

COHERENT STRUCTURES AND SYMMETRY PROPERTIES IN
NONLINEAR MODELS USED IN THEORETICAL PHYSICS

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I hereby certify that this is my original work and has not been submitted before for any other degree.

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ABSTRACT

This thesis is devoted to two aspects of nonlinear PDEs which are fundamental for the understanding of the order and coherence observed in the underlying physical systems. These are symmetry properties and soliton solutions. We analyse these fundamental aspects for a number of models arising in various branches of theoretical physics and applied mathematics.

We start with a fluid model of a plasma in the case of a general polytropic process. We propose a method of the analysis of unmagnetized travelling structures, alternative to the conventional formalism of Sagdeev's pseudopotential. This method is then utilized to obtain the existence domain for compressive solitons and to establish the absence of rarefactive solitons and monotonic double layers in a two-component plasma.

The second class of models under consideration arises in $(2+1)$ -dimensional condensed matter physics. These are the Abelian gauge theories with Chern-Simons term, which are currently considered as candidates for the description of high- T_c superconductivity and fractional quantum Hall effect. The emphasis here is on non-relativistic theories. The standard model of a self-gravitating gas of nonrelativistic bosons coupled to the Chern-Simons gauge field is capable of describing asymptotically vanishing field configurations, such as lump-like solitons. We formulate an alternative model, which describes systems of repulsive particles with a background electric charge and allows to incorporate asymptotically nonvanishing configurations, such as condensate and its topological excitations. We demonstrate the absence of

the condensate state in the standard nonrelativistic gauge theory and relate this fact to the inadequate Lagrangian formulation of its nongauged precursor. Using an appropriate modification of this Lagrangian as a basis for the gauge theory naturally leads to the new model. Reformulating it as a constrained Hamiltonian system allows us to find two self-duality limits and construct a large variety of self-dual solutions. We demonstrate the equivalence of the model with the background charge and the standard model in the external magnetic field. Finally we discuss nontopological bubble solutions in Chern–Simons–Maxwell theories and demonstrate their absence in nonrelativistic theories.

Finally, we consider a model of a nonhomogeneous nonlinear string. We continue the group theoretical classification of the string equations initiated by Ibragimov *et al.* and present their preliminary group classification with respect to a countable-dimensional subalgebra of their equivalence algebra. This subalgebra is an extension of the 10-dimensional subalgebra considered by Ibragimov *et al.* Our main result here is a table of non-equivalent equations possessing an additional symmetry.

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CHAPTER I

INTRODUCTION

A large number of physical models gives rise to nonlinear partial differential equations. For a long time, however, it was accepted that for nearly all practical purposes the corresponding linear models could be used. Typically one would assume that nonlinear effects can be neglected for small variations of physical variables. The resultant linearized equations then can be solved using functional-analytic and operator methods. However, in the middle of this century it was realized that linear equations often give inadequate and even incorrect description of the inherently nonlinear physical reality. The systematic analysis of nonlinear PDEs and the explosive growth of what nowadays is known as nonlinear science began in the late sixties with the discovery of amazingly stable localized structures, solitons. At approximately the same time there was a revival of interest in Lie symmetries of nonlinear differential equations and their group-theoretical analysis.

The role of solitons and Lie symmetries in the analysis of nonlinear models cannot be overestimated. In dynamical models, for instance, solitons are indispensable for the determination of nonlinear evolution. More precisely in completely integrable systems every localised initial condition breaks up asymptotically into a set of solitons, propagating with constant velocities, accompanied by the background radiation that typically dies off as $1/\sqrt{t}$. Since collisions of solitons in integrable systems are elastic, the number of solitons remains constant. Hence the evolution of any initial data in integrable systems reduces asymptotically to soliton dynamics. The existence of stable solitons in a nonintegrable system is even more significant. In this case collisions of solitons are no longer elastic. They produce the background radiation that can be

absorbed by other solitons. This usually leads to the loss of “mass” by some solitons in the system. The implication of this behaviour for systems in a bounded region, where solitons are confined to keep on colliding, is that after some transient period there remains only one stable soliton. Thus solitons in nonintegrable systems act as statistical attractors (even in the absence of dissipation) [58].

Another type of solitons, the topological solitons, is especially important for static models. These localized configurations are characterized by a quantised topological charge, whose conservation is a consequence of the topology of the underlying space. Topological solitons cannot be smoothly deformed into the background configuration and hence play a role of topological defects.

On the whole solitons represent islands of regular and coherent behaviour in the midst of surrounding chaos and serve as footholds for the nonlinear analysis. The symmetries play a conceptually similar role. On the one hand the presence of a symmetry is a manifestation of a conservation law. On the other hand, since equations admitting similar algebras of symmetries can be transformed into each other by a change of variables, symmetries serve as a basis for the classification of nonlinear equations. Group analysis can reveal a hidden symmetry structure of a nonlinear equation thus providing a way for its possible simplification, identification of particular solutions and, sometimes, complete integration.

A remarkable feature of Lie symmetries and soliton solutions is that both of them have a deep physical content furnishing a mathematical formulation of major physical concepts. The Lie symmetries provide a natural mathematical description of physical concepts of invariance. Solitons, as localized nondispersive packets of energy, can be identified as real objects of particle and condensed matter physics.

A. Solitons and soliton-bearing systems

The concept of soliton takes its origin in the discovery of solitary wave by Scott–Russell over a century ago [52]. He experimentally observed a localised water wave propagating without a change of shape and velocity. The explanation of this phenomenon was given by Korteweg and deVries in 1895 [36], who introduced an equation describing shallow water waves:

$$\phi_t + \alpha\phi\phi_x + \phi_{xxx} = 0, \quad \alpha = \text{const.}$$

A solitary wave solution of this equation can be easily found. This is a solution which is stationary in the coordinate system moving with velocity u :

$$\phi(x, t) = \phi_{\text{sw}}(\xi), \quad \xi = x - ut. \quad (1.1)$$

The second defining property of a solitary wave is that it is an essentially localised solution. The rate of its approach to the constant asymptotic state at spatial infinity is exponential. Note that the asymptotic values can be different at different infinities. This is characteristic for topological solitons, such as kinks in a one-dimensional space, vortices in two dimensions and monopoles in 3D.

Although solitary waves were discovered over a hundred years ago, their significance was recognized only in the early seventies. Before that it was generally accepted that solitary waves are rather exceptional and tricky particular solutions that may occur only under very special initial conditions, just as it has been thought that the nonlinearity responsible for their formation will lead to their destruction after they collide. The process towards the real understanding of solitary waves and recognition of their importance started with numerical experiments by Kruskal and Zabusky [57]. These numerical experiments demonstrated that solitary waves can emerge after a

collision unchanged. Solitary waves that survive a collision have been called solitons.

A major breakthrough in the understanding of the soliton dynamics came with the discovery of the inverse scattering transform method (IST) [22, 37]. The IST method rigorously reduces the solution of a nonlinear problem to a sequence of linear computations. Many other equations amenable to inverse scattering have since then been identified [51]. Using IST one can prove that not only do solitary waves of these equations emerge unchanged after collision, but that any initial condition breaks up asymptotically into a set of solitons propagating with constant velocities. Equations that can be solved by the inverse scattering are known as *completely integrable*. Unfortunately, like most problems that can be solved analytically, completely integrable equations are an exception, even though they model a wide range of physical phenomena. As a rule systems arising in mathematical physics, especially higher dimensional ones, are nonintegrable.

One of the major sources of soliton-bearing equations is the physics of fluids and plasma physics in particular. Another large supplier of soliton equations is particle physics. Here solitons arise as solutions of classical field equations and serve as a basis for the construction of quantum objects. Solitons also play an important role in condensed matter physics. In this thesis we investigate Abelian gauge theories in (2+1) dimensions which are used to describe quasiplanar systems of condensed matter. On the one hand these theories provide a phenomenological description of normal and high- T_c superconductivity, and systems exhibiting the fractional quantum Hall effect. On the other hand they serve as a basis for quantum field theories that provide a second-quantized description of these models.

The hydrodynamic model of plasma is the first system with which we are concerned. As a medium with a large number of oscillating modes, plasma supports a rich variety of soliton-like nonlinear structures. These are obtained both theoretically

and experimentally. The most general description of plasma can be given in terms of the distribution function governed by the Boltzmann equation. Such description constitutes the foundations of the kinetic theory. The analysis of the full kinetic model is, however, extremely difficult and one has to invoke various approximations. The most important one is the hydrodynamic model. In the hydrodynamic description plasma is treated as if it were composed of several intermingled nonviscous fluids, corresponding to different kinds of particles. The velocity of any one fluid at any point in space is assumed to be the average velocity characteristic of the particles comprising that fluid at that particular point. The collisionless plasma (no interaction between fluids) is hence governed simply by the equations of multifluid magnetohydrodynamics [17]:

$$\nabla \cdot \mathbf{E} = 4\pi \sum_j n_j q_j \quad (1.2a)$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \quad (1.2b)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.2c)$$

$$c^2 \nabla \times \mathbf{B} = 4\pi \sum_j n_j q_j \mathbf{v}_j + \dot{\mathbf{E}} \quad (1.2d)$$

$$m_j n_j \left[\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla) \mathbf{v}_j \right] = q_j n_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \nabla p_j \quad (1.2e)$$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}_j) = 0 \quad (1.2f)$$

$$p_j = C(m_j n_j)^{\gamma_j}. \quad (1.2g)$$

The first four equations in the system (1.2) are Maxwell's equations relating electromagnetic fields \mathbf{E} , \mathbf{B} to particle densities, n_j , and velocities, v_j , in each of the fluids. Eq. (1.2e) is a momentum equation for the j -th fluid in the electromagnetic field. Eq. (1.2f) is a continuity equation for j -th fluid. Constants q_j and m_j stand for the charge and mass of particles so that $n_j m_j$ and $n_j q_j$ express the mass and charge

densities, respectively. The last equation in the system is a thermodynamic equation of state for a polytropic process. It implies that the heat flow can be neglected. In the case of an n -component plasma (1.2) is a set of $5n + 8$ nonlinear scalar equations for $5n + 6$ unknowns. However, two of the Maxwell equations, eqs. (1.2a) and (1.2c) are superfluous since they can be recovered from eqs. (1.2d) and (1.2b). Thus (1.2) is a self-consistent system of $5n + 6$ scalar PDEs for $5n + 6$ unknowns. The complexity of this system is reflected by the large number of soliton equations it contains as reductions. The KdV, nonlinear Schrödinger, Boussinesq, Kadomtsev-Petviashvili equations, as well as their nonintegrable modifications, are all derived from (1.2) (see e.g. [42] for a review and references). These equations approximate the system (1.2) in a variety of situations. However, the assumption of small amplitude is essential in all of these situations. The study of large-amplitude solutions requires the analysis of the general system (1.2). We study these in the next chapter. More precisely we investigate the existence conditions of planar travelling structures in the absence of the magnetic field.

Another class of models that we are concerned with are models of condensed matter physics, namely those processes that are described by the $U(1)$ gauge theory. The state of a superconductor, for instance, is characterized by the local density of the superconducting component, $\rho(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2$. (Here $\psi(\mathbf{x}, t)$ is a complex wave function.) The interaction with the electromagnetic field is introduced by the minimal coupling of ψ to the corresponding vector potential $A^\alpha(\mathbf{x}, t)$: $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$. This produces Lagrangian $\mathcal{L}(\psi, A^\alpha)$ invariant under local gauge transformations:

$$\begin{aligned}\psi &\rightarrow e^{i\chi(\mathbf{x}, t)} \\ \mathbf{A} &\rightarrow \mathbf{A} + \frac{1}{e} \nabla \chi(\mathbf{x}, t) \\ A^0 &\rightarrow A^0 - \partial_t \chi(\mathbf{x}, t).\end{aligned}$$

Gauge theories of which the Abelian model is a simple example play a very important role in contemporary physics, providing a mathematical foundation for the modern unified theories. These theories are based on more complicated multidimensional nonabelian models. However, the $U(1)$ theories already possess many properties inherent in more general gauge fields. Thus, apart from applications in condensed matter, this class of models has even greater significance as a nice laboratory for studying more complicated gauge fields.

The $U(1)$ model is especially interesting on the plane. The planar case possesses two important features that are absent in (3+1) dimensions. First of all in (3+1) dimensions the most general quadratic kinetic term has a conventional Maxwell form: $-(\mu/4)F_{\alpha\beta}F^{\alpha\beta}$, where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. In (2+1) dimensions the Chern–Simons (CS) term $(\kappa/4)\epsilon^{\gamma\alpha\beta}A_\gamma F_{\alpha\beta}$ provides a possible alternative. The resultant Chern–Simons “electrodynamics” drastically differs from the conventional electrodynamics of Maxwell, as charged particles interacting with Chern–Simons field carry a magnetic flux. The Chern–Simons model was proposed for the description of high- T_c superconductivity and fractional quantum Hall effect. It is currently argued that anyonic quasiparticles, objects exhibiting fractional statistics, might be responsible for these phenomena. Fractional statistics, on the other hand, implies that anyons must carry magnetic flux and hence obey the Chern–Simons electrodynamics [41].

Another very important feature of the planar model is the occurrence of *topological* solutions, which is a consequence of the nontriviality of the homotopy group of mappings of the circle: $S^1 \rightarrow S^1$. Finite energy configurations in this model all fall into a countable set of non-overlapping topological classes, characterised by a conserved topological charge $n \in \mathbf{Z}$. Indeed, let us assume that the corresponding

energy is given by

$$H = \int \left[|\mathbf{D}\psi|^2 + \frac{\mu}{4} F_{\mu\nu}^2 + U(\rho) \right] d^2x.$$

Suppose that $U'(\rho_0) = U(\rho_0) = 0$, where $\rho_0 \neq 0$. The finiteness of the energy requires that $|\psi|^2 \rightarrow \rho_0$ as $x \rightarrow \infty$. The phase of the field at the infinity can however vary and hence the allowed asymptotic states of ψ form a circle $S_1^{(\text{int})}$, defined by $|\psi|^2 = \rho_0$. On the other hand, since the physical space is two-dimensional, its boundary essentially is another circle $S_2^{(\text{phy})}$. Hence the set of boundary conditions on ψ compatible with the finiteness of energy is the set of all nonsingular mappings $S_1^{(\text{phy})} \rightarrow S_1^{(\text{int})}$. These mappings fall into a countable number of homotopy classes characterised by a topological index n . It has a meaning of winding number as it describes the number of times $S_1^{(\text{int})}$ is traversed when $S_1^{(\text{phys})}$ is traversed once. Clearly mappings with different winding numbers can not be continuously deformed into each other. A consequence of this fact is the conservation of the topological index (or topological charge) with time. Since the time evolution is a continuous deformation, a field configuration belonging to some topological sector stays within that sector as time evolves.

Soliton solutions with $n = 0$ and $\rho_0 \neq 0$ are often called solitonic bubbles. These solutions reside in the same topological class as the background field, $\psi = \sqrt{\rho_0}$. The solitons with $n \neq 0$ are topological in the sense that they cannot be continuously deformed into the background configuration. These solutions are usually referred to as vortices and their topological index n as vorticity.

In this thesis we consider Abelian gauge theories in Chapter 2. We formulate a new nonrelativistic Chern-Simons model with vortex solutions and present its detailed analysis.

B. Symmetries and invariance

Rather intuitive concepts of invariance and symmetry have their mathematical foundation in the theory developed by Sophus Lie in the previous century. The continuous groups introduced by Lie (Lie groups) comprise both algebraic and topological structures and naturally describe physical concepts of invariance, such as the homogeneity and isotropy of space and time, dynamical similarity, Galilean and Lorentz invariance. A major part of Lie's work was devoted to the study of symmetry properties of differential equations. The subject of studying differential equations from the Lie-group point of view is nowadays known as the *group analysis of differential equations*. Since many physical models are formulated as systems of differential equations, this is probably one of the most important applications of the Lie group theory.

A concept of the group admitted by a given differential equation is central to the group analysis. We say that a family G of transformations of \mathbf{R}^n :

$$\bar{q} = \phi(q, a), \quad q \in \mathbf{R}^n, \quad a \in \mathbf{R}^r, \quad (1.4)$$

is an (r -parameter) Lie group if

- 1) The composition of any two transformations from G also belongs to G .
- 2) G contains the identity transformation, which corresponds to $a = a_0$.
- 3) For any transformation in G there exists its inverse, also belonging to G .
- 4) $\phi(x, a)$ is an analytical function in some open set of parameters (q, a) containing a_0 .

If $a \in \mathbf{R}$, then the Lie group is *one-parameter*. A one-parametric group can be

completely recovered if we know its *generator*:

$$X = \xi \frac{\partial}{\partial q},$$

where $\xi = \partial\phi/\partial a|_{a=a_0}$. Transformations of the form (1.4) are recovered as solutions of the initial value problem for the Lie equation:

$$\frac{d\phi}{da} = \xi(\phi), \quad \phi(a_0) = q. \quad (1.5)$$

Let us suppose now that q represents a vector of independent (x) and dependent (y) variables of a differential equation:

$$q = (x, y) \quad x \in \mathbf{R}^k, \quad y \in \mathbf{R}^l$$

and the generator of the Lie group of transformations (1.4) is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial y^\alpha}. \quad (1.6)$$

A differential equation describes a surface in the extended space, spanned by x, y and by the derivatives of y with respect to x . The general form of, for instance, a second order equation is

$$F(x^i, y^\alpha, y_i^\alpha, y_{ij}^\alpha) = 0, \quad (1.7)$$

where F is a \mathbf{R}^m -valued function, $y_i^\alpha = \partial y^\alpha / \partial x^i$ and $y_{ij}^\alpha = \partial^2 y^\alpha / \partial x^i \partial x^j$. In order to formulate the invariance condition for a differential equation one needs to know the transformations (or the corresponding generator) induced by the transformations (1.4) on the extended space.

The generator of the transformation that acts on the space spanned by $(x^i, y^\alpha, y_i^\alpha, y_{ij}^\alpha)$ is called a second prolongation or extension, X_2 , and is obtained by the

following prolongation formulas:

$$X_2 = X + \zeta_i^\alpha \frac{\partial}{\partial y_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial y_{ij}^\alpha},$$

where

$$\zeta_i^\alpha = D_i(\eta^\alpha) - y_j^\alpha D_i(\xi^j)$$

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - y_{ik}^\alpha D_j(\xi^k).$$

Here

$$D_i = \frac{\partial}{\partial x^i} + y_i^\alpha \frac{\partial}{\partial y^\alpha} + y_{ij}^\alpha \frac{\partial}{\partial y_j^\alpha} + \dots$$

is the total derivative. We now say that the group G with the generator (1.6) is *admitted* by the equation (1.7) if

$$(X_2 F)_{F=0} = 0. \tag{1.8}$$

For an n th order equation, the n th prolongation of X , X_n , should be used instead of X_2 . It can be obtained in a similar fashion.

The group admitted by a differential equation acts on the solutions of the DE and hence endows the set of solutions with an algebraic structure. This can be used, for instance, to describe the general properties of solutions, to find solutions that possess certain invariance properties, and to generate new solutions from the known ones.

An important problem in the group analysis is classification of differential equations according to their admitted algebras. Equations appearing in mathematical physics are often not rigidly defined, but contain some arbitrary functions and parameters, i.e. they come as members of a certain family. In this case group analysis can be used to divide members of a family into classes, nonequivalent with respect to

a change of variables, and find equations with certain desirable features, such as the simplest structure or the widest admitted group of symmetries.

Symmetries and invariance are dealt with in all chapters of this thesis. In the next chapter we study *self-similar* solutions of the hydrodynamic equations of plasma physics. In Chapter 3 we discuss *relativistic* (i.e. invariant under Lorentz transformations) and *nonrelativistic* (admitting the Galilei group) models. We use gauge and rotational symmetries to obtain a simpler form of solution. These symmetries together with translations are also used in the context of Noether's theorem in order to define correctly momenta of a vortex. Chapter 4 is completely devoted to the Lie symmetries of differential equations. We use methods of the group analysis to perform a group classification of equations describing a nonlinear nonhomogeneous vibrating string.

CHAPTER II

ARBITRARY AMPLITUDE ELECTROSTATIC TRAVELLING WAVES IN A
PLASMA

The study of arbitrary amplitude travelling structures (such as solitons, double layers etc.) in plasmas has been a subject of considerable interest in the recent years. The fact that the amplitude can be arbitrarily large makes the KdV equation inapplicable and one should use the more general fluid model. Starting from the paper [49] the travelling structure solutions of fluid equations were usually analysed using the formalism of the Sagdeev pseudopotential. It was shown [55] (in the approximation of Boltzmann electrons and cold ions) that plasmas consisting of single ion and electron components do not admit rarefactive solitons, although the question of their existence in a more general case remained open. Recently several publications appeared where travelling structures are studied in multi-component plasmas [12, 3, 4]. A question of special interest has been the domain of existence of such solutions. Numerical investigation [5, 48] has shown that they are possible only in rather restricted regions of the parameter space of the model.

It was found both theoretically [13, 38, 50, 56] and experimentally [40, 18, 19] that particle distribution within a monotonic transition layer (which is a kink-type soliton) can be classified into “*trapped*” and “*free*” groups. This implies that in a fluid model of a two-component plasma kinks do not occur, although a rigorous mathematical proof of this fact has not been given.

In this chapter we study the existence conditions for travelling structures without using the conventional method of the Sagdeev pseudopotential. In the next section we introduce the fluid model of a plasma. In sec. B we perform a partial integration

of the model using a travelling wave *ansatz* (1.1). This results in two constraints on the density configurations. The first one defines a line and the second defines a region in the density space. In sec. C we restrict ourselves to a two-component plasma and formulate a necessary and sufficient condition for the existence of solitons and kinks in terms of the mutual geometrical locations of the constraints on the density plane. As a by-product of this consideration we establish the non-existence of rarefactive solitons. In sec. D we show that a two-component plasma with the same thermodynamic properties of the components cannot support double layers. The domain of existence of compressive solitons is found in sec. E. Finally in sec. F some concluding remarks are made.

A. The model

The plasma is assumed to be infinite, homogeneous, collisionless, unmagnetised and quasineutral. The system of plasma fluid equations (1.2) for one-dimensional configurations then becomes

$$\frac{\partial n_j}{\partial t} + \frac{\partial(n_j v_j)}{\partial x} = 0, \quad (2.1a)$$

$$m_j n_j \left(\frac{\partial v_j}{\partial t} + v_j \frac{\partial v_j}{\partial x} \right) + \frac{\partial p_j}{\partial x} = -e_j n_j \frac{\partial \phi}{\partial x}, \quad (2.1b)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi \sum_j e_j n_j. \quad (2.1c)$$

Here n_j, m_j, e_j, v_j, p_j are the density, mass, charge, velocity and pressure of the j -th species respectively. To obtain a closed system one should add an equation of state.

We assume a general polytropic process

$$\frac{p_j}{n_j^{\gamma_j}} = \text{const}, \quad (2.2)$$

where γ_j is the polytropic index.

Also we assume the following boundary conditions:

$$\phi, \frac{\partial \phi}{\partial x}, v_j \rightarrow 0; \quad n_j \rightarrow n_{j0}; \quad p_j \rightarrow n_{j0} T_j, \quad (2.3)$$

as $x \rightarrow -\infty$. Here T_j is the background temperature and the unperturbed density, n_{j0} , satisfies the quasineutrality condition:

$$\sum_j e_j n_{j0} = 0. \quad (2.4)$$

B. Arbitrary amplitude solitary waves

We look for solitary wave solutions propagating with a constant velocity u . It is advantageous to pass into a moving frame by means of the transformation $\xi = x - ut$.

Equation (2.1a) can then be readily integrated to yield

$$v_j = u \left(1 - \frac{n_{j0}}{n_j} \right).$$

Substituting this into eq. (2.1b) we obtain

$$\left[\gamma_j T_j \left(\frac{n_j}{n_{j0}} \right)^{\gamma_j - 1} - m_j u^2 \left(\frac{n_{j0}}{n_j} \right)^2 \right] \frac{\partial n_j}{\partial \xi} = -e_j n_j \frac{\partial \phi}{\partial \xi}. \quad (2.5)$$

Equation (2.5) can be integrated to give

$$T_j \ln x_j + \frac{m_j u^2}{2} (x_j^{-2} - 1) = -e_j \phi, \quad (2.6a)$$

for an isothermal process ($\gamma_j = 1$) or

$$\frac{\gamma_j T_j}{\gamma_j - 1} (x_j^{\gamma_j - 1} - 1) + \frac{m_j u^2}{2} (x_j^{-2} - 1) = -e_j \phi, \quad (2.6b)$$

for an anisothermal process ($\gamma_j \neq 1$), where $x_j = n_j/n_{j0}$ are normalised densities.

Eliminating ϕ from eqs. (2.6) we obtain

$$\begin{aligned} & \sum_{\gamma_j \neq 1} \left[\frac{\gamma_j T_j}{\gamma_j - 1} (x_j^{\gamma_j - 1} - 1) + \frac{m_j u^2}{2} (x_j^{-2} - 1) \right] n_{j0} \\ & + \sum_{\gamma_j = 1} \left[T_j \ln x_j + \frac{m_j u^2}{2} (x_j^{-2} - 1) \right] n_{j0} = 0. \end{aligned} \quad (2.7)$$

This equation defines a curve in a space of normalised densities, x_j , which we will further refer to as a trajectory of the solution. We denote the LHS of eq. (2.7) by $\mathcal{T}(\mathbf{x})$.

On the other hand, summing eqs. (2.5) over all types of species, using the Poisson equation (2.1c) and integrating, we obtain

$$\sum_j \left[T_j (x_j^{\gamma_j} - 1) + m_j u^2 (x_j^{-1} - 1) \right] n_{j0} = \frac{1}{8\pi} \left(\frac{\partial \phi}{\partial \xi} \right)^2. \quad (2.8)$$

We denote the LHS of eq. (2.8) as $\mathcal{B}(\mathbf{x})$. Since $(\partial \phi / \partial \xi)^2 \geq 0$, configurations with the densities satisfying $\mathcal{B} < 0$ are not allowed. The boundary of this “no dynamics” region is defined by

$$\sum_j \left[T_j (x_j^{\gamma_j} - 1) + m_j u^2 (x_j^{-1} - 1) \right] n_{j0} = 0. \quad (2.9)$$

C. Two-component plasma

We restrict ourselves to the plasma consisting of two kinds of species, $j = 1, 2$. In this case both the trajectory and the boundary are curves in the plane of the densities (x_1, x_2) . Let k and l be fixed indices, $k = 1$ or 2 , $l = 3 - k$. The trajectory and the boundary can be represented as solutions of autonomous differential equations

$$\frac{dx_{tl}}{dx_k} = \frac{e_l}{e_k} \left[\frac{\gamma_k T_k x_k^{\gamma_k - 2} - m_k u^2 x_k^{-3}}{\gamma_l T_l x_{tl}^{\gamma_l - 2} - m_l u^2 x_{tl}^{-3}} \right] \quad (2.10)$$

and

$$\frac{dx_{bl}}{dx_k} = \frac{e_l}{e_k} \left[\frac{\gamma_k T_k x_k^{\gamma_k-1} - \dot{m}_k u^2 x_k^{-2}}{\gamma_l T_l x_{bl}^{\gamma_l-1} - m_l u^2 x_{bl}^{-2}} \right] \quad (2.11)$$

respectively, with initial condition $x_{tl}(0) = x_{bl}(0) = 1$.

The initial point $x_1 = x_2 = 1$ belongs both to the trajectory and the boundary.

This is also a point at which they have a common tangent,

$$\left. \frac{dx_{tl}}{dx_k} \right|_{x_k=1} = \left. \frac{dx_{bl}}{dx_k} \right|_{x_k=1} = \frac{e_l}{e_k} \left[\frac{\gamma_k T_k - m_k u^2}{\gamma_l T_l - m_l u^2} \right]. \quad (2.12)$$

We denote $[m_j u^2 / (\gamma_j T_j)]^{1/(\gamma_j+1)}$ by α_j . For $0 < \alpha_{k,l} < +\infty$ the curves defined by eq. (2.7) and (2.9) are closed. The extrema of x_j are achieved at $x_{3-j} = \alpha_{3-j}$, since here the derivatives dx_j/dx_{3-j} vanish (fig. 1). According to eq. (2.5) the derivatives $dx_j/d\xi$ become infinite when $x_j = \alpha_j$. Therefore, the solution lives only on the part of the trajectory falling into the quadrant bounded by the lines $x_j = \alpha_j$ and containing the initial point (permissible quadrant).

We shall now estimate the “escape time” from the points of the boundary. By the escape time we mean the range of ξ needed for the finite change of densities x_j . Expression of $\partial\phi/\partial\xi$ from eq. (2.8) and substitution of it into eq. (2.5) results in

$$\left(\gamma_j T_j x_j^{\gamma_j-2} - m_j u^2 x_j^{-3} \right) \frac{dx_j}{d\xi} = \mp e_j \sqrt{8\pi \sum_i n_{i0} \left[T_i (x_i^{\gamma_i} - 1) + m_i u^2 (x_i^{-1} - 1) \right]}. \quad (2.13)$$

Let (κ_1, κ_2) be a point of the boundary and $\kappa_j \neq \alpha_j$. Introducing new variables $\nu_j = x_j - \kappa_j$ and taking into account that

$$\nu_{3-j} = \nu_j \frac{e_{3-j}}{e_j} \left[\frac{\gamma_j T_j \kappa_j^{\gamma_j-2} - m_j u^2 \kappa_j^{-3}}{\gamma_{3-j} T_{3-j} \kappa_{3-j}^{\gamma_{3-j}-2} - m_{3-j} u^2 \kappa_{3-j}^{-3}} \right] + O(\nu_j^2),$$

due to eq. (2.10), we can write eqs. (2.13) as

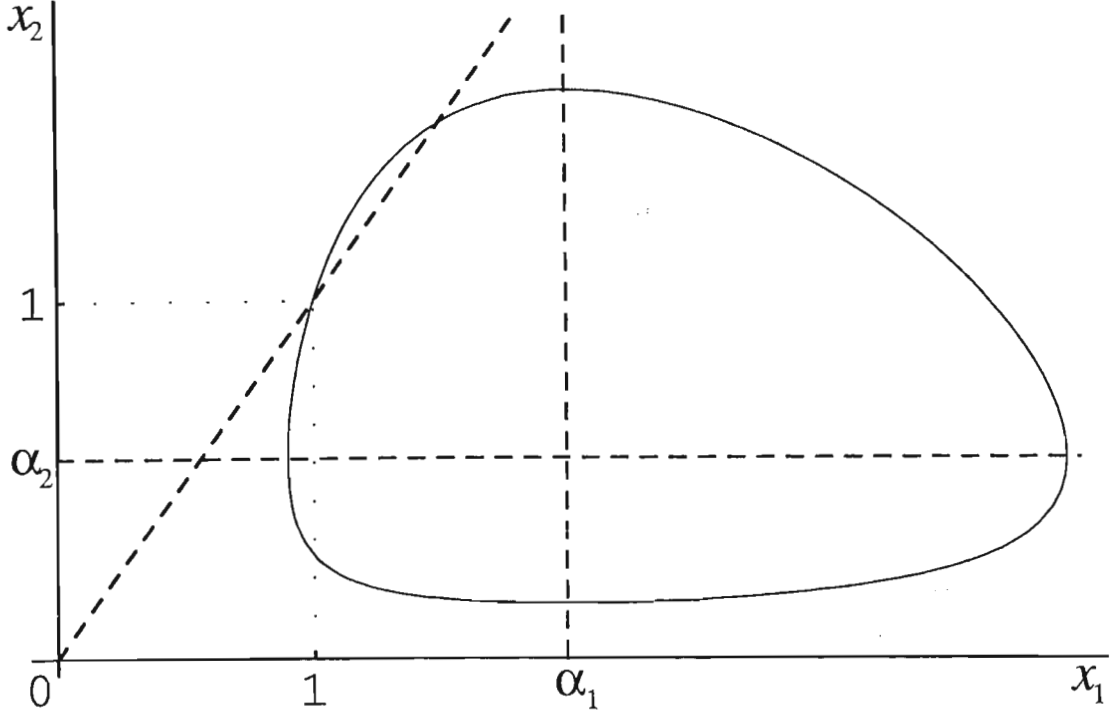


Fig. 1. A typical form of the “boundary” on the x_1, x_2 plane in the case $\alpha_2 < 1, \alpha_1 > 1$.

$$\begin{aligned} (\gamma_j T_j \kappa_j^{\gamma_j - 2} - m_j u^2 \kappa_j^{-3}) \frac{d\nu_j}{d\xi} = \\ \mp e_j \sqrt{8\pi n_{j0} \nu_j (\kappa_{3-j} - \kappa_j) (m_e u^2 \kappa_j^{-3} - T_j \gamma_j \kappa_j^{\gamma_j - 2}) + O(\nu_j^2)}. \end{aligned}$$

We observe that for $\kappa_1 = \kappa_2$

$$\frac{d\nu_j}{d\xi} = \pm a_j \nu_j + O(\nu_j^2), \quad a_j > 0, \quad (2.14)$$

and for $\kappa_1 \neq \kappa_2$

$$\frac{d\nu_j}{d\xi} = \pm b_j \sqrt{|\nu_j|} + O(|\nu_j|^{3/2}). \quad (2.15)$$

Eq. (2.14) implies that ν_j behaves like an exponential function. It vanishes when $\xi \rightarrow \infty$. Hence the trajectory may leave or enter points with $\kappa_1 = \kappa_2$ in an infinite “time”. On the other hand eq. (2.15) yields a finite “time” of enter (escape) for the points with $\kappa_1 \neq \kappa_2$. Assuming $x_1 = x_2$ in eq. (2.9) one can verify that the boundary

and the bisector $x_1 = x_2$ can have at most two common points. The infiniteness of the escape time for the point $(1, 1)$ proves the consistency of the boundary conditions (2.3).

At the infinity the solution can have two different types of behaviour. It can either end at the second point with infinite escape time ($x_1 = x_2 \neq 1$) or it reaches a turning point on the boundary where $x_1 \neq x_2$ (at this point the sign of the RHS of eq. (2.13) changes) and comes back to the point $(1, 1)$. In the first case the solution will have the form of a kink while the second case corresponds to a symmetric (with respect to ξ) lump-like soliton. In both cases the solution satisfying boundary conditions (2.3) exists if and only if the following two conditions hold.

1. The trajectory touches the region $\mathcal{B} < 0$ at the initial point $(1, 1)$ on the outer side. (Escape condition.)
2. The trajectory intersects (touches at $x_1 = x_2 \neq 1$ in the case of a kink) the boundary in the permissible quadrant.

The “escape condition” can be written in terms of second derivatives at the initial point,

$$\left. \frac{d^2 x_{tl}}{dx_k^2} \right|_{x_k=1} < \left. \frac{d^2 x_{bl}}{dx_k^2} \right|_{x_k=1} \quad \text{if } \alpha_l > 1, \quad (2.16a)$$

$$\left. \frac{d^2 x_{tl}}{dx_k^2} \right|_{x_k=1} > \left. \frac{d^2 x_{bl}}{dx_k^2} \right|_{x_k=1} \quad \text{if } \alpha_l < 1. \quad (2.16b)$$

In view of eqs. (2.10), (2.11) and (2.12) these conditions can be reduced to

$$0 < \left. \frac{dx_{tl}}{dx_k} \right|_{x_k=1} = \left. \frac{dx_{bl}}{dx_k} \right|_{x_k=1} < 1 \quad \text{for } \alpha_l > 1, \alpha_k < 1. \quad (2.17)$$

In the case $\alpha_1, \alpha_2 > 1$ the escape condition can never be satisfied while in the case $\alpha_1, \alpha_2 < 1$ it always holds.

Let us suppose that the escape condition is satisfied, i.e in the neighbourhood of the initial point the trajectory is in the region $\mathcal{B} > 0$. The requirement of intersection will then be fulfilled if the trajectory is in the region $\mathcal{B} < 0$ when it reaches one of the lines $\alpha_j = 0$ (boundaries of the permissible quadrant). To show that this condition is also necessary we prove that the trajectory can intersect the boundary in the permissible quadrant only once.

Let

$$f(x_1) = \frac{d(x_{t2} - x_{b2})}{dx_1} = \frac{e_2 \gamma_1 T_1 x_1^{\gamma_1 - 2} - m_1 u^2 x_1^{-3}}{e_1 \gamma_2 T_2 x_{t2}^{\gamma_2 - 2} - m_2 u^2 x_{t2}^{-3}} - \frac{e_2 \gamma_1 T_1 x_1^{\gamma_1 - 1} - m_1 u^2 x_1^{-2}}{e_1 \gamma_2 T_2 x_{b2}^{\gamma_2 - 1} - m_2 u^2 x_{b2}^{-2}}. \quad (2.18)$$

$f(x_1)$ is the derivative of the difference $g(x_1) = x_{t2}(x_1) - x_{b2}(x_2)$ which is a single-valued function of x_1 in the permissible quadrant. Note that $f(1) = 0$. Assuming that escape conditions (2.16a), (2.16b) hold we obtain that

$$\text{sgn } f(1 + \delta) = \text{sgn}(\delta) \text{sgn}(1 - \alpha_2) \quad (2.19)$$

for sufficiently small δ . On the other hand at a common point of the boundary and trajectory where $x_{t2} = x_{b2} = x_2$,

$$f(x_1) = \frac{e_2 \gamma_1 T_1 x_2^3}{e_1 \gamma_2 T_2 x_1^3} \frac{x_1^{\gamma_1 + 1} - \alpha_1^{\gamma_1 + 1}}{x_2^{\gamma_2 + 1} - \alpha_2^{\gamma_2 + 1}} \left(1 - \frac{x_1}{x_2}\right). \quad (2.20)$$

We first show that in the case $\alpha_2, \alpha_1 < 1$ the trajectory and the boundary have no points of intersection in the permissible quadrant which is now defined by $(x_1 > \alpha_1, x_2 > \alpha_2)$. Let (x_1^*, x_2^*) be the point of intersection of the boundary and the trajectory immediately to the right of the initial point, $x_2^* < 1 < x_1^*$. Eq. (2.19) then implies that $f(1 + 0) > 0$, i.e. $g(x_1)$ emerges from zero while increasing and it has to decrease when it reaches its next zero, and so $f(x_1^*) < 0$. Hence according to eq. (2.20)

$x_2^* > x_1^*$, which contradicts the assumption made above. The same argument can be applied if the point of intersection (x_1^*, x_2^*) is assumed to be immediately to the left of the initial point.

Let us now analyse the case $\alpha_2 < 1, \alpha_1 > 1$ (Fig. 1). Again let (x_1^*, x_2^*) be the point of intersection immediately to the right of the initial point, $x_1^* > 1$. Again eq. (2.19) implies that $f(1+0) > 0$ and so $f(x_1^*) < 0$. However, this time eq. (2.20) yields $x_2^* < x_1^*$. Note that condition (2.17) implies that $dx_2/dx_1 > 1$ and hence the bisector $x_1 = x_2$ intersects the boundary between $x_1 = 1$ and $x_1 = x_1^*$. Suppose there exists another point of intersection (x_1^{**}, x_2^{**}) to the right of (x_1^*, x_2^*) , $x_1^{**} > x_1^*$. Then $g(x_1)$ reaches its zero at x_1^{**} while increasing, i.e. $f(x_1^{**}) > 0$, and consequently $x_2^{**} > x_1^{**}$. This means that the line $x_1 = x_2$ should again intersect the boundary, which is impossible.

If (x_1^*, x_2^*) is the point of intersection immediately to the left, $x_1^* < 1$, then, since $f(1-0) < 0$, we have $f(x_1^*) > 0$. Hence $x_2^* > x_1^*$. This, however, contradicts the condition $dx_2/dx_1 > 1$ which implies that this point lies below the bisector, i.e. $x_2^* < x_1^*$.

The case $\alpha_2 > 1, \alpha_1 < 1$ can be obtained from the previous case by formal exchange of indices.

As a result the trajectory and the boundary can have only one point of intersection in the permissible quadrant and only in the case when $\alpha_l < 1, \alpha_k > 1$. Moreover, if (x_1^*, x_2^*) is such a point of intersection, $x_{1,2}^* > 1$. In this case the sufficient condition of intersection which now becomes also necessary can be expressed as the following inequality

$$x_{tl}(\alpha_k) < x_{bl}(\alpha_k). \quad (2.21)$$

Note that the condition $x_{1,2}^* > 1$ means that densities can only increase, i.e.

rarefactive solitons are not allowed.

D. A no-go result for monotonic double layers

Apart from lump solitons, the analysis of the sec. 4 gave us another possible type of a travelling structure, *viz.* a kink. Such solution would exist if and only if the second common point of the boundary and trajectory in the permissible quadrant belongs also to the line $x_1 = x_2$. We shall further assume $\gamma_1 = \gamma_2 = \gamma$. The Assumption that $x_1 = x_2 = x$ in equations (2.7) and (2.9) for the trajectory and the boundary results in

$$x^{\gamma-1} - 1 + \frac{\gamma-1}{\gamma}a(x^{-2} - 1) = 0 \quad (2.22a)$$

and

$$x^\gamma - 1 + a(x^{-1} - 1) = 0, \quad (2.22b)$$

where

$$a = \frac{m_1 n_{10} + m_2 n_{20}}{T_1 n_{10} + T_2 n_{20}} u^2.$$

A kink can exist only for such γ and a that a system of equations (2.22a), (2.22b) has a solution $x \neq 1$. Multiplication of equation (2.22a) by x^2 and subtraction of it from equation (2.22b) multiplied by x results in

$$x^2 \left(1 + \frac{\gamma-1}{2\gamma}a \right) - x(a+1) + \frac{\gamma+1}{2\gamma}a = 0. \quad (2.23)$$

The roots of this quadratic equation are

$$x_1 = 1, \quad x_2 = \frac{(\gamma+1)a}{2\gamma + (\gamma-1)a}.$$

The system (2.22) has the solution x_2 if and only if it solves eq. (2.22b). Expressing a through x_2 ,

$$a = \frac{2x_2\gamma}{(\gamma + 1) - x_2(\gamma - 1)},$$

we rewrite eq. (2.22b) as

$$g(x_2) \stackrel{\text{def}}{=} x_2^\gamma \left(x_2 - \frac{\gamma + 1}{\gamma - 1} \right) - 1 + x_2 \frac{\gamma + 1}{\gamma - 1} = 0. \quad (2.24)$$

Since $g(1) = 0$ while $g'(x_2) \neq 0$ for $x_2 \neq 1$; $x_2 = 1$ is the only root in accordance with Rolle's theorem. Consequently $x = 1$ is the only common root of equations (2.22a) and (2.22b).

For an isothermal process ($\gamma = 1$) instead of eqs. (2.22a) and (2.22b) we have

$$\ln x + \frac{a}{2} (x^{-2} - 1) = 0, \quad (2.25a)$$

$$x - 1 + a (x^{-1} - 1) = 0. \quad (2.25b)$$

Equation (2.25b) has two roots $x_1 = 1$, $x_2 = a$. Substituting $x = a$ into eq. (2.25a) we obtain

$$\ln a = \frac{1}{2} \left(a - \frac{1}{a} \right). \quad (2.26)$$

Using the same consideration as for eq. (2.24) we find that $a = 1$ is the only root of eq. (2.26).

This proves that the model in question can not support monotonic transition layers.

E. Domain of existence for compressive ion-acoustic solitons

Let us call the j th type of particles in the plasma ions (i) if $\alpha_j > 1$ or electrons (e) if $\alpha_j < 1$. A condition necessary and sufficient for an existence of compressive solitons is given by eqs. (2.17) and (2.21). We introduce the following dimensionless

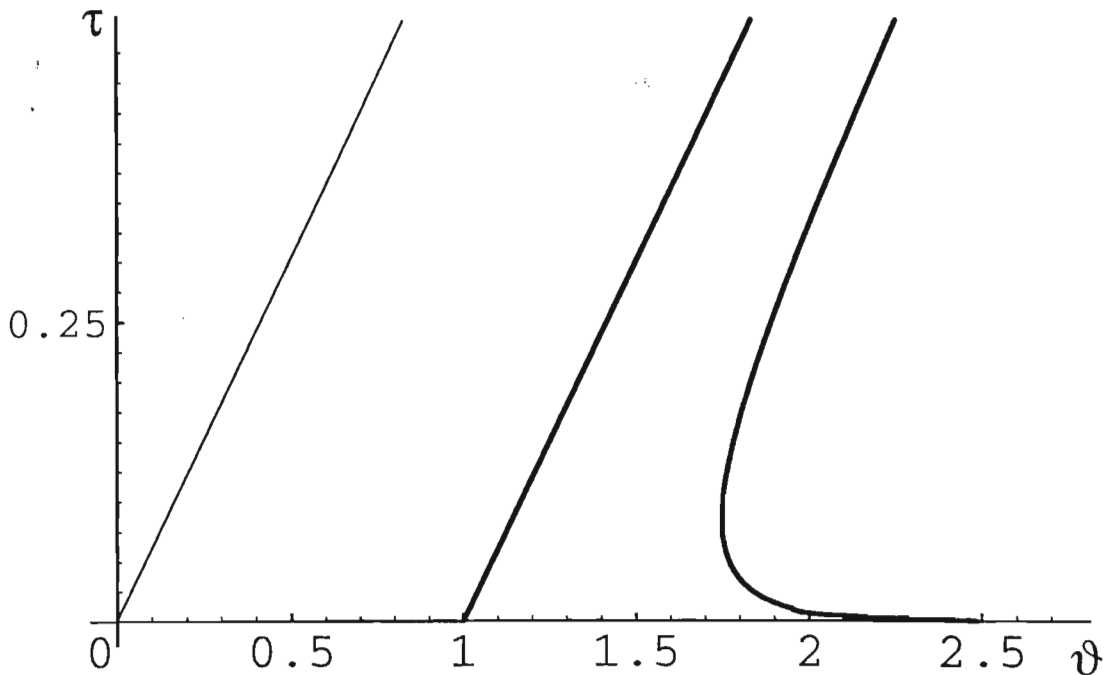


Fig. 2. The domain of existence of compressive ion-acoustic solitons for an electron-proton plasma on the τ, ϑ -plane, $\tau = T_i/T_e$, $\vartheta = m_i u^2/T_e = u^2/V_s^2$. The domain is bounded by the thick lines.

parameters, $\mu = m_e/m_i$, $\eta = n_{i0}/n_{e0}$, $\tau = T_i/T_e$, $\theta = u^2/V_s^2 = m_i u^2/T_e$, where V_s is ion acoustic speed. Since $\alpha_e = (\theta\mu/\gamma_e)^{1/(\gamma_e+1)} < 1$ and $\alpha_i = [\theta/(\gamma_i\tau)]^{1/(\gamma_i+1)} > 1$, θ and τ satisfy: $\tau < \theta/\gamma_i$, $\theta < \gamma_e/\mu$.

The escape condition (2.17) written in our dimensionless parameters is

$$\tau < \frac{\theta}{\gamma_i} \left(1 + \frac{\mu}{\eta} \right) - \frac{\gamma_e}{\gamma_i \eta}. \quad (2.27)$$

Eq. (2.27) gives the left margin of the existence domain (thick straight line in Fig. 2).

The boundary of the region defined by eq. (2.21) is given by

$$x_{te}(\alpha_i) = x_{be}(\alpha_i), \quad (2.28)$$

where $x_{be}(\alpha_i)$ and $x_{te}(\alpha_i)$ are the maximal roots of equations $\mathcal{T}(x_{te}, \alpha_i) = 0$ and

$\mathcal{B}(x_{be}, \alpha_i) = 0$ respectively. (We should choose the maximal roots, since we consider only those parts of the trajectory and the boundary that fall into the permissible quadrant.) Hence eq. (2.28) is a compatibility condition of these equations.

Ion-acoustic solitons are stable and propagate with a constant speed only if $\tau \ll 1$, otherwise their amplitude and hence speed decrease due to Landau damping. Then $V_{T_e} \gg u$ ($V_{T_j} = \sqrt{2T_j/m_j}$ is a thermal velocity of particle species j) and hence the variations in the electron temperature are very small. Accordingly the polytropic index for electrons is very close to that of an isothermal process, $\gamma_e = 1$. The ion-acoustic velocity, on the other hand, although less, can be of the same order as a structure speed. Hence the process for ions can be naturally approximated as an adiabatic.

$x_{te}(\alpha_i)$ and $x_{be}(\alpha_i)$, for an isothermal electron, $\gamma_e = 1$, and for an anisothermal ion, $\gamma_i = \gamma \neq 0$, are found from

$$\begin{aligned} \mathcal{T}(x_{te}, \alpha_i) = \ln x_e + \frac{\theta\mu}{2} (x_e^{-2} - 1) + \frac{\gamma\tau\eta}{\gamma-1} (\alpha_i^{\gamma-1} - 1) \\ + \frac{\theta\eta}{2} (\alpha_i^{-2} - 1) = 0 \end{aligned} \quad (2.29)$$

and

$$\begin{aligned} \mathcal{B}(x_{be}, \alpha_i) = x_e - 1 + \theta\mu (x_e^{-1} - 1) + \tau\eta (\alpha_i^\gamma - 1) \\ + \theta\eta (\alpha_i^{-1} - 1) = 0, \end{aligned} \quad (2.30)$$

according to eqs. (2.7), (2.9). Eq. (2.30) produces a quadratic equation

$$x_e^2 - x_e b + c = 0, \quad (2.31)$$

where $b = 1 + \mu\theta + \theta\eta (1 - \alpha_i^{-1}) + \tau\eta (1 - \alpha_i^\gamma) > 0$, $c = \theta\mu$. To prove the positiveness

of b it suffices to show that

$$\theta(1 - \alpha_i^{-1}) + \tau(1 - \alpha_i^\gamma) \geq 0, \quad (2.32)$$

where $\alpha_i = [\theta/(\tau\gamma)]^{1/(\gamma+1)}$. On the introduction of a new variable $q = \theta/\tau$, eq. (2.32) becomes

$$\left[1 + q - q^{\frac{\gamma}{\gamma+1}}(\gamma^{\frac{1}{\gamma+1}} + \gamma^{-\frac{\gamma}{\gamma+1}})\right] \tau = p(q)\tau \geq 0. \quad (2.33)$$

Eq. (2.33) always holds since, at $q = \gamma$, $p(q)$ achieves its minimum, $p(\gamma) = 0$.

Hence the maximal root of eq. (2.31) is $x_+ = b + \sqrt{b^2 - 4ac}/2a$. Accordingly, eq. (2.28) is equivalent to

$$\begin{aligned} \ln x_+ + \frac{\theta\mu}{2} (x_e^{-2} - 1) + \frac{\gamma\tau\eta}{\gamma-1} (\alpha_i^{\gamma-1} - 1) \\ + \frac{\theta\eta}{2} (\alpha_i^{-2} - 1) = 0 \end{aligned} \quad (2.34)$$

Eq. (2.34), as one can verify, is satisfied on a line segment $\tau = \theta/\gamma$, $0 < \theta \leq \gamma/\mu$ (thin straight line in Fig. 2). Numerical solution shows that eq. (2.34) is also satisfied on another line that gives the right margin of the existence domain (Fig. 2) (thick curve in fig. 2).

F. Concluding Remarks

We have obtained the following results for a fluid model of a two-component plasma:

1. The model does not admit monotonic transition layers in the case of a plasma consisting of two types of particles with the same thermodynamic properties ($\gamma_1 = \gamma_2$),
2. The model does not admit rarefactive solitons,

3. Compressive ion-acoustic solitons exist only in the domain defined by eqs. (2.27), (2.34).

In Fig. 2 we present the domains of existence of compressive solitons in a plasma consisting a single isothermal electron species and a single adiabatic proton species, $\mu = 1/1836$, $\gamma_e = 1$, $\gamma_i = 5/3$. These solitons are supersonic, $\vartheta \geq 1$. Let us note that the right boundary of the domain for $\tau = 0$ (cold ions) corresponds to the speed $u \approx 1.58V_s$ which coincides with the standard result of Sagdeev [49].

For low ion temperature the domain of existence sharply shrinks with the increase in temperature. Consequently even a comparatively low ion temperature cannot be neglected.

We have assumed that the polytropic indices are constants and do not depend on other parameters. This is in fact a simplification of the model, since polytropic indices do depend on parameters such as ratio of structure speed and thermal velocity of particles, structure amplitude etc. The natural way to deal with this would be to use the kinetic theory, where there is no need to make assumptions about the dependence of polytropic indices. Although this approach is more general, it does not seem to be feasible to obtain even the most simple general conclusions about the properties of travelling structures.

We have not considered solutions with trajectories intersecting the lines $x_j = \alpha_j$. However, we note that such solutions might be possible and would correspond to shock waves. The analysis of shock waves requires the use of the appropriate integral relations and lies beyond the scope of the present investigation.

CHAPTER III

TOPOLOGICAL VORTICES IN A CONDENSATE OF NONRELATIVISTIC
 BOSONS COUPLED TO THE MAXWELL AND CHERN–SIMONS FIELDS

Vortices, topologically nontrivial localised structures, lie at the heart of all theories of particles with fractional statistics. It is these collective excitations of the field quanta that are considered as candidates for anyonic objects in quasi-planar condensed matter physics. More precisely, in the case of charged matter interacting with a Maxwell field, the anyon is a bound state of an (electrically neutral) vortex and a field quantum, a “flux” and a “charge”. If the gauge field is of Chern–Simons type, the vortex is no more electrically neutral and behaves as an anyon itself.

Both Maxwell and Chern–Simons vortices are discussed in the literature, for both relativistic and nonrelativistic matter fields. So far the relativistic model has been amenable for a more thorough analysis. It admits a nontrivial ground state over which topological vortices can be superimposed.

The vortex configurations of a scalar field minimally coupled to the Maxwell field were discovered by Abrikosov within the (2+0)-dimensional Ginzburg–Landau model of superconductivity [1]. This model can be treated as the static limit of the (2+1)-dimensional Higgs, or more precisely, Higgs–Maxwell theory. Therefore one can think of Abrikosov’s solutions as of *relativistic* vortices in the sense that they are static solutions of the (2+1)-dimensional relativistic Higgs–Maxwell theory [43]. The compatibility of Abrikosov’s vortices and of the Ginzburg–Landau model itself with relativistic dynamics follows from the fact that this model does not contain the Gauss law, $\Delta A_0 = J_0$. This omission is compatible with the assumption of the electrical neutrality ($A_0 = 0$) only when J_0 vanishes for static fields, i.e., when it is

the *relativistic* charge: $J_0 = ie(\bar{\phi}D_0\phi - \phi\overline{D_0\phi})$.

The possibility of existence of *Chern–Simons–Maxwell* vortices was first discussed in [45] within the relativistic Higgs model. (For the actual numerical solutions of this model see [16].) Multivortex solutions and the selfduality limit were discovered (in the pure Chern–Simons case) in [26] and [33, 29]. In addition to these asymptotically nonvanishing solutions with quantized flux, the relativistic model exhibits nontopological solitons for which the matter field, ϕ , vanishes at infinity [29]. (We call such bell-shaped solitons “lumps” in this thesis.) Although the two sets of solutions are supported by the same scalar potential $U(\phi) = (|\phi|^2 - \rho_0)^2(|\phi|^2 - \alpha)$, they obviously pertain to different regimes of the self–interaction, vortices to repulsion, lumps to attraction.

The *nonrelativistic* model, i.e. the gauged nonlinear Schrödinger equation [31], is presumably more relevant for condensed matter applications. However, it was found to exhibit only asymptotically vanishing solitons [31, 30, 32]. Although these collective excitations are also referred to as topological vortices, as their flux is quantized, they are clearly distinct from the relativistic topological vortices which have the form of defects interpolating between topologically distinct vacua. Solutions of the latter type do not arise in the standard nonrelativistic model as formulated by Jackiw and Pi [31]. There is even no condensate state, i.e. nontrivial vacuum solution in this model.

As in the ordinary, nongauged nonlinear Schrödinger equation, the above–said bell–like solitons arise in the case of the *self-attractive* boson gas (coupled to the Chern–Simons field). By analogy with the nongauged equation, one could expect the condensate state to emerge in the case of the nonlinear Schrödinger equation with repulsion. However, as is shown below, the condensate can be incorporated into the “standard” model by no choice of the scalar self–interaction. It turns out that this

model is applicable only for the description of asymptotically vanishing fields. Our aim is to modify it to incorporate the condensate (and the topological defects).

We demonstrate that the necessary modification can be attained by the revision of the Lagrangian formulation of the nongauged precursor of the standard model. Using the revised Lagrangian as a basis for the gauge theory, we arrive at a new version of the gauged nonlinear Schrödinger equation which is completely compatible with the nonvanishing (“condensate”) boundary conditions.

A. The standard model and asymptotically nonvanishing fields

The gauged nonlinear Schrödinger equation was formulated by Jackiw and Pi [31, 30, 32]:

$$i\psi_t - eA_0\psi + \mathbf{D}^2\psi - U'(\rho)\psi = 0, \quad (3.1a)$$

where $\rho = |\psi|^2$, $U'(\rho) = dU/d\rho$, $\mathbf{D} = \nabla - ie\mathbf{A}$, A_0 is the scalar and \mathbf{A} the vector potential. The gauge field $A^\alpha = (A_0, \mathbf{A})$ satisfies its own equation for which the conserved matter current, $J^\alpha = (J_0, \mathbf{J})$, serves as a source. (Note that there are no *external* gauge fields.) The most general linear equation for A^α in (2+1) dimensions comprises both Maxwell and Chern-Simons terms:

$$\mu\partial_\beta F^{\beta\alpha} + \frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} = eJ^\alpha. \quad (3.1b)$$

Here $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$; Greek and Latin indices run over 0,1,2 and 1,2, respectively; and the metric signature is (+, -, -). The matter current of Jackiw and Pi has the following form:

$$J_0 = \rho = |\psi|^2, \quad (3.2a)$$

$$\mathbf{J} = \frac{1}{i}(\bar{\psi}\mathbf{D}\psi - \psi\overline{\mathbf{D}\psi}). \quad (3.2b)$$

Next the parameters μ and κ control the relative contributions of the Maxwell and Chern-Simons terms in the corresponding Lagrangian:

$$\begin{aligned} \mathcal{L} = & \frac{i}{2}(\bar{\psi} D_0 \psi - \psi \overline{D_0 \psi}) - \overline{D_k \psi} D_k \psi \\ & - \frac{\mu}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{\kappa}{4} \epsilon^{\gamma\alpha\beta} A_\gamma F_{\alpha\beta} - U(\rho). \end{aligned} \quad (3.3)$$

In eq. (3.3) $D_0 = \partial_0 + ieA_0$; μ is ≥ 0 , and κ will be considered nonnegative as well. The case of negative κ is recovered simply by the parity transformation ($x^1 \rightleftharpoons x^2$, $A^1 \rightleftharpoons A^2$). Finally, the scalar self-interaction $U(\rho)$ is taken to be completely arbitrary in this section. We demonstrate that, no matter what $U(\rho)$ is, the system (3.1)–(3.3), which we will call the standard model, does not admit the condensate solution.

Componentwise, the gauge field equation (3.1b) can be written as

$$\mu \operatorname{div} \mathbf{E} - \kappa B = eJ_0, \quad (3.4a)$$

$$- \mu \mathbf{E}_t + \mu \operatorname{curl} B - \kappa \tilde{\mathbf{E}} = e\mathbf{J}, \quad (3.4b)$$

where $E^i = F^{i0}$ and $B = F^{21}$ are the electric and magnetic fields respectively, $\operatorname{curl} B = (\partial B / \partial y, -\partial B / \partial x)$. $\tilde{\mathbf{E}}$ stands for the dual to \mathbf{E} : $\tilde{E}^i = \epsilon_{ij} E^j$.

We will call a *condensate* any solution to the equations (3.1a), (3.4) with the following properties: (i) ψ is a time-independent nonsingular matter field distribution with a uniform density, i.e. $\psi = \sqrt{\rho_0} e^{i\chi(\mathbf{x})}$ with $\rho_0 = \text{const}$ and $\chi(\mathbf{x})$ single-valued; (ii) the electric and magnetic fields are also static, $\mathbf{E} = \mathbf{E}(\mathbf{x})$, $B = B(\mathbf{x})$, and bounded.

In the pure Maxwell case ($\kappa = 0$), the nonexistence of the solution with the properties above is straightforward. Eq. (3.4b) is then simply $\operatorname{div} \mathbf{E} \sim \rho_0$ and so the electric field has to grow indefinitely. In the pure Chern-Simons case ($\mu = 0$), eq. (3.4b) becomes $B \sim \rho_0$ and the nonexistence of the condensate is not so obvious. Below we prove this fact for the most general situation when both the Maxwell and

Chern-Simons terms are present.

Our argument appears somewhat shorter if we choose the real gauge, $\chi(\mathbf{x}) = 0$. The imaginary part of eq. (3.1a) amounts to $\text{div } \mathbf{A} = 0$ and hence \mathbf{A} is a curl:

$$\mathbf{A} = \text{curl } \omega. \quad (3.5)$$

The current (3.2b) is then a curl as well:

$$\mathbf{J} = -2e\rho_0 \text{curl } \omega. \quad (3.6)$$

The requirement $\mathbf{E}_t = B_t = 0$ does not yet ensure $\mathbf{A}_t = 0$ and so the electric field,

$$\mathbf{E} = -\nabla A_0 - \mathbf{A}_t, \quad (3.7)$$

may have, in general, both conservative and solenoidal components. However, taking the divergence of eq. (3.4b) with $\mathbf{E}_t = 0$, we get

$$-\kappa \nabla \cdot \tilde{\mathbf{E}} = -\kappa \nabla \times \mathbf{E} = 0, \quad (3.8)$$

which means that \mathbf{E} is a pure gradient. Eq. (3.7) yields then

$$\mathbf{E} = -\nabla A_0. \quad (3.9)$$

Evaluating the curl of both sides of eq. (3.4b) and subtracting eq. (3.4a) with an appropriate coefficient, it is not difficult to obtain a stationary nonhomogeneous Klein-Gordon equation for the magnetic field:

$$-\mu^2 \Delta B + (\kappa^2 + 2e^2 \mu \rho_0) B = -e \kappa \rho_0. \quad (3.10)$$

Here we also used the fact that

$$B = -\Delta \omega; \quad (3.11)$$

this is a simple consequence of eq. (3.5).

The general solution of the equation (3.10) is the sum of a particular solution of the nonhomogeneous equation, for instance

$$B(\mathbf{x}) = B_0 = -\frac{e\kappa\rho_0}{\kappa^2 + 2e^2\mu\rho_0}, \quad (3.12)$$

and the general solution of the corresponding homogeneous equation,

$$-\mu^2\Delta B + (\kappa^2 + 2e^2\mu\rho_0)B = 0. \quad (3.13)$$

The general solution of eq. (3.13) is, of course, well known:

$$B(r, \theta) = \sum_{m=0}^{\infty} e^{im\theta} \left[b_m K_m \left(\frac{r}{r_0} \right) + \tilde{b}_m I_m \left(\frac{r}{r_0} \right) \right], \quad (3.14)$$

where K_m and I_m are the modified Bessel functions, b_m and \tilde{b}_m arbitrary complex coefficients and

$$r_0^2 = \frac{\mu^2}{\kappa^2 + 2e^2\mu\rho_0}.$$

Unless all $b_m = \tilde{b}_m = 0$, this solution grows exponentially and/or has singularities in a finite part of the (x, y) -plane. (In fact, to see that $B = 0$ is the only square integrable solution, we do not even need the explicit formula (3.14). All one needs to do is to observe that the operator $L = -\mu^2\Delta + \kappa^2 + 2e^2\mu\rho_0$ on the left hand side of (3.13) is positive definite, and apply L^{-1} to both sides.)

Thus the only bounded solution of eq. (3.10) is the constant magnetic field (3.12).

For constant B , eq. (3.4b) becomes

$$\mathbf{E} = \frac{2e^2\rho_0}{\kappa} \nabla\omega. \quad (3.15)$$

Comparing it to (3.9) we observe that ω is equal to

$$\omega = -\frac{\kappa}{2e^2\rho_0} A_0 \quad (3.16)$$

plus, probably, some function of time. (We do not keep track of it as it cancels in what follows.)

Eliminating ω between eqs. (3.11) and (3.16), we obtain

$$\Delta A_0 = \frac{2e^2\rho_0}{\kappa} B_0, \quad (3.17a)$$

with B_0 given by the expression (3.12). The real part of the nonlinear Schrödinger equation (3.1a) is another equation for A_0 :

$$eA_0 + e^2 \left(\frac{\kappa}{2e^2\rho_0} \right)^2 (\nabla A_0)^2 = -U'(\rho_0). \quad (3.17b)$$

In what follows we find it useful to rewrite the equations (3.17) as

$$\partial\bar{\partial}A_0 = \frac{2e^2\rho_0}{\kappa} B_0, \quad (3.18)$$

$$A_0 + e \left(\frac{\kappa}{2e^2\rho_0} \right)^2 \partial A_0 \bar{\partial} A_0 = -\frac{1}{e} U'(\rho_0). \quad (3.19)$$

Here z and \bar{z} are the Laplace coordinates on the plane,

$$z = \frac{x + iy}{2}, \quad \bar{z} = \frac{x - iy}{2};$$

$\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$.

Up to now we were implicitly assuming that $\kappa \neq 0$. If $\kappa = 0$, the substitution of (3.7) and (3.5) into (3.4a) produces exactly the equation (3.17a), where we should only replace B_0/κ by its limiting value:

$$\lim_{\kappa \rightarrow 0} \frac{B_0}{\kappa} = -\frac{1}{2e\rho_0}.$$

Solution of the Poisson equation (3.18) is

$$A_0 = \frac{2e^2\rho_0}{\kappa} B_0 z \bar{z} + f(z) + \overline{f(z)}, \quad (3.20)$$

where $f(z)$ is an arbitrary complex function, analytic in the finite part of the complex plane. The first term in (3.20) is a solution of the inhomogeneous equation, while $f(z) + \overline{f(z)}$ represents the general solution of the corresponding homogeneous equation. The nonexistence of the condensate solution is obvious already from eq. (3.20). (A_0 comprises a term quadratic in coordinates and so the electric field grows at least linearly as $r \rightarrow \infty$.) However, we prove a stronger assertion. We demonstrate that equations (3.18) and (3.19) are, in general, incompatible and, with a single exception, there are no even *polynomially* growing solutions.

Substitution of eq. (3.20) into (3.19) produces

$$\begin{aligned} & \frac{2e^2\rho_0}{\kappa} B_0 z \bar{z} + e B_0^2 z \bar{z} + e \left(\frac{\kappa}{2e^2\rho_0} \right)^2 f'(z) \overline{f'(z)} \\ & + \left[f(z) + \frac{\kappa B_0}{2e\rho_0} z f'(z) + \text{c. c.} \right] = -\frac{1}{e} U'(\rho_0). \end{aligned} \quad (3.21)$$

We expand the analytic function $f(z)$ in a Taylor series in the neighbourhood of $z = 0$:

$$f(z) = f_0 + f_1 z + f_2 z^2 + \dots \quad (3.22)$$

Substituting this into eq. (3.21) we have

$$\begin{aligned} & \left(\frac{2e^2\rho_0}{\kappa} B_0 + e B_0^2 \right) z \bar{z} + e \left(\frac{\kappa}{2e^2\rho_0} \right)^2 \\ & \times \left[f_1 \overline{f'(z)} + \bar{f}_1 f'(z) - |f_1|^2 + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} C_{ij} z^i \bar{z}^j \right] \\ & + \left[f(z) + \frac{\kappa B_0}{2e\rho_0} z f'(z) + \text{c. c.} \right] = -\frac{1}{e} U'(\rho_0) \end{aligned} \quad (3.23)$$

with $C_{ij} = \text{const.}$ In eq. (3.23) there are terms which are functions of z only; terms which are functions only of \bar{z} ; and those which are products of powers of z and \bar{z} . For the equation to be satisfied, each group of terms must be set equal to a constant. In

particular, assembling all functions of z , we obtain

$$e \left(\frac{\kappa}{2e^2\rho_0} \right)^2 \bar{f}_1 f'(z) + \frac{\kappa B_0}{2e\rho_0} z f'(z) + f(z) = C_0, \quad (3.24)$$

where C_0 is the corresponding separation constant. Eq. (3.24) is separable and so its solution is straightforward:

$$f(z) = C_1(z - z_0)^n + C_0, \quad (3.25)$$

where

$$n = -\frac{2e\rho_0}{\kappa B_0}, \quad (3.26)$$

$z_0 = -\kappa \bar{f}_1 / (2e^2\rho_0 B_0)$ and C_1 is an arbitrary complex constant. If C_1 is zero, eq. (3.21) reduces simply to

$$\kappa^2 + 4e^2\mu\rho_0 = 0, \quad (3.27)$$

which is clearly impossible for $\mu \geq 0$. Hence, $C_1 \neq 0$.

For $C_1 \neq 0$, the analyticity requires n to be a positive integer. Returning to equation (3.21), it is elementary to see now that z_0 is necessarily zero, and the only admissible value of n is 2. The value of $|C_1|$ is then found to be $e^3\rho_0^2/\kappa^2$.

Invoking eq. (3.26) we find that the condition $n = 2$ is equivalent to $\mu = 0$. This means that a solution to the system (3.17) exists only in the pure Chern-Simons case, $\mu = 0$. However, this solution is quadratic in the co-ordinates,

$$A_0 = -\frac{2e^3\rho_0^2}{\kappa^2} z\bar{z} + C_1 z^2 + \bar{C}_1 \bar{z}^2 + \text{const}, \quad (3.28)$$

and so the corresponding electric field grows linearly as x or $y \rightarrow \pm\infty$. As we have already mentioned, this solution is clearly not admissible as the condensate state. Returning to the original system (3.1), this implies that the condensate exists for no μ and κ .

B. The regularised model

At first glance it may seem that there is just one way to couple minimally electromagnetism to the nonlinear Schrödinger field, namely the way Jackiw and Pi did it. In this prescription one substitutes $\partial_\mu \rightarrow D_\mu$ in the corresponding Lagrangian,

$$\mathcal{L} = \frac{i}{2}(\psi_t \bar{\psi} - \bar{\psi}_t \psi) - |\nabla \psi|^2 - U(\rho). \quad (3.29)$$

However, the Lagrangian for the NLS equation,

$$i\psi_t + \Delta \psi - U'(\rho)\psi = 0 \quad (3.30)$$

is not defined uniquely. One may add a time derivative to (3.29), for example. It is natural to ask if all possible Lagrangians produce the same gauge theory and, if not, then on what grounds should we choose one or another.

To understand the impact of the specific choice of the Lagrangian on the theory, consider the simplest model, which is the ordinary nongauged NLS equation on a line:

$$i\psi_t + \psi_{xx} - U'(\rho)\psi = 0. \quad (3.31)$$

It is easy to check that the Lagrangian

$$\mathcal{L} = \frac{i}{2}(\psi_t \bar{\psi} - \bar{\psi}_t \psi) - |\psi_x|^2 - U(\rho), \quad (3.32)$$

which is normally associated with eq. (3.31), does not *automatically* produce correct integrals of motion for solutions with $|\psi|^2 \rightarrow \rho_0$ at infinity. Indeed, the number of particles integral,

$$N = \int \left(\frac{\partial \mathcal{L}}{\partial \psi_t} i\psi - \frac{\partial \mathcal{L}}{\partial \bar{\psi}_t} i\bar{\psi} \right) dx, \quad (3.33)$$

takes the form $N = \int \rho dx$ and obviously diverges. The regularized number of parti-

cles,

$$N = \int (\rho - \rho_0) dx, \quad (3.34)$$

has to be obtained by the *ad hoc* subtraction of the background contribution. The conventional definition of momentum,

$$P = \int \left(\psi_x \frac{\partial \mathcal{L}}{\partial \psi_t} + \bar{\psi}_x \frac{\partial \mathcal{L}}{\partial \bar{\psi}_t} \right) dx, \quad (3.35)$$

does not yield the correct expression either. For the Lagrangian (3.32) one obtains

$$P = \frac{i}{2} \int (\psi_x \bar{\psi} - \bar{\psi}_x \psi) dx \quad (3.36)$$

which is not compatible with the Hamiltonian formulation of the model [11]. Indeed, varying this P gives

$$\delta P = i \int (\psi_x \delta \bar{\psi} - \bar{\psi}_x \delta \psi) dx - \rho_0 \delta \text{Arg} \psi \Big|_{-\infty}^{+\infty}.$$

Due to the appearance of the boundary term we cannot compute the functional derivatives $\delta P / \delta \psi$ and $\delta P / \delta \bar{\psi}$. As a result, the Poisson bracket of P with some other functional (e.g. Hamiltonian) can not be evaluated. To obtain a definition compatible with the Hamiltonian structure we again have to make an *a posteriori* regularization [11] (see also [14]):

$$\begin{aligned} P &= \frac{i}{2} \int (\bar{\psi} \psi_x - \psi \bar{\psi}_x) dx + \rho_0 \text{Arg} \psi \Big|_{-\infty}^{+\infty} \\ &= \frac{i}{2} \int (\psi_x \bar{\psi} - \bar{\psi}_x \psi) \left(1 - \frac{\rho_0}{\rho} \right) dx. \end{aligned} \quad (3.37)$$

Proceeding to two dimensions the standard definition $\mathbf{P} = \int \mathcal{P} d^2x$, with

$$\mathcal{P} = \frac{i}{2} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \quad (3.38)$$

is even less suitable since for asymptotically nonvanishing configurations this func-

tional is not only nondifferentiable but also ill-defined. Assuming, for instance, the vortex boundary conditions $\psi(\mathbf{x}) \rightarrow \sqrt{\rho_0} e^{in\theta}$ as $|\mathbf{x}| \rightarrow \infty$ we have $\mathcal{P} \rightarrow (-n)\mathbf{x} \times \rho_0/r^2$ and so the integral $\int \mathcal{P} d^2x$ converges only in the sense of the principal value. Another important conserved quantity in two dimensions is the angular momentum:

$$L = \int \mathbf{x} \times \mathcal{P} d^2x. \quad (3.39)$$

As $|\mathbf{x}| \rightarrow \infty$, the integrand in (3.39) approaches $n\rho_0$ and so the angular momentum, in general, diverges. Thus we have to conclude that nonvanishing boundary conditions at infinity (or nonvanishing background) are not compatible with the internal structure of the standard model. On the one hand this incompatibility manifests itself in the fact that there is no condensate solution. On the other hand, even if the condensate existed (as in the case of the non-gauged nonlinear Schrödinger), we would still have problems with integrals of motion. We turn to a nongauged model to see if the problem can be overcome at this more elementary level.

Adding the time derivative of some function to the standard one-dimensional Lagrangian (3.32) obviously does not change its Euler-Lagrange equation (3.31). This function can be chosen in such a way that the corresponding integrals of motion *automatically* arise in the regularized form given by (3.34) and (3.37). The appropriate time derivative is $\rho_0 \partial/\partial t \text{Arg } \psi$, and the regularized Lagrangian is

$$\tilde{\mathcal{L}} = \frac{i}{2}(\psi_t \bar{\psi} - \bar{\psi}_t \psi) \left(1 - \frac{\rho_0}{\rho}\right) - |\psi_x|^2 - U(\rho). \quad (3.40)$$

Let us see if the same prescription can be extended to two dimensions. The Lagrangian

$$\tilde{\mathcal{L}} = \frac{i}{2}(\psi_t \bar{\psi} - \bar{\psi}_t \psi) \left(1 - \frac{\rho_0}{\rho}\right) - |\nabla \psi|^2 - U(\rho) \quad (3.41)$$

produces the two-dimensional nonlinear Schrödinger equation,

$$i\psi_t + \Delta\psi - U'(\rho)\psi = 0, \quad (3.42)$$

while the conserved quantities (as evaluated by Noether's theorem) are:

$$N = \int (\rho - \rho_0) d^2x, \quad (3.43)$$

$$\mathbf{P} = \int \mathcal{P} d^2x, \quad (3.44a)$$

with

$$\mathcal{P} = \frac{i}{2}(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}) \left(1 - \frac{\rho_0}{\rho}\right), \quad (3.44b)$$

and

$$L = \int \mathbf{x} \times \mathcal{P} d^2x. \quad (3.45)$$

It is easy to see that these integrals are convergent for fields with ρ approaching ρ_0 rapidly enough as $r \rightarrow \infty$. As far as differentiability is concerned, \mathbf{P} , eq. (3.44a), and L , eq. (3.45) are differentiable in the $n = 0$ topological sector, i.e. for fields with boundary conditions $\psi(\mathbf{x}) \rightarrow \sqrt{\rho_0}$ as $r \rightarrow \infty$. (Solitonic bubbles in the Bose condensate [11] furnish an example of configurations of this type. Another example is a vortex-antivortex pair.) However, the momentum (3.44a) is not differentiable on a vortex-like configuration, i.e. a configuration of the form $\psi(\mathbf{x}) = \psi(r)e^{in\theta}$ with $\psi(r) \rightarrow \sqrt{\rho_0}$ as $r \rightarrow \infty$, and $\psi(r) \sim r^n$ for $r \sim 0$. Indeed let us evaluate the variation of the momentum:

$$\delta\mathbf{P} = i \int (\delta\bar{\psi}\nabla\psi - \delta\psi\nabla\bar{\psi}) d^2x \quad (3.46)$$

$$- \int \nabla[(\rho - \rho_0)\delta \text{Arg} \psi] d^2x. \quad (3.47)$$

Similarly to the one-dimensional situation, the quantity $(\rho - \rho_0)\delta \text{Arg} \psi$ approaches

zero as $r \rightarrow \infty$ and it is tempting to expect the second term in (3.47) to vanish as well after the double integral has been transformed to a line integral over an infinitely remote contour. However, Green's theorem is not applicable for vortices as $\text{Arg } \psi$ is not differentiable at the centre of the vortex. (For instance, if the vortex is placed at the origin, $\text{Arg } \psi = n\theta = n \arctan(y/x)$.) To apply Green's theorem we would have to integrate not only over the infinitely remote contour but also over a small contour encircling the vortex core. The result will clearly be nonzero. Note that this nondifferentiability has a nature different from the nondifferentiability of momentum (3.36) in one dimension. The latter arises from the nonvanishing boundary conditions at infinity; this discontinuity is an attribute of the condensate. The expressions (3.37) and (3.44a) are free from this drawback, both in $D = 1$, and $D = 2$. However, in two dimensions the price we pay for the elimination of one discontinuity is the appearance of another one.

Hence the integrals (3.44a) and (3.45) are not compatible with the Hamiltonian formulation of the model in the $n \neq 0$ topological sector. However, this fact does not devalue the modified expressions (3.44a) and (3.45). The point is that the nongauged NLS (3.42) does not have ($n \neq 0$)-solutions with finite energy [47, 21]. The celebrated Ginsburg-Pitaevsky vortex [23, 46] has a logarithmically divergent energy. (This is why, in numerical experiments with finite energy initial conditions, vortices nucleate only in vortex-antivortex pairs [35].) This implies that the nongauged NLS does not admit a Hamiltonian formulation in nontrivial topological sectors and there is simply no need for the differentiability of momentum in topological configurations.

Thus the Lagrangian (3.41) automatically produces finite, differentiable conservation laws in the space of finite energy configurations with nonvanishing boundary conditions at infinity. It is natural to expect that, if we couple ψ minimally to the gauge field, the resulting gauge theory will not be so hostile to asymptotically non-

vanishing fields. Indeed gauging the Lagrangian (3.41) yields

$$\begin{aligned} \tilde{\mathcal{L}} = & \frac{i}{2} (\bar{\psi} D_0 \psi - \psi \overline{D_0 \psi}) \left(1 - \frac{\rho_0}{\rho} \right) - \overline{D_k \psi} D_k \psi \\ & - \frac{\mu}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{\kappa}{4} \epsilon^{\gamma\alpha\beta} A_\gamma F_{\alpha\beta} - U(\rho), \end{aligned} \quad (3.48a)$$

which can also be rewritten as

$$\tilde{\mathcal{L}} = \mathcal{L} + \rho_0 (\partial_0 \text{Arg } \psi + e A_0), \quad (3.48b)$$

where \mathcal{L} is the standard Lagrangian of Jackiw and Pi (eq. (3.3)). The time derivative $\rho_0 \partial_0 \text{Arg } \psi$ can be dropped from the Lagrangian without any consequences for the equations of motion and the final form of the regularized model is

$$\begin{aligned} \tilde{\mathcal{L}} = & \frac{i}{2} (\bar{\psi} D_0 \psi - \psi \overline{D_0 \psi}) - \overline{D_k \psi} D_k \psi + e \rho_0 A_0 \\ & - \frac{\mu}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{\kappa}{4} \epsilon^{\gamma\alpha\beta} A_\gamma F_{\alpha\beta} - U(\rho). \end{aligned} \quad (3.49)$$

It is only one term, $e \rho_0 A_0$, that makes eq. (3.49) different to the standard model (3.3). However, this term has a transparent physical interpretation (it describes the gauge field coupling to the uniformly charged static background) and produces a drastic change in the structure of solutions to the model. The equations of motion look the same, eqs. (3.1), with the vector \mathbf{J} being given by the standard expression (3.2b). It is only the number density, J_0 , that is changed by the addition of the term $\rho_0 A_0$ to the Lagrangian. This time J_0 is given not by eq. (3.2a) but by

$$J_0 = \rho - \rho_0. \quad (3.50)$$

It is straightforward to check that the model regularized in this way does possess the condensate solution: $\psi = \sqrt{\rho_0}$, $A_0 = -(1/e)U'(\rho_0)$, $\mathbf{A} = 0$.

C. The constrained Hamiltonian formulation

The Hamiltonian formulation of the new model (as well as of the standard model of Jackiw and Pi) is crucial for the construction of self-dual solutions (see sec. E.2). One immediately notices that the Lagrangian (3.49) is singular (degenerate in velocities). In its treatment we follow the Dirac-Bergmann approach (see [20], [53] for a review and references).

The momenta conjugate to the matter field ψ and its complex conjugate $\bar{\psi}$ are given by

$$\begin{aligned}\pi &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \frac{i}{2} \bar{\psi} \left(1 - \frac{\rho_0}{\rho}\right), \\ \bar{\pi} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = -\frac{i}{2} \psi \left(1 - \frac{\rho_0}{\rho}\right),\end{aligned}$$

and give rise to two primary constraints:

$$\xi_1 = \pi - \frac{i}{2} \bar{\psi} \left(1 - \frac{\rho_0}{\rho}\right) = 0, \quad (3.51a)$$

$$\xi_2 = \bar{\pi} + \frac{i}{2} \psi \left(1 - \frac{\rho_0}{\rho}\right) = 0. \quad (3.51b)$$

The momenta conjugate to the fields A^α are

$$\Pi_\alpha = \frac{\kappa}{2} \epsilon_{\alpha\beta 0} A^\beta - \mu F^{\alpha 0}. \quad (3.52)$$

Π_0 vanishes. Therefore $\Pi_0 = 0$ is a third primary constraint. The canonical Hamiltonian density is the energy density T^{00} [see eq. (3.64a) below] expressed in terms of canonical variables $\psi, \bar{\psi}, \pi, \bar{\pi}, A^\mu$ and Π_μ . Integrating by parts we write the canonical Hamiltonian as

$$\begin{aligned}H_c &= \int \left\{ |\mathbf{D}\psi|^2 + U(\rho) + \frac{\mu}{2} (\mathbf{E}^2 + B^2) \right\} d^2x \\ &\quad - \int A_0 \{ (\mu \operatorname{div} \mathbf{E} - \kappa B - e(\rho - \rho_0)) \} d^2x, \quad (3.53)\end{aligned}$$

where $E^i = (\kappa\epsilon^{ij}A^j/2 - \Pi_i)/\mu$, and $B = \partial_1 A^2 - \partial_2 A^1$. The Poisson bracket between two functionals is

$$\{S, T\} = \int \left[\frac{\delta S}{\delta\psi} \frac{\delta T}{\delta\pi} - \frac{\delta S}{\delta\pi} \frac{\delta T}{\delta\psi} + \frac{\delta S}{\delta\bar{\psi}} \frac{\delta T}{\delta\bar{\pi}} - \frac{\delta S}{\delta\bar{\pi}} \frac{\delta T}{\delta\bar{\psi}} + \frac{\delta S}{\delta A^\alpha} \frac{\delta T}{\delta \Pi_\alpha} - \frac{\delta S}{\delta \Pi_\alpha} \frac{\delta T}{\delta A^\alpha} \right] d^2x.$$

The constraints (3.51) are second class, $\{\xi_1(\mathbf{x}, t), \xi_2(\mathbf{y}, t)\} = -i\delta^{(2)}(\mathbf{x} - \mathbf{y})$, and can be accommodated by the introduction of the Dirac bracket,

$$\{S, T\}_D = \{S, T\} - \frac{1}{i} \int \left[\{S, \xi_1(\mathbf{z})\} \{ \xi_2(\mathbf{z}), T \} - \{S, \xi_2(\mathbf{z})\} \{ \xi_1(\mathbf{z}), T \} \right] d^2z. \quad (3.54)$$

In fact one can express π and $\bar{\pi}$ from (3.51) and consider S and T as functionals of ψ and $\bar{\psi}$ only:

$$S = S(\psi, \bar{\psi}, \pi(\psi, \bar{\psi}), \bar{\pi}(\psi, \bar{\psi})),$$

$$T = T(\psi, \bar{\psi}, \pi(\psi, \bar{\psi}), \bar{\pi}(\psi, \bar{\psi})).$$

(They also depend on A_i, Π_i , of course.) As a result the Dirac bracket is

$$\{S, T\}_D = \int \left[\frac{1}{i} \left(\frac{\tilde{\delta} S}{\delta\psi} \frac{\tilde{\delta} T}{\delta\bar{\psi}} - \frac{\tilde{\delta} S}{\delta\bar{\psi}} \frac{\tilde{\delta} T}{\delta\psi} \right) + \frac{\delta S}{\delta A^\alpha} \frac{\delta T}{\delta \Pi_\alpha} - \frac{\delta S}{\delta \Pi_\alpha} \frac{\delta T}{\delta A^\alpha} \right] d^2x, \quad (3.55)$$

where

$$\frac{\tilde{\delta}}{\delta\psi} = \frac{\delta}{\delta\psi} + \frac{\partial\pi}{\partial\psi} \frac{\delta}{\delta\pi} + \frac{\partial\bar{\pi}}{\partial\psi} \frac{\delta}{\delta\bar{\pi}}. \quad (3.56)$$

The constraint $\Pi_0 = 0$ is first class and the requirement of its conservation leads to a

secondary constraint

$$\eta = \{\Pi_0, H_c\}_D = \mu \operatorname{div} \mathbf{E} - \kappa B - e(\rho - \rho_0) = 0. \quad (3.57)$$

The secondary constraint $\eta = 0$ does not lead to further constraints since, as one can check, $\{\eta, H_c\}_D = 0$.

The total Hamiltonian is now $H_T = H_c + \int v \Pi_0 d^2x$, where v is an arbitrary multiplier. One can now see that the variables A^0 and Π_0 have no physical significance. $\Pi_0 = 0$ all the time while A^0 can take arbitrary values. Accordingly we may drop them out from the set of dynamical variables of the model. This can be accomplished by discarding the term $v \Pi_0$ in H_T , the only role of which is to let A^0 vary arbitrarily, and by treating A^0 as an arbitrary multiplier. As a result the Hamiltonian formulation of our model can be given in terms of three pairs of canonical fields, $\psi, \bar{\psi}; A^i, \Pi_i$. The dynamics is described by the Hamiltonian, H_c , with the canonical bracket (3.55) (now $\alpha=1,2$). The initial state should satisfy the constraint (3.57). Hamilton's equations will contain an arbitrary function A^0 . It can be determined only for a static problem, in which case the problem reduces to finding the stationary points of energy

$$H = \int \left[|\mathbf{D}\psi|^2 + U(\rho) + \frac{\mu}{2}(\mathbf{E}^2 + B^2) \right] d^2x \quad (3.58)$$

on the constraint manifold (3.57). In this case A^0 plays the rôle of a Lagrange multiplier and H_c is a Lagrange function.

So far we have assumed $\mu \neq 0$. For $\mu = 0$ (pure CS model) the situation is somewhat different. The theory has two more second class primary constraints, *viz.*

$$\xi_3 = \Pi_1 - \frac{\kappa}{2} A^2 = 0, \quad (3.59a)$$

$$\xi_4 = \Pi_2 + \frac{\kappa}{2} A^1 = 0, \quad (3.59b)$$

$\{\xi_3(\mathbf{x}, t), \xi_4(\mathbf{y}, t)\} = -\kappa\delta^{(2)}(\mathbf{x} - \mathbf{y})$, resulting from the definition of momenta Π_i . This can be accommodated by modifying the bracket:

$$\begin{aligned} \{S, T\}_D &= \{S, T\} \\ &+ \int \left[-\frac{1}{i}\{S, \xi_1(\mathbf{z})\}\{\xi_2(\mathbf{z}), T\} + \frac{1}{i}\{S, \xi_2(\mathbf{z})\}\{\xi_1(\mathbf{z}), T\} \right. \\ &\quad \left. - \frac{1}{\kappa}\{S, \xi_3(\mathbf{z})\}\{\xi_4(\mathbf{z}), T\} + \frac{1}{\kappa}\{S, \xi_4(\mathbf{z})\}\{\xi_3(\mathbf{z}), T\} \right] d^2z. \end{aligned}$$

The reduced phase space in this case is spanned by $\psi, \bar{\psi}, A^1$ and A^2 , and the Dirac bracket takes the form:

$$\begin{aligned} \{S, T\}_D &= \int \left[\frac{1}{i} \left(\frac{\tilde{\delta}S}{\delta\psi} \frac{\tilde{\delta}T}{\delta\bar{\psi}} - \frac{\tilde{\delta}S}{\delta\bar{\psi}} \frac{\tilde{\delta}T}{\delta\psi} \right) \right. \\ &\quad \left. + \frac{1}{\kappa} \left(\frac{\tilde{\delta}S}{\delta A^1} \frac{\tilde{\delta}T}{\delta A^2} - \frac{\tilde{\delta}S}{\delta A^2} \frac{\tilde{\delta}T}{\delta A^1} \right) \right] d^2x, \end{aligned} \quad (3.60)$$

where

$$\frac{\tilde{\delta}}{\delta A^i} = \frac{\delta}{\delta A^i} + \frac{\partial \Pi_j}{\partial A^i} \frac{\delta}{\delta \Pi_j}. \quad (3.61)$$

In fact in the pure Chern-Simons case further reduction is possible. In this case the constraint (3.57) can be explicitly resolved which makes it possible to consider $\psi, \bar{\psi}$ as the only canonical variables. This is the approach utilised by Jackiw and Pi [31, 30, 32]. In the general case, however, the fact that the constraint involves both Π_i and A^i makes such reduction impossible.

Our final remark in this section is that the fact that the model can be formulated as a *constrained* Hamiltonian system is a consequence of the local gauge invariance. The generator of the local gauge transformations is nothing but the first class constraint (3.57). The linear part of the transformation is

$$\begin{aligned} \psi(\mathbf{x}) &\rightarrow \psi(\mathbf{x}) + a(\mathbf{x}_0)\{\psi(\mathbf{x}), \eta(\mathbf{x}_0)\} \\ &= \psi(\mathbf{x}) + iea(\mathbf{x}_0)\delta^{(2)}(\mathbf{x} - \mathbf{x}_0)\psi(\mathbf{x}) \end{aligned} \quad (3.62a)$$

$$\begin{aligned}\bar{\psi}(\mathbf{x}) &\rightarrow \bar{\psi}(\mathbf{x}) + a(\mathbf{x}_0)\{\bar{\psi}(\mathbf{x}), \eta(\mathbf{x}_0)\} \\ &= \psi(\mathbf{x}) - iea(\mathbf{x}_0)\delta^{(2)}(\mathbf{x} - \mathbf{x}_0)\bar{\psi}(\mathbf{x})\end{aligned}\quad (3.62b)$$

$$\begin{aligned}A^i(\mathbf{x}) &\rightarrow A^i(\mathbf{x}) + a(\mathbf{x}_0)\{A^i(\mathbf{x}), \eta(\mathbf{x}_0)\} \\ &= A^i(\mathbf{x}) + a(\mathbf{x}_0)\frac{\partial}{\partial x^i}\delta^{(2)}(\mathbf{x} - \mathbf{x}_0)\end{aligned}\quad (3.62c)$$

$$\begin{aligned}\Pi_i(\mathbf{x}) &\rightarrow \Pi_i(\mathbf{x}) + a(\mathbf{x}_0)\{\Pi_i(\mathbf{x}), \eta(\mathbf{x}_0)\} \\ &= \Pi_i(\mathbf{x}) + \frac{\kappa}{2}a(\mathbf{x}_0)\epsilon_{ij}\frac{\partial}{\partial x^j}\delta^{(2)}(\mathbf{x} - \mathbf{x}_0).\end{aligned}\quad (3.62d)$$

The formulas (3.62) pertain to the case where both Maxwell and Chern-Simons terms are present ($\mu \neq 0$). The bracket here is the Dirac bracket (3.55). When A_μ is the pure Chern-Simons gauge field, eqs. (3.62a-c) are still in place, while (3.62d) should be discarded. The bracket then is defined by eq. (3.60).

D. Momentum

We start with the derivation of the energy-momentum tensor for the modified model (3.49). The corresponding results for the standard Jackiw-Pi's model are recovered by simply letting $\rho_0 = 0$.

By Noether's theorem, the canonical energy-momentum tensor is

$$T^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\partial^\beta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\bar{\psi})}\partial^\beta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\gamma)}\partial^\beta A^\gamma - \mathcal{L}g^{\alpha\beta}, \quad (3.63)$$

where $g^{\alpha\beta} = \text{diag}\{1, -1, -1\}$. For the Lagrangian (3.49) eq. (3.63) yields

$$\begin{aligned}T^{00} &= |D_k\psi|^2 + U(\rho) + \frac{\mu}{4}F_{\alpha\beta}^2 \\ &\quad + A_0(e(\rho - \rho_0) + \kappa F_{21} - \mu\partial_k F_{0k}) \\ &\quad + \partial_k \left[A_0 \left(\mu F_{0k} - \frac{\kappa}{2}\epsilon_{km}A^m \right) \right], \\ T^{0i} &= \frac{i}{2}(\psi\overline{D_i\psi} - \bar{\psi}D_i\psi) + e\rho_0\epsilon_{ij}x^j B + \mu\epsilon_{ij}F_{0j}F_{21}\end{aligned}\quad (3.64a)$$

$$\begin{aligned}
& + A^i (-\mu \partial_k F_{0k} + \kappa F_{21} + e(\rho - \rho_0)) \\
& + \partial_k \left[e \rho_0 \epsilon_{mk} \epsilon_{ij} x^j A^m + A^i \left(\mu F_{0k} - \frac{\kappa}{2} \epsilon_{km} A^m \right) \right],
\end{aligned} \tag{3.64b}$$

$$\begin{aligned}
T^{i0} & = -\overline{D_i \psi} D_0 \psi - D_i \psi \overline{D_0 \psi} + \mu \epsilon_{im} F_{0m} F_{21} \\
& - \partial_0 \left[A_0 \left(\mu F_{0i} - \frac{\kappa}{2} \epsilon_{im} A^m \right) \right] \\
& + \epsilon_{ik} \partial_k \left\{ \mu A_0 F_{21} + \frac{\kappa}{2} (A_0)^2 \right\},
\end{aligned} \tag{3.64c}$$

$$\begin{aligned}
T^{ij} & = \overline{D_i \psi} D_j \psi + D_i \psi \overline{D_j \psi} - \mu F_{0i} F_{0j} \\
& + \left(\frac{\mu}{4} F_{\alpha\beta}^2 - \frac{\Delta\rho}{2} + U'(\rho)\rho - U + e\rho_0 A_0 \right) \delta_{ij} \\
& - \partial_0 \left[A^j \left(\mu F_{0j} - \frac{\kappa}{2} \epsilon_{jk} A^k \right) \right] \\
& + \epsilon_{ik} \partial_k \left\{ \mu A^j F^{21} + \frac{\kappa}{2} A^0 A^j \right\}.
\end{aligned} \tag{3.64d}$$

The expressions for the fluxes, eqs (3.64c) and (3.64d), were simplified using the equations of motion. In the conserved densities, eqs (3.64a) and (3.64b), the multiples of $\mu \operatorname{div} \mathbf{E} + \kappa B + e(\rho - \rho_0)$ vanish as well. However,

$$\mu \operatorname{div} \mathbf{E} + \kappa B + e(\rho - \rho_0) \approx 0$$

is a *weak* equality in Dirac's sense and can be implemented only after all Poisson brackets have been calculated.

The last terms in the fluxes T^{i0} are two-dimensional curls. We can drop them as they cancel in the local conservation laws,

$$\partial_0 T^{0\alpha} + \partial_k T^{k\alpha} = 0; \quad \alpha = 0, 1, 2. \tag{3.65}$$

Next, terms in the square brackets in the conserved densities (3.64a) and (3.64b)

cause the divergence or nondifferentiability of the corresponding integrals of motion. Fortunately the terms with the square brackets in (3.64a,b,c,d) are the components of (2+1)-dimensional curls and so can be dropped without violating the local conservation laws (3.65).

Thus we arrive at the following expression for the linear momentum:

$$P^i = \int \left\{ \frac{i}{2} (\psi \overline{D}_i \overline{\psi} - \overline{\psi} D_i \psi) + e \rho_0 \epsilon_{ij} x^j B + \mu \epsilon_{ij} F_{0j} B + (A^i - \lambda^i) [-\mu \partial_k F_{0k} + \kappa B + e(\rho - \rho_0)] \right\} d^2 x. \quad (3.66)$$

Here λ^i ($i = 1, 2$) are arbitrary multipliers reflecting the gauge freedom in the constrained Hamiltonian formulation. One can verify that P^i is indeed a differentiable functional. It generates translations supplemented by local gauge transformations:

$$\begin{aligned} \psi(\mathbf{x}) &\rightarrow \psi_{a_s}(\mathbf{x}) = \psi(\mathbf{x}) + a_s \{ \psi(\mathbf{x}), P^s \} \\ &= \psi(\mathbf{x}) - a_s \partial_s \psi(\mathbf{x}) + i e a_s \lambda(\mathbf{x}, a) \psi(\mathbf{x}), \end{aligned} \quad (3.67a)$$

$$\begin{aligned} \overline{\psi}(\mathbf{x}) &\rightarrow \overline{\psi}_{a_s}(\mathbf{x}) = \overline{\psi}(\mathbf{x}) + a_s \{ \overline{\psi}(\mathbf{x}), P^s \} \\ &= \overline{\psi}(\mathbf{x}) - a_s \partial_s \overline{\psi}(\mathbf{x}) + i e a_s \lambda(\mathbf{x}, a) \overline{\psi}(\mathbf{x}), \end{aligned} \quad (3.67b)$$

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &\rightarrow \mathbf{A}_{a_s}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + a_s \{ \mathbf{A}(\mathbf{x}), P^s \} \\ &= \mathbf{A}(\mathbf{x}) - a_s \partial_s \mathbf{A}(\mathbf{x}) + a_s \nabla \lambda, \end{aligned} \quad (3.67c)$$

$$\begin{aligned} \mathbf{\Pi}(\mathbf{x}) &\rightarrow \mathbf{\Pi}_{a_s}(\mathbf{x}) = \mathbf{\Pi}(\mathbf{x}) + a_s \{ \mathbf{\Pi}(\mathbf{x}), P^s \} \\ &= \mathbf{\Pi}(\mathbf{x}) - a_s \partial_s \mathbf{\Pi}(\mathbf{x}) + \frac{\kappa}{2} a_s \nabla \times \lambda, \end{aligned} \quad (3.67d)$$

where $s = 1, 2$.

The explicit dependence of the momentum density, eq. (3.66), on x and y results in a nonstandard commutation relation between P^1 and P^2 :

$$\{P^1, P^2\} = \frac{d}{da_2} P^1 [\psi_{a_2}, \overline{\psi}_{a_2}, \mathbf{A}_{a_2}, \mathbf{\Pi}_{a_2}] \Big|_{a_2=0} = 2\pi \rho_0 n, \quad (3.68)$$

where n is the topological charge of the configuration ψ , *viz.*

$$n = \frac{e}{2\pi} \int B d^2x = \frac{1}{2\pi} \oint d \text{Arg } \psi. \quad (3.69)$$

$$= \frac{1}{2\pi i \rho_0} \int (\partial_1 \bar{\psi} \partial_2 \psi - \partial_1 \psi \partial_2 \bar{\psi}) d^2x. \quad (3.70)$$

In fact the bracket (3.68) can be evaluated without invoking the explicit expressions for the momenta, eq. (3.66). One only needs to know the Poisson brackets (3.67). This implies that the linear momentum in our theory should *necessarily* be translationally noninvariant.

E. Vortices in the condensate

1. Background solution and asymptotic behaviour

The condensate solution has the form: $\psi = \sqrt{\rho_0}$, $A_0 = -(1/e)U'(\rho_0)$, $\mathbf{A} = 0$. This solution exists even when the potential $U(\rho)$ does not possess a symmetry-breaking minimum at ρ_0 . We can, however, confine ourselves to potentials with $U'(\rho_0) = 0$ as this condition can always be accomplished by the transformation $A_0 \rightarrow A_0 - U'(\rho_0)/e$, $U(\rho) \rightarrow U(\rho) + U'(\rho_0)\rho$. The condensate solution is then $\psi = \sqrt{\rho_0}$, $A_0 = 0$, $\mathbf{A} = 0$, and it corresponds to an extremum of $U(\rho)$. By a singular gauge transformation the condensate generates a set of singular solutions,

$$\psi = \sqrt{\rho_0}e^{in\theta}, \quad A_0 = 0, \quad \mathbf{A} = \frac{n}{e} \frac{\mathbf{e}_\theta}{r} \quad (3.71)$$

which serve as $r \rightarrow \infty$ asymptotes for vortices. The magnetic flux of the vortex is quantized: $\Phi = \int B d^2x = \oint A_\theta r d\theta = 2\pi n/e$. Integrating eq. (3.57) over the entire plane we observe that the flux and the (regularized) number of particles in the vortex are related: $-\kappa\Phi = e \int (\rho - \rho_0) d^2x$. Hence N is quantized as well:

$$N = \int (\rho - \rho_0) d^2x = -\frac{2\pi\kappa}{e^2}n. \quad (3.72)$$

Confining consideration to radially symmetric configurations we write

$$\psi(\mathbf{x}) = \psi(r)e^{in\theta}, \quad (3.73a)$$

$$A^j(\mathbf{x}) = \epsilon^{jk}x_k \frac{\phi(r)}{r^2}, \quad j, k = 1, 2, \quad (3.73b)$$

$$A_0(\mathbf{x}) = A_0(r). \quad (3.73c)$$

Regularity at the origin requires $\psi(0) = 0$ when $n \neq 0$. For finite $\mathbf{A}(0)$ we should

also have $\phi(0) = 0$. Hence

$$\phi(r) = \int_0^r B(r)rdr$$

and the quantity $2\pi\phi(r)$ acquires a simple interpretation as the magnetic flux through a disc of radius r . As $r \rightarrow \infty$, $\phi(r) \rightarrow \Phi/(2\pi) = n/e$. The system (3.1) reduces to

$$-\Delta\psi + \frac{e^2}{r^2} \left(\phi - \frac{n}{e} \right)^2 \psi + eA_0\psi + U'(\rho)\psi = 0, \quad (3.74a)$$

$$-\mu \left(\phi_{rr} - \frac{\phi_r}{r} \right) - \kappa r \frac{dA_0}{dr} + 2e^2\rho \left(\phi - \frac{n}{e} \right) = 0, \quad (3.74b)$$

$$-\mu\Delta A_0 - \kappa \frac{\phi_r}{r} - e(\rho - \rho_0) = 0. \quad (3.74c)$$

The system (3.74) may exhibit both topological vortices, for which $n \neq 0$, and vorticity-free solutions, *bubbles*. Eq. (3.72) indicates that the bubbles have to be nodal in our model, i.e. $\rho(r) - \rho_0$ should necessarily change sign. In the pure Maxwell case ($\kappa = 0$) both bubbles and vortices must be nodal.

For small r the solutions to (3.74) can be sought as

$$\psi(r) = r^p(\psi_0 + \psi_1 r + \psi_2 r^2 + \dots), \quad p \geq 0; \quad (3.75a)$$

$$\phi(r) = r^m(\phi_0 + \phi_1 r + \phi_2 r^2 + \dots), \quad m > 0; \quad (3.75b)$$

$$A(r) = r^l(\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \dots), \quad l \geq 0, \quad (3.75c)$$

where $\phi_0, \alpha_0, \psi_0 \neq 0$. Substituting these into (3.74) one readily verifies that there are two possibilities corresponding to bubble and vortex solutions respectively, *viz.*

$$\psi(r) = \begin{cases} \psi_0 + O(r^2) & \text{(bubble)} \\ \psi_0 r^{|n|} + O(r^{|n|+1}) & \text{(vortex, } n \neq 0); \end{cases}$$

$$\phi(r) = O(r^2);$$

$$A_0(r) = A_0(0) + O(r^2).$$

(Here $\psi_0 \neq 0$.) It is also worth looking at the asymptotic behaviour at infinity.

Assuming exponentially localised solutions we write

$$\psi(r) = \sqrt{\rho_0} \left[1 + r^p(\psi_0 + \psi_1 r^{-1} + \psi_2 r^{-2} + \dots) e^{-\gamma r} \right], \quad (3.76a)$$

$$\phi(r) = \frac{n}{e} + r^m(\phi_0 + \phi_1 r^{-1} + \phi_2 r^{-2} + \dots) e^{-\gamma r}, \quad (3.76b)$$

$$A_0(r) = r^l(\alpha_0 + \alpha_1 r^{-1} + \alpha_2 r^{-2} + \dots) e^{-\gamma r}, \quad (3.76c)$$

where $\psi_0, \phi_0, \alpha_0 \neq 0$ and $\text{Re } \gamma > 0$.

When $\kappa \neq 0$, one ends up with the following cubic equation for the exponent squared:

$$\gamma^2(c^2 - \gamma^2) \left(\frac{1}{r_0^2} - \gamma^2 \right) = \frac{2e^2 \rho_0}{\mu} \left(\frac{2e^2 \rho_0}{\mu} - \gamma^2 \right), \quad (3.77)$$

where $c^2 = 2\rho_0 U''(\rho_0)$ and $1/r_0^2 = (2e^2 \rho_0 \mu + \kappa^2)/\mu^2$. We assume that $U(\rho)$ possesses a *minimum* at ρ_0 : $U''(\rho_0) > 0$. Then eq. (3.77) can obviously have no negative or zero roots, γ^2 . This means that there are three exponents $\gamma_1, \gamma_2, \gamma_3$ with positive (and three with negative) real parts. This justifies our exponential localisation assumption.

At the Chern-Simons limit ($\mu = 0$) eq. (3.77) reduces to a biquadratic equation. (The reason is that eqs. (3.74b) and (3.74c) are of the *first* order now.) The exponents can readily be found,

$$\gamma^2 = U''(\rho_0) \rho_0 \pm \rho_0 \sqrt{U''(\rho_0)^2 - 4 \frac{e^4}{\kappa^2}}. \quad (3.78)$$

Clearly for $U''(\rho_0) \geq 2e^2/\kappa$ the solution approaches the condensate monotonically, while for $0 < U''(\rho_0) < 2e^2/\kappa$ it undergoes an oscillatory decay to the background.

The pure Maxwell case ($\kappa = 0$) is exceptional. In this case we have, instead of eq. (3.77),

$$\gamma^2(c^2 - \gamma^2) = \frac{e^2 \rho_0}{\mu}. \quad (3.79)$$

This time we arrive at a quadratic equation because asymptotically eq. (3.74b) decou-

ples from eqs. (3.74a) and (3.74c) and we effectively have a system of *two* second-order equations. Provided $c^2 > 0$, γ^2 can not be a negative real number and so we again have exponentially localised solutions.

2. Self-dual limits

Static solutions correspond to stationary points of the Hamiltonian (3.58) on the constraint manifold (3.57). In the mixed Chern–Simons–Maxwell case, using the flux-vorticity relation, $\Phi = 2\pi n/e$, and the Bogomol’nyi decomposition,

$$|\mathbf{D}\psi|^2 = |(D_1 \pm iD_2)\psi|^2 \pm \frac{1}{2}\nabla \times \mathbf{J} \pm eB\rho, \quad (3.80)$$

the energy (3.58) takes the form

$$\begin{aligned} H = \int \left\{ |(D_1 \pm iD_2)\psi|^2 + \frac{\mu}{2} \left[B \pm \frac{e}{\mu}(\rho - \rho_0) \right]^2 \right. \\ \left. - \frac{e^2}{2\mu}(\rho - \rho_0)^2 + U(\rho) + \frac{\mu}{2}(\nabla A_0)^2 \pm \frac{1}{2}\nabla \times \mathbf{J} \right\} d^2x \\ \pm 2\pi\rho_0 n. \end{aligned} \quad (3.81)$$

For $U(\rho) = e^2/(2\mu)(\rho - \rho_0)^2$ (which corresponds to the Bose gas with δ -function pairwise repulsion) and fields approaching the condensate background (3.71) the energy can be rewritten as

$$\begin{aligned} H = \int \left\{ |(D_1 \pm iD_2)\psi|^2 + \frac{\mu}{2} \left[B \pm \frac{e}{\mu}(\rho - \rho_0) \right]^2 \right. \\ \left. + \frac{\mu}{2}(\nabla A_0)^2 \right\} d^2x \pm 2\pi\rho_0 n. \end{aligned} \quad (3.82)$$

The lower bound of energy, $H = \pm 2\pi\rho_0 n$ is saturated when the following self-duality equations are satisfied:

$$(D_1 \pm iD_2)\psi = 0, \quad (3.83a)$$

$$B \pm \frac{e}{\mu}(\rho - \rho_0) = 0, \quad (3.83b)$$

$$A_0 = 0. \quad (3.83c)$$

The upper (lower) sign should be associated with the positive (negative) vorticity n . This can be concluded, for example, simply from the fact that the energy is positive for positive $U(\rho)$ (see eq. (3.58)). Comparing (3.83b) with (3.57) we see that solutions of eqs. (3.83) lie on the constraint manifold only for $\kappa = \mu$ and only in the case of the upper sign. Thus, for $\kappa > 0$, only vortices with positive vorticities may exist. Eq. (3.83a) yields

$$A^i = \pm \frac{1}{2e} \epsilon_{ij} \partial_j \ln \rho + \frac{1}{e} \partial_i \text{Arg} \psi \quad (3.84)$$

and we should retain only the upper sign here. Substituting this into (3.83b) we arrive at

$$\nabla^2 \ln \rho = 2 \frac{e^2}{\mu} (\rho - \rho_0). \quad (3.85a)$$

The corresponding magnetic field is then

$$B = -\frac{e}{\mu} (\rho - \rho_0). \quad (3.85b)$$

Eq. (3.85a) appeared previously in the self-dual limit of the relativistic Higgs model with Maxwell term [15] and is known to possess solutions with the “topological vortex” asymptotic behaviour: $\rho(\infty) = \rho_0$, $\rho(0) \sim r^{2n}$, $n \geq 1$. No explicit solutions of this equation for $\rho_0 \neq 8$ have been found. Nevertheless eq. (3.85a) was proved to possess multivortex solutions [54] for which no closed form representations exist but which can be found, for instance, numerically. In Fig.3, we plot the $n = 1$ solution of eq. (3.85a) together with the corresponding $B(r)$, eq. (3.85b).

The self-duality reduction is also possible in the pure Chern-Simons case ($\mu = 0$). Making use of the identity (3.80) and the constraint (3.57) the energy (3.58) can be

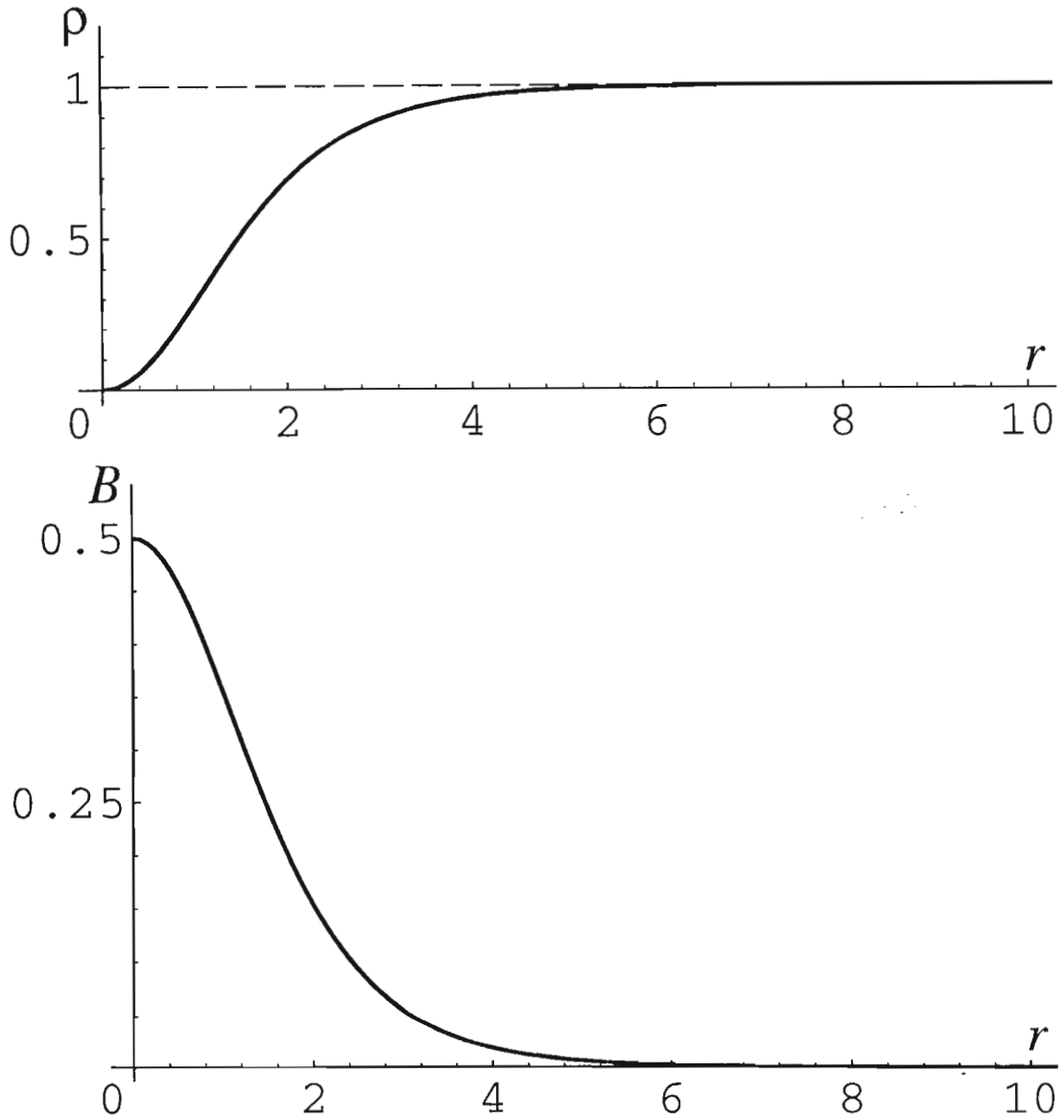


Fig. 3. The ($n = 1$) self-dual vortex. 1(a) and 1(b) correspond to both the Maxwell-Chern-Simons ($\mu = \kappa$) and the pure Chern-Simons ($\mu = 0$) case. Fig. 1(a) shows the solution of the Liouville-like equation (3.85a) while on 1(b) the corresponding magnetic field is plotted, eq. (3.85b).

represented as

$$H = \int \left[|(D_1 \pm iD_2)\psi|^2 \mp \frac{e^2}{\kappa}(\rho - \rho_0)\rho + U(\rho) \right] d^2x. \quad (3.86)$$

Invoking the integral form of the constraint, eq. (3.72), we rewrite this as

$$H = \int \left[|(D_1 \pm iD_2)\psi|^2 \mp \frac{e^2}{\kappa}(\rho - \rho_0)^2 + U(\rho) \right] d^2x \\ \pm 2\pi\rho_0 n. \quad (3.87)$$

With the choice of $U(\rho) = \pm(e^2/\kappa)(\rho - \rho_0)^2$ one observes that the energy is minimal provided $(D_1 \pm iD_2)\psi = 0$ whence we have, as before, eq. (3.84). Substituting this into the constraint equation (3.57) we arrive at

$$\nabla^2 \ln \rho = \pm 2 \frac{e^2}{\kappa} (\rho - \rho_0) \quad (3.88a)$$

and

$$B = -\frac{e}{\kappa}(\rho - \rho_0). \quad (3.88b)$$

Note that no equation for A_0 arises here. This is not surprising as A_0 is not a dynamical variable. (In our Hamiltonian formulation, it is just a Lagrangian multiplier.) For any combination of the Maxwell and Chern-Simons terms it can be determined from the spatial part of eq. (3.1b):

$$\mu \nabla \times B + \kappa \nabla \times A_0 = e\mathbf{J}. \quad (3.89)$$

The matter current is calculated from eq. (3.84): $\mathbf{J} = \mp \nabla \times \rho$. Substituting this into eq. (3.89) for $\mu = 0$ we can readily solve for A_0 :

$$A_0 = \mp \frac{e}{\kappa}(\rho - \rho_0). \quad (3.90)$$

In contrast to the mixed Chern-Simons-Maxwell case, *both* signs are allowed in

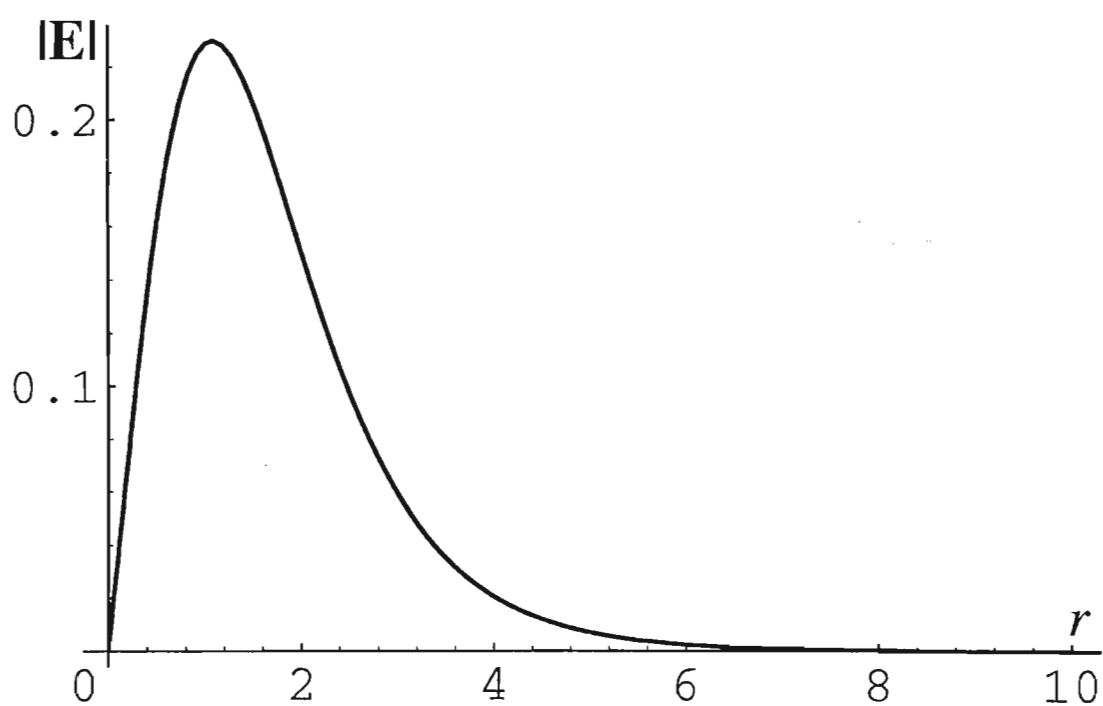


Fig. 4. Electric field carried by $n = 1$ vortex in the pure Chern-Simons case, $\mathbf{E} = -\nabla A_0$ with $A_0 = -(e/\kappa)(\rho - \rho_0)$. (In the Maxwell-Chern-Simons case, the vortex carries *no* electric field.)

(3.88a). As a result the pure Chern-Simons model exhibits a wider variety of self-dual solutions. Besides conventionally shaped topological vortices of positive vorticity, arising in the case of the repulsive potential $U(\rho) = e^2(\rho - \rho_0)^2/\kappa$ (figs. 3, 4), it admits a new type of solutions corresponding to a self-gravitating gas, $U(\rho) = -e^2(\rho - \rho_0)^2/\kappa$ [negative sign in (3.88a)]. These solutions approach the condensate in an oscillatory fashion:

$$\rho(r) \rightarrow \rho_0 \exp \left\{ \frac{C_1}{\sqrt{r}} \cos \left(\sqrt{\frac{2e^2\rho_0}{\kappa}} r - C_2 \right) \right\}, \quad (3.91)$$

as $r \rightarrow \infty$. Here $C_1, C_2 = \text{const}$. At the origin $\rho(0)$ can be both zero and nonzero; more precisely, as $r \sim 0$,

$$\psi(r) = \sqrt{h_0} \left[1 + \frac{e^2(\rho_0 - h_0)}{4\kappa} r^2 + \dots \right] \quad (3.92a)$$

or

$$\psi(r) = \sqrt{h_0} r^{2n} \left(1 + \frac{e^2\rho_0}{4\kappa} r^2 + \dots \right) e^{-in\theta}, \quad (3.92b)$$

with $h_0 = \text{const}$ and n positive integer. These two types of oscillating solutions are presented in Figs. 5a and 5b, respectively. Because of the slow approach of these solutions to ρ_0 (see eq. (3.91)) the corresponding flux, number of particles and, consequently, energy are infinite. Naturally this fact reduces the interest in these solutions.

When $\rho_0 \rightarrow 0$, the period of oscillation in (3.91) becomes infinite and the oscillatory solutions pass to the lumps of Jackiw and Pi [31, 30, 32]. The vorticity-free solutions, eq. (3.92a) (Fig. 5a), pass to the lumps with $n = 0$, whereas solutions of the second type, eq. (3.92b) (Fig. 5b) pass to the lumps with negative vorticity.

We close this section by mentioning that eq. (3.85a) arising in both self-dual limits and pertaining to the repulsive potential $U = e^2(\rho - \rho_0)^2/\kappa$ admits also vorticity-free lump-like solutions, $\rho(r) \rightarrow 0$ as $r \rightarrow \infty$. A straightforward phase space analysis

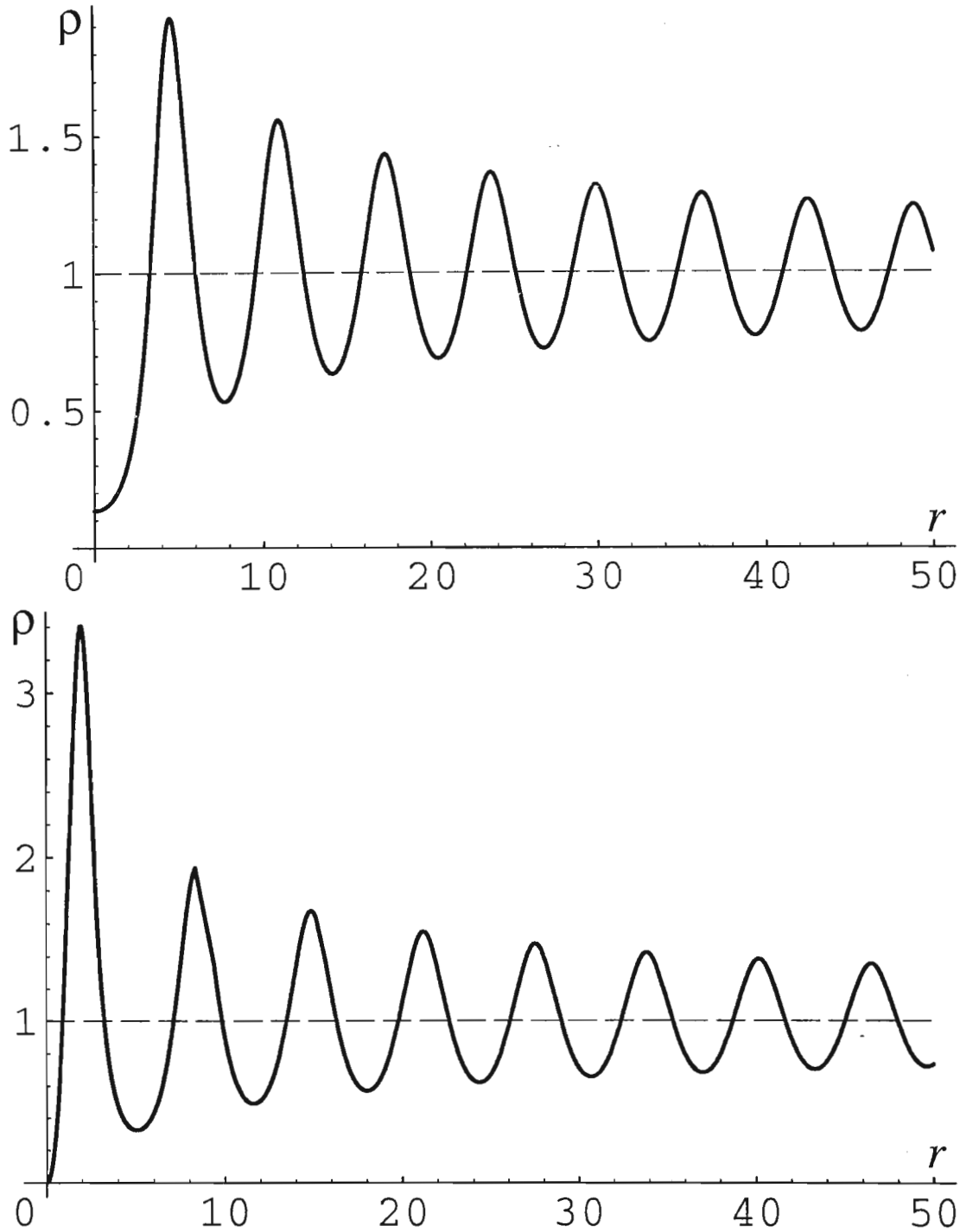


Fig. 5. Oscillating solutions in the case of attraction, $U = -e^2(\rho - \rho_0)^2/\kappa$. 1a. Vorticity-free solution. 1b. Solution with a negative vorticity.

shows that $\rho(0)$ can be any number between 0 and ρ_0 . As $\rho_0 \rightarrow 0$, these solutions disappear.

F. Equivalence of the background charge and the external magnetic field

This equivalence can be demonstrated on the level of Lagrangians. Suppose we have Jackiw–Pi’s model (3.1) where, in addition to the gauge field A_μ whose source is the matter field ψ , there is an external potential A_μ^{ext} . The Lagrangian is

$$\begin{aligned} \mathcal{L} = & \frac{i}{2}(\bar{\psi}D_0\psi - \psi\overline{D_0\psi}) - |D_k\psi|^2 \\ & - \frac{\mu}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\kappa}{4}\epsilon^{\gamma\alpha\beta}A_\gamma F_{\alpha\beta} - U(\rho) \end{aligned} \quad (3.93)$$

and the only distinction from eq. (3.3) is that in (3.93),

$$D_\mu = \partial_\mu + ieA_\mu + ieA_\mu^{\text{ext}}. \quad (3.94)$$

(Note that $F_{\alpha\beta}$ is defined as before: $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.) Now let us define a new gauge field,

$$\tilde{A}_\mu = A_\mu + A_\mu^{\text{ext}}. \quad (3.95)$$

This substitution does not involve derivatives and hence can be implemented directly in the Lagrangian. Inserting (3.95) into eq. (3.93) yields:

$$\begin{aligned} \mathcal{L} = & \frac{i}{2}(\bar{\psi}D_0\psi - \psi\overline{D_0\psi}) - |D_k\psi|^2 \\ & - \frac{\mu}{4}\tilde{F}_{\alpha\beta}\tilde{F}^{\alpha\beta} + \frac{\kappa}{4}\epsilon^{\gamma\alpha\beta}\tilde{A}_\gamma\tilde{F}_{\alpha\beta} - U(\rho) + \mathcal{L}_1, \end{aligned} \quad (3.96a)$$

where

$$\mathcal{L}_1 = \frac{\mu}{2}\tilde{F}^{\alpha\beta}F_{\alpha\beta}^{\text{ext}} - \frac{\kappa}{2}\epsilon^{\gamma\alpha\beta}\tilde{A}_\gamma F_{\alpha\beta}^{\text{ext}}, \quad (3.96b)$$

$$D_\mu = \partial_\mu + ie\tilde{A}_\mu, \quad (3.96c)$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \quad (3.96d)$$

and we have dropped the term

$$-\frac{\mu}{4} F_{\alpha\beta}^{\text{ext}} F^{\text{ext}\alpha\beta} + \frac{\kappa}{4} \epsilon^{\gamma\alpha\beta} A_\gamma^{\text{ext}} F_{\alpha\beta}^{\text{ext}} - \frac{\kappa}{2} \epsilon^{\gamma\alpha\beta} \partial_\alpha (A_\gamma^{\text{ext}} A_\beta)$$

the variation of which is zero. We can rewrite \mathcal{L}_1 as

$$\mathcal{L}_1 = (\kappa \tilde{A}_0 + \mu \tilde{B}) B^{\text{ext}} - \kappa \tilde{\mathbf{A}} \times \mathbf{E}^{\text{ext}} - \mu \tilde{\mathbf{E}} \cdot \mathbf{E}^{\text{ext}}. \quad (3.97)$$

Here the quantities with tildes correspond to the field \tilde{A}_μ . In particular $\tilde{\mathbf{E}}$ stands for the corresponding electric field (and not for the dual to \mathbf{E}). If $B^{\text{ext}} = \text{const}$, the term $\mu \tilde{B} B^{\text{ext}}$ may be dropped from \mathcal{L}_1 . Consequently, in the case of purely magnetic ($\mathbf{E}^{\text{ext}} = 0$) constant external field, \mathcal{L}_1 reduces simply to $\kappa \tilde{A}_0 B^{\text{ext}}$. If we rename the constant κB^{ext} as $e\rho_0$, eq. (3.96a) becomes nothing but our model with the background charge, eq. (3.49), for the fields ψ and \tilde{A}_μ , *viz.*

$$\begin{aligned} \mathcal{L} = & \frac{i}{2} (\bar{\psi} D_0 \psi - \psi \overline{D_0 \psi}) - |D_k \psi|^2 - \frac{\mu}{4} \tilde{F}_{\alpha\beta} \tilde{F}^{\alpha\beta} \\ & + \frac{\kappa}{4} \epsilon_{\gamma\alpha\beta} \tilde{A}_\gamma \tilde{F}_{\alpha\beta} - U(\rho) + e\rho_0 \tilde{A}_0. \end{aligned}$$

G. Solitonic bubbles in Chern–Simons theories.

In this section we examine the existence of nontopological solitons in the Chern–Simons model, namely solitonic bubbles. The bubbles are characteristic for both relativistic and nonrelativistic scalar models with competing self-interactions [10, 6]. The simplest examples of such models are the Higgs theory with a sextic potential (the so-called ψ^6 theory),

$$\mathcal{L} = |\partial_\mu \psi|^2 - (|\psi|^2 - \rho_0)^2 (|\psi|^2 - A) \quad (3.98)$$

and its nonrelativistic counterpart, the cubic-quintic nonlinear Schrödinger equation:

$$i\psi_t + \Delta\psi - \alpha_1\psi + \alpha_3\psi|\psi|^2 - \alpha_5\psi|\psi|^4 = 0. \quad (3.99)$$

The solitonic bubbles can be defined [6] as *vorticity-free* axially symmetric configurations with the following properties:

$$\psi(x) = \psi(r) \rightarrow \sqrt{\rho_0} \quad \text{as } r \rightarrow \infty, \quad (3.100a)$$

$$\psi(0) > 0, \quad \psi_r(0) = 0. \quad (3.100b)$$

Similar to vortices the bubbles have the form of localised rarefaction domains on the background of the spatially uniform solution. However, as opposed to topological vortices, they are inherently unstable. Invoking the boson gas interpretation of eq. (3.99), bubble-like solitons have the meaning of the low density cavities in the constant density condensate.

Physically the importance of solitonic bubbles stems from the fact that these unstable structures can seed first-order phase transitions [11, 35]. Indeed the energy functional for static configurations of the models (3.98) and (3.99) is

$$E = \int \{|\nabla\psi|^2 + U(|\psi|^2)\} d^2x, \quad (3.101)$$

where $U(|\psi|^2) = (|\psi|^2 - \rho_0)^2(|\psi|^2 - A)$. When $0 < A < \rho_0$, the homogeneous solution $\psi = \sqrt{\rho_0}$ realises only a local minimum of the energy, while the global minimum is achieved on a homogeneous solution $\psi = 0$ (Fig. 6). Thermodynamically this implies that the state $\psi = 0$ is stable while the state with $\psi = \sqrt{\rho_0}$ is *metastable*. The bubble solution represent the configuration with $\psi = \sqrt{\rho_0}$ everywhere except for a finite domain, where ψ is close to zero. Accordingly it is interpreted as a nucleus of the stable phase in the metastable one. It has been shown that solitonic bubbles are

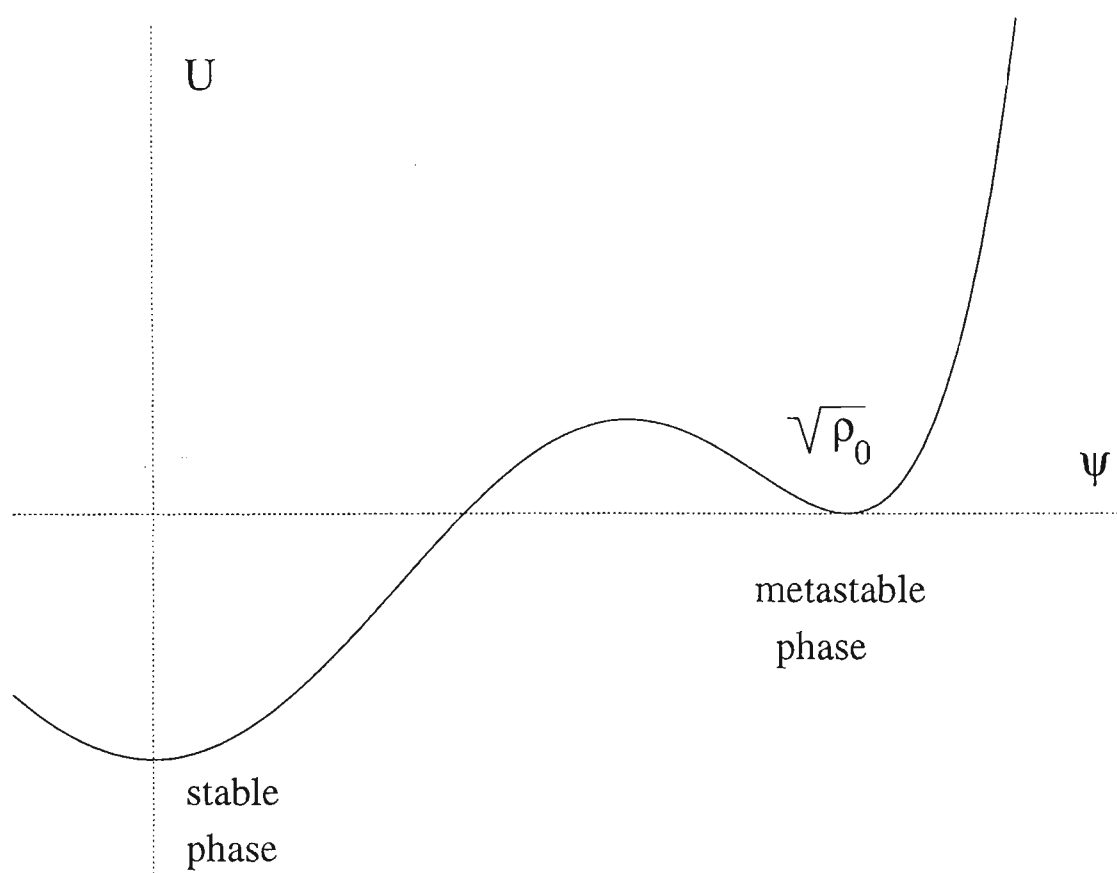


Fig. 6. The energy of homogeneous states in the ψ^3 - ψ^5 model.

inherently unstable, no matter what is the number of dimensions or the form of the nonlinearity [11]. Therefore, after a solitonic bubble has nucleated, it will instantly start to grow and in the process convert the entire space into the stable state.

In the context of gauge theories the idea of inhomogeneous nucleation is not unknown either. Jensen and Steinhardt [34] proposed that phase transitions in superfluid ^3He – ^4He mixtures can be mediated by dissociating vortices. (The model they discussed was the ψ^6 Higgs field coupled to the Maxwell field.)

An important question is whether solitonic bubbles survive interaction with gauge fields. If so, these would be natural candidates for the role of nucleation seeds in gauge theories.

The problem of the existence of bubbles appears to be quite nontrivial. So far it has not been completely solved. There seem to be principal differences between relativistic and nonrelativistic theories in this respect. It turns out that relativistic theories cannot support bubble-like solitons. However, we are not able to reach a similar negative conclusion within our nonrelativistic theory.

In what follows we demonstrate the absence of bubbles in the relativistic model. The Lagrangian density for the relativistic theory, comprising both Maxwell and Chern-Simons terms, is given by

$$\mathcal{L} = (D_\mu \psi)^* (D^\mu \psi) - \frac{1}{4} \mu F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} \kappa \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} - U(\rho). \quad (3.102)$$

Variation of the corresponding action yields the Euler-Lagrange equations

$$\bar{D}_\mu D^\mu \psi + \frac{dU}{d\rho} \psi = 0 \quad (3.103a)$$

$$\mu \partial_\beta F^{\beta\alpha} + \kappa \epsilon^{\alpha\beta\gamma} \partial_\beta A_\gamma = J^\alpha. \quad (3.103b)$$

Here J^α is the relativistic conserved matter current,

$$J^\alpha = ie\{\psi^*(D^\alpha\psi) - \psi(D^\alpha\psi)^*\}. \quad (3.104)$$

When $e = 0$, the gauge field decouples and for static fields ($\psi_t = 0$) eq. (3.103a) becomes simply

$$-\Delta\psi + \left. \frac{dU}{d\rho} \right|_{\rho=\psi^2} \psi = 0. \quad (3.105)$$

It can be shown [6, 10] that eq. (3.105) has a bubble solution provided $U(\psi^2)$ has at least two minima (say, at $\psi = 0$ and $\psi = \sqrt{\rho_0}$) such that $U(0) < U(\rho_0)$.

Coming back to the model with the gauge field, eqs. (3.103), we look for static solutions with the bubble-type boundary conditions on ψ (3.100). We restrict ourselves to solutions with radially symmetric charge density and electric and magnetic fields:

$$\psi(\mathbf{x}) = \psi(r)e^{i\chi(r,\theta)}, \quad (3.106a)$$

$$A^j = \epsilon^{jk} x_k \frac{\phi(r)}{r^2} \quad j, k = 1, 2, \quad (3.106b)$$

$$A_0 = A_0(r). \quad (3.106c)$$

The spatial part of eq. (3.103b) is then

$$-\mu\Delta\mathbf{A} - \kappa \frac{dA_0}{dr} \mathbf{e}_\theta - 2e\rho\nabla\psi + 2e^2\rho\mathbf{A} = 0, \quad (3.107)$$

where $\mathbf{A} = \{A^1, A^2\} = (\phi(r)/r)\mathbf{e}_\theta$ and $\mathbf{e}_r, \mathbf{e}_\theta$ are the unit vectors in r and θ directions respectively. Now since

$$\Delta\mathbf{A} = \left(\frac{\phi_{rr}}{r} - \frac{\phi_r}{r^2} \right) \mathbf{e}_\theta$$

and

$$\nabla\chi = \chi_r \mathbf{e}_r + \frac{\chi_\theta}{r} \mathbf{e}_\theta,$$

the r -component of eq. (3.107) is simply $\chi_r = 0$ and hence $\chi = \chi(\theta)$. Substituting

this into eq. (3.107) we obtain

$$\mu r \left(\frac{\phi_r}{r} \right)' + \kappa \frac{dA_0}{dr} - 2e^2 \rho \frac{\phi}{r} = -\frac{2e}{r} \rho \frac{d\chi}{d\theta}. \quad (3.108)$$

The LHS of (3.108) depends only on r while the RHS is, up to an r -dependent factor, a function of θ . Consequently $d\chi/d\theta = n$, a constant, which can always be chosen to be positive. (The case of negative n s can be easily recovered by the transformation $a \rightarrow -a$, $A_0 \rightarrow -A_0$.) In order that $\psi(\mathbf{x})$ be a single-valued function this constant is required to be an integer.

With this choice of ψ and A^μ eqs. (3.103) take the form

$$-\Delta\psi + (n - e\phi)^2 \frac{\psi}{r^2} - e^2 A_0^2 \psi + \frac{dU}{d\rho} \psi = 0, \quad (3.109a)$$

$$\mu r \left(\frac{\phi_r}{r} \right)' + \kappa r A_0' + 2e(n - e\phi)\rho = 0, \quad (3.109b)$$

$$-\frac{\mu}{r} (rA_0')' - \kappa \frac{\phi_r}{r} + 2e^2 \rho A_0 = 0. \quad (3.109c)$$

Note that now our *Ansatz* is exactly the same as the one we used for vortices (eqs 3.73), *viz.*

$$\psi = \psi(r)e^{in\theta}, \quad A^j = \epsilon^{jk} x_k \frac{\phi(r)}{r^2}, \quad A_0 = A_0(r). \quad (3.110)$$

In order to avoid a singularity at the origin either $\psi(0)$ or n must be set equal to zero. Since we are interested in solutions with the boundary condition $\psi(0) > 0$, we have to put $n = 0$.

In actual fact there is no loss of generality in choosing $n = 0$. The reason is that we can define a new $\tilde{\phi}$: $\tilde{\phi} = \phi - n/e$ and this $\tilde{\phi}$ will satisfy the same system (3.109) but this time with $n = 0$. This fact is a manifestation of the gauge invariance of the system (3.103). If the pair A_μ, ψ is a solution of this system, then

$$\tilde{A}_\mu = A_\mu - \frac{1}{e} \partial_\mu \chi, \quad \tilde{\psi} = \psi e^{i\chi} \quad (3.111)$$

is a solution as well. The substitution of $\tilde{\phi} = \phi - n/e$ instead of ϕ amounts to the choice $\chi = -n\theta$, i. e. to the transformation to real ψ 's.

For the sake of convenience we rewrite the system (3.109) for the case $n = 0$:

$$-\Delta\psi + e^2\phi^2\frac{\psi}{r^2} - e^2A_0^2\psi + \frac{dU}{d\rho}\psi = 0, \quad (3.112a)$$

$$\mu r \left(\frac{\phi_r}{r}\right)' + \kappa r A_0' - 2e^2\phi\rho = 0, \quad (3.112b)$$

$$-\frac{\mu}{r}(rA_0')' - \kappa\frac{\phi_r}{r} + 2e^2\rho A_0 = 0. \quad (3.112c)$$

Now let us specify the boundary conditions. As $r \rightarrow \infty$, we have $\rho \rightarrow \rho_0$ and eqs. (3.112a)–(3.112b) become

$$\mu r \left(\frac{\phi_r}{r}\right)' + \kappa r A_0' - 2e^2\rho_0\phi = 0, \quad (3.113a)$$

$$\frac{\mu}{r}(rA_0')' + \kappa\frac{\phi_r}{r} - 2e^2\rho_0A_0 = 0. \quad (3.113b)$$

Physically only solutions with finite electric and magnetic fields make sense, where

$$\mathbf{E} = -\nabla A_0 = -\frac{dA_0}{dr}\mathbf{e}_r, \quad B = \nabla \times \mathbf{A} = \frac{\phi_r}{r}.$$

Hence we shall not allow $a(r)$ to grow faster than r^2 and $A_0(r)$ to grow faster than r at infinity. Let us look for solutions in the form

$$\phi(r) = r^m(\phi_0 + \phi_1 r^{-1} + \phi_2 r^{-2} + \dots)e^{-\gamma r}, \quad (3.114a)$$

$$A_0(r) = r^l(\alpha_0 + \alpha_1 r^{-1} + \alpha_2 r^{-2} + \dots)e^{-\gamma r}. \quad (3.114b)$$

Substituting this into eqs. (3.113), we find after some algebra that $m = 1/2$, $l = -1/2$, and

$$\gamma_{1,2,3,4} = \sigma_1 \frac{\kappa}{2\mu} + \sigma_2 \sqrt{\left(\frac{\kappa}{2\mu}\right)^2 + 2e^2\frac{\rho_0}{\mu}},$$

where σ_1 and σ_2 are uncorrelated sign factors, $\sigma_{1,2} = \pm 1$. Accordingly there are two

exponentially growing and two exponentially decaying solutions. Power-law growth and asymptotically constant solutions are thereby excluded.

Some doubts can be raised only regarding the pure Chern-Simons limit, $\kappa/2\mu \rightarrow \infty$, where two of the four γ 's tend to zero. However, it is not difficult to demonstrate that in this case asymptotically constant or polynomially growing solutions cannot emerge either. Indeed expression of $A_0(r)$ from eq. (3.112c) and substitution of it into eq. (3.112b) yield a scalar equation

$$-\frac{d}{dr} \left(\frac{1}{\rho r} \frac{d\phi}{dr} \right) + \left(\frac{2e^2}{\kappa} \right)^2 \frac{\rho}{r} \phi = 0. \quad (3.115)$$

Writing

$$\phi(r) = r^{1/2} b(r),$$

and letting $r \rightarrow \infty$, we obtain from eq. (3.115)

$$b_{rr} - \left(\frac{2e^2 \rho_0}{\kappa} \right)^2 b = 0,$$

the solutions of which are, of course, two exponentials.

Consequently, the only admissible boundary conditions at infinity are

$$|\psi(r)|^2 \rightarrow \rho_0, \quad \phi(r) \rightarrow 0, \quad \text{and} \quad A_0(r) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \quad (3.116)$$

Note that these conditions are compatible with eq. (3.112a).

To find the boundary conditions at the origin we expand a , A_0 and ψ as

$$\phi(r) = r^m (\phi_0 + \phi_1 r + \phi_2 r^2 + \dots), \quad (3.117a)$$

$$A_0(r) = r^l (\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \dots), \quad (3.117b)$$

$$\psi(r) = \psi_0 + \psi_1 r + \psi_2 r^2 + \dots, \quad \psi_0 \neq 0. \quad (3.117c)$$

Physically we are prepared to tolerate solutions with *local* singularities. Hence neg-

ative m and l should not be *a priori* excluded. Substituting these expressions into eqs. (3.112) we find two possibilities:

$$(i) \quad l = 0 \text{ or } 2, \quad m = 2, \quad \phi_1 = \alpha_1 = \psi_1 = 0, \quad \text{and}$$

$$(ii) \quad l = 0, \quad m \geq 4, \quad \alpha_1 = \psi_1 = 0.$$

Thus the boundary conditions at the origin are

$$\psi(r) = \psi(0) + O(r^2), \quad \phi(r) = \phi_0 r^2 + O(r^4), \quad A_0(r) = A_0(0) + O(r^2). \quad (3.118)$$

Now let us demonstrate that the system (3.112) does not possess solutions with the boundary conditions (3.116) and (3.118) other than the vacuum solution,

$$\phi(r) \equiv A_0(r) \equiv 0, \quad \psi \equiv \sqrt{\rho_0}. \quad (3.119)$$

To this end note that eqs. (3.112b)–(3.112c) can be represented simply as

$$L y(r) = 0, \quad (3.120)$$

where $y(r)$ is a vector-function,

$$y(r) = \begin{pmatrix} \phi(r) \\ A_0(r) \end{pmatrix},$$

and L is a linear differential operator in the space of vector-functions:

$$L y = \begin{pmatrix} -\mu r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} + 2e^2 \rho & -\kappa r \frac{d}{dr} \\ -\kappa \frac{1}{r} \frac{d}{dr} & -\mu \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + 2e^2 \rho \end{pmatrix} \begin{pmatrix} \phi \\ A_0 \end{pmatrix}. \quad (3.121)$$

It turns out that the operator L is positive and hence does not have zero eigenvalues in the space of vector-functions with boundary conditions (3.116) and (3.118). Indeed,

denoting

$$(y^{(1)}, y^{(2)}) = \int_0^\infty \left(\frac{\phi^{(1)}\phi^{(2)}}{r} + A_0^{(1)}A_0^{(2)}r \right) dr, \quad (3.122)$$

we have

$$(y, Ly) = \mu \int_0^\infty \left[\frac{\phi_r^2}{r} + \left(\frac{dA_0}{dr} \right)^2 r \right] dr + 2e^2 \int_0^\infty \left(\frac{\phi^2}{r} + A_0^2 r \right) \rho dr > 0 \quad (3.123)$$

and hence eq. (3.120) cannot be satisfied.

It is worthwhile to mention that, in the pure Chern-Simons case ($\mu = 0$), the nonexistence of bubbles is quite obvious. In this case eq. (3.120) reduces to eq. (3.115) which is nothing but a Sturm-Liouville equation with a positive potential or, speaking in quantum-mechanical terms, with a potential barrier. Localised solutions (bound states) clearly cannot exist in such a potential. As we have shown above, the asymptotically constant or slowly growing solutions cannot arise either.

The argument above does not forbid the existence of *vortices*, i. e. solutions of eqs. (3.109) with $n > 0$, though it may seem to do so. Although the vortices are among solutions of the very same system (3.112) (due to the gauge invariance of the model), the boundary conditions at the origin for vortices differ from those we imposed on bubbles, eqs. (3.118). More precisely vortex solutions arise if we discard the requirement $\psi_0 \neq 0$ in eq. (3.117c) and take $\psi(r)$ in the form

$$\psi(r) = r^n(\psi_0 + \psi_1 r + \psi_2 r^2 + \dots). \quad (3.124)$$

Substituting (3.117a), (3.117b) and (3.124) into eqs. (3.112) we find that $m = 0$, $l \geq 2$ or $l = 0$, and $\phi_1 = 0$, while $\phi_0 = n/e$. (In the case $l = 0$ we have, in addition, $\alpha_1 = 0$.) Hence, if we treat vortices as solutions of eqs. (3.112), the boundary conditions should be

$$|\psi(r)|^2 \rightarrow \rho_0, \quad \phi(r) \rightarrow 0, \quad A_0(r) \rightarrow 0 \quad (3.125a)$$

exponentially as $r \rightarrow \infty$ and

$$\psi(r) = \psi_0 r^n + O(r^{n+1}), \quad \phi(r) = -\frac{n}{e} + O(r^2), \quad A_0(r) = A(0) + O(r^2) \quad (3.125b)$$

as $r \rightarrow 0$. Now the functions $\psi(r)$, $\tilde{\phi}(r) = \phi(r) + n/e$ and $A_0(r)$ will satisfy eqs. (3.109) with boundary conditions

$$|\psi(r)|^2 \rightarrow \rho_0, \quad \phi(r) \rightarrow n/e, \quad A_0(r) \rightarrow 0 \quad (3.126a)$$

exponentially as $r \rightarrow \infty$ and

$$\psi(r) = \psi_0 r^n + O(r^{n+1}), \quad \phi(r) = O(r^2), \quad A_0(r) = A(0) + O(r^2) \quad (3.126b)$$

as $r \rightarrow 0$. (We have omitted the tilde.)

In the space of functions with the boundary conditions (3.125) the operator L is no longer positive. The reason is simply that (y, Ly) cannot be represented in the form (3.123) since the boundary term $\mu\phi\phi_r/r|_{r=0}$ will not vanish when we integrate by parts. Consequently our argument on the nonexistence of the bubbles does not extend to the case of vortices.

However, this boundary term does vanish in the space of functions with boundary conditions (3.126) and hence we may try to perform our analysis on the system (3.109) instead. The last two equations, eq. (3.109b) and (3.109c), can be written as

$$Ly(r) = f(r), \quad (3.127)$$

where

$$f(r) = \begin{pmatrix} 2en\rho(r) \\ 0 \end{pmatrix}.$$

The solvability condition for eq. (3.127) is that $f(r)$ should be orthogonal to the

solution of the conjugate homogeneous equation,

$$L^\dagger y_0(r) = 0, \quad (3.128)$$

where L^\dagger is obtained from L by the substitution $\frac{d}{dr} \rightarrow -\frac{d}{dr}$:

$$L^\dagger = \begin{pmatrix} -\mu r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} + 2e^2 \rho & \kappa r \frac{d}{dr} \\ \kappa \frac{1}{r} \frac{d}{dr} & -\mu \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + 2e^2 \rho \end{pmatrix}. \quad (3.129)$$

To check this one can simply compute the scalar product (y_0, f) :

$$(y_0, f) = (y_0, L y) = (y, L^\dagger y_0) = 0.$$

We have already seen that the operator L is positive and so is L^\dagger . (L^\dagger results from L simply by substituting $\kappa \rightarrow -\kappa$ while the representation (3.123) does not involve κ at all.) Consequently eq. (3.128) does not have solutions with the boundary conditions (3.125) and eq. (3.127) is always solvable.

We have thus demonstrated that, unlike relativistic vortices, relativistic bubbles cannot nucleate in the presence of a gauge field, no matter whether the latter is of the Chern-Simons or Maxwell type or a combination of the two.

In the nonrelativistic case, however, a similar argument does not lead to the no-go conclusion. This leaves the hope that bubble-like solitons might arise in the nonrelativistic theory. Note that they will have to be nodal since the equation (3.72), relating the number of particles to the topological charge, implies

$$\int (\rho - \rho_0) d^2 x = 0$$

for vorticity-free solutions

CHAPTER IV

PRELIMINARY GROUP CLASSIFICATION OF NONHOMOGENEOUS WAVE
EQUATIONS

The problem of classification of partial differential equations according to their symmetries was first considered by Sophus Lie [39]. In his paper he studied a class of linear second-order equations with two independent variables. Since then a great number of works giving the group classification for concrete equations has been published (see [28] for a comprehensive list). The classification is performed up to the transformations of variables that do not change the structure of the equation (equivalence transformations) and requires a knowledge of the general solution of the determining equation. Unfortunately it is quite common that the general solution of the determining equation cannot be found. For such problems a new method, the so-called preliminary group classification, was proposed [2]. The idea of the new method is to look for the extensions of the principal Lie algebra admitted by a class of differential equations among the elements of its equivalence algebra. Since the algebra of equivalence transformations can usually be found quite easily, the problem of the classification reduces to the algebraic problem of constructing the optimal system of subalgebras, which in the case of a finite-dimensional equivalence algebra can be effectively solved [44]. However, in general there are no effective algorithms for constructing the optimal system of an infinite-dimensional algebra, so that one can effectively carry out the classification only relative to the finite-dimensional subalgebras of the full algebra of equivalence transformations.

This method was applied [27] to the equations of the type

$$v_{tt} = f(x, v_x)v_{xx} + g(x, v_x) \quad (4.1)$$

governing the longitudinal vibrations of elastic nonhomogeneous strings or bars. As this class of equations possesses an infinite-dimensional equivalence algebra $L_{\mathcal{E}}$, the authors of [27] confined themselves to the classification relative to its 7-dimensional subalgebra L_7 . The result of the classification is a table of 29 non-equivalent equations.

We continue the analysis of the equations (4.1). The question we now have in mind is how we can utilize the equivalence transformations that are not contained in L_7 . Hence we investigate a countable-dimensional subalgebra $L_{\#}$ of $L_{\mathcal{E}}$ or rather a countable number of n -dimensional extensions L_n of L_7 . Firstly we use the inner automorphisms of $L_{\#}$ as external automorphisms for L_7 and reduce the number of one-dimensional subalgebras in the optimal system, found in [27], by four. Secondly we fulfill the preliminary classification of eqs. (4.1) relative to the countable-dimensional subalgebra $L_{\#}$.

A. The problem of group classification

The problem of a group classification of differential equations belonging to a certain family consists of dividing equations of the family into classes, nonequivalent with respect to a change of variables, such that members of the same class have equivalent algebras of symmetries. The algebras of symmetry generators

$$X = \xi^1(t, x, v) \frac{\partial}{\partial t} + \xi^2(t, x, v) \frac{\partial}{\partial x} + \eta(t, x, v) \frac{\partial}{\partial v}, \quad (4.2)$$

admitted by equations (4.1) is found from the invariance condition (1.8) which in this case reads

$$X_2 \left[v_{tt} - f(x, v_x) v_{xx} - g(x, v_x) \right] \Big|_{(4.1)} = 0. \quad (4.3)$$

Here X_2 is the second prolongation of X , given by

$$X_2 = X + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_x} + \zeta_{11} \frac{\partial}{\partial v_{tt}} + \zeta_{22} \frac{\partial}{\partial v_{xx}}$$

with

$$\zeta_1 = D_t(\eta) - v_t D_t(\xi^1) - v_x D_t(\xi^2)$$

$$\zeta_2 = D_x(\eta) - v_t D_x(\xi^1) - v_x D_x(\xi^2)$$

$$\zeta_{11} = D_t(\zeta_1) - v_{tt} D_t(\xi^1) - v_{tx} D_t(\xi^2)$$

$$\zeta_{22} = D_x(\zeta_2) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2).$$

The operators D_t and D_x denote the total derivatives with respect to t and x :

$$D_t = \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} + \dots$$

$$D_x = \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v} + v_{tx} \frac{\partial}{\partial v_t} + v_{xx} \frac{\partial}{\partial v_x} + \dots$$

After use is made of the above equations the invariance condition (4.3) reduces to the following *determining equation*:

$$[\zeta_{11} - f\zeta_{22} - v_{xx}(\xi^2 f_x + \zeta_2 f_{vx}) - \xi^2 g_x - \zeta_2 g_{vx}]|_{(4.1)} = 0. \quad (4.4)$$

The algebra of the symmetries admitted by every equation of the family (4.1) can be easily found. Assuming that (4.4) holds for arbitrary f and g we obtain

$$\xi^2 = 0, \quad \zeta_2 = 0, \quad \zeta_{22} = \zeta_{11} = 0$$

or

$$\xi^1 = 0, \quad \xi^2 = 0, \quad \eta = C_2 + C_3 t.$$

Hence eq. (4.1) admits a three-dimensional *principal algebra* $L_{\mathcal{P}}$ with the basis

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial v}, \quad X_3 = t \frac{\partial}{\partial v}.$$

Now, in order to perform the classification we should specify f and g such that the corresponding equation (4.1) admits an additional symmetry. Two members of the family are considered to be in the same class if they can be transformed into each other by a change of variables. The transformations of variables acting on the members of the family are called *equivalence transformations* and are used to choose the simplest representative of each class. Clearly such classification requires the knowledge of the admitted algebra for an arbitrary choice of f and g , i.e. we need to know the general solution of (4.4), which cannot be found due to the high arbitrariness in the choice of the coefficients f and g . The full group classification of eqs (4.1) therefore cannot be fulfilled. We can however perform a partial group classification using the method of *preliminary group classification* recently proposed by Ibragimov *et al.* [27]. The main idea of the new method is that instead of solving the determining equation additional symmetries are sought only among the elements of the equivalence algebra. The classification is partial since equations that do not admit symmetries from the equivalence algebra are not classified.

B. The equivalence algebra and preliminary group classification

This section is devoted to the equivalence algebra for eq. (4.1). An equivalence transformation is a change of variables that takes any equation belonging to the family into an equation of the same family. Usually the group of equivalence transformations can be easily found. In the case of eq. (4.1) it consists of discrete transformations:

$$t \rightarrow -t$$

$$x \rightarrow -x$$

$$v \rightarrow -v, \quad g \rightarrow -g,$$

and of continuous transformations constituting a subgroup E_c . The generators of the continuous subgroup E_c span an *equivalence algebra* and can be sought in the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g},$$

where $\xi^i = \xi^i(t, x, v)$, $\eta = \eta(t, x, v)$, $\mu^k = \mu^k(t, x, v, v_t, v_x, f, g)$. The invariance condition (1.8) for the generators Y should be written for the system

$$v_{tt} - f v_{xx} - g = 0$$

$$f_t = f_v = f_{v_t} = g_t = g_v = g_{v_t} = 0.$$

Solving the corresponding determining equation we find that the equivalence algebra $L_{\mathcal{E}}$ is spanned by

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, & Y_2 &= t \frac{\partial}{\partial v}, & Y_3 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \\ Y_4 &= -\frac{t}{2} \frac{\partial}{\partial t} + f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}, & Y_5 &= \frac{t^2}{2} \frac{\partial}{\partial v} + \frac{\partial}{\partial g}, \\ V_\phi &= \phi(x) \frac{\partial}{\partial x} + 2\phi'(x) f \frac{\partial}{\partial f} + \phi''(x) v_x f \frac{\partial}{\partial g}, \\ W_F &= F(x) \frac{\partial}{\partial v} - F''(x) f \frac{\partial}{\partial g}, \end{aligned} \tag{4.5}$$

where $\phi(x)$ and $F(x)$ are arbitrary functions. Hence the equivalence algebra is *infinite-dimensional*. In order to perform a preliminary group classification with respect to one-dimensional subalgebras we have to specify f and g such that the corresponding equations (4.1) possess an additional symmetry from $L_{\mathcal{E}}$. For a given symmetry generator $Y \in L_{\mathcal{E}}$ such f and g can be found simply as differential invariants of Y .

Clearly if an equivalence transformation with the generator Y leaves equations

$$f = \Phi(x, v_x) \quad \text{and} \quad g = \Gamma(x, v_x) \quad (4.6)$$

invariant, then the projection of Y on the space (t, x, v) will be admitted by the corresponding equation (4.1). Hence the remaining part of the preliminary group classification is to divide generators $Y \in L_{\mathcal{E}}$ into classes such that members of different classes can not be transformed into each other by equivalence transformations. Consequently, equations (4.1) obtained as invariants of the generators Y belonging to different classes will be non-equivalent with respect to the transformation of variables. Since $L_{\mathcal{E}}$ acts on itself as an *inner automorphism* the preliminary classification reduces to the problem of constructing all classes of nonsimilar one-dimensional subalgebras of $L_{\mathcal{E}}$ (optimal system of one-dimensional subalgebras). Since only those Y that have nonzero projections on the space of (x, f, g) yield nontrivial invariants (4.6), it is sufficient to consider the optimal system for a subalgebra $L'_{\mathcal{E}}$ spanned by

$$\begin{aligned} Z_1 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \\ Z_2 &= -\frac{t}{2} \frac{\partial}{\partial t} + f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}, \quad Z_3 = \frac{t^2}{2} \frac{\partial}{\partial v} + \frac{\partial}{\partial g}, \\ V_{\phi} &= \phi(x) \frac{\partial}{\partial x} + 2\phi'(x) f \frac{\partial}{\partial f} + \phi''(x) v_x f \frac{\partial}{\partial g}, \\ W_F &= F(x) \frac{\partial}{\partial v} - F''(x) f \frac{\partial}{\partial g}. \end{aligned} \quad (4.7)$$

C. The countable-dimensional subalgebra $L_{\#}$

There exists an effective and simple algorithm for the construction of an optimal system of subalgebras of a *finite-dimensional* Lie algebra [44]. Unfortunately this algorithm loses its effectiveness when an infinite-dimensional algebra is considered, which is the case with which we should deal. ($L'_{\mathcal{E}}$ is infinite-dimensional.) A way to

Table I. Table of commutators of $L'_\mathcal{E}$.

	Z_1	Z_2	Z_3	V_ψ	W_G
Z_1	0	0	0	$V_{x\psi'-\psi}$	$W_{xG'-2G}$
Z_2	0	0	$-Z_3$	0	0
Z_3	0	Z_3	0	0	0
V_ϕ	$V_{\phi-x\phi'}$	0	0	$V_{\phi\psi'-\psi\phi'}$	$W_{\phi G'}$
W_F	$W_{2F-xF'}$	0	0	$-W_{\psi F'}$	0

get round this is to perform a classification relative to a subalgebra of $L'_\mathcal{E}$ for which the construction of the optimal system is feasible. It will be desirable, of course, to choose a largest possible subalgebra. Fortunately $L'_\mathcal{E}$ possesses a large (in fact a countable-dimensional) subalgebra $L_\#$ for which an optimal system of subalgebras can be constructed. The classification with respect to a countable-dimensional $L_\#$ considerably improves the classification of Ibragimov *et al.* [27] who used a 7-dimensional subalgebra L_7 .

$L_\#$ can be defined as follows. From the table of commutators (Table I) one can easily see that subalgebras of arbitrary finite dimensions $n + 5, n \geq 1$, can be constructed by restricting ϕ and F in (4.7) to

$$\phi = 1, x;$$

$$F = x, \frac{1}{2}x^2, \dots, \frac{1}{n}x^n.$$

We denote the corresponding generators V_ϕ and W_F as V_1, V_2 and W_1, W_2, \dots, W_n . The algebra L_7 examined in [27] corresponds to the case $n = 2$. All these subalgebras L_{n+5} are contained in the countable-dimensional subalgebra $L_\#$ that corresponds to

Table II. Table of commutators of L_n .

	Z_1	Z_2	Z_3	V_1	V_2	W_1	W_2	...	W_n
Z_1	0	0	0	$-V_1$	0	$-W_1$	0	...	$\frac{n-2}{n}W_n$
Z_2	0	0	$-Z_3$	0	0	0	0	...	0
Z_3	0	Z_3	0	0	0	0	0	...	0
V_1	V_1	0	0	0	V_1	0	W_1	...	W_{n-1}
V_2	0	0	0	$-V_1$	0	W_1	W_2	...	W_n
W_1	W_1	0	0	0	$-W_1$	0	0	...	0
W_2	0	0	0	$-W_1$	$-W_2$	0	0	...	0
...
W_n	$-\frac{n-2}{n}W_n$	0	0	$-W_{n-1}$	$-W_n$	0	0	...	0

the choice of F as an analytic function of x . In fact, since every analytic function $F(x)$ can be represented as a power series $F(x) = \sum_{i=0}^{\infty} \nu_i x^i/n$, every operator W_F can be expressed as a linear combination of the basis vectors W_i : $W_F = \sum_{i=1}^{\infty} \nu_i W_i$.

The commutator relations for the subalgebras L_{n+5} are given in Table II.

D. Automorphisms

Our aim now is to divide the elements of the equivalence algebra $L_{\#}$ into classes nonequivalent with respect to the corresponding group $E_{\#}$ of equivalence transformations. Let us consider a transformation $T \in E_{\#}$ and its generator $\xi \in L_{\#}$. The transformation induced by T on $L_{\#}$ is called an inner automorphism of the algebra $L_{\#}$ and its generator is given as a linear operator

$$A: X \rightarrow [\xi, X], \quad X \in L_{\#}. \quad (4.8)$$

The set of all inner automorphisms is called an *adjoint group* $E_{\#}^*$ and the corresponding algebra of the generators is called an *adjoint algebra* $L_{\#}^*$.

According to (4.8) each line of Table II can be considered as the coordinates of the generators A_i of the adjoint algebra $L_{\#}^*$:

$$\begin{aligned} A_1 &= -V_1 \frac{\partial}{\partial V_1} - W_1 \frac{\partial}{\partial W_1} + \dots + \frac{n-2}{n} W_n \frac{\partial}{\partial W_n}, \\ A_2 &= -Z_3 \frac{\partial}{\partial Z_3}, \quad A_3 = Z_3 \frac{\partial}{\partial Z_2}, \\ A_4 &= V_1 \frac{\partial}{\partial V_1} + V_1 \frac{\partial}{\partial V_2} + W_1 \frac{\partial}{\partial W_2} + \dots + W_{n-1} \frac{\partial}{\partial W_n}, \\ A_5 &= -V_1 \frac{\partial}{\partial V_1} + W_1 \frac{\partial}{\partial W_1} + W_2 \frac{\partial}{\partial W_2} + \dots + W_n \frac{\partial}{\partial W_n}, \\ A_6 &= W_1 \frac{\partial}{\partial Z_1} - W_1 \frac{\partial}{\partial V_2}, \\ A_7 &= -W_1 \frac{\partial}{\partial V_1} - W_2 \frac{\partial}{\partial V_2}, \\ &\dots, \\ A_{n+5} &= -\frac{n-2}{n} W_n \frac{\partial}{\partial Z_1} - W_{n-1} \frac{\partial}{\partial V_1} - W_n \frac{\partial}{\partial V_2}. \end{aligned}$$

The inner automorphisms of $L_{\#}$, T_i , can be easily recovered from the corresponding generators, A_i , using the Lie equation (1.5). (We denote the group parameter of T_i as A_i .) For example, T_1 is given as the following transformation of the basis:

$$\begin{aligned} Z'_1 &= Z_1, \quad Z'_2 = Z_2, \quad Z'_3 = Z_3, \\ V'_1 &= a_1 V_1, \quad V'_2 = V_2, \\ W'_1 &= a_1 W_1, \quad \dots, \quad W'_n = a_1^{-(n-2)/n} W_n, \quad a_1 > 0. \end{aligned}$$

We will be interested in the automorphisms T_i with $i \geq 8$, which can be represented

as

$$\begin{aligned}
Z'_1 &= Z_1 - \frac{i-7}{i-5}W_{i-5}a_i, \\
V'_1 &= V_1 - W_{i-6}a_i, \\
V'_2 &= V_2 - W_{i-5}a_i.
\end{aligned} \tag{4.9}$$

The composition of T_1, T_2, \dots, T_7 gives the 7-parameter group E_7 used in [27], while The composition of T_1, \dots, T_n, \dots gives the group of inner automorphisms $E_{\#}^*$. The problem of constructing the optimal system of one-dimensional subalgebras consists of finding all classes of the operators

$$Z = e^1 Z_1 + e^2 Z_2 + e^3 Z_3 + e^4 V_1 + e^5 V_2 + \sum_{i=1}^{\infty} e^{i+5} W_i, \tag{4.10}$$

non-equivalent with respect to $E_{\#}^*$. This task can be made easier by making use of the optimal system of one-dimensional subalgebras of L_7 constructed in [27]. In our basis this system is written as the following 27 vectors:

$$\begin{aligned}
Z^{(1)} &= V_1, & Z^{(2)} &= W_1, & Z^{(3)} &= Z_1, \\
Z^{(4)} &= \alpha Z_1 + Z_2, & Z^{(5)} &= Z_3, \\
Z^{(6)} &= (\frac{1}{2} + \alpha)Z_1 + (1 + \beta)Z_2 - \frac{1}{2}V_2, \\
Z^{(7)} &= W_2, & Z^{(8)} &= Z_2 + V_1, & Z^{(9)} &= Z_3 + V_1, \\
Z^{(10)} &= \frac{1}{2}Z_1 + (1 + \beta)Z_2 + V_1 - \frac{1}{2}V_2, \\
Z^{(11)} &= V_1 + W_2, & Z^{(12)} &= Z_2 + W_1, & Z^{(13)} &= Z_3 + W_1, \\
Z^{(15)} &= Z_1 + Z_3, & Z^{(16)} &= Z_1 + W_2, & Z^{(17)} &= Z_3 + W_2, \\
Z^{(18)} &= Z_3 - W_2, & Z^{(19)} &= V_1 + Z_2 + W_2, \\
Z^{(20)} &= V_1 + Z_3 + W_2, & Z^{(21)} &= V_1 + Z_3 - W_2, \\
Z^{(25)} &= \alpha Z_1 + Z_2 + W_2, & Z^{(26)} &= Z_1 + Z_3 + W_2,
\end{aligned} \tag{4.11}$$

$$Z^{(27)} = Z_1 + Z_3 - W_2, \quad Z^{(28)} = \frac{1}{2}Z_1 + Z_3 + V_1 - \frac{1}{2}V_2,$$

$$Z^{(29)} = -\frac{1}{2}Z_1 + (1 + \beta)Z_2 - \frac{1}{2}V_2 + W_1,$$

$$Z^{(30)} = (\frac{1}{2} + \alpha)Z_1 + Z_3 - \frac{1}{2}V_2,$$

$$Z^{(31)} = -\frac{1}{2}Z_1 + Z_3 - \frac{1}{2}V_2 + W_1.$$

We have omitted vectors $Z^{(14)}$, $Z^{(22)}$, $Z^{(13)}$, $Z^{(24)}$ since they are in fact equivalent to $Z^{(7)}$, $Z_{\alpha=0}^{(25)}$, $Z^{(17)}$ and $Z^{(18)}$. In [27] these vectors appear due to an unfortunate mistake in the second of the eqs. (5.5). The correct form is $\bar{e}^2 = a_3[-a_7e^1 + a_6e^2 + (a_2a_6 - a_1a_7)e^3 + a_2a_6e^6 + a_1a_6e^7]$.

Let us represent transformations T_i as the transformations of coordinates. For $i \geq 8$ they can be written as

$$\begin{aligned} \bar{e}^{i-1} &= e^{i-1} - a_i e^4, \\ \bar{e}^i &= e^i - a_i e^5 - \frac{i+3}{i-5} a_i e^1, \\ \bar{e}^j &= e^j, \quad j \neq i-1, i. \end{aligned} \tag{4.12}$$

From (4.12) one observes that T_i with $i \geq 9$ leaves vectors of L_7 invariant. The transformation T_8 however changes the e^7 component of those $l \in L_7$ that have a nonzero component e^4 . Hence T_8 can be used to further simplify the optimal system (4.11) of L_7 .

Optimal system of L_7 has four vectors with non-zero component e^4 , namely $Z^{(11)}$, $Z^{(19)}$, $Z^{(20)}$ and $Z^{(21)}$. Since these vectors have $e^5 = e^1 = 0$, T_8 changes only e^7 , and thus leaves them in L_7 . Since e^7 can be annulled by taking $a_8 = e^7/e^4$, the subalgebras $Z^{(11)}$, $Z^{(19)}$, $Z^{(20)}$ and $Z^{(21)}$ considered as elements of $L_{\#}$ are in fact equivalent to the subalgebras $Z^{(1)}$, $Z^{(8)}$ and $Z^{(9)}$. Hence only the following vectors of

L_7 are nonequivalent with respect to the automorphisms of $L_\#$:

$$\begin{aligned}
\tilde{Z}^{(1)} &= V_1, & \tilde{Z}^{(2)} &= W_1, & \tilde{Z}^{(3)} &= Z_1, \\
\tilde{Z}^{(4)} &= \alpha Z_1 + Z_2, & \tilde{Z}^{(5)} &= Z_3, \\
\tilde{Z}^{(6)} &= (\tfrac{1}{2} + \alpha)Z_1 + (1 + \beta)Z_2 - \tfrac{1}{2}V_2, \\
\tilde{Z}^{(7)} &= W_2, & \tilde{Z}^{(8)} &= Z_2 + V_1, & \tilde{Z}^{(9)} &= Z_3 + V_1, \\
\tilde{Z}^{(10)} &= \tfrac{1}{2}Z_1 + (1 + \beta)Z_2 + V_1 - \tfrac{1}{2}V_2, \\
\tilde{Z}^{(11)} &= Z_2 + W_1, & \tilde{Z}^{(12)} &= Z_3 + W_1, & \tilde{Z}^{(13)} &= Z_1 + Z_3, \\
\tilde{Z}^{(14)} &= Z_1 + W_2, & \tilde{Z}^{(15)} &= Z_3 + W_2, & \tilde{Z}^{(16)} &= Z_3 - W_2, \\
\tilde{Z}^{(17)} &= \alpha Z_1 + Z_2 + W_2, & \tilde{Z}^{(18)} &= Z_1 + Z_3 + W_2, \\
\tilde{Z}^{(19)} &= Z_1 + Z_3 - W_2, & \tilde{Z}^{(20)} &= \tfrac{1}{2}Z_1 + Z_3 + V_1 - \tfrac{1}{2}V_2, \\
\tilde{Z}^{(21)} &= -\tfrac{1}{2}Z_1 + (1 + \beta)Z_2 - \tfrac{1}{2}V_2 + W_1, \\
\tilde{Z}^{(22)} &= (\tfrac{1}{2} + \alpha)Z_1 + Z_3 - \tfrac{1}{2}V_2, \\
\tilde{Z}^{(23)} &= -\tfrac{1}{2}Z_1 + Z_3 - \tfrac{1}{2}V_2 + W_1.
\end{aligned}$$

Consequently there exist equivalence transformations that transform equations 11, 19, 20 and 21 listed in Table II in [27] to the cases 1, 8 and 9, thereby reducing the number of equations in the table by four.

E. Optimal system of subalgebras for $L_\#$

We can now assume that the transformations $T_i, i \leq 7$ have been already employed and the vectors (4.10) are divided into classes, nonequivalent with respect to E_7 , represented by

$$Z^{[k]} = \tilde{Z}^{(k)} + \sum_{j=3} e^{5+j} W_j. \quad (4.13)$$

These vectors can now be further simplified and divided into classes nonequivalent with respect to the countable-dimensional group of inner automorphisms $E_{\#}$ using transformations T_i with $i \geq 8$, eq. (4.12). Firstly one can easily see that in the case of vectors $Z^{[1]}, Z^{[3]}, Z_{\alpha \neq 0}^{[4]}, Z^{[8]}, Z^{[9]}, Z^{[13]}, Z^{[14]}, Z_{\alpha \neq 0}^{[17]}, Z^{[18]}, Z^{[19]}, Z^{[21]}, Z^{[23]}$ and $Z^{[6]}, Z^{[22]}$ with $\alpha \neq \frac{1}{i-7}$, the transformations T_i , eqs. (4.12), only change either the component e^{i-1} :

$$\begin{aligned}\bar{e}^{i-1} &= e^i - a_i e^4, \\ \bar{e}^j &= e^j, \quad j \neq i-1, \quad i \geq 8, \\ e^4 &\neq 0;\end{aligned}$$

or the component e^i :

$$\begin{aligned}\bar{e}^i &= e^i + a_i \xi(e^1, e^5), \\ \bar{e}^j &= e^j, \quad j \neq i, \quad i \geq 8, \\ \xi &\neq 0.\end{aligned}$$

With the appropriate choice of a_i the components e^i , $i \geq 8$, become zero, while the components e^i , $i \leq 7$, remain the same. Note that we need to perform only a finite number of transformations to eliminate the components e^i with $i \geq 8$, provided the sum in (4.12) is finite.

In the same way the vectors $Z^{[6]}$ and $Z^{[22]}$ with $\alpha = \frac{1}{i-7}$, $i \geq 8$, can be brought to the form

$$\tilde{Z}^{(i)} + \mu W_{i-5}, \quad i = 6, 22.$$

In the case of the vectors $Z^{[10]}$ and $Z^{[20]}$ the transformations T_i change two

components:

$$\begin{aligned}
\bar{e}^{i-1} &= e^{i-1} - a_i, \\
\bar{e}^i &= e^i + \frac{1}{i-5}a_i, \\
\bar{e}^j &= e^j, \quad j \neq i-1, i, \quad i \geq 8.
\end{aligned} \tag{4.14}$$

Suppose that the last component in (4.13) is e^N . We annul it by the transformation T_N and use the transformations T_{N-1}, \dots, T_8 to bring the vectors $Z^{[10]}$ and $Z^{[20]}$ to the form

$$Z^{[i]} = \tilde{Z}^{(i)} + \delta W_2, \quad i = 10, 20.$$

Now $Z^{[10]}$ and $Z^{[20]}$ are in L_7 and are similar [27] to $\tilde{Z}^{(10)}$ and $\tilde{Z}^{(20)}$ respectively.

The remaining vectors $Z^{[i]}$ do not change under the transformations (4.12). Hence, we have constructed the following optimal system of one-dimensional subalgebras of $L_\#$:

$$\begin{aligned}
\tilde{Z}^{[1]} &= V_1, \quad \tilde{Z}^{[2]} = Z_1 + \alpha Z_2, \\
\tilde{Z}^{[3]} &= \left(\frac{1}{2} + \alpha\right)Z_1 + (1 + \beta)Z_2 - \frac{1}{2}V_2, \\
\tilde{Z}^{[4]} &= Z_2 + V_1, \quad \tilde{Z}^{[5]} = Z_3 + V_1, \\
\tilde{Z}^{[6]} &= \frac{1}{2}Z_1 + (1 + \beta)Z_2 + V_1 - \frac{1}{2}V_2, \\
\tilde{Z}^{[7]} &= Z_1 + Z_3, \quad \tilde{Z}^{[8]} = Z_1 + W_2, \\
\tilde{Z}^{[9]} &= Z_1 + \alpha Z_2 + \alpha W_2, \quad \tilde{Z}^{[10]} = Z_1 + Z_3 + W_2, \\
\tilde{Z}^{[11]} &= Z_1 + Z_3 - W_2, \quad \tilde{Z}^{[12]} = \frac{1}{2}Z_1 + Z_3 + V_1 - \frac{1}{2}V_2, \\
\tilde{Z}^{[13]} &= -\frac{1}{2}Z_1 + (1 + \beta)Z_2 - \frac{1}{2}V_2 + W_1, \\
\tilde{Z}^{[14]} &= \left(\frac{1}{2} + \alpha\right)Z_1 + Z_3 - \frac{1}{2}V_2, \\
\tilde{Z}^{[15]} &= -\frac{1}{2}Z_1 + Z_3 - \frac{1}{2}V_2 + W_1.
\end{aligned}$$

$$\begin{aligned}\tilde{Z}^{[16]} &= \frac{n}{2}Z_1 + (n-2+\beta)Z_2 - \frac{n-2}{2}V_2 + \mu W_n, \quad (n \geq 3), \\ \tilde{Z}^{[17]} &= \frac{n}{2}Z_1 + (n-2)Z_3 - \frac{n-2}{2}V_2 + \mu W_n, \quad (n \geq 3), \\ \tilde{Z}^{[18]} &= W_{F(x)}, \quad \tilde{Z}^{[19]} = Z_2 + W_{F(x)}, \quad \tilde{Z}^{[20]} = Z_3 + W_{F(x)}.\end{aligned}$$

Here in the case of vectors $\tilde{Z}^{[18]}$ and $\tilde{Z}^{[19]}$, $F(x)$ is an arbitrary analytic function with either $F'(0) = 0$, $F''(0) = 0$ or $F'(0) = 1$, $F''(0) = 0$ or $F'(0) = 0$, $F''(0) = 1$; and in the case of vector $\tilde{Z}^{[20]}$, $F(x)$ is an arbitrary analytic function with either $F'(0) = 0$, $F''(0) = 0$ or $F'(0) = 1$, $F''(0) = 0$ or $F'(0) = 0$, $F''(0) = \pm 1$.

The corresponding equations can be found in the same way as in Ref. [27]. The classification is listed in Table III. In addition to the principal algebra $L_{\mathcal{P}}$ admitted by all equations of the type (4.1) equations of the Table III also possess an extra symmetry X_4 . The constitutive laws corresponding to such choice of parameters f and g thus produce models with a more interesting algebraic structure. Note that most of the equations in the Table III have an invariant solution corresponding to the additional operator X_4 . For example, in the case of equations corresponding to $N = 2$ the invariant solutions should be sought in the form

$$\frac{v}{x^2} = f\left(\frac{x^{(1-\alpha/2)}}{t}\right),$$

as $I_1 = v/x^2$ and $I_2 = x^{(1-\alpha/2)}/t$ constitute a basis of invariants for the operator X_4 .

Table III. The result of the classification ($\sigma = 1/\alpha$, $\gamma = \beta/\alpha$, Φ and Γ are arbitrary functions of λ).

N	Z	Invariant λ	Equation	Additional operator X_4
1	$\tilde{Z}^{[1]}$	v_x	$v_{tt} = \Phi v_{xx} + \Gamma$	$\frac{\partial}{\partial x}$
2	$\tilde{Z}^{[2]}$	v_x/x	$v_{tt} = x^\alpha \{\Phi v_{xx} + \Gamma\}$	$(1 - \frac{\alpha}{2})t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}$
3	$\tilde{Z}_{\alpha=0}^{[3]}$	x	$v_{tt} = v_x^\beta \{\Phi v_{xx} + \Gamma v_x\}$	$\beta t \frac{\partial}{\partial t} - 2v \frac{\partial}{\partial v}$
4	$\tilde{Z}_{\alpha \neq 0}^{[3]}$	$v_x/x^{\sigma+1}$	$v_{tt} = x^\gamma \{\Phi v_{xx} + x^\sigma \Gamma\}$	$(2 - \gamma)t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2(\sigma + 2)v \frac{\partial}{\partial v}$
5	$\tilde{Z}^{[4]}$	v_x	$v_{tt} = e^x \{\Phi v_{xx} + \Gamma\}$	$t \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial x}$
6	$\tilde{Z}^{[5]}$	v_x	$v_{tt} = \Phi v_{xx} + \Gamma + x$	$2 \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial v}$
7	$\tilde{Z}^{[6]}$	$e^{-x} v_x$	$v_{tt} = v_x^\beta \{\Phi v_{xx} + \Gamma v_x\}$	$\beta t \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial x} - 2v \frac{\partial}{\partial v}$
8	$\tilde{Z}^{[7]}$	v_x/x	$v_{tt} = \Phi v_{xx} + \Gamma + \ln x $	$2t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + (t^2 + 4v) \frac{\partial}{\partial v}$
9	$\tilde{Z}^{[8]}$	$v_x/x - \ln x $	$v_{tt} = \Phi v_{xx} + \Gamma - \Phi \ln x $	$2t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + (x^2 + 4v) \frac{\partial}{\partial v}$
10	$\tilde{Z}^{[9]}$	$v_x/x - \alpha \ln x $	$v_{tt} = x^\alpha \Phi \{v_{xx} - \alpha \ln x + \Gamma\}$	$(2 - \alpha)t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + (\alpha x^2 + 4v) \frac{\partial}{\partial v}$
11	$\tilde{Z}^{[10]}$	$v_x/x - \ln x $	$v_{tt} = \Phi v_{xx} + (1 - \Phi) \ln x + \Gamma$	$2t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + (t^2 + x^2 + 4v) \frac{\partial}{\partial v}$
12	$\tilde{Z}^{[11]}$	$v_x/x + \ln x $	$v_{tt} = \Phi v_{xx} + (1 + \Phi) \ln x + \Gamma$	$2t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + (t^2 - x^2 + 4v) \frac{\partial}{\partial v}$
13	$\tilde{Z}^{[12]}$	$e^{-x} v_x$	$v_{tt} = e^{-x} \Phi v_{xx} + \Gamma + x$	$t \frac{\partial}{\partial t} + 2 \frac{\partial}{\partial x} + (t^2 + 2v) \frac{\partial}{\partial v}$
14	$\tilde{Z}^{[13]}$	$v_x + \ln x $	$v_{tt} = x^{-\beta} \{\Phi v_{xx} + x^{-1} \Gamma\}$	$(\beta + 2)t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2(-x + v) \frac{\partial}{\partial v}$
15	$\tilde{Z}_{\alpha=0}^{[14]}$	x	$v_{tt} = \Phi v_x^{-1} v_{xx} + \Gamma + \ln v_x $	$t \frac{\partial}{\partial t} + (t^2 + 2v) \frac{\partial}{\partial v}$
16	$\tilde{Z}_{\alpha \neq 0}^{[14]}$	$x^{-(1+\sigma)} v_x$	$v_{tt} = x^{-\sigma} \Phi v_{xx} + \Gamma + \sigma \ln x $	$(1 + \frac{\sigma}{2})t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + [\frac{\sigma}{2}t^2 + (2 + \sigma)v] \frac{\partial}{\partial v}$
17	$\tilde{Z}^{[15]}$	$v_x + \ln x $	$v_{tt} = x \Phi v_{xx} + \Gamma + v_x$	$t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + (-t^2 - 2x + 2v) \frac{\partial}{\partial v}$
18	$\tilde{Z}^{[16]}$	$v_x/x^{n-1} - \mu \ln x $	$v_{tt} = x^\beta \Phi \{v_{xx} - x^{n-2} [\mu(n-1) \ln x + \Gamma]\}$	$(1 - \frac{\beta}{2})t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + (nv + \frac{x^n}{n} \mu) \frac{\partial}{\partial v}$
19	$\tilde{Z}^{[17]}$	$v_x/x^{n-1} - \mu \ln x $	$v_{tt} = x^{2-n} \Phi v_{xx} + [n-2-\mu(n-1)\Phi] \ln x + \Gamma$	$\frac{n}{2}t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + (nv + \frac{n-2}{2}t^2 + \mu \frac{x^n}{n}) \frac{\partial}{\partial v}$
20	$\tilde{Z}^{[18]}$	x	$v_{tt} = \Phi v_{xx} - \frac{F''}{F'} \Phi v_x + \Gamma$	$F \frac{\partial}{\partial v}$
21	$\tilde{Z}^{[19]}$	x	$v_{tt} = e^{v_x/F'} \Phi \{v_{xx} - \frac{F''}{F'} v_x + \Gamma\}$	$-\frac{t}{2} \frac{\partial}{\partial t} + F \frac{\partial}{\partial v}$
22	$\tilde{Z}^{[20]}$	x	$v_{tt} = \Phi v_{xx} + \frac{1-F''\Phi}{F'} v_x + \Gamma$	$(\frac{t^2}{2} + F) \frac{\partial}{\partial v}$

CONCLUSIONS

In this thesis we have reported the following main results:

The existence of soliton solutions in the hydrodynamic model of plasma has been analysed. We have obtained the domains of existence of compressive ion-acoustic solitons using an analytical method. We also proved that the model does not support rarefactive solitons and monotonic transition layers.

A new nonrelativistic Chern-Simons theory possessing vortex solutions has been proposed. The absence of the nonvanishing background solution in the standard model was cured by adding a background charge. The new model was reformulated as a constrained Hamiltonian system. This allowed the determination of explicit vortex solutions in two self-duality limits and the correct definition of the linear momentum. The model with the background charge has been shown to be equivalent to the model with the external magnetic field. Finally the existence of nontopological solutions has been considered and the absence of solitonic bubbles in relativistic Chern-Simons-Maxwell theories established.

The preliminary group classification of nonhomogeneous wave equations has been given. The classification has been performed with respect to a countable-dimensional subalgebra of the full equivalence algebra of these equations.

Some of the results presented in this thesis have also been reported in [25, 7, 8, 9, 24]

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