved before a search for a relationship similar to the linear case can be commenced. It would seem that Gat [9] has provided some classification for third order nonlinear equations that could be used as a starting point.

As a further illustration of the importance of first integrals of third order equations we mention that (6.7) can be solved using the first integrals (6.9). Setting

$$X = I_3 - 2x \tag{6.13a}$$

$$Y = I_2 - 2y \tag{6.13b}$$

we have (using the ratio of I_2 to I_3 and integrating)

$$KX = \frac{1}{(1 + V^2)^{1/2}} \exp(\arctan V), \tag{6.14}$$

where K is a constant of integration and $Y = VX$. The solution (6.14) is implicit but is still an improvement over the parametric solution provided in [7].

7 Conclusion

We have shown that the symmetries of first integrals of third order linear equations are related to the number of symmetries of the equation. For the maximal case two first integrals have four symmetries and one has three. When the equation has five symmetries, all three first integrals have three symmetries and for four symmetries the first integrals have two symmetries each. The relationship is rather intriguing and bears further investigation in a generalization of the result to higher order linear equations. Unfortunately the relationship for nonlinear equations is not as yet obvious.

In the case of second order ordinary differential equations with the maximal symmetry, $sl(3, R)$, there are three first integrals which each have three symmetries [1]. To take the example of the free particle with equation

$$y'' = 0 \tag{7.1}$$

those integrals are

$$I_1 = y' \tag{7.2a}$$

$$I_2 = y - xy' \tag{7.2b}$$

$$I_3 = \frac{y}{y'} - x. \tag{7.2c}$$

In each case the algebra of the symmetries is $A_{3,3}$ (or $D \oplus_s T_2$). We note that the first integrals with this property are $y(0), y'(0)$ and their ratio.

Third order linear equations (and others transformable to one by a point transformation) differ in two respects. In the first instance all such equations do not have the same number of symmetries, but have four, five or seven depending on the internal structure of the equation. In the case of four the maximum number of symmetries for the first integrals is two; five, three and seven, four. The last of these is the closest to the second order case in that it is the case of maximal symmetry. It could be anticipated that the maximum number of symmetries would be four. What is unexpected is that this occurs only for two first integrals because the equation, (2.18), acts as a constraint. The third integral has only three symmetries. It is also of interest to note that the maximum number of symmetries does not necessarily occur for the initial condition integrals (cf the second of the maximal symmetry examples). The property of the ratio of the two first integrals also having the same number of symmetries is also lost as can easily be seen from an analysis of I_1/I_2 of the second of the maximal symmetry examples. In fact the ratio has only the two symmetries

$$G_1 = \sin x \frac{\partial}{\partial y} \tag{7.3a}$$

$$G_2 = y \frac{\partial}{\partial y}. \tag{7.3b}$$

The significance and theoretical basis behind this are not obvious, but will presumably become more transparent under an investigation of the first integrals of higher order linear equations.

Acknowledgements

The Foundation for Research Development of South Africa and the University of Natal are thanked for their ongoing support.

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An elementary demonstration
of the existence of $s\lambda(3, R)$
symmetry for all second order
linear ordinary differential
equations

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(preprint: Department of Mathematics and Applied Mathematics, University
of Natal, Durban, 1994)

**An elementary demonstration of the existence of
 $sl(3, R)$ symmetry for all second order linear
ordinary differential equations**

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Short Title: Equivalence of linear sodes

Abstract

All second order linear ordinary differential equations are shown in an elemen-
tary way to possess the symmetry algebra $sl(2, R)$.

Despite the comment in Mahomed and Leach [7] that it is now well-known that all second order linear equations have the symmetry algebra $sl(3, R)$ [5, 1, 6], it appears that these general proofs are not as widely known as the more restricted results of Anderson and Davison [1] and Wulfman and Wybourne [8]. It is the intention of this note to make the result more widely known and to provide a simpler demonstration than given in Mahomed [5], Aguirre and Krause [1] and Mahomed and Leach [6].

The general second order linear differential equation

$$y'' + a(x)y' + b(x)y = c(x) \quad (1)$$

is transformed to

$$\bar{v} (ut'^2) + \dot{v} (2u't' + ut'' + aut') + v(u'' + au' + bu) + w'' + aw' + bw = c \quad (2)$$

under the generalized Kummer-Liouville transformation [4]

$$y = u(x)v(t) + w(x) \quad t = t(x), \quad (3)$$

where, as usual, ' denotes d/dx and $\dot{}$ denotes d/dt . In (2) the coefficient of \dot{v} can be made zero if

$$t' = u^{-2} \exp \left[- \int a(x) dx \right] \quad (4)$$

which establishes $t(x)$ once $u(x)$ is determined. The coefficient of v is zero if

$$u'' + au' + b = 0 \quad (5)$$

and the nonhomogeneous term vanishes if

$$w'' + aw' + bw = c. \quad (6)$$

Both (5) and (6) have continuous solutions provided the functions $a(x)$, $b(x)$ and $c(x)$ are continuous and satisfy a Lipschitz condition [3].

Hence (1) is equivalent to

$$\bar{v} = 0 \quad (7)$$

under a point transformation. (This may be only local, but we are concerned here with the algebra and not the group.) Eq (7) has the symmetry algebra $sl(3, R)$ and so does the original equation.

Acknowledgement

The Foundation for Research Development of South Africa and the University of Natal are thanked for their ongoing support.

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Integrability analysis of the Emden-Fowler equation

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(preprint: Department of Mathematics, University of the Aegean, Karlovassi
83 200, Greece, 1994)

Integrability analysis of the Emden–Fowler equation

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Key words. Emden-Fowler, integrability, Lie symmetries, Painlevé analysis

AMS subject classification: 34A25, 58F07

Short title: The Emden–Fowler equation

Abstract

The Emden–Fowler equation of index n is studied utilising the techniques of Lie and Painlevé analysis. For general n information about the integrability of this equation is obtained. The link between these two types of analyses is explored. The special cases of $n = -3, 2$ are also examined. As a result of the Painlevé analysis new second order equations possessing the Painlevé property are found.

1 Introduction

The Emden–Fowler equation [1, 2, 3, 4, 5, 6] has attracted much attention over the years. Wong, in his review of 1976 [7], contains over 100 references, but even these were selective. Subsequently a plethora of papers has appeared devoted to a study of this ubiquitous equation. The most general form studied today is

$$Y'' + p(X)Y' + q(X)Y = r(X)Y^n. \quad (1.1)$$

However, it is well-known [8] that a Kummer–Louville transformation [9] converts (1.1) into standard form, viz.

$$y'' = f(x)y^n. \quad (1.2)$$

It is this form of the equation to which we confine our analysis. Eq (1.2) has become increasingly important as it arises in the modelling of many physical systems. It is perhaps best known for its occurrence as the quintessential equation in the study of the shear-free spherically symmetric perfect fluid motion in Cosmology when $n = 2$ [10, 11, 12, 13].

We study (1.2) from the viewpoints of Lie symmetries and the Painlevé analysis. In general, (1.2) does not possess any Lie point symmetries nor can one easily say anything about its possession of the Painlevé property. However, for an appropriate $f(x)$, (1.2) does possess at least one Lie point symmetry. We analyse (1.2) for these instances and also consider the conditions for it to possess more than one Lie point symmetry, thereby enabling the reduction to quadratures. (See also in this respect [14, 15].) In addition we show under what conditions (1.2) (with only one Lie point symmetry) can be reduced to quadratures.

We also undertake a Painlevé analysis of (1.2) (suitably transformed) and discuss its possession of the full Painlevé property. We comment on a possible link between possession of the Painlevé property and explicit integration of the equation. We also consider the special cases of $n = -3, 2$ and show how they affect the analysis.

2 Lie Analysis

An n th order differential equation

$$E(x, y, y', \dots, y^{(n)}) = 0 \quad (2.1)$$

is said to possess the Lie point symmetry

$$G = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (2.2)$$

if

$$G^{[n]} E|_{E=0} = 0, \quad (2.3)$$

where [16]

$$G^{[n]} = G + \sum_{j=1}^n \left(\eta^{(j)} - \sum_{k=0}^{j-1} \binom{j}{k} y^{(k+1)} \xi^{(j-k)} \right) \frac{\partial}{\partial y^{(j)}} \quad (2.4)$$

is the n th extension of G needed to contend with the n th derivatives of y in (2.1). Note that we are restricting our analysis to scalar ordinary differential equations. The concept of Lie point symmetries applies equally to systems of equations and to partial differential equations. In those instances the ξ, η, x and y take on suitable indices. (See eg. [17, 18].) The action of $G^{[n]}$ on (2.1), (ie. (2.3)) gives a system of partial differential equations which is solved to obtain G . Thereafter G can be used to either transform (2.1) appropriately (usually into autonomous form) or reduce the order. The aim, of course, is to reduce the n th order equation to zeroth order.

In the case of the Emden–Fowler equation

$$y'' = f(x)y^n \quad (2.5)$$

it is easily verified that ξ and η in (2.2) must have the form

$$\xi = a(x) \quad \eta = c(x)y + d(x). \quad (2.6)$$

We therefore begin the Lie analysis of (2.5) by assuming the form

$$G = a(x) \frac{\partial}{\partial x} + (c(x)y + d(x)) \frac{\partial}{\partial y} \quad (2.7)$$

for a symmetry of (2.5). The action of $G^{[2]}$ on (2.5) results in the system

$$-2fa' + cf = af' + ncf \quad (2.8)$$

$$nfd = 0 \quad (2.9)$$

$$c'' = 0 \quad (2.10)$$

$$d'' = 0 \quad (2.11)$$

$$2c' - a'' = 0. \quad (2.12)$$

We immediately integrate (2.12) to obtain

$$c = \frac{1}{2}(a' + \alpha) \quad (2.13)$$

and observe that d in (2.9) is zero. Thus (2.11) is identically satisfied. Note that (2.9) and (2.10) coalesce in the case $n = 2$ and c and f are related via d . Note also that (2.8) can be rewritten as

$$af' + \left(\frac{n-1}{2}(a' + \alpha) + 2a' \right) f = 0 \quad (2.14)$$

which is special in the case $n = -3$ and $\alpha = 0$. We return to the cases $n = 2, -3$ later. The cases $n = 0, 1$ are equivalent as the equation is then linear. Linear second order differential equations have eight Lie point symmetries which form the Lie algebra $s\ell(3, R)$. (See [19] for a recent proof and references therein.)

For general n we write (2.14) as

$$\frac{f'}{f} = -\frac{n+3}{2} \frac{a'}{a} - \frac{n-1}{2} \frac{\alpha}{a} \quad (2.15)$$

from which

$$f = K a^{-(n+3)/2} \exp \left[-\frac{(n-1)\alpha}{2} \int \frac{dx}{a} \right]. \quad (2.16)$$

The differential equation for a is (from (2.10) and (2.13))

$$a''' = 0, \quad (2.17)$$

whence

$$a = A_0 + A_1x + A_2x^2. \quad (2.18)$$

Eq (2.5) has the symmetry

$$G_1 = a \frac{\partial}{\partial x} + \frac{1}{2}(a' + \alpha)y \frac{\partial}{\partial y} \quad (2.19)$$

if $f(x)$ is given by (2.16).

It is best to proceed with the analysis after writing (2.5) in autonomous form. This is obtained by transforming the symmetry (2.19) to

$$Z_1 = \frac{\partial}{\partial X}. \quad (2.20)$$

The transformation is

$$X = \int \frac{dx}{a} \quad (2.21)$$

$$Y = \frac{y}{a^{\frac{1}{2}}} \exp\left(-\alpha \int \frac{dx}{2a}\right) \quad (2.22)$$

and the differential equation transforms to

$$Y'' + \alpha Y' + \left(\Delta + \frac{\alpha^2}{4}\right) Y = KY^n, \quad (2.23)$$

where

$$\Delta = A_0 A_2 - \frac{1}{4} A_1^2. \quad (2.24)$$

Reduction via Z_1 results in an Abel's equation of the second kind, viz.

$$vv' = Ku^n - \alpha v - \left(\Delta + \frac{\alpha^2}{4}\right) u, \quad (2.25)$$

where

$$u = Y \quad v = Y', \quad (2.26)$$

the solution of which (though it exists in principle) is unobvious.

To reduce (2.23) to quadratures we require that the equation which arises after the first reduction of order possesses at least one Lie point symmetry. (The fact that this reduced equation is of first order and so possesses an infinite number of Lie point symmetries is moot. The determination of these symmetries remains an intractable problem [17].)

If (2.23) possesses two Lie point symmetries, G_1 and G_2 say, and $[G_1, G_2] = \lambda G_1$ (λ an arbitrary constant usually 1 or 0) the reduction via G_1 will result in a first order equation with G_2 (suitably extended) as a point symmetry [18]. (Note that the reduction via G_2 causes the transformation of G_1 into a nonlocal symmetry.) We therefore examine (2.23) to determine under which circumstances it possesses two point symmetries.

Setting

$$G = \bar{a}(X) \frac{\partial}{\partial X} + \bar{c}(X) Y \frac{\partial}{\partial Y}, \quad (2.27)$$

where we have removed $\bar{d}(X)$ in the coefficient of $\partial/\partial Y$ since the form of (2.23) implies $\bar{d}(X) = 0$, we require

$$G^{[2]} N|_{N=0} = 0, \quad (2.28)$$

where we have rewritten (2.23) as $N(Y, Y', Y'') = 0$. The operation (2.28) results in the system

$$\bar{c} - 2\bar{a}' = n\bar{c} \quad (2.29)$$

$$2\bar{c}' + \alpha\bar{a}' - \bar{a}'' = 0 \quad (2.30)$$

$$\bar{c}'' + 2M\bar{a}' + \alpha\bar{c}' = \bar{c}, \quad (2.31)$$

where

$$M = \Delta + \frac{\alpha^2}{4}. \quad (2.32)$$

The function \bar{c} is determined from (2.29), viz.

$$\bar{c} = -\frac{2\bar{a}'}{n-1}. \quad (2.33)$$

The differential equation for \bar{a} now becomes (via (2.30) and (2.33))

$$\frac{n+3}{n-1} \bar{a}'' - \alpha\bar{a}' = 0 \quad (2.34)$$

and so \bar{a} is given by

$$\bar{a} = \bar{A}_0 + \bar{A}_1 \exp\left(\left(\frac{n-1}{n+3}\right) \alpha X\right). \quad (2.35)$$

Eq (2.31) now becomes the consistency condition

$$\frac{\tilde{a}'''}{n-1} - M\tilde{a}' + \frac{\alpha\tilde{a}''}{n-1} = 0. \quad (2.36)$$

Invoking (2.34), (2.36) is only satisfied if

$$M = \frac{2\alpha^2(n+1)}{(n+3)^2} \quad (2.37)$$

from which (via (2.32))

$$\Delta = - \left[\frac{\alpha(n-1)}{2(n+3)} \right]^2. \quad (2.38)$$

This implies

$$\frac{1}{4}A_1^2 - A_0A_2 = \left[\frac{\alpha(n-1)}{2(n+3)} \right]^2 \quad (2.39)$$

which further implies that the equation for a , viz. (2.18), has real roots!

From (2.27) and (2.34), (2.23) has the two Lie point symmetries

$$G_1 = \frac{\partial}{\partial X} \quad (2.40)$$

$$G_2 = \exp \left(\left(\frac{n-1}{n+3} \right) \alpha X \right) \left(\frac{\partial}{\partial X} - \frac{2\alpha Y}{n+3} \frac{\partial}{\partial Y} \right) \quad (2.41)$$

provided (2.39) holds.

Under the transformation

$$\mathcal{X} = \frac{\exp \left(\left(\frac{n-1}{n+3} \right) \alpha X \right)}{\left(\frac{n-1}{n+3} \right) \alpha} \quad \mathcal{Y} = Y \exp \left(\left(\frac{2\alpha}{n+3} \right) X \right) \quad (2.42)$$

(2.23) becomes

$$\mathcal{Y}'' = K\mathcal{Y}^n, \quad (2.43)$$

(2.40)-(2.41) transform to

$$X_1 = (1-n)\mathcal{X} \frac{\partial}{\partial \mathcal{X}} + 2\mathcal{Y} \frac{\partial}{\partial \mathcal{Y}} \quad (2.44)$$

$$X_2 = \frac{\partial}{\partial \mathcal{X}} \quad (2.45)$$

and

$$[X_1, X_2] = (1-n)X_2. \quad (2.46)$$

We can now evaluate f as

$$f = K A_2^{-\frac{n+3}{2}} \left(x + \frac{A_1}{2A_2} - \frac{\alpha}{2A_2} \frac{n-1}{n+3} \right)^{-(n+3)}. \quad (2.47)$$

Eq (2.43) can be reduced (taking the relationship (2.46) into account) using X_2 via

$$u = \mathcal{Y} \quad v = \mathcal{Y}' \quad (2.48)$$

to

$$vv' = K u^n, \quad (2.49)$$

which now has the point symmetry

$$\tilde{X}_1 = 2u \frac{\partial}{\partial u} + (n+1)v \frac{\partial}{\partial v}. \quad (2.50)$$

We can immediately integrate (2.49) (or reduce it via (2.50)) to obtain

$$v^2 = \frac{K}{n+1} u^{n+1} + K_1. \quad (2.51)$$

Thus the solution to (2.43) can be expressed as the quadrature

$$\mathcal{X} - \mathcal{X}_0 = \pm \int \frac{d\mathcal{Y}}{\left(\frac{K}{n+1} \mathcal{Y}^{n+1} + K_1 \right)^{\frac{1}{2}}}, \quad (2.52)$$

where \mathcal{X}_0 and K_1 are arbitrary constants of integration.

We observe that (2.23) can also be reduced to quadratures in the case $\alpha = 0$. This fact suggests that the Lie theory of point transformations [20] is not exhaustive in its treatment of the integrable cases of (2.23). However, an extension of the Lie theory to nonlocal symmetries [21] reveals that

$$y'' + My = Ky^n, \quad (2.53)$$

possesses, in addition to (2.20), the 'useful' nonlocal symmetry [21, 22]

$$Z_{nl} = \int \frac{H(\frac{1}{2}y'^2 - \frac{K}{n+1}y^{n+1} + \frac{M}{2}y^2)}{y'^2} dx \frac{\partial}{\partial x}. \quad (2.54)$$

The reduced equation, via

$$u = y \quad v = y' \quad (2.55)$$

is

$$vv' + Mu = Ku^n \quad (2.56)$$

which possesses the Lie point symmetry

$$Z_1 = \frac{H(\frac{1}{2}v^2 - \frac{K}{n+1}u^{n+1} + \frac{M}{2}u^2)}{v} \frac{\partial}{\partial v}. \quad (2.57)$$

(Note that in practice H is usually 1.) The function f now becomes

$$f = K(A_0 + A_1x + A_2x^2)^{-\frac{n+3}{2}}. \quad (2.58)$$

(See also [23] for a treatment of the two symmetry case for $f = x^m$.)

3 Painlevé Analysis

From §2 it is evident that the Lie theory of differential equations (and its extensions) is rather exhaustive in its integrability of the Emden–Fowler equation. A more popular criterion for an equation to be integrable is the possession of the Painlevé property. (See the excellent report [24] for a lucid introduction to this technique of analysis.) An ordinary differential equation is said to possess the Painlevé property if its general solution has no critical points [24]. Considering the evidence encountered in the literature, it is conjectured that an equation possessing the Painlevé property is integrable. The method of analysis we employ is not due to Painlevé [25] but was introduced in 1889 by Kowalevski [27, 28]. It is the method of pole-like expansions that has recently been popularised by Ablowitz *et al* [29].

Before we proceed with the analysis some comments are in order. It is well-known that Painlevé worked on the classification of second order ordinary differential equations of the first degree to determine those which possess the Painlevé property (though he presumably did not term it so). The work, completed by Gambier [30], constitutes a complete classification of all first degree second order differential equations that are rational in the both the dependent variable and its first derivative

and analytic in the independent variable. However, we note that (2.43) does not naturally fall into the classes of equations listed in [30, 31]. Thus a Painlevé analysis of (2.43) should highlight interesting properties of this equation. Further, the classification in [30] omits mention of algebraic branch points. Thus equations possessing the so-called ‘weak’ Painlevé property, which is also of interest, is excluded.

Having motivated the need for the Painlevé analysis of (2.43) we study the equation in the (more accessible) form

$$y'' = y^n, \quad (3.1)$$

where the K in (2.43) has been removed through the rescaling of y . The analysis essentially involves assuming a Laurent series expansion for the dependent variable about some point $x - x_0$. The procedure has three accepted parts. The first is the determination of the leading order behaviour by the substitution of

$$y = \alpha \chi^p, \quad (3.2)$$

where α and p are constants to be determined and

$$\chi = x - x_0, \quad (3.3)$$

where x_0 is the location of the moveable pole, into the equation of interest. After the calculation of p and α the expression

$$y = \alpha \chi^p + \beta \chi^{i+p} \quad (3.4)$$

is substituted into the dominant terms of the equation to determine the indices i (at which the remaining constants of integration arise) by requiring that the coefficient of terms linear in β is zero. Finally the truncated Laurent expansion

$$y = \alpha \chi^p + \delta_1 \chi^{p+1} + \dots + \delta_{i+p} \chi^{i+p} \quad (3.5)$$

is substituted into the original equation to verify that no incompatibilities arise that violate the arbitrariness of the the constants of integration which arise at the indices.

Remark: The determination of the index is a convenient mechanism that points to the non-possession of the Painlevé property (eg. for complex or irrational i) and also

aids computation. However, this part of the analysis can cause some properties of the equation(s) to be obscured and should be implemented with due caution. An appropriate example is that of Lotka-Volterra and Quadratic Systems [32]. Fortunately we are concerned with a relatively uncomplicated equation and these considerations do not apply.

The substitution of (3.2) into (3.1) yields

$$p = -\frac{2}{n-1} \quad \alpha^{n-1} = \frac{2(n+1)}{(n-1)^2} \quad (3.6)$$

with both terms in (3.1) being obviously dominant, and (3.4) into (3.1) yields

$$i = -1, \frac{2(n+1)}{n-1}. \quad (3.7)$$

The -1 is to be expected [24] and the second constant of integration (in addition to x_0) arises as the coefficient of $\chi^{2(n+1)/(n-1)}$ in the Laurent expansion for y . Note that for the implementation of the method p in (3.6) must be a negative integer. This arises for the values $n = 2, 3$. The index arises at $i = 6, 4$ respectively. We do not need to substitute the truncated Laurent expansion into (3.1) to check for incompatibilities at the index as all the terms in (3.1) are dominant. We note that (2.52) can be easily evaluated for these values of n .

For $n \neq 2, 3$ p and i are rational and the possibility that the solution of (3.1) possesses algebraic branch points exists. This is suggestive of the so-called 'weak' Painlevé property [33]. However, we can transform the denominator of p away by setting either

$$Y = y^{n-1} \quad X = x \quad (3.8)$$

or

$$Y = y \quad X = x^{1/(n-1)}. \quad (3.9)$$

The transformation (3.9) is homographic (and thereby preserves the Painlevé property [24, 30]) and results in

$$Y'' = (n-2)\frac{Y'}{X} + (n-1)^2 X^{2(n-2)} Y^n. \quad (3.10)$$

Our analysis of (3.10) yields the two cases

$$\text{i) } p = -2, i = 1 - n, 2(1 + n), \alpha^{n-1} = \frac{2(n+1)}{(n-1)^2}$$

$$\text{ii) } p = n - 1, i = -1, 0, \alpha - \text{arbitrary.}$$

The first case does not have $i = -1$ and so the analysis halts. An alternative route need to be sought.

While (3.8) is not homographic, it does preserve the polynomial form of (3.1) (for integer n) and so is an acceptable transformation [24]. The equation becomes

$$Y'' = (n - 1)Y^2 + \frac{n - 2}{n - 1} \frac{Y'^2}{Y}. \quad (3.11)$$

Again two cases arise:

$$\text{i) } p = -2, i = -1, 2(n + 1)/(n - 1), \alpha = \frac{2(n+1)}{(n-1)^2}$$

$$\text{ii) } p = n - 1, i = -1, 0, \alpha - \text{arbitrary.}$$

Case ii) arises in the instance that the two derivative terms in (3.11) are dominant only. However, this is equivalent to only the first term in (3.1) being dominant. The full implication of this needs further thought. In case i) i is a positive integer only when $n = -3, -1, 2, 3, 5$ (with the corresponding i values 1, 0, 6, 4, 3 respectively). The case $n = -1$ can be immediately discounted as $i = 0$ implies α is arbitrary. However, we note that α , in case i), is fixed (and is in fact zero!). This points to the introduction of logarithmic terms in the expansion for Y which violates the integrability of (3.11).

We need to examine the family $p = n - 1$ for the remaining values of n . For $n = 2, 3, 5$, $p > 0$. Ordinarily this would suggest the transformation

$$\mathcal{Y} = \frac{1}{Y} \quad (3.12)$$

to make p negative. However, as we have specific values for n we can resort to looking up the appropriate equations in [30, 31]. We find that the equations corresponding to $n = -3, 2, 3, 5$ are eqq (22), (2), (18) and (21) of [30] respectively. Thus (3.1) has the Painlevé property for $n = -3, 2, 3, 5$. For other integer values of n , i will be rational and the best one can hope for is possession of the weak Painlevé property.

Note that, as n is a physical constant related to the ratio of specific heats in the astrophysical context [34], it can be rational. In the subsequent analysis we expressly ignore integer n . A study of the relationship (3.6) reveals that p is a negative integer for $1 < n < 5/3$. For $p = -2/(n-1) \in \mathbb{Z} < 0$, $i = 2(n+1)/(n-1) = 2-2p \in \mathbb{Z} > 0$ (in (3.7)). This points to (3.1) possessing the full Painlevé property. In this instance we do not have recourse to the lists in [30, 31] as these are concerned with rational functions of the dependent variables. It should be noted that no such restriction was originally intended by Painlevé [25, 35]. We do not have to substitute the truncated Laurent expansion (3.5) into (3.1) to verify that no incompatibilities arise at the index as both terms are dominant. Thus we introduce equations of the form

$$y'' = y^{(p+2)/p}, \quad p \in \mathbb{N}, p > 2 \quad (3.13)$$

into the literature as part of the class of second order ordinary differential equations possessing the Painlevé property.

4 The special cases $n = -3, 2$

We have seen above that in the case $n = -3, 2$, (1.2) can be reduced to a quadrature that can be evaluated. However, these cases have a deeper significance that the Lie analysis in §2 did not reveal.

For $n = -3$ and $\alpha = 0$ we solve (2.14) to obtain

$$f = \tilde{K}, \quad (4.1)$$

where \tilde{K} is an arbitrary constant. The solution for a (obtained from (2.10) and (2.13)) is

$$a = A_0 + A_1x + A_2x^2 \quad (4.2)$$

with c given by

$$c = \frac{A_1}{2} + A_2x \quad (4.3)$$

and $d = 0$ as before. Eq (1.2) now has the form

$$y'' = \tilde{K}y^{-3} \quad (4.4)$$

(which is the well-known Ermakov-Pinney equation [36, 37]) and has the three Lie point symmetries

$$G_1 = \frac{\partial}{\partial x} \quad (4.5)$$

$$G_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad (4.6)$$

$$G_3 = -x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \quad (4.7)$$

which form the Lie algebra $sl(2, R)$. (See also [38, 39, 40].) This Lie algebra is not solvable, but as we are only concerned with a second order equation, it is sufficient to reduce (4.4) to quadratures.

For $n = 2$, (2.9) and (2.1) coalesce into

$$2fd - c'' = 0. \quad (4.8)$$

In the case $d = 0$ the analysis for general n applies. We observe that the functions f for which (1.2) can be integrated fall into the class given by Srivastava [10].

The case of $d \neq 0$ has two further subcases. The first is of constant d , ie $d = D_0$. In this case f is given by

$$f = \frac{c''}{2d}, \quad (4.9)$$

with

$$c = \frac{1}{2}a' - \alpha. \quad (4.10)$$

To find the explicit form for f in (4.9) we need the function a which is obtained by solving

$$2aa^{iv} + 5a'a''' - 2\alpha a''' = 0. \quad (4.11)$$

For $\alpha = 0$ (4.11) has three symmetries and can be formally integrated. When $a \neq 0$ there are only two symmetries and (4.11) can only be partially integrated.

In the case $d = D_0 + D_1x$, f has the same form as in (4.9). The equivalent of (4.11) is now

$$2aa^{iv}d + 5a'a'''d - 2aa'''d' - 2\alpha a'''d = 0. \quad (4.12)$$

Using the transformation

$$\eta = \frac{D_1}{D_0 + D_1x} \quad \zeta = \frac{aD_1^2}{(D_0 + D_1x)^2} \quad (4.13)$$

we rewrite (4.12) as (with 's denoting differentiation with respect to η)

$$2\zeta\zeta^{iv} + 5\zeta'\zeta''' + 2\alpha\zeta''' = 0 \quad (4.14)$$

which has the same form as (4.11). Thus the cases of $d = D_0$ and $d = D_0 + D_1x$ reduce to the analysis of the single equation (4.11).

Given a solution to (4.11) we can find f using (4.9) and subsequently transform (1.2) to autonomous form. The resulting equations will be of the form (2.23) and the analysis that follows (2.23) will apply.

However, we still need to solve the equation for a . We first consider eq (4.11) with $\alpha = 0$:

$$2aa^{iv} + 5a'a''' = 0. \quad (4.15)$$

It is fairly simple to integrate (4.15) formally to

$$x - x_0 = \int \frac{du}{(-Ku^3/6 - Lu^2/2 - 2Pu - 2Q)^{3/2}}, \quad (4.16)$$

where K, L, P and Q are constants of integration and

$$u = \int \frac{1}{a^{3/2}}. \quad (4.17)$$

It is interesting to note that the trivial cases of setting all except one (in turn) of K, L, P and Q to zero produce functions f that are subclasses of that of Srivastava [10].

In spite of the fact that (4.15) only has the three symmetries [43]

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= x \frac{\partial}{\partial x} \\ G_3 &= a \frac{\partial}{\partial a} \end{aligned} \quad (4.18)$$

we can still obtain (4.16) via a symmetry reduction. If we reduce (4.15) using G_1 , G_2 and G_3 in turn, we obtain an Abel's equation of the second kind, the solution of which is unobvious. The proper route to the reduction is via hidden symmetries [44, 45].

The reduction of (4.15) via G_1 results in

$$2u(v^2v''' + 4vv'v'' + v'^3) + 5(v^2v'' + vv'^2) = 0, \quad (4.19)$$

where

$$u = a, \quad v = a'. \quad (4.20)$$

The Lie analysis of (4.19) produces three instead of the expected two symmetries, viz.

$$\begin{aligned} G'_1 &= u \frac{\partial}{\partial u} \\ G'_2 &= v \frac{\partial}{\partial v} \\ G'_3 &= 2u^2 \frac{\partial}{\partial u} + uv \frac{\partial}{\partial v}. \end{aligned} \quad (4.21)$$

The 'new' point symmetry G'_3 is not a descendent of any of the point symmetries in (4.18), but comes from the nonlocal symmetry [46]

$$G_4 = 3 \left(\int a dx \right) \frac{\partial}{\partial x} + 2a^2 \frac{\partial}{\partial a} \quad (4.22)$$

and is hence a Type II hidden symmetry [44]. This hidden symmetry is the appropriate one for further reduction of (4.19). Using

$$t = vu^{-1/2} \quad w = \frac{1}{2}(v'u^{3/2} - \frac{1}{2}vu^{-1/2})^2 \quad (4.23)$$

we obtain

$$w'' + 3w' + 2w = 0 \quad (4.24)$$

which is trivially solved. Reversing the transformations we obtain (4.16).

When $\alpha \neq 0$, eq (4.11) has only the two point symmetries

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= x \frac{\partial}{\partial x} + a \frac{\partial}{\partial a}. \end{aligned} \quad (4.25)$$

Unfortunately reduction using G_1 does not produce any hidden symmetries. However, it is of interest to test (4.11) for integrability using the Painlevé analysis. We remove the α in (4.11) by rescaling a to obtain

$$2aa^{iv} + 5a'a''' + 2a''' = 0. \quad (4.26)$$

In the case of all terms dominant, $p = 1$. The other possible leading order behaviour is that of the first two terms being dominant. This gives rise to $p = 0, 6/7, 1, 2$. As the p values are positive we invoke the homographic transformation

$$v = \frac{1}{a} \quad X = x \quad (4.27)$$

to transform (4.26) to

$$2v^2v^{iv} - 21v^2v'v''' + 2v^4v'''' - 12v^2v''^2 + 102vv'^2v'' - 12v^3v'v'' - 78v'^4 + 12v^2v'^3 = 0. \quad (4.28)$$

In the case of all terms dominant, $p = -1$ and α (the coefficient of the pole in this case) is arbitrary. However, the indices are $i = -1, 0, 1, -(1 + 2\alpha)/2$. To pass the Painlevé test i must be (at least) rational. This fixes α which contradicts the implication of $i = 0$ that α is arbitrary. Thus (4.28) (and hence (4.26)) does not possess the full Painlevé property. However, it may possess (and the indices do suggest this) either

the partial Painlevé property [47] or the pseudo-partial Painlevé property [13]. It has been observed [48] that some information about the partial solution of (4.26) can be obtained from considering the different families of expansions for y . However, this information is naturally contained in (4.16).

5 Discussion

The Emden-Fowler equation

$$y'' = f(x)y^n \quad (5.1)$$

has been shown to be integrable (for certain functions $f(x)$) for all n (including rational values) by considering a Lie analysis. It was further shown that, if (5.1) possessed two Lie symmetries, it could always be transformed to

$$Y'' = Y^n. \quad (5.2)$$

This point must be emphasised – all Emden-Fowler equations with two symmetries (and the Lie algebra A_2) can be transformed to (5.2) and the solution to the original equation is obtained from the solution of (5.2) via the same transformation.

The Painlevé analysis of (5.2) reveals integrability only for restricted values of n . It is remarkable that, for these restricted integer values, the quadrature (2.52) can be evaluated. This does reinforce the close relationship between the Lie and Painlevé analyses observed previously [13, 49]. In the case of rational n , it was shown that, for specific values of n in the range $(1, \frac{5}{3}]$, (5.2) possessed the Painlevé property. While the quadrature (2.52) cannot, as yet, be evaluated for these values, noting the results in [41] and [42] we believe that the evaluation thereof is only a matter of time and effort.

The method used to implement the Painlevé analysis in this paper was that of ‘pole-like’ expansions. While it is a convenient mechanism, it does contain certain pitfalls into which the unwary practitioner may fall. Mention was made previously of the

obscuring of valuable information in systems of equations [32]. Further, the presence of negative indicies (apart from -1) has only recently been appreciated and used to advantage [50]. A final remark is due to the requirement of negative p for the order of the pole. This requirement is frequently the first step in the analysis and suggests an immediate transformation of the equation under consideration when p is positive. While the resulting equation may possess the Painlevé property, it must be noted that it could well fall outside the class of equations listed in [30, 31]. The reason is simple: these lists of equations do *not* insist on negative p ! An example is the case $n = -3$ considered in §3. Eq (3.11) possess the full Painlevé property, but does not occur in [30, 31] in its present form. Invoking (3.12) results in an equation that *does* arise in [30, 31] with p now being positive. Thus due caution must be observed in the implementation of the algorithm and the analysis of the results. It is little wonder that Painlevé did not see the need for ‘le procédé connu de Madame Kowaleski’ [35].

It was of interest to observe that previous classifications of ordinary differential equations with solutions free of moveable critical points were confined to those equations that were rational in the dependent variable. This does suggest that even the recent exhaustive analyses [51, 52] are not sufficient to complete the classification problem, even for second order equations. The fact that we were able to obtain equations algebraic in the dependent variable that do possess the Painlevé property would suggest that this avenue should be investigated further.

We finally note that the Painlevé analysis was restricted to equations of the form (5.2) by requiring that (5.1) possess two Lie point symmetries. However, noting that some of the equations in [30, 31] do not possess at least two Lie point symmetries, the investigation of the equation in the form (5.1) would be of some interest. Work on this already been started [53].

Acknowledgements

It is a pleasure to thank Robert Conte for interesting discussions and clarifications. We also thank the Foundation for Research Development of South Africa and the University of Natal for their ongoing support.

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Characterisation of the algebraic properties of first integrals of scalar ordinary differential equations

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(preprint: Department of Mathematics, University of the Aegean, Karlovassi
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Characterisation of the Algebraic Properties of First Integrals of Scalar Ordinary Differential Equations of Maximal Symmetry

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Abstract

We undertake a study of the first integrals of linear n th order scalar ordinary differential equations with maximal symmetry. We establish patterns for the first integrals associated with these equations. It is shown that second and third order equations are the pathological cases in the study of higher order differential equations. The equivalence of contact symmetries for third order equations to nonCartan symmetries of second order equations is highlighted.

1 Introduction

A scalar ordinary differential equation

$$E(x, y, y', \dots, y^{(n)}) = 0, \quad (1)$$

where ' denotes differentiation of the dependent variable, y , with respect to the independent variable, x , and $y^{(n)}$ the n th derivative, possesses a Lie point symmetry

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \quad (2)$$

if

$$G^{[n]} E|_{E=0} = 0, \quad (3)$$

where $G^{[n]}$ is the n th extension of G given by [1]

$$G^{[n]} = G + \sum_{i=1}^n \left\{ \eta^{(i)} - \sum_{j=0}^{i-1} \binom{i}{j} y^{(j+1)} \xi^{(i-j)} \right\} \frac{\partial}{\partial y^{(i)}}. \quad (4)$$

(The extension is needed to give the infinitesimal transformations in the derivatives up to $y^{(n)}$ induced by the infinitesimal transformations which G produces in x and y .) The symmetries of (1) constitute a Lie algebra under the operation of taking the Lie Bracket

$$[G_1, G_2] = G_1 G_2 - G_2 G_1. \quad (5)$$

A first integral of (1) associated with the symmetry (2) is a function, $f(x, y, y', \dots, y^{(n-1)})$, in which the dependence on $y^{(n-1)}$ is nontrivial, satisfying the two conditions

$$G^{[n-1]} f = 0 \quad (6)$$

and

$$\frac{df}{dx}|_{E=0} = 0. \quad (7)$$

The association of f with G , as stated in (6), and (7) means that f is a first integral of (1). Equally a first integral, $f(x, y, y', \dots, y^{(n-1)})$, of (1) has a symmetry of the form of (2) if

$$G^{[n-1]} f = 0. \quad (8)$$

In recent years a number of papers has been devoted to the algebraic properties of first integrals of scalar ordinary differential equations associated with its symmetries. The

number of symmetries associated with a differential equation depends upon its internal structure up to an upper limit which is fixed by the order of the equation and the type of symmetry under consideration. In this paper we are concerned with point and contact symmetries only. The symmetry (2) is a Lie point symmetry if the coördinate functions ξ and η are functions of x and y only. It is a contact symmetry if ξ and η depend upon x , y and y' subject to the constraint that [2, p 94]

$$\frac{\partial \eta}{\partial y'} = y' \frac{\partial \xi}{\partial y'} \quad (9)$$

which means that the first extension of G also depends upon x , y and y' only. A point symmetry is always a contact symmetry. Note that we do not restrict ξ to be a function of x only which is the case for the so-called Cartan symmetries [3].

Lie showed that the maximum number of point symmetries of a scalar ordinary differential equation was infinite for equations of the first order [4, p 114], eight for equations of the second order [4, p 405] and $n + 4$ for equations of the n th order [5, p 298]. He also [6] classified all the invariance algebras of dimension one, two and three for second order equations. Mahomed and Leach [1] showed that higher order linear equations could have $n + 1$ or $n + 2$ point symmetries instead of the $n + 4$ for the maximal symmetry case. Note that, whenever reference is made to linear equations, we include nonlinear equations which are linearisable by a point (resp contact) transformation when point (resp contact) symmetries are being considered. Lie also showed that second order equations possessed an infinite number of contact symmetries [2, p 84] and third order at most ten [2, p 241]. For higher order (in standard form) it has been shown [7] that equations of maximal symmetry only admit $n + 4$ contact symmetries. Abraham-Shrauner *et al* [8] showed that the maximal number of contact symmetries of third order equations could be found in equations which did not have seven (the maximum) point symmetries in addition to the Kummer-Schwarz equation given by Lie [2, p 148].

Leach and Mahomed [9] discussed the algebraic properties of the first integrals of equations of maximal point symmetry which are represented by the single equivalence class

$$y'' = 0. \quad (10)$$

It was found that the two functionally independent first integrals

$$I_1 = xy' - y \quad (11)$$

$$I_2 = y' \quad (12)$$

each possessed three symmetries with the three-dimensional algebra $A_{3,3}$ in the Mubarakzhanov classification [10]. Furthermore the related integral

$$I_3 = \frac{I_1}{I_2} \quad (13)$$

also possessed three symmetries with the same algebra. In the case of (13) the three symmetries are

$$G_1 = y \frac{\partial}{\partial y} \quad (14)$$

$$G_2 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} \quad (15)$$

$$G_3 = y \frac{\partial}{\partial x} \quad (16)$$

G_1 is the homogeneity symmetry and for any integral to possess it the integral must be homogeneous of degree zero in y . G_2 and G_3 are the nonCartan symmetries of (10). Note that, although the integrals have nine symmetries in all, there are only eight linearly independent symmetries. The interesting algebraic structure of the symmetries associated with the linear integrals of (10) led to the study of the corresponding algebras of third order equations with four, five and seven symmetries [11] and n th order equations with $n + 1$, $n + 2$ and $n + 4$ symmetries [12]. The pattern of the second order equations was not maintained. The demonstration by Abraham-Shrauner *et al* [8] that contact symmetries are the appropriate ones to be used in treatments of third order equations led to a study [13] of the algebraic properties of the contact symmetries associated with the integrals of third order equations with the symmetry algebra $sp(4)$ which was more extensive than the comparable study of second order equations by Leach and Mahomed [9].

Second and third order ordinary differential equations have properties which are peculiar to each and which mark them off from the general n th order equation. This is the case even for the representative equations of maximal symmetry. For

$$y'' = 0 \quad (17)$$

there are eight point symmetries, six of which are of Cartan form and two of which are not. In the case of

$$y''' = 0 \quad (18)$$

the seven point symmetries are all of Cartan form, but there are three additional purely contact symmetries and the consideration of third order equations without contact symmetries is as incomplete as the consideration of second order equations without the nonCartan point symmetries. However,

$$y^{(n)} = 0 \tag{19}$$

has only $n + 4$ point symmetries all of which are of Cartan type [5, p 298]. It is this variation in the type, rather than in the number (apart from the fact that it is maximal), of symmetries which suggests that the generic behaviour for ordinary differential equations of maximal symmetry is not to be found at the second or third order, but at the fourth order. It is the intention of this paper to demonstrate this feature and to describe the generic properties of the algebras of the symmetries of first integrals of scalar ordinary differential equations.

2 Methodology

There is a certain ambiguity in the treatment of the symmetries of the first integrals. In the case of the second order equation, (10), we recalled that the three first integrals, (11), (12) and (13) each had three point symmetries and that the algebras were isomorphic. However, this is not the case for any combination of (11) and (12) [9]. We do seek those first integrals which have, in some sense, a maximal number of symmetries. To make comparison from one order to another feasible it is necessary to use as much of the common structure for the algebras of the differential equations as is possible. To this end we use the form given by Mahomed and Leach [1], $nA_1 \oplus (sl(2, R) \oplus A_1)$, to which the two nonCartan symmetries are to be added when $n = 2$ and the three purely contact symmetries are to be added when $n = 3$. To each of these symmetries there corresponds a first integral and it is the symmetries of these integrals which we consider. For an n th order equation there are $n - 1$ integrals associated with each symmetry of the n th order equation and the first integral mentioned above is an arbitrary function of these. We select $n - 1$ of these arbitrary independent functions in such a way as to have the maximum number of symmetries possible.

In the case of (18) there are three functionally independent first integrals which we take

Table 1: The first integrals associated with the ten contact symmetries of $y''' = 0$

Symmetry	p	q
G_1	I_2	I_3
G_2	I_1	I_3
G_3	I_1	I_2
G_4	I_3	$I_1 I_3 - \frac{1}{2} I_2^2$
G_5	I_2	$I_1 I_3 - \frac{1}{2} I_2^2$
G_6	I_1	$I_1 I_3 - \frac{1}{2} I_2^2$
G_7	I_2/I_3	I_3/I_1
G_8	I_2/I_3	$(I_1 I_3 - \frac{1}{2} I_2^2)/I_3$
G_9	I_3/I_1	$(I_1 I_3 - \frac{1}{2} I_2^2)/I_1$
G_{10}	I_1/I_2	$(I_1 I_3 - \frac{1}{2} I_2^2)/I_2$

to be [13]

$$I_1 = \frac{1}{2} x^2 y'' - x y' + y \quad (20)$$

$$I_2 = x y'' - y' \quad (21)$$

$$I_3 = y'' \quad (22)$$

which have been shown [11] to possess maximal symmetry. In terms of I_1 , I_2 and I_3 the integrals associated with the symmetries of (18) are given [13] in Table 1. Note that each symmetry has two integrals (denoted by p and q) since the equation is of the third order. In Table 2 we list the symmetries [13] associated with each integral given in Table I in terms of the symmetries of (18) (and combinations thereof) and the corresponding algebra according to the Mubarakzyanov classification [10]. We observe that the algebras are either three-dimensional or four-dimensional. The latter is always $A_{4,9}^1$, but the former are either $A_{3,4}$ (also known as $E(1,1)$, the algebra of the pseudo-Euclidean group in the plane) or $A_{3,8}$ (much better known as $sl(2, R)$). This already indicates two departures from the results for (17) in that the dimensions of the algebras are not the same and also the three-dimensional algebras differ not only from that of the second order case, $A_{3,3}$, but within themselves. Perhaps even more surprising is that $A_{3,3}$ is not a subalgebra of $A_{4,9}^1$ [14].

With the example of the third order equation before us we shall present the results for

the fourth order equation of maximal symmetry, viz.

$$y^{iv} = 0, \quad (23)$$

in §§3 and 4. In §5 our concluding remarks address, amongst a number of observations, the matter of the general equation

$$y^{(n)} = 0. \quad (24)$$

3 Symmetries and their integrals

Eq (23) is the representative equation of the fourth order with the maximum number of symmetries which is eight with the algebraic structure $(A_1 \oplus sl(2, R)) \oplus_s 4A_1$. The symmetries are

$$\begin{aligned} G_1 &= \frac{\partial}{\partial y} \\ G_2 &= x \frac{\partial}{\partial y} \\ G_3 &= \frac{1}{2} x^2 \frac{\partial}{\partial y} \\ G_4 &= \frac{1}{6} x^3 \frac{\partial}{\partial y} \\ G_5 &= \frac{\partial}{\partial x} \\ G_6 &= x \frac{\partial}{\partial x} + \frac{3}{2} y \frac{\partial}{\partial y} \\ G_7 &= x^2 \frac{\partial}{\partial x} + 3xy \frac{\partial}{\partial y} \\ G_8 &= y \frac{\partial}{\partial y}. \end{aligned} \quad (25)$$

The first four symmetries are called the solution symmetries since the coefficient of $\partial/\partial y$ is a solution of the original differential equation. (This is one of the banes of linear differential equations. It is necessary to be able to solve the equation before most of the symmetries can be determined. In this respect nonlinear equations are more amenable to treatment.) G_5 through G_7 are the elements of $sl(2, R)$ appropriate to (23). For an n th order equation of maximal symmetry they have the form [1]

$$G = a(x) \frac{\partial}{\partial x} + \frac{n-1}{2} a'(x) y \frac{\partial}{\partial y}, \quad (26)$$

Table 2: The contact symmetries and algebras of the integrals

Integral	Symmetries	Nonzero Lie Brackets	Algebra
I_1	$X_{11} = G_2$ $X_{12} = G_3$ $X_{13} = G_7$ $X_{14} = G_6 - G_4$	$[X_{11}, X_{13}] = X_{12}$ $[X_{11}, X_{14}] = -X_{11}$ $[X_{12}, X_{14}] = -2X_{12}$ $[X_{13}, X_{14}] = -X_{13}$	$A_{4,9}^1$
I_2	$X_{21} = G_1$ $X_{22} = G_3$ $X_{23} = G_6$	$[X_{21}, X_{23}] = X_{21}$ $[X_{22}, X_{23}] = -X_{22}$	$A_{3,4}(E(1,1))$
I_3	$X_{31} = G_1$ $X_{32} = G_2$ $X_{33} = G_5$ $X_{34} = G_6 + G_4$	$[X_{31}, X_{34}] = 2X_{31}$ $[X_{32}, X_{33}] = -X_{31}$ $[X_{32}, X_{34}] = X_{32}$ $[X_{33}, X_{34}] = X_{33}$	$A_{4,9}^1$
$I_4 = I_3/I_2$	$X_{41} = G_1$ $X_{42} = G_4$ $X_{13} = G_8$	$[X_{41}, X_{42}] = X_{41}$ $[X_{42}, X_{43}] = X_{43}$	$A_{3,4}(E(1,1))$
$I_5 = I_3/I_1$	$X_{51} = G_4$ $X_{52} = G_2$ $X_{53} = G_9$	$[X_{51}, X_{52}] = -X_{52}$ $[X_{51}, X_{53}] = X_{53}$ $[X_{52}, X_{53}] = -2X_{51}$	$A_{3,8}(sl(2, R))$
$I_6 = I_2/I_1$	$X_{61} = G_4$ $X_{62} = G_3$ $X_{63} = G_{10}$	$[X_{61}, X_{62}] = -X_{62}$ $[X_{61}, X_{63}] = X_{63}$	$A_{3,4}(E(1,1))$
$I_7 = I_1 I_3 - \frac{1}{2} I_2^2$	$X_{71} = G_5$ $X_{72} = G_6$ $X_{73} = G_7$	$[X_{71}, X_{72}] = X_{71}$ $[X_{71}, X_{73}] = 2X_{72}$ $[X_{72}, X_{73}] = X_{73}$	$A_{3,8}(sl(2, R))$
$I_8 = I_7/I_3$	$X_{81} = G_5$ $X_{82} = G_6 - G_4$ $X_{83} = G_8$ $X_{84} = G_9$	$[X_{81}, X_{82}] = X_{81}$ $[X_{81}, X_{84}] = 2X_{83}$ $[X_{82}, X_{83}] = -X_{83}$ $[X_{82}, X_{84}] = -X_{84}$	$A_{4,9}^1$
$I_9 = I_7/I_2$	$X_{91} = G_8$ $X_{92} = G_6$ $X_{93} = G_{10}$	$[X_{91}, X_{92}] = X_{91}$ $[X_{92}, X_{93}] = X_{93}$	$A_{3,4}(E(1,1))$
$I_{10} = I_7/I_1$	$X_{101} = G_6 + G_4$ $X_{102} = G_7$ $X_{103} = G_9$ $X_{104} = G_{10}$	$[X_{101}, X_{102}] = X_{102}$ $[X_{101}, X_{103}] = X_{103}$ $[X_{101}, X_{104}] = 2X_{104}$ $[X_{102}, X_{103}] = -2X_{104}$	$A_{4,9}^1$

where $a(x)$ is one of the three solutions of the self-adjoint equation

$$\frac{(n+1)!}{(n-2)!4!}a''' + B_{n-2}a' + \frac{1}{2}B'_{n-2}a = 0, \quad (27)$$

where B_{n-2} is the coefficient of $y^{(n-2)}$ when the equation is cast into normal form. The final symmetry, G_8 , follows from the homogeneity of the differential equation which happens to coincide with linearity in this case.

In calculating the first integrals associated with each of G_1 through G_8 according to eqq (6) and (7) we find that four functionally independent linear first integrals occur. To make the reportage of our results more compact we express all other integrals in terms of them. They are

$$\begin{aligned} I_1 &= \frac{1}{6}x^3y''' - \frac{1}{2}x^2y'' + xy' - y \\ I_2 &= \frac{1}{2}x^2y''' - xy'' + y' \\ I_3 &= xy''' - y'' \\ I_4 &= y'''. \end{aligned} \quad (28)$$

With each symmetry there will be associated three functionally independent first integrals. This follows from the solutions to the two first order partial differential equations (6) and (7) for the first integral associated with a particular symmetry. In (6) there are five variables, x , y , y' , y'' and y''' , and so four characteristics. This means that (7) has four variables and hence three characteristics each of which is a first integral. The integrals belonging to the symmetries are listed in Table 3.

In comparison with the integrals listed in Table 1 (less those associated with the purely contact symmetries, $G_8 - G_{10}$) for $y''' = 0$ we note a number of similarities and differences. The solution symmetries, $G_1 - G_4$, simply form a permutation with three of the linear integrals. (The labelling was chosen to highlight this feature.) The homogeneity symmetry, G_8 , has the three independent ratios of the linear integrals. Any other independent set of three could be equally be chosen. The ratios are the only integrals possible for G_8 since any integral associated with it must be of zero degree in y .

These are the anticipated generalisations of the corresponding results for $y''' = 0$ and, indeed, for $y'' = 0$. For (24) we may infer that each of the n solution symmetries will have $n - 1$ of the n functionally independent linear integrals associated with it and that, by a suitable choice of labels as was made for (18) and (23), the one symmetry label and

Table 3: The first integrals associated with the eight point symmetries of $y^{iv} = 0$

Symmetry	First integrals		
G_1	I_2	I_3	I_4
G_2	I_3	I_4	I_1
G_3	I_4	I_1	I_2
G_4	I_1	I_2	I_3
G_5	I_4	$I_2I_4 - \frac{1}{2}I_3^2$	$I_2I_3I_4 - \frac{1}{3}I_3^3 - I_1I_4^2$
G_6	I_1I_4	$I_2I_4^{\frac{1}{3}}$	$I_3I_4^{-\frac{1}{3}}$
G_7	I_1	$I_1I_3 - \frac{2}{3}I_2^2$	$I_1I_2I_3 - \frac{4}{3}I_2^3 - I_1^2I_4$
G_8	I_1/I_2	I_2/I_3	I_3/I_4

the $n - 1$ integral labels will be a cyclic permutation of the integers 1 through n . Equally confidently we infer that the homogeneity symmetry, G_{n+4} , will have $n - 1$ independent ratios of the linear integrals associated with it.

In the case of the representation of $sl(2, R)$, which is common to all linear equations of maximal symmetry, the situation is not so clear. To give more scope for observation we list the corresponding relationships for $y'' = 0$. They are

$$\begin{aligned}
 G_3 &= \frac{\partial}{\partial x} & I_2 \\
 G_4 &= x \frac{\partial}{\partial x} + \frac{1}{2}y \frac{\partial}{\partial y} & I_1I_2 \\
 G_5 &= x^2 \frac{\partial}{\partial x} + \frac{1}{2}xy \frac{\partial}{\partial y} & I_1,
 \end{aligned} \tag{29}$$

where, in the spirit adopted for (18) and (23), the $sl(2, R)$ symmetries have been listed after the solution symmetries and

$$\begin{aligned}
 I_1 &= xy' - y \\
 I_2 &= y'.
 \end{aligned} \tag{30}$$

One point should be made before we continue. Since the first and third of the two symmetries in $sl(2, R)$ are related by the transformation

$$x \longrightarrow -\frac{1}{x} \quad y \longrightarrow \frac{y}{x^{n-1}} \tag{31}$$

and the second is invariant under (31), we need only consider the first two symmetries, invariance under translation in the independent variable and a self-similar transformation.

(The integrals of the so-called conformal symmetry follow from those of invariance under translations in x by the application of (31).)

There are two problems. The first is the form of the expressions for the more complicated integrals associated with $\partial/\partial x$. Under the labelling scheme adopted I_n can always be taken as the first representative. The first representative for the self-similar symmetry differs from the even order equations to the odd order equation. This provides the necessary hint and we have

Proposition 1: *One of the integrals associated with the self-similar symmetry*

$$G = x \frac{\partial}{\partial x} + \frac{n-1}{2} y \frac{\partial}{\partial y} \quad (32)$$

of

$$y^{(n)} = 0 \quad (33)$$

is

$$J_1 = I_1 I_n \quad (34)$$

when n is even and

$$J_1 = I_{(n+1)/2} \quad (35)$$

when n is odd, where the numbering of the functionally independent linear first integrals of (33) is according to the scheme

$$I_i = \sum_{k=0}^{n-i} \frac{(-1)^k}{(n-i-k)!} x^{n-i-k} y^{(n-1-k)}, \quad i = 1, n. \quad (36)$$

The result follows trivially from **Proposition 3**.

Associated with the self-similar symmetry there is another problem. We choose this symmetry so that the subalgebra $sl(2, R)$ occurs naturally within the list of symmetries of the equation. However, the homogeneity symmetry, G_{n+4} , can be added without changing the nature of the self-similar symmetry.

4 Integrals and their symmetries

The Lie point symmetries associated with each of the first integrals listed in Table 3 are calculated following the prescription of (6) using the symbolic code Program LIE [15]. The integrals, symmetries, nonzero Lie Brackets and algebras are given in Table 4. The

symmetries are written as X_{ij} in which the label i refers to the integral number and the label j to the number of the symmetry within the integral's algebra. The relationship of these symmetries to those of the differential equation (23) is also given.

5 Discussion

The linear integrals, as expected [12], have either five or four symmetries. Three of these are solution symmetries and, in the way the labelling has been arranged, the solution symmetry not included is the one of the number of the integral. (The action of the omitted symmetry on the integral is a constant, +1 for I_2 and I_4 and -1 for I_1 and I_3 due to the way the integrals have been defined. The integrals with five symmetries have either G_5 or G_7 which are equivalent under the transformation

$$x \longrightarrow -\frac{1}{x} \quad y \longrightarrow \frac{y}{x^3}. \quad (37)$$

The remaining symmetry is a combination of the self-similar G_6 and the homogeneity G_8 of the form.

$$X_i = G_6 + \frac{2i-5}{2}G_8, \quad i = 1, 4. \quad (38)$$

Similar combinations occur for some of the other integrals. This does suggest that the choice of the form of the self-similar symmetry is at our disposal and that we should not be constrained by the form of G_6 as it occurs in the list of symmetries for (23). This has some impact on the expressions for the integrals associated with the self-similar symmetry. If we take it to be of the form

$$G = x \frac{\partial}{\partial x} - ay \frac{\partial}{\partial y}, \quad (39)$$

where a is a parameter, the three integrals are

$$J_1 = I_4^{-\frac{a}{a+3}} I_1 \quad (40)$$

$$J_2 = I_4^{-\frac{a+1}{a+3}} I_2 \quad (41)$$

$$J_3 = I_4^{-\frac{a+2}{a+3}} I_3. \quad (42)$$

Equivalently other combinations may be taken. For example, if I_1 were taken as the integral to have the fractional power associated with it, we would have

$$K_1 = I_1^{-\frac{a+3}{a}} I_4 \quad (43)$$

Table 4: The point symmetries and algebras of the integrals

Integral	Symmetries	Nonzero Lie Brackets	Algebra
I_1	$X_{11} = G_2$ $X_{12} = G_3$ $X_{13} = G_4$ $X_{14} = G_6 - \frac{3}{2}G_8$ $X_{15} = G_7$	$[X_{11}, X_{14}] = -X_{11}$ $[X_{11}, X_{15}] = 4X_{12}$ $[X_{12}, X_{14}] = -2X_{12}$ $[X_{12}, X_{15}] = \frac{3}{2}X_{13}$ $[X_{14}, X_{15}] = X_{15}$	$A_{4,9}^1$
I_2	$X_{21} = G_1, X_{22} = G_3$ $X_{23} = G_4$ $X_{24} = G_6 - \frac{1}{2}G_8$	$[X_{21}, X_{24}] = X_{21}$ $[X_{22}, X_{24}] = -X_{22}$ $[X_{23}, X_{24}] = -2X_{23}$	$A_{3,4}(E(1,1))$
I_3	$X_{31} = G_1, X_{32} = G_2$ $X_{33} = G_4$ $X_{34} = G_6 + \frac{1}{2}G_8$	$[X_{31}, X_{34}] = 2X_{31}$ $[X_{32}, X_{34}] = X_{32}$ $[X_{33}, X_{34}] = -X_{33}$	$A_{4,9}^1$
I_4	$X_{41} = G_1$ $X_{42} = G_2$ $X_{43} = G_3$ $X_{44} = G_5$ $X_{45} = G_6 + \frac{3}{2}G_8$	$[X_{41}, X_{45}] = 3X_{41}$ $[X_{42}, X_{44}] = -X_{41}$ $[X_{42}, X_{45}] = 2X_{42}$ $[X_{43}, X_{44}] = -2X_{42}$ $[X_{43}, X_{45}] = X_{43}$ $[X_{44}, X_{45}] = X_{44}$	$A_{4,9}^1$
$I_5 = I_1/I_2$	$X_{51} = G_3, X_{52} = G_4$ $X_{53} = G_8$	$[X_{51}, X_{52}] = X_{51}$ $[X_{52}, X_{53}] = X_{52}$	$A_{3,4}(E(1,1))$
$I_6 = I_2/I_3$	$X_{61} = G_1, X_{62} = G_4$ $X_{63} = G_8$	$[X_{61}, X_{63}] = X_{61}$ $[X_{62}, X_{63}] = X_{62}$	$A_{3,3}(D \otimes, T_2)$
$I_7 = I_3/I_4$	$X_{71} = G_1, X_{72} = G_2$ $X_{73} = G_8$	$[X_{71}, X_{73}] = X_{71}$ $[X_{72}, X_{73}] = X_{73}$	$A_{3,4}(E(1,1))$
$I_8 = I_2I_4 - \frac{1}{2}I_2^2$	$X_{81} = G_1, X_{82} = G_5$ $X_{83} = \frac{1}{2}(G_6 + \frac{1}{2}G_8)$	$[X_{81}, X_{83}] = X_{81}$ $[X_{82}, X_{83}] = \frac{1}{2}X_{82}$	$A_{3,5}^{\frac{1}{2}}$
$I_9 = I_2I_3I_4 - \frac{1}{3}I_3^3 - I_1I_4^2$	$X_{91} = G_5, X_{92} = G_6 + \frac{1}{2}G_8$	$[X_{91}, X_{92}] = X_{91}$	$2A_1$
$I_{10} = I_1I_4$	$X_{101} = G_2, X_{102} = G_3$ $X_{103} = 2G_6$	$[X_{101}, X_{103}] = X_{101}$ $[X_{102}, X_{103}] = -X_{102}$	$A_{3,4}(E(1,1))$
$I_{11} = I_1^{\frac{1}{3}}I_3$	$X_{111} = G_1, X_{112} = G_3$ $X_{113} = \frac{2}{3}G_6$	$[X_{111}, X_{113}] = X_{111}$ $[X_{112}, X_{113}] = \frac{1}{3}X_{112}$	$A_{3,5}^{-\frac{1}{3}}$
$I_{12} = I_1^{-\frac{1}{3}}I_2$	$X_{121} = G_1, X_{122} = G_2$ $X_{123} = \frac{2}{3}G_6$	$[X_{121}, X_{123}] = X_{121}$ $[X_{122}, X_{123}] = \frac{1}{3}X_{122}$	$A_{3,5}^{\frac{1}{3}}$
$I_{13} = I_1I_3 - \frac{2}{3}I_2^2$	$X_{131} = G_4, X_{132} = G_7$ $X_{133} = -\frac{1}{2}(G_6 - \frac{1}{2}G_8)$	$[X_{131}, X_{133}] = X_{131}$ $[X_{132}, X_{133}] = \frac{1}{2}X_{133}$	$A_{3,5}^{\frac{1}{2}}$
$I_{14} = I_1I_2I_3 - \frac{4}{3}I_2^3 - I_1^2I_4$	$X_{141} = G_6 - \frac{1}{2}G_8, X_{142} = G_7$	$[X_{141}, X_{142}] = X_{142}$	$2A_1$

$$K_2 = I_1^{-\frac{a+2}{a}} I_3 \quad (44)$$

$$K_3 = I_1^{-\frac{a+1}{a}} I_2. \quad (45)$$

In the cases of the four forms of the self-similarity symmetry found in the algebras we find the following sets of integrals

$$\begin{aligned} X_{14}: \quad J_1 &= I_1 \\ J_2 &= I_4^{-\frac{1}{3}} I_2 \\ J_3 &= I_4^{-\frac{2}{3}} I_3 \end{aligned}$$

$$\begin{aligned} X_{24}: \quad J_1 &= I_4^{\frac{1}{2}} I_1 \\ J_2 &= I_2 \\ J_3 &= I_4^{-\frac{1}{2}} I_3 \end{aligned}$$

(46)

$$\begin{aligned} X_{34}: \quad J_1 &= I_4^2 I_1 \\ J_2 &= I_4 I_2 \\ J_3 &= I_4 I_3 \end{aligned}$$

$$\begin{aligned} X_{44}: \quad J_1 &= I_4 \\ J_2 &= I_1^{-\frac{2}{3}} I_2 \\ J_3 &= I_1^{-\frac{1}{3}} I_3. \end{aligned}$$

The ratio integrals all have the same algebras of their symmetries. There are three symmetries with the algebra $A_{3,3}$ which represents dilatations and translations in the plane. Two of the symmetries are the solution symmetries of subscript not that of the two integrals in the ratio and the third is the expected homogeneity symmetry, G_8 . We note that this algebra is the same as for the ratio integral of (17) although that algebra used nonCartan symmetries. The three ratio integrals of (18) are different. Although they are three-dimensional, the algebras are $A_{3,4}$ (the algebra of the pseudo-Euclidean group $E(1,1)$) for the ratios I_1/I_2 and I_3/I_2 . For I_1/I_3 it is $A_{3,8}$ (or $sl(2, R)$). These algebras involved one truly contact symmetry. Given the conflicting information from the three equations as far as the algebras of their ratio integrals are concerned it is not immediately evident what the general situation is. However, if we consider just the Cartan symmetries, we find the following pattern for the algebras of the ratio integrals. If we denote a ratio integral by

R_{ij}^k , where i and j are the labels of the two linear integrals comprising the ratio integral and k is the order of the equation, we have

$$\begin{aligned} R_{12}^2 : \{G\} &= \{y \frac{\partial}{\partial y}\} \\ R_{ij}^3 : \{G\} &= \{\frac{\partial}{\partial y}\} + \{s_m \frac{\partial}{\partial y}; m \neq i, j\} \\ R_{ij}^4 : \{G\} &= \{\frac{\partial}{\partial y}\} + \{s_m \frac{\partial}{\partial y}, s_n \frac{\partial}{\partial y}; m, n \neq i, j\}, \end{aligned}$$

where s_m and s_n are solutions of (18) or (23). Thus we are led to

Proposition 2: *The number of symmetries associated with the ratio of any two linear first integrals of an n th order scalar ordinary differential equation is $n - 1$, where $n \geq 4$. The symmetries consist of the homogeneity symmetry $y \frac{\partial}{\partial y}$ and $n - 2$ solution symmetries. The Lie algebra of the symmetries is $(n - 2)A_1 \oplus_s A_1$.*

Proof: Since the linear integrals are each homogeneous of degree one in y , the ratio integral is of degree zero in y and so possesses $y \frac{\partial}{\partial y}$ as a symmetry.

Let the linear integrals be labelled $I_1 - I_n$ and the solution symmetries $G_1 - G_n$ in such a way that

$$G_i^{[n-1]} I_j = 0 \quad i \neq j$$

(as noted above $G_i^{[n-1]} I_i = \pm 1$ depending on the value of i). Then

$$\begin{aligned} G_i^{[n-1]} \left(\frac{I_j}{I_k} \right) &= \frac{(G_i^{[n-1]} I_j) I_k - I_j (G_i^{[n-1]} I_k)}{I_k^2} \\ &= 0 \end{aligned}$$

provided

$$(G_i^{[n-1]} I_j) I_k - I_j (G_i^{[n-1]} I_k) = 0. \quad (47)$$

If $i \neq j, k$, this is true. Hence there are at least $n - 2$ solution symmetries for I_j/I_k . There are also at most $n - 2$ solution symmetries since $j \neq k$ and, if $i = k$ (say), (47) would require I_j to be zero. The symmetries of the ratio integral comprise solution symmetries of the form $s_m(x) \frac{\partial}{\partial y}$, where $s_m(x)$ is one of the fundamental solutions of $y^{(n)} = 0$, and the homogeneity symmetry, $y \frac{\partial}{\partial y}$. The algebra of the solution symmetries admitted by the ratio integral is $(n - 2)A_1$ and, since

$$[s_m(x) \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}] = s_m(x) \frac{\partial}{\partial y},$$

the algebra is $(n - 2)A_1 \oplus_s A_1$. QED

We have now been able to explain the algebras of the linear and ratio integrals in general. It now remains to deal with those associated with the three symmetries of $sl(2, R)$. It is well to recall them. For $y'' = 0$ we have

$$\begin{aligned} G_3 &= \frac{\partial}{\partial x} & J_1 &= y' \\ G_4 &= x \frac{\partial}{\partial x} + \frac{1}{2}y \frac{\partial}{\partial y} & J_2 &= y'(xy' - y) = J_1 J_3 \\ G_5 &= x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} & J_3 &= xy' - y. \end{aligned}$$

J_1 and J_3 are the two linear integrals and have three Cartan symmetries whereas J_2 has only one Cartan symmetry. They are

$$\begin{aligned} J_1 : \quad X_{11} &= \frac{\partial}{\partial y} \\ X_{12} &= \frac{\partial}{\partial x} \\ X_{13} &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \\ \\ J_2 : \quad X_{21} &= x \frac{\partial}{\partial x} + \frac{1}{2}y \frac{\partial}{\partial y} \\ \\ J_3 : \quad X_{31} &= x \frac{\partial}{\partial y} \\ X_{32} &= x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \\ X_{33} &= x \frac{\partial}{\partial x}. \end{aligned}$$

The algebras are $A_{3,3}$ for the symmetries of J_1 and J_3 and A_1 for that of J_2 .

The situation for the third order equation presents an anomaly due to the behaviour of the symmetries of $sl(2, R)$ and the solution integrals under the transformation (35) for which

$$\begin{aligned} G_5 &\longrightarrow G_7 & G_6 &\longrightarrow G_6 & G_7 &\longrightarrow G_5 \\ I_1 &\longrightarrow I_3 & I_2 &\longrightarrow I_2 & I_3 &\longrightarrow I_1. \end{aligned}$$

This does not have much effect on the single integrals, I_1 , I_2 and I_3 , but it does on the quadratic integral, $I_1 I_3 - \frac{1}{2} I_2^2$, which is invariant under the transformation (37). (The reader will appreciate the resemblance to the generalised Kummer-Schwarz equation [16].) This is why this integral possesses the $sl(2, R)$ algebra of its symmetries in contrast to the $A_{3,3}$, $A_{3,4}$ and $A_{3,5}^{\frac{1}{2}}$ found for the integrals of the second and fourth order equations. In fact this labelling is a little misleading. The nonzero Lie Brackets of the three algebras,

$A_{3,3}$, $A_{3,4}$ and $A_{3,5}^{\frac{1}{2}}$, are respectively

$$[G_1, G_3] = G_1 \quad [G_2, G_3] = G_2$$

$$[G_1, G_3] = G_1 \quad [G_2, G_3] = -G_2$$

$$[G_1, G_3] = G_1 \quad [G_2, G_3] = aG_2 \quad 0 < |a| < 1.$$

By expanding the range of definition of a to unit modulus these three algebras are variations of the one form which we denote by $A_{3,5}^a$ with Lie Brackets

$$[G_1, G_3] = G_1 \quad [G_2, G_3] = \alpha G_2 \quad 0 < |\alpha| \leq 1.$$

to avoid confusion with the more standard notation. In this way we obviate a plethora of labels for what is essentially the one algebra. The numerical coefficient other than 1 depends upon the order of the equation and the number of the integral being considered.

We have noted that for $n \geq 4$ the treatment of the $\frac{\partial}{\partial x}$ and $x^2 \frac{\partial}{\partial x} + (n-1)xy \frac{\partial}{\partial y}$ symmetries and their associated integrals is equivalent under the transformation (37). Consequently the only two symmetries with which we must deal are $\frac{\partial}{\partial x}$ and $x \frac{\partial}{\partial x} + \frac{n-1}{2}y \frac{\partial}{\partial y}$. It is not surprising that these should be the most complex of all the symmetries. In fact there is insufficient information in the cases considered and we add the results for $y^{(v)} = 0$ and $y^{(vi)} = 0$. We firstly note some of the properties of the linear integrals as they are defined in (36). For the linear integrals of $y^{(n)} = 0$ we have

$$G_i^{[n-1]} I_j = (-1)^{n+j} \delta_{ij} \quad i, j = 1, n \quad (48)$$

for G_i a solution symmetry,

$$G_{n+1}^{[n-1]} I_j = \begin{cases} I_{j+1} & 1 \leq j \leq n-1 \\ 0 & j = n, \end{cases} \quad (49)$$

where $G_{n+1} = \partial/\partial x$, and

$$G_{n+2}^{[n-1]} I_j = \frac{n+1-2j}{2} I_j, \quad (50)$$

where $G_{n+2} = x\partial/\partial x + (n-1)y/2\partial/\partial y$. This accounts for the two members of $sl(2, R)$ with which we must treat. In addition we have

$$G_{s1}^{[n-1]} I_j = [n-j] I_j, \quad (51)$$

Table 5: Integrals and associated algebras of G_{n+1} for $y^{iv} = 0$, $y^v = 0$ and $y^{vi} = 0$

Equation	Integrals	Algebras
$y^{iv} = 0$	I_4 $I_2 I_4 - \frac{1}{2} I_3^2$ $I_2 I_3 I_4 - \frac{1}{3} I_3^3 - I_1 I_4^2$	$3A_1 \oplus_s A_2$ $A_1 \oplus_s A_2$ A_2
$y^v = 0$	I_5 $I_3 I_5 - \frac{1}{2} I_4^2$ $I_3 I_4 I_5 - \frac{1}{3} I_4^3 - I_2 I_5^2$ $\frac{1}{2} I_3 I_4^2 I_5 - \frac{1}{8} I_4^4 - I_2 I_4 I_5^2 + I_1 I_5^3$	$4A_1 \oplus_s A_2$ $2A_1 \oplus_s A_2$ $A_1 \oplus_s A_2$ A_2
$y^{vi} = 0$	I_6 $I_4 I_6 - \frac{1}{2} I_5^2$ $I_4 I_4 I_6 - \frac{1}{3} I_5^3 - I_3 I_6^2$ $\frac{1}{2} I_4 I_5^2 I_6 - \frac{1}{8} I_5^4 - I_3 I_5 I_6^2 + I_2 I_6^3$ $\frac{1}{6} I_4 I_5^3 I_6 - \frac{1}{30} I_5^5 - I_3 I_5^2 I_6^2 + I_2 I_5 I_6^2 - I_1 I_6^4$	$5A_1 \oplus_s A_2$ $3A_1 \oplus_s A_2$ $2A_1 \oplus_s A_2$ $A_1 \oplus_s A_2$ A_2

where $G_{s1} = x\partial/\partial x + (n-1)y\partial/\partial y$, and

$$G_{s2}^{[n-1]} I_j = [n-j-1] I_j, \quad (52)$$

where $G_{n+2} = x\partial/\partial x + (n-2)y\partial/\partial y$. The necessity for considering G_{s1} and G_{s2} becomes evident shortly.

As we do not have labels for all of the algebras to be considered, we simply label them by the two subalgebras they contain. The first subalgebra consists of solution symmetries and the second of elements of $sl(2, R)$. In Table 5 we list the integrals associated with G_{n+1} for $n = 4, 5$ and 6 and the algebras associated with the integrals. The A_2 algebra consists of G_{n+1} and G_{s1} for the G_{n+1} first integral and G_{n+1} and G_{s2} for the subsequent integrals. We note that all of the integrals have both G_{n+1} and G_{n+2} as symmetries. The number of solution symmetries decreases as the number of linear integrals included in each integral increases. To take the case of $y^{(v)} = 0$, for example, I_5 is invariant under four solution symmetries, the second integral contains three of the linear integrals and these can only have two solution symmetries in common. Hence the number of these drops from four to two. The third integral introduces the linear integral I_2 and the number of solution

Table 6: Integrals and associated algebras of G_{n+2} for $y^{iv} = 0$, $y^v = 0$ and $y^{vi} = 0$

Equation	Integrals	Algebras
$y^{iv} = 0$	$I_1 I_4$	$2A_1 \oplus_s G_6(A_{3,4})$
	$I_2 I_4^{\frac{1}{3}}$	$2A_1 \oplus_s G_6(A_{3,5}^{\frac{1}{3}})$
	$I_3 I_4^{-\frac{1}{3}}$	$2A_1 \oplus_s G_6(A_{3,5}^{\frac{1}{3}})$
$y^v = 0$	$I_1 I_5$	$3A_1 \oplus_s G_7$
	$I_2 I_5^{\frac{1}{2}}$	$3A_1 \oplus_s G_7$
	I_3	$4A_1 \oplus_s G_7$
	$I_4 I_5^{-\frac{1}{2}}$	$3A_1 \oplus_s G_7$
$y^{vi} = 0$	$I_1 I_6$	$4A_1 \oplus_s G_8$
	$I_2 I_6^{\frac{3}{5}}$	$4A_1 \oplus_s G_8$
	$I_3 I_6^{\frac{1}{5}}$	$4A_1 \oplus_s G_8$
	$I_4 I_6^{-\frac{1}{5}}$	$4A_1 \oplus_s G_8$
	$I_5 I_6^{-\frac{3}{5}}$	$4A_1 \oplus_s G_8$

symmetries is reduced to one. This single symmetry is lost in the fourth integral when I_1 is introduced.

The pattern for the algebraic structures of the integrals associated with G_{n+1} is clear from Table 5. It is also evident that, as the order of the equation increases by one, an additional integral is added to the list for the previous equation and that the subscripts for the existing ones are increased by one. Hence the problem of determining the integrals is reduced to finding a homogeneous polynomial of degree $n - 1$ in $I_1 - I_n$ which is invariant under the action of $G_{n+1}^{[n-1]}$.

We turn now to the self-similar symmetry, G_{n+2} . The integrals and the algebras of their symmetries are listed in Table 6, also for the three equations $y^{iv} = 0$, $y^v = 0$ and $y^{vi} = 0$. Recalling that in the case of $y'' = 0$ the integral was $I_1 I_2$ with the single Cartan symmetry $G_4 = x\partial/\partial x + \frac{1}{2}y\partial/\partial y$ and for $y''' = 0$ the integral was I_2 with the anomalous $A_{3,4}$ algebra the pattern for the self-similar symmetry is clear to see. We have

Proposition 3: For

$$y^{(n)} = 0 \quad n \geq 4 \quad (53)$$

there are $n - 1$ integrals associated with the symmetry

$$G = x \frac{\partial}{\partial x} + \frac{n-1}{2} y \frac{\partial}{\partial y} \quad (54)$$

of the form

$$J_i = I_i I_n^{(n+1-2i)/(n-1)}, \quad i = 1, \dots, n-1. \quad (55)$$

Proof: We first confirm that (54) is a symmetry of (53). The n th extension of (54) is (using (4)),

$$G^{[n]} = x \frac{\partial}{\partial x} + \sum_{i=0}^n \left(\frac{n-1}{2} - i \right) y^{(i)} \frac{\partial}{\partial y^{(i)}}. \quad (56)$$

Operating on (53) with (56) gives

$$\left(\frac{n-1}{2} - n \right) y^{(n)} = 0, \quad (57)$$

which, given (53) is identically satisfied.

For the determination of the first integrals associated with (54) we require the $(n-1)$ th extension of G , viz.

$$G^{[n-1]} = x \frac{\partial}{\partial x} + \sum_{i=0}^{n-1} \left(\frac{n-1}{2} - i \right) y^{(i)} \frac{\partial}{\partial y^{(i)}}. \quad (58)$$

If we assume the form

$$I = f(x, y, y', \dots, y^{(n-1)}) \quad (59)$$

for the first integral, the associated Lagrange's system of

$$G^{[n-1]}I = 0 \quad (60)$$

is

$$\frac{dx}{x} = \frac{dy}{(n-1)y/2} = \dots = \frac{dy^{(i)}}{((n-1)/2 - i)y^{(i)}} = \dots = \frac{dy^{(n-1)}}{-(n-1)y^{(n-1)}/2}. \quad (61)$$

The first set of (n) characteristics are (taking combinations of the first and i th terms in (61))

$$\begin{aligned} u_1 &= x^{(1-n)/2} y \\ u_2 &= x^{(3-n)/2} y' \\ &\vdots \\ u_i &= x^{(2i-n-1)/2} y^{(i-1)} \end{aligned}$$

$$\begin{aligned}
u_{i+1} &= x^{(2i-n+1)/2} y^{(i)} \\
&\vdots \\
u_{n-1} &= x^{(n-3)/2} y^{(n-2)} \\
u_n &= x^{(n-1)/2} y^{(n-1)}.
\end{aligned} \tag{62}$$

The first integral (59) now has the form

$$I = g(u_i), \quad i = 1, \dots, n \tag{63}$$

where the u_i are given in (62).

The final requirement

$$\left. \frac{dI}{dx} \right|_{y^{(n)}=0} = 0 \tag{64}$$

gives the linear first order partial differential equation

$$\begin{aligned}
&\left[\left(\frac{1-n}{2} \right) x^{(1-n)/2-1} y + x^{(1-n)/2} y' \right] \frac{\partial f}{\partial u_1} + \left[\left(\frac{3-n}{2} \right) x^{(3-n)/2-1} y' \right. \\
&+ \left. x^{(3-n)/2} y'' \right] \frac{\partial f}{\partial u_2} + \dots + \left[\left(\frac{2i-n-1}{2} \right) x^{(2i-n-1)/2-1} y^{(i-1)} + x^{(2i-n-1)/2} y^{(i)} \right] \frac{\partial f}{\partial u_i} \\
&+ \left[\left(\frac{2i-n+1}{2} \right) x^{(2i-n+1)/2-1} y^{(i)} + x^{(2i-n+1)/2} y^{(i+1)} \right] \frac{\partial f}{\partial u_{i+1}} + \dots \\
&+ \left[\left(\frac{n-3}{2} \right) x^{(n-3)/2-1} y^{(n-2)} + x^{(n-3)/2} y^{(n-1)} \right] \frac{\partial f}{\partial u_{n-1}} \\
&+ \left[\left(\frac{n-1}{2} \right) x^{(n-1)/2-i} y^{(n-1)} \right] \frac{\partial f}{\partial u_n} = 0,
\end{aligned} \tag{65}$$

where we have substituted for $y^{(n)}$ from (53). Multiplying (65) by x and using (62) we obtain

$$\begin{aligned}
&\left[\left(\frac{1-n}{2} \right) u_1 + u_2 \right] \frac{\partial f}{\partial u_1} + \left[\left(\frac{3-n}{2} \right) u_2 + u_3 \right] \frac{\partial f}{\partial u_2} + \dots \\
&+ \left[\left(\frac{2i-n-1}{2} \right) u_i + u_{i+1} \right] \frac{\partial f}{\partial u_i} + \left[\left(\frac{2i-n+1}{2} \right) u_{i+1} + u_{i+2} \right] \frac{\partial f}{\partial u_{i+1}} + \dots \\
&+ \left[\left(\frac{n-3}{2} \right) u_{n-2} + u_n \right] \frac{\partial f}{\partial u_{n-1}} + \left[\left(\frac{n-1}{2} \right) u_n \right] \frac{\partial f}{\partial u_n} = 0
\end{aligned} \tag{66}$$

with the associated Lagrange's system

$$\begin{aligned}
\frac{du_1}{(1-n)u_1/2 + u_2} &= \frac{du_2}{(3-n)u_2/2 + u_3} = \dots = \frac{du_i}{(2i-n-1)u_i/2 + u_{i+1}} \\
&= \frac{du_{i+1}}{(2i-n+1)u_{i+1}/2 + u_{i+2}} = \dots = \frac{du_{n-1}}{(n-3)u_{n-1}/2 + u_n} = \frac{du_n}{(n-1)u_n}.
\end{aligned} \tag{67}$$

The second set of $(n - 1)$ characteristics are obtained by taking combinations of the i th and final terms in (67). Starting with $i = n - 1$ we have to solve

$$\frac{du_{n-1}}{du_n} = \frac{n-3}{n-1} \frac{u_{n-1}}{u_n} + \frac{2}{n-1}. \quad (68)$$

This linear equation is easily integrated to

$$u_{n-1} u_n^{(3-n)/(n-1)} = u_n^{2/(n-1)} + v_{n-1}, \quad (69)$$

where v_{n-1} is the constant of integration which we take to be the first characteristic

$$v_{n-1} = u_{n-1} u_n^{(3-n)/(n-1)} - u_n^{2/(n-1)}. \quad (70)$$

Substituting for u_i from (62) we have

$$\begin{aligned} v_{n-1} &= x^{(n-3)/2} y^{(n-2)} \left(x^{(n-1)/2} y^{(n-1)} \right)^{(3-n)/(n-1)} - \left(x^{(n-1)/2} y^{(n-1)} \right)^{2/(n-1)} \\ &= (y^{n-1})^{(3-n)/(n-1)} \left(y^{(n-2)} - x y^{(n-1)} \right) \\ &= -I_{n-1} I_n^{(3-n)/(n-1)}, \end{aligned} \quad (71)$$

where we have used (36) to determine the I_i , which is just (55) with $i = n - 1$.

In general, the equation to be solved for the i th characteristic is

$$\frac{du_i}{du_n} = \frac{2i-n-1}{n-1} \frac{u_i}{u_n} + \frac{2}{n-1} \frac{u_{i+1}}{u_n}. \quad (72)$$

We can write u_{i+1} in terms of v_i ($i = i + 1, \dots, n - 1$) and u_n . Thus (72) is always a linear equation in u_i . However, it is not at all obvious as to how the general formula for u_{i+1} can be determined as a repeated substitution needs to be effected. It can be verified, from (67), that (55) holds by using a symbolic manipulation package, for example, Mathematica [17].

For our purposes we will simply demonstrate that (54) is a symmetry of (55). To this end we operate on (55) with the $(n - 1)$ th extension of (54), viz.

$$\begin{aligned} G^{[n-1]} J_i &= G^{[n-1]} \left(I_i I_n^{(n+1-2i)/(n-1)} \right) \\ &= \left(G^{[n-1]} I_i \right) I_n^{(n+1-2i)/(n-1)} + \frac{n+1-2i}{n-1} I_i I_n^{(2-2i)/(n-1)} \left(G^{[n-1]} I_n \right) \\ &= I_n^{(2-2i)/(n-1)} \left[\left(G^{[n-1]} I_i \right) I_n + \frac{n+1-2i}{n-1} I_i \left(G^{[n-1]} I_n \right) \right] \\ &= \left(y^{(n-1)} \right)^{(2-2i)/(n-1)} \left\{ \sum_{k=0}^{n-i} \frac{(-1)^k}{(n-i-k)!} \left[(n-i-k) x^{(n-i-k)} y^{(n-1-k)} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2k - n + 1}{2} \right) x^{(n-i-k)} y^{(n-1-k)} \Big] \\
& + \left(\frac{n + 1 - 2i}{n - 1} \right) \sum_{k=0}^{n-1} \frac{(-1)^k}{(n - i - k)!} x^{(n-i-k)} \left(\frac{1 - n}{2} \right) y^{(n-1-k)} \Big\} \\
= & 0.
\end{aligned} \tag{73}$$

QED

Each of the integrals, J_i has $n - 1$ symmetries with the 'algebra' $(n - 2)A_1 \oplus_s G_{n+2}$ unless n is an odd integer in which case $J_{(n+1)/2}$ has the 'algebra' $(n - 1)A_1 \oplus_s G_{n+2}$. The algebraic properties follow directly from the symmetries of the constituent linear integrals. The solution symmetries are those apart from the two (one if n is odd) associated with the two (one) integrals in the expression for J_i . We note that the structure given for the integrals is not unique. For example we could equally use $I_4 I_1$, $I_3 I_1^{1/3}$ and $I_2 I_1^{-1/3}$ for $y^{(iv)} = 0$.

6 Conclusion

In this paper we have treated at length the integrals and the algebras of the symmetries associated with them in the case of scalar ordinary differential equations of maximal symmetry. The pattern for the general equation $y^{(n)} = 0$ has been established with the exception of a formula for the integral associated with the symmetry $\partial/\partial x$ of homogeneous degree $n - 1$. To elaborate this formula does not appear to be a feasible proposition. We have seen that the cases $n = 2$ and $n = 3$ are anomalous and that the pattern is established at $n = 4$ when the only symmetries are of Cartan form. However, there is still a distinction between equations of even and odd degree of a number theoretic origin. In the context of linear equations this problem has, in a sense, been the easier one since equations of maximal symmetry are equivalent to $y^{(n)} = 0$ and so the solution symmetries are trivial to determine. This is not the case with linear equations of lower symmetry, particularly in the case of the linear equation of least symmetry. That one had to contend with the additional complications of the $s\ell(2; R)$ -subalgebra did compensate for the ease of solution of the equations.

Acknowledgements

GPF thanks the Research Committee of the University of the Aegean for its continuing support. KSG and PGLL thank the Foundation for Research Development of South Africa and the University of Natal for their continuing support.

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