SPHERICALLY SYMMETRIC COSMOLOGICAL SOLUTIONS

by

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Abstract

This thesis examines the role of shear in inhomogeneous spherically symmetric spacetimes in the field of general relativity. The Einstein field equations are derived for a perfect fluid source in comoving coordinates. By assuming a barotropic equation of state, two classes of nonaccelerating solutions are obtained for the Einstein field equations. The first class has equation of state $p = \frac{1}{3}\mu$ and the second class, with equation of state $p = \mu$, generalises the models of Van den Bergh and Wils (1985). For a particular choice of a metric potential a new class of solutions is found which is expressible in terms of elliptic functions of the first and third kind in general. A class of nonexpanding cosmological models is briefly studied. The method of Lie symmetries of differential equations generates a self–similar variable which reduces the field and conservation equations to a system of ordinary differential equations. The behaviour of the gravitational field in this case is governed by a Riccati equation which is solved in general. Another class of solutions is obtained by making an *ad hoc* choice for one of the gravitational potentials. It is demonstrated that for a stiff fluid a particular case of the generalised Emden–Fowler equation arises.

To

Shamla Govender

For her constant encouragement, infinite patience and unending inspiration.

Preface and Declaration

The study described in this thesis was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, during the period May 1992 to May 1996. This thesis was completed under the supervision of Professor S D Maharaj and Professor P G L Leach.

The research contained in this study represents original work by the author. It has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

1 Introduction

In this thesis we study spherically symmetric gravitational fields with a perfect fluid energy–momentum tensor. We consider in particular the case of nonvanishing shear as this case has not been studied comprehensively before. Spherically symmetric spacetimes are physically significant and are extensively utilised in relativistic astrophysics and cosmology (Shapiro and Teukolsky 1983). The Robertson–Walker models, the standard cosmological models, are homogeneous and isotropic. To study many physical situations we need to incorporate anisotropies and/or inhomogeneities. Anisotropic cosmologies have been studied in detail by Ryan and Shepley (1975) which extends the seminal treatment of Ellis and MacCallum (1969) and MacCallum and Ellis (1970). However, there is increasing evidence, at least on the cosmological scale for the microwave background radiation, of a high degree of isotropy; see for example the recent analyses of Maartens *et al* (1995a, 1995b) and Stoeger *et al* (1995). In contrast certain observational evidence suggests that inhomogeneities are significant at large scales (Krasinski 1996). Consequently in this thesis we are mainly concerned with inhomogeneous cosmologies. The simplest inhomogeneous models are the spherically symmetric spacetimes which are invariant under the action of a three dimensional Lie algebra of rotational Killing vectors. Many exact solutions of the Einstein field equations for inhomogeneous spherically symmetric gravitational fields are listed by Kramer *et al* (1980). If these inhomogeneous models are also invariant under a three dimensional Lie algebra of translational Killing vectors, we regain the isotropic Robertson–Walker models.

We characterise spherically symmetric models in terms of the nonvanishing kinematical quantities: shear, acceleration and expansion. Most of the spherically symmetric models studied thus far have had vanishing shear. These shear–free models are mathematically simpler to study than their counterparts with shear. An early treatment of shear–free and expanding spherically symmetric perfect fluid solutions is given by Kustaanheimo and Qvist (1948). Maharaj *et al* (1996), using the method of Lie symmetries for differential equations, have shown that the evolution of shear–free models can be reduced to an Emden–Fowler equation. Shear–free solutions with an equation of state have been discussed by Wyman (1946) and special cases by Taub (1968). Broad classes of shear–free solutions have also been presented by Srivastava (1987) and Sussman (1988a, 1988b). Kramer *et al* (1980) have listed and categorised most shear–free spherically symmetric solutions. These solutions are normally given in comoving coordinates which facilitate the physical interpretation of the models. The relevance of shear–free solutions as viable inhomogeneous cosmological models has been discussed by Krasinski (1996).

There are few spherically symmetric solutions having nonzero shear because these studies involve systems of highly nonlinear partial differential equations which are difficult to integrate. Some early known solutions with shear, which have generated considerable interest, are those of Gutman and Bespal'ko (1967) and of Wesson (1978); these admit a stiff equation of state in comoving coordinates. McVittie and Wiltshire (1975, 1977), Szafron (1977), Szekeres (1975) and Vaidya (1968) also found some shearing solutions, but these were given in noncomoving coordinates and are consequently more difficult to interpret. Van den Bergh and Wils (1985) found exact solutions for nonstatic perfect fluid spheres with shear and a barotropic equation of state. All their models are expanding and some are also accelerating. Herrera and Ponce de Leon (1985a) produced some exact analytical solutions of the field equations by using spherically symmetric perfect fluids admitting a one–parameter group of conformal motions. Collins and Lang (1987) studied spacetimes which are locally rotationally symmetric and in which there is a nontrivial similarity vector that is orthogonal to the fluid flow. They showed that their spherically symmetric perfect fluid solutions obeying a barotropic equation of state are related to those obtained by Gutman and Bespal'ko (1967), Hajj–Boutros (1985), Herrera and Ponce de Leon (1985a), Van den Bergh and Wils (1985) and Wesson (1978). Kitamura (1989, 1994, 1995a, 1995b), using the 'characteristic system' method devised by Takeno (1966), derived exact solutions for a perfect fluid with shear, expansion and acceleration which also involve the Gutman and Bespal'ko (1967), Sussman (1991), Van den Bergh and Wils (1985) and Wesson (1978) solutions as special cases. Maharaj *et al* (1993) found a general class of solutions in terms of elementary functions with shear, expansion and acceleration. They incorporated the solutions of Gutman and Bespal'ko (1967), Hajj–Boutros (1985), Lake (1983), Shaver and Lake (1988) and Wesson (1978). The models of Maharaj *et al* (1993) obey an equation of state which is a generalisation of the stiff equation of state. In addition they admit a conformal Killing vector which acts in the radial direction (Maharaj and Maharaj 1994). Qadir and Zaid (1995) presented a complete classification of spherically symmetric spacetimes according to their isometries and line elements by solving the Killing equations. All the solutions listed above are in comoving coordinates. Noncomoving coordinates are more to difficult to analyse physically (see *eg* Knutsen 1995). This brief overview indicates that shearing solutions for spherically symmetric gravitational fields arise in a variety of applications and motivates a systematic analysis of the relevance of shear in general relativity.

The study of inhomogeneous and anisotropic cosmological models has been motivated by a number of phenomena relating to the observable universe. Khlopov and Polnarev (1983) have observed that inhomogeneities in the early universe might have evolved due to the creation of particles with thermal energy (kT) of the order equivalent to their rest mass energy (mc^2), and lifetime exceeding the cosmological time scale. These initial inhomogeneities could have developed into the present observable universe. The formation of clusters and superclusters of galaxies, on the scales 10–100 Mps, is another evidence in the support of inhomogeneities in the universe. The possibility of the existence of primordial gravitational waves could also lead to anisotropy in the universe (Krauss 1986). Many researchers have investigated the relationship between inhomogeneous shear anisotropy and inflation in relativistic cosmology. In particular Krishna Rao (1995) showed that spherically symmetric cosmological models with inhomogeneities and anisotropies exhibit inflationary behaviour subject to certain conditions on the Weyl tensor.

In Chapter 2 of this thesis we consider the kinematical and dynamical features of spherically symmetric spacetimes. We only consider those elements of differential geometry and general relativity that are relevant to this thesis. The general spherically symmetric line element for a perfect fluid source is given in comoving coordinates and the kinematical quantities are derived in §2.2. The nonvanishing components of the connection coefficients, the Ricci tensor, Ricci scalar and the

Einstein tensor are explicitly calculated. In §2.3 the energy–momentum tensor is coupled to the Einstein tensor to derive the Einstein field equations. The Einstein field equations, for a perfect fluid energy–momentum tensor, are generated explicitly for a spherically symmetric line element. Some known solutions of the field equations for nonzero shear in comoving coordinates are also discussed in §2.4. In particular we are concerned with their equations of state.

New shearing solutions to the Einstein field equations are derived in Chapter 3. In §3.2 we investigate nonaccelerating solutions with expansion. These solutions are divided into two types. For the first type a barotropic equation of state is assumed and the field equations are integrated completely for radiation and stiff fluid models. A class of solutions, corresponding to radiation, is obtained and this is compared to the models of Kantowski and Sachs (1966). Another class of solutions, corresponding to stiff fluid models, is derived and this is shown to contain the solutions of Van den Bergh and Wils (1985). The second type of nonaccelerating solution is expressed in terms of quadratures and explicitly solved in terms of elementary and elliptic functions when the pressure is assumed to be constant. Nonexpanding solutions with shear and acceleration are briefly investigated in §3.3. The solutions are expressed in terms of quadratures when the energy density is taken to be constant. This chapter illustrates that imposing an equation of state leads to inhomogeneous models with shear.

In Chapter 4 Lie symmetries of differential equations are used to derive a new class of solutions to the Einstein field equations. The theory of Lie symmetries and its application to partial differential equations are briefly discussed in §4.2. With the aid of a suitable infinitesimal generator a self-similar variable is chosen in §4.3 to simplify the field equations. This self-similar variable is used to conflate the time and radial coordinates. The Einstein field equations and the conservation equations then reduce to a system of ordinary differential equations. In §4.4 the integration of the system of ordinary differential equations is reduced to a single nonlinear equation for a particular case. This equation is integrated and the metric potentials, energy density and pressure are found. In a suitable limit we regain a static model which corresponds to a special case of the self-similar Tolman–Bondi models. This chapter indicates that the systematic method of Lie symmetries is useful in generating new solutions.

New solutions to the field equations are obtained in Chapter 5 by assuming that one of the metric potentials is constant. This enables us to reduce the integration of the Einstein field equations to a single partial differential equation with one dependent variable. We first assume an *ad hoc* form for the dependent variable and generate a class of solutions which has vanishing energy–momentum. Lie symmetries are then utilised to reduce the partial differential equation to an ordinary differential equation. The solution in general is expressed in terms of elliptic functions. As a special case in §5.4 we assume a stiff equation of state and the field equations are reduced to an Emden–Fowler equation. The integrability of the Emden–Fowler equation is briefly discussed.

Chapter 6 outlines the conclusions arrived in this thesis. The main results of the investigations are highlighted and possible extensions arising from these results are discussed.

2 Spherically Symmetric Spacetimes

2.1 Introduction

In this chapter we consider, in general, the kinematical and dynamical features of spherically symmetric spacetimes. In particular we obtain the Einstein field equations for the general spherically symmetric line element for a perfect fluid source. The elements of differential geometry that are relevant to our work are briefly introduced in §2.2. The spherically symmetric line element is given in comoving coordinates and the kinematical quantities, namely, acceleration, vorticity, expansion and shear are derived. The nonvanishing components of the connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor are explicitly calculated. These agree with the results of Stephani (1990) and have also been verified with MATHEMATICA (Wolfram 1991). In §2.3 we introduce the energy–momentum tensor for a perfect fluid matter distribution and consider the relevance of the barotropic equation of state in cosmology. The energy–momentum tensor is coupled to the Einstein tensor to derive the Einstein field equations which enable us to consider the dynamical features of inhomogeneous universes. In §2.4 we list some known exact solutions of the field equations for nonzero shear in comoving coordinates and discuss their interdependence. Some of these solutions possess an equation of state which is related to the stiff equation of state and arise as special cases in later chapters.

2.2 Spacetime Geometry

In this section we briefly introduce those elements of differential geometry that are necessary to set up the Einstein field equations. For a more detailed study of differential geometry applicable to general relativity, the reader is referred to texts by de Felice and Clark (1990), Hawking and Ellis (1973) and Misner *et al* (1973). We take spacetime to be a four-dimensional differentiable manifold with local coordinates (x^a) where x^0 is timelike and x^1, x^2, x^3 are spacelike. The manifold is endowed with a metric tensor field $\mathbf{g}$ with signature $(- + ++)$. The invariant distance between neighbouring points in the manifold is defined by the line element

$$ds^2 = g_{ab}dx^a dx^b,$$

where $\mathbf{g}$ is the symmetric, nondegenerate metric tensor field. The metric connection $\mathbf{\Gamma}$, also called the Christoffel symbol of the second kind, is defined in terms of the metric tensor field and its derivatives by

$$\Gamma^a{}_{bc} = \tfrac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}), \tag{2.1}$$

where commas denote partial differentiation. The statement that there exists a unique metric connection $\mathbf{\Gamma}$ which preserves inner products under parallel transport is called the fundamental theorem of Riemannian geometry (do Carmo 1992).

The Riemann curvature tensor is a type (1,3) tensor and is defined as

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{2.2}$$

in terms of the connection coefficients (2.1). Upon contraction of a with c in (2.2) we obtain the Ricci tensor

$$R_{ab} = \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^e{}_{ab}\Gamma^d{}_{ed} - \Gamma^e{}_{ad}\Gamma^d{}_{eb}. \tag{2.3}$$

A contraction of (2.3) yields the Ricci or curvature scalar

$$R = R^a{}_a. \tag{2.4}$$

The Einstein tensor,

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab}, \tag{2.5}$$

is defined in terms of the Ricci tensor (2.3) and the Ricci scalar (2.4). The Einstein tensor $\mathbf{G}$ has zero divergence:

$$G^{ab}{}_{;b} = 0 \tag{2.6}$$

which follows directly from the definition (2.5). The Bianchi identity (2.6) generates the conservation of energy–momentum via the Einstein field equations.

We consider the general case of spherically symmetric spacetimes. These spacetimes are invariant under the action of a three–dimensional Lie algebra of rotational Killing vectors; therefore they may be used to model inhomogeneous universes for spherically symmetric spacetimes (Kramer *et al* 1980). With comoving coordinates, $(x^a) = (t, r, \theta, \phi)$, the line element takes the form

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)\left[d\theta^2 + \sin^2\theta d\phi^2\right], \tag{2.7}$$

where the gravitational potentials ν, λ and Y are functions of the spacetime coordinates t and r. It should be noted that the coordinates used in the line element (2.7) are not isotropic. In the comoving frame of reference, the four–velocity $\mathbf{u}$ has the form

$$u^a = \left(e^{-\nu}, 0, 0, 0\right).$$

For the spherically symmetric metric (2.7) the kinematical quantities are given by

$$\omega_{ab} = 0 \tag{2.8a}$$

$$\dot{u}^a = (0, \nu', 0, 0) \tag{2.8b}$$

$$\Theta = e^{-\nu}\left(\dot{\lambda} + \frac{2\dot{Y}}{Y}\right) \tag{2.8c}$$

$$\sigma^1_1 = \sigma^2_2 = -\tfrac{1}{2}\sigma^3_3 = \tfrac{1}{3}e^{-\nu}\left(\frac{\dot{Y}}{Y} - \dot{\lambda}\right) \tag{2.8d}$$

relative to the four–velocity $\mathbf{u}$, where dots and primes denote partial differentiation with respect to t and r respectively. In (2.8) ω_{ab} is the vorticity tensor, $\dot{u}^a$ is the acceleration vector, Θ is the expansion scalar (or rate of expansion) and σ is the magnitude of the shear (or rate of shear). The vorticity vanishes since the spacetime is spherically symmetric. The acceleration, expansion and shear are nonzero in general. Note that most of the exact solutions corresponding to the metric (2.7) are categorised in terms of the kinematical quantities. Since most of our work in this thesis is concerned with nonzero shear our solutions have to satisfy the condition

$$\frac{\dot{Y}}{Y} - \dot{\lambda} \neq 0.$$

If the shear vanishes ($\sigma = 0$), then, after a suitable coordinate transformation, (2.7) assumes the form

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{2.9}$$

(Kramer *et al* 1980). It is only in the case of vanishing shear that we can find coordinates which are simultaneously comoving and isotropic and which enable us to express the line element in the form (2.9). Consequently this line element has received considerably greater attention than the more complicated line element (2.7). Essentially the evolution of the model is governed by a single nonlinear differential equation, an Emden–Fowler equation, when $\sigma = 0$. For a recent comprehensive treatment of exact solutions in the shear–free spacetimes (2.9) see Maharaj *et al* (1996).

The nonvanishing connection coefficients (2.1) for the line element (2.7) are given by

$$\Gamma^0{}_{00} = \dot{\nu} \qquad \Gamma^0{}_{01} = \nu'$$

$$\Gamma^0{}_{11} = e^{2(\lambda-\nu)}\dot{\lambda} \qquad \Gamma^0{}_{22} = e^{-2\nu}Y\dot{Y}$$

$$\Gamma^0{}_{33} = \sin^2\theta\; e^{-2\nu}Y\dot{Y} \qquad \Gamma^1{}_{00} = e^{2(\nu-\lambda)}\nu'$$

$$\Gamma^1{}_{01} = \dot{\lambda} \qquad \Gamma^1{}_{11} = \lambda'$$

$$\Gamma^1{}_{22} = -e^{-2\lambda}YY' \qquad \Gamma^1{}_{33} = -\sin^2\theta\; e^{-2\lambda}YY'$$

$$\Gamma^2{}_{02} = \frac{\dot{Y}}{Y} \qquad \Gamma^2{}_{12} = \frac{Y'}{Y}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{03} = \frac{\dot{Y}}{Y}$$

$$\Gamma^3{}_{13} = \frac{Y'}{Y} \qquad \Gamma^3{}_{23} = \cot\theta.$$

These connection coefficients have also been verified from the Euler–Lagrange equations and MATHEMATICA (Wolfram 1991).

The nonzero Ricci tensor components (2.3) take the form

$$R_{00} = -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\lambda}\dot{\nu} + 2\dot{\nu}\frac{\dot{Y}}{Y} - 2\frac{\ddot{Y}}{Y}$$

$$+\, e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\nu'\frac{Y'}{Y}\right) \tag{2.10a}$$

$$R_{01} = 2\left(\dot{\lambda}\frac{Y'}{Y} + \nu'\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y}\right) \tag{2.10b}$$

$$R_{11} = -\nu'' - \nu'^2 + \lambda'\nu' + 2\lambda'\frac{Y'}{Y} - 2\frac{Y''}{Y}$$

$$+\, e^{2(\lambda-\nu)}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y}\right) \tag{2.10c}$$

$$R_{22} = e^{-2\nu}Y\dot{Y}\left(\dot{\lambda} - \dot{\nu} + \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{\dot{Y}}\right)$$

$$+\, e^{-2\lambda}YY'\left(\lambda' - \nu' - \frac{Y'}{Y} - \frac{Y''}{Y'}\right) + 1 \tag{2.10d}$$

$$R_{33} = \sin^2\theta\, R_{22} \tag{2.10e}$$

for the connection coefficients listed above.

With the Ricci tensor components (2.10) and the definition (2.4) we generate the Ricci scalar

$$R = 2e^{-2\nu}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y} - 2\dot{\nu}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + 2\frac{\ddot{Y}}{Y}\right)$$

$$-2e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' - 2\lambda'\frac{Y'}{Y} + 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{2}{Y^2}. \quad (2.11)$$

For the line element (2.7) we substitute (2.10) and (2.11) in (2.5) to obtain the nonvanishing Einstein tensor components

$$G_{00} = 2\dot{\lambda}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} - e^{2(\nu-\lambda)}\left(-2\lambda'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{e^{2\nu}}{Y^2} \quad (2.12a)$$

$$G_{01} = 2\dot{\lambda}\frac{Y'}{Y} + 2\nu'\frac{\dot{Y}}{Y} - 2\frac{\dot{Y}'}{Y} \quad (2.12b)$$

$$G_{11} = 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + e^{2(\lambda-\nu)}\left(2\dot{\nu}\frac{\dot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - 2\frac{\ddot{Y}}{Y}\right) - \frac{e^{2\lambda}}{Y^2} \quad (2.12c)$$

$$G_{22} = -e^{-2\nu}\left[\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right)Y^2 + \left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)Y\right]$$

$$+e^{-2\lambda}\left[\left(\nu'' + \nu'^2 - \nu'\lambda'\right)Y^2 + \left(\nu'Y' - \lambda'Y' + Y''\right)Y\right] \quad (2.12d)$$

$$G_{33} = \sin^2\theta\, G_{22} \quad (2.12e)$$

for the metric (2.7). The components (2.12) represent the curvature and are necessary for deriving the Einstein field equations.

2.3 Field Equations

The general form of the energy–momentum tensor $\mathbf{T}$ for a perfect fluid matter distribution is given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}, \tag{2.13}$$

where μ is the energy density and p is the isotropic pressure. These quantities are measured relative to a fluid four–velocity $\mathbf{u}$ $(u^a u_a = -1)$ with

$$u^a = e^{-\nu}\delta^a_0$$

for the spacetime (2.7). It is possible to generate exact solutions with an anisotropic energy–momentum tensor $\mathbf{T}$; such solutions tend to be applied in relativistic astrophysics and not in cosmology. Some solutions with anisotropic $\mathbf{T}$ have been given by Herrera *et al* (1984), Herrera and Ponce de Leon (1985b), Maartens and Maharaj (1990), Maharaj and Maartens (1989) and Maharaj (1993). However, for the purpose of this thesis we restrict our attention to the perfect fluid matter distribution (2.13).

The energy–momentum tensor (2.13) is coupled to the Einstein tensor (2.5) via the Einstein field equations

$$G_{ab} = T_{ab}, \tag{2.14}$$

where the choice of units is such that the speed of light and the coupling constant are taken to be unity. Following frequent practice in studies in cosmology and relativistic astrophysics we often suppose in this thesis that the fluid satisifies the barotropic equation of state $p = p(\mu)$ (Collins and Wainwright 1983). In an attempt to find physically reasonable solutions to the Einstein field equations, we often assume the simple equation of state

$$p = (\gamma - 1)\mu, \tag{2.15}$$

where $1 \leq \gamma \leq 2$. Other values of γ are normally not permitted for conventional matter as causality is violated: the requirements for the energy conditions may not be fulfilled and the speed of sound may exceed the speed of light (Hawking and Ellis 1973). Note that the case $\gamma = 2$ corresponds to the stiff equation of state for which the speed of sound equals the speed of light. The case $\gamma = \frac{4}{3}$ corresponds to radiation. When $\gamma = 1$ the pressure vanishes and we have a dust solution. Many exact solutions of the field equations, obeying the equation of state (2.15), are known which may be used in studies of cosmology (Kramer *et al* 1980). Sometimes in applications the polytropic equation of state relating p and μ is used. This is given by

$$p = k\mu^{1+1/n},$$

where k and n are constants (Shapiro and Teukolsky 1983). However, for the purposes of this thesis we largely restrict ourselves to the equation of state (2.15).

For the line element (2.7) the energy–momentum tensor (2.13) has the particular form

$$T_{ab} = \mathrm{diag}\left(\mu e^{2\nu},\ pe^{2\lambda},\ pY^2,\ p\sin^2\theta\ Y^2\right)$$

relative to $u^a = e^{-\nu}\delta^a_0$. Then on substitution of these components of the energy–momentum tensor and the Einstein tensor components (2.12) into (2.14) we generate the Einstein field equations

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{2.16a}$$

$$p = -\frac{1}{Y^2} + \frac{2}{Y}e^{-2\lambda}\left(\nu' Y' + \frac{Y'^2}{2Y}\right) - \frac{2}{Y}e^{-2\nu}\left(\ddot{Y} - \dot{\nu}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{2.16b}$$

$$p = e^{-2\lambda}\left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{Y}\left(\nu' Y' - \lambda' Y' + Y''\right)\right]$$

$$- e^{-2\nu}\left[\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)\right] \tag{2.16c}$$

$$0 = \dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda} \tag{2.16d}$$

for the general spherically symmetric metric (2.7). In this thesis we are primarily concerned with generating exact solutions of the field equations (2.16) with nonvanishing shear. There are very few known solutions of (2.16) with $\sigma \neq 0$; most solutions given by Kramer *et al* (1980), and others, have $\sigma = 0$. A simple class of shearing metrics satisfying (2.16) has been found by Maharaj *et al* (1993); these solutions admit a conformal Killing vector acting in the radial direction (Maharaj and Maharaj 1994).

The conservation of matter

$$T^{ab}{}_{;b} = 0$$

follows from the field equations (2.14) and the Bianchi identity (2.6). From this conservation law we generate the two first order differential equations

$$p' = -(\mu + p)\nu' \tag{2.17a}$$

$$\dot{\mu} = -(\mu + p)\left(\dot{\lambda} + 2\frac{\dot{Y}}{Y}\right) \tag{2.17b}$$

which may also be obtained directly from the field equations (2.16). The seminal treatment of the general spherically symmetric field equations (2.16)–(2.17) is by

Misner and Sharp (1964). They expressed the field equations in a form which is useful for studying the physical features of the model.

2.4 Exact Solutions

Most exact solutions of the field equations (2.16) that have been studied in the past correspond to $\sigma = 0$. In this section we list some of the known solutions for which $\sigma \neq 0$ in comoving coordinates; these solutions have a simple form and some are contained as special cases in our general classes presented in later chapters.

Maharaj *et al* (1993) presented a general class of accelerating, expanding and shearing metrics. This is a simple class of exact solutions which are expressible in terms of elementary functions, and contain, as special cases, many solutions found previously. Maharaj *et al* found that it was convenient to distinguish between the three cases $k = 0$, $k < 0$, $k > 0$ that arise in their solution. The metric (2.7) then assumes the following forms

$k = 0:$

$$ds^2 = -a_1^2 r^2 dt^2 + \left(\frac{1}{a_2 r^2}\right) dr^2 + r^2 \left(-a_1^2 t^2 + a_3 t + a_4\right) \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.18a}$$

$k = -n^2 < 0:$

$$ds^2 = -a_1^2 r^2 dt^2 + \left(\frac{1}{-n^2 + a_2 r^2}\right) dr^2$$

$$+r^2 \left(a_3 \sin(2a_1 nt) + a_4 cos(2a_1 nt) - \frac{1}{2n^2}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

(2.18b)

$k = n^2 > 0$:

$$ds^2 = -a_1^2 r^2 dt^2 + \left(\frac{1}{n^2 + a_2 r^2}\right) dr^2$$

$$+r^2 \left(a_3 e^{2a_1 nt} + a_4 e^{-2a_1 nt} + \frac{1}{2n^2}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (2.18c)$$

where a_1, a_2, a_3 and a_4 are constants of integration. It must be noted that $a_2 > 0$ for $k \leq 0$ and $r \geq \sqrt{-k/a_2}$ for $k < 0$. These models obey the relationship

$$p = \mu + 6a_2$$

which is a generalisation of the stiff equation of state $p = \mu$. In addition they admit a conformal Killing vector

$$\mathbf{X} = \frac{\partial}{\partial r}$$

which acts in the radial direction (Maharaj and Maharaj 1994). Thus the models of Maharaj *et al* (1993) have a clear thermodynamical interpretation, in terms of an equation of state, and a geometrical interpretation, in terms of a conformal symmetry.

If we set $a_1 = 1/2, a_2 = 0, k = 1$, then (2.18c) gives the particular case

$$ds^2 = -\left(\frac{r^2}{4}\right) dt^2 + dr^2 + r^2\left(a_3 e^t + a_4 e^{-t} + \frac{1}{2}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (2.19)$$

which was first found by Gutman and Bespal'ko (1967). As $a_2 = 0$, we have that the Gutman and Bespal'ko solution (2.19) satisfies the stiff equation of state $p = \mu$. Another shearing solution with the stiff equation of state $p = \mu$ was reported by Wesson (1978). However, the solutions of Wesson and of Gutman and Bespal'ko are equivalent and are contained in the general class (2.18) (for details see Maharaj *et al* 1993). In the literature the Wesson solution has sometimes been taken to be distinct from (2.19) (see p173 of Kramer *et al* 1980). Special cases of (2.18) were also reported by Hajj–Boutros (1985) who presented a class of solutions in terms of the third Painlevé transcendent. However, those forms of the metric are redundant since they can be transformed to the metrics (2.18) (see Maharaj *et al* 1993). Shaver and Lake (1988) considered separable metrics with spherical, plane and hyperbolic symmetries which obey the weak and strong energy conditions and which do not contain scalar polynomial singularities. Their results contain those of Lake (1983) in the special case of spherical symmetry and are related to the line element (2.18).

Van den Bergh and Wils (1985), working with nonstatic perfect fluid spheres and a linear equation of state, found a number of shearing solutions. Their first solution, which is directly relevant to our work, is given by

$$ds^2 = -m^2t^2\left(1 - m^2t^2\right)^{-1} dt^2 + dr^2 + t^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.20}$$

in terms of their coordinates and where m is a constant of integration. This solution was derived for a nonaccelerating stiff fluid. The exact solution (2.20) is related to the Wesson (1978) solution and belongs to a group of solutions obtained by McVittie and Wiltshire (1975). The second solution derived by Van den Bergh and Wils is an accelerating solution with an equation of state $p = \mu - 2\Lambda$ where Λ is an arbitrary constant which may be interpreted as the cosmological constant. After redefinition of the coordinates the solution is given by

$$ds^2 = -r^2t^2\left(mt^4 - t^2 + n\right)^{-1} dt^2 + \left(m - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2t^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (2.21)$$

with m and n constants. Van den Bergh and Wils demonstrated that it is possible to generate (2.20) from (2.21) by a limit transition. Van den Bergh and Wils also presented a third solution

$$ds^2 = -4e^{-6\lambda}\lambda_x^2 dt^2 + e^{2\lambda}dr^2 + e^{-4\lambda}\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (2.22)$$

where $x = -t_0t^2 + r$ and t_0 is a constant. These solutions are accelerating and λ is determined by a second order differential equation

$$2(e^{-\lambda})_{xx} = 3t_0xe^{7\lambda} - e^{5\lambda}.$$

Note that the model (2.22) cannot admit the linear equation of state (2.15).

Kitamura (1994) recently presented a new class of shearing solutions utilising a method proposed by Takeno (1966). This class arises from conditions for spherically symmetric spacetimes to yield perfect fluid models and is imbeddable in five–dimensional pseudo–Euclidean space (Kitamura 1989). His solutions are all of the form

$$ds^2 = -\frac{1}{[H(r) + G(t)]^2}\left[-dt^2 + dr^2 + b(t)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \quad (2.23)$$

where $H(r)$, $G(t)$ and $b(t)$ are specified functions . In the appropriate limit Kitamura regains the solutions of Gutman and Bespal'ko (1967), Van den Bergh and Wils (1985) and Wesson (1978). The line element (2.23) admits a conformal Killing vector acting in the radial direction as established by Kitamura (1995a, 1995b). This is similar to the result of Maharaj and Maharaj (1994) for the class of line elements (2.18).

A number of solutions, special cases of which may be nonshearing, have been found in noncomoving coordinates. These include the solutions of McVittie and Wiltshire (1975, 1977) and Vaidya (1968) and subcases of the general class of solutions of Szafron (1977) and Szekeres (1975). A physical analysis of solutions in noncomoving coordinates has been attempted by Bonnor and Knutsen (1993), Knutsen (1992, 1995) and Ray (1978). However, as it is easier to analyse the physical features of the solutions in comoving coordinates rather than in noncomoving coordinates, we choose the former in this thesis.

3 Nonaccelerating and Nonexpanding Solutions to the Field Equations

3.1 Introduction

In this chapter we find new shearing solutions to the field equations derived in Chapter 2. Because of the complexity of the equations we make assumptions about the acceleration or the expansion in order to arrive at some feasible solutions. In §3.2 we investigate solutions having both shear and expansion but without acceleration. Following Kramer *et al* (1980) this class of solutions is divided into two categories, according to whether $Y' = 0$ or $Y' \neq 0$. Firstly the case $Y' = 0$ is studied in §3.2.1. By assuming the barotropic equation of state $p = (\gamma - 1)\mu$ we reduce the solution of the Einstein field equations to an Abel's equation of the first kind. Although the equation is of first order, it is not easily integrated due to its nonlinearity. The integration procedure is completely performed for two special cases of the constant γ: $\gamma = \frac{4}{3}$ (radiating solutions) and $\gamma = 2$ (stiff fluid solutions). The radiating solutions are related to the Kantowski–Sachs model (Kantowski and Sachs 1966). The stiff fluid solutions contain, as a special case, the result of Van den Bergh and Wils (1985). Secondly the case $Y' \neq 0$ for nonaccelerating solutions is studied in §3.2.2. The solution of the field equations is reduced to quadratures and explicitly solved for

$p =$ constant in terms of known functions. We investigate nonexpanding solutions, with shear and acceleration, in §3.3. We observe that, by choosing $\mu =$ constant, we can integrate for the potential Y in terms of quadratures.

3.2 Nonaccelerating Solutions

In this section we study solutions having both shear and expansion, but assume that the acceleration $\dot{u}^a = 0$. From (2.8b) this condition implies

$$\nu = \nu(t). \tag{3.1}$$

Kramer *et al* (1980) point out that the nonaccelerating solutions can be classified according to whether $Y' = 0$ or $Y' \neq 0$. We firstly consider the case $Y' = 0$ in §3.2.1. Later in §3.2.2 we investigate the case $Y' \neq 0$.

3.2.1 The Case $Y' = 0$

If we consider the case $Y' = 0$, then we can take

$$Y = t \tag{3.2}$$

without any loss of generality. Then the field equations (2.16) reduce to

$$\mu = \frac{1}{t^2} + \frac{2}{t}e^{-2\nu}\left(\dot{\lambda} + \frac{1}{2t}\right) \tag{3.3a}$$

$$p = -\frac{1}{t^2} + \frac{2}{t}e^{-2\nu}\left(\dot{\nu} - \frac{1}{2t}\right) \tag{3.3b}$$

$$p = -\left[\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + \frac{1}{t}\left(\dot{\lambda} - \dot{\nu}\right)\right]e^{-2\nu} \tag{3.3c}$$

which is a system of three equations. Note that the field equation (2.16d) is identically satisfied for $\nu' = 0$ and $Y = t$.

From (3.3b), or (2.17a), it follows that

$$p = p(t).$$

In order to find solutions to the nonaccelerating field equations (3.3) we impose the linear equation of state

$$p = (\gamma - 1)\mu.$$

This immediately implies that

$$\mu = \mu(t).$$

From (2.17b) we have

$$\dot{\lambda} = -\frac{\dot{\mu}}{\gamma\mu} - \frac{2}{t}. \tag{3.4}$$

On substituting (3.4) into (3.3a) and using

$$\tilde{\mu} = t^2\mu$$

we generate an equation independent of λ which can be solved for t to give

$$t = \frac{\gamma\tilde{\mu}\left(\tilde{\mu} - 1 + 3e^{-2\nu}\right) - 4\tilde{\mu}e^{-2\nu}}{-2\dot{\tilde{\mu}}e^{-2\nu}}. \tag{3.5}$$

Now from (3.3b) we also obtain an equation for t:

$$t = \frac{(\gamma - 1)\tilde{\mu} + 1 + e^{-2\nu}}{2\dot{\nu}e^{-2\nu}}. \tag{3.6}$$

Then equations (3.5) and (3.6) imply the relationship

$$\frac{d\nu}{d\tilde{\mu}} = \frac{\left[1 + e^{2\nu}\left(1 + (\gamma - 1)\tilde{\mu}\right)\right]}{\tilde{\mu}\left[4 - 3\gamma + \gamma e^{2\nu}(1 - \tilde{\mu})\right]} \tag{3.7}$$

in the variables ν and $\tilde{\mu}$. We have thus reduced the solutions of the Einstein field equations to the single equation (3.7). Note that (3.3c), a consistency condition, generates the condition of pressure isotropy. Equation (3.7) is a first order differential equation which is highly nonlinear, but has the advantage that the coordinate t has been eliminated. Equation (3.7) was also derived by Van den Bergh and Wils (1985). They presented a particular solution in the case $\gamma = 2$. We attempt to integrate (3.7) in general. In the above analysis we have used the same notation as Van den Bergh and Wils to ease comparison with their results.

As the integration procedure is nontrivial, we present the important steps in the solution. With the transformations

$$y = e^{2\nu} \tag{3.8a}$$

$$x = \tilde{\mu}, \tag{3.8b}$$

equation (3.7) becomes

$$\left[y + \frac{4-3\gamma}{\gamma(1-x)}\right]\frac{dy}{dx} = \frac{2[1+(\gamma-1)x]}{\gamma x(1-x)}y^2 + \frac{2y}{\gamma x(1-x)}. \tag{3.9}$$

Even though (3.9) is nonlinear, its form is more transparent as we have eliminated the exponential function. Equation (3.9) is a special case of Abel's equation of the second kind (Kamke 1983). This equation is highly nonlinear and difficult to solve in closed form; there are only a few known exact solutions to Abel's equation of the second kind. If we make the transformation

$$\frac{1}{w(x)} = \left(y + \frac{4-3\gamma}{\gamma(1-x)}\right)(1-x)^2 x^{-2/\gamma}, \tag{3.10}$$

then (3.9) can be written as

$$\frac{dw}{dx} = \frac{2(1-x)(4-3\gamma)}{\gamma^3 x^{1+4/\gamma}}\left[(2-\gamma)(2-3\gamma)x - 4(1-\gamma)\right]w^3$$

$$-\frac{1}{\gamma^2 x^{1+2/\gamma}}\left[(9\gamma-8)(\gamma-2)x + 2(7\gamma-8)\right]w^2. \tag{3.11}$$

Equation (3.11) is a special case of Abel's equation of the first kind and is also difficult to integrate. It is convenient to transform (3.11) to an equivalent form. Following the transformation suggested by Kamke (1983) we let

$$w(x) = \eta(\xi),$$

where

$$\xi = \frac{(7\gamma-8)}{\gamma}x^{-2/\gamma} - \frac{(9\gamma-8)}{\gamma}x^{1-2/\gamma}. \tag{3.12}$$

Then (3.11) becomes

$$\frac{d\eta}{d\xi} = g(\xi)\eta^3 + \eta^2, \tag{3.13}$$

where

$$g(\xi) = \frac{-2(1-x)(4-3\gamma)\left[(2-\gamma)(2-3\gamma)x - 4(1-\gamma)\right]}{\gamma x^{2/\gamma}\left[(9\gamma-8)(\gamma-2)x - 2(8-7\gamma)\right]} \tag{3.14}$$

and ξ is defined in terms of x by (3.12). If we now introduce the relationship

$$\frac{d\xi}{dz} = -\frac{1}{z\eta(\xi)}, \tag{3.15}$$

then (3.13) becomes

$$z^2\frac{d^2\xi}{dz^2} + g(\xi) = 0. \tag{3.16}$$

Thus the differential equation (3.11) has been transformed to the simpler form (3.16). Even though (3.16) is simpler the remaining integration is nontrivial and there is the added complication of performing the inversion in (3.15) in practice. However, it is possible to make progress for particular values of γ. It is not easy to integrate (3.15) for all values of $1 \leq \gamma \leq 2$ mainly because of the technical difficulties experienced in the inversion. We have completed the integration in the two cases of $\gamma = \frac{4}{3}$ and $\gamma = 2$ which correspond to radiation and a stiff equation of state, respectively.

We first consider the case $\gamma = \frac{4}{3}$. With this value of γ (3.14) implies that $g(\xi) = 0$ and then (3.16) becomes

$$\frac{d^2\xi}{dz^2} = 0.$$

This has the general solution

$$\xi = a_1 + a_2 z,$$

where a_1 and a_2 are constants of integration. We can then use (3.15) to express η as

$$\eta(\xi) = (a_1 - \xi)^{-1}. \tag{3.17}$$

With the aid of (3.12) the function $w(x)$ then becomes

$$w(x) = \left(a_1 - x^{-3/2} + 3x^{-1/2}\right)^{-1}$$

and from (3.10) we have

$$y = \frac{a_1 x^{3/2} + 3x - 1}{(1-x)^2}$$

which has the equivalent form in the original variables t and μ:

$$e^{2\nu} = \frac{a_1 t^3 \mu^{3/2} + 3t^2\mu - 1}{(1 - t^2\mu)^2}, \tag{3.18}$$

where we have utilised (3.8). To find the second metric coefficient λ we substitute $\gamma = \frac{4}{3}$ into (3.4) and integrate to obtain

$$e^{2\lambda} = b_1 \mu^{-3/2} t^{-4}, \tag{3.19}$$

where b_1 is a constant of integration. It remains to find the energy density μ. On substituting (3.17) into (3.5) for $\gamma = \frac{4}{3}$ we obtain

$$\dot{\tilde{\mu}}t = -\frac{2}{3}\frac{a_1\tilde{\mu}^{5/2} + 3\tilde{\mu}^2 - \tilde{\mu}}{\tilde{\mu} - 1}. \tag{3.20}$$

If we make the transformation

$$z = \sqrt{\tilde{\mu}}$$

then (3.20) can be written as

$$\begin{aligned}\frac{dt}{t} &= -\frac{3(z^2-1)dz}{a_1z^4 + 3z^3 - z}\\ &= -3\left[\frac{1}{z} - \frac{a_1z^2 + 2z}{a_1z^3 + 3z^2 - 1}\right]dz\end{aligned}$$

which can be integrated to give the result

$$t^4\mu^{3/2} = c_1\left(a_1t^3\mu^{3/2} + 3t^2\mu - 1\right) \tag{3.21}$$

in the original variables μ and t, where c_1 is an integration constant. We have verified this result with the help of the software package MATHEMATICA (Wolfram 1991). It is not possible to write μ explicitly as a function of the spacetime coordinate t. Equation (3.21) is an implicit representation of μ in terms of t. The pressure is given by

$$p = \tfrac{1}{3}\mu \tag{3.22}$$

for our value of $\gamma = \frac{4}{3}$. The metric for this solution has the form

$$ds^2 = -\left(\frac{a_1t^3\mu^{3/2} + 3t^2\mu - 1}{(1-t^2\mu)^2}\right)dt^2 + b_1\mu^{-3/2}t^{-4}dr^2 + t^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.23}$$

The general solution of (3.3), for $\nu' = 0$, $Y = t$ and $\gamma = \frac{4}{3}$, is given by the line element (3.23) and the quantities (3.21), (3.22).

The geometry of the line element (3.23) is related to that of the Kantowski–Sachs models (Kantowski and Sachs 1966). Solutions corresponding to pure radiation for the Kantowski–Sachs models were considered by Kompaneets and Chernov (1965), Kantowski and Sachs (1966), McVittie and Wiltshire (1975) and Ruban (1983). Those solutions were given in a complicated parametric form; our solution has the advantage of having a simple form. The solutions published previously for the case $\gamma = \frac{4}{3}$ involve a redefinition of the t coordinate ($dt = Cd\Psi$ with C being a function of Ψ) (Kramer *et al* 1980). Consequently these solutions are harder to interpret; our solution (3.23) is presented in the original coordinates (t, r, θ, ϕ) and is easier to interpret physically. In addition the solutions presented here follow from a systematic analysis of Abel's equations without *ad hoc* assumptions on the behaviour of the gravitational field.

We now consider the case $\gamma = 2$ which corresponds to the stiff equation of state $p = \mu$. For this value of γ (3.12) and (3.14) become

$$\xi = \frac{3}{x}$$

and

$$g(\xi) = \frac{2}{3}\left(\frac{\xi}{3} - 1\right)$$

respectively. For these values of ξ and $g(\xi)$ we obtain

$$z^2\frac{d^2\xi}{dz^2} + \frac{2}{9}\xi = \frac{2}{3} \qquad (3.24)$$

from (3.16). Equation (3.24) is linear and can be easily integrated to give

$$\xi(z) = a_3 z^{2/3} + a_4 z^{1/3} + 3, \qquad (3.25)$$

where a_3 and a_4 are constants of integration. Even though (3.25) is not a polynomial in z we can generate an expression for z:

$$z = \left(\frac{-a_4 \pm \sqrt{a_4^2 - 4a_3(3-\xi)}}{2a_3}\right)^3.$$

. With the above values of ξ and z we find that (3.15) implies

$$\eta(\xi) =$$

$$-6a_3\left[\left(-a_4 \pm \sqrt{a_4^2 - 4a_3\xi\left(\frac{3}{\xi}-1\right)}\right)^2 + a_4\left(-a_4 \pm \sqrt{a_4^2 - 4a_3\xi\left(\frac{3}{\xi}-1\right)}\right)\right]^{-1}.$$

Hence we are in a position to find the function $w(x)$, and it has the form

$$w(x) = -6\left(C \mp C\sqrt{1 - \frac{12}{Cx}(x-1)} - \frac{12}{x}(x-1)\right)^{-1},$$

where we have put

$$C = \frac{a_4^2}{a_3}.$$

Then (3.10) gives the function

$$y = -\frac{Cx}{6(1-x)^2}\left(1 \mp \sqrt{1 - \frac{12}{Cx}(x-1)}\right) + \frac{1}{x-1}.$$

We are now in a position to generate the potential ν. In terms of the original variables t and μ we obtain the potential

$$e^{2\nu} = -\frac{t^2\mu}{6(1-t^2\mu)^2}\left(C \mp \sqrt{C^2 - \frac{12C}{t^2\mu}(t^2\mu - 1)}\right) + \frac{1}{t^2\mu - 1}, \tag{3.26}$$

where we have utilised the transformation (3.8). To find the remaining metric coefficient λ we substitute $p = \mu$ into (3.4) and integrate to obtain

$$e^{2\lambda} = b_2\mu^{-1}t^{-4}, \tag{3.27}$$

where b_2 is a constant of integration. The metric for this solution has the form

$$ds^2 = \left[\frac{t^2\mu}{6(1-t^2\mu)^2}\left(C \mp \sqrt{C^2 - \frac{12C}{t^2\mu}(t^2\mu - 1)}\right) - \frac{1}{t^2\mu - 1}\right] dt^2$$

$$+ b_2\mu^{-1}t^{-4}dr^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.28}$$

where we have used (3.26) and (3.27). The general solution of (3.3), for $\nu' = 0$ and $Y = t$, is given by the line element (3.28) for the stiff equation of state $p = \mu$; the explicit form for μ is derived later in this section. We believe that (3.28) is a new solution to the Einstein field equations (2.16).

When $C = 0$, (3.28) becomes

$$ds^2 = -\frac{dt^2}{t^2\mu - 1} + b_2\mu^{-1}t^{-4}dr^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This can be written as

$$ds^2 = -m^2t^2\left(1 - m^2t^2\right)^{-1} dt^2 + dr^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.29}$$

where we have chosen a new radial coordinate such that

$$r \longrightarrow \sqrt{b_2}mr$$

and the energy density μ is given by

$$\mu = \frac{1}{m^2t^4}, \qquad m \text{ a constant.}$$

The metric (3.29) was also found by Van den Bergh and Wils (1985). Our solution (3.28) contains their solution as a special case. As pointed out in §2.4 the special case (3.29) is also related to the results of Maharaj *et al* (1993) and Kitamura (1994). Thus we have demonstrated that a systematic analysis of Abel's equations has generated a new solution of the Einstein field equations when $\gamma = 2$. It has the advantage of containing the well-known Van den Bergh and Wils solution which arises in a variety of models in cosmology.

To complete our solution for $\gamma = 2$ we have to determine the energy density μ. As this is not a simple procedure, we provide details of the calculation for completeness. The energy density can be found on substitution of (3.26) into (3.5) with $\gamma = 2$. This yields the first order differential equation

$$\dot{\tilde{\mu}}t = \frac{\tilde{\mu}^2}{6(\tilde{\mu}-1)}\left(C \mp \sqrt{C^2 - \frac{12C}{\tilde{\mu}}(\tilde{\mu}-1)}\right) - 2\tilde{\mu}. \tag{3.30}$$

To simplify the integration in (3.30) we let

$$z = \sqrt{\tilde{\mu}}.$$

Then the variables t and z separate to give

$$\frac{dt}{t} = 12\left[\frac{zdz}{z^2(C-12)+12 \mp z\sqrt{z^2(C^2-12C)+12C}}\right]$$

$$-12\left[\frac{dz}{z^3(C-12)+12z \mp z^2\sqrt{z^2(C^2-12C)+12C}}\right]. \tag{3.31}$$

Equation (3.31) may be written as

$$\ln t = 12I_1 - 12I_2 + \text{constant} \;,$$

where I_1 and I_2 are given by

$$I_1 = \int \frac{zdz}{z^2(C-12)+12 \mp z\sqrt{z^2(C^2-12C)+12C}}$$

$$I_2 = \int \frac{dz}{z^3(C-12) + 12z \mp z^2\sqrt{z^2(C^2-12C)+12C}}.$$

Even though the integrands in I_1 and I_2 are complicated, we are in a position to complete the integration in closed form. The integration of (3.31) is conducted for the different cases $C = 0$, $C > 12$, $C = 12$, $12 > C > 0$ and $C < 0$ respectively. To avoid duplication we consider only the negative square root appearing in the integrands of I_1 and I_2; the solutions have a similar form with the positive square root.

Case 1 : $C = 0$

This case corresponds to the Van den Bergh and Wils (1985) solution. When $C = 0$, (3.31) can be easily integrated to produce

$$t = (c_1 t\sqrt{\mu})^{-1},$$

where c_1 is an integration constant. The energy density is given by

$$\mu = \frac{1}{c_1^2 t^4}. \tag{3.32}$$

Our results are consistent with those obtained by Van den Bergh and Wils.

Case 2 : $C > 12$

To complete the integration for I_1 it is convenient to introduce the new variable

$$z = \sqrt{\frac{12}{C-12}}\sinh\eta \qquad \text{and} \qquad C = 12\cosh^2\alpha.$$

Then the integral I_1 can be written as

$$I_1 = -\frac{1}{12\sinh\alpha}\int\frac{\sinh\eta d\eta}{\cosh\alpha\sinh\eta - \sinh\alpha\cosh\eta}.$$

With the aid of the table of integrals in Gradshteyn and Ryzhik (1994, 2.447(1), p133) the quantity I_1 can then be written as

$$I_1 = -\frac{1}{12\sinh\alpha}\left[\frac{\eta\cosh\alpha - \sinh\alpha\ln\left[\sinh\left(\eta + \operatorname{arctanh}\left(-\frac{\sinh\alpha}{\cosh\alpha}\right)\right)\right]}{\cosh^2\alpha - \sinh^2\alpha}\right]$$

which can then be expressed in terms of z as

$$I_1 = -\frac{1}{12}\sqrt{\frac{C}{C-12}}\ln\left(\frac{1}{\sqrt{C}}\left(z\sqrt{C-12} + \sqrt{z^2(C-12)+C}\right)\right)$$

$$+\frac{1}{12}\ln\left[\sqrt{\frac{3}{C}}\frac{\left(z\sqrt{C-12} + \sqrt{z^2(C-12)+C}\right)}{\left(\sqrt{C}+\sqrt{C+12}\right)}\right.$$

$$\left.-\sqrt{\frac{C}{3}}\frac{\left(\sqrt{C}+\sqrt{C+12}\right)}{\left(z\sqrt{C-12} + \sqrt{z^2(C-12)+C}\right)}\right].$$

Similarly to evaluate I_2 we define

$$z = \frac{\sinh\eta}{\sinh\alpha} \qquad \text{where} \qquad C = 12\cosh^2\alpha.$$

The quantity I_2 simplifies to

$$I_2 = \frac{1}{12}\int\frac{\operatorname{cosech}^2\eta d\eta}{\coth\eta - \tanh\alpha}.$$

Upon integration we obtain

$$I_2 = -\frac{1}{12}\ln(\coth\eta - \tanh\alpha)$$

which is equivalent to

$$I_2 = -\frac{1}{12}\ln\left(\frac{\sqrt{z^2(C-12)C+12C} - z(C-12)}{z\sqrt{(C-12)C}}\right).$$

Combining the two integrals I_1 and I_2 we have for the case $C > 12$ that

$$t = c_2\left[\frac{1}{\sqrt{C}}\left(t\sqrt{\mu(C-12)} + \sqrt{t^2\mu(C-12)+C}\right)\right]^{-\sqrt{C/(C-12)}}$$

$$\times\left[\sqrt{\frac{3}{C}}\frac{t\sqrt{\mu(C-12)} + \sqrt{t^2\mu(C-12)+C}}{\sqrt{C}+\sqrt{C+12}} - \sqrt{\frac{C}{3}}\frac{\sqrt{C}+\sqrt{C+12}}{t\sqrt{\mu(C-12)} + \sqrt{t^2\mu(C-12)+C}}\right]$$

$$\times\left[\frac{\sqrt{t^2\mu(C-12)C+12C} - t\sqrt{\mu}(C-12)}{t\sqrt{\mu(C-12)C}}\right] \tag{3.33}$$

in terms of the original variables t and μ. Here c_2 is an integration constant. Equation (3.33) is the solution of (3.30) for $C > 12$.

Case 3 : $C = 12$

This is a simple case and the general solution is given by

$$t = \frac{\exp(c_3 - \sqrt{\mu}t)}{\sqrt{\mu}t}, \tag{3.34}$$

where c_3 is an integration constant.

Case 4 : $0 < C < 12$

In this case we introduce the variable

$$z = \sqrt{\frac{12}{12-C}}\sin\eta \qquad \text{and} \qquad C = 12\cos^2\alpha.$$

Then I_1 can be written as

$$I_1 = \frac{\csc^2\alpha}{12}\int\frac{\sin\eta\, d\eta}{\cos\eta - \cot\alpha\sin\eta}.$$

As before I_1 can be evaluated with the aid of the table of integrals in Gradshteyn and Ryzhik (1994, 2.557(1), p181) and yields

$$I_1 = \frac{\csc^2\alpha}{12}\left[\frac{-\eta\cot\alpha - \ln\sin\left(\eta + \arctan\left(\frac{-1}{\cot\alpha}\right)\right)}{1+\cot^2\alpha}\right]$$

which can then be expressed in terms of z as

$$I_1 = -\frac{1}{12}\sqrt{\frac{C}{12-C}}\arcsin\left(z\sqrt{\frac{12-C}{12}}\right)$$

$$+\frac{1}{12}\ln\frac{1}{12}\left(z\sqrt{(12-C)C} - \sqrt{(12-(12-C)z^2)(12-C)}\right).$$

Similarly to evaluate I_2 we define

$$z = \frac{\cos\eta}{\sin\alpha} \qquad \text{where} \qquad C = 12\cos^2\alpha.$$

The quantity I_2 then simplifies to

$$I_2 = \frac{1}{12}\int\frac{\sec^2\eta\, d\eta}{\cot\alpha + \tan\eta}.$$

This can be integrated to give

$$I_2 = \frac{1}{12}\ln\left(\frac{12 - z^2(12-C) + z\sqrt{12C}}{z\sqrt{12(12-C)}}\right)$$

in terms of the variable z. The combination of the two integrals (3.31), for the case $0 < C < 12$, can be expressed in terms of t and μ as

$$t = \frac{12c_4 t\sqrt{12\mu(12-C)}}{\left(12 - t^2\mu(12-C) + t\sqrt{12\mu C}\right)\left(t\sqrt{\mu(12-C)C} - \sqrt{(12-C)(12-(12-C)t^2\mu)}\right)}$$

$$\times\exp\left[-\sqrt{\frac{C}{12-C}}\arcsin\left(t\sqrt{\frac{\mu(12-C)}{12}}\right)\right], \qquad (3.35)$$

where c_4 is an integration constant. Therefore (3.35) is the solution for (3.30) in the case $0 < C < 12$.

Case 5 : $C < 0$

In this case we let $C = -k$ where $k > 0$ and introduce the new variable

$$z = \sqrt{\frac{12}{12+k}}\cosh\eta \qquad \text{and} \qquad k = 12\sinh^2\alpha$$

which will be used to transform both I_1 and I_2. Then I_1 can be written as

$$I_1 = -\frac{1}{12\cosh\alpha}\int \frac{\cosh\eta\, d\eta}{\sinh\alpha\cosh\eta + \cosh\alpha\sinh\eta}.$$

Again this integral can be evaluated with the aid of the table of integrals in Gradshteyn and Ryzhik (1994, 2.448(1), p133) and gives

$$I_1 = -\frac{1}{12\cosh\alpha}\left[\frac{-\eta\sinh\alpha + \cosh\alpha\ln\sinh\left(\eta + \operatorname{arctanh}(\frac{\sinh\alpha}{\cosh\alpha})\right)}{\cosh^2\alpha - \sinh^2\alpha}\right].$$

Expressing I_1 in terms of z we have

$$I_1 = \frac{1}{12}\sqrt{\frac{k}{k+12}}\ln\left(z\sqrt{\frac{12}{k}} + \sqrt{\frac{12}{k}z^2 - 1}\right)$$

$$-\frac{1}{12}\ln\frac{1}{2}\left[\left(z\sqrt{\frac{12}{k}} + \sqrt{\frac{12}{k}z^2 - 1}\right)\left(\frac{\sqrt{k}+\sqrt{k+12}}{\sqrt{12}}\right)\right.$$

$$\left.-\left(\left(z\sqrt{\frac{12}{k}} + \sqrt{\frac{12}{k}z^2 - 1}\right)\left(\frac{\sqrt{k}+\sqrt{k+12}}{\sqrt{12}}\right)\right)^{-1}\right].$$

Using the same transformation as for I_1, the second integral I_2 can be expressed in terms of η and α as

$$I_2 = -\frac{1}{12}\int \frac{\operatorname{sech}^2\eta\, d\eta}{\tanh\eta + \tanh\alpha}.$$

This can be easily integrated to give

$$I_2 = -\frac{1}{12} \ln \left[\frac{\sqrt{z^2(k+12) - 12} + z\sqrt{k}}{z\sqrt{k+12}} \right]$$

in terms of the variable z. On combining the two integrals we have for $C < 0$ that

$$t = \frac{c_5 \left(\frac{1}{\sqrt{k}}(t\sqrt{12\mu} + \sqrt{12t^2\mu - k})\right)^{\sqrt{k/(k+12)}} \left(\sqrt{t^2\mu(k+12) - 12} + t\sqrt{\mu k}\right)}{\left(t\sqrt{12\mu} + \sqrt{12t^2\mu - k}\right)\left(\sqrt{k} + \sqrt{k+12}\right) - 12k\left(\left(t\sqrt{12\mu} + \sqrt{12t^2\mu - k}\right)\left(\sqrt{k} + \sqrt{k+12}\right)\right)^{-1}}$$

$$\times \frac{2\sqrt{12k}}{t\sqrt{\mu(k+12)}} \tag{3.36}$$

is the solution for (3.30) in terms of the original variables t and μ. The quantity c_5 is an integration constant.

It is remarkable that the integrals I_1 and I_2 may be evaluated explicitly and the solutions are expressible in terms of elementary functions. Equations (3.32)–(3.36) are implicit representations of the energy density μ in terms of t for the different values of the constant C. The pressure is then given by

$$p = \mu \tag{3.37}$$

for the value of $\gamma = 2$. The general solution of the field equations (3.3), for $\nu' = 0$, $Y = t$ and $\gamma = 2$, is the line element (3.28), μ is given by (3.32)–(3.36) and p satisfies (3.37).

In this section we generated new nonaccelerating solutions to the field equations by using a barotropic equation of state. One class of solutions corresponds to radiating solutions ($\gamma = \frac{4}{3}$) and the other class to stiff fluid solutions ($\gamma = 2$). The gravitational potentials were explicitly found and expressions for the energy density and pressure were derived. These solutions are of importance in the study of inhomogeneous models in cosmology.

3.2.2 The Case $Y' \neq 0$

We now consider the second possibility $Y' \neq 0$ in the class of nonaccelerating solutions. If $Y' \neq 0$, then it is possible to make the choice

$$\nu = 0$$

as pointed out by Kramer *et al* (1980). The field equations (2.16) then become

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{3.38a}$$

$$p = -\frac{1}{Y^2} + e^{-2\lambda}\left(\frac{Y'^2}{Y^2}\right) - \frac{2}{Y}\left(\ddot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{3.38b}$$

$$p = \frac{e^{-2\lambda}}{Y}\left(Y'' - \lambda' Y'\right) - \left[\ddot{\lambda} + \dot{\lambda}^2 + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} + \ddot{Y}\right)\right] \tag{3.38c}$$

$$0 = \dot{Y}' - Y'\dot{\lambda}. \tag{3.38d}$$

We now investigate the integration of the system (3.38).

Integration of (3.38d) gives an expression for the potential λ in terms of

Y. It is convenient to express the result as

$$e^{2\lambda} = \frac{Y'^2}{1 - \epsilon f^2(r)} \qquad \epsilon = 0, \pm 1, \tag{3.39}$$

where $f(r)$ is a function of integration. A linear combination of (3.38a), (3.38b) and (3.38c) yields

$$\mu + 3p = -2\left(\ddot{\lambda} + \dot{\lambda}^2\right) - 4\frac{\ddot{Y}}{Y}.$$

We use (3.38d) to eliminate the variable λ appearing in the last equation. Thus we obtain the energy density μ in terms of Y and p:

$$\mu = -3p - 4\frac{\ddot{Y}}{Y} - 2\frac{\ddot{Y}'}{Y'}. \tag{3.40}$$

We notice from (2.17a) that the pressure p is a function of t only. In order to obtain an expression for p in terms of Y we substitute (3.39) into (3.38b) which results in

$$Y^2 p(t) = -2Y\ddot{Y} - \dot{Y}^2 - \epsilon f^2(r). \tag{3.41}$$

Note that (3.38c) follows from (3.41) by differentiation with respect to the coordinate r. Thus the quantities λ, μ and p, given by (3.39)–(3.41), satisfy the system (3.38). The metric in this case is given by

$$ds^2 = -dt^2 + \frac{Y'^2}{1 - \epsilon f^2(r)} dr^2 + Y^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.42}$$

where $\epsilon = 0, \pm 1$. It remains to determine Y for the line element (3.42). To find Y we must specify $p(t)$ explicitly in (3.41). We make the choice

$$p = \text{constant}.$$

This choice allows us to solve (3.41) by quadratures.

It seems that the explicit integration of (3.41) is not possible as Y depends on both t and r, and f is arbitrary. However, as (3.41) contains derivatives with respect to t only and p is constant, it is possible to make progress. In order to facilitate the integration procedure in (3.41) it is convenient to define

$$t = \frac{1}{\sqrt{p}}\tau \qquad \text{and} \qquad Y = \frac{f}{\sqrt{p}}y.$$

Then (3.41) can be expressed as

$$2yy_{\tau\tau} + {y_\tau}^2 + y^2 + \epsilon = 0, \tag{3.43}$$

where the subscript refers to differentiation with respect to τ. Note that, even though $y = y(t, \tau)$, we may essentially treat (3.43) as an ordinary differential equation. We shall now solve (3.43) for the three different values of ϵ.

Case 1 : $\epsilon = 0$

We let $\epsilon = 0$ and substitute

$$y = u^{2/3}$$

in (3.43) to obtain the simpler differential equation

$$4u_{\tau\tau} + 3u = 0.$$

This may be easily integrated to give

$$u = a_1 \sin\frac{\sqrt{3}}{2}\tau + a_2 \cos\frac{\sqrt{3}}{2}\tau,$$

where $a_1(r)$ and $a_2(r)$ are functions of integration. The solution of (3.41) then becomes

$$Y = \frac{f(r)}{\sqrt{p}}\left[a_1 \sin\frac{\sqrt{3p}}{2}t + a_2 \cos\frac{\sqrt{3p}}{2}t\right]^{2/3} \tag{3.44}$$

for the case $\epsilon = 0$.

Case 2 : $\epsilon = -1$

In this case it is easy to see that (3.43) admits the first integral

$$y y_\tau^{\,2} = a_3 + y - \frac{1}{3} y^3,$$

where $a_3(r)$ is an integration constant. This is a first order equation and the variables separate. Then a second integration results in

$$\tau - \tau_0 = \int \frac{\sqrt{y}dy}{\left[a_3 + y - \frac{1}{3}y^3\right]^{1/2}},$$

where $\tau_0(r)$ is a function of integration. At this stage it is desirable to introduce the new variable

$$v = \frac{1}{y}$$

to bring the integral into standard form. The above integral can then be written as

$$\tau - \tau_0 = -\int \frac{dv}{v\left[a_3 v^3 + v^2 - \frac{1}{3}\right]^{1/2}}. \tag{3.45}$$

The cubic expression $a_3 v^3 + v^2 - \frac{1}{3}$ can be factorised and with the aid of MATHEMATICA (Wolfram 1991) we can write (3.45) as

$$\tau - \tau_0 = \int_v^\infty \frac{dw}{w\sqrt{(w - \mathcal{B})(w - \mathcal{C})(w - \mathcal{D})}}, \tag{3.46}$$

where

$$\mathcal{B} = -\frac{1}{3a_3} + \frac{2^{1/3}}{a_3\left[-54 + 243a_3{}^2 + \sqrt{-2916 + (-54 + 243a_3{}^2)^2}\right]^{1/3}}$$

$$+\frac{1}{9.2^{1/3}a_3}\left[-54+243{a_3}^2+\sqrt{-2916+(-54+243{a_3}^2)^2}\right]^{1/3}$$

$$\mathcal{C}=-\frac{1}{3a_3}-\frac{\left(1+i\sqrt{3}\right)}{2^{2/3}a_3\left[-54+243a_3^2+\sqrt{-2916+(-54+243{a_3}^2)^2}\right]^{1/3}}$$

$$-\frac{1}{18.2^{1/3}a_3}\left(1-i\sqrt{3}\right)\left[-54+243{a_3}^2+\sqrt{-2916+(-54+243{a_3}^2)^2}\right]^{1/3}$$

$$\mathcal{D}=-\frac{1}{3a_3}-\frac{\left(1-i\sqrt{3}\right)}{2^{2/3}a_3\left[-54+243a_3^2+\sqrt{-2916+(-54+243{a_3}^2)^2}\right]^{1/3}}$$

$$-\frac{1}{18.2^{1/3}a_3}\left(1+i\sqrt{3}\right)\left[-54+243{a_3}^2+\sqrt{-2916+(-54+243{a_3}^2)^2}\right]^{1/3}$$

and

$$v\geq\mathcal{B}\geq\mathcal{C}\geq\mathcal{D}.$$

The quantity $\mathcal{B}$ is always real but the quantities $\mathcal{C}$ and $\mathcal{D}$ contain terms involving $i=\sqrt{-1}$. However, it is possible to obtain real values of $\mathcal{C}$ and $\mathcal{D}$ for the range $-2/3<a_3<2/3$. After some tedious calculations it is possible to rewrite $\mathcal{C}$ and $\mathcal{D}$ as

$$\mathcal{C}=-\frac{1}{3a_3}+\frac{1}{6a_3}\left[(1+i\sqrt{3})\left(1-\frac{9}{2}a_3^2+3a_3i\sqrt{1-\left(\frac{3a_3}{2}\right)^2}\right)^{1/3}\right.$$

$$\left.+(1-i\sqrt{3})\left(1-\frac{9}{2}a_3^2-3a_3i\sqrt{1-\left(\frac{3a_3}{2}\right)^2}\right)^{1/3}\right]$$

$$\mathcal{D}=-\frac{1}{3a_3}+\frac{1}{6a_3}\left[(1-i\sqrt{3})\left(1-\frac{9}{2}a_3^2+3a_3i\sqrt{1-\left(\frac{3a_3}{2}\right)^2}\right)^{1/3}\right.$$

$$+(1+i\sqrt{3})\left(1-\frac{9}{2}a_3^2-3a_3 i\sqrt{1-\left(\frac{3a_3}{2}\right)^2}\right)^{1/3}\Bigg] .$$

Observe that the terms in square brackets, for both $\mathcal{C}$ and $\mathcal{D}$, are sums of complex conjugates. Thus on the interval $-2/3 < a_3 < 2/3$, the constants $\mathcal{C}$ and $\mathcal{D}$ are real. The integral (3.46) can now be expressed in terms of elliptic functions (Gradshteyn and Ryzhik 1994, 3.137(8), p276) as

$$\tau-\tau_0=\frac{2}{\mathcal{D}\sqrt{\mathcal{B}-\mathcal{D}}}\left[\Pi\left(\beta,\frac{-\mathcal{D}}{\mathcal{B}-\mathcal{D}},q\right)-F(\beta,q)\right]$$

where

$$\beta=\arcsin\sqrt{\frac{\mathcal{B}-\mathcal{D}}{v-\mathcal{D}}}, \qquad q=\sqrt{\frac{\mathcal{C}-\mathcal{D}}{\mathcal{B}-\mathcal{D}}}.$$

The functions F and Π are elliptic integrals of the first and third kind respectively. In terms of the original variables the solution is

$$\sqrt{p}t-\tau_0=\frac{2}{\mathcal{D}\sqrt{\mathcal{B}-\mathcal{D}}}\left[\Pi\left(\beta,\frac{-\mathcal{D}}{\mathcal{B}-\mathcal{D}},q\right)-F(\beta,q)\right]$$

with

$$\beta=\arcsin\sqrt{\frac{\mathcal{B}-\mathcal{D}}{f/(\sqrt{p}Y)-\mathcal{D}}}.$$

The quantities $\mathcal{B},\mathcal{C},\mathcal{D}$ are defined in terms of a_3, and are all real for the range $-2/3 < a_3 < 2/3$.

A special case of the above solution leads to a simple form for the pressure p. It is possible for a particular value of a_3 in (3.45) to express the solution in terms of elementary functions; this corresponds to instances when two of the factors of the cubic expression $a_3v^3+v^2-\frac{1}{3}$ are repeated. For the special case with $a_3=2/3$ (3.45) becomes

$$\tau-\tau_0=-\sqrt{3}\int\frac{dv}{v(v+1)\sqrt{2v-1}}$$

which can be easily integrated to give

$$\tau - \tau_0 = -2\sqrt{3}\arctan\sqrt{2v-1} + 2\arctan\sqrt{\frac{2v-1}{3}}.$$

The solution in terms of the original variables Y and t then becomes

$$\sqrt{p}t - \tau_0 = 2\arctan\left(\frac{1}{\sqrt{3}}\sqrt{\frac{2f}{\sqrt{p}Y} - 1}\right) - 2\sqrt{3}\arctan\left(\sqrt{\frac{2f}{\sqrt{p}Y} - 1}\right). \tag{3.47}$$

This is the special solution for (3.41) when $a_3 = 2/3$, $\epsilon = -1$ and p is constant.

Case 3 : $\epsilon = 1$

In this case it is easy to show that (3.43) can be integrated to give

$$y{y_\tau}^2 = a_4 - y - \frac{1}{3}y^3,$$

where $a_4(r)$ is an integration constant. The variables separate in this first order differential equation and a second integration results in

$$\tau - \tau_0 = \int \frac{\sqrt{y}dy}{\left[a_4 - y - \frac{1}{3}y^3\right]^{1/2}}.$$

As for *Case 2* we utilise the transformation

$$v = \frac{1}{y}$$

to convert the above integral to

$$\tau - \tau_0 = -\int \frac{dv}{v\left[a_4 v^3 - v^2 - \frac{1}{3}\right]^{1/2}}. \tag{3.48}$$

The cubic expression $a_3 v^3 - v^2 - \frac{1}{3}$ can be factorised as in *Case 2* and the integral expressed in terms of elliptic functions (see Gradshteyn and Ryzhik 1994, 3.137 (8), p276). However, it must be pointed out that two of the factors in the integrand (corresponding to $\mathcal{C}$ and $\mathcal{D}$ of *Case 2*) are complex in nature. These factors are not real for any value of a_4.

We have presented solutions to the Einstein field equations in terms of elementary functions and elliptic integrals. They correspond to vanishing acceleration in spherically symmetric spacetimes with nonvanishing shear. An analysis of the literature indicates that these explicit solutions have not been found previously. Kramer *et al* (1980) observed that this class of solutions may exist, but no explicit solutions were presented. The solutions found here will assist in increasing our understanding of the physics of inhomogeneous cosmological models. This is an area for future research.

3.3 Nonexpanding Solutions

In this section we briefly study solutions that are shearing and accelerating, but are expansion–free. If the cosmological model is nonexpanding, then it can be seen from (2.8c) that the condition

$$\dot{\lambda} = -2\frac{\dot{Y}}{Y}$$

must hold, where $\dot{Y} \neq 0$. An integration of this equation gives the metric potential

$$e^{2\lambda} = Y^{-4},$$

where we have eliminated the constant of integration by a coordinate transformation in the metric. In order to find an expression for the metric potential ν we substitute this form of $e^{2\lambda}$ into (2.16d) and then integrate the resulting equation to obtain

$$e^{2\nu} = f^2(t)Y^4\dot{Y}^2,$$

where $f(t)$ is a function of integration. We observe from (2.17b) that the energy density μ, for nonexpanding solutions, is a function of r only. Equation (2.16a) can be written as

$$2Y^5Y'' + 5Y^4Y'^2 + 3f^{-2}(t)Y^{-4} + \mu(r)Y^2 - 1 = 0 \qquad (3.49)$$

which relates the two metric potentials μ and Y. Equation (2.16b) can be used to express the pressure p in term of Y:

$$p = -Y^{-2} + 2Y^3\dot{Y}'\dot{Y}^{-1}Y' + 5Y^2Y'^2 + 2f^{-3}(t)\dot{f}(t)Y^{-5}\dot{Y}^{-1} + 3f^{-2}(t)Y^{-6}. \quad (3.50)$$

Thus we have reduced the solution of the field equations to (3.49) and (3.50). (Note that (2.16c) is satisfied if equations (3.49) and (3.50) are satisfied.) The metric can now be written solely in terms of Y as

$$ds^2 = -f^2(t)Y^4\dot{Y}^2dt^2 + Y^{-4}dr^2 + Y^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

To complete the solution we need only determine the metric potential Y.

If Y is known and $\mu(r)$ and $f(t)$ prescribed, then p can be computed from (3.50). To find Y we must specify $\mu(r)$ explicitly in (3.49). We make the choice

$$\mu = \text{constant}.$$

With this choice of μ (3.49) can be integrated to give

$$Y^5Y'^2 - f^{-2}(t)Y^{-3} + \frac{\mu}{3}Y^3 - Y = A(t),$$

where $A(t)$ is an integration constant. This differential equation may be reduced to quadratures. For a qualitative treatment see Skripkin (1960). We can write the above differential equation equivalently as

$$r - r_0 = \int \frac{Y^4 dY}{\left[\frac{1}{f^2} + AY^3 + Y^4 - \frac{\mu}{3}Y^6\right]^{1/2}}. \tag{3.51}$$

We observe that, in the special case when $A = 0$, this integral may be expressed in terms of elliptic functions (Gradshteyn and Ryzhik 1994). As this class of non-expanding solutions is unlikely to produce viable cosmological models, we do not pursue this case further.

4 Self–Similar Solutions

4.1 Introduction

In Chapter 3 we presented shearing solutions with vanishing acceleration or expansion. The most general class of solutions in spherical symmetry will have no vanishing kinematical quantities, apart from the vorticity. In this chapter we seek viable solutions with nonvanishing acceleration, expansion and shear. We find a simple class of solutions by the introduction of a self–similarity variable. In §4.2 we provide a brief exposé of Lie symmetries and their relationship to differential equations. This is essential for determining the self–similarity variable that we seek. The infinitesimal generator and its extensions are defined in terms of Lie groups of transformations. We apply the theory of Lie symmetries to determine a linear infinitesimal generator which leads to the choice of a suitable self–similarity variable in §4.3. The Einstein field equations and conservation equations are then expressed in terms of this self–similarity variable and are subsequently reduced to a system of ordinary differential equations. In §4.4 we present solutions to the field equations for a special case by relating the behaviour of two of the potentials. With the aid of a series of transformations we are able to reduce the field equations to a single second order differential master equation in one variable. This equation is reducible to the nonlinear Riccati equation. This Riccati equation is integrated in general and an implicit representation of the potential Y is found. Expressions for the other metric potentials, the energy density and the pressure are then given in terms of Y. A particular solution is obtained by setting one of the constants to zero in the equation for Y. This leads to a stiff fluid solution which is a special case of the self–similar Tolman–Bondi metrics.

4.2 Lie Symmetries and the Similarity Generator

In this section we briefly introduce those elements of the Lie analysis which will assist in finding solutions to the field equations. The Lie analysis of differential equations is a major area of research and a comprehensive treatment is not possible here. For a more detailed study of Lie symmetries and their applications to differential equations the reader is referred to texts by Bluman and Kumei (1989), Olver (1993) and Stephani (1989). The Lie approach is proving to be a very useful technique in finding exact solutions in general relativity. In spherical symmetry, with vanishing shear, solutions have been categorised in terms of their symmetry generators by Leach *et al* (1992), Maharaj *et al* (1996) and Stephani (1983). This allows us to interpret geometrically the broad classes of solutions of Srivastava (1987) and Sussman (1986, 1987, 1988a, 1988b), amongst others, in terms of invariance transformations of differential equations. We expect the invariance transformations to be also helpful in the case of nonzero shear investigated in this thesis.

We shall first define some important concepts concerning Lie symmetries of algebraic equations and then proceed to Lie symmetries of partial differential equations. Let $\mathbf{x} = (x^1, x^2, \ldots, x^n)$ lie in the region $D \subset \Re^n$ and ϵ belong to $C \subset \Re$ and consider the one–parameter (ϵ) Lie group of transformations

$$\mathbf{x}^* = G(\mathbf{x}; \epsilon). \tag{4.1}$$

We define the infinitesimal generator G as the operator

$$G = G(\mathbf{x}) = \xi(\mathbf{x}) \cdot \bigtriangledown = \sum_{i=1}^{p} \xi^i(\mathbf{x}) \frac{\partial}{\partial x^i},$$

where $\bigtriangledown$ is the gradient operator

$$\bigtriangledown = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^p} \right)$$

and

$$\xi(\mathbf{x}) = \frac{\partial G}{\partial \epsilon}(\mathbf{x}; \epsilon) \mid_{\epsilon=0} .$$

If $F(\mathbf{x})$ is an infinitely differentiable function, then it is said to be an invariant function of the Lie group of transformations (4.1) if and only if

$$GF(\mathbf{x}) \equiv 0. \tag{4.2}$$

The associated characteristic system of equations of (4.2) is given by

$$\frac{dx^1}{\xi^1(x)} = \frac{dx^2}{\xi^2(x)} = \cdots = \frac{dx^p}{\xi^p(x)}.$$

The general solution of (4.2) can then be written as

$$F^1(x^1, \ldots, x^p) = c_1, \ldots, F^{p-1}(x^1, \ldots, x^p) = c_{p-1},$$

where the $c_1, \ldots, c_{p-1}$ are constants of integration.

We now repeat similar results for systems of partial differential equations. Consider a system of n–th order partial differential equations, with p independent variables $x = (x^1, \dots, x^p)$ and q dependent variables $u = (u^1, \dots, u^q)$ and with derivatives of u with respect to x up to order n, given by

$$F_\beta(x, u^{(n)}) = 0. \tag{4.3}$$

Let $g_\epsilon = \exp(\epsilon G)$ be a one–parameter(ϵ) Lie group of the system with transformations given by

$$(x^*, u^*) = g_\epsilon \cdot (x, u) = (\Psi_\epsilon(x, u), \Phi_\epsilon(x, u)). \tag{4.4}$$

Suppose that the generator given by

$$G = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{4.5}$$

with

$$\xi^i(x, u) = \frac{d}{d\epsilon}\Psi^i_\epsilon(x, u) \mid_{\epsilon=0}, \qquad i = 1, \dots, p$$

$$\phi_\alpha(x, u) = \frac{d}{d\epsilon}\Phi^\alpha_\epsilon(x, u) \mid_{\epsilon=0}, \qquad \alpha = 1, \dots, q$$

is an infinitesimal generator of the system (4.3). Then $G^{[n]}$, the n–th extension or prolongation of G given by (4.5), is defined as

$$G^{[n]} = G + \sum_{\alpha=1}^{q} \sum_{J} \left[\left(D_J \left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \right) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i} \right) \frac{\partial}{\partial u^\alpha_J} \right], \tag{4.6}$$

where

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \qquad u^\alpha_{J,i} = \frac{\partial u^\alpha_J}{\partial x^i}$$

and the total derivative D_J being defined by

$$D_J = D_{j_1} D_{j_2} \dots D_{j_k}.$$

It must be noted that the summation is over all multi–indices $J = (j_1, \ldots, j_k)$ with $1 \leq j_k \leq p$, $1 \leq k \leq n$. The transformations defined by (4.4) leave the system of partial differential equations (4.3) invariant if and only if

$$G^{[n]} F_\beta|_{F=0} = 0,$$

where $G^{[n]}$ is given by (4.6).

Lie symmetries are helpful in generating solutions to differential equations. Many of the exact solutions obtained using the Lie symmetry approach may not have otherwise been easily found using other techniques. Stephani (1989) provides numerous examples of the applicability of Lie symmetries in problems that arise in mathematical physics.

4.3 Self–Similar Form of the Field Equations

We shall now apply the above theory of Lie symmetries to the field equations (2.16) to reduce the number of independent variables appearing in them. We need to choose an infinitesimal generator G such that its extension leaves the field equations invariant. Our choice is guided by our objective to introduce a self–similar variable. A number of choices are possible for the quantities ξ^i and ϕ_α in (4.5). The simplest possibility is when these quantities are linear *ie* the self–similar case. Consequently we choose the infinitesimal generator

$$G = \mathcal{A}r\frac{\partial}{\partial r} + \mathcal{B}t\frac{\partial}{\partial t} + \mathcal{C}\mu\frac{\partial}{\partial \mu} + \mathcal{D}p\frac{\partial}{\partial p} + \mathcal{E}Y\frac{\partial}{\partial Y} + \mathcal{F}\frac{\partial}{\partial \lambda} + \mathcal{G}\frac{\partial}{\partial \nu}, \tag{4.7}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ are constants. Note that we cannot use $\mathcal{F}\lambda\frac{\partial}{\partial\lambda}$ and $\mathcal{G}\nu\frac{\partial}{\partial\nu}$ because this will introduce a multiplier of λ/ν in the exponential terms appearing in (2.16). The $1st$ and $2nd$ extensions of G follow from (4.6). With the help of (4.6) we obtain for the $1st$ extension

$$G^{[1]} = G + (\mathcal{C} - \mathcal{A})\mu'\frac{\partial}{\partial\mu'} + (\mathcal{C} - \mathcal{B})\dot{\mu}\frac{\partial}{\partial\dot{\mu}} + (\mathcal{D} - \mathcal{A})p'\frac{\partial}{\partial p'} + (\mathcal{D} - \mathcal{B})\dot{p}\frac{\partial}{\partial\dot{p}}$$

$$+(\mathcal{E} - \mathcal{A})Y'\frac{\partial}{\partial Y'} + (\mathcal{E} - \mathcal{B})\dot{Y}\frac{\partial}{\partial\dot{Y}} - \mathcal{A}\lambda'\frac{\partial}{\partial\lambda'} - \mathcal{B}\dot{\lambda}\frac{\partial}{\partial\dot{\lambda}} - \mathcal{A}\nu'\frac{\partial}{\partial\nu'} - \mathcal{B}\dot{\nu}\frac{\partial}{\partial\dot{\nu}} \tag{4.8}$$

and for the $2nd$ extension

$$G^{[2]} = G^{[1]} + (\mathcal{C} - 2\mathcal{A})\mu''\frac{\partial}{\partial\mu''} + (\mathcal{C} - \mathcal{A} - \mathcal{B})\dot{\mu}'\frac{\partial}{\partial\dot{\mu}'} + (\mathcal{C} - 2\mathcal{B})\ddot{\mu}\frac{\partial}{\partial\ddot{\mu}}$$

$$+(\mathcal{D} - 2\mathcal{A})p''\frac{\partial}{\partial p''} + (\mathcal{D} - \mathcal{A} - \mathcal{B})\dot{p}'\frac{\partial}{\partial\dot{p}'} + (\mathcal{D} - 2\mathcal{B})\ddot{p}\frac{\partial}{\partial\ddot{p}}$$

$$+(\mathcal{E} - 2\mathcal{A})Y''\frac{\partial}{\partial Y''} + (\mathcal{E} - \mathcal{A} - \mathcal{B})\dot{Y}'\frac{\partial}{\partial\dot{Y}'} + (\mathcal{E} - 2\mathcal{B})\ddot{Y}\frac{\partial}{\partial\ddot{Y}}$$

$$-2\mathcal{A}\lambda''\frac{\partial}{\partial\lambda''} - (\mathcal{A} + \mathcal{B})\dot{\lambda}'\frac{\partial}{\partial\dot{\lambda}'} - 2\mathcal{B}\ddot{\lambda}\frac{\partial}{\partial\ddot{\lambda}} - 2\mathcal{A}\nu''\frac{\partial}{\partial\nu''} - (\mathcal{A} + \mathcal{B})\dot{\nu}'\frac{\partial}{\partial\dot{\nu}'} - 2\mathcal{B}\ddot{\nu}\frac{\partial}{\partial\ddot{\nu}}. \tag{4.9}$$

In order to determine the constants $\mathcal{A}, ..., \mathcal{G}$ appearing in G we apply (4.7) and its extensions (4.8), (4.9) to the system of field equations (2.16) and the conservation equations (2.17). This generates the following system of equations:

$$\mathcal{C}\mu = (-2\mathcal{E})\frac{1}{Y^2} - (-2\mathcal{F} - 2\mathcal{A})\frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda'Y' + \frac{Y'^2}{2Y}\right)$$

$$+(-2\mathcal{B}-2\mathcal{G})\frac{2}{Y}e^{-2\nu}\left(\dot{\lambda}\dot{Y}+\frac{\dot{Y}^2}{2Y}\right)$$

$$\mathcal{D}p=(-2\mathcal{E})\frac{1}{Y^2}+(-2\mathcal{F}-2\mathcal{A})\frac{2}{Y}e^{-2\lambda}\left(\nu'Y'+\frac{Y'^2}{2Y}\right)$$

$$-(-2\mathcal{G}-2\mathcal{B})\frac{2}{Y}e^{-2\nu}\left(\ddot{Y}-\dot{\nu}\dot{Y}+\frac{\dot{Y}^2}{2Y}\right)$$

$$(\mathcal{D}+\mathcal{E})pY=(-2\mathcal{F}+\mathcal{E}-2\mathcal{A})e^{-2\lambda}\left[\nu''+\nu'^2-\nu'\lambda'+\frac{1}{Y}(\nu'Y'-\lambda'Y'+Y'')\right]$$

$$-(-2\mathcal{G}+\mathcal{E}-2\mathcal{B})e^{-2\nu}\left[\ddot{\lambda}+\dot{\lambda}^2-\dot{\lambda}\dot{\nu}+\frac{1}{Y}\left(\dot{\lambda}\dot{Y}-\dot{\nu}\dot{Y}+\ddot{Y}\right)\right]$$

$$0=(\mathcal{E}-\mathcal{B}-\mathcal{A})\dot{Y}'-(\mathcal{E}-\mathcal{B}-\mathcal{A})\dot{Y}\nu'-(\mathcal{E}-\mathcal{B}-\mathcal{A})Y'\dot{\lambda}$$

$$(\mathcal{D}-\mathcal{A})p=-\left[(\mathcal{C}-\mathcal{A})\mu\nu'+(\mathcal{D}-\mathcal{A})p\nu'\right]$$

$$(\mathcal{C}-\mathcal{B})\dot{\mu}=-\left[(\mathcal{C}-\mathcal{B})\mu\dot{\lambda}+(\mathcal{D}-\mathcal{B})p\dot{\lambda}+(\mathcal{C}-\mathcal{B})\mu\frac{2\dot{Y}}{Y}+(\mathcal{D}-\mathcal{B})p\frac{2\dot{Y}}{Y}\right].$$

This set of constraints on the constants $\mathcal{A},\ldots,\mathcal{G}$ will be satisfied if we set

$$\mathcal{A}=\mathcal{B}=\mathcal{E}$$

$$\mathcal{C}=\mathcal{D}=-2\mathcal{A}$$

$$\mathcal{F}=\mathcal{G}=0.$$

The above set of conditions is consistent with the field equations (2.16) and the conservation equations (2.17). We set

$$\mathcal{A} = 1$$

without any loss of generality. Thus the infinitesimal generator (4.7) becomes

$$G = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t} - 2\mu\frac{\partial}{\partial \mu} - 2p\frac{\partial}{\partial p} + Y\frac{\partial}{\partial Y} \tag{4.10}$$

which is the relevant self–similar infinitesimal generator.

We utilise the form of G given in (4.10) to simplify the field equations. To reduce the number of variables in the field equations we introduce the invariant $F(u, w, y, z)$ as pointed out in §4.2. The associated characteristic system is then given by

$$\frac{dr}{r} = \frac{dt}{t} = \frac{dY}{Y} = \frac{d\mu}{-2\mu} = \frac{dp}{-2p}$$

which upon integration yields

$$u = \frac{r}{t}$$

$$y = \frac{Y}{r}$$

$$w = pr^2$$

$$z = \mu r^2,$$

where u, y, w and z are characteristics. The characteristic $u = r/t$ suggests that we introduce a new independent variable which will lead to a simplification of the field equations. This has the advantage of reducing the field equations to a system of ordinary differential equations. If we now make our new independent variable r/t (the 'similarity' variable), then we can replace the variables t and r appearing in the field equations by the single variable u. With $u = r/t$ as our independent variable the metric potentials, energy density and pressure can be expressed as

$$\lambda = \lambda(u) \qquad \nu = \nu(u)$$

$$Y = ry(u) \qquad \mu = \frac{z(u)}{r^2}$$

$$p = \frac{w(u)}{r^2}$$

in terms of the similarity variable u. Note that the new functions y, z, w are functions of u only.

The infinitesimal generator G given by (4.10) suggests that we introduce a new independent variable which reduces the field equations to a simpler form. The introduction of the similarity variable u allows us to express the field equations (2.16) as the following system

$$z = \frac{1}{y^2} - \frac{2}{e^{2\lambda}y}\left(2uy_u + u^2y_{uu} - (y + uy_u)u\lambda_u + \frac{(y + uy_u)^2}{2y}\right)$$

$$+ \frac{2u^4}{e^{2\nu}y}\left(y_u\lambda_u + \frac{y_u^2}{2y}\right) \tag{4.11a}$$

$$w = -\frac{1}{y^2} + \frac{2}{e^{2\lambda}y}\left((y + uy_u)u\nu_u + \frac{(y + uy_u)^2}{2y}\right)$$

$$-\frac{2u^3}{e^{2\nu}y}\left(uy_{uu}+2y_u-uy_u\nu_u+\frac{uy_u^2}{2y}\right) \tag{4.11b}$$

$$wy=\frac{u}{e^{2\lambda}}\left[uy\left(\nu_{uu}+\nu_u^2-\nu_u\lambda_u\right)+(y+uy_u)(\nu_u-\lambda_u)+2y_u+uy_{uu}\right]$$

$$-\frac{u^3}{e^{2\nu}}\left[y\left(2\lambda_u+u\lambda_{uu}+u\lambda_u^2-u\lambda_u\nu_u\right)+2y_u+uy_{uu}+uy_u(\lambda_u-\nu_u)\right] \tag{4.11c}$$

$$0=2y_u+uy_{uu}+uy_u\nu_u-\lambda_u(y+uy_u), \tag{4.11d}$$

where the subscripts denote differentiation with respect to u. The conservation equations (2.17) transform to

$$uw_u-2w=-u\nu_u(z+w) \tag{4.12a}$$

$$z_u=-(w+z)\left(\lambda_u+\frac{2y_u}{y}\right). \tag{4.12b}$$

We use the conservation laws (4.12) together with the field equations (4.11) in an attempt to generate solutions with a self–similar variable. The conservation equations do not provide new information. However, for the purposes of this calculation, they help to simplify the integration and we utilise them in addition to the field equations.

It is possible to further simplify the form of the differential equations (4.11)–(4.12) by introducing a new independent variable η. This is given by

$$u=e^{\eta}.$$

Then equations (4.11) and (4.12) transform to

$$z = \frac{1}{y^2} - \frac{2}{e^{2\lambda}y}\left[y_{\eta\eta} + y_\eta - (y + y_\eta)\lambda_\eta + \frac{(y + y_\eta)^2}{2y}\right]$$

$$+\frac{2}{e^{2(\nu-\eta)}y}\left[y_\eta\lambda_\eta + \frac{{y_\eta}^2}{2y}\right] \tag{4.13a}$$

$$w = -\frac{1}{y^2} + \frac{2}{e^{2\lambda}y}\left[(y + y_\eta)\nu_\eta + \frac{(y + y_\eta)^2}{2y}\right]$$

$$-\frac{2}{e^{2(\nu-\eta)}y}\left[y_{\eta\eta} + y_\eta - y_\eta\nu_\eta + \frac{{y_\eta}^2}{2y}\right] \tag{4.13b}$$

$$wy = \frac{1}{e^{2\lambda}}\left[(\nu_{\eta\eta} - \nu_\eta + {\nu_\eta}^2 - \nu_\eta\lambda_\eta)y + (y + y_\eta)(\nu_\eta - \lambda_\eta) + y_{\eta\eta} + y_\eta\right]$$

$$-\frac{1}{e^{2(\nu-\eta)}}\left[(\lambda_{\eta\eta} + \lambda_\eta + {\lambda_\eta}^2 - \lambda_\eta\nu_\eta)y + y_\eta(\lambda_\eta - \nu_\eta) + y_{\eta\eta} + y_\eta\right] \tag{4.13c}$$

$$y_{\eta\eta} + y_\eta = y_\eta\nu_\eta + (y + y_\eta)\lambda_\eta \tag{4.13d}$$

$$w_\eta - 2w = -\nu_\eta(w + z) \tag{4.13e}$$

$$z_\eta = -(w + z)\left(\lambda_\eta + \frac{2y_\eta}{y}\right). \tag{4.13f}$$

The field equations may be simplified further by redefining the gravitational potential ν. This new potential σ is related to the old potential ν by

$$\sigma = \nu - \eta.$$

Now we are in a position to express the field equations as the system

$$z = \frac{1}{y^2} - \frac{2}{e^{2\lambda}y}\left[y_{\eta\eta} + y_\eta - (y+y_\eta)\lambda_\eta + \frac{(y+y_\eta)^2}{2y}\right]$$

$$+\frac{2}{e^{2\sigma}y}\left[y_\eta\lambda_\eta + \frac{{y_\eta}^2}{2y}\right] \tag{4.14a}$$

$$w = -\frac{1}{y^2} + \frac{2}{e^{2\lambda}y}\left[(y+y_\eta)(\sigma_\eta+1) + \frac{(y+y_\eta)^2}{2y}\right]$$

$$-\frac{2}{e^{2\sigma}y}\left[y_{\eta\eta} - y_\eta\sigma_\eta + \frac{{y_\eta}^2}{2y}\right] \tag{4.14b}$$

$$wy = \frac{1}{e^{2\lambda}}\left[(\sigma_{\eta\eta} + \sigma_\eta + {\sigma_\eta}^2 - \lambda_\eta(\sigma_\eta+1))y + (y+y_\eta)(\sigma_\eta + 1 - \lambda_\eta) + y_{\eta\eta} + y_\eta\right]$$

$$-\frac{1}{e^{2\sigma}}\left[(\lambda_{\eta\eta} + {\lambda_\eta}^2 - \lambda_\eta\sigma_\eta)y + y_\eta(\lambda_\eta - \sigma_\eta - 1) + y_{\eta\eta} + y_\eta\right] \tag{4.14c}$$

$$y_{\eta\eta} + y_\eta = y_\eta(\sigma_\eta + 1) + (y + y_\eta)\lambda_\eta \tag{4.14d}$$

$$w_\eta - 2w = -(\sigma_\eta + 1)(w + z) \tag{4.14e}$$

$$z_\eta = -(w+z)\left(\lambda_\eta + \frac{2y_\eta}{y}\right) \tag{4.14f}$$

which is essentially equivalent to (4.13).

The system of field equations (4.14) governs the behaviour of the gravitational field with a self–similarity variable. This is a system of ordinary differential equations which includes the conservation equations. We need to integrate (4.14) to find forms for the gravitational potentials (λ, σ, y), the energy density (z) and the pressure (w). A class of solutions satisfying (4.14) is presented in §4.4.

4.4 Some Solutions of the Self–Similar Equations

The system (4.14) is highly nonlinear and there is no obvious technique that could be utilised to find solutions in general. We can make progress for a special case by assuming that

$$\sigma = \lambda.$$

Then the system of ordinary differential equations (4.14) reduces to

$$z = \frac{1}{y^2} - \frac{2}{e^{2\sigma} y}\left[y_{\eta\eta} + 2y_\eta + \frac{y}{2} - (y + 2y_\eta)\sigma_\eta\right] \tag{4.15a}$$

$$w = -\frac{1}{y^2} + \frac{2}{e^{2\sigma} y}\left[-y_{\eta\eta} + 2\sigma_\eta y_\eta + 2y_\eta + \sigma_\eta y + \frac{3y}{2}\right] \tag{4.15b}$$

$$wy = \frac{1}{e^{2\sigma}}\left[2y_\eta + y\right] \tag{4.15c}$$

$$y_{\eta\eta} = (y + 2y_\eta)\,\sigma_\eta \tag{4.15d}$$

$$w_\eta - 2w = -(\sigma_\eta + 1)(w + z) \tag{4.15e}$$

$$z_\eta = -(w + z)\left(\sigma_\eta + \frac{2y_\eta}{y}\right) \tag{4.15f}$$

which is clearly simpler than (4.14). We now investigate the integration of (4.15) in general.

On elimination of $y_{\eta\eta}$ from (4.15b), with the help of (4.15d), we obtain

$$w = -\frac{1}{y^2} + \frac{2}{e^{2\sigma}y}\left[2y_\eta + \frac{3y}{2}\right].$$

We eliminate $e^{-2\sigma}$ in the above equation with the help of (4.15c) which results in

$$w = \frac{2y_\eta + y}{2y^2\left(y_\eta + y\right)}. \tag{4.16}$$

Differentiating (4.15c) with respect to η leads to

$$w_\eta = -2\sigma_\eta w + e^{-2\sigma}\left[\frac{2y_{\eta\eta}}{y} - \frac{2{y_\eta}^2}{y^2}\right]. \tag{4.17}$$

We need to eliminate w, $e^{-2\sigma}$ and σ_η from (4.17). We substitute (4.16) for w, (4.15c) for $e^{-2\sigma}$ and (4.15d) for σ_η into the master equation (4.17) to obtain a second order differential equation

$$y_{\eta\eta}y^2 - 5{y_\eta}^2 y - 2{y_\eta}^3 - 2y_\eta y^2 = 0 \tag{4.18}$$

in the variable y. We have therefore reduced the evolution of spherically symmetric gravitational fields to a single ordinary differential in one dependent variable. This equation is homogeneous and accordingly integrable. Once we solve for y, we can find the other quantities.

We can reduce the order in (4.18) by making the substitution

$$q = y$$

$$Q = y_\eta.$$

This leads to a first order equation of the form

$$Q_q = \left(\frac{2}{q^2}\right) Q^2 + \left(\frac{5}{q}\right) Q + 2, \tag{4.19}$$

where the subscript denotes differentiation with respect to q. We note that (4.18) has been reduced to a Riccati equation. By inspection a particular solution for (4.19) is given by

$$Q = -q.$$

We replace Q with a new variable V that linearises (4.19). On using the transformation

$$Q = -q + \frac{1}{V}$$

equation (4.19) can be written as

$$\frac{dV}{dq} = -\frac{V}{q} - \frac{2}{q^2}$$

which is linear in V. This equation is easily integrated to yield

$$V = \frac{1}{q}(c_1 - 2\ln q),$$

where c_1 is an integration constant. Then the solution in terms of the variables q and Q is given by

$$Q = -q + \frac{q}{c_1 - 2\ln q}.$$

Therefore we have generated a first integral of (4.18); the solution in terms of the variable y is then given by

$$y_\eta = -y + \frac{y}{c_1 - 2\ln y}. \tag{4.20}$$

We can express (4.20) as

$$d\eta = \frac{c_1 - \ln y^2}{y\,(1 - c_1 + \ln y^2)} dy$$

since the variables separate. This can be written as

$$d\eta = \frac{1}{2}\left[1 - \frac{1}{1-U}\right] dU$$

if we introduce the variable

$$U = c_1 - \ln y^2.$$

Integration then yields

$$1 - c_1 + \ln y^2 = c_2 e^{2\eta} y^2, \tag{4.21}$$

where c_2 is an integration constant. Equation (4.21) is the general solution of the master equation (4.18).

We now proceed to determine the pressure and energy density. From (4.16) and (4.20) we have

$$w = \frac{2 - c_1 + \ln y^2}{2y^2}.$$

The logarithmic term may be eliminated with the help of (4.21) to give

$$w = \frac{1}{2y^2} + \frac{c_2}{2} e^{2\eta}$$

for the pressure. Equations (4.15d), (4.16), (4.20) and (4.21) enable us to write (4.15a) as

$$z = \frac{1}{y^2}\left[1 - \frac{1}{2}\left(\frac{1 + 3c_2 y^2 e^{2\eta}}{1 + c_2 y^2 e^{2\eta}}\right)\right]$$

which is the energy density. It remains to find the metric potentials. We firstly determine the variable σ from (4.15c):

$$\sigma = \frac{1}{2}\ln\left(\frac{2y^2}{1 - y^2 c_2 e^{2\eta}}\right),$$

where we have utilised (4.16), (4.20) and (4.21). The metric potentials ν and λ can then be expressed as

$$\nu = \eta + \frac{1}{2}\ln\left(\frac{2y^2}{1 - y^2 c_2 e^{2\eta}}\right)$$

and

$$\lambda = \frac{1}{2}\ln\left(\frac{2y^2}{1 - y^2 c_2 e^{2\eta}}\right)$$

respectively.

We have solved the field equations and expressed quantities in terms of one of the gravitational potentials, namely Y. The potential Y is related in terms of the coordinates t and r as follows

$$1 - c_1 + \ln\left(\frac{Y^2}{r^2}\right) = c_2 \frac{Y^2}{t^2}. \tag{4.22}$$

Note that this is an implicit relationship; it is not possible to write Y explicitly in terms of t and r. The other potentials ν and λ can be expressed as

$$\nu = \ln\left[\frac{\sqrt{2}Y}{(t^2 - c_2 Y^2)^{1/2}}\right]$$

and

$$\lambda = \ln\left[\frac{\sqrt{2}Yt}{r\,(t^2 - c_2 Y^2)^{1/2}}\right]$$

in terms of Y, t and r. The line element for this class of solutions has the form

$$ds^2 = -\frac{2Y^2}{t^2 - c_2 Y^2}dt^2 + \frac{2Y^2 t^2}{r^2(t^2 - c_2 Y^2)}dr^2 + Y^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.23}$$

The energy density is given by

$$\mu = \frac{1}{Y^2}\left[1 - \frac{1}{2}\left(\frac{t^2 + 3c_2 Y^2}{t^2 + c_2 Y^2}\right)\right] \tag{4.24}$$

and the pressure has the form

$$p = \frac{1}{2}\left[\frac{1}{Y^2} + \frac{c_2}{t^2}\right]. \tag{4.25}$$

Thus we have found a new solution, given by (4.22)–(4.25), of the Einstein field equations with a self–similar variable. This class of solutions is expanding, accelerating and shearing as the metric depends on Y which is a function of t and r. Our class of solutions, with line element (4.23), arises because we have essentially imposed the condition

$$\nu - \lambda = \ln\frac{r}{t}$$

relating the potentials ν and λ. Other classes of solutions, with a self–similar variable, are possible, but it is difficult to determine the precise relation between ν and λ that allows the field equations to be integrated. The metric (4.23) is a new class of solutions of the Einstein field equations which have nonvanishing kinematical quantities $\sigma, \Theta, \dot{u}^a$. We have demonstrated that the Lie symmetry analysis helps to isolate new classes of solutions for inhomogeneous models in the difficult case when the model is shearing.

A particular solution can be obtained by setting $c_2 = 0$ in (4.22). This results in

$$\frac{Y}{r} = k,$$

where $k = e^{(c_1-1)/2}$ is a constant. The line element for this particular solution is given by

$$ds^2 = -\frac{2k^2r^2}{t^2}dt^2 + 2k^2dr^2 + k^2r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.26}$$

in terms of the constant k and the coordinates r and t. From (4.24) and (4.25) we observe that the energy density and pressure are equal and are given by

$$\mu = p = \frac{1}{2Y^2}.$$

Thus this particular solution has a stiff equation of state. This is clearly an exceptional case as the general solution (4.23) does not admit a barotropic equation of state. If we make the replacement

$$kr \longrightarrow \frac{r}{\sqrt{2}}$$

$$\frac{dt}{t} \longrightarrow \frac{dt}{2},$$

then the line element (4.26) can be written as

$$ds^2 = -\frac{r^2}{4}dt^2 + dr^2 + \frac{r^2}{2}\left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (4.27)$$

We note that (4.27) is static and is a special case of the Gutman–Bespal'ko solution (2.19) with $a_3 = a_4 = 0$. In fact (4.27) is a special case of the self–similar Tolman–Bondi metrics (Carot and Sintes 1994, Maartens and Maharaj 1990, Maartens *et al* 1995, 1996, Wainwright 1985). It is a pleasing feature that our new solutions found by utilising the Lie analysis regain well–known particular cases. This suggests that the Lie analysis of differential equations may help in finding more general classes of physically viable inhomogeneous models.

There are few analyses of spherically symmetric spacetimes, in particular for nonzero shear, involving a self–similar variable. A fairly comprehensive analysis of self–similar spacetimes for perfect fluids, including spherically symmetric spacetimes as a special case, was conducted by Collins and Lang (1987). Their spacetimes admit a four–parameter similarity group, whose orbits are spacelike hypersurfaces which are orthogonal to the fluid flow. Sussman (1991), also working with spherically symmetric spacetimes for perfect fluids admitting a self-similar motion orthogonal to the four-velocity vector, investigated the global properties and causal structure of such spacetimes. Examples of spherically symmetric spacetimes with a self-similar variable are the Gutman and Bespal'ko (1967) models and the Wesson (1978) solution. Generalisations of the Gutman-Bespal'ko and Wesson solutions were considered by Lake (1983), Shaver and Lake (1988) and Van den Bergh and Wils (1985). These solutions admit a conformal motion and an equation of state of the form $p = \mu +$ constant. Hajj-Boutros (1985), Herrera and Ponce de Leon (1985a) and Maharaj *et al* (1993) considered more general classes of solutions admitting a conformal motion, but not necessarily a barotropic equation of state. As far as we are aware there has been no systematic attempt to find solutions of the Einstein field equations with a similarity variable with a nonzero shear; only particular cases have been isolated before. We believe that our results in this chapter are new and represent a class of solutions for nonvanishing shear with a self-similar variable. It is important to note that our solutions were obtained without assuming an equation of state.

5 Solutions with Constant Potential

5.1 Introduction

In the previous chapter we found solutions of the field equations having shear, expansion and acceleration by introducing a self–similar variable. This reduced the field equations to a system of ordinary differential equations. It is possible to generate shearing, expanding and accelerating solutions using other techniques. In this chapter we present a new class of solutions using a different variable. To simplify the Einstein field equations we assume, in §5.2, that one of the gravitational potentials is constant. We then find that the field equations reduce to a single partial differential equation in the dependent variable Y. In §5.3 we first assume an *ad hoc* form for Y and generate a class of solutions which has vanishing energy–momentum. In order to obtain other more physically relevant solutions to the partial differential equation, we use Lie symmetries to reduce the equation to an ordinary differential equation. We show that in general the solution can be expressed in terms of elliptic functions. As a particular case we assume a stiff equation of state in §5.4. The field equations imply an additional restriction, namely an ordinary differential equation of the Emden–Fowler type. Solutions to this Emden–Fowler equation are briefly investigated. Note that in both §5.3 and §5.4 our objective is to seek a new independent variable that reduces the field equations to an ordinary differential equation which is easier to integrate. The solutions presented in this chapter indicate that it is fruitful to investigate variables other than the self–similar type to find new solutions.

5.2 Field Equations for Constant λ

In this section we consider a simple ansatz that produces a new family of shearing solutions that are expanding and accelerating. We take the potential

$$\lambda = \text{constant}$$

and consequently $e^{2\lambda}$ can be absorbed by a redefinition of r in the metric. With this value for λ the field equation (2.16d) simplifies to

$$\dot{Y}' - \dot{Y}\nu' = 0$$

which immediately integrates to give

$$e^{2\nu} = \dot{Y}^2,$$

where we have set a constant of integration to be unity, after redefinition of the coordinate t. Thus we may take the line element to be

$$ds^2 = -\dot{Y}^2 dt^2 + dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.1}$$

Clearly we require $\dot{Y} \neq 0$. From (2.8) the acceleration, the expansion and the shear are given respectively by

$$\dot{u}^a = (0, \frac{\dot{Y}'}{\dot{Y}}, 0, 0,)$$

$$\Theta = \frac{2}{Y}$$

$$\sigma^1_1 = \sigma^2_2 = -\frac{1}{2}\sigma^3_3 = \frac{1}{3Y}$$

for the line element (5.1). Thus the cosmological models for the metric (5.1) will have nonvanishing kinematics ($\dot{u}^a \neq 0, \Theta \neq 0, \sigma \neq 0$) in general. This is true in spite of the simple form of the metric (5.1) with the assumption $\lambda =$ constant.

The field equations (2.16), for the choice (5.1), reduce to the system of partial differential equations

$$\mu = \frac{2}{Y^2} - \frac{2}{Y}\left(Y'' + \frac{Y'^2}{2Y}\right) \tag{5.2a}$$

$$p = -\frac{2}{Y^2} + \frac{2}{Y}\left(\frac{\dot{Y}'}{\dot{Y}}Y' + \frac{Y'^2}{2Y}\right) \tag{5.2b}$$

$$p = \frac{\dot{Y}''}{\dot{Y}} + \frac{\dot{Y}'}{\dot{Y}}\frac{Y'}{Y} + \frac{Y''}{Y}. \tag{5.2c}$$

We observe from (5.2a) and (5.2b) that μ and p are given completely in terms of the function Y. Equation (5.2c), a consistency condition, generates the condition of pressure isotropy. Thus, if Y is specified, we have the solution to the system (5.2). Equations (5.2b) and (5.2c) generate the partial differential equation

$$\frac{Y''}{Y} + \frac{\dot{Y}''}{\dot{Y}} - \frac{\dot{Y}'}{\dot{Y}}\frac{Y'}{Y} - \frac{Y'^2}{Y^2} + \frac{2}{Y^2} = 0 \tag{5.3}$$

which contains the single dependent function Y. Therefore we have reduced the evolution of spherically symmetric gravitational fields, with λ = constant, to the single equation (5.3). In the following sections we seek solutions to (5.3).

5.3 Solutions to the Field Equations

Equation (5.3) is a nonlinear third order differential equation which is difficult to integrate in general. However, we can generate a simple solution by assuming an *ad hoc* form for Y. We assume that

$$Y = g(t) + h(r)$$

where g and h are functions of t and r respectively. Then (5.3) implies that $h(r)$ is a linear function as $\dot{Y} \neq 0$. We obtain

$$Y = g(t) + \sqrt{2}r \tag{5.4}$$

as a solution to (5.3). These solutions are nonaccelerating and the line element has the simple form

$$ds^2 = -\left[\dot{g}(t)\right]^2 dt^2 + dr^2 + \left[g(t) + \sqrt{2}r\right]^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{5.5}$$

However, we observe that, for the solution (5.5), μ and p vanish from (5.2). Thus the solutions to the Einstein field equations with $Y = g(t) + \sqrt{2}r$, from (5.4), have vanishing energy–momentum. Clearly we require a more realistic form for the metric function $Y(t,r)$ to find nonvacuum models.

In order to find a general solution to (5.3) we use the method of Lie symmetries as was the case in Chapter 4. We recall that (5.3) is said to possess the Lie point symmetry

$$G = \xi(r,Y)\frac{\partial}{\partial r} + \eta(r,Y)\frac{\partial}{\partial Y}$$

if the following equation

$$G^{[2]}\left(Y''Y + \frac{\dot{Y}''}{\dot{Y}}Y^2 - \frac{\dot{Y}'}{\dot{Y}}YY' - Y'^2 + 2\right) = 0$$

as well as (5.3) are satisfied, where $G^{[2]}$ is the $2nd$ extension of G. It can be verified that (5.3) has the three Lie point symmetries

$$G_1 = \frac{\partial}{\partial r}$$

$$G_2 = g(t)\frac{\partial}{\partial t}$$

$$G_3 = r\frac{\partial}{\partial r} + Y\frac{\partial}{\partial Y},$$

where g is an arbitrary function of t. There are two nonequivalent similarity reductions which could be used to reduce (5.3) to an ordinary differential equation. However, given the number and forms of the symmetries we do not always expect to reduce the resulting ordinary differential equations to quadratures.

The nature of the similarity generators G_1 and G_2 suggests that we introduce a new variable u:

$$u = g(t) + r$$

with

$$Y = y(u).$$

Then (5.3) can be reduced to

$$y^2 y_{uuu} - y_u^3 + 2y_u = 0, \tag{5.6}$$

where the subscript refers to differentiation with respect to u. Thus we have reduced the partial differential equation (5.3) to an ordinary differential equation (5.6). This is a third order equation and we observe that it can be written as

$$\left(y^2 y_{uu}\right)_u - \left(y y_u^2\right)_u + 2y_u = 0.$$

This is easily integrated to yield

$$y^2 y_{uu} - y y_u^2 + 2y = A \tag{5.7}$$

with A being an integration constant. We can rewrite (5.7) as the first order equation

$$\frac{y^2}{2}\frac{dy_u^2}{dy} - y y_u^2 + 2y = A$$

which upon integration gives

$$y_u^2 = -\frac{2A}{3y} + By^2 + 2,$$

where B is an integration constant. On integration the above equation gives the expression

$$u - u_0 = \int \frac{\sqrt{3y}\,dy}{\sqrt{3By^3 + 6y - 2A}},$$

where u_0 is an integration constant. The spatial coordinate r can then be expressed as

$$r - r_0 = \int^Y \frac{\sqrt{y}\,dy}{\sqrt{By^3 + 2y - 2/3A}} - g(t). \tag{5.8}$$

In order to evaluate the integral in (5.8) we make the transformation

$$v = \frac{1}{y}.$$

Equation (5.8) can then be written as

$$r - r_0 = -\int \frac{dv}{v\,[B + 2v^2 - 2/3Av^3]^{1/2}} - g(t). \tag{5.9}$$

The integral in (5.9) is similar to that in (3.45) of §3.2.2. We can therefore follow a procedure similar to that of §3.2.2 in expressing the integral in (5.9) in terms of elliptic functions.

We have solved the partial differential equation (5.3) in terms of the quadrature (5.8). The solution may be given in terms of elliptic functions. Thus the general solution of the Einstein field equations (5.2), with $\lambda =$ constant, is given by the line element (5.1). The potential Y is related to the spacetime coordinates t and r by (5.8). For the special case $A = B = 0$ the metric potential Y can be expressed as

$$Y = \sqrt{2}r + \sqrt{2}f(t)$$

which recovers the special vacuum case (5.4) considered earlier.

Other reductions may lead to new solutions. A different reduction of (5.3) is obtained by the transformation

$$u = \exp\left(-\int f(t)dt\right)$$

$$Y = k(u).$$

Then (5.3) is reduced to

$$(u^2 k_{uuu} + 3uk_{uu})k^2 - 2uk_u^2 k - (k^2 + u^2 k_u^2 - 2)k_u = 0 \tag{5.10}$$

which is an equation of Euler type and admits the Lie point symmetry

$$G_1 = u\frac{\partial}{\partial u}.$$

Under the reduction

$$w = uk_u, \qquad z = k$$

equation (5.10) becomes

$$z^2 w w_{zz} = -z^2 w_z^2 + w^2 - 2z^2 - 2z + 2zw.$$

Since this equation has no Lie point symmetries we do not pursue the integration any further. In the next section we consider a stiff fluid model, which arises as a special case of the analysis in this section.

5.4 Stiff Fluid Solutions

In this section we present solutions to the field equations for constant λ with $p = \mu$. In this case a second differential equation arises in addition to (5.3). If we let $\mu = p$, then (5.2a) and (5.2b) can be equated to obtain

$$Y''Y + \frac{\dot{Y}'}{\dot{Y}}YY' + Y'^2 - 2 = 0. \tag{5.11}$$

On adding (5.11) to (5.3) we get

$$\frac{\dot{Y}''}{Y''} = -2\frac{\dot{Y}}{Y}$$

which upon integration yields

$$Y'' = f(r)Y^{-2}, \tag{5.12}$$

where $f(r)$ results from the integration process. Equation (5.12) is a particular case of the Emden–Fowler equation

$$Y'' = f(r)Y^n. \tag{5.13}$$

For a detailed treatment of the Emden–Fowler equation and its integrability, the reader is referred to Govinder and Leach (1996) and the references therein. When $n = 2$ the Emden–Fowler equation appears in the study of spherically symmetric cosmological models involving shear–free expanding perfect fluids. Maharaj *et al*

(1996) have studied the integrability properties of the Emden–Fowler equation in such models. For $n = -3$ Govinder and Leach (1996) have been able to reduce (5.13) to quadratures.

In order to find solutions for (5.12) we again use the techniques of Lie analysis of differential equations discussed in §4.2. We seek a Lie point symmetry

$$G = \xi(r, Y)\frac{\partial}{\partial r} + \eta(r, Y)\frac{\partial}{\partial Y}$$

such that the action of $G^{[2]}$ leaves the differential equation (5.12) invariant. It can be verified that G takes the form

$$G = A(r)\frac{\partial}{\partial r} + [B(r)Y + C(r)]\frac{\partial}{\partial Y}. \tag{5.14}$$

The $1st$ and $2nd$ extensions of G are then given respectively by

$$G^{[1]} = G + [B'Y + (2B' - A)Y' + C']\frac{\partial}{\partial Y'} \tag{5.15}$$

and

$$G^{[2]} = G^{[1]} + [B''Y + (2B' - A'')Y' + (B - 2A')Y'' + C'']\frac{\partial}{\partial Y''}. \tag{5.16}$$

The action of $G^{[2]}$ on (5.12) gives the condition

$$B''Y + (2B' - A'')Y' + (B - 2A')Y'' + C'' = Af'\frac{1}{Y^2} - 2(BY + C)f\frac{1}{Y^2}. \tag{5.17}$$

Using (5.12) in (5.17) and setting the coefficients of the different powers of Y and Y' to zero, we see that the functions f, A, B and C must satisfy the conditions:

$$B'' = 0 \tag{5.18a}$$

$$2B' - A'' = 0 \tag{5.18b}$$

$$C'' = 0 \tag{5.18c}$$

$$(B - 2A')f = Af' - 2Bf \tag{5.18d}$$

$$2Cf = 0. \tag{5.18e}$$

We notice that (5.18e) implies that $C = 0$ for nonzero f. Integration of (5.18b) results in

$$B = \frac{1}{2}(A' + \alpha), \tag{5.19}$$

where α is an integration constant. From (5.18a) and (5.19) we have that

$$A''' = 0$$

and hence A can be written as

$$A = a_1 + a_2 r + a_3 r^2, \tag{5.20}$$

where a_1, a_2 and a_3 are constants of integration. It is now possible to find f from (5.18d) if A is given by (5.20). We generate the expression

$$f(r) = \beta A^{-1/2} \exp\left(\frac{3\alpha}{2}\int\frac{dr}{A}\right), \tag{5.21}$$

where β is a constant. This form of $f(r)$ allows us to reduce the equation (5.12) to first order. Clearly, to set $\alpha = 0$ provides a simple case.

The form of $f(r)$ given in (5.21) reduces the differential equation (5.12) to first order. For a detailed treatment of the various cases see Govinder and Leach

(1996). Note that the solutions generated from (5.21) have to satisfy (5.3), a consistency condition. We do not pursue the stiff fluid solutions any further as the more general case has been investigated in §5.3. It is interesting to observe that the stiff equation of state generates an Emden–Fowler equation which is an additional restriction on the field equations. Maharaj *et al* (1996) illustrate that the Emden–Fowler equation arises in shear-free spherically symmetric gravitational fields independent of an equation of state.

6 Conclusion

The research conducted in this thesis investigated the role of shear in spherically symmetric spacetimes; we believe that such a study has not been systematically done before. On the assumption of a perfect fluid source, new classes of shearing solutions were obtained for the Einstein field equations. These classes contain solutions found previously, *eg* the Van den Bergh and Wils (1985) models. Some of the methods employed in generating the new solutions were

- imposition of an equation of state,
- the Lie analysis of differential equations,
- self–similar variables,
- the *ad hoc* choice of gravitational potentials

in an attempt to simplify the Einstein field equations. The inhomogeneous exact solutions found are important as they can be applied to situations in cosmology and astrophysics where shear cannot be neglected.

We list the important results found in this thesis:

- The spherically symmetric line element was given in comoving coordinates and those elements of differential geometry relevant to this thesis were briefly revised. In particular the Einstein field equations were derived *ab initio* for spherically symmetric spacetimes.

- Firstly nonaccelerating fluids obeying a barotropic equation of state were studied by assuming $Y = t$. The resulting field equations were transformed to an Abel's equation of the first kind and solutions were obtained in two particular cases. The first solution corresponds to radiating models which is related to that of Kantowski and Sachs (1966). The second solution, being that for a stiff fluid, is new and generalises the solutions obtained by Van den Bergh and Wils (1985).

- Another class of nonaccelerating solutions was obtained by assuming that the metric potential $\nu = 0$. The line element was then expressed completely in terms of the remaining potential Y. Then the field equations could be solved by assuming that the pressure p was constant. In general the solution of the field equations is expressible in terms of elliptic functions of the first and third kinds. In a particular case the solution is expressible in terms of elementary functions. We believe that these solutions are new and have not been reported before.

- A class of nonexpanding cosmological models which were both accelerating and shearing was briefly studied. If the energy density is constant, the solution may be given in terms of quadratures.

- A new class of solutions was obtained by using the theory of Lie symmetries of differential equations. A self-similar variable was used to reduce the Einstein field equations and the conservation equations to a system of ordinary differential equations. On the assumption of a relationship between the two metric potentials λ and ν, this system of ordinary differential equations was transformed to a single differential equation in Y. Essentially the Einstein field equations were reduced to a Riccati equation. The solution to this equation was found explicitly in terms of the variables t, r and Y. Then the line element, pressure and energy density were expressed in terms of Y. As far as we are aware, this class of solutions, which was obtained without an equation of state, is new. In the appropriate limit we regained the Tolman–Bondi stiff fluid solution.

- By assuming the metric potential λ to be constant we were able to generate another new class of solutions. In this case the line element was expressed in terms of the potential Y and the field equations were integrated to find solutions for Y. An *ad hoc* choice for Y yielded a vacuum solution. However, a more realistic class of models was found using the Lie analysis of differential equations. The general solution for Y can be expressed in terms of elliptic functions. In particular, stiff fluid solutions for constant λ resulted in a further condition to be satisfied, namely an Emden–Fowler equation. The integrability conditions for the Emden–Fowler equation were investigated.

We now briefly discuss possible areas for future investigation that arise as a result of the findings in this thesis. The form of Abel's equation derived in Chapter 3, namely equation (3.12), could be solved for values of γ other than $\gamma = \frac{4}{3}$ and $\gamma = 2$. Other relationships of the potentials λ and ν, besides the one discussed in Chapter 4, could be used to generate new solutions. By assuming different *ad hoc* forms for the metric potentials it may also be possible to find new solutions as was the case in Chapter 5. Another possible area of research is to apply the new solutions generated in this thesis to the areas of cosmology and astrophysics. This is a formidable problem and is a research initiative in its own right. Other spacetimes, besides spherically symmetric spacetimes, with a different set of Killing symmetries could be investigated to gain a deeper insight into the role of inhomogeneities in relativistic physics. The relationship between comoving and noncomoving coordinates for shearing solutions could be another avenue of research. One could also investigate the relationship between inhomogeneous shear anisotropy and inflation in cosmological models (Krishna Rao 1995).

In conclusion it is hoped that this thesis has contributed significantly to the study of spherically symmetric spacetimes and in particular to the role of shear in such spacetimes.

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