Coagulation-Fragmentation Dynamics in Size and Position Structured Population Models

By

SUARES CLOVIS OUKOUOMI NOUTCHIE

SUBMITTED IN FULFILLMENT OF THE ACADEMIC
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN THE
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF KWAZULU-NATAL
DURBAN

DECEMBRE 2008

PREFACE

The work described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, from August 2006 to October 2008, under the supervision of Professor Jacek Banasiak.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Abstract

One of the most interesting features of fragmentation models is a possibility to breach the mass conservation principle through 'shattering'; that is, the formation of a dust of 'zero-size' particles. A similar phenomenon may occur in the evolution of the number of particles in the system which, to some extent, is intertwined with the evolution of its total mass. To investigate these phenomena we consider the fragmentation equation in the space of densities yielding both a finite number of particles and a finite mass of the ensemble, and show, in particular, that in a non-shattering fragmentation typically one can control the total number of particles in the system. On the other hand, in the shattering fragmentation both mass and particles can disappear from the system. A possible explanation of such cases is that the fragmentation equation alone does not offer the full description of the dynamics of the problem.

We extend existing results on coagulation-fragmentation to some models with fragmentation rate unbounded at 0 and growing faster than the size x at infinity. We investigate the dynamical behavior of coagulation - multiple fragmentation processes in biological populations. In particular, we prove existence and uniqueness of conservative solutions for general $b(x|y)$, (where $b(x|y)$ is a non-negative measurable function describing the distribution of particles of size x spawned by the fragmentation of a particle of size y) and arbitrary fragmentation rate a.

A kinetic-type nonlinear integro-differential equation describing the evolution of aggregates of phytoplankton is analyzed. For single cells growth rate ($b(x_0) > 0$), the McKendrick-von Foerster renewal boundary condition is prescribed to incorporate the effects of cell division. We make use of substochastic semigroup perturbations techniques and semilinear abstract Cauchy problems theory to determine the existence of a strong solution to the evolution equation. In particular, we provide sufficient conditions for honesty of the model.

An initial-value problem describing multiple fragmentation processes, where the fragmentation rate is size-position dependent and new particles are spatially randomly distributed according to some probability density is investigated by means of substochastic semigroup theory and approximation techniques. The existence of a semigroup is established and, under natural conditions on certain coefficients, the generator of this semigroup is identified. In particular we prove the existence and uniqueness of nonnegative mass-conserving solutions and provide sufficient conditions for honesty.

DECLARATION 1 - PLAGIARISM

I, Suares Clovis OUKOUOMI NOUTCHIE declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.
2. This thesis has not been submitted for any degree or examination at any other university.
3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.
4. This thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:
    - Their words have been re-written but the general information attributed to them has been referenced.
    - Where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.
5. This thesis does not contain text, graphics or tables copied and pasted from the internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed: ....

DECLARATION 2 - PUBLICATIONS

Paper submitted

— Controlling number of particles in fragmentation equations

Papers in preparation

— A theoretical approach for phytoplankton dynamics

— Nonlocal continuous fragmentation processes

— Coagulation-Fragmentation models

— Honesty in nonlocal fragmentation models

Signed: ....

I dedicate this dissertation to my beloved mother

FLORENTINE KOUAMO OUKOUOMI

and to the memory of my father

SAMUEL OUKOUOMI.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor Professor Jacek Banasiak for his guidance and encouragements. Apart from his excellent supervision, his moral and financial support made the task enjoyable and rewarding.

I also thank Professor R. Rudnicki and Doctor W. Lamb for their time and remarks. The warm hospitality received from Prof Rudnicki during my stay in Poland was greatly appreciated. The helpful discussions with Dr Lamb deepened my interest in fragmentation-coagulation processes.

Thanks to my mother who has always been very supportive.

I am grateful to my beloved wife Christelle Oukouomi Noutchie, my son Peres Oukouomi and my daughter Leticia Oukouomi for their love.

Thanks to my brothers Yves Oukouomi, William Oukouomi and my sisters Judith Oukouomi, Christelle Oukouomi, Ghislene Oukouomi for their support and prayers.

Thanks to my friends Amartya Goswami, Stella Mugisha, Sunday Faleye, Ludovic Tomgwo, Jules Djoko Kamdem, Riette Maritz, Justin Munganga, Jules-Raymond Tapamo, Franck Emile Ngoufo and Ferdinand Kamga for their precious assistance.

The financial support received from NRF (National Research Foundation) is deeply acknowledged.

Finally, I extend my gratitude to the staff of the Department of Mathematical Sciences of University of KwaZulu-Natal for offering me Academic Support Tutor Position and for the kindly environment in which we have been working.

Contents

Chapter 1

Introduction

In many ramifications of applied sciences, the evolution of a dynamical system is described by a concentration function $(t, \xi) \rightarrow u(t, \xi)$, where t is the time and ξ is an element of some state space Ω which identifies an individual uniquely. The function u is then interpreted as the probability (density) of finding an individual which at the time t enjoys the property ξ. An intrinsic property of the dynamical process is that all the particles must be accounted for or, in other words:

$$\int_{\Omega} u(t, \xi) d\mu_{\xi} = \int_{\Omega} u(0, \xi) d\mu_{\xi}, \tag{1.1}$$

for any time t, where $d\mu_{\xi}$ is an appropriate measure in the state space. Therefore from the physical point of view, the natural spaces for studying such problems are L_1 spaces. In fragmentation-coagulation theory, ξ could be for example the mass of a particle, its spatial location or a combination of the above. In this chapter we introduce several mathematical models that we intend to examine via the theory of semigroups of linear operators and the theory of evolution systems.

1.1 Pure fragmentation

Fragmentation processes can be observed in natural sciences and engineering. To provide just a few examples we mention the study of stellar fragments in astrophysics, rock fracture, degradation of large polymer chains, DNA fragmentation, evolution of phytoplankton aggregates, liquid droplet breakup or breakup of solid drugs in organisms. Though mathematical study of fragmentation processes can be traced back to papers by Melzak [44] (from the analytical point of view) and Filippov [33] (from the probabilistic one), it was not until the 1980s that a systematic investigation of them was undertaken, mainly by Ziff and his students, e.g. [55, 56], who provided explicit solutions to a large class of fragmentation equations of the form

$$\frac{\partial}{\partial t} u(t, x) = -a(x) u(t, x) + \int_{x}^{\infty} a(y) b(x|y) u(t, y) dy, \quad x \geq 0, t > 0, \tag{1.2}$$

with power law fragmentation rates $a(x) = x^\alpha, \alpha \in \mathbb{R}$ and where $b(x|y)$, the distribution of particle masses x spawned by the fragmentation of a particle of mass $y > x$, also was given by a power law

$$b(x|y) = (\nu + 2)\frac{x^\nu}{y^{\nu+1}}, \tag{1.3}$$

with $\nu \in (-2, 0]$ (see also [41] for a more detailed discussion of this case). Here $u(t, x)$ is the density of particles having mass x at time t.

Later a comprehensive probabilistic theory of fragmentation processes was developed by Bertoin and Haas, see e.g. [22, 23, 24, 35, 36], while a development of functional-analytic methods and, in particular, of the semigroup theory, helped to put many earlier phenomenological results on a firm mathematical ground, see e.g. [16, 10, 11, 15, 19, 26, 43].

Fragmentation processes are difficult to analyze as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it. That is why, though linear, they display nonlinear features such as phase transition which, in this case, is called shattering and consists in the formation of a 'dust' of particles of zero size carrying, nevertheless, a non-zero mass. Quantitatively we can identify this process by disappearance of mass from the system even though it is conserved in each fragmentation event. Probabilistically, shattering is an example of an explosive, or dishonest Markov process, see e.g. [6, 45] and from this point of view it has been exhaustively analyzed in [22, 23, 24, 33, 35, 36, 54]. In natural sciences shattering was rediscovered in [57] where the loss of mass was noticed by analyzing explicit solutions of fragmentation equations with power-law fragmentation rates. In a general case shattering was explained analytically in [16, 13, 10, 11] by linking it to the characterization of the generator of the dynamical system associated with the fragmentation process; these results were compared with the probabilistic approach in [15].

If u is a solution to (1.2), the total mass of the ensemble at a time t is given by the first moment of u; that is, $M(t) = \int_0^\infty xu(t, x)dx$. From the physical point of view the total mass of fragmenting particles cannot increase, thus fragmentation equations are usually investigated in the space

$$X_1 := L_1(\mathbb{R}_+, xdx) = \left\{u;\ \int_0^\infty |u(x)|xdx < +\infty\right\}. \tag{1.4}$$

The reason for this is that the process in this space should be dissipative which typically results in simpler analysis. However, as we mentioned earlier, fragmentation events result in an increase of number of particles in the system, which is not tracked by the norm in X_1. Apart from an inherent interest in knowing how the number of particles evolves, there is also a practical angle to this question: fragmentation events are often coupled with, in some sense reverse to, coagulation processes which are most easily analyzed in the finite number of particles space:

$$X_0 := L_1(\mathbb{R}_+, dx) = \left\{u;\ \int_0^\infty |u(x)|dx < +\infty\right\}. \tag{1.5}$$

Hence, analysis of the combined fragmentation-coagulation equation requires well posedness of the fragmentation equation in

$$X_{0,1} := L_1(\mathbb{R}_+, (1+x)dx) = \left\{u;\ \int_0^\infty |u(x)|(1+x)dx < +\infty\right\}. \tag{1.6}$$

Nonlocal fragmentation models are investigated in detail in Chapter 6. A special emphasis is placed on the honesty of these models. We recover some fundamental properties from local fragmentation models.

1.2 Coagulation fragmentation equations (CFE)

Coagulation-fragmentation processes describe the evolution of systems in which particles react in either fusing together or breaking apart. The first pure coagulation equation was derived in the early part of the twentieth century by Smoluchowski [52, 53] who applied the theory of Brownian motion to the problem of the collision of hard, non-interacting spheres which are thermally agitated in a continuum. The problem was considered as a diffusion process and the approach resulted in a discrete model involving an infinite set of non-linear differential equations. In the late 1920s, Muller extended the results of Smoluchowski by considering a continuous mass density function. As a result, this was probably the first instance in which the pure coagulation was considered as a continuous problem and modelled as an integro-differential equation.

The fragmentation equation was introduced into the models of evolving systems in the 1950s. The coagulation-fragmentation equation was first derived by Melzak [44] in 1957. The equation was formulated in such a way as to ensure that mass was a conserved quantity. The equation had the form

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) = &- u(t,x)\int_0^x \frac{y}{x}\gamma(x,y)dy + \int_x^\infty \gamma(y,x)u(t,y)dy \\ &+ \frac{1}{2}\int_0^x k(x-y,y)u(t,x-y)u(t,y)dy, \\ &- u(t,x)\int_0^\infty k(x,y)u(t,y)dy,\end{aligned} \tag{1.7}$$

where $u(t,x)$ represented the density of particles of mass x at time t. We recall that in the continuous version it is assumed that the number of particles is large enough to justify the use of a density function $u(t,x)$. The product $u(t,x)dx$ is then the average number of particles with mass in the interval $(x, x+dx)$ at time t. The fragmentation kernel, $\gamma(x, y)$, describes the rate at which particles of mass y are produced from the fragmentation of particles of size x and the coagulation kernel, $k(x, y)$, describes the rate at which particles of mass x coalesce with particles of mass y. The fragmentation kernel γ, introduced above, is often referred to as the *multiple* fragmentation kernel as this model allows for particles to split into many pieces at each fragmentation process. In his work, Melzak assumed that

(i) $k(x, y)$ is a continuous, symmetric, non-negative, bounded function;

(ii) $\gamma(x, y)$ is a continuous, non-negative, bounded function.

Furthermore, solutions $u(t, x)$ to the coagulation-fragmentation equation were sought in the form

$$u(t, x) = \sum_{n=0}^{\infty} a_n(x) t^n, \tag{1.8}$$

for some sequence of functions $a_n, \quad n = 0, 1, \ldots.$ Under these assumptions, global existence and uniqueness of continuous, non-negative, bounded solutions to (1.7) were established. Melzak also obtained results on the solution of the coagulation fragmentation equation for the case in which the kernels, γ and k, are time-dependent. Various results on the existence, uniqueness and asymptotic behavior of solutions have been established under appropriate hypothesis on the kernels and many distinct approaches have been used to obtain them. The case where the fragmentation and coagulation kernels are both constant has been analyzed via semigroup techniques by Aizenman and Bak [5]. Asymptotic analysis of coagulation-fragmentation equations may be found in [29], [39] and [51]. In [39], Philippe Laurençot investigated a model for the dynamics of a system of particles undergoing simultaneously coalescence and break-up. The equation describing his model was similar to (1.7). He showed existence of solutions to the corresponding evolution integral partial differential equation for product-type coagulation kernels with a weak fragmentation. Further information on the development of the coagulation-fragmentation equation may be found in the comprehensive review article by Drake [28]. Although fragmentation equations are often studied in a form involving a single multiple fragmentation kernel, it is also possible to write the fragmentation operator in terms of rate functions. We define the rate functions a and b, via

$$a(x) := \int_0^x \frac{y}{x} \gamma(x, y) dy \tag{1.9}$$

and

$$b(x|y) := \frac{\gamma(y, x)}{a(y)} \tag{1.10}$$

respectively where (1.9) describes the overall rate of break-up of an x-particle and (1.10) denotes the distribution of particles of size x formed during the break-up of larger particles of size y. This formulation coincides with the fragmentation problem derived by McGrady and Ziff [55, 56, 57] and will be used in the thesis. Note that $b(x|y) = 0$ for $y < x$ as it is not physically possible for a solid of size $x > y$ to be produced during the break-up of a y-sized solid.

1.3 Phytoplankton aggregates

Phytoplankton is a generic name for a great variety of micro-organisms (algae) that live in the ocean and in lakes. Phytoplankton populations are large contributors to the production in the ocean. They are, in particular, the main food available to the early larval stages of many fish species, including the anchovy. An important observation is that phytoplankton cells tend to form aggregates; that is, groups of cells living together. In phytoplankton dynamics, a system of particles called TEP (Transparent Exopolymer Particles) play a major role. They are a by-product of the growth of phytoplankton and their stickiness causes that cells will remain together upon contact [27, 47]. On the other hand, the low level of concentration of TEP results in fragmentation of the aggregate due to external causes, like currents or turbulence on one hand, and internal unspecified forces of biotic nature on the other. The distribution of aggregates can be studied at different levels. Individual-based models, which can be thought of as providing 'microscopic' properties, track the random motion and division of individual particles [50]. A 'macroscopic' description known to ecologists by advection-diffusion -reaction equations works with approximations of densities by empirical concentrations of particles [40] and is heavily used in simulations [4]. The model which we study in this thesis was considered by O. Arino and R. Rudnicki in [14]. It can be looked at as lying somewhere in between, on a 'mesoscopic' scale, in that it describes the role played by the phytoplankton aggregates of cells which are treated as individual building blocks of the system. The aggregates are structured by size and the phytoplankton consists of aggregates of all possible sizes. The aggregate size can change due to splitting, death, growth or combining of aggregates into bigger ones. To include the effects of cell division, we incorporate the McKendrick-von Foerster renewal condition. The resulting model consists of a partial differential equation with two integral terms responsible for the fragmentation and coagulation processes, the McKendrick-von Foerster renewal boundary condition and the initial condition.

1.4 Outline of thesis

The purpose of this thesis is to develop and expand existing results on various problems related to the coagulation-fragmentation equation. This work has been carried out using techniques from functional analysis and the theory of semigroups of operators. In spite of the fact that many of the methods which we apply are relatively well known, our analysis often required some possibly less familiar results. Hence, in Chapter 2 a discussion of these subsidiary results is given.

The aim of Chapter 3 is to examine (1.2) in the space $X_{0,1}$ for arbitrary fragmentation rate a and a class of separable $b(x|y)$ which is more general than (1.3). Of particular interest is honesty of the process in $X_{0,1}$; that is, whether the evolution of the mass and number of particles, given by the solution u to (1.2), coincides with the one predicted by the local laws (3.6) and (3.7), used to construct (1.2). One of the main results is that (1.2) is well-posed and honest in $X_{0,1}$ for fragmentation rate a bounded at 0 and with b yielding a finite number of daughter particles at 0. On the other hand, shattering fragmentation (corresponding, roughly speaking, to a unbounded at 0) is associated with an accelerating infinite cascade of fragmentation events of smaller and smaller particles leading to the creation of dust. Hence, intuitively, in shattering fragmentation we should observe the appearance of an infinite number of particles. We shall demonstrate that this intuition is not necessarily correct. Another counterintuitive result observed in this chapter is related to the case when the number of daughter particles produced in each fragmentation event is infinite and, at the same time, the fragmentation is strongly shattering (e.g. if $\nu \in (-2, 1]$ and $\alpha < -1$ in the power law case). Despite this, we observe that we still have evolution in $X_{0,1}$ (at least for a class of initial densities).

Under the assumption that the fragmentation rate is linearly bounded, an existence result for the fragmentation-coagulation model was obtained by Lamb, McBride and McLaughlin in their paper [42]. In Chapter 4, we make use of the analysis performed in Chapter 3 for a separable kernel $b(x|y) = \beta(x)\gamma(y)$ to expand this result to general fragmentation rates. In the real world there is always a lowest size of objects beyond which a particle cannot reach without encountering quantum effects. As a result, it is realistic to assume that in some populations the individual size is in the range (x_0, ∞) for some $x_0 > 0$. With this assumption, the work done in Chapter 3 for the separable kernel $b(x|y)$ can be extended to arbitrary $b(x|y)$ including $b(x|y) = y^{-1}h(x/y)$ for some suitable function h. In particular, we prove existence and uniqueness of conservative solutions for the coagulation-fragmentation equation.

In Chapter 5, we present a theoretical model for phytoplankton dynamics. It consists of a kinetic-type nonlinear integro-differential equation with two integral terms responsible for the 'multiple' fragmentation and coagulation processes, the McKendrick-von Foerster renewal boundary condition and the initial condition. In the derivation of the model, a new coagulation kernel is formulated. We make use of substochastic semigroup methods, perturbations techniques and semilinear abstract Cauchy problems theory to show the existence of a strong solution to the evolution equation. In particular, we provide sufficient conditions for honesty of the model.

Chapter 6 revolves around nonlocal fragmentation models. The work contained in this chapter is noteworthy as no previous results are known to exist for 'honesty' in nonlocal fragmentation models. Newborn particles are spatially randomly distributed according to some probability density function and the fragmentation rate depends on the size of the particle and its spatial location. The existence of a semigroup is established and, under natural conditions on the kernels, the generator of this semigroup is identified. In particular we prove the existence and uniqueness of nonnegative mass-conserving solutions and provide sufficient conditions for honesty.

Chapter 2

Preliminary and Auxiliary Results

In this chapter, we gather results, definitions and theorems which will be used in our later analysis. For much of this thesis we shall apply techniques from the calculus of vector-valued functions and so we begin by giving a brief introduction to some functional analysis concepts which will be used in subsequent chapters.

2.1 Calculus of vector-valued functions and Banach lattices

2.1.1 Spaces and opeartors

Definition 2.1.1.

A vector-valued function u from some abstract set I to a Banach space X is a mapping $t \to u(t)$ from I into X, where to each point $t \in I$ there corresponds a unique vector $u(t) \in X$.

In the case where the Banach space is the space of bounded linear operators from X into Y, denoted by $\mathcal{L}(X, Y)$ with norm $\|\cdot\|_{\mathcal{L}(X,Y)}$, we call the function an operator valued function. (When $X = Y$ we write $\mathcal{L}(X)$ with norm $\|\cdot\|_{\mathcal{L}(X)}$.)

Definition 2.1.2. (**Strong Derivative**)

Let X be a Banach space with norm $\|\cdot\|_X$ and let the function u be an X-valued function of $t \in [0, \infty)$. Then the strong derivative $\dfrac{du(t)}{dt}$ of u at $t > 0$ is defined to be an element $\overline{u}(t)$ such that

$$\lim_{h \to 0} \|h^{-1}[u(t+h) - u(t)] - \overline{u}(t)\|_X = 0 \tag{2.1}$$

provided that the limit exists.

Definition 2.1.3.

Let Π denote any partition $a = t_0 < t_1 < t_2 ... < t_n = b$ of the closed interval $[a, b]$ together with the arbitrary points $s_k \in [t_{\varsigma-1}, t_\varsigma]$, $\varsigma = 1, 2, ..., n$ and let the norm $|\Pi| = \max_\varsigma (t_\varsigma - t_{\varsigma-1})$. If, for a vector-valued function $u : [a, b] \to X$, there exists $v \in X$ (independently of the manner in which $|\Pi| \to 0^+$) such that

$$\lim_{|\Pi| \to 0^+} \left\| \sum_{\varsigma=1}^{n} u(s_\varsigma)(t_\varsigma - t_{\varsigma-1}) - v \right\|_X = 0,$$

then v is the strong Riemann integral and is denoted by

$$\int_a^b u(t)dt.$$

Theorem 2.1.4.

If u is a strongly continuous vector-valued function on $[a, b]$ to X, then the strong Riemann integral over $[a, b]$ exists. Moreover, if $A : X \supseteq D(A) \to Y$ is a closed linear operator, $u(t) \in D(A)$ for each $t \in [a, b]$ and if Au is strongly continuous on $[a, b]$, then

$$A \left[\int_a^b u(t)dt \right] = \int_a^b [Au](t)dt.$$

Proof. [34, Theorem 3.3.2]. □

Definition 2.1.5.

A Banach space X is of type L if it consists of equivalence classes of numerically-valued functions defined on a set Ω and if it has the following two properties:

(i) If u is a continuous X-valued function defined on $I = [\alpha, \beta]$, then there exists a function ψ measurable on the product $I \times \Omega$ such that $u(t) = \phi(t, \cdot)$ for each $t \in [\alpha, \beta]$. Note $u(t) = \psi(t, \cdot)$ means equality in X.

(ii) If u is continuous on $I = [\alpha, \beta]$ and ψ is any function that is measurable on $I \times \Omega$ and satisfies $u(t) = \psi(t, \cdot)$ for each $t \in [\alpha, \beta]$, then

$$\left[\int_\alpha^\beta u(t)dt \right] (\cdot) = \int_\alpha^\beta \psi(t, \cdot)dt, \tag{2.2}$$

where the integral on the left-hand side is the abstract Riemann integral and the integral on the right-hand side is the Lebesgue integral of numerically-valued functions.

Theorem 2.1.6. *Any space $L_p(\Omega)$, $1 \leq p < \infty$ is of type L.*

Proof. [16, Theorem 2.39]. □

Theorem 2.1.7.

Let X be a Banach space of type L. If u is a vector-valued function on $I = [a, b]$ to X and if u is n-times continuously strongly differentiable, then there exists a numerically-valued function v measurable on $I \times \Omega$ such that

(i) for $0 \leq s \leq n-1$, $\frac{\partial^s}{\partial t^s} v(t,x)$ is absolutely continuous for each $x \in \Omega$ and

$$\frac{\partial^s}{\partial t^s} v(t, \cdot) = \left[\frac{d^s}{dt^s} u(t)\right](\cdot)$$

for each $t \in I$;

(ii) $\frac{\partial^n}{\partial t^n} v(t,x)$ exists almost everywhere in $I \times \Omega$ and

$$\frac{\partial^n}{\partial t^n} v(t, \cdot) = \left[\frac{d^n}{dt^n} u(t)\right](\cdot)$$

for almost all $t \in I$.

Proof. See [34, Theorem 3.4.2]. □

Note that in case the Banach space X is a space of numerically-valued functions defined on some abstract set Ω, the relation between the differential equation $\frac{d}{dt} u(t) = g(t, u(t))$ (in strong sense) and the partial differential equation $\frac{\partial}{\partial t} u(t,x) = g(t, u(t,x))$ depends on the nature of X.

Theorem 2.1.8.

Let $\{\psi_n\}$ be a Cauchy sequence in $L_p(\Omega)$ that converges strongly to ψ. Then there exists a subsequence $\{\psi_{n_\varsigma}\}$ that converges pointwise almost everywhere on Ω to a limit function ψ.

Proof. See [48, Corollary 5.11]. □

Theorem 2.1.9.

Let $\{\psi_n\}$ be a sequence of Lebesgue-integrable functions over $\Omega \subseteq \mathbb{R}^n$ such that

(i) $\{\psi_n\}$ increases almost everywhere on Ω;

(ii) $\lim_{n\to\infty} \int_\Omega \psi_n(x)dx$ exists.

Then $\{\psi_n\}$ converges almost everywhere to a limit function $\psi \in L_1(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega \psi_n(x)dx = \int_\Omega \psi(x)dx.$$

Proof. See [7, Theorem 10.24]. □

2.1.2 Banach lattices and positive operators

Definition 2.1.10.

Let X be an arbitrary set. A partial order (or simply, an order) on X is a binary relation, denoted here by '$\geq$', which is reflexive, transitive, and antisymmetric, that is,

(1) $x \geq x$ for each $x \in X$;

(2) $x \geq y$ and $y \geq x$ imply $x = y$ for any $x, y \in X$;

(3) $x \geq y$ and $y \geq z$ imply $x \geq z$ for any $x, y, z \in X$.

Definition 2.1.11.

An ordered vector space is a vector space X equipped with partial order which is compatible with its vector structure in the sense that

(4) $x \geq y$ implies $x + z \geq y + z$ for all $x, y, z \in X$;

(5) $x \geq y$ implies $\alpha x \geq \alpha y$ for any $x, y \in X$ and $\alpha \geq 0$.

The set $X_+ = \{x \in X; \quad x \geq 0\}$ is referred to as the positive cone of X.

We say that X is a lattice if every pair of elements (and so every finite collection of them) has both supremum and infimum.

If an ordered vector space X is also a lattice, then it is called a vector lattice or a Riesz space. Typical examples of Riesz spaces are provided by spaces of functions. If X is a vector space of real-valued functions on a set Ω, then we can introduce a pointwise order in X by saying that $f \leq g$ in X if $f(x) \leq g(x)$ for any $x \in S$. Equipped with such an order, X becomes an ordered vector space. We recall that if Ω is a measure space, then all above considerations are valid when the pointwise order is replaced by $f \leq g$ if $f(x) \leq g(x)$ almost everywhere. With this understanding, $L_0(\Omega)$ and $L_p(\Omega)$ spaces with $1 \leq p \leq \infty$ become function spaces and are thus Riesz spaces.

For an element x in a Riesz space X we can define its positive and negative part, and its absolute value, respectively, by

$$x_+ = \sup\{x, 0\}, \quad x_- = \sup\{-x, 0\}, \quad |x| = \sup\{x, -x\}.$$

Proposition 2.1.12.

If x is an element of a Riesz space, then

$$x = x_+ - x_-, \quad |x| = x_+ + x_-$$

Thus, in particular, the positive cone in a Riesz space is generating.

Proof. See [16, Proposition 2.46]. □

In the next step, we investigate the relation between the lattice structure and the norm when X is both a normed and an ordered vector space.

Definition 2.1.13.

A norm on a vector lattice X is called a lattice norm if

$$|x| \leq |y| \quad \textit{implies} \quad \|x\| \leq \|y\|. \tag{2.3}$$

A Riesz space X complete under a lattice norm is called a Banach lattice.

Property (2.3) gives the important identity:

$$\|x\| = \||x|\|, \quad x \in X. \tag{2.4}$$

Proposition 2.1.14.

If X is a normed lattice, then all lattice operations are uniformly continuous in the norm of X with respect to all variables involved.

Proof. [16, Proposition 2.55]. □

2.1.3 Positive operators

Definition 2.1.15.

A linear operator A from a Banach lattice X into a Banach lattice Y is called positive, denoted by $A \geq 0$, if $Ax \geq 0$ for any $x \geq 0$.

Positive operators are fully determined by their behaviour on the positive cone. Precisely speaking, we have the following theorem.

Theorem 2.1.16.

If $A : X_+ \to Y_+$ is additive, then A extends uniquely to a positive linear operator from X to Y. Keeping the notation A for the extension, we have, for each $x \in X$,

$$Ax = Ax_+ - Ax_-. \tag{2.5}$$

Proof. [16, Theorem 2.64] □

We point out here an easy and often used result on positive operators.

Proposition 2.1.17.

If A is positive, then

$$\|A\| = \sup_{x \geq 0,\, \|x\| \leq 1} \|Ax\|. \tag{2.6}$$

Proof. [16, Theorem 2.67] □

Definition 2.1.18.

We say that a Banach lattice X is a KB-space (Kantorovic Banach space) if every increasing norm bounded sequence of elements of X_+ converges in norm in X.

The next theorem characterizes the KB-spaces and is very useful in applications.

Theorem 2.1.19.

Assume that X is a weakly sequentially complete Banach lattice. If $(x_n)_{n\in\mathbb{N}}$ is increasing and $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded, then there is $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \tag{2.7}$$

in X. In other words, weakly sequentially complete and, in particular, reflexive Banach lattices are KB-spaces.

Proof. [16, Theorem 2.82]. □

2.2 Linear semigroups

In this section we deal with methods of finding solutions of a Cauchy problem.

Definition 2.2.1.

Given a Banach space X and a linear operator $\mathcal{A}$ with domain $D(\mathcal{A})$ and range $Im\mathcal{A}$ contained in X and also given an element $u_0 \in X$, find a function $u(t) = u(t, u_0)$ such that

(1) $u(t)$ *is continuous on* $[0, \infty)$ *and continuously differentiable on* $(0, \infty)$*,*

(2) *for each* $t > 0$, $u(t) \in D(\mathcal{A})$ *and*

$$\frac{du}{dt}(t) = \mathcal{A}u(t), \quad t > 0, \tag{2.8}$$

(3)

$$\lim_{t\to 0} u(t) = u_0 \tag{2.9}$$

in the norm of X.

A function satisfying all conditions above is called the classical (or strict) solution of (2.8), (2.9).

Definition 2.2.2.

A family $(S(t))_{t\geq 0}$ of bounded linear operators on X is called a C_0-semigroup, or a strongly continuous semigroup, if

(i) $S(0) = I$*;*

(ii) $S(t+s) = S(t)S(s)$ *for all* $t, s \geq 0$*;*

(iii) $\lim_{t\to 0^+} S(t)x = x$ *for any* $x \in X$.

A linear operator A *is called the (infinitesimal) generator of* $(S(t))_{t\geq 0}$ *if*

$$Ax = \lim_{h\to 0^+} \frac{S(h)x - x}{h}, \tag{2.10}$$

where the domain of A, $D(A)$, *is defined as the set of all* $x \in X$ *for which this limit exists. Typically the semigroup generated by* A *is denoted by* $(S_A(t))_{t\geq 0}$.

Note that if A is the generator of $(S(t))_{t\geq 0}$, then for $x \in D(A)$ the function $t \to S(t)x$ is a classical solution of the following Cauchy problem,

$$\begin{aligned} \frac{du}{dt}(t) &= A(u(t)) \\ \lim_{t\to 0^+} u(t) &= x \end{aligned} \qquad t \geq 0 \tag{2.11}$$

For $x \in X \setminus D(A)$, however, the function $u(t) = S(t)x$ is continuous but, in general, not differentiable, nor $D(A)$-valued, and, therefore, not a classical solution. Nevertheless, the integral $v(t) = \int_0^t u(s)ds \in D(A)$ and it is a strict solution of the integrated version of (2.11):

$$\begin{aligned} \frac{dv}{dt}(t) &= A(v(t)) + x \\ \lim_{t\to 0^+} v(t) &= 0 \end{aligned} \qquad t \geq 0 \tag{2.12}$$

or equivalently,

$$u(t) = A\int_0^t u(s)ds + x. \tag{2.13}$$

We say that a function u satisfying (2.12) (or, equivalently, (2.13)) is a mild solution or integral solution of (2.11).

Proposition 2.2.3.

Let $(S(t))_{t\geq 0}$ *be the semigroup generated by* $(A, D(A))$. *Then* $t \to S(t)x, \quad x \in D(A)$, *is the only solution of* (2.11) *taking values in* $D(A)$. *Similarly, for* $x \in X$, *the function* $t \to S(t)x$ *is the only mild solution to* (2.11).

Proof. [16, Proposition 3.4] □

Thus, if we have a semigroup, we can identify the Cauchy problem of which it is a solution. Usually, however, we are interested in the reverse question, that is, in finding the semigroup for a given equation. The answer is given by the Hille-Yosida theorem.

Theorem 2.2.4. (Hille-Yosida Theorem)

$A \in \mathcal{G}(M,\omega)$ if and only if

(a) A is closed and densely defined,

(b) there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega,\infty) \in \rho(A)$ and for all $n \geq 1$, $\lambda > \omega$,

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda-\omega)^n}. \tag{2.14}$$

where $\rho(A)$ is the resolvent set of the operator A and is defined as follow:

$$\rho(A) = \{\lambda \in \mathbb{R}; \quad \lambda I - A : D(A) \to X \text{ is invertible and } \quad (\lambda I - A)^{-1} \in \mathcal{L}(X)\}. \tag{2.15}$$

Proof. [16, Theorem 3.5] □

Theorem 2.2.5.

Assume that the closure $(\overline{A}, D(\overline{A}))$ of an operator $(A, D(A))$ generates a C_0-semigroup in X. If $(B, D(B))$ is also a generator, such that $B|_{D(A)} = A$, then $(B, D(B)) = (\overline{A}, D(\overline{A}))$.

Proof. [16, Proposition 3.8] □

The Lumer-Phillips Theorem gives an alternative characterization of the infinitesimal generator of a C_0-semigroup of contractions. Before stating the theorem we define the term dissipative.

Definition 2.2.6.

Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in X. The operator $\mathcal{A}$ is dissipative if $\|(\lambda I - \mathcal{A})\psi\|_X \geq \lambda\|\psi\|_X$ for all $\psi \in D(\mathcal{A})$ and $\lambda > 0$.

Theorem 2.2.7. (Lumer-Phillips)

Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in X.

(i) If $\mathcal{A}$ is dissipative and if there exists $\lambda_0 \in \mathbb{C}$ such that the range $Im(\lambda_0 I - \mathcal{A})$ of $\lambda_0 I - \mathcal{A}$ is X, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X.

(ii) If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X then $\mathcal{A}$ is dissipative and for all $\lambda > 0$, $Im(\lambda I - \mathcal{A}) = X$.

Proof. [46, Theorem 4.3, p14]. □

It is not always necessary to know the infinitesimal generator on its whole domain.

Definition 2.2.8. *Let A be a closed operator in a Banach space X. A core of A is a dense subspace D of X such that A is the closure of its restriction to D i.e. $\overline{A|_D} = A$.*

Theorem 2.2.9. (Core) *Let A be the generator of the semigroup $(S_A(t))_{t\geq 0}$ on a Banach space X and let D be a dense set contained within the domain of A, i.e. $D \subset D(A)$. If the set D is invariant under the semigroup $(S_A(t))_{t\geq 0}$, then D is a core for A.*

Proof. [41, Theorem 2.1.1]. □

Next we consider a case of restrictions of $(S(t))_{t\geq 0}$, acting in a Banach space X, to a subspace Y which is continuously embedded in X and which is invariant under $(S(t))_{t\geq 0}$. The restriction $(S_Y(t))_{t\geq 0}$ of $(S(t))_{t\geq 0}$ to Y is obviously a semigroup but not necessarily a C_0-semigroup. If, however, it is strongly continuous, then we can identify the generator of $(S_Y(t))_{t\geq 0}$ as the part in Y of the generator A of $(S(t))_{t\geq 0}$.

Proposition 2.2.10.

Let $(A, D(A))$ generate a C_0-semigroup $(S(t))_{t\geq 0}$ in a Banach space X and let Y be a subspace continuously embedded in X, invariant under $(S(t))_{t\geq 0}$. If the restricted semigroup $(S_Y(t))_{t\geq 0}$ is strongly continuous in Y then its generator is the part A_Y of A in Y . Moreover, if Y is closed in X, then $(S_Y(t))_{t\geq 0}$ is automatically strongly continuous and A_Y is the restriction of A to the domain $D(A) \cap Y$.

Proof. [16, Proposition 3.12] □

Next we consider resolvent positive operators.

Definition 2.2.11.

Let X be a Banach lattice. We say that the semigroup $(S(t))_{t\geq 0}$ on X is positive if for any $x \in X_+$ and $t \geq 0$,

$$S(t)x \geq 0.$$

We say that an operator $(A, D(A))$ is resolvent positive if there is ω such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega$.

Note that a strongly continuous semigroup is positive if and only if its generator is resolvent positive.

Let A be a resolvent positive operator. We introduce the following notation:

$$s(A) = \inf\{\omega \in \mathbb{R} : \quad (\omega, \infty) \subset \rho(A) \text{ and } R(\lambda, A) \geq 0 \text{ for all } \lambda > \omega\},$$

where $\rho(A)$ is the resolvent set of A.

Theorem 2.2.12. (Arendt-Robinson-Batty)

Let A be a densely defined resolvent positive operator. If there exist $\lambda_0 > s(A)$, $c > 0$ such that for all $\psi \geq 0$,

$$\|R(\lambda_0, A)\psi\| \geq c\|\psi\|, \tag{2.16}$$

then A generates a positive semigroup $(S_A(t))_{t\geq 0}$ on X and $s(A) = \omega_0(S_A)$, where $\omega_0(S_A)$ is the uniform growth bound of the semigroup $(S_A(t))_{t\geq 0}$.

Proof. [16, Theorem 3.39] □

2.3 Some classical perturbation results

Let $(A, D(A))$ be a generator of a C_0-semigroup on a Banach space X and $(B, D(B))$ be another operator in X. The purpose of the perturbation theory is to find conditions that ensure that there is an extension G of $A + B$ that generates a C_0-semigroup on X and characterize this extension.

2.3.1 Bounded perturbation theorem

The simplest and possibly the most often used perturbation result can be obtained if the operator B is bounded. The following theorem holds.

Theorem 2.3.1. (Bounded perturbation)

Let $(A, D(A)) \in \mathcal{G}(M, \omega)$; that is, it generates a C_0-semigroup $(S_A(t))_{t\geq 0}$ satisfying $\|S_A(t)\| \leq Me^{\omega t}$ for some $\omega \in \mathbb{R}$ and $M \geq 1$. If $B \in \mathcal{L}(X)$, then

$$(A + B, D(A)) \in \mathcal{G}(M, \omega + M\|B\|).$$

Proof. [16, Theorem 4.9] □

2.3.2 Kato-Voigt perturbations

The Kato-Voigt theorem is useful in the sense that it allows us to establish the existence of a *smallest* substochastic semigroup associated with a specific Cauchy problem. We begin with the definition of the terms *stochastic* and *substochastic* semigroups.

Definition 2.3.2. *The strongly continuous semigroup of operators $(S(t))_{t\geq 0}$ on the Banach space $X = L_1(\Omega, \mu)$ is said to be*

(i) substochastic if $S(t) \geq 0$ and $\|S(t)\| \leq 1$ for all $t \geq 0$,

(ii) stochastic if, in addition, it satisfies $\|S(t)\psi\| = \|\psi\|$ for all non-negative $\psi \in X$.

Theorem 2.3.3. *Let A be the generator of a C_0-semigroup in $X = L_1(\Omega)$ and let $B \in \mathcal{L}(D(A), X)$ be a positive operator. If for some $\lambda > s(A)$ the operator $\lambda I - A - B$ is resolvent positive, then $(A + B, D(A))$ generates a positive C_0-semigroup on X.*

Proof. [16, Theorem 5.13] □

Corollary 2.3.4. *Let $(S(t))_{t\geq 0}$ be the semigroup generated by $(A + B, D(A))$. Then $(S(t))_{t\geq 0}$ satisfies the Duhamel equation*

$$S(t)x = S_A(t)x + \int_0^t S(t-s)BS_A(s)x\,ds, \quad x \in D(A). \tag{2.17}$$

Proof. [16, Corollary 5.15] □

Theorem 2.3.5. *Let $X = L_1(\Omega)$ and suppose that the operators A and B satisfy:*

(1) $(A, D(A))$ *generates a substochastic semigroup* $(S_A(t))_{t\geq 0}$*;*

(2) $D(B) \supset D(A)$ *and* $Bu \geq 0$ *for* $u \in D(B)_+$*;*

(3) *For all* $u \in D(A)_+$,

$$\int_\Omega (Au + Bu)d\mu \leq 0. \tag{2.18}$$

Then there exists a smallest substochastic semigroup, $(S_G(t))_{t\geq 0}$*, generated by an extension,* G*, of* $A + B$*. Moreover,* G *is characterized by*

$$(I - G)^{-1}\psi = \sum_{n=0}^{\infty}(I - A)^{-1}[B(I - A)^{-1}]^n\psi, \quad \forall \psi \in X. \tag{2.19}$$

Proof. [16, Corollary 5.17] □

Proposition 2.3.6. *Let D be a core of A. If $(S(t))_{t\geq 0}$ is another semigroup generated by an extension of $(A + B, D)$, then $S(t) \geq S_G(t)$.*

Proof. [16, Proposition 5.7] □

2.4 Semilinear semigroups

The success of linear semigroup theory in solving linear evolution equations has stimulated extensions of the linear ideas, which provide opportunity for examination of semilinear problems. Unlike the linear case, semilinear semigroup theory is not complete, yet it remains a useful and powerful method of analyzing more difficult evolution equations.

Definition 2.4.1. (**Semilinear Abstract Cauchy Problem**)
Let X be a Banach space and let $(G, D(G))$ be an operator in X with associated semigroup $(S_G(t))_{t\geq 0}$. Furthermore, let N be a nonlinear operator which maps a subset D of X into X where $D(G) \cap D$ is not empty. Then the abstract problem

$$\frac{du}{dt}(t) = Gu(t) + Nu(t), \quad (t > 0); \quad u(0) = u_0 \in D(G) \cap D, \tag{2.20}$$

is called a semilinear *abstract Cauchy problem (ACP).*

Definition 2.4.2.
A function u is said to be a strong solution to the semilinear ACP (2.20) *on $[0, t_0)$ if u is continuous on $[0, t_0)$, differentiable on $(0, t_0)$ and is such that $u(t) \in D(G) \cap D$ for all $t \in [0, t_0)$ and u satisfies* (2.20).

Proposition 2.4.3.

Let u be a strong solution on $[0, t_0)$ of the semilinear ACP (2.20)*. Then u satisfies the integral equation*

$$u(t) = S_G(t)u_0 + \int_0^t S_G(t-s)N(u(s))ds, \quad 0 \le t < t_0, \tag{2.21}$$

where $(S_G(t))_{t\ge 0}$ is the semigroup associated with the linear operator G.

Proof. [21, p. 108]. □

Definition 2.4.4.

$u : [0, t_0) \to X$ is said to be a mild solution to the semilinear ACP (2.20) *if*

1. *u is continuous on $[0, t_0)$,*
2. *$u(t) \in D$ for all $t \in [0, t_0)$,*
3. *u satisfies* (2.21)*.*

We now introduce some definitions which are required in the theorems which follow.

Definition 2.4.5. (**Local Lipschitz Condition**)

An operator N on a Banach space X is said to satisfy a local Lipschitz condition if for any given $u_0 \in X$, there exists a closed ball,

$$\overline{B}(u_0, r) = \{f \in X : \quad \|f - u_0\| \le r\},$$

such that $\|Nf - Ng\| \le C\|f - g\|$ for all $f, g \in \overline{B}(u_0, r)$ where C depends on u_0 and r.

Definition 2.4.6. (**Fréchet Derivative**)

If a linear operator $N_f \in \mathcal{L}(X)$ exists such that $N(f + \delta) = Nf + N_f\delta + \mathcal{H}(f, \delta)$ where $\mathcal{H}$ satisfies

$$\lim_{\delta \to 0} \left(\frac{\|\mathcal{H}(f, \delta)\|}{\|\delta\|} \right) = 0,$$

then N is Fréchet differentiable at f and N_f is the Fréchet derivative.

Theorem 2.4.7.

Let $(G, D(G))$ be the generator of the strongly continuous semigroup $(S_G(t))_{t\ge 0}$ on X, let N be a nonlinear operator and let X be a Banach space. If N satisfies a local Lipschitz condition on X, then the semilinear ACP has a unique, local in time, mild solution.

Proof. [21, Theorem 3.20, p. 119]. □

Theorem 2.4.8.

Let $(G, D(G))$ generate the strongly continuous semigroup $(S_G(t))_{t\geq 0}$ on X and let N satisfy the local Lipschitz condition

$$\|N(f) - N(g)\| \leq \kappa\|f - g\|$$

for all f, g in the closed ball $\overline{B}(u_0, r) \subseteq D = D(N)$. If

1. *N is Fréchet differentiable at any $f \in B(u_0, r)$ and the Fréchet derivative N_f is such that $\|N_f g\| \leq \kappa_1\|g\|$ for all $f \in B(u_0, r)$, $g \in X$ where κ_1 is a positive constant independent of f and g,*

2. *the Fréchet derivative is continuous with respect to $f \in B(u_0, r)$ such that*

$$\|N_f g - N_{f_1} g\| \to 0 \quad \text{as} \quad \|f - f_1\| \to 0 \quad \text{where} \quad f, f_1 \in B(u_0, r),$$

   for any given $g \in X$,

3. *$u_0 \in D(G)$,*

then there exists $t_1 > 0$ such that the continuous solution on $[0, t_1)$ of (2.21) is strongly differentiable on $[0, t_1)$ and satisfies the equation (2.20).

Proof. [21, Theorems 3.30 and 3.32]. □

Chapter 3

Particles Control in Fragmentation Equations

3.1 Preliminaries

As we mentioned in the introduction, we are concerned with the initial value problem for the kinetic type rate equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= -a(x)u(t,x) + \int_x^\infty a(y)b(x|y)u(t,y)dy, \qquad x,t>0 \\ u(0,x) &= u_0(x), \end{aligned} \tag{3.1}$$

which describes the evolution of the density u of particles having mass x at time t; the particles undergo fragmentation at a rate a. We assume that a is a positive and continuous function on $(0,\infty)$. Further, b describes the distribution of daughter particles masses x spawned by the fragmentation of a parent particle of mass $y > x$. In absence of any other mechanism, the mass of all daughter particles must be equal to the mass of the parent. This 'local' conservation mass principle mathematically is expressed by

$$\int_0^y xb(x|y)dx = y. \tag{3.2}$$

Similarly, the expected number of particles produced by a particle of mass y is given by

$$n(y) = \int_0^y b(x|y)dx. \tag{3.3}$$

We note that $n(y)$ may be infinite. For a comprehensive discussion of the physical background of the model as well as the properties of the function b we refer the reader to [57, 10] and [16, Section 8.2]. Solvability of (3.1) in X_1 has been established by means of the substochastic semigroup theory, [16, Chapters 5 and 8]. Let us recall that a semigroup is called *substochastic* if it is a semigroup of positive contractions. To formulate the result we introduce some additional notation used also in the remaining part of the thesis. By $\mathcal{A}$ we denote the pointwise multiplication $\phi(x) \to -a(x)\phi(x)$ defined on a set of, say, measurable functions. Similarly, by $\mathcal{B}$ we denote the expression

$$[\mathcal{B}\phi](x) = \int_x^\infty a(y)b(x|y)\phi(y)dy, \tag{3.4}$$

defined first on all positive measurable functions for which the above integral is finite almost everywhere and then extended by linearity to a suitable linear subspace of measurable functions. The formal expressions $\mathcal{A}$ and $\mathcal{B}$ may define various operators. We start with A defined by $Au = \mathcal{A}u$ on $D(A) = \{u \in X_1;\ \mathcal{A}u \in X_1\}$; $\mathcal{B}$ restricted to $D(A)$ is a well-defined positive operator, denoted by B. Then (3.1) can be written as an abstract Cauchy problem in X_1:

$$\begin{aligned} \frac{d}{dt}u &= Au + Bu, \quad t > 0 \\ u(0) &= u_0. \end{aligned} \tag{3.5}$$

Unfortunately, $(A+B, D(A))$ does not necessarily generate a semigroup on X_1. However, the substochastic semigroup theory (Theorem 2.3.5) yields the existence of a smallest substochastic semigroup $(S_G(t))_{t\geq 0}$ generated by an extension G of $A + B$.

The semigroup $(S_G(t))_{t\geq 0}$ can be obtained as the strong limit in X_1 of semigroups $(S_{G_r}(t))_{t\geq 0}$ generated by $(A + rB, D(A))$ as $r \nearrow 1^-$; the limit is monotonic on nonnegative data. The fact that, in general, G is a proper extension of $\overline{A + B}$ has far reaching consequences which we explain below.

Local conservation principles (3.2) and (3.3) render formal conservation principles by integration of (3.1):

$$\frac{d}{dt}M(t) = \int_0^\infty \frac{\partial}{\partial t}u(t,x)xdx = 0, \tag{3.6}$$

$$\frac{d}{dt}N(t) = \int_0^\infty \frac{\partial}{\partial t}u(t,x)dx = \int_0^\infty a(x)(n(x)-1)u(t,x)dx. \tag{3.7}$$

If the equations (3.6), (3.7) or (3.6, 3.7) are satisfied by all nonnegative solutions to (3.1), then the process is called *honest* (in the respective space X_1, X_0 or $X_{0,1}$). However, validity of either equation depends on properties of u, namely whether each term on the right hand side of the integro-differential equation in (3.1) can be separately integrated with respect to the prescribed measure. Since the solution semigroup is generated by an extension of $A+B$, the right hand side of (3.5) should be treated as a single operator G and thus such separation of integrals is not always possible. Hence the global conservation principles are not always satisfied, in which instance the process is called *dishonest*.

Honesty of fragmentation processes in X_1 has been extensively studied. Here we recall the main results specified to the particular form of the function b which is also our choice for analysis in this chapter. Namely, we assume that b can be written as

$$b(x|y) = \beta(x)\gamma(y) \tag{3.8}$$

where, to satisfy the local principle of mass conservation,

$$\gamma(y) = \frac{y}{\int_0^y s\beta(s)ds}. \tag{3.9}$$

We assume that β is a non-negative continuous function on $(0, \infty)$. Eq. (3.8) is a natural generalization of the power law b described in (1.3) and has the advantage of allowing the number of daughter particles

$$n(y) = \frac{y \int_0^y \beta(s)ds}{\int_0^y s\beta(s)ds}, \tag{3.10}$$

to vary with the parent size y, [10]. An important role in the analysis is played by the function

$$b(x|x) = \beta(x)\gamma(x) = \frac{x\beta(x)}{\int_0^x s\beta(s)ds} = \frac{d}{dx} \ln \int_0^x s\beta(s)ds. \tag{3.11}$$

Honesty in X_1 (that is, the validity of the global mass conservation principle (3.6)) turns out to be equivalent to G being equal to the closure of $A + B$, $G = \overline{A + B}$. When b is given by (3.8), the following theorem settles this question.

Theorem 3.1.1. *Assume that* $\lim_{x \to 0^+} a(x)$ *exists (finite or infinite). Then* $G = \overline{A + B}$ *if and only if there exists* $\delta > 0$ *such that* $b(x|x)/a(x) \notin L_1([0, \delta])$.

Proof. See [10]. □

Let us turn our attention to the space $X_{0,1}$. We cannot, however, expect the process in $X_{0,1}$ to be dissipative as the number of particles grows rapidly, see (3.7), and thus we shall not be able to employ the substochastic semigroup theory. Instead, we use the theory of resolvent positive operators. The starting point is to find the resolvent of the generator.

3.2 Resolvent in X_1

To find the formula for the resolvent of the generator G, first we consider the Miyadera perturbation of $A + rB$ of A, as in [16, Theorem 5.2]. Defining $G_r = A + rB$ with $0 < r < 1$ we know, by *op.cit.*, that $(G_r, D(A))$ generates a positive semigroup of contractions and thus the resolvent $R(\lambda, G_r) = (\lambda I - A - rB)^{-1}$ exists for all $\lambda > 0$. Let us define $Q_{r\lambda} := \lambda I - A - rB$. To find the formula for $R(\lambda, G_r)$, we start by solving

$$f(x) = \lambda u_r(x) + a(x)u_r(x) - r \int_x^\infty a(y)b(x|y)u_r(y)dy, \;\; 0 < r < 1.$$

Choosing the constant in the general solution so as to have solutions converging to zero for $x \to +\infty$ (at least for regular f), we obtain

$$[R_r(\lambda)f](x) = u_r(x) = \frac{f(x)}{\lambda + a(x)} + \frac{r\beta(x)}{\lambda + a(x)} e^{-\xi_r(x)} \int_x^\infty \frac{a(s)\gamma(s)}{\lambda + a(s)} e^{\xi_r(s)} f(s) ds, \tag{3.12}$$

where

$$\xi_r(x) = r \int_1^x \frac{a(s)\gamma(s)\beta(s)}{\lambda + a(s)} ds.$$

This formula can be derived by using calculations similar to [16, Page 217-218] and actually describes the resolvent $R(\lambda, G_r)$. To prove this, we re-write (3.12) in a more convenient form. Define

$$B_\lambda(x) = \frac{b(x|x)}{\lambda + a(x)}, \quad \lambda > 0. \tag{3.13}$$

By (3.11)

$$\begin{aligned} \xi_r(x) &= r \left(\int_1^x b(s|s) ds - \lambda \int_1^x B_\lambda(s) ds \right) \\ &= r \left(\ln \int_0^x s\beta(s) ds - \ln \int_0^1 s\beta(s) ds - \lambda \int_1^x B_\lambda(s) ds \right), \end{aligned} \tag{3.14}$$

and

$$e^{\xi_r(x)} = e^{-C} \left(\int_0^x s\beta(s) ds \right)^r e^{-\lambda r \int_1^x B_\lambda(z) dz},$$

where $C = r \ln \int_0^1 s\beta(s) ds$. Thus

$$\begin{aligned} u_r(x) = [R_r(\lambda)f](x) = \frac{f(x)}{\lambda + a(x)} + \\ \frac{rB_\lambda(x)}{x} \Gamma(x)^{1-r} e^{r\lambda \int_1^x B_\lambda(s) ds} \int_x^\infty \frac{sa(s)f(s)}{\lambda + a(s)} \Gamma(s)^{r-1} e^{-r\lambda \int_1^s B_\lambda(z) dz} ds, \end{aligned} \tag{3.15}$$

where $\Gamma(x) = \int_0^x s\beta(s) ds$ is a positive and increasing function. We start with the following estimate.

Lemma 3.2.1. *For any $f \in D(A)$, $x > 0, 0 \le r < 1$*

$$\int_x^\infty a(s)\gamma(s) e^{\xi_r(s)} |f(s)| ds < \infty.$$

Proof. Since for $f \in D(A)$ and $x > 0$

$$\int_x^\infty a(s)\gamma(s) e^{\xi_r(s)} |f(s)| ds = \int_x^\infty (sa(s)|f(s)|) \frac{\gamma(s)}{s} e^{\xi_r(s)} ds,$$

it is enough to show that $\frac{\gamma(s)}{s} e^{\xi_r(s)}$ is bounded at ∞. For $r > 0$, $s \ge 1$ we have

$$e^{\xi_r(s)} = e^{-C} \Gamma^r(s) e^{-\lambda r \int_1^s B_\lambda(z) dz} \le e^{-C} \Gamma^r(s),$$

and for $r = 0$ it is bounded by 1. Using (3.9), properties of $\Gamma(s)$ and $r < 1$, we obtain

$$\frac{\gamma(s)}{s} e^{\xi_r(s)} \leq e^{-C} \Gamma^{r-1}(s) \leq e^{-C} \Gamma^{r-1}(1).$$

□

Proposition 3.2.2. *Let* $0 < r < 1,\ \lambda > 0$. *Then*

$$R(\lambda, A + rB) = R_r(\lambda).$$

Proof. Consider (3.15). The first term on the right hand side, $f \to f/(\lambda + a)$, clearly defines a bounded operator provided $\lambda > 0$. Let $f \geq 0$ and consider the norm of the second term:

$$\begin{aligned}
E_2 &= \int_0^\infty \left(\frac{rB_\lambda(x)}{x} \Gamma(x)^{1-r} e^{r\lambda \int_1^x B_\lambda(r)dr} \int_x^\infty \frac{sa(s)f(s)}{\lambda + a(s)} \Gamma(s)^{r-1} e^{-r\lambda \int_1^s B_\lambda(z)dz} ds \right) xdx \\
&= r \int_0^\infty \left(sf(s) \frac{a(s)}{\lambda + a(s)} \Gamma(s)^{r-1} e^{-r\lambda \int_1^s B_\lambda(z)dz} \int_0^s B_\lambda(x) \Gamma(x)^{1-r} e^{r\lambda \int_1^x B_\lambda(z)dz} dx \right) ds \\
&\leq r \int_0^\infty \left(sf(s) \frac{a(s)}{\lambda + a(s)} e^{-r\lambda \int_1^s B_\lambda(z)dz} \int_0^s B_\lambda(x) e^{r\lambda \int_1^x B_\lambda(z)dz} dx \right) ds \\
&= \frac{1}{\lambda} \int_0^\infty sf(s) \frac{a(s)}{\lambda + a(s)} e^{-r\lambda \int_1^s B_\lambda(z)dz} \int_0^s \left(\frac{d}{dx} e^{r\lambda \int_1^x B_\lambda(z)dz} \right) dxds \\
&= \frac{1}{\lambda} \int_0^\infty sf(s) \frac{a(s)}{\lambda + a(s)} e^{-r\lambda \int_1^s B_\lambda(z)dz} \left(e^{r\lambda \int_1^s B_\lambda(z)dz} - \lim_{\epsilon \to 0^+} e^{-r\lambda \int_\epsilon^1 B_\lambda(z)dz} \right) ds, \quad (3.16)
\end{aligned}$$

where we used $\Gamma(x)^{1-r} \leq \Gamma(s)^{1-r}$ for $0 \leq x \leq s$ and $r \leq 1$. Now, B_λ is a positive function, so $\int_0^s B_\lambda(z)dz$ always exists and can be either finite or $+\infty$. In either case

$$e^{-r\lambda \int_1^s B_\lambda(z)dz} \left(e^{r\lambda \int_1^s B_\lambda(z)dz} - \lim_{\epsilon \to 0^+} e^{-r\lambda \int_\epsilon^1 B_\lambda(z)dz} \right) \leq 1.$$

and $E_2 \leq \lambda^{-1} \|f\|$, thus $R_r(\lambda)$ is a bounded operator on X_1. Since we know that $R(\lambda, G_r)$ exists, to show that $R(\lambda, G_r) = R_r(\lambda)$ it is enough to prove that $R_r(\lambda)$ is the left inverse of $Q_{r\lambda} = \lambda I - A - rB$. For $0 \leq f \in D(A)$ we have

$$\begin{aligned}
([R_r(\lambda) Q_{r\lambda}] f)(x) &= \frac{[Q_{r\lambda} f](x)}{\lambda + a(x)} + \frac{r\beta(x)}{\lambda + a(x)} e^{-\xi_r(x)} \int_x^\infty \frac{a(s)\gamma(s)}{\lambda + a(s)} e^{\xi_r(s)} [Q_{r\lambda} f](s) ds \\
&=: I_1(x) + I_2(x),
\end{aligned}$$

where we used the fact that each term in $R_r(\lambda)$ is a bounded operator. Then

$$\begin{aligned}
I_1(x) &= \frac{(\lambda + a(x))f(x) - r\int_x^\infty a(y)b(x|y)f(y)dy}{\lambda + a(x)} \\
&= f(x) - \frac{r}{\lambda + a(x)} \int_x^\infty a(y)b(x|y)f(y)dy,
\end{aligned}$$

and

$$
\begin{aligned}
I_2(x) &= \frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty \frac{a(s)\gamma(s)}{\lambda+a(s)}e^{\xi_r(s)}[Q_{r\lambda}f](s)ds \\
&= \frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty \frac{a(s)\gamma(s)}{\lambda+a(s)}e^{\xi_r(s)}\,[(\lambda+a(s))f(s) \\
&\quad -r\int_s^\infty a(y)b(s|y)f(y)dy]\;ds \\
&= \frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty a(s)\gamma(s)e^{\xi_r(s)}f(s)ds \\
&\quad -\frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty r\frac{a(s)\gamma(s)}{\lambda+a(s)}e^{\xi_r(s)}\left(\int_s^\infty a(y)b(s|y)f(y)dy\right)ds \\
&=: J_1(x)-J_2(x),
\end{aligned}
$$

where we could split the integral thanks to the integrability of the first term ensured by Lemma 3.2.1. Changing the order of integration by the Fubini theorem, we get that

$$
\begin{aligned}
J_2(x) &= \frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty r\frac{a(s)\gamma(s)}{\lambda+a(s)}e^{\xi_r(s)}\left(\int_s^\infty a(y)b(s|y)f(y)dy\right)ds \\
&= \frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty r\beta(s)\frac{a(s)\gamma(s)}{\lambda+a(s)}e^{\xi_r(s)}\left(\int_s^\infty a(y)\gamma(y)f(y)dy\right)ds \\
&= \frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty a(y)\gamma(y)f(y)\left(\int_x^y r\frac{a(s)\gamma(s)\beta(s)}{\lambda+a(s)}e^{\xi_r(s)}ds\right)dy \\
&= \frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty a(y)\gamma(y)f(y)\left(e^{\xi_r(y)}-e^{\xi_r(x)}\right)dy \\
&= \frac{r\beta(x)}{\lambda+a(x)}e^{-\xi_r(x)}\int_x^\infty a(y)\gamma(y)e^{\xi_r(y)}f(y)dy-\frac{r\beta(x)}{\lambda+a(x)}\int_x^\infty a(y)\gamma(y)f(y)dy,
\end{aligned}
$$

where again we used Lemma 3.2.1 to split the integral in the penultimate line. It follows that

$$
I_2(x)=J_1(x)-J_2(x)=\frac{r\beta(x)}{\lambda+a(x)}\int_x^\infty a(y)\gamma(y)f(y)dy.
$$

Thus, for $0\le f\in D(A)$,

$$
\begin{aligned}
&([R_r(\lambda)Q_{r\lambda}]f)(x)=I_1(x)+I_2(x)= \\
&f(x)-\frac{r}{\lambda+a(x)}\int_x^\infty a(y)b(x|y)f(y)dy+\frac{r\beta(x)}{\lambda+a(x)}\int_x^\infty a(y)\gamma(y)f(y)dy=f(x).
\end{aligned}
$$

Since $D(A)$ is a weighted L_1 space, an arbitrary $f\in D(A)$ can be written as $f=f_+-f_-$ where $f_+,f_-\in D(A)$ are non-negative and the above equality extends to $D(A)$, thus proving the proposition. □

Let us introduce the formal expression

$$
[R(\lambda)f](x)=\frac{f(x)}{\lambda+a(x)}+\frac{B_\lambda(x)}{x}e^{\lambda\int_1^x B_\lambda(s)ds}\int_x^\infty \frac{sa(s)f(s)}{\lambda+a(s)}e^{-\lambda\int_1^s B_\lambda(r)dr}ds. \tag{3.17}
$$

Theorem 3.2.3. *Under assumptions of this section, the resolvent $R(\lambda, G)$ of the generator G in X_1 is given by*

$$[R(\lambda, G)f](x) = [R(\lambda)f](x) \tag{3.18}$$

Proof. We use the fact that $R(\lambda, G)$ is the strong limit in X_1 of the family $(R(\lambda, G_r))_{0<r<1}$ as $r \nearrow 1^-$, see [16, Theorem 5.2].

Denote the right hand side of (3.17) by $R(\lambda)$. First, using the same argument as in (3.16) with $\Gamma(x) = 1$, we see that $R(\lambda)$ defines a bounded operator on X_1. Next, from Proposition 3.2.2 we know that $R_r(\lambda) = R(\lambda, A+rB)$ is the resolvent of $A+rB$. Consider now $\lim_{r\nearrow 1} u_r(x)$. It is clear that

$$\lim_{r\nearrow 1} \frac{rB_\lambda(x)}{x}\Gamma(x)^{1-r}e^{r\lambda\int_1^x B_\lambda(s)ds} = \frac{B_\lambda(x)}{x}e^{\lambda\int_1^x B_\lambda(s)ds}.$$

Further, taking f with $supp f \subset [x_0, M]$ where $x_0 > 0$ we have

$$\int_{x_0}^{\infty} \frac{sa(s)f(s)}{\lambda + a(s)}\Gamma(s)^{r-1}e^{-r\lambda\int_1^s B_\lambda(z)dz}ds \;=\; \int_{x_0}^{M} \frac{sa(s)f(s)}{\Gamma(s)(\lambda + a(s))}\left(\frac{\Gamma(s)}{e^{\lambda\int_1^s B_\lambda(z)dz}}\right)^r ds$$

and the integrand is bounded by a constant which is integrable on this interval. Thus, we can pass to the limit for any $x > 0$ getting

$$\begin{aligned}\lim_{r\nearrow 1}[R(\lambda, A+rB)f](x) &= [R(\lambda)f](x)\\ &= \frac{f(x)}{\lambda + a(x)} + \frac{B_\lambda(x)}{x}e^{\lambda\int_1^x B_\lambda(s)ds}\int_x^{\infty}\frac{sa(s)f(s)}{\lambda + a(s)}e^{-\lambda\int_1^s B_\lambda(r)dr}ds,\end{aligned}$$

for $f \geq 0$ with bounded support and, by linearity, for any function with bounded support. Thus $R(\lambda)f = R(\lambda, G)f$ on a dense subset of X_1 and, since the expression for $R(\lambda)$ defines a bounded positive operator on X_1, the representation (3.17) extends to the whole space. □

3.3 Resolvent in $X_{0,1}$

Our aim is to prove the existence of solutions to the fragmentation equation (3.1) in $X_{0,1} = L_1(\mathbb{R}_+, (1+x)dx) = X_0 \cap X_1 = L_1(\mathbb{R}_+, dx) \cap L_1(\mathbb{R}_+, xdx)$. Since $X_{0,1} \subset X_1$ the resolvent, if it exists, again must be given by (3.17). Define

$$\Delta_\lambda(x) = e^{-\lambda\int_x^1 B_\lambda(s)ds}. \tag{3.19}$$

Theorem 3.3.1. *The expression $R(\lambda)$, $\lambda > 0$, defines an operator on $X_{0,1}$ if and only if*

$$\Xi_\lambda(x) := \frac{xa(x)}{\lambda + a(x)}\Delta_\lambda^{-1}(x)\int_0^x \frac{B_\lambda(s)(1+s)}{s}\Delta_\lambda(s)ds \quad \textit{is bounded at } 0. \tag{3.20}$$

Proof. First we observe that a necessary condition for (3.20) to hold is that the inner integral be finite; that is,

$$\int_{0^+} \frac{B_\lambda(x)}{x}\Delta_\lambda(x)dx = \frac{1}{\lambda}\int_{0^+} \frac{1}{x}\frac{d}{dx}\Delta_\lambda(x)dx < +\infty. \tag{3.21}$$

Since $X_{0,1}$ inherits the lattice structure from X_1, $R(\lambda) : X_{0,1} \to X_{0,1}$ if and only if $\|R(\lambda)f\|_{0,1} < +\infty$ for any $0 \le f \in X_{0,1}$. Since in this case $R(\lambda)f$ is a sum of two positive terms, for $0 \le f \in X_{0,1}$ we have

$$\int_0^\infty [R(\lambda)f](x)(1+x)dx = \int_0^\infty f(x)(1+x)\left(\frac{1}{\lambda + a(x)} + \frac{a(x)}{\lambda + a(x)}\frac{x}{1+x}e^{-\lambda\int_1^x B_\lambda(s)ds}\int_0^x \frac{B_\lambda(s)(1+s)}{s}e^{\lambda\int_1^s B_\lambda(r)dr}ds\right)dx. \tag{3.22}$$

Since $1/(\lambda + a(x))$ is bounded, the first term is finite and $\|R(\lambda)f\|_{0,1} < +\infty$ if and only if the second term is bounded on $\mathbb{R}$. Hence, if $\|R(\lambda)f\|_{0,1} < +\infty$ then, in particular, (3.20) is satisfied (as the behaviour of $x/(1+x)$ is the same as that of x as $x \to 0$). This, moreover, yields (3.21).

To prove the opposite implication, assume (3.20). Then (3.21) is also satisfied. Consider the behaviour of the second term in (3.22), which equals $(1+x)^{-1}\Xi_\lambda(x)$, as $x \to \infty$. First observe that, by (3.21), $\int_\alpha^\infty B_\lambda(x)(1+x)x^{-1}\Delta_\lambda(x)dx$ either exists or does not exist irrespectively of $\alpha \ge 0$ and

$$\int_1^x \frac{B_\lambda(s)(1+s)}{s}\Delta_\lambda(s)ds \le \frac{2}{\lambda}(\Delta_\lambda(x) - 1).$$

If the improper integral exists, then $(1+x)^{-1}\Xi_\lambda(x)$ is bounded at infinity on account of the boundedness of $\Delta_\lambda(x)^{-1}$. If the integral is infinite then, by the above, $\Delta_\lambda(x)$ tends to infinity as $x \to \infty$ and hence $\Delta_\lambda(x)^{-1}$ tends to 0. Then we can use the l'Hospital rule getting

$$\lim_{x\to\infty} e^{-\lambda\int_1^x B_\lambda(s)ds}\int_0^x \frac{B_\lambda(s)(1+s)}{s}e^{\lambda\int_1^s B_\lambda(r)dr}ds = \lim_{x\to\infty}\frac{B_\lambda(x)(1+x)\Delta_\lambda(x)}{\lambda x B_\lambda(x)\Delta_\lambda(x)} = \frac{1}{\lambda}.$$

Since the other two multipliers are bounded, $(1+x)^{-1}\Xi(x)$ is bounded as $x \to \infty$ and hence (3.20) alone ensures finiteness of $\|R(\lambda)f\|_{0,1}$. □

Corollary 3.3.2. *If there is $\lambda_0 > 0$ such that (3.20) is satisfied for all $\lambda > \lambda_0$, then $R(\lambda)$ defines a positive resolvent of an operator $G_{0,1}$ which is the part of G in $X_{0,1}$.*

Proof. Since $\widetilde{R}(\lambda) := R(\lambda)|_{X_{0,1}}$ is a pseudo-resolvent (see [46, Theorem 9.3]), we can define the operator

$$\widetilde{G}f = \lambda f - \widetilde{R}(\lambda)^{-1}f$$

for $f \in D(\widetilde{G}) := \text{Range}\widetilde{R}(\lambda)$. Recall that the part $G_{0,1}$ is defined as the restriction of G to $D(G_{0,1}) = \{f \in D(G) \cap X_{0,1};\ Gf \in X_{0,1}\}$. If $f \in D(\widetilde{G}) \subset X_{0,1}$, then clearly f is in the range of $R(\lambda, G) = D(G)$. Hence $f \in D(G_{0,1})$ and, since $R(\lambda, G)^{-1}f = \widetilde{R}(\lambda)^{-1}f$ and $\widetilde{G}f = G_{0,1}f$, we have $\widetilde{G} \subset G_{0,1}$. On the other hand, if $f \in D(G_{0,1})$, then $f = R(\lambda, G)g = R(\lambda)g$ for some $g \in X$ and $f \in X_{0,1}$. Then

$$X_{0,1} \ni Gf = \lambda R(\lambda, G)g - g = \lambda f - g,$$

thus $g \in X_{0,1}$ and f is in the range of $\widetilde{R}(\lambda)$, which is $D(\widetilde{G})$, and

$$G_{0,1}f = \lambda f - R(\lambda, G)^{-1}f = \lambda f - \widetilde{R}(\lambda)^{-1}f = \widetilde{G}f.$$

□

Due to the interplay of possible singularities in B_λ and Δ_λ, it seems to be difficult to give more explicit necessary and sufficient conditions ensuring that $R(\lambda)$ defines an operator in $X_{0,1}$. We can, however, provide a set of easy to check sufficient conditions which cover most standard cases.

Corollary 3.3.3. *Let one of the conditions be satisfied:*

1.
$$\lim_{x\to 0^+} xB_\lambda(x) = L_\lambda \tag{3.23}$$
with $1 < \lambda L_\lambda < +\infty$ and $xB(x)$, extended by continuity to $x = 0$, be Hölder continuous, or

2.
$$\lim_{x\to 0^+} B_\lambda(x) = 0 \tag{3.24}$$
and $B_\lambda(x)$, extended by continuity to 0, is Hölder continuous, or

3.
$$\lim_{x\to 0^+} xB_\lambda(x) = \infty. \tag{3.25}$$
with $M_1/x^\beta \le xB_\lambda(x) \le M_2/x^\alpha$ close to $x = 0$, where $\alpha, \beta > 0$ and may depend on λ and $M_1, M_2 > 0$ may depend on β and α, respectively.

Then condition (3.20) is satisfied.

Proof. First we consider (3.23) so that $|xB_\lambda(x) - L_\lambda| \le Mx^\alpha$ for some $\alpha > 0$. We have

$$e^{-\lambda \int_x^1 B_\lambda(s)ds} = e^{-\lambda L_\lambda \int_x^1 \frac{1}{s}ds} e^{-\lambda \int_x^1 \left(B_\lambda(s) - \frac{L_\lambda}{s}\right)ds} \tag{3.26}$$

with

$$\int_x^1 \left|B_\lambda(s) - \frac{L_\lambda}{s}\right| ds \le M \int_x^1 s^{\alpha-1}ds \tag{3.27}$$

bounded as $x \to 0^+$. On the other hand, $e^{-\lambda L_\lambda \int_x^1 \frac{1}{s}ds} = x^{\lambda L_\lambda}$ and, by the above,

$$\int_0^x \frac{B_\lambda(s)}{s}\Delta_\lambda(s)ds \quad\leq\quad M'\left(L_\lambda\int_0^x s^{\lambda L_\lambda - 2}ds + \int_0^x |sB_\lambda(s) - L_\lambda|s^{\lambda L_\lambda-2}ds\right) < +\infty$$

provided $\lambda L_\lambda > 1$. This gives (3.21).

Further, in this case, $\Delta_\lambda^{-1}(x)$ tends to infinity as $x^{-\lambda L_\lambda}$ since the second factor in (3.26) is also bounded from below by (3.27), and thus $x\Delta_\lambda^{-1}$ also tends to infinity since $\lambda L_\lambda > 1$. Hence, by the l'Hospital rule

$$\lim_{x\to 0^+} x\Delta_\lambda^{-1}(x)\int_0^x \frac{B_\lambda(s)(1+s)}{s}\Delta_\lambda(s)ds = \lim_{x\to 0^+}\frac{x(1+x)B_\lambda(x)}{\lambda x B_\lambda(x) - 1} = \frac{L_\lambda}{\lambda L_\lambda - 1} < \infty. \quad (3.28)$$

Assume now (3.24) is satisfied. Then, by Hölder continuity of B_λ, $B_\lambda(x)/x$ (and also $B_\lambda(x)$) are integrable at 0 and both conditions (3.21) and (3.20) are satisfied.

Finally, let (3.25) be satisfied. In this case we have

$$C_\alpha e^{-\frac{C'_\alpha}{x^\alpha}} \leq \Delta_\lambda(x) \leq C_\beta e^{-\frac{C'_\beta}{x^\beta}} \quad (3.29)$$

for some constants $C_\alpha, C'_\alpha, C_\beta, C'_\beta$ and

$$\frac{B_\lambda(x)}{x}\Delta_\lambda(x) \leq M_2 C_\beta \frac{1}{x^{1+\alpha}}e^{-\frac{C'_\beta}{x^\beta}} \to 0$$

for $x \to 0^+$. Eqn. (3.29) also ensures that $\Delta_\lambda^{-1}(x) \geq C_\beta e^{C'_\beta/x^\beta}$ and thus $x\Delta_\lambda^{-1}(x) \to \infty$ for $x \to 0^+$. Hence, applying the l'Hospital rule as in the first part of the proof gives

$$\lim_{x\to 0^+} x\Delta_\lambda^{-1}(x)\int_0^x \frac{B_\lambda(s)(1+s)}{s}\Delta_\lambda(s)ds = \lim_{x\to 0^+}\frac{x(1+x)B_\lambda(x)}{\lambda x B_\lambda(x) - 1} = \frac{1}{\lambda}, \quad (3.30)$$

on account of $xB_\lambda(x) \to \infty$. □

In what follows we shall work with the additional assumption

$$\lim_{x\to 0^+} a(x) = a_0 \in [0, \infty]. \quad (3.31)$$

We note that this assumption is mostly technical and several results below can be proved without it. This, however, would require a more detailed analysis of particular cases.

Lemma 3.3.4. *Assume (3.31). If any of the conditions (3.23), (3.24) or (3.25) holds for some $\lambda_0 > 0$, then it holds for any $\lambda \geq \lambda_0$.*

Proof. First consider (3.23). Then from

$$\frac{xb(x|x)}{\lambda + a(x)} - \frac{xb(x|x)}{\mu + a(x)} = \frac{(\mu - \lambda)xb(x|x)}{(\lambda + a(x))(\mu + a(x))}$$

we see that if the limit exists for some $\lambda > 0$ then it exists for any $\lambda > 0$. Let $a_0 < \infty$. Then

$$\lambda L_\lambda = \frac{\lambda \lim_{x\to 0^+} xb(x|x)}{\lambda + a_0}$$

is an increasing function of λ so that if $\lambda L_\lambda > 1$ for some λ_0 then it is true for any $\lambda > \lambda_0$. If $a_0 = \infty$, then

$$\lambda L_\lambda = \lambda \lim_{x\to 0} \frac{xb(x|x)}{a(x)},$$

which is also monotonic in λ. Moving to (3.24) we see that

$$0 \le B_\mu(x) \le B_\lambda(x)$$

for $\mu > \lambda > 0$ and the lemma holds true. If (3.25) holds, then either $xb(x|x) \to \infty$ if $a_0 < +\infty$ or $xb(x|x)/a(x) \to +\infty$ if $a_0 = \infty$ and in either case the condition is independent of λ. □

Corollary 3.3.5. *Under assumption (3.31), if any of the conditions (3.23) - (3.25) holds for some $\lambda_0 > 0$, then $R(\lambda)$ for $\lambda \ge \lambda_0$ is the resolvent of $G_{0,1}$.*

Proposition 3.3.6. *Assume that (3.31) holds and let either (3.23), or (3.25) with the additional condition: $a(x) \le x^{-\kappa}$ as $x \to 0^+$ for some $\kappa > 0$, be satisfied. Then $\beta(x)$ is integrable on $[0, M]$, $M < \infty$, and thus the number of particles produced in each fragmentation event is finite.*

Proof. Let us fix some λ for which (3.23) holds. First assume that (3.23) is satisfied and $a(x) \to a_0 < \infty$ as $x \to 0^+$. Then

$$\lim_{x\to 0^+} xb(x|x) = L_\lambda(\lambda + a_0) \ge \lambda L_\lambda > 1$$

and $xb(x|x) \ge L > 1$ for x sufficiently close to zero. This yields

$$\frac{x\beta(x)}{\int_0^x s\beta(s)ds} \ge \frac{L}{x}$$

which, upon integration from $x > 0$ to some sufficiently small α, gives

$$\ln \frac{\int_0^\alpha s\beta(s)ds}{\int_0^x s\beta(s)ds} \ge \ln\left(\frac{\alpha}{x}\right)^L, \tag{3.32}$$

which can be written as $\int_0^x s\beta(s)ds \le C_\alpha x^L$, for some constant C_α and small $x > 0$. Since

$$\beta(x) = xb(x|x)\frac{1}{x^2}\int_0^x s\beta(s)ds, \tag{3.33}$$

$\beta(x)$ behaves as x^{L-2} which is integrable at 0. Let now $a_0 = \infty$. We can write

$$xB_\lambda(x) = \frac{x^2}{a(x)}\frac{b(x|x)}{1 + \lambda/a(x)}$$

so that

$$\lim_{x\to 0^+}\frac{\lambda x^2 b(x|x)}{a(x)} = \lambda L_\lambda > 1.$$

Let us take $\alpha > 0$ such $a(x) > \lambda$ and

$$\frac{x\beta(x)}{\int_0^x s\beta(s)ds} \geq \frac{La(x)}{\lambda x}$$

for some $L > 1$ for $x \in (0, \alpha)$. As before, this gives

$$\ln\frac{\int_0^\alpha s\beta(s)ds}{\int_0^x s\beta(s)ds} \geq \frac{L}{\lambda}\int_x^\alpha \frac{a(s)}{s}ds \geq L\int_x^\alpha \frac{ds}{s} = \ln\left(\frac{\alpha}{x}\right)^L$$

which is the same as (3.32).

In the last case we have

$$\frac{x\beta(x)}{\int_0^x s\beta(s)ds} \geq \frac{M_1(\lambda + a(x))}{x^{\beta+1}}$$

and, following the steps of the previous cases, we obtain $\int_0^x s\beta(s)ds \leq C_1 e^{-C_2 x^{-\beta}}$ for some constants C_1 and C_2, which, by (3.33), yields

$$\beta(x) \leq C_1 x b(x|x) x^{-2} e^{-C_2 x^{-\beta}} \leq M x^{-\kappa-\alpha-2} e^{-C_2 x^{-\beta}},$$

for some M. It is now clear that $\beta(x) \to 0$ as $x \to 0^+$. □

Remark 1. *In general, (3.24) does not yield the result of the above proposition. Indeed, taking* $a(x) = x^{-2}$ *and* $b(x|y) = 2^{-1}y^{1/2}x^{-3/2}$, *we see that* $B_\lambda(x) = 1/(2x(\lambda + x^{-2})) = x/2(\lambda x^2 + 1) \to 0$. *However,* $\beta(x) = x^{-3/2}$ *is not integrable and hence the expected number of particles in each fragmentation event is infinite.*

3.4 Dynamics in $X_{0,1}$

Theorem 3.4.1.

Let the assumptions of Proposition 3.3.6 be satisfied. Then $(G_{0,1}, D(G_{0,1}))$ *generates a positive semigroup in* $X_{0,1}$.

Proof. First we prove that $G_{0,1}$ is densely defined. We note that $C_0^\infty(\mathbb{R}_+) \subset D(A) \cap X_{0,1} \subset D(G) \cap X_{0,1}$ and thus, for $\phi \in C_0^\infty(\mathbb{R}_+)$ with support in $[m, M]$

$$G\phi = A\phi + B\phi.$$

It is clear that $A\phi \in X_{0,1}$ and

$$\begin{aligned}\int_0^\infty [B\phi](x)(1+x)dx &= \int_0^\infty \left(\int_x^M a(y)b(x|y)\phi(y)dy\right)(1+x)dx \\ &= \int_m^M a(y)\phi(y)\gamma(y)\left(\int_0^x \beta(x)(1+x)dx\right)dy \\ &= \int_m^M a(y)\phi(y)(y+n(y))dy < \infty,\end{aligned}$$

where $n(y)$ is given by (3.10).

Let us return to the expression (3.22) for the norm of the resolvent in $X_{0,1}$ for $f \geq 0$:

$$\int_0^\infty [R(\lambda, G_{0,1})f](x)(1+x)dx = \int_0^\infty f(x)(1+x)\left(\frac{1}{\lambda + a(x)}\right. \tag{3.34}$$
$$\left. + \frac{a(x)}{\lambda + a(x)}\frac{x}{1+x}e^{-\lambda\int_1^x B_\lambda(s)ds}\int_0^x \frac{B_\lambda(s)(1+s)}{s}e^{\lambda\int_1^s B_\lambda(r)dr}ds\right)dx.$$

The term within the brackets is a sum of two terms which are positive on $(0,\infty)$. Let us first discuss the behavior of it as $x \to \infty$. We note that $a(x)$ may have no limit as $x \to \infty$. Our analysis is valid irrespective of the behavior of $a(x)$ as $x \to \infty$. Let $s \geq 1$, we have $1 \leq 1 + s^{-1} \leq 2$. It follows that

$$\frac{1}{\lambda}(\Delta_\lambda(x) - 1) \leq \int_1^x \frac{B_\lambda(s)(1+s)}{s}\Delta_\lambda(s)ds \leq \frac{2}{\lambda}(\Delta_\lambda(x) - 1).$$

We recall that the limit of $\Delta_\lambda(x)$ as $x \to \infty$ always exists (finite or infinite). If $\Delta_\lambda(x)$ is finite as $x \to \infty$, then the improper integral exists and is non zero. If $\Delta_\lambda(x)$ is infinite, then $\Delta_\lambda(x)^{-1} \to 0$ and, by the above, $\int_1^x B_\lambda(s)\Delta_\lambda(s)(1+s)s^{-1}ds$ tends to infinity as $x \to \infty$. Hence we can use the l'Hospital rule obtaining

$$\lim_{x\to\infty} e^{-\lambda\int_1^x B_\lambda(s)ds}\int_0^x \frac{B_\lambda(s)(1+s)}{s}e^{\lambda\int_1^s B_\lambda(r)dr}ds = \lim_{x\to\infty}\frac{B_\lambda(x)(1+x)\Delta_\lambda(x)}{\lambda x B_\lambda(x)\Delta_\lambda(x)} = \frac{1}{\lambda}.$$

Therefore at infinity $\frac{x}{1+x}e^{-\lambda\int_1^x B_\lambda(s)ds}\int_0^x \frac{B_\lambda(s)(1+s)}{s}e^{\lambda\int_1^s B_\lambda(r)dr}ds$ is bounded away from zero. As a result the expression within the brackets in (3.34) is greater than the expression $\frac{1}{\lambda+a(x)} + \alpha\frac{a(x)}{\lambda+a(x)}$ as x approaches infinity, for some constant $\alpha > 0$. Furthermore

$$\frac{1}{\lambda + a(x)} + \alpha\frac{a(x)}{\lambda + a(x)} \geq \min\left(\frac{1}{\lambda}, \alpha\right)\left[\frac{\lambda}{\lambda + a(x)} + \frac{a(x)}{\lambda + a(x)}\right] = \min\left(\frac{1}{\lambda}, \alpha\right).$$

Consequently the expression within the brackets in (3.34) is bounded away from zero for large x.

Next we discuss the behavior of the expression within the brackets in (3.34) as $x \to 0$. If $a(x)$ is bounded as $x \to 0$, then $a(x) \to a_0 < \infty$ and $\frac{1}{\lambda+a_0} > 0$. It follows that $\frac{1}{\lambda+a(x)}$ is bounded away from 0 as x approaches 0.
Now assume $a(x)$ becomes infinite as $x \to 0$. If (3.23) holds then, as in (3.28), $\Delta_\lambda^{-1}(x)$ tends to infinity as $x^{-\lambda L_\lambda}$ and

$$\lim_{x\to 0^+} x\Delta_\lambda^{-1}(x)\int_0^x \frac{B_\lambda(s)(1+s)}{s}\Delta_\lambda(s)ds = \frac{L_\lambda}{\lambda L_\lambda - 1},$$

which is finite and non-zero by (3.23). Finally, if (3.25) is satisfied then, again following the proof of Corollary 3.3.3 and (3.30), we obtain

$$\lim_{x\to 0^+} x\Delta_\lambda^{-1}(x)\int_0^x \frac{B_\lambda(s)(1+s)}{s}\Delta_\lambda(s)ds = \frac{1}{\lambda} > 0.$$

Therefore, there is $c > 0$ such that for any $x \in [0,\infty)$

$$\frac{1}{\lambda+a(x)} + \frac{a(x)}{\lambda+a(x)}\frac{x}{1+x}e^{-\lambda\int_1^x B_\lambda(s)ds}\int_0^x \frac{B_\lambda(s)(1+s)}{s}e^{\lambda\int_1^s B_\lambda(r)dr}ds \geq c \tag{3.35}$$

yielding

$$\|R(\lambda, G_{0,1})f\|_{0,1} \geq c\|f\|_{0,1}$$

for $\lambda > \lambda_0$ and $f \geq 0$ and by the Arendt-Robinson-Batty theorem, see e.g. [8, Theorem 2.5], $G_{0,1}$ generates a positive semigroup. □

Remark 2. An important role in the application of Arendt-Batty-Robinson in the previous theorem is played by the density of the domain of $G_{0,1}$ in $X_{0,1}$. We observe that, in general, it is far from obvious. Let us take a model with infinite $n(y)$; that is, with non-integrable β. If a function $0 \neq \phi \geq 0$ belongs to $D(G_{0,1})$, then $X_{0,1} \ni G_{0,1}\phi = G\phi$. If ϕ has support in $[m, M]$, then

$$G_{0,1}\phi = G\phi = A\phi + B\phi.$$

The function $[A\phi](x) = a(x)\phi(x)$ has compact support hence it belongs to $X_{0,1}$. Thus, for $\phi \in D(G_{0,1})$, $B\phi$ also must be in $X_{0,1}$. However,

$$[B\phi](x) = \beta(x)\int_x^M a(y)\gamma(y)\phi(y)dy$$

and close to zero $\int_x^M a(y)\gamma(y)\phi(y)dy = \int_m^M a(y)\gamma(y)\phi(y)dy$ is finite and independent of x, hence $B\phi$ is integrable close to 0 if and only if β has the same property. Hence in this case positive compactly supported functions are not in $D(G_{0,1})$ and, at present, we do not know whether $G_{0,1}$ with non-integrable β is densely defined in $X_{0,1}$.

Also, in general, (3.24) does not allow the estimate (3.35) which was instrumental in getting the generation result. However, it follows that we still have some nontrivial dynamics in this case.

Corollary 3.4.2. *Let assumption (3.24) be satisfied. Then $(G_{0,1}, D(G_{0,1}))$ generates a once integrated positive semigroup $X_{0,1}$ if $D(G_{0,1})$ is dense in $X_{0,1}$ and twice integrated semigroup if $D(G_{0,1})$ is not dense in $X_{0,1}$. Consequently, the problem (3.1) has classical solutions for $u_0 \in D(G_{1,0}^2)$ in the first case and if $G_{1,0}^2 u_0 \in \overline{D(G_{0,1})}^{X_{0,1}}$ (hence, in particular, $u_0 \in D(G_{0,1}^3)$), in the second.*

Proof. The proof is a direct consequence of [9, Corollary 4.5, Proposition 5.5]. In particular, the second part follows from the fact that if $G_{0,1}$ is a resolvent positive operator with a non-dense domain, then its part in $\overline{D(G_{0,1})}^{X_{0,1}}$ generates there a once-integrated semigroup and the argument from the first part can be applied for this restriction. □

Theorem 3.4.3. *Assume that either (3.23) or (3.25) are satisfied and $n(y)$ is bounded as $y \to 0$, (then, in particular, β is integrable close to 0). Then*

$$G_{0,1} = \overline{(A+B)|_{X_{0,1}}}^{X_{0,1}}. \tag{3.36}$$

Moreover, if $a(x) \to a_0 < +\infty$ as $x \to 0^+$, then for any $0 \leq \; u_0 \in D(G_{0,1})$

$$\frac{d}{dt}\|S_{G_{0,1}}(t)\, u_0\|_{0,1} = \int_0^\infty a(x)(n(x)-1)[S_{G_{0,1}}(t)\, u_0](x)dx$$

so that the semigroup is honest in $X_{0,1}$.

Proof. To prove the first part, we use [16, Theorem 4.3]. Let us define $A_{0,1}$ to be the part of A in $X_{0,1}$ which, since $X_{0,1} \subset X_1$, is the restriction of multiplication by a to

$$D(A_{0,1}) = \{u \in X_{0,1}, au \in X_{0,1}\}.$$

Next we observe that boundedness of $n(y)$ at zero implies that $n(y)/(1+y)$ is bounded on $[0,\infty)$. Indeed, let $n(y) \leq N_1$ for $y \leq 1$. Then for $y > 1$ we get

$$\frac{n(y)}{y} = \frac{\int_0^y \beta(s)ds}{\int_0^y s\beta(s)ds} = \frac{C_1 + \int_1^y \beta(s)ds}{C_2 + \int_1^y s\beta(s)ds} \leq \frac{C_1 + \int_1^y \beta(s)ds}{C_2 + \int_1^y \beta(s)ds} \leq C$$

for some constant C. If $\mathcal{B}$ denotes the integral expression (3.4), then for $0 \leq u \in D(A_{0,1})$ we have

$$\|\mathcal{B}u\|_{0,1} = \int_0^\infty a(y)(y+n(y))u(y)dy \leq \sup_{y\in\mathbb{R}_+} \frac{y+n(y)}{1+y} \int_0^\infty a(y)u(y)(1+y)dy < \infty$$

hence we can define $B_{0,1}$ by restricting $\mathcal{B}$ to $D(A_{0,1})$. Let $u \in D(A_{0,1}) \subset D(A) \cap X_{0,1} \subset D(G) \cap X_{0,1}$. But then

$$X_{0,1} \ni (A_{0,1} + B_{0,1})u = Au + Bu = Gu,$$

hence $u \in D(G_{0,1})$ and $G_{0,1} \supset A_{0,1} + B_{0,1}$. Since both $G_{0,1}$ and $A_{0,1}$ generate semigroups on $X_{0,1}$, we can use Theorem 4.3 of [16], which states that $G_{0,1} = \overline{(A+B)|_{X_{0,1}}}^{X_{0,1}}$ if and only if 1 is not an eigenvalue of $(B_{0,1}R(\lambda, A_{0,1}))^*$, in the same way as in the proof of Theorem 8.13 in *op. cit.* (see also [10],[16, Corollary 6.15]).

We identify the dual to $X_{0,1}$ with $X_\infty := L_{\infty,(1+x)^{-1}}(\mathbb{R}_+)$ so that the duality pairing is the same as between L_1 and L_∞. Thanks to the considerations in the previous paragraph, the operator $B_{0,1}R(\lambda, A_{0,1})$ is bounded and, as in *op. cit.*, we find that the adjoint is given by the expression

$$[(B_{0,1}R(\lambda, A_{0,1}))^* g](y) = \frac{a(y)\gamma(y)}{\lambda + a(y)} \int_0^y \beta(x)g(x)dx = \frac{a(y)b(y|y)}{\beta(y)(\lambda + a(y))} \int_0^y \beta(x)g(x)dx.$$

Thus, assume 1 is an eigenvalue of $(B_{0,1}R(\lambda, A_{0,1}))^*$; that is, there is $0 \neq h \in X_\infty$ satisfying

$$\beta(y)h(y) - \frac{a(y)b(y|y)}{\lambda + a(y)} \int_0^y \beta(x)h(x)dx = 0. \tag{3.37}$$

Denoting

$$\varphi(y) = \int_0^y \beta(x)h(x)dx \tag{3.38}$$

we find from (3.37) that φ is differentiable on $(0, \infty)$ and the equation can be converted to

$$\varphi'(y) = \frac{a(y)b(y|y)}{\lambda + a(y)}\varphi(y).$$

It has a solution

$$\varphi(y) = e^{\int_1^y \frac{a(s)b(s|s)}{\lambda + a(s)}ds}. \tag{3.39}$$

Differentiating and using $b(s|s) = s\beta(s)/\int_0^s r\beta(r)dr$ and (3.14), we obtain

$$\begin{aligned} h(y) &= \frac{\varphi'(y)}{\beta(y)} = \frac{1}{\beta(y)}\frac{a(y)b(y|y)}{\lambda + a(y)}e^{\int_1^y \frac{a(s)b(s|s)}{\lambda+a(s)}ds} \\ &= \frac{1}{\beta(y)}\frac{y\beta(y)a(y)}{(\lambda + a(y))\int_0^y s\beta(s)ds}e^{\int_1^y b(s|s)ds}e^{-\lambda\int_1^y \frac{b(s|s)}{\lambda+a(s)}ds} = C\frac{ya(y)}{\lambda + a(y)}e^{\lambda\int_y^1 \frac{sB_\lambda(s)}{s}ds}, \end{aligned}$$

where $C = 1/\ln\int_0^1 \beta(s)ds$. Assume that (3.23) is satisfied. Then $\lambda sB_\lambda(s) \to \lambda L_\lambda > 1$ as $s \to 0$ and thus $\lambda sB_\lambda(s) \geq L > 1$ on $(0, \alpha)$ for sufficiently small α.

Hence

$$e^{\lambda\int_y^1 \frac{sB_\lambda(s)}{s}ds} = e^{\lambda\int_y^\alpha \frac{sB_\lambda(s)}{s}ds}e^{\lambda\int_\alpha^1 \frac{sB_\lambda(s)}{s}ds} \geq C'_\alpha\alpha^L y^{-L} = C_\alpha y^{-L}.$$

Thus

$$h(y) \geq \frac{a(y)y^{1-L}}{\lambda + a(y)}$$

close to zero and clearly h is not bounded at 0 if $a(y) \to a_0 > 0$ (including $a_0 = \infty$) as $y \to 0$ contradicting the assumption that $h \in X_\infty$. If $a_0 = 0$, then we have two cases to consider: either $a(y)y^{1-L}$ is unbounded which, leads to the previous conclusion, or $a(y)y^{1-L}$ is bounded. Then the exponent in (3.39) can be written as

$$\int_1^y \frac{a(s)b(s|s)}{\lambda + a(s)}ds = \int_1^y \frac{a(s)s^{1-L}}{s^{2-L}}\frac{sb(s|s)}{\lambda + a(s)}ds$$

which means that

$$\varphi(0) = e^{-\int_0^1 \frac{a(s)b(s|s)}{\lambda + a(s)}} \neq 0$$

contrary to the construction (3.38).

If we assume (3.25), then in a similar way we obtain that

$$h(y) \geq C_1 \frac{a(y)y}{\lambda + a(y)} e^{\frac{C_2}{y^\beta}}$$

close to zero, where C_1 and C_2 are constants. As before, as long as $a_0 > 0$ then $h(y)$ is unbounded at zero since ye^{C_2/y^β} is unbounded. If $a_0 = 0$, then the numerator may be unbounded in which case the previous argument applies. Otherwise, we have $a(y) \leq Cy^{-1}e^{-C_2/y^\beta}$ and

$$\frac{a(s)b(s|s)}{\lambda + a(s)} \leq CM_2 \frac{e^{-C_2/s^\beta}}{s^{\alpha+2}}$$

with the right hand side tending to zero as $s \to 0$. This yields integrability on $(0, 1)$ of the exponent in (3.39) and $\varphi(0) \neq 0$, contradicting (3.38).

To prove the second part, first we note that $\{S_{G_{0,1}}(t)\}_{t\geq 0}$ is the restriction of the semigroup $\{S_G(t)\}_{t\geq 0}$, generated by G on X_1, to $X_{0,1}$. This follows from the resolvent formula for semigroups and the analogous statement for the generators. Therefore $\{S_{G_{0,1}}(t)\}_{t\geq 0}$ leaves the spaces $L_1([0, M], (1 + x)dx)$, $M > 0$, invariant (these are isometric to $L_1([0, M], dx)$). Clearly, if we take $u \geq 0$ with $supp\, u \subset [0, M]$, then $S_{G_{0,1}}(t)u \in D(A_{0,1})$ and, by direct integration,

$$\frac{d}{dt}\|S_{G_{0,1}}(t)u\|_{0,1} = \int_0^\infty (n(x) - 1)a(x)[S_{G_{0,1}}(t)u](x)dx. \tag{3.40}$$

It will be more convenient to work with the integrated version of (3.40):

$$\|S_{G_{0,1}}(t)]u\|_{0,1} = \|u\|_{0,1} + \int_0^\infty (n(x) - 1)a(x)\left(\int_0^t [S_{G_{0,1}}(s)u](x)ds\right) dx, \tag{3.41}$$

where the change of the order of integration is justified by positivity.

Now, given $0 \leq u_0 \in X_{0,1}$, we approximate it by $u_n := \chi_{[0,n]}u_0 \nearrow u_0$ in $X_{0,1}$. Then, for any $t \geq 0$, we have

$$S_{G_{0,1}}(t)u_n \nearrow S_{G_{0,1}}(t)u_0.$$

Consider

$$S(t)u_n = \int_0^t S_{G_{0,1}}(s)u_n ds.$$

By the dominated convergence (or monotonic as well), we have

$$S(t)u_n \nearrow S(t)u_0$$

in $X_{0,1}$ for any $t \geq 0$. Rewriting (3.41) for u_n, we see

$$\|S_{G_{0,1}}(t)u_n\|_{0,1} = \|u_n\|_{0,1} + \int_0^\infty (n(x)-1)a(x)\left(\int_0^t [S_{G_{0,1}}(s)u_n](x)ds\right)dx, \tag{3.42}$$

we see that the convergence of the norm terms imply convergence of the integral and, since the multiplication by $a(x)$ does not change monotonicity of the sequence we obtain

$$\|S_{G_{0,1}}(t)u_0\|_{0,1} = \|u_0\|_{0,1} + \int_0^\infty (n(x)-1)a(x)\left(\int_0^t [S_{G_{0,1}}(s)u_0](x)ds\right)dx, \tag{3.43}$$

Because $X_{0,1}$ is an L-space, see [16, Theorem 2.39], we can represent $[S_{G_{0,1}}(s)u_0](x)$ as a measurable function of two variables $\phi(x,s)$ and the strong integral with respect to s as the Lebesgue integral with respect to one variable s. Multiplication by $(n(x)-1)a(x)$ does not change the measurability hence, by Fubini theorem, we get

$$\begin{aligned}
\|S_{G_{0,1}}(t)u_0\|_{0,1} &= \|u_0\|_{0,1} + \int_0^\infty (n(x)-1)a(x)\left(\int_0^t [S_{G_{0,1}}(s)u_0](x)ds\right)dx \\
&= \|u_0\|_{0,1} + \int_0^\infty (n(x)-1)a(x)\left(\int_0^t \phi(x,s)ds\right)dx \\
&= \|u_0\|_{0,1} + \int_0^t \left(\int_0^\infty (n(x)-1)a(x)\phi(x,s)dx\right)ds \\
&= \|u_0\|_{0,1} + \int_0^t \left(\int_0^\infty (n(x)-1)a(x)[S_{G_{0,1}}(s)u_0](x)dx\right)ds.
\end{aligned}$$

If $u_0 \in D(G_{0,1})$, then the left hand side is differentiable and, since the inner integral in the last line is clearly integrable with respect to s, the derivative of the right hand side is this integrand (at least almost everywhere); that is

$$\frac{d}{dt}\|S_{G_{0,1}}(t)u_0\|_{0,1} = \int_0^\infty (n(x)-1)a(x)[S_{G_{0,1}}(t)u_0](x)dx, \quad a.e.$$

This, however, shows that $t \to \int_0^\infty (n(x)-1)a(x)[S_{G_{0,1}}(t)u_0](x)dx$ is continuous, and thus the above extends to all t. □

We note that while the theory for the non-shattering case has been developed up to a reasonably complete level, the shattering case and the case with infinite production of daughter particles still contain gaps and open problems. Therefore we present the results pertaining to the latter rather in the form of examples and comments; the research to fill the gaps and answer the open questions is ongoing.

3.5 Examples

Conditions (3.23)-(3.25) seem to be quite technical but they prove to be sharp for a large, and best understood, class of fragmentation processes governed by power laws; that is, for

$$a(x) = x^{\alpha}, \qquad b(x|y) = (\nu+2)\frac{x^{\nu}}{y^{\nu+1}}. \tag{3.44}$$

Here, $\alpha \in \mathbb{R}$; we exclude, however, the case $\alpha = 0$ which yields boundedness of all involved operators. On the other hand, the range of the parameter ν is restricted to $\nu \in (-2, 0]$. The reason for this is that for $\nu > 0$ the expected number of daughter particles after each fragmentation event is smaller than 2, which is nonphysical. On the other hand, if $\nu \leq -2$, then $\int_0^y xb(x|y)dx = \infty$, yielding an infinite mass of daughter particles after each split.

We note that in the power law case the expected number of daughter particles in each fragmentation event does not depend on the size y of the parent and equals

$$n(y) = \frac{\nu+2}{\nu+1}, \quad \nu > -1. \tag{3.45}$$

In this framework we have

$$B_{\lambda}(x) = \frac{b(x|x)}{\lambda + a(x)} = \frac{\nu+2}{x(\lambda + x^{\alpha})}$$

and (3.23) corresponds to $\alpha > 0$ and $-1 < \nu \leq 0$, whereas (3.24) is yielded by $\alpha < -1$ with arbitrary $\nu \in (-2, 0]$. The case (3.25) cannot be realized in the present framework, that is, for power law kernels.

We can state the following.

Corollary 3.5.1. *Let $\alpha > 0$ and $-1 < \nu \leq 0$. Then $G_{0,1}$ generates a positive semigroup $\{S_{G_{0,1}}(t)\}_{t\geq 0}$ on $X_{0,1}$ which, moreover, is honest; that is, for any $0 \leq u_0 \in D(G_{0,1})$*

$$\frac{d}{dt}\|S_{G_{0,1}}(t)\, u_0\|_{0,1} = \frac{1}{\nu+1}\int_0^{\infty} x^{\alpha}[S_{G_{0,1}}(t)\, u_0](x)dx.$$

Furthermore, $G_{0,1} = \overline{(A+B)|_{X_{0,1}}}^{X_{0,1}}$.

A more interesting result is contained in the next corollary. Usually shattering is associated with an infinite cascade of fragmentation events creating a dust of dimensionless particles which, however, carry some mass. Implicit in this interpretation is that we should have an infinite number of particles. The following result shows that such an interpretation is, in general, erroneous.

Corollary 3.5.2. *Let $\alpha < -1$.*

1. *If* $-1 < \nu \leq 0$*, then* $G_{0,1}$ *generates a once integrated semigroup in* $X_{0,1}$ *and therefore* $\|G_{0,1}(t)u_0\|_{0,1} < +\infty$ *for all* $t \geq 0$ *and* $u_0 \in D(G_{0,1}^2)$ *(e.g. with compact support).*

2. *If* $-2 < \nu \leq -1$*, then* $G_{0,1}$ *generates a twice integrated semigroup in* $X_{0,1}$ *and thus* $\|S_{G_{0,1}}(t)u_0\|_{0,1} < +\infty$ *for all* $t \geq 0$ *and* $u_0 \in D(G_{0,1}^3)$*.*

Thus, in both cases there are (many) trajectories along which the number of particles in the system remains finite for all times.

The fact that for $\alpha < -1$ and $\nu < -1$ we may have a finite number of particles in the system despite the expected number of particles in each split being infinite was noticed in [32] where the authors commented that

> *...a finite fraction of the total mass would be transferred to a finite number of particles with zero or infinitesimal mass! We conclude that a physically acceptable situation corresponds to* $\alpha > -1$*.*

Our interpretation of this case is different: in our opinion the fragmentation equation (3.1) 'sees' only a part of the system where only a finite number of 'physical' particles remains, while the mass is carried away by the dust which is beyond the resolution of (3.1).

The statement in Corollary 3.5.2.1 is rather due to our failure to prove the existence of the semigroup. However, it is possible to show the existence of a semigroup. For illustration, the next subsection provides a detail analysis of a binary fragmentation model with $\alpha = -2$. We shall prove that in this example $G_{0,1}$ generates a strongly continuous semigroup in $X_{0,1}$.

3.5.1 Binary fragmentation with $\alpha = -2$

The method we employ is similar to the analysis performed in [15, Example 6.5] and was used successfully to prove the generation of a semigroup for the binary fragmentation model with $\alpha = -1$. The aim of this subsection is to prove a similar result for a case with $\alpha < -1$, and we have focused on $\alpha = -2$ in detail.

We consider the binary fragmentation equation

$$\frac{\partial}{\partial t}u(t,x) = -x^{-2}u(t,x) + 2\int_x^{\infty} y^{-3}u(t,y)dy. \tag{3.46}$$

The formula for a solution of this equation [55] is,

$$u(t,x) = e^{-\frac{t}{x^2}}u_0(x) + 2t\int_x^{\infty} y^{-3}e^{-\frac{t}{y^2}}u_0(y)dy, \tag{3.47}$$

where u_0 represents the initial condition.

Proposition 3.5.3. *If $t \to u(t,\cdot)$ is a continuous $L_1([0,\infty), xdx)$-valued function which satisfies pointwise the integrated version of* (3.46)*:*

$$u(t,x) \;=\; u_0(x) - x^{-2}\int_0^t u(s,x)ds + 2\int_x^\infty y^{-3}\left(\int_0^t u(s,y)ds\right)dy,$$

then u is a mild solution of (3.46) *defined on its maximal domain and therefore must be given by the semigroup $(S_G(t))_{t\geq 0}$.*

Proof. See ([15, Theorem 4.1]) □

Lemma 3.5.4. *For any $t \geq 0$ and $y \in \mathbb{R}_+$, we have that*

$$ty^{-2}e^{-ty^{-2}} \leq e^{-1}. \tag{3.48}$$

Proof. For any $z \in \mathbb{R}_+$, we have

$$0 \leq ze^{-z} \leq e^{-1} \tag{3.49}$$

since $(ze^{-z})' = e^{-z} - ze^{-z} = e^{-z}(1-z)$. Setting $z = ty^{-2}$, the thesis follows. □

Proposition 3.5.5. *The semigroup $(S_G(t))_{t\geq 0}$ is given by*

$$S_G(t)u_0(x) = e^{-\frac{t}{x^2}}u_0(x) + 2t\int_x^\infty y^{-3}e^{-\frac{t}{y^2}}u_0(y)dy \tag{3.50}$$

for any $t \geq 0$ and any initial condition $u_0 \in D(G)$.

Proof. The proof is based on Proposition 3.5.3. We start by showing that $t \to u(t,\cdot)$ is a continuous $L_1([0,\infty), xdx)$-valued function, where $u(t,\cdot)$ is defined via (3.47). Clearly,

$$\|u(t,\cdot) - u(t_0,\cdot)\|_1 \;=\; \int_0^\infty x\left|\,[u(t,\cdot) - u(t_0,\cdot)](x)\right|dx = \int_0^\infty x|u(t,x) - u(t_0,x)|\,dx.$$

— If $t_0 = 0$, by the Fubini theorem,

$$\begin{aligned}
&\int_0^\infty x|u(t,x) - u(0,x)|\,dx \\
\leq\; &\int_0^\infty x\left|e^{-\frac{t}{x^2}} - 1\right||u_0(x)|\,dx + 2t\int_0^\infty\int_x^\infty xy^{-3}e^{-\frac{t}{y^2}}|u_0(y)|dy\,dx \\
=\; &\int_0^\infty x\left(1 - e^{-\frac{t}{x^2}}\right)|u_0(x)|\,dx + \int_0^\infty ty^{-2}e^{-\frac{t}{y^2}}(y|u_0(y)|)dy.
\end{aligned}$$

As a result $\lim_{t\to 0}\|u(t,\cdot) - u(0,\cdot)\|_1 = 0$, where we used the dominated convergence theorem.

— If $t_0 \neq 0$, by the Fubini theorem,

$$\begin{aligned}
&\int_0^\infty x|u(t,x)-u(t_0,x)|\,dx \\
\leq\ & \int_0^\infty x\left|e^{-\frac{t}{x^2}}-e^{-\frac{t_0}{x^2}}\right||u_0(x)|\,dx + 2|t-t_0|\int_0^\infty\int_x^\infty xy^{-3}e^{-\frac{t}{y^2}}|u_0(y)|dy\,dx \\
&+2t_0\int_0^\infty\int_x^\infty xy^{-3}\left|e^{-\frac{t}{y^2}}-e^{-\frac{t_0}{y^2}}\right||u_0(y)|dy\,dx \\
\leq\ & \int_0^\infty \left|e^{-\frac{t}{x^2}}-e^{-\frac{t_0}{x^2}}\right|(x|u_0(x)|)\,dx + \frac{|t-t_0|}{t}e^{-1}\|u_0\|_1 \\
&+t_0\int_0^\infty y^{-2}\left|e^{-\frac{t}{y^2}}-e^{-\frac{t_0}{y^2}}\right|(y|u_0(y)|)dy \\
=\ & S_1(t)+S_2(t)+S_3(t),
\end{aligned}$$

where we used (3.48). Because $0 \leq e^{-\frac{t}{y^2}} \leq 1$ for any positive t, we have that $\left|e^{-\frac{t}{y^2}}-e^{-\frac{t_0}{y^2}}\right| \leq 1$. By the dominated convergence theorem, $\lim_{t\to t_0} S_1(t) = 0$. Also since $t_0 \neq 0$, we have $\lim_{t\to t_0} S_2(t) = \lim_{t\to t_0}\frac{|t-t_0|}{t}e^{-1}\|u_0\|_1 = 0$. To complete the proof, we notice that for any $t \in [\frac{t_0}{2},\frac{3t_0}{2}]$, we have that $\left|e^{-\frac{t}{y^2}}-e^{-\frac{t_0}{y^2}}\right| \leq \max\left(e^{-\frac{t}{y^2}},e^{-\frac{t_0}{y^2}}\right) \leq e^{-\frac{t}{3y^2}}$. Thus

$$\int_0^\infty y^{-2}\left|e^{-\frac{t}{y^2}}-e^{-\frac{t_0}{y^2}}\right|(y|u_0(y)|)dy \leq \frac{6}{t_0}\int_0^\infty \frac{t}{3y^2}e^{-\frac{t}{3y^2}}(y|u_0(y)|)dy \leq \frac{6}{t_0}e^{-1}\|u_0\|_1,$$

where we used the fact that $t \geq \frac{t_0}{2}$ and (3.49) (with $z = \frac{t}{3y^2}$) respectively. Once again by the dominated convergence theorem, $\lim_{t\to t_0} S_3(t) = 0$. Therefore $t \to u(t,\cdot)$ is a continuous $L_1([0,\infty),xdx)$-valued function. It remains to show that

$$u(t,x) = u_0(x) - x^{-2}\int_0^t u(s,x)ds + 2\int_x^\infty y^{-3}\left(\int_0^t u(s,y)ds\right)dy.$$

By Fubini theorem,

$$\begin{aligned}
\int_0^t u(s,x)ds &= u_0(x)\int_0^t e^{-sx^{-2}}ds + 2\int_x^\infty z^{-3}u_0(z)\left(\int_0^t se^{-sz^{-2}}ds\right)dz \\
&= x^2\left(1-e^{-tx^{-2}}\right)u_0(x) \\
&\quad +2\int_x^\infty z^{-3}u_0(z)\left(-tz^2e^{-tz^{-2}}-z^4e^{-tz^{-2}}+z^4\right)dz
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^\infty y^{-3}\int_y^\infty z^{-3}u_0(z)\left(-tz^2e^{-tz^{-2}}-z^4e^{-tz^{-2}}+z^4\right)dzdy \\
=\ &\int_x^\infty\int_x^z y^{-3}z^{-3}u_0(z)\left(-tz^2e^{-tz^{-2}}-z^4e^{-tz^{-2}}+z^4\right)dydz \\
=\ &\int_x^\infty z^{-3}u_0(z)\left(-tz^2e^{-tz^{-2}}-z^4e^{-tz^{-2}}+z^4\right)\left(\frac{x^{-2}}{2}-\frac{z^{-2}}{2}\right)dz.
\end{aligned}$$

It follows that

$$\begin{aligned}
& u_0(x) - x^{-2}\int_0^t u(s,x)ds \\
= \; & u_0(x)e^{-tx^{-2}} - 2x^{-2}\int_x^\infty z^{-3}u_0(z)\left(-tz^2e^{-tz^{-2}} - z^4e^{-tz^{-2}} + z^4\right)dz
\end{aligned}$$

and

$$\begin{aligned}
& \int_x^\infty y^{-3}\left(\int_0^t u(s,y)ds\right)dy \\
= \; & \int_x^\infty y^{-1}\left(1-e^{-ty^{-2}}\right)u_0(y)dy \\
& + \int_x^\infty z^{-3}u_0(z)\left(-tz^2e^{-tz^{-2}} - z^4e^{-tz^{-2}} + z^4\right)\left(x^{-2}-z^{-2}\right)dz.
\end{aligned}$$

Thus

$$\begin{aligned}
& u_0(x) - x^{-2}\int_0^t u(s,x)ds + 2\int_x^\infty y^{-3}\left(\int_0^t u(s,y)ds\right)dy \\
= \; & u_0(x)e^{-tx^{-2}} + 2\int_x^\infty y^{-1}\left(1-e^{-ty^{-2}}\right)u_0(y)dy \\
& -2\int_x^\infty z^{-5}u_0(z)\left(-tz^2e^{-tz^{-2}} - z^4e^{-tz^{-2}} + z^4\right)dz \\
= \; & u(t,x).
\end{aligned}$$

Therefore $u(t,x) = e^{-\frac{t}{x^2}}u_0(x) + 2t\int_x^\infty y^{-3}e^{-\frac{t}{y^2}}u_0(y)dy$ satisfies pointwise the integrated version of (3.46). □

Lemma 3.5.6. *For any $f \in X_{0,1}$, the inequality below is satisfied:*

$$t\int_0^1\left(\int_x^\infty y^{-3}e^{-\frac{t}{y^2}}|f(y)|dy\right)dx \le (t+e^{-1})\|f\|_0. \tag{3.51}$$

Proof. Let $f \in X_{0,1}$. By the Fubini theorem, we have

$$\begin{aligned}
& \int_0^1\left(\int_x^\infty y^{-3}e^{-\frac{t}{y^2}}|f(y)|dy\right)dx \\
= \; & \int_0^1\left(\int_0^y y^{-3}e^{-\frac{t}{y^2}}|f(y)|dx\right)dy + \int_1^\infty\left(\int_0^1 y^{-3}e^{-\frac{t}{y^2}}|f(y)|dx\right)dy \\
\le \; & \int_0^1 y^{-2}e^{-\frac{t}{y^2}}|f(y)|dy + \int_0^\infty |f(y)|dy,
\end{aligned}$$

since $0 \le y^{-3}e^{-\frac{t}{y^2}} \le y^{-3} \le 1$ when $y > 1$. It follows that

$$t\int_0^1\left(\int_x^\infty y^{-3}e^{-\frac{t}{y^2}}|f(y)|dy\right)dx \le (t+e^{-1})\|f\|_0$$

where we used (3.48). □

Proposition 3.5.7. *The Banach space $X_{0,1}$ is an invariant subspace of $(S_G(t))_{t\geq 0}$.*

Proof. Let $t \geq 0$ and $f \in X_{0,1}$, we have

$$\begin{aligned}
\|S_G(t)f\|_0 &\leq \|S_G(t)f\|_1 + \int_0^1 \left| e^{-\frac{t}{x^2}} f(x) + 2t \int_x^\infty y^{-3} e^{-\frac{t}{y^2}} f(y) dy \right| dx \\
&\leq \|S_G(t)f\|_1 + \int_0^1 e^{-\frac{t}{x^2}} |f(x)| dx + 2t \int_0^1 \left(\int_x^\infty y^{-3} e^{-\frac{t}{y^2}} |f(y)| dy \right) dx \\
&\leq \|S_G(t)f\|_1 + \int_0^1 |f(x)| dx + 2(t + e^{-1}) \|f\|_0 \quad \text{by (3.51)} \\
&\leq \|S_G(t)f\|_1 + (1 + 2e^{-1} + 2t) \|f\|_0.
\end{aligned}$$

Since $(S_G(t))_{t\geq 0}$ is a substochastic semigroup in X_1, we have $\|S_G(t)f\|_1 \leq \|f\|_1$. It follows that

$$\begin{aligned}
\|S_G(t)f\|_{0,1} &\leq \|S_G(t)f\|_0 + \|S_G(t)f\|_1 \\
&\leq 2\|S_G(t)f\|_1 + (1 + 2e^{-1} + 2t)\|f\|_0 \\
&\leq 2\|f\|_1 + (1 + 2e^{-1} + 2t)\|f\|_0 \\
&= (2 + 2e^{-1} + 2t)\|f\|_{0,1} \\
&\leq 3e^t \|f\|_{0,1}.
\end{aligned}$$

□

Proposition 3.5.8. *The restriction $(S_{G_{0,1}}(t))_{t\geq 0}$ of the semigroup $(S_G(t))_{t\geq 0}$ to the Banach Space $X_{0,1}$ is a strongly continuous semigroup of bounded operators.*

Proof. Note that the restriction of the semigroup $(S_G(t))_{t\geq 0}$ to the space $X_{0,1}$ is a semigroup since $X_{0,1}$ is an invariant subspace of $(S_G(t))_{t\geq 0}$. Thus it is enough to show that $(S_G(t))_{t\geq 0}$ is strongly continuous at $t = 0$ in $X_{0,1}$. Let $f \in X_{0,1}$, we have

$$\begin{aligned}
&\|S_G(t)f - f\|_{0,1} = \|S_G(t)f - f\|_0 + \|S_G(t)f - f\|_1 \\
&\leq 2\|S_G(t)f - f\|_1 + \int_0^1 \left| \left(e^{-\frac{t}{x^2}} f(x) + 2t \int_x^\infty y^{-3} e^{-\frac{t}{y^2}} f(y) dy \right) - f(x) \right| dx \\
&\leq 2\|S_G(t)f - f\|_1 + \int_0^1 (1 - e^{-\frac{t}{x^2}}) |f(x)|\, dx + 2t \int_0^1 \left(\int_x^\infty y^{-3} e^{-\frac{t}{y^2}} |f(y)| dy \right) dx \\
&\to 0 \quad \text{as} \quad t \to 0^+.
\end{aligned}$$

Indeed, since $(S_G(t))_{t\geq 0}$ is a substochastic semigroup in X_1, we have $\lim_{t\to 0^+} \|S_G(t)f - f\|_1 = 0$. Furthermore, at any time $t \geq 0$, we have that $(1 - e^{-\frac{t}{x^2}}) \leq 1$. Because $f \in X_{0,1}$, the dominated convergence theorem yields $\lim_{t\to 0^+} \int_0^1 (1 - e^{-\frac{t}{x^2}}) |f(x)|\, dx = 0$. In the same way, by Lemma 3.5.6 and the dominated convergence theorem,

$$\lim_{t\to 0^+} t \int_0^1 \left(\int_x^\infty y^{-3} e^{-\frac{t}{y^2}} |f(y)| dy \right) dx = 0.$$

It follows that $\lim_{t\to 0^+} \|S_G(t)f - f\|_{0,1} = 0$. □

Hence the solution (3.47) of the fragmentation equation (3.46) is given by the semigroup $S_{G_{0,1}}(t))_{t\geq 0}$ generated by $G_{0,1}$ in the space $X_{0,1}$. Therefore for some specific models with $\alpha < -1$, the once integrated semigroup generated by $G_{0,1}$ and described in Corollary 3.5.1 is a strongly continuous semigroup of bounded linear operators.

So far we have not discussed the case when $\alpha \in [-1, 0)$ and $\nu \in (-2, 0]$ or $\alpha > 0$ and $\nu \in (-2, -1]$. This is done in the last subsection.

3.5.2 Case $\alpha \in [-1, 0)$ and $\nu \in (-2, 0]$ or $\alpha > 0$ and $\nu \in (-2, -1]$

It is easy to see that in both cases

$$\lim_{x\to 0} \lambda x B_\lambda(x) = \lim_{x\to 0} \frac{\lambda(\nu+2)}{\lambda + x^\alpha} \leq 1$$

(0 in the first case and $\nu + 2 < 1$ in the second one) and, at the same time,

$$\lim_{x\to 0} B_\lambda(x) = \infty$$

so this case is not covered by Corollary 3.3.3. It turns out that there is a reason for this since the integral condition (3.21) reads now

$$\int_0^\epsilon \frac{B_\lambda(x)}{x}\Delta_\lambda(x)dx = (\nu+2)\int_0^\epsilon e^{-\lambda(\nu+2)\int_x^1 \frac{ds}{s(\lambda+s^\alpha)}} \frac{dx}{x^2(\lambda + x^\alpha)}. \tag{3.52}$$

Direct integration shows that $\int_x^1 \frac{ds}{s(\lambda+s^\alpha)} = -\ln\left[\frac{x^{\frac{1}{\lambda}}(\lambda+1)^{\frac{1}{\alpha\lambda}}}{(\lambda+x^\alpha)^{\frac{1}{\alpha\lambda}}}\right]$, where we used the method of substitution with $w = s^\alpha$. It follows that the integrand on the right-hand side of (3.52) is equal to $(\lambda+1)^{\frac{\nu+2}{\alpha}} x^\nu (\lambda + x^\alpha)^{-(1+\frac{\nu+2}{\alpha})}$. As a result, it behaves as x^ν for $\alpha > 0$ and as $x^{-\alpha-2}$ for $\alpha < 0$. Hence (3.21) is not satisfied in either case discussed in this remark and therefore $R(\lambda)X_{0,1} \nsubseteq X_{0,1}$. Therefore the restriction of G to $X_{0,1}$ cannot generate any reasonable dynamics there.

This is supported by the following example. The case with $\alpha = -1$ and $\nu = 0$ gives the equation

$$\frac{\partial}{\partial t}u(x,t) = -x^{-1}u(x,t) + 2\int_x^\infty y^{-2}u(y,t)dy,$$

for which we have

$$B_\lambda(x) = \frac{2}{x(\lambda + x^{-1})} \to 2 \neq 0$$

and

$$xB_\lambda(x) = \frac{2}{\lambda + x^{-2}} \to 0 < 1.$$

By [15, Example 6.5], the X_1-semigroup is represented by

$$u(x,t) = e^{-\frac{t}{x}}u_0(x) + 2t\int_x^\infty \frac{e^{-\frac{t}{y}}}{y^2}e^{-\frac{t}{y}}u_0(y)dy + t^2\int_x^\infty \frac{e^{-\frac{t}{y}}}{y^2}\left(\frac{1}{x}-\frac{1}{y}\right)u_0(y)dy.$$

For $u_0 \geq 0$, each term is nonnegative. Taking the X_0 norm of the last term we obtain

$$2t^2\int_0^\infty \frac{e^{-\frac{t}{y}}}{y^2}u_0(y)\left(\int_0^y\left(\frac{1}{x}-\frac{1}{y}\right)dx\right)dy = \infty \tag{3.53}$$

for any nonzero $u_0 \geq 0$ and any $t > 0$, irrespectively whether $\|u_0\|_0 < \infty$ or not. Hence, the number of particles in this case immediately becomes infinite and stays infinite for all times.

Chapter 4

Coagulation-Fragmentation models

4.1 Preliminaries

The dynamical behaviour of a system of particles that can combine to form larger particles or break up to produce smaller particles is given by the integro-differential equation

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) = &- a(x)u(t,x) + \int_{x+x_0}^{\infty} a(y)b(x|y)u(t,y)dy \\ &+ \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)u(t,x-y)u(t,y)dy, \\ &- u(t,x)\int_{x_0}^{\infty} k(x,y)u(t,y)dy,\end{aligned} \tag{4.1}$$

where χ_U is the characteristic function of the interval $U = [2x_0, \infty)$. In this equation u is the particle mass distribution function, a is the fragmentation rate and $b(x|y)$ is the distribution of particle masses x spawned by the fragmentation of a particle of mass y. The coagulation kernel $k(x,y)$ is the rate at which particles of mass x coalesce with particles of mass y. The characteristic function χ_U ensures no particle of mass $x < 2x_0$ can emerge as a result of coagulation.

The terms on the right side of (4.1) describe, from left to right, the reduction in the number of particles in the mass range $(x; x+dx)$ due to the fragmentation of particles in the same range, the increase in the number of particles in the range due to fragmentation of larger particles, the increase in the number of particles of mass $x \geq 2x_0$ as the result of particles of mass $x-y$ and mass $y \quad (x_0 \leq y \leq x-x_0)$ merging to form a particle of mass x and the last term accounts for the loss of particles of mass x because they have coalesced with particles of mass $y, \quad y \geq x_0$. Note that the factor $1/2$ takes into account that either a particle of mass $x-y$ coalesces with one of mass y or vice versa.

The main purpose of this chapter is to make use of substochastic semigroup theory and semilinear ACP techniques to analyze the coagulation-fragmentation equation in the space $X_1 = L_1([x_0,\infty), xdx) = \left\{\psi : \ \|\psi\|_1 := \int_{x_0}^{\infty} x|\psi(x)|\,dx < \infty\right\}$. This has been done in the past only in the space $X_{0,1} = L_1([x_0,\infty), (1+x)dx)$ since the fragmentation equation is known to behave well in the space X_1 and the coagulation operator in the space $X_0 = L_1([x_0,\infty), dx)$. We shall prove existence and uniqueness of conservative solutions for the coagulation-fragmentation equation in a biological population. Another important aspect of this chapter covers the case $x_0 = 0$. We extend some existing results on coagulation-fragmentation to some models with fragmentation rate unbounded at 0 and growing faster than x at infinity.

Before proceeding to the abstract setting of (4.1), we provide some assumptions that we use throughout the rest of this chapter. We assume that the fragmentation rate a is essentially bounded on compact subintervals of (x_0,∞); ie

$$a \in L_{\infty,loc}((x_0,\infty)). \tag{4.2}$$

Particles of sizes less than $2x_0$ do not fragment since the minimum size of a particle is x_0. Therefore we assume that

$$a(x) = 0 \quad \text{for} \quad x < 2x_0, \tag{4.3}$$

and

$$b(x|y) = 0 \quad \text{for} \quad y < x + x_0. \tag{4.4}$$

Normally it is expected that the total mass in the system is a conserved quantity during fragmentation, and hence b is usually assumed to satisfy the condition

$$\int_{x_0}^{y-x_0} xb(x|y)dx = y, \quad \text{for each } y > 2x_0. \tag{4.5}$$

We assume as well that the coagulation kernel k is a nonnegative function in $L_\infty((x_0,\infty)\times(x_0,\infty))$ with

$$k_0 := \operatorname{ess\,sup}\{k(x,y); \quad (x,y) \in (x_0,\infty)\times(x_0,\infty)\}. \tag{4.6}$$

4.1.1 Abstract reformulation

The idea is to analyze the problem by rephrasing it in abstract form (ACP) as an ordinary differential equation. Let $\mathcal{A}$, $\mathcal{B}$ denote the first expressions appearing on the right-hand side of (4.1); that is

$$(\mathcal{A}u)(x) = -a(x)u(x), \tag{4.7}$$

$$(\mathcal{B}u)(x) = \int_{x+x_0}^{\infty} a(y)b(x|y)u(y)\,dy. \tag{4.8}$$

With the expressions of $\mathcal{A}$, $\mathcal{B}$ we associate operators A and B in X_1 defined by

$$[Au](x) = [\mathcal{A}u](x), \qquad [Bu](x) = [\mathcal{B}u](x).$$

The operator A is defined on $D(A) = \{u \in X_1;\ \ au \in X_1\}$. Direct integration shows that for $0 \leq u \in D(A)$, $Bu \in X_1$, so that we can take $D(B) = D(A)$ and $(A + B, D(A))$ is well-defined.

Accordingly, we define the coagulation operator K on X_1 by

$$\begin{aligned}(K\psi)(x) &:= \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)\psi(x-y)\psi(y)dy - \psi(x)\int_{x_0}^{\infty} k(x,y)\psi(y)dy \\ &= \mathcal{K}_1[\psi,\psi](x) - \mathcal{K}_2[\psi,\psi](x) \\ &= \mathcal{K}[\psi,\psi](x),\end{aligned}$$

where for $\psi, \phi \in X_1$,

$$\begin{aligned}\mathcal{K}_1[\psi,\phi](x) &= \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)\psi(x-y)\phi(y)dy \\ \mathcal{K}_2[\psi,\phi](x) &= \psi(x)\int_{x_0}^{\infty} k(x,y)\phi(y)dy.\end{aligned}$$

Theorem 4.1.1. *There exists a smallest substochastic semigroup* $(S_G(t))_{t\geq 0}$ *on* X_1 *generated by an extension* G *of* $(A + B, D(A))$.

Proof. The proof is based on Kato-Voigt Perturbation theorem. It is obvious that A generates a substochastic semigroup in X_1. We also have that $Bu \geq 0$ for $u \in D(B)_+$. Furthermore $\int_{x_0}^{\infty}[Au+Bu](x)xdx \leq 0$ for any $u \in D(A)_+$. In fact by the Fubini theorem,

$$\begin{aligned}\int_{x_0}^{\infty}[Bu](x)xdx &= \int_{x_0}^{\infty}\left(\int_{x+x_0}^{\infty} a(y)b(x|y)u(y)\,dy\right)xdx \\ &= \int_{2x_0}^{\infty} a(y)u(y)\left(\int_{x_0}^{y-x_0} xb(x|y)\,dx\right)dy \\ &= \int_{2x_0}^{\infty} ya(y)u(y)dy,\end{aligned}$$

where we made use of (4.5). The desired result follows from (4.3) and Theorem 2.3.5. □

Theorem 4.1.2. *The semigroup* $(S_G(t))_{t\geq 0}$ *is honest.*

Proof. The argument we use is similar to the one used in the proof of [16, Theorem 8.5]. Note that for any $u \in D(A)_+$ we have that $\displaystyle\int_{x_0}^{\infty}[Au + Bu](x)xdx = 0$. By [16, Theorem 6.22], it is enough to prove that for any $\psi \in X_{1+}$ such that $-a\psi + \mathcal{B}\psi \in X_1$, we have the inequality

$$\int_{x_0}^{\infty}[-a\psi + \mathcal{B}\psi](x)xdx \geq 0.$$

By (4.2), the function $a\psi$ satisfies $a\psi \in L_1([x_0, R], xdx)$ for any $x_0 < R < \infty$, therefore the same is true for $\mathcal{B}\psi$. Thus

$$\begin{aligned} & \int_{x_0}^{\infty} [-a\psi + \mathcal{B}\psi](x)xdx \\ = & \lim_{R\to\infty} \left(-\int_{x_0}^{R} a(x)\psi(x)xdx + \int_{x_0}^{R} \left(\int_{x+x_0}^{\infty} a(y)b(x|y)\psi(y)\,dy \right) xdx \right). \end{aligned}$$

Next, by (4.5) and the Fubini theorem,

$$\begin{aligned} & \int_{x_0}^{R} \left(\int_{x+x_0}^{\infty} a(y)b(x|y)\psi(y)\,dy \right) xdx \\ = & \int_{2x_0}^{R+x_0} a(y)\psi(y) \left(\int_{x_0}^{y-x_0} xb(x|y)\,dx \right) dy + \int_{R+x_0}^{\infty} a(y)\psi(y) \left(\int_{x_0}^{R} xb(x|y)\,dx \right) dy \\ = & Z_R + \int_{2x_0}^{R} ya(y)\psi(y)dy, \end{aligned}$$

where

$$Z_R = \int_{R+x_0}^{\infty} a(y)\psi(y) \left(\int_{x_0}^{R} xb(x|y)\,dx \right) dy + \int_{R}^{R+x_0} ya(y)\psi(y)dy \geq 0.$$

Combining we see that

$$\int_{x_0}^{\infty} [-a\psi + \mathcal{B}\psi](x)xdx \;\; = \;\; \lim_{R\to\infty} Z_R \geq 0,$$

where we used (4.3). The thesis follows from [16, Theorem 6.13 and Theorem 6.22]. □

4.2 Analysis of the evolution equation for $x_0 \neq 0$

In this section, we intend to analyze the semi-linear problem

$$\frac{du}{dt}(t) = G[\,u(t)] + K[\,u(t)]\,, t > 0, \quad u(0) = u_0\,, \tag{4.9}$$

within the framework of the Banach space X_1. This approach is new in the sense that previous investigations of the coagulation-fragmentation equation using semilinear ACP theory have been carried out in the space $X_{0,1}$. The main difference is that thanks to $x_0 \neq 0$ we are able to prove global existence of the solutions without assuming that the fragmentation rate is linearly bounded.

Remark 3. In the previous chapter, the generation of a semigroup by the fragmentation operator in the space $X_{0,1}$ was proved for the separable kernel $b(x|y) = \beta(x)\gamma(y)$. This can be extended to the case $x_0 \neq 0$ with arbitrary kernel b, including $b(x|y) = y^{-1}h(x/y)$

for some suitable function h. This result simply follows from the fact that the space X_1 is topologically equivalent to the space $X_{0,1}$. It ensures that the semigroup $(S_G(t))_{t\geq 0}$ introduced in Theorem 4.1.1 is a strongly continuous semigroup generated by G in the space $X_{0,1}$.

Next we explore the properties of the coagulation operator K.

Proposition 4.2.1. *$K(X_1) \subset X_1$ with $\|K\psi\|_1 \leq 2\frac{k_0}{x_0}\|\psi\|_1^2$ for all $\psi \in X_1$.*

Proof. Let $\psi, \phi \in X_1$, we have

$$\begin{aligned}
&\int_{x_0}^{\infty}\int_{x_0}^{x-x_0} x|\psi(x-y)\phi(y)|\,dy\,dx \\
&= \int_{x_0}^{\infty}\int_{y+x_0}^{\infty} x|\psi(x-y)\phi(y)|\,dx\,dy \\
&= \int_{x_0}^{\infty}\int_{x_0}^{\infty} (z+y)|\psi(z)\phi(y)|\,dz\,dy,
\end{aligned}$$

where $x > 2x_0$. It follows that

$$\begin{aligned}
\|\mathcal{K}_1[\psi,\phi]\|_1 &\leq \int_{x_0}^{\infty} x\frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)|\psi(x-y)\phi(y)|\,dy\,dx \\
&\leq \frac{k_0}{2}\int_{x_0}^{\infty}\int_{x_0}^{x-x_0} x|\psi(x-y)\phi(y)|\,dy\,dx \\
&\leq \frac{k_0}{x_0}\left(\int_{x_0}^{\infty}\int_{x_0}^{\infty} zy|\psi(z)\phi(y)|\,dz\,dy\right) \\
&= \frac{k_0}{x_0}\|\psi\|_1\|\phi\|_1.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|\mathcal{K}_2[\psi,\phi]\|_1 &\leq \int_{x_0}^{\infty}\int_{x_0}^{\infty} xk(x,y)|\psi(x)\phi(y)|dy\,dx \\
&\leq k_0\int_{x_0}^{\infty}\int_{x_0}^{\infty} x|\psi(x)\phi(y)|dy\,dx \\
&\leq \frac{k_0}{x_0}\|\psi\|_1\|\phi\|_1.
\end{aligned}$$

The result follows. □

Proposition 4.2.2. *K is locally Lipschitz on X_1.*

Proof. Let $u_0 \in X_1$ and $\psi, \phi \in \overline{B}(u_0, \rho) := \{h \in X_1 : \|h - u_0\|_1 \leq \rho\}$, then

$$\begin{aligned} \|K\psi - K\phi\|_1 &= \|\mathcal{K}[\psi,\psi] - \mathcal{K}[\phi,\phi]\|_1 \\ &= \|\mathcal{K}[\psi-\phi,\psi] - \mathcal{K}[\phi,\psi-\phi]\|_1 \\ &\leq 2\frac{k_0}{x_0}(\|\psi-\phi\|_1\|\psi\|_1 + \|\phi\|_1\|\psi-\phi\|_1) \\ &= 2\frac{k_0}{x_0}\|\psi-\phi\|_1(\|\psi\|_1 + \|\phi\|_1) \\ &\leq \varrho_{\rho,u_0}\|\psi-\phi\|_1, \end{aligned}$$

where

$$\varrho_{\rho,u_0} = 4\frac{k_0}{x_0}(\rho + \|u_0\|_1), \tag{4.10}$$

and so K is locally Lipschitz on X_1. □

Proposition 4.2.3. *K is Fréchet differentiable on X_1 and for any $\psi \in X_1$, the Fréchet derivative K_ψ is expressed by*

$$K_\psi\phi := \mathcal{K}[\psi,\phi] + \mathcal{K}[\phi,\psi], \quad \forall\ \phi \in X_1.$$

Moreover the Fréchet derivative is continuous with respect to ψ.

Proof. Let $\psi, \delta \in X_1$. The bilinearity of $\mathcal{K}$ leads to

$$\begin{aligned} K(\psi+\phi) &= \mathcal{K}[\psi+\phi,\psi+\phi] \\ &= \mathcal{K}[\psi,\psi] + \mathcal{K}[\psi,\phi] + \mathcal{K}[\phi,\psi] + \mathcal{K}[\phi,\phi]. \end{aligned}$$

For fixed ψ , $\mathcal{K}[\psi,\cdot] + \mathcal{K}[\cdot,\psi]$ is a bounded operator on X_1 with

$$||\,\mathcal{K}[\psi,\delta] + \mathcal{K}[\delta,\psi]\,||_1 \leq 4\frac{k_0}{x_0}\,||\psi||_1\,||\delta||_1 \quad \forall\delta \in X_1\,.$$

Also

$$\frac{||K\delta||_1}{||\delta||_1} \leq 2\frac{k_0}{x_0}\,||\delta||_1 \;\; \to 0 \;\text{ as }\; ||\delta||_1 \to 0\,.$$

Hence K is Fréchet differentiable at each $\psi \in X_1$ and the Fréchet derivative K_ψ at ψ is given by

$$K_\psi\phi := \mathcal{K}[\psi,\phi] + \mathcal{K}[\phi,\psi] \quad \forall\phi \in X_1\,.$$

Consequently,

$$\|K_\psi\phi\|_1 \leq \varrho_{\rho,u_0}\|\phi\|_1, \quad \forall\ \phi \in X_1,\ \psi \in \overline{B}(u_0,\rho).$$

Also, for $\psi_1, \psi_2, \phi \in X_1$,

$$\begin{aligned} \|K_{\psi_1}\phi - K_{\psi_2}\phi\|_1 &= \|\mathcal{K}[\psi_1,\phi] + \mathcal{K}[\phi,\psi_1] - \mathcal{K}[\psi_2,\phi] - \mathcal{K}[\phi,\psi_2]\|_1 \\ &= \|\mathcal{K}[\psi_1-\psi_2,\phi] + \mathcal{K}[\phi,\psi_1-\psi_2]\|_1 \\ &\leq 4\frac{k_0}{x_0}\|\phi\|_1\|\psi_1-\psi_2\|_1 \to 0 \quad \text{as} \quad \|\psi_1-\psi_2\|_1 \to 0. \end{aligned}$$

Hence, the Fréchet derivative is continuous with respect to ψ. □

4.2.1 Local existence

Theorem 4.2.4. Local existence of a solution
There exist positive constants ρ_0, t_0 and a strongly differentiable function

$$u : [0, t_0) \to B(u_0, \rho_0) := \{h \in X_1 : \ \|h - u_0\|_1 < \rho_0\}$$

such that

$$\frac{du}{dt}(t) = G[u(t)] + K[u(t)], \ \ 0 < t < t_0; \ u(0) = u_0 \in D(G) \cap X_{1+}, \tag{4.11}$$

where $X_{1+} = \{\psi \in X_1 : \ \psi \geq 0 \ a.e. \ on \, (x_0, \infty)\}$.

Proof. Considering the properties of the nonlinear operator K and the fact that G is the generator of a strongly continuous semigroup, the theorem follows from standard results on semilinear ACPs. □

To show that this local (in time) solution is in X_{1+} for all $t \in [0, t_0)$, we adopt the argument used in [20, Chapter 8]. This was applied to coagulation models by Banasiak and Lamb in their recent paper [18]. First we note that the solution u of (4.11) is also the unique strongly differentiable solution of

$$\frac{du}{dt}(t) = (G[u(t)] - \alpha u(t)) + (\alpha u(t) + K[u(t)]) \tag{4.12}$$

for any $\alpha \in \mathbb{R}$. Hence u is the unique solution of the integral equation

$$u(t) = e^{-\alpha t} S_G(t) u_0 + \int_0^t e^{-\alpha(t-s)} S_G(t-s) K_\alpha[u(s)] ds, \ 0 \leq t < t_0, \tag{4.13}$$

where $K_\alpha := K + \alpha I$.

Lemma 4.2.5. *Let $\alpha \geq \frac{k_0}{x_0}(\|u_0\|_1 + \rho_0)$. Then $K_\alpha \psi \in X_{1+}$ for all $\psi \in B(u_0, \rho_0) \cap X_{1+}$.*

Proof. By definition, we have

$$K_\alpha \psi = \alpha \psi + \mathcal{K}_1[\psi, \psi] - \mathcal{K}_2[\psi, \psi].$$

Clearly $\mathcal{K}_1[\psi, \psi] \in X_{1+}$ for all $\psi \in X_{1+}$. Also, for $\psi \in B(u_0, \rho_0) \cap X_{1+}$,

$$\begin{aligned} \psi(x) \int_{x_0}^{\infty} k(x, y) \psi(y) dy \ &\leq \ \frac{k_0}{x_0} \psi(x) \|\psi\|_1 \\ &\leq \ \frac{k_0}{x_0} (\|u_0\|_1 + \rho_0) \psi(x). \end{aligned}$$

Hence

$$\begin{aligned} \alpha \psi(x) - \mathcal{K}_2[\psi, \psi](x) \ &\geq \ \alpha \psi(x) - \frac{k_0}{x_0} (\|u_0\|_1 + \rho_0) \psi(x) \\ &\geq \ 0 \text{ provided that } \alpha \geq \frac{k_0}{x_0} (\|u_0\|_1 + \rho_0). \end{aligned}$$

□

Theorem 4.2.6. *Let $u_0 \in D(G) \cap X_{1+}$ and let $u : [0, t_0) \to B(u_0, \rho_0)$ be the unique strict solution of* (4.11)*. Then there exists $t_1 \in (0, t_0]$ such that $u(t) \in X_{1+}$ for all $t \in [0, t_1)$.*

Proof. Let $Y := C([0, t_1], X)$ with norm $\|v\|_Y := \max\{\|v(t)\|_1 : 0 \leq t \leq t_1\}$. Moreover, let

$$\Psi := \{v \in Y : v(t) \in \overline{B}(u_0, \rho_1) \cap X_{1+} \ \forall t \in [0, t_1]\}$$

where $0 < \rho_1 < \rho_0$, and define

$$\begin{aligned} (\mathcal{Q}v)(t) &:= e^{-\alpha t} S_G(t) u_0 + \int_0^t e^{-\alpha(t-s)} S_G(t-s) K_\alpha[v(s)] ds, \ 0 \leq t \leq t_1, \\ D(\mathcal{Q}) &:= \Psi, \end{aligned}$$

with $\alpha \geq \frac{k_0}{x_0}(\|u_0\|_1 + \rho_0)$. Then $\mathcal{Q}(\Psi) \subset Y$ and $(\mathcal{Q}v)(t) \in X_{1+}$ for all $t \in [0, t_1]$. Also, for all $v, w \in \Psi$,

$$\begin{aligned} \|(\mathcal{Q}v)(t) - (\mathcal{Q}w)(t)\|_1 &\leq \int_0^t e^{-\alpha(t-s)} \|S_G(t-s)\|_{\mathcal{L}(X_1)} \|K_\alpha[v(s)] - K_\alpha[w(s)]\|_1 \, ds \\ &\leq \int_0^t e^{(-\alpha)(t-s)} \|K_\alpha[v(s)] - K_\alpha[w(s)]\|_1 \, ds \\ &\leq (\varrho_{\rho_0, u_0} + \alpha) \int_0^t e^{(-\alpha)(t-s)} \|v(s) - w(s)\|_1 \, ds, \end{aligned}$$

where ϱ_{ρ_0, u_0} is defined via (4.10), $\mathcal{L}(X_1)$ is the set of bounded linear operator on X_1 and we used the fact that the semigroup $(S_G(t))_{t \geq 0}$ is substochastic, see Theorem 4.1.1. Hence

$$\|\mathcal{Q}v - \mathcal{Q}w\|_Y \leq (\varrho_{\rho_0, u_0} + \alpha) t_1 \|v - w\|_Y.$$

Similarly,

$$\begin{aligned} & \|(\mathcal{Q}v)(t) - u_0\|_1 \\ \leq & \ \|e^{-\alpha t} S_G(t) u_0 - u_0\|_1 + \int_0^t e^{-\alpha(t-s)} \|S_G(t-s) K_\alpha[v(s)]\|_1 \, ds \\ \leq & \ \|e^{-\alpha t} S_G(t) u_0 - u_0\|_1 + \int_0^t e^{(-\alpha)(t-s)} \|K_\alpha[v(s)]\|_1 \, ds. \end{aligned} \tag{4.14}$$

Now

$$\begin{aligned} \|K_\alpha[v(s)]\|_1 &= \|K_\alpha[v(s)] - K_\alpha u_0 + K_\alpha u_0\|_1 \\ &\leq \|K_\alpha[v(s)] - K_\alpha u_0\|_1 + \|K_\alpha u_0\|_1 \\ &\leq (\varrho_{\rho_0, u_0} + \alpha) \|v(s) - u_0\|_1 + \|K u_0\|_1 + \alpha \|u_0\|_1 \\ &\leq (\varrho_{\rho_0, u_0} + \alpha) \rho_1 + \|K u_0\|_1 + \alpha \|u_0\|_1. \end{aligned}$$

Hence the expression in (4.14) is bounded above by

$$\|e^{-\alpha t} S_G(t) u_0 - u_0\|_1 + ((\varrho_{\rho_0, u_0} + \alpha) \rho_1 + \|K u_0\|_1 + \alpha \|u_0\|_1) t_1.$$

If we now define

$$\zeta(t_1) := \frac{1}{\rho_1}\max_{0\le t\le t_1}\{\|e^{-\alpha t}S_G(t)u_0 - u_0\|_1\} + \frac{1}{\rho_1}((\varrho_{\rho_0,u_0}+\alpha)\rho_1 + \|Ku_0\|_1 + \alpha\|u_0\|_1)t_1,$$

then it follows that

$$\|(\mathcal{Q}v)(t) - u_0\|_1 \le \rho_1\zeta(t_1), \quad \forall \in [0,t_1] \qquad \text{and}$$

$$\|\mathcal{Q}v - \mathcal{Q}w\|_Y \le \zeta(t_1)\|v-w\|_Y, \ \ \forall v,w \in \Psi.$$

Since $\zeta(t_1) \to 0^+$ as $t_1 \to 0^+$, we can choose t_1 so that $0 < \zeta(t_1) < 1$, in which case $\mathcal{Q}(\Psi) \subset \Psi$. Hence there exists a unique solution $u \in \Psi$ of $u = \mathcal{Q}u$ and so the integral equation (4.13) has a unique solution $u \in C([0,t_1], X_{1+})$. □

Corollary 4.2.7. *Let the maximal interval of existence of the strict solution u of* (4.11) *be $[0,\widehat{\tau})$. Then $u(t) \in X_{1+}$ for all $t \in [0,\widehat{\tau})$ whenever $f \in D(G) \cap X_{1+}$.*

Proof. Let $\tau_0 \in (0,\widehat{\tau})$ be arbitrarily fixed and define

$$\tau_{\max} := \sup\{0 < \tau < \tau_0 : \ \ u(t) \in X_{1+} \ \text{for all}\, t \in [0,\tau]\}.$$

Suppose that $\tau_{\max} < \tau_0$ and consider the semi-linear problem

$$\frac{dv}{dt}(t) = G[v(t)] + K[v(t)], \ t > 0; \ v(0) = u(\tau_{\max}). \tag{4.15}$$

The solution of (4.15) on $[0,\tau_0-\tau_{\max}]$ is $v(t) = u(t+\tau_{\max})$. Since X_{1+} is closed, $u(\tau_{\max}) \in X_{1+}$ and the previous analysis shows that $u(t+\tau_{\max}) \in X_{1+}$ for sufficiently small t. This contradicts the definition of $\tau_{\max}$ and therefore $u(t) \in X_{1+}$ for all $t \in [0,\tau_0]$. □

4.2.2 Global existence

To prove the global (in time) existence of a strict non-negative solution to (4.11) we shall establish that the local solution cannot blow up in finite time [46].

Lemma 4.2.8. *If $\psi \in D(G) \cap X_{1+}$ then*

$$\int_{x_0}^{\infty} x(G\psi)(x)dx = 0 \tag{4.16}$$

Proof. The result follows directly from the honesty of the semigroup $(S_G(t))_{t\ge 0}$ generated by the operator G, see Theorem 4.1.2. □

Lemma 4.2.9. *If $\psi \in X_{1+}$, then*

$$\int_{x_0}^{\infty} x(K\psi)(x)dx = 0.$$

Proof. Let $\psi \in X_{1+}$. We have

$$\begin{aligned}
&\int_{x_0}^{\infty}\int_{x_0}^{x-x_0} xk(x-y,y)\psi(x-y)\psi(y)\chi_U(x)dydx \\
=& \int_{x_0}^{\infty}\int_{y+x_0}^{\infty} xk(x-y,y)\psi(x-y)\psi(y)dxdy \\
=& \int_{x_0}^{\infty}\int_{x_0}^{\infty} (z+y)k(z,y)\psi(z)\psi(y)dzdy \\
=& \; 2\int_{x_0}^{\infty}\int_{x_0}^{\infty} xk(x,y)\psi(x)\psi(y)dydx,
\end{aligned}$$

where we used the fact that $k(x,y) = k(y,x)$. It follows that

$$\begin{aligned}
\int_{x_0}^{\infty} x(K\psi)(x)dx =& \frac{1}{2}\int_{x_0}^{\infty}\left(\int_{x_0}^{x-x_0} xk(x-y,y)\psi(x-y)\psi(y)\chi_{U(x)}dy\right)dx \\
&-\int_{x_0}^{\infty}\int_{x_0}^{\infty} xk(x,y)\psi(x)\psi(y)dydx \\
=&0.
\end{aligned} \tag{4.17}$$

□

Theorem 4.2.10. *The abstract Cauchy problem* (4.11) *has a unique, global, non-negative conservative solution u for each $u_0 \in D(G) \cap X_{1+}$.*

Proof. Because the local solution u is a non-negative solution of (4.11), it follows from the previous two lemmas that

$$\frac{d}{dt}\|u(t)\| \;=\; \int_{x_0}^{\infty} x(G[u(t)])(x)dx + \int_{x_0}^{\infty} x(K[u(t)])(x)dx = 0$$

for $0 \le t < \widehat{\tau}$. Therefore $\|u(t)\|_1 = \|u_0\|_1$ for all $t \in [0, \widehat{\tau})$. Consequently u does not blow up in finite time. The result follows. □

4.3 Analysis of the evolution equation for $x_0 = 0$

In this section, we assume that we have a separable distribution rate kernel $b(x|y) = \beta(x)\gamma(y)$. By Theorem 4.1.1, the fragmentation operator G defined in X_1 generates a C_0-semigroup. From the analysis performed in the previous chapter, the part $G_{0,1}$ of the fragmentation operator G on the space $X_{0,1}$ generates a strongly continuous semigroup $(S_{G_{0,1}}(t))_{t\ge 0}$. This result shall be used to expand the work of Lamb, McBride and McLaughlin [42] on coagulation-fragmentation models with linearly bounded fragmentation rates to some models with arbitrary fragmentation rates.

With some abuse of notation, the part $K_{0,1}$ of the coagulation operator K in $X_{0,1}$ will still be denoted K. Note that because $x_0 = 0$, the coagulation operator does not behave well in the space X_1 . As a result, it is necessary to analyze the semi-linear problem

$$\frac{du}{dt}(t) = G_{0,1}[\,u(t)] + K[\,u(t)]\,, t > 0, \quad u(0) = u_0\,, \tag{4.18}$$

within the framework of the Banach space $X_{0,1}$. In this view, the following lemma is important.

Lemma 4.3.1. *The operator $\mathcal{K} : X_{0,1} \times X_{0,1} \to X_{0,1}$ is bilinear and*

$$||\,\mathcal{K}[\psi,\phi]\,||_{0,1} \le 2k_0\,||\psi||_{0,1}\,||\phi||_{0,1} \quad \forall \psi,\phi \in X_{0,1}\,. \tag{4.19}$$

Proof. The operator $\mathcal{K}$ is clearly bilinear. Let $\psi, \phi \in X$,

$$\begin{aligned}
\|\mathcal{K}_1[\psi,\phi]\|_{0,1} &\le \frac{k_0}{2}\int_0^\infty\int_0^x (1+x)|\psi(x-y)\phi(y)|\,dy\,dx \\
&= \frac{k_0}{2}\int_0^\infty\int_y^\infty (1+x)|\psi(x-y)\phi(y)|\,dx\,dy \\
&= \frac{k_0}{2}\int_0^\infty\int_0^\infty (1+z+y)|\psi(z)|\,|\phi(y)|\,dz\,dy \\
&\le \frac{k_0}{2}\left(\int_0^\infty\int_0^\infty (1+z)(1+y)|\psi(z)|\,|\phi(y)|\,dz\,dy\right) \\
&= \frac{k_0}{2}\|\psi\|_{0,1}\|\phi\|_{0,1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\mathcal{K}_2[\psi,\phi]\|_{0,1} &\le \int_0^\infty\int_0^\infty (1+x)k(x,y)|\psi(x)\phi(y)|dy\,dx \\
&\le k_0\int_0^\infty\int_0^\infty (1+x)|\psi(x)\phi(y)|dy\,dx \\
&\le k_0\|\psi\|_{0,1}\|\phi\|_{0,1}.
\end{aligned}$$

The result follows. □

Theorem 4.3.2.

(i) $K : X_{0,1} \to X_{0,1}$ *;*

(ii) K *is locally Lipschitz on* $X_{0,1}$ *;*

(iii) K *is Fréchet differentiable on* $X_{0,1}$ *and the Fréchet derivative* K_ψ *is such that*

$$\|K_\psi\phi\|_{0,1} \le \varrho\|\phi\|_{0,1}, \quad \forall\ \phi \in X_{0,1},\ \psi \in \overline{B}(u_0,r),$$

where ϱ *is a positive constant.*

(iv) The Fréchet derivative is continuous with respect to $\psi \in \overline{B}(u_0, r)$ *in the sense that*

$$\|K_{\psi_1}\phi - K_{\psi_2}\phi\|_{0,1} \to 0 \quad as \quad \|\psi_1 - \psi_2\|_{0,1} \to 0 \ \ where \ \ \psi_1, \psi_2 \in \overline{B}(u_0, r),$$

for any given $\phi \in X_{0,1}$.

Proof. (i), (ii). On applying Lemma 4.3.1, we obtain

$$||K\psi||_{0,1} = ||\,\mathcal{K}[\psi,\psi]\,||_{0,1} \le 2k_0\,||\psi||_{0,1}^2 \quad \forall \psi \in X_{0,1}$$

and

$$\begin{aligned} ||K\psi - K\phi||_{0,1} &= ||\,\mathcal{K}[\psi,\psi] - \mathcal{K}[\phi,\phi]\,||_{0,1} \\ &= ||\,\mathcal{K}[\psi-\phi,\psi] + \mathcal{K}[\phi,\psi-\phi]\,||_{0,1} \\ &\le 2k_0\,||\psi-\phi||_{0,1}\,(\,||\psi||_{0,1} + ||\phi||_{0,1}\,)\,. \end{aligned}$$

Consequently, if $u_0 \in X_{0,1}$ is fixed then

$$||K\psi - K\phi||_{0,1} \le \varrho(r,u_0)\,||\psi-\phi||_{0,1} \quad \forall \psi,\phi \in \bar{B}(u_0,r)\,,$$

where

$$\begin{aligned} \bar{B}(u_0,r) &:= \{\, f \in X_{0,1} : \, ||f - u_0\,||_{0,1} \le r\,\}\ , \text{ and} \\ \varrho(r,u_0) &:= 4k_0\,(\,r + ||u_0||_{0,1}\,)\,. \end{aligned} \tag{4.20}$$

(iii) Let $\psi, \delta \in X_{0,1}$. Then

$$K[\psi+\delta] = K\psi + \mathcal{K}[\psi,\delta] + \mathcal{K}[\delta,\psi] + K\delta\,.$$

For fixed ψ, $\mathcal{K}[\psi,\cdot] + \mathcal{K}[\cdot,\psi]$ is a bounded operator in $X_{0,1}$ with

$$||\,\mathcal{K}[\psi,\delta] + \mathcal{K}[\delta,\psi]\,||_{0,1} \le 4k_0\,||\psi||_{0,1}\,||\delta||_{0,1} \quad \forall \delta \in X_{0,1}\,.$$

Also

$$\frac{||K\delta||_{0,1}}{||\delta||_{0,1}} \le 2k_0\,||\delta||_{0,1} \ \to 0 \ \text{ as } ||\delta||_{0,1} \to 0\,.$$

Hence K is Fréchet differentiable at each $\psi \in X_{0,1}$ and the Fréchet derivative K_ψ at ψ is given by

$$K_\psi\phi := \mathcal{K}[\psi,\phi] + \mathcal{K}[\phi,\psi] \quad \forall \phi \in X_{0,1}\,.$$

Consequently,

$$\|K_\psi\phi\|_{0,1} \le \varrho(r,u_0)\|\phi\|_{0,1}, \quad \forall\ \phi \in X_{0,1},\ \psi \in \overline{B}(u_0,r).$$

(iv) Let $\psi_1, \psi_2, \phi \in X_{0,1}$,

$$\begin{aligned} \|K_{\psi_1}\phi - K_{\psi_2}\phi\|_{0,1} &= \|\mathcal{K}[\psi_1,\phi] + \mathcal{K}[\phi,\psi_1] - \mathcal{K}[\psi_2,\phi] - \mathcal{K}[\phi,\psi_2]\|_{0,1} \\ &= \|\mathcal{K}[\psi_1-\psi_2,\phi] + \mathcal{K}[\phi,\psi_1-\psi_2]\|_{0,1} \\ &\le 4k_0\|\phi\|_{0,1}\|\psi_1-\psi_2\|_{0,1} \to 0 \quad \text{as} \quad \|\psi_1-\psi_2\|_{0,1} \to 0. \end{aligned}$$

Hence, the Fréchet derivative is continuous with respect to ψ. □

4.3.1 Local existence

Theorem 4.3.3. Local existence
There exist positive constants r, t_0 and a strongly differentiable positive function

$$u : [0, t_0) \to B(u_0, r) := \{\psi \in X_{0,1} : \ \|\psi - u_0\|_{0,1} < r\}$$

such that

$$\frac{du}{dt}(t) = G_{0,1}[u(t)] + K[u(t)], \ \ 0 < t < t_0; \ u_0 \in D(G_{0,1}) \cap X_{0,1_+}, \tag{4.21}$$

where $G_{0,1}$ is the part of G in $X_{0,1}$.

Proof. Similar to the proof for the existence of a local solution developed in the previous section for $x_0 \neq 0$. □

4.3.2 Global existence

Global existence of the solution can be proved easily under the assumption that the fragmentation rate is linearly bounded by playing with Gronwall's inequality. This has been done by several authors and the method is quite standard. In this section our aim was to extend the investigation to general fragmentation kernels. Making use of the analysis performed in the previous chapter, we have been able to prove the existence of a local (in time) solution in various situations including fragmentation rate unbounded at 0 and growing faster than x at infinity. Global existence however seems difficult to prove due to the fact that the Gronwall's inequality does not help in showing that the local solution does not blow up in a finite time. A possibility to avoid this problem is to analyze the evolution equation in the space X_1. We have done this successfully in the previous section thanks to the topological equivalence of the spaces X_1 and $X_{0,1}$ as $x_0 \neq 0$. The problem in adopting this idea in this framework is that since $x_0 = 0$, the coagulation operator behaves very badly in the space X_1.

Chapter 5

Phytoplankton Dynamics

5.1 Introduction

In phytoplankton dynamics, a system of particles called TEP (Transparent Exopolymer Particles) plays a major role. They are by-product of the growth of phytoplankton and their stickiness causes cells to remain together upon contact [27, 47]. On the other hand, the low level of concentration of TEP results in fragmentation of the aggregate due to external causes, like currents or turbulence on one hand, and internal unspecified forces of biotic nature on the other. A conservative model describing the influence of the TEP on the phytoplankton population was derived and introduced by O. Arino and R. Rudnicki in [14]. To include the effects of cell division, the McKendrick-von Foerster renewal condition is incorporated. The aggregates are structured by size and the phytoplankton consists of aggregates of all possible sizes. The aggregate size can change due to splitting, death, growth or combining of aggregates into bigger ones. The resulting model consists of a kinetic-type nonlinear integro-differential equation with two integral terms responsible for the fragmentation and coagulation processes, the McKendrick-von Foerster renewal boundary condition and the initial condition. We make use of substochastic semigroup perturbations techniques and semilinear abstract Cauchy problems theory to show the existence of a strong solution to the evolution equation. In particular, we provide sufficient conditions for honesty of the model.

5.2 Description of the model and assumptions

We describe the dynamics of phytoplankton using the aggregate density function $u(t, x)$. Here $x \in [x_0, \infty)$ is a variable that represents the size of the aggregate, $x_0 \geq 0$ is the minimum single cell size, the variable t represents time and $u(t, x)$ is the concentration of aggregates of size x at time t. We assume that for each $t \geq 0$ the function $x \mapsto u(t, x)$

is from the space

$$X_1 = L_1([x_0,\infty),\, xdx) = \left\{\psi : \quad \|\psi\|_1 := \int_{x_0}^{\infty} x|\psi(x)|\, dx < \infty\right\}. \tag{5.1}$$

The space is X_1 chosen in a natural way because $\int_{x_0}^{\infty} x|\psi(x)|\, dx$ is the total number of cells in the population.

5.2.1 Growth and mortality

We assume that both processes depend on the size of the aggregate. Phytoplankton cells may die, for example, by sinking to the seabed, or whatever cause. We denote by d the death rate. We assume that it is a non-negative function and

$$d \in L_\infty((x_0,\infty)). \tag{5.2}$$

Aggregates grow as a result of divisions of phytoplankton cells. The growth rate is denoted by r. We assume that r is a non-negative function, differentiable at x_0 and

$$r \in AC((x_0,\infty)) \cap X_\infty, \tag{5.3}$$

where X_∞ is the dual space of X_1 and $r \in AC((x_0,\infty))$ means that r is absolutely continuous in the standard sense on each compact subinterval of (x_0,∞). We denote by $\|\cdot\|_\infty$ the norm of X_∞ and we recall that

$$\|\psi\|_\infty = \text{ess sup}_{x_0\le x<\infty}\frac{|\psi(x)|}{x},$$

and the duality pairing is the normal integral

$$< \psi, \omega >= \int_{x_0}^{\infty} \psi(x)\omega(x)dx.$$

If growth and mortality were the only processes taking place, the equation for the dynamics would read

$$\frac{\partial}{\partial t}u(t,x) = -\partial_x[r(x)u(t,x)] - d(x)u(t,x).$$

5.2.2 Fragmentation

During a small time interval Δt, a fraction $a(x)\Delta t$ of the aggregates of size x are undergoing breakup, i.e. a is the fragmentation rate. We assume that it is a non-negative function and

$$a \in L_{\infty,\,loc}((x_0,\infty)). \tag{5.4}$$

The size distribution of daughter particles after fragmentation is denoted by b. We assume that

$$\int_{x_0}^{y-x_0} xb(x|y)dx = y, \quad y > 2x_0 \tag{5.5}$$

which accounts for mass conservation after any fragmentation event. If the dynamics were just the result of fragmentation, the equation would read:

$$\frac{\partial}{\partial t}u(t,x) = -a(x)u(t,x) + \int_{x+x_0}^{\infty} a(y)b(x|y)u(t,y)\,dy.$$

5.2.3 Coagulation equation

The classical coagulation kernel $k(x,y)$ used in the previous chapter is defined as the rate at which particles of mass x coalesce with particles of mass y. This kernel is derived by assuming that the average number of coalescences between particles having mass in $(x; x+dx)$ and those having mass in $(y; y+dy)$ is $k(x,y)u(t,x)u(t,y)dxdydt$ during the time interval $(t; t+dt)$. In this chapter, we use a different coagulation model. This model is suitable for populations where an individual is viewed as a collection of joined cells. In what follows we provide a full description.

Following [14], we assume that only a part of the aggregates has the competence to join. This could for example be due to the fact that only cells of some species have the necessary devices to glue or to attach to others. The coefficient of competence is a function $g(x)$. We assume that g is a positive and bounded function,

$$g \in L_\infty((x_0, \infty)). \tag{5.6}$$

The number of cells in all aggregates that, at time t, are implicated in the coagulation process is given by:

$$J(t) := \int_{x_0}^{\infty} zg(z)u(t,z)\,dz,$$

and

$$j(t,x) := \frac{xg(x)u(t,x)}{J(t)}$$

is the fraction of cells in size-x aggregates competent for the coagulation process with respect to the total population of cells in aggregates prone to join. In terms of the quantities introduced so far, we can express the time rate of cells forming aggregates of size x:

$$J(t)\chi_U(x) \int_{x_0}^{x-x_0} j(t,x-y)j(t,y)\,dy,$$

where χ_U is the characteristic function of the interval $U = [2x_0, \infty)$.

Again, if coagulation were the only process, the equation would read:

$$\frac{\partial}{\partial t}xu(t,x) = J(t)\chi_U(x) \int_{x_0}^{x-x_0} j(t,x-y)j(t,y)dy - xg(x)u(t,x),$$

which, after obvious algebra, leads to:

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) =& \chi_U(x)\frac{\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)dy}{x\int_{x_0}^{\infty} zg(z)u(t,z)dz} \\ &- g(x)u(t,x).\end{aligned} \tag{5.7}$$

Proposition 5.2.1. *The coagulation model described by* (5.7) *is formally conservative.*

Proof. Our purpose is to show that

$$\frac{d}{dt}M(t) = \frac{d}{dt}\int_{x_0}^{\infty} xu(t,x)\,dx = \int_{x_0}^{\infty} x\frac{\partial}{\partial t}u(t,x)\,dx = 0.$$

Because $g \in L_\infty((x_0,\infty))$, it is enough to prove that

$$\begin{aligned}&\int_{x_0}^{\infty}\chi_U(x)\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)\,dy\;dx \\ &\quad= \int_{x_0}^{\infty} xg(x)u(t,x)\,dx\cdot\int_{x_0}^{\infty} zg(z)u(t,z)\,dz.\end{aligned} \tag{5.8}$$

By the Fubini integration theorem,

$$\begin{aligned}&\int_{x_0}^{\infty}\left(\chi_U(x)\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)\,dy\right)dx \\ &\quad= \int_{2x_0}^{\infty}\left(\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)\,dy\right)dx \\ &\quad= \int_{x_0}^{\infty} yg(y)u(t,y)\left(\int_{y+x_0}^{\infty}(x-y)g(x-y)u(t,x-y)\,dx\right)dy \\ &\quad= \int_{x_0}^{\infty} yg(y)u(t,y)\left(\int_{x_0}^{\infty} zg(z)u(t,z)\,dz\right)dy \\ &\quad= \int_{x_0}^{\infty} zg(z)u(t,z)\,dz\times\int_{x_0}^{\infty} yg(y)u(t,y)\,dy.\end{aligned}$$

□

5.2.4 Boundary conditions

The McKendrick-von Foerster renewal condition reads

$$\lim_{x\to x_0^+} r(x)u(t,x) = \int_{x_0}^{\infty}\beta(y)u(t,y)\,dy,$$

where $\beta \in X_\infty$. The function $\beta(y)$ describes the number of single cells that fall off an aggregate of size y and join the single cell population [1]. The boundary condition represents the addition of newborn single cells to the single cell population.

5.2.5 The whole model

The model of the phytoplankton dynamics should incorporate the afore-mentioned processes, thus it has the following form:

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) = &-\frac{\partial}{\partial x}[r(x)u(t,x)] - d(x)u(t,x) - g(x)u(t,x)\\ &- a(x)u(t,x) + \int_{x+x_0}^{\infty} a(y)b(x|y)u(t,y)\,dy\\ &+ \chi_U(x)\frac{\int_{x_0}^{x-x_0} yg(y)u(t,y)(x-y)g(x-y)u(t,x-y)dy}{x\int_{x_0}^{\infty} zg(z)u(t,z)\,dz},\end{aligned} \tag{5.9}$$

where χ_U is the characteristic function of the interval $U = [2x_0, \infty)$, $x_0 \geq 0$. The nonlinear integro-differential equation is supplemented with the initial condition

$$u(0,x) = u_0(x), \tag{5.10}$$

where $u_0 \in X_1$ and with the McKendrick-von Foerster renewal boundary condition [3, 2]:

$$\lim_{x\to x_0^+} r(x)u(t,x) = \int_{x_0}^{\infty} \beta(y)u(t,y)\,dy. \tag{5.11}$$

5.2.6 Abstract reformulation

The analysis is performed in the space X_1. In what follows we denote by $\mathcal{T}$, $\mathcal{B}$ and $\mathcal{N}$ the expressions appearing on the right-hand side of the equations (5.9); that is,

$$[\mathcal{T}\psi](x) = -\frac{d}{dx}[r(x)\psi(x)] - q(x)\psi(x) \tag{5.12}$$

where $q = a + d + g$,

$$[\mathcal{B}\psi](x) = \int_{x+x_0}^{\infty} a(y)b(x|y)\psi(y)dy, \tag{5.13}$$

and

$$[\mathcal{N}\psi](x) = \chi_U(x)\frac{\int_{x_0}^{x} y\mathrm{g}(y)\psi(y)(x-y)\mathrm{g}(x-y)\psi(x-y)dy}{x\int_{x_0}^{\infty} z\mathrm{g}(z)\psi(z)dz}, \tag{5.14}$$

for non-zero positive ψ and $\mathcal{N}0 = 0$. $\mathcal{T}, \mathcal{B}$ and $\mathcal{N}$ are defined on measurable and finite almost everywhere functions ψ for which they make pointwise (almost everywhere) sense.

For each fixed $t \geq 0$, we define a function $u(t) : (x_0, \infty) \to \mathbb{R}$ of the "mass" variable x by

$$u(t)(x) = u(t,x), \text{ for a.e. } x > x_0,\ t \geq 0. \tag{5.15}$$

Hence u is the function from $[0, \infty)$ into the space X_1. Since X_1 is a Banach space of type L, $\dfrac{\partial u}{\partial t}$ can be thought of as the derivative with respect to t of the function

$u : [0, \infty) \to X_1$ defined by (5.15). For fixed $t > 0$, we can write the right-hand side of (5.9) as $(\mathcal{T} + \mathcal{B} + \mathcal{N})u(t)$ defined on its maximal domain. The initial condition (5.10) becomes $u(0) = u_0$.

A vital role in the analysis of the model is played by the integrability of $1/r(x)$ at x_0. Indeed if $1/r(x)$ is integrable at x_0, the characteristics do reach the line $x = x_0$ and therefore the boundary condition becomes crucial for the uniqueness investigation. If not, the characteristics do not reach the line $x = x_0$ and the prescription of a boundary condition is of no use. Therefore, we consider two cases according to the integrability of $1/r(x)$.

5.3 Case of r^{-1} non-integrable at x_0

5.3.1 The streaming semigroup

With respect to the above, the transport problem reads:

$$\frac{du}{dt}(t) = Tu(t)$$
$$u(0) = u_0,$$

where T is the realization of $\mathcal{T}$ (defined via (5.12)) on X_1. It turns out that direct estimates of the resolvent of T are not easy. For this reason, we start with the operator F expressed by

$$[F\psi](x) := -\frac{d}{dx}[r(x)\psi(x)], \;\; x \in (x_0, \infty), \tag{5.16}$$

on the domain

$$D(F) = \{\psi \in X_1; \;\; r\psi \in AC((x_0, \infty)) \text{ and } (r\psi)_x \in X_1\}.$$

We denote by R a fixed antiderivative of $1/r$, say

$$R(x) = \int_{x_0+\varepsilon}^{x} \frac{ds}{r(s)},$$

where $\varepsilon > 0$ is a given positive number. We see, due to $r \in X_\infty$ and the non-integrability of r^{-1} at x_0, that

$$\lim_{x\to\infty} R(x) = \infty \;\text{ and }\; \lim_{x\to x_0} R(x) = -\infty. \tag{5.17}$$

Since R is a strictly increasing function, it follows that R is globally invertible on $\mathbb{R}$. Hence if we define

$$Y(t, x) := R^{-1}(R(x) - t), \quad x > x_0, \; t \in \mathbb{R},$$

we can prove as in [16, Theorem 9.4] that $(F,\ D(F))$ generates a C_0-semigroup $(S_F(t))_{t\geq 0}$ expressed by

$$[S_F(t)u_0](x) = \frac{r(Y(t,x))u_0(Y(t,x))}{r(x)}, \quad t \geq 0, x > x_0,$$

where u_0 is any fixed element of $D(F)$. In particular, we have

$$\begin{aligned}\|S_F(t)u_0\|_1 &\leq \int_{x_0}^{\infty} \frac{r(Y(t,x))u_0(Y(t,x))}{r(x)} x dx \\ &= \int_{x_0}^{\infty} u_0(z)Y(-t,z)dz \\ &\leq e^{\|r\|_\infty t}\|u_0\|_1,\end{aligned}$$

where we used the change of variables $z = Y(t,x)$ so that

$$\frac{dz}{r(z)} = \frac{dx}{r(x)} \quad \text{and} \quad Y(t,x_0) = x_0,\ Y(t,\infty) = \infty \quad \text{by (5.17)}.$$

The last estimate follows due to the fact that $x(t) = Y(-t,z)$ is the solution to the Cauchy problem

$$\frac{dx}{dt} = r(x), \qquad x(0) = z,$$

so that

$$x(t) = z + \int_0^t r(x(s))ds \leq z + \int_0^t \|r\|_\infty x(s)ds$$

and, by Gronwall's lemma, we obtain

$$Y(-t,z) \leq z e^{\|r\|_\infty t}.$$

In particular, by the Hille-Yosida theorem, we obtain for $\psi \in X_1$ and $\lambda > \|r\|_\infty$,

$$\|R(\lambda,F)\psi\|_1 \leq \frac{1}{\lambda - \|r\|_\infty}\|\psi\|_1. \tag{5.18}$$

Let us denote by T the operator realization of $\mathcal{T}$ (defined via (5.12)) on the maximal domain :

$$D(T) = \{\psi \in X_1;\ q\psi \in X_1,\ \ r\psi \in AC((x_0,\infty)) \text{ and } (r\psi)_x \in X_1\}.$$

With the above, we can prove the following result for the semigroup solving (5.16).

Theorem 5.3.1. *The operator T defined above generates a positive semigroup, say $(S_T(t))_{t\geq 0}$, satisfying for any $\psi \in X_1$:*

$$\|S_T(t)\psi\|_1 \leq e^{\|r\|_\infty t}\|\psi\|_1. \tag{5.19}$$

Furthermore for any $\lambda > \|r\|_\infty$, the resolvent $R(\lambda, T)$ of the operator T is expressed as follows:

$$[R(\lambda, T)\psi](x) = \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda R(y)+Q(y)}\psi(y)dy, \tag{5.20}$$

where $\lambda R + Q$ is a fixed antiderivative of $(\lambda + q(s))/r(s)$, say

$$R(x) = \int_{x_0+\varepsilon}^{x} \frac{ds}{r(s)} \quad \text{and} \quad Q(x) = \int_{x_0+\varepsilon}^{x} \frac{q(s)}{r(s)}ds, \qquad \varepsilon > 0.$$

Proof. We consider the resolvent equation of T. Making use of the method of variation of constants, we see that a good candidate for the resolvent of T is a solution of the equation

$$\lambda u(x) + \frac{d}{dx}[r(x)u(x)] + q(x)u(x) = \psi(x), \tag{5.21}$$

given by

$$[R_\lambda \psi](x) = \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda R(y)+Q(y)}\psi(y)dy. \tag{5.22}$$

By the Fubini theorem,

$$\begin{aligned}
\int_{x_0}^{\infty} |[R_\lambda\psi](x)|x\,dx &\le \int_{x_0}^{\infty} \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} \left(\int_{x_0}^{x} e^{\lambda R(y)+Q(y)}|\psi(y)|\,dy\right) x\,dx \\
&= \int_{x_0}^{\infty} |\psi(y)|e^{\lambda R(y)+Q(y)} \left(\int_{y}^{\infty} \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} x\,dx\right) dy \\
&\le (\lambda - \|r\|_\infty)^{-1} \int_{x_0}^{\infty} |\psi(y)|y\,dy,
\end{aligned}$$

where we made use of (5.18) and the monotonicity of $e^{-Q(x)}$. Since

$$\begin{aligned}
\frac{xq(x)}{r(x)} e^{-\lambda R(x)-Q(x)} &\le \frac{x(\lambda + q(x))}{r(x)} e^{-\lambda R(x)-Q(x)} \\
&= e^{-\lambda R(x)-Q(x)} - \frac{d}{dx}\left(xe^{-\lambda R(x)-Q(x)}\right)
\end{aligned}$$

we have

$$\begin{aligned}
\|qR_\lambda\psi\|_1 &\le \int_{x_0}^{\infty} \left(\frac{e^{\lambda R(y)+Q(y)}}{y} \int_{y}^{\infty} \frac{xq(x)e^{-\lambda R(x)-Q(x)}}{r(x)}\,dx\right) |\psi(y)|y\,dy \\
&\le \int_{x_0}^{\infty} \left(1 + \frac{e^{\lambda R(y)+Q(y)}}{y} \int_{y}^{\infty} e^{-\lambda R(x)-Q(x)}\,dx\right) |\psi(y)|y\,dy \\
&\le (1 + (\lambda - \|r\|_\infty)^{-1})\|\psi\|_1,
\end{aligned}$$

where again we used (5.18) and the fact that $e^{-Q(x)}$ is non-increasing.

Next we notice that for $\psi \in X_1$,

$$r(x)[R_\lambda \psi](x) = e^{-\lambda R(x) - Q(x)} \int_{x_0}^{x} e^{\lambda R(y)+Q(y)} \psi(y)\, dy,$$

and both $e^{-\lambda R(x)-Q(x)}$ and the integral (as a function of its upper limit) are absolutely continuous and bounded over any fixed interval $[\alpha, \beta] \subset (x_0, \infty)$. Hence it follows that the product is absolutely continuous on $[\alpha, \beta]$ and therefore $r[R_\lambda \psi]$ is absolutely continuous there.

Furthermore

$$\begin{aligned} (r(x)[R_\lambda \psi](x))_x &= -\frac{\lambda + q(x)}{r(x)} e^{-\lambda R(x)-Q(x)} \int_{x_0}^{x} e^{\lambda R(y)+Q(y)} \psi(y)\, dy + \psi(x) \\ &= -(\lambda + q(x))[R_\lambda \psi](x) + \psi(x) \in X, \end{aligned}$$

therefore $R_\lambda(X_1) \subset D(T)$.

In addition, direct substitution shows that $[(\lambda I - T)R_\lambda]\psi = \psi$.

In order to show that R_λ is the resolvent of T, it remains to show that $\lambda I - T$ is injective on $D(T)$. It is clear that the only solution (up to a multiplicative constant) of $\lambda u(x) + q(x)u(x) + (r(x)u(x))_x = 0$ is

$$R_\lambda(0) = \frac{e^{-\lambda R(x)-Q(x)}}{r(x)},$$

By contradiction let us assume that the multiplicative constant is different from 0.

If $x_0 \neq 0$, we have

$$\begin{aligned} \|R_\lambda(0)\|_1 &= \int_{x_0}^{\infty} \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} x\, dx \\ &\geq \int_{x_0}^{x_0+\varepsilon} \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} x\, dx \\ &\geq x_0 \int_{x_0}^{x_0+\varepsilon} \frac{dx}{r(x)} = \infty \end{aligned}$$

where we used the fact that $-\lambda R(x) - Q(x) \geq 0$ in the interval $(x_0, x_0 + \varepsilon)$ and the non-integrability of r^{-1} at x_0.

If $x_0 = 0$, we first notice that for $x \in (0, \varepsilon)$,

$$e^{-\lambda R(x)} = \exp\left(\lambda \int_x^{\varepsilon} \frac{ds}{r(s)}\right) \geq \exp\left(\frac{\lambda}{\|r\|_\infty} \int_x^{\varepsilon} \frac{ds}{s}\right) = \left(\frac{\varepsilon}{x}\right)^{\lambda/\|r\|_\infty}.$$

It follows that

$$\begin{aligned}
\|R_\lambda(0)\|_1 &= \int_0^\infty \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} x\,dx \\
&\geq \int_0^\varepsilon \frac{e^{-\lambda R(x)}}{r(x)} x\,dx \\
&\geq \frac{1}{\|r\|_\infty} \int_0^\varepsilon \left(\frac{\varepsilon}{x}\right)^{\lambda/\|r\|_\infty} dx = \infty
\end{aligned}$$

for any $\lambda > \|r\|_\infty$. This is impossible since $R_\lambda(0) \in X$. Therefore $\lambda I - T$ is injective. Hence the resolvent $R(\lambda, T)$ of the operator T is equal to R_λ. Since the resolvent is a positive operator, by the Hille-Yosida theorem, $(T, D(T))$ generates a positive semigroup satisfying (5.19). □

Theorem 5.3.1 implies that the operator

$$((\widetilde{T}, D(T)) = (T - \|r\|_\infty I, D(T))) \tag{5.23}$$

generates a positive semigroup of contractions expressed by

$$S_{\widetilde{T}}(t)\psi = e^{-\|r\|_\infty t} S_T(t)\psi. \tag{5.24}$$

As a result, to prove the existence of a semigroup solving the growth-fragmentation equation, we can make use of the Kato-Voigt perturbation theorem.

5.3.2 Growth-fragmentation equation

Let us define B as the realization of $\mathcal{B}$ (see (5.13)) on the domain

$$D(B) = D(T) = \{\psi \in X_1;\ q\psi \in X_1,\ \ r\psi \in AC((x_0, \infty)) \text{ and } (r\psi)_x \in X_1\}. \tag{5.25}$$

The corresponding Cauchy problem reads:

$$\begin{aligned}
\frac{du}{dt}(t) &= [T + B]u(t) \quad\ t > 0 \\
u(0) &= u_0,
\end{aligned} \tag{5.26}$$

where

$$[(T + B)\psi](x) = -\frac{d}{dx}[r(x)\psi(x)] - q(x)\psi(x) + \int_{x+x_0}^\infty a(y)b(x|y)\psi(y)\,dy.$$

Lemma 5.3.2. *For any $\psi \in D(T)_+$ we have*

$$\int_{x_0}^\infty [T\psi + B\psi](x)\,x\,dx = \int_{x_0}^\infty r(x)\psi(x)\,dx - \int_{x_0}^\infty [d + g](x)\psi(x)x\,dx. \tag{5.27}$$

Proof. Thanks to the Fubini theorem (see proof of Theorem 4.1.1), it is clear that

$$\int_{x_0}^{\infty} [-q\psi + B\psi](x)\, x\, dx = -\int_{x_0}^{\infty} [d+g](x)\psi(x) x\, dx$$

for any $\psi \in D(T)$. It remains to show that $\int_{x_0}^{\infty} [F\psi](x)\, x\, dx = \int_{x_0}^{\infty} r(x)\psi(x)\, dx$, where F is the operator described by (5.16). The approach we consider is similar to the analysis performed in the proof of [16, Lemma 9.7] for the model of fragmentation with decay.

Let $\lambda > \|r\|_\infty$ and $\psi \in D(T)_+$. Then $\psi = R(\lambda, T)\phi$ for some $\phi \in X_1$. Direct calculation shows that

$$[FR(\lambda, T)\phi](x) = -\phi(x) + (\lambda + q(x))[R(\lambda, T)\phi](x). \tag{5.28}$$

Now

$$\begin{aligned} &\int_{x_0}^{\infty} ((\lambda + q(x))[R(\lambda, T)\phi](x))\, x dx \\ &\quad = \int_{x_0}^{\infty} e^{\lambda R(y)+Q(y)}\phi(y) \left(\int_y^{\infty} \frac{x(\lambda + q(x))}{r(x)} e^{-\lambda R(x)-Q(x)} dx \right) dy. \end{aligned} \tag{5.29}$$

Also for any $y > x_0$, we have

$$\begin{aligned} \int_y^{\infty} \frac{x(\lambda + q(x))}{r(x)} e^{-\lambda R(x)-Q(x)} dx &= -\int_y^{\infty} x \left(\frac{d}{dx} e^{-\lambda R(x)-Q(x)} \right) dx \\ &= \int_y^{\infty} e^{-\lambda R(x)-Q(x)} dx + y e^{-\lambda R(y)-Q(y)} - \lim_{x\to\infty} x e^{-\lambda R(x)-Q(x)}, \end{aligned}$$

where we used integration by parts. Note that $\lim_{x\to\infty} x e^{-\lambda R(x)-Q(x)} = 0$. In fact

$$0 \le x e^{-\lambda R(x)-Q(x)} \le x e^{-\lambda R(x)} \le x \exp\left(-\lambda \int_{x_0+\varepsilon}^{x} \frac{ds}{r(s)} \right) \le \left| \frac{x_0 + \varepsilon}{x} \right|^{\frac{\lambda}{\|r\|_\infty}} \to 0$$

as $x \to \infty$, where we used $r(x) \le \|r\|_\infty x$ and $\lambda > \|r\|_\infty$ respectively. Hence

$$\begin{aligned} \int_{x_0}^{\infty} [F\psi](x)\, x\, dx &= \int_{x_0}^{\infty} [FR(\lambda, T)\phi](x)\, x\, dx \\ &= \int_{x_0}^{\infty} e^{\lambda R(y)+Q(y)}\phi(y) \left(\int_y^{\infty} e^{-\lambda R(x)-Q(x)} dx \right) dy \\ &= \int_{x_0}^{\infty} e^{-\lambda R(x)-Q(x)} \left(\int_{x_0}^{x} e^{\lambda R(y)+Q(y)}\phi(y) dy \right) dx \\ &= \int_{x_0}^{\infty} r(x)\psi(x) dx. \end{aligned}$$

□

Proposition 5.3.3. *There is an extension $\widetilde{G}$ of the operator $\widetilde{T}+B$ that generates a substochastic semigroup $(S_{\widetilde{G}}(t))_{t\geq 0}$ of bounded linear operators on X_1. This semigroup, for arbitrary $\psi \in D(\widetilde{G})$ and $t>0$, satisfies:*

$$\frac{d}{dt}S_{\widetilde{G}}(t)\psi = \widetilde{G}S_{\widetilde{G}}(t)\psi \tag{5.30}$$

$(S_{\widetilde{G}}(t))_{t\geq 0}$ can be obtained as a strong limit in X_1 of semigroups $(S_r(t))_{t\geq 0}$ generated by $(\widetilde{T}+rB, D(\widetilde{T}))$ as $r \to 1^-$; if $\psi \in X_{1+}$, then the limit is monotonic.
The generator $\widetilde{G}$ of $(S_{\widetilde{G}}(t))_{t\geq 0}$ is characterized by:

$$(\lambda I - \widetilde{G})^{-1}\psi = \sum_{n=0}^{\infty}(\lambda I - \widetilde{T})^{-1}[B(\lambda I - \widetilde{T})^{-1}]^n\psi \tag{5.31}$$

for $\psi \in X_1$.

Proof. By (5.23), the operator $(\widetilde{T}, D(\widetilde{T}))$ generates a substochastic semigroup $(S_{\widetilde{T}}(t))_{t\geq 0}$. We obviously have that $B\psi \geq 0$ for any $\psi \in D(\widetilde{T})_+$. Also by (5.27),

$$\begin{aligned}\int_{x_0}^{\infty}(\widetilde{T}\psi + B\psi)x\,dx \leq & -\int_{x_0}^{\infty}(\|r\|_\infty x - r(x))\psi(x)\,dx \\ & -\int_{x_0}^{\infty}[d+g](x)\psi(x)x\,dx \\ \leq & 0.\end{aligned}$$

Consequently the assumptions of Theorem 2.3.5 are satisfied. □

Theorem 5.3.4. *There is an extension G of the operator $T+B$ given by*

$$(G, D(G)) = (\widetilde{G} + \|r\|_\infty I, D(\widetilde{G}))$$

which generates a positive semigroup $(S_G(t))_{t\geq 0} = (e^{\|r\|_\infty t}S_{\widetilde{G}}(t))_{t\geq 0}$ in X_1. Moreover, the generator G is characterized by:

$$(\lambda I - G)^{-1}\psi = \sum_{n=0}^{\infty}(\lambda I - T)^{-1}[B(\lambda I - T)^{-1}]^n\psi \tag{5.32}$$

for $\psi \in X_1$ and $\lambda > \|r\|_\infty$.

Proof. The argument used follows similar lines to that used in [16, Proposition 9.29]. Formula (5.32) is obtained directly from (5.31). In fact since $\lambda I - G = (\lambda - \|r\|_\infty)I - \widetilde{G}$, it is clear that $R(\lambda, G) = R(\lambda', \widetilde{G})$ for $\lambda > \|r\|_\infty$, where $\lambda' = \lambda - \|r\|_\infty$. To prove the first part of the theorem, we note that operator $\widetilde{T}$ was constructed from T by the substraction of the bounded operator $\|r\|_\infty I$. Also the approximating semigroups $(S_r(t))_{t\geq 0}$ mentioned in the previous proposition are generated by $(T - \|r\|_\infty I + rB, D(T))$, $0 < r < 1$. In addition,

$$\lim_{r \to 1^-} S_r(t)\psi = S_{\widetilde{G}}(t)\psi \tag{5.33}$$

in X_1, uniformly in t on bounded intervals. We introduce the semigroups $(S'_r(t))_{t\geq 0} := (e^{\|r\|_\infty t} S_r(t))_{t\geq 0}$ generated by $T + rB$. Since multiplication by $e^{\|r\|_\infty t}$ does not affect convergence, (5.33) implies that $(S'_r(t))_{t\geq 0}$ converges strongly to the semigroup $(S_G(t))_{t\geq 0} = (e^{\|r\|_\infty t} S_{\widetilde{G}}(t))_{t\geq 0}$ generated by $G = \widetilde{G} + \|r\|_\infty I$. □

5.4 Case of r^{-1} integrable at x_0

The approach in this section is analogous to the work of [18] where the abstract space $X_{0,1}$ was used and the linear boundedness of the fragmentation rate assumed. Working in the bigger space X_1, we can extend the work of [18] to general fragmentation rate kernels.

5.4.1 Transport semigroup

The transport problem reads:

$$\begin{aligned} \frac{du}{dt}(t) &= Tu(t) \\ \lim_{x \to x_0^+} r(x)[u(t)(x)] &= \int_{x_0}^{\infty} \beta(y)[u(t)(y)]\, dy, \\ u(0) &= u_0 \end{aligned} \tag{5.34}$$

The first step is to restrict the operator T to a domain in which the boundary condition is satisfied. In this respect we introduce T_β as T restricted to

$$D(T_\beta) = \left\{ \psi \in D(T) : \quad \lim_{x \to x_0^+} r(x)\psi(x) = \int_{x_0}^{\infty} \beta(y)\psi(y)\, dy \right\}. \tag{5.35}$$

The general solution of the resolvent equation

$$\lambda u(x) + \frac{d}{dx}[r(x)u(x)] + q(x)u(x) = \psi(x),$$

is in the form

$$u(x) = [\widetilde{R}_\lambda \psi](x) + c\frac{e^{-\lambda \widetilde{R}(x) - \widetilde{Q}(x)}}{r(x)}, \tag{5.36}$$

where c is a suitable scalar and

$$[\widetilde{R}_\lambda \psi](x) = \frac{e^{-\lambda \widetilde{R}(x) - \widetilde{Q}(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda \widetilde{R}(y) + \widetilde{Q}(y)} \psi(y)\, dy, \tag{5.37}$$

where

$$\widetilde{R}(x) = \int_{x_0}^{x} \frac{ds}{r(s)}, \quad \text{and} \quad \widetilde{Q}(x) = \int_{x_0}^{x} \frac{q(s)}{r(s)}\, ds. \tag{5.38}$$

In the following lemma, we collect some identities and estimates that appear throughout this chapter.

Lemma 5.4.1. *Let $\lambda > \|r\|_\infty$. Then*

(a) For any $x_0 \le x \le x'' < \infty$,

$$I(x, x'') := \int_{x}^{x''} \frac{e^{-\lambda \widetilde{R}(s)}}{r(s)} s\, ds \le \frac{1}{\lambda - \|r\|_\infty} x e^{-\lambda \widetilde{R}(x)}; \tag{5.39}$$

(b)

$$\int_{x_0}^{\infty} e^{-\lambda \widetilde{R}(s) - \widetilde{Q}(s)} ds \le \int_{x_0}^{\infty} e^{-\lambda \widetilde{R}(s)}\, ds < \infty; \tag{5.40}$$

(c) For any $x_0 \le x \le x'' < \infty$,

$$\begin{aligned} J(x, x'') &:= \int_{x}^{x''} \frac{(\lambda + q(s)) e^{-\lambda \widetilde{R}(s) - \widetilde{Q}(s)}}{r(s)} s\, ds \\ &= x e^{-\lambda \widetilde{R}(x) - \widetilde{Q}(x)} - x'' e^{-\lambda \widetilde{R}(x'') - \widetilde{Q}(x'')} + \int_{x}^{x''} e^{-\lambda \widetilde{R}(s) - \widetilde{Q}(s)}\, ds; \end{aligned} \tag{5.41}$$

(d) In particular,

$$J(x_0, \infty) < \infty. \tag{5.42}$$

Proof. [18, Lemma 2.1]. □

We denote by T_0 the operator T with zero boundary conditions.

Lemma 5.4.2. *Under the adopted assumptions, if $\lambda > \|r\|_\infty$, then $R(\lambda, T_0) = \widetilde{R}_\lambda$ defines the resolvent of $(T_0, D(T_0))$ and satisfies the estimate*

$$\|R(\lambda, T_0)\|_1 \le \frac{1}{\lambda - \|r\|_\infty}. \tag{5.43}$$

Proof. [18, Lemma 2.2]. □

Next we turn our attention to the problem with $\beta \neq 0$ and we set

$$\kappa := x_0 \|\beta\|_\infty + \|r\|_\infty. \tag{5.44}$$

Lemma 5.4.3. *For any $\lambda > \kappa$, the resolvent $R(\lambda, T_\beta)$ of the operator $(T_\beta, D(T_\beta))$ satisfies*

$$R(\lambda, T_\beta) = R(\lambda, T_0) + \Phi_{\lambda,\beta} R(\lambda, T_0), \tag{5.45}$$

where

$$\Phi_{\lambda,\beta}\psi = \frac{\varepsilon_\lambda}{r} \frac{< \beta, \psi >}{(1- < \beta, r^{-1}\varepsilon_\lambda >)} \quad \textit{and} \quad \varepsilon_\lambda(x) := e^{-\lambda \widetilde{R}(x) - \widetilde{Q}(x)}.$$

Furthermore the resolvent $R(\lambda, T_\beta)$ satisfies the estimate

$$\|R(\lambda, T_\beta)\|_1 \leq \frac{1}{\lambda - \kappa}. \tag{5.46}$$

Proof. [18, Lemma 2.4]. □

5.4.2 Growth-fragmentation equation

The growth-fragmentation equation reads:

$$\begin{aligned} \frac{du}{dt}(t) &= [T_\beta + B]u(t) \quad t > 0 \\ u(0) &= u_0, \end{aligned} \tag{5.47}$$

Note that if $a \in L_\infty((x_0, \infty))$, the operator B is bounded. There is a semigroup $(S_{T_\beta+B}(t))_{t\geq 0}$ on X_1 associated with (5.47) generated by $(T_\beta + B, D(T_\beta))$. In their article [18], the authors proved a similar result under the assumption that the fragmentation rate is linearly bounded. In this section we extend this result to general case $a \in L_{\infty,\mathrm{loc}}((x_0, \infty))$, but for the coagulation term of the form (5.7).

Lemma 5.4.4.

$$D(T_\beta) \subseteq D(B). \tag{5.48}$$

Proof. Indeed let $\psi \in D(T_\beta)$, then by definition of $D(T_\beta)$, it is clear that $a\psi \in X_1$. It follows that

$$\begin{aligned} \|B\psi\|_1 &= \int_{x_0}^{\infty} \left| \int_x^\infty a(y) b(x|y)\psi(y)\, dy \right| x\, dx \\ &\leq \int_{x_0}^{\infty} |\psi(y)| a(y) \left(\int_{x_0}^{y} b(x|y) x\, dx \right) dy \\ &\leq \int_{x_0}^{\infty} |\psi(y)| a(y) y\, dy = \|a\psi\|_1 < \infty, \end{aligned}$$

where we used (5.5). Therefore $\psi \in D(B)$. □

Let us define

$$(\widetilde{T}_\beta, D(T_\beta)) := (T_\beta - \kappa I, D(T_\beta)). \tag{5.49}$$

Lemma 5.4.5. *For any $\psi \in D(T_\beta)_+$, the operator $\widetilde{T}_\beta + B := T_\beta - \kappa I + B$ satisfies*

$$\int_{x_0}^{\infty} [(\widetilde{T}_\beta + B)(\psi)](x)x\,dx \le 0.$$

Proof. Let $\psi \in D(T_\beta)_+$. Integrating by parts, we have

$$\int_{x'}^{x''} \frac{d}{dx}[r(x)\psi(x)]\,x\,dx = x''r(x'')\psi(x'') - x'r(x')\psi(x') - \int_{x'}^{x''} r(x)\psi(x)\,dx,$$

for any $x_0 < x' < x'' < \infty$.

Because $(r\psi)_x \in X_1$, the left hand side converges to $\int_{x_0}^{\infty} \partial_x[r(x)\psi(x)]x\,dx$ and $x'r(x')\psi(x') \to x_0 \int_{x_0}^{\infty} \beta(x)\psi(x)dx$ as $x' \to x_0$ and $x'' \to \infty$. Since $r \in X_\infty$, $r\psi$ is integrable on (x_0, ∞) and so the last integral on the right hand side converges to $\int_{x_0}^{\infty} r(x)u(x)dx$. It follows that $x''r(x'')\psi(x'')$ converges to a limit l as $x'' \to \infty$. Following [18, Lemma 2.5] we suppose that $l \neq 0$. Then $r(x)\psi(x) \ge \nu x^{-1}$ for some $\nu > 0$ and for large enough x, which contradicts the integrability of $r\psi$. Thus

$$\lim_{x'' \to \infty} x''r(x'')\psi(x'') = 0. \tag{5.50}$$

Therefore

$$\begin{aligned}\int_{0}^{\infty} (T_\beta\psi + B\psi)x\,dx &= x_0 \int_{x_0}^{\infty} \beta(x)\psi(x)\,dx + \int_{x_0}^{\infty} r(x)\psi(x)\,dx \\ &\quad - \int_{x_0}^{\infty} [d+g](x)\psi(x)x\,dx.\end{aligned} \tag{5.51}$$

Hence

$$\begin{aligned}\int_{x_0}^{\infty} (\widetilde{T}_\beta\psi + B\psi)x\,dx \le &- \int_{x_0}^{\infty} (\kappa x - x_0\beta(x) - r(x))\psi(x)\,dx \\ &- \int_{x_0}^{\infty} [d+g](x)\psi(x)x\,dx \le 0.\end{aligned}$$

□

Theorem 5.4.6. *There is an extension G_β of $T_\beta + B$ given by*

$$(G_\beta, D(G_\beta)) = (\widetilde{G}_\beta + \kappa I, D(\widetilde{G}_\beta))$$

that generates a positive semigroup $(S_{G_\beta}(t))_{t\ge0} = (e^{\kappa t}S_{\widetilde{G}_\beta}(t))_{t\ge0}$. Moreover, the generator G_β is characterized by:

$$(\lambda I - G_\beta)^{-1}\psi = \sum_{n=0}^{\infty} (\lambda I - T_\beta)^{-1}[B(\lambda I - T_\beta)^{-1}]^n\psi \tag{5.52}$$

for $\psi \in X_1$ and $\lambda > \kappa$.

Proof. Similar to the proof of Theorem 5.3.4. □

5.5 Honesty

The theory of honesty of substochastic semigroup was developed in [11, 12, 16, 13, 17]. This section provides some applications of this concept to our model. The purpose of this section is to provide sufficient conditions for honesty of the semigroups $(S_G(t))_{t\geq 0}$ and $(S_{G_\beta}(t))_{t\geq 0}$ defined via Theorem 5.3.4 and Theorem 5.4.6 respectively. We shall expand the analysis of honesty developed in [12] to populations with $x_0 \neq 0$ as we analyze honesty for $(S_G(t))_{t\geq 0}$. Furthermore we shall prove honesty of the semigroup $(S_{G_\beta}(t))_{t\geq 0}$ without assuming linear boundedness of the fragmentation and imposing finite number of daughter particles as in [18] thanks to the fact that we work in the space X_1. However the ideas and calculations for honesty of $(S_G(t))_{t\geq 0}$ and $(S_{G_\beta}(t))_{t\geq 0}$ are analogous to the analysis in [12] and [18] respectively. In this view we require the theory of extensions of operators.

Define by $\mathcal{E}$ the set of measurable functions that are defined on (x_0, ∞) and take values in $\mathbb{R} \cup \{-\infty, \infty\}$, and , by E, the subspace of $\mathcal{E}$ consisting of functions that are finite almost everywhere. The space $\mathcal{E}$ is a vector lattice with respect to the usual relation "$\leq$ almost everywhere". Moreover, $X_1 \subset E \subset \mathcal{E}$ with X_1 and E being sublattices of $\mathcal{E}$.

We consider the operator $\mathcal{T}$ (given by (5.12)) on the domain

$$D(\mathcal{T}) = \{\psi \in X_1;\ \ r\psi \in \mathrm{AC}((x_0, \infty)) \text{ and } \partial_x[r\psi] + q\psi \in E\},$$

and the restriction $\mathcal{T}_\beta$ of $\mathcal{T}$ in the domain

$$D(\mathcal{T}_\beta) = \{\psi \in D(\mathcal{T});\ \ \lim_{x\to x_0^+} r(x)\psi(x) = \int_{x_0}^{\infty} \beta(y)\psi(y)\,dy\}.$$

Similarly, we denote by $\mathcal{B}$ the operator defined by the expression (5.13) on $D(\mathcal{B}) = \{\psi \in X_1;\ \ \mathcal{B}\psi \in E\}$. Using these concepts, we can define operators that can be thought of as the maximal extension of $T + B$ and $T_\beta + B$ in X_1 as follow:

$$[\mathcal{G}\psi](x) := [\mathcal{T}\psi](x) + [\mathcal{B}\psi](x), \quad [\mathcal{G}_\beta\psi](x) := [\mathcal{T}_\beta\psi](x) + [\mathcal{B}\psi](x),$$

defined on the domain $D(\mathcal{G}) = \{\psi \in D(\mathcal{T}) \cap D(\mathcal{B});\quad x \to [\mathcal{G}\psi](x)\ \in X_1\}$ and $D(\mathcal{G}_\beta) = \{\psi \in D(\mathcal{T}_\beta) \cap D(\mathcal{B});\quad x \to [\mathcal{G}_\beta\psi](x)\ \in X_1\}$ respectively. Accordingly we consider the operators $\mathcal{R}(\lambda)$ and $\mathcal{R}_\beta(\lambda)$ extending $R(\lambda, T)$ for $\lambda > \widetilde{r}$ and $R(\lambda, T_\beta)$ for $\lambda > \kappa$ respectively and defined by the following expressions:

$$[\mathcal{R}(\lambda)\psi](x) = \frac{e^{-\lambda R(x) - Q(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda R(y) + Q(y)}\psi(y)\,dy, \tag{5.53}$$

on the domain $D(\mathcal{R}(\lambda)) = \{\psi \in \mathcal{E};\quad x \to [\mathcal{R}(\lambda)\psi](x)\ \in E\}$ and

$$\mathcal{R}_\beta(\lambda)\psi = \mathcal{R}_0(\lambda)\psi + \frac{\varepsilon_\lambda}{r}\frac{< \beta, \mathcal{R}_0(\lambda)\psi >}{1 - < \beta, r^{-1}\varepsilon_\lambda >} \tag{5.54}$$

on $D(\mathcal{R}_\beta(\lambda)) = \{\psi \in \mathcal{E}; \quad x \to [\mathcal{R}_\beta(\lambda)\psi](x) \ \in E\}$, where

$$[\mathcal{R}_0(\lambda)\psi](x) = \frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}\psi(y)\,dy.$$

Note that since the kernels of $\mathcal{B}$, $\mathcal{R}(\lambda)$ and $\mathcal{R}_\beta(\lambda)$ are nonnegative, the existence of the respective integrals is equivalent to the existence of the positive and negative parts of the integrands. It can be shown as in [16, Section 9.3] that $G \subset \mathcal{G}$ and $G_\beta \subset \mathcal{G}_\beta$, so that the extensions are defined correctly.

The usefulness of the concept of extensions is illustrated by the following observation.

Proposition 5.5.1. *(i) Any function $\psi \in D(G)$ is continuous on (x_0, ∞).*
(ii) Any function $\psi \in D(G_\beta)$ is continuous on (x_0, ∞).

Proof. Similar to the proof of [12, Proposition 5.2]. □

5.5.1 Honesty of the semigroup $(S_G(t))_{t\geq 0}$

A vital role in the following considerations is played by the following lemma.

Lemma 5.5.2. *Let $\mathcal{B}$ and $\mathcal{R}(\lambda)$ be the extensions introduced above. If for some $\psi \in D(\mathcal{R}(\lambda))_+$, both ψ and $\mathcal{B}\mathcal{R}(\lambda)\psi$ belong to $L_1([\alpha,\eta], xdx)$, where $x_0 \leq \alpha < \eta \leq \infty$, then:*

$$\begin{aligned}
&\int_\alpha^\eta (-\psi(x) + [\mathcal{B}\mathcal{R}(\lambda)\psi](x) + \lambda[\mathcal{R}(\lambda)\psi](x))x\,dx \\
&= -\,\eta r(\eta)[\mathcal{R}(\lambda)\psi](\eta) + \int_\eta^\infty a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_\alpha^\eta b(x|y)x\,dx\right)dy \\
&\quad + \int_\alpha^\eta r(x)[\mathcal{R}(\lambda)\psi](x)\,dx - \int_\alpha^\eta d(x)[\mathcal{R}(\lambda)\psi](x)x\,dx \\
&\quad - \int_\alpha^\eta g(x)[\mathcal{R}(\lambda)\psi](x)x\,dx + \alpha b(\alpha)[\mathcal{R}(\lambda)\psi](\alpha) \\
&\quad - \int_\alpha^\eta a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_{x_0}^\alpha xb(x|y)\,dx\right)dy.
\end{aligned} \tag{5.55}$$

Proof. The method we use is analogous to the proof of [16, Lemma 9.12] for fragmentation with mass loss. Changing the order of integration by the Fubini theorem we obtain

$$\begin{aligned}
&\int_\alpha^\eta [\mathcal{B}\mathcal{R}(\lambda)\psi](x)xdx \\
= &\int_\alpha^\eta a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_\alpha^y xb(x|y)dx\right)dy + \int_\eta^\infty a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_\alpha^\eta xb(x|y)dx\right)dy \\
= &\int_\alpha^\eta a(y)[\mathcal{R}(\lambda)\psi](y))ydy + \int_\eta^\infty a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_\alpha^\eta xb(x|y)dx\right)dy \\
&-\int_\alpha^\eta a(y)[\mathcal{R}(\lambda)\psi](y))\left(\int_{x_0}^\alpha xb(x|y)dx\right)dy.
\end{aligned}$$

Thus we can write

$$\begin{aligned}
&\int_\alpha^\eta [\mathcal{B}\mathcal{R}(\lambda)\psi](x)xdx \\
= &\int_\alpha^\eta (\lambda + a(y))[\mathcal{R}(\lambda)\psi](y)ydy - \lambda\int_\alpha^\eta [\mathcal{R}(\lambda)\psi](y)ydy - \int_\alpha^\eta a(y)[\mathcal{R}(\lambda)\psi](y)ydy \\
&+\int_\alpha^\eta a(y)[\mathcal{R}(\lambda)\psi](y)ydy + \int_\eta^\infty a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_\alpha^\eta xb(x|y)dx\right)dy \\
&-\int_\alpha^\eta a(y)[\mathcal{R}(\lambda)\psi](y))\left(\int_{x_0}^\alpha xb(x|y)dx\right)dy \\
= &I_1 - I_2 - I_3 + I_4 + I_5 - I_6. \qquad (5.56)
\end{aligned}$$

Next,

$$\begin{aligned}
I_1 = &\int_\alpha^\eta \left(\frac{(\lambda + a(y))e^{-\lambda R(y)-Q(y)}}{r(y)}\int_{x_0}^y e^{\lambda R(s)+Q(s)}\psi(s)ds\right)y\,dy \\
= &\int_\alpha^\eta e^{\lambda R(s)+Q(s)}\psi(s)\left(\int_\eta^s y\frac{d}{dy}e^{-\lambda R(y)-Q(y)}dy\right)ds \\
&+\int_{x_0}^\alpha e^{\lambda R(s)+Q(s)}\psi(s)\left(\int_\eta^\alpha y\frac{d}{dy}e^{-\lambda R(y)-Q(y)}dy\right)ds \\
= &\int_\alpha^\eta e^{\lambda R(s)+Q(s)}\psi(s)\left(se^{-\lambda R(s)-Q(s)} - \eta e^{-\lambda R(\eta)-Q(\eta)} + \int_s^\eta e^{-\lambda R(y)-Q(y)}dy\right)ds \\
&+\int_{x_0}^\alpha e^{\lambda R(s)+Q(s)}\psi(s)\left(\alpha e^{-\lambda R(\alpha)-Q(\alpha)} - \eta e^{-\lambda R(\eta)-Q(\eta)} + \int_\alpha^\eta e^{-\lambda R(y)-Q(y)}dy\right)ds.
\end{aligned}$$

It follows that

$$\begin{aligned} I_1 &= \int_{\alpha}^{\eta} \psi(s)sds - \eta e^{-\lambda R(\eta)-Q(\eta)} \int_{x_0}^{\eta} e^{\lambda R(s)+Q(s)}\psi(s)ds \\ &\quad + \int_{\alpha}^{\eta} e^{\lambda R(s)+Q(s)}\psi(s) \left(\int_{s}^{\eta} e^{-\lambda R(y)-Q(y)}dy \right) ds \\ &\quad + \alpha e^{-\lambda R(\alpha)-Q(\alpha)} \int_{x_0}^{\alpha} e^{\lambda R(s)+Q(s)}\psi(s)ds \\ &\quad + \int_{x_0}^{\alpha} e^{\lambda R(s)+Q(s)}\psi(s) \left(\int_{\alpha}^{\eta} e^{-\lambda R(y)-Q(y)}dy \right) ds \\ &= \int_{\alpha}^{\eta} \psi(s)sds - \eta r(\eta)[\mathcal{R}(\lambda)\psi](\eta) + \alpha r(\alpha)[\mathcal{R}(\lambda)\psi](\alpha) + \int_{\alpha}^{\eta} r(s)[\mathcal{R}(\lambda)\psi](s)ds, \end{aligned}$$

since

$$\begin{aligned} \int_{\alpha}^{\eta} r(y)[\mathcal{R}(\lambda)\psi](y)dy &= \int_{\alpha}^{\eta} \left(e^{-\lambda R(y)-Q(y)} \int_{x_0}^{y} e^{\lambda R(s)+Q(s)}\psi(s)ds \right) dy \\ &= \int_{\alpha}^{\eta} e^{\lambda R(s)+Q(s)}\psi(s) \left(\int_{s}^{\eta} e^{-\lambda R(y)-Q(y)}dy \right) ds \\ &\quad + \int_{x_0}^{\alpha} e^{\lambda R(s)+Q(s)}\psi(s) \left(\int_{\alpha}^{\eta} e^{-\lambda R(y)-Q(y)}dy \right) ds. \end{aligned}$$

Combining with (5.56), we obtain

$$\begin{aligned} &\int_{\alpha}^{\eta} (-\psi(x) + [\mathcal{F}\mathcal{R}(\lambda)\psi](x) + \lambda[\mathcal{R}(\lambda)\psi](x))xdx \\ &= \alpha b(\alpha)[\mathcal{R}(\lambda)\psi](\alpha) - \eta b(\eta)[\mathcal{R}(\lambda)\psi](\eta) + \int_{\eta}^{\infty} a(y)[\mathcal{R}(\lambda)\psi](y) \left(\int_{\alpha}^{\eta} b(x|y)xdx \right) dy \\ &\quad + \int_{x_0}^{\eta} r(x)[\mathcal{R}(\lambda)\psi](x)dx - \int_{x_0}^{\eta} q(x)[\mathcal{R}(\lambda)\psi](x)xdx + \int_{x_0}^{\eta} a(x)[\mathcal{R}(\lambda)\psi](x)xdx \\ &\quad - \int_{\alpha}^{\eta} a(y)[\mathcal{R}(\lambda)\psi](y)) \left(\int_{x_0}^{\alpha} xb(x|y)dx \right) dy. \end{aligned}$$

□

Theorem 5.5.3. *If $u \in D(G)$, then there are sequences $\alpha_k \to x_0^+$ and $\eta_k \to \infty$ as $k \to \infty$ such that:*

$$\begin{aligned} \int_{x_0}^{\infty} [Gu](x)x\,dx = \lim_{k\to\infty} \Bigg(&- \int_{\alpha_k}^{\eta_k} a(y)u(y) \left(\int_{x_0}^{\alpha_k} xb(x|y)\,dx \right) dy \\ &+ \int_{\eta_k}^{\infty} a(y)u(y) \left(\int_{\alpha_k}^{\eta_k} b(x|y)x\,dx \right) dy \Bigg) \\ &+ \int_{x_0}^{\infty} r(x)u(x)\,dx - \int_{x_0}^{\infty} d(x)u(x)\,dx - \int_{x_0}^{\infty} g(x)u(x)\,dx. \end{aligned} \tag{5.57}$$

Proof. By using the previous lemma, this result can be proved as in [12, Theorem 5.1]. □

We recall that G is defined via Theorem 5.3.4. Using the above the following result can be established.

Theorem 5.5.4.
If $\lim_{x\to x_0} q(x) = \lim_{x\to x_0} a(x) + d(x) + g(x) < +\infty$*, then* $G = \overline{T+F}$*, thus the semigroup* $(S_G(t))_{t\geq 0}$ *is honest.*

Proof. Similar to the proof of [12, Theorem 5.2]. □

5.5.2 Honesty of the semigroup $(S_{G_\beta}(t))_{t\geq 0}$

The following lemma plays an important role in the proof of the next theorem.

Lemma 5.5.5. *(a)* *If* $\psi \in D(\mathcal{R}_\beta(\lambda))$*, then*
(i) $\psi \in L_1((x_0, \eta))$ *for any* $\eta < \infty$;
(ii) $\mathcal{R}_\beta(\lambda)\psi$ *is continuous on* (x_0, ∞);
(iii)

$$\lim_{x\to x_0^+} r(x)[\mathcal{R}_\beta(\lambda)\psi](x) = \int_{x_0}^{\infty} \beta(x)[\mathcal{R}_\beta(\lambda)\psi](x)\, dx. \tag{5.58}$$

(b) $r^{-1}\varepsilon_\lambda \in D(A) := \{\psi \in X_1;\ \ a\psi \in X_1\}$.

Proof. [18, Lemma 2.6]. □

We recall that S_{G_β} is defined via Theorem 5.4.6.

Theorem 5.5.6.
Assume $q = a + d + g$ *is bounded at* x_0*, then*

$$G_\beta = \overline{T_\beta + F}.$$

Proof. We make use of the theory of substochastic semigroup. By (5.51) and [16, Theorem 6.13], it is enough to check that

$$\begin{aligned}\int_{x_0}^{\infty} [G_\beta u](x)x\, dx \ \geq\ & \int_{x_0}^{\infty} r(x)u(x)\, dx - \int_{x_0}^{\infty} d(x)u(x)x\, dx \\ & - \int_{x_0}^{\infty} g(x)u(x)x\, dx + x_0 \int_{x_0}^{\infty} \beta(x)u(x)\, dx,\end{aligned}$$

on elements of the form $u = R(\lambda, G_\beta)\psi$, $\psi \in X_{1+}$, $\lambda > \|r\|_\infty$. We recall (see [16, Remark 6.21]) that if $u = R(\lambda, G_\beta)\psi$, $\psi \in X_{1+}$, then there exists $\omega \in E_+$ such that $u = \mathcal{R}_\beta(\lambda)\omega$ and $G_\beta u = \lambda\mathcal{R}_\beta(\lambda)\omega - \omega + \mathcal{F}\mathcal{R}_\beta(\lambda)\omega$. Now, if $X_1 \ni u = \mathcal{R}_\beta(\lambda)\omega$, then $\omega \in D(\mathcal{R}_\beta(\lambda))$ and, by Lemma 5.5.5 (a)(i), $\omega \in L_1((x_0, \eta), xdx)$ and therefore $\mathcal{F}\mathcal{R}_\beta(\lambda)\omega \in L_1((x_0, \eta), xdx)$ as all other terms of the equality above are integrable. Next, consider the decomposition

$$\mathcal{R}_\beta(\lambda)\omega = \mathcal{R}_0(\lambda)\omega + \mathcal{R}_\beta^{'}\omega = \mathcal{R}_0(\lambda)\omega + \frac{\varepsilon_\lambda}{r}\frac{<\beta, \mathcal{R}_0(\lambda)\omega>}{1-<\beta, r^{-1}\varepsilon_\lambda>}.$$

By Lemma 5.5.5(b), $\mathcal{F}\mathcal{R}_\beta^{'}\omega = F\mathcal{R}_\beta^{'}\omega \in X_1$ and therefore $\mathcal{R}_0(\lambda)\omega \in L_1((x_0, \eta), xdx)$. Hence we can write

$$G_\beta u = \lambda\mathcal{R}_0(\lambda)\omega - \omega + \mathcal{F}\mathcal{R}_0(\lambda)\omega + \lambda\mathcal{R}'_\beta\omega + \mathcal{F}\mathcal{R}'_\beta\omega,$$

where each of the first three terms on the right-hand side is in $L_1((x_0, \eta), xdx)$ and the last two are both in X_1. Since $G_\beta u \in X_1$, we can write

$$\begin{aligned}\int_{x_0}^{\infty}[G_\beta u](x)x\,dx &= \lim_{\eta\to\infty}\int_{x_0}^{\eta}[G_\beta u](x)x\,dx \\ &= \lim_{\eta\to\infty}\int_{x_0}^{\eta}(\lambda[\mathcal{R}_0(\lambda)\omega](x) - \omega(x) + [\mathcal{F}\mathcal{R}_0(\lambda)\omega](x))x\,dx \\ &\quad + \int_{x_0}^{\eta}(\lambda[\mathcal{R}'_\beta\omega](x) + [\mathcal{F}\mathcal{R}'_\beta\omega](x))x\,dx \qquad (5.59)\end{aligned}$$

where the limit on the right-hand side exists. Since the integral over (x_0, η) of each term within this limit exists, we can evaluate

$$\begin{aligned}&\int_{x_0}^{\eta}[\mathcal{F}\mathcal{R}_0(\lambda)\omega](x)x\,dx = \int_{x_0}^{\eta}\left(\int_x^{\infty}a(y)b(x|y)[\mathcal{R}_0(\lambda)\omega](y)\,dy\right)x\,dx \\ =\;& \int_{x_0}^{\eta}(\lambda + q(y))[\mathcal{R}_0(\lambda)\omega](y)y\,dy - \lambda\int_{x_0}^{\eta}[\mathcal{R}_0(\lambda)\omega](y)y\,dy - \int_{x_0}^{\eta}q(y)[\mathcal{R}_0(\lambda)\omega](y)y\,dy \\ &+ \int_{x_0}^{\eta}a(y)[\mathcal{R}_0(\lambda)\omega](y)y\,dy + \int_{\eta}^{\infty}a(y)[\mathcal{R}_0(\lambda)\omega](y)\left(\int_{x_0}^{\eta}xb(x|y)\,dx\right)dy \\ =\;& I_1 - I_2 - I_3 + I_4 + I_5.\end{aligned}$$

Using (5.41), we evaluate

$$\begin{aligned}I_1 &= \int_{x_0}^{\eta}\left(\frac{(\lambda + q(y))e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}\int_{x_0}^{y}e^{\lambda\widetilde{R}(s)+\widetilde{Q}(s)}\omega(s)\,ds\right)y\,dy \\ &= \int_{x_0}^{\eta}e^{\lambda\widetilde{R}(s)+\widetilde{Q}(s)}\omega(s)J(s,\eta)\,ds \\ &= \int_{x_0}^{\eta}e^{\lambda\widetilde{R}(s)+\widetilde{Q}(s)}\omega(s)\left(se^{-\lambda\widetilde{R}(s)-\widetilde{Q}(s)} - \eta e^{-\lambda\widetilde{R}(\eta)-\widetilde{Q}(\eta)} + \int_s^{\eta}e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}\,dy\right)ds \\ &= \int_{x_0}^{\eta}\omega(s)s\,ds - \eta r(\eta)[\mathcal{R}_0(\lambda)\omega](\eta) + \int_{x_0}^{\eta}r(y)[\mathcal{R}_0(\lambda)\omega](y)\,dy.\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{x_0}^{\eta}(-\omega(x)+[\mathcal{F}\mathcal{R}_0(\lambda)\omega](x)+\lambda[\mathcal{R}_0(\lambda)\omega](x))x\,dx\\
= \quad & -\eta r(\eta)[\mathcal{R}_0(\lambda)\omega](\eta)-\int_{x_0}^{\eta}q(y)[\mathcal{R}_0(\lambda)\omega](y)y\,dy+\int_{x_0}^{\eta}a(y)[\mathcal{R}_0(\lambda)\omega](y)y\,dy\\
&+\int_{x_0}^{\eta}r(y)[\mathcal{R}_0(\lambda)\omega](y)\,dy+\int_{\eta}^{\infty}a(y)[\mathcal{R}_0(\lambda)\omega](y)\left(\int_{x_0}^{\eta}xb(x|y)\,dx\right)dy.
\end{aligned}$$

Let $u_0 := [\mathcal{R}_0(\lambda)\omega]$. Since $u_0 = u - \text{const}(r^{-1}\varepsilon_\lambda)$, we see that $u_0 \in X_1$. Thus $ru_0 \in L_1((x_0,\infty))$ and both du_0, gu_0 are in X. Furthermore, there exists a sequence $\eta_k \to \infty$ as $k \to \infty$ for which $\eta_k r(\eta_k)u_0(\eta_k) \to 0$. Indeed, otherwise $xr(x)u_0(x) \geq \varepsilon > 0$ for some ε and all sufficient large x. But then $r(x)u_0(x) \geq \varepsilon x^{-1}$ which would contradict the integrability of ru_0. Hence

$$\begin{aligned}
&\lim_{k\to\infty}\int_{x_0}^{\eta_k}(-\omega(x)+[\mathcal{B}u_0](x)+\lambda u_0(x))x\,dx\\
= \quad & -\int_{x_0}^{\infty}q(y)u_0(y)y\,dy+\int_{x_0}^{\infty}a(y)u_0(y)y\,dy\\
&+\int_{x_0}^{\infty}r(y)u_0(y)\,dy+\lim_{k\to\infty}\int_{\eta_k}^{\infty}a(y)u_0(y)\left(\int_{x_0}^{\eta_k}xb(x|y)\,dx\right)dy.
\end{aligned}$$

To deal with the last two terms in (5.59), we note that ω enters the expression through a constant scalar multiplier and hence first evaluate

$$\begin{aligned}
&\int_{x_0}^{\infty}\left(\int_{x}^{\infty}a(y)b(x|y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}dy\right)x\,dx\;+\;\lambda\int_{x_0}^{\infty}\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy\\
= \quad & \int_{x_0}^{\infty}a(y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy\;+\;\lambda\int_{x_0}^{\infty}\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy\\
= \quad & J(x_0,\infty)-\int_{x_0}^{\infty}q(y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{a(y)}y\,dy+\int_{x_0}^{\infty}a(y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy\\
= \quad & x_0+\int_{x_0}^{\infty}e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}\,dy-\int_{x_0}^{\infty}[q-a](y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy,
\end{aligned}$$

where we used (5.41) with (5.50). Hence we obtain

$$\begin{aligned}
&\int_{x_0}^{\infty}(\lambda[\mathcal{R}'_\beta\omega](x)+[\mathcal{B}\mathcal{R}'_\beta\omega](x))x\,dx\\
= \quad & \int_{x_0}^{\infty}(\lambda+q(y))[\mathcal{R}'_\beta\omega](y)y\,dy-\int_{x_0}^{\infty}q(y)[\mathcal{R}'_\beta\omega](y)y\,dy+\int_{x_0}^{\infty}a(y)[\mathcal{R}'_\beta\omega](y)y\,dy\\
= \quad & x_0<\beta,\mathcal{R}_\beta(\lambda)\omega>+\int_{x_0}^{\infty}r(y)[\mathcal{R}'_\beta\omega](y)\,dy-\int_{x_0}^{\infty}q(y)[\mathcal{R}'_\beta\omega](y)y\,dy\\
&+\int_{x_0}^{\infty}a(y)[\mathcal{R}'_\beta\omega](y)y\,dy,
\end{aligned}$$

where we used

$$< \beta, \mathcal{R}_\beta(\lambda)\omega >= \frac{< \beta, \mathcal{R}_0(\lambda)\omega >}{1- < \beta, r^{-1}\varepsilon_\lambda >}.$$

Combining the above results, we see that there is a sequence $(\eta_k)_{k\in\mathbb{N}}$ such that

$$\begin{aligned}
&\int_{x_0}^{\infty} [G_\beta u](x)x\,dx = \lim_{\eta\to\infty} \int_{x_0}^{\eta} [G_\beta u](x)x\,dx \\
= \;& x_0 \int_{x_0}^{\infty} \beta(x)u(x)\,dx + \int_{x_0}^{\infty} r(x)u(x)\,dx - \int_{x_0}^{\infty} q(x)u(x)x\,dx \\
&+ \int_{x_0}^{\infty} a(x)u(x)x\,dx + \lim_{k\to\infty} \int_{\eta_k}^{\infty} a(y)u_0(y) \left(\int_{x_0}^{\eta_k} xb(x|y)\,dx \right) dy \\
\geq \;& x_0 \int_{x_0}^{\infty} \beta(x)u(x)\,dx + \int_{x_0}^{\infty} r(x)u(x)\,dx - \int_{x_0}^{\infty} [g+d](x)u(x)x\,dx
\end{aligned}$$

which proves the thesis. □

5.6 Global solution for the evolution equation

The combined mortality, coagulation and mass-growth fragmentation equation reads as:

$$\begin{aligned}
\frac{du}{dt}(t) &= [T+B+N]u(t) \\
u(0) &= u_0,
\end{aligned}$$

where N defined on the set $X_{1+} = \{\psi \in X_1 : \psi \geq 0\}$ is the realization of the operator $\mathcal{N}$ (defined via (5.14)) on the space X_1. We recall that $N(0) = 0$ and for any $\psi \in X_{1+} \setminus \{0\}$, we have

$$(N\psi)(x) := \chi_U(x) \frac{\int_{x_0}^{x-x_0} yg(y)\psi(y)(x-y)g(x-y)\psi(x-y)\,dy}{x\int_{x_0}^{\infty} zg(z)\psi(z)\,dz}, \tag{5.60}$$

where χ_U is the characteristic function of the interval $U = [2x_0, \infty)$.

Lemma 5.6.1. *The operator N satisfies a global Lipschitz condition on the set X_{1+}.*

Proof. The proof is similar to the analysis performed in the appendix of [14]. Adopting the notation

$$\Theta\phi(x) = xg(x)\phi(x) \quad \text{and} \quad \alpha(\phi) = \int_{x_0}^{\infty} \Theta\phi(x)\,dx,$$

the operator N can be expressed as $N\phi(x) = \chi_U(x)\dfrac{(\Theta\phi * \Theta\phi)(x)}{x\alpha(\phi)}$, where $\phi \in X_{1+} \setminus \{0\}$ and

$$(\Theta\phi * \Theta\phi)(x) := \int_{x_0}^{x-x_0} \Theta\phi(y)\Theta\phi(x-y)\,dy.$$

Fix a function $\psi_0 \in X_{1+} \setminus \{0\}$, set $c := \operatorname{ess\,sup}\{g(x) : \; x_0 < x < \infty\}$ and $\varepsilon := \alpha(\psi_0)c^{-1}$. Let ψ be any function from $X_{1+} \setminus \{0\}$ such that $\|\psi - \psi_0\| \leq \varepsilon$. Then

$$\alpha(\psi) = \alpha(\psi_0) + \alpha(\psi - \psi_0) \leq 2\alpha(\psi_0). \tag{5.61}$$

Note that

$$\begin{aligned} & N\psi(x) - N\psi_0(x) \\ = \;& \chi_U(x)\frac{[(\Theta\psi * \Theta\psi)(x)]\alpha(\psi_0 - \psi)}{x\alpha(\psi_0)\alpha(\psi)} + \chi_U(x)\frac{(\Theta\psi * \Theta\psi)(x)}{x\alpha(\psi_0)} + \chi_U(x)\frac{(\Theta\psi_0 * \Theta\psi_0)(x)}{x\alpha(\psi_0)} \\ = \;& \chi_U(x)\frac{[(\Theta\psi * \Theta\psi)(x)]\alpha(\psi_0 - \psi)}{x\alpha(\psi_0)\alpha(\psi)} + \chi_U(x)\frac{[\Theta(\psi + \psi_0) * \Theta(\psi - \psi_0)](x)}{x\alpha(\psi_0)}, \end{aligned}$$

where we used linearity of α and properties of the convolution. It follows that

$$\begin{aligned} \|N\psi - N\psi_0\|_1 \;\leq\; & \frac{\alpha(|\psi_0 - \psi|)\displaystyle\int_{x_0}^{\infty}(\Theta\psi * \Theta\psi)(x)\,dx}{\alpha(\psi_0)\alpha(\psi)} \\ & + \frac{\displaystyle\int_{x_0}^{\infty}[\Theta(\psi + \psi_0) * |\Theta(\psi - \psi_0)|\,](x)\,dx}{\alpha(\psi_0)}. \end{aligned} \tag{5.62}$$

Because

$$\int_{x_0}^{\infty}(\Theta\psi * \Theta\psi)(x)\,dx = \left[\int_{x_0}^{\infty}(\Theta\psi)(x)\,dx\right]^2 = [\alpha(\psi)]^2$$

and

$$\int_{x_0}^{\infty}[\Theta(\psi + \psi_0) * |\Theta(\psi - \psi_0)|](x)dx = \alpha(\psi + \psi_0)\alpha(|\psi - \psi_0|),$$

the previous inequality yields

$$\begin{aligned} \|N\psi - N\psi_0\|_1 \;\leq\; & \frac{\alpha(\psi)\alpha(|\psi_0 - \psi|)}{\alpha(\psi_0)} + \frac{\alpha(\psi + \psi_0)\alpha(|\psi - \psi_0|)}{\alpha(\psi_0)} \\ \leq\; & 5\alpha(|\psi - \psi_0|) \\ \leq\; & 5c\|\psi - \psi_0\|_1, \end{aligned} \tag{5.63}$$

where we used linearity of α and applied (5.61).

Now we check this inequality for all $\phi, \psi \in X_{1+} \setminus \{0\}$. Fix $\phi, \psi \in X_{1+} \setminus \{0\}$ and let $\phi_t = (1-t)\phi + t\psi$ for $t \in [0,1]$. Since the function $t \mapsto \alpha(\phi_t)$ is continuous and $\alpha(\phi_t) > 0$ for each $t \in [0,1]$ we have $\inf_t \alpha(\phi_t) > 0$. Let $\overline{\varepsilon} = c^{-1}\inf_t \alpha(\phi_t)$. Then (5.63) implies that

$$\|N\phi_s - N\phi_t\|_1 \leq 5c\|\phi_s - \phi_t\|_1 \quad \text{provided that} \quad \|\phi_s - \phi_t\|_1 \leq \overline{\varepsilon}.$$

Let n be an integer such that $n \geq \|\phi - \psi\|_1/\bar{\varepsilon}$ and let $t_i = i/n$ for $i = 0, 1, ..., n$. Then $\|\phi_{t_i} - \phi_{t_{i-1}}\|_1 \leq \bar{\varepsilon}$ and consequently:

$$\begin{aligned} \|N\phi - N\psi\|_1 &\leq \sum_{i=1}^{n} \|N\phi_{t_i} - N\phi_{t_{i-1}}\|_1 \\ &\leq 5c \sum_{i=1}^{n} \|\phi_{t_i} - \phi_{t_{i-1}}\|_1 \\ &= 5c\|\phi - \psi\|_1, \end{aligned} \tag{5.64}$$

where we used the fact that $\phi_{t_i} - \phi_{t_{i-1}} = \dfrac{\psi - \phi}{n}$ for any $i = 0, 1, ..., n$. Furthermore by (5.8), $\|N\psi\|_1 \leq \int_{x_0}^{\infty} xg(x)\psi(x)\,dx \leq c\|\psi\|_1$ for any $\psi \in X_{1+}$. As a result the operator N is continuous at 0. Therefore inequality (5.64) passes to the limit at $\phi = 0$ or $\psi = 0$. □

Theorem 5.6.2. *(i) Let $u_0 \in D(G) \cap X_{1+}$. Subject to the initial condition $u(0) = u_0$, the equation*

$$\frac{du}{dt}(t) = G[u(t)] + N[u(t)] \tag{5.65}$$

has a global unique solution.

Proof. First we recall that the solution u of (5.65) is the unique solution of the integral equation

$$u(t) = S_G(t)u_0 + \int_0^t S_G(t-s)N[u(s)]\,ds,\ t \geq 0, \tag{5.66}$$

where $(S_G(t))_{t\geq 0}$ is the semigroup generated by the operator G.

Let

$$Y := C([0, t_1], X_{1+})$$

with norm

$$\|v\|_Y := \max\{\|v(t)\|_1 :\ 0 \leq t \leq t_1\}.$$

Moreover, let

$$\Upsilon := \{v \in Y :\ v(t) \in \overline{B}(u_0, r_1) \cap X_{1+}\ \forall t \in [0, t_1]\},$$

where r_1 is a non-negative real number.

Define the mapping $\mathcal{Q}$ on Υ as follow:

$$(\mathcal{Q}v)(t) \ := \ S_G(t)f + \int_0^t S_G(t-s)N[v(s)]\,ds,\ 0 \leq t \leq t_1.$$

Then $\mathcal{Q}(\Upsilon) \subset Y$ and $(\mathcal{Q}v)(t) \in X_{1+}$ for all $t \in [0, t_1]$. The proof of the existence of a unique solution $u \in \Upsilon$ to the equation $u = \mathcal{Q}u$ is similar to the calculations performed in Theorem 4.2.6. Consequently the integral equation (5.66) has a unique solution $u \in C([0, t_1], X_{1+})$. Since N is globally Lipschitz, the existence of a global strong solution to problem (5.65) follows. □

Theorem 5.6.3. *Subject to the initial condition* $u(0) = u_0 \in D(G_\beta) \cap X_{1+}$*, the equation*

$$\frac{du}{dt}(t) = G_\beta[u(t)] + N[u(t)] \tag{5.67}$$

has a global unique solution.

Proof. The proof is similar to the analysis above with the operator G. □

Chapter 6

Nonlocal Continuous Fragmentation Processes

6.1 Introduction

An initial-value integro-differential problem describing multiple fragmentation processes, where the fragmentation rate is size and position dependent and new particles are spatially randomly distributed according to some probability density is investigated by means of substochastic semigroup theory and approximation techniques. The existence of a semigroup is established and, under natural conditions on certain coefficients, the generator of this semigroup is identified. In particular we prove the existence and uniqueness of a nonnegative mass-conserving solution and provide sufficient conditions for honesty.

6.2 Description of the model and assumptions

We focus on continuous models; that is, we assume that the mass of a particle can be an arbitrary positive real number. The starting point is to describe the state variable of the problem. The state at a given time t is the repartition at that time of all aggregates according to their size m and their position x. In terms of m and x, the state of the system is characterized at any moment t by the particle-mass-position distribution $u = u(t, m, x)$, (u is also called the *density* or *concentration* of particles), where $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}_+$. Thus,

$$\int_n^p \int_{\mathbb{R}^3} u(t, m, x) dx dm$$

is the number of particles having mass between n and p and

$$\int_n^p \int_{\mathbb{R}^3} u(t, m, x) m dx dm$$

is the mass contained in particles in $\mathbb{R}^3$ having mass within this range.

Definition 6.2.1. *The fragmentation rate $a = a(m,x)$ describes the ability of aggregates of size m and position x to break into smaller particles.*

During the unit time, a fraction $a(m,x)$ of aggregates of size m and located at x are undergoing breakup. We assume that

$$a \in L_{\infty,\,\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3). \tag{6.1}$$

Once an aggregate of mass s and position x breaks, the expected number of daughter particles of size m is a non-negative measurable function $b(m,s,x)$ defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^3$ with support in the set

$$\{(m,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : m < s\} \times \mathbb{R}^3.$$

After the fragmentation of a mass s particle, the sum of masses of all daughter particles should again be s, hence it follows that for any $s > 0$, $x \in \mathbb{R}^3$

$$\int_0^s nb(n,s,x)dn = s. \tag{6.2}$$

Furthermore the expected number of daughter particles produced by fragmentation of a mass s particle (with position x) is, by definition, given by

$$\int_0^s b(n,s,x)dn. \tag{6.3}$$

In case of binary fragmentation [14, 38], it is straightforward that for a.a $x \in \mathbb{R}^3$, $b(m,s,x) = b(s-m,s,x)$ for all $m, s, s > m$, and

$$\int_0^s b(m,s,x)dm = 2 \ \text{ for all } \ s > 0. \tag{6.4}$$

After cluster fragmentation new originating clusters have different centers distributed according to a given probabilistic law $\widetilde{b}(\cdot\,,m,s,y)$. This is the probability density that after a break up of an (s)- aggregate (with the center at y), the new formed m-aggregate will be located at x. Therefore

$$\int_{\mathbb{R}^3} \widetilde{b}(x,m,s,y)dx = 1. \tag{6.5}$$

The equation describing the evolution of the particle-mass-size distribution function for a continuous system undergoing fragmentation can be derived by balancing loss and gain of particles of mass m (with position x) over a short period of time. From the definitions, at any time t, the loss term is $a(m,x)u(t,m,x)$ and the gain term

$$\int_m^\infty \int_{\mathbb{R}^3} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)u(t,s,y)dyds.$$

Therefore the whole equation reads

$$\begin{aligned}\frac{\partial u}{\partial t}(t,m,x) =& - a(m,x)u(t,m,x)\\ &+ \int_m^\infty \int_{\mathbb{R}^3} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)u(t,s,y)dyds.\end{aligned} \tag{6.6}$$

The total mass of the ensemble at time t is the quantity

$$\int_0^\infty \int_{\mathbb{R}^3} u(t,m,x)mdxdm, \tag{6.7}$$

thus the natural space for analysis is

$$X = L_1(\mathbb{R}_+ \times \mathbb{R}^3, mdxdm). \tag{6.8}$$

In order to make use of the semigroup theory of linear operators we need to complement (6.6) with the initial mass-position distribution

$$u(0,m,x) = u_0(m,x), \quad \text{a.e } (m,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \tag{6.9}$$

where $u_0 \in X$.

In what follows we denote by $\mathcal{A}$ and $\mathcal{B}$ the expressions appearing on the right-hand side of the equations (6.6); that is,

$$[\mathcal{A}\psi](m,x) = -a(m,x)\psi(m,x), \tag{6.10}$$

and

$$[\mathcal{B}\psi](m,x) = \int_m^\infty \int_{\mathbb{R}^3} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)\psi(s,y)dyds, \tag{6.11}$$

defined on all measurable and finite almost everywhere functions ψ for which they make pointwise (almost everywhere) sense.

6.3 Analysis

We introduce operators A and B in X defined by

$$[Au](m,x) = [\mathcal{A}u](m,x), \qquad [Bu](m,x) = [\mathcal{B}u](m,x) \tag{6.12}$$

and set $D(A) = \{\psi \in X; \quad a\psi \in X\}$.

Lemma 6.3.1. *$(A+B,\ D(A))$ is a well defined operator.*

Proof. In order to prove the first part of the theorem, we need to show that $\mathcal{B}D(A) \subset X$. Let $u \in D(A)_+$, changing the order of integration by the Fubini theorem, we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} \mathcal{B}u(m,x)mdxdm \\
= &\int_{\mathbb{R}^3} \int_0^\infty \left[\int_m^\infty a(s,y)b(m,s,y)\left(\int_{\mathbb{R}^3} \widetilde{b}(x,m,s,y)dx\right) u(s,y)ds\right] mdmdy \\
= &\int_{\mathbb{R}^3} \left(\int_0^\infty \int_m^\infty ma(s,y)b(m,s,y)u(s,y)dsdm\right) dy \\
= &\int_{\mathbb{R}^3} \left(\int_0^\infty \int_0^s ma(s,y)b(m,s,y)u(s,y)dmds\right) dy \\
= &\int_0^\infty \int_{\mathbb{R}^3} a(s,y)u(s,y)sdyds,
\end{aligned}$$

where we used (6.5) and (6.2) respectively. Because $u \in D(A)_+$ it follows that

$$\int_0^\infty \int_{\mathbb{R}^3} \mathcal{B}u(m,x)mdxdm < +\infty.$$

The result follows from the fact that any arbitrary element u of $D(A)$ can be written in the form $u = u_+ - u_-$, where $u_+, u_- \in D(A)_+$. ☐

Theorem 6.3.2. *There is an extension G of $A + B$ that generates a positive semigroup of contractions $(S_G(t))_{t\geq 0}$ on X. Moreover, for each $u_0 \in D(G)$ there is a measurable representation $u(t,m,x)$ of $S_G(t)u_0$ which is absolutely continuous with respect to $t \geq 0$ for almost any (m,x) and such that (6.6) is satisfied almost everywhere.*

Proof. We claim that $(A,\, D(A))$ generates a positive semigroup of contractions. In fact because the operator A is a multiplication operator on X induced by the measurable function a, it is closed and densely defined [31]. Also for any $\lambda > 0$, it is obvious that $\lambda I - A$ is bijective and the resolvent $R(\lambda,\, A)$ of A satisfies the estimate

$$\|R(\lambda,\, A)\psi\| \leq \frac{1}{\lambda}\|\psi\| \tag{6.13}$$

for any $\psi \in X$. Furthermore for any positive λ, the operator $R(\lambda,\, A)$ is nonnegative. Therefore $(A,\, D(A))$ generates a positive semigroup of contractions.

It is clear that $(B,\, D(B))$ is positive. Furthermore for any $u \in D(A)$, by the calculations in the previous lemma, we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} (\mathcal{A}u + \mathcal{B}u)(m,x)mdxdm \\
= &\int_0^\infty \int_{\mathbb{R}^3} [\mathcal{A}u](m,x)mdxdm + \int_0^\infty \int_{\mathbb{R}^3} [\mathcal{B}u](m,x)mdxdm \\
= &\int_0^\infty \int_{\mathbb{R}^3} -a(m,x)u(m,x)mdxdm + \int_0^\infty \int_{\mathbb{R}^3} a(s,y)u(s,y)sdyds = 0.
\end{aligned}$$

Thus the assumptions of Theorem 2.3.5 are satisfied. Therefore there is an extension G of $A+B$ generating a substochastic semigroup $(S_G(t))_{t\geq 0}$. Also for any $u_0 \in D(G)$, the function

$$t \to S_G(t)u_0$$

is a C^1-function in the norm of X and satisfies the equation

$$\frac{d}{dt}S_G(t)u_0 = GS_G(t)u_0, \tag{6.14}$$

where the equality holds for any $t > 0$ in the sense of equality in X. The initial condition is satisfied in the following sense

$$\lim_{t\to 0^+} S_G(t)u_0 = u_0, \tag{6.15}$$

where the convergence is in the X-norm.

In order to prove the second part of this theorem we make use of the theory of extensions and the theory of L spaces [16]. Let E be the set of finite almost everywhere measurable functions defined on $(0,\infty)\times\mathbb{R}^3$. We recall that E is a lattice with respect to the usual relation ($\leq$ almost everywhere), $X \subset E$ and X is a sublattice of E. We denote by X_+ and E_+ the positive cones of X and E respectively. Also we introduce the operator B defined such that for any nondecreasing sequence $(\psi_n)_{n\in\mathbb{N}}$ in X_+ with $\sup_{n\in\mathbb{N}}\psi_n = \psi \in E_+$,

$$\mathrm{B}\psi := \sup_{n\in\mathbb{N}} B\psi_n. \tag{6.16}$$

Since B is an integral operator with positive kernel, Lebesgue's monotone convergence theorem yields that $\mathrm{B} = \mathcal{B}$. Thus, [16, Theorem 6.20] yields

$$G \subset \mathcal{A} + \mathcal{B}.$$

Hence $S_G(t)u_0$ satisfies

$$\left[\frac{d}{dt}S_G(t)u_0\right](m,x) = [\mathcal{A}S_G(t)u_0](m,x) + [\mathcal{B}S_G(t)u_0](m,x), \tag{6.17}$$

for each fixed $t > 0$, where the right hand side does not depend (in the sense of equality almost everywhere) on what representation of the solution $S_G(t)u_0$ is considered.

Making use of the fact that X is an L-space, from Theorem 2.1.7, we have that since the function $S_G(t)u_0$ is strongly differentiable, there is a representation $u(t,m,x)$ of $S_G(t)u_0$ that is absolutely continuous with respect to $t \geq 0$ for almost any $(m,x) \in \mathbb{R}_+\times\mathbb{R}^3$, and that satisfies

$$\frac{\partial}{\partial t}u(t,m,x) = \left[\frac{d}{dt}S_G(t)u_0\right](m,x)$$

for any $t \geq$ and almost any (m,x). Hence, taking this representation, we obtain that

$$\begin{aligned}\frac{\partial u}{\partial t}(t,m,x) &= -a(m,x)u(t,m,x) \\ &\quad + \int_m^\infty \int_{\mathbb{R}^3} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)u(t,s,y)dyds\end{aligned} \tag{6.18}$$

holds almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^3$. Moreover, the continuity of $u(t, m, x)$ with respect to t for almost every (m, x) shows that

$$\lim_{t \to 0^+} u(t, m, x) = \overline{u}(m, x)$$

exists almost everywhere. From (6.15) we see that there is a sequence $(t_n)_{n \in \mathbb{N}}$ converging to 0 such that

$$\lim_{n \to \infty} u(t_n, m, x) = u_0(m, x),$$

for almost every (m, x). Here we can use the same representation as above because we are dealing with a (countable) sequence. Indeed, changing the representation on a set of measure zero for each n and further taking the union of all these sets still produces a set of measure zero. Thus $u_0 = \overline{u}$ almost everywhere. □

On account of this result we use the same notation for the abstract X-valued functions of t and their representations as scalar functions of several variables, bearing in mind that we select a 'proper' representation. Thus, for example, for $u(t) = S_G(t)u_0$ (with $u_0 \in D(G)$), by $u(t, m, x)$ we mean the representation satisfying (6.18).

In general, the function $S_G(t)u_0$ is not differentiable if $u_0 \in X \backslash D(G)$. Therefore it cannot be a classical solution of the Cauchy problem (6.14), (6.15). However it is a mild solution, that is, it is a continuous function such that

$$\int_0^t u(\tau) d\tau \in D(G) \quad \text{for any } t \geq 0,$$

satisfying the integrated version of (6.14), (6.15):

$$u(t) = u_0 + G \int_0^t u(\tau) d\tau. \tag{6.19}$$

Corollary 6.3.3. *If $u_0 \in X \backslash D(G)$, then $u(t, m, x) = [S_G(t)u_0](m, x)$ satisfies the equation*

$$\begin{aligned} u(t, m, x) &= u_0(m, x) - a(m, x) \int_0^t u(\tau, m, x) d\tau + \\ &\int_m^\infty \int_{\mathbb{R}^3} a(s, y) b(m, s, y) \widetilde{b}(x, m, s, y) \left(\int_0^t u(\tau, s, y) d\tau \right) dy ds. \end{aligned} \tag{6.20}$$

Proof. Because u is continuous in the norm of X, we can use (2.2) to claim that $a(m, x) \int_0^t u(\tau, m, x) d\tau$ is defined for almost any (m, x) and any t, and hence we can write

$$\begin{aligned} \left[(\mathcal{A} + \mathcal{B}) \int_0^t u(\tau) d\tau \right] (m, x) &= -a(m, x) \int_0^t u(\tau, m, x) d\tau \\ + \int_0^\infty \int_{\mathbb{R}^3} a(s, y) b(m, s, y) \widetilde{b}(x, m, s, y) &\left(\int_0^t u(\tau, s, y) d\tau \right) dy ds. \end{aligned} \tag{6.21}$$

Thus, combining the result used in the previous theorem that $G \subset \mathcal{A} + \mathcal{B}$ with (6.19) we obtain (6.20). □

Next we provide a fairly general condition for honesty of $(S_G(t))_{t\geq 0}$.

6.4 Honesty

In the physical processes modeled by the fragmentation equation the total mass of the system is expected to remain constant throughout the evolution. It is formally expressed by (6.6) as the mass rate equation can be found by multiplying (6.6) by m and integrating over $(0, \infty) \times \mathbb{R}^3$. Thus by (6.2) and (6.7) we obtain

$$\frac{d}{dt}\int_0^\infty \int_{\mathbb{R}^3} mu(t, m, x)dxdm = 0, \tag{6.22}$$

which agrees with the physics of the process as fragmentation should simply rearrange the distribution of masses of the particles without altering the total mass of the system.

However, the validity of (6.22) depends on certain properties of the solution u that we tacitly assumed during the integration and which are far from obvious. In fact, by analyzing models with specific coefficients, several authors have observed that the local version of (6.22) is not valid [57]. In other words, there occurs an unexpected mass loss in the system. In local fragmentation, the unaccounted for mass loss was termed *shattering fragmentation* and was attributed to the phase transition in which a dust of particles with zero size and nonzero mass is formed. The presence of x in (6.22) suggests that honesty in nonlocal fragmentation depends also on the spatial properties of the fragmentation kernels. In this section we provide sufficient conditions for the fragmentation semigroup to be honest for general coefficients.

Lemma 6.4.1. *Assume that for any $g \in X_+$ such that $-ag + Bg \in X$ we have the inequality*

$$\int_0^\infty \int_{\mathbb{R}^3} (-a(m, x)g(m, x) + [Bg](m, x))mdxdm \geq 0, \tag{6.23}$$

then $G = \overline{A + B}$. Thus the solution $u(t) = S_G(t)u_0$ satisfies

$$\frac{d}{dt}\int_0^\infty \int_{\mathbb{R}^3} S_G(t)u_0(m, x)mdxdm = \frac{d}{dt}\|S_G(t)u_0\| = 0$$

and for any $0 \leq u_0 \in D(G)$.

Proof. The method we employ is analogous to that used in [16, Theorem 8.5]. Assume that for any $g \in X_+$ such that $-ag + Bg \in X$ the inequality (6.23) holds. By ([16], Theorem 6.13 and Theorem 6.22), it is enough to show that for any $h \in \mathrm{F}_+$ such that $-h + \mathrm{BL}h \in X$ the following inequality holds,

$$\int_0^\infty \int_{\mathbb{R}^3} [\mathrm{L}h](m,x) m dx dm + \int_0^\infty \int_{\mathbb{R}^3} (-h(m,x) + [\mathrm{BL}h](m,x)) m dx dm \geq 0,$$

where $\mathrm{F} := \{h \in \mathrm{E}\,;\ (1+a)^{-1}h \in X\}$, $L : F_+ \to X$ is defined such that $\mathrm{L}h := (1+a)^{-1}h$ and B is defined via (6.16). We recall that $\mathrm{B}\psi = B\psi$ for any $\psi \in D(B)$. Now let $h \in \mathrm{F}_+$ such that $-h + B\mathrm{L}h \in X$, let us set $g := \mathrm{L}h$, it is clear that $g \in X_+$. Furthermore

$$-ag + Bg = -a\mathrm{L}h + B\mathrm{L}h = \mathrm{L}h + (-h + B\mathrm{L}h) \in X.$$

Since g satisfies the assumption, we have that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} [Lh](m,x) m dx dm + \int_0^\infty \int_{\mathbb{R}^3} (-h(m,x) + [BLh](m,x)) m dx dm \\
&= \int_0^\infty \int_{\mathbb{R}^3} (g(m,x) - (1 + a(m,x))g(m,x) + [Bg](m,x)) m dx dm \\
&= \int_0^\infty \int_{\mathbb{R}^3} (-a(m,x)g(m,x) + [Bg](m,x)) m dx dm \geq 0.
\end{aligned}$$

□

Theorem 6.4.2. *Assume that there exists a real-valued function* $\widetilde{a} = \widetilde{a}(m) \in L_{\infty,\,loc}(\overline{\mathbb{R}}_+)$ *such that* $a(m,x) \leq \widetilde{a}(m)$ *for almost all* $(m,x) \in \mathbb{R}_+ \times \mathbb{R}^3$*, then the semigroup* $(S_G(t))_{t\geq 0}$ *is honest.*

Proof. Making use of the previous lemma, it is enough to prove that for any $g \in X_+$ such that $-ag + Bg \in X$, the inequality

$$\int_0^\infty \int_{\mathbb{R}^3} (-a(m,x)g(m,x) + [Bg](m,x)) m dx dm \geq 0$$

is satisfied. We have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} (-a(m,x)g(m,x) + [Bg](m,x)) m dx dm \\
&= \lim_{R\to\infty} \left(\int_0^R \int_{\mathbb{R}^3} -a(m,x)g(m,x) m dx dm + \int_0^R \int_{\mathbb{R}^3} [Bg](m,x) m dx dm \right).
\end{aligned}$$

Also by (6.5),

$$\begin{aligned}
&\int_0^R \int_{\mathbb{R}^3} [Bg](m,x) m dx dm \\
=\ &\int_0^R \int_m^\infty \int_{\mathbb{R}^3} a(s,y) b(m,s,y) \left(\int_{\mathbb{R}^3} \widetilde{b}(m,x,s,y) dx \right) g(s,y) m\, dy\, ds\, dm \\
=\ &\int_{\mathbb{R}^3} \left(\int_0^R \int_m^\infty m a(s,y) b(m,s,y) g(s,y) ds dm \right) dy.
\end{aligned}$$

Furthermore by (6.2),

$$\begin{aligned}
& \int_0^R \int_m^\infty ma(s,y)b(m,s,y)g(s,y)dsdm \\
= \quad & W_R(y) + \int_0^R \int_0^s ma(s,y)b(m,s,y)g(s,y)dmds \\
= \quad & W_R(y) + \int_0^R sa(s,y)g(s,y)ds,
\end{aligned}$$

where for any $y \geq 0$, $W_R(y) = \displaystyle\int_R^\infty \int_0^R ma(s,y)b(m,s,y)g(s,y)dmds \geq 0$. Combining, for any $R > 0$ we have

$$\begin{aligned}
& \int_0^R \int_{\mathbb{R}^3} (-a(m,x)g(m,x) + [Bg](m,x))mdxdm \\
&= \int_0^R \int_{\mathbb{R}^3} (-[ag](m,x)mdxdm + \int_{\mathbb{R}^3} \left(W_R(y) + \int_0^R s[ag](s,y)ds \right) dy \\
&= \int_{\mathbb{R}^3} W_R(y)dy \geq 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}^3} (-a(m,x)g(m,x) + [Bg](m,x))mdxdm \\
= \quad & \lim_{R\to\infty} \int_{\mathbb{R}^3} W_R(y)dy \geq 0.
\end{aligned}$$

□

The previous theorem shows that the spatial distribution of the particles does not influence conservativeness of the system when the fragmentation rate is bounded by a size only dependent function. In other words nonlocal models with fragmentation rate $a(m,x)$ bounded as $|x|$ approaches infinity behave like local models, therefore are conservative provided that the fragmentation rate is bounded as m approaches zero. A major problem arises when the fragmentation rate $a(m,x)$ becomes infinite as $|x|$ is close to infinity. The next theorem gives a sufficient condition for honesty in that case.

Theorem 6.4.3. *Assume that*

$$a \in L_{\infty,\, loc}(\overline{\mathbb{R}}_+ \times \mathbb{R}^3) \tag{6.24}$$

and there exists $C > 0$ such that

$$a(s,y) \int_{|x|>|y|} \widetilde{b}(m,x,s,y)dx < C \tag{6.25}$$

for almost any $m, s, y \in \mathbb{R}^3$, then the semigroup $(S_G(t))_{t\geq 0}$ is honest.

Proof. Considering (6.24), for any $0 < R_1, R_2 < \infty$ we have that $ag \in L_1([0, R_1] \times B(O, R_2), mdxdm)$, where $B(O, R_2) = \{x \in \mathbb{R}^3; |x| \leq R_2\}$. Since $-ag + Bg \in X$, it follows that $Bg \in L_1([0, R_1] \times B(O, R_2), mdxdm)$. In this respect,

$$\int_0^\infty \int_{\mathbb{R}^3} (-a(m,x)g(m,x) + [Bg](m,x))mdxdm =$$
$$\lim_{R_1, R_2 \to \infty} \left(\int_0^{R_1} \int_{B(O,R_2)} -a(m,x)g(m,x)mdxdm + \int_0^{R_1} \int_{B(O,R_2)} [Bg](m,x)mdxdm \right).$$

We have

$$\begin{aligned} &\int_0^{R_1} \int_{B(O,R_2)} [Bg](m,x)mdxdm \\ = \;& \int_0^{R_1} \int_{B(O,R_2)} \left(\int_m^\infty \int_{\mathbb{R}^3} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)g(s,y)dyds \right) mdxdm \\ = \;& H(R_1, R_2) + \int_0^{R_1} \int_{\mathbb{R}^3} \int_0^s \int_{B(O,R_2)} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)g(s,y)mdxdmdyds, \end{aligned}$$

where

$$H(R_1, R_2) = \int_{R_1}^\infty \int_{\mathbb{R}^3} \int_0^{R_1} \int_{B(O,R_2)} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)g(s,y)mdxdmdyds \geq 0.$$

It follows that

$$\begin{aligned} &\int_0^{R_1} \int_{B(O,R_2)} [Bg](m,x)mdxdm \\ \geq \;& \int_0^{R_1} \int_{\mathbb{R}^3} a(s,y)g(s,y) \left(\int_0^s \int_{B(O,R_2)} b(m,s,y)\widetilde{b}(x,m,s,y)mdxdm \right) dyds \\ \geq \;& \int_0^{R_1} \int_{B(O,R_2)} a(s,y)g(s,y) \left(\int_0^s \int_{B(O,R_2)} b(m,s,y)\widetilde{b}(x,m,s,y)mdxdm \right) dyds. \end{aligned}$$

Thus

$$\begin{aligned} &\int_0^{R_1} \int_{B(O,R_2)} [Bg](m,x)mdxdm \geq \int_0^{R_1} \int_{B(O,R_2)} a(s,y)g(s,y)sdyds \\ &- \int_0^{R_1} \int_{B(O,R_2)} a(s,y)g(s,y) \left(\int_0^s \int_{|x|>R_2} b(m,s,y)\widetilde{b}(x,m,s,y)mdxdm \right) dyds. \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^{R_1} \int_{B(O,R_2)} -a(m,x)g(m,x)mdxdm + \int_0^{R_1} \int_{B(O,R_2)} [Bg](m,x)mdxdm \\ \geq \;& - \int_0^{R_1} \int_{B(O,R_2)} a(s,y)g(s,y) \left(\int_0^s \int_{|x|>R_2} b(m,s,y)\widetilde{b}(x,m,s,y)mdxdm \right) dyds. \end{aligned}$$

By the assumption, for any $y \in B(O, R_2)$, we have

$$a(s,y)\int_{|x|>R_2}\widetilde{b}(m,x,s,y)dx \le a(s,y)\int_{|x|>|y|}\widetilde{b}(m,x,s,y)dx < C.$$

This implies that

$$\begin{aligned}&\int_0^\infty\int_{B(O,R_2)} a(s,y)g(s,y)\left(\int_0^s\int_{|x|>R_2} b(m,s,y)\widetilde{b}(x,m,s,y)mdxdm\right)dyds\\ \le\ & C\int_0^\infty\int_{\mathbb{R}^3} g(s,y)\left(\int_0^s mb(m,s,y)dm\right)dyds\\ \le\ & C\int_0^\infty\int_{\mathbb{R}^3} sg(s,y)dyds < \infty.\end{aligned}$$

By the dominated convergence theorem,

$$\begin{aligned}&\lim_{R_1,R_2\to\infty}\int_0^{R_1}\int_{B(O,R_2)} a(s,y)g(s,y)\left(\int_0^s\int_{|x|>R_2} b(m,s,y)\widetilde{b}(x,m,s,y)mdxdm\right)dyds\\ =\ & \int_0^\infty\int_{\mathbb{R}^3}\int_0^s ma(s,y)g(s,y)b(m,s,y)\left(1-\lim_{R_2\to\infty}\int_{B(O,R_2)}\widetilde{b}(x,m,s,y)dx\right)dmdyds\\ =\ & 0.\end{aligned}$$

Therefore

$$\int_0^\infty\int_{\mathbb{R}^3}(-a(m,x)g(m,x)+[Bg](m,x))mdxdm \ge 0.$$

□

This theorem demonstrates that the process is honest if at infinity daughter particles tend to move back into the system with a high probability described by the inequality (6.25). Note that the analysis developed in this thesis in the space $\mathbb{R}^3$ is also valid in one - and two- dimensional spaces $\mathbb{R}$ and $\mathbb{R}^2$. We shall illustrate the usefulness of Theorem 6.4.3 by the following examples in the physical space $\mathbb{R}$:

- In the first example daughter particles from the position $y > 0$ are uniformly distributed in the unit ball centered at $y+\gamma(y)-\frac{1}{2}$ and daughter particles from the position $y < 0$ are uniformly distributed in the unit ball centered at $y-\gamma(y)+\frac{1}{2}$, where $0 \le \gamma(y) \le 1$ is a function. The fragmentation rate a is a position only dependent function. By Theorem 6.4.3 the existence of a constant $\widetilde{\gamma}$ such that $\gamma(y) \le \frac{\widetilde{\gamma}}{a(y)}$ (for almost any $y \in \mathbb{R}$) guarantees honesty of the model.

- We consider the model $a(x) = |x|^n$, $n > 0$ and $\widetilde{b}(x,y) = \frac{|y|}{2}\exp(-|xy|)$. It is obvious that $a(x) \to \infty$ as $|x|$ approaches infinity. We have

$$\lim_{y\to\infty}|y|^n\int_{|x|>|y|}\frac{|y|}{2}\exp(-|xy|)dx = \lim_{y\to\infty}|y|^n\exp(-y^2) = 0.$$

  Therefore the assumptions of Theorem 6.4.3 are satisfied.

Throughout the rest of the thesis, we assume there exists a real-valued function $\widetilde{a} = \widetilde{a}(m) \in L_{\infty,\text{loc}}(\overline{\mathbb{R}_+})$ such that $a(m,x) \leq \widetilde{a}(m)$ for almost all $(m,x) \in \mathbb{R}_+ \times \mathbb{R}^3$. This obviously implies that for any $N \in \mathbb{N}$ there exists a positive Λ_N such that

$$\text{ess sup}_{(0,N)\times\mathbb{R}^3}\ a(m,x) \leq \Lambda_N. \tag{6.26}$$

6.5 Approximation techniques

The strategy used in this section is analogous to that employed in [16, Subsection 8.3.2] for the local fragmentation model. The idea of this method is to approximate the solution of (6.6) by a sequence of solutions of cut-off problems of a similar form.

6.5.1 The truncated problem

For any given $N \in \mathbb{N}$ we introduce the projection operator defined for a function $\psi \in X = L_1(\mathbb{R}_+ \times \mathbb{R}^3, mdxdm)$ by

$$(P_N\psi)(m,x) = \begin{cases} \psi(m,x) & \text{if} \quad 0 < m < N, \ \text{ and } x \in \mathbb{R}^3, \\ 0 & \text{otherwise.} \end{cases} \tag{6.27}$$

The projection P_N acts onto the closed subspace

$$X_N = \{\phi \in X : \ \phi(m,x) \equiv 0 \ \text{ on } (N,\infty) \times \mathbb{R}^3\} \tag{6.28}$$

of X. The truncated problem consists of finding a solution to the truncated ACP

$$\begin{aligned} \frac{du}{dt}(t) &= KP_N u(t), \qquad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{6.29}$$

where K is the realization of $\mathcal{A} + \mathcal{B}$ in X (see (6.10, 6.11)). We set

$$A_N = AP_N, B_N = BP_N \ \text{ and } \ K_N = KP_N.$$

6.5.2 The limit semigroup

The technique we employ is based on a modified version of an old method of Reuter and Lederman [25, 49] for solving Kolmogorov equations. It has been used lately for local fragmentation [16].

With some abuse of notation we consider A_N and B_N both in X_N and X.

Lemma 6.5.1. *For each N, K_N generates a positive uniformly continuous semigroup of contractions on X_N, say $(S_N(t))_{t\geq 0}$, which is conservative on $X_{N,+}$. Moreover, for any $M \geq N$ and $t \geq 0$,*

$$P_N S_M(t) P_N = S_N(t).$$

Proof. The operator A_N is bounded by (6.26). Changing the order of integration by the Fubini theorem and making use of (6.5), we get

$$\begin{aligned}\|B_N u\|_{X_N} &= \int_0^N \int_{\mathbb{R}^3} m|BP_N u(m,x)|dxdm \\ &\leq \int_0^N \int_{\mathbb{R}^3} a(s,y)|u(s,y)|sdyds \\ &\leq \Lambda_N \|u\|_{X_N},\end{aligned}$$

so that B_N is also bounded. Hence K_N generates a uniformly continuous semigroup. We denote this semigroup by $(S_N(t))_{t\geq 0}$. Clearly, A_N generates a positive semigroup of contractions and B_N is a positive operator. Moreover, by similar calculations as above,

$$\begin{aligned}&\int_0^N \int_{\mathbb{R}^3} \left(\int_m^N \int_{\mathbb{R}^3} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)u(t,s,y)dyds \right) mdxdm \\ &= \int_0^N \int_{\mathbb{R}^3} a(s,y)u(t,s,y) \left(\int_0^s mb(m,s,y)dm \right) dyds \\ &= \int_0^N \int_{\mathbb{R}^3} a(s,y)u(t,s,y)sdyds;\end{aligned}$$

thus the assumptions of Theorem 2.3.5 hold. Therefore there is an extension K_N' of K_N which generates a substochastic semigroup. Because $a(m,x)$ is bounded in $(0,N)\times\mathbb{R}^3$, this substochastic semigroup is honest, it follows that $K_N' = \overline{K}_N$, where $\overline{K}_N$ is the closure of K_N. Since K_N generates a uniformly (and hence strongly) continuous semigroup, K_N is a closed operator. Therefore we have that $K_N' = K_N$, consequently, the uniformly continuous semigroup $(S_N(t))_{t\geq 0}$ is a positive strongly continuous semigroup of contractions, furthermore $(S_N(t))_{t\geq 0}$ is honest.

To prove the last statement we notice first that since

$$BP_N u(m,x) = \int_m^N \int_{\mathbb{R}^3} a(s,y)b(m,s,y)\widetilde{b}(x,m,s,y)u(t,s,y)dyds$$

for $0 \leq m \leq N$ and $BP_N u = 0$ for $m > N$, we have that $BP_N u = P_N BP_N u$ on $[0,N]\times\overline{B}(0_{\mathbb{R}^3}, N)$. Furthermore, it is clear that $AP_N u = P_N AP_N u$, hence we have also

$$(A+B)P_N = AP_N + BP_N = P_N AP_N + P_N BP_N = A_N + B_N = K_N.$$

Next, by $P_N P_M = P_M P_N = P_N$ we have

$$P_N K_M P_N = P_N P_M K P_M P_N = P_N K P_N = K_N$$

and, by induction, if we assume that $P_N(K_M)^{n-1}P_N = (K_N)^{n-1}$, then

$$\begin{aligned}P_N(K_M)^n P_N &= P_N(K_M)^{n-1}K_M P_N \\ &= P_N(K_M)^{n-1}P_M K P_M P_N \\ &= P_N(K_M)^{n-1}P_M P_N K P_N \\ &= P_N(K_M)^{n-1}P_N K_N \\ &= (K_N)^n.\end{aligned}$$

Owing to the fact that K_M is a bounded operator, the semigroup generated by K_M is given by the exponential formula

$$\begin{aligned} P_N S_M(t) P_N &= \sum_{n=0}^{\infty} \frac{t^n P_N (K_M)^n P_N}{n!} \\ &= \sum_{n=0}^{\infty} \frac{t^n (K_N)^n}{n!} \\ &= S_N(t), \end{aligned}$$

and the Lemma is proved.

□

Theorem 6.5.2. *The truncated problem* (6.29) *has a unique, strongly continuously differentiable, positive, mass-conserving solution for all initial data* $u_0 \in X_N$*. The solution is given by* $u(t) = S_N(t)u_0$*,* $t \geq 0$*.*

Proof. This follows immediately by Lemma 6.5.1 and Proposition 2.2.3. □

The family $(S_N(t))_{t\geq 0}$ defined in lemma 6.5.1 can be extended to the uniformly continuous family of operators defined on X by

$$\overline{S}_N(t) = P_N S_N(t) P_N.$$

Note that $\overline{S}_N(0) \neq I_X$, therefore $\overline{S}_N(t)$ is no longer a semigroup. On the other hand, the operator K_N, as a bounded operator on X, generates a uniformly continuous semigroup, denoted by $(S_{K_N}(t))_{t\geq 0}$. As the restriction of K_N to the complement of X_N is the zero operator, it generates there a constant semigroup and we have

$$S_{K_N}(t) = P_N S_N(t) P_N + (I_X - P_N), \tag{6.30}$$

where I_X is the identity on X. Thus

$$S_{K_N}(t) P_N u = \overline{S}_N(t) u.$$

Proposition 6.5.3. *The families* $(S_{K_N}(t))_{t\geq 0}$ *and* $(\overline{S}_N(t))_{t\geq 0}$ *have the following properties.*

1. *For any fixed* t *the family* $(\overline{S}_N(t))_{t\geq 0}$ *is increasing with* N*;*

2. *There is a positive* C_0*-semigroup of contractions, say* $(S(t))_{t\geq 0}$*, such that for* $u \in X$*, and* $t \geq 0$

$$S(t)u = \lim_{N\to\infty} \overline{S}_N(t)u = \lim_{N\to\infty} S_{K_N}(t)u \quad \text{in } X; \tag{6.31}$$

3. *Both limits in* (6.31) *are uniform in t on bounded intervals. In particular, for $u_0 \in X_N$,*

$$S(t)u_0 = P_M S_M(t) P_M u_0 \quad \textit{for any } M \geq N. \tag{6.32}$$

Proof. (*1*) Let $u \geq 0$ and define

$$u_N(t) = P_N S_N(t) P_N u = \overline{S}_N(t)u \geq 0.$$

By the monotonicity of the projection operators we have

$$(P_{N+1} - P_N)u_{N+1}(t) \geq 0.$$

On the other hand, because

$$\frac{d}{dt}u_{N+1} = K_{N+1}u_{N+1},$$

we obtain

$$\frac{d}{dt}P_N u_{N+1} = P_N K_{N+1} P_N u_{N+1} + P_N K_{N+1}(P_{N+1} - P_N)u_{N+1}.$$

However, $P_N K_{N+1} P_N = K_N$ and $P_N A_{N+1} = P_N A_N$ so that

$$\begin{aligned} P_N K_{N+1}(P_{N+1} - P_N)u_{N+1} &= P_N A_N (P_{N+1} - P_N)u_{N+1} + P_N B_{N+1}(P_{N+1} - P_N)u_{N+1} \\ &= P_N B_{N+1}(P_{N+1} - P_N)u_{N+1} \geq 0, \end{aligned}$$

and

$$P_N u_{N+1}(0) = P_N u = u_N(0).$$

Thus, by the Duhamel formula in X_N,

$$\begin{aligned} P_N u_{N+1}(t) &= S_N(t)P_N u + \int_0^t S_N(t-\tau)P_N B_{N+1}(P_{N+1} - P_N)u_{N+1}(\tau)d\tau \\ &\geq S_N(t)P_N u \end{aligned}$$

and

$$P_N u_{N+1}(t) = P_N P_N u_{N+1}(t) \geq P_N S_N(t) P_N u = \overline{S}_N(t)u.$$

Combining the estimates, we get

$$\overline{S}_{N+1}(t)u = u_{N+1}(t) = P_{N+1}u_{N+1}(t) \geq P_N u_{N+1}(t) \geq \overline{S}_N(t)u.$$

The result follows.

(*2*) The family $(S_{K_N}(t))_{t\geq 0}$ is not increasing with N; we have, however, $S_{K_N} \geq \overline{S}_N$. Because the space X is a KB-space and the sequence $(\overline{S}_N(t))_{t\geq 0}$ is increasing with

$$\|\overline{S}_N(t)u\|_X = \|S_N(t)u\|_{X_N} = \|P_N u\|_{X_N} \leq \|u\|_X$$

provided $u \geq 0$, we can define

$$S(t)u = \lim_{N\to\infty} \overline{S}_N(t)u, \ \ t \geq 0 \ \ u \geq 0,$$

and by linearity this definition can be extended to arbitrary $u \in X$. Moreover by (6.30) we get

$$S_{K_N}(t) - \overline{S}_N(t) = (I_X - P_N)$$

and because

$$\lim_{N\to\infty}(I_X - P_N)u = 0$$

for any fixed u, we obtain

$$S(t)u = \lim_{N\to\infty} S_{K_N}(t)u, \quad t \geq 0 \quad \text{for any } u \in X.$$

Therefore, $(S(t))_{t\geq 0}$ is the strong limit of a sequence of uniformly bounded positive semigroup of contractions. We need to show that $(S(t))_{t\geq 0}$ is a positive strongly continuous semigroup of contractions. The semigroup relation $S(t+s)u = S(t)S(s)u$ is just the limit relation for $(S_{K_N}(t))_{t\geq 0}$. For any $u \in X$, we set $u_N = P_N u$ for some fixed N; Then for $M > N$ we have

$$S_{K_M}(t)u = \overline{S}_M(t)u$$

and for such M, as $t \to 0^+$,

$$\begin{aligned} \|S(t)u_N - u_N\| &\leq \|S(t)u_N - \overline{S}_M(t)u_N\| + \|\overline{S}_M(t)u_N - u_N\| \\ &= \|S(t)u_N\| - \|\overline{S}_M(t)u_N\| + \|\overline{S}_M(t)u_N - u_N\| \\ &\leq \|u_N\| - \|S_{K_M}(t)u_N\| + \|S_{K_M}(t)u_N - u_N\| \to 0. \end{aligned}$$

For arbitrary u we make use of the density of compactly supported functions in X and the boundedness of $(S(t))_{t\geq 0}$.

(*3*) The uniform convergence of $(\overline{S}_N(t))_{t\geq 0}$ follows by the classical argument of Dini, as in ([37], Lemma 4). To prove this statement for $(S_{K_N}(t))_{t\geq 0}$ it is enough to note that the difference between $(S_{K_N}(t))_{t\geq 0}$ and $(\overline{S}_N(t))_{t\geq 0}$ is independent of t. Equation (6.32) follows directly from the last statement of theorem 6.5.1. □

6.6 Uniqueness

The following lemma proves the minimality of $(S(t))_{t\geq 0}$ and this is crucial for the uniqueness investigations.

Lemma 6.6.1. *Let* $(t, m, x) \to u(t, m, x)$ *be a function integrable on*

$$[0, T] \times (0, N) \times \mathbb{R}^3$$

with respect to the measure $mdtdmdx$ *for any* $N, T > 0$ *and assume that* u *satisfies for almost all* (t, m, x) *the integral version of* (6.6)*:*

$$u(t,m,x) \quad = \quad u_0(m,x) - \int_0^t a(m,x)u(\tau,m,x)d\tau + \int_0^t [Bu](\tau,m,x)d\tau, \quad (6.33)$$

where $u_0 \in X$*. Then for all* $t > 0$ *and almost all* (m, x)*,*

$$u(t,m,x) \geq (S(t)u_0)(m,x). \quad (6.34)$$

Proof. We first observe that $[Bu](t, m, x)$ is finite for almost every t and (m, x). Next, integrating both side of (6.33), we obtain that for any $0 \leq N < \infty$ and $0 \leq t \leq T < \infty$

$$\int_0^N \int_{\mathbb{R}^3} \int_0^t |-a(m,x)u(\tau,m,x) + [Bu](\tau,m,x)| md\tau dxdm < +\infty.$$

In fact, by integrability assumption on u and (6.26) we have

$$\int_0^N \int_{\mathbb{R}^3} \left(\int_0^t a(m,x)|u|(\tau,m,x)d\tau \right) mdxdm = \int_0^t \left(\int_0^N \int_{\mathbb{R}^3} a(m,x)|u|(\tau,m,x)mdxdm \right) d\tau < +\infty,$$

hence also

$$\begin{aligned} &\int_0^N \int_{\mathbb{R}^3} \left(\int_0^t |Bu|(\tau,m,x)d\tau \right) mdxdm \\ &\leq \int_0^t \left(\int_0^N \int_{\mathbb{R}^3} a(m,x)|u|(\tau,m,x)mdxdm \right) d\tau \\ &< +\infty \end{aligned} \quad (6.35)$$

where in both cases the change of order of integration is justified by the positivity of the integrands and the Fubini theorem. In particular, this shows that

$$-au + Bu \in L_1([0,T] \times (0,N) \times \mathbb{R}^3, mdtdmdx). \quad (6.36)$$

Defining $u_N = P_N u$, from the fact that

$$\begin{aligned} P_N Bu &= P_N B(P_N + I - P_N)u \\ &= P_N B P_N P_N u + P_N B(I - P_N)u \\ &= B_N u_N + P_N B(I - P_N)u \end{aligned}$$

we see that u_N satisfies

$$\begin{aligned} u_N(t,m,x) =& P_N u_0(m,x) + \int_0^t -a(m,x)u_N(\tau,m,x)d\tau \\ &+ \int_0^t g(\tau,m,x)d\tau + \int_0^t [B_N u_N](\tau,m,x)d\tau, \end{aligned} \quad (6.37)$$

where

$$g(\tau, m, x) = \begin{cases} B(I - P_N)u(\tau, m, x) \text{ if } (m, x) \in (0, N) \times \mathbb{R}^3 \\ 0 \quad \text{if not.} \end{cases} \tag{6.38}$$

By (6.35) we have

$$\begin{aligned} &\int_0^N \int_{\mathbb{R}^3} \left(\int_0^T |g(\tau, m, x)| d\tau \right) m dx dm \\ &\quad = \int_0^N \int_{\mathbb{R}^3} \left(\int_0^T |B(I - P_N)u(\tau, m, x)|\, d\tau \right) m dx dm \\ &\quad \le \int_0^N \int_{\mathbb{R}^3} \left(\int_0^T |Bu|(\tau, m, x) d\tau \right) m dx dm \\ &\quad < +\infty. \end{aligned}$$

hence $g \in L_1(\,[0, T],\ L_1(\mathbb{R}_+ \times \mathbb{R}^3,\ mdxdm))$. Now, considering

$$\|u_N(t + \varsigma) - u_N(t)\|_X \le \int_t^{t+\varsigma} \int_0^N \int_{\mathbb{R}^3} |-au(s, m, x) + [Bu](s, m, x)| m dx dm d\tau$$

we see that because $-au + Bu \in L_1([0, T] \times (0, N) \times \mathbb{R}^3,\ mdtdxdm)$ and the measure of $[t, t+\varsigma] \times (0, N) \times \mathbb{R}^3$ goes to 0 as $\varsigma \to 0$, the function $t \to u_N(t)$ is an X_N continuous function for any $N < \infty$. Hence (6.37) can be written as

$$u_N(t) \;=\; P_N u_0 + \int_0^t (A_N + B_N) u_N(\tau) d\tau + \int_0^t g(\tau) d\tau, \tag{6.39}$$

where g, given by (6.38), is an $L_1([0, T],\ X)$ function. Because $A_N + B_N$ is a bounded operator, u_N is a mild solution to the Cauchy problem

$$\frac{du_N}{dt} = (A_N + B_N) u_N + g, \qquad u_N(0) = P_N u_0$$

and must therefore be given by the Duhamel formula

$$u_N(t) = S_N(t) P_N u_0 + \int_0^t S_N(t - \tau) g(\tau) d\tau.$$

Thus, for any N, $u_N(t, m, x) \ge (S_N(t) P_N u_0)(m, x)$. As $(S_N(t) P_N u_0)$ converges to $S(t)u_0$ and u_N converges to u, we get (6.34). □

The following Proposition shows that the semigroup $(S(t))_{t\ge 0}$ constructed in Proposition 6.5.3 coincides with the semigroup $(S_G(t))_{t\ge 0}$ of Theorem 6.3.2.

Proposition 6.6.2. *Under the assumptions of this section*

$$S_G(t)u_0 = S(t)u_0, \qquad t \ge 0,\ u_0 \in X.$$

Proof. In the first step we make use of Proposition 2.3.6. Clearly, because a satisfies (6.26), the subspace $X_0 := \bigcup_{N=0}^{\infty} X_N$ of all functions of X that have bounded support is a core for the multiplication operator A. From (6.32) it follows that X_0 is a subset of the domain of the generator of $(S(t))_{t\geq 0}$ because $S(t)|_{X_N} = S_N(t)$ is a uniformly bounded semigroup and therefore differentiable on the whole space. Thus Proposition 2.3.6 yields

$$S(t)u_0 \geq S_G(t)u_0 \quad \text{for any } \; u_0 \in X_+.$$

On the other hand, taking $u_0 \in D(A)_+ \subseteq D(G)$ and integrating (6.18) with respect to t, we see that $[S_G(t)u_0](m,x)$ satisfies (6.33) and therefore by (6.34),

$$S(t)u_0 \leq S_G(t)u_0.$$

Hence, for $u_0 \in D(A)_+$, we obtain

$$S(t)u_0 = S_G(t)u_0.$$

Since any element in $D(A)$ can be expressed as a difference of two nonnegative elements, we can extend this equality to $D(A)$ and, by density, to X. □

Theorem 6.6.3. *Assume $u(t,m,x)$ is a nonnegative function integrable in $[0,T] \times (0,\infty) \times \mathbb{R}^3$, $T < \infty$ with respect to the measure $mdtdxdm$, that satisfies*

$$u(t,m,x) = u_0(m,x) - \int_0^t a(m,x)u(\tau,m,x)d\tau + \int_0^t [Bu](\tau,m,x)d\tau, \tag{6.40}$$

where $u_0 \in X_+$, and

$$\int_0^\infty \int_{\mathbb{R}^3} u(t,m,x)mdmdx = \int_0^\infty \int_{\mathbb{R}^3} u_0(m,x)mdmdx \tag{6.41}$$

for any $t > 0$, then

$$u(t,m,x) = [S_G(t)u_0](m,x) \tag{6.42}$$

for any $t > 0$ and almost any $(m,x) \in (0,\infty) \times \mathbb{R}^3$.

Proof. By Lemma 6.6.1 and Proposition 6.6.2 we have

$$u(t,m,x) \geq [S(t)u_0](m,x) = [S_G(t)u_0](m,x).$$

On the other hand, for any $t > 0$,

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^3} (u(t,m,x) - [S_G(t)u_0](m,x))mdxdm \\ = & \int_0^\infty \int_{\mathbb{R}^3} u(t,m,x)mdxdm - \int_0^\infty \int_{\mathbb{R}^3} [S_G(t)u_0](m,x)mdxdm \\ = & \int_0^\infty \int_{\mathbb{R}^3} u_0(m,x)mdxdm - \int_0^\infty \int_{\mathbb{R}^3} u_0(m,x)mdxdm = 0. \end{aligned}$$

From Theorem 6.4.2 and because the integrand on the left-hand side is nonnegative, we obtain (6.42). □

Chapter 7

Conclusion

The main aim of this work was to extend existing results related to the coagulation-fragmentation equation. This has been achieved in various ways.

In the first chapters of the thesis, we analyzed the fragmentation equation (1.2) in the space $X_{0,1}$ for arbitrary fragmentation kernels a and $b(x|y)$. Our main focus was on the formation of a 'dust' of particles of zero size carrying, nevertheless, a non-zero mass. Shattering fragmentation (corresponding, roughly speaking, to a unbounded at 0) is associated with an accelerating infinite cascade of fragmentation events of smaller and smaller particles leading to the creation of dust. Hence, intuitively, in shattering fragmentation we should observe the appearance of an infinite number of particles. We demonstrated that this intuition is not necessarily correct – for sufficiently fast fragmentation of small particles (e.g., for $\alpha < -1$ and $\nu \in (-1, 0)$ in the power law case) the total number of them remains finite (though we do not know whether the process is honest in $X_{0,1}$). We note that this phenomenon was noticed in [32] for power law rates by analyzing explicit solutions and in [24] for homogeneous fragmentation using probabilistic methods. To explain this phenomenon one could conjecture that a full description of the dynamics in the shattering regime requires two compartments: one for the 'physical' particles, which are visible within the model governed by (1.2), and one for the 'dust'. In this interpretation shattering would be a flow of particles from the former to the latter with the speed related to the fragmentation rate a close to zero. If a close to zero is just large enough for shattering to occur, we may observe an accumulation of small particles in the 'physical' compartment and, if a becomes larger, the flow between compartments becomes fast enough to keep the number of physical particles finite for all times. This approach is a subject of current research. Another counterintuitive result observed is related to the case when the number of daughter particles produced in each fragmentation event is infinite and, at the same time, the fragmentation is strongly shattering (e.g. if $\nu \in (-2, 1]$ and $\alpha < -1$ in the power law case). Despite this, we observed that we still have evolution in $X_{0,1}$ (at least for a class of initial densities). This phenomenon also could be explained by the conjecture discussed in the previous paragraph.

We note that while the theory for the non-shattering case has been developed up to a reasonably complete level and as such is presented here, the shattering case and the case with infinite production of daughter particles still contain gaps and open problems. Therefore we decided to present the results pertaining to the latter rather in the form of examples and comments; the research to fill the gaps and answer the open questions is ongoing.

In chapter 5, the semigroup approach allowed us to extend existing results on phytoplankton dynamics. We included unbounded fragmentation rates, growth rate $r \in X_\infty$ (where X_∞ is the dual space of X_1)and to account for birth of particles through the McKendrick-von Foerster renewal boundary condition. We established the existence of non-negative solutions to the evolution equation derived by O. Arino and R. Rudnicki [14].

An integro-differential equation describing multiple fragmentation processes in the $3-D$ space was considered in chapter 6. New particles were spatially randomly distributed according to some probability density. By means of substochastic semigroup theory and approximation techniques, we recovered some main results known for local fragmentation. In particular, if the fragmentation rate is bounded by a locally integrable size-dependent function, we were able to prove the existence and uniqueness of nonnegative mass-conserving solutions. Another crucial aspect developed in this chapter covers the notion of shattering in nonlocal fragmentation. We showed that if the fragmentation rate a is bounded by a size only dependent function, the spatial distribution of the particles does not influence conservativeness of the system. Also we provided a general condition for honesty of nonlocal fragmentation system.

Although we have made some advances in the use of semigroups techniques, there are still areas in which further analysis could prove fruitful. In particular, it would be interesting to follow up the investigation of 'shattering' in nonlocal fragmentation and to provide in this context a reasonable physical interpretation.

Bibliography

[1] A.S. Ackleh, *Parameter Estimation in a Structured Algal Coagulation Fragmentation Model,* Non. Ana. Theo. Methods Appl. **28** (5), (1997), 837–854.

[2] A.S. Ackleh, K. Deng, *On a First Order Hyperbolic Coagulation Model*, Math. Methods Appl. Sci., **26**(8), (2003), 703–715.

[3] A.S. Ackleh, B.G. Fitzpatrick, *Modeling Aggregation and Growth Processes in an Algal Population Model: Analysis and Computations*, J. Math. Biol., **35**(4), (1997), 480–502.

[4] R. Adler, *Superprocesses and Phytoplankton Dynamics: Monte Carlo Simulations in Oceanography*, Proceedings of 'Aha Huliko'a Hawaiian Winter Workshop, University of Hawaii at Manoa, 1997, 121–128.

[5] M. Aizenman, T.A. Bak, *Convergence to Equilibrium in a System of Reacting Polymers,* Commun. Math. Phys **65**, 203–230, 1979.

[6] W. J. Anderson, *Continuous-Time Markov Chains. An Applications-Oriented Approach,*, Springer Verlag, New York, 1991.

[7] T.M. Apostol, *Mathematical Analysis,*(Second Edition) World Student Series, Addison-Wesley Publishing Company, Inc., 1974.

[8] W. Arendt, *Resolvent positive operators*, Proc. Lond. Math. Soc. (3), **54**(2), (1987), 321–349.

[9] W. Arendt, *Vector-Valued Laplace Transforms and Cauchy Problems*, Israel J. Math., **59** (3), (1987), 327–352.

[10] J. Banasiak, *Conservative and Shattering Solutions for some Classes of Fragmentation Equations*, Math. Models Methods Appl Sci., **14**(4), 2004, 483–501. (3), (1987), 327-352.

[11] J. Banasiak, *On an Extension of Kato-Voigt Perturbation Theorem for Substochastic Semigroups and its Applications,* Taiw. J. Math. **5** (1) (2001) 169–191.

[12] J. Banasiak, *On Conservativity and Shattering for an Equation of Phytoplankton Dynamics* C.R. Biologies **327** (2004) 1025–1036.

[13] L. Arlotti, J. Banasiak, *Strictly Substochastic Semigroups with Application to Conservative and Shattering Solutions to Fragmentation equations with mass-loss,* J. Math. Anal. Appl. **293** (2004) 693–720.

[14] O. Arino, R. Rudnicki, *Phytoplankton Dynamics*, C.R. Biologies **327** (2004) 961–969.

[15] J. Banasiak, *Shattering and Non-uniqueness in Fragmentation Models—An Analytic Approach,* Phys. D **222**(1-2), (2006), 63–72.

[16] J. Banasiak, L. Arlotti, *Perturbations of Positive Semigroups with Applications*, Springer Monographs in Mathematics.

[17] J. Banasiak, W. Lamb, *On the Application of Substochastic Semigroup Theory to Fragmentation Models with Mass Loss,* J. Math. Anal. Appl. **284**(1), (2003) 9–30.

[18] J. Banasiak, W. Lamb, *Coagulation, Fragmentation and Growth Processes in a Size Structured Population,* http://www.cs.und.ac.za/ banasiak/NewPublications/banasiaklambDCDS.pdf.

[19] G. Baumann, M. Freyberger, W.G. Glöckle, T.F. Nonnenmacher, *Similarity Solutions in Fragmentation Kinetics,* J. Phys A: Math. Gen., **24**, 5085-5096, 1991.

[20] A. Belleni-Morante, *A Concide Guide to Semigroups*, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.

[21] A. Belleni-Morante, A.C. McBride, *Applied Nonlinear Semigroups*, Willey Mathematical Method in Practice, Chichester, 1998.

[22] J. Bertoin, *Random Fragmentation and Coagulation Processes,* Cambridge University Press, Cambridge, 2006.

[23] J. Bertoin, *Self-Similar Fragmentations,* Ann. Inst. H. Poincar Probab. Statist., **38**(3) (2002) 319–340.

[24] J. Bertoin, *The Asymptotic Behaviour of Fragmentation Processes,* J. Eur. Math. Soc., **4** (2006) 395–416.

[25] A.T. Bharucha-Reid, *Elements of the Theory of Markov Processes and their Applications*, Mc Graw-Hill, New York, 1960.

[26] M. Cai, B.F. Edwards, H. Han, *Exact ans Asymptotic Scaling Solutions for Fragmentation with Mass Loss*, Phys. Rev. A, **43**(2), 656–662, 1991.

[27] H.E. Dam, D.T. Drapeau, *Coagulation Efficiency, Organic-Matter Glues and the Dynamics of Particle During a Phytoplankton Bloom in a Mesocosm Study,* Deep-Sea Res. II **42**(1), (1995), 111–123.

[28] R. L. Drake, *A General Mathematical Survey of the Coagulation Equation,* Topics in current aerosol research: International Review in International Physics and Chemistry **3** part 2, Hidy and Brock (editors), Pergamon Press, Oxford, 1972.

[29] P.B. Dubovskii, I.W. Stewart, *Trend to Equilibrium for the Coagulation Fragmentation Equation*, Math. Meth. Appl. Sci. **19**, 761–772, 1996.

[30] K. J. Engel, R. Nagel, *A Short Course on Operator Semigroups*, Universitext, Springer, New York, 2006.

[31] K. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, Springer.

[32] M. H. Ernst, G. Szamel, *Fragmentation Kinetics,* J. Phys. A: Math. Gen., **26** (1993) 6085–6091.

[33] I. Filippov, *On the Distribution of the Sizes of Particles which Undergo Splitting,* Theory Probab. Appl., **6** (1961) 275–293.

[34] E. Hille, R.S. Phillips, *Functional Analysis and Semigroups,* Colloquium Publications, V.31, American Mathematical Society, Providence, RI, 1957.

[35] B. Haas, *Appearance of Dust in Fragmentation,* Comm. Math. Sci, Supp. Issue, **1** (2004) 66–73.

[36] B. Haas, *Loss of Mass in Deterministic and Random Fragmentation,* Stochastic Process. Appl., **106**(2) (2003) 245–277.

[37] T. Kato, *Perturbation Theory for Linear Operators* Second Edition. Springer Verlag, Berlin, 1984.

[38] M. Lachowicz, D. Wrzosek, *A Nonlocal Coagulation-Fragmentation Model*, Math. Sub. Class.-35K57,45K06, 92E20, (1991).

[39] P. Laurençot, *On a Class of Continuous Coagulation-fragmentation Equation*, J. Diff. Eq. **167**, 245–274, 2000.

[40] S. R. Massel, *Fluid Dynamics for Marine Biologists*, Springer Verlag, Berlin, 1999.

[41] D.J. McLaughlin, *Coagulation and Fragmentation Models: A Semigroup Approach*, PhD Thesis, Strathclyde University 1995.

[42] D.J. McLaughlin, W. Lamb, A.C. Mc.Bride, *An Existence and Uniqueness Result for a Coagulation and Multiple-Fragmentation Equation*, Siam J. Math. Anal. **28**(5), 1173–1190, Sep 1997.

[43] D.J. McLaughlin, W. Lamb, A.C. McBride, *A Semigroup Approach to Fragmentation Models*, Siam J. Math. Anal. **28**(5), 1158–1172, Sep 1997.

[44] Z. A. Melzak, *A Scalar Transport Equation,* Trans. Amer. Math. Soc. **85** (1957) 547–560.

[45] J. R. Norris, *Markov Chains,* Cambridge University Press, Cambridge, 1998.

[46] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences.

[47] U. Passow, A.L. Alldredge, *Aggregation of a Diatom Bloom in a Mesocosm: The Role of Transparent Exopolymer Particles (TEP),* Deep-Sea Res. II **42**(1), (1995) 99–109.

[48] J.D. Pryce, *Basic Methods of Linear Functional Analysis*, Hutchinson and Co Ltd, London, J.C. Shepherdson (ed), 1973.

[49] G.E.H. Reuter, W. Lederman, *On differential Equations for the Transition Probabilities of Markov Processes with Denumerably many States*, Proc. Cambridge. Phil. Soc., **49**, (1953), 247–242.

[50] R. Rudnicki, R. Wieczorek, *Fragmentation-Coagulation Models of Phytoplankton*, Bull. Polish Acad. Sci. Math., (2006), 175–191.

[51] I.W. Stewart, P.B. Dubovskii, *Approach to Equilibrium for the Coagulation-Fragmentation Equation via a Lyapunov Functional*, Math. Meth. Appl. Sci. **19**, 171–185, 1996.

[52] M. Von. Smoluchowski, *Drei Vortrage uber Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen,* Physik. Z. **17** (1916), P 557–585.

[53] M. Von. Smoluchowski, *Versuch einer Mathematischen Theorie der Koagulationskinetic Kolloider Lösunger,* Z. Phys. Chem **92** (1917), 129–168.

[54] W. Wagner, *Explosion Phenomena in Stochastic Coagulation-Fragmentation Models,* Ann. App. Probab., **15**(3) (2005) 2081–2112.

[55] R.M. Ziff, E.D. McGrady, *Kinetics of Polymer Degradation,* Macromolecules, **19**, 2513–2519, 1986.

[56] R.M. Ziff, E.D. McGrady, *The Kinetics of Cluster Fragmentation and Depolymerization,* J. Phys. A: Math. Gen. **18**, (1985), 3027–3037.

[57] R.M. Ziff, E.D. McGrady, *"Shattering" Transition in Fragmentation,* **58**(9), Physical Review Letters, 2 March 1987.