RADIATING SOLUTIONS WITH HEAT FLOW IN GENERAL RELATIVITY

by

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To Trevor Chetty

For his infinite time and patience.

Preface

The study described in this thesis was carried out in the Department of Physics, University of Natal, Durban, during the period January 1994 to December 1994. This thesis was completed under the excellent supervision of Professor A. R. W. Hughes and Professor S. D. Maharaj.

The research contained in this study represents original work by the author. It has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Abstract

In this thesis we model spherically symmetric radiating stars dissipating energy in the form of a radial heat flux. We assume that the spacetime for the interior matter distribution is shear–free. The junction conditions necessary for the matching of the exterior Vaidya solution to an interior radiating line element are obtained. In particular we show that the pressure at the boundary of the star is nonvanishing when the star is radiating (Santos 1985). The junction conditions, with a nonvanishing cosmological constant, were obtained. This generalises the results of Santos (1985) and we believe that this is an original result. The Kramer (1992) model is reviewed in detail and extended. The evolution of this model depends on a function of time which has to satisfy a nonlinear second order differential equation. We solve this differential equation in general and thereby completely describe the temporal behaviour of the Kramer model. Graphical representations of the thermodynamical and gravitational variables are generated with the aid of the software package MATHEMATICA Version 2.0 (Wolfram 1991). We also analyse two other techniques to generate exact solutions to the Einstein field equations for modelling radiating stars. In the first case the particle trajectories are assumed to be geodesics. We indicate how the model of Kolassis *et al* (1988) may be extended by providing an ansatz to solve a second order differential equation. In the second case we review the models of de Oliveira *et al* (1985, 1986, 1988) where the gravitational potentials are separable functions of the spatial and temporal coordinates.

1 Introduction

General relativity is a global theory of gravitation which is based on a covariant formalism in four–dimensional, curved spacetime and has as its limiting case Newton's theory of gravitation. In general relativity the metric function describes not only the metric properties of space, as well as its causal properties, but also the gravitational field. Hence general relativity unifies the matter content of spacetime to the gravitational field. The geometry of spacetime is represented by the Einstein tensor which is defined in terms of the Ricci tensor, the Ricci scalar and the metric tensor. Mechanical and electromagnetic systems are described by the symmetric energy–momentum tensor which is coupled to the gravitational field via the Einstein field equations. Unlike Newtonian or Maxwellian field equations, the Einstein field equations are a system of highly nonlinear partial differential equations. Exact solutions to the Einstein field equations are important in modelling astrophysical and cosmological phenomena in the observable universe. However exact solutions to the field equations in closed form are in general very difficult to obtain. In this thesis we model spherically symmetric radiating stars undergoing gravitational collapse with heat flow within the framework of general relativity.

The study of static spheres is an idealised problem since astronomical observations indicate that most, if not all gravitating systems are non–static and radiative processes are vital mechanisms of energy dissipation in such systems. The surface of a collapsing star divides spacetime into two distinct regions, the interior region and the exterior region. Since the star is radiating the exterior spacetime is no longer described by the exterior Schwarzschild solution but is now represented by the Vaidya solution for pure outgoing radiation. The interior matter distribution is described by a spherically symmetric, shear–free line element for a generalised energy–momentum tensor with heat flow. By utilising Raychaudhuri's equation (Raychaudhuri 1957) we can show that the slowest possible collapse is for shear–free matter distributions. The interior spacetime has to be matched to the exterior spacetime at the boundary of the radiating star. Hence to obtain a complete picture of the gravitational collapse of a star it is necessary to adequately describe the interior and exterior spacetimes and to provide the matching conditions for them. The problem of gravitational collapse has many interesting applications in astrophysics where formation of compact stellar objects such as white dwarfs and neutron stars are usually preceded by a period of radiative collapse. It is well known that during the late stages of gravitational collapse the temperature inside the star is so intense that production of neutrinos is possible. Hence the model of gravitational collapse with a neutrino flux is important in relativistic astrophysics and has been investigated in the past (Glass 1990). The cosmic censorship hypothesis due to Penrose (Penrose 1978, 1979) which asserts that a physically realizable collapse of matter which obeys the usual energy conditions will not lead to the formation of naked singularities, can be investigated using the theory of gravitational collapse.

Historically the problem of radiative gravitational collapse was first addressed by Oppenheimer and Snyder (1939) in which they presented a model based on a spherically symmetric dust cloud undergoing gravitational collapse. Here the exterior spacetime is described by the exterior Schwarzschild solution and the interior spacetime is represented by a Friedmann–like solution for an isotropic and homogeneous universe. Vaidya (1951, 1953) derived the line element which describes the exterior gravitational field of a radiating sphere. Then it became possible to model the interior of radiating stars by matching such solutions to the exterior Vaidya spacetime (see for example Glass 1981). Santos (1985) obtained the junction conditions for a spherically symmetric radiating star. He was able to show with the use of the junction conditions that the pressure on the boundary of a radiating sphere cannot vanish. This important result has since become a crucial requirement for spherically symmetric, shear–free radiative collapse. Several physically reasonable models of radiative spherical collapse with heat flow have been proposed by utilising the junction conditions derived by Santos (1985). One such approach presented by Kramer (1992) makes use of a static interior solution in which one of the parameters is allowed to become a function of time. This solution is then matched to the exterior spacetime described by the Vaidya solution for pure radiation. Another method is to assume separability of the gravitational potentials into their spatial and temporal components. Such a model was investigated by Kolassis *et al* (1988) and de Oliveira *et al* (1985, 1986, 1988). More realistic models based on anisotropic pressure (Grammenos *at al* 1992) and neutrino outbursts (Glass 1990) have been recently studied. It is interesting to note that Bonnor (1987) has studied the arrow of time in a gravitational context by making use of a radiating star undergoing gravitational collapse.

In chapter 2 we briefly consider differential geometry, curvature and the

Einstein field equations in the spacetime of general relativity and we derive the junction conditions across a spherically symmetric hypersurface by matching two different regions of spacetime. The energy–momentum tensor includes terms corresponding to heat flow which enables us to model radiating stars. In §2.3 spherically symmetric, shear–free spacetimes are considered in detail as they describe the interior of radiating stars. The interior Einstein field equations are derived for a sphere undergoing gravitational collapse. In §2.4 we analyse the Vaidya solution (with vanishing cosmological constant) in null coordinates which describes the exterior spacetime of the radiating star. The junction conditions matching the interior and exterior spacetimes at the boundary of the star are derived in §2.5. We establish the important result that the pressure does not vanish at the boundary of a radiating star. The luminosity profile and the surface redshift of the star are briefly considered.

In chapter 3 we investigate the problem of spherically symmetric, shear–free gravitational collapse with heat flow when the Einstein field equations are generalised to include the cosmological constant. The Einstein field equations, with nonzero cosmological constant, for a spherically symmetric shear–free spacetime are derived in §3.2. The form of the heat flow is not directly affected by the inclusion of the cosmological constant. In §3.3 we analyse the Vaidya solution with cosmological constant for a null fluid and obtain the Einstein field equations for the exterior spacetime. We present all the relevant details as the Vaidya solution with cosmological constant is not well known. In §3.4 the junction conditions are obtained for the case with nonvanishing cosmological constant by matching the generalised Vaidya solution to the interior spacetime. The results obtained in this chapter generalise the results obtained earlier in §2.5.

In chapter 4 we review a radiating model of gravitational collapse proposed by Kramer (1992). A nonstatic model is generated from a known static model by allowing certain parameters to become functions of time. In §4.2 we consider the interior spacetime which is described by the interior Schwarzschild solution in isotropic coordinates. The mass function in the interior Schwarzschild solution becomes a function of time. The Einstein field equations, with vanishing cosmological constant, are derived. In section §4.3 we utilise the junction conditions, derived in §2.5, to generate an ordinary differential equation which governs the temporal evolution of this model. This equation is nonlinear and of second order. Kramer (1992) provides a first integral of this equation. We fully integrate this differential equation in §4.4 and describe completely the temporal behaviour of the model. The physical properties of the model are investigated in §4.5. Since the forms of the expressions for the thermodynamical variables and gravitational potentials are complicated we provide graphical plots of these functions for a chosen interval.

In chapter 5 we investigate two different methods of generating solutions to the Einstein field equations with vanishing cosmological constant. Firstly we consider in §5.2 the model of Kolassis *et al* (1988) in which the fluid trajectories of the collapsing star are assumed to be geodesics. We provide a mathematical justification for the existence of their solution. In addition we provide an ansatz that generalises the Kolassis *et al* (1988) solution. Secondly in §5.3 we review the model investigated by de Oliveira *et al* (1985, 1986, 1988) in which the gravitational potentials are separated into spatial and temporal components. The spatial component is directly related to the initial static configuration of the star which corresponds to a given perfect fluid solution. The temporal component which governs the nonstatic evolution is determined from the junction conditions.

The results obtained in this thesis are briefly reviewed in the conclusion. The original results obtained are highlighted in particular. We point out areas of future investigations emanating from the solutions obtained.

2 Differential Geometry, Field Equations and the Junction Conditions

2.1 Introduction

In this chapter we consider only those aspects of differential geometry and general relativity that are relevant to this thesis. For a more detailed account of differential geometry and tensor calculus the reader is referred to de Felice and Clarke (1990), Hawking and Ellis (1973) and Misner *et al* (1973). In §2.2 we introduce the metric tensor field, connection coefficients, the Riemann tensor and associated tensors. The general form of the energy–momentum tensor is presented and its coupling to the Einstein tensor via the Einstein field equations is briefly discussed. In §2.3 we consider spacetimes which are spherically symmetric and shear–free. The Einstein field equations are derived in detail under these conditions for an energy–momentum tensor with heat flow. The Vaidya solution (1951, 1953) is introduced in §2.4 by transforming the Schwarzschild exterior solution to Eddington–Finkelstein coordinates. The Vaidya solution, or some other radiating solution, must be matched to an interior line element to model a radiating star. In §2.5 we derive the junction conditions in detail at the boundary of the radiating sphere by matching the Vaidya solution to the interior of a radiating star undergoing gravitational collapse because of heat dissipation. In particular we obtain the expression for the local conservation of momentum at the surface. Some physical properties of the radiating star are briefly considered. Our results in this chapter are valid only for a vanishing cosmological constant; a generalisation with a nonvanishing cosmological constant is considered in chapter 3.

2.2 Differential Geometry

The spacetime of special relativity is a four–dimensional manifold which may be covered by a single coordinate neighbourhood. In general relativity we take spacetime to be a differentiable manifold with local coordinates (x^a) where x^0 is timelike and x^1, x^2, x^3 are spacelike. In contrast to special relativity it is not possible to cover all of spacetime with a single coordinate neighbourhood because of gravitational effects. For our purposes it is sufficient to assume that spacetime is a four–dimensional, differentiable, connected, Hausdorff, oriented manifold. To discuss metrical properties it is necessary to introduce a differentiable metric tensor field $\mathbf{g}$ on the manifold. The invariant distance between neighbouring points on a curve in the manifold is defined by the line element

$$ds^2 = g_{ab}dx^a dx^b$$

where $\mathbf{g}$ is the symmetric, nondegenerate metric field with signature $(-+++)$. The metric connection $\mathbf{\Gamma}$ is defined in terms of the metric tensor field and its derivatives by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.1}$$

where commas denote partial differentiation.

The Riemann tensor provides a measure of the curvature of a manifold, that is, it provides a measure of deviation from flatness of the spacetime of special relativity. The Riemann tensor is a type (1,3) tensor and is defined as

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{2.2}$$

in terms of the connection coefficients (2.1). In flat Minkowski spacetime we have that $R^a{}_{bcd} = 0$ and for a curved spacetime $R^a{}_{bcd}$ is nonvanishing in general. Upon contraction of (2.2) we obtain the Ricci tensor

$$R_{ab} = R^c{}_{acb}$$

$$= \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^e{}_{ab}\Gamma^d{}_{ed} - \Gamma^e{}_{ad}\Gamma^d{}_{eb} \tag{2.3}$$

A contraction of (2.3) yields the scalar

$$R = g^{ab}R_{ab}$$

$$= R^a{}_a \tag{2.4}$$

called the Ricci scalar. The Einstein tensor

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} \tag{2.5}$$

is defined in terms of the Ricci tensor (2.3) and the Ricci scalar (2.4). The Einstein tensor **G** has the property of being divergence–free:

$$G^{ab}{}_{;b} = 0 \tag{2.6}$$

which follows directly from the definition (2.5). This property of the Einstein tensor is sometimes referred to in the literature as the Bianchi identity and generates the conservation laws through the Einstein field equations.

The general form for the energy–momentum tensor $\mathbf{T}$ for uncharged matter is given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{2.7}$$

where μ is the energy density, p is the isotropic pressure, q_a is the heat flow vector and π_{ab} represents the stress tensor. These quantities are measured relative to a fluid four–velocity $\mathbf{u}$ ($u^a u_a = -1$). The heat flow vector and stress tensor satisfy the conditions

$$q^a u_a = 0$$

$$\pi^{ab} u_b = 0$$

The energy–momentum tensor (2.7) is coupled to the Einstein tensor (2.5) via the Einstein field equations

$$G_{ab} = T_{ab} \tag{2.8}$$

in appropriate units. We are using units in which the speed of light and the coupling constant are taken to be unity. The field equations (2.8) relates the gravitational field to the matter content. This is a system of coupled partial differential equations which are highly nonlinear. For a more comprehensive discussion of differential geometry applicable to general relativity and further information on the field equations the reader is referred to de Felice and Clarke (1990), Hawking and Ellis (1973) and Stephani (1990). The case of the Einstein field equations with a cosmological constant is treated in chapter 3.

2.3 Shear–free Spacetimes

We consider the particular case of spherically symmetric, shear–free spacetimes. This is a reasonable assumption when modelling a relativistic star. In this case there exists coordinates for which the line element may be expressed in a form that is simultaneously isotropic and comoving. With the coordinates $(x^a) = (t, r, \theta, \phi)$ the line element takes the form

$$ds^2 = -A^2dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.9}$$

where $A = A(t,r)$ and $B = B(t,r)$ are metric functions. It is also possible to include the effects of shear. However this leads to a more complex model and we neglect the effects of shear; this is an area for future investigation. For the line element (2.9) the nonvanishing connection coefficients (2.1) are given by

$$\Gamma^0{}_{00} = \frac{A_t}{A} \qquad \Gamma^0{}_{01} = \frac{A_r}{A}$$

$$\Gamma^0{}_{11} = \frac{BB_t}{A^2} \qquad \Gamma^0{}_{22} = r^2\frac{BB_t}{A^2}$$

$$\Gamma^0{}_{33} = r^2\sin^2\theta\frac{BB_t}{A^2} \qquad \Gamma^1{}_{00} = \frac{AA_r}{B^2}$$

$$\Gamma^1{}_{11} = \frac{B_r}{B} \qquad \Gamma^1{}_{22} = -r^2\left(\frac{B_r}{B} + \frac{1}{r}\right)$$

$$\Gamma^1{}_{33} = -r^2\sin^2\theta\left(\frac{B_r}{B} + \frac{1}{r}\right) \qquad \Gamma^1{}_{01} = \frac{B_t}{B}$$

$$\Gamma^2{}_{02} = \frac{B_t}{B} \qquad \Gamma^3{}_{03} = \frac{B_t}{B}$$

$$\Gamma^2{}_{12} = \frac{B_r}{B} + \frac{1}{r} \qquad \Gamma^3{}_{13} = \frac{B_r}{B} + \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{23} = \cot\theta$$

where subscripts denote partial differentiation. The non–zero Ricci tensor components (2.3) take the form

$$R_{00} = \frac{AA_{rr}}{B^2} + AA_r\frac{B_r}{B^3} - 3\frac{B_{tt}}{B} + 3\frac{A_t}{A}\frac{B_t}{B} + \frac{2}{r}\frac{AA_r}{B^2} \tag{2.10a}$$

$$R_{01} = 2\frac{B_rB_t}{B^2} - 2\frac{B_{rt}}{B} + \frac{A_rA_t}{A^2} + 3\frac{A_r}{A}\frac{B_t}{B} - \frac{1}{A^2}\left(A_t{}^2 + A_r{}^2\right) \tag{2.10b}$$

$$R_{11} = 2\frac{B_t{}^2}{A^2} + \frac{A_r}{A}\frac{B_r}{B} - \frac{2}{r}\frac{B_r}{B} - \frac{A_t}{A^3}BB_t - \frac{A_{rr}}{A}$$

$$+\frac{BB_{tt}}{A^2} + 2\frac{B_r{}^2}{B^2} - 2\frac{B_{rr}}{B} \tag{2.10c}$$

$$R_{22} = r^2\frac{BB_{tt}}{A^2} - r^2\frac{A_t}{A^3}B_tB + 2r^2\frac{B_t{}^2}{A^2} - r^2\frac{A_r}{A}\frac{B_r}{B} - r\frac{A_r}{A}$$

$$-3r\frac{B_r}{B} - r^2\frac{B_{rr}}{B} \tag{2.10d}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.10e}$$

for the connection coefficients listed above. Utilising the components (2.10) and the definition (2.4) we generate the Ricci scalar

$$R = -2\frac{A_{rr}}{A}\frac{1}{B^2} - \frac{4}{r}\frac{A_r}{A}\frac{1}{B^2} + \frac{6}{A^2}\frac{B_t{}^2}{B^2} - \frac{8}{r}\frac{B_r}{B^3} + 2\frac{B_r{}^2}{B^4} - 2\frac{A_r}{A}\frac{B_r}{B^3} - 4\frac{B_{rr}}{B^3}$$

$$-6\frac{A_t}{A^3}\frac{B_t}{B} + 6\frac{B_{tt}}{B} \tag{2.11}$$

for the line element (2.9). On substituting (2.10) and (2.11) in (2.5) we obtain the nonvanishing Einstein tensor components

$$G_{00} = 3\frac{B_t{}^2}{B^2} - \frac{A^2}{B^2}\left(2\frac{B_{rr}}{B} - \frac{B_r{}^2}{B^2} + \frac{4}{r}\frac{B_r}{B}\right) \tag{2.12a}$$

$$G_{01} = -\frac{2}{B^2}\left(BB_{rt} - B_rB_t - \frac{A_r}{A}BB_t\right) \tag{2.12b}$$

$$G_{11} = \frac{1}{A^2}\left(-2BB_{tt} - B_t{}^2 + 2\frac{A_t}{A}BB_t\right)$$

$$+\frac{1}{B^2}\left(B_r{}^2 + 2\frac{A_r}{A}BB_r + \frac{2}{r}\frac{A_r}{A}B^2 + \frac{2}{r}BB_r\right) \tag{2.12c}$$

$$G_{22} = -2r^2\frac{BB_{tt}}{A^2} + 2r^2\frac{A_t}{A^3}BB_t - r^2\frac{B_t{}^2}{A^2} + r\frac{A_r}{A}$$

$$+r\frac{B_r}{B} + r^2\frac{A_{rr}}{A} - r^2\frac{B_r{}^2}{B^2} + r^2\frac{B_{rr}}{B} \tag{2.12d}$$

$$G_{33} = \sin^2\theta G_{22} \tag{2.12e}$$

for the spherically symmetric line element (2.9).

In this thesis we consider a model which represents a spherically symmetric, shear–free fluid configuration with heat conduction. This is a reasonable approximation for many applications in relativistic astrophysics (Shapiro and Teukolsky 1983). An example is the model of gravitational collapse in a radiating star because the star is losing mass corresponding to outgoing radiation. For our model we take $\pi_{ab} = 0$ and (2.7) becomes

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{2.13}$$

The fluid four–velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta^a_0$$

The heat flow vector takes the form

$$q^a = (0, q, 0, 0)$$

since $q^a u_a = 0$ and the heat is assumed to flow in the radial direction on physical grounds because of spherical symmetry. Then on substituting (2.12) and (2.13) in (2.8) we obtain the field equations

$$\mu = \frac{3}{A^2}\frac{{B_t}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B_{rr}}{B} - \frac{{B_r}^2}{B^2} + \frac{4}{r}\frac{B_r}{B}\right) \tag{2.14a}$$

$$p = \frac{1}{A^2}\left(-2\frac{B_{tt}}{B} - \frac{{B_t}^2}{B^2} + 2\frac{A_t}{A}\frac{B_t}{B}\right)$$

$$+\frac{1}{B^2}\left(\frac{{B_r}^2}{B^2} + 2\frac{A_r}{A}\frac{B_r}{B} + \frac{2}{r}\frac{A_r}{A} + \frac{2}{r}\frac{B_r}{B}\right) \tag{2.14b}$$

$$p = -2\frac{1}{A^2}\frac{B_{tt}}{B} + 2\frac{A_t}{A^3}\frac{B_t}{B} - \frac{1}{A^2}\frac{{B_t}^2}{B^2} + \frac{1}{r}\frac{A_r}{A}\frac{1}{B^2}$$

$$+\frac{1}{r}\frac{B_r}{B^3} + \frac{A_{rr}}{A}\frac{1}{B^2} - \frac{{B_r}^2}{B^4} + \frac{B_{rr}}{B^3} \tag{2.14c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{B_{rt}}{B} + \frac{B_r B_t}{B^2} + \frac{A_r}{A}\frac{B_t}{B}\right) \tag{2.14d}$$

for the line element (2.9). This is a system of coupled partial differential equations in the variables A, B, μ, p and q. The equations (2.14) describe the interior of a radiating spherically symmetric star without shear.

If we eliminate p from (2.14b) and (2.14c) we obtain the following partial differential equation

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right) \tag{2.15}$$

called the condition of pressure isotropy. It is interesting to observe that if we redefine the radial coordinate r by

$$x = r^2$$

then we generate the differential equation

$$(AB^{-1})_{xx} = 2A(B^{-1})_{xx}$$

which is a more compact form of (2.15).

2.4 The Vaidya Solution

The Vaidya solution (1951, 1953) represents the exterior gravitational field of a radiating star. It is possible to generate the Vaidya solution directly from the exterior Schwarzschild solution using an appropriate coordinate transformation. In coordinates $(t, \mathsf{r}, \theta, \phi)$ the exterior Schwarzschild solution is given by

$$ds^2 = -\left(1 - \frac{2M}{\mathsf{r}}\right)dt^2 + \left(1 - \frac{2M}{\mathsf{r}}\right)^{-1} d\mathsf{r}^2 + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.16}$$

where M is a constant. To generate the Vaidya line element from (2.16) we utilise the Eddington–Finkelstein coordinate transformation in which

$$v = ct + \mathsf{r} + 2M \ln\left(\frac{\mathsf{r}}{2M} - 1\right) \tag{2.17}$$

where v becomes the new coordinate variable. With the help of the transformation (2.17) we can express (2.16) in the form

$$ds^2 = -\left(1 - \frac{2M}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

To obtain the Vaidya solution we have to make the interpretation that $M = m(v)$. Then we have that

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.18}$$

which is the Vaidya line element. The quantity $m(v)$ represents the Newtonian mass of the gravitating body as measured by an observer at infinity. This solution was originally generated using a different approach by Vaidya (1951, 1953). The solution (2.18) is the unique spherically symmetric solution of the Einstein field equations (2.8) for radiation in the form of a null fluid. The Vaidya solution is often used to describe the exterior gravitational field of a radiating star in applications: de Oliveira *et al* (1985), Kolassis *et al* (1988) and Kramer (1992) are some of the authors who have used the Vaidya solution in applications.

The metric tensor for the Vaidya solution (2.18) is

$$g_{ab} = \begin{pmatrix} -\left(1 - \frac{2m(v)}{\mathsf{r}}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \mathsf{r}^2 & 0 \\ 0 & 0 & 0 & \mathsf{r}^2\sin^2\theta \end{pmatrix}$$

with inverse

$$g^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 - \frac{2m(v)}{\mathsf{r}} & 0 & 0 \\ 0 & 0 & \mathsf{r}^{-2} & 0 \\ 0 & 0 & 0 & \mathsf{r}^{-2}\sin^{-2}\theta \end{pmatrix}$$

The connection coefficients of the Vaidya solution (2.18) are necessary to generate the field equations and are also used in the junction conditions in §2.5. The nonvanishing connection coefficients (2.1) for the metric (2.18) are given by

$$\Gamma^0{}_{00} = -\frac{m}{\mathsf{r}^2} \qquad \Gamma^0{}_{22} = \mathsf{r}$$

$$\Gamma^0{}_{33} = \mathsf{r}\sin^2\theta \qquad \Gamma^1{}_{00} = -\frac{1}{\mathsf{r}}\frac{dm}{dv} + (\mathsf{r} - 2m)\frac{m}{\mathsf{r}^3}$$

$$\Gamma^1{}_{01} = \frac{m}{\mathsf{r}^2} \qquad \Gamma^1{}_{22} = 2m - \mathsf{r}$$

$$\Gamma^1{}_{33} = (2m - \mathsf{r})\sin^2\theta \qquad \Gamma^2{}_{12} = \frac{1}{\mathsf{r}}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{13} = \frac{1}{\mathsf{r}}$$

$$\Gamma^3{}_{23} = \cot\theta$$

The Ricci tensor (2.3) for the Vaidya solution (2.18) is nonvanishing. The only nonvanishing Ricci tensor component is given by

$$R_{00} = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv}$$

where we have utilised the connection coefficients given above. However the Ricci scalar (2.4) for the line element (2.18) vanishes:

$$R = 0$$

Consequently the Einstein tensor (2.5) takes the simple form

$$G_{ab} = -\frac{2}{\mathrm{r}^2}\frac{dm}{dv}\delta^0_a\delta^0_b \tag{2.19}$$

The energy–momentum tensor for pure radiation is given by Kramer *et al* (1980):

$$T_{ab} = \Phi^2 k_a k_b \tag{2.20}$$

where $\mathbf{k}$ is a null vector ($k^a k_a = 0$). The quantity Φ^2 is the energy density of the radiation measured relative to the null vector k_a. For the Vaidya solution (2.18) the null vector is given by $k_a = (1, 0, 0, 0)$. Thus from (2.19) and (2.20) we have that

$$\Phi^2 = -\frac{2}{\mathrm{r}^2}\frac{dm}{dv} \tag{2.21}$$

for the energy density of the radiation.

The Vaidya solution (2.18) is completely determined by the mass function $m(v)$. In order that the exterior spacetime of the radiating star is not unphysical the function $m(v)$ must be a nonincreasing function: $\dfrac{dm}{dv} \leq 0$. In other words the mass of the star is decreasing because of the energy being carried away in the form of radiation. We briefly consider the luminosity profile of a radiating star at the end of §2.5.

2.5 Junction Conditions

In this section we generate the junction conditions to match two spherically symmetric spacetimes on a hypersurface Σ (Bonnor *et al* 1989, Santos 1985). We only provide those details relevant for later chapters. In this section the results obtained are valid for the Einstein field equations (2.8) with a vanishing cosmological constant. For a comprehensive treatment of junction conditions in general relativity see Israel (1967) and Lake (1987). We consider a spherical surface described by a timelike three–space Σ. The surface Σ divides spacetime into two distinct regions $\mathcal{M}^-$ and $\mathcal{M}^+$. Let g_{ij} be the intrinsic metric to Σ so that

$$ds^2_\Sigma = g_{ij} d\xi^i d\xi^j \tag{2.22}$$

The intrinsic coordinates to Σ are given by ξ^i where $i = 1, 2, 3$. The line elements in the regions $\mathcal{M}^\pm$ are of the form

$$ds^2{}_\pm = g_{ab} d\mathcal{X}^a_\pm d\mathcal{X}^b_\pm \tag{2.23}$$

The coordinates in $\mathcal{M}^\pm$ are $\mathcal{X}^a_\pm$ where $a = 0, 1, 2, 3$. We require that the metrics (2.22) and (2.23) match smoothly across Σ. This generates the first junction condition

$$(ds^2_-)_\Sigma = (ds^2_+)_\Sigma = ds^2_\Sigma \tag{2.24}$$

We are using the notation $(\ )_\Sigma$ to represent the value of $(\ )$ on Σ. Consequently the coordinates of Σ in $\mathcal{M}^\pm$ are given by $\mathcal{X}^a_\pm = \mathcal{X}^a_\pm(\xi^i)$. The second junction condition is obtained by requiring continuity of the extrinsic curvature of Σ across the boundary. This gives

$$K^+_{ij} = K^-_{ij} \tag{2.25}$$

where

$$K_{ij}^{\pm} \equiv -n_a^{\pm}\frac{\partial^2 \mathcal{X}_{\pm}^a}{\partial \xi^i \partial \xi^j} - n_a^{\pm}\Gamma^a{}_{cd}\frac{\partial \mathcal{X}_{\pm}^c}{\partial \xi^i}\frac{\partial \mathcal{X}_{\pm}^d}{\partial \xi^j} \tag{2.26}$$

and $n_a^{\pm}(\mathcal{X}_{\pm}^b)$ are the components of the vector normal to Σ. We should point out that the junction conditions (2.24) and (2.25) are equivalent to the junction conditions generated by Lichnerowicz (1955) and O' Brien and Synge (1952). Lake (1987) provides a comprehensive review of the junction conditions for boundary surfaces and surface layers with applications to general relativity and cosmology.

The intrinsic metric to Σ is given by

$$ds_{\Sigma}^2 = -d\tau^2 + \mathcal{R}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.27}$$

with coordinates $\xi^i = (\tau, \theta, \phi)$ and $\mathcal{R} = \mathcal{R}(\tau)$. Note that the time coordinate τ is defined only on the surface Σ. In comoving coordinates we take the interior spacetime $\mathcal{M}^-$ to be given by the shear–free line element (2.9):

$$ds^2 = -A^2dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.28}$$

The surface Σ is the boundary of the interior matter distribution in this case, and is given by

$$f(r,t) = r - r_{\Sigma} = 0$$

where r_{Σ} is a constant. We note that $\dfrac{\partial f}{\partial \mathcal{X}_-^a}$ is a vector orthogonal to Σ which is given by

$$\frac{\partial f}{\partial \mathcal{X}_-^a} = (0,1,0,0)$$

Hence the unit vector normal to Σ must be of the form

$$n_a^- = [0, B(r_{\Sigma}, t), 0, 0] \tag{2.29}$$

For $\mathcal{M}^-$ the first junction condition (2.24), for the metrics (2.27) and (2.28), yields the restrictions

$$A(r_\Sigma, t)\dot{t} = 1 \tag{2.30a}$$

$$r_\Sigma B(r_\Sigma, t) = \mathcal{R}(\tau) \tag{2.30b}$$

where dots represent differentiation with respect to τ. The extrinsic curvature K_{ij}^- of Σ can be obtained using (2.26), (2.28) and (2.29) after a lengthy and tedious calculation. The nonvanishing components are presented below

$$K_{\tau\tau}^- = \left(-\frac{1}{B}\frac{A_r}{A}\right)_\Sigma \tag{2.31a}$$

$$K_{\theta\theta}^- = (r(rB)_r)_\Sigma \tag{2.31b}$$

$$K_{\phi\phi}^- = \sin^2\theta \, K_{\theta\theta}^- \tag{2.31c}$$

valid on the surface Σ.

We take the exterior spacetime $\mathcal{M}^+$ to be described by the Vaidya line element (2.18):

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}}\right) dv^2 - 2dv d\mathsf{r} + \mathsf{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.32}$$

For the exterior spacetime the equation of the surface Σ is given by

$$f(\mathsf{r}, v) = \mathsf{r} - \mathsf{r}_\Sigma(v) = 0$$

Hence we have that the vector orthogonal to the surface Σ is given by

$$\frac{\partial f}{\partial \mathcal{X}_+^a} = \left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0\right)$$

It follows that the unit normal to Σ is of the form

$$n_a^+ = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{-\frac{1}{2}} \left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0\right) \tag{2.33}$$

For $\mathcal{M}^+$ the first junction condition (2.24) for the line elements (2.27) and (2.32), generates the equations

$$\mathsf{r}_\Sigma(v) = \mathcal{R}(\tau) \tag{2.34a}$$

$$\left(1 - \frac{2m}{\mathsf{r}} + 2\frac{d\mathsf{r}}{dv}\right)_\Sigma = \left(\frac{1}{\dot{v}^2}\right)_\Sigma \tag{2.34b}$$

Note that on using (2.34b) we can rewrite the unit normal vector (2.33) in the more compact form

$$n_a^+ = (-\dot{\mathsf{r}}, \dot{v}, 0, 0) \tag{2.35}$$

The nonvanishing components of the extrinsic curvature to Σ can be calculated with the aid of (2.26), (2.32) and (2.35) after a lengthy calculation. These are given by

$$K_{\tau\tau}^+ = \left(\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\mathsf{r}^2}\right)_\Sigma \tag{2.36a}$$

$$K_{\theta\theta}^+ = (\dot{v}(\mathsf{r} - 2m) + \mathsf{r}\dot{\mathsf{r}})_\Sigma \tag{2.36b}$$

$$K_{\phi\phi}^+ = \sin^2\theta \, K_{\theta\theta}^+ \tag{2.36c}$$

valid on the surface Σ.

The first junction condition (2.24) generates the equations (2.30) and (2.34). Collecting these results we have

$$A(r_\Sigma, t)\dot{t} = 1 \tag{2.37a}$$

$$r_\Sigma B(r_\Sigma, t) = \mathcal{R}(\tau) \tag{2.37b}$$

$$\mathrm{r}_\Sigma(v) = \mathcal{R}(\tau) \tag{2.37c}$$

$$\left(1 - \frac{2m}{\mathrm{r}} + 2\frac{d\mathrm{r}}{dv}\right)_\Sigma = \left(\frac{1}{\dot{v}^2}\right)_\Sigma \tag{2.37d}$$

The variable τ was defined only as an intermediary and may be eliminated from these equations. *Hence we find that the necessary and sufficient conditions on the spacetimes for the first junction condition (2.24) to be valid are that*

$$A(r_\Sigma, t)dt = \left(1 - \frac{2m}{\mathrm{r}_\Sigma} + 2\frac{d\mathrm{r}_\Sigma}{dv}\right)^{\frac{1}{2}} dv \tag{2.38a}$$

$$r_\Sigma B(r_\Sigma, t) = \mathrm{r}_\Sigma(v) \tag{2.38b}$$

This result is generalised in §3.4.

The second junction condition (2.25) is obtained by equating the appropriate extrinsic curvature components (2.31) and (2.36), and we have

$$\left(-\frac{1}{B}\frac{A_r}{A}\right)_\Sigma = \left(\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\mathrm{r}^2}\right)_\Sigma \tag{2.39a}$$

$$(r(rB)_r)_\Sigma = (\dot{v}(\mathrm{r} - 2m) + \mathrm{r}\dot{\mathrm{r}})_\Sigma \tag{2.39b}$$

The junction conditions (2.39) may be expressed in an equivalent form which is convenient for applications. We can obtain an equation for $m(v)$ given in terms of A and B only, from (2.39b) after eliminating r, $\dot{\mathrm{r}}$ and $\dot{v}$. Relation (2.39b) can be rewritten, with the help of (2.30) and (2.34), after lengthy algebra as

$$m(v) = \left(\frac{r^3 B}{2A^2}{B_t}^2 - r^2 B_r - \frac{r^3}{2B}{B_r}^2\right)_\Sigma \tag{2.40}$$

We may interpret $m(v)$ as representing the total gravitational mass within the surface Σ. The expression (2.40) corresponds to the mass function of Cahill and McVittie (1970) (also see Hernandez and Misner 1966) for spheres of radius r inside Σ. From (2.30) and (2.34a) we can write

$$\dot{r}_\Sigma = \left(\frac{r}{A}B_t\right)_\Sigma$$

Using this expression for $\dot{r}_\Sigma$ and on substituting (2.40) in (2.39b) we have that

$$\dot{v}_\Sigma = \left(1 + r\frac{B_r}{B} + r\frac{B_t}{A}\right)^{-1}_\Sigma \tag{2.41}$$

If we now differentiate (2.41) with respect to τ and make use of (2.30a) we can write

$$\ddot{v}_\Sigma = \left[\frac{1}{A}\left(1 + r\frac{B_r}{B} + r\frac{B_t}{A}\right)^{-2} \times\right.$$

$$\left.\times\left(r\frac{B_r B_t}{B^2} - r\frac{B_{rt}}{B} + r\frac{A_t B_t}{A^2} - r\frac{B_{tt}}{A}\right)\right]_\Sigma \tag{2.42}$$

Substituting (2.30b), (2.34a), (2.40), (2.41) and (2.42) into (2.39a) we obtain

$$\left(-\frac{1}{B}\frac{A_r}{A}\right)_\Sigma = \left[\left(1 + r\frac{B_r}{B} + r\frac{B_t}{A}\right)^{-1}\right.$$

$$\left(\frac{r}{A}\frac{B_t B_r}{B^2} - \frac{r}{A}\frac{B_{rt}}{B} + r\frac{A_t}{A^3}B_t - \frac{r}{A^2}B_{tt}\right.$$

$$\left.\left.+\frac{B_r}{B^2} + \frac{r}{2}\frac{{B_r}^2}{B^3} - \frac{r}{2A^2}\frac{{B_t}^2}{B}\right)\right]_\Sigma$$

On multiplying this equation by $1+r\left(\frac{B_r}{B}\right)+r\left(\frac{B_t}{A}\right)$ and simplifying we obtain the following result

$$-\frac{2}{A^2}\frac{B_{tt}}{B}+2\frac{A_t}{A^3}\frac{B_t}{B}-\frac{1}{A^2}\frac{{B_t}^2}{B^2}+\frac{2}{r}\frac{A_r}{A}\frac{1}{B^2}+\frac{2}{r}\frac{B_r}{B^3}+\frac{{B_r}^2}{B^4}+2\frac{A_r}{A}\frac{B_r}{B^3}$$

$$=-\frac{2}{AB}\left(-\frac{B_{rt}}{B}+\frac{B_rB_t}{B^2}+\frac{A_r}{A}\frac{B_t}{B}\right)$$

which is equivalent to

$$p_\Sigma=(qB)_\Sigma$$

where we have utilised the field equations (2.14b) and (2.14d). This important result relating the isotropic pressure p to the heat flow q was first established by Santos (1985). *Hence we have established that the necessary and sufficient conditions on the spacetimes for the second junction condition (2.25) to be valid are that*

$$m(v)=\left(\frac{r^3B}{2A^2}{B_t}^2-r^2B_r-\frac{r^3}{2B}{B_r}^2\right)_\Sigma \tag{2.43a}$$

$$p_\Sigma=(qB)_\Sigma \tag{2.43b}$$

This result is generalised in §3.4.

The equations (2.38) and (2.43) are the general matching conditions for the spherically symmetric spacetimes $\mathcal{M}^+$ and $\mathcal{M}^-$. Relation (2.43b) implies that the isotropic pressure p is proportional to the magnitude of the heat flow q which is nonvanishing in general. The pressure p_Σ on the boundary can only be zero when q_Σ becomes zero. In this case there is no radial heat flow and the exterior spacetime consequently is not the Vaidya spacetime but is the exterior Schwarzschild spacetime. Note that the result (2.43b) has been established in general for spherically symmetric, shear–free spacetimes without assuming any particular forms for the metric functions. This result may be summarised as the theorem (Santos 1985): *For a spherically symmetric, shear–free distribution of a collapsing fluid undergoing dissipation in the form of heat flow, the isotropic pressure at the surface of discontinuity cannot be zero.* In the past authors have erroneously assumed that for isotropic collapsing fluids with radial heat flow $p_\Sigma = 0$. For an example of a treatment that makes such an assumption see Glass (1981). We generalise (2.38) and (2.43) to include the cosmological constant in chapter 3. We should point out that the junction conditions for shearing spacetimes have been obtained by Tomimura and Nunes (1993). The case with a nonvanishing electromagnetic field was investigated by de Oliveira and Santos (1987).

We can give a physical interpretation to (2.43b) by deriving a new relationship. As the expression (2.40) also gives the total energy for a sphere of radius r within Σ we can write

$$m(t,r) = \left(\frac{r^3 B}{2A^2}{B_t}^2 - r^2 B_r - \frac{r^3}{2B}{B_r}^2\right)_\Sigma$$

Differentiating partially with respect to t we obtain

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = \left(\frac{r^3}{2A^2}{B_t}^3 + \frac{r^3}{A^2}BB_tB_{tt} - r^3\frac{A_t}{A^3}B{B_t}^2\right.$$
$$\left. - r^2 B_{rt} + \frac{r^3}{2}\frac{B_t}{B^2}{B_r}^2 - r^3\frac{B_r}{B}B_{rt}\right)_\Sigma \tag{2.44}$$

On using the field equations (2.14b) and (2.14d) we can rewrite $\left(\frac{\partial m}{\partial t}\right)_\Sigma$ as

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = -\frac{1}{2}\left(pr^3B^2B_t + qr^2AB^2\left(B + rB_r\right)\right)_\Sigma \tag{2.45}$$

Substituting (2.43b) in (2.45) we obtain

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = -\frac{p}{2}\left[r^2 AB^2\left(1 + r\frac{B_r}{B} + \frac{r}{A}B_t\right)\right]_\Sigma \tag{2.46}$$

Since the radial coordinate is comoving with respect to Σ we can write

$$\left(\frac{\partial m}{\partial t}\right)_\Sigma = \left(\frac{dm}{dt}\right)_\Sigma = \left(\frac{\dot{v}}{\dot{t}}\frac{dm}{dv}\right)_\Sigma \tag{2.47}$$

and by considering (2.30a), (2.30b), (2.34a), (2.47) and (2.46) we obtain

$$\left(-\frac{2}{\mathrm{r}^2}\frac{dm}{dv}\dot{v}^2\right)_\Sigma = p_\Sigma \tag{2.48}$$

The radial flux of momentum of the radiation on both sides of Σ is given by

$$F^\pm = e_0^{\pm a} n^{\pm b} T^\pm_{ab}$$

where

$$e_0^{+a} = \left(1 - \frac{2m}{\mathrm{r}_\Sigma} + 2\frac{d\mathrm{r}_\Sigma}{dv}\right)^{-\frac{1}{2}}\left(\delta_0^a + \frac{d\mathrm{r}_\Sigma}{dv}\delta_1^a\right)$$

$$e_0^{-a} = A_\Sigma{}^{-1}\delta_0^a$$

are the unit tangent vectors in the τ–direction of Σ. For details of this result see Lindquist *et al* (1965). Then it is easy to show that

$$F^+ = \left[\frac{2}{\mathrm{r}^2}\frac{dm}{dv}\dot{v}^2\right]_\Sigma$$

$$F^- = [-qB]_\Sigma$$

so that $F^+ = F^-$ which is equivalent to the junction condition (2.43b). Therefore the result (2.43b) corresponds to the continuity of the radial flux of momentum of the radiation across the surface Σ, that is it expresses the local conservation of momentum.

We are now in a position to discuss the luminosity of the star. Lindquist *et al* (1965) define the total luminosity for an observer at rest at infinity by

$$L_{\infty}(v) = -\frac{dm}{dv} = \lim_{\mathrm{r}\to\infty} 4\pi \mathrm{r}^2 \Phi^2 \tag{2.49}$$

An observer with four–velocity $v^a = (\dot{v}, \dot{\mathrm{r}}, 0, 0)$ located on Σ has proper time τ related to the time t by $d\tau = Adt$. The energy density that this observer measures on Σ is

$$\Phi^2{}_{\Sigma} = \frac{1}{4\pi}\left(-\frac{\dot{v}^2}{\mathrm{r}^2}\frac{dm}{dv}\right)_{\Sigma}$$

and the luminosity observed on Σ is

$$L_{\Sigma} = 4\pi \mathrm{r}^2 \Phi^2{}_{\Sigma}$$

which is not the same as (2.49). The boundary redshift z_{Σ} of the radiation emitted by the star is given by

$$1 + z_{\Sigma} = \frac{dv}{d\tau}$$

which generates the formula

$$1 + z_{\Sigma} = \left(\frac{L_{\Sigma}}{L_{\infty}}\right)^{\frac{1}{2}} \tag{2.50}$$

Equation (2.50) relates the luminosities L_{Σ} to L_{∞}.

3 Heat Flow with Cosmological Constant

3.1 Introduction

In this chapter we seek to model a radiating star when the field equations (2.8) are generalised to include the cosmological constant. In the previous chapter we considered a spherically symmetric, shear–free star undergoing gravitational collapse because of heat loss. Also note that this model may be generalised to include the effects of shear and the electromagnetic field. Tomimura and Nunes (1993) and de Oliveira and Santos (1987) have proposed models with nonvanishing shear and the electromagnetic field respectively. We generalise the results of the preceding chapter by taking into consideration the cosmological constant. We believe that this is an original result. In §3.2 we present the Einstein field equations with nonzero cosmological constant for a spherically symmetric, shear–free matter distribution which is a generalisation of the equations in §2.3. The Vaidya solution is generalised in §3.3 with cosmological constant. Following the procedure set out in §2.5 we derive the junction conditions at the boundary of the star by matching the generalised Vaidya solution to the interior spacetime in §3.4. The generalised junction conditions reduce to those in §2.5 when the cosmological constant vanishes.

3.2 Interior Spacetime

In this section we consider the Einstein field equations for spherically symmetric, shear–free spacetimes in which the cosmological constant is taken to be nonzero. This generalises the results presented in §2.3. The cosmological constant is of importance in many cosmological scenarios (Misner *et al* 1973) and in particular is utilised in theories of the early universe (Maharaj and Beesham 1988). The Einstein field equations governing the interior spacetime with the cosmological constant are

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab}$$

$$G_{ab} + \Lambda g_{ab} = T_{ab} \tag{3.1}$$

where Λ is a constant. The energy–momentum tensor T_{ab} is given by (2.13).

As before the line element for the interior spacetime is described by

$$ds^2 = -A^2dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.2}$$

where $A = A(t,r)$ and $B = B(t,r)$ are metric functions. Even though the line element has the same form as (2.9) we note that A and B are different as the field equations are now (3.1). The nonzero $G_{ab} + \Lambda g_{ab}$ components are given by

$$G_{00} + \Lambda g_{00} = 3\frac{{B_t}^2}{B^2} - \frac{A^2}{B^2}\left(2\frac{B_{rr}}{B} - \frac{{B_r}^2}{B^2} + \frac{4}{r}\frac{B_r}{B}\right) - \Lambda A^2 \tag{3.3a}$$

$$G_{01} + \Lambda g_{01} = -\frac{2}{B^2}\left(BB_{rt} - B_rB_t - \frac{A_r}{A}BB_t\right) \tag{3.3b}$$

$$G_{11} + \Lambda g_{11} = \frac{1}{A^2}\left(-2BB_{tt} - B_t{}^2 + 2\frac{A_t}{A}BB_t\right)$$

$$+\frac{1}{B^2}\left(B_r{}^2 + 2\frac{A_r}{A}BB_r + \frac{2}{r}\frac{A_r}{A}B^2 + \frac{2}{r}BB_r\right) + \Lambda B^2 \qquad (3.3c)$$

$$G_{22} + \Lambda g_{22} = -2r^2\frac{BB_{tt}}{A^2} + 2r^2\frac{A_t}{A^3}BB_t - r^2\frac{B_t{}^2}{A^2} + r\frac{A_r}{A}$$

$$+r\frac{B_r}{B} + r^2\frac{A_{rr}}{A} - r^2\frac{B_r{}^2}{B^2} + r^2\frac{B_{rr}}{B} + \Lambda B^2 r^2 \qquad (3.3d)$$

$$G_{33} + \Lambda g_{33} = \sin^2\theta(G_{22} + \Lambda g_{22}) \qquad (3.3e)$$

for the line element (3.2). In the above we have utilised the components of G_{ab} from (2.12).

Substituting (3.3) and (2.13) into the Einstein field equations (3.1) we obtain

$$\mu = 3\frac{1}{A^2}\frac{B_t{}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B_{rr}}{B} - \frac{B_r{}^2}{B^2} + \frac{4}{r}\frac{B_r}{B}\right) - \Lambda \qquad (3.4a)$$

$$p = \frac{1}{A^2}\left(-2\frac{B_{tt}}{B} - \frac{B_t{}^2}{B^2} + 2\frac{A_t}{A}\frac{B_t}{B}\right)$$

$$+\frac{1}{B^2}\left(\frac{B_r{}^2}{B^2} + 2\frac{A_r}{A}\frac{B_r}{B} + \frac{2}{r}\frac{A_r}{A} + \frac{2}{r}\frac{B_r}{B}\right) + \Lambda \qquad (3.4b)$$

$$p = -2\frac{1}{A^2}\frac{B_{tt}}{B} + 2\frac{A_t}{A^3}\frac{B_t}{B} - \frac{1}{A^2}\frac{B_t{}^2}{B^2} + \frac{1}{r}\frac{A_r}{A}\frac{1}{B^2}$$

$$+\frac{1}{r}\frac{B_r}{B^3}+\frac{A_{rr}}{A}\frac{1}{B^2}-\frac{{B_r}^2}{B^4}+\frac{B_{rr}}{B^3}+\Lambda \tag{3.4c}$$

$$q=-\frac{2}{AB^2}\left(-\frac{B_{rt}}{B}+\frac{B_rB_t}{B^2}+\frac{A_r}{A}\frac{B_t}{B}\right) \tag{3.4d}$$

for the spherically symmetric line element (3.2). The equations (3.4) are the Einstein field equations with cosmological constant Λ for a spherically symmetric, shear–free line element. If we set $\Lambda = 0$ then we regain the field equations (2.14) considered in §2.3. Note from (3.4d) that the cosmological constant Λ does not directly appear in the expression for the heat flow q.

3.3 Exterior Spacetime

The Vaidya solution with cosmological constant Λ is given by

$$ds^2=-\left(1-\frac{2m(v)}{\mathsf{r}}-\frac{1}{3}\Lambda\mathsf{r}^2\right)dv^2-2dvd\mathsf{r}+\mathsf{r}^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \tag{3.5}$$

which describes the exterior spacetime of the collapsing fluid. If $\Lambda = 0$ then (3.5) reduces to (2.18). The line element (3.5) was derived by Plebanski and Stachel (1968) by considering the eigenvectors of the Einstein tensor in spherically symmetric spacetimes. Note that the Schwarzschild solution and the Reissner–Nordström solution with $\Lambda \neq 0$ in null coordinates have a similar form to the line element (3.5) (Kramer *et al* 1980). We present the various quantities associated with the line element (3.5) as this is not well documented in the literature.

The metric tensor of (3.5) assumes the form

$$g_{ab} = \begin{pmatrix} -\left(1-\frac{2m(v)}{\mathsf{r}}-\frac{1}{3}\Lambda\mathsf{r}^2\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \mathsf{r}^2 & 0 \\ 0 & 0 & 0 & \mathsf{r}^2\sin^2\theta \end{pmatrix}$$

with the inverse

$$g^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1-\frac{2m(v)}{\mathsf{r}}-\frac{1}{3}\Lambda\mathsf{r}^2 & 0 & 0 \\ 0 & 0 & \mathsf{r}^{-2} & 0 \\ 0 & 0 & 0 & \mathsf{r}^{-2}\sin^{-2}\theta \end{pmatrix}$$

These equations assist in the calculation of the connection coefficients. The nonzero connection coefficients (2.1) are given by

$$\Gamma^0{}_{00} = -\frac{m}{\mathsf{r}^2}+\frac{\Lambda\mathsf{r}}{3} \qquad \Gamma^0{}_{22} = \mathsf{r}$$

$$\Gamma^0{}_{33} = \mathsf{r}\sin^2\theta$$

$$\Gamma^1{}_{00} = -\frac{1}{\mathsf{r}}\frac{dm}{dv}+\left(1-\frac{2m}{\mathsf{r}}-\frac{1}{3}\Lambda\mathsf{r}^2\right)\left(\frac{m}{\mathsf{r}^2}-\frac{1}{3}\Lambda\mathsf{r}\right) \qquad \Gamma^1{}_{01} = \frac{m}{\mathsf{r}^2}-\frac{1}{3}\Lambda\mathsf{r}$$

$$\Gamma^1{}_{22} = 2m-\mathsf{r}+\frac{1}{3}\Lambda\mathsf{r}^3 \qquad \Gamma^1{}_{33} = (2m-\mathsf{r}+\frac{1}{3}\Lambda\mathsf{r}^3)\sin^2\theta$$

$$\Gamma^2{}_{12} = \frac{1}{\mathsf{r}} \qquad \Gamma^2{}_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3{}_{13} = \frac{1}{\mathsf{r}} \qquad \Gamma^3{}_{23} = \cot\theta$$

for the line element (3.5). This generalises the connection coefficients for the Vaidya solution presented in §2.4. The nonzero Ricci tensor components (2.3) assume the form

$$R_{00} = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv} - \Lambda\left(1 - \frac{2m}{\mathsf{r}} - \frac{1}{3}\Lambda\mathsf{r}^2\right) \tag{3.6a}$$

$$R_{01} = 0 \tag{3.6b}$$

$$R_{11} = 0 \tag{3.6c}$$

$$R_{22} = \Lambda\mathsf{r}^2 \tag{3.6d}$$

$$R_{33} = \sin^2\theta R_{22} \tag{3.6e}$$

for the connection coefficients presented above. Utilising the results of (3.6) and the form of the Ricci scalar given in (2.4) we obtain

$$R = 4\Lambda$$

for the line element (3.5). This differs from the result in §2.4 where the Ricci scalar vanishes. Consequently we may write

$$G_{ab} + \Lambda g_{ab} = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv}\delta_a^0\delta_b^0 \tag{3.7}$$

for the line element (3.5) with cosmological constant Λ.

The Einstein field equations governing the exterior spacetime with cosmological constant Λ are

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab}$$

$$G_{ab} + \Lambda g_{ab} = \Phi^2 k_a k_b \tag{3.8}$$

where $T_{ab} = \Phi^2 k_a k_b$ is the energy–momentum tensor of radiation. With the help of (3.7) the field equations (3.8) reduce to

$$\Phi^2 = -\frac{2}{r^2}\frac{dm}{dv}$$

for the exterior spacetime. The form of the above result is similar to (2.21) but note that $m(v)$ is now related to the line element (3.5) which contains the cosmological constant.

3.4 Junction Conditions

We generate the junction conditions by matching the line element (3.2) and (3.5) across a three–dimensional, spherically symmetric hypersurface Σ which separates spacetime into two distinct regions $\mathcal{M}^-$ and $\mathcal{M}^+$. In this case we utilise the field equations (3.4) and 3.8) so that our results generalise those in §2.5. As the derivation is analogous to §2.5 we do not present all the steps in the proof.

As before the intrinsic metric to Σ is given by

$$ds^2_\Sigma = -d\tau^2 + \mathcal{R}^2(\tau)(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.9}$$

The defining equation of the interior matter distribution at the surface Σ is

$$r - r_\Sigma = 0$$

where r_Σ is a constant. Then the unit vector normal to the hypersurface Σ assumes the form

$$n_a^- = [0, B(r_\Sigma, t), 0, 0] \tag{3.10}$$

For $\mathcal{M}^-$ the first junction condition (2.24), for the metrics (3.2) and (3.9), yields

$$A(r_\Sigma, t)\dot{t} = 1 \tag{3.11a}$$

$$r_\Sigma B(r_\Sigma, t) = \mathcal{R}(\tau) \tag{3.11b}$$

where dots represent differentiation with respect to τ. With the help of (3.2) and (3.10) we calculate the nonzero components of the extrinsic curvature (2.26) for the interior spacetime

$$K^-_{\tau\tau} = \left(-\frac{1}{B}\frac{A_r}{A}\right)_\Sigma \tag{3.12a}$$

$$K^-_{\theta\theta} = (r(rB)_r)_\Sigma \tag{3.12b}$$

$$K^-_{\phi\phi} = \sin^2\theta\, K^-_{\theta\theta} \tag{3.12c}$$

on the hypersurface Σ.

The equation of the surface Σ for the exterior spacetime $\mathcal{M}^+$ is given by

$$\mathsf{r} - \mathsf{r}_\Sigma(v) = 0$$

Following the analysis in §2.5 we can write the unit normal to Σ as

$$n_a^+ = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} - \frac{1}{3}\Lambda \mathsf{r}^2{}_\Sigma + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{-\frac{1}{2}} \left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0\right) \tag{3.13}$$

The first junction condition (2.24) for the line elements (3.5) and (3.9) yields the restrictions

$$\mathsf{r}_\Sigma(v) = \mathcal{R}(\tau) \tag{3.14a}$$

$$\left(1-\frac{2m}{\mathsf{r}}-\frac{1}{3}\Lambda\mathsf{r}^2+2\frac{d\mathsf{r}}{dv}\right)_\Sigma=\left(\frac{1}{\dot{v}^2}\right)_\Sigma \tag{3.14b}$$

for the exterior spacetime $\mathcal{M}^+$. Utilising (3.14b) we can rewrite the unit normal vector in the more compact form

$$n_a^+=(-\dot{\mathsf{r}},\dot{v},0,0) \tag{3.15}$$

as for $\mathcal{M}^-$. With the help of (3.5) and (3.15) we obtain the nonvanishing components of the extrinsic curvature (2.26) in $\mathcal{M}^+$

$$K_{\tau\tau}^+=\left(\frac{\ddot{v}}{\dot{v}}-\dot{v}\frac{m}{\mathsf{r}^2}+\frac{1}{3}\Lambda\mathsf{r}\dot{v}\right)_\Sigma \tag{3.16a}$$

$$K_{\theta\theta}^+=\left(\dot{v}(\mathsf{r}-2m)+\mathsf{r}\dot{\mathsf{r}}-\frac{1}{3}\Lambda\mathsf{r}^3\dot{v}\right)_\Sigma \tag{3.16b}$$

$$K_{\phi\phi}^+=\sin^2\theta\, K_{\theta\theta}^+ \tag{3.16c}$$

on the hypersurface Σ.

From (3.11) and (3.14) we generate the first set of junction conditions. *We find that the necessary and sufficient conditions on the spacetimes for the first junction condition (2.24) to be valid are that*

$$A(r_\Sigma,t)dt=\left(1-\frac{2m}{\mathsf{r}_\Sigma}-\frac{1}{3}\Lambda{\mathsf{r}^2}_\Sigma+2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{\frac{1}{2}}dv \tag{3.17a}$$

$$r_\Sigma B(r_\Sigma,t)=\mathsf{r}_\Sigma(v) \tag{3.17b}$$

When $\Lambda=0$ the junction conditions (3.17) reduce to (2.38) of §2.5. By equating the appropriate extrinsic curvature components (3.12) and (3.16) we can calculate the second junction condition (2.26):

$$\left(-\frac{1}{B}\frac{A_r}{A}\right)_\Sigma = \left(\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\mathsf{r}^2} + \frac{1}{3}\Lambda \mathsf{r}\dot{v}\right)_\Sigma \tag{3.18a}$$

$$(r(rB)_r)_\Sigma = (\dot{v}(\mathsf{r} - 2m) + \mathsf{r}\dot{\mathsf{r}} - \frac{1}{3}\Lambda \mathsf{r}^3\dot{v})_\Sigma \tag{3.18b}$$

We rewrite (3.18) so that comparison with our results of §2.5 are easy.

Relation (3.18b) can be rewritten, with the help of (3.17), as

$$m(v) = \left(\frac{r^3 B}{2A^2}{B_t}^2 - r^2 B_r - \frac{r^3}{2B}{B_r}^2 - \frac{1}{6}\Lambda r^3 B^3\right)_\Sigma \tag{3.19}$$

If $\Lambda = 0$ then (3.19) becomes (2.40) so that $m(v)$ represents the generalisation of the mass function for spheres of radius r within Σ with a cosmological constant. We now seek an expression for p at the boundary of the sphere. Utilising (3.11) and (3.14a) we obtain

$$\dot{\mathsf{r}}_\Sigma = \left(\frac{r}{A}B_t\right)_\Sigma$$

Substituting the above expression together with (3.19) into (3.18b) we have that

$$\dot{v}_\Sigma = \left(1 + r\frac{B_r}{B} + r\frac{B_t}{A}\right)^{-1}_\Sigma \tag{3.20}$$

Differentiating (3.20) with respect to τ and utilising (3.11a) we have

$$\ddot{v}_\Sigma = \left[\frac{1}{A}\left(1 + r\frac{B_r}{B} + r\frac{B_t}{A}\right)^{-2}\right.$$

$$\left.\left(r\frac{B_r B_t}{B^2} - r\frac{B_{rt}}{B} + r\frac{A_t B_t}{A^2} - r\frac{B_{tt}}{A}\right)\right]_\Sigma \tag{3.21}$$

Substituting (3.11b), (3.14a), (3.19), (3.20) and (3.21) into (3.18a) we obtain

$$\left(-\frac{1}{B}\frac{A_r}{A}\right)_\Sigma = \left[\left(1 + r\frac{B_r}{B} + r\frac{B_t}{A}\right)^{-1}\right.$$

$$\left(\frac{r}{A}\frac{B_tB_r}{B^2} - \frac{r}{A}\frac{B_{rt}}{B} + r\frac{A_t}{A^3}B_t - \frac{r}{A^2}B_{tt}\right.$$

$$\left.\left. + \frac{B_r}{B^2} + \frac{r}{2}\frac{{B_r}^2}{B^3} - \frac{r}{2A^2}\frac{{B_t}^2}{B} + \frac{\Lambda r B}{2}\right)\right]_\Sigma$$

On multiplying this equation by $1 + r\left(\frac{B_r}{B}\right) + r\left(\frac{B_t}{A}\right)$ we obtain the following result

$$\frac{1}{A^2}\left(-2\frac{B_{tt}}{B} - \frac{{B_t}^2}{B^2} + 2\frac{A_t}{A}\frac{B_t}{B}\right) + \frac{1}{B^2}\left(\frac{{B_r}^2}{B^2} + 2\frac{A_r}{A}\frac{B_r}{B} + \frac{2}{r}\frac{A_r}{A} + \frac{2}{r}\frac{B_r}{B}\right) + \Lambda$$

$$= -\frac{2}{AB}\left(-\frac{B_{rt}}{B} + \frac{B_rB_t}{B^2} + \frac{A_r}{A}\frac{B_t}{B}\right)$$

which is the same as

$$p_\Sigma = (qB)_\Sigma$$

where we have made use of the field equations (3.4b) and (3.4d). This junction condition has the same form as (2.43b) in §2.5. However note that the pressure p_Σ is now given by (3.4b) which includes the cosmological constant.

The expressions for $m(v)$ and p that we have established above are equivalent to (3.18). *Hence we have shown that the necessary and sufficient conditions on the spacetimes for the second junction condition (2.25) to be valid are that*

$$m(v) = \left(\frac{r^3B}{2A^2}{B_t}^2 - r^2B_r - \frac{r^3}{2B}{B_r}^2 - \frac{1}{6}\Lambda r^3B^3\right)_\Sigma \tag{3.22a}$$

$$p_\Sigma = (qB)_\Sigma \tag{3.22b}$$

The junction conditions (3.17) and (3.22) generalise the junction conditions (2.38) and (2.43) of §2.5. If we set $\Lambda = 0$ then we regain the results of §2.5. *Hence our equa-*

tions (3.17) and (3.22) are the junction conditions across a spherically symmetric hypersurface with a cosmological constant. This is an original result and generalises that of Santos (1985). Even though (3.22b) has the same form as (2.43b) the gravitational potential B now has to be a solution of the Einstein field equations (3.4) with cosmological constant Λ. Our results are important in the modelling of radiating spheres when the equations contain a cosmological constant Λ. As far as we are aware no exact solutions with a cosmological constant for a radiating sphere have been presented in the literature. We intend pursuing this problem in the future.

4 The Kramer Model

4.1 Introduction

Our intention in this chapter is to apply the equations derived in chapter 2 to a particular model of physical interest. We review the model proposed by Kramer (1992) for a spherically symmetric, shear–free star undergoing gravitational collapse because of radial heat dissipation. Essentially a nonstatic model is generated from a static model by allowing certain parameters to become functions of time. The interior static model is taken to be the interior Schwarzschild solution. In §4.2 we consider the interior Schwarzschild solution in isotropic coordinates. A parameter is allowed to become a function of time and the Einstein field equations, without a cosmological constant, is presented in terms of this time–dependent parameter. In §4.3 we present the junction conditions for this particular model by matching the interior spacetime to the exterior spacetime, described by the Vaidya solution, across a spherically symmetric hypersurface. The junction conditions generate a nonlinear ordinary differential equation that governs the temporal evolution of this model. This equation was also derived by Kramer (1992). In §4.4 we completely integrate the nonlinear equation in terms of elementary and special functions so that the gravitational behaviour of the Kramer (1992) model is completely specified. Some physical properties of this model are investigated in §4.5. Graphs illustrating the behaviour of the thermodynamic variables and gravitational potentials are presented.

4.2 Interior Spacetime

We generate an exact solution of the Einstein field equations (with vanishing cosmological constant) that models the interior of a radiating star. As the spacetime is spherically symmetric and shear–free we can utilise the results of §2.3. In this chapter we use the technique of Kramer (1992) to generate a nonstatic solution from a known static model. This is possible because the isotropy condition does not contain time derivatives. We take advantage of this feature to generate a solution to the field equations. If we start with a given static perfect fluid solution and allow the parameters in this solution to become functions of time then the isotropy condition is immediately satisfied. We apply this procedure to the interior Schwarzschild solution in isotropic coordinates.

The interior Schwarzschild solution in coordinates $(x^a) = (t, r, \theta, \phi)$ has the form

$$
\begin{aligned}
ds^2 = &- \left[\frac{3}{2}\left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{2Mr^2}{R^3}\right)^{\frac{1}{2}}\right] dt^2 + \left[1 - \frac{1}{3}\mu_0 r^2\right]^{-1} dr^2 \\
&+ r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (4.1)
\end{aligned}
$$

where M is the total mass contained within a sphere of radius R and the constant μ_0 is the energy density. The isotropic pressure p for the interior Schwarzschild solution is related to μ_0 by

$$p = \mu_0 \left[\frac{(1 - 2Mr^2/R^3)^{1/2} - (1 - 2M/R)^{1/2}}{3\,(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}} \right]$$

For our application it is necessary to write (4.1) in isotropic form. As the isotropic form of the Schwarzschild interior solution is not well known we present the transformation explicitly. This means that we have to introduce a new coordinate ρ which replaces the radial coordinate r. The appropriate transformation is given by

$$r = \frac{M}{{\rho_0}^3} \left(\frac{3}{\mu_0} \right)^{\frac{1}{2}} \left(\frac{M^2}{4{\rho_0}^6} + \rho^2 \right)^{-1} \rho$$

$$\mu_0 = \frac{6M}{R^3}$$

$$R = \rho_0 \left(1 + \frac{M}{2\rho_0} \right)^2$$

where ρ_0 is a constant. With the assistance of this transformation we are in a position to rewrite (4.1) as

$$ds^2 = - \left(1 + \frac{M\rho^2}{2{\rho_0}^3} \right)^{-2} \left[\left(\frac{1 - \frac{M}{\rho_0} + \left(\frac{M\rho^2}{2{\rho_0}^3} \right) \left(2 - \frac{M}{2\rho_0} \right)}{1 + \frac{M}{2\rho_0}} \right)^2 \right] dt^2$$

$$+ \left(1 + \frac{M\rho^2}{2{\rho_0}^3} \right)^{-2} \left[\left(1 + \frac{M}{2\rho_0} \right)^6 \left[d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \right] \qquad (4.2)$$

which is the desired form of the interior Schwarzschild solution in isotropic coordinates. The form of the line element (4.2) agrees with that given by Kramer (1992).

It is possible to write (4.2) in simpler form. To simplify this expression we define a new constant

$$y = \frac{M}{2\rho_0}$$

and we introduce dimensionless coordinates by scaling ρ and t as follows

$$\frac{\rho}{\rho_0} \rightarrow r \qquad\qquad \frac{t}{\rho_0} \rightarrow t$$

Then (4.2) takes the following compact form

$$ds^2 = -\frac{(1 + 2yr^2 - 2y - r^2y^2)^2}{(1+y)^2(1+yr^2)^2}dt^2 + \frac{(1+y)^6}{(1+yr^2)^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (4.3)$$

We utilise the form of the line element (4.3) to generate a solution to the Einstein field equations (2.8). As (4.3) is in isotropic form we make the identification

$$A^2 = \frac{(1 + 2yr^2 - 2y - r^2y^2)^2}{(1+y)^2(1+yr^2)^2}$$

$$B^2 = \frac{(1+y)^6}{(1+yr^2)^2}$$

on comparison with (2.9). This means that (4.3) is a special case of (2.9) and we can utilise the results established in §2.3 and §2.5.

To generate an analytic model with nonvanishing heat flow from the interior Schwarzschild solution we consider y as a function of time. That is we take $y = y(t)$ so that the t–dependence of A and B is not specified but the r–dependence is given *ab initio*. This procedure was first utilised by Kramer (1992) and may be considered as a particular method to generate exact solutions to the Einstein field equations (2.8). Since the metric (4.3) is in isotropic form and the isotropy condition does not contain time derivatives this procedure yields a solution. *The isotropy condition (2.15) is automatically satisfied for a known static perfect fluid solution where the* *parameters in that solution now become functions of time.* Then on substituting the above expressions for A and B in (2.14) we obtain the energy density, the heat flow and pressure respectively:

$$\mu = \frac{12y}{(1+y)^6} + 3\left(\frac{dy}{dt}\right)^2 \left(\frac{2yr^2 - r^2 + 3}{1 + 2yr^2 - 2y - y^2r^2}\right)^2 \tag{4.4a}$$

$$q = -\frac{4r(1+yr^2)^2}{(1+y)^4(1+2yr^2-2y-y^2r^2)^2}\frac{dy}{dt} \tag{4.4b}$$

$$\begin{aligned} p = {} & \frac{12(1-r^2)y^2}{(1+y)^6(1+2yr^2-2y-y^2r^2)} - \frac{2(1+y)(1+yr^2)(2yr^2-r^2+3)}{(1+2yr^2-2y-y^2r^2)^2}\frac{d^2y}{dt^2} \\ & - \left[\frac{4[3(1+yr^2)^2 - r^2(1+y)(2yr^2-r^2+3)] + (2yr^2-r^2+3)^2}{(1+2yr^2-2y-y^2r^2)^2}\right]\left(\frac{dy}{dt}\right)^2 \\ & - \left[\frac{2(3y^2r^4 - y^2r^2 + 4yr^2 - r^2 + 3)(2yr^2 - r^2 + 3)}{(1+2yr^2-2y-y^2r^2)^3}\right]\left(\frac{dy}{dt}\right)^2 \end{aligned} \tag{4.4c}$$

for the line element (4.3). Therefore the equations (4.3) and (4.4) are an exact solution to the Einstein field equations (2.14) which may be utilised to model the interior of a spherically symmetric star with heat flow. The gravitational and matter variables depend on the quantity y. The function $y = y(t)$ is an arbitrary function of time; the junction conditions will govern the behaviour of y.

4.3 Junction Conditions

The interior spacetime of the radiating star is given by (4.3). The exterior spacetime describes outgoing null radiation and we take this to be the Vaidya solution (2.18).

The general conditions for two line elements to match smoothly across a hypersurface Σ have been derived in §2.5. To match the spacetimes (2.18) and (4.3) continuously across Σ the junction conditions (2.38) and (2.43) must be satisfied. For our model these become

$$\frac{1 + 2yr_\Sigma{}^2 - 2y - r_\Sigma{}^2y^2}{(1+y)(1+yr_\Sigma{}^2)}dt = \left(1 - \frac{2m}{\mathsf{r}_\Sigma} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{\frac{1}{2}} dv \tag{4.5a}$$

$$r_\Sigma \frac{(1+y)^3}{1+yr_\Sigma{}^2} = \mathsf{r}_\Sigma(v) \tag{4.5b}$$

$$p_\Sigma = \left[q\frac{(1+y)^3}{1+yr^2}\right]_\Sigma \tag{4.5c}$$

$$m(v) = \left[\frac{r^3}{2}\frac{(1+y)^9}{(1+yr^2)^3}\frac{(3-r^2+2r^2y)^2}{(1+2y(r^2-1)-r^2y^2)^2}\left(\frac{dy}{dt}\right)^2\right.$$

$$\left. + \frac{2r^3y(1+y)^3}{(1+yr^2)^2}\right]_\Sigma \tag{4.5d}$$

From equations (4.4) and (4.5) we obtain a second order differential equation that determines the function y.

Using (4.4b), (4.4c) and the junction condition (4.5c) we obtain

$$\frac{d^2y}{dt^2} + \frac{2(2-y)}{(1+y)(1-y)}\left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+y)^3}\frac{dy}{dt} \tag{4.6}$$

which is a nonlinear ordinary differential equation. Kramer (1992) presents a first integral of (4.6) by inspection. As this integration is nontrivial we present the relevant steps in the integration process. Multiplying (4.6) by $(1+y)^3/(1-y)$ we obtain

$$\frac{(1+y)^3}{1-y}\frac{d^2y}{dt^2} + \frac{2(2-y)(1+y)^2}{(1-y)^2}\left(\frac{dy}{dt}\right)^2 = \frac{1}{(1-y)}\frac{dy}{dt}$$

which can be rewritten as

$$\frac{d}{dt}\left(\frac{(1+y)^3}{1-y}\frac{dy}{dt}\right) = -\frac{d}{dt}\left[\ln(1-y)\right]$$

which is a simpler form of (4.6). Integrating the above equation we obtain

$$\frac{dy}{dt} = -\frac{1-y}{(1+y)^3}\ln\frac{1-y}{1-y_0} \tag{4.7}$$

where $\ln(1-y_0)$ is a constant of integration. The result (4.7) is a first integral of the differential equation (4.6); we have proved that it is the most general first integral. The nonlinear first order equation (4.7) was also presented by Kramer (1992). He did not pursue the integration of (4.7) further because of the nonlinearity. By inspection it seems that it is not posssible to complete the integration of (4.7) in closed form. *However we are in a position to fully solve (4.7) and determine the behaviour of $y = y(t)$ in general.* This is pursued in the next section.

4.4 General Behaviour of $y = y(t)$

It is possible to integrate (4.7) in general in terms of elementary functions and special functions. We rewrite (4.7) as

$$\frac{(1+y)^3}{1-y}\left(\ln\frac{1-y}{1+y_0}\right)^{-1} dy = -dt$$

where the variables y and t have separated. This enables us to integrate by parts to obtain

$$\frac{(1+y)^3}{1-y_0}\ln\left(\ln\frac{1-y}{1-y_0}\right)$$

$$-\frac{3}{1-y_0}\left[\int \ln\left(\ln\frac{1-y}{1-y_0}\right)dy + 2\int y\ln\left(\ln\frac{1-y}{1-y_0}\right)dy\right.$$

$$\left.+\int y^2\ln\left(\ln\frac{1-y}{1-y_0}\right)dy\right] = -t + y_1 \tag{4.8}$$

where y_1 is the second constant of integration. To complete the solution we need to evaluate the three remaining integrals on the left handside of (4.8). We consider the integrals separately for clarity.

Integrating by parts we obtain

$$\int \ln\left(\ln\frac{1-y}{1-y_0}\right)dy = (1-y_0)\left[\frac{1-y}{1-y_0}\ln\left(\ln\frac{1-y}{1-y_0}\right)\right.$$

$$\left.-(1-y_0)\mathrm{Li}\left(\frac{1-y}{1-y_0}\right)\right]$$

for the first integral. Again integrating by parts we obtain the second integral

$$\int y\ln\left(\ln\frac{1-y}{1-y_0}\right)dy = \frac{1}{2}(1-y)^2\ln\left(\ln\frac{1-y}{1-y_0}\right)$$

$$-\frac{(1-y_0)^3}{2}\mathrm{Li}\left(\frac{1-y}{1-y_0}\right)^2$$

To evaluate the remaining integral we use integration by parts to obtain

$$\int y^2\ln\left(\ln\frac{1-y}{1-y_0}\right)dy = \frac{1}{3}(1-y)^3\ln\left(\ln\frac{1-y}{1-y_0}\right)$$

$$-\frac{(1-y_0)^4}{3}\mathrm{Li}\left(\frac{1-y}{1-y_0}\right)^3$$

In these expressions Li represents a special function. The Li function is defined by the integral

$$\mathrm{Li}(z^{n+1}) = \int \frac{z^n}{\ln z}dz$$

Here the constant n takes on three values $n = 0, 1, 2$ which correspond to the three integrals that arise in equation (4.8). For the properties of the Li function the reader is referred to Gradshteyn and Ryzhik (1994) and Lebedev (1972). It is interesting to note that the logarithmic integral function $\mathrm{Li}(z)$ is widely utilised in the study of the distribution of primes in number theory (Wolfram 1991) and here we find it useful in relativistic astrophysics.

On substituting the above three integrals into (4.8) we generate the following solution

$$\begin{aligned} t + y_1 = {} & (1+y)^3 \ln\left(\ln\frac{1-y}{1-y_0}\right) + 12\left[(1-y)\ln\left(\ln\frac{1-y}{1-y_0}\right) - (1-y_0)\mathrm{Li}\left(\frac{1-y}{1-y_0}\right)\right] \\ & -6\left[(1-y)^2\ln\left(\ln\frac{1-y}{1-y_0}\right) - (1-y_0)^3\mathrm{Li}\left(\frac{1-y}{1-y_0}\right)^2\right] \\ & +(1-y)^3\ln\left(\ln\frac{1-y}{1-y_0}\right) - (1-y_0)^4\mathrm{Li}\left(\frac{1-y}{1-y_0}\right)^3 \end{aligned} \tag{4.9}$$

Thus (4.9) represents the general solution of the nonlinear, first order differential equation (4.7). The general solution depends on the special function Li. As far as we are aware result (4.9) is new and has not been published previously. We have succeeded in fully describing the temporal behaviour of the model proposed by Kramer for a radiating star. *Thus the general solution of the field equations is given by (4.3) and (4.4), where $y(t)$ is governed by (4.9), in the Kramer model for a radiating star.*

4.5 Physical Properties

In this section we briefly consider some of the physical properties of our radiating model. The matter variables μ, p and q may be expressed in terms of y only with the help of (4.4) and (4.7):

$$\mu = \frac{12y}{(1+y)^6} + 3\frac{(1-y)^2}{(1+y)^6}\left(\ln\frac{1-y}{1-y_0}\right)^2\left(\frac{2yr^2-r^2+3}{1+2yr^2-2y-y^2r^2}\right)^2 \tag{4.10a}$$

$$q = \frac{4r(1+yr^2)^2}{(1+y)^4(1+2yr^2-2y-y^2r^2)^2}\frac{1-y}{(1+y)^3}\ln\frac{1-y}{1-y_0} \tag{4.10b}$$

$$\begin{aligned}
p = {} & \frac{12(1-r^2)y^2}{(1+y)^6(1+2yr^2-2y-y^2r^2)} + \frac{2(1+y)(1+yr^2)(2yr^2-r^2+3)}{(1+2yr^2-2y-y^2r^2)^2} \times \\
& \frac{1-y}{(1+y)^6}\ln\frac{1-y}{1-y_0}\left(1+2\frac{(2-y)}{1+y}\ln\frac{1-y}{1-y_0}\right) \\
& - \left[\left(\frac{4[3(1+yr^2)^2 - r^2(1+y)(2yr^2-r^2+3)] + (2yr^2-r^2+3)^2}{(1+2yr^2-2y-y^2r^2)^2}\right)\right. \\
& + \left.\left(\frac{2(3y^2r^4-y^2r^2+4yr^2-r^2+3)(2yr^2-r^2+3)}{(1+2yr^2-2y-y^2r^2)^3}\right)\right] \\
& \left[\frac{(1-y)^2}{(1+y)^6}\left(\ln\frac{1-y}{1-y_0}\right)^2\right]
\end{aligned} \tag{4.10c}$$

The behaviour of y is completely determined by (4.9). Then the general solution of the Einstein field equations (2.14) for the line element (4.3) is given by (4.9) and (4.10).

We observe from (4.10a) that the energy density is positive if $y > 0$ for the interior matter distribution. From (4.10b) (or (4.4b)) we must have $\frac{dy}{dt} < 0$ which ensures that $q > 0$ and that the heat flow is directed outwards. The behaviour of the pressure p in (4.10c) is complicated and has to be plotted graphically. Note that the pressure may become negative. The positive constant y_0 may be given a physical interpretation. The constant of integration y_0 gives the initial value of the function $y(t)$ for $t \to -\infty$ when the solution approaches the static Schwarzschild limit. From (4.5d) we obtain

$$\frac{m(v)}{2r_0} = y + \left(\ln \frac{1-y}{1-y_0}\right)^2$$

which yields the mass parameter $m(v)$ in the Vaidya solution (2.18) When $y = y_0$ this expression becomes

$$\frac{m}{2r_0} = y_0$$

which coincides with the interior Schwarzschild mass M. The requirement that $\frac{dy}{dt} < 0$ and the singularity in the metric at $y = -1$ places the following restriction on $y(t)$:

$$y_0 \geq y > -1$$

for a consistent model.

In general relativity the heat flow vector $\mathbf{q}$ is related to the temperature T

by

$$q^a = -Kh^{ab}(T_{,b} + Tu_{b;c}u^c)$$

where K is the thermal conductivity and h^{ab} is the projection tensor defined by $h^{ab} = g^{ab} + u^a u^b$ (Stephani 1990). For the line element (4.3) the magnitude of the heat flow is

$$q = -\left(\frac{K}{AB^2}\right)(TA)_r$$

In principle it is possible to determine the temperature T from the above expression since A, B and q are known. However the integration process is difficult for this particular model and we hope to pursue this aspect in future work. We should point out that the Kramer solution presented above belongs to the general class of conformally flat solutions with heat flow derived by Maiti (1982) and Banerjee *et al* (1989).

We note that it is not trivial to determine the radial and temporal evolution of the gravitational and thermodynamical variables from an inspection of (4.3) and (4.10). Even though an analytic treatment is difficult it is possible to obtain a graphical description of the behaviour of the various functions for the Kramer (1992) model. With the aid of the software package MATHEMATICA Version 2.0 (Wolfram 1991) we provide the following graphical plots for the thermodynamical variables, gravitational potentials and the behaviour of the function $y(t)$:

- Figure a: Plot of Energy Density versus Radial and Temporal Coordinates

- Figure b: Plot of Pressure versus Radial and Temporal Coordinates

- Figure c: Plot of Heat Flux versus Radial and Temporal Coordinates

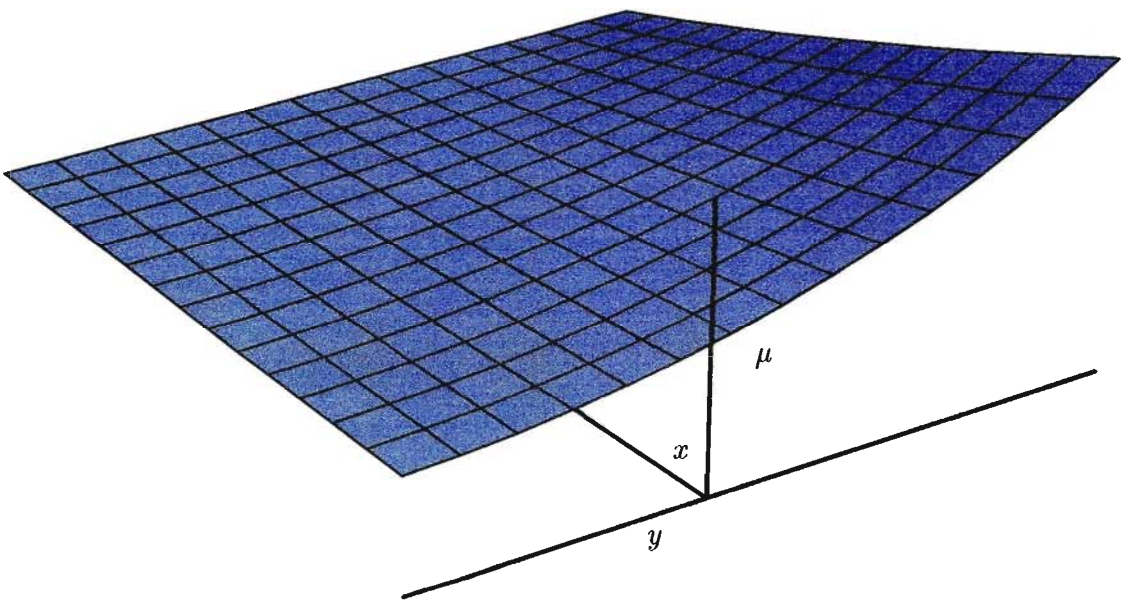
- Figure d: Plot of the Gravitational Potential $A(t,r)$ versus Radial and Temporal Coordinates

- Figure e: Plot of the Gravitational Potential $B(t,r)$ versus Radial and Temporal Coordinates

- Figure f: Plot of the behaviour of $y(t)$

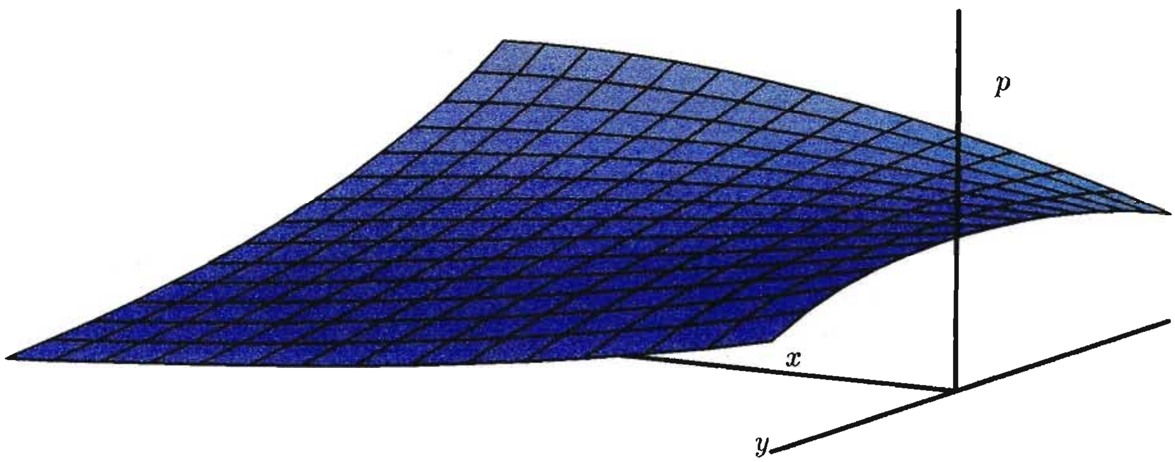
The graphical plots indicate that this model is well–defined for the chosen intervals of r and t. We should point out that there exists other intervals for which this solution is well–behaved. We have included the MATHEMATICA code that was written to generate the figures a–f. The figures a–f show that the matter variables and the gravitational potentials are well–behaved over the interval plotted. The behaviour of $y(t)$ is complicated and difficult to interpret. Figure f illustrates that there exists a finite interval for which the behaviour of $y(t)$ is reasonable. From figure f we observe that $y(t)$ is a decreasing function of time over the chosen interval. This implies that the mass of the radiating star decreases with increasing time which is physically reasonable. Clearly there are other intervals for which this is true. Our treatment here is a verification that the Kramer (1992) solution is a physically reasonable model for a radiating star.

Figure a: Plot of Energy Density versus Radial and Temporal Coordinates



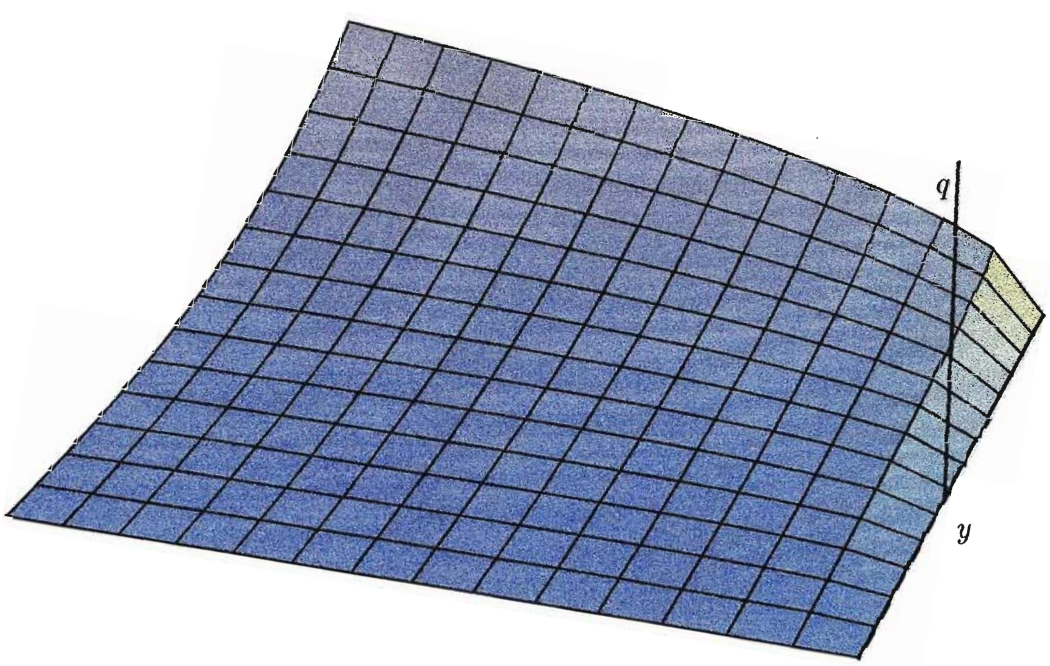
$0 \leq x \leq 1 \qquad 0 \leq y(t) \leq 0.25$

Figure b: Plot of Pressure versus Radial and Temporal Coordinates



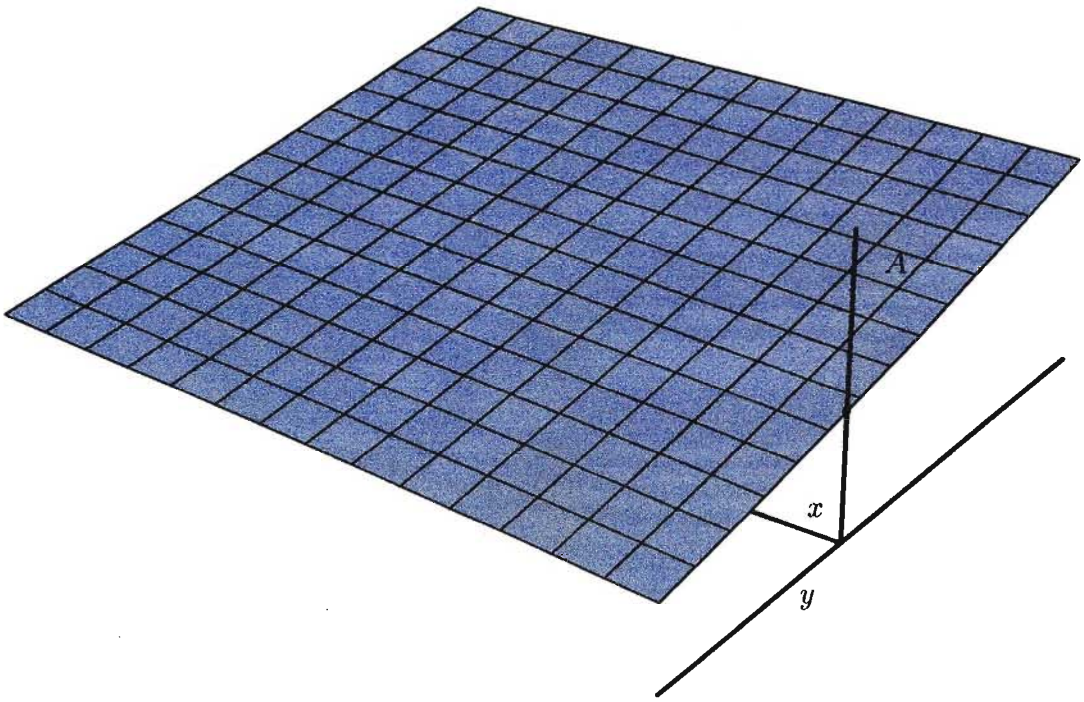
$0 \leq x \leq 1 \qquad 0 \leq y(t) \leq 0.25$

Figure c: Plot of Heat Flux versus Radial and Temporal Coordinates



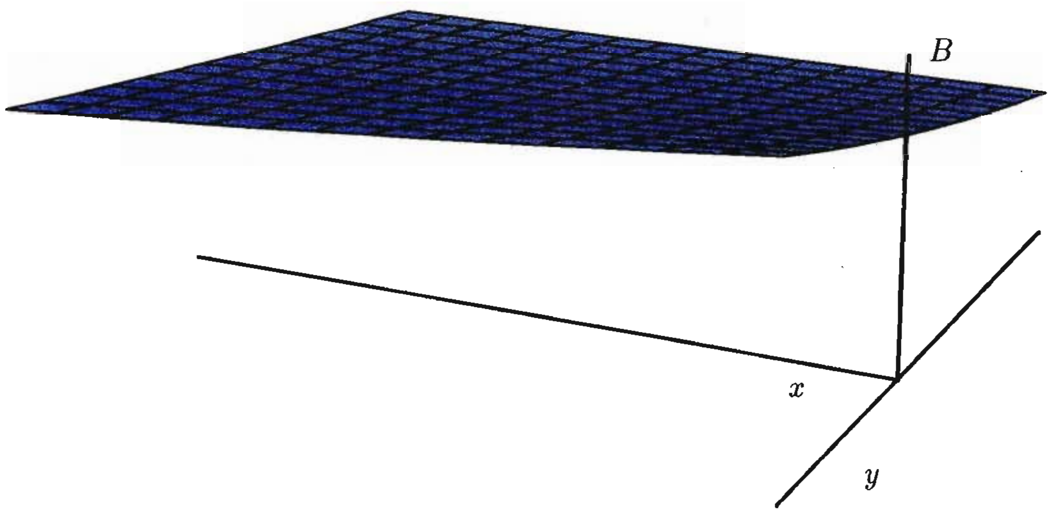
$0 \leq x \leq 1 \qquad 0 \leq y(t) \leq 0.25$

Figure d: Plot of the Gravitational Potential $\mathbf{A(t, r)}$ versus Radial and Temporal Coordinates



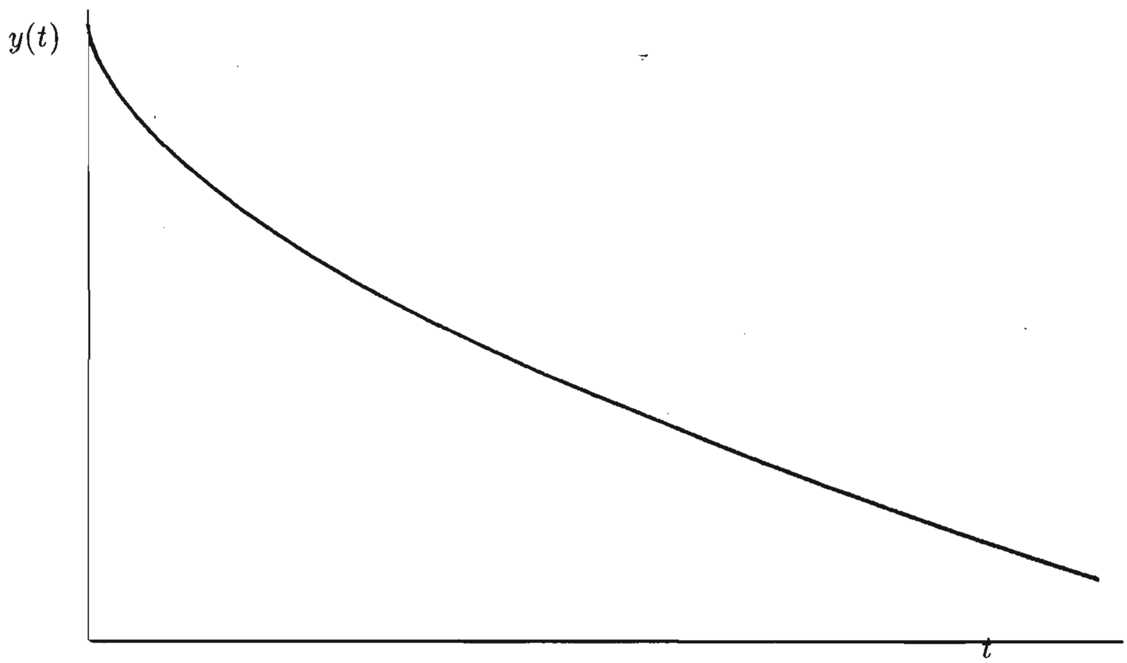
$0 \leq x \leq 1 \qquad 0 \leq y(t) \leq 0.25$

Figure e: Plot of the Gravitational Potential B(t, r) versus Radial and Temporal Coordinates



$0 \leq x \leq 1 \qquad 0 \leq y(t) \leq 0.25$

Figure f: Plot of the Behaviour of $y(t)$



$0 \leq y(t) \leq 0.065$

```
(*Mathematica package that generates the plots for the
 gravitational potentials, thermodynamical variables
 and the behaviour  of  y(t)*)


ClearAll[y0,y,x,mu,q,p,a,b,c,d];


y0 = 0.25;



(* We define frequently used expressions a,b,c,d *)



a[y_] := (1-y)/(1+y)^3;

b[y_] := Log[(1-y)/(1-y0)];

c[x_,y_] := 1 + 2y x - 2y - y^2 x;

d[x_,y_] := 2y x - x + 3;



(* Here are the three main expressions *)

mu[x_,y_] := 12y/(1+y)^6 + 3 a[y]^2 b[y]^2 (d[x,y]/c[x,y])^2;

q[x_,y_] := 4 Sqrt[x] (1+y x)^2 / ((1+y)^4 c[x,y]^2) a[y] b[y];

p[x_,y_] := 12(1-x)y^2/((1+y)^6 c[x,y]) +
          2(1+y)(1+y x)d[x,y]/c[x,y]^2 (1-y)/(1+y)^6 b[y]
          (1+2(2-y)/(1+y) b[y])-
          ((4(3(1+y x)^2-x(1+y)d[x,y])+d[x,y]^2)/c[x,y]^2
          + 2(3y^2 x^2 - y^2 x + 4y x - x + 3)d[x,y]/c[x,y]^3)
          a[y]^2 b[y]^2;



figure a1 = Plot3D[mu[x, y], {x, 0, 1}, {y, 0, .25},
            ClipFill-> None,
            ViewPoint->{-2.612,1.755,1.245},
            Boxed-> False, Axes-> False]
```

```
figure a2 = Graphics3D[{Line[{{0, .15, 0}, {0, .15, 2}}],
            Line[{{0, .15, 0}, {1, .15, 0}}],
            Line[{{0, 0, 0}, {0, .25, 0}}]}, Boxed->False]


figure b1 = Plot3D[p[x, y], {x, 0, 1}, {y, 0, .25},
            ClipFill-> None,Boxed-> False, Axes-> False,
            ViewPoint->{-1.810,2.840,0.335}]

figure b2 = Graphics3D[{Line[{{0, .15, 0}, {0, .15, 1.5}}],
            Line[{{0, .15, 0},{1, .15, 0}}],
            Line[{{0, 0, 0}, {0, .25, 0}}]},Boxed->False]


figure c1 = Plot3D[q[x, y], {x, 0, 1}, {y, 0, .25},
            ViewPoint->{-1.070,2.987,1.175},
            Boxed -> False, Axes->False, ClipFill-> None]

figure c2 = Graphics3D[{Line[{{0, .15, 0}, {0, .15, 1.5}}],
            Line[{{0, .15, 0},{1, .15, 0}}],
            Line[{{0, 0, 0}, {0, .25, 0}}]}, Boxed ->False]


figure d1 = Plot3D[(1 + 2 y x - 2 y - x y^2)/((1 + y)*(1 + x y)),
            {x, 0, 1}, {y, 0, .25}, Boxed -> False,
            Axes -> None, ViewPoint->{-1.799,2.491,1.418}]

figure d2 = Graphics3D[{Line[{{0, .15, 0}, {0,.15, 2}}],
            Line[{{0, .15, 0}, {1, .15, 0}}],
            Line[{{0, 0, 0}, {0, .25, 0}}]}, Boxed->False]


figure e1 = Plot3D[(1 + y)^3/(1 + y x), {x, 0, 1}, {y, 0, .25},
            ClipFill-> None, Axes-> False, Boxed -> False,
            ViewPoint->{-1.406,2.912,0.997}]


figure e2 = Graphics3D[{Line[{{0, .15, 0}, {0, .15, 2}}],
            Line[{{1, .15, 0}, {0, .15, 0}}],
            Line[{{0, 0, 0}, {0, .25, 0}}]}, Boxed->False]
```

```
(*We now define y(t)*)

y0 = 1/4;

x = (1 - y)/(1 - y0);

t =  (1 + y)^3/(1 - y0)*Log[Log[x]] - 3*(x*Log[Log[x]] - (1
     - y0)*LogIntegral[x]) - 3/(1 - y0)*((1 - y)^2*Log[Log[x]] -
     (1 - y0)^3*(LogIntegral[x])^2) - 1/(1 - y0)*((1 - y)^3*Log[Log[x]] -
     (1 - y0)^4*(LogIntegral[x])^3);

figure f = ParametricPlot[{t,y},{y,0, 0.065}]
```

5 Other Solutions

5.1 Introduction

In addition to the Kramer (1992) approach there are other techniques available to generate solutions for radiating stars. In this chapter we present two different methods of generating solutions to the Einstein field equations, with vanishing cosmological constant, for radiating spheres undergoing gravitational collapse. These approaches were originally proposed by Kolassis *et al* (1988) and de Oliveira *et al* (1985, 1986, 1988). Our intention is to show how these models may be constructed mathematically. A physical analysis of the models proposed here will be undertaken in the future. In §5.2 we assume that the particle trajectories of the collapsing fluid are geodesics. We provide a method of solving the junction condition for this particular model and we show that this solution corresponds to the Friedmann–like model of Kolassis *et al* (1988). Furthermore we provide an ansatz which generalises the Kolassis *et al* (1988) solution when the particle trajectories are geodesics. In section §5.3 we assume separability of the gravitational potentials into their spatial and temporal components. In this section we review the results of de Oliveira *et al* (1985, 1986, 1988). The temporal evolution of the model is determined from the junction conditions.

5.2 Friedmann–like Radiating Model

In chapter 4 we analysed a method due to Kramer (1992) to generate a model of a radiating sphere. We now review a method of generating solutions to the Einstein field equations which was first applied by Kolassis *et al* (1988). We consider a spherically symmetric, shear–free collapsing fluid configuration with the interior spacetime being described by (2.9). We make the simplifying assumption that the fluid trajectories are geodesics. It is easy to show that this requirement is equivalent to setting $A = A(t)$. Then using the freedom in the coordinate system we can choose the temporal coordinate t such that

$$A = 1$$

Then the line element (2.9) becomes

$$ds^2 = -dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

and $B = B(t,r)$ is the remaining gravitational potential.

The Einstein field equations (2.14) for this line element simplifies to

$$\mu = 3\frac{{B_t}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B_{rr}}{B} - \frac{{B_r}^2}{B^2} + \frac{4}{r}\frac{B_r}{B}\right) \tag{5.1a}$$

$$p = -2\frac{B_{tt}}{B} - \frac{{B_t}^2}{B^2} + \frac{1}{B^2}\left(\frac{{B_r}^2}{B^2} + \frac{2}{r}\frac{B_r}{B}\right) \tag{5.1b}$$

$$p = -2\frac{B_{tt}}{B} - \frac{{B_t}^2}{B^2} + \frac{1}{r}\frac{B_r}{B^3} - \frac{{B_r}^2}{B^4} + \frac{B_{rr}}{B^3} \tag{5.1c}$$

$$q = -\frac{2}{B^2}\left(-\frac{B_{rt}}{B} + \frac{B_r B_t}{B^2}\right) \tag{5.1d}$$

where we have set $A = 1$. Also the isotropy condition (2.15) reduces to

$$\left(\frac{1}{B}\right)_{rr} = \frac{1}{r}\left(\frac{1}{B}\right)_{r} \tag{5.2}$$

The junction condition (2.43b), together with (5.1b) and (5.1d), yields

$$\left\{2B\left(\frac{B_r}{B}\right)_t - \left(\frac{B_r}{B}\right)^2 - \frac{2}{r}\frac{B_r}{B} + B^2\left[2\left(\frac{B_t}{B}\right)_t + 3\left(\frac{B_t}{B}\right)^2\right]\right\}_{\Sigma} = 0 \tag{5.3}$$

A solution of the isotropy condition (5.2) and the junction condition (5.3) is required to fully describe the model.

Integrating (5.2) we obtain

$$B(r,t) = -\frac{\left(\frac{M}{6}\right)^{\frac{1}{3}} b}{C_1(t)r^2 - C_2(t)} \tag{5.4}$$

where $C_1(t)$ and $C_2(t)$ are constants of integration and b, M are constants. We have chosen the constant of integration so that comparison with the Kolassis *et al* (1988) paper is simplified. On substituting (5.4) in (5.3) we obtain

$$-4\alpha r_\Sigma[\dot{C}_1 C_2 - C_1\dot{C}_2](C_1 {r_\Sigma}^2 - C_2) + 4C_1C_2(C_1{r_\Sigma}^2 - C_2)^2$$

$$-2\alpha^2(\ddot{C}_1{r_\Sigma}^2 - \ddot{C}_2)(C_1{r_\Sigma}^2 - C_2) + 5\alpha^2(\dot{C}_1{r_\Sigma}^2 - \dot{C}_2)^2 = 0 \tag{5.5}$$

where $\alpha = \left(\frac{M}{6}\right)^{\frac{1}{3}} b$ is a constant and r_Σ is the value of r at the boundary Σ of the sphere. In the above equation dots represent differentiation with respect to the coordinate t. We remark that Kolassis *et al* (1988) have succeeded in obtaining a particular solution of (5.5) by inspection. We present an ansatz for generating their solution. The advantage of our technique of solution is that it may be generalised to generate other new solutions. A particular solution of (5.5) can be obtained in the following manner. Taking the boundary of the star to be $r_\Sigma = b$ in (5.5) we obtain

$$-4\alpha b[\dot{C}_1 C_2 - C_1\dot{C}_2](C_1 b^2 - C_2) + 4C_1C_2(C_1b^2 - C_2)^2$$

$$-2\alpha^2(\ddot{C}_1 b^2 - \ddot{C}_2)(C_1 b^2 - C_2) + 5\alpha^2(\dot{C}_1 b^2 - \dot{C}_2)^2 = 0$$

This is an ordinary differential equation for the two functions C_1 and C_2. This differential equation will have an infinite family of solutions. In particular it will be satisfied if we set

$$-4\alpha b[\dot{C}_1 C_2 - C_1\dot{C}_2](C_1 b^2 - C_2) + 4C_1C_2(C_1b^2 - C_2)^2 = 0 \tag{5.6a}$$

$$-2\alpha^2(\ddot{C}_1 b^2 - \ddot{C}_2)(C_1 b^2 - C_2) + 5\alpha^2(\dot{C}_1 b^2 - \dot{C}_2)^2 = 0 \tag{5.6b}$$

It is convenient to introduce a new variable

$$y = C_1 b^2 - C_2$$

Then (5.6b) can be written as

$$2\ddot{y}y - 5\dot{y}^2 = 0 \tag{5.7}$$

which contains the single dependent function y. Integrating this equation we obtain

$$y = \left[-\frac{3}{2}(\eta t + \psi)\right]^{-\frac{2}{3}}$$

where ψ and η are integration constants. Hence we have established the relationship between C_1 and C_2:

$$C_1 b^2 - C_2 = \left[-\frac{3}{2}(\eta t + \psi)\right]^{-\frac{2}{3}} \tag{5.8}$$

Substituting (5.8) into (5.6a) we obtain

$$\dot{C}_2 = \left\{\eta\left[-\frac{3}{2}(\eta t+\psi)\right]^{-1} - \frac{1}{\alpha b}\left[-\frac{3}{2}(\eta t+\psi)\right]^{-\frac{2}{3}}\right\} C_2$$

$$-\frac{1}{\alpha b}{C_2}^2$$

Even though this is a nonlinear equation it can be written in the linear form

$$\left(\frac{1}{C_2}\right)^{\cdot} + \left\{\eta\left[-\frac{3}{2}(\eta t+\psi)\right]^{-1} - \frac{1}{\alpha b}\left[-\frac{3}{2}(\eta t+\psi)\right]^{-\frac{2}{3}}\right\}\left(\frac{1}{C_2}\right)$$

$$-\frac{1}{\alpha b} = 0$$

The general solution is given by

$$C_2(t) = \left[-\frac{3}{2}(\eta t+\psi)\right]^{-\frac{2}{3}}\left[1 + \mathcal{C}e^{-\frac{2}{\alpha b\eta}[-\frac{3}{2}(\eta t+\psi)]^{\frac{1}{3}}}\right]^{-1}$$

where $\mathcal{C}$ is a constant of integration. On setting $\eta = -2\left(\frac{1}{3}\right)^{-\frac{1}{2}} b^{-3}$, $\psi = 0$ and $\mathcal{C} = -ab^2$ we obtain

$$C_2 = \frac{1}{3}b^2\left[\frac{1}{1 - ab^2\exp\left(\frac{6t}{M}\right)^{\frac{1}{3}}}\right]t^{-2/3} \tag{5.9}$$

Then the relationship $C_1 b^2 - C_2 = \left[-\frac{3}{2}(\eta t+\psi)\right]^{-\frac{2}{3}}$ implies that

$$C_1 = \frac{1}{3}\left[\frac{1}{1 - ab^2\exp\left(\frac{6t}{M}\right)^{\frac{1}{3}}} - 1\right]t^{-2/3} \tag{5.10}$$

Therefore the temporal behaviour of the model has been completely determined as $C_1(t)$ and $C_2(t)$ are known functions.

Hence the line element for the interior spacetime assumes the form

$$ds^2 = -dt^2 + \frac{\left(\frac{M}{6}\right)^{\frac{2}{3}} b^2}{[C_1(t)r^2 - C_2(t)]^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{5.11}$$

The functions $C_1(t)$ and $C_2(t)$ are given by (5.10) and (5.9) respectively. This solution was also found by Kolassis *et al* (1988) using a different approach. We have provided a mathematical justification for their solution. By substituting (5.4) into (5.1) we obtain expressions for the thermodynamical variables in terms of $C_1(t)$ and $C_2(t)$:

$$\mu = 3\left(\frac{\dot{C}_1 r^2 - \dot{C}_2}{C_1 r^2 - C_2}\right)^2 - \frac{12}{b^2}\left(\frac{6}{M}\right)^{\frac{2}{3}} C_1 C_2 \tag{5.12a}$$

$$p = 2\left(\frac{\dot{C}_1 r^2 - \dot{C}_2}{C_1 r^2 - C_2}\right)^{\cdot} - 3\left(\frac{\dot{C}_1 r^2 - \dot{C}_2}{C_1 r^2 - C_2}\right)^2 + \frac{4}{b^2}\left(\frac{6}{M}\right)^{\frac{2}{3}} C_1 C_2 \tag{5.12b}$$

$$q = \frac{4}{b^2}\left(\frac{6}{M}\right)^{\frac{2}{3}}\left(\dot{C}_1 C_2 - C_1 \dot{C}_2\right) r \tag{5.12c}$$

If we now set the radial heat flux equal to zero (that is $a = 0$) we obtain

$$C_1 = 0 \tag{5.13a}$$

$$C_2 = \frac{b^2}{3} t^{-2/3} \tag{5.13b}$$

$$B = \frac{3}{b}\left(\frac{M}{6}\right)^{\frac{1}{3}} t^{2/3} \tag{5.13c}$$

$$\mu = \frac{4}{3} t^{-2} \tag{5.13d}$$

$$p = 0 \tag{5.13e}$$

$$q = 0 \tag{5.13f}$$

We observe from (5.13) that the collapsing fluid is in the form of dust ($p = 0$) and its metric is the $k = 0$ Robertson–Walker model

$$ds^2 = -dt^2 + \left(\frac{3}{b}\right)^2 \left(\frac{M}{6}\right)^{\frac{2}{3}} t^{4/3} \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

This model is also called the Einstein–de Sitter model (Stephani 1990). This result follows because we required that the particle trajectories should be geodesics. When $q = 0$ the pressure p vanishes and the fluid collapses freely. On the other hand when $q \neq 0$ the pressure has to be nonzero to balance the radial heat flux to continue to allow the fluid to collapse freely. The luminosity radius of the collapsing fluid can be calculated from junction condition (2.38b) together with (5.13c) and is given by

$$(rB)_\Sigma = 3\left(\frac{M}{6}\right)^{\frac{1}{3}} t^{2/3}$$

which is independent of the constant a. Thus the luminosity radius $(rB)_\Sigma$ does not depend on the heat flow q: it has the same functional dependence whether $q = 0$ or $q \neq 0$. When $a = 0$ the junction condition (2.43a) yields

$$m(v) = M$$

which is the total gravitational mass inside Σ in which case the exterior spacetime is the exterior Schwarzschild solution. Hence in the case $a = 0$ our solution reduces to that of Oppenheimer and Snyder (1939).

We now provide a generalisation of the method presented above for integrating (5.5). This may assist in finding new solutions to describe the interior of a radiating star when $A(t) = 1$. Equation (5.5) is satisfied if we set

$$4\alpha b[\dot{C}_1 C_2 - C_1 \dot{C}_2](C_1 b^2 - C_2) + 4C_1 C_2 (C_1 b^2 - C_2)^2 = g(C_1 b^2 - C_2) \quad (5.14a)$$

$$2(\ddot{C}_1 b^2 - \ddot{C}_2)(C_1 b^2 - C_2) - 5(\dot{C}_1 b^2 - \dot{C}_2)^2 = -g(C_1 b^2 - C_2) \quad (5.14b)$$

where $g(C_1 b^2 - C_2)$ is an arbitrary function. As before we let

$$y = C_1 b^2 - C_2$$

Then (5.14b) can be rewritten as

$$2\ddot{y}y - 5\dot{y}^2 = -g(y)$$

We have obtained a differential equation with only one dependent variable, and consequently our ansatz will lead to a new solution. Integrating this equation we obtain

$$\dot{y} = \sqrt{-y^5 \int \frac{1}{y^6} g(y) dy + \zeta y^5} \quad (5.15)$$

where ζ is a constant of integration. If the function $g(y)$ is specified then (5.15) may be integrated in principle to obtain y. Then the equations (5.14) yield forms for $C_1(t)$ and $C_2(t)$. In this way we can generate a number of new solutions to the Einstein field equations by choosing particular forms for $g(y)$. Note that the case $g(y) = 0$ corresponds to the solution of Kolassis *et al* (1988). Thus we have shown how *solutions of the Einstein field equations for radiating stars may be generated when the particle trajectories are geodesics. Our ansatz regains the solution of Kolassis et al (1988) as a special case.* We do not pursue solutions to (5.14) since this is outside the scope of this chapter. It would be interesting to find those functions $g(y)$ which generate physically reasonable models.

5.3 Initial Static Configuration

In this section we briefly consider a third method of generating solutions to the Einstein field equations for radiating spheres which is different from that studied in chapter 4 and §5.2. This method has been applied by de Oliveira *et al* (1985, 1986, 1988) in relativistic astrophysics. In this approach the solution has an initial static configuration before the sphere starts gradually to collapse. We again assume a spherically symmetric, shear–free line element described by (2.9). In this approach we require separability of the gravitational potentials into their spatial and temporal components:

$$A = A_0(r) \tag{5.16a}$$

$$B = B_0(r)f(t) \tag{5.16b}$$

where $f(t)$ is a positive function of t.

Utilising (5.16) the isotropy condition (2.15) can be written as

$$\left(\frac{A_0{}'}{A_0} + \frac{B_0{}'}{B_0}\right)' - \left(\frac{A_0{}'}{A_0} + \frac{B_0{}'}{B_0}\right)^2 - \frac{1}{r}\left(\frac{A_0{}'}{A_0} + \frac{B_0{}'}{B_0}\right) + 2\left(\frac{A_0{}'}{A_0}\right)^2 = 0$$

where primes denote differentiation with respect to r. The functions $A_0(r)$ and $B_0(r)$ must satisfy this isotropy condition for a solution. If A_0 and B_0 describe a static perfect fluid then the Einstein field equations (2.14) are satisfied and the energy density μ_0 and the pressure p_0 are given by

$$\mu_0 = -\frac{1}{B_0{}^2}\left[2\left(\frac{B_0{}'}{B_0}\right)' + \left(\frac{B_0{}'}{B_0}\right)^2 + \frac{4}{r}\frac{B_0{}'}{B_0}\right] \tag{5.17a}$$

$$p_0 = \frac{1}{{B_0}^2}\left[\left(\frac{{B_0}'}{B_0}\right)^2 + 2\frac{{A_0}'}{A_0}\frac{{B_0}'}{B_0} + \frac{2}{r}\left(\frac{{A_0}'}{A_0} + \frac{{B_0}'}{B_0}\right)\right] \tag{5.17b}$$

The above static solution matches with the exterior Schwarzschild spacetime across a spherical hypersurface Σ. At this junction the pressure p_0 vanishes for some finite radius $r = r_\Sigma$:

$$(p_0)_\Sigma = 0$$

Note that the pressure p is nonzero in general because of the heat flow. It is only for the initial static configuration that $(p_0)_\Sigma = 0$. The Einstein field equations (2.14) together with (5.16) and (5.17) yield

$$\mu = \frac{1}{f^2}\left(\mu_0 + \frac{3}{{A_0}^2}\dot{f}^2\right) \tag{5.18a}$$

$$p = \frac{1}{f^2}\left(p_0 - \frac{1}{{A_0}^2}(2f\ddot{f} + \dot{f}^2)\right) \tag{5.18b}$$

$$q = -\frac{2{A_0}'}{{A_0}^2{B_0}^2}\frac{\dot{f}}{f^3} \tag{5.18c}$$

This solution describes a radiating star with an initial static configuration. The functions $A_0(r)$ and $B_0(r)$ arise from a static perfect fluid solution. The remaining function $f(t)$ is determined by the junction condition $p_\Sigma = (qB)_\Sigma$.

The junction conditions (2.38) and (2.43) become

$$(rB_0f)_\Sigma = r_\Sigma \tag{5.19a}$$

$$p_\Sigma = (qB_0f)_\Sigma \tag{5.19b}$$

$$(r(rB_0f)_r)_\Sigma = (\dot{v}(\mathrm{r} - 2m) + \dot{\mathrm{r}}\mathrm{r})_\Sigma \tag{5.19c}$$

$$m(v) = \left(\frac{r^3 {B_0}^3 f\dot{f}^2}{2{A_0}^2} - r^2 B_0' f - \frac{r^3}{2}\frac{{B_0'}^2 f}{B_0}\right)_\Sigma \tag{5.19d}$$

$$\left(\frac{dm}{dv}\right)_\Sigma = \left(-\frac{p}{2}\frac{\mathrm{r}^2}{\dot{v}^2}\right)_\Sigma \tag{5.19e}$$

$$\dot{v}_\Sigma = \left(1 + r\frac{B_0'}{B_0} + r\frac{B_0}{A_0}\dot{f}\right)^{-1}_\Sigma \tag{5.19f}$$

for the gravitational potentials given in (5.16). Using the junction condition (5.19b) together with (5.18b) and (5.18c), and taking into account $(p_0)_\Sigma = 0$, we obtain

$$2f\ddot{f} + \dot{f}^2 - 2a\dot{f} = 0 \tag{5.20}$$

which governs the behaviour of f. The constant

$$a = \left(\frac{A_0'}{B_0}\right)_\Sigma \tag{5.21}$$

is positive because the static solution (A_0, B_0) must match with the exterior Schwarzschild metric. A first integral of (5.20) is given by

$$\dot{f} = -2a\left(\frac{b}{\sqrt{f}} - 1\right) \tag{5.22}$$

where the constant of integration is $-2ab$. Note that p_Σ is nonnegative so on utilising the result $(p_0)_\Sigma = 0$, (5.18b) and (5.20) we obtain

$$\dot{f}(t) \leq 0$$

This implies that the only possible evolution of the system is contraction. Then on using (5.22) and the fact that $f(t)$ is positive we have

$$0 \leq f(t) \leq b^2$$

Integrating (5.22) we obtain

$$t = \frac{1}{a}\left[\frac{1}{2}f + b\sqrt{f} + b^2 \ln\left(1 - \frac{\sqrt{f}}{b}\right)\right] \tag{5.23}$$

where the constant of integration has been absorbed by means of rescaling the time coordinate as $t \to t + \text{constant}$.

An analysis of (5.23) shows that the function $f(t)$ decreases monotonically from its value b^2 at $t = -\infty$ to zero at $t = 0$ where a physical singularity is encountered. Physically this implies that the collapse starts at $t = -\infty$ from a static perfect fluid sphere with its interior described by the solution $(A_0, b^2 B_0)$ and whose energy density and pressure are given by (5.17) provided that the right hand side of these equations are divided by a factor of b^2. For convenience we set

$$b = 1$$

The initial mass of the static sphere can be obtained using (5.16) and (5.19d):

$$m_0 = -\left(r^2 B_0{}' + r^3 \frac{B_0{}'^2}{2B_0}\right)_\Sigma$$

where primes denote differentiation with respect to r. Its initial 'luminosity radius' is given by

$$\mathfrak{r}_0 = (rB_0)_\Sigma$$

At $t = -\infty$ the exterior spacetime is described by the vacuum Schwarzschild solution in isotropic coordinates, that is

$$ds^2 = -\left(\frac{1 - \frac{m_0}{2r}}{1 + \frac{m_0}{2r}}\right)^2 dt^2 + \left(1 + \frac{m_0}{2r}\right)^4 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

We match the static perfect fluid solution to the exterior Schwarzschild solution in isotropic coordinates. Considering the junction condition (2.24) we obtain

$$A'_{0\Sigma} = \frac{m_0}{\mathfrak{r}_0{}^2}\left(1 + \frac{m_0}{2r_\Sigma}\right)^2 \tag{5.24a}$$

$$B_{0\Sigma} = \left(1 + \frac{m_0}{2r_\Sigma}\right)^2 \tag{5.24b}$$

We can rewrite $a = \left(\frac{A_0{}'}{B_0}\right)_\Sigma$ in terms of the initial quantities m_0 and $\mathfrak{r}_0$:

$$a = \frac{m_0}{\mathfrak{r}_0{}^2}$$

Thus a is determined by the mass m_0 of the initial static configuration and the initial luminosity radius $\mathfrak{r}_0$. For a discussion of the physical properties of the matter variables and gravitational potentials see Bonnor *et al* (1989). In this section we have demonstrated that *a radiating star may be modelled by starting with an initial static configuration.*

6 Conclusion

The research presented in this thesis addresses the phenomenon of gravitational collapse with heat flow in the context of general relativity. The spacetime is spherically symmetric and the energy–momentum tensor includes heat flow. Our objective was to seek interior solutions to the Einstein field equations which match to the exterior Vaidya solution (1951, 1953). To achieve this we need to satisfy the junction conditions across a spherically symmetric hypersurface. A number of radiating models were investigated and some original results were obtained.

We now provide an overview of the main results obtained during the course of our investigations:

- The Einstein field equations, with vanishing cosmological constant, were obtained. The interior line element representing a spherically symmetric spacetime has to be matched to the Vaidya solution which represents directed radiation. The Vaidya line element was derived and we considered some of its properties.

- The junction conditions necessary for the matching of the interior and exterior spacetimes at the surface of the star were investigated in detail. We obtained an expresssion for the total energy contained within a sphere of radius r. In particular we succeeded in proving that the pressure at the surface of the radiating star is proportional to the radial heat flux which confirms the result due to Santos (1985).

- We also considered the scenario of radiative gravitational collapse with non-vanishing cosmological constant. The results obtained generalises the work of Santos (1985). The results obtained for the case of nonvanishing cosmological constant are original and to our knowledge have not been published elsewhere.

- We investigated the Kramer (1992) model in detail. In his paper Kramer (1992) failed to provide a second integral of the nonlinear differential equation governing the temporal evolution of the model. We integrated this equation in general in terms of elementary and Li functions and thereby completely specified the model's temporal evolution. Graphical plots of the thermodynamical and gravitational variables were provided and we showed that the Kramer model is a physically reasonable model for a radiating star.

- Two other different methods of obtaining solutions to the Einstein field equations were also investigated. The first method is due to Kolassis *et al* (1988) in which the particle trajectories of the collapsing fluid are assumed to be geodesics. We provided a mathematical justification for their solution. Furthermore we presented an ansatz that generalised this method of finding solutions to the Einstein field equations. In the second method we assumed separability of the gravitational potentials into spatial and temporal components and reviewed the results of de Oliveira *et al* (1985, 1986, 1988). We found that given an initial static configuration we can in principle determine the temporal evolution of the model by utilising the junction conditions.

In the above we have presented an overview of our results contained in this thesis.

We now consider possible extensions of the work presented in this thesis for future research. Kramer (1992) succeeded in obtaining a nonstatic solution from the static Schwarzschild interior solution by allowing the mass parameter to become a function of time. The isotropy condition is immediately satisfied for this method and the junction condition $p_\Sigma = (qB)_\Sigma$ gives the temporal evolution of the model. It would be interesting to utilise other existing static solutions in isotropic coordinates for the interior spacetime of the star to generate new models of radiative gravitational collapse. No specific model for a radiating star with a nonvanishing cosmological constant has been studied before; the results of chapter 3 will help generate such models. We may also investigate the problem of radiative gravitational collapse with nonvanishing cosmological constant and a spherically symmetric electromagnetic field in the future. This will lead immediately to a generalisation of the results contained in this thesis. Our ansatz for generating the Kolassis *et al* (1988) model should produce other physically reasonable models. This is the subject of future research. Another possibility is the study of gravitational collapse with heat flow and anisotropic pressure due to neutrino outbursts (Glass 1990).

We hope that we have succeeded in demonstrating that the problem of gravitational collapse with heat flow is a fruitful area of research. There are many outstanding problems pertaining to radiating stars which deserve further attention.

7 References

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