Anisotropic Stars in General Relativity

by

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Abstract

In this thesis we seek new solutions to the anisotropic Einstein field equations which are important in the study of highly dense stellar structures. We first adopt the approach used by Maharaj & Maartens (1989) to obtain an exact anisotropic solution in terms of elementary functions for a particular choice of the energy density. This class of solution contains the Maharaj & Maartens (1989) and Gokhroo & Mehra (1994) models as special cases. In addition, we obtain six other new solutions following the same approach for different choices of the energy density. All the solutions in this section reduce to one with the energy density profile $\mu \propto r^{-2}$. Two new algorithms are generated, Algorithm $\mathcal{A}$ and Algorithm $\mathcal{B}$, which produce a new anisotropic solution to the Einstein field equations from a given isotropic solution. For any new anisotropic solution generated with the help of these algorithms, the original isotropic seed solution is regained as a special case. Two examples of known isotropic solutions are used to demonstrate how Algorithm $\mathcal{A}$ and Algorithm $\mathcal{B}$ work, and to obtain new anisotropic solutions for the Einstein and de Sitter models. Anisotropic isothermal sphere models are generated given the corresponding isotropic ($\mu \propto r^{-2}$) solution of the Einstein field equations. Also, anisotropic interior Schwarzschild sphere models are found given the corresponding isotropic ($\mu \propto$ constant) solution of the field equations. The exact solutions and line elements are given in each case for both Algorithm $\mathcal{A}$ and Algorithm $\mathcal{B}$. Note that the solutions have a simple form and are all expressible in terms of elementary functions. Plots for the anisotropic factor $S = \sqrt{3}(p_r - p_\perp)/2$ (where p_r and $p_\perp$ are radial and tangential pressure respectively) are generated and these point to physically viable models.

To

'Mamohale and our little girls, Reitu and Refiloe

for the joy, sense of purpose and belonging they brought into my life.

Mahali, and my late grannies Tjopi and 'Malokisi, and the rest of the family for invaluable guidance and support over the years.

Declaration

This study represents original work by the author. It has not been submitted in any form to another University. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

List of Tables

List of Figures

Introduction

Schwarzschild (1916) constructed, within the framework of Einstein's general relativity, the simplest model of astrophysical relevance. He gave an exact solution for Einstein equations with the energy-momentum stress tensor for a perfect fluid, with a constant energy density μ, of a static spherically symmetric star surrounded by vacuum. The implication of these rather stringent assumptions imposed on the matter properties is that the mass to radius ratio of a stellar body should be bounded for it to remain stable and escape collapse to a black hole. In addition there is a bound on the maximum surface redshift from the object (Barraco *et al* 2003). These conditions also hold for isotropic stars with variable energy densities under reasonable assumptions. The purpose of this thesis is to find anisotropic relativistic models which are related to isotropic solutions which have a clear physical interpretation. To this end we choose models for which $\mu \propto r^{-2}$ (isotropic isothermal spheres) and $\mu \propto$ constant (isotropic interior Schwarzschild spheres).

From as far back as 1933, close to two decades after the Schwarzschild interior solution became known, a number of researchers raised the possibility that matter properties may need to be relaxed to accommodate higher values than those imposed by isotropy on the maximum value of the surface gravitational potential (Füzfa *et al* 2002, Herrera & Santos 1997, Lemaître 1933, Ivanov 2002). However, it was not until the pioneering work of Bowers & Liang (1974), applying anisotropic fluid models to neutron stars, that some formalism for achieving this was laid out. Anisotropic pressure is a physical phenomenon where the radial pressure p_r may be different from the tangential pressure $p_\perp$, that is $p_r - p_\perp \neq 0$.

Bowers & Liang (1974), Chan (1993), Chan *et al* (1993), Consenza *et al* (1981), Dev & Gleiser (2002, 2003), Gokhroo & Mehra (1994), Guven & Ó Muchandha (1999), Herrera *et al* (1979), Herrera & Santos (1995), Heintzmann & Hillebrandt (1975), Hillebrandt & Steinmetz (1976), Maharaj & Maartens (1989), Mak & Harko (2002, 2003), Rago (1991) (and in references therein) have examined how anisotropic matter affects the critical mass, critical surface redshift and stability of highly compact bodies. Using various *ansatze* they all established that in many cases the maximum equilibrium mass and surface redshift of anisotropic matter increase without any upper bound, over the isotropic values. Certain models were found to be more stable while others became less stable if the matter was anisotropic (Stewart 1982).

Although much work has been done on various aspects of anisotropy in stellar bodies, its existence, let alone direct application has been questioned over the years (Bowers & Liang 1974, Gleiser 1988). This may be due to the fact that the existence of anisotropy was initially suggested by theoretical work on more realistic stellar models (Canuto 1974, 1975, Ruderman 1972). Up to this point in time a consensus has not been fully reached on what causes anisotropy or where it occurs naturally. There is however mounting evidence, again substantiated theoretically, that allowing local anisotropy to exist in stellar models is probably the best route to take in trying to understand the physics of highly compact bodies with high densities. Ivanov (2002) argues that in many papers it is stressed that arbitrarily large redshifts are obtained when tangential pressure grows to infinity with no or little consideration of its physical conditions. The author deals with the problem in a manner that leads to finite values for the maximum surface redshift when the tangential pressure satisfies either the dominant or strong energy condition, and shows that the bounds can be relaxed but cannot be saturated. This concern is to some extent indirectly corroborated by Bondi (1992, 1999) who shows that with all pressures positive and $Q = (p_r + 2p_\perp)/\mu \leq 1$, a constant Q (implying $p_r \neq p_\perp$) relatively gives the highest possible redshifts that remain finite. In view of all these restrictions the question arises as to which of these conditions of 'physical reasonableness' is more acceptable. General relativity has so far given no unique and universally accepted answer (Florides 1974).

Amidst all the uncertainties related to anisotropy, there is yet another school of thought that also came to the fore in the early 1970's (Bodmer 1971), again theoretically motivated, of what are called strange matter stars (Farhi & Jaffe 1984, Kettner *et al* 1995, Witten 1984). This comes about in an attempt to explain the ever increasing likelihood that there are bound and stable stellar bodies, more dense than a neutron star, but which have not collapsed to a black hole. Again pressure anisotropy may be the physical feature to consider in the quest to understanding the role of strange matter in 'denser than the neutron star' stellar bodies (Mak & Harko 2002, Sharma & Mukherjee 2002). In the extreme case that the strange matter hypothesis (Witten 1984) stands, some neutron stars could actually be strange stars and pulsars would be interpreted as rotating strange stars rather than rotating neutron stars (Kettner *et al* 1995). So this concept of strange matter stars on the one hand is seen to be an area where looking at systems with pressure anisotropy could be applied, while on the other it compounds the issue of applications of systems with pressure anisotropy by raising a lot more questions about the nature of matter than there were already.

Herrera & Santos (1997) have compiled a very thorough overview of trends in the general area of local anisotropy and possible areas in which the theory may be applicable. The reference can be consulted for more on these topics. On the issue of physical reasonableness, Delgaty & Lake (1998) and Finch & Skea (1998) completed a comprehensive physical analysis review on isotropic solutions. Although the reviews are on isotropic solutions, they do put into some context the question of physical considerations when dealing with Einstein solutions of stellar bodies, and this can be easily extended to anisotropic stars.

As strange as all these may sound, experience shows that extreme situations tend to occur in the Universe where least expected (Bondi 1992). What appears as apparent theoretical studies like those done in works referred to above, and the work presented in this thesis may therefore have future applications. We hope the work presented here makes a contribution to the quest of mathematically modelling, and perhaps to facilitate the understanding of the physics of ultradense stellar bodies.

Chapter 1

Spacetime Geometry and Field Equations

1.1 Introduction

In this chapter we introduce the basic elements of differential geometry which are used as building blocks for the Einstein field equations. Spacetime is taken to be a four-dimensional, differentiable manifold endowed with a metric tensor field $\mathbf{g}$ of signature $(- + + +)$. Points in the four-dimensional manifold are labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where $x^0 = ct$ is the timelike coordinate (with the speed of light $c = 1$) and x^1, x^2, x^3 are the spacelike coordinates. In §1.2 we introduce the concept of curvature, the Riemann tensor, the Einstein tensor, and the general energy-momentum tensor. The Einstein field equations are given. The spacetime manifold is then restricted to be static and spherically symmetric on physical grounds. In §1.3 the Einstein field equations for anisotropic matter are derived in detail for static spacetimes with spherical geometry. This system is rewritten as a system of first order differential equations which is the form that is integrated in later chapters. In §1.4 the physical conditions for a realistic gravitating relativistic sphere are briefly reviewed. A more detailed discussion on spacetime geometry and the field equations in general relativity is given in d'Inverno (1992), Misner *et al* (1973) and Stephani (1982).

1.2 Spacetime Geometry

The line element defining the invariant infinitesimal separation between neighbouring points is

$$ds^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{1.1}$$

An additional structure in spacetime, the metric connection $\mathbf{\Gamma}$ is used to characterise the curvature of spacetime. The metric connection $\mathbf{\Gamma}$, also known as the Christoffel symbol of the second kind, is defined in terms of the metric tensor $\mathbf{g}$ and its derivatives by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(g_{dc,b} + g_{bd,c} - g_{bc,d}\right) \tag{1.2}$$

where a comma denotes partial differentiation. The metric tensor $\mathbf{g}$ in equations (1.1) and (1.2) is a function of the spacetime coordinates x^a.

The notion of curvature arises from the path dependence of parallel transport and is directly related to the non-commutativity of the covariant derivative. The Riemann tensor $\mathbf{R}$, also called the curvature tensor, provides a measure of the curvature of spacetime. The components of $\mathbf{R}$ are given by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed} \tag{1.3}$$

On contracting the Riemann tensor (1.3) we obtain

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac} \end{aligned} \tag{1.4}$$

which is the Ricci tensor. A further contraction of the Riemann tensor, that is contraction of R_{ab} in (1.4), yields

$$R = R^a{}_a \tag{1.5}$$

The quantity R represents the trace of the Ricci tensor and is called the Ricci or curvature scalar.

The components of the Einstein tensor $\mathbf{G}$ are given by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{1.6}$$

which is defined in terms of the Ricci tensor (1.4) and the Ricci scalar (1.5). The Einstein tensor is constructed such that it has zero divergence:

$$G^{ab}{}_{;b} = 0 \tag{1.7}$$

where the semicolon denotes covariant differentiation.

In general relativity the matter distribution for neutral fluids is described by the energy-momentum tensor $\mathbf{T}$, the components of which are given by

$$T^{ab} = (\mu + p)\, u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab} \tag{1.8}$$

where the energy density μ, the isotropic pressure p, the energy flux vector $q^a(q^a u_a = 0)$, and the stress or anisotropic pressure tensor $\pi^{ab}(\pi^{ab}u_a = \pi^a{}_a = 0)$ are measured relative to the four-velocity u^a. The four-velocity $\mathbf{u}$ is timelike and unit so that $u^a u_a = -1$. For perfect fluids the stress tensor and energy flux vector vanish, and equation (1.8) becomes

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab} \tag{1.9}$$

Normally the perfect fluid form (1.9) is assumed to describe a relativistic gravitating system in cosmology. However in this thesis we will take $\pi^{ab} \neq 0$ in equation (1.8) for describing relativistic anisotropic stars. In many applications we require that

$$p = p(\mu)$$

so that there is a barotropic equation of state.

The energy-momentum tensor $\mathbf{T}$ is coupled to the Einstein tensor $\mathbf{G}$ via the Einstein field equations which are given by

$$G^{ab} = T^{ab} \tag{1.10}$$

where we have chosen units in which the value of the gravitational coupling constant is unity. These field equations were first formulated by Einstein to provide a description of gravitating systems, and constitute a system of nonlinear partial differential equations which determine the gravitational field $\mathbf{g}$. From equations (1.7) and (1.10) it follows that

$$T^{ab}{}_{;b} = 0 \tag{1.11}$$

which generates the conservation law for energy-momentum.

1.3 Static Spherically Symmetric Spacetimes

The generic line element for static spherically symmetric spacetimes is given by

$$\mathrm{d}s^2 = -e^{\nu}\mathrm{d}t^2 + e^{\lambda}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{1.12}$$

where $\nu(r)$ and $\lambda(r)$ are related to the gravitational potentials. The non-vanishing components of the connection coefficients for the line element (1.12) are

$$\begin{aligned}
\Gamma^0{}_{01} &= \tfrac{1}{2}\nu' & \Gamma^1{}_{00} &= \tfrac{1}{2}\nu' e^{-\lambda}e^{\nu} \\
\Gamma^1{}_{11} &= \tfrac{1}{2}\lambda' & \Gamma^1{}_{22} &= -re^{-\lambda} \\
\Gamma^1{}_{33} &= -re^{-\lambda}\sin^2\theta & \Gamma^2{}_{12} &= 1/r \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta & \Gamma^3{}_{13} &= 1/r \\
\Gamma^3{}_{23} &= \cot\theta
\end{aligned}$$

where primes denote differentiation with respect to the coordinate r. Then the non-vanishing Ricci tensor components (1.4) become

$$\begin{aligned} R_{00} &= \frac{1}{2}e^{-\lambda}e^{\nu}\left(\nu'' - \frac{\nu'\lambda'}{2} + \frac{(\nu')^2}{2} + \frac{2\nu'}{r}\right) && (1.13a)\\ R_{11} &= -\frac{\nu''}{2} + \frac{\lambda'\nu'}{4} - \frac{(\nu')^2}{4} + \frac{\lambda'}{r} && (1.13b)\\ R_{22} &= 1 + e^{-\lambda} + r\lambda' e^{-\lambda} - \frac{1}{2}re^{-\lambda}\left(\nu' + \lambda' + \frac{4}{r}\right) && (1.13c)\\ R_{33} &= \sin^2\theta R_{22} && (1.13d) \end{aligned}$$

The Ricci tensor components (1.13) imply that the Ricci scalar (1.5) has the form

$$R = \frac{2}{r^2} + \frac{2}{r}\lambda' e^{-\lambda} - \frac{1}{2}e^{-\lambda}\left(2\nu'' - \nu'\lambda' + (\nu')^2 + \frac{4\nu'}{r} + \frac{4}{r^2}\right) \tag{1.14}$$

Then the components of the Einstein tensor (1.6), from equations (1.13) and (1.14), assume the form

$$\begin{aligned} G_{00} &= -\frac{1}{r^2}e^{-\lambda}e^{\nu}\left(1 - \lambda' r - e^{\lambda}\right) && (1.15a)\\ G_{11} &= \frac{1}{r^2}\left(1 - e^{\lambda} + r\nu'\right) && (1.15b)\\ G_{22} &= \frac{1}{4}r^2 e^{-\lambda}\left(2\nu'' - \nu'\lambda' + (\nu')^2 + \frac{2\nu'}{r} - \frac{2\lambda'}{r}\right) && (1.15c)\\ G_{33} &= \sin^2\theta G_{22} && (1.15d) \end{aligned}$$

We study non-radiating relativistic spheres with anisotropic stress ($q^a = 0$, $\pi^{ab} \neq 0$) and write the energy-momentum tensor (1.8) in the form

$$T^{ab} = \mu u^a u^b + p h^{ab} + \pi^{ab} \tag{1.16}$$

In (1.16) we have introduced the projection tensor $h^{ab} = u^a u^b + g^{ab}$ which is measured relative to the four-velocity u^a. It is convenient to express the anisotropic stress in the form

$$\pi^{ab} = \sqrt{3}S(r)\left(c^a c^b - \frac{1}{3}h^{ab}\right)$$

where the unit spacelike vector $\mathbf{c}$ is orthogonal to the fluid four-velocity $\mathbf{u}$ and $|S(r)|$ is the magnitude of the stress tensor. This representation for π^{ab} is a consequence of the symmetries of the static spherically symmetric spacetimes (Maharaj & Maartens 1986). The quantity S is a useful device to introduce

$$\begin{aligned} p_r &= p + 2S/\sqrt{3} \\ p_\perp &= p - S/\sqrt{3} \end{aligned}$$

which are the radial and tangential pressures respectively. Note that for isotropic matter $S = 0$ and $p_r = p_\perp = p$. The magnitude S provides a measure of anisotropy.

We assume that the fluid four-velocity is comoving. This assumption implies that

$$\begin{aligned} u^a &= e^{-\nu/2}\delta^a_0 \\ c^a &= e^{-\lambda/2}\delta^a_1 \end{aligned}$$

for the vectors $\mathbf{u}$ and $\mathbf{c}$ from the line element (1.12). Then the non-vanishing energy-momentum tensor components (1.16) are

$$T_{00} = e^{\nu}\mu \tag{1.17a}$$

$$T_{11} = e^{\lambda}p_r \tag{1.17b}$$

$$T_{22} = r^2 p_\perp \tag{1.17c}$$

$$T_{33} = \sin^2\theta T_{22} \tag{1.17d}$$

From (1.15) and (1.17) we find that the Einstein field equations (1.10) become

$$-\frac{e^{-\lambda}}{r^2}\left(1 - \lambda' r - e^{\lambda}\right) = \mu \tag{1.18a}$$

$$\frac{e^{-\lambda}}{r^2}\left(1 - e^{\lambda} + r\nu'\right) = p_r \tag{1.18b}$$

$$\frac{e^{-\lambda}}{4}\left(2\nu'' - \nu'\lambda' + (\nu')^2 + \frac{2\nu'}{r} - \frac{2\lambda'}{r}\right) = p_\perp \tag{1.18c}$$

for static spherically symmetric anisotropic matter.

The momentum conservation equation (1.11) becomes

$$(\mu + p_r)\,\nu' + 2p_r' + \frac{4}{r}(p_r - p_\perp) = 0 \tag{1.19}$$

for the spacetime (1.12). Note that (1.19) can also be obtained directly from the field equations (1.18). We define the mass function as

$$m(r) = \frac{1}{2}\int_0^r x^2\mu(x)\mathrm{d}x \tag{1.20}$$

following the treatment of Stephani (1982). With the help of (1.19) and (1.20) we can replace the field equations (1.18) by the equivalent system

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{1.21a}$$

$$r(r-2m)\nu' = p_r r^3 + 2m \tag{1.21b}$$

$$(\mu + p_r)\,\nu' + 2p_r' = -\frac{4}{r}(p_r - p_\perp) \tag{1.21c}$$

The system (1.21) has the advantage of being a first order system of differential equations. For certain applications it is easier to use (1.21) rather than the original second order system (1.18), which is the approach that we follow in this thesis.

1.4 Physical Conditions

For physical viability, any stellar interior solution should match smoothly to the appropriate exterior spacetime. The spacetime surrounding a static, spherically symmetric body is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{1.22}$$

known as the exterior Schwarzschild line element. In (1.22), M is the mass of the stellar body measured by an observer at infinity which is given by $M = m(R)$ from (1.20) where R is the stellar radius. In addition, it is often assumed that realistic stellar models for isotropic matter should satisfy:

(a) The energy density μ and pressure p should be positive and finite throughout the interior of the star. The radial pressure should vanish at the boundary $r = R$.

(b) The gradients μ' and p' should be negative for barotropic matter.

(c) The speed of sound should remain subluminal throughout the interior of the star, that is $0 \leq \mathrm{d}p/\mathrm{d}\mu \leq 1$. This condition is necessary to preserve causality.

(d) The metric functions e^{ν} and e^{λ} should be positive and non-singular throughout the interior of the star.

(e) At the boundary the metric functions should match smoothly to the exterior Schwarzschild solution:

$$\begin{aligned} e^{\nu} &= e^{-\lambda} \\ &= 1 - \frac{2M}{R} \end{aligned}$$

(f) The solutions should be stable with respect to radial perturbations.

It should be noted that not all relativistic stellar models satisfy the above conditions. Extensive reviews on this aspect for isotropic stars can be found in papers by Delgaty & Lake (1998) and Finch & Skea (1998). The physical analysis for anisotropic relativistic stars is more complicated because $p_r \neq p_{\perp}$, as one can see in the treatments of Bowers & Liang (1974), Dev & Gleiser (2002, 2003) and Mak & Harko (2003). Note that it is possible to study the behaviour of anisotropic matter in the presence of an electromagnetic field which represents a charged anisotropic star (Sharma *et al* 2001). It is advisable to put any exact solution through the test of the conditions listed above because they provide qualitative features which represent many physical stars. Exact solutions to the Einstein field equations which do not satisfy all of conditions (a)-(f) are still of interest because they provide useful qualitative features which assist in the analysis of relativistic stars.

Chapter 2

Exact Solutions: Maharaj & Maartens Algorithm

2.1 Introduction

We seek explicit solutions to the Einstein field equations that describe anisotropic relativistic stars by utilising an algorithm that was initially proposed by Maharaj & Maartens (1989). In their approach they expressed the field equations as the first order system of differential equations (1.21). The energy density μ and the radial pressure p_r are chosen on physical grounds. The remaining relevant quantities $(e^{\nu}, e^{\lambda}, m, p_{\perp}, S)$ then follow from the field equations. In §2.2 we generate a new class of anisotropic exact solutions for a particular form of the energy density using this approach. This class of solution contains particular cases studied previously. All details of the calculations are given. Six additional classes are tabulated in §2.3 which were also obtained by following the Maharaj & Maartens (1989) algorithm. The tables list the energy density functions, the radial pressure functions, gravitational potentials and the matter variables. The physical features of the various solutions found in this chapter are briefly considered in §2.4. The software package Mathematica® 5 (Wolfram 2003) was a useful tool in helping to verify the accuracy of the solutions found.

2.2 A class of solutions

We demonstrate that the argument given in §2.1 does indeed lead to *new* solutions of the Einstein field equations (1.21) for anisotropic matter which are physically reasonable. It is convenient to make the following choice for the energy density

$$\mu = \frac{j}{r^2} + k + \ell r^2 \tag{2.1}$$

where j, k and ℓ are constants. This form for μ contains particular cases studied previously. Then (1.20) yields the expression

$$m = \frac{r}{2}\left(j + \frac{k}{3}r^2 + \frac{\ell}{5}r^4\right) \tag{2.2}$$

for the mass function and the particular energy density (2.1). Equation (1.21a) gives

$$e^{-\lambda} = 1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4 \tag{2.3}$$

for the particular mass function (2.2), and the gravitational potential λ has been determined.

With the help of (2.2), we can write (1.21b) as

$$\nu' = \frac{r p_r}{1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4} + \frac{j + \frac{k}{3}r^2 + \frac{\ell}{5}r^4}{r\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)} \tag{2.4a}$$

$$= \frac{r p_r}{1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4} + \frac{j}{r\,(1-j)} + \frac{\frac{k}{3}r + \frac{\ell}{5}r^3}{(1-j)\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)} \tag{2.4b}$$

where we have used partial fractions to simplify the last term in (2.4a). On integration, (2.4b) can be expressed as

$$\nu = I_1 + \frac{j}{1-j}\ln r + \frac{1}{1-j}I_2 + \ln B \tag{2.5}$$

where $\ln B$ is a constant of integration and we have set

$$\begin{aligned} I_1 &= \int \frac{rp_r}{1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4}\mathrm{d}r \\ I_2 &= \int \frac{\frac{k}{3}r+\frac{\ell}{5}r^3}{1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4}\mathrm{d}r \end{aligned}$$

for simplicity. To continue it is necessary to make a choice for the radial pressure p_r. A number of choices are possible which are physically reasonable. We make the choice

$$p_r = \frac{C}{1-j}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)\left(1-\frac{r^2}{R^2}\right)^n \tag{2.6}$$

When $j = k = \ell = 0$, we obtain the radial pressure

$$p_r = C\left(1-\frac{r^2}{R^2}\right)^n$$

which was the form postulated by Maharaj and Maartens (1989). The form (2.6) for p_r is physically reasonable because $p_r > 0$ in the interval $(0, R)$, $p_r = C$ at the centre $r = 0$, $p_r = 0$ at the boundary $r = R$, and p_r is continuous and well behaved in the interval $[0, R]$. The first integral I_1 then simplifies to

$$\begin{aligned} I_1 &= \frac{C}{1-j}\int\left(1-\frac{r^2}{R^2}\right)^n r\mathrm{d}r \\ &= -\frac{CR^2}{2(1-j)(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1} \end{aligned}$$

To evaluate the second integral I_2 we need to consider two cases: $\ell = 0$ and $\ell \neq 0$.

Case I: $\ell = 0$

With $\ell = 0$ the integral I_2 becomes

$$\begin{aligned} I_2 &= \int \frac{\frac{k}{3}r}{1-j-\frac{k}{3}r^2}\mathrm{d}r \\ &= -\frac{1}{2}\ln\left\{1-j-\frac{k}{3}r^2\right\} \end{aligned}$$

Case II: $\ell \neq 0$

When $\ell \neq 0$ we let

$$
\begin{aligned}
u &= r^2 + \frac{5k}{6\ell} \\
q^2 &= 1 - j + \frac{5k^2}{36\ell}
\end{aligned}
$$

and obtain

$$
\begin{aligned}
I_2 &= \int \frac{\frac{\ell r}{5}\left(r^2 + \frac{5k}{3\ell}\right)}{1 - j + \frac{5k^2}{36l} - \frac{\ell}{5}\left(r^2 + \frac{5k}{6\ell}\right)^2} \mathrm{d}r \\
&= \frac{\ell}{10} \int \frac{u + \frac{5k}{6\ell}}{q^2 - \frac{\ell}{5}u^2} \mathrm{d}u \\
&= \frac{\ell}{10}\left(-\frac{5}{2\ell}\ln\left\{q^2 - \frac{\ell}{5}u^2\right\} + \frac{5k}{6\ell}\left(\frac{\sqrt{5}}{q\sqrt{\ell}}\right)\tanh^{-1}\left\{\frac{u\sqrt{\ell}}{q\sqrt{5}}\right\}\right) \\
&= -\frac{1}{4}\ln\left\{1 - j + \frac{5k^2}{36\ell} - \frac{\ell}{5}\left(r^2 + \frac{5k}{6\ell}\right)^2\right\} \\
&\quad + \left(\frac{5}{\ell}\right)^{\frac{1}{2}}\left(\frac{k}{12\sqrt{1 - j + \frac{5k^2}{36\ell}}}\right)\tanh^{-1}\left\{\left(\frac{\ell}{5}\right)^{\frac{1}{2}} \frac{r^2 + \frac{5k}{6\ell}}{\sqrt{1 - j + \frac{5k^2}{36\ell}}}\right\}
\end{aligned}
$$

We can collectively write for both *Case I* and *Case II* that

$$
I_2 = \begin{cases}
-\frac{1}{2}\ln\left\{1 - j - \frac{k}{3}r^2\right\}, \text{ for } \ell = 0 \\[2em]
-\frac{1}{4}\ln\left\{1 - j + \frac{5k^2}{36\ell} - \frac{\ell}{5}\left(r^2 + \frac{5k}{6\ell}\right)^2\right\} \\
\quad + \left(\frac{5}{\ell}\right)^{\frac{1}{2}}\left(\frac{k}{12\sqrt{1-j+\frac{5k^2}{36\ell}}}\right)\tanh^{-1}\left\{\left(\frac{\ell}{5}\right)^{\frac{1}{2}} \frac{r^2+\frac{5k}{6\ell}}{\sqrt{1-j+\frac{5k^2}{36\ell}}}\right\}, \text{ for } \ell \neq 0
\end{cases} \tag{2.7}
$$

On substituting I_1 in (2.5) we obtain

$$
e^{\nu} = Br^{\frac{j}{1-j}} \exp\left\{\frac{I_2}{1-j} - \frac{CR^2}{2(1-j)(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right\} \tag{2.8}
$$

for the gravitational potential e^{ν} where $I_2 = I_2(r)$ has the functional representation given above in (2.7) for $\ell = 0$ and $\ell \neq 0$.

The last field equation (1.21c) then gives the tangential pressure $p_\perp$:

$$
\begin{aligned}
p_\perp = {} & p_r + \frac{C}{2(1-j)}\left(j - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n \\
& + \frac{r^2}{2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^{-1} \\
& \times \left\{ \frac{C^2}{2(1-j)^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{2n} \right. \\
& - \frac{2nC}{(1-j)R^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{n-1} \\
& \left. + \frac{1}{2r^2}\left(\frac{j}{r^2} + k + \ell r^2\right)\left(j + \frac{k}{3}r^2 + \frac{\ell}{5}r^4\right)\right\}
\end{aligned}
\tag{2.9}
$$

where we have used (2.1), (2.6) and (2.8). The anisotropic factor $S(r)$ is given by

$$
\begin{aligned}
S = {} & -\frac{C}{2\sqrt{3}\,(1-j)}\left(j - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n \\
& - \frac{r^2}{2\sqrt{3}}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^{-1} \\
& \times \left\{ \frac{C^2}{2(1-j)^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{2n} \right. \\
& - \frac{2nC}{(1-j)R^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{n-1} \\
& \left. + \frac{1}{2r^2}\left(\frac{j}{r^2} + k + \ell r^2\right)\left(j + \frac{k}{3}r^2 + \frac{\ell}{5}r^4\right)\right\}
\end{aligned}
\tag{2.10}
$$

which follows from (2.6) and (2.9).

Thus we have generated a class of solutions to the Einstein field equations (1.21). Collecting the various results given above we can express the exact solution as

$$
\begin{aligned}
\mu &= \frac{j}{r^2} + k + \ell r^2 && \text{(2.11a)}\\
p_r &= \frac{C}{1-j}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n && \text{(2.11b)}\\
p_\perp &= p_r + \frac{C}{2\,(1-j)}\left(j - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n + \frac{r^2}{2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^{-1}\\
&\quad \times \left\{ \frac{C^2}{2(1-j)^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{2n} \right.\\
&\quad - \frac{2nC}{(1-j)R^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{n-1}\\
&\quad \left. + \frac{1}{2r^2}\left(\frac{j}{r^2} + k + \ell r^2\right)\left(j + \frac{k}{3}r^2 + \frac{\ell}{5}r^4\right)\right\} && \text{(2.11c)}\\
e^\nu &= B r^{\frac{j}{1-j}} \exp\left\{\frac{I_2}{1-j} - \frac{CR^2}{2(1-j)(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right\} && \text{(2.11d)}\\
e^\lambda &= \frac{1}{1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4} && \text{(2.11e)}
\end{aligned}
$$

where I_2 (given in (2.7)) contains the two cases, $\ell = 0$ and $\ell \neq 0$. The exact solution (2.11) represents the interior of an anisotropic star corresponding to the energy density $\mu = j/r^2 + k + \ell r^2$. We believe that (2.11) is a *new* solution to the Einstein field equations. This solution has to match the Schwarzschild exterior solution (1.22). The matching condition $e^{\nu(R)} = e^{-\lambda(R)} = 1 - 2M/R$ yields

$$
B = R^{-\frac{j}{1-j}}\left(1 - \frac{2M}{R}\right)\exp\left\{-\frac{I_2\,(R)}{1-j}\right\}
$$

which fixes the constant of integration B where I_2 is as given in (2.7), $M = m(R)$ and R is the stellar radius.

We consider three special cases of the above new class of solution which lead to two particular models that have been studied previously.

Case a: $j = 0$

When $j = 0$, (2.11) becomes

$$\begin{aligned}
\mu &= k + \ell r^2 && (2.12a)\\
p_r &= C\left(1 - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n && (2.12b)\\
p_\perp &= p_r - \frac{\ell C}{10}r^4\left(1 - \frac{r^2}{R^2}\right)^n + \frac{r^2}{2}\left(1 - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^{-1} \\
&\quad \times \left\{\frac{C^2}{2}\left(1 - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{2n}\right. \\
&\quad \left. - \frac{2nC}{R^2}\left(1 - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{n-1} + \frac{1}{2}\left(k + \ell r^2\right)\left(\frac{k}{3} + \frac{\ell}{5}r^2\right)\right\} && (2.12c)\\
e^\nu &= B\exp\left\{I_2 - \frac{CR^2}{2(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right\} && (2.12d)\\
e^\lambda &= \frac{1}{1 - \frac{k}{3}r^2 - \frac{\ell}{5}r^4} && (2.12e)
\end{aligned}$$

where I_2 (given in (2.7)) contains the two cases, $\ell = 0$ and $\ell \neq 0$. The particular solution (2.12) was found by Gokhroo & Mehra (1994). Their solution is regained when we set

$$\begin{aligned}
k &= \rho_0\\
\ell &= -\frac{\rho_0 K}{a^2}
\end{aligned}$$

in (2.12) above. If we require $\mu' < 0$ in (2.12a) then the constant $\ell < 0$ for a monotonically decreasing energy density as we approach the boundary $r = R$ from the centre.

Case b: $j = \ell = 0$

When $j = \ell = 0$, (2.11) becomes

$$\mu = k \tag{2.13a}$$

$$p_r = C\left(1 - \frac{k}{3}r^2\right)\left(1 - \frac{r^2}{R^2}\right)^n \tag{2.13b}$$

$$\begin{aligned} p_\perp = {} & p_r + \frac{r^2}{2}\left(1 - \frac{k}{3}r^2\right)^{-1}\left\{\frac{C^2}{2}\left(1 - \frac{k}{3}r^2\right)^2\left(1 - \frac{r^2}{R^2}\right)^{2n}\right. \\ & \left. -\frac{2nC}{R^2}\left(1 - \frac{k}{3}r^2\right)^2\left(1 - \frac{r^2}{R^2}\right)^{n-1} + \frac{k^2}{6}\right\} \end{aligned} \tag{2.13c}$$

$$e^\nu = B\left(1 - \frac{k}{3}r^2\right)^{-\frac{1}{2}}\exp\left\{-\frac{CR^2}{2(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right\} \tag{2.13d}$$

$$e^\lambda = \frac{1}{1 - \frac{k}{3}r^2} \tag{2.13e}$$

The particular solution (2.13) was found by Maharaj & Maartens (1989). Their solution is regained when we let

$$k = \frac{6M}{R^3}$$

in (2.13) above. Note that μ is constant in (2.13a) and we may interpret this solution as an anisotropic generalisation of the incompressible Schwarzschild interior sphere; however note that the anisotropy factor $S(r) \neq 0$ everywhere except at the centre $r = 0$.

Case c: $k = \ell = 0$

When $k = \ell = 0$, (2.11) assumes the form

$$\mu = \frac{j}{r^2} \tag{2.14a}$$

$$p_r = C\left(1 - \frac{r^2}{R^2}\right)^n \tag{2.14b}$$

$$\begin{aligned} p_\perp = {} & p_r + \frac{jC}{2\,(1-j)}\left(1 - \frac{r^2}{R^2}\right)^n + \frac{r^2}{2}\left\{\frac{C^2}{2(1-j)}\left(1 - \frac{r^2}{R^2}\right)^{2n}\right. \\ & \left. -\frac{2nC}{R^2}\left(1 - \frac{r^2}{R^2}\right)^{n-1} + \frac{j^2}{2r^4(1-j)}\right\} \end{aligned} \tag{2.14c}$$

$$e^\nu = Br^{\frac{j}{1-j}}\,(1-j)^{-\frac{1}{2(1-j)}}\exp\left\{-\frac{CR^2}{2(1-j)(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right\} \tag{2.14d}$$

$$e^\lambda = \frac{1}{1-j} \tag{2.14e}$$

Since $\mu \propto r^{-2}$ we may interpret (2.14) as an anisotropic generalisation of the isothermal sphere. Even though (2.14) has a very simple form, we believe that it is a *new* anisotropic solution to the Einstein field equations and has not been published before.

Note that Dev & Gleiser (2002), Herrera & Santos (1997) and Petri (2003) found solutions to the anisotropic Einstein field equations involving $\mu \propto r^{-2}$. In each of these papers a different additional assumption to that utilised in this chapter was used; in our treatment we have chosen a form for the radial pressure p_r. In each of these works, a different set of assumptions and a different integration technique was used. Therefore their solutions are necessarily different from (2.14) for the corresponding energy density choice $\mu = jr^{-2}$. In addition, the general solution obtained by Dev & Gleiser (2002) for the choice $\mu = jr^{-2} + k$ is given in terms of hypergeometric functions. Our corresponding solution has the advantage of being expressed in terms of elementary functions (see Case II in Tables 2.3-2.4).

The physical features of the exact solution (2.11) are considered in §2.4.

2.3 Additional classes of solutions

In this section we find other solutions to the Einstein field equations (1.21) by following a similar procedure to that outlined in §2.2. We do so by considering other possible forms for the energy density function μ. The following is a list of forms for energy densities that have been studied:

Case	$\mu(r)$
I	$\frac{j}{r^2}$
II	$\frac{j}{r^2} + k$
III	$\frac{j}{r^2} + k + \ell r^2$
IV	$\frac{1}{r^2}(1-a) - (p+1)br^{p-2}$
V	$\frac{1}{r^2}\left(\frac{a-1+br^2}{a+br^2}\right) + \frac{2b}{(a+br^2)^2}$
VI	$\frac{1}{r^2}\left(\frac{a-c+(b-d)r^2}{a+br^2}\right) - 2\frac{b(a+br^2)-d(c+dr^2)}{(a+br^2)^2}$
VII	$\frac{1}{r^2}\left(\frac{a-1+br^2+cr^4}{a+br^2+cr^4}\right) - 2\frac{b+2cr^2}{(a+br^2+cr^4)^2}$

Table 2.1: Energy density functions

The forms of energy density μ in Table 2.1 were chosen so that the Einstein field equations could be fully integrated, and the gravitational potentials and matter variables written in closed form. To complete the integration of the Einstein field equations (1.21) we also need to make a choice for the radial pressure p_r. Clearly a variety of choices for p_r is possible; our choice is made on physical grounds. The following is a list of forms for the radial pressure:

Case	$p_r(r)$
I	$C\left(1-\frac{r^2}{R^2}\right)^n$
II	$\frac{C}{1-j}\left(1-j-\frac{k}{3}r^2\right)\left(1-\frac{r^2}{R^2}\right)^n$
III	$\frac{C}{1-j}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)\left(1-\frac{r^2}{R^2}\right)^n$
IV	$\frac{C}{a}\left(a+br^p\right)\left(1-\frac{r^2}{R^2}\right)^n$
V	$\frac{aC}{a+br^2}\left(1-\frac{r^2}{R^2}\right)^n$
VI	$\frac{a}{c}C\left(\frac{c+dr^2}{a+br^2}\right)\left(1-\frac{r^2}{R^2}\right)^n$
VII	$\frac{aC}{a+br^2+cr^4}\left(1-\frac{r^2}{R^2}\right)^n$

Table 2.2: Radial pressure functions

The forms of p_r selected in Table 2.2 are all reducible to the expression

$$p_r = C\left(1-\frac{r^2}{R^2}\right)^n$$

(which was first used by Maharaj & Maartens (1989)) with appropriate choices for the parameters.

Table 2.3 contains the gravitational potentials e^{ν} and e^{λ}. Tables 2.4 and 2.5 list the corresponding matter variables: the mass function m, the energy density μ, the radial pressure p_r and the tangential pressure $p_\perp$. We have not presented details of the calculations as the method of integration is similar to that in §2.2. All the results were carefully checked and we believe that they are correct. In addition the solutions were all verified with the help of the software package Mathematica® 5 (Wolfram 2003).

We believe that the families of solutions for Cases I-VII presented in Tables 2.1-2.5 are *new* solutions to the field equations, apart from particular cases, for relativistic anisotropic matter. These solutions are amenable to a physical analysis because they have a simple form, and in all cases the gravitational potentials and matter variables are given in terms of elementary functions. Note that the solution discussed in §2.2 corresponds to Case III where I_2 is given in (2.7). We have included Case III in the tables for completeness. The parameters j, k, and ℓ in Cases I-III are constants. The quantities a, b, c, d and p are also constants in Cases IV-VII.

Case	e^{λ}	e^{ν}
I	$(1-j)^{-1}$	$Br^{\frac{j}{1-j}}(1-j)^{-\frac{1}{2(1-j)}}\exp\left\{-\frac{CR^2}{2(1-j)(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}\right\}$
II	$\left(1-j-\frac{k}{3}r^2\right)^{-1}$	$\frac{Br^{\frac{j}{1-j}}}{\left(1-j-\frac{k}{3}r^2\right)^{\frac{1}{2(1-j)}}}\exp\left\{-\frac{CR^2}{2(1-j)(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}\right\}$
III	$\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^{-1}$	$Br^{\frac{j}{1-j}}\exp\left\{\frac{I_2}{1-j}-\frac{CR^2}{2(1-j)(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}\right\}$
IV	$(a+br^p)^{-1}$ $p>0$	$\frac{Br^{\frac{1-a}{a}}}{(a+br^p)^{\frac{1}{ap}}}\exp\left\{-\frac{CR^2}{2a(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}\right\}$
V	$a+br^2$	$Br^{a-1}\exp\left\{-\frac{aCR^2}{2(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}+\frac{br^2}{2}\right\}$
VI	$\frac{a+br^2}{c+dr^2}$	$Br^{\frac{a-c}{c}}(c+dr^2)^{\frac{bc-ad}{2cd}}\exp\left\{-\frac{aCR^2}{2c(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}\right\}$
VII	$a+br^2+cr^4$	$Br^{a-1}\exp\left\{-\frac{aCR^2}{2(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}+\frac{br^2}{2}+\frac{cr^4}{4}\right\}$

Table 2.3: Gravitational potentials

Case	μ	m	$p_r\ (C=p_r(0), n\geq 1)$	$p_\perp$
I	$\frac{j}{r^2}$	$\frac{j}{2}r$	$C\left(1-\frac{r^2}{R^2}\right)^n$	$p_r+\frac{jC}{2(1-j)}\left(1-\frac{r^2}{R^2}\right)^n$ $+\frac{r^2}{2}\left\{\frac{C^2}{2(1-j)}\left(1-\frac{r^2}{R^2}\right)^{2n}-\frac{2nC}{R^2}\left(1-\frac{r^2}{R^2}\right)^{n-1}+\frac{j^2}{2r^4(1-j)}\right\}$
II	$\frac{j}{r^2}+k$	$\frac{r}{2}\left(j+\frac{k}{3}r^2\right)$	$\frac{C}{1-j}\left(1-j-\frac{k}{3}r^2\right)$ $\times\left(1-\frac{r^2}{R^2}\right)^n$	$p_r+\frac{jC}{2(1-j)}\left(1-\frac{r^2}{R^2}\right)^n+\frac{r^2}{2}\left(1-j-\frac{k}{3}r^2\right)^{-1}$ $\times\left\{\frac{C^2}{2(1-j)^2}\left(1-j-\frac{k}{3}r^2\right)^2\left(1-\frac{r^2}{R^2}\right)^{2n}\right.$ $-\frac{2nC}{(1-j)R^2}\left(1-j-\frac{k}{3}r^2\right)^2\left(1-\frac{r^2}{R^2}\right)^{n-1}$ $\left.+\frac{1}{2r^2}\left(\frac{j}{r^2}+k\right)\left(j+\frac{k}{3}r^2\right)\right\}$
III	$\frac{j}{r^2}+k+\ell r^2$	$\frac{r}{2}\left(j+\frac{k}{3}r^2+\frac{\ell}{5}r^4\right)$	$\frac{C}{1-j}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)$ $\times\left(1-\frac{r^2}{R^2}\right)^n$	$p_r+\frac{C}{2(1-j)}\left(j-\frac{\ell}{5}r^4\right)\left(1-\frac{r^2}{R^2}\right)^n+\frac{r^2}{2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^{-1}$ $\times\left\{\frac{C^2}{2(1-j)^2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^2\left(1-\frac{r^2}{R^2}\right)^{2n}\right.$ $-\frac{2nC}{(1-j)R^2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^2\left(1-\frac{r^2}{R^2}\right)^{n-1}$ $\left.+\frac{1}{2r^2}\left(\frac{j}{r^2}+k+\ell r^2\right)\left(j+\frac{k}{3}r^2+\frac{\ell}{5}r^4\right)\right\}$

Table 2.4: Matter variables

Case	μ	m	$p_r\ (C = p_r(0), n \geq 1)$	$p_\perp$
IV	$\frac{1}{r^2}(1-a)$ $-(p+1)br^{p-2}$	$\frac{r}{2}(1-a-br^p)$ $p>0$	$\frac{C}{a}(a+br^p)$ $\times\left(1-\frac{r^2}{R^2}\right)^n$	$p_r + \frac{C}{a}\left(\frac{1-a}{2} + b\left(\frac{p}{4}-\frac{1}{2}\right)r^p\right)\left(1-\frac{r^2}{R^2}\right)^n + \frac{r^2}{2}(a+br^p)^{-1}$ $\times\left\{\frac{C^2}{2a^2}(a+br^p)^2\left(1-\frac{r^2}{R^2}\right)^{2n} - \frac{2nC}{aR^2}(a+br^p)^2\left(1-\frac{r^2}{R^2}\right)^{n-1}\right.$ $\left.+\frac{1}{2r^4}(1-a-br^p)(1-a-b(p+1)r^p)\right\}$
V	$\frac{1}{r^2}\left(\frac{a-1+br^2}{a+br^2}\right)$ $+\frac{2b}{(a+br^2)^2}$	$\frac{r}{2}\left(1-\frac{1}{a+br^2}\right)$	$\frac{aC}{a+br^2}$ $\times\left(1-\frac{r^2}{R^2}\right)^n$	$p_r + \frac{r}{4}\left\{\frac{a-1+br^2}{r^2(a+br^2)} + \frac{2b}{(a+br^2)^2} + \frac{aC}{a+br^2}\left(1-\frac{r^2}{R^2}\right)^n\right\}$ $\times\left\{aCr\left(1-\frac{r^2}{R^2}\right)^n + \frac{1}{r}(a-1+br^2)\right\}$ $-\frac{abCr^2}{(a+br^2)^2}\left(1-\frac{r^2}{R^2}\right)^n - \frac{aCnr^2}{R^2(a+br^2)}\left(1-\frac{r^2}{R^2}\right)^{n-1}$
VI	$\frac{1}{r^2}\left(\frac{a-c+(b-d)r^2}{a+br^2}\right)$ $-2\frac{b(a+br^2)-d(c+dr^2)}{(a+br^2)^2}$	$\frac{r}{2}\left(1-\frac{c+dr^2}{a+br^2}\right)$	$\frac{a}{c}C\left(\frac{c+dr^2}{a+br^2}\right)$ $\times\left(1-\frac{r^2}{R^2}\right)^n$	$p_r + \frac{r}{4}\left\{\frac{a-c+(b-d)r^2}{r^2(a+br^2)} - \frac{ad-bc}{(a+br^2)^2} + \frac{aC}{c}\left(\frac{c+dr^2}{a+br^2}\right)\left(1-\frac{r^2}{R^2}\right)^n\right\}$ $\times\left\{\frac{aCr}{c}\left(1-\frac{r^2}{R^2}\right)^n + \frac{a-c+(b-d)r^2}{r(c+dr^2)}\right\}$ $+\frac{aCr^2}{c}\frac{ad-bc}{(a+br^2)^2}\left(1-\frac{r^2}{R^2}\right)^n - \frac{anCr^2}{cR^2}\left(\frac{c+dr^2}{a+br^2}\right)\left(1-\frac{r^2}{R^2}\right)^{n-1}$
VII	$\frac{1}{r^2}\left(\frac{a-1+br^2+cr^4}{a+br^2+cr^4}\right)$ $-2\frac{b+2cr^2}{(a+br^2+cr^4)^2}$	$\frac{r}{2}\left(1-\frac{1}{a+br^2+cr^4}\right)$	$\frac{aC}{a+br^2+cr^4}$ $\times\left(1-\frac{r^2}{R^2}\right)^n$	$p_r + \frac{r}{4}\left\{\frac{a-1+br^2+cr^4}{r^2(a+br^2+cr^4)} + \frac{b+2cr^2}{2(a+br^2+cr^4)} + \frac{aC}{a+br^2+cr^4}\left(1-\frac{r^2}{R^2}\right)^n\right\}$ $\times\left\{aCr\left(1-\frac{r^2}{R^2}\right)^n + \frac{1}{r}(a-1+br^2+cr^4)\right\}$ $-aCr^2\frac{b+2cr^2}{(a+br^2+cr^4)^2}\left(1-\frac{r^2}{R^2}\right)^n - \frac{aCnr^2}{R^2(a+br^2+cr^4)}\left(1-\frac{r^2}{R^2}\right)^{n-1}$

Table 2.5: Matter variables

2.4 Physical features

The gravitational potentials e^{λ} are finite for all Cases I-VII at the centre $r = 0$ and at the boundary $r = R$. The functions e^{λ} are well behaved in the interior of the relativistic star. The gravitational potentials e^{ν} for all Cases I-VII are continuous and well behaved in the interior and finite at the boundary of the star $r = R$. From Table 2.3 we observe that there is a singularity at the centre $r = 0$ in general for all Cases I-VII in the potential e^{ν}. The singularity in e^{ν} is removable for specific choices of parameter values. This singularity is eliminated by setting

- $j = 0$ in Cases I, II and III
- $a = 1$ in Cases IV, V and VII
- $a = c$ in Case VI

It should be pointed out that in Case I, we are left with a point mass if we set $j = 0$. This form of μ is usually used in domains where it is not possible to use a single equation of state; particularly where the origin is excluded, like a body with a constant density core and matter density distribution around the core going like r^{-2} (Dev & Gleiser 2002, Sharma & Mukherjee 2002).

The radial pressure p_r is continuous and well behaved in the interior of the star. Also $p_r > 0$ in the interval $(0, R)$, regular at the centre ($p_r(r = 0) = C$), and vanishes at the boundary ($p_r(r = R) = 0$) in all considered cases. The tangential pressure $p_{\perp}$ in the studied cases has a singularity at the centre, but is otherwise well behaved throughout the interior of the star and finite at the boundary. The singularity in $p_{\perp}$ may be eliminated by suitable particular choices for parameter values. In general the tangential pressure is not zero at the boundary of the star ($p_{\perp}(r = R) \neq 0$) which is different from the radial pressure ($p_r(r = R) = 0$). It is also important to observe that the magnitude of the stress tensor

$$S = \frac{1}{\sqrt{3}}(p_r - p_\perp)$$

is a nonzero function in general for all Cases I-VII. Hence this class of solutions is generally anisotropic and does not have an isotropic limit (the isotropic limit results when we set particular values for the constants in our ansatz). It is not possible to eliminate S and obtain an isotropic counterpart. This means that the model remains anisotropic. An analogous situation rises in Einstein-Maxwell solutions modelling charged relativistic stars in which the electric field is always present. An example of such a charged star is given by Hansraj (1999).

The energy density μ for all cases contains the limiting case

$$\mu = r^{-2}$$

which arises in isothermal spheres for isotropic matter for both Newtonian and relativistic stars (Saslaw *et al* 1996). The energy densities studied here are physically reasonable and describe important phenomena. For example, Misner & Zapolsky (1964) propose that Case I models the physical configuration of a relativistic Fermi gas for some particular value of the parameter j. Another example is due to Dev & Gleiser (2002) who suggest that for some particular value of j and $k \neq 0$ the energy density function in Case II describes a relativistic Fermi gas core immersed in a constant density background.

One of the original reasons for studying anisotropic matter was to generate models that permit redshifts higher than the critical redshift z_c of isotropic matter (Bowers & Liang 1974). Observational results indicate that certain isolated objects have redshifts higher than z_c. The surface redshift is given by

$$z = \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} - 1$$

The critical redshift $z_c = 2$ is the limiting value for the perfect fluid spheres, and is attained when $2M/R = 8/9$. For the range of values $8/9 < 2M/R < 1$ the redshift is greater than z_c; this phenomenon may be explained by allowing for anisotropy. For values of $2M/R$ close to unity, the surface redshift becomes infinitely large. The feasibility of higher redshifts for anisotropic matter, in both Newtonian and relativistic models, was firmly established by Bondi (1992). It is interesting to note that Bondi (1992), Binney & Tremaine (1987), Cuddeford (1991) and Michie (1963) emphasise the significance of anisotropies in stellar clusters and galaxies, in addition to individual stars.

The solutions presented in this chapter have the feature that $p_r \neq p_\perp$ in general so that the anisotropy factor $S \neq 0$ and the solutions do not have an isotropic limit as indicated earlier. This means that we cannot obtain isotropic stars from the anisotropic stars presented here, for example we cannot regain the constant density interior Schwarzschild solution. This may be viewed as an undesirable feature because we would expect isolated matter to eventually isotropise. In subsequent chapters we present algorithms that generate solutions to the Einstein field equations for anisotropic matter that do have an isotropic limit.

Chapter 3

Algorithms to generate Anisotropic from Isotropic Solutions

3.1 Introduction

In this chapter we follow a different approach from that presented in Chapter 2 to solve the field equations. Two new algorithms which generate *new* anisotropic solutions for the Einstein field equations from known isotropic solutions are established. The original seed isotropic solution is regained when the anisotropic term vanishes (this is not possible in general for the solutions found in Chapter 2). The first algorithm, Algorithm $\mathcal{A}$, is presented in §3.2. In this case we set

$$\nu = \nu_0 + \beta(r)$$

for the gravitational potential ν, ν_0 is given, and $\beta(r)$ is the anisotropy inducing term. The Einstein field equations are formally integrated, and we generate a representation for the exact solution. The second algorithm, Algorithm $\mathcal{B}$, is presented in §3.3. In this case we set

$$\begin{aligned}\lambda &= \lambda_0 + x(r)\\ m &= m_0 + y(r)\end{aligned}$$

for the gravitational potential λ and the mass function m, λ_0 and m_0 are given, and $x(r)$ and $y(r)$ are the anisotropy inducing terms. We show that the Einstein field equations can be formally integrated, and we present a representation for the exact solution. In §3.4 we illustrate Algorithms $\mathcal{A}$ and $\mathcal{B}$ with the help of two known isotropic solutions, namely the Einstein and de Sitter models.

3.2 Algorithm $\mathcal{A}$

We consider the Einstein field equations with isotropic pressure distribution so that $p_r = p_\perp = p$. Then we can write the field equations in the form

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{3.1a}$$

$$r(r-2m)\nu' = pr^3 + 2m \tag{3.1b}$$

$$\left(\frac{2m'}{r^2} + p\right)\nu' + 2p' = 0 \tag{3.1c}$$

from (1.20) and (1.21). Suppose an explicit solution to (3.1) is known where

$$\nu = \nu_0 \tag{3.2a}$$

$$\lambda = \lambda_0 \tag{3.2b}$$

$$m = m_0 \tag{3.2c}$$

$$p = p_0 \tag{3.2d}$$

are given. Then the equations in (3.1) are satisfied and we can write

$$
\begin{aligned}
e^{-\lambda_0} &= 1-\frac{2m_0}{r} && \text{(3.3a)}\\
r(r-2m_0)\nu_0' &= p_0 r^3 + 2m_0 && \text{(3.3b)}\\
\left(\frac{2m_0'}{r^2}+p_0\right)\nu_0' + 2p_0' &= 0 && \text{(3.3c)}
\end{aligned}
$$

The equations in (3.3) correspond to an isotropic relativistic sphere.

The Einstein field equations (1.21) with anisotropic matter can be written in the form

$$
\begin{aligned}
e^{-\lambda} &= 1-\frac{2m}{r} && \text{(3.4a)}\\
r(r-2m)\nu' &= pr^3 + \alpha r^3 + 2m && \text{(3.4b)}\\
\left(\frac{2m'}{r^2}+p+\alpha\right)\nu' + 2p' + 2\alpha' &= -\frac{6}{r}\alpha && \text{(3.4c)}
\end{aligned}
$$

where we have used (1.20). Also we have set

$$
\begin{aligned}
p_r &= p+\alpha\\
p_\perp &= p-\frac{1}{2}\alpha
\end{aligned}
$$

for the radial and tangential pressures respectively. Observe that

$$
\alpha = \frac{2}{\sqrt{3}}S
$$

which connects the function $\alpha = \alpha(r)$ to the anisotropic factor $S = S(r)$ defined in §1.3.

We seek a solution to the system (3.4). To this end we propose the possible solution

$$\begin{aligned} \nu &= \nu_0 + \beta(r) && (3.5a) \\ \lambda &= \lambda_0 && (3.5b) \\ m &= m_0 && (3.5c) \\ p_r &= p_0 + \alpha(r) && (3.5d) \\ p_\perp &= p_0 - \frac{1}{2}\alpha(r) && (3.5e) \end{aligned}$$

where α and β are arbitrary functions. With the assumed solution (3.5), the system (3.4) becomes

$$\begin{aligned} e^{\lambda_0} &= 1 - \frac{2m_0}{r} && (3.6a) \\ r(r-2m_0)\nu_0' + r(r-2m_0)\beta' &= p_0 r^3 + \alpha r^3 + 2m_0 && (3.6b) \\ \left(\frac{2m_0'}{r^2} + p_0 + \alpha\right)(\nu_0' + \beta') + 2p_0' + 2\alpha' &= -\frac{6}{r}\alpha && (3.6c) \end{aligned}$$

The systems (3.3) and (3.6) imply that

$$\begin{aligned} (r-2m_0)\beta' &= \alpha r^2 && (3.7a) \\ \left(\frac{2m_0'}{r^2} + p_0\right)\beta' + \alpha(\nu_0' + \beta') + 2\alpha' &= -\frac{6}{r}\alpha && (3.7b) \end{aligned}$$

We need to integrate (3.7) to obtain expressions for α and β. Two cases arise: $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ then (3.7) has the solution

$$\begin{aligned} \alpha &= 0 && (3.8a) \\ \beta &= \text{constant} && (3.8b) \end{aligned}$$

Equations (3.8) correspond to the isotropic case. Thus Algorithm $\mathcal{A}$ regains the isotropic solution in the relevant limit. If $\alpha \neq 0$ then we can write (3.7b) as

$$\left(\frac{2m_0'}{r^2} + p_0\right)\frac{r^2}{r - 2m_0} + \nu_0' + \beta' + 2\frac{\alpha'}{\alpha} = -\frac{6}{r}$$

with the help of (3.7a). This differential equation can be integrated to give

$$I_\alpha + \nu_0 + \beta + 2\ln\alpha = -6\ln r + 2\ln k \tag{3.9}$$

where $2\ln k$ is a constant of integration, $k \neq 0$, and we have set

$$I_\alpha = \int \left(\frac{2m_0'}{r^2} + p_0\right)\frac{r^2}{r - 2m_0}\mathrm{d}r$$

The solution (3.9) above can be rewritten as

$$\alpha = \frac{k}{r^3}\exp\left\{-\frac{1}{2}\left(I_\alpha + \nu_0 + \beta\right)\right\} \tag{3.10}$$

so that α is the subject of the formula. From (3.7a) and (3.10) we generate the nonlinear differential equation in β:

$$e^{\beta/2}\beta' = \frac{k}{r(r - 2m_0)}\exp\left\{-\frac{1}{2}\left(I_\alpha + \nu_0\right)\right\}$$

This first order differential equation is integrable and we generate the result

$$\begin{aligned}\beta &= 2\ln\left\{\frac{k}{2}\int \frac{\exp\left\{-\frac{1}{2}\left(I_\alpha + \nu_0\right)\right\}}{r\left(r - 2m_0\right)}\mathrm{d}r + \ell\right\} \\ &= 2\ln\left\{\frac{k}{2}I_\beta + \ell\right\}\end{aligned} \tag{3.11}$$

where ℓ is a constant of integration and we have set

$$I_\beta = \int \frac{\exp\left\{-\frac{1}{2}\left(I_\alpha + \nu_0\right)\right\}}{r\left(r - 2m_0\right)}\mathrm{d}r$$

in (3.11). Equations (3.10) and (3.11) correspond to anisotropic matter.

Thus if given an isotropic solution (3.2) we can generate a *new* anisotropic solution (3.5) to the Einstein field equations where

$$\alpha = \frac{k}{r^3}\exp\left\{-\frac{1}{2}\left(I_\alpha + \nu_0 + \beta\right)\right\} \tag{3.12a}$$

$$\beta = 2\ln\left\{\frac{k}{2}I_\beta + \ell\right\} \tag{3.12b}$$

The integrals I_α and I_β in (3.12) are given by

$$I_\alpha = \int\left(\frac{2m_0'}{r^2} + p_0\right)\frac{r^2}{r - 2m_0}\mathrm{d}r$$

$$I_\beta = \int\frac{\exp\left\{-\frac{1}{2}\left(I_\alpha + \nu_0\right)\right\}}{r\left(r - 2m_0\right)}\mathrm{d}r$$

The integrations in I_α and I_β can be performed explicitly as ν_0, p_0 and m_0 are specified in the isotropic solution functions in (3.2). Note that (3.12) applies to both cases $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ we can set $k = 0$ and regain the isotropic result (3.8). When $\alpha \neq 0$ then $k \neq 0$ and we regain the anisotropic equations (3.10) and (3.11).

3.3 Algorithm $\mathcal{B}$

The Einstein field equations for isotropic matter are

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{3.13a}$$

$$r(r - 2m)\nu' = pr^3 + 2m \tag{3.13b}$$

$$\left(\frac{2m'}{r^2} + p\right)\nu' + 2p' = 0 \tag{3.13c}$$

as given in §3.2. Again we assume the explicit solution

$$\nu = \nu_0 \tag{3.14a}$$

$$\lambda = \lambda_0 \tag{3.14b}$$

$$m = m_0 \tag{3.14c}$$

$$p = p_0 \tag{3.14d}$$

to (3.13). The equations in (3.13) are satisfied and we can write

$$e^{-\lambda_0} = 1 - \frac{2m_0}{r} \tag{3.15a}$$

$$r(r-2m_0)\nu_0' = p_0 r^3 + 2m_0 \tag{3.15b}$$

$$\left(\frac{2m_0'}{r^2} + p_0\right)\nu_0' + 2p_0' = 0 \tag{3.15c}$$

which, as indicated earlier, corresponds to the isotropic sphere.

We now write the Einstein field equations (1.21) with anisotropic matter in the form

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{3.16a}$$

$$r(r-2m)\nu' = pr^3 + \alpha r^3 + 2m \tag{3.16b}$$

$$\left(\frac{2m'}{r^2} + p + \alpha\right)\nu' + 2p' + 2\alpha' = -\frac{6}{r}\alpha \tag{3.16c}$$

where we have used (1.20) and we set

$$\begin{aligned} p_r &= p_0 + \alpha \\ p_\perp &= p_0 - \frac{1}{2}\alpha \end{aligned}$$

which suggests

$$\alpha = \frac{2}{\sqrt{3}} S(r)$$

as in the previous section.

We seek a solution to (3.16). To this end we propose the solution

$$\nu = \nu_0 \tag{3.17a}$$

$$\lambda = \lambda_0 + x(r) \tag{3.17b}$$

$$m = m_0 + y(r) \tag{3.17c}$$

$$p_r = p_0 + \alpha(r) \tag{3.17d}$$

$$p_\perp = p_0 - \frac{1}{2}\alpha(r) \tag{3.17e}$$

where α, x and y are arbitrary functions. With the assumed solution (3.17) the system (3.16) becomes

$$e^{-(\lambda_0+x)} = 1 - \frac{2m_0 + 2y}{r} \tag{3.18a}$$

$$r(r - 2m_0 - 2y)\nu_0' = p_0 r^3 + \alpha r^3 + 2m_0 + 2y \tag{3.18b}$$

$$\left(\frac{2m_0' + 2y'}{r^2} + p_0 + \alpha\right)\nu_0' + 2p_0' + 2\alpha' = -\frac{6}{r}\alpha \tag{3.18c}$$

The systems (3.15) and (3.18) lead to

$$x = -\ln\left\{1 - \frac{2y}{r}e^{\lambda_0}\right\} \tag{3.19a}$$

$$y = -\frac{\alpha r^3}{2(1 + r\nu_0')} \tag{3.19b}$$

$$\left(\frac{2y'}{r^2} + \alpha\right)\nu_0' + 2\alpha' = -\frac{6\alpha}{r} \tag{3.19c}$$

We need to integrate (3.19c) to find the function α. The remaining functions x and y are defined in terms of α. Two cases arise: $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ then (3.19) has the solution

$$\alpha = 0 \tag{3.20a}$$

$$x = 0 \tag{3.20b}$$

$$y = 0 \tag{3.20c}$$

Equations (3.20) correspond to the isotropic case. Thus Algorithm $\mathcal{B}$ also regains the isotropic solution in the appropriate limit. If $\alpha \neq 0$ then we eliminate y to get

$$\frac{2\nu_0'}{r^2}\left\{-\frac{\alpha' r^3}{2(1+r\nu_0')} - \frac{3\alpha r^2}{2(1+r\nu_0')} + \frac{\alpha r^3}{2}\frac{\nu_0' + r\nu_0''}{(1+r\nu_0')^2}\right\} + \alpha\nu_0' + 2\alpha' = -\frac{6\alpha}{r}$$

from (3.19b) and (3.19c). This differential equation can be written as

$$\frac{\alpha'}{\alpha} - \frac{\nu_0'}{2+r\nu_0'}\left\{3 - r\frac{\nu_0' + r\nu_0''}{1+r\nu_0'}\right\} = -\left(\nu_0' + \frac{6}{r}\right)\left(\frac{1+r\nu_0'}{2+r\nu_0'}\right) \tag{3.21}$$

after some simplification. On integration (3.21) leads to

$$\ln\alpha = J_\alpha + \ln k \tag{3.22}$$

where $\ln k$ is a constant of integration, $k \neq 0$, and we have set

$$J_\alpha = \int\left\{\frac{\nu_0'}{2+r\nu_0'}\left(\frac{3 + 2r\nu_0' - r^2\nu_0''}{1+r\nu_0'}\right) - \left(\nu_0' + \frac{6}{r}\right)\left(\frac{1+r\nu_0'}{2+r\nu_0'}\right)\right\}\mathrm{d}r$$

We can write (3.22) in the compact form

$$\alpha = ke^{J_\alpha} \tag{3.23}$$

Equations (3.19a), (3.19b) and (3.23) correspond to anisotropic matter.

Thus if given a known isotropic solution (3.14) we can generate a *new* anisotropic solution (3.17) where

$$\alpha = ke^{J_\alpha} \tag{3.24a}$$

$$x = -\ln\left\{1 - \frac{2y}{r}e^{\lambda_0}\right\} \tag{3.24b}$$

$$y = -\frac{\alpha r^3}{2(1+r\nu_0')} \tag{3.24c}$$

and the integral J_α is given by

$$J_\alpha = \int\left\{\frac{\nu_0'}{2+r\nu_0'}\left(\frac{3+2r\nu_0'-r^2\nu_0''}{1+r\nu_0'}\right) - \left(\nu_0' + \frac{6}{r}\right)\left(\frac{1+r\nu_0'}{2+r\nu_0'}\right)\right\}\mathrm{d}r$$

The integration in J_α can be explicitly performed as ν_0 is specified in the isotropic solution (3.14). Note that (3.24) applies to both cases $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ we can set $k = 0$ and regain the isotropic result (3.20). When $\alpha \neq 0$ then $k \neq 0$ and we regain the anisotropic equations (3.19a), (3.19b) and (3.23).

3.4 Simple Examples

In this section we show how isotropic solutions lead to anisotropic solutions by utilising Algorithms $\mathcal{A}$ and $\mathcal{B}$. We choose the simple Einstein and de Sitter models as illustrations. In later chapters we consider other examples which have greater physical significance for the description of stars.

Example 1 : Einstein model

The line element for the Einstein model is

$$ds^2 = -c^2 dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.25}$$

The relevant isotropic functions for (3.25) are

$$\nu_0 = 2\ln c \tag{3.26a}$$

$$\lambda_0 = -\ln\left\{1 - \frac{r^2}{R^2}\right\} \tag{3.26b}$$

$$m_0 = \frac{r^3}{2R^2} \tag{3.26c}$$

$$p_0 = -\frac{1}{R^2} \tag{3.26d}$$

Algorithm $\mathcal{A}$

The integrals I_α and I_β for Algorithm $\mathcal{A}$ respectively become

$$\begin{aligned} I_\alpha &= \int \frac{2r}{R^2(1 - r^2/R^2)} dr \\ &= -\ln\left\{1 - \frac{r^2}{R^2}\right\} \\ I_\beta &= \int \frac{1}{cr^2\sqrt{1 - r^2/R^2}} dr \\ &= -\frac{1}{cr}\sqrt{1 - \frac{r^2}{R^2}} \end{aligned}$$

for the functions (3.26). Then (3.12) gives

$$\alpha = \frac{k}{cr^3}\sqrt{1 - \frac{r^2}{R^2}}\left(\ell - \frac{k}{2cr}\sqrt{1 - \frac{r^2}{R^2}}\right)^{-1} \tag{3.27}$$

$$\beta = 2\ln\left\{\ell - \frac{k}{2cr}\sqrt{1 - \frac{r^2}{R^2}}\right\} \tag{3.28}$$

Hence the new anisotropic solution to the Einstein field equations is

$$\nu = 2\ln c + 2\ln\left\{\ell - \frac{k}{2cr}\sqrt{1-\frac{r^2}{R^2}}\right\} \tag{3.29a}$$

$$\lambda = -\ln\left\{1-\frac{r^2}{R^2}\right\} \tag{3.29b}$$

$$m = \frac{r^3}{2R^2} \tag{3.29c}$$

$$p_r = -\frac{1}{R^2} + \frac{k}{cr^3}\sqrt{1-\frac{r^2}{R^2}}\left(\ell - \frac{k}{2cr}\sqrt{1-\frac{r^2}{R^2}}\right)^{-1} \tag{3.29d}$$

$$p_\perp = -\frac{1}{R^2} - \frac{k}{2cr^3}\sqrt{1-\frac{r^2}{R^2}}\left(\ell - \frac{k}{2cr}\sqrt{1-\frac{r^2}{R^2}}\right)^{-1} \tag{3.29e}$$

with the corresponding line element

$$\begin{aligned} ds^2 &= -c^2\left(\ell - \frac{k}{2cr}\sqrt{1-\frac{r^2}{R^2}}\right)^2 dt^2 + \left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 \\ &\quad + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned} \tag{3.30}$$

The isotropic Einstein model (3.25) generates the anisotropic Einstein model (3.30), where (3.29) is a *new* exact solution to the anisotropic field equations, by Algorithm $\mathcal{A}$.

Algorithm $\mathcal{B}$

The integral J_α for Algorithm $\mathcal{B}$ is

$$\begin{aligned} J_\alpha &= -3\int \frac{dr}{r} \\ &= -3\ln r \end{aligned}$$

for the function (3.26a). Then (3.24) gives

$$\begin{aligned} \alpha &= \frac{k}{r^3} \\ x &= -\ln\left\{1 + \frac{k}{r\left(1 - r^2/R^2\right)}\right\} \\ y &= -\frac{k}{2} \end{aligned}$$

Hence the new anisotropic solution to the field equations is

$$\begin{aligned}
\nu &= 2\ln c && (3.31a)\\
\lambda &= -\ln\left\{1-\frac{r^2}{R^2}\right\} - \ln\left\{1+\frac{k}{r\left(1-r^2/R^2\right)}\right\} && (3.31b)\\
m &= \frac{r^3}{2R^2}-\frac{k}{2} && (3.31c)\\
p_r &= -\frac{1}{R^2}+\frac{k}{r^3} && (3.31d)\\
p_\perp &= -\frac{1}{R^2}-\frac{k}{2r^3} && (3.31e)
\end{aligned}$$

with the line element

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \left(1-\frac{r^2}{R^2}+\frac{k}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \qquad (3.32)$$

The isotropic Einstein model (3.25) gives the anisotropic Einstein model (3.32) where (3.31) is the new exact solution for Algorithm $\mathcal{B}$.

Example 2 : de Sitter model

The line element for the de Sitter model is

$$\mathrm{d}s^2 = -\left(1-\frac{r^2}{R^2}\right)\mathrm{d}t^2 + \left(1-\frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \qquad (3.33)$$

The relevant isotropic functions for (3.33) are

$$\begin{aligned}
\nu_0 &= \ln\left\{1-\frac{r^2}{R^2}\right\} && (3.34a)\\
\lambda_0 &= -\ln\left\{1-\frac{r^2}{R^2}\right\} && (3.34b)\\
m_0 &= \frac{r^3}{2R^2} && (3.34c)\\
p_0 &= -\frac{3}{R^2} && (3.34d)
\end{aligned}$$

Algorithm $\mathcal{A}$

The integrals I_α and I_β for Algorithm $\mathcal{A}$ become

$$\begin{aligned} I_\alpha &= 0 \\ I_\beta &= \int \frac{1}{r^2\left(1 - r^2/R^2\right)^{\frac{3}{2}}} \mathrm{d}r \\ &= \sqrt{1 - \frac{r^2}{R^2}} \left(\frac{r}{R^2(1 - r^2/R^2)} - \frac{1}{r} \right) \end{aligned}$$

for the functions (3.34). Then (3.12) gives

$$\begin{aligned} \alpha &= \frac{k}{r^3}\sqrt{1 - \frac{r^2}{R^2}} \left(\ell + \frac{k}{2}\sqrt{1 - \frac{r^2}{R^2}} \left(\frac{r}{R^2(1 - r^2/R^2)} - \frac{1}{r} \right) \right)^{-1} \\ \beta &= 2 \ln \left\{ \ell + \frac{k}{2}\sqrt{1 - \frac{r^2}{R^2}} \left(\frac{r}{R^2(1 - r^2/R^2)} - \frac{1}{r} \right) \right\} \end{aligned}$$

Hence the new anisotropic solution to the field equations is

$$\nu = \ln \left\{ 1 - \frac{r^2}{R^2} \right\} + 2 \ln \left\{ \ell + \frac{k}{2}\sqrt{1 - \frac{r^2}{R^2}} \left(\frac{r}{R^2(1 - r^2/R^2)} - \frac{1}{r} \right) \right\} \tag{3.35a}$$

$$\lambda = -\ln \left\{ 1 - \frac{r^2}{R^2} \right\} \tag{3.35b}$$

$$m = \frac{r^3}{2R^2} \tag{3.35c}$$

$$p_r = -\frac{3}{R^2} + \frac{k}{r^3}\sqrt{1 - \frac{r^2}{R^2}} \left(\ell + \frac{k}{2}\sqrt{1 - \frac{r^2}{R^2}} \left(\frac{r}{R^2(1 - r^2/R^2)} - \frac{1}{r} \right) \right)^{-1} \tag{3.35d}$$

$$p_\perp = -\frac{3}{R^2} - \frac{k}{2r^3}\sqrt{1 - \frac{r^2}{R^2}} \left(\ell + \frac{k}{2}\sqrt{1 - \frac{r^2}{R^2}} \left(\frac{r}{R^2(1 - r^2/R^2)} - \frac{1}{r} \right) \right)^{-1} \tag{3.35e}$$

with the corresponding line element

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1 - \frac{r^2}{R^2} \right) \left(\ell + \frac{k}{2}\sqrt{1 - \frac{r^2}{R^2}} \left(\frac{r}{R^2(1 - r^2/R^2)} - \frac{1}{r} \right) \right)^2 \mathrm{d}t^2 \\ &\quad + \left(1 - \frac{r^2}{R^2} \right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \end{aligned} \tag{3.36}$$

The isotropic de Sitter model (3.33) generates the anisotropic de Sitter model (3.36) where (3.35) is a *new* exact solution to the field equations by Algorithm $\mathcal{A}$.

Algorithm $\mathcal{B}$

The integral J_α for Algorithm $\mathcal{B}$ is

$$\begin{aligned} J_\alpha &= -\int \frac{27r^4 - 16r^2R^2 + 3R^4}{6r^5 - 5r^3R^2 + rR^4}\mathrm{d}r \\ &= -3\ln r - \frac{7}{4}\ln\left\{2r^2 - R^2\right\} \\ &\quad + \ln\left\{3r^2 - R^2\right\} \end{aligned}$$

for the function (3.34a). Then (3.24) gives

$$\begin{aligned} \alpha &= k\left(\frac{3r^2 - R^2}{r^3\left(2r^2 - R^2\right)^{\frac{7}{4}}}\right) \\ x &= -\ln\left\{1 + \frac{kR^2}{r\left(R^2 - 2r^2\right)^{\frac{7}{4}}}\right\} \\ y &= k\left(\frac{R^2 - r^2}{2\left(2r^2 - R^2\right)^{\frac{7}{4}}}\right) \end{aligned}$$

Hence the new anisotropic solution to field equations is

$$\nu = \ln\left\{1 - \frac{r^2}{R^2}\right\} \tag{3.37a}$$

$$\lambda = -\ln\left\{1 - \frac{r^2}{R^2}\right\} - \ln\left\{1 + \frac{kR^2}{r\left(R^2 - 2r^2\right)^{\frac{7}{4}}}\right\} \tag{3.37b}$$

$$m = \frac{r^3}{2R^2} + k\left(\frac{R^2 - r^2}{2\left(2r^2 - R^2\right)^{\frac{7}{4}}}\right) \tag{3.37c}$$

$$p_r = -\frac{3}{R^2} + k\left(\frac{3r^2 - R^2}{r^3\left(2r^2 - R^2\right)^{\frac{7}{4}}}\right) \tag{3.37d}$$

$$p_\perp = -\frac{3}{R^2} - \frac{k}{2}\left(\frac{3r^2 - R^2}{r^3\left(2r^2 - R^2\right)^{\frac{7}{4}}}\right) \tag{3.37e}$$

with the line element

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1 - \frac{r^2}{R^2}\right)\mathrm{d}t^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1}\left(1 + \frac{kR^2}{r\left(R^2 - 2r^2\right)^{\frac{7}{4}}}\right)^{-1}\mathrm{d}r^2 \\ &\quad + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \end{aligned} \tag{3.38}$$

The isotropic de Sitter model (3.33) gives the anisotropic de Sitter model (3.38) where (3.37) is the new exact solution by Algorithm $\mathcal{B}$.

Chapter 4

Anisotropic Isothermal Sphere

4.1 Introduction

Isothermal structures feature in many physical phenomena in general relativity and astrophysics. In this chapter we generate two new exact solutions to the Einstein field equations for anisotropic isothermal spheres utilising Algorithm $\mathcal{A}$ and Algorithm $\mathcal{B}$ established in Chapter 3. The isotropic isothermal model is presented in §4.2. The isotropic isothermal model is characterised by a linear barotropic equation of state and the energy density $\mu \propto r^{-2}$. Algorithm $\mathcal{A}$ is used to generate a *new* solution to the anisotropic Einstein field equations in §4.3. Another *new* solution is obtained with the help of Algorithm $\mathcal{B}$ in §4.4 for anisotropic matter. The anisotropic factor $S(r)$ for each of the new solutions is plotted and the physical implications of their behaviour are briefly discussed. Both anisotropic solutions found reduce to the conventional isotropic isothermal sphere in the appropriate limit. The solutions are given in closed form and are expressed in terms of simple elementary functions.

4.2 Isotropic Model

The line element for the isothermal model (Saslaw *et al* 1996) is

$$ds^2 = -r^{\frac{4c}{1+c}}dt^2 + \left(1 + \frac{4c}{(1+c)^2}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.1}$$

where c is a constant. The relevant isotropic functions for (4.1) are

$$\nu_0 = \frac{4c}{1+c}\ln r \tag{4.2a}$$

$$\lambda_0 = \ln\left\{1 + \frac{4c}{1+c}\right\} \tag{4.2b}$$

$$m_0 = \frac{2cr}{4c + (1+c)^2} \tag{4.2c}$$

$$p_0 = \frac{1}{r^2}\frac{4c^2}{4c + (1+c)^2} \tag{4.2d}$$

The energy density function, that generates the mass function (4.2c), has the form

$$\mu_0 = \frac{4c}{4c + (1+c)^2}\frac{1}{r^2} \tag{4.3}$$

From (4.3) and (4.2d) we observe that

$$p_0 = c\mu_0 \tag{4.4}$$

which is a linear barotropic equation of state. Isothermal spheres with $\mu \propto r^{-2}$ and the equation of state (4.4) arise in both Newtonian and relativistic stars. They have a long history in astrophysics as an equilibrium approximation to more complicated systems which are close to a dynamically relaxed state (Saslaw 1985).

4.3 Algorithm $\mathcal{A}$

The integrals I_α and I_β for Algorithm $\mathcal{A}$ respectively become

$$
\begin{aligned}
I_\alpha &= \frac{4c}{1+c}\int \frac{\mathrm{d}r}{r} \\
&= \frac{4c}{1+c}\ln r && (4.5) \\
I_\beta &= \int \left(1+\frac{4c}{(1+c)^2}\right) r^{-\frac{2+6c}{1+c}}\mathrm{d}r \\
&= -\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+5c}{1+c}} && (4.6)
\end{aligned}
$$

for the functions (4.2). Then (3.12) gives

$$
\begin{aligned}
\alpha &= k\left(\ell - \frac{k}{2}\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+5c}{1+c}}\right)^{-1} r^{-\frac{3+7c}{1+c}} \\
\beta &= 2\ln\left\{\ell - \frac{k}{2}\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+5c}{1+c}}\right\}
\end{aligned}
$$

Hence the new anisotropic solution to the field equations is

$$
\begin{aligned}
\nu &= \frac{4c}{1+c}\ln r + 2\ln\left\{\ell - \frac{k}{2}\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+5c}{1+c}}\right\} && (4.7a) \\
\lambda &= \ln\left\{1+\frac{4c}{(1+c)^2}\right\} && (4.7b) \\
m &= \frac{2cr}{4c+(1+c)^2} && (4.7c) \\
p_r &= \frac{1}{r^2}\frac{4c^2}{4c+(1+c)^2} + k\left(\ell - \frac{k}{2}\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+5c}{1+c}}\right)^{-1} r^{-\frac{3+7c}{1+c}} && (4.7d) \\
p_\perp &= \frac{1}{r^2}\frac{4c^2}{4c+(1+c)^2} - \frac{k}{2}\left(\ell - \frac{k}{2}\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+5c}{1+c}}\right)^{-1} r^{-\frac{3+7c}{1+c}} && (4.7e)
\end{aligned}
$$

with the corresponding line element

$$
\begin{aligned}
\mathrm{d}s^2 &= -r^{\frac{4c}{1+c}}\left(\ell - \frac{k}{2}\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+5c}{1+c}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{4c}{(1+c)^2}\right)\mathrm{d}r^2 \\
&\quad + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) && (4.8)
\end{aligned}
$$

for the equations (4.7).

The isotropic isothermal sphere model (4.1) generates the anisotropic isothermal sphere model (4.8). With the parameter values

$$
\begin{aligned}
k &= 0 \\
\ell &= 1
\end{aligned}
$$

we regain the conventional isothermal sphere. The degree of anisotropy is

$$
S = \frac{k}{2}\sqrt{3}\left(\ell - \frac{k}{2}\frac{4c+(1+c)^2}{(1+c)(1+5c)}r^{-\frac{1+5c}{1+c}}\right)^{-1} r^{-\frac{3+7c}{1+c}} \tag{4.9}
$$

A graph of the anisotropy factor (4.9) was plotted with the help of Mathematica® 5 (Wolfram 2003). This is given in Figure 4.1 for the particular values of the parameters shown. The anisotropy factor S is plotted against the radial distance on the interval $0 < r \leq 1$. It is worth noting from this graph that the anisotropy has the feature that it is a monotonically decreasing function as r approaches the boundary subject to a particular choice of parameters. There is a singularity at $r = 0$ which S shares with the other dynamical and metric functions. However there are other choices of parameters that could be made such that S is a monotonically increasing function if the physics of the problem demanded such behaviour. We have provided an illustration of such a profile in Figure 4.2 for a particular choice of parameters. The profile for S in Figure 4.2 could actually turn out to be physically more relevant for boson stars as pointed out by Dev & Gleiser (2002); however note that their analysis was performed for a constant energy density. The monotonically increasing profile will also fit in well with the physical constraint of Dev & Gleiser (2002) that p_r and $p_\perp$ should coincide at the centre of the stellar body, and that $p_\perp$ is not subjected to any constraint at the boundary. The simple behaviour of S reflected in these two graphs indicates that a full physical investigation of this solution is possible; we will perform this investigation in future work.

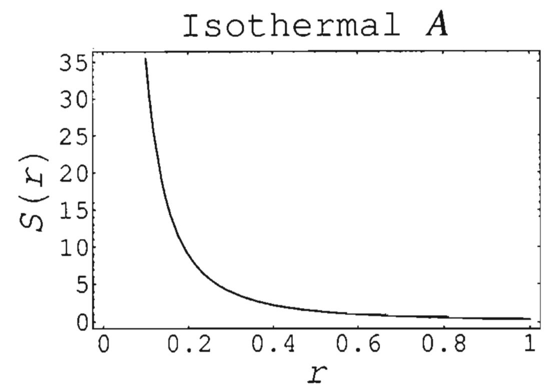

Figure 4.1: $S(r)$ for anisotropic isothermal sphere (Alg. $\mathcal{A}$); $c = -1.1$, $k = 1$, and $\ell = 0.2$

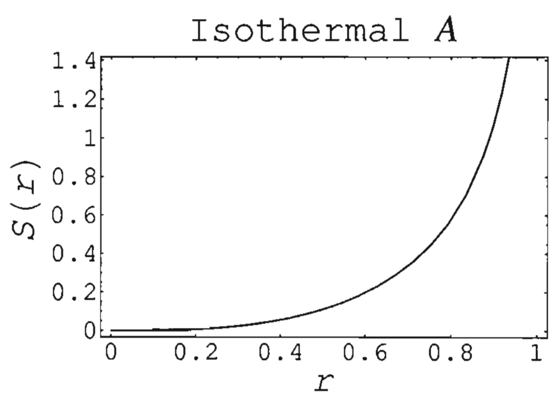

Figure 4.2: $S(r)$ for anisotropic isothermal sphere (Alg. $\mathcal{A}$); $c = -0.6$, $k = 1$, and $\ell = 1$

4.4 Algorithm $\mathcal{B}$

The integral J_α for Algorithm $\mathcal{B}$ is

$$\begin{aligned} J_\alpha &= -\int \left(\frac{3+14c+19c^2}{1+4c+3c^2} \right) \frac{\mathrm{d}r}{r} \\ &= -\frac{3+14c+19c^2}{1+4c+3c^2} \ln r \end{aligned}$$

for the function (4.2a). Then (3.24) gives

$$\begin{aligned} \alpha &= kr^{-\frac{3+14c+19c^2}{1+4c+3c^2}} \\ x &= -\ln\left\{ 1 + k\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+6c+13c^2}{1+4c+3c^2}} \right\} \\ y &= -\frac{k}{2}\left(\frac{1+c}{1+5c} \right) r^{-\frac{2c+10c^2}{1+4c+3c^2}} \end{aligned}$$

Hence the new anisotropic solution to the field equations is

$$\nu = \frac{4c}{1+c} \ln r \tag{4.10a}$$

$$\lambda = \ln\left\{ \frac{(1+c)^2}{4+(1+c)^2} \right\} - \ln\left\{ 1 + k\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+6c+13c^2}{1+4c+3c^2}} \right\} \tag{4.10b}$$

$$m = \frac{2cr}{4c+(1+c)^2} - \frac{k}{2}\left(\frac{1+c}{1+5c} \right) r^{-\frac{2c+10c^2}{1+4c+3c^2}} \tag{4.10c}$$

$$p_r = \frac{1}{r^2}\left(\frac{4c^2}{4c+(1+c)^2} \right) + kr^{-\frac{3+14c+19c^2}{1+4c+3c^2}} \tag{4.10d}$$

$$p_\perp = \frac{1}{r^2}\left(\frac{4c^2}{4c+(1+c)^2} \right) - \frac{k}{2} r^{-\frac{3+14c+19c^2}{1+4c+3c^2}} \tag{4.10e}$$

with the line element

$$\begin{aligned} \mathrm{d}s^2 = {} & -r^{\frac{4c}{1+c}}\mathrm{d}t^2 + \left(1 + \frac{4c}{(1+c)^2} \right) \left(1 + k\frac{4c+(1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+6c+13c^2}{1+4c+3c^2}} \right)^{-1} \mathrm{d}r^2 \\ & + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \end{aligned} \tag{4.11}$$

for the equations (4.10).

The isotropic isothermal sphere model (4.1) gives the anisotropic isothermal model (4.11). With the parameter value

$$k = 0$$

we regain the conventional isothermal sphere. The degree of anisotropy is

$$S = \frac{k}{2}\sqrt{3}r^{-\frac{3+14c+19c^2}{1+4c+3c^2}} \tag{4.12}$$

Again a graph of the anisotropy factor (4.12) was plotted with the help of Mathematica® 5 (Wolfram 2003). This plot is shown in Figure 4.3 for the chosen particular values of the parameters. The anisotropy factor S plotted against the radial distance on the interval $0 < r \leq 1$ (note that Figures 4.1 and 4.3 yield similar graphs; this is not surprising as they both contain the isotropic isothermal sphere). We again note that subject to the choice of the parameters, the anisotropy is a monotonically decreasing function as r approaches the boundary. A different selection of parameters produces a monotonically increasing function as given in Figure 4.4. The profile in Figure 4.4 of the anisotropy factor S is consistent with the behaviour of boson stars as illustrated by Dev & Gleiser (2002) for a constant energy density. A complete analysis of the physical features of this model will be pursued in future work.

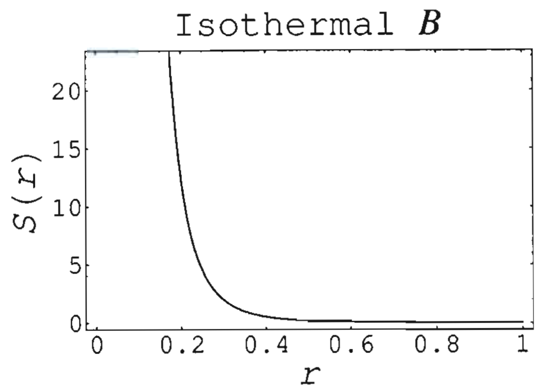

Figure 4.3: $S(r)$ for anisotropic isothermal sphere (Alg. $\mathcal{B}$); $c = 1$ and $k = 0.01$

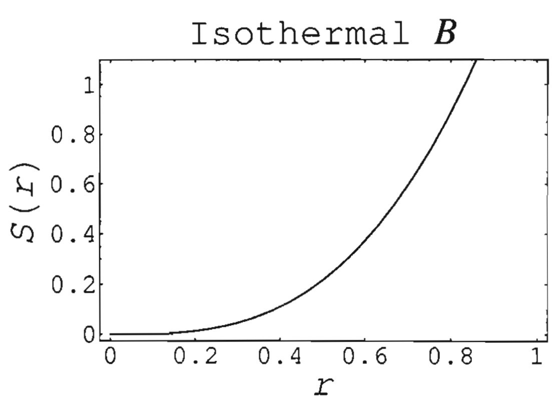

Figure 4.4: $S(r)$ for anisotropic isothermal sphere (Alg. $\mathcal{B}$); $c = -0.5$ and $k = 2$

Chapter 5

Anisotropic Schwarzschild Sphere

5.1 Introduction

With the isotropic Schwarzschild sphere being one of the widely studied models with astrophysical relevance, often used as a benchmark for analytical models, we felt it would be justified to apply the new algorithms, Algorithm $\mathcal{A}$ and Algorithm $\mathcal{B}$ (presented in Chapter 3), to the interior Schwarzschild sphere solution. In §5.2 the isotropic Schwarzschild sphere solution is given. The Schwarzschild interior solution is important for the description of dense, nearly uniform stars. Algorithm $\mathcal{A}$ is used to obtain a *new* anisotropic Schwarzschild interior solution in §5.3. In §5.4 another *new* solution is generated with the help of Algorithm $\mathcal{B}$ for anisotropic matter. Plots of the respective anisotropy factors $S(r)$ are shown in each case and physical implications of their behaviour are briefly discussed. Both anisotropic solutions found reduce to the isotropic interior Schwarzschild sphere in the relevant limit. Note that the solutions are presented in closed form and are expressed in terms of simple elementary functions.

5.2 Isotropic Model

The line element for the interior Schwarzschild model is

$$ds^2 = -\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)^2 dt^2 + \left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (5.1)$$

where A and B are constants. The relevant isotropic functions for (5.1) are

$$\nu_0 = 2\ln\left\{A - B\sqrt{1-\frac{r^2}{R^2}}\right\} \quad (5.2a)$$

$$\lambda_0 = -\ln\left\{1-\frac{r^2}{R^2}\right\} \quad (5.2b)$$

$$m_0 = \frac{r^3}{2R^2} \quad (5.2c)$$

$$p_0 = -\frac{1}{R^2}\left(\frac{A - 3B\sqrt{1-\frac{r^2}{R^2}}}{A - B\sqrt{1-\frac{r^2}{R^2}}}\right) \quad (5.2d)$$

The energy density function, that generates the mass function (5.2c), has the form

$$\mu_0 = \frac{3}{R^2} \quad (5.3)$$

From (5.3) we observe that

$$\mu_0 = \text{constant} \quad (5.4)$$

which essentially replaces the equation of state (4.4) of Chapter 4. The interior of dense neutron stars and superdense relativistic stars are of near uniform density (Maharaj & Leach 1996, Rhoades & Ruffini 1974). Consequently the assumption (5.4) of uniform energy density is often made in the modelling process (Bowers & Liang 1974, Dev & Gleiser 2002, Maharaj & Maartens 1989).

5.3 Algorithm $\mathcal{A}$

The integral I_α for Algorithm $\mathcal{A}$ becomes

$$\begin{aligned}
I_\alpha &= \frac{2Br/R^2}{\left(1-\frac{r^2}{R^2}\right)\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)}\mathrm{d}r \\
&= \int\left(\frac{2Br/R^2}{\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)\sqrt{1-\frac{r^2}{R^2}}}+\frac{2r/R^2}{1-\frac{r^2}{R^2}}\right)\mathrm{d}r \\
&= 2\ln\left\{A-B\sqrt{1-\frac{r^2}{R^2}}\right\}-\ln\left\{1-\frac{r^2}{R^2}\right\}
\end{aligned}$$

which was easily integrated. However the integral I_β takes the form

$$I_\beta = \int\frac{1}{r^2\sqrt{1-\frac{r^2}{R^2}}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^2}\mathrm{d}r \tag{5.5}$$

We need to investigate two cases: $A \neq B$ and $A = B$.

Case I: $A \neq B$

To carry out the integration in (5.5) for $A \neq B$ we make the substitution

$$\sin\vartheta = \frac{r}{R}$$

so that I_β becomes

$$\begin{aligned}
I_\beta &= \frac{1}{R}\int\frac{1}{\sin^2\vartheta(A-B\cos\vartheta)^2}\mathrm{d}\vartheta \\
&= \frac{1}{R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\frac{\vartheta}{2}}{\sqrt{B^2-A^2}}\right\}\right. \\
&\quad \left.-\left(2AB+A^2\cos\vartheta+B^2\cos\vartheta\right)\csc\vartheta-\frac{B^3\sin\vartheta}{A-B\cos\vartheta}\right)
\end{aligned}$$

in terms of ϑ. In terms of the original radial coordinate r we have

$$
\begin{aligned}
I_\beta &= \frac{1}{R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right. \\
&\quad \left.-\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\cos\left\{\sin^{-1}\frac{r}{R}\right\}\right)-\frac{B^3r}{R\left(A-B\cos\left\{\sin^{-1}\frac{r}{R}\right\}\right)}\right) \\
&= \frac{1}{R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right. \\
&\quad \left.-\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right)-\frac{B^3r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)
\end{aligned}
$$

Then (3.12) gives

$$
\begin{aligned}
\alpha &= \frac{k\sqrt{1-\frac{r^2}{R^2}}}{r^3\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^2} \\
&\quad \times\left[\frac{k}{2R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right.\right. \\
&\quad -\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right) \\
&\quad \left.\left.-\frac{B^3r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)+\ell\right]^{-1} \\
\beta &= 2\ln\left\{\frac{k}{2R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right.\right. \\
&\quad -\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right) \\
&\quad \left.\left.-\frac{B^3r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)+\ell\right\}
\end{aligned}
$$

for the anisotropic functions α and β.

Hence the new anisotropic solution, with $A \neq B$, to the field equations is

$$\begin{aligned}
\nu &= 2\ln\left\{A - B\sqrt{1-\frac{r^2}{R^2}}\right\} \\
&\quad +2\ln\left\{\frac{k}{2R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right.\right. \\
&\quad -\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right) \\
&\quad \left.\left. -\frac{B^3 r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)+\ell\right\} \qquad (5.6a) \\
\lambda &= -\ln\left\{1-\frac{r^2}{R^2}\right\} \qquad (5.6b) \\
m &= \frac{r^3}{2R^2} \qquad (5.6c) \\
p_r &= -\frac{1}{R^2}\left(\frac{A-3B\sqrt{1-\frac{r^2}{R^2}}}{A-B\sqrt{1-\frac{r^2}{R^2}}}\right)+\frac{k\sqrt{1-\frac{r^2}{R^2}}}{r^3\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^2} \\
&\quad \times\left[\frac{k}{2R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right.\right. \\
&\quad -\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right) \\
&\quad \left.\left. -\frac{B^3 r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)+\ell\right]^{-1} \qquad (5.6d) \\
p_\perp &= -\frac{1}{R^2}\left(\frac{A-3B\sqrt{1-\frac{r^2}{R^2}}}{A-B\sqrt{1-\frac{r^2}{R^2}}}\right)-\frac{k\sqrt{1-\frac{r^2}{R^2}}}{2r^3\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^2} \\
&\quad \times\left[\frac{k}{2R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right.\right. \\
&\quad -\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right) \\
&\quad \left.\left. -\frac{B^3 r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)+\ell\right]^{-1} \qquad (5.6e)
\end{aligned}$$

with the corresponding line element

$$
\begin{aligned}
\mathrm{d}s^2 &= -\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)^2 \\
&\quad \times \left[\frac{k}{2R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right.\right. \\
&\quad -\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right) \\
&\quad \left.\left. -\frac{B^3 r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)+\ell\right]^2 \mathrm{d}t^2 \\
&\quad +\left(1-\frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)
\end{aligned} \tag{5.7}
$$

for the equations (5.6).

With the parameter values

$$
\begin{aligned}
k &= 0 \\
\ell &= 1
\end{aligned}
$$

we regain the original interior Schwarzschild sphere. The degree of anisotropy is

$$
\begin{aligned}
S &= \frac{k\sqrt{3}\sqrt{1-\frac{r^2}{R^2}}}{2r^3\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^2} \\
&\quad \times \left[\frac{k}{2R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right.\right. \\
&\quad -\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right) \\
&\quad \left.\left. -\frac{B^3 r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)+\ell\right]^{-1}
\end{aligned} \tag{5.8}
$$

The graph of the anisotropy factor (5.8) was plotted with the assistance of Mathematica® 5 (Wolfram 2003). This is shown in Figure 5.1 for the particular values of the parameters. The interval for the plot of S against r is $0 < r \leq 1$. The quantity S is a monotonically decreasing function. Subject to the choice of the parameters, the anisotropy can be constructed such that it is a monotonically decreasing function as r approaches the boundary. The function S vanishes at the boundary. Other choices of the parameters A, B, k and ℓ may generate different behaviour for S. The singularity at the centre does not seem to be 'removable' by any choice of parameters (as was the case with Figures 4.2 and 4.4 presented in the previous chapter). A comparison of this case with Dev & Gleiser (2002) is not possible as the anisotropy factor vanishes at the boundary $r = R$. This will be investigated further and a detailed analysis of the anisotropy factor S and the dynamical variables for this anisotropic solution will be pursued in the future.

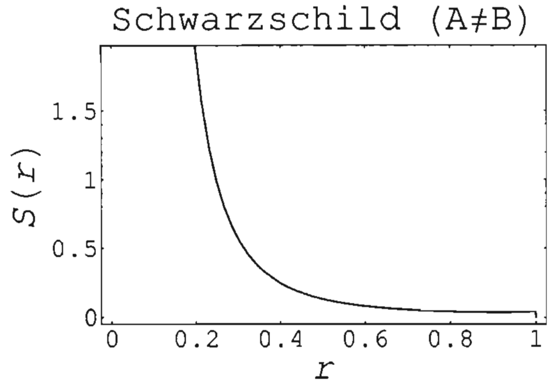

Figure 5.1: $S(r)$ for anisotropic Schwarzschild sphere (Alg. $\mathcal{A}$ $(A \neq B)$); $A = 1$, $B = -10$, $k = 2$, $\ell = 1$ and $R = 1$.

Case II: $A = B$

The integral I_α when $A = B$ becomes

$$I_\alpha = 2\ln\left\{1 - \sqrt{1 - \frac{r^2}{R^2}}\right\} - \ln\left\{1 - \frac{r^2}{R^2}\right\}$$

To carry out the integration in (5.5) with $A = B$ we again make the substitution

$$\sin\vartheta = \frac{r}{R}$$

to have

$$\begin{aligned} I_\beta &= \int \frac{1}{r^2\sqrt{1-\frac{r^2}{R^2}}\left(1-\sqrt{1-\frac{r^2}{R^2}}\right)^2}\mathrm{d}r \\ &= \frac{1}{R}\int \frac{1}{\sin^2\vartheta\,(1-\cos\vartheta)^2}\mathrm{d}\vartheta \\ &= -\frac{1}{80R}\csc^5\frac{\vartheta}{2}\sec\frac{\vartheta}{2}\,(5\cos\vartheta - 4\cos 2\vartheta + \cos 3\vartheta) \end{aligned}$$

The terms $\cos 2\vartheta$ and $\cos 3\vartheta$ can be simplified with basic trigonometric identities and I_β, in terms of the original radial coordinate r, becomes

$$\begin{aligned} I_\beta &= -\frac{1}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\} \\ &\quad\times\left(5\sqrt{1-\frac{r^2}{R^2}} - 4\left(1-\frac{r^2}{R^2}-\frac{r^2}{R^2}\right) + \sqrt{1-\frac{r^2}{R^2}}\left(1-\frac{r^2}{R^2}-3\frac{r^2}{R^2}\right)\right) \\ &= -\frac{1}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\} \\ &\quad\times\left(\left(6-\frac{4r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}} - 4\left(1-\frac{2r^2}{R^2}\right)\right) \end{aligned}$$

Then (3.12) gives

$$
\begin{aligned}
\alpha &= \frac{k\sqrt{1-\frac{r^2}{R^2}}}{r^3\left(1-\sqrt{1-\frac{r^2}{R^2}}\right)^2}\left[\ell-\frac{k}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\right.\\
&\quad\left.\times\left(\left(3-\frac{2r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}}-2\left(1-\frac{2r^2}{R^2}\right)\right)\right]^{-1}\\
\beta &= 2\ln\left\{\ell-\frac{k}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\right.\\
&\quad\left.\times\left(\left(3-\frac{2r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}}-2\left(1-\frac{2r^2}{R^2}\right)\right)\right\}
\end{aligned}
$$

Hence the new anisotropic Schwarzschild sphere solution, with $A=B$, to the field equations is

$$
\begin{aligned}
\nu &= 2\ln\left\{1-\sqrt{1-\frac{r^2}{R^2}}\right\}\\
&\quad+2\ln\left\{\ell-\frac{k}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\right.\\
&\quad\left.\times\left(\left(3-\frac{2r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}}-2\left(1-\frac{2r^2}{R^2}\right)\right)\right\} &\text{(5.9a)}\\
\lambda &= -\ln\left\{1-\frac{r^2}{R^2}\right\} &\text{(5.9b)}\\
m &= \frac{r^3}{2R^2} &\text{(5.9c)}\\
p_r &= -\frac{1}{R^2}\left(\frac{1-3\sqrt{1-\frac{r^2}{R^2}}}{1-\sqrt{1-\frac{r^2}{R^2}}}\right)\\
&\quad+\frac{k\sqrt{1-\frac{r^2}{R^2}}}{r^3\left(1-\sqrt{1-\frac{r^2}{R^2}}\right)^2}\left[\ell-\frac{k}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\right.\\
&\quad\left.\times\left(\left(3-\frac{2r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}}-2\left(1-\frac{2r^2}{R^2}\right)\right)\right]^{-1} &\text{(5.9d)}
\end{aligned}
$$

$$
\begin{aligned}
p_{\perp} &= -\frac{1}{R^2}\left(\frac{1-3\sqrt{1-\frac{r^2}{R^2}}}{1-\sqrt{1-\frac{r^2}{R^2}}}\right) \\
&\quad -\frac{k\sqrt{1-\frac{r^2}{R^2}}}{2r^3\left(1-\sqrt{1-\frac{r^2}{R^2}}\right)^2}\left[\ell-\frac{k}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\right. \\
&\quad \left.\times\left(\left(3-\frac{2r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}}-2\left(1-\frac{2r^2}{R^2}\right)\right)\right]^{-1} \qquad (5.9e)
\end{aligned}
$$

with the corresponding line element

$$
\begin{aligned}
\mathrm{d}s^2 &= -\left(1-\sqrt{1-\frac{r^2}{R^2}}\right)^2 \\
&\quad \times\left[\ell-\frac{k}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\right. \\
&\quad \left.\times\left(\left(3-\frac{2r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}}-2\left(1-\frac{2r^2}{R^2}\right)\right)\right]^2 \mathrm{d}t^2 \\
&\quad +\left(1-\frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \qquad (5.10)
\end{aligned}
$$

for equations (5.9).

With the parameter values

$$
\begin{aligned}
k &= 0 \\
\ell &= 1
\end{aligned}
$$

we regain the original interior Schwarzschild sphere (5.1) with $A = B$. The degree of anisotropy is

$$
\begin{aligned}
S &= \frac{k\sqrt{3}\sqrt{1-\frac{r^2}{R^2}}}{2r^3\left(1-\sqrt{1-\frac{r^2}{R^2}}\right)^2}\left[\ell-\frac{k}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\right. \\
&\quad \left.\times\left(\left(3-\frac{2r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}}-2\left(1-\frac{2r^2}{R^2}\right)\right)\right]^{-1} \qquad (5.11)
\end{aligned}
$$

The graph of the anisotropy factor (5.11) was plotted with the assistance of Mathematica® 5 (Wolfram 2003). This is shown in Figure 5.2 for the particular values of the parameters. The interval for the plot of S against r is $0 < r \leq 1$. The quantity S is a monotonically decreasing function. The behaviour of S is similar to the case $A \neq B$ given in Figure 5.1. However observe that the behaviour in Figure 5.2 is more restricted for *Case II* as A and B are fixed.

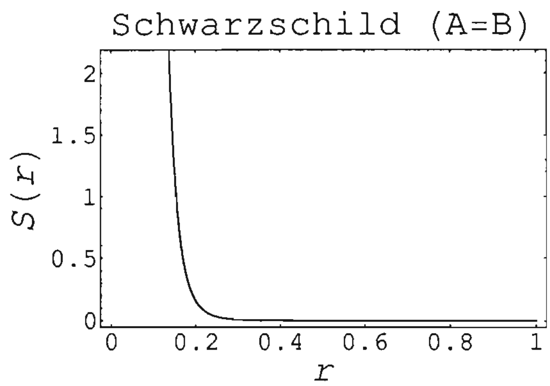

Figure 5.2: $S(r)$ for anisotropic Schwarzschild sphere (Alg. $\mathcal{A}$ $(A = B)$); $k = 0.00006$, $\ell = 100$ and $R = 1$

5.4 Algorithm $\mathcal{B}$

The integral J_α for Algorithm $\mathcal{B}$ is

$$
\begin{aligned}
J_\alpha \;=\; \int \Bigg\{ \Bigg[& 2Br \left(3 - r^2 \left(\frac{2Br^2}{R^4 \left(\sqrt{1-\frac{r^2}{R^2}}\right)^3 \left(A + B\sqrt{1-\frac{r^2}{R^2}}\right)} \right.\right. \\
& \left. - \frac{2B^2r^2}{R^4\left(1-\frac{r^2}{R^2}\right)\left(A + B\sqrt{1-\frac{r^2}{R^2}}\right)^2} + \frac{2B}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A + B\sqrt{1-\frac{r^2}{R^2}}\right)} \right) \\
& \left. + \frac{4Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A + B\sqrt{1-\frac{r^2}{R^2}}\right)} \right) \Bigg] \Bigg/ \\
& \left[R^2\sqrt{1-\frac{r^2}{R^2}}\left(A + B\sqrt{1-\frac{r^2}{R^2}}\right) \left(1 + \frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A + B\sqrt{1-\frac{r^2}{R^2}}\right)} \right) \right. \\
& \left. \times \left(2 + \frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A + B\sqrt{1-\frac{r^2}{R^2}}\right)} \right) \right] \\
& - \left(\frac{6}{r} + \frac{2Br}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)} \right) \\
& \times \left(1 + \frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)} \right) \\
& \times \left(2 + \frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)} \right)^{-1} \Bigg\} \, \mathrm{d}r
\end{aligned}
$$

With the substitution

$$u \;=\; \sqrt{1-\frac{r^2}{R^2}}$$

this integral becomes

$$
\begin{aligned}
J_\alpha &= \int \left(\frac{(B+3Au-4Bu^2)\,(2B+Au-3Bu^2)}{(1-u^2)\,(A-Bu)\,(B+Au-2Bu^2)} \right. \\
&\quad - \frac{3Bu^2\,(A-Bu)}{(2B+Au-3Bu^2)\,(B+Au-2Bu^2)} \\
&\quad \left. + \frac{2B^2\,(1-u^2)\,(A-2Bu-2Au^2+3Bu^3)}{u\,(A-Bu)\,(2B+Au-3Bu^2)\,(B+Au-2Bu^2)} \right) du
\end{aligned}
$$

The above integral is very complicated but can be simplified with the help of partial fractions. We obtain

$$
\begin{aligned}
J_\alpha &= \int \left(\frac{1}{u} + \frac{3}{2\,(1-u)} - \frac{3}{2\,(1+u)} + \frac{B}{A-Bu} \right. \\
&\quad \left. + \frac{A-6Bu}{2B+Au-3Bu^2} - \frac{2A-7Bu}{B+Au-2Bu^2} \right) du \\
&= \ln u - \frac{3}{2}\ln\{1-u\} - \frac{3}{2}\ln\{1+u\} - \ln\{A-Bu\} \\
&\quad + \ln\left\{2B+Au-3Bu^2\right\} \\
&\quad - \int \left(\frac{A/B}{\frac{A^2+8B^2}{16B^2} - \left(u-\frac{A}{4B}\right)^2} - \left(\frac{7}{2}\right) \frac{u-\frac{A}{4B}}{\frac{A^2+8B^2}{16B^2} - \left(u-\frac{A}{4B}\right)^2} \right) du \\
&= \ln\left\{ \frac{u\,(2B+Au-3Bu^2)}{(A-Bu)\,(1-u^2)^{\frac{3}{2}}} \right\} \\
&\quad - \frac{A}{4\sqrt{A^2+8B^2}} \ln\left\{ \frac{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(u-\frac{A}{4B}\right)}{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(u-\frac{A}{4B}\right)} \right\} \\
&\quad - \frac{7}{4}\ln\left\{B+Au-2Bu^2\right\} \\
&= \ln\left\{ \frac{u\,(2B+Au-3Bu^2)}{(A-Bu)\,(1-u^2)^{\frac{3}{2}}\,(B+Au-2Bu^2)^{\frac{7}{4}}} \right\} \\
&\quad + \frac{A}{4\sqrt{A^2+8B^2}} \ln\left\{ \frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(u-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(u-\frac{A}{4B}\right)} \right\}
\end{aligned}
$$

We have completed the integration and obtained J_α in terms of the intermediate variable u. In terms of the original variable r we can write J_α as

$$J_\alpha = \ln\left\{\frac{\sqrt{1-\frac{r^2}{R^2}}\left(2B+A\sqrt{1-\frac{r^2}{R^2}}-3B\left(1-\frac{r^2}{R^2}\right)\right)}{\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)\left(1-\left(1-\frac{r^2}{R^2}\right)\right)^{\frac{3}{2}}\left(B+A\sqrt{1-\frac{r^2}{R^2}}-2B\left(1-\frac{r^2}{R^2}\right)\right)^{\frac{7}{4}}}\right\}$$
$$+\frac{A}{4\sqrt{A^2+8B^2}}\ln\left\{\frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}\right\}$$

Then the function α in (3.24a) becomes

$$\begin{aligned}\alpha &= \frac{kR^3\sqrt{1-\frac{r^2}{R^2}}}{r^3\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)}\left(2B+A\sqrt{1-\frac{r^2}{R^2}}-3B\left(1-\frac{r^2}{R^2}\right)\right)\\ &\quad\times\left(B+A\sqrt{1-\frac{r^2}{R^2}}-2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}}\\ &\quad\times\left(\frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}\end{aligned} \tag{5.12a}$$

and (3.24b) and (3.24c) respectively lead to

$$\begin{aligned}x &= -\ln\left\{1+\frac{kR^3}{r\left(1-\frac{r^2}{R^2}\right)}\left(1+\frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)}\right)^{-1}\right.\\ &\quad\times\frac{\sqrt{1-\frac{r^2}{R^2}}}{\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)}\left(2B+A\sqrt{1-\frac{r^2}{R^2}}-3B\left(1-\frac{r^2}{R^2}\right)\right)\\ &\quad\times\left(B+A\sqrt{1-\frac{r^2}{R^2}}-2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}}\\ &\quad\left.\times\left(\frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}\right\}\end{aligned} \tag{5.12b}$$

$$
\begin{aligned}
y &= -\frac{kR^3}{2}\left(1+\frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)}\right)^{-1} \\
&\quad\times\frac{\sqrt{1-\frac{r^2}{R^2}}}{\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)}\left(2B+A\sqrt{1-\frac{r^2}{R^2}}-3B\left(1-\frac{r^2}{R^2}\right)\right) \\
&\quad\times\left(B+A\sqrt{1-\frac{r^2}{R^2}}-2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}} \\
&\quad\times\left(\frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}
\end{aligned} \tag{5.12c}
$$

Hence the new anisotropic solution to field equations is

$$
\nu = 2\ln\left\{A-B\sqrt{1-\frac{r^2}{R^2}}\right\} \tag{5.13a}
$$

$$
\begin{aligned}
\lambda &= -\ln\left\{1-\frac{r^2}{R^2}\right\} \\
&\quad-\ln\left\{1+\frac{kR^3}{r\left(1-\frac{r^2}{R^2}\right)}\left(1+\frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)}\right)^{-1}\right. \\
&\quad\times\frac{\sqrt{1-\frac{r^2}{R^2}}}{\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)}\left(2B+A\sqrt{1-\frac{r^2}{R^2}}-3B\left(1-\frac{r^2}{R^2}\right)\right) \\
&\quad\times\left(B+A\sqrt{1-\frac{r^2}{R^2}}-2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}} \\
&\quad\left.\times\left(\frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}}-\frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}\right\}
\end{aligned} \tag{5.13b}
$$

$$
\begin{aligned}
m &= \frac{r^3}{2R^2} - \frac{kR^3}{2}\left(1 + \frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)}\right)^{-1} \\
&\times \frac{\sqrt{1-\frac{r^2}{R^2}}}{\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)}\left(2B + A\sqrt{1-\frac{r^2}{R^2}} - 3B\left(1-\frac{r^2}{R^2}\right)\right) \\
&\times \left(B + A\sqrt{1-\frac{r^2}{R^2}} - 2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}} \\
&\times \left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}
\end{aligned} \tag{5.13c}
$$

$$
\begin{aligned}
p_r &= -\frac{1}{R^2}\left(\frac{A - 3B\sqrt{1-\frac{r^2}{R^2}}}{A - B\sqrt{1-\frac{r^2}{R^2}}}\right) + \frac{kR^3\sqrt{1-\frac{r^2}{R^2}}}{r^3\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)} \\
&\times \left(2B + A\sqrt{1-\frac{r^2}{R^2}} - 3B\left(1-\frac{r^2}{R^2}\right)\right) \\
&\times \left(B + A\sqrt{1-\frac{r^2}{R^2}} - 2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}} \\
&\times \left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}
\end{aligned} \tag{5.13d}
$$

$$
\begin{aligned}
p_\perp &= -\frac{1}{R^2}\left(\frac{A - 3B\sqrt{1-\frac{r^2}{R^2}}}{A - B\sqrt{1-\frac{r^2}{R^2}}}\right) - \frac{kR^3\sqrt{1-\frac{r^2}{R^2}}}{2r^3\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)} \\
&\times \left(2B + A\sqrt{1-\frac{r^2}{R^2}} - 3B\left(1-\frac{r^2}{R^2}\right)\right) \\
&\times \left(B + A\sqrt{1-\frac{r^2}{R^2}} - 2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}} \\
&\times \left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}
\end{aligned} \tag{5.13e}
$$

and with the line element

$$
\begin{aligned}
ds^2 &= -\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)^2 dt^2 \\
&+ \left[1 + \frac{kR^3}{r\left(1-\frac{r^2}{R^2}\right)}\left(1 + \frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)}\right)^{-1}\right. \\
&\times \frac{\sqrt{1-\frac{r^2}{R^2}}}{\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)}\left(2B + A\sqrt{1-\frac{r^2}{R^2}} - 3B\left(1-\frac{r^2}{R^2}\right)\right) \\
&\times \left(B + A\sqrt{1-\frac{r^2}{R^2}} - 2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}} \\
&\times \left.\left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}\right]^{-1} \\
&\times \left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (5.14)
\end{aligned}
$$

for the equations (5.13).

With the parameter value

$$
k = 0
$$

we regain the original interior Schwarzschild sphere. The degree of anisotropy has the form

$$
\begin{aligned}
S &= \frac{kR^3\sqrt{3}\sqrt{1-\frac{r^2}{R^2}}}{2r^3\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)} \\
&\times \left(2B + A\sqrt{1-\frac{r^2}{R^2}} - 3B\left(1-\frac{r^2}{R^2}\right)\right) \\
&\times \left(B + A\sqrt{1-\frac{r^2}{R^2}} - 2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}} \\
&\times \left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}} \qquad (5.15)
\end{aligned}
$$

The graph of the anisotropy factor (5.15) was plotted with the help of Mathematica® 5 (Wolfram 2003). This is presented in Figure 5.3 for chosen particular values of the parameters. The plot of S against r is in the interval $0 < r \leq 1$. The quantity S is a monotonically decreasing function (note that Figures 5.1, 5.2 and 5.3 produce similar graphs; this is not surprising as they all reduce to the isotropic Schwarzschild interior solution). Subject to the choice of the parameters, the anisotropy S can be constructed so that it is a decreasing function as r approaches the boundary. The function S vanishes at the boundary. Other choices of the parameters A, B, and k may produce a different behaviour for S. In future work we intend to fully investigate the behaviour of this solution.

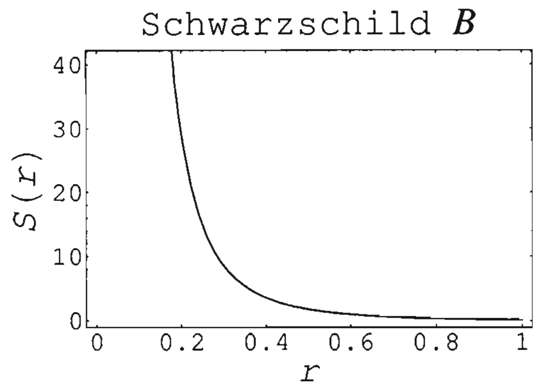

Figure 5.3: $S(r)$ for anisotropic Schwarzschild sphere (Alg. $\mathcal{B}$); $A = 3$, $B = 1$, $k = 1$ and $R = 1$

Conclusion

The main objective of this thesis was to find new solutions to the anisotropic Einstein field equations which can be used to describe a relativistic anisotropic star. We believe that we have met this objective by utilising existing methods and also generating new techniques of integration which we have formulated as algorithms. We demonstrated that the algorithms work with a number of examples of exact solutions that have a simple analytical form and have physical relevance.

In the Introduction we briefly outlined the reasons for studying anisotropic matter in general relativity and applications for relativistic anisotropic stars. In Chapter 1 we provide only those aspects of the Einstein field equations necessary for later work.

In Chapter 2 we applied the Maharaj & Maartens (1989) integration procedure to generate new exact solutions to the anisotropic Einstein field equations. We chose the energy density function to be

$$\mu = \frac{j}{r^2} + k + \ell r^2$$

on physical grounds to produce a class of solutions in terms of elementary funtions. This new category of solutions contains models found previously as special cases. When $j = 0 = \ell$, we regain the Maharaj & Maartens (1989) solution. When $j = 0$, we regain the solution of Gokhroo & Mehra (1994). Six more classes of solution were found utilising the algorithm of Maharaj & Maartens (1989). The energy density for all cases contains the limiting case

$$\mu \ \propto \ r^{-2}$$

which is significant for isothermal spheres in Newtonian and relativistic stars (Saslaw *et al* 1996). Note also that the radial pressure is reducible to

$$p_r \ = \ C\left(1-\frac{r^2}{R^2}\right)^n$$

for appropriate choice of parameters. This form of p_r was first used by Maharaj & Maartens (1989) and ensures $p_r(r=R)=0$. The physical features of the solutions found were briefly considered.

The Maharaj & Maartens (1989) approach yields an exact solution that is anisotropic in general and the anisotropy factor $S \neq 0$ throughout the interior of the star. In Chapter 3, we sought to overcome this difficulty and to produce models that have an isotropic limit. We derived two new algorithms with each producing a new solution to the anisotropic Einstein field equations from a given isotropic solution. In Algorithm $\mathcal{A}$, given an isotropic solution $(\nu_0, \lambda_0, m_0, p_0)$, we generated the anisotropic solution $(\nu_0+\beta, \lambda_0, m_0, p_0+\alpha, p_0-\alpha/2)$ where

$$\begin{aligned}
\alpha &= \frac{k}{r^3}\exp\left\{-\frac{1}{2}\left(I_\alpha+\nu_0+\beta\right)\right\} \\
\beta &= 2\ln\left\{\frac{k}{2}I_\beta+\ell\right\}
\end{aligned}$$

The integrals I_α and I_β have the form

$$\begin{aligned}
I_\alpha &= \int\left(\frac{2m_0'}{r^2}+p_0\right)\frac{r^2}{r-2m_0}\mathrm{d}r \\
I_\beta &= \int\frac{\exp\left\{-\frac{1}{2}\left(I_\alpha+\nu_0\right)\right\}}{r\left(r-2m_0\right)}\mathrm{d}r
\end{aligned}$$

While in Algorithm $\mathcal{B}$, given an isotropic solution $(\nu_0, \lambda_0, m_0, p_0)$, we generated the anisotropic solution $(\nu_0, \lambda_0 + x, m_0 + y, p_0 + \alpha, p_0 - \alpha/2)$ where

$$\begin{aligned}
\alpha &= k e^{J_\alpha} \\
x &= -\ln\left\{1 - \frac{2y}{r}e^{\lambda_0}\right\} \\
y &= -\frac{\alpha r^3}{2(1 + r\nu_0')}
\end{aligned}$$

The integral J_α has the form

$$J_\alpha = \int \left\{ \frac{\nu_0'}{2 + r\nu_0'} \left(\frac{3 + 2r\nu_0' - r^2\nu_0''}{1 + r\nu_0'} \right) - \left(\nu_0' + \frac{6}{r} \right) \left(\frac{1 + r\nu_0'}{2 + r\nu_0'} \right) \right\} \mathrm{d}r$$

We considered two known isotropic solutions, the Einstein and de Sitter models to demonstrate that the algorithms work, and to explicitly show that they indeed generate new anisotropic solutions.

We generated anisotropic isothermal sphere models in Chapter 4, given the corresponding isotropic solution of the Einstein field equations. Algorithm $\mathcal{A}$ was used on the isotropic isothermal sphere solution to produce the new line element

$$\begin{aligned}
\mathrm{d}s^2 &= -r^{\frac{4c}{1+c}} \left(\ell - \frac{k}{2} \frac{4c + (1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+5c}{1+c}} \right)^2 \mathrm{d}t^2 + \left(1 + \frac{4c}{(1+c)^2} \right) \mathrm{d}r^2 \\
&\quad + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right)
\end{aligned}$$

The conventional isothermal sphere model is regained when $k = 0$ and $\ell = 1$. Algorithm $\mathcal{B}$ was used on the isotropic isothermal solution to produce another anisotropic solution depicted by the line element

$$\begin{aligned}
\mathrm{d}s^2 &= -r^{\frac{4c}{1+c}} \mathrm{d}t^2 + \left(1 + \frac{4c}{(1+c)^2} \right) \left(1 + k \frac{4c + (1+c)^2}{(1+c)(1+5c)} r^{-\frac{1+6c+13c^2}{1+4c+3c^2}} \right)^{-1} \mathrm{d}r^2 \\
&\quad + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right)
\end{aligned}$$

The conventional isothermal sphere is regained when we set $k = 0$. The anisotropy factor S was plotted for both of the above line elements and shown to exhibit a wide range of interesting behaviour.

We generated anisotropic Schwarzschild interior sphere models in Chapter 5, given the corresponding isotropic solution of the Einstein field equations. This was more complicated technically and the integrations were rather involved. Algorithm $\mathcal{A}$ led to the line element

$$
\begin{aligned}
\mathrm{d}s^2 \;=\; & -\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)^2 \\
& \times\left[\frac{k}{2R\left(A^2-B^2\right)^2}\left(\frac{6AB^2}{\sqrt{B^2-A^2}}\tanh^{-1}\left\{\frac{(A+B)\tan\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}}{\sqrt{B^2-A^2}}\right\}\right.\right. \\
& -\frac{R}{r}\left(2AB+\left(A^2+B^2\right)\sqrt{1-\frac{r^2}{R^2}}\right) \\
& \left.\left.-\frac{B^3 r}{R}\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)^{-1}\right)+\ell\right]^2\mathrm{d}t^2 \\
& +\left(1-\frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)
\end{aligned}
$$

for the anisotropic Schwarzschild sphere solution. With the parameter values $k = 0$ and $\ell = 1$ we regain the original isotropic interior Schwarzschild sphere line element. The above line element holds for $A \neq B$. With $A = B$, we generated the metric

$$
\begin{aligned}
\mathrm{d}s^2 \;=\; & -\left(1-\sqrt{1-\frac{r^2}{R^2}}\right)^2 \\
& \times\left[\ell-\frac{k}{80R}\csc^5\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\sec\left\{\frac{1}{2}\sin^{-1}\frac{r}{R}\right\}\right. \\
& \times\left.\left(\left(3-\frac{2r^2}{R^2}\right)\sqrt{1-\frac{r^2}{R^2}}-2\left(1-\frac{2r^2}{R^2}\right)\right)\right]^2\mathrm{d}t^2 \\
& +\left(1-\frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)
\end{aligned}
$$

using Algorithm $\mathcal{A}$. Algorithm $\mathcal{B}$ was used on the isotropic Schwarzschild interior solution to produce another anisotropic solution given by

$$
\begin{aligned}
\mathrm{d}s^2 &= -\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)^2 \mathrm{d}t^2 \\
&+ \left[1 + \frac{kR^3}{r\left(1-\frac{r^2}{R^2}\right)} \left(1 + \frac{2Br^2}{R^2\sqrt{1-\frac{r^2}{R^2}}\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)}\right)^{-1}\right. \\
&\times \frac{\sqrt{1-\frac{r^2}{R^2}}}{\left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)} \left(2B + A\sqrt{1-\frac{r^2}{R^2}} - 3B\left(1-\frac{r^2}{R^2}\right)\right) \\
&\times \left(B + A\sqrt{1-\frac{r^2}{R^2}} - 2B\left(1-\frac{r^2}{R^2}\right)\right)^{-\frac{7}{4}} \\
&\times \left.\left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\sqrt{1-\frac{r^2}{R^2}} - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}\right]^{-1} \\
&\times \left(1-\frac{r^2}{R^2}\right)^{-1} \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)
\end{aligned}
$$

With the parameter $k = 0$ we regain the original isotropic interior Schwarzschild sphere. The anisotropy factor S was plotted for the above three line elements and all showed a monotonically decreasing behaviour with $S(r = R) = 0$.

We hope that we have established that the Maharaj & Maartens (1989) approach and our new algorithms produce valuable new anisotropic solutions. We expect that these exact solutions will lead to physically viable models of anisotropic relativistic stars. We have only briefly considered the physical features in our treatment. In future work we intend to fully study the physical behaviour and stability of these using analyses such as that found in Dev & Gleiser (2003).

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