Realistic Charged Stellar Models

by

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Abstract

In this thesis we seek exact solutions to the isotropic Einstien-Maxwell system that model the interior of relativistic stars. The field equations are transformed to a simpler form using the transformation of Durgapal and Bannerji (1983); the integration of the system is reduced to solving the condition of pressure isotropy. This condition is a recurrence relation with variable rational coefficients which can be solved in general. New classes of solutions of linearly independent functions are obtained in terms of special functions and elementary functions for different spatial geometries. Our results contain models found previously including the superdense Tikekar (1990) neutron star model, the uncharged isotropic Maharaj and Leach (1996) solutions, the Finch and Skea (1989) model and the Durgapal and Bannerji (1983) superdense neutron star. Our general class of solutions also contain charged relativistic spheres found previously, including the model of Hansraj and Maharaj (2006) and the model of Thirukkanesh and Maharaj (2006). In addition, two exact analytical solutions describing the interior of a charged strange quark star are obtained by applying the MIT bag equation of state. We regain the Mak and Harko (2004) solution for a charged quark star as a special case.

To

My wife Inthu, and our little boy Gobeesan,

for all the joy they bring to my life.

Preface and Declaration

The study described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban. This thesis was completed under the supervision of Professor S D Maharaj.

The research contained in this thesis represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

Chapter 1

Introduction

The theory of general relativity provides a very satisfactory explanation of the behaviour of the gravitational field. The predictions of general relativity have been demonstrated to be in harmony with observational data in relativistic astrophysics and cosmology. In general relativity the spacetime is taken to be a four-dimensional, differentiable manifold endowed with a symmetric, nondegenerate metric tensor field. The spacetime geometry of general relativity only locally resembles that of special relativity. However, globally the geometries differ in that the differentiable manifold is not flat. The Riemann tensor describes the curvature of the spacetime manifold. The Einstein tensor, which is obtained via the Riemann tensor, describes the geometry of the gravitational field. The matter content and the electromagnetic contribution of the universe can be treated as a relativistic fluid and is given by the symmetric energy momentum tensor. The influence of the gravitational field on the matter distribution is expressed by the Einstein field equations which are a nonlinear coupled system of partial differential equations, and they are difficult to solve in general. In the presence of an electromagnetic field these equations have to be supplemented by the Einstein-Maxwell equations.

The first exact solution to the Einstein field equations discovered was the exterior Schwarzschild solution (Schwarzschild 1916a) which describes the gravitational field outside a static spherically symmetric body. This solution is essential for a discussion of the classical tests of general relativity (d'Inverno 1992, Wald 1984, Will 1981). There has been a huge effort to find interior solutions which match to the exterior

Schwarzschild solution because they are important in the description of relativistic spheres in astrophysics. The models generated may be used to describe highly compact objects where the gravitational field is strong, as in the case of neutron stars. The first and most famous of these is the Schwarzschild interior solution (Schwarzschild 1916b) which describes the interior gravitational field of a static incompressible fluid sphere. It is a good model of dense stars in which the pressures are not too large. A large number of static, spherically symmetric solutions are known today. For a comprehensive list of known exact solutions, the reader is refered to Stephani *et al* (2003), Finch and Skea (1998) and Delgaty and Lake (1998), among others. Most of these solutions, however, do not stand all the tests of physical reality. Some of the exact solutions, that qualify all the physical requirements include those of Durgapal and Bannerji (1983), Durgapal and Fuloria (1985), Finch and Skea (1989), Tikekar (1990), Maharaj and Leach (1996) and Lake (2003). It is interesting to observe that a set of these isotropic solutions may be used as seed solutions to produce anisotropic solutions to the Einstein field equations with the help of a new algorithm proposed by Maharaj and Chaisi (2006).

The unique exterior metric for a spherically symmetric charged distribution of matter is the Reissner-Nordstrom (Reissner 1916, Nordstrom 1918) solution. This solution reduces to the Schwarzschild exterior solution in the limit of vanishing electromagnetic field. Interior regular charged perfect fluid solutions are far from unique and have been studied by different authors. The original Schwarzschild idea of constant density has also been tested in the charged case for a perfect fluid (Wilson 1969, Mehra and Bohra 1979, de Felice *et al* 1999). Exact solutions of the Einstein-Maxwell field equations are important in the description of relativistic astrophysical processes. These solutions may be utilised to model a charged relativistic star as they are matchable to the Reissner-Nordstrom exterior at the boundary. It is for this reason that many investigators use a variety of techniques to attain exact solutions. A comprehensive list of Einstein-Maxwell solutions, satisfying a variety of criteria for physical admissability, is provided by Ivanov (2002). It is interesting to observe that, in the presence of charge, the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided (Krasinski 1997). In this situation the gravitational attraction is counterbalanced by the repulsive Coulombian force, in addition to the pressure gradient. Einstein-Maxwell solutions are also important in studies involving the cosmic censorship hypothesis and the formation of naked singularities (Joshi 1993). The presence of charge affects the values for redshifts, luminosities, and maximum mass for stars. Consequently the Einstein-Maxwell system, for a charged star, has attracted considerable attention in various physical investigations including those of Patel *et al* (1997) and Sharma *et al* (2001).

In an attempt to generate exact solutions in some cases, Vaidya and Tikekar (1982) proposed that the geometry of the spacelike hypersurfaces generated by $\{t = \text{constant}\}$ is that of the 3-spheroid. This spheroidal condition provides a clear geometrical interpretation which is not the case in many other exact solutions. Knutsen (1984) was the first to consider the pressure gradients of stars with spheroidal geometry and showed that they were negative. Also note that spheroidal geometries exhibit the important physical feature of being stable with respect to radial pulsations (Knutsen 1988). Tikekar (1990) comprehensively studied a particular spheroidal geometry and showed that it could be applied to superdense neutron stars with densities in the range of 10^{14} g cm^{-3}. Maharaj and Leach (1996) found all spheroidal solutions, for uncharged stars, that could be expressed in terms of elementary functions. Mukherjee *et al* (1997) showed that it was possible to express the general solution in terms of Gegenbauer functions; an alternate form of the general solution was found by Gupta and Jasim (2004). These uncharged solutions can be extended to models in the presence of electromagnetic field. Spheroidal models in the presence of an electric field have been extensively studied by Sharma *et at* (2001), Patel and Koppar (1987), Patel *et al* (1997), and Gupta and Kumar (2005). These investigations have been motivated on the grounds that restricting the geometry of the hypersurfaces $\{t = \text{constant}\}$ to be spheroidal produces neutral and charged stars which are consistent with observations for dense astronomical objects. Models with spheroidal geometry can be directly related to particular physical intuitions: the maximum mass is in agreement with values for cold compact objects (Sharma *et al* 2006); values for densities are consistent with strange matter (Tikekar and Jotania 2005); the equation of state is consistent with a compact X-ray binary pulsar Her X-1 (Sharma and Mukherjee 2001); relevance to equation of state for stars compared of quark-diquark mixtures in equilibrium (Sharma and

Mukherjee 2002); and uniform charged dust in equilibrium (Tikekar 1984). Spheroidal geometries are relevant in core-envelope stellar models, core consisting of isotropic fluid and an envelope of anisotropic fluid, as shown by Thomas *et al* (2005), and Tikekar and Thomas (1998). These references provide a sample as to why the Einstein-Maxwell system, describing the interior of a charged star, has attracted the attention of many researchers. In this thesis we are concerned with generating exact solutions to the Einstein-Maxwell system in spheroidal and other spacetime geometries.

In order to integrate the field equations, various restrictions have been placed on the geometry of spacetime and the matter content. Mainly two distinct procedures have been adopted to solve these equations for the spherically symmetric and static manifolds. Firstly, the coupled differential equations are solved by computation after choosing an equation of state. This procedure was first adopted by Oppenheimer and Volkoff (1939). There exist several reviews of the problems associated with an equation of state for nuclear matter and corresponding properties of neutron stars. Secondly, the exact Einstein-Maxwell solution can be obtained by specifying the geometry and the form of the electromagnetic field. We follow the latter technique in an attempt to find solutions in terms of special functions and elementary functions that are suitable for the description of relativistic charged stars.

This thesis is organised as follows:

- Chapter 1: Introduction.
- Chapter 2: In this chapter we present an overview and background material necessary for later Chapters.
- Chapter 3: We specify a physically reasonable choice for the electric field intensity to solve the field equations for the charged Tikekar (1990) model. A general series solution is obtained to the Einstein-Maxwell system using the method of Frobenius. In addition, we obtain polynomials and algebraic functions as solutions by restricting particular parameters. We regain the Maharaj and Leach (1996) solutions in the uncharged limit. We establish a new algorithm which enables us to find new solutions to the field equations from a given seed solution. This algorithm is illustrated with two simple examples. The work contained in

this chapter has been published (Komathiraj and Maharaj 2007a).

- Chapter 4: We find new classes of exact solutions to the Einstein-Maxwell system of equations for a charged sphere with a particular choice of the electric field intensity and one of the gravitational potentials. Solutions are presented in terms of hypergeometric functions and elementary functions. Uncharged solutions are regainable with our choice of electric field; in particular we generate the Einstein universe for particular parameter values. This chapter represents original work and has been submitted for publication (Komathiraj and Maharaj 2007b).

- Chapter 5: In this chapter we present new solutions to the Einstein-Maxwell field equations. The condition of pressure isotropy yields a difference equation with variable, rational coefficients. In an earlier treatment this condition was integrated by first transforming it to a hypergeometric equation. We demonstrate that it is possible to obtain a more general class of solutions. Our result contain models found previously including the neutron star solution of Finch and Skea (1989) and the superdense stellar solution of Durgapal and Bannerji (1983). The results of this Chapter have been submitted for publication (Maharaj and Komathiraj 2007).

- Chapter 6: We find two new classes of exact solutions to the Einstein-Maxwell system of equations. The matter content satisfies a linear equation of state consistent with quark matter; a particular form of one of the gravitational potentials is specified to generate solutions. The exact solutions can be written in terms of elementary functions, and these can be related to quark matter in the presence of an electromagnetic field. The first class of solutions generalises the Mak and Harko (2004) model. The second class of solutions does not admit any singularities in the matter and gravitational potentials at the centre. The results of this chapter have been submitted for publication (Komathiraj and Maharaj 2007c).

- Chapter 7: Conclusion.

Chapter 2

Differential geometry and field equations

2.1 Introduction

In this chapter we briefly review those aspects of differential geometry, the general theory of relativity and electromagnetism crucial to this thesis. A more detailed discussion on differentiable manifolds and tensor analysis is given in d'Inverno (1992), Misner *et al* (1973), Stephani (1990) and Wald (1984). In §2.2 we introduce the concepts of the line element, the metric tensor field and the metric connection on the spacetime manifold. This makes it possible to define the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor. The spacetime manifold is then restricted to be static and spherically symmetric on physical grounds. In §2.3 a covariant formulation of Maxwell's equations is described in a curved background. The matter content and the electromagnetic field are then coupled to the gravitational field by means of the Einstein-Maxwell field equations. In §2.4 we consider the effect of the spacetime geometry on the energy momentum tensor, and evaluate the relevant expressions for both neutral and charged perfect fluids. The field equations are then derived for neutral and charged perfect fluids, in detail, for static spacetimes with spherical geometry. This system is rewritten in an equivalent form by utilising a change of coordinates and redefining two metric functions. The transformed field equations are easier to integrate in particular situations. Finally, in §2.5, the physical conditions required of interior solutions to the field equations are briefly reviewed and the exterior spacetimes for isolated matter distributions are presented.

2.2 Differential geometry

In the general theory of relativity we take spacetime **M** to be a four-dimensional differentiable manifold endowed with the symmetric and nonsingular metric tensor field **g** with signature $(- + + +)$. A manifold with an indefinite metric tensor field, as is the case in general relativity, is termed a pseudo-Riemannian manifold. Points in **M** are labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike and x^1, x^2, x^3 are spacelike. For convenience we use units in which the speed of light $c = 1$. For a rigorous definition of differentiable manifolds and for material on differential geometry the reader is referred to texts by Bishop and Goldberg (1968), de Felice and Clarke (1990), Hawking and Ellis (1973), Misner *et al* (1973) and Wald (1984).

The metric tensor field **g** is fundamental to the invariant definition of the length of a curve in **M**. The invariant distance between neighbouring points is defined by the line element

$$ds^2 = g_{ab} dx^a dx^b \tag{2.2.1}$$

where g_{ab} are the covariant components of **g**. The fundamental theorem of Riemannian geometry guarantees the existence of a unique symmetric connection that preserves the inner product under parallel transport. This is called the metric connection Γ, or the Christoffel symbol of the second kind, which is given by

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2.2}$$

where commas denote partial differentiation. The line element (2.2.1) and the connection coefficients (2.2.2) are the basic building blocks in differential geometry as applied to the spacetime manifold.

The Riemann tensor is a type (1,3) tensor and is defined as

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{2.2.3}$$

in terms of the connection coefficients (2.2.2). The Riemann tensor components R_{abcd}

satisfy the following useful identities

$$
\begin{aligned}
R_{abcd} &= -R_{bacd} \\
&= -R_{abdc} \\
&= R_{cdab} \\
R_{abcd} + R_{acdb} + R_{adbc} &= 0 \\
R_{abcd;e} + R_{abde;c} + R_{abec;d} &= 0
\end{aligned}
$$

These identities assist in calculations that involve the curvature of the manifold and are important in the formulation of the field equations. On contracting the Riemann tensor (2.2.3) we obtain

$$
\begin{aligned}
R_{ab} &= R^{c}{}_{acb} \\
&= \Gamma^{c}{}_{ab,c} - \Gamma^{c}{}_{ac,b} + \Gamma^{c}{}_{dc}\Gamma^{d}{}_{ab} - \Gamma^{c}{}_{db}\Gamma^{d}{}_{ac}
\end{aligned}
\tag{2.2.4}
$$

where R_{ab} is the Ricci tensor which is symmetric. A second contraction of the Riemann tensor (2.2.3), i.e. contraction of the Ricci tensor (2.2.4), yields

$$
\begin{aligned}
R &= g^{ab}R_{ab} \\
&= R^{a}{}_{a}
\end{aligned}
\tag{2.2.5}
$$

where R is the Ricci or curvature scalar.

The Einstein tensor $\boldsymbol{G}$ is constructed in terms of the Ricci tensor (2.2.4) and the Ricci scalar (2.2.5) as follows

$$
G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{2.2.6}
$$

which is necessarily symmetric. A distinguishing feature of the Einstein tensor is that it has zero divergence so that

$$
G^{ab}{}_{;b} = 0 \tag{2.2.7}
$$

The property (2.2.7) of the Einstein tensor is called the contracted Bianchi identity and generates the conservation of energy momentum via the Einstein field equations.

The most general line element (2.2.1) in the case of static, spherically symmetric spacetimes in standard coordinates $(x^a) = (t, r, \theta, \phi)$ is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2.8}$$

where the functions $\nu(r)$ and $\lambda(r)$ are related to the gravitational potentials. The connection coefficients, associated with the line element (2.2.8), are determined from (2.2.2), and the nonvanishing components are given by

$$\Gamma^0{}_{01} = \nu' \qquad \Gamma^2{}_{12} = \tfrac{1}{r}$$

$$\Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)} \qquad \Gamma^2{}_{33} = -\sin\theta\cos\theta$$

$$\Gamma^1{}_{11} = \lambda' \qquad \Gamma^3{}_{13} = \tfrac{1}{r}$$

$$\Gamma^1{}_{22} = -re^{-2\lambda} \qquad \Gamma^3{}_{23} = \cot\theta$$

$$\Gamma^1{}_{33} = -re^{-2\lambda}\sin^2\theta$$

where primes denote differentiation with respect to the radial coordinate r. It is now possible to calculate the Ricci tensor (2.2.4), for the line element (2.2.8), using the above connection coefficients. The Ricci tensor components take the form

$$R_{00} = e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\frac{\nu'}{r}\right) \tag{2.2.9a}$$

$$R_{11} = -\left(\nu'' + \nu'^2 - \nu'\lambda' - 2\frac{\lambda'}{r}\right) \tag{2.2.9b}$$

$$R_{22} = 1 - e^{-2\lambda}(1 + r\nu' - r\lambda') \tag{2.2.9c}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.2.9d}$$

$$R_{ab} = 0, \; a \neq b$$

With the help of the Ricci tensor components (2.2.9) and the definition (2.2.5) we obtain the Ricci scalar

$$R = 2\left[\frac{1}{r^2} - e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\frac{\nu'}{r} - 2\frac{\lambda'}{r} + \frac{1}{r^2}\right)\right] \tag{2.2.10}$$

for static, spherically symmetric spacetimes. The components of the Einstein tensor (2.2.6), for the line element (2.2.8), are then given by

$$G_{00} = \frac{1}{r^2}e^{2\nu}(1-e^{-2\lambda}) + 2\frac{\lambda'}{r}e^{2(\nu-\lambda)} \tag{2.2.11a}$$

$$G_{11} = -\frac{1}{r^2}e^{2\lambda}(1-e^{-2\lambda}) + 2\frac{\nu'}{r} \tag{2.2.11b}$$

$$G_{22} = r^2e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{\nu'}{r} - \frac{\lambda'}{r}\right) \tag{2.2.11c}$$

$$G_{33} = \sin^2\theta G_{22} \tag{2.2.11d}$$

$$G_{ab} = 0, \ a \neq b$$

which follow from the Ricci tensor components (2.2.9) and the Ricci scalar (2.2.10).

2.3 Matter and charge distribution

The distribution of matter is specified by the energy momentum tensor $\boldsymbol{M}$ which is given by

$$M_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{2.3.1}$$

for neutral matter. In the above ρ is the energy density, p is the isotropic pressure, q_a is the heat flow vector and π_{ab} represents the stress tensor. These quantities are measured relative to the four-velocity $\boldsymbol{u}$ ($u^a u_a = -1$). The heat flow vector and stress tensor satisfy the conditions

$$q^a u_a = 0$$

$$\pi^{ab} u_b = 0$$

In the simpler case of a perfect fluid, which is the case for most cosmological models, the energy flux vector and the stress tensor vanish and (2.3.1) becomes

$$M_{ab} = (\rho + p)u_a u_b + p g_{ab} \tag{2.3.2}$$

For many applications we require that

$$p = p(\rho)$$

so that there is a barotropic equation of state.

The total energy momentum tensor $\boldsymbol{T}$ for the charged fluid is given by

$$T_{ab} = M_{ab} + E_{ab} \tag{2.3.3}$$

where $\boldsymbol{E}$ is the contribution of the electromagnetic field, and which is given by

$$E_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \tag{2.3.4}$$

Here the components of the skew-symmetric electromagnetic field tensor $\boldsymbol{F}$ may be given in terms of a four-potential $\boldsymbol{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b} \tag{2.3.5}$$

(Misner *et al* 1973, Stephani 1990).

The gravitational behaviour of the charged matter distribution that we have considered is governed by a relevant set of field equations. The Einstein-Maxwell field equations can be expressed as the system

$$\begin{aligned} G_{ab} &= T_{ab} \\ &= M_{ab} + E_{ab} && \text{(2.3.6a)} \\ F_{ab;c} + F_{bc;a} + F_{ca;b} &= 0 && \text{(2.3.6b)} \\ F^{ab}{}_{;b} &= J^a && \text{(2.3.6c)} \end{aligned}$$

In the above system $\boldsymbol{J}$ represents the four-current density defined by

$$J^a = \sigma u^a$$

where σ is the proper charge density. We are using units in which the coupling constant in (2.3.6a) is unity. The Maxwell equations (2.3.6b) and (2.3.6c) are the fundamental equations of electrodynamics in a curved space. From equations (2.2.7) and (2.3.6a) it follows that

$$T^{ab}{}_{;b} = 0 \tag{2.3.7}$$

which is a conservation law. The equations (2.3.6) constitute a system of nonlinear differential equations which determine the behaviour of a gravitating system in the presence of an electromagnetic field. For neutral matter only the Einstein field equations (2.3.6a) are required with $\boldsymbol{E} = 0$.

2.4 Field equations

The field equations (2.3.6) are a set of highly nonlinear differential equations which are difficult to integrate without simplifying assumptions. One approach is to impose a symmetry requirement on the spacetime manifold (Castejon-Amenedo and Coley 1992, Maharaj *et al* 1991). Our intention in this thesis is to generate solutions to the field equations which are static and spherically symmetric without this restriction. In other words we are assuming the line element (2.2.8), and we intend to solve the field equations (2.3.6) directly. We are considering a spacetime manifold that admits a Lie algebra spanned by four Killing vectors, so that it is time independent (static) and invariant under rotations. Such solutions are applicable in relativistic astrophysics (Shapiro and Teukolsky 1983). In this situation we formulate the field equations for the case of both neutral and charged perfect fluids with the comoving four-velocity $u^a = e^{-\nu}\delta^a_0$ for the line element (2.2.8).

The energy momentum tensor $\boldsymbol{M}$ for neutral matter ($\boldsymbol{E} = 0$) has the form

$$M_{ab} = \begin{pmatrix} e^{2\nu}\rho & 0 & 0 & 0 \\ 0 & e^{2\lambda}p & 0 & 0 \\ 0 & 0 & r^2 p & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta p \end{pmatrix} \tag{2.4.1}$$

where we have used the perfect fluid form (2.3.2).

In the case of charged perfect fluids, for static metrics, it is customary to choose the four-potential as

$$A_a = (\phi(r), 0, 0, 0)$$

This choice is made by Humi and Mansour (1984) and Pant and Sah (1979), among others. The above choice is the simplest choice possible and has the advantage of generating only one nonvanishing component, and its skew-symmetric counterpart, of the electromagnetic field tensor; this component is given by

$$F_{01} = -\phi'(r)$$

where we have used (2.3.5). Using the above expression for F_{01} it is easy to verify that (2.3.6b) is identically satisfied. The corresponding contravariant component has the form

$$F^{01} = e^{-2(\nu+\lambda)}\phi'(r) = e^{-(\nu+\lambda)}E(r)$$

where we have defined

$$E(r) = e^{-(\nu+\lambda)}\phi'(r)$$

following the treatment of Herrera and Ponce de Leon (1985). The quantity $E(r)$ may be interpreted as the electric field intensity. The equation (2.3.6c) is identically satisfied for $a = 1, 2, 3$. When $a = 0$ we obtain

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2 E)' \tag{2.4.2}$$

for the proper charge density.

On using (2.3.4) we calculate the electromagnetic contribution to the energy momentum tensor which is given by

$$E_{ab} = \frac{1}{2}E^2 \begin{pmatrix} e^{2\nu} & 0 & 0 & 0 \\ 0 & -e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.4.3}$$

The nonvanishing components of the total energy momentum tensor for charged ($\boldsymbol{E} \neq$

0) perfect fluids are then given by

$$T_{ab} =$$

$$\begin{pmatrix} e^{2\nu}\left(\rho+\frac{1}{2}E^2\right) & 0 & 0 & 0 \\ 0 & e^{2\lambda}\left(p-\frac{1}{2}E^2\right) & 0 & 0 \\ 0 & 0 & r^2\left(p+\frac{1}{2}E^2\right) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta\left(p+\frac{1}{2}E^2\right) \end{pmatrix} \tag{2.4.4}$$

which follow from (2.3.3), (2.4.1) and (2.4.3). Clearly, when $E = 0$ we regain (2.4.1) for uncharged matter.

2.4.1 Neutral fluids

It is now possible to obtain the field equations for the case of a neutral perfect fluid. The Einstein tensor components (2.2.11) and the energy momentum tensor (2.4.1) generate the Einstein field equations

$$\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\lambda'}{r}e^{-2\lambda} = \rho \tag{2.4.5a}$$

$$\frac{-1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p \tag{2.4.5b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p \tag{2.4.5c}$$

for a static, spherically symmetric spacetime. Observe that as a consequence of the conservation law (2.3.7) we have

$$p' = -(\rho+p)\nu'$$

which may be used in place of one of the field equations in (2.4.5). The system of equations (2.4.5) governs the gravitational behaviour of a neutral perfect fluid. This system has three equations with four variables ν, λ, ρ and p. In order to obtain a solution for the system (2.4.5) it is necessary to impose an additional condition.

The field equations (2.4.5) may be expressed in a variety of equivalent forms which, for particular applications, may make the integration process simpler. We introduce the following transformation, which has been used by Durgapal (1982), Durgapal and Bannerji (1983), Durgapal and Fuloria (1985), and Finch and Skea (1989), to generate new solutions in the case of neutral matter. It is convenient to introduce a new coordinate x and two new metric functions $y(x)$ and $Z(x)$ defined as follows

$$
\begin{aligned}
x &= Cr^2 \\
Z(x) &= e^{-2\lambda(r)} \\
A^2y^2(x) &= e^{2\nu(r)}
\end{aligned}
$$

where A and C are constants. This transformation enables us to write (2.4.5) as the new system

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{2.4.6a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} \tag{2.4.6b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0 \tag{2.4.6c}$$

where dots denote differentiation with respect to x. The new metric functions y and Z now depend on the coordinate x.

2.4.2 Charged fluids

We now obtain the Einstein -Maxwell field equations for a static, spherically symmetric spacetime in the presence of the electromagnetic field. By utilising equations (2.2.11), (2.4.4) and (2.4.2) we obtain

$$\frac{1}{r^2}(1-e^{-2\lambda})+\frac{2\lambda'}{r}e^{-2\lambda} = \rho+\frac{1}{2}E^2 \tag{2.4.7a}$$

$$\frac{-1}{r^2}(1-e^{-2\lambda})+\frac{2\nu'}{r}e^{-2\lambda} = p-\frac{1}{2}E^2 \tag{2.4.7b}$$

$$e^{-2\lambda}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right) = p+\frac{1}{2}E^2 \tag{2.4.7c}$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)' \tag{2.4.7d}$$

The system of equations (2.4.7) governs the behaviour of the gravitational field for a charged perfect fluid. When $E=0$, (2.4.7) reduces to (2.4.5). The conservation law (2.3.7) generates the equation

$$p'+(\rho+p)\nu' = \frac{E}{r^2}(r^2E)'$$

which is also a direct consequence of the field equations.

Utilising the new independent variable $x=Cr^2$ introduced in §2.4.1 we can write (2.4.7) in the equivalent form

$$\frac{1-Z}{x}-2\dot{Z} = \frac{\rho}{C}+\frac{E^2}{2C} \tag{2.4.8a}$$

$$4Z\frac{\dot{y}}{y}+\frac{Z-1}{x} = \frac{p}{C}-\frac{E^2}{2C} \tag{2.4.8b}$$

$$4Zx^2\ddot{y}+2\dot{Z}x^2\dot{y}+\left(\dot{Z}x-Z+1-\frac{E^2x}{C}\right)y = 0 \tag{2.4.8c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}(x\dot{E}+E)^2 \tag{2.4.8d}$$

It is clear that when $E=0$ we regain the system (2.4.6) from (2.4.8). In the above we have a system of four equations in the six unknowns ρ, p, E, σ, y and Z. The result is that (2.4.8c) is the master equation that determines the solution of the system. Once the metric functions Z and E are specified, the remaining metric function y can be found by directly integrating (2.4.8c) which is second order and linear. The remaining variables are then obtained from the rest of the system. This is the approach that we follow in later chapters to obtain solutions. The system (2.4.8) can also be supplemented with an equation of state to generate exact solutions.

2.5 Physical conditions

For physical viability, any solution applicable to the interior of the stellar body should match smoothly to the appropriate exterior spacetime. We consider two famous exact exterior solutions, applicable to relativistic astrophysics for static stars. The gravitational field outside a static, spherically symmetric body with neutral matter is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5.1}$$

known as the exterior Schwarzschild solution (1916a). Here the constant m represents the mass of the stellar body measured by an observer at infinity. In (2.5.1), the metric components become singular when $r = 0$ and $r = 2m$. The singularity at $r = 2m$ is not a true singularity of the spacetime structure but represents a breakdown in the coordinates that have been used to obtain the general form of the line element (2.2.8). We may avoid this coordinate singularity by utilising the Kruskal-Szekeres coordinates which covers all of spacetime (Misner *et al* 1973, Stephani 1990). The exterior gravitational field to a static, charged spherically symmetric matter distribution has the form

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5.2}$$

Here q is the constant related to the total charge of the sphere. The line element (2.5.2) is the exterior Reissner-Nordstrom solution (Reissner 1916, Nordstrom 1918). For (2.5.2) the radial electric field is

$$E = \frac{q}{r^2}$$

and consequently the proper charge density is

$$\sigma = 0$$

by (2.4.2). Thus the four current density $\boldsymbol{J} = 0$ which is consistent with an exterior spacetime with no barotropic matter. When $q = 0$, (2.5.2) becomes the exterior Schwarzschild line element (2.5.1).

In addition, it is often assumed that realistic stellar models for isotropic matter should satisfy:

(a) The energy density and the pressure should be positive and finite throughout the interior of the star including the origin. The radial pressure should vanish at the boundary $r = a$:

$$0 < \rho < \infty, \qquad 0 < p < \infty, \qquad p(a) = 0$$

(b) The energy density and the pressure should be monotonic decreasing functions of the radial coordinate r:

$$\frac{d\rho}{dr} \leq 0, \qquad \frac{dp}{dr} \leq 0$$

(c) Causality should be preserved. The speed of sound should remain subluminal throughout the interior of the star which leads to the condition:

$$0 \leq \frac{dp}{d\rho} \leq 1$$

(d) The metric functions $e^{2\nu}$ and $e^{2\lambda}$ and the electric field intensity E should be positive and nonsingular throughout the stellar interior.

(e) At the boundary the interior metric functions should match smoothly to the exterior line elements (2.5.1) and (2.5.2) for the case of neutral and charged spherically symmetric solutions respectively. This generates the following conditions on the gravitational potentials:

$$e^{2\nu(a)} = e^{-2\lambda(a)} = 1 - \frac{2m}{a} \text{ for neutral stars,}$$

$$e^{2\nu(a)} = e^{-2\lambda(a)} = 1 - \frac{2m}{a} + \frac{q^2}{a^2} \text{ for charged stars.}$$

(f) Continuity of the electric field intensity across the boundary for the case of charged solutions:

$$E(a) = \frac{q}{a^2}$$

(g) The solutions should be stable with respect to radial perturbations.

It should be noted that not all relativistic stellar models satisfy all the conditions listed above throughout the stellar interior and may be valid only in some regions of spacetime. For example some of the Tolman solutions (Tolman 1939) become singular at the centre. Such solutions have to be treated as an envelope of the core of the star and have to be matched to some other solution valid for the core; see for example the treatment of Thomas *et al* (2005). Some of the conditions (a)-(g) may be overly restrictive. For example, observational evidence suggests that in particular stars the energy density ρ is not a strictly monotonically decreasing function (Shapiro and Teukolsky 1983). It is advisable to put any exact solution through the test of the above conditions because they provide qualitative features which may represent many physical stars. Note that it is interesting to study the behaviour of anisotropic matter because in this case the radial pressure may be different from the tangential pressure. Such cases were examined by Chaisi and Maharaj (2005), and Dev and Gleiser (2002, 2003) in the case of neutral spheres; while Herrera and Ponce de Leon (1985) considered tangential pressures in the presence of charge. Exact solutions to the field equations which do not satisfy all of the conditions (a)-(g) are still of interest because they provide useful qualitative features which assist in the analysis of relativistic stars.

Chapter 3

Exact solutions for the charged Tikekar superdense stars

3.1 Introduction

It is clear that stars with spheroidal geometry have a number of different physical applications and therefore require deeper investigation. In this chapter our objective is to generate a class of charged spheroidal solutions corresponding to a physically reasonable form for the electric field intensity. Our intention is to obtain simple forms for the solution that highlight the role of the spheroidal parameter. In §3.2 we obtain a simpler form of the condition of pressure isotropy extended to the electromagnetic field, with the assistance of an appropriate transformation. Upon specifying a choice for the electric field, we obtain a second order linear differential equation which facilitates the integration process. We assume a solution in a series form which yields recurrence relations, which we manage to solve from first principles in §3.3. It is then possible to exhibit exact solutions to the Einstein-Maxwell system. In §3.4 we present polynomials as first solutions by restricting the spheroidal parameter K and the electric field intensity parameter α. We find products of polynomials and algebraic functions as second solutions for restricted values of K and α in §3.5. In §3.6 we show that it is possible to express the general solutions in terms of elementary functions. We demonstrate that solutions found previously are special cases of our general treatment. We briefly discuss the physical viability of our solutions. We emphasise that our simple approach of utilising the series method of Frobenius yields a rich family of Einstein-Maxwell solutions in terms of elementary functions. In addition, a new algorithm which generates new solutions to the field equations from a specified seed solution is established in §3.7. We illustrate this algorithm with two simple examples. The results of this chapter have been published in Komathiraj and Maharaj (2007a).

3.2 Charged Tikekar stars

To integrate the system (2.4.7) it is necessary to choose two of the variables $\nu,\ \lambda,\ \rho,\ p$ or E. In our approach in this chapter we specify λ and E. In the integration procedure we make the choice

$$e^{2\lambda(r)} = \frac{1 - Kr^2/R^2}{1 - r^2/R^2} \tag{3.2.1}$$

where K is an arbitrary constant. The form (3.2.1) for the gravitational potential λ restricts the geometry of the 3-dimensional hypersurfaces $\{t = \text{constant}\}$ to be spheroidal. When $K = 0$ the hypersurfaces $\{t = \text{constant}\}$ become spherical. For the choice $K = 0$ familiar spacetimes are regainable for particular forms of the metric function $e^{2\nu}$, e.g., the choice $\nu = 0$ gives the metric of Einstein's universe.

On eliminating p from (2.4.7b) and (2.4.7c), for the particular form (3.2.1), we obtain

$$\begin{aligned} \left(1 - \frac{Kr^2}{R^2}\right)^2 E^2 \;&=\; \left(1 - \frac{Kr^2}{R^2}\right)\left(1 - \frac{r^2}{R^2}\right)\left(\nu'' + \nu'^2 - \frac{\nu'}{r}\right) \\ &-\; (1-K)\left(\frac{r}{R^2}\right)\left(\nu' + \frac{1}{r}\right) + \frac{1-K}{R^2}\left(1 - \frac{Kr^2}{R^2}\right) \end{aligned} \tag{3.2.2}$$

which is the condition of pressure isotropy with the nonzero electromagnetic field. It is convenient at this point to introduce the transformation

$$\psi(x) \;=\; e^{\nu(r)} \tag{3.2.3a}$$

$$x^2 \;=\; 1 - \frac{r^2}{R^2} \tag{3.2.3b}$$

Then the condition of pressure isotropy (3.2.2) becomes

$$(1 - K + Kx^2)\ddot{\psi} - Kx\dot{\psi} + \left(\frac{(1-K+Kx^2)^2R^2E^2}{x^2-1} + K(K-1)\right)\psi = 0 \tag{3.2.4}$$

in terms of the new variables ψ and x; dots denote differentiation with respect to x.

The Einstein-Maxwell system (2.4.7) implies

$$\rho = \frac{1-K}{R^2}\frac{(3-K+Kx^2)}{(1-K+Kx^2)^2} - \frac{1}{2}E^2 \tag{3.2.5a}$$

$$p = \frac{1}{R^2(1-K+Kx^2)}\left(-2x\frac{\dot{\psi}}{\psi} + K - 1\right) + \frac{1}{2}E^2 \tag{3.2.5b}$$

$$\sigma^2 = \frac{[2xE-(1-x^2)\dot{E}]^2}{R^2(1-x^2)(1-K+Kx^2)} \tag{3.2.5c}$$

in terms of the variables E and ψ. Thus ρ, p and σ are defined in terms of E in (3.2.5). The solution of the Einstein-Maxwell system depends on the integrability of (3.2.4). Clearly (3.2.4) is integrable once E is specified. A variety of choices for E is possible; however only a few are physically reasonable. We also need to choose E such that closed form solutions are possible. We make the choice

$$E^2 = \frac{\alpha K(x^2-1)}{(1-K+Kx^2)^2R^2} \tag{3.2.6}$$

where α is constant. A similar form of E was also used by Sharma *et al* (2001). The electric field intensity E in (3.2.6) vanishes at the centre of the star, and remains continuous and bounded in the interior of the star for a wide range of values of the parameters α and K. Thus this choice for E is physically reasonable and useful in the study of the gravitational behaviour of charged stars. On substituting (3.2.6) into (3.2.4) we obtain

$$(1-K+Kx^2)\ddot{\psi} - Kx\dot{\psi} + K(\alpha+K-1)\psi = 0 \tag{3.2.7}$$

This is a second order differential equation which is linear in ψ. We expect that our investigation of (3.2.7) will produce viable models of charged stars since the special case $\alpha = 0$ yields models consistent with neutron stars.

3.3 General series solution

It is possible to express the solution of (3.2.7) in terms of special functions namely the Gegenbauer functions as demonstrated by Sharma *et al* (2001). However, that form of the solution is not particularly useful because of the analytic complexity of the special functions involved. In addition, the role of parameters of physical interest, such as the spheroidal parameter K, is lost or obscured in the representation as Gegenbauer functions. The representation of the solutions in a simple form is necessary for a detailed physical analysis. Consequently we attempt to obtain a general solution to the differential equation (3.2.7) in a series form using the method of Frobenius. Later we will indicate that it is possible to extract solutions in terms of polynomials and algebraic functions for particular parameter values.

As the point $x = 0$ is a regular point of (3.2.7), there exists two linearly independent solutions having the form of a power series with centre $x = 0$. Thus we assume

$$\psi(x) = \sum_{i=0}^{\infty} a_i x^i \tag{3.3.1}$$

where the constants a_i are the coefficients of the series. For a legitimate solution we need to determine the coefficients a_i explicitly. On substituting (3.3.1) in to (3.2.7) we obtain, after simplification

$$\{(1-K)2.1a_2 + K(\alpha + K - 1)a_0\} + \{(1-K)3.2a_3 + K(\alpha + K - 2)a_1\}x$$

$$+\sum_{i=2}^{\infty}\{(1-K)(i+1)(i+2)a_{i+2} + K[\alpha + K - 1 + i(i-2)]a_i\}x^i = 0 \tag{3.3.2}$$

in increasing powers of x. For this equation to be valid for all x in the interval of convergence we require

$$(1-K)2.1a_2 + K(\alpha + K - 1)a_0 = 0 \tag{3.3.3a}$$

$$(1-K)3.2a_3 + K(\alpha + K - 2)a_1 = 0 \tag{3.3.3b}$$

$$(1-K)(i+1)(i+2)a_{i+2} + K[\alpha + K - 1 + i(i-2)]a_i = 0, i \geq 2 \tag{3.3.3c}$$

It remains to obtain the coefficients a_i from the system (3.3.3). Equation (3.3.3c) is the linear recurrence relation governing the structure of the solution.

The recurrence relation (3.3.3c) consists of variable, rational coefficients. It does not fall in the known classes of difference equations and has to be solved from first principles. It is possible to solve (3.3.3c) using the principle of mathematical induction. We first consider the even coefficients $a_0,\ a_2,\ a_4,\ \ldots$. Equation (3.3.3a) implies

$$\begin{aligned} a_{2.1} &= \left(\frac{K}{K-1}\right)\frac{1}{(2.1)!}\{\alpha+K-1+(2-2)(2-4)\}a_0 \\ &= \left(\frac{K}{K-1}\right)^1\frac{1}{(2.1)!}\prod_{q=1}^{1}\{\alpha+K-1+(2q-2)(2q-4)\}a_0 \end{aligned}$$

where we have utilised the conventional symbol $\prod$ to denote multiplication for the first term. We now assume the pattern

$$a_{2p} = \left(\frac{K}{K-1}\right)^p\frac{1}{(2p)!}\prod_{q=1}^{p}\{\alpha+K-1+(2q-2)(2q-4)\}a_0$$

for the coefficient a_{2p} which is the inductive step. We now establish that this is true for the next coefficient $a_{2(p+1)}$. Replacing i with $2p$ in (3.3.3c) we obtain

$$\begin{aligned} a_{2(p+1)} &= \left(\frac{K}{K-1}\right)\left(\frac{\alpha+K-1+2p(2p-2)}{(2p+1)(2p+2)}\right)a_{2p} \\ &= \left(\frac{K}{K-1}\right)\left(\frac{\alpha+K-1+2p(2p-2)}{(2p+1)(2p+2)}\right) \\ &\times \left(\frac{K}{K-1}\right)^p\frac{1}{(2p)!}\prod_{q=1}^{p}\{\alpha+K-1+(2q-2)(2q-4)\}a_0 \\ &= \left(\frac{K}{K-1}\right)^{p+1}\frac{1}{[2(p+1)]!}\prod_{q=1}^{p+1}\{\alpha+K-1+(2q-2)(2q-4)\}a_0 \end{aligned}$$

where we have used the above assumed pattern for a_{2p}. Hence, by mathematical induction, all the even coefficients a_{2i} can be written in terms of the leading coefficient a_0. These coefficients generate a pattern

$$a_{2i} = \left(\frac{K}{K-1}\right)^i\frac{1}{(2i)!}\prod_{q=1}^{i}\{\alpha+K-1+(2q-2)(2q-4)\}a_0 \tag{3.3.4}$$

for the even coefficients $a_0,\ a_2,\ a_4,\ \ldots$.

We can obtain a similar formula for the odd coefficients $a_1,\ a_3,\ a_5,\ \ldots$. From (3.3.3b) we have

$$\begin{aligned} a_{2.1+1} &= \left(\frac{K}{K-1}\right)\frac{1}{(2.1+1)!}\{\alpha+K-1+(2.1-1)(2.1-3)\}a_1 \\ &= \left(\frac{K}{K-1}\right)^1\frac{1}{(2.1+1)!}\prod_{q=1}^{1}\{\alpha+K-1+(2q-1)(2q-3)\}a_1 \end{aligned}$$

for the first term. We now assume

$$a_{2p+1} = \left(\frac{K}{K-1}\right)^{p} \frac{1}{(2p+1)!} \prod_{q=1}^{p} \{\alpha + K - 1 + (2q-1)(2q-3)\} a_1$$

for the coefficient a_{2p+1}. We then establish that this is true for the next coefficient $a_{(2p+1)+1}$. Replacing i with $(2p+1)$ in (3.3.3c)we obtain

$$\begin{aligned}
a_{2(p+1)+1} &= \left(\frac{K}{K-1}\right)\left(\frac{\alpha + K - 1 + (2p+1)(2p-1)}{(2p+2)(2p+3)}\right) a_{2p+1} \\
&= \left(\frac{K}{K-1}\right)\left(\frac{\alpha + K - 1 + (2p+1)(2p-1)}{(2p+2)(2p+3)}\right) \\
&\times \left(\frac{K}{K-1}\right)^{p} \frac{1}{(2p+1)!} \prod_{q=1}^{p} \{\alpha + K - 1 + (2q-1)(2q-3)\} a_1 \\
&= \left(\frac{K}{K-1}\right)^{p+1} \frac{1}{[2(p+1)+1]!} \prod_{q=1}^{p+1} \{\alpha + K - 1 + (2q-1)(2q-3)\} a_1
\end{aligned}$$

on utilising the assumed formula for a_{2p+1}. Hence, by using mathematical induction all the odd coefficients a_{2i+1} can be written in terms of the leading coefficient a_1. These coefficients generate a pattern which is clearly of the form

$$a_{2i+1} = \left(\frac{K}{K-1}\right)^{i} \frac{1}{(2i+1)!} \prod_{q=1}^{i} \{\alpha + K - 1 + (2q-1)(2q-3)\} a_1 \qquad (3.3.5)$$

for the odd coefficients $a_1,\ a_3,\ a_5,\ \ldots$.

The coefficients a_{2i} are generated from (3.3.4). The coefficients a_{2i+1} are generated from (3.3.5). Hence, the difference equation (3.3.3c) has been solved and all nonzero coefficients are expressible in terms of the leading coefficients a_0 and a_1. From (3.3.1), (3.3.4), and (3.3.5) we establish that

$$\begin{aligned}
\psi(x) &= \left(a_0 + \sum_{i=1}^{\infty} a_{2i} x^{2i}\right) + \left(a_1 x + \sum_{i=1}^{\infty} a_{2i+1} x^{2i+1}\right) \\
&= a_0 \left(1 + \sum_{i=1}^{\infty} \left(\frac{K}{K-1}\right)^{i} \frac{1}{(2i)!} \prod_{q=1}^{i} \{\alpha + K - 1 + 4(q-1)(q-2)\} x^{2i}\right) \\
&+ a_1 \left(x + \sum_{i=1}^{\infty} \left(\frac{K}{K-1}\right)^{i} \frac{1}{(2i+1)!} \prod_{q=1}^{i} \{\alpha + K - 1 + (2q-1)(2q-3)\} x^{2i+1}\right)
\end{aligned} \qquad (3.3.6)$$

where a_0 and a_1 are arbitrary constants. Clearly (3.3.6) is of the form

$$\psi(x) = a_0\psi_1(x) + a_1\psi_2(x)$$

where

$$\psi_1(x) = \left(1 + \sum_{i=1}^{\infty}\left(\frac{K}{K-1}\right)^i \frac{1}{(2i)!}\prod_{q=1}^{i}\{\alpha + K - 1 + 4(q-1)(q-2)\}x^{2i}\right) \quad (3.3.7a)$$

$$\psi_2(x) = \left(x + \sum_{i=1}^{\infty}\left(\frac{K}{K-1}\right)^i \frac{1}{(2i+1)!}\prod_{q=1}^{i}\{\alpha + K - 1 + (2q-1)(2q-3)\}x^{2i+1}\right) \quad (3.3.7b)$$

are linearly independent solutions of (3.2.7). Thus we have found the general series solution to the differential equation (3.2.7) for the choice of the electromagnetic field E given in (3.2.6). The solution (3.3.6) is expressed in terms of a series with real arguments unlike the complex arguments given by software packages. Series (3.3.7a) and (3.3.7b) converge if there exists a nonnegative value for the radius of convergence. Note that the radius of convergence of the series is not less than the distance from the centre ($x = 0$) to the nearest root of the leading coefficient of the differential equation (3.2.7). Clearly this is possible for a wide range of values for K.

It is interesting to observe that the series in (3.3.7) terminates for restricted values of the parameters α and K. This will happen when $K+\alpha$ takes on specific integer values. Utilising this feature it is possible to generate solutions in terms of elementary functions by determining the specific restriction on α and K for a terminating series. Solutions in terms of polynomials and algebraic functions can be found. We use the recurrence relation (3.3.3c), rather than the series (3.3.7), to find the elementary solutions as this is simpler.

3.4 Polynomial solutions

We first consider polynomials of even degree. It is convenient to set

$$\begin{aligned} i &= 2(j-1) \\ K + \alpha &= 2 - (2n-1)^2 \end{aligned}$$

where $n > 1$ is a fixed integer in (3.3.3c). This leads to

$$\begin{aligned} a_{2j} &= -4\left[\frac{(2n-1)^2-2+\alpha}{(2n-1)^2-1+\alpha}\right]\frac{(n+j-2)(n-j+1)}{2j(2j-1)}a_{2j-2} \\ &= -\gamma\frac{(n+j-2)(n-j+1)}{2j(2j-1)}a_{2j-2} \qquad (3.4.1) \end{aligned}$$

where we have set $\gamma = 4 - 4/[4n(n-1)+\alpha]$. We note that (3.4.1) implies $a_{2(n+1)} = 0$. Consequently the remaining coefficients $a_{2(n+2)},\ a_{2(n+3)},\ a_{2(n+4)},\ \ldots$ vanish. Equation (3.4.1) may be solved to yield

$$a_{2j} = (-\gamma)^j \frac{(n+j-2)!}{(n-j)!(2j)!},\ 0 \le j \le n \qquad (3.4.2)$$

where we have set $a_0 = 1/n(n-1)$. With the help of (3.4.2) we can express the polynomial in even powers of x in the form

$$f_1(x) = \sum_{j=0}^{n}(-\gamma)^j \frac{(n+j-2)!}{(n-j)!(2j)!}x^{2j} \qquad (3.4.3)$$

for $K + \alpha = 2 - (2n-1)^2$.

We now consider polynomials of odd degree. For this case we let

$$\begin{aligned} i &= 2(j-1)+1 \\ K+\alpha &= 2(1-2n^2) \end{aligned}$$

where $n > 0$ is a fixed integer in (3.3.3c). We obtain

$$\begin{aligned} a_{2j+1} &= -4\left[\frac{4n^2-2+\alpha}{4n^2-1+\alpha}\right]\frac{(n+j-1)(n-j+1)}{2j(2j+1)}a_{2j-1} \\ &= -\mu\frac{(n+j-1)(n-j+1)}{2j(2j+1)}a_{2j-1} \qquad (3.4.4) \end{aligned}$$

where we set $\mu = 4 - 4/(4n^2-1+\alpha)$. We observe that (3.4.4) implies $a_{2(n+1)+1} = 0$. Consequently the remaining coefficients $a_{2(n+2)+1},\ a_{2(n+3)+1},\ a_{2(n+4)+1},\ \ldots$ vanish. Equation (3.4.4) can be solved to yield

$$a_{2j+1} = (-\mu)^j \frac{(n+j-1)!}{(n-j)!(2j+1)!},\ 0 \le j \le n \qquad (3.4.5)$$

where we have set $a_1 = 1/n$. With the assistance of (3.4.5) we can express the polynomial in odd powers of x as

$$g_1(x) = \sum_{j=0}^{n} (-\mu)^j \frac{(n+j-1)!}{(n-j)!(2j+1)!} x^{2j+1} \tag{3.4.6}$$

for $K + \alpha = 2(1 - 2n^2)$.

The polynomial solutions (3.4.3) and (3.4.6) comprise the first solution of (3.2.7) for appropriate values of $K + \alpha$.

3.5 Algebraic solutions

We take the second solution of (3.2.7) to be of the form

$$\psi_(x) = u(x)(1 - K + Kx^2)^{3/2}$$

when $u(x)$ is an arbitrary polynomial. Particular solutions found in the past are special cases of this general form; the factor $(1 - K + Kx^2)^{3/2}$ helps to simplify the integration process. This motivates the above algebraic form for ψ as a generic solution to the differential equation (3.2.7). On substituting ψ in (3.2.7) we obtain after simplification

$$(1 - K + Kx^2)\frac{d^2u}{dx^2} + 5Kx\frac{du}{dx} + K(\alpha + K + 2)u = 0 \tag{3.5.1}$$

which is a linear differential equation for $u(x)$.

As in §3.4 we can find two classes of polynomial solutions for $u(x)$, in even powers of x and in odd powers of x, for certain values of $K + \alpha$. As the point $x = 0$ is a regular point of (3.5.1), there exist two linearly independent solutions of the form of power series with centre $x = 0$. Therefore we can write

$$u(x) = \sum_{i=0}^{\infty} b_i x^i \tag{3.5.2}$$

where b_i are the coefficients of the series. Substituting (3.5.2) in (3.5.1) we obtain

$$\{(1-K)2.1b_2 + K(\alpha + K + 2)b_0\} + \{(1-K)3.2b_3 + K(\alpha + K + 7)b_1\}x$$

$$+ \sum_{i=2}^{\infty} \{(1-K)(i+2)(i+1)b_{i+2} + K[\alpha + K + 2 + i(i+4)]b_i\}x^i = 0$$

For this equation to hold true for all x we require

$$(1-K)2.1b_2 + K(\alpha + K + 2)b_0 = 0 \tag{3.5.3a}$$

$$(1-K)3.2b_3 + K(\alpha + K + 7)b_1 = 0 \tag{3.5.3b}$$

$$(1-K)(i+2)(i+1)b_{i+2} + K[\alpha + K + 2 + i(i+4)]b_i = 0, i \geq 2 \tag{3.5.3c}$$

which governs the coefficients.

We first consider even powers of x. Replacing i with $2(j-1)$ and assuming $K+\alpha = 2(1-2n^2)$ in (3.5.3c), where $n > 0$ is fixed integer, we obtain

$$\begin{aligned} b_{2j} &= -4\left[\frac{4n^2-2+\alpha}{4n^2-1+\alpha}\right]\frac{(n+j)(n-j)}{2j(2j-1)}b_{2j-2} \\ &= -\mu\frac{(n+j)(n-j)}{2j(2j-1)}b_{2j-2} \end{aligned} \tag{3.5.4}$$

where we have set $\mu = 4 - 4/(4n^2 - 1 + \alpha)$. From (3.5.4) we have that $b_{2n} = 0$ and subsequent coefficients $b_{2(n+1)},\ b_{2(n+2)},\ b_{2(n+3)},\ \ldots$ vanish. Then (3.5.4) has the solution

$$b_{2j} = (-\mu)^j\frac{(n+j)!}{(n-j-1)!(2j)!},\ 0 \leq j \leq n-1 \tag{3.5.5}$$

where we have set $b_0 = n$. On using (3.5.2) and (3.5.5) the polynomial in even powers of x leads to the expression

$$g_2(x) = (1-K+Kx^2)^{3/2}\sum_{j=0}^{n-1}(-\mu)^i\frac{(n+j)!}{(n-j-1)!(2j)!}x^{2j} \tag{3.5.6}$$

for $K + \alpha = 2(1-2n^2)$.

We now consider odd powers of x. Replacing i with $2(j-1)+1$ and assuming $K + \alpha = 2 - (2n-1)^2$ in (3.5.3c), where $n > 1$ is fixed integer, we obtain

$$\begin{aligned} b_{2j+1} &= -4\left[\frac{(2n-1)^2-2+\alpha}{(2n-1)^2-1+\alpha}\right]\frac{(n+j)(n-j-1)}{2j(2j+1)}b_{2j-1} \\ &= -\gamma\frac{(n+j)(n-j-1)}{2j(2j+1)}b_{2j-1} \end{aligned} \tag{3.5.7}$$

where we have set $\gamma = 4-4/[4n(n-1)+\alpha]$. From (3.5.7) we have that $b_{2(n-1)+1} = 0$ and subsequent coefficients $b_{2n+1},\ b_{2(n+1)+1},\ b_{2(n+2)+1},\ \ldots$ vanish. Then equation (3.5.7) has the solution

$$b_{2j+1} = (-\gamma)^j \frac{(n+j)!}{(n-j-2)!(2j+1)!},\ 0 \le j \le n-2 \tag{3.5.8}$$

where we have set $b_1 = n(n-1)$. On using (3.5.2) and (3.5.8) the polynomial in odd powers of x leads to the result

$$f_2(x) = (1-K+Kx^2)^{3/2} \sum_{j=0}^{n-2} (\gamma)^j \frac{(n+j)!}{(n-j-2)!(2j+1)!} x^{2j+1} \tag{3.5.9}$$

for $K+\alpha = 2-(2n-1)^2$.

The algebraic solutions (3.5.6) and (3.5.9) comprise the second solution of (3.2.7) for appropriate values of $K+\alpha$. The solutions (3.5.6) and (3.5.9) are expressed as products of algebraic functions and polynomials, and they are clearly linearly independent from (3.4.6) and (3.4.3), respectively.

3.6 General solutions with elementary functions

We have obtained two classes of polynomial solutions (3.4.3) and (3.4.6) in §3.4 to the differential equation (3.2.7). Also we have found two classes of algebraic solutions (3.5.6) and (3.5.9) in §3.5. By collecting these results we can express the solution to (3.2.7) in two categories. The first category of solution for $\psi(x) = f(x)$ is given by

$$\begin{aligned} f(x) &= Af_1(x) + Bf_2(x) \\ &= A\sum_{j=0}^{n} (-\gamma)^j \frac{(n+j-2)!}{(n-j)!(2j)!} x^{2j} \\ &+ B(1-K+Kx^2)^{3/2} \sum_{j=0}^{n-2} (-\gamma)^j \frac{(n+j)!}{(n-j-2)!(2j+1)!} x^{2j+1} \end{aligned} \tag{3.6.1}$$

for the values

$$\begin{aligned} \gamma &= 4 - \frac{4}{4n(n-1)+\alpha} \\ K+\alpha &= 2-(2n-1)^2 \end{aligned}$$

The second category of solution for $\psi(x) = g(x)$ has the form

$$\begin{aligned} g(x) &= Ag_1(x) + Bg_2(x) \\ &= A\sum_{j=0}^{n}(-\mu)^j \frac{(n+j-1)!}{(n-j)!(2j+1)!}x^{2j+1} \\ &+ B(1-K+Kx^2)^{3/2}\sum_{j=0}^{n-1}(-\mu)^j \frac{(n+j)!}{(n-j-1)!(2j)!}x^{2j} \qquad (3.6.2) \end{aligned}$$

for the values

$$\begin{aligned} \mu &= 4 - \frac{4}{4n^2 - 1 + \alpha} \\ K + \alpha &= 2(1 - 2n^2) \end{aligned}$$

where A and B are arbitrary constants.

It is remarkable that these solutions are expressed completely as combinations of polynomials and algebraic functions. From our general class of solutions (3.6.1) and (3.6.2) it is possible to generate particular solutions found previously. Consider one example with $\alpha = 0$ and $K = -7 (n = 2)$. Then $\gamma = \frac{7}{2}$ and it is easy to verify that equation (3.6.1) becomes

$$\psi = A\left(1 - \frac{7}{2}x^2 + \frac{49}{24}x^4\right) + Bx\left(1 - \frac{7}{8}x^2\right)^{3/2}$$

Thus we have regained the Tikekar (1990) solution for a superdense neutron star from our general solutions. Many other particular solutions found in the literature are also contained in our general solutions, e.g., the model of Patel and Koppar (1987). The solutions (3.6.1) and (3.6.2) reduce to the Maharaj and Leach (1996) model when $\alpha = 0$. Our solutions are applicable to a charged superdense star with spheroidal geometry. When $\alpha = 0$ we obtain uncharged relativistic stars which model ultradense barotropic matter.

In the general solution of Sharma *et al* (2001) it is not possible to isolate the spheroidal parameter K as that solution is given in terms of special functions. Our solutions are in terms of simple elementary functions which facilitate a study of the physical features, in particular the role of K. The exact solutions (3.6.1) and (3.6.2) make it possible to analyse the role of the spheroidal parameter K and the connection to the electromagnetic field. In particular it is possible to make the following comment about the special role that the spheroidal parameter K has in charged solutions. The form of the solution for the uncharged relativistic star is similar to (3.6.1) and (3.6.2); however the models are different because the coefficients of the polynomials (namely γ and μ) differ by the parameter α. If the parameter $\alpha \geq 0$ then we observe that

$$K_{(\alpha\neq 0)} < K_{(\alpha=0)}$$

Hence the presence of charge directly affects the spheroidal geometry through the parameter K. The geometry of the hypersurfaces $\{t = \text{constant}\}$ in the spacetime manifold is related to the electromagnetic field via the relationships $K = 2-(2n-1)^2-\alpha$ and $K = 2(1-2n^2)-\alpha$. Such explicit relationships connecting the spacetime geometry to the energy momentum (or electromagnetic field) are rare in exact solutions. The presence of charge α decreases the value of the spheroidal parameter K in our solutions.

We make a few brief comments about the physics of the models found in this chapter. If $0 < K < 1$ ($\alpha < 0$) then ρ remains positive in the region

$$(1-x^2) < \frac{3(1-K)}{K(1-K-\alpha/2)} \Longrightarrow r^2 < \frac{3R^2(1-K)}{K(1-K-\alpha/2)}$$

which restricts the size of the configuration. When $K < 0$ ($\alpha > 0$) there is no restriction on ρ. Hence ρ is positive in the interior of the star. It is clear from (3.2.5a) and (3.2.6) that $d\rho/dr < 0$ for $K < 0$ ($\alpha > 0$). Consequently the energy density decreases from the centre to the boundary. For the pressure to be vanish at the boundary $r = a$ we require that

$$\left(\frac{2}{R^2\sqrt{1-a^2/R^2}}\left[\frac{\dot{\psi}}{\psi}\right]_{r=a} - \frac{1}{a^2}\right)\frac{1-a^2/R^2}{1-Ka^2/R^2} + \frac{1}{a^2} + \frac{\alpha K a^2/R^2}{2R^2(1-Ka^2/R^2)^2} = 0$$

where ψ is given by (3.6.1) or (3.6.2). This will constrain the values of the constants A and B. To match the line element (2.2.8) with the Reissner-Nordstrom metric (2.5.2) across the boundary at $r = a$ we require the continuity of the gravitational potentials and of the radial electric field at $r = a$. Continuity of the gravitational potentials yields the relationships between the constants $A,\ B,\ K,\ a$ and R as

$$\left(1-\frac{2m}{a}+\frac{q^2(a)}{a^2}\right) \quad = \quad [A\psi_1(a)+B\psi_2(a)]^2$$

$$\left(1-\frac{2m}{a}+\frac{q^2(a)}{a^2}\right)^{-1} \quad = \quad \frac{1-Ka^2/R^2}{1-a^2/R^2}$$

The continuity of electric field yields the form

$$q^2(a) = -\frac{\alpha K a^6/R^4}{(1-Ka^2/R^2)^2}$$

for the charge at the boundary. This shows that continuity of the metric functions across the boundary $r=a$ is easily achieved. The matching conditions at $r=a$ may place restrictions on the metric coefficients ν and its first derivative for uncharged matter; and the pressure may be nonzero if there is a surface layer of charge. However, there are sufficient free parameters to satisfy the necessary conditions that arise for a particular spheroidal model. It is interesting to note that our solutions may be interpreted as models for relativistic anisotropic stars where the parameter α plays the role of the anisotropy factor. Isotropic and uncharged stars can be regained when $\alpha=0$. Chaisi and Maharaj (2005), Dev and Gleiser (2002, 2003), and Maharaj and Chaisi (2006) provide some recent treatments involving the physics of anisotropic matter.

3.7 A solution-generating algorithm

It is possible to utilise a variety of approaches to generate solutions to a system of equations. Many different methods are listed in Stephani *et al* (2003). Our aim in this section is to introduce a new algorithm which is easy to use. This algorithm enables us to find another new solution to the field equations (3.2.5) for a specified seed solution.

3.7.1 The algorithm

Suppose an explicit solution to (3.2.5) is known where

$$(\psi,\ \rho,\ p,\ E^2) = (\psi_0,\ \rho_0,\ p_0,\ E_0^2) \tag{3.7.1}$$

and the functions $\psi_0,\ \rho_0,\ p_0$ and E_0 are explicitly given. Then the equations in (3.2.5) and (3.2.7) are satisfied and we obtain

$$\rho_0 = \frac{1-K}{R^2}\frac{(3-K+Kx^2)}{(1-K+Kx^2)^2} - \frac{1}{2}E_0^2 \tag{3.7.2a}$$

$$p_0 = \frac{1}{R^2(1-K+Kx^2)}\left(-2x\frac{\dot{\psi}_0}{\psi_0} + K - 1\right) + \frac{1}{2}E_0^2 \tag{3.7.2b}$$

$$0 = (1-K+Kx^2)\ddot{\psi}_0 - Kx\dot{\psi}_0 + K(\alpha + K - 1)\psi_0 \tag{3.7.2c}$$

Now we seek a new solution of the form

$$(\psi,\ \rho,\ p,\ E^2) = (\psi_0 W,\ \rho_0,\ p_0 + P,\ E_0^2) \tag{3.7.3}$$

where W and P are arbitrary functions of x to be determined and $(\psi_0,\ \rho_0,\ p_0,\ E_0^2)$ are given by (3.7.1). When $P = 0$ and $W = 1$ we note that the new solution (3.7.3) reduces to the seed solution (3.7.1). For (3.7.3) to be a solution, (3.2.5) and (3.2.7) must be satisfied and we have

$$\rho_0 = \frac{1-K}{R^2}\frac{(3-K+Kx^2)}{(1-K+Kx^2)^2} - \frac{1}{2}E_0^2 \tag{3.7.4a}$$

$$p_0 + P = \frac{1}{R^2(1-K+Kx^2)}\left[-2x\left(\frac{\dot{\psi}_0}{\psi_0} + \frac{\dot{W}}{W}\right) + K - 1\right] + \frac{1}{2}E_0^2 \tag{3.7.4b}$$

$$\begin{aligned} 0 &= (1-K+Kx^2)(\ddot{\psi}_0 W + 2\dot{\psi}_0\dot{W} + \psi_0\ddot{W}) \\ &- Kx(\dot{\psi}_0 W + \psi_0\dot{W}) + K(\alpha + K - 1)\psi_0 W \end{aligned} \tag{3.7.4c}$$

It remains to integrate (3.7.4) and obtain P and W. Comparing (3.7.2) and (3.7.4) we generate a pair of differential equations restricting P and W:

$$P = -2x\frac{\dot{W}}{W}\frac{1}{R^2(1-K+Kx^2)} \tag{3.7.5a}$$

$$0 = (1-K+Kx^2)\psi_0\ddot{W} - Kx\psi_0\dot{W} + 2(1-K+Kx^2)\dot{\psi}_0\dot{W} \tag{3.7.5b}$$

We can rewrite (3.7.5b) as

$$\frac{\ddot{W}}{\dot{W}} = \frac{Kx}{(1-K+Kx^2)} - 2\frac{\dot{\psi}_0}{\psi_0}$$

which may be integrated to yield

$$\dot{W} = C\frac{(1-K+Kx^2)^{1/2}}{\psi_0^2}$$

where C is the first constant of integration. Since the variables W and x in this differential equation separate we have

$$W = C\int \frac{(1-K+Kx^2)^{1/2}}{\psi_0^2}dx + D \tag{3.7.6}$$

where D is the second constant of integration. From (3.7.5a) and (3.7.6) we can find a form for P given by

$$\begin{aligned} P &= -\frac{2}{R^2}\left[\frac{C(1-K+Kx^2)^{1/2}}{\psi_0^2}\right]\left[C\int \frac{(1-K+Kx^2)^{1/2}}{\psi_0^2}dx + D\right]^{-1} \\ &\times \frac{x}{(1-K+Kx^2)} \end{aligned} \tag{3.7.7}$$

Hence a new solution to the Einstein-Maxwell system is given by (3.7.3) where W and P are given by (3.7.6) and (3.7.7) respectively. The integral in (3.7.7) can be evaluated once a particular solution ψ_0 in (3.7.1) is specified. For the values $C = 0$ and $D = 1$ we have $W = 1$ and $P = 0$, and we have regained the seed solution (3.7.1).

3.7.2 Some examples

We illustrate the algorithm in §3.7.1 with two simple examples corresponding to uncharged and charged relativistic stars respectively. As a first example we consider uncharged stars so that $E = 0$. For the parameter value $K = -2$ ($\alpha = 0$) we obtain a particular solution from (3.6.2) for ψ as

$$\psi_0 = (3-2x^2)^{3/2}$$

where we have set $A = 0$ and $B = 1$. Hence the particular solution of the Einstein field equations becomes

$$\psi_0 = (3-2x^2)^{3/2} \tag{3.7.8a}$$

$$\rho_0 = \frac{3(5-2x^2)}{R^2(3-2x^2)^2} \tag{3.7.8b}$$

$$p_0 = \frac{9(2x^2-1)}{R^2(3-2x^2)^2} \tag{3.7.8c}$$

Note that (3.7.8) is contained in the seed solution (3.7.1). Then from (3.7.3), (3.7.6) and (3.7.7) we generate the new solution

$$\psi = (3-2x^2)^{3/2}\left[C\left(\frac{x(9-4x^2)}{27(3-2x^2)^{3/2}}\right)+D\right] \tag{3.7.9a}$$

$$\rho = \frac{3(5-2x^2)}{R^2(3-2x^2)^2} \tag{3.7.9b}$$

$$p = \frac{9(2x^2-1)}{R^2(3-2x^2)^2} - \frac{2Cx}{R^2(3-2x^2)^{7/2}}\left[C\left(\frac{x(9-4x^2)}{27(3-2x^2)^{3/2}}\right)+D\right]^{-1} \tag{3.7.9c}$$

for uncharged matter.

Other solutions are possible in the presence of the electromagnetic field. As a second example we analyse a charged star so that $E \neq 0$. For the parameter value $K = -3$ and $\alpha = 1$ we observe from (3.6.2) that

$$\psi_0 = x - \frac{x^3}{2} \tag{3.7.10a}$$

$$\rho_0 = \frac{3(15-7x^2)}{2R^2(4-3x^2)^2} \tag{3.7.10b}$$

$$p_0 = \frac{143x^2-3(30+19x^4)}{2R^2(2-x^2)(4-3x^2)^2} \tag{3.7.10c}$$

$$E_0^2 = \frac{3(1-x^2)}{R^2(4-3x^2)^2} \tag{3.7.10d}$$

is a particular solution of the Einstein-Maxwell field equations. Note that (3.7.10) corresponds to the seed solution (3.7.1). Then from (3.7.3), (3.7.6) and (3.7.7) we obtain the new solution

$$\psi = \left(x - \frac{x^3}{2}\right)\left[C\left(\frac{(4-3x^2)^{3/2}}{2x(-2+x^2)}\right) + D\right] \tag{3.7.11a}$$

$$\rho = \frac{3(15-7x^2)}{2R^2(4-3x^2)^2} \tag{3.7.11b}$$

$$\begin{aligned} p = {} & \frac{143x^2 - 3(30+19x^4)}{2R^2(2-x^2)(4-3x^2)^2} - \\ & \frac{8C}{R^2x(-2+x^2)^2(4-3x^2)^{1/2}}\left[C\left(\frac{(4-3x^2)^{3/2}}{2x(-2+x^2)}\right) + D\right]^{-1} \end{aligned} \tag{3.7.11c}$$

$$E^2 = \frac{3(1-x^2)}{R^2(4-3x^2)^2} \tag{3.7.11d}$$

for charged matter.

Therefore, using the algorithm of §3.7.1, we have generated two new solutions (3.7.9) and (3.7.11) from specified seed solutions. Clearly this algorithm can be used on other seed solutions to find new solutions to the Einstein and Einstein-Maxwell systems. This algorithm will work as long as the integration in (3.7.7) is possible.

Chapter 4

Classes of exact Einstein-Maxwell solutions

4.1 Introduction

The objective of this chapter is to provide systematically a rich family of Einstein-Maxwell solutions similar to the recent treatment of Komathiraj and Maharaj (2007a) and John and Maharaj (2006). Komathiraj and Maharaj (2007a) presented a general class of Einstein-Maxwell solutions that contains Tikekar (1990) spheroidal stars as a special case, and which are physically viable neutron star models. John and Maharaj (2006) found an uncharged star which approximates a polytrope close to the centre. Hence, the approach followed in this chapter has proved to be a fruitful avenue for generating new exact solutions for describing the interior spacetimes of charged spheres. In §4.2 we choose particular forms for one of the gravitational potentials $Z(x)$ and the electric field intensity $E(x)$, which enables us to obtain the condition of pressure isotropy in the remaining gravitational potential $y(x)$. Two cases arise: $K = \alpha \neq 0$ and $K \neq \alpha$. In §4.3 we consider the special case $K = \alpha \neq 0$ which has solutions in terms of elementary functions. For the general case $K \neq \alpha$, the solution to the Einstein-Maxwell system is reduced to a hypergeometric differential equation which can be integrated using the method of Frobenius as shown in §4.4. It is then possible to find exact solutions in terms of hypergeometric functions. In §4.5 we generate two linearly independent classes of solutions by determining the specific restriction on the parameters for a terminating series. We show the general solution can be written explicitly in terms of elementary functions in §4.6. We demonstrate that uncharged solutions are regained in the appropriate limit. We show that other solutions exist to the Einstein-Maxwell system, outside the class considered in this chapter. Some brief comments relating to physical features of the model are made in §4.7. The results of this chapter have been submitted for publication in Komathiraj and Maharaj (2007b).

4.2 Particular choice for $Z(x)$

We study a particular form of the Einstein-Maxwell system by making explicit choices for Z and E. For the metric function Z we make the choice

$$Z(x) = \frac{(1+kx)^2}{(1+x)} \tag{4.2.1}$$

where k is a real constant. For the choice (4.2.1) the line element (2.2.8) becomes

$$ds^2 = -A^2y^2dt^2 + \frac{(1+x)}{4Cx(1+kx)}dx^2 + \frac{x}{C}(d\theta^2 + \sin^2\theta d\phi^2)$$

and we need to find the function $y(x)$. Note that the choice (4.2.1) ensures that the metric function $e^{2\lambda}$ is regular and finite at the centre of the sphere. When $k = 1$, in the absence of charge, we regain the interior Schwarzschild metric. Also observe that when $k = 0$ we regain the metric function considered by Hansraj and Maharaj (2006) which generalises the Finch and Skea neutron star model (1989). We have chosen the above form as it provides for a wider range of possibilities than the solutions of Hansraj and Maharaj (2006), and it produces charged and uncharged solutions which are necessary for a realistic model. On substituting (4.2.1) in (2.4.8c) we obtain

$$\begin{aligned} & 4(1+kx)^2(1+x)\ddot{y} + 2(1+kx)(2k-1+kx)\dot{y} \\ + & \left[(1-k)^2 - \frac{E^2(1+x)^2}{Cx}\right]y = 0 \end{aligned} \tag{4.2.2}$$

It is convenient at this point to introduce the following transformation

$$\frac{1}{k} + x = KX \quad (4.2.3a)$$

$$\frac{1-k}{k} = K \quad (4.2.3b)$$

$$y(x) = Y(X) \quad (4.2.3c)$$

This transformation enables us to transform the second order differential equation (4.2.2) to a simpler form. Under the transformation (4.2.3), equation (4.2.2) becomes

$$4X^2(X-1)\frac{d^2Y}{dX^2} + 2X(X-2)\frac{dY}{dX} + \left[K - \frac{E^2K(K+1)^2(X-1)^2}{C[K(X-1)-1]}\right]Y = 0 \quad (4.2.4)$$

in terms of the new dependent and independent variables Y and X respectively.

It is necessary to specify the electric field intensity E to integrate the equation (4.2.4). A variety of choices for E is possible but only a few are physically reasonable and generate closed form solutions. We can reduce (4.2.4) to simpler form if we let

$$\frac{E^2}{C} = \frac{\alpha[K(X-1)-1]}{K(K+1)^2(X-1)^2} = \frac{\alpha K x}{(K+1)^2(1+x)^2} \quad (4.2.5)$$

where α is a constant. The form (4.2.5) for E^2 is physically palatable because E remains regular and continuous throughout the sphere. In addition, the field intensity E vanishes at the stellar centre and has positive values in the interior of the star for relevant choices of the constants α and K. Upon substituting the choice (4.2.5) in equation (4.2.4) we obtain

$$4X^2(X-1)\frac{d^2Y}{dX^2} + 2X(X-2)\frac{dY}{dX} + (K-\alpha)Y = 0 \quad (4.2.6)$$

which is the master equation for the system (2.4.8). When $\alpha = 0$ there is no charge. The differential equation (4.2.6) has to be integrated to find an exact model for a charged sphere.

4.3 Special case $K = \alpha \neq 0$: Elementary functions

We can immediately integrate (4.2.6) for the special case $K = \alpha \neq 0$. Equation (4.2.6) is separable and we obtain the solution

$$Y(X) = c_1(\sqrt{X-1} - \arctan\sqrt{X-1}) + c_2 \quad (4.3.1)$$

where c_1 and c_2 are constants of integration. In terms of the independent variable x we can write

$$y(x) = c_1 \left(\sqrt{\frac{1+x}{K}} - \arctan \sqrt{\frac{1+x}{K}} \right) + c_2 \tag{4.3.2}$$

Hence, the complete solution of the Einstein-Maxwell system (2.4.8) is then given by

$$e^{2\lambda} = \frac{(K+1)^2(1+x)}{(K+1+x)^2} \tag{4.3.3a}$$

$$e^{2\nu} = A^2 \left[c_1 \left(\sqrt{\frac{1+x}{K}} - \arctan \sqrt{\frac{1+x}{K}} \right) + c_2 \right]^2 \tag{4.3.3b}$$

$$\frac{\rho}{C} = \frac{K^2(6+x) - 6(1+x)^2}{2(K+1)^2(1+x)^2} \tag{4.3.3c}$$

$$\begin{aligned} \frac{p}{C} &= \frac{2c_1(K+1+x)}{\sqrt{K}(K+1)^2\sqrt{1+x}\left[c_1\left(\sqrt{\frac{1+x}{K}} - \arctan\sqrt{\frac{1+x}{K}}\right) + c_2\right]} \\ &+ \frac{2(1+x)^2 - K^2(2+x)}{2(K+1)^2(1+x)^2} \end{aligned} \tag{4.3.3d}$$

$$\frac{E^2}{C} = \frac{K^2 x}{(K+1)^2(1+x)^2} \tag{4.3.3e}$$

Note that the charged solution (4.3.3) does not have an uncharged analogue as the electric field intensity E cannot vanish (except at the centre). This effect essentially results from our condition that $\alpha = K(\neq 0)$. This means that this solution models a sphere that is always charged and hence cannot attain a neutral state. A particular class in the family of solutions found by Hansraj and Maharaj (2006) also demonstrates the same feature and $E \neq 0$. The model (4.3.3) is a simple solution of the Einstein-Maxwell system which is expressed in terms of elementary functions.

4.4 General case $K \neq \alpha$: Hypergeometric series

With $K \neq \alpha$ the master equation (4.2.6) is difficult to solve. However it can be transformed to a hypergeometric differential equation which can be integrated using the method of Frobenius. We now introduce a new function $U(X)$ such that

$$Y(X) = X^a U(X) \tag{4.4.1}$$

where a is constant. On substituting (4.4.1) in (4.2.6) we obtain

$$\begin{aligned} & 4X^2(X-1)\frac{d^2U}{dX^2} + 2X[(4a+1)X - 2(2a+1)]\frac{dU}{dX} \\ + & \quad [2a(2a-1)X + K - \alpha - 4a^2]U = 0 \end{aligned} \tag{4.4.2}$$

We observe that there is considerable simplification if we make the choice

$$K - \alpha = 4a^2$$

This then gives

$$2X(X-1)\frac{d^2U}{dX^2} + [(4a+1)X - 2(2a+1)]\frac{dU}{dX} + a(2a-1)U = 0 \tag{4.4.3}$$

which is a hypergeometric equation in terms of the new dependent variable U and independent variable X. When $a = 0$ then $\alpha = K$ and we regain the result of §4.3. Therefore we take $a \neq 0$ in this section to ensure that $\alpha \neq K$.

As the point $X = 1$ is a regular singular point of (4.4.3), there exist two linearly independent solutions of the form of a power series with centre $X = 1$. We therefore assume

$$U = \sum_{i=0}^{\infty} c_i (X-1)^{i+b}, c_0 \neq 0 \tag{4.4.4}$$

where c_i are the coefficients of the series and b is a constant. For a legitimate solution we need to determine the coefficients c_i and the parameter b explicitly. On substituting (4.4.4) into (4.4.3) we obtain

$$\begin{aligned} & c_0 b(2b-3)(X-1)^{b-1} \\ + \sum_{i=1}^{\infty} \{ & c_i(i+b)(2i+2b-3) + c_{i-1}[(i+b-1)(2i+2b+4a-3) \\ & + a(2a-1)]\}(X-1)^{i+b-1} = 0 \end{aligned} \tag{4.4.5}$$

The coefficients of the various powers of $(X-1)$ must vanish. Equating the coefficient of $(X-1)^{b-1}$ in (4.4.5) to zero we obtain

$$c_0 b(2b-3) = 0$$

Since $c_0 \neq 0$ we must have $b = 0$ or $b = 3/2$. Equating the coefficient of $(X-1)^{i+b-1}$ in (4.4.5) to zero we obtain

$$c_i = -\frac{[(i+b-1)(2i+2b+4a-3)+a(2a-1)]}{(i+b)(2i+2b-3)}c_{i-1}, \quad i \geq 1 \tag{4.4.6}$$

The relation (4.4.6) is the recurrence formula, or difference equation, governing the structure of the solution.

It is possible to express the coefficients $c_1, c_2, c_3, \ . \ . \ .$ in terms of the leading coefficient c_0 by establishing a general structure for the coefficients by considering the leading terms. We generate the expression

$$c_i = (-1)^i \prod_{p=1}^{i} \frac{[(p+b-1)(2p+2b+4a-3)+a(2a-1)]}{(p+b)(2p+2b-3)}c_0 \tag{4.4.7}$$

where the conventional symbol $\prod$ denotes multiplication. It is easy to establish that the result (4.4.7) holds for all positive integers p using the principle of mathematical induction. From (4.4.6), we have

$$c_1 = (-1)^1 \prod_{p=1}^{1} \frac{[(p+b-1)(2p+2b+4a-3)+a(2a-1)]}{(p+b)(2p+2b-3)}c_0$$

Now we assume the form for c_q as

$$c_q = (-1)^q \prod_{p=1}^{q} \frac{[(p+b-1)(2p+2b+4a-3)+a(2a-1)]}{(p+b)(2p+2b-3)}c_0$$

which is the inductive step. From (4.4.6), we have

$$\begin{aligned}
c_{q+1} &= -\frac{[(q+b)(2q+2b+4a-1)+a(2a-1)]}{(q+1+b)(2q+2b-1)}c_q \\
&= -\frac{[(q+b)(2q+2b+4a-1)+a(2a-1)]}{(q+1+b)(2q+2b-1)} \\
&\times \ (-1)^q \prod_{p=1}^{q} \frac{[(p+b-1)(2p+2b+4a-3)+a(2a-1)]}{(p+b)(2p+2b-3)}c_0 \\
&= (-1)^{q+1} \prod_{p=1}^{q+1} \frac{[(p+b-1)(2p+2b+4a-3)+a(2a-1)]}{(p+b)(2p+2b-3)}c_0
\end{aligned}$$

Hence (4.4.7) is true for all positive integers p.

We can now generate two linearly independent solutions to (4.4.3) with the help of (4.4.4) and (4.4.7). For the parameter value $b = 0$ we obtain the first solution

$$U_1(X) = c_0 \left[1 + \sum_{i=1}^{\infty} (-1)^i \prod_{p=1}^{i} \frac{[(p-1)(2p+4a-3) + a(2a-1)]}{p(2p-3)} (X-1)^i \right]$$

For the parameter value $b = 3/2$ we obtain the second solution

$$U_2(X) = c_0 (X-1)^{3/2} \left[1 + \sum_{i=1}^{\infty} (-1)^i \prod_{p=1}^{i} \frac{[(2p+1)(p+2a) + a(2a-1)]}{p(2p+3)} (X-1)^i \right]$$

Since the functions U_1 and U_2 are linearly independent we have found the general solution to (4.4.3). In terms of the original variable x, the functions U_1 and U_2 given above become

$$\begin{aligned} y_1(x) &= c_0 \left(\frac{K+1+x}{K} \right)^a \\ &\times \left[1 + \sum_{i=1}^{\infty} (-1)^i \prod_{p=1}^{i} \frac{[(p-1)(2p+4a-3) + a(2a-1)]}{p(2p-3)} \left(\frac{1+x}{K} \right)^i \right] \end{aligned} \tag{4.4.8}$$

and

$$\begin{aligned} y_2(x) &= c_0 \left(\frac{K+1+x}{K} \right)^a \left(\frac{1+x}{K} \right)^{3/2} \\ &\times \left[1 + \sum_{i=1}^{\infty} (-1)^i \prod_{p=1}^{i} \frac{[(2p+1)(p+2a) + a(2a-1)]}{p(2p+3)} \left(\frac{1+x}{K} \right)^i \right] \end{aligned} \tag{4.4.9}$$

where we have used (4.2.3) and (4.4.1). Thus the general solution to the differential equation (4.2.2), for the choice of the electric field (4.2.5), is given by

$$y(x) = A_1 y_1(x) + A_2 y_2(x) \tag{4.4.10}$$

where A_1 and A_2 are arbitrary constants, $K = (1-k)/k$, $a^2 = (K-\alpha)/4$ and y_1, y_2 are given by (4.4.8) and (4.4.9) respectively. From (4.4.10) and (2.4.8) we can write the exact solution of the Einstein-Maxwell system in the form

$$e^{2\lambda} = \frac{(K+1)^2(1+x)}{(K+1+x)^2} \tag{4.4.11a}$$

$$e^{2\nu} = A^2y^2 \tag{4.4.11b}$$

$$\frac{\rho}{C} = \frac{(K^2-1)(3+x)-x(5+3x)}{(K+1)^2(1+x)^2} - \frac{\alpha Kx}{2(K+1)^2(1+x)^2} \tag{4.4.11c}$$

$$\frac{p}{C} = \frac{4(K+1+x)^2}{(K+1)^2(1+x)}\frac{\dot{y}}{y} + \frac{1-K^2+x}{(K+1)^2(1+x)} + \frac{\alpha Kx}{2(K+1)^2(1+x)^2} \tag{4.4.11d}$$

$$\frac{E^2}{C} = \frac{\alpha Kx}{(K+1)^2(1+x)^2} \tag{4.4.11e}$$

Unlike the solution presented in §4.3, the models found in this section, in general, cannot be written in terms of elementary functions as the series (4.4.8) and (4.4.9) do not terminate. However terminating series are possible for particular values of a, which leads to elementary functions, as we show in §4.5.

4.5 Terminating series

The general solution (4.4.10) is given in the form of a series and can be expressed in terms of hypergeometric functions which are special functions. It is well known that hypergeometric functions can be written in terms of elementary functions for particular parameter values. This statement is also true for the solution found in §4.4 for particular values of the parameter a as the two series then terminate. Consequently two sets of general solutions in terms of elementary functions can be found by restricting the range of values of a so that the series terminates.

4.5.1 The first solution

On substituting $b = 0$ in (4.4.6) and setting $a = -n$, for integer values of n, we obtain after simplification

$$c_i = -\frac{(n-i+1)(2n-2i+3)}{i(2i-3)}c_{i-1}, \ i \geq 1 \tag{4.5.1}$$

where n is a fixed integer. We observe from (4.5.1) that $c_{n+1} = 0$. Consequently the remaining coefficients $c_{n+2},\ c_{n+3},\ c_{n+4},\ \ldots$ vanish. Equation (4.5.1) may be solved to yield

$$c_i = (-1)^{i-1}\frac{(2i-1)(2n+1)!}{(2i)!(2n-2i+1)!}c_0,\ 0 \leq i \leq n \tag{4.5.2}$$

Then from (4.4.4) (when $b = 0$) and (4.5.2), we obtain

$$U_1(X) = c_0\sum_{i=0}^{n}(-1)^{i-1}\frac{(2i-1)(2n+1)!}{(2i)!(2n-2i+1)!}(X-1)^i \tag{4.5.3}$$

where $a = -n$.

On substituting $b = 0$ in (4.4.6) and setting $a = \frac{1}{2} - n$ we obtain

$$c_i = -\frac{(n-i+1)(2n-2i+1)}{i(2i-3)}c_{i-1},\ i \geq 1 \tag{4.5.4}$$

where n is fixed integer. It is easy to see from (4.5.4) that $c_{n+1} = 0$. Clearly the subsequent coefficients $c_{n+2},\ c_{n+3},\ c_{n+4},\ \ldots$ vanish. Equation (4.5.4) has the solution

$$c_i = (-1)^{i-1}\frac{(2i-1)(2n)!}{(2i)!(2n-2i)!}c_0,\ 0 \leq i \leq n \tag{4.5.5}$$

Now from (4.4.4) (when $b = 0$) and (4.5.5), we obtain

$$U_1(X) = c_0\sum_{i=0}^{n}(-1)^{i-1}\frac{(2i-1)(2n)!}{(2i)!(2n-2i)!}(X-1)^i \tag{4.5.6}$$

where $a = \frac{1}{2} - n$.

The polynomials (4.5.3) and (4.5.6) generate the first solution of the differential equation (4.4.3) for appropriate values of a.

4.5.2 The second solution

On substituting $b = 3/2$ in (4.4.6) and setting $a = -n$, where n is fixed integer, we obtain after simplification

$$c_i = -\frac{(n-i)(2n-2i-1)}{i(2i+3)}c_{i-1},\ i \geq 1 \tag{4.5.7}$$

From (4.5.7) we have $c_n = 0$. The subsequent coefficients $c_{n+1},\ c_{n+2},\ c_{n+3},\ \ldots$ vanish. Equation (4.5.7) has the solution

$$c_i = 6(-1)^i\frac{(i+1)(2n-2)!}{(2i+3)!(2n-2i-2)!}c_0,\ 0 \leq i \leq n-1 \tag{4.5.8}$$

From (4.4.4) (when $b = 3/2$) and (4.5.8) we obtain

$$U_2(X) = 6c_0(X-1)^{3/2}\sum_{i=0}^{n-1}(-1)^i\frac{(i+1)(2n-2)!}{(2i+3)!(2n-2i-2)!}(X-1)^i \tag{4.5.9}$$

where $a = -n$.

On substituting $b = 3/2$ in (4.4.6) and setting $a = \frac{1}{2} - n$ we obtain after simplification

$$c_i = -\frac{(n-i-1)(2n-2i-1)}{i(2i+3)}c_{i-1},\ i \geq 1 \tag{4.5.10}$$

From (4.5.10) we have that $c_{n-1} = 0$. The remaining coefficients $c_n,\ c_{n+1},\ c_{n+2},\ \ldots$ vanish. Equation (4.5.10) has the solution

$$c_i = 6(-1)^i\frac{(i+1)(2n-3)!}{(2i+3)!(2n-2i-3)!}c_0,\ 0 \leq i \leq n-2 \tag{4.5.11}$$

From (4.4.4) (when $b = 3/2$) and (4.5.11) we obtain

$$U_2(X) = 6c_0(X-1)^{3/2}\sum_{i=0}^{n-2}(-1)^i\frac{(i+1)(2n-3)!}{(2i+3)!(2n-2i-3)!}(X-1)^i \tag{4.5.12}$$

where $a = \frac{1}{2} - n$.

The products of polynomials and algebraic functions given in (4.5.9) and (4.5.12) comprise the second solution of the differential equation (4.4.3) for appropriate values of a. Clearly they are linearly independent from (4.5.3) and (4.5.6) respectively.

4.6 Elementary functions

Thus we have found the general solution to (4.4.3) by restricting the values of a so that only elementary functions appear. The elementary functions are expressible as polynomials and products of polynomials with algebraic functions. From (4.5.3) and (4.5.9) we can express the first category of solution as

$$\begin{aligned} U(X) &= A_1\sum_{i=0}^{n}(-1)^{i-1}\frac{(2i-1)}{(2i)!(2n-2i+1)!}(X-1)^i \\ &+ A_2(X-1)^{3/2}\sum_{i=0}^{n-1}(-1)^i\frac{(i+1)}{(2i+3)!(2n-2i-2)!}(X-1)^i \end{aligned} \tag{4.6.1}$$

where $a = -n$. From (4.5.6) and (4.5.12) the second category of solution is given by

$$\begin{aligned} U(X) &= A_1 \sum_{i=0}^{n} (-1)^{i-1} \frac{(2i-1)}{(2i)!(2n-2i)!} (X-1)^i \\ &+ A_2 (X-1)^{3/2} \sum_{i=0}^{n-2} (-1)^i \frac{(i+1)}{(2i+3)!(2n-2i-3)!} (X-1)^i \end{aligned} \tag{4.6.2}$$

where $a = \frac{1}{2} - n$. In terms of the original variable x, we can write (4.6.1) as

$$\begin{aligned} y(x) &= \\ & A_1 \left(\frac{K}{K+1+x}\right)^n \sum_{i=0}^{n} (-1)^{i-1} \frac{(2i-1)}{(2i)!(2n-2i+1)!} \left(\frac{1+x}{K}\right)^i \\ + & A_2 \left(\frac{K}{K+1+x}\right)^n \left(\frac{1+x}{K}\right)^{3/2} \sum_{i=0}^{n-1} (-1)^i \frac{(i+1)}{(2i+3)!(2n-2i-2)!} \left(\frac{1+x}{K}\right)^i \end{aligned} \tag{4.6.3}$$

where

$$K - \alpha = 4n^2$$

relates the constants K and n. Also, in terms of x, we can show that (4.6.2) becomes

$$\begin{aligned} y(x) &= \\ & A_1 \left(\frac{K}{K+1+x}\right)^{n-1/2} \sum_{i=0}^{n} (-1)^{i-1} \frac{(2i-1)}{(2i)!(2n-2i)!} \left(\frac{1+x}{K}\right)^i \\ + & A_2 \left(\frac{K}{K+1+x}\right)^{n-1/2} \left(\frac{1+x}{K}\right)^{3/2} \sum_{i=0}^{n-2} (-1)^i \frac{(i+1)}{(2i+3)!(2n-2i-3)!} \left(\frac{1+x}{K}\right)^i \end{aligned} \tag{4.6.4}$$

where

$$K - \alpha = 4n(n-1) + 1$$

relates the constants K and n.

Therefore, we have shown that two categories of solutions in terms of elementary functions can be extracted from the general series in §4.4. The solutions in (4.6.3) and

(4.6.4) have a simple form, and they have been expressed completely as combinations of polynomials and algebraic functions. This has the advantage of simplifying the investigation into the physical properties of a dense charged star. As the metric function (4.2.1) and the electric field intensity (4.2.5) have not been considered before, we believe that the Einstein-Maxwell solutions found here have not been published previously. It is interesting to observe that our treatment has brought together the charged and uncharged models for a relativistic star. If we set $\alpha = 0$ in the Einstein-Maxwell solutions (4.6.3) and (4.6.4) then we obtain the solutions for the uncharged case directly. Thus our approach has the welcome feature of producing uncharged solutions when $E = 0$; it is possible that the uncharged solutions produced in this procedure may be new.

We illustrate this feature with an example. We observe that when $K - \alpha = 4$ $(n = 1)$, (4.6.3) becomes

$$y(x) = \frac{a_1(K+3+3x) + a_2(1+x)^{3/2}}{K+1+x} \tag{4.6.5}$$

where a_1 and a_2 are constants. On substituting (4.6.5) in (4.4.11) we obtain the general solution to the Einstein-Maxwell system of equations as

$$e^{2\lambda} = \frac{(K+1)^2(1+x)}{(K+1+x)^2} \tag{4.6.6a}$$

$$e^{2\nu} = A^2\left[\frac{a_1(K+3+3x) + a_2(1+x)^{3/2}}{K+1+x}\right]^2 \tag{4.6.6b}$$

$$\frac{\rho}{C} = \frac{6(K^2-1) + x[(K+6)(K-2) - 6x]}{2(K+1)^2(1+x)^2} \tag{4.6.6c}$$

$$\begin{aligned}\frac{p}{C} &= \frac{2(K+1+x)[4a_1K + a_2\sqrt{1+x}(3K+1+x)]}{(K+1)^2(1+x)[a_1(K+3+3x) + a_2(1+x)^{\frac{3}{2}}]} \\ &+ \frac{2(1+x)(1-K^2+x) + K(K-4)x}{2(K+1)^2(1+x)^2}\end{aligned} \tag{4.6.6d}$$

$$\frac{E^2}{C} = \frac{K(K-4)x}{(K+1)^2(1+x)^2} \tag{4.6.6e}$$

for our chosen parameter values. When $\alpha = 0$ $(K = 4)$, the electromagnetic field vanishes and we get

$$e^{2\lambda} = \frac{25(1+x)}{(5+x)^2} \tag{4.6.7a}$$

$$e^{2\nu} = A^2\left[\frac{a_1(7+3x)+a_2(1+x)^{3/2}}{5+x}\right]^2 \tag{4.6.7b}$$

$$\frac{\rho}{C} = \frac{45+x(10-3x)}{25(1+x)^2} \tag{4.6.7c}$$

$$\frac{p}{C} = \frac{a_1[3x(x-2)+55]+a_2\sqrt{1+x}[x(22+3x)+115]}{25(1+x)[a_1(7+3x)+a_2(1+x)^{3/2}]} \tag{4.6.7d}$$

Thus we have generated the uncharged solution (4.6.7) from the charged solution (4.6.6).

4.7 Discussion

We have found new solutions (4.3.3) to the Einstein-Maxwell system (2.4.8), by utilising the coordinate transformation (4.2.3). These do not have an uncharged analogue. A systematic series analysis using (4.2.3) produced recurrence relations with real, rational coefficients that could be solved in general. This produced new exact solutions (4.4.11) to the Einstein-Maxwell field equations in terms of special functions, namely hypergeometric functions. The electromagnetic field may vanish in the general series solutions and we can regain uncharged solutions. It is possible for hypergeometric functions to be expressed in terms of elementary functions for particular parameter values. We used this feature to find two classes of exact solutions (4.6.3) and (4.6.4) to the Einstein-Maxwell system in terms of polynomials and the product of polynomials and algebraic functions. The simple form of the solutions found facilitates the analysis of the physical features of a charged sphere.

We should emphasise that the solutions found in this chapter depend crucially on the transformation (4.2.3) in which $k \neq 0$ and $k \neq 1$. Consequently we cannot regain the Schwarzschild interior metric ($k = 1$) or the family of metrics of Hansraj and Maharaj (2006) ($k = 0$). A different coordinate transformation from (4.2.3), allowing for $k = 0$ and $k = 1$, must be utilised to regain previously known solutions.

We outline this further new class of Einstein-Maxwell solutions in the next chapter. Clearly such solutions are possible as the following example illustrates. For the choice of metric function (4.2.1), we can show that the system (2.4.8) admits the particular exact solution

$$e^{2\lambda} = \frac{1+x}{(1+kx)^2} \tag{4.7.1a}$$

$$e^{2\nu} = 1 \tag{4.7.1b}$$

$$\rho = \frac{C[6(1-2k)+x(1-2k-11k^2)-6k^2x^2]}{2(1+x)^2} \tag{4.7.1c}$$

$$p = \frac{C[2(2k-1)+x(2k+3k^2-1)+2k^2x^2]}{2(1+x)^2} \tag{4.7.1d}$$

$$E^2 = \frac{C(1-k)^2x}{(1+x)^2} \tag{4.7.1e}$$

When $k=1$ then $E=0$ and we have uncharged matter with the line element

$$ds^2 = -dt^2 + \frac{1}{1+Cr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.7.2}$$

with equation of state $\rho+3p=0$. Thus we have regained the familiar Einstein universe.

We make some brief comments relating to the physics found in this chapter. In the general solution (4.4.11), when studying models of charged spheres, we should consider only those values of K for which the energy density ρ, the pressure p and the electric field intensity E are positive. Our choice of the gravitational potential (4.2.1) is clearly positive for a wide range of the parameter values of K. Since $y(x) = A_1y_1(x)+A_2y_2(x)$ given in (4.6.3) or (4.6.4) is well defined function on the interval $[0,d]$ where $d = CR^2$ and R is the stellar radius, the quantities ν, λ, ρ, p and E are nonsingular and continuous. If $K>1$ ($\alpha>0$) or $K<-1$ ($\alpha<0$), then it is clear from (4.4.11c) that ρ remains positive in the region

$$\frac{x(10+\alpha K+6x)}{(3+x)} < 2(K^2-1)$$

for positive constant C, which restricts the size of the configuration. We require that the pressure must vanish across the boundary $r = R$ which implies that

$$4\frac{(K+1+CR^2)}{(K+1)^2(1+CR^2)}\left[\frac{\dot{y}}{y}\right]_{x=CR^2} + \frac{1-K^2+CR^2}{(K+1)^2(1+CR^2)}$$

$$+ \quad \frac{\alpha KCR^2}{2(K+1)^2(1+CR^2)^2} = 0$$

where y is given by (4.6.3) or (4.6.4). Essentially this places a restriction on the constants A_1 and A_2. The interior metric (2.2.8) must match to the exterior Reissner-Nordstrom line element (2.5.2) at the boundary $r = R$. This requirement implies that

$$\begin{aligned} 1 - \frac{2m}{r} + \frac{q^2}{r^2} &= A^2[A_1 y_1(CR^2) + A_2 y_2(CR^2)]^2 \\ \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} &= \frac{1+CR^2}{(1+kCR^2)^2} \end{aligned}$$

This gives the relationships between the constants $A_1,\ A_2,\ k$ (or K), A and C. We must have

$$q^2(R) = \frac{\alpha KC^2R^6}{(K+1)^2(1+CR^2)^2}$$

to ensure the continuity of the electric field intensity across the boundary. This shows that continuity of the metric coefficients and matter variables across the boundary of the star is easily achieved. The matching condition at the boundary may place restrictions on the metric coefficients ν and its first derivative for uncharged matter; and the pressure may be nonzero if there is a surface layer of charge. However, there are sufficient free parameters to satisfy the necessary condition that arises from a particular physical model under consideration.

Chapter 5

Generalised compact spheres in electric fields

5.1 Introduction

Our intention in this chapter is two-fold. Firstly, we seek to model a charged relativistic sphere which is physically acceptable. Secondly, we seek to regain an uncharged solution of Einstein equations which satisfies the relevant physical criteria when the electric field vanishes. This ensures that a neutral relativistic star is regainable as a stable equilibrium state. Our approach here complements the approach of Thirukkanesh and Maharaj (2006) who introduced a transformation that reduces the condition of pressure isotropy to a hypergeometric equation. The transformation utilised by Komathiraj and Maharaj (2007b) does produce new exact models but restricts the classes of solutions that are possible because of constraints placed on particular parameters. In this treatment we do not transform the condition of pressure isotropy to the hypergeometric equation but are still in a position to integrate the field equations. A new class of Einstein-Maxwell solutions are found that contain familiar uncharged models which are regainable for different choices of the metric function and the electric field. In §5.2 we choose particular forms for one of the gravitational potentials and the electric field intensity. This enables us to obtain the condition of pressure isotropy in the remaining gravitational potential. This is the master equation which determines the integrability of the system. The solutions to the Einstein-Maxwell equations fall in to two classes: $k = 1$ and $k \neq 1$. The case $k = 1$ is considered and is related to the Euler-Cauchy equation. When $k \neq 1$, we assume a solution in a series form which yields recurrence relations, which we manage to solve from first principles. It is then possible to exhibit exact solutions to the Einstein-Maxwell system. Solutions in terms of elementary functions are possible for particular parameter values. In §5.3 we present two linearly independent classes of solutions as combination of polynomials and algebraic functions. In addition we show that it is possible to express the general solution of the Einstein-Maxwell system in terms of elementary functions. We regain some physically reasonable solutions found previously from our general solutions in §5.4. The results of this chapter have been published in Maharaj and Komathiraj (2007).

5.2 Master equation

In the integration procedure we make the choice

$$Z = \frac{1 + kx}{1 + x} \tag{5.2.1}$$

where k is a real constant. On substituting (5.2.1) in (2.4.8c) we obtain

$$4(1 + kx)(1 + x)\ddot{y} + 2(k - 1)\dot{y} + \left[(1 - k) - \frac{E^2(1 + x)^2}{Cx}\right] y = 0 \tag{5.2.2}$$

which is the condition of pressure isotropy. The solution of the Einstein-Maxwell system (2.4.8), for the form (5.2.1), depends on the integrability of (5.2.2). It is necessary to specify the electric field intensity E in order to integrate (5.2.2). A variety of choices for E is possible but only a few are physically reasonable and generate closed form solutions. We can reduce (5.2.2) to simpler form if we let

$$\frac{E^2}{C} = \frac{\alpha x}{(1 + x)^2} \tag{5.2.3}$$

where α is a constant. When $\alpha = 0$ there is no charge. The form for E^2 in (5.2.3) vanishes at the centre of the star, and remains continuous and bounded in the interior of the star for a wide range of values of the parameter α. Note that the same form for the electric field intensity (5.2.3) was utilised by Hansraj and Maharaj (2006) and Thirukkanesh and Maharaj (2006). Upon substituting the choice (5.2.3) in equation (5.2.2) we obtain

$$4(1 + kx)(1 + x)\ddot{y} + 2(k - 1)\dot{y} + (1 - k - \alpha)y = 0 \tag{5.2.4}$$

which is the master equation for the system (2.4.8).

Thirukkanesh and Maharaj (2006) introduced the transformation

$$1 + x = KX$$

$$K = \frac{k-1}{k}$$

$$Y(X) = y(x)$$

so that (5.2.4) becomes

$$4X(1-X)\frac{d^2Y}{dX^2} - 2\frac{dY}{dX} + (K + \tilde{\alpha})Y = 0, \quad \tilde{\alpha} = \alpha/k$$

which is a special case of the hypergeometric equation. It is possible to integrate this hypergeometric equation. However it is important to note that the possible solutions are restricted as $k \neq 0$ and $k \neq 1$ because of the transformation used. We demonstrate in this section that we can accommodate $k = 0$ and $k = 1$ in a wider class of solutions.

It is convenient to introduce the new variable $z = 1 + x$ in (5.2.4) to yield

$$4z(1 - k + kz)\frac{d^2\tilde{y}}{dz^2} + 2(k-1)\frac{d\tilde{y}}{dz} + (1 - k - \alpha)\tilde{y} = 0 \tag{5.2.5}$$

where we have set $\tilde{y} = y(z)$. Two categories of solution are possible: $k = 1$ and $k \neq 1$.

Case I : $k = 1$

In this case (5.2.5) becomes the Euler-Cauchy equation with solution

$$y = \begin{cases} \sqrt{1+x}\ [c_1(1+x)^{\mu} + c_2(1+x)^{-\mu}] & \text{if } \alpha > -1; \\ \\ \sqrt{1+x}\ [c_1 + c_2 \ln(1+x)] & \text{if } \alpha = -1; \\ \\ \sqrt{1+x}\ \{c_1 \sin[\mu \ln(1+x)] + c_2 \cos[\mu \ln(1+x)]\} & \text{if } \alpha < -1 \end{cases} \tag{5.2.6}$$

where $\mu = \sqrt{(1+\alpha)}/2$. From (2.4.8) and (5.2.1) we can show that the general solution to the Einstein-Maxwell system becomes

$$e^{2\lambda} = 1 \tag{5.2.7a}$$

$$e^{2\nu} = A^2y^2 \tag{5.2.7b}$$

$$\rho = -\frac{\alpha C x}{2(1+x)^2} \tag{5.2.7c}$$

$$p = 4C\frac{\dot{y}}{y} + \frac{\alpha C x}{2(1+x)^2} \tag{5.2.7d}$$

$$E^2 = \frac{\alpha C x}{(1+x)^2} \tag{5.2.7e}$$

in terms of the variable x; where y is given by (5.2.6). Observe that this case is not regainable from Thirukkanesh and Maharaj (2006) as $k \neq 1$ in their transformation. We do not pursue this case further as either $\rho < 0$ or $E^2 < 0$.

Case II : $k \neq 1$

As the point $z = 0$ is a regular singular point of (5.2.5), there exist two linearly independent solutions of the form of a power series with centre $z = 0$. These solutions can be generated using the method of Frobenius. Therefore we can write

$$\tilde{y} = \sum_{i=0}^{\infty} a_i z^{i+b}, \quad a_0 \neq 0 \tag{5.2.8}$$

where a_i are the coefficients of the series and b is a constant. For a legitimate solution we need to determine the coefficients a_i as well as the parameter b. On substituting (5.2.8) into (5.2.5) we obtain

$$\begin{aligned} & 2a_0b(1-k)(2b-3)z^{b-1} \\ + & \sum_{i=0}^{\infty}\{2(1-k)(i+b+1)(2i+2b-1)a_{i+1} \\ + & [4k(i+b)(i+b-1)+(1-k-\alpha)]a_i\}z^{b+i} = 0 \end{aligned} \tag{5.2.9}$$

in increasing powers of z. For equation (5.2.9) to hold for all powers of z in the interval of convergence we require

$$2a_0b(1-k)(2b-3) \quad = \quad 0 \tag{5.2.10a}$$

$$a_{i+1} \quad = \quad \frac{4k(i+b)(i+b-1)+(1-k-\alpha)}{2(k-1)(i+b+1)(2i+2b-1)}a_i, \;\; i \geq 0 \tag{5.2.10b}$$

Since $a_0 \neq 0$ and $k \neq 1$, we have from (5.2.10a) that $b = 0$ or $b = 3/2$. Equation (5.2.10b) is the basic difference equation governing the structure of the solution. It is possible to express the general coefficient a_i in terms of the leading coefficient a_0 by establishing a general structure for the coefficients by considering the leading terms. These coefficients generate the pattern

$$a_{i+1} = \prod_{p=0}^{i} \frac{4k(p+b)(p+b-1)+(1-k-\alpha)}{2(k-1)(p+b+1)(2p+2b-1)} a_0 \tag{5.2.11}$$

where we have utilised the conventional symbol $\prod$ to denote multiplication. It is easy to establish that the result (5.2.11) holds for all positive integers p using the principle of mathematical induction.

Now it is possible to generate two linearly independent solutions to (5.2.5) with the help of (5.2.8) and (5.2.11). For the parameter value $b = 0$ we obtain the first solution

$$\tilde{y}_1 = a_0 \left[1 + \sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{4kp(p-1)+(1-k-\alpha)}{2(k-1)(p+1)(2p-1)} z^{i+1} \right] \tag{5.2.12}$$

For the parameter value $b = 3/2$ we obtain the second solution

$$\tilde{y}_2 = a_0 z^{3/2} \left[1 + \sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{k(2p+3)(2p+1)+(1-k-\alpha)}{(k-1)(2p+5)(2p+2)} z^{i+1} \right] \tag{5.2.13}$$

Thus we can write the general solution to the differential equation (5.2.5), for the choice of the electric field given in (5.2.3), as

$$\tilde{y} = \tilde{c}_1 \tilde{y}_1 + \tilde{c}_2 \tilde{y}_2 \tag{5.2.14}$$

where $\tilde{c}_1, \; \tilde{c}_2$ are arbitrary constants, and $\tilde{y}_1$ and $\tilde{y}_2$ are given in (5.2.12) and (5.2.13) respectively. In terms of the variable x we can write (5.2.14) as

$$
\begin{aligned}
y &= c_1\left[1+\sum_{i=0}^{\infty}\prod_{p=0}^{i}\frac{4kp(p-1)+(1-k-\alpha)}{2(k-1)(p+1)(2p-1)}(1+x)^{i+1}\right] \\
&+ c_2(1+x)^{3/2}\left[1+\sum_{i=0}^{\infty}\prod_{p=0}^{i}\frac{k(2p+3)(2p+1)+(1-k-\alpha)}{(k-1)(2p+5)(2p+2)}(1+x)^{i+1}\right] \\
&= c_1y_1(x)+c_2y_2(x) \qquad (5.2.15)
\end{aligned}
$$

where we have set $c_1 = \tilde{c}_1 a_0$ and $c_2 = \tilde{c}_2 a_0$ for simplicity. The general solution to the Einstein-Maxwell system (2.4.8) can now be written as

$$e^{2\lambda} = \frac{1+x}{1+kx} \qquad (5.2.16a)$$

$$e^{2\nu} = A^2y^2 \qquad (5.2.16b)$$

$$\frac{\rho}{C} = \frac{(1-k)(3+x)}{(1+x)^2} - \frac{\alpha x}{2(1+x)^2} \qquad (5.2.16c)$$

$$\frac{p}{C} = 4\frac{(1+kx)}{(1+x)}\frac{\dot{y}}{y} + \frac{(k-1)}{(1+x)} + \frac{\alpha x}{2(1+x)^2} \qquad (5.2.16d)$$

$$\frac{E^2}{C} = \frac{\alpha x}{(1+x)^2} \qquad (5.2.16e)$$

in terms of the variable x. The form of the exact solution (5.2.16) has a similar structure to the general solution of Thirukkanesh and Maharaj (2006); however it is important to realise that our solution is a new result because the series in (5.2.15) is different. In addition note that $k = 0$ is allowed in (5.2.16) unlike the result of Thirukkanesh and Maharaj (2006); our result can be interpreted as a generalisation.

5.3 Elementary solutions

The general solution (5.2.15) is given in the form of a series which may be used to define special functions. For particular values of the parameters involved it is possible for the general solution to be written in terms of elementary functions which is a more desirable form for the physical description of a charged relativistic star. In this section, we find two linearly independent solutions, in terms of elementary functions, for the differential equation (5.2.5).

5.3.1 Polynomials and algebraic functions

On substituting $b = 0$ and setting $1 - k - \alpha = -4kn(n-1)$ in (5.2.10b) we obtain

$$a_{i+1} = \frac{4k}{1-k} \frac{(n-i)(n+i-1)}{(2i+2)(2i-1)} a_i, \quad i \geq 0 \tag{5.3.1}$$

It is easy to see from (5.3.1) that $a_{n+1} = 0$. Clearly the subsequent coefficients $a_{n+2},\ a_{n+3},\ a_{n+4}, \ldots$ vanish. Equation (5.3.1) has the solution

$$a_i = -\left(\frac{4k}{1-k}\right)^i \frac{n(n-1)(2i-1)(n+i-2)!}{(2i)!(n-i)!} a_0, \quad 0 \leq i \leq n \tag{5.3.2}$$

Then from (5.2.8) (when $b = 0$) and (5.3.2), we obtain

$$\tilde{y}_1 = -a_0 \sum_{i=0}^{n} \left(\frac{4k}{1-k}\right)^i \frac{n(n-1)(2i-1)(n+i-2)!}{(2i)!(n-i)!} z^i \tag{5.3.3}$$

for $1 - k - \alpha = -4kn(n-1)$.

On substituting $b = 3/2$ and setting $1 - k - \alpha = -k(2n+3)(2n+1)$ in (5.2.10b) we obtain

$$a_{i+1} = \frac{4k}{1-k} \frac{(n-i)(n+i+2)}{(2i+5)(2i+2)} a_i, \quad i \geq 0 \tag{5.3.4}$$

We observe from (5.3.4) that $a_{n+1} = 0$. Consequently the remaining coefficients $a_{n+2},\ a_{n+3},\ a_{n+4},\ \ldots$ vanish. Equation (5.3.4) may be solved to yield

$$a_i = 3\left(\frac{4k}{1-k}\right)^i \frac{(2i+2)(n+i+1)!}{(n+1)(2i+3)!(n-i)!} a_0, \quad 0 \leq i \leq n \tag{5.3.5}$$

Now from (5.2.8) (when $b = 3/2$) and (5.3.5), we obtain

$$\tilde{y}_1 = 3a_0 z^{3/2} \sum_{i=0}^{n} \left(\frac{4k}{1-k}\right)^i \frac{(2i+2)(n+i+1)!}{(n+1)(2i+3)!(n-i)!} z^i \tag{5.3.6}$$

for $1 - k - \alpha = -k(2n+3)(2n+1)$.

The polynomial and algebraic functions (5.3.3) and (5.3.6) comprise the first solution to the differential equation (5.2.5) for appropriate values of k and α.

5.3.2 Algebraic functions

We can find the second solution of (5.2.5) using the method of reduction of order in principle. However this to be difficult in practice because of the complicated form of the first solution given in (5.3.3) and (5.3.6). We utilise a transformation to first simplify (5.2.5) before seeking the second solution. We take the second solution of (5.2.5) to be of the form

$$\tilde{y} = (1 - k + kz)^{1/2} u(z) \tag{5.3.7}$$

when $u(z)$ is an arbitrary polynomial. Special cases of (5.3.7) are known to solve (5.2.5) which motivates our ansatz for the algebraic form for $\tilde{y}$ as a generic second solution to the differential equation (5.2.5). On substituting $\tilde{y}$ in (5.2.5) we obtain

$$4z(1 - k + kz)\frac{d^2u}{dz^2} + [4kz + 2(k-1)]\frac{du}{dz} + (1 - 2k - \alpha)u = 0 \tag{5.3.8}$$

which is a linear differential equation for $u(z)$.

As in §5.3.1 it is possible to find solutions in terms of polynomials, and product of polynomials with algebraic functions for $u(z)$ for certain values of the parameters k and α. As the point $z = 0$ is a regular singular point of (5.3.8), there exist two linearly independent solutions of the form of the power series with centre $z = 0$. Thus we assume

$$u = \sum_{i=0}^{\infty} c_i z^{i+d} \tag{5.3.9}$$

where the constants c_i are the coefficients of the series and d is the constant. Substituting (5.3.9) in (5.3.8) we obtain

$$\begin{aligned} & 2c_0 d(1-k)(2d-3)z^{d-1} \\ + & \sum_{i=0}^{\infty}\{2(1-k)(i+d+1)(2i+2d-1)c_{i+1} \\ + & [4k(i+d)^2 + (1-2k-\alpha)]c_i\}z^{i+d} = 0 \end{aligned} \tag{5.3.10}$$

For equation (5.3.10) to hold true for all z we require that

$$2c_0 d(1-k)(2d-3) = 0 \tag{5.3.11a}$$

$$c_{i+1} = \frac{4k(i+d)^2 + (1-2k-\alpha)}{2(k-1)(i+d+1)(2i+2d-1)}c_i, \quad i \geq 0 \tag{5.3.11b}$$

Since $c_0 \neq 0$ and $k \neq 1$ we have from (5.3.11a) that $d = 0$ or $d = 3/2$. Equation (5.3.11b) is the linear recurrence relation governing the structure of the solution.

On substituting $d = 0$ and setting $1 - k - \alpha = -k(2n+1)(2n+3)$ in (5.3.11b) we obtain

$$c_{i+1} = \left(\frac{4k}{1-k}\right) \frac{(n-i+1)(n+i+1)}{(2i+2)(2i-1)} c_i, \quad i \geq 0 \tag{5.3.12}$$

From (5.3.12) we have that $c_{n+2} = 0$ and subsequent coefficients $c_{n+3},\ c_{n+4},\ c_{n+5},\ \ldots$ vanish. Then (5.3.12) has the solution

$$c_i = -\left(\frac{4k}{1-k}\right)^i \frac{(n+1)(2i-1)(n+i)!}{(2i)!(n-i+1)!} c_0, \quad 0 \leq i \leq n+1 \tag{5.3.13}$$

Then from (5.3.9) (when $d = 0$) and (5.3.13) we obtain

$$\tilde{y}_2 = -c_0(1-k+kz)^{1/2} \sum_{i=0}^{n+1} \left(\frac{4k}{1-k}\right)^i \frac{(n+1)(2i-1)(n+i)!}{(2i)!(n-i+1)!} z^i \tag{5.3.14}$$

for $1 - k - \alpha = -k(2n+1)(2n+3)$.

On substituting $d = 3/2$ and setting $1 - k - \alpha = -4kn(n-1)$ in (5.3.11b) we obtain

$$c_{i+1} = \left(\frac{4k}{1-k}\right) \frac{(n+i+1)(n-i-2)}{(2i+2)(2i+5)} c_i, \quad i \geq 0 \tag{5.3.15}$$

From (5.3.15) we have that $c_{n-1} = 0$ and subsequent coefficients $c_n,\ c_{n+1},\ c_{n+2},\ \ldots$ vanish. The equation (5.3.15) has the solution

$$c_i = 3\left(\frac{4k}{1-k}\right)^i \frac{(2i+2)(n+i)!}{n(n-1)(2i+3)!(n-i-2)!} c_0, \quad 0 \leq i \leq n-2 \tag{5.3.16}$$

From (5.3.9) (when $d = 3/2$) and (5.3.16) we obtain

$$\tilde{y}_2 = 3c_0(1-k+kz)^{1/2} z^{3/2} \sum_{i=0}^{n-2} \left(\frac{4k}{1-k}\right)^i \frac{(2i+2)(n+i)!}{n(n-1)(2i+3)!(n-i-2)!} z^i \tag{5.3.17}$$

for $1 - k - \alpha = -4kn(n-1)$.

The algebraic solutions (5.3.14) and (5.3.17) comprise the second solution of the differential equation (5.2.5) which are clearly independent from (5.3.6) and (5.3.3).

5.3.3 Elementary functions

We have obtained one class of polynomial solution (5.3.3) and three classes of solutions (5.3.6), (5.3.14), and (5.3.17) in terms of products of polynomials and algebraic functions. The polynomial solution (5.3.3) and the product of polynomials with an algebraic function (5.3.6) generate the first solution. The second linearly independent solution is given by (5.3.14) and (5.3.17) which are products of polynomials and algebraic functions. By collecting these results we can express the general solution to (5.2.5) in two categories. We express the general solution in terms of the independent variable x. From (5.3.3) and (5.3.17) we write the first category as

$$
\begin{aligned}
y &= \\
&A\sum_{i=0}^{n}\left(\frac{4k}{1-k}\right)^{i}\frac{n(n-1)(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+x)^{i}+ \\
&B(1+kx)^{1/2}(1+x)^{3/2}\sum_{i=0}^{n-2}\left(\frac{4k}{1-k}\right)^{i}\frac{(2i+2)(n+i)!}{n(n-1)(2i+3)!(n-i-2)!}(1+x)^{i}
\end{aligned}
\tag{5.3.18}
$$

for $1-k-\alpha=-4kn(n-1)$. From (5.3.6) and (5.3.14), the second category of solution has the form

$$
\begin{aligned}
y &= \\
&A(1+x)^{3/2}\sum_{i=0}^{n}\left(\frac{4k}{1-k}\right)^{i}\frac{(2i+2)(n+i+1)!}{(n+1)(2i+3)!(n-i)!}(1+x)^{i}+ \\
&B(1+kx)^{1/2}\sum_{i=0}^{n+1}\left(\frac{4k}{1-k}\right)^{i}\frac{(n+1)(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+x)^{i}
\end{aligned}
\tag{5.3.19}
$$

for $1-k-\alpha=-k(2n+1)(2n+3)$. In the above A and B are arbitrary constants. Consequently we have demonstrated that elementary functions can be extracted from the general series (5.2.15) by restricting the parameter values α and k. The general solutions (5.3.18) and (5.3.19) have a very simple form. It is important to observe that the Einstein-Maxwell solutions (5.3.18) and (5.3.19) apply to both charged and uncharged relativistic stars. We regain neutral exact solutions, which may possibly be new, by setting $\alpha=0$.

5.4 Known cases

We may generate individual models for charged and uncharged stars found previously from our general class of solutions. These can be explicitly regained from the general series solution (5.2.15) or the elementary functions (5.3.18) and (5.3.19). We demonstrate that this is possible in the following classes:

Case I : Thirukkanesh and Maharaj charged stars

A simple transformation leads us to the class of solutions in Thirukkanesh and Maharaj (2006). If we set $K = (k-1)/k, \quad \hat{\alpha} = \alpha/k, \quad c_1 = d_1$ and $c_2 = K^{-3/2} d_2$ then (5.2.15) can be written as

$$\begin{aligned} y &= d_1 \left[1 + \sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{4p(p-1) - (K + \tilde{\alpha})}{2(p+1)(2p-1)} \left(\frac{1+x}{K} \right)^{i+1} \right] \\ &+ d_2 \left(\frac{1+x}{K} \right)^{3/2} \left[1 + \sum_{i=0}^{\infty} \prod_{p=0}^{i} \frac{(2p+3)(2p+1) - (K + \tilde{\alpha})}{(2p+5)(2p+2)} \left(\frac{1+x}{K} \right)^{i+1} \right] \end{aligned} \tag{5.4.1}$$

The Einstein-Maxwell solution (5.4.1) corresponds to the charged model of Thirukkanesh and Maharaj (2006). However note that $k \neq 0$ and $k \neq 1$ in (5.4.1). In our wider class of solutions (5.2.15) it is permitted that $k = 0$; the exact solution (5.2.6) corresponds to the case $k = 1$. The Thirukkanesh-Maharaj charged stars (2006) contain neutron star models found previously including the Durgapal and Bannerji model (1983), and they are consequently physically reasonable. This is a desirable feature in exact solutions to the Einstein-Maxwell system.

Case II : Hansraj and Maharaj charged stars

The Hansraj and Maharaj (2006) models appear to be generically different but can be shown to be a special case of our solutions. If we set $k = 0$ then it is possible after some manipulation, to write (5.2.15) in the form

$$
\begin{aligned}
y &= c_1\left[1+\sum_{i=0}^{\infty}\frac{(-1)^i(2i+1)}{(2i+2)!}[(1-\alpha)(1+x)]^{i+1}\right] \\
&+ c_2(1+x)^{3/2}\left[1+\sum_{i=0}^{\infty}\frac{3(-1)^{i+1}(2i+4)}{(2i+5)!}[(1-\alpha)(1+x)]^{i+1}\right] \\
&= c_1\left[1-\sum_{i=0}^{\infty}(-1)^i\left(\frac{1}{(2i+2)!}-\frac{1}{(2i+1)!}\right)[\sqrt{(1-\alpha)(1+x)}]^{2(i+1)}\right] \\
&+ \frac{3c_2}{(1-\alpha)^{3/2}}[\sqrt{(1-\alpha)(1+x)}]^3 \\
&\times \left[\frac{1}{3}-\sum_{i=0}^{\infty}(-1)^{i+1}\left(\frac{1}{(2i+5)!}-\frac{1}{(2i+4)!}\right)[\sqrt{(1-\alpha)(1+x)}]^{2(i+1)}\right] \quad (5.4.2)
\end{aligned}
$$

It is interesting to observe that the last equation above can be expressed in terms of trigonometric functions. Then it is easy to show that (5.4.2) can be written in the closed form

$$
\begin{aligned}
y &= c_1[\cos\sqrt{(1-\alpha)(1+x)}+\sqrt{(1-\alpha)(1+x)}\sin\sqrt{(1-\alpha)(1+x)}] \\
&+ \frac{3c_2}{(1-\alpha)^{3/2}}[\sin\sqrt{(1-\alpha)(1+x)}-\sqrt{(1-\alpha)(1+x)}\cos\sqrt{(1-\alpha)(1+x)}] \\
&= [d_1+d_2\sqrt{(1-\alpha)(1+x)}]\sin\sqrt{(1-\alpha)(1+x)} \\
&+ [d_2-d_1\sqrt{(1-\alpha)(1+x)}]\cos\sqrt{(1-\alpha)(1+x)} \quad (5.4.3)
\end{aligned}
$$

where we have set $d_1 = 3c_2/(1-\alpha)^{3/2}$ and $d_2 = c_1$. The Einstein-Maxwell solution (5.4.3) is the same as the charged model of Hansraj and Maharaj (2006). [Note that our solution (5.4.3) corrects a minor misprint before equation (29) of Hansraj and Maharaj (2006).] The Hansraj-Maharaj charged stars were comprehensively studied and it was demonstrated that their model produced a charged relativistic sphere that satisfies all physical criteria. In particular the speed of sound is less that the speed of light and causality is maintained.

Case III : Finch and Skea neutron stars

If we set $k = 0$ and $\alpha = 0$, and follow the procedure outlined for Case II, then (5.2.15) becomes

$$y = [d_1 + d_2\sqrt{(1+x)}]\sin\sqrt{(1+x)} + [d_2 - d_1\sqrt{(1+x)}]\cos\sqrt{(1+x)} \quad (5.4.4)$$

where we have set $d_1 = 3c_2$ and $d_2 = c_1$. Alternatively we can obtain the result (5.4.4) directly from (5.4.3) with $\alpha = 0$. The exact solution (5.4.4) is the neutron star model of Finch and Skea (1989). The Finch-Skea model satisfies all the physical conditions for an isolated spherically symmetric stellar source, and consequently has been utilised by many researchers to model neutron stars.

Case IV : Durgapal and Bannerji neutron stars

If we set $\alpha = 0$ and $k = -\frac{1}{2}(n = 0)$, then (5.3.19) becomes

$$y = c(1+x)^{3/2} + d(2-x)^{1/2}(5+2x) \quad (5.4.5)$$

where we have set $c = A/3$ and $d = -B/3\sqrt{2}$. The exact solution (5.4.5) was first found by Durgapal and Bannerji (1983). The Durgapal-Bannerji solution has been widely applied as a relativistic model for neutral stars with superdense matter.

Chapter 6

Analytical models for quark stars

6.1 Introduction

The existence of strange stars consisting of quark matter has stimulated much interest in the last few decades since this could represent the most energetically favourable state of baryon matter. Matter consisting of u, d and s quarks may be the absolute ground state of matter at zero pressure and temperature as suggested by Bodmer (1971). It is expected that strange stars form during the collapse of the core of a massive star after a supernova explosion (Cheng *et al* 1998). In regions of low temperatures and sufficiently high densities hadrons are crushed into quark matter with color superconducting phases which occur in the dense cores of neutron stars as remarked by Alford (2001). Consequently the core of a proto-neutron star or neutron star provides the appropriate environment for ordinary matter to convert to strange quark matter. Another possibility is that a rapidly spinning dense star can accrete sufficient mass to undergo a phase transition to form a strange star.

As the physics of ultrahigh densities for quark matter is not well understood, researchers restrict their attention to the phenomenological MIT bag model (Chodos *et al* 1974, Farhi and Jaffe 1984, Witten 1984). In the bag model, the strange matter equation of state has a simple linear form given by

$$p = \frac{1}{3}(\rho - 4B) \tag{6.1.1}$$

where ρ is the energy density, p is the isotropic pressure and B is the bag constant. The quark confinement is determined by the vacuum pressure B, in the bag model, that equilibrates the pressure of quarks thereby stabilising the system. Studies of particular compact astronomical objects indicate that they could be strange stars with quark matter (Bombaci 1997, Pons *et al* 2002, Usov 2004). A candidate for a strange star may have been observed using the deep Chandra LETG+HRC-S observations; Drake *et al* (2002) suggested that the X-ray source RXJ1856.5-3754 may be such an object. Sotani *et al* (2004) have used observational data on gravitational waves to obtain the equation of state for quark matter. Harko and Cheng (2000) considered collapsing strange matter in spherically symmetric fields. Yilmaz and Baysal (2005) studied charged strange matter in rotating fields. The role of anisotropy, with the linear equation of state (6.1.1), was pursued by Mak and Harko (2002) and Sharma and Maharaj (2007) who demonstrated exact analytical solutions.

In a recent treatment Mak and Harko (2004) found a charged strange quark star under the assumption of spherical symmetry and the existence of a conformal Killing vector. In this chapter we consider the Einstein-Maxwell system of equations with the linear equation of state (6.1.1) and apply them to strange stars. The existence of a conformal symmetry is not an assumption that we make. We demonstrate that exact analytical solutions to the field equations are possible that contain the Mak-Harko model (2004). In §6.2, we rewrite the Einstein-Maxwell field equations (2.4.8) for the static spherically line element as a new set of differential equations using the bag equation of state (6.1.1) for strange matter. On specifying an explicit form for one of the gravitational potentials, we obtain a first order differential equation in the remaining potential. In §6.3 we find a new class of exact solutions to the Einstein-Maxwell system. The model of Mak and Harko (2004) is regained as a special case. In §6.4 we present a second class of exact solutions that satisfy the Einstein-Maxwell system. This category of solutions has the desirable feature of not admitting singularities at the centre. The results of this chapter have been submitted for publication in Komathiraj and Maharaj (2007c).

6.2 Integration procedure

We can replace the system of field equations (2.4.8), including the bag equation of state (6.1.1), by the system

$$\rho = 3p + 4B \tag{6.2.1a}$$

$$\frac{p}{C} = Z\frac{\dot{y}}{y} - \frac{1}{2}\dot{Z} - \frac{B}{C} \tag{6.2.1b}$$

$$\frac{E^2}{2C} = \frac{1-Z}{x} - 3Z\frac{\dot{y}}{y} - \frac{1}{2}\dot{Z} - \frac{B}{C} \tag{6.2.1c}$$

$$0 = 4Zx^2\ddot{y} + (6xZ + 2x^2\dot{Z})\dot{y} + \left[2x\left(\dot{Z} + \frac{B}{C}\right) + Z - 1\right]y \tag{6.2.1d}$$

$$\sigma = 2\sqrt{\frac{CZ}{x}}(E + x\dot{E}) \tag{6.2.1e}$$

The system of equations (6.2.1) governs the gravitational behaviour of a charged quark star. We describe one possible integration procedure that leads to an exact solution of the Einstein-Maxwell system (6.2.1). Note that other procedures are possible; our approach has the advantage of producing a first order equation that has solutions in terms of elementary functions. We observe from (6.2.1a) that ρ and p are related. Therefore in the system (6.2.1) there are five independent variables ($Z,\ y,\ p$ or $\rho,\ E,\ \sigma$) and only four independent equations. We have freedom to choose only one of the quantities involved. In our approach we specify $y(x)$ on physical grounds. A number of choices for the gravitational potential $y(x)$ are possible; clearly we should choose a form that is likely to lead to a physically reasonable solution. To make the above set of equations tractable, we choose the metric function in the particular form

$$y(x) = (a + x^m)^n \tag{6.2.2}$$

where a, m and n are constants. The form chosen ensures that the metric function y is continuous and well behaved in the interior of the star for the wide range of values of parameters m and n. The function y yields a finite value at the centre of the star. This is a very desirable feature for the model on physical grounds. It is interesting to observe that many of the solutions found previously do not share this feature.

Substitution of (6.2.2) into (6.2.1d) leads to the first order equation

$$\dot{Z} + \frac{a^2 + 2a[1 + mn(2m+1)]x^m + [2mn(2mn+1)+1]x^{2m}}{2x(a+x^m)[a+(1+mn)x^m]} Z$$

$$- \quad \frac{(1 - \frac{2B}{C}x)(a+x^m)}{2x[a+(1+mn)x^m]} = 0$$

This first order equation is linear so that it can be integrated in principle. The complicated rational coefficient of Z can be simplified using partial fractions and we obtain

$$\dot{Z} + \left[\frac{1}{2x} + \frac{2m(n-1)x^{m-1}}{(a+x^m)} + \frac{m[4(1+mn)-3n]x^{m-1}}{2[a+(1+mn)x^m]}\right] Z$$

$$- \quad \frac{(1 - \frac{2B}{C}x)(a+x^m)}{2x[a+(1+mn)x^m]} = 0 \qquad (6.2.3)$$

Note that we have essentially reduced the solution of the field equations (6.2.1) to integrating (6.2.3). Once the potential Z in (6.2.3) is found the remaining relevant quantities ρ, p and E then follow from (6.2.1a), (6.2.1b) and (6.2.1c) respectively. It is possible to find exact solutions to the Einstein-Maxwell field equations with the linear equation of state for different values of m and n in (6.2.3). We illustrate this with two simple examples in terms of elementary functions. Other exact solutions are possible but the form of the solution becomes more complicated and could involve special functions.

6.3 Generalised Mak-Harko model

An exact solution of (6.2.1) can be found with $m = 1/2$ and $n = 1$. In this case (6.2.2) gives the first metric function

$$y(x) = (a + \sqrt{x})$$

Equation (6.2.3) becomes

$$\dot{Z} + \left[\frac{1}{2x} + \frac{3}{2\sqrt{x}(2a+3\sqrt{x})}\right] Z - \frac{(1 - \frac{2B}{C}x)(a+\sqrt{x})}{x(2a+3\sqrt{x})} = 0$$

which can be integrated to give the second metric function

$$Z = \frac{3(2a+\sqrt{x}) - \frac{B}{C}x(4a+3\sqrt{x})}{3(2a+3\sqrt{x})} \qquad (6.3.1)$$

Hence we can generate the exact analytical model

$$e^{2\nu} = A^2(a+\sqrt{x})^2 \tag{6.3.2a}$$

$$e^{2\lambda} = \frac{3(2a+3\sqrt{x})}{3(2a+\sqrt{x})-\frac{B}{C}x(4a+3\sqrt{x})} \tag{6.3.2b}$$

$$\rho = f(x)+\frac{B(16a^3+47a^2\sqrt{x}+48ax+18x^{\frac{3}{2}})}{2(a+\sqrt{x})(2a+3\sqrt{x})^2} \tag{6.3.2c}$$

$$3p = f(x)-\frac{B(16a^3+81a^2\sqrt{x}+120ax+54x^{\frac{3}{2}})}{2(a+\sqrt{x})(2a+3\sqrt{x})^2} \tag{6.3.2d}$$

$$E^2 = \frac{C(-2a^2-2a\sqrt{x}+3x)+Bx(a^2+2a\sqrt{x})}{\sqrt{x}(a+\sqrt{x})(2a+3\sqrt{x})^2} \tag{6.3.2e}$$

For simplicity we have set

$$f(x)=\frac{3C(6a^2+10a\sqrt{x}+3x)}{2\sqrt{x}(a+\sqrt{x})(2a+3\sqrt{x})^2}$$

in (6.3.2).

The exact model (6.3.2) satisfies the Einstein-Maxwell system (6.2.1). Note that, if we set $a=0$, then the system (6.3.2) becomes

$$e^{2\nu} = D^2r^2 \tag{6.3.3a}$$

$$e^{2\lambda} = \frac{3}{1-Br^2} \tag{6.3.3b}$$

$$\rho = \frac{1}{2r^2}+B \tag{6.3.3c}$$

$$p = \frac{1}{6r^2}-B \tag{6.3.3d}$$

$$E^2 = \frac{1}{3r^2} \tag{6.3.3e}$$

where we have set $D^2=A^2C$. The particular solution (6.3.3) was found by Mak and Harko (2004) under the assumption of spherical symmetry and the existence of a one-parameter group of conformal motions. It is interesting to observe that, on substituting the values $a = 0$ and $B = 0$ for the constants, the solution (6.3.2) becomes identical to that obtained by Misner and Zapolsky (1964). The physical features of the solutions (6.3.3) were studied by Mak and Harko (2004) and shown to be consistent with the interior of a quark star with charged material. This corresponds to a single stable quark configuration with radius $R = 9.46\ km$ and mass $M = 2.86 M_\odot$; these figures are consistent with values obtained using numerical methods by other researchers (Haensel *et al* 1986, Haensel and Zdunik 1989, Gourgoulhon *et al* 1999). Consequently, our more general class of solutions is likely to produce charged quark models consistent with stellar evolution and observational data. We comment that our new class of solutions (6.3.2) has a singularity in the charge density and mass density at the centre; the Mak and Harko (2004) model also shares this feature. The singularity in the charge density and mass density is physically acceptable since the total charge and mass remain finite. However, our gravitational potentials $e^{2\nu}$ and $e^{2\lambda}$ remain finite at the centre which contrasts with the singularities in the metric functions of Mak and Harko when $x = 0$.

6.4 Nonsingular quark model

Another exact solution of (6.2.1) can be found with $m = 1$ and $n = 2$. For this case (6.2.2) gives the first metric function

$$y(x) = (a + x)^2$$

Equation (6.2.3) becomes

$$\dot{Z} + \left[\frac{1}{2x} + \frac{2}{a+x} + \frac{3}{a+3x}\right] Z - \frac{(1 - \frac{2B}{C}x)(a+x)}{2x(a+3x)} = 0$$

which can be integrated to give the second metric function

$$Z = \frac{9(35a^3 + 35a^2x + 21ax^2 + 5x^3) - \frac{2B}{C}x(105a^3 + 189a^2x + 135ax^2 + 35x^3)}{315(a+x)^2(a+3x)} \quad (6.4.1)$$

Therefore we can find the exact analytical model

$$e^{2\nu} = A^2(a+x)^4 \tag{6.4.2a}$$

$$e^{2\lambda} = \frac{315(a+x)^2(a+3x)}{9(35a^3+35a^2x+21ax^2+5x^3) - \frac{2B}{C}x(105a^3+189a^2x+135ax^2+35x^3)} \tag{6.4.2b}$$

$$\rho = g(x) + \frac{2B[3(35a^5+133a^4x+246a^3x^2)+5(254a^2x^3+209ax^4+63x^5)]}{105(a+x)^3(a+3x)^2} \tag{6.4.2c}$$

$$\begin{aligned} 3p &= g(x) - \frac{2B[3(35a^5+497a^4x+1854a^3x^2)+5(1678a^2x^3+1177ax^4+315x^5]}{105(a+x)^3(a+3x)^2} \\ &= (\rho - 4B) \end{aligned} \tag{6.4.2d}$$

$$\begin{aligned} E^2 &= 4x[9C(49a^3+363a^2x+339ax^2+105x^3) \\ &- 2B(21a^4+162a^3x+816a^2x^2+910ax^3+315x^4)]/315(a+x)^3(a+3x)^2 \end{aligned} \tag{6.4.2e}$$

Again, for simplicity, we have set

$$g(x) = \frac{6C(70a^4+217a^3x+159a^2x^2+75ax^3+15x^4)}{35(a+x)^3(a+3x)^2}$$

in (6.4.2).

The exact model (6.4.2) satisfies the Einstein-Maxwell system (6.2.1) and constitutes a new family of analytical solutions for a quark star with charged material. The gravitational potentials $e^{2\nu}$ and $e^{2\lambda}$ in (6.4.2) have the advantage of having a simple analytic form, and they are written in terms of polynomials and rational functions. Consequently the matter variables and the electric field intensity have a simple analytic representation. The function $e^{2\nu}$ is continuous and well behaved in the interior and finite at the centre $x = 0$. The function $e^{2\lambda}$ is well behaved and has a constant value at the centre $x = 0$. The energy density ρ is positive throughout the interior, and regular at the centre with value $\rho_0 = 2(\frac{6C}{a} + B)$. The pressure p is regular at the centre with value $p_0 = 2(\frac{2C}{a} - \frac{B}{3}) = \frac{1}{3}(\rho_0 - 4B)$. The electric field intensity E is continuous in the interior and vanishes at the centre. Hence, the matter variables and gravitational potentials comply with usual conditions for a stellar source. The finiteness of $e^{2\nu}$, $e^{2\lambda}$, ρ, p and E at the origin $x = 0$ is a very welcome feature which is absent in the previous class of solutions. Consequently, the exact solutions (6.4.2) are likely to produce charged quark stars with physically acceptable interiors. A recent attempt in this direction is the strange star model of Jotania and Tikekar (2006) admitting compact configurations with mass to size ratio consistent with strange matter.

Chapter 7

Conclusion

The main objective of this thesis was to find new exact solutions to the isotropic Einstein-Maxwell field equations which can be used to describe a relativistic dense star. Solutions of the complicated system of nonlinear partial differential equations were sought by specifying physically reasonable forms for one of the gravitational potentials and the electric field intensity. A change of variables was effected which made the condition of pressure isotropy equation more tractable. A number of new simple solutions to the Einstein-Maxwell system, which we believe to be physically reasonable, were obtained explicitly in terms of special functions and elementary functions. In particular we generated a class of new solutions which are applicable to charged superdense stars with spheroidal geometry. It was also possible to find other categories of solutions by specifying other types of spatial geometries. In addition, we have obtained solutions to the Einstein-Maxwell field equations for a charged strange quark star described by the MIT bag model.

We now provide an overview of the main results obtained during the course of our investigations:

- In Chapter 2, we briefly introduced aspects of differential geometry applicable to general relativity that were necessary for later sections. In particular we developed the Einstein-Maxwell field equations for a perfect fluid source in the presence of charge in spherically symmetric spacetimes. We obtained an equivalent form of the field equations with the transformation used by Durgapal and Bannerji (1983).

- Our intention in Chapter 3 was to obtain new solutions to the field equations for the charged Tikekar (1990) model by specifying the electric field intensity

$$E^2 = \frac{\alpha K(x^2-1)}{R^2(1-K+Kx^2)^2}$$

on physical grounds. We obtained a general series solution with real arguments unlike the complex arguments given by software packages. The solutions in terms of polynomials and algebraic functions were also given by restricting the parameters K and α. We demonstrated that our general solutions reduce to the Maharaj and Leach (1996) model in the limit of vanishing electric field intensity and for integer values of K. We determined an explicit relationship between the spheroidal parameter K and the charge α, and showed that

$$K_{(\alpha\neq 0)} < K_{(\alpha=0)}$$

Thus, the presence of charge decreases the value of the spheroidal parameter K and the spacetime geometry is affected directly. We also generated a new algorithm to find a new solution to the field equations from a specified seed solution. A pair of differential equations were obtained which we solved in general. This procedure is illustrated with two simple examples for uncharged and charged stars.

- The objective in Chapter 4 was to devise realistic models of charged spheres. We chose the metric function

$$Z(x) = \frac{(1+kx)^2}{1+x}$$

to produce classes of Einstein-Maxwell solutions in terms of elementary functions. We observed that the condition of pressure isotropy can be written as

$$4X^2(X-1)\frac{d^2Y}{dX^2} + 2X(X-2)\frac{dY}{dX} + (K-\alpha)Y = 0$$

in terms of the new dependent and independent variables Y and X respectively; here α denotes the electric field intensity parameter and $K = (1-k)/k$. The exact solutions obtained were categorised according to the behaviour of the parameters K and α. For the case $K = \alpha \neq 0$, our solution was given in terms of elementary functions but did not have an uncharged counterpart. The second case $K \neq \alpha$ yields a series as solution and this can be expressed in terms of hypergeometric functions. The general solutions are generated in terms of products of polynomials with an algebraic function for appropriate values of $K - \alpha$. Our solutions contain the Einstein universe. The transformation used to simplify the condition of pressure isotropy limits the solutions that are possible. Consequently many well known models are not regainable from the results of this chapter.

- In Chapter 5, we sought to overcome the problem faced in Chapter 4, and to produce models that contain well known solutions found in the past. Solutions of the Einstein-Maxwell field equations were obtained after specifying the electric field intensity and one of the gravitational fields. Two cases arise depending on the parameter k: $k = 1$ and $k \neq 1$. When $k = 1$ we reduced the condition of pressure isotropy to a Euler-Cauchy equation with solutions in terms of elementary functions. When $k \neq 1$, we demonstrate that it is possible to obtain a more general class of solutions both in terms of special functions and elementary functions. We regained models found previously, including the charged solution of Thirukkanesh and Maharaj (2006), the charged solution of Hansraj and Maharaj (2006), the neutron star solution of Finch and Skea (1989) and the superdense stellar solution of Durgapal and Bannerji (1983)

- In Chapter 6, we obtained two solutions of the gravitational field equations for a charged strange quark star described by the MIT bag model for the metric functions

$$\begin{aligned} y(x) &= a + \sqrt{x} \\ y(x) &= (a + x)^2 \end{aligned}$$

The model reduces to an integration of a first order differential equation. The first solution generalised the model of Mak and Harko (2004) for a quark star in an electromagnetic field. The second solution has the advantage of not containing any singularities at the stellar centre.

In the above we have highlighted only those items of principal interest.

This thesis represents an attempt to find exact solutions to the Einstein-Maxwell field equations. We have demonstrated a number of new exact solutions which generalise earlier models. We have only briefly considered the physical features in our treatment. In future work we intend to fully study the physical properties and stability of the solutions obtained here in greater detail following the approach of Knutsen (1984, 1988), Tikekar (1990), Finch and Skea (1989) and Durgapal and Bannerji (1983). We hope that we have demonstrated that the study of relativistic spherical stars is a fertile area of research, and that further investigation of the solutions presented here and other known solutions should be pursued.

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