A NEW APPROACH TO

ILL-POSED

EVOLUTION EQUATIONS:

C-regularized and B-bounded semigroups

by

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Declaration

I hereby declare that the contents of this thesis, entitled, A new approach to ill-posed evolution equations: C-regularized and B-bounded semigroups, is the result of my sincere work, save and except, where due reference has been made. To my knowledge it has not been submitted before for any degree to this or any other institution.

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For

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and

mum, Sharada Singh

Abstract

The theory of semigroups of linear operators forms an integral part of Functional Analysis with substantial applications to many fields of the natural sciences. In this study we are concerned with the application to equations of mathematical physics. The theory of semigroups of bounded linear operators is closely related to the solvability of evolution equations in Banach spaces that model time dependent processes in nature.

Well-posed evolution problems give rise to a semigroup of bounded linear operators. However, in many important and interesting cases the problem is ill-posed making it inaccessible to the classical semigroup theory. One way of dealing with this problem is to generalize the theory of semigroups.

In this thesis we give an outline of the theory of two such generalizations, namely, C-regularized semigroups and B-bounded semigroups, with the inter-relations between them and show a number of applications to ill-posed problems.

Notation

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{K}$	set $\mathbb{C}$ or set $\mathbb{R}$
$\mathrm{Re}\, z$	real part of a complex number z
$\mathrm{Im}\, z$	imaginary part of a complex number z
$\|z\|$	absolute value of the complex number z
$x \in A$	x is an element of set A
$x \notin A$	x is not an element of set A
iff	if and only if
$A \Leftrightarrow B$	A iff$B$
$A \Rightarrow B$	A implies B
$A \subset B$	set A is contained in set B or operator B is an extension of operator$A$
$\overline{A}$	closure of set A or the closure of an operator A
$\emptyset$	empty set
∂	derivative operator
A^{-1}	inverse to operator A
$D(A)$	domain of the operator A
$\mathrm{Im}(A)$	image of the operator A
$A\mid_S$	restriction of operator A to a subspace S
$\ker A$	kernel of A
$R_\lambda(A)$	resolvent operator of A
$\rho(A)$	resolvent set for operator A

$\rho_C(A)$	C-resolvent set for operator A
$A^*, A^{\#}$	adjoint and formal adjoint operator of A
$\|x\|_X$	norm of x
$\|\cdot\|_G$	graph-norm
$[[X]]$	space X endowed with the graph-norm
v^+, v^-	positive and negative parts of v
$A \in \mathcal{G}(\cdot,\cdot,X)$	A generates a C_0-semigroup
$A \in B - \mathcal{G}(\cdot,\cdot,X)$	A and B generate a B–bounded semigroup
$[x]$, $\underline{x}$	cosets
$A\backslash B$	$\{x \in A;\ x \notin B\}$
X/S	quotient space
$A \hookrightarrow B$	A is continuously embedded in B

Contents

Chapter 1

Generation and representation

1.1 Introduction

Fluid dynamics, electricity, optics, magnetism, heat-flow, etc., just to name a few, are physical phenomena that can be described by partial differential equations. Most of the laws of physics such as Maxwell's equations, Navier-Stokes equations, etc. are stated in terms of partial differential equations, i.e., these laws describe physical phenomena by relating space and time derivatives. Derivatives occur in these equations because they represent rate of change (like velocity, acceleration, etc.), and thus play a vital role in the translation of a physical problem into a mathematical model.

In these models, we have equations relating partial derivatives of some unknown quantity that we wish to find. This entails solving these equations and as these mathematical models become more complex it is natural that the mathematical methods needed for their solution should increase in number and complexity.

The notion of a semigroup is the most important method for describing time-dependent processes in nature in terms of functional analysis. It is difficult to tell when the epoch of the theory of semigroups began. The recognition of the theory of semigroups began in the 1930's, perhaps inspired by the realization that the theory had immediate applications to partial differential equations, [16, p.499]. The abstract theory of semigroups of bounded linear operators now forms an integral part of functional analysis. Since the formulation of the generation theorem by Hille and Yosida in the year 1948, it has become an extensive mathematical subject with substantial applications to many fields of analysis, [16, p.508].

The theory of semigroups of bounded linear operators is closely related to the solution of ordinary differential equations in Banach spaces. Usually, each well-posed initial value problem gives rise to a semigroup of bounded linear operators. Most of the theory deals with a first order equation for the simple reason that linear higher order equations can be reduced to first order systems and then by changing the underlying Banach space, obtain a first order equation.

In the theory of differential equations, one of the first differential equations one solves is the following

$$\partial_t u(t) = \alpha u(t), \qquad \alpha \in \mathbb{C}, \tag{1.1}$$

with initial condition $u(0) = u_0$. It is not difficult to verify that $u(t) = e^{t\alpha} u_0$ is a solution of equation (1.1).

As early as in 1887, [16, p.503], G.P. Peano showed that the system of linear ordinary differential equations with constant coefficients,

$$\begin{aligned} \partial_t u_1 &= \alpha_{11} u_1 + ... + \alpha_{1n} u_n, \\ \cdots & \qquad \cdots\cdots \\ \partial_t u_n &= \alpha_{n1} u_1 + ... + \alpha_{nn} u_n, \end{aligned} \tag{1.2}$$

can be written in a matrix form as

$$\partial_t u(t) = A u(t), \tag{1.3}$$

where A is an $n \times n$ matrix and u is an n-vector whose components are unknown functions, and solved it using the explicit formula

$$u(t) = e^{tA} u_0, \tag{1.4}$$

where the matrix exponential e^{tA} is defined by

$$e^{tA} = I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \dots . \tag{1.5}$$

Taking a norm on $\mathbb{C}^n$ and the corresponding matrix-norm on $M_n(\mathbb{C})$, the space of all complex $n \times n$ matrices, one shows that the partial sums of the series (1.5) form a Cauchy sequence and converge. Moreover, the map $t \longrightarrow e^{tA}$ is continuous and satisfies the properties, Proposition 2.3 of [16]:

$$\begin{aligned} e^{(t+s)A} &= e^{tA}e^{sA} \text{ for all } t, s \geq 0, \\ e^{0A} &= I. \end{aligned} \tag{1.6}$$

Thus the one-parameter family $\{e^{tA}\}_{t\geq 0}$, satisfies the semigroup properties and forms what is termed a semigroup.

The representation (1.5) can be used to obtain a solution of the abstract Cauchy problem

$$\begin{aligned} \partial_t u(t) &= Au(t), \\ u(0) &= u_0, \end{aligned} \tag{1.7}$$

where $A : X \longrightarrow X$ is a bounded linear operator, as in this case the series in (1.5) is still convergent with respect to the norm in $\mathcal{L}(X)$.

Unfortunately, in general, operators coming from applications, like e.g. differential operators, are not bounded and (1.5) cannot be used to obtain a solution of the abstract Cauchy problem (1.7). This is due to the fact that the domain of the operator A in such cases is a proper subspace of X and since (1.5) involves iterates of A, their common domain could shrink to the trivial subspace $\{0\}$. For the same reason, another common representation of the exponential function

$$e^{tA} = \lim_{n\to\infty} \left(1 + \frac{tA}{n}\right)^n, \tag{1.8}$$

cannot be used.

It turns out that for a large class of unbounded operators a variation of the latter makes the representation (1.4) meaningful and e^{tA} can be calculated according to the formula

$$e^{tA} = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} = \lim_{n\to\infty} \left[\frac{n}{t}\left(\frac{n}{t} - A\right)^{-1}\right]^n. \tag{1.9}$$

For this class of operators also other possible interpretations were given eg., in terms of the Yosida approximations of A; see [25, Theorem 5.5]. Such a defined one-parameter family of operators $\{e^{tA}\}_{t\geq 0}$ forms a semigroup of bounded operators and could be used to solve the Cauchy problem (1.7).

In many cases, however, the natural setting of a given problem leads to it being ill-posed, i.e., the solution may not exist for all data, or it is not unique, or else it does not depend continuously on initial data, etc.. To deal with these problems the theory of semigroups was generalized in various ways leading to e.g., integrated semigroups, B-bounded semigroups, etc. In this thesis we shall concern ourselves with two of these generalizations, namely B-bounded semigroups and C-regularized semigroups.

In recent years, the concept of a B-bounded semigroup was introduced by A. Belleni-Morante, [11]. The original idea, prompted by a problem coming from a transport problem with multiplying boundary conditions, was to obtain some explicit estimates of the semigroup obtained by nonconstructive positivity methods. In subsequent papers [4-7], it was shown that B-bounded semigroups could be used to regularize ill-posed evolution problems by embedding them in new spaces related to some appropriately chosen operator, playing the role of a "regularizer", and also that they can be treated as a generalization of Showalter's method, [30], for solving implicit evolution problems.

The other recent generalization of the theory of strongly continuous semigroups is the theory of C-regularized semigroups, [15]. In general, the evolution of a system may be described by a family of operators $\{T(t)\}_{t\geq 0}$ that map the initial value to the states of the system at later times. However, contrary to the theory of strongly continuous semigroups, such a family of operators in general may be unbounded. To deal with this problem a possible strategy would be to find an operator C, such that $\{T(t)C\}_{t\geq 0}$ is a strongly continuous family of bounded linear operators on X. This family is termed a C-existence family. The idea is that C "smooths" whatever uncontrolled behavior $T(t)$ may have; the more ill-posed the abstract Cauchy problem is, the more smoothing is required by the operator C, i.e., the smaller the image of C. When C commutes with A, this family of operators may be characterized by the algebraic properties similar to strongly continuous semigroups and is termed a C-regularized semigroup.

We discuss both of these theories with applications and show that despite some similarities to the concept of C-regularized semigroups and evolution families, B-bounded semigroups are completely different. Extending the quotation from deLaubenfel's monograph, in C-regularized semigroups "a different yardstick is used to measure the initial data, than is used to measure the solution" (the changes in the initial values are measured in the topology of the range of the regularizing operator C) whereas for B-bounded semigroups we use the same yardstick, related to the regularizing operator B, both for solutions and the initial data. Another important difference is that the operator B is not required to commute with the operator A and it can be unbounded, non-invertible, not even closable, which gives much more flexibility.

We shall now present a summary of the thesis:

In Chapter 1, we begin by presenting the statements of pertinent definitions and theorems used in the thesis. In particular, in Section 1.3, we give an outline of the theory of strongly continuous semigroups.

In Chapter 2, we present the transport problem with multiplying boundary conditions and show that generation results, obtained in [10-11], could be extended to accommodate the case when one part of the boundary is multiplying and the other absorbing.

In Chapter 3, we present the main generation results for B-bounded semigroups, originally found in [5], however, under much weaker assumptions imposed on the operators A and B, as developed in [3], [6] and [8].

In Chapter 4, we present an outline of the basic theory of C-regularized semigroups and, in particular, discuss the relations between various concepts of the generators. Furthermore, we discuss the inhomogeneous Cauchy problem in the case when A is a generator of a C-regularized semigroup and provide conditions that should be satisfied by the inhomogeneity in order for this problem to have mild and strict solutions. This part can be regarded as a translation of Section 4.2 of [25] into the context of C-regularized semigroups.

Finally, in Chapter 5, we discuss in detail the relationship between B-bounded and C-regularized semigroups, as presented in [8].

1.2 Preliminaries

In this section, we wish to present the notation and terminology used throughout the thesis.

Unless otherwise stated, let $(X, ||\cdot||)$ denote a Banach space over the field $\mathbb{K}$.

Definition 2.1 An operator $A : D(A) \longrightarrow X$, $D(A) \subset X$ is *linear* iff

$$A(\alpha u + \beta v) = \alpha Au + \beta Av, \tag{1.10}$$

for all α, $\beta \in \mathbb{K}$ and u, $v \in D(A)$.

Unless otherwise stated, it shall be assumed that A is a linear operator from $D(A)$ into X.

Definition 2.2 A linear operator A is *bounded* iff there exists a positive constant, α such that

$$||Au|| \leq \alpha \, ||u|| \, , \tag{1.11}$$

for all $u \in D(A)$.

Denote the set of all bounded linear operators on X by $\mathcal{L}(X)$.

Definition 2.3 A linear operator A is *closed* iff

$$u_n \longrightarrow u \text{ and } Au_n \longrightarrow v, \tag{1.12}$$

implies $u \in D(A)$ and $Au = v$.

Denote the set of all closed linear operators in X by $\mathcal{C}(X)$.

Definition 2.4 A linear operator $B : D(B) \longrightarrow X$, $D(B) \subset X$, is an *extension* of a linear operator A, written as $A \subset B$, iff

$$Au = Bu \quad \text{for all} \quad u \in D(A) \subset D(B). \tag{1.13}$$

Definition 2.5 An operator $A \mid_S : S \longrightarrow X$, defined as follows

$$A \mid_S u = Au, \, u \in S \subset D(A), \tag{1.14}$$

is a *restriction* of the operator A to a subspace S.

Definition 2.6 If a linear operator A has a closed linear extension $B : D(B) \longrightarrow X$, $D(B) \subset X$, then A is called a *closable* operator. The *closure* operator, denoted by $\overline{A}$, is the smallest closed extension of A, i.e., if there exists a closed linear operator $C : D(C) \longrightarrow X$, $D(C) \subset X$, such that $C \subset B$ then $\overline{A} \subset C$.

In applications, we are usually interested in the properties of the family of operators $\{A_\lambda\}_{\lambda\in\mathbb{C}}$ defined as

$$A_\lambda x = (\lambda I - A)x, \quad x \in D(A). \tag{1.15}$$

Definition 2.7 Any point $\lambda \in \mathbb{C}$, for which there is a continuous inverse of A_λ defined on X, is said to belong to the *resolvent set* of A and the inverse A_λ^{-1} is called the *resolvent* of A and denoted by $R_\lambda(A)$. The resolvent set is denoted by $\rho(A)$ and thus

$$\rho(A) = \{\lambda \in \mathbb{C},\ R_\lambda(A) \in \mathcal{L}(X)\}\,. \tag{1.16}$$

Definition 2.8 The complement of the resolvent is called the *spectrum* of A, denoted by $\sigma(A)$.

To distinguish different cases for which A_λ is not invertible, the spectrum is subdivided into three disjoint sets: the point spectrum of A, $\sigma_p(A)$, the continuous spectrum of A, $\sigma_c(A)$ and the residual spectrum of A, $\sigma_r(A)$.

Definition 2.9 A complex number λ_0 belongs to the point spectrum of A iff the equation $Ax = \lambda_0 x$ has a non-trivial solution x_0. The complex number λ_0 is called an *eigenvalue* of A and x_0 is called the *eigenvector*.

It is important to develop a theory of evolution equations in spaces with some partial order, since in many cases the only physically reasonable data and solutions for (1.7) are non-negative quantities e.g., density. Fortunately, most function spaces have a natural order which is compatible with the norm.

Definition 2.10 Let X be a linear space over the field $\mathbb{K}$, with partial relation $\leq$ between its elements. The space X is a *lattice* with respect to $\leq$, if for all $u,v \in X$ there exists a least upper bound, $u \vee v$ and a greater lower bound, $u \wedge v$.

Definition 2.11 Let X be a lattice. Then X is a *Banach lattice* if the following relations among the linear structure, its norm and the order relation $\leq$ are satisfied:

i) if $u \leq v$, then $u + w \leq v + w$ for $w \in X$,

ii) if $u \leq v$, then $\alpha u \leq \alpha v$ for $\alpha \geq 0$,

iii) if $|u| \leq |v|$, then $\|u\| \leq \|v\|$,

where $|u|$, the *modulus* of u, is the element defined by $|u| = u \vee (-u)$.

It follows that if we define the positive and negative parts of each element $u \in X$ by $u^+ = u \vee 0$ and $u^- = (-u) \vee 0$, respectively, then any $u \in X$ can be represented as follows $u = u^+ - u^-$ and the modulus by $|u| = u^+ + u^-$. Furthermore, we have that for any $u \in X$, $\|u\| = \|u^+\| + \|u^-\|$.

The space $L_1(\mathbb{R})$, of all integrable functions on $\mathbb{R}$, is an example of a Banach lattice that carries the almost everywhere pointwise ordering.

Definition 2.12 Let X be a Banach lattice, with partial order $\leq$. Then the closed subset X^+,

$$X^+ = \{u \in X;\, u \geq 0\}\,, \tag{1.17}$$

is called a *positive cone*.

Definition 2.13 A bounded linear operator A on a Banach lattice X is *positive* if $u \geq 0$ implies $Au \geq 0$.

It now makes sense to compare positive operators: we say $A \leq B$, if $B - A$ is positive. For the positive operator A, since $u \leq |u|$, we get that for any $u \in X$

$$|Au| \;\leq A\,|u|\,,$$

so that if $A \leq B$

$$\|Au\| = \||Au|\| \leq \|A\,|u|\| \leq \|B\,|u|\| \leq \|B\|\,\||u|\| = \|B\|\,\|u\|$$

and this implies

$$\|A\| := \sup_{\|u\|\leq 1} \|Au\| \leq \|B\|\,. \tag{1.18}$$

Definition 2.14 Let X be a Banach lattice, then an operator A is *resolvent positive*, if there is a constant $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and for all $\lambda > \omega$

$$R_\lambda(A) = (\lambda - A)^{-1} \geq 0. \tag{1.19}$$

Definition 2.15 Let $A : D(A) \longrightarrow X$ be a densely defined operator. The *adjoint* of A is the operator $A^* : D(A^*) \subset X \longrightarrow X$ given by the formula

$$(Au, v) = (u, A^* v), \tag{1.20}$$

for all $u \in D(A)$ and $v \in D(A^*)$, where $D(A^*)$ is defined by

$D(A^*) = \{v \in X;$ there exists $w \in X$ such that $(Au, v) = (u, w)$ for all $u \in D(A)\}$.

Let the interval (a, b), for a,$b \in \mathbb{R}$ be denoted by I.

Definition 2.16 A function $u : I \longrightarrow X$ is *strongly continuous* on I, if for every $t_0 \in I$,

$$\lim_{t \to t_0} \|u(t) - u(t)\| = 0. \tag{1.21}$$

Denote the space of all continuous function on I by $C(I)$. Unless otherwise stated, it shall be assumed that u is a function from the interval I into the Banach space X.

Definition 2.17 A function $u : I \longrightarrow X$ is *strongly differentiable* on I if for every $t_0 \in I$, there is an element $u'(t_0) \in X$ such that

$$\lim_{h \to 0} \left\| \frac{u(t_0 + h) - u(t_0)}{h} - u'(t_0) \right\| = 0. \tag{1.22}$$

Definition 2.18 Denote by $C^k(I)$ the space of k-*times continuously differen-*

tiable functions $u : I \longrightarrow X$ and

$$C^{\infty}(I) := \bigcap_{k \in \mathbb{N}} C^{k}(I). \tag{1.23}$$

Definition 2.19 A function u has a *compact support* if there exists a compact set $\Omega \subset I$ such that $u(t) = 0$ whenever $t \in I \backslash \Omega$. The closure of the set $\overline{\{t \in I \mid u(t) \neq 0\}}$ is the smallest of such sets Ω and is called the support of u.

Definition 2.20 The space of *infinitely differentiable* functions on I having compact support contained in I is denoted by $C_0^{\infty}(I)$.

Definition 2.21 A function u is *Lipschitz continuous* iff there is a constant $L > 0$ such that

$$\|u(t) - u(s)\| \leq L \; |t - s| \text{ for all } t,s \in I. \tag{1.24}$$

Definition 2.22 For $0 \leq \alpha < 1$, a function $u : I \longrightarrow X$ is *Hölder continuous* of exponent α if the quantity

$$\sup \left\{ \frac{\|u(t) - u(t_0)\|}{|t - t_0|^{\alpha}}, \; t, \, t_0 \in I, \, t \neq t_0 \right\}, \tag{1.25}$$

is finite.

It follows that the function u is *locally Hölder continuous* if every $t \in I$ has a neighborhood in which u is Hölder continuous.

In the definitions that follows, we shall use Lebesgue measure m on the interval I.

Definition 2.23 Let $\{I_i\}_{i=1}^m$ be a finite collection of mutually disjoint, measurable subsets of I, such that $I = \bigcup\limits_{i=1}^{m} I_i$ and $\{x_i\}_{i=1}^m$ be a collection of points of X. A function $u : I \longrightarrow X$, defined by

$$u(t) = \sum_{i=1}^{m} x_i \chi_{I_i}(t), \tag{1.26}$$

where χ_{I_i}, is the characteristic function of I_i (i.e., $\chi_{I_i} = 1$ on I_i and $\chi_{I_i} = 0$ otherwise), is called a *simple* function.

Definition 2.24 A function $u : I \longrightarrow X$, defined almost everywhere on I is called *measurable* on I if there exist a sequence $(u_n)_{n\in\mathbb{N}}$ of simple functions such that

$$\lim_{n\to\infty} \|u_n(t) - u(t)\| = 0, \tag{1.27}$$

almost everywhere on I.

Definition 2.25 If u is a simple function then we define its integral by

$$\int_I u(t)dt = \sum_{i=1}^{m} x_i m(I_i). \tag{1.28}$$

Definition 2.26 If for a function u we can choose a sequence of simple functions

$(u_n)_{n\in\mathbb{N}}$ in such a way that

$$\lim_{n\to\infty}\int_I \|u_n(t)-u(t)\|\,dt = 0, \tag{1.29}$$

then we say that u is (*Bochner*) *integrable* on I and define the *Bochner integral* by

$$\int_I u(t)dt = \lim_{n\to\infty}\int_I u_n(t)dt. \tag{1.30}$$

It follows that a measurable function u is Bochner integrable on I iff $t \longrightarrow \|u(t)\|$ is Lebesgue integrable on I and we have

$$\left\|\int_I u(t)dt\right\| \leq \int_I \|u(t)\|\,dt. \tag{1.31}$$

If A is a closed linear operator and $u(t)$, $Au(t)$ are continuous on the interval I, then $\int_I u(t)dt \in D(A)$ and

$$A\int_I u(t)dt = \int_I Au(t)dt. \tag{1.32}$$

Definition 2.27 For $1 \leq p < \infty$, let $L_p(I)$ denote the classical Banach space of equivalence classes consisting of measurable functions u that are p-*integrable* and differ only on a subset of Lebesgue measure zero.

The norm on $L_p(I)$ is defined by

$$\|u\|_p = \left(\int_I \|u(t)\|^p\,dt\right)^{\frac{1}{p}}. \tag{1.33}$$

In particular, if $X = \mathbb{R}$, we can define the generalized derivative of real-valued functions belonging to $L_p(I)$ as follows:

Definition 2.28 Let $1 \leq p < \infty$, $u \in L_p(I)$ and if there exists $u' \in L_p(I)$ such that

$$\int_I v(t)u'(t)dt = -\int_I v'(t)u(t)dt, \tag{1.34}$$

for all $v \in C_0^\infty(I)$ then u' is referred to as the *generalized derivative* of u. Denote the space of all functions for which the generalized derivative exists on I by $W^1(I)$.

Definition 2.29 For $1 \leq p < \infty$, let

$$W_p^1(I) = \left\{u \in W^1(I) \bigcap L_p(I) \mid u' \in L_p(I)\right\}. \tag{1.35}$$

A norm is introduced by

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|u'\|_p^p\right)^{\frac{1}{p}}. \tag{1.36}$$

For real-valued functions every Lipschitz continuous function is differentiable almost everywhere, however, the same cannot be said for X-valued functions. A class of Banach spaces for which this is true are spaces having the Radon-Nikodym property.

Theorem 2.30 [2] A Banach space X has the *Radon-Nikodym* property iff every Lipschitz continuous function $u : [0,1] \longrightarrow X$ is differentiable almost everywhere.

1.3 Strongly continuous semigroups

The key to solving an abstract Cauchy problem is to obtain a family of operators that mimic the properties of the exponential function of a single variable. One particularly useful definition of such a family is that of a strongly continuous semigroup.

Definition 3.1 A family of bounded linear operators on X, $\{T(t)\}_{t\geq 0}$ is called a *strongly continuous semigroup*, denoted by C_0-semigroup, if it satisfies the following conditions

$$\begin{array}{lllll} \text{i)} & T(t+s) & = & T(t)T(s) & \text{for all } t,s \geq 0, \\ \text{ii)} & T(0) & = & I, & \\ \text{iii)} & t \longrightarrow T(t)x & \in & C\left([0,\infty),\ X\right) & \text{for all } x \in X. \end{array}$$

By the application of the Uniform boundedness principle, it can be shown; see [25, Theorem 2.2], that the family $\{T(t)\}_{t\geq 0}$ is exponentially bounded ie., there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that for any $t \geq 0$,

$$\|T(t)\| \leq Me^{\omega t}. \tag{1.37}$$

The semigroup $\{T(t)\}_{t\geq 0}$ is called a uniformly bounded semigroup if $\omega \leq 0$, and if, in addition, $M = 1$, then the semigroup is called a contractive semigroup.

Definition 3.2 A linear operator $A : D(A) \longrightarrow X$, $D(A) \subset X$, defined by

$$Ax := \lim_{t\to 0^+} \frac{1}{t}\left(T(t)x - x\right), \tag{1.38}$$

for $x \in D(A)$, with

$$D(A) = \left\{ x \in X;\ \lim_{t\to 0^+} \frac{1}{t}\left(T(t)x - x\right) \text{ exists} \right\}, \tag{1.39}$$

is called the *infinitesimal generator* of the semigroup $\{T(t)\}_{t\geq 0}$.

If A generates a semigroup satisfying estimate (1.37), then we write $A \in \mathcal{G}(M, \omega, X)$.

Theorem 3.3 (Theorem 2.4 of [25]) Let A be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Then

i) $D(A)$ is dense in X,

ii) A is a closed linear operator,

iii) for any $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x, \tag{1.40}$$

iv) for any $x \in X$, $\int_0^t T(s)x ds \in D(A)$ and

$$A\left(\int_0^t T(s)x ds\right) = T(t)x - x, \tag{1.41}$$

v) for any $x \in D(A)$, $T(t)x \in D(A)$ and

$$\partial_t T(t)x = AT(t)x = T(t)Ax, \tag{1.42}$$

vi) for any $x \in D(A)$

$$\int_s^t T(r)Ax dr = T(t)x - T(s)x. \tag{1.43}$$

For $A \in \mathcal{G}(M, \omega, X)$ and for an arbitrary $x \in D(A)$, the function $t \longrightarrow u(t, x) = T(t)x$ solves the abstract Cauchy problem (1.7).

Definition 3.4 A *classical* (or *strict*) solution of the abstract Cauchy problem is a function $u : [0, \infty) \longrightarrow X$ that satisfies the following:

i) $u(t)$ is strongly continuous, for all $t \in [0, \infty)$,

ii) $u(t)$ is strongly differentiable, for all $t \in (0, \infty)$,

iii) $u(t) \in D(A)$, for all $t \in (0, \infty)$,

iv) $u(t)$ satisfies (1.7).

The semigroup is uniquely determined by the generator A, Theorem 2.6 of [25], and it follows that $T(t)x$ is a unique solution of the Cauchy problem (1.7).

In general, for $x \in X \backslash D(A)$, $T(t)x$ is continuous but it is neither differentiable nor belongs to $D(A)$ and therefore it is not a solution of (1.7).

Therefore, it follows that the existence of a solution of the abstract Cauchy problem (1.7) is guaranteed, provided we restrict the initial data to a dense subspace, namely to the domain of the generator A.

For $A \in \mathcal{G}(M, \omega, X)$, the resolvent $R_\lambda(A)$, is the Laplace transform of $\{T(t)\}_{t \geq 0}$ ie.,

$$R_\lambda(A) = \int_0^\infty e^{-\lambda t} T(t) dt, \tag{1.44}$$

for $\lambda \in \rho(A)$. We conclude this section by stating the Hille-Yosida theorem, which addresses a fundamental problem in the theory of strongly continuous semigroups to characterize properties of the semigroup in terms of the its generator and resolvent.

Theorem 3.5 (Theorem 5.3 of [25]) An operator A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ iff

i) A is closed and $D(A)$ is dense in X,

ii) the resolvent set $\rho(A) \supset (\omega, \infty)$ and for every $\lambda > \omega$

$$\| R_\lambda(A)^n \| \leq \frac{M}{(\lambda - \omega)^n}, \tag{1.45}$$

for all $\lambda > \omega$, $n \in \mathbb{N}$.

The condition that every real $\lambda > \omega$ is in the resolvent set of A, together with estimate (1.45), imply that every complex λ satisfying $\mathrm{Re}\,\lambda > \omega$ is in the resolvent set of A and

$$\| R_\lambda(A)^n \| \leq \frac{M}{(\mathrm{Re}\,\lambda - \omega)^n} \tag{1.46}$$

for all $\mathrm{Re}\,\lambda > \omega$, $n \in \mathbb{N}$. One major disadvantage of the above characterization is that a direct verification of estimate (1.45) in most cases is tedious, if not impossible.

In the following section we shall discuss a class of semigroups for which it is sufficient to verify (1.45) once, but for complex values of λ.

1.4 Analytic semigroups

In this section, we present the conditions imposed on the resolvent and the resolvent set of the generator of a uniformly bounded strongly continuous semigroup that ensures the extension of the semigroup and the parameter domain to an analytic semigroup in some sector containing the positive real axis.

Definition 4.1 Let $\Delta_\delta = \{z;\, |\arg z| < \delta\}$ for some $\delta > 0$ and $\{T(z)\}_{z\in\Delta_\delta}$ be a family of bounded linear operators on X. The family $\{T(z)\}_{z\in\Delta_\delta}$ is an *analytic semigroup* in the sector Δ_δ if

i) $z \longrightarrow T(z)$ is analytic in Δ_δ,

ii) $\lim\limits_{\substack{z\to 0,\\ z\in\Delta_\delta}} T(z)x = x,$ for every $x \in X$,

iii) $T(0) = I,$ $T(z+w) = T(z)T(w)$ $z,\, w \in \Delta_\delta$.

Let $A : D(A) \longrightarrow X$ be a densely defined operator,

$$\rho(A) \supset \Delta_{\frac{\pi}{2}+\delta} = \left\{\lambda;\ |\arg\lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\}, \tag{1.47}$$

for some $0 < \delta < \dfrac{\pi}{2}$ and

$$\|\, R_\lambda(A)\| \ \le \ \frac{M}{|\lambda|}, \tag{1.48}$$

for all $0 \ne \lambda \in \Delta_{\frac{\pi}{2}+\delta}$. Then A is the infinitesimal generator of a uniformly bounded, C_0-semigroup $\{T(t)\}_{t\ge 0}$; see [25, Theorem 7.7]. The family of operators $\{T(t)\}_{t\ge 0}$ is given by the Dunford type integral

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R_\lambda(A)\, d\lambda, \tag{1.49}$$

where Γ is the path composed from the two rays $\rho e^{i\gamma}$ and $\rho e^{-i\gamma}$, $0 < \rho < \infty$ and $\dfrac{\pi}{2} < \gamma < \dfrac{\pi}{2} + \delta$.

Furthermore, the semigroup $\{T(t)\}_{t\geq 0}$ generated by the operator A satisfying (1.47) and (1.48) can be extended to an analytic semigroup in the sector Δ_δ; see [25, Theorem 5.2].

If A is the generator of an analytic semigroup $\{T(z)\}_{z\in\Delta_\delta}$, then $t \longrightarrow T(t)$ has derivatives of arbitrary orders on $(0,\infty)$ in the uniform operator topology and $T(t) \in \mathcal{L}(X, [[D(A^n)]])$ for any $t > 0$ and $n \in \mathbb{N}$.

Moreover, $\partial_t^n T(t) = A^n T(t)$ for $t > 0$, $n \geq 0$ and so in particular it follows that $t \longrightarrow T(t)x$ solves (1.7) for an arbitrary $x \in X$. This is a significant improvement on that of a C_0-semigroup, for which $x \in D(A)$ was a requirement.

Chapter 2

An evolution system

In this chapter, we consider a particle transport problem in a slab under the assumption that particles are reflected and either multiplied by both boundary planes, absorbed by both boundary planes or multiplied by one boundary plane and absorbed by the other. As in [10, Section 3] we show that the abstract version of the problem has a unique positive strict solution and derive an inequality of physical interest. A part of this chapter contains an extension of the results of Belleni-Morante, who considered only the transport with purely multiplying boundary conditions (the case with purely absorbing boundary conditions being standard). It should be emphasized that to avoid going into physical details we confined ourselves to the simplest geometry of a homogeneous slab, whereas

Belleni examined the problem in a general convex region $V \subset \mathbb{R}^3$ bounded by a closed C^1 surface.

2.1 A particle transport problem

In this section we shall briefly examine how a linear problem of particle transport in a slab can be transformed into an abstract evolution problem.

Consider a mathematical model of particles, e.g., electrons, photons, particles of a contaminant etc., moving in a homogeneous slab S bounded by the boundary planes $x = -a$ and $x = a$, $(0 < a < \infty)$. The particles on contact with the boundary planes are either absorbed by both boundaries, multiplied by both boundaries or multiplied by one boundary plane and absorbed by the other (the latter referred to as mixed boundary planes), under the assumption of plane symmetry. This boundary behaviour at $x = a$ and $x = -a$ can be modeled by the equations:

$$\begin{aligned} yu(-a, y, t) &= \alpha \int_{-1}^{0} |y'|\, u(-a, y', t)dy',\ y \in (0, 1), \\ |y|\, u(a, y, t) &= \beta \int_{0}^{1} y'u(a, y', t)dy',\ y \in (-1, 0), \end{aligned} \tag{2.1}$$

where the positive constants α, β are either the absorption coefficients, i.e., α, $\beta < 1$, the multiplication coefficients, i.e., α, $\beta > 1$ or one is the absorption coefficient and the other the multiplication coefficient. In conditions (2.1) we denote by $u(x, y, t)$ the (numerical) density of the particles which at time t are at location $x \in [-a, a]$ and have a velocity $\mathbf{v}$ such that $y = \mathbf{v}.\mathbf{i}/|\mathbf{v}|$, where $\mathbf{i}$ is a unit vector in the positive direction of the x-axis.

The transport problem in the homogeneous slab leads to the following integro-differential system:

$$\frac{\partial}{\partial t}u(x,y,t) = -vy\frac{\partial}{\partial x}u(x,y,t) - v\sigma u(x,y,t) + \frac{1}{2}v\sigma_s\int_{-1}^{1} u(x,y',t)dy',\ t > 0 \tag{2.1a}$$

$$u(x,y,0) = u^{(0)}(x,y),\ x \in (-a,a),\ y \in (-1,1),$$

supplemented by the boundary conditions (2.1). The positive constant v is the speed of the particles in the slab S, $u^{(0)}(x, y)$ is a non-negative function and $\sigma = \sigma_c + \sigma_s$, where σ_c, $\sigma_s > 0$ are the macroscopic cross-sections for the capture and scattering respectively.

Since $u(x, y, t)$ is the particle density at (x, y), it follows that $[u(x, y, t)dxdy]$ is the expected number of particles that at time t are between x and $x + dx$ and have velocities $\mathbf{v}$ such that the cosine of the angle between $\mathbf{v}$ and the x axis is between y and $y + dy$. The transport equation is a balance equation for the particles characterized by the the state variables x and y and shows that during the infinitesimal time interval dt, the change $\left[\left(\frac{\partial}{\partial t}u\right) dxdydt\right]$ of the number of such particles is due to four processes that are represented by:

a) The free streaming term $\left[\left(vy\frac{\partial}{\partial x}u\right) dxdydt\right]$ that gives the number of (x, y) particles which enter or leave the region between x and $x + dx$ during the time interval dt, without interacting with the material of the slab S.

b) The attenuation term $[(v\sigma_c u)\, dxdydt]$ that refers to (x, y) particles that are removed during the time interval dt because they are captured and disappear for good. Note that $[(v\sigma_c u)\, dxdydt]$ is the expected number of captures during the time interval dt because $(v\sigma_c u)\, dt$ is the probability of a capture event during the time interval dt.

c) The loss term $[(v\sigma_s u)\, dxdydt]$ that gives the number of (x, y) particles that are scattered and re-appear with a different speed.

d) The gain term $\left[\left(\frac{1}{2}v\sigma_s \int_{-1}^{1} u(x, y', t)dy'\right) dxdydt\right]$ that gives the contribution to the family of (x, y) particles of the scattering phenomenon. In other words, (x, y') particles with $y' \in (-1, 1)$ are scattered and reappear as (x, y) particles. Since the number of (x, y') particles is $u(x, y', t)dxdy'$ and since $(v\sigma_s)\, dt$ is the probability of a scattering event during the interval dt, $[(v\sigma_s u)\, dxdy'dt]$ is the number of (x, y') particles scattered during the time interval dt and by integration we get the contribution of all (x, y') classes to the class (x, y).

Following [10], we write the particles density $u(x, y, t)$ as follows:

$$u(x, y, t) = \sum_{j=0}^{\infty} u_j(x, y, t),$$

where $u_j(x, y, t)$ is the density of particles which at time t, "remember" just j reflections from the boundary surface. To explain the definition of $u_j(x, y, t)$, we shall call "mother" a particle just before undergoing a reflection and "daughters" the particles generated in the reflection. Assume that the daughter "remembers" both the reflection during which they were generated and all the reflection events remembered by the their mother. Then, $u_j(x, y, t)$ is the density of the particles which remember just j reflections.

The partial densities $u_j(x, y, t)$ satisfy the following system

$$\begin{aligned} \frac{\partial}{\partial t}u_j(x,y,t) &= -vy\frac{\partial}{\partial x}u_j(x,y,t) - v\sigma u_j(x,y,t) + \\ & \frac{1}{2}v\sigma_s \int_{-1}^{1} u_j(x,y',t)dy',\ t>0 \end{aligned} \tag{2.2}$$

$$\begin{aligned} u_j(x,y,0) &= u_j^{(0)}(x,y),\ x\in(-a,a),\ y\in(-1,1). \\ yu_j(-a,y,t) &= \alpha\int_{-1}^{0} |y'|\, u_{j-1}(-a,y',t)dy',\ y\in(0,1), \\ |y|\, u_j(a,y,t) &= \beta\int_{0}^{1} y'u_{j-1}(a,y',t)dy',\ y\in(-1,0), \end{aligned} \tag{2.3}$$

where $j = 0,1,2,\ldots.$ The boundary conditions (2.3) become the non re-entry boundary conditions for u_0 if $j=0$, because $u_{-1}=0$ by assumption. The initial condition at the instant when $t=0$ is chosen so that $u_j^{(0)}(x,y)=0$ for all $j=0,1,2,3,\ldots.$

In order to apply the theory of semigroups, the physical system must be transformed into an evolution problem in a suitable Banach space. The choice of the space X often depends on which quantity is considered to be the most relevant from a physical viewpoint (in this case the density of particles). For the above model, we consider the Banach space

$$X = \left\{ u = \begin{pmatrix} u_0 \\ u_1 \\ \dots \end{pmatrix}, \ u_j \in L_1\left((-a,a)\times(-1,1)\right), \ \sum_{j=0}^{\infty} \|u_j\|_1 \text{ is finite} \right\}, \tag{2.4}$$

with norm defined as follows

$$\|u\| = \sum_{j=0}^{\infty} \|u_j\|_1, \qquad \|u_j\|_1 = \int_{-a}^{a} dx \int_{-1}^{1} |u_j(x,y)|\, dy. \tag{2.5}$$

In what follows we shall focus only on the part of (2.2) corresponding to the free streaming phenomenon. The free streaming operator A is defined as follows

$$Au = \begin{pmatrix} A_0 u_0 \\ A_1 u_1 \\ \dots \end{pmatrix}, \ A_j u_j = -vy\frac{\partial u_j}{\partial x} \tag{2.6}$$

for $j = 0$,1,2,..., with domain

$$D(A) = \{u \in X \colon Au \in X;\ u \text{ satisfies } (2.3)\},$$

where $\dfrac{\partial u_j}{\partial x}$ is the generalized derivative. Using (2.6) this part can be written as

$$
\begin{aligned}
\partial_t u(t) &= Au(t), \\
u(0) &= u^{(0)} = \begin{pmatrix} u_{00} \\ u_{10} \\ \dots \end{pmatrix} \in D(A),
\end{aligned}
\tag{2.7}
$$

where $u(t)$ is a function from $\mathbb{R}$ into X and $\partial_t u(t)$ is the strong derivative.

We shall now consider the resolvent equation

$$
(\lambda I - A)u = w, \tag{2.8}
$$

where $w \in X$, $u \in D(A)$. Using equation (2.6), we obtain

$$
vy\frac{\partial}{\partial x}u_j(x,y) + \lambda u_j(x,y) = w_j(x,y), \tag{2.9}
$$

for all $j =$ 0,1,2,... and solving for u_j, we get for $y \in (0,1)$

$$
u_j(x,y) = \frac{1}{vy}C_j^{+}e^{-\frac{\lambda(a+x)}{vy}} + \frac{1}{vy}\int_{-a}^{x} e^{-\frac{\lambda(x-x')}{vy}} w_j(x',y)dx' \tag{2.10a}
$$

and for $y \in (-1,0)$

$$
u_j(x,y) = \frac{1}{v\,|y|}C_j^{-}e^{\frac{\lambda(a-x)}{vy}} + \frac{1}{v\,|y|}\int_{x}^{a} e^{-\frac{\lambda(x-x')}{vy}} w_j(x',y)dx', \tag{2.10b}
$$

where C_j^+, C_j^- are constants and are determined from the boundary conditions as follows:

$$C_j^+ = \alpha C_{j-1}^- \int_{-1}^{0} e^{\frac{\lambda 2a}{vy'}} dy' + \qquad (2.11a)$$

$$\alpha \int_{-1}^{0} dy' \int_{-a}^{a} e^{\frac{\lambda(a+x')}{vy'}} w_{j-1}(x', y') dx',$$

$$C_j^- = \beta C_{j-1}^+ \int_{0}^{1} e^{-\frac{\lambda 2a}{vy'}} dy' + \qquad (2.11b)$$

$$\beta \int_{0}^{1} dy' \int_{-a}^{a} e^{-\frac{\lambda(a-x')}{vy'}} w_{j-1}(x', y') dx',$$

where $j = 0,1,2,...$ and C_{-1}^+, $C_{-1}^- = 0$, $w_{-1} = 0$, [11]. Relations (2.11) give C_j^+, C_j^- explicitly in terms of w_{j-1}, for $j = 0,1,2,...$, and thus in particular, if $w \in X^+$ then $C_j^+ \geq 0$, $C_j^- \geq 0$ for $j = 0,1,2,....$

Next, following [10], we show that the resolvent $R_\lambda(A)$, belongs to the space $\mathcal{L}(X)$, for all $\lambda > \lambda_0$, where λ_0 is a constant depending on α and β.

Setting $\chi = \max\{\alpha, \beta\}$, we obtain

$$\sum_{j=0}^{\infty} \left[\, |C_j^+| + |C_j^-| \,\right] \qquad (2.12)$$

$$\leq \chi e^{-\frac{2a\lambda}{v}} \sum_{j=0}^{\infty} \left[\, |C_j^+| + |C_j^-| \,\right] + \chi \|w\|$$

and so, we get

$$\sum_{j=0}^{\infty} \left[\, \left|C_j^+\right| + \left|C_j^-\right| \,\right] \;\leq\; \xi(\lambda)\, \|w\| \,, \tag{2.13}$$

where

$$\xi(\lambda) = \frac{\chi}{1 - \chi e^{-\frac{2a\lambda}{v}}}, \tag{2.14}$$

provided $1 - \chi e^{-\frac{2a\lambda}{v}}$ is positive i.e., $\lambda > \lambda_0 = \dfrac{v}{2a} \ln \chi$. From (2.10) we obtain

$$\begin{aligned} \|u_j\|_1 &= \int_{-a}^{a} dx \int_{-1}^{1} |u_j(x,y)|\, dy \\ &\leq \frac{1}{\lambda} \left\{ \left|C_j^+\right| + \left|C_j^-\right| + \|w_j\|_1 \right\} \end{aligned} \tag{2.15}$$

and this implies

$$\|u\| \;\leq\; \frac{\xi(\lambda) + 1}{\lambda} \|w\| \,, \tag{2.16}$$

for all $w \in X$, $\lambda > \lambda_0$. Since $(\lambda I - A)^{-1} w = u$, it follows that $(\lambda I - A)^{-1} \in \mathcal{L}(X)$ with

$$\|(\lambda I - A)^{-1}\| \;\leq\; \frac{\xi(\lambda) + 1}{\lambda}, \tag{2.17}$$

provided $\lambda > \lambda_0$. Estimate (2.17) is not sufficient to show that A generates a strongly continuous semigroup, because $\xi(\lambda) + 1 > 1$. But, if we take

$w \in X^+$, then equation (2.8) has a unique solution $u_j \in (L_1)^+$, and so we have

$$\begin{aligned}
\lambda \left\| u_j \right\|_1 &= -v \int_{-1}^{+1} dy \left[y \int_{-a}^{+a} \frac{\partial}{\partial x} u_j(x, y) dx \right] + \left\| w_j \right\|_1 \\
&= v \left[\int_{-1}^{0} |y| \, u_j(a, y) dy \; + \int_{0}^{1} y u_j(-a, y) dy \right] \qquad (2.18) \\
&\quad -v \left[\int_{0}^{1} y u_j(a, y) dy + \int_{-1}^{0} |y| \, u_j(-a, y) dy \right] + \left\| w_j \right\|_1 .
\end{aligned}$$

Since $u \in D(A)$, from the boundary conditions we obtain

$$\begin{aligned}
&\lambda \left\| u_j \right\|_1 \\
&= \left\| w_j \right\|_1 + v \left[\beta \int_{0}^{1} y' u_{j-1}(a, y') dy' \; + \alpha \int_{-1}^{0} |y'| \, u_{j-1}(-a, y') dy' \right] \qquad (2.19) \\
&- v \left[\int_{0}^{1} y u_j(a, y) dy + \int_{-1}^{0} |y| \, u_j(-a, y) dy \right] .
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\lambda \left\| u \right\| &= \left\| w \right\| + v(\beta - 1) \sum\nolimits_{j=0}^{\infty} \int_{0}^{1} y u_j(a, y) dy + \\
&\qquad v(\alpha - 1) \sum\nolimits_{j=0}^{\infty} \int_{-1}^{0} |y| \, u_j(-a, y) dy,
\end{aligned} \qquad (2.20)$$

because $u_{-1} = 0$.

In the subsections that follow, equation (2.20) is used to show that A generates a semigroup in all three cases: purely absorbing boundaries ($\alpha \leq 1$, $\beta \leq 1$), purely multiplying boundaries ($\alpha > 1$, $\beta > 1$), and mixed (one of the parameters greater and the other smaller that 1). We note that the analysis of the first two cases is based on [10] but the last one is new.

Before we proceed we introduce the following notation: a coefficient γ appears as a subscript to A if $\gamma \leq 1$, i.e., A_γ and as a superscript to A if $\gamma > 1$, i.e., A^γ.

2.1.1 Generation for non-multiplying boundary conditions

In this subsection, we show that the free streaming operator A generates a semigroup of contractions, provided the coefficients α, $\beta \leq 1$.

Since $\chi = \max\{\alpha, \beta\} \leq 1$, $\lambda_0 \leq 0$ and from equation (2.20), it follows that

$$\|R_\lambda(A_{\alpha\beta})\, w\| \leq \frac{1}{\lambda}\|w\|, \tag{2.21}$$

for all $w \in X^+$, $\lambda > 0$. Now from inequality (2.21) for all $w \in X$

$$\begin{aligned} \|R_\lambda(A_{\alpha\beta})\, w\| &\leq \|R_\lambda(A_{\alpha\beta})\, w^+\| + \|R_\lambda(A_{\alpha\beta})\, w^-\| \\ &\leq \frac{1}{\lambda}\left(\|w^+\| + \|w^-\|\right) = \frac{1}{\lambda}\|w\|. \end{aligned} \tag{2.22}$$

Next we shall show that $D(A)$ is dense in X. Let $w = \begin{pmatrix} w_0 \\ w_1 \\ \dots \end{pmatrix} \in X$. Then for each j, $w_j \in L_1\left((-a,a) \times (-1,1)\right)$. If we denote by

$$C_0 = \left\{ u = \begin{pmatrix} u_0 \\ u_1 \\ \dots \end{pmatrix}, \ u_j \in C_0^\infty\left((-a,a) \times (-1,1)\right), \ \sum_{j=0}^{\infty} \|u_j\|_1 \text{ is finite} \right\},$$

then for $\epsilon > 0$ fixed and for each $w_j \in L_1\left((-a,a) \times (-1,1)\right)$ there exists $u_j \in C_0^\infty\left((-a,a) \times (-1,1)\right)$ such that $\|w_j - u_j\|_1 < \frac{\epsilon}{2^{j+1}}$. From the triangle inequality for each j, $\|u_j\|_1 \leq \|w_j\|_1 + \frac{\epsilon}{2^{j+1}}$, thus $\sum_{j=0}^{\infty} \|u_j\|_1 \leq \|w\| + \epsilon$ and so $u = \begin{pmatrix} u_0 \\ u_1 \\ \dots \end{pmatrix} \in X$. Thus C_0 is dense in X. As for functions from C_0 they vanish on the boundaries. Thus the boundary conditions are satisfied and so $C_0 \subset D(A)$. Thus $D(A)$ is dense in X. By the Hille-Yosida theorem, it follows that $A \in \mathcal{G}(1, 0, X)$.

2.1.2 Generation for multiplying boundary conditions

In this subsection, we show that if the coefficients α, $\beta > 1$, then A generates a (positive) semigroup.

By the use of equation (2.20), and the assumption α, $\beta > 1$, we get

$$||R_\lambda(A^{\alpha\beta})w|| \geq \frac{1}{\lambda}||w|| \tag{2.23}$$

for all $w^+ \in X^+$, $\lambda > \lambda_0$. By [1, Theorem 2.5], it follows that A generates a (positive) semigroup $\{e^{tA}\}_{t\geq 0}$ such that

$$||e^{tA}w|| \leq Me^{\mu t}||w||, \tag{2.24}$$

for all $w \in X$, $t \geq 0$. The constants M and μ are positive constants and from [1], we obtain that $\mu \leq \dfrac{v}{2a}\ln\chi$, but as far as the positive constant M is concerned one can only state that it exists and if required it has be evaluated by some suitable technique.

2.1.3 Generation for mixed boundary conditions

In this subsection, we show that for the mixed boundary surfaces we still obtain the generation of a semigroup.

Let us assume that $\alpha > 1$, $\beta < 1$ and $w \in X^+$. Then it follows that

$C_j^+ \geq 0$ and $C_j^- \geq 0$, for $j = 0,1,2,...$ and that, if $\left(R_\lambda(A_\beta^\alpha)\right)_j$ represents the j^{th} component of the resolvent, then for $y \in (0,1)$

$$\left(R_\lambda(A_\beta^\alpha)\right)_j w_j = \frac{1}{vy} C_j^+ e^{-\frac{\lambda(a+x)}{vy}} + \frac{1}{vy} \int_{-a}^{x} e^{-\frac{\lambda(x-x')}{vy}} w_j(x',y)dx' \geq 0, \quad (2.25)$$

and for $y \in (-1,0)$

$$\left(R_\lambda(A_\beta^\alpha)\right)_j w_j = \frac{1}{v|y|} C_j^- e^{\frac{\lambda(a-x)}{vy}} + \frac{1}{v|y|} \int_{x}^{a} e^{-\frac{\lambda(x-x')}{vy}} w_j(x',y)dx' \geq 0, \quad (2.26)$$

for $j = 0,1,2,...$ and thus we get

$$R_\lambda(A_\beta^\alpha) \geq 0. \quad (2.27)$$

For simplicity, we shall compare componentwise the resolvent of $R_\lambda(A_\beta^\alpha)$ with that of the resolvent $R_\lambda(A^{\alpha 1})$: for $y \in (0,1)$

$$0 \leq \left(R_\lambda(A_\beta^\alpha)\right)_0 w_0 \leq \left(R_\lambda(A^{\alpha 1})\right)_0 w_0, \quad (2.28)$$

and

$$0 \leq \left(R_\lambda(A_\beta^\alpha)\right)_1 w_1 \leq \left(R_\lambda(A^{\alpha 1})\right)_1 w_1, \quad (2.29)$$

and for the third component from equation (2.25) we obtain

$$\left(R_\lambda(A_\beta^\alpha)\right)_2 w_2 = \frac{1}{vy} C_2^+ e^{-\frac{\lambda(a+x)}{vy}} + \frac{1}{vy} \int_{-a}^{x} e^{-\frac{\lambda(x-x')}{vy}} w_2(x',y)dx' \quad (2.30)$$

and from equation (2.11)

$$\left(R_\lambda(A_\beta^\alpha)\right)_2 w_2$$

$$= \frac{1}{vy}\left(\alpha C_1^- \int_{-1}^{0} e^{\frac{\lambda 2a}{vy'}} dy' + \alpha \int_{-1}^{0} dy' \int_{-a}^{a} e^{\frac{\lambda(a+x')}{vy'}} w_1(x',y')dx'\right) \quad (2.31)$$

$$\times\, e^{-\frac{\lambda(a+x)}{vy}} + \frac{1}{vy}\int_{-a}^{x} e^{-\frac{\lambda(x-x')}{vy}} w_2(x',y)dx'.$$

Now writing $\left(R_\lambda(A_\beta^\alpha)\right)_2 w_2$ in terms of C_0^+, we get

$$\left(R_\lambda(A_\beta^\alpha)\right)_2 w_2$$

$$= \frac{1}{vy}\left(\alpha\left[\beta C_0^+ \int_0^1 e^{-\frac{\lambda 2a}{vy'}} dy' \ + \right.\right.$$

$$\left.\beta \int_0^1 dy' \int_{-a}^{a} e^{-\frac{\lambda(a-x')}{vy'}} w_0(x',y')dx'\right] \times \int_{-1}^{0} e^{\frac{\lambda 2a}{vy'}} dy' \qquad (2.32)$$

$$\left. +\ \alpha \int_{-1}^{0} dy' \int_{-a}^{a} e^{\frac{\lambda(a+x')}{vy'}} w_1(x',y')dx'\right) \times e^{-\frac{\lambda(a+x)}{vy}}$$

$$+ \frac{1}{vy}\int_{-a}^{x} e^{-\frac{\lambda(x-x')}{vy}} w_2(x',y)dx'.$$

Since $C_0^+ = 0$ it follows that $\left(R_\lambda(A^\alpha_\beta)\right)_2 w_2$ can be written as

$$\left(R_\lambda(A^\alpha_\beta)\right)_2 w_2$$

$$= \frac{1}{vy}\left(\left[\alpha\beta\int_0^1 dy' \int_{-a}^{a} e^{-\frac{\lambda(a-x')}{vy'}} w_0(x',y')dx'\right]\right.$$

$$\left.\times \int_{-1}^{0} e^{\frac{\lambda 2a}{vy'}} dy' \ + \ \alpha \int_{-1}^{0} dy' \int_{-a}^{a} e^{\frac{\lambda(a+x')}{vy'}} w_1(x',y')dx'\right) \tag{2.33}$$

$$\times e^{-\frac{\lambda(a+x)}{vy}} + \frac{1}{vy}\int_{-a}^{x} e^{-\frac{\lambda(x-x')}{vy}} w_2(x',y)dx'.$$

Now since $\beta < 1$, it follows that

$$\left(R_\lambda(A^\alpha_\beta)\right)_2 w_2$$

$$\leq \frac{1}{vy}\left(\left[\alpha\int_0^1 dy' \int_{-a}^{a} e^{-\frac{\lambda(a-x')}{vy'}} w_0(x',y')dx'\right] \int_{-1}^{0} e^{\frac{\lambda 2a}{vy'}} dy'\right.$$

$$\left. + \ \ \alpha \int_{-1}^{0} dy' \int_{-a}^{a} e^{\frac{\lambda(a+x')}{vy'}} w_1(x',y')dx'\right) e^{-\frac{\lambda(a+x)}{vy}} \tag{2.34}$$

$$+ \frac{1}{vy}\int_{-a}^{x} e^{-\frac{\lambda(x-x')}{vy}} w_2(x',y)dx' \ = \left(R_\lambda(A^{\alpha 1})\right)_2 w_2$$

Proceeding in this manner we obtain that for $y \in (0.1)$

$$0 \ \leq \ \left(R_\lambda(A^\alpha_\beta)\right)_j \ \leq \ (R_\lambda(A^{\alpha 1}))_j \, , \tag{2.35}$$

for $j = 0,1,2....$ A similar result holds for $y \in (-1, 0)$. Thus we conclude that

$$0 \ \leq \ R_\lambda(A^\alpha_\beta) \ \leq \ R_\lambda(A^{\alpha 1}), \tag{2.36}$$

where $R_\lambda(A^\alpha_\beta)$ and $R_\lambda(A^{\alpha 1})$ are positive operators. From the relation (2.36) and inequality (1.18), we obtain that

$$\begin{aligned} \left\|R_\lambda(A^\alpha_\beta)\, u\right\| \ &\leq \ \|R_\lambda(A^{\alpha 1}) \ |u|\| \\ &\leq \ \|R_\lambda(A^{\alpha 1})\| \ \||u|\| \qquad (2.37) \\ &= \ \|R_\lambda(A^{\alpha 1})\| \ \|u\| \, . \end{aligned}$$

It follows that

$$\left\|R_\lambda(A^\alpha_\beta)\right\| \ \leq \ \|R_\lambda(A^{\alpha 1})\| \, . \tag{2.38}$$

Now, if we assume that the following relation holds for $k \in \mathbb{N}$, that is

$$\left(R_\lambda(A^\alpha_\beta)\right)^k \ \leq \ (R_\lambda(A^{\alpha 1}))^k \, , \tag{2.39}$$

then for $u \in X^+$, we obtain that

$$\begin{aligned}
\left(R_\lambda(A^\alpha_\beta)\right)^{k+1} u &= \left(R_\lambda(A^\alpha_\beta)\right)^k \left(R_\lambda(A^\alpha_\beta)\right) u \\
&\leq \left(R_\lambda(A^{\alpha 1})\right)^k \left(R_\lambda(A^\alpha_\beta)\right) u \qquad (2.40) \\
&\leq \left(R_\lambda(A^{\alpha 1})\right)^k \left(R_\lambda(A^{\alpha 1})\right) u.
\end{aligned}$$

Thus, by induction, we conclude that the relation

$$\left(R_\lambda(A^\alpha_\beta)\right)^n \leq \left(R_\lambda(A^{\alpha 1})\right)^n, \qquad (2.41)$$

holds for all $n \in \mathbb{N}$, and therefore

$$\left\|\left(R_\lambda(A^\alpha_\beta)\right)^n\right\| \leq \left\|\left(R_\lambda(A^{\alpha 1})\right)^n\right\|. \qquad (2.42)$$

It follows from Subsection 2.1.2 and the Hille-Yosida theorem that there exists a positive constant M such that

$$\left\|\left(R_\lambda(A^{\alpha 1})\right)^n\right\| \leq \frac{M}{(\lambda - \lambda_0)^n}, \qquad (2.43)$$

$\lambda > \lambda_0 = \dfrac{v}{2a} = \ln \chi$, where $\chi = \max\{\alpha, 1\} = \alpha$. Thus by (2.42) also $R_\lambda(A^\alpha_\beta)$ satisfies Hille-Yosida estimates and it follows that A generates a (positive) semigroup $\{e^{tA}\}_{t\geq 0}$ such that

$$||e^{tA}u|| \leq Me^{\mu t}||u||, \tag{2.44}$$

for all $u \in X$, $t \geq 0$, where μ and M are positive constants.

2.2 A control of the evolution process

We observe, in the cases of multiplying and mixed boundary conditions, that while the constant μ can be estimated by λ_0, the constant M is completely unknown. As a result of this it is of interest for applications to derive some other inequality involving the semigroup. The following calculations for the case of multiplying boundary conditions can be found in [10]; for the sake of completeness we provide them here for the mixed case.

As in Subsection 2.1.3, we assume that $\alpha > 1$ and $\beta < 1$. For $u_j \in (L_1)^+ \bigcap D(A_j)$ it follows from equations (2.6) and (2.7) that

$$\partial_t u_j(t) = A_j u_j(t). \tag{2.45}$$

Integrating with respect to x and y gives

$$\begin{aligned}\frac{d}{dt}\|u_j(t)\|_1 &= -v\int_{-1}^{+1} dy\left[y\int_{-a}^{+a}\frac{\partial}{\partial x}u_j(x,y)dx\right] \\ &= v\left[\int_{-1}^{0}|y|\,u_j(a,y)dy + \int_{0}^{1} yu_j(-a,y)dy\right] \qquad (2.46) \\ &\quad -v\left[\int_{0}^{1} yu_j(a,y)dy + \int_{-1}^{0}|y|\,u_j(-a,y)dy\right]\end{aligned}$$

and since $u \in D(A)$, we can use the boundary conditions to obtain

$$\begin{aligned}&\frac{d}{dt}\|u_j(t)\|_1 \\ &= v\left[\beta\int_{0}^{1} y'u_{j-1}(a,y')dy' + \alpha\int_{-1}^{0}|y'|\,u_{j-1}(-a,y')dy'\right] \qquad (2.47) \\ &- v\left[\int_{0}^{1} yu_j(a,y)dy + \int_{-1}^{0}|y|\,u_j(-a,y)dy\right].\end{aligned}$$

Since $\chi > 1$,

$$\begin{aligned}&\frac{d}{dt}\|u_j(t)\|_1 \\ &< v\chi\left[\int_{0}^{1} y'u_{j-1}(a,y')dy' + \int_{-1}^{0}|y'|\,u_{j-1}(-a,y')dy'\right] \qquad (2.48) \\ &- v\left[\int_{0}^{1} yu_j(a,y)dy + \int_{-1}^{0}|y|\,u_j(-a,y)dy\right].\end{aligned}$$

Now, multiplying inequality (2.48) by χ^{-j}, we obtain

$$\frac{d}{dt}\chi^{-j}\left\|u_j(t)\right\|_1$$

$$< v\chi^{-j+1}\left[\int_0^1 y'u_{j-1}(a,y')dy' \; + \int_{-1}^0 |y'|\, u_{j-1}(-a,y')dy'\right] \tag{2.49}$$

$$-\, v\chi^{-j}\left[\int_0^1 yu_j(a,y)dy + \int_{-1}^0 |y|\, u_j(-a,y)dy\right].$$

Finally, summing over j and using the assumption $u_{-1} = 0$, we get

$$\frac{d}{dt}\sum\nolimits_{j=0}^{n}\chi^{-j}\left\|u_j(t)\right\|_1$$

$$< v\sum\nolimits_{j=0}^{n}\chi^{-j+1}\int_0^1 y'u_{j-1}(a,y')dy' \; - v\sum\nolimits_{j=0}^{n}\chi^{-j}\int_0^1 yu_j(a,y)dy$$

$$+\, v\sum\nolimits_{j=0}^{n}\chi^{-j+1}\int_{-1}^0 |y'|\, u_{j-1}(-a,y')dy' - v\sum\nolimits_{j=0}^{n}\chi^{-j}\int_{-1}^0 |y|\, u_j(-a,y)dy$$

$$= -v\chi^{-n}\left[\int_0^1 yu_n(a,y)dy + \int_{-1}^0 |y|\, u_n(-a,y)dy\right] \le 0, \tag{2.50}$$

for all $n =$ 0,1,2,.... Thus we have

$$\sum\nolimits_{j=0}^{n}\chi^{-j}\left\|u_j(t)\right\|_1 \le \sum\nolimits_{j=0}^{n}\chi^{-j}\left\|u_{j0}\right\|_1. \tag{2.51}$$

Letting n approach infinity, the above leads to

$$\sum_{j=0}^{\infty} \chi^{-j} \left\| u_j(t) \right\|_1 \leq \sum_{j=0}^{\infty} \chi^{-j} \left\| u_{j0} \right\|_1 . \tag{2.52}$$

Since $\chi > 1$, it follows that

$$\sum_{j=0}^{\infty} \chi^{-j} \left\| u_{j0} \right\|_1 \leq \left\| u^{(0)} \right\| \text{ is finite.} \tag{2.53}$$

It follows that inequality (2.52) can be put in the form

$$\left\| Bu(t) \right\| \leq \left\| Bu^{(0)} \right\|, \tag{2.54}$$

for all $u^{(0)} \in D(A) \bigcap X^{+}$, where

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \chi^{-1} & 0 & \dots \\ 0 & 0 & \chi^{-2} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{2.55}$$

The operator B has the following properties:

i) $D(B) = X$,

ii) $B \in \mathcal{L}(X)$ with $\|B\| = 1$,

iii) B^{-1}exists, $B^{-1} \notin \mathcal{L}(X)$,

iv) B does not commute with A,

v) $B\left(X^{+}\right) \subset X^{+}$.

From Subsection 2.1.3, it follows that A generates a semigroup and thus inequality (2.54) can be written as follows

$$\left\|Be^{tA}u^{(0)}\right\| \leq \left\|Bu^{(0)}\right\|, \text{ for all } t \geq 0. \tag{2.56}$$

We shall now show that inequality (2.56) can be extend to the whole space. Firstly we note that from the proof of the density of C_0^{∞} in L_1, positive functions of L_1 are approximated by positive differentiable functions. Thus the positive cone $(C_0^{\infty})^{+}$ is dense in X^{+}. Hence, $(C_0^{\infty})^{+} \subset D(A) \bigcap X^{+}$ and $D(A) \bigcap X^{+}$ is dense in X^{+}. We conclude that we can extend inequality (2.56) to the whole space X^{+}

$$\left\|Be^{tA}u\right\| \leq \|Bu\|, \text{ for all } u \in X^{+}. \tag{2.57}$$

Now, from inequality (2.57), we have

$$\begin{aligned}\left\|Be^{tA}u\right\| &\leq \left\|Be^{tA}u^{+}\right\| + \left\|Be^{tA}u^{-}\right\| \\ &\leq \|Bu^{+}\| + \|Bu^{-}\| = \|Bu\| \leq \|u\|,\end{aligned} \tag{2.58}$$

for all $u \in X$.

The operator B multiplies $u_j(t)$ by χ^{-j} and thus in some sense annihilates the multiplying effect of the boundary surfaces. In this sense it provides the additional information about the evolution that, in some instances, can be used to replace the missing estimates of the constant M, discussed at the beginning of this section.

It turns out that evolutions obeying the estimate (2.57) can be of independent interest and that is the reason as to why an abstract definition of evolution families satisfying (2.57) has been introduced in [11]. The next chapter is devoted to them.

Chapter 3

A generalization: B-bounded semigroups

3.1 An overview of the development of B -bounded semigroups

The notion of a B-bounded semigroup was originally introduced to provide a tool for estimating the growth rate of solutions to abstract Cauchy problems. However, in further developments, it turned out that B-bounded semigroups can be used to regularize ill-posed problems and in this way can be seen as a complement to B-evolution families. Also, they proved to be an efficient tool for solving some implicit evolution equations and in this context complement the theories of B-evolutions and empathy introduced by N. Sauer in [28-29] and the Showalter approach [30]. This shows how important it is to obtain a full characterization of the generators of a B-bounded semigroup.

The original definition of a B-bounded semigroup has been modified several times and we shall not discuss all the intermediate steps. One of the most used recent versions reads as follows:

Definition 1.1 Let $(Z, \|.\|_Z)$ be a Banach space and $A : D(A) \longrightarrow X$, $D(A) \subset X$ and $B : D(B) \longrightarrow Z$, $D(B) \subset X$ be a pair of linear operators with

i) $D(A) \subset D(B)$,

ii) for some $\omega \in \mathbb{R}$ the resolvent set of A satisfies, $\rho(A) \supset (\omega, \infty)$.

The one-parameter family $\{Y(t)\}_{t\geq 0}$ of operators from the Banach space X into the Banach space Z which satisfies the following conditions:

a) $D(Y(t)) =: \Omega \supseteq D(B)$, and for $t \geq 0$ and any $x \in D(B)$

$$\| Y(t)x \|_Z \quad \leq \quad M \exp(\omega t)\|Bx\|_Z, \tag{3.1}$$

where M is a constant that may depend on the operators A and B,

b) $t \longrightarrow Y(t)x \in C([0, \infty),\ Z)$,

c) For any $x \in D_\Omega(A) = \{x \in D(A) \bigcap D(B);\ Ax \in \Omega\}$ and $t \geq 0$

$$Y(t)x = Bx + \int_0^t Y(s)Ax\, ds, \tag{3.2}$$

is called a B-*quasi bounded semigroup* generated by the pair A and B.

In the case when B is bounded, $D(B) = X = Z$, $M = 1$, $\omega = 0$ one obtains the original definition in [11], while the case $X = Z$ gives Belleni's generalization, [35]. To shorten notation, if A and B generate a B-bounded semigroup satisfying the above conditions, then we write $A \in B - \mathcal{G}(M, \omega, X, Z)$.

A first characterization of the generation theorem, under the assumption that operators A and B satisfy all the conditions of Definition 1.1, was obtained in [5]. The author introduced there the extrapolation space X_B, which represents the completion of the space X with respect to the seminorm $\|B\cdot\|$. In [3], independently the characterization theorem for B-quasi bounded semigroups was proved directly using a constructive procedure which can be seen as a generalization of Yosida's method for constructing a strongly continuous semigroup. It was proved that the pair of operators A and B generates the B-quasi bounded semigroup $\{Y(t)\}_{t\geq 0}$ with $Y(0) = B$, iff by putting for each $y \in B(D_B(A))$,

$$\mathcal{A}(y) = \{z \in \operatorname{Im} B,\ z = BAx \text{ for } x \in D_B(A) \text{ with } Bx = y\}$$

where $D_B(A) := \{u \in D(A) \bigcap D(B);\ Au \in D(B)\}$, we obtain a single-valued mapping $\mathcal{A}$: $B(D_B(A)) \longrightarrow \operatorname{Im} B$; such an operator is linear and closable in $\overline{\operatorname{Im} B}$ and its closure generates a C_0-semigroup.

In this dissertation we shall focus on the development of the theory as presented in [5–8] without going into details of Arlotti's approach that is to some extent parallel.

Next we shall present the definition of the extrapolation space and provide a number of examples in which the extrapolation space can be identified with a subspace of X.

Let us note that most of the theory presented below has been developed for both invertible and not invertible operators B. However, practical applications of the non-invertible results are rather limited at this stage and thus we shall focus on the case when B is invertible (but unbounded).

Definition 1.2 Let us consider the set χ of all sequences $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in D(B)$ for all $n \in \mathbb{N}$ and $(Bx_n)_{n\in\mathbb{N}}$ is convergent in $(Z,\ \|.\|_Z)$. Partition the sequence space χ into disjoint subsets with the aid of the following equivalence relation

$$(x_n)_{n\in\mathbb{N}} \sim (x'_n)_{n\in\mathbb{N}} \quad \text{iff} \quad \lim_{n\to\infty} \|Bx_n - Bx'_n\| = 0. \tag{3.3}$$

Define X_B to be the set of all equivalence classes in χ. The space X_B is a complete normed space; see [21, Theorem 2.3-2], with norm

$$\|[(x_n)_{n\in\mathbb{N}}]\|_{X_B} = \lim_{n\to\infty} \|Bx_n\|_Z. \tag{3.4}$$

If B is an invertible operator, then X_B coincides with the completion of $D(B)$ in the norm $\|B\cdot\|_Z$. By construction $(D(B), \|B\cdot\|_Z)$ is isometric to a dense subspace of X_B, denoted by $\widetilde{D(B)}$.

Definition 1.3 Denote by $\mathfrak{i}$ the isometry from $\widetilde{D(B)}$ into $D(B)$ defined by

$$\mathfrak{i}[(x, x, ...)] := x. \tag{3.5}$$

The operator B is shifted to the space X_B by the formula $\tilde{B} = B \circ \mathfrak{i}$. From equation (3.4) it follows that on $\widetilde{D(B)}$ we get $\|[(x, x, ...)]\|_{X_B} = \|Bx\|_Z$. The operator $\tilde{B} \in \mathcal{L}(\widetilde{D(B)}, Z)$ is an isometry, since we have $\|[(x, x, ...)]\|_{X_B} = \|Bx\|_Z = \left\|\tilde{B}[(x, x, ...)]\right\|$, and has a unique extension by continuity to an operator $\mathfrak{B} \in \mathcal{L}(X_B, Z)$ that is defined through the formula:

$$\mathfrak{B}[(x_n)_{n\in\mathbb{N}}] := \lim_{n\to\infty} Bx_n. \tag{3.6}$$

Furthermore, the operator $\mathfrak{B}$ is an isometric isomorphism of X_B onto $\overline{\operatorname{Im} B}$, [5, Lemma 2.1].

In a similar way we shall shift the operator A to X_B.

Definition 1.4 The *shift* of A, denoted by $\tilde{A}$, is defined as follows:

$$
\begin{aligned}
\tilde{A}[(x,x,...)] &= \mathrm{i}^{-1}A\mathrm{i}[(x,x,..)] \\
&= [(Ax,Ax,...)] \quad \text{for all } [(x,x,...)] \in D_B(\widetilde{A})
\end{aligned}
\tag{3.7}
$$

where $D_B(\widetilde{A}) := \mathrm{i}^{-1}D_B(A)$.

It can be also proved from [5] that if $A \in B - (M,\omega,X,Z)$, then the operator $\tilde{A}$ is closable. Denoting the closure by $\mathfrak{A}$, we also have that the resolvent set $\rho(\mathfrak{A}) \supset (\omega,\infty)$.

The main result of [5, Theorem 4.1] reads as follows.

Theorem 1.5 If $A \in B-\mathcal{G}(M,\omega,X,Z)$ and $B(D_B(A))$ is dense in $\overline{\mathrm{Im}\, B}$, then the operator $\mathfrak{A}$ generates a semigroup on X_B, ie., $\mathfrak{A} \in \mathcal{G}(M,\omega,X_B)$. Conversely, if there is an extension $\mathcal{A}$ of $\tilde{A}$ such that $\mathcal{A} \in \mathcal{G}(M,\omega,X_B)$ then $\mathcal{A} = \mathfrak{A}$ and $A \in B-\mathcal{G}(M,\omega,X,Z)$. Furthermore, the B-bounded semigroup $\{Y(t)\}_{t\geq 0}$ for $x \in D(B)$ is given by

$$Y(t)x = \exp(t\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1})Bx = \mathfrak{B}\exp(t\mathfrak{A})\mathrm{i}^{-1}x. \tag{3.8}$$

The assumption that $B(D_B(A))$ is dense in $\overline{\mathrm{Im}(B)}$ can be discarded if Z (and consequently $\overline{\mathrm{Im}(B)}$) are reflexive spaces; see [5, Corollary 4.1]. Recently Arlotti, [3, Theorem 2.1], proved that if the B-bounded semigroup satisfies the additional condition

$$Y(0)x = Bx \tag{3.9}$$

for all $x \in D(B)$, then $B(D_B(A))$ is dense in $\overline{\mathrm{Im}(B)}$ (or equivalently, $D_B(\widetilde{A})$ is dense in X_B). Therefore if (3.9) holds, then the density assumption in Theorem 1.5 can be omitted.

Since the space X_B is in many cases rather difficult to handle, Theorem 1.5 is most often used in the following version.

Theorem 1.6 Let the operators A and B satisfy the conditions of Definition 1.1. Then A is the generator of a B-quasi bounded semigroup satisfying (3.9) iff the following conditions hold :

i) $B(D_B(A))$ is dense in $\overline{\mathrm{Im}(B)}$,

ii) there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for any $x \in D(B)$, $\lambda > \omega$ and $n \in \mathbb{N}$

$$\|B(\lambda I - A)^{-n}x\|_Z \;\le\; \frac{M}{(\lambda-\omega)^n}\|Bx\|_Z. \tag{3.10}$$

The main point in the proof of the Theorem 1.6, is the observation that (3.10) can be extended to hold on the entire space X_B. This allows a useful corollary.

Corollary 1.7 Let the operators A and B satisfy the conditions of Definition 1.1 and let us assume in addition that conditions i) and ii) of Theorem 1.6 are satisfied.

i) If the estimate (3.10) is satisfied for $n = 1$ with $M = 1$ and $\omega = 0$, then $\mathfrak{A}$ generates a semigroup of contractions in X_B and consequently $A \in B - \mathcal{G}(1, 0, X, Z)$.

ii) If the estimate

$$\|B(\lambda I - A)^{-1}x\|_Z \leq \frac{M}{|\lambda - \omega|}\|Bx\|_Z, \tag{3.11}$$

holds for $\lambda \in S_\theta = \{\lambda \in \mathbb{C};\ |\arg \lambda| \leq \frac{\pi}{2} + \theta,\ \theta > 0\}$, then $\mathfrak{A}$ generates an analytic semigroup in X_B and consequently $A \in B - \mathcal{G}(M', \omega, X, Z)$ for some constant M'.

The extrapolation space

Next we shall discuss some cases when X_B can be identified with a subspace of X.

Example 1.8 Let B be a bounded linear operator from X into a Banach space Z that is bijective. Then X is isometrically isomorphic to X_B.

Example 1.9 Let $(Z, \|.\|_Z)$ be a Banach space and $B : D(B) \longrightarrow Z$, $D(B) \subset X$, be an unbounded linear operator with the domain of B contained in the domain of an injective bounded linear operator $A : D(A) \longrightarrow X$, and the image of B is contained in the domain of an injective bounded linear operator C, $C : D(C) \longrightarrow Z$, $D(C) \subset Z$, such that $CB \in \mathcal{L}(D(B), Z)$ and $AB^{-1} \in \mathcal{L}(\operatorname{Im} B, X)$. Then B is closable and X_B can be identified with a subspace of X, namely $D(\overline{B})$.

Proof The operator B is closable: Let $\lim_{n\to\infty} x_n = 0$, $x_n \in D(B)$ and $\lim_{n\to\infty} Bx_n = y$. Then since CB is bounded, $\lim_{n\to\infty}(CB)x_n = 0$. But, we also have that $\lim_{n\to\infty} C(Bx_n) = Cy$, since C is bounded, and by the uniqueness of limits we conclude that $Cy = 0$. Since C is injective, this implies that $y = 0$.

Furthermore, the closure of B, $\overline{B}$ is injective: If $\overline{B}x = 0$, then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $D(B)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Bx_n = 0$. Now, $\lim_{n\to\infty} B^{-1}(Bx_n) = x$ and since A is bounded, $\lim_{n\to\infty} AB^{-1}(Bx_n) = Ax$. Also, since AB^{-1} is bounded, $\lim_{n\to\infty}(AB^{-1})(Bx_n) = 0$ and it follows that $x = 0$.

Each element of

$$D(\overline{B}) := \left\{x \in X;\ x = \lim_{n\to\infty} x_n,\ x_n \in D(B),\ (Bx_n)_{n\in\mathbb{N}} \text{ is Cauchy in } Z\right\},$$

determines a unique class consisting of sequences $(x_n)_{n\in\mathbb{N}}$ converging to x such that $(Bx_n)_{n\in\mathbb{N}}$ is convergent in Z. Conversely, each class $[(x_n)_{n\in\mathbb{N}}]$ can be identified with $x = \overline{B}^{-1}\left(\lim_{n\to\infty} Bx_n\right)$, where $y = \lim_{n\to\infty} Bx_n$ is independent of the choice of sequence $(x_n)_{n\in\mathbb{N}}$ in the class. $\square$

For the case when the operator B is closed or closable, we can identify X_B with the subspace $D(\overline{B})$, [6, Theorem 2.8].

Theorem 1.10 Let $(Z, \ \|.\|_Z)$ be a Banach space and $B : D(B) \longrightarrow Z$, $D(B) \subset X$, be an injective operator. Then the following statements are equivalent:

1) X_B has the following properties:

(i) each coset $\underline{x} \in X_B$ contains a sequence $(x_n)_{n\in\mathbb{N}}$ converging in norm of X to some $x \in X$, and x is the limit of any other X-Cauchy sequence belonging to $\underline{x}$,

(ii) if $(x_n)_{n\in\mathbb{N}} \in \underline{x} \in X_B$, $(y_n)_{n\in\mathbb{N}}$ satisfy $\|x_n - y_n\|_X \longrightarrow 0$ as $n \rightarrow \infty$, and $(y_n)_{n\in\mathbb{N}} \in \underline{y}$ for some $\underline{y} \in X_B$, then $\underline{x} = \underline{y}$.

2) The operator B is closable and B^{-1} is bounded.

3) There exists an isometric isomorphism $i : X_B \longrightarrow X'_B$, where X'_B is a subspace of X such that $X'_B \hookrightarrow X$ and satisfies the following

$i \mid_{\widetilde{D(B)}} =$ identity. $\square$

The final generation theorem

We shall present the final form of the definition of a B-bounded semigroup and the generation theorem based on this definition. Firstly, however, we shall discuss two examples showing the need for such generalizations.

In [3] and [6], it was proved that the characterization theorem holds even if the assumption ii) of Definition 1.1 is replaced by the weaker assumption:

iii) for $\lambda > \omega$ the operator

$$A_\lambda : D_B(A) \longrightarrow D(B), \tag{3.12}$$

where we recall that

$$D_B(A) = \{u \in D(A) \bigcap D(B);\ Au \in D(B)\},$$

is bijective.

The need for such a generalization is demonstrated in the example below, where we show that there is an evolution family $\{Y(t)\}_{t \geq 0}$ satisfying a)-c) of Definition 1.1 but the associated operators A and B satisfy neither i) nor ii) . However, (3.12) is satisfied.

Example 1.11 Let $v(x) = e^{-x^2}$ and $X = L_2(\mathbb{R}, vdx)$ be the space of all Lebesgue square-integrable functions on the real line with weight v. Then define the operator A,

$$Au := \partial_x u, \tag{3.13}$$

on the maximal domain $D(A) \subset X$, and define the operator $B : D(B) \longrightarrow X$, $D(B) \subset X$, as follows:

$$Bu = v^{-\frac{1}{2}}u. \tag{3.14}$$

The operator $B : D(B) \longrightarrow X$, is an unbounded operator and since

$$||Bu|| = ||u||_{L_2(\mathbb{R})}, \tag{3.15}$$

we obtain that $D(B) = L_2(\mathbb{R})$. Since $B(D(B)) = X$, we obtain that

$$X_B = D(B) = L_2(\mathbb{R}), \tag{3.16}$$

by Theorem 1.10. Then

$$D_B(A) = \{x \in D(A) \bigcap D(B);\ Ax \in D(B)\} = W_2^1(\mathbb{R}). \tag{3.17}$$

Moreover A generates a contraction semigroup $\{e^{tA}\}_{t\geq 0}$ in $L_2(\mathbb{R})$ and

$$(Y(t)u)(x) = Be^{tA}u(x) = e^{\frac{x^2}{2}}u(t+x)$$

is a family satisfying a)-c) of Definition 1.1. □

In the example that follows we shall show that the assumption (3.12) is still too restrictive, that is, there are evolution families satisfying a)-c) of Definition 1.1 but for which the operator $A_\lambda : D_B(A) \longrightarrow D(B)$ is not a bijective operator. This example furthermore indicates that we should be able to replace (3.12) by a weaker condition that would require only the bijectivity of a suitable extension of A.

Example 1.12 Let $v(x) = e^{x^2}$ and $X = L_2(\mathbb{R}, v dx)$ be the space of all Lebesgue square-integrable functions on the real line with weight v. Let $A : D(A) \longrightarrow X$, $D(A) \subset X$, be the differential operator,

$$Au = \partial_x u, \tag{3.18}$$

on its maximal domain and let $B : D(B) \longrightarrow X$, $D(B) \subset X$, be the operator

$$Bu = v^{-\frac{1}{2}} u, \tag{3.19}$$

with $D(B) = X$. For any $u \in X$,

$$\begin{aligned} \|Bu\| &= \left(\int_{-\infty}^{\infty} \left| e^{-\frac{x^2}{2}} u(x) \right|^2 e^{x^2} dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_{-\infty}^{\infty} |u(x)|^2 e^{x^2} dx \right)^{\frac{1}{2}} = \|u\| \end{aligned} \tag{3.20}$$

and so the operator $B \in \mathcal{L}(X)$.

The space $(X, \|\cdot\|)$ is continuously imbedded in $(L_2(\mathbb{R}), \|\cdot\|_{L_2(\mathbb{R})})$. If we denote the inclusion map by $i : X \longrightarrow L_2(\mathbb{R})$, then for any $u \in X$,

$$\|iu\|_{L_2(\mathbb{R})} \leq \left(\int_{-\infty}^{\infty} |u(x)|^2 e^{x^2} dx\right)^{\frac{1}{2}} = \|u\|. \tag{3.21}$$

Furthermore, since B is invertible, $(X, \|B\cdot\|)$ is a normed space and if $i_d : X \longrightarrow X$, denotes the identity map, then for any $u \in X$

$$\begin{aligned} \|i_d u\|_{L_2(\mathbb{R})} &= \left(\int_{-\infty}^{\infty} |u(x)|^2 e^{x^2} e^{-x^2} dx\right)^{\frac{1}{2}} \\ &= \left(\int_{-\infty}^{\infty} \left|e^{\frac{-x^2}{2}} u(x)\right|^2 e^{x^2} dx\right)^{\frac{1}{2}} = \|Bu\|. \end{aligned} \tag{3.22}$$

Thus we have that the normed space $(X, \|B\cdot\|)$ is isometrically isomorphic to the normed space $(X, \|.\|_{L_2(\mathbb{R})})$. Since the space of infinitely differentiable functions with compact support, $C_0^\infty(\mathbb{R})$, is dense in $L_2(\mathbb{R})$ and $C_0^\infty(\mathbb{R}) \subseteq X$, we conclude that $(X, \|B\cdot\|)$ is isometrically isomorphic to a dense subspace of $L_2(\mathbb{R})$ and so we have that $(L_2(\mathbb{R}), \|.\|_{L_2(\mathbb{R})})$ is a completion for $(X, \|B\cdot\|)$. Since we have that $(X_B, \|.\|_{X_B})$ is also a completion, we conclude that we can identify X_B with $L_2(\mathbb{R})$.

Let us now consider the closure $\mathfrak{A}$ of A ie., we take all sequences $(u_n)_{n\in\mathbb{N}}$ of elements of $D(A)$ such that $u_n \longrightarrow u$ and $\partial_x u_n \longrightarrow w$ as $n \longrightarrow \infty$ in $L_2(\mathbb{R})$. However, this is the same as the closure of $D(A)$ in $W_2^1(\mathbb{R})$ and as

$C_0^\infty(\mathbb{R}) \subset D(A)$ is dense in $W_2^1(\mathbb{R})$, we obtain that $\mathfrak{A}u = \partial_x u$ for $u \in W_2^1(\mathbb{R})$. Thus $\mathfrak{A}$ generates a semigroup of contractions in X_B and therefore $(Y(t)u)(x) = e^{-\frac{x^2}{2}}u(t+x)$ satisfies conditions a)-c) of Definition 1.1.

We shall now show that $A_\lambda : D(A) \longrightarrow X$ is not bijective. We begin by finding the formal adjoint of A in X. For any $w \in C_0^\infty(\mathbb{R})$ and using integration by parts, we find a formula for the formal adjoint, $A^\#$

$$\begin{aligned}
(Au, w) &:= \int_{-\infty}^{\infty} Au(x)w(x)e^{x^2}dx \\
&= \int_{-\infty}^{\infty} u'(x)w(x)e^{x^2}dx \\
&= u(x)w(x)e^{x^2} \mid_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\partial_x(w(x)e^{x^2})dx \\
&= -\int_{-\infty}^{\infty} u(x)[2xw(x) + w'(x)]e^{x^2}dx = (u, A^\# w).
\end{aligned} \tag{3.23}$$

Therefore $A^\# w = -[2xw(x) + w'(x)]$ is the formal adjoint of A. Let us consider the equation

$$(\lambda I - A^\#)w(x) = \lambda w(x) + 2xw(x) + w'(x) = 0, \tag{3.24}$$

in X. We find that

$$w(x) = e^{-\lambda x - x^2}, \tag{3.25}$$

is a solution. We shall prove that $w \in D(A^*)$: let $\phi_n \in C_0^\infty(\mathbb{R})$, $n = 1,2,3,...$ be such that

$$\phi_n = \begin{cases} 1 & \text{for } |x| \le n \\ 0 & \text{for } |x| \ge n+1, \end{cases} \tag{3.26}$$

and that $|\phi_n'(x)| \le M$ for $n \le |x| \le n+1$. Then we have that for any $u \in D(A)$

$$\begin{aligned} \int_{-\infty}^{\infty} \partial_x u(x) w(x) \phi_n(x) e^{x^2} dx &= \int_{-\infty}^{\infty} \partial_x u(x) e^{-\lambda x} \phi_n(x) dx \\ &= -\int_{n \le |x| \le n+1} u(x) \partial_x \phi_n(x) e^{-\lambda x} dx \\ &\quad + \lambda \int_{-\infty}^{\infty} u(x) \phi_n(x) e^{-\lambda x} dx. \end{aligned} \tag{3.27}$$

It follows that u, $\partial_x u \in L_1(\mathbb{R})$. Hence passing to the limit with $n \to \infty$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \partial_x u(x) e^{-\lambda x} dx &= \lambda \int_{-\infty}^{\infty} u(x) e^{-\lambda x} dx \\ &= -\int_{-\infty}^{\infty} u(x) \Big(\partial_x e^{-\lambda x - x^2} + \\ &\quad 2x e^{-\lambda x - x^2} \Big) e^{x^2} dx. \end{aligned} \tag{3.28}$$

This shows that $w \in D(A^*)$ and therefore $A_\lambda : D(A) \longrightarrow X$ is not a bijective operator, moreover, it is not a surjection onto any dense subspace of X. $\square$

In the proof of the main generation result, [5, Theorem 4.1], the resolvent set, $\rho(A) \supset (\omega, \infty)$, was used to show that $\rho(\mathfrak{A}) \supset (\omega, \infty)$, where $\mathfrak{A}$ is the closure of $\widetilde{A}$. Thus what we really need is that the Hille-Yosida estimate holds on some dense subspace, $\mathfrak{X}$ of X_B. Moreover, as we are using the pseudoresolvent identity, we need that $\mathfrak{D}_\lambda := (\lambda I - \widetilde{A})^{-1}\mathfrak{X} \subset \mathfrak{X}$ for $\lambda > \omega$. As our starting point is the space X and the operators defined in it, the space $\mathfrak{X}$ must be accessible from X in the sense of the operator closure of X_B. Before we formulate the suitable assumption, we note that the above requirements make our choice limited to a certain extent. We have the following proposition.

Proposition 1.13 Let $\mathfrak{X}$ be a dense subspace of X_B such that $\mathfrak{D}_\lambda := (\lambda I - \widetilde{A})^{-1}\mathfrak{X} \subset \mathfrak{X}$, where $(\lambda I - \widetilde{A}) : D_B(\widetilde{A}) \longrightarrow X_B$ for all $\lambda > \omega$. Then

$$\mathfrak{D}_\lambda = \mathfrak{D} \text{ iff } \mathfrak{D}_\lambda \subset \mathfrak{X}. \tag{3.29}$$

Proof For $\lambda \neq \lambda' > \omega$ and any $\underline{x'} \in \mathfrak{D}_{\lambda'}$ there exists $\underline{x} \in \mathfrak{D}_\lambda$ such that for some $\underline{y} \in \mathfrak{X}$

$$\begin{aligned} \lambda I \underline{x} - \widetilde{A}\underline{x} &= \underline{y}, \\ \lambda' I \underline{x'} - \widetilde{A}\underline{x'} &= \underline{y}. \end{aligned} \tag{3.30}$$

This can be written as

$$\lambda I(\underline{x} - \underline{x'}) - \widetilde{A}(\underline{x} - \underline{x'}) = (\lambda' - \lambda) I \underline{x'}. \tag{3.31}$$

Now assume that $\mathfrak{D}_{\lambda'} \subset \mathfrak{D}_{\lambda}$. Then $\underline{x} - \underline{x'} \in \mathfrak{D}_{\lambda}$, and therefore $\lambda I(\underline{x} - \underline{x'}) - \widetilde{A}(\underline{x} - \underline{x'}) \in \mathfrak{X}$. Thus, $\underline{x'} \in \mathfrak{X}$. Since $\underline{x'}$ is arbitrary, $\mathfrak{D}_{\lambda'} \subset \mathfrak{X}$. Clearly the converse is also true. Since the argument is symmetric with respect to primed and un-primed objects we conclude the proof. $\square$

These considerations lead to the following new assumption on A.

iv) The shifted operator $\tilde{A}$ is closable in X_B ie., if the sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $D_B(A)$ is such that $Bx_n \longrightarrow 0$ and $BAx_n \longrightarrow y$ in Z as $n \longrightarrow \infty$, then $y = 0$. Denoting $\mathfrak{A} = \overline{\tilde{A}}$, we assume further that there exist subspaces $\mathfrak{X}$, satisfying $\widetilde{D(B)} \subseteq \mathfrak{X} \subseteq X_B$, and $D_B(\widetilde{A}) \subseteq \mathfrak{D} \subseteq \mathfrak{X} \bigcap D(\mathfrak{A})$ such that $(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}}) : \mathfrak{D} \longrightarrow \mathfrak{X}$ is bijective for all $\lambda > \omega$.

As a result of the assumption iv), the main result of [6, Theorem 2.4] now reads as follows:

Theorem 1.14 (Theorem 3.1 of [8]) Let $A : D(A) \longrightarrow X$, $D(A) \subset X$, be a linear operator in a Banach space X and $B : D(B) \longrightarrow Z$, $D(B) \subset X$, be a linear operator from X to a Banach space Z satisfying the conditions of Definition 1.1 with ii) replaced by assumption iv). Then $A \in B - \mathcal{G}(M, \omega, X, Z)$ and satisfies equation (3.9) iff

i) $B(\mathfrak{D})$ is dense in $\overline{\operatorname{Im} B}$,

ii) there exists a constant $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|\mathfrak{B}(\lambda I - \mathfrak{A}\,|_{\mathfrak{D}})^{-n}\eta\|_Z \;\leq\; \frac{M}{(\lambda-\omega)^n}\|\mathfrak{B}\eta\|_Z, \tag{3.32}$$

for any $\eta \in \mathfrak{X}$, $\lambda > \omega$ and $n \in \mathbb{N}$.

Proof We begin by employing the operators $J_n(\lambda) : \Omega \longrightarrow Z$, defined as follows

$$J_n(\lambda)x := \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\exp(-\lambda t)Y(t)x\,dt, \tag{3.33}$$

where Ω is the common domain of the family of operators $\{Y(t)\}_{t\geq 0}$. The integral exists since the function $t \longrightarrow Y(t)x$ is continuous. In particular, for $x \in D(B)$, we have from the definition, that

$$\begin{aligned} \|J_n(\lambda)x\|_Z &\leq \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\exp(-\lambda t)\,\|Y(t)x\|_Z\,dt \\ &\leq \frac{M}{(\lambda-\omega)^n}\,\|Bx\|\,. \end{aligned} \tag{3.34}$$

By (3.1) and (3.34) we can extend by continuity the operators $J_n(\lambda)$, $n = 1,2,3,...,\ \lambda > \omega$ and $Y(t)$, $t \geq 0$, to bounded linear operators $\mathfrak{J}_n(\lambda) : X_B \longrightarrow Z$ and $\mathfrak{Y}(t) : X_B \longrightarrow Z$, respectively. Let $\underline{x_n} \longrightarrow \mathfrak{x}$ in X_B for $\underline{x_n} \in \widetilde{D(B)}$. Then $\left\|Y(t) \circ \mathrm{i}\ \underline{x_n} - \mathfrak{Y}(t)\mathfrak{x}\right\|_Z \leq Me^{\omega t} \left\|\mathfrak{B}(\underline{x_n} - \mathfrak{x})\right\|$ and thus $t \longrightarrow \mathfrak{Y}(t)\mathfrak{x}$ is continuous for any $\mathfrak{x} \in X_B$. Moreover, by (3.34) we can pass to the limit in (3.33) to get

$$\mathfrak{J}_n(\lambda)\mathfrak{x} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} \exp(-\lambda t)\mathfrak{Y}(t)\mathfrak{x}\, dt \tag{3.35}$$

for any $\mathfrak{x} \in X_B$ and $\lambda > \omega$. Now, for all $\mathfrak{x} \in D(\mathfrak{A})$

$$\begin{aligned}
\mathfrak{J}_1(\lambda)(\lambda I - \mathfrak{A})\mathfrak{x} &= \lambda \int_0^\infty \exp(-\lambda t)\mathfrak{Y}(t)\mathfrak{x}\, dt - \int_0^\infty \exp(-\lambda t)\mathfrak{Y}(t)\mathfrak{A}\mathfrak{x}\, dt \\
&= \lambda \int_0^\infty \exp(-\lambda t) \left[\mathfrak{B}\mathfrak{x} + \int_0^t \mathfrak{Y}(s)\mathfrak{A}\mathfrak{x}\, ds\right] dt \\
&\quad - \int_0^\infty \exp(-\lambda t)\mathfrak{Y}(t)\mathfrak{A}\mathfrak{x}\, dt \\
&= \mathfrak{B}\mathfrak{x} + \lambda \int_0^\infty \int_s^\infty \exp(-\lambda t)dt\ \mathfrak{Y}(s)\mathfrak{A}\mathfrak{x}\, ds \\
&\quad - \int_0^\infty \exp(-\lambda t)\mathfrak{Y}(t)\mathfrak{A}\mathfrak{x}\, dt,
\end{aligned} \tag{3.36}$$

where the first integral is evaluated by the use of an extension of (3.2) and we finally obtain the following equation

$$\mathfrak{J}_1(\lambda)(\lambda I - \mathfrak{A})\mathfrak{x} = \mathfrak{B}\mathfrak{x}. \tag{3.37}$$

Using the assumption iv), we obtain that if $(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})\mathfrak{x} = \eta$ then

$$\mathfrak{J}_1(\lambda)\eta = \mathfrak{B}(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\eta \tag{3.38}$$

for all $\eta \in \mathfrak{X}$ and by estimate (3.34) we obtain that for those η, we have

$$\|(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\eta\|_Z \;\; \leq \;\; \frac{M}{(\lambda - \omega)}\|\eta\|_{X_B}. \tag{3.39}$$

Next, using (3.35) and the extension of (3.2)

$$\mathfrak{Y}(t)\mathfrak{x} = \mathfrak{B}\mathfrak{x} + \int_0^t \mathfrak{Y}(s)\mathfrak{A}\mathfrak{x}\,ds, \quad t \geq 0, \tag{3.40}$$

which is valid for all $\mathfrak{x} \in D(\mathfrak{A})$, we obtain

$$\mathfrak{J}_2(\lambda)\mathfrak{x} = \int_0^\infty t\exp(-\lambda t)\left[\mathfrak{B}\mathfrak{x} + \int_0^t \mathfrak{Y}(s)\mathfrak{A}x\,ds\right]\,dt \tag{3.41}$$

for all $\mathfrak{x} \in D(\mathfrak{A})$ and $\lambda > \omega$.

Hence,

$$\begin{aligned}
\mathfrak{J}_2(\lambda)\mathfrak{x} &= \frac{1}{\lambda^2}\mathfrak{B}\mathfrak{x} + \int_0^\infty \left[\int_s^\infty t\exp(-\lambda t)dt\right] \mathfrak{Y}(s)\mathfrak{A}\mathfrak{x}\, ds \\
&= \frac{1}{\lambda^2}\mathfrak{B}\mathfrak{x} + \int_0^\infty \left[\frac{1}{\lambda}s\exp(-\lambda s) \;+\right. \\
&\qquad \left.\frac{1}{\lambda^2}\exp(-\lambda s)\right] \mathfrak{Y}(s)\mathfrak{A}\mathfrak{x}\, ds,
\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}
\mathfrak{J}_2(\lambda)(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})\mathfrak{x} &= \frac{1}{\lambda}\left[\mathfrak{B}\mathfrak{x} + \int_0^\infty \exp(-\lambda s)\mathfrak{Y}(s)\mathfrak{A}\mathfrak{x}\, ds\right] \\
&= \frac{1}{\lambda}\left[\mathfrak{B}\mathfrak{x} + \mathfrak{B}(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\mathfrak{A}\mathfrak{x}\right],
\end{aligned} \tag{3.43}$$

where we have used equations (3.38) and (3.35). Thus, we have

$$\begin{aligned}
\mathfrak{J}_2(\lambda)(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})\mathfrak{x} &= \frac{1}{\lambda}\mathfrak{B}\left[I + (\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\mathfrak{A}\right]\mathfrak{x} \\
&= \frac{1}{\lambda}\mathfrak{B}\left[(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}}) + (\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\mathfrak{A}\right]\mathfrak{x} \\
&= \mathfrak{B}(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\mathfrak{x}
\end{aligned} \tag{3.44}$$

for $\mathfrak{x} \in D(\mathfrak{A})$. From the equation $(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})\mathfrak{x} = \eta$, we get

$$\mathfrak{J}_2(\lambda)\eta = \mathfrak{B}(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-2}\eta \tag{3.45}$$

for all $\eta \in \mathfrak{X}$.

By iterating the above procedure, we have

$$\mathfrak{J}_n(\lambda)\eta = \mathfrak{B}(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-n}\eta \tag{3.46}$$

for $\eta \in \mathfrak{X}$, $\lambda > \omega$. Finally from equations (3.46), (3.34) and inequality (3.35), we obtain

$$\begin{aligned} \|\mathfrak{B}(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-n}\eta\|_Z &\leq \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\exp(-\lambda t)\|\mathfrak{Y}(t)\eta\|_Z \, dt \\ &\leq \frac{M}{(\lambda-\omega)^n}\|\eta\|_{X_B}, \end{aligned} \tag{3.47}$$

which is part ii). To prove property i) we extend the argument of [11]. Let $\mathfrak{x} \in \mathfrak{X}$ and define $\eta_\lambda = \lambda(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\mathfrak{x} \in \mathfrak{D}$. We have by (3.38)

$$\mathfrak{B}\mathfrak{x} - \mathfrak{B}\eta_\lambda = \mathfrak{B}\mathfrak{x} - \lambda\mathfrak{B}(\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\mathfrak{x} = \int_0^\infty \lambda\exp(-\lambda t)(\mathfrak{B} - \mathfrak{Y}(t))\mathfrak{x}dt. \tag{3.48}$$

The property (3.9) gives $Y(0)x = Bx$ for all $x \in D(B)$. Since both operators can be extended by continuity to X_B and $\widetilde{D(B)}$ is dense in X_B, we have that $\mathfrak{Y}(0)\mathfrak{x} = \mathfrak{B}\mathfrak{x}$ for all $\mathfrak{x} \in \mathfrak{X}$. The function $t \longrightarrow \mathfrak{Y}(t)\mathfrak{x}$ is continuous, hence for any $\varepsilon > 0$ we can find a $\delta > 0$ such that $\sup_{0 \leq t \leq \delta} \|(\mathfrak{B} - \mathfrak{Y}(t))\mathfrak{x}\|_Z \leq \varepsilon$. Thus

$$\begin{aligned}
\|\mathfrak{B}\mathfrak{x} - \mathfrak{B}\eta_\lambda\|_Z &\leq \int_0^\infty \lambda \exp(-\lambda t) \|(\mathfrak{B} - \mathfrak{Y}(t))\mathfrak{x}\|_Z \, dt \\
&\leq \varepsilon + (1+M) \int_\delta^\infty \lambda \exp\left(-(\lambda - \omega)\right) \|\mathfrak{x}\|_{X_B} \, dt \qquad (3.49) \\
&\leq 2\varepsilon,
\end{aligned}$$

provided λ is sufficiently large. Hence, $\mathfrak{B}(\mathfrak{D})$ is dense in $\mathfrak{B}(\mathfrak{X})$ in the Z topology or $\mathfrak{D}$ is dense in $\mathfrak{X}$ in the X_B topology. However, as $\widetilde{D(B)} \subset \mathfrak{X}$ is dense in X_B, $\mathfrak{X}$ is dense in X_B and therefore $\overline{\mathfrak{D}} = X_B$. This proves the necessity of conditions i) and ii).

To prove the sufficiency we note first that the resolvent equation is of purely algebraic character and therefore for λ, $\mu \in (\omega, \infty)$ and $\mathfrak{x} \in \mathfrak{X}$, we have

$$(\lambda I - \mathfrak{A}|_{\mathfrak{D}})^{-1}\mathfrak{x} - (\mu I - \mathfrak{A}|_{\mathfrak{D}})^{-1}\mathfrak{x} = (\mu - \lambda)(\lambda I - \mathfrak{A}|_{\mathfrak{D}})^{-1}(\lambda I - \mathfrak{A}|_{\mathfrak{D}})^{-1}\mathfrak{x} \quad (3.50)$$

where we used the assumption that $\mathfrak{D} \subset \mathfrak{X}$.

Since $B(\mathfrak{D})$ is dense in $\overline{\mathrm{Im}\, B}$, we see that $\mathfrak{D}$, and consequently $\mathfrak{X}$, are dense in X_B. From the assumption (3.32) it follows that for each $\lambda > \omega$ the operator $(\lambda I - \mathfrak{A}|_{\mathfrak{D}})^{-1}$ can be extended by continuity to a bounded operator

$\mathfrak{R}_\lambda : X_B \longrightarrow X_B$ which satisfies

$$||\mathfrak{R}_\lambda \eta||_{X_B} \leq \frac{M}{(\lambda - \omega)} ||\eta||_{X_B} \tag{3.51}$$

for any $\eta \in X_B$. Thus, equation (3.50) can be extended onto the whole of X_B preserving its structure, and hence the family of operators $\mathfrak{R}_\lambda$ is a pseudoresolvent. The range of each $\mathfrak{R}_\lambda$ contains $\mathfrak{D}$, and therefore is dense in X_B. As a result of (3.51) and the use of Theorem 9.4 of [25], we can conclude that $\mathfrak{R}_\lambda$ is the resolvent of a unique densely defined closed operator in X_B. Denote this operator by $\mathcal{A}$. Since $\left((\lambda I - \mathcal{A})^{-1}\right)^{-1} \mathfrak{x} = \left((\lambda I - \mathfrak{A}\mid_{\mathfrak{D}})^{-1}\right)^{-1} \mathfrak{x}$ for $\mathfrak{x} \in \mathfrak{D}$ and

$$\mathcal{A} = \lambda I - \left((\lambda I - \mathcal{A})^{-1}\right)^{-1} \tag{3.52}$$

we obtain that $\mathcal{A}$ is an extension of $\mathfrak{A} \mid_{\mathfrak{D}} =: \mathfrak{A}_{\mathfrak{D}}$, and since $\mathfrak{A}_{\mathfrak{D}}$ is closable we get $\overline{\mathfrak{A}_{\mathfrak{D}}} \subset \mathcal{A}$. Let now $\mathfrak{x} \in D(\mathcal{A})$. Then $\mathfrak{x} = (\lambda I - \mathcal{A})^{-1}\eta$ for some $\eta \in X_B$. This means that

$$\mathfrak{x} = \lim_{n \to \infty} (\lambda I - \mathfrak{A}_{\mathfrak{D}})^{-1} \eta_n \tag{3.53}$$

for $\eta_n \in \mathfrak{X}$ and $\eta_n \longrightarrow \eta$. In other words, $\mathfrak{x}_n = (\lambda I - \mathfrak{A}_{\mathfrak{D}})^{-1}\eta_n \in \mathfrak{D}$ converges to $\mathfrak{x}$. Solving this equation we get $\mathfrak{A}_{\mathfrak{D}}\mathfrak{x}_n = \lambda \mathfrak{x}_n - \eta_n$ and $(\mathfrak{A}_{\mathfrak{D}}\mathfrak{x}_n)_{n \in \mathbb{N}}$ converges, hence $\mathfrak{x} \in D(\overline{\mathfrak{A}_{\mathfrak{D}}})$ and $\overline{\mathfrak{A}_{\mathfrak{D}}}\mathfrak{x} = \lambda \mathfrak{x} - \eta = \lambda \mathfrak{x} - \left((\lambda I - \mathcal{A})^{-1}\right)^{-1} \mathfrak{x} = \mathcal{A}\mathfrak{x}$.

Therefore inequality (3.51) can be written as

$$\left|\left|(\lambda I - \overline{\mathfrak{A}_{\mathfrak{D}}})^{-1}\mathfrak{x}\right|\right|_{X_B} \leq \frac{M}{(\lambda - \omega)} ||\mathfrak{x}||_{X_B}, \tag{3.54}$$

valid for any $\mathfrak{x} \in X_B$ and $\lambda > \omega$. Writing

$$(\lambda I - \mathfrak{A}_{\mathfrak{D}})^{-n} = (\lambda I - \mathfrak{A}_{\mathfrak{D}})^{-1}(\lambda I - \mathfrak{A}_{\mathfrak{D}})^{-n+1} \tag{3.55}$$

and using induction in $n \in \mathbb{N}$, we find that

$$\left\|(\lambda I - \overline{\mathfrak{A}_{\mathfrak{D}}})^{-n}\mathfrak{x}\right\|_{X_B} \leq \frac{M}{(\lambda - \omega)^n}\|\mathfrak{x}\|_{X_B} \tag{3.56}$$

for all $n \in \mathbb{N}$ and $\mathfrak{x} \in X_B$ which shows that $\overline{\mathfrak{A}_{\mathfrak{D}}}$ generates a semigroup in X_B. Due to the assumption $D_B(\widetilde{A}) \subset \mathfrak{D} \subset D(\overline{\mathfrak{A}_{\mathfrak{D}}})$, the family $\{Y(t)\}_{t\geq 0} = \{\mathfrak{B}e^{t\overline{\mathfrak{A}_{\mathfrak{D}}}}\}_{t\geq 0}$ satisfies the condition of the Definition 1.1, which can be proved as in the proof of Theorem 1.5. □

The assumption that $D_B(\widetilde{A}) \subset \mathfrak{D}$ may seem too restrictive as what we need and use is that $D_B(\widetilde{A}) \subset D(\overline{\mathfrak{A}_{\mathfrak{D}}})$ (otherwise the condition iii) of Definition 1.1 would be satisfied on a smaller set that required). However, the proposition below shows that this is precisely what we need. Let us consider the relations between the operators appearing in this theorem. We have the original operator A, its shift $\widetilde{A}$, its closure $\mathfrak{A}$ of $\widetilde{A}$, the restriction of $\mathfrak{A}$ to $\mathfrak{D}$ and the generator $\mathcal{A} = \overline{\mathfrak{A}_{\mathfrak{D}}}$. We can prove the following proposition.

Proposition 1.15 (Proposition 3.2 of [8]) The following are equivalent

i) $\mathcal{A} = \mathfrak{A}$,

ii) $\widetilde{A} \subset \overline{\mathfrak{A}_{\mathfrak{D}}}$,

iii) for some $\lambda > \omega$ the operator $\lambda I - \mathfrak{A}$ is injective,

iv) $\widetilde{A} \subset \mathfrak{A}_{\mathfrak{D}}$.

Proof i)$\Leftrightarrow$ii) Let $\mathcal{A} = \mathfrak{A}$. Then $\mathcal{A} = \overline{\mathfrak{A}_{\mathfrak{D}}} = \mathfrak{A}$ yields $\widetilde{A} \subset \overline{\mathfrak{A}_{\mathfrak{D}}}$. Conversely, we have $\mathcal{A} \subset \overline{\mathfrak{A}_{\mathfrak{D}}} \subset \overline{\mathfrak{A}} = \mathfrak{A}$ and $\mathfrak{A} = \overline{\widetilde{A}} \subset \overline{\overline{\mathfrak{A}_{\mathfrak{D}}}} = \mathcal{A}$. Hence $\mathcal{A} = \mathfrak{A}$.

i)$\Leftrightarrow$iii) To prove i)$\Leftrightarrow$iii) we assume that $\mathfrak{A}_\lambda$ is one-to-one and let $\mathfrak{A} \supset \mathcal{A}$; then also $\mathfrak{A}_\lambda \supset \mathcal{A}_\lambda$. However $\mathcal{A}_\lambda$ acts onto X_B and hence for any $\underline{x}' \in D(\mathfrak{A})$ there is $\underline{x} \in D(\mathcal{A})$ such that $\mathfrak{A}_\lambda \underline{x}' = \mathcal{A}_\lambda\, \underline{x}$. By i) $\mathfrak{A}_\lambda \underline{x} = \mathcal{A}_\lambda\, \underline{x}$, and therefore by the injectivity of $\mathfrak{A}$ we obtain that $\underline{x} = \underline{x}'$. The converse is obvious.

i)$\Leftrightarrow$iv) For i)$\Leftrightarrow$iv) we see that if $\widetilde{A} \subset \mathfrak{A}_{\mathfrak{D}}$ then $\mathfrak{A} = \overline{\widetilde{A}} \subset \overline{\mathfrak{A}_{\mathfrak{D}}} = \mathcal{A}$. Conversely, if $\mathcal{A} = \mathfrak{A}$ then by iii) $\mathfrak{A}_\lambda$ is a one-to-one operator for some λ and therefore $\widetilde{A}_\lambda$ and $(\mathfrak{A}_{\mathfrak{D}})_\lambda$ are one-to-one. Let $\underline{x} \in D(\widetilde{A}) \setminus \mathfrak{D}$. Then $\widetilde{A}_\lambda \underline{x} = \underline{y} \in \widetilde{D(B)}$ and since $\widetilde{D(B)} \subset \mathfrak{X}$ there is $\underline{x} \neq \underline{x}' \in \mathfrak{D}$ such that $(\mathfrak{A}_{\mathfrak{D}})_\lambda\, \underline{x}' = \underline{y}$. However, since both these operators are restrictions of the injective operator $\mathfrak{A}_\lambda$, this is impossible. Thus $D(\widetilde{A}) \subset \mathfrak{D}$. $\square$

This proposition shows that the adopted assumption $D_B(\widetilde{A}) \subset \mathfrak{D}$ is necessary and sufficient for the semigroup $\{e^{tA}\}_{t\geq 0}$ to define a B-bounded semigroup. Another important consequence of this proposition is that the B-bounded semigroup

$\{Y(t)\}_{t\geq 0}$ is uniquely determined by A and B, being defined by the semigroup generated by the B-closure of A restricted to $D_B(A)$.

3.2 A heat transfer problem

In this section we show that B-bounded semigroups can be used to solve certain types of implicit evolution equations. The heat transfer problem that follows was considered by N. Sauer; see [28, Section 8], as an example of application of B-evolution theory.

Consider the following problem: solve

$$\partial_t u(x,t) = \partial_x^2 u(x,t) \quad 0 < x < 1,\ t > 0 \tag{3.57}$$

with the following initial and dynamic boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ \partial_t u(1,t) &= -\partial_x u(1,t), \text{ for } t > 0 \\ u(x,0) &= a(x), \text{ for } 0 < x < 1 \\ u(1,0) &= \alpha. \end{aligned} \tag{3.58}$$

A possible way of finding a solution to the problem is to express it as an implicit evolution equation:

$$\frac{d}{dt}(Ku) = Lu, \quad \lim_{t \to 0^+} (Ku)(t) = u^\circ \tag{3.59}$$

where $K : Z \longrightarrow X$, $L : Z \longrightarrow X$, and Z, X are Banach spaces and K, L are linear operators.

The system of equations (3.57), (3.58) may be written in the form (3.59) in the following way: Let $X = L_2(0,1) \times \mathbb{C}$ with the inner product defined as $(\langle f, \alpha \rangle, \langle g, \beta \rangle) = (f, g) + \alpha\overline{\beta}$, where $(\cdot, \cdot)$ denotes the usual inner product in L_2. The norm of $\langle f, \alpha \rangle \in X$ is defined as $\|\langle f, \alpha \rangle\| = \left(\|f\|^2 + |\alpha|^2\right)^{1/2}$. Let $Z = L_2(0,1)$. We introduce the trace operators

$$\gamma_0 u = u(1); \;\; \gamma_1 u = \partial_x u(1), \tag{3.60}$$

and define the linear operators L and K as follows:

$$Lv = \left\langle \partial_x^2 v, -\gamma_1 v \right\rangle \tag{3.61}$$

on $D(L) = \{v;\, v \in W_2^2(0,1), v(0) = 0\}$ and

$$Kv = \langle v, \gamma_0 v \rangle \tag{3.62}$$

on $D(K) = \{v;\, v \in W_2^1(0,1), v(0) = 0\}$, so that problem (3.57), (3.58) takes the required form.

A "natural" way of approaching (3.59) would be to factor out K, and provided it is invertible, to consider a standard Cauchy problem with the operator $K^{-1}L$ on the right-hand side. In some cases, however, the operator K is not closable and therefore there is no way the "time derivative" and the limit at $t = 0^+$ can commute with K. In fact, this is the case for the operator K defined in (3.62), even though it has a bounded inverse and $\operatorname{Im} K$ is dense in Z; see [28, Proposition 8.1]. To circumvent this difficulty, in [28] a special evolution family, called a B-evolution, is introduced.

Definition 2.1 Let X and Z be Banach spaces and let B be a linear operator with $D(B) \subset X$ and $\operatorname{Im} B \subset Z$. A family $\{S(t)\}_{t>0}$ of bounded linear operators defined on Z is called a B-*evolution* if

$$S(t)Z \subset D(B) \text{ for all } t > 0 \tag{3.63}$$

and

$$S(t+s) = S(s)BS(t) \text{ for all } s,\, t > 0. \tag{3.64}$$

Associated with any B-evolution is a semigroup $\{T(t)\}_{t>0}$ of linear operators in Z defined by

$$T(t) = BS(t) \text{ for all } t > 0. \tag{3.65}$$

Using the B-evolution theory, we see from Theorem 5.1, Proposition 7.3 and

Proposition 8.2 of [28] that to obtain solvability of the problem it is sufficient to prove that LK^{-1} generates a holomorphic semigroup in X. To do this we consider for $\lambda \in \mathbb{C}$ the equation

$$(\lambda K - L) K^{-1} \langle u, \gamma_0 u \rangle = \langle f, \alpha \rangle, \quad u \in D(L), \tag{3.66}$$

which is equivalent to the system of equations

$$\begin{aligned} \lambda u - u_{xx} &= f, \\ \lambda u(1) + u_x(1) &= \alpha, \\ u(0) &= 0, \end{aligned} \tag{3.67}$$

for arbitrary $\langle f, \alpha \rangle \in X$. Taking the inner product of the left-hand side of the first two equations with $\langle u, u(1) \rangle$ we obtain

$$\begin{aligned} &\lambda \|u\|^2 - \int_0^1 u_{xx} \bar{u} dx + \lambda |u(1)|^2 + u_x(1)\overline{u(1)} \\ &= \lambda \left(\|u\|^2 + |u(1)|^2 \right) + \|u_x\|^2 . \end{aligned} \tag{3.68}$$

To shorten notation, denote the left hand side by Q. Then we obtain

$$
\begin{aligned}
|Q|^2 &= \left|\lambda\left(\|u\|^2 + |u(1)|^2\right) + \|u_x\|^2\right|^2 \\
&= (\operatorname{Re}\lambda)^2\left(\|u\|^2 + |u(1)|^2\right)^2 + (\operatorname{Im}\lambda)^2\left(\|u\|^2 + |u(1)|^2\right)^2 + \\
&\quad 2\operatorname{Re}\lambda\left(\|u\|^2 + |u(1)|^2\right)\|u_x\|^2 + \|u_x\|^4 \qquad (3.69) \\
&= |\lambda|^2\left(\left(\|u\|^2 + |u(1)|^2\right)^2 + \|u_x\|^4 |\lambda|^{-2} + \right. \\
&\quad \left. 2|\lambda|^{-1}\cos\theta\left(\|u\|^2 + |u(1)|^2\right)\|u_x\|^2\right),
\end{aligned}
$$

where $\theta = \arg\lambda$.

Since for $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, we have that $\cos\theta \geq 0$ and from equation (3.69) we obtain

$$
|Q|^2 \geq |\lambda|^2\left(\|u\|^2 + |u(1)|^2\right)^2. \qquad (3.70)
$$

Using

$$
\begin{aligned}
2|\lambda|^{-1}\cos\theta\left(\|u\|^2 + |u(1)|^2\right)\|u_x\|^2 \;&\geq\; -\|u_x\|^4|\lambda|^{-2} - \qquad (3.71) \\
&\cos^2\theta\left(\|u\|^2 + |u(1)|^2\right)^2,
\end{aligned}
$$

we obtain from equation (3.69) that

$$|Q|^2 \geq |\lambda|^2 \left(\|u\|^2 + |u(1)|^2\right)^2 \sin^2\theta. \tag{3.72}$$

For $\frac{\pi}{2} \leq |\theta| \leq \frac{3\pi}{4}$, inequality (3.72) yields

$$|Q|^2 \geq \frac{1}{2} |\lambda|^2 \left(\|u\|^2 + |u(1)|^2\right)^2. \tag{3.73}$$

From inequalities (3.70) and (3.73) it follows that in the sector $-\frac{3\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

$$\begin{aligned} \|\langle u, \gamma_0 u\rangle\|^2 &\leq \frac{\sqrt{2}}{|\lambda|} \left|\left(\langle u, \gamma_0 u\rangle, \langle f, \alpha\rangle\right)\right| \\ &\leq \frac{\sqrt{2}}{|\lambda|} \|\langle u, \gamma_0 u\rangle\| \, \|\langle f, \alpha\rangle\|. \end{aligned} \tag{3.74}$$

Taking into account that by (3.66) we have $(\lambda K - L)\, K^{-1} \langle u, \gamma_0 u\rangle = \langle f, \alpha\rangle$, we obtain that in the sector $-\frac{3\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ the estimate

$$\left\|(\lambda I - LK^{-1})^{-1} \langle f, \alpha\rangle\right\| \leq \frac{\sqrt{2}}{|\lambda|} \|\langle f, \alpha\rangle\| \tag{3.75}$$

is valid, which shows that LK^{-1}satisfies the Hille-Yosida estimate, and since X is a Hilbert (reflexive) space, this is sufficient for the generation of a holomorphic semigroup. □

The theory of B-bounded semigroups provides another convenient way of performing this "impossible" commutation by passing to a specifically constructed space related to K.

Definition 2.2 Let $X \hookrightarrow X_B$ and $\tilde{L}$ be an extension of operator L and $\tilde{K}$ be an extension of the operator K in the space X_B. A map $t \longrightarrow u(t)$ is called a X_B-*solution* of problem (3.59), if it is a classical solution of the problem

$$\frac{d}{dt}(\tilde{K}u) = \tilde{L}u, \quad \lim_{t \to 0^+}(\tilde{K}u)(t) = u^\circ. \tag{3.76}$$

Theorem 2.3 (Theorem 3.4 of [6]) Assume that

i) $D(K^{-1}L) = \{y \in D(L) \bigcap D(K), Ly \in \operatorname{Im} K\}$ is dense in $\overline{D(K)}$,

ii) The operator $(K^{-1}L, D(K^{-1}L))$ is closable in Z and there exists spaces: $\mathfrak{X}$ satisfying $D(K) \subset \mathfrak{X} \subset Z$, and $\mathfrak{D}$ satisfying $D(K^{-1}L) \subset \mathfrak{D} \subset D(\overline{K^{-1}L})$ such that for $x \in \mathfrak{X}$ either

$$\left\| \left((\lambda I - \overline{K^{-1}L} \mid_{\mathfrak{D}}\right)^{-1} x \right\|_Z \leq \frac{1}{\lambda} \; \|x\|_Z \text{ , for } \lambda > 0 \tag{3.77}$$

or

$$\left\| \left(\lambda I - \overline{K^{-1}L} \mid_{\mathfrak{D}}\right)^{-1} x \right\|_Z \leq \frac{M}{|\lambda - \omega|} \; \|x\|_Z \text{ ,} \tag{3.78}$$

for $\lambda \in S_\theta = \{\lambda \in \mathbb{C};\ |\arg \lambda| \leq \dfrac{\pi}{2} + \theta,\ \theta > 0\}$. Then for any $x \in D(LK^{-1})$ the function $t \longrightarrow Y(t)x$ is an X_B -solution of (3.59). For $x \in D(LK^{-1})$ the classical solution is given by

$$u(t,x) = e^{t\overline{K^{-1}L}}K^{-1}x. \tag{3.79}$$

In reflexive spaces assumption i) of Theorem 2.3 is superfluous. □

Retaining the same spaces as in the B-evolution approach, let us try the B-bounded semigroup approach. We should prove the estimate

$$\left\|K^{-1}(\lambda I - LK^{-1})^{-1}\langle f,\alpha\rangle\right\| \leq \frac{M}{|\lambda|}\left\|K^{-1}\langle f,\alpha\rangle\right\|, \tag{3.80}$$

where $Z = L_2(0,1)$, $X = L_2(0,1)\times\mathbb{C}$ and $\langle f,\alpha\rangle \in D(K^{-1})$. Thus $f \in W_2^1(0,1)$, $f(0) = 0$, $f(1) = \alpha$. However, using the results of the B-evolution approach, we see that knowing that

$$\|\langle u, u(1)\rangle\| \leq \frac{\sqrt{2}}{|\lambda|}\|\langle f, f(1)\rangle\|, \tag{3.81}$$

we are required to prove that

$$\|u\| \;\leq\; \frac{M}{|\lambda|}\|f\|, \tag{3.82}$$

where u is the solution of (3.67) with f described above. This seems unlikely, as $\alpha = u(1)$ can be estimated only by the norm of f in $W_2^1(0,1)$ and not in $L_2(0,1)$.

Let us now change the setting and take $X = \{v \in W_2^1(0,1);\ v(0) = 0\}$. Then K is well-defined and bounded on X with range in $Y = X \times \mathbb{C}$ defined as $\mathrm{Im}\, K = \{\langle v, \alpha\rangle\, ; v \in X, v(1) = \alpha\}$. Since this set is closed in Y, K is an isomorphism onto its image. According to Theorem 2.3, we should define L on such a domain that $Lu = \langle u_{xx}, -u_x(1)\rangle \in \mathrm{Im}\, K$. This requires $u \in W_2^3(0,1)$, $u_{xx}(0) = 0$, and $u_{xx}(1) = -u_x(1)$. In other words

$$D(K^{-1}L) = \{u \in W_2^3(0,1); u(0) = u_{xx}(0) = 0,\ u_{xx}(1) + u_x(1) = 0\}. \quad (3.83)$$

Now, $K^{-1}Lu = u_{xx}$ and to prove the generation theorem, we must solve the problem

$$\begin{aligned} \lambda u - u_{xx} &= f, \\ u_x(1) + u_{xx}(1) &= 0, \\ u(0) &= 0, \end{aligned} \quad (3.84)$$

where $f \in X$. Now, differentiating the first equation with respect to x, defining $v = u_x$ and using $u_{xx}(0) = u(0) = 0$, we rewrite (3.84) in the form

$$\begin{aligned} \lambda v - v_{xx} &= f_x, \\ v(1) + v_x(1) &= 0, \\ v_x(0) &= 0. \end{aligned} \quad (3.85)$$

Next, we have

$$
\begin{aligned}
(\lambda v - v_{xx}, v)_{L_2(0,1)} &= \lambda \|v\|^2 - \int_0^1 v_{xx}\overline{v}dx \\
&= \lambda \|v\|^2 - v_x(1)\overline{v(1)} + v_x(0)\overline{v(0)} + \|v_x\|^2 \qquad (3.86) \\
&= \lambda \|v\|^2 + |v(1)|^2 + \|v_x\|^2 .
\end{aligned}
$$

Since the last two terms are non-negative, the right-hand side of (3.86) has exactly the same structure as $|Q|^2$ in (3.69) so that we can repeat the estimates (3.69) – (3.73) to obtain

$$
|(\lambda v - v_{xx}, v)|^2 \geq \frac{1}{2} |\lambda|^2 \|v\|^4 \qquad (3.87)
$$

for $-\dfrac{3\pi}{4} \leq \arg\lambda \leq \dfrac{3\pi}{4}$. From the Cauchy Schwarz inequality, we obtain

$$
\begin{aligned}
\|v\|^2 &\leq \frac{\sqrt{2}}{|\lambda|} |(\lambda v - v_{xx}, v)| \\
&= \frac{\sqrt{2}}{|\lambda|} |(f_x, v)| \qquad (3.88) \\
&\leq \frac{\sqrt{2}}{|\lambda|} \|f_x\| \, \|v\| .
\end{aligned}
$$

Since we defined $v = u_x$, it follows that the estimate

$$
\|u_x\|_{L_2(0,1)} \leq \frac{\sqrt{2}}{|\lambda|} \|f_x\|_{L_2(0,1)} , \qquad (3.89)
$$

is valid in the sector $-\frac{3\pi}{4} \leq \arg\lambda \leq \frac{3\pi}{4}$. As a result of the fact that for an element $f \in X$, we have $f(0) = 0$, we obtain immediately that the norms above are equivalent to the norms in X, thus we have the estimate

$$\|u\| \;\leq\; \frac{M}{|\lambda|}\,\|f\| \tag{3.90}$$

for some M, valid in the same sector. Since we are working in Hilbert (reflexive) spaces, we get the density of $D(K^{-1}L)$ in the space X and therefore the problem is solvable. Here $X_B = X$, the semigroup $t \longrightarrow e^{tK^{-1}L}$ acts in X, and the solution operator is defined as $t \longrightarrow e^{tK^{-1}L}K^{-1}\langle a, a(1)\rangle$. Thus our solution, contrary to the empathy approach, requires compatibility $a(1) = \alpha$. $\square$

Chapter 4

A generalization: C-regularized semigroups

4.1 C-regularized semigroups

Generating a strongly continuous semigroup corresponds to the abstract Cauchy problem having a unique mild solution, for all initial data in X, [16, p 146]. By a mild solution we mean a map $t \longrightarrow u(t)$ that is a strict solution of the "integrated" version of the abstract Cauchy problem. What should be done when the abstract Cauchy problem is ill-posed, for instance, if it does not have a mild solution for all initial data ? One possible approach would be to search for initial data in the original space that yield solutions. This approach entails finding a bounded operator C, such that the abstract Cauchy problem (1.7) has a mild solution for all initial data in the image of C, and this corresponds to the problem having a mild C-existence family, [15, Theorem 4.13].

Definition 1.1 A *mild* solution of the abstract Cauchy problem is a function $u(t)$, satisfying $u(t) \in C\left([0,\infty), X\right)$, $\int_0^t u(s)ds \in D(A)$, $t \geq 0$, and

$$A\left(\int_0^t u(s)ds\right) = u(t) - x, \quad u(0) = x.$$

In what follows let us assume that $C \in \mathcal{L}(X)$ and that it is injective.

Definition 1.2 The family of operators $\{W(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$ is a *mild C-existence* family for A if

i) the map $t \longrightarrow W(t)x \in C([0,\infty), X)$, for all $x \in X$;

ii) for all $x \in X$, $\int_0^t W(s)xds \in D(A)$, $t > 0$, with

$$A\left(\int_0^t W(s)xds\right) = W(t)x - Cx. \tag{4.1}$$

In particular, $u(t, Cy) = W(t)y$, $x = Cy$, is a mild solution of the abstract Cauchy problem, for all $x \in \operatorname{Im} C$; see Theorem 2.6 of [15]. The existence of a mild solution of the abstract Cauchy problem, for all initial data in the image of C, corresponds to the problem having a mild C-existence family; see Theorem 4.13 of [15].

Definition 1.3 The family of operators $\{W(t)\}_{t\geq 0} \subseteq \mathcal{L}([[D(A)]])$ is called a C-*existence* family for A if

i) $t \longrightarrow W(t)x \in C([0,\infty),[[D(A)]])$, for all $x \in D(A)$

ii) for all $x \in D(A)$, $t \geq 0$

$$\int_0^t AW(s)xds = W(t)x - Cx, \tag{4.2}$$

where $A : D(A) \longrightarrow X$ is a closed linear operator. In particular, $u(t, Cy) = W(t)y$, $x = Cy$, is a solution of the abstract Cauchy problem, for all $x \in C(D(A))$; see Theorem 2.6 of [15].

Definition 1.4 The mild C-existence family for A, $\{W(t)\}_{t\geq 0}$ is called a *strong* C-*existence* family for A if $\{W(t)\mid_{[[D(A)]]}\}_{t\geq 0}$ is a C-existence family for A.

Next, we shall address the matter of uniqueness; see Lemma 2.10 and Proposition 2.9 of [15]. For a closed linear operator $A : D(A) \longrightarrow X$ that has no eigenvalues in (ω, ∞), $\omega \in \mathbb{R}$, a mild solution $u(t)$ of the abstract Cauchy problem, that is exponentially bounded, is unique. Furthermore, for $\operatorname{Re} z > \omega$, $\int_0^\infty e^{-zt}u(t)dt \in D(A)$ and

$$(z - A)\int_0^\infty e^{-zt}u(t)dt = x. \tag{4.3}$$

Definition 1.5 The strongly continuous family $\{W(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$ is called a

C-*regularized semigroup* if

i) $W(0) = C$,

ii) $CW(t+s) = W(t)W(s)$ for all $s,t \geq 0$.

Definition 1.6 An operator $A : D(A) \longrightarrow X$, $D(A) \subset X$, is the *generator* of a C-regularized semigroup $\{W(t)\}_{t\geq 0}$, if

$$CAx = \lim_{t \to 0^+} \frac{1}{t} \left(W(t)x - Cx \right), \tag{4.4}$$

with $D(A) = \{x;$ the limit exists and is in $\operatorname{Im} C\}$.

Theorem 1.7 (Theorem 3.4 of [15]) If $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup generated by A, then

i) A is closed,

ii) $\operatorname{Im} C \subseteq \overline{D(A)}$,

iii) for all $t \geq 0$, $W(t)A \subseteq AW(t)$ and

$$W(t)x = Cx + \int_0^t W(s)Axds, \tag{4.5}$$

for all $x \in D(A)$.

iv) if $f : [0, \infty) \longrightarrow X$ is continuously differentiable and $t \geq 0$ then $\int_0^t W(s)f(s)ds \in D(A)$ with

$$A\left(\int_0^t W(s)f(s)ds\right) = W(t)f(t) - Cf(0) - \int_0^t W(s)f'(s)ds.$$

Proposition 1.8 (Proposition 3.11 of [15]) Suppose an extension of A, $\tilde{A}$ generates a C-regularized semigroup. Then the following are equivalent:

$$\begin{aligned} \text{i)} \quad & C\left(D(\tilde{A})\right) && \subseteq && D(A). \\ \text{ii)} \quad & C^{-1}AC && = && \tilde{A}. \end{aligned}$$

Definition 1.9 A complex number $\lambda \in \mathbb{C}$ is in $\rho_C(A)$, the *C-resolvent set* of A, if $(\lambda I - A)$ is injective and $\operatorname{Im} C \subset \operatorname{Im}(\lambda I - A)$.

Proposition 1.10 (Corollary 3.12 of [15]) Suppose A is closed, $A \subseteq \tilde{A}$, $\tilde{A}$ generates a C-regularized semigroup and $\rho_C(A) \cap \rho_C(\tilde{A})$ is nonempty. Then $C^{-1}AC = \tilde{A}$.

We conclude this section with an example; see [15, Example 3.2].

Example 1.11 Let

$$(Af)x = xf(x), \quad \text{for } x \in \mathbb{R} \tag{4.6}$$

on the set of all continuous real-valued functions on $\mathbb{R}$ that vanish at infinity, denoted by $C^0(\mathbb{R})$, with sup-norm. The operator A does not generate a strongly continuous semigroup, for

$$(e^{tA}f)x = e^{tx}f(x), \tag{4.7}$$

defines an unbounded operator, since e^{tx} is an unbounded function. However, there are solutions of the abstract Cauchy problem for all initial data in a dense set, namely for any initial data f such that $\lim_{|x|\to\infty} e^{tx} f(x) = 0$. The multiplication of the unbounded function e^{tx} with the bounded function e^{-x^2} yields a bounded function of x, since $e^{-x^2}e^{tx} = e^{-(x-\frac{t}{2})^2}e^{\frac{t^2}{4}} \le e^{\frac{t^2}{4}}$. Setting

$$(W(t)f)(x) = e^{-x^2}e^{tx}f(x) \tag{4.8}$$

and $(Cf)(x) = e^{-x^2}f(x)$, we get that for $0 < t \le t_0$, $x \in \mathbb{R}$ and $f \in C^0(\mathbb{R})$,

$$\begin{aligned}
&\left|\frac{1}{t}\left(e^{-x^2}e^{tx}f(x) - e^{-x^2}f(x)\right) - e^{-x^2}xf(x)\right| \\
&= \left|e^{-x^2}f(x)\frac{e^{tx}-1-tx}{t}\right| \\
&= \left|e^{-x^2}f(x)\frac{1}{t}\left(x\int_0^t x\left(\int_0^\zeta e^{\tau x}d\tau\right)d\zeta\right)\right| \qquad (4.9)\\
&= \left|f(x)\frac{e^{-x^2}x^2}{t}\int_0^t\int_0^\zeta e^{\tau x}d\tau d\zeta\right| \\
&\le \frac{x^2e^{-x^2}e^{t_0x}t}{2}\sup_{x\in\mathbb{R}}|f(x)|
\end{aligned}$$

and since $\dfrac{e^{-x^2}e^{t_0 x}t}{2} \leq t\dfrac{e^{-\left(x-\frac{t_0}{2}\right)^2}e^{\frac{t_0^2}{4}}}{2} \leq t\dfrac{e^{\frac{t_0^2}{4}}}{2} \longrightarrow 0$ as $t \longrightarrow 0^+$ it follows that

$$\lim_{t\to 0^+}\left\|\frac{W(t)f - Cf}{t} - CAf\right\| = 0. \tag{4.10}$$

Thus A generates a C-regularized semigroup.

4.2 An equivalent representation of the generator

Regularized semigroups were introduced by G. Da Prato, [26], in which the author defined their generator as in Section 4.1. Regularized semigroups have been introduced independently by E.B Davies and M. M. Pang, [12], and the authors defined their generator using the Laplace transform. In this section we show that in the case of the generation of an exponentially bounded C-regularized semigroup the generators defined in [12] and [26] are equivalent.

Definition 2.1 The strongly continuous family $\{W(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$ is an *exponentially bounded* C-regularized semigroup if it satisfies i) and ii) of Definition 1.5 of Section 4.1 and there exists constants $M > 0$ and $\omega \in \mathbb{R}$ such that $\|W(t)\| \leq M\exp(\omega t)$, for all $t \geq 0$.

Definition 2.2 For $\omega \in \mathbb{R}$, $\lambda > \omega$, define the operators $L_\lambda : X \longrightarrow X$ as follows:

$$L_\lambda x = \int_0^\infty e^{-\lambda t} W(t) x dt, \tag{4.11}$$

where $x \in X$ and $\{W(t)\}_{t\geq 0}$ is an exponentially bounded C-regularized semigroup with $\operatorname{Im} C$ dense in X.

Since $\{W(t)\}_{t\geq 0}$ is exponentially bounded it follows that $L_\lambda \in \mathcal{L}(X)$ for all $\lambda > \omega$.

We shall show that the operators L_λ are invertible and satisfy the resolvent identity with an accuracy to within C; see Proposition 3.1 of [22].

Proposition 2.3 (Proposition 3.1 of [22]) Let $\{W(t)\}_{t\geq 0}$ be an exponentially bounded C-regularized semigroup, then the identity

$$(\lambda - \mu) L_\lambda L_\mu = L_\mu C - L_\lambda C, \tag{4.12}$$

is satisfied for λ, $\mu > \omega$ and there exists a closed operator

$$Zx := (\lambda - L_\lambda^{-1} C) x, \tag{4.13}$$

that is independent of the scalar $\lambda > \omega$ with $D(Z) = \{x \in X; Cx \in \operatorname{Im}(L_\lambda)\}$.

Proof Let $x \in X$ and $\lambda,\mu > \omega$. Then from the C-semigroup property ii) of Definition 1.5 of Section 4.1 it follows that

$$
\begin{aligned}
L_\lambda L_\mu x &= \int_0^\infty \int_0^\infty \exp\left(-(\lambda s + \mu t)\right) W(s+t) C x dt ds \\
&= \int_0^\infty \int_0^\tau \exp\left(-\lambda(\tau - t) + \mu t)\right) W(\tau) C x dt d\tau \\
&= (\mu - \lambda)^{-1} \left\{ \int_0^\infty \exp\left(-\lambda\tau\right) W(\tau) C x d\tau - \right. \qquad (4.14) \\
&\quad \left. \int_0^\infty \exp\left(-\mu\tau\right) W(\tau) C x d\tau \right\} \\
&= (\mu - \lambda)^{-1} \left\{ L_\lambda C - L_\mu C \right\}.
\end{aligned}
$$

Let us show that L_λ is invertible for $\lambda > \omega$. It follows from relation (4.12), that if $L_\lambda x = 0$, then $CL_\mu x = 0$ and $L_\mu x = 0$, i.e., ker$L_\lambda =$ kerL_μ. From the definition of L_λ and the uniqueness theorem for the Laplace transformation, it follows that if $L_\lambda x = 0$, then $W(0)x = Cx = 0$ and $x = 0$, i.e., the operator L_λ is invertible for $\lambda > \omega$. Let $Cx \in \mathrm{Im}(L_\lambda)$. It follows from (4.12), that

$$
L_\mu \left(\lambda L_\lambda - C\right) = L_\lambda \left(\mu L_\mu - C\right) \qquad (4.15)
$$

and $L_\lambda^{-1}(\lambda L_\lambda - C)x = L_\mu^{-1}(\mu L_\mu - C)x$. This means that the operator

$$Zx := L_\lambda^{-1}(\lambda L_\lambda - C)x = \left(\lambda - L_\lambda^{-1}C\right)x, \tag{4.16}$$

does not depend on λ and, since the composition of a closed operator and a bounded operator is closed, it follows that Z is closed. We obtain that

$$(\lambda - Z)^{-1} x = C^{-1} L_\lambda x, \tag{4.17}$$

where $D((\lambda - Z)^{-1}) = \{x \in X;\, L_\lambda x \in \operatorname{Im} C\}$ and

$$L_\lambda x = C(\lambda - Z)^{-1} x$$

for $x \in D\left((\lambda - Z)^{-1}\right)$. $\square$

In [12] the operator Z is called the generator of the exponentially bounded C-regularized semigroup $\{W(t)\}_{t\geq 0}$. In what follows we shall show that the operator Z from Proposition 2.3 is identical to the generator A defined in Definition 1.6 of Section 4.1. We require the following lemma which also holds for the operator Z; see Theorem 11 of [12].

Lemma 2.4 Suppose $\{W(t)\}_{t\geq 0}$ is an exponentially bounded C-regularized semigroup generated by A, then for $\lambda > \omega$, $x \in X$, $\int_0^t e^{-\lambda s} W(s)xds \in D(A)$ and

$$(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t} W(t)xdt. \tag{4.18}$$

Proof Since A generates an exponentially bounded C-regularized semigroup, $\int_0^t W(s)xds \in D(A)$; see Theorem 1.7 of Section 4.1, and from Theorem 3.8 of [15], it follows that $\{W(t)\}_{t\geq 0}$ is a mild C-existence family for A and the mild solution is given by

$$u(t, Cx) = W(t)x,\ x \in X. \tag{4.19}$$

Fix $\lambda > \omega$, and let $L = \int_0^\infty e^{-\lambda t} u(t,y)dt$, $y = Cx$. Then integration by parts shows that

$$L = \lambda \int_0^\infty e^{-\lambda t} \int_0^t u(s,y)dsdt.$$

Since A is closed, $L \in D(A)$ and

$$\begin{aligned} AL &= \lambda \int_0^\infty e^{-\lambda t} A \left(\int_0^t u(s,y)ds \right) dt \\ &= \lambda \int_0^\infty e^{-\lambda t} \left(u(t,y) - y \right) dt \qquad (4.20) \\ &= \lambda L - y \end{aligned}$$

and so we get

$$(\lambda - A)\int_0^\infty e^{-\lambda t}u(t,y)dt = y. \tag{4.21}$$

From equation (4.19) and $Cx = y$, it follows that

$$(\lambda - A)\int_0^\infty e^{-\lambda t}W(t)xdt = Cx \tag{4.22}$$

for all $x \in X$, $\lambda > \omega$. We conclude by showing that $(\lambda I - A)$ is injective, whenever $\lambda > \omega$. Suppose $(\lambda I - A)x = 0$, $x \in D(A)$ and $\lambda > \omega$. Then since A is closed

$$0 = \int_0^\infty e^{-\lambda t}W(t)(\lambda I - A)xdt = (\lambda I - A)\int_0^\infty e^{-\lambda t}W(t)xdt = Cx. \tag{4.23}$$

Since C is injective this implies that $x = 0$. After applying $(\lambda - A)^{-1}$ to both sides of equation (4.22),

$$\int_0^\infty e^{-\lambda t}W(t)xdt = (\lambda - A)^{-1}Cx \tag{4.24}$$

for all $x \in X$, $\lambda > \omega$. $\square$

Theorem 2.5 Let $\{W(t)\}_{t\geq 0}$ be an exponentially bounded C-regularized semigroup generated by A in the sense of (4.4) and let $\{S(t)\}_{t\geq 0}$ be an exponentially bounded C-regularized semigroup generated by Z in the sense of Proposition 2.3. If $W(t)x = S(t)x$ for all $x \in \operatorname{Im} C$, then $A = Z$.

Proof Let $x \in D(A)$. Then from Proposition 1.8 and Proposition 1.10 of Section 4.1, it follows that $Cx \in D(A)$. Thus $(\lambda I - A)Cx$, defines an element belonging to X, for all $\lambda > \omega$. Now, since $CA \subset AC$, we obtain that $(\lambda I - A)Cx \in \operatorname{Im} C$ and thus for some $y \in X$,

$$(\lambda I - A)Cx = Cy. \tag{4.25}$$

From equation (4.18), and Definition 2.2, it follows that

$$(\lambda I - A)Cx = (\lambda I - A)L_\lambda y. \tag{4.26}$$

Since $(\lambda I - A)$ is injective, this implies that

$$Cx = L_\lambda y. \tag{4.27}$$

From equations (4.25), (4.27) and since L_λ is injective, we get

$$C^{-1}(\lambda I - A)Cx = y = L_\lambda^{-1}Cx. \tag{4.28}$$

Finally, we obtain from Proposition 1.8 of Section 4.1, that $A = C^{-1}AC$, and so equation (4.28), reduces to

$$(\lambda I - A)x = (\lambda I - C^{-1}AC)x = L_\lambda^{-1}Cx = (\lambda I - Z)x. \tag{4.29}$$

Furthermore, from equation (4.27), $Cx \in \mathrm{Im}(L_\lambda)$. Thus $x \in D(Z)$.

Let $x \in D(Z)$. Then $Cx \in \mathrm{Im}(L_\lambda)$, and thus for some $y \in X$,

$$Cx = L_\lambda y. \tag{4.30}$$

From equation (4.18), it follows that $Cx \in D(A)$ and

$$C^{-1}(\lambda I - A)Cx = L_\lambda^{-1}Cx.$$

By Proposition 1.8 of Section 4.1, $A = C^{-1}AC$. Since $Cx \in D(A)$ this implies that $x \in D(A)$ and so

$$(\lambda I - Z)x = L_\lambda^{-1}Cx = C^{-1}(\lambda I - A)Cx = (\lambda I - A)x. \tag{4.31}$$

Thus $x \in D(A)$. The proof is complete. $\square$

Below we present an alternative proof of Theorem 2.5 that was found in [22], Proposition 3.2.

Let $x \in D(A)$. Then

$$\begin{aligned}
\frac{d}{dt}W(t)Cx &= \lim_{h\to 0}\frac{W(t)\,(W(h)-C)\,x}{h} \\
&= W(t)\lim_{h\to 0}\frac{(W(h)-C)\,x}{h} \qquad (4.32)\\
&= W(t)CAx
\end{aligned}$$

and

$$\begin{aligned}
CL_\lambda(\lambda - A)x &= L_\lambda C(\lambda - A)x \\
&= \lambda L_\lambda Cx - \int_0^\infty \exp(-\lambda t)\, W(t)CAxdt \\
&= \lambda L_\lambda Cx - \int_0^\infty \exp(-\lambda t)\, \frac{d}{dt}W(t)Cxdt \qquad (4.33)\\
&= C^2x.
\end{aligned}$$

Hence, $L_\lambda(\lambda - A)x = Cx$, i.e., $Cx \in \operatorname{Im} L_\lambda$ and $Zx = (\lambda - L_\lambda^{-1}C)x = Ax$.

If we now assume that $x \in D(Z)$ then there exists $y \in X$ such that $Cx = L_\lambda y$.

Let us consider the following:

$$
\begin{aligned}
\frac{W(h)x - Cx}{h} &= \frac{W(h)C^{-1}L_\lambda y - L_\lambda y}{h} \\
&= \frac{1}{h}\left\{C^{-1}W(h)\int_0^\infty \exp(-\lambda t)\, W(t)y dt \;-\right. \\
&\qquad \left.\int_0^\infty \exp(-\lambda t)\, W(t)y dt\right\} \\
&= \frac{1}{h}\left\{\int_0^\infty \exp(-\lambda t)\, W(t+h)y dt \;-\right. \\
&\qquad \left.\int_0^\infty \exp(-\lambda t)\, W(t)y dt\right\} \\
&= \frac{\exp(\lambda h) - 1}{h}\int_h^\infty \exp(-\lambda t) W(t)y dt - \\
&\qquad \frac{1}{h}\int_0^h \exp(-\lambda t)\, W(t)y dt \\
&\longrightarrow \lambda L_\lambda y - Cy = C(\lambda x - y) \in \operatorname{Im} C
\end{aligned}
\tag{4.34}
$$

as $h \to 0$. It follows that $x \in D(A)$, $D(A) = D(Z)$ and $Ax = Zx$. $\square$

In the paper [12, Definition 3], the authors introduced an operator G that was used as a technical tool. The operator G is defined as follows:

$$Gx = \lim_{t \to 0^+} \left(\frac{C^{-1}W(t)x - x}{t} \right), \tag{4.35}$$

with

$$D(G) = \left\{ x \in \operatorname{Im} C;\ \lim_{t \to 0^+} \left(\frac{C^{-1}W(t)x - x}{t} \right) \text{ exists} \right\}.$$

We shall now show the relation that exists between the operator A and the operator G. Let $x \in D(G)$. Then

$$CGx = C \lim_{t \to 0^+} \left(\frac{C^{-1}W(t)x - x}{t} \right) = \lim_{t \to 0^+} \left(\frac{W(t)x - Cx}{t} \right) = CAx$$

and since C is injective, it follows that $x \in D(A)$ and so $G \subset A$.

Since A is a closed operator, it follows that G is closable. The closure of the operator G is called a complete infinitesimal generator of the C-regularized semigroup. The inclusion $\overline{G} \subset A$, holds for the complete infinitesimal generator. In [12, Theorem 33, Corollary 36], it was shown that when C is accretive or when the resolvent set $\rho(A)$, contains a subinterval of $\mathbb{R}$, then $\overline{G} = A$. The theorem that follows establishes this relationship between these operators, but, instead of requiring the resolvent set of A to contain a subinterval of $\mathbb{R}$, we show that it is sufficient that the resolvent set of A be nonempty for the complete infinitesimal generator to coincide with the generator.

Theorem 2.6 If A generates an exponentially bounded C-regularized semigroup $\{W(t)\}_{t\geq 0}$ and $\rho(A) \neq \emptyset$, then $\overline{G} = A$.

Proof Since $\rho(A) \neq \emptyset$, there exists $\lambda \in \rho(A)$ such that

$$(\lambda I - A)^{-1} Cx = L_\lambda x, \tag{4.36}$$

for $x \in X$. Setting $y = Cx$, we obtain that

$$(\lambda I - A)^{-1} y = L_\lambda C^{-1} y. \tag{4.37}$$

From Lemma 31 of [12], we have that $\overline{G}$ satisfies a similar relation

$$\left(\lambda I - \overline{G}\right)^{-1} y = L_\lambda C^{-1} y \tag{4.38}$$

for $y \in \operatorname{Im} C$. Thus it follows that

$$(\lambda I - A)^{-1} y = L_\lambda C^{-1} y = \left(\lambda I - \overline{G}\right)^{-1} y, \tag{4.39}$$

for $y \in \operatorname{Im} C$. Furthermore, $\operatorname{Im} C$ is dense in X and by a variation of Kato's result; see [5, Lemma 4.1], $(\lambda I - A)^{-1}$ has a unique extension by continuity to $\left(\lambda I - \overline{A}\right)^{-1}$ and in a similar manner we can extend $\left(\lambda I - \overline{G}\right)^{-1}$ to the whole space X and obtain the following relation

$$\left(\lambda I - \overline{A}\right)^{-1} y = \left(\lambda I - \overline{G}\right)^{-1} y \tag{4.40}$$

for all $y \in X$. It follows that $A = \overline{G}$. $\square$

In [12] it was shown that if we assume that the resolvent set of the complete infinitesimal generator is nonempty then the complete infinitesimal generator coincides with the generator.

Theorem 2.7 (Proposition 3.3 of [22]) If A generates an exponentially bounded C-regularized semigroup $\{W(t)\}_{t\geq 0}$ and $\rho(\overline{G}) \neq \emptyset$, then $\overline{G} = A$. $\square$

4.3 The inhomogeneous initial value problem

In order to guarantee the existence of a mild/strict solution for the non-homogeneous abstract Cauchy problem one has to impose condition/s on the non-homogeneous part. In this section we focus on these condition/s, and investigate under what conditions is a mild solution a classical solution.

In this section, we consider the inhomogeneous initial value problem

$$u'(t) = Au(t) + f(t), \quad u(0) = x \tag{4.41}$$

where $f : [0, t_0) \longrightarrow \mathrm{Im}\, C$, $x \in \mathrm{Im}\, C$, $t_0 > 0$.

Definition 3.1 A function $u : [0, t_0) \longrightarrow X$ is a classical solution of (4.41) on $[0, t_0)$ if u is continuous on $[0, t_0)$, continuously differentiable on $[0, t_0)$, $u(t) \in D(A)$ for $0 < t < t_0$ and (4.41) is satisfied on $[0, t_0)$.

Definition 3.2 Let $\{W(t)\}_{t\geq 0}$ be a C-regularized semigroup generated by A and $C^{-1}f(s) \in L_1([0, t_0), X)$. Then the function $u \in C([0, t_0), X)$ given by

$$u(t)x = W(t)C^{-1}x + \int_0^t W(t-s)C^{-1}f(s)ds \tag{4.42}$$

for $x \in \operatorname{Im} C$ is a mild solution of the initial valued problem (4.41) on $[0, t_0)$.

Let $\{W(t)\}_{t\geq 0}$ be a C-regularized semigroup generated by A and let $u(t)$ be a classical solution of equation (4.41). Since the derivative $u'(t)$ exists, we get

$$u(t+h) = u(t) + hu'(t) + \varepsilon(h), \tag{4.43}$$

where $\varepsilon(h) \to 0$ as $h \to 0$. For $0 < s < t < t_0$, and using the semigroup property of Definition 1.5 of Section 4.1,

$$\frac{1}{h}[CW(t-s+h)u(s+h) - CW(t-s)u(s)]$$

$$= W(t-s)\left\{\frac{W(h)-C}{h}\right\}u(s) + CW(t-s+h)u'(s) + \tag{4.44}$$

$$CW(t-s+h)\varepsilon(h).$$

The family of operators $\{W(t)\}_{t\geq 0}$ is bounded for $0 \leq t \leq \epsilon$, where ϵ is some positive constant. In fact, if this statement is false, then there is a sequence $(t_n)_{n\in\mathbb{N}}$ satisfying $t_n \geq 0$, $\lim_{n\longrightarrow\infty} t_n = 0$ and $\|W(t_n)\| \geq n$. From the uniform boundedness theorem, it then follows that for some $x \in X$, $\|W(t_n)x\|$ is unbounded contrary to the fact that this family is strongly continuous. Thus, it follows that $\|W(t)\| \leq \alpha$ for $0 \leq t \leq \epsilon$. Similarly, it can be shown that there exists a constant α such that $\|W(t)\| \leq \alpha$, for every compact subset of $[0, t_0)$. Thus, we get $\|CW(t-s+h)\varepsilon(h)\| \leq \alpha' \|\varepsilon(h)\| \longrightarrow 0$ as $h \rightarrow 0$. Hence we obtain

$$\begin{aligned}&\lim_{h\rightarrow 0} \frac{1}{h}\left[CW(t-s+h)u(s+h) - CW(t-s)u(s)\right] \\ &= -W(t-s)CAu(s) + CW(t-s)u'(s).\end{aligned} \tag{4.45}$$

Thus the function $s \rightarrow CW(t-s)u(s)$ is differentiable for $0 < s < t < t_0$ and since $W(t)C = CW(t)$, it follows that

$$\begin{aligned}
\frac{d}{ds}CW(t-s)u(s) &= -W(t-s)CAu(s) + CW(t-s)u'(s) \\
&= -W(t-s)CAu(s) + CW(t-s)\times \\
&\quad (Au(s) + f(s)) \\
&= CW(t-s)f(s).
\end{aligned} \tag{4.46}$$

If $C^{-1}f(s) \in L_1([0,t_0), X)$, then $W(t-s)C^{-1}f(s)$ is integrable and integrating equation (4.46) from 0 to t, $t < t_0$ yields

$$\begin{aligned}
CW(0)u(t) - C^2W(t)C^{-1}x &= \int_0^t CW(t-s)f(s)ds \\
&= C^2\int_0^t W(t-s)C^{-1}f(s)ds.
\end{aligned} \tag{4.47}$$

Since C is injective, we get

$$u(t) = W(t)C^{-1}x + \int_0^t W(t-s)C^{-1}f(s)ds, \tag{4.48}$$

which is a mild solution of the initial value problem (4.41) for all $x \in \operatorname{Im} C$. In the homogeneous case we know that not every mild solution is a solution and thus we investigate under what condition/s imposed on f, will the mild solution become a strict solution. In order to prove the existence of solutions of (4.41) we require more than just the continuity of f. We commence with a general criterion for the existence of solutions of (4.41).

Theorem 3.3 Let A be the infinitesimal generator of a C-regularized semigroup $\{W(t)\}_{t\geq 0}$, and let $C^{-1}f(s) \in L_1([0,t_0),X)$ be continuous on $[0,t_0)$ and define

$$v(t) = \int_0^t W(t-s)C^{-1}f(s)ds. \tag{4.49}$$

The initial value problem (4.41) has a solution u on $[0,t_0)$ for every $x \in C(D(A))$ iff one of the following conditions is satisfied:

i) $v(t)$ is continuously differentiable on $(0,t_0)$.

ii) $v(t) \in D(A)$ for $0 < t < t_0$ and $Av(t)$ is continuous on $(0,t_0)$.

Proof If the initial value problem (4.41) has a solution u for some $x \in C(D(A))$, then the solution is given by (4.42). Consequently

$$v(t) = u(t) - W(t)C^{-1}x, \tag{4.50}$$

is differentiable for $t > 0$ as a difference of two such differentiable functions and

$$v'(t) = u'(t) - W(t)AC^{-1}x, \tag{4.51}$$

is continuous on $(0, t_0)$. Therefore i) is satisfied. Also if $x \in C(D(A))$, $W(t)C^{-1}x \in D(A)$ for $t \geq 0$ and therefore $v(t) = u(t) - W(t)C^{-1}x \in D(A)$ for $t > 0$ and

$$Av(t) = Au(t) - AW(t)C^{-1}x = u'(t) - f(t) - W(t)AC^{-1}x \tag{4.52}$$

is continuous on $[0, t_0)$. Thus also ii) is satisfied.

On the other hand for $h > 0$, we obtain the following identity

$$\frac{W(h) - C}{h} v(t) = C \left\{ \frac{v(t+h) - v(t)}{h} - \frac{1}{h} \int_t^{t+h} W(t+h-s)C^{-1}f(s)ds \right\}. \tag{4.53}$$

From the continuity of $C^{-1}f(s)$ it follows that the second term on the right hand side of (4.53) has the limit $f(t)$ as $h \longrightarrow 0$. If $v(t)$ is continuously differentiable on $(0, t_0)$ then it follows from (4.53) that $v(t) \in D(A)$ for $0 < t < t_0$, and $Av(t) = v'(t) - f(t)$. Since $v(0) = 0$ it follows that $u(t) = W(t)C^{-1}x + v(t)$ is the solution of the initial value problem (4.41), for $x \in C(D(A))$. If $v(t) \in D(A)$ it follows from (4.53), that $v(t)$ is differentiable from the right at t and the right derivative $D^+v(t)$ of v satisfies $D^+v(t) = Av(t) + f(t)$. Since $D^+v(t)$ is continuous, $v(t)$ is continuously differentiable and $v'(t) = Av(t) + f(t)$, [25]. Since $v(0) = 0$, $u(t) = W(t)C^{-1}x + v(t)$ is the solution of (4.41) for $x \in C(D(A))$ and the proof is complete. $\square$

From Theorem 3.3 we get the following useful corollaries.

Corollary 3.4 Suppose $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup generated by A, $f : [0, t_0) \longrightarrow \operatorname{Im} C$ has the property that $t \longrightarrow C^{-1}f(t)$ is continuously differentiable on $[0, t_0)$, and $x \in C(D(A))$. Then (4.41) has a unique solution on $[0, t_0)$, for every $x \in C(D(A))$.

Proof We have that

$$\begin{aligned} v(t) &= \int_0^t W(t-s)C^{-1}f(s)\,ds \\ &= \int_0^t W(s)C^{-1}f(t-s)\,ds \end{aligned} \tag{4.54}$$

From equation (4.54) it follows that $v(t)$ is differentiable for $t > 0$ and that its derivative

$$\begin{aligned} v'(t) &= W(t)C^{-1}f(0) + \int_0^t W(s)C^{-1}f'(t-s)\,ds \\ &= W(t)C^{-1}f(0) + \int_0^t W(t-s)C^{-1}f'(s)\,ds \end{aligned} \tag{4.55}$$

is continuous on $(0, t_0)$, (sum of continuous functions is continuous). The result therefore follows from Theorem 3.3 i). Since A is the generator of a C-regularized semigroup we are guaranteed uniqueness of solutions; see [15, Lemma 25.26]. □

Corollary 3.5 Let A be the infinitesimal generator of a C-regularized semigroup $\{W(t)\}_{t\geq 0}$. Let $C^{-1}f(s) \in L_1([0,t_0),X)$ be continuous on $(0,t_0)$. If $f(s) \in C(D(A))$ for $0 < s < t_0$, and $AC^{-1}f(s) \in L_1([0,t_0),X)$, then for every $x \in C(D(A))$ the initial value problem (4.41) has a solution on $[0,t_0)$.

Proof Since $f(s) \in C(D(A))$, it follows that for $s > 0$, $W(t-s)C^{-1}f(s) \in D(A)$ and that $AW(t-s)C^{-1}f(s) = W(t-s)AC^{-1}f(s)$ is integrable. Therefore $v(t)$ defined by (4.50) satisfies $v(t) \in D(A)$ for $t > 0$ and

$$Av(t) = A\int_0^t W(t-s)C^{-1}f(s)ds = \int_0^t W(t-s)AC^{-1}f(s)ds \qquad (4.56)$$

is continuous. The result follows from the Theorem 3.1 ii). $\square$

As a consequence of the previous results we can prove the following:

Theorem 3.6 Let A be the infinitesimal generator of a C-regularized semigroup $\{W(t)\}_{t\geq 0}$ and let $C^{-1}f \in L_1((0,t_0),X)$. If u is a mild solution of (4.41) on $[0,t_0]$, then for every $t_0' < t_0$, u is the uniform limit on $[0,t_0']$ of solutions of (4.41).

Proof Assume that $C^{-1}x_n \in D(A)$ satisfy $C^{-1}x_n \longrightarrow C^{-1}x$ and let $C^{-1}f_n \in C^1((0,t_0),X)$ satisfy $C^{-1}f_n \longrightarrow C^{-1}f$ in $L_1((0,t_0),X)$. From Corollary 3.4, it follows that for each $n \geq 1$ the initial value problem

$$u_n'(t) = Au_n(t) + f_n(t), \qquad u_n(0) = x_n \tag{4.57}$$

has a solution $u_n(t)$ on $[0, t_0)$ satisfying

$$u_n(t) = W(t)C^{-1}x_n + \int_0^t W(t-s)C^{-1}f_n(s)ds. \tag{4.58}$$

If u is the mild solution of (4.41) on $[0, t_0]$ then

$$\begin{aligned} &\|u_n(t) - u(t)\| \\ &\leq \alpha \left[\|C^{-1}x_n - C^{-1}x\| + \int_0^{t_0'} \|C^{-1}f_n(s) - C^{-1}f(s)\| \, ds \right] \end{aligned} \tag{4.59}$$

and the result follows from (4.59). □

We see that if one imposes further conditions on f, then the mild solution (4.42), becomes a classical solution i.e., a continuous differentiable solution of (4.41). If A is the infinitesimal generator of a holomorphic C-semigroup, we have stronger results.

Definition 3.7 Let $\Delta_\delta = \{z; |\arg z| < \delta\}$ for some $0 < \delta < \dfrac{\pi}{2}$. A *holomorphic C-regularized semigroup* is a family of bounded linear operators $\{W(z)\}_{z\in\Delta_\delta}$

satisfying the following:

i) $z \longrightarrow W(z)$, from Δ_δ into $\mathcal{L}(X)$ is holomorphic,

ii) $W(0) = C$, $W(z)W(w) = CW(z+w)$ for all $z,\ w \in \Delta_\delta$,

iii) $\lim\limits_{\substack{z\to 0,\\ z\in\Delta_{\delta-\epsilon}}} W(z)x = Cx$ for $x \in X$ and $\epsilon \in (0,\delta)$.

Theorem 3.8 Let $0 < \delta < \dfrac{\pi}{2}$ and let A be the infinitesimal generator of a holomorphic C-regularized semigroup $\{W(t)\}_{t\geq 0}$ of angle δ and let $C^{-1}f \in L_p([0,t_0),X)$ with $1 < p < \infty$. If u is a mild solution of (4.41) then u is Hölder continuous with exponent $1 - \dfrac{1}{p}$ on $[\epsilon, t_0]$ for every $\epsilon > 0$.

Proof Let $\|W(t)\| \leq \alpha$, on $[0,t_0]$, where

$$W(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{\lambda t}(\lambda - A)^{-1}Cd\lambda \tag{4.60}$$

and Γ is the path composed from the rays $\rho e^{i\theta}$ and $\rho e^{-i\theta}$, $0 < \rho < \infty$ and $\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2} + \delta$. The path Γ is oriented so that $\operatorname{Im}\lambda$ increases along Γ. Since $\{W(t)\}_{t\geq 0}$ is holomorphic, differentiating (4.60) with respect to t yields

$$W'(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda e^{\lambda t}(\lambda - A)^{-1}Cd\lambda. \tag{4.61}$$

From the proof of [15, Theorem 3.4], we obtain that $AW(t) = W'(t)$ for $t \in (0,t_0]$ and

$$\|AW(t)\| \leq \frac{\alpha}{\pi}\int\limits_0^\infty e^{-\rho|\cos\theta|t}d\rho = \left(\frac{\alpha}{\pi\,|\cos\theta|}\right)\frac{1}{t} \quad \text{for } t \in (0,t_0]. \tag{4.62}$$

This implies that $W(t)C^{-1}x$ is Lipschitz continuous on $[\epsilon, t_0]$ for every $\epsilon > 0$. If $x \in C(D(A))$, $W(t)C^{-1}x$ is Lipschitz continuous on $[0, t_0]$. It suffices to show that if $C^{-1}f \in L_p([0, t_0), X)$ then $v(t) = \displaystyle\int_0^t W(t-s)C^{-1}f(s)ds$ is Hölder continuous with exponent $1 - \dfrac{1}{p}$ on $[0, t_0)$. For $h > 0$, we have

$$\begin{aligned} v(t+h) - v(t) &= \int_t^{t+h} W(t+h-s)C^{-1}f(s)ds \\ &+ \int_0^t \left(W(t+h-s) - W(t-s)\right)C^{-1}f(s)ds. \end{aligned} \tag{4.63}$$

For the first integral, using Hölders inequality, we obtain

$$\begin{aligned} &\left\| \int_t^{t+h} W(t+h-s)C^{-1}f(s)ds \right\| \\ &\leq \alpha \int_t^{t+h} \|C^{-1}f(s)\| \, ds \\ &\leq \alpha \left(\int_t^{t+h} \|C^{-1}f(s)\|^p \, ds \right)^{\frac{1}{p}} h^{1-\frac{1}{p}}. \end{aligned} \tag{4.64}$$

In order to estimate the second integral, we note that for $h > 0$,

$$\|W(t+h) - W(t)\| \leq 2\alpha, \text{ for } t, t+h \in [0, t_0] \tag{4.65}$$

and for t,$t+h \in (0, t_0]$

$$\begin{aligned}
\|W(t+h) - W(t)\| &\leq \int_t^{t+h} \|AW(s)\|\, ds \\
&\leq \left(\frac{\alpha}{\pi\,|\cos\theta|}\right)\frac{h}{t}.
\end{aligned} \tag{4.66}$$

Therefore, for t,$t+h \in [0, t_0]$

$$\|W(t+h) - W(t)\| \leq \max\left(2\alpha, \frac{\alpha}{\pi\,|\cos\theta|}\right)\min\left(1, \frac{h}{t}\right). \tag{4.67}$$

Using Hölder's inequality

$$\begin{aligned}
&\left\|\int_0^t \left(W(t+h-s) - W(t-s)\right) C^{-1} f(s) ds\right\| \\
&\leq \max\left(2\alpha, \frac{\alpha}{\pi\,|\cos\theta|}\right)\int_0^t \min\left(1, \frac{h}{t-s}\right)\|C^{-1}f(s)\|\, ds \\
&\leq \max\left(2\alpha, \frac{\alpha}{\pi\,|\cos\theta|}\right)\left(\int_0^{t_0} \|C^{-1}f(s)\|^p\, ds\right)^{1/p} \times \\
&\left(\int_0^t \min\left(1, \frac{h}{t-s}\right)^{p/(p-1)} ds\right)^{(p-1)/p},
\end{aligned} \tag{4.68}$$

but since, $\min\left(1,\dfrac{h}{t}\right) \geq 0$, we get

$$\begin{aligned}\int_0^t \left(\min\left(1,\frac{h}{t-s}\right)\right)^{p/(p-1)} ds &= \int_0^t \left(\min\left(1,\frac{h}{\tau}\right)\right)^{p/(p-1)} d\tau \\ &\leq \int_0^\infty \left(\min\left(1,\frac{h}{\tau}\right)\right)^{p/(p-1)} d\tau = ph.\end{aligned} \tag{4.69}$$

Finally, we get $\left\|\int_0^t \left(W(t+h-s)-W(t-s)\right)C^{-1}f(s)ds\right\| \leq$ const.$h^{(p-1)/p}$, and the proof is complete. □

Finally, we turn to conditions imposed on f that will ensure that a mild solution is a classical solution.

Theorem 3.9 Let A be the infinitesimal generator of a holomorphic C-regularized semigroup $\{W(t)\}_{t\geq 0}$. Let $C^{-1}f(s) \in L_1\left((0,t_0),X\right)$ and assume that for every $0<t<t_0$, there exists $\delta_t>0$ and a continuous real valued function $\theta_t(\tau):[0,\infty) \longrightarrow [0,\infty)$ such that

$$\left\|C^{-1}f(t)-C^{-1}f(s)\right\| \leq \theta_t(|t-s|) \tag{4.70}$$

and

$$\int_0^{\delta_t} \theta_t(\tau)/\tau \; d\tau < \infty. \tag{4.71}$$

Then for every $x \in \operatorname{Im} C$ the mild solution of (4.41) is a classical solution.

Proof Since $W(t)$ is a holomorphic C-regularized semigroup, $W(t)C^{-1}x$ is a solution of the homogenous equation with initial data $x \in \operatorname{Im} C$. To prove the theorem it is therefore sufficient to show that

$v(t) = \int_0^t W(t-s)C^{-1}f(s)ds \in D(A)$ and that $Av(t)$ is continuous. To this end, we write

$$
\begin{aligned}
v(t) &= v_1(t) + v_2(t) \\
&= \int_0^t W(t-s)\left(C^{-1}f(s) - C^{-1}f(t)\right) ds + \qquad (4.72) \\
&\quad \int_0^t W(t-s)C^{-1}f(t)ds.
\end{aligned}
$$

It follows that $v_2(t) \in D(A)$ and that $Av_2(t) = (W(t) - C)C^{-1}f(t)$. Since the assumptions of our theorem imply that $C^{-1}f$ is continuous on $(0, t_0)$, it follows that $Av_2(t)$ is continuous. To prove the same conclusion for $v_1(t)$ we define

$$v_{1,\varepsilon}(t) = \int_0^{t-\varepsilon} W(t-s)\left(C^{-1}f(s) - C^{-1}f(t)\right) ds, \quad t \geq \varepsilon \qquad (4.73)$$

and

$$v_{1,\varepsilon}(t) = 0 \text{ for } t < \varepsilon. \qquad (4.74)$$

From this definition it follows that $v_{1,\varepsilon}(t) \longrightarrow v_1(t)$ as $\varepsilon \longrightarrow 0$. It also follows that $v_{1,\varepsilon}(t) \in D(A)$ and for $t \geq \varepsilon$

$$Av_{1,\varepsilon}(t) = \int_0^{t-\varepsilon} AW(t-s)\left(C^{-1}f(s) - C^{-1}f(t)\right) ds. \tag{4.75}$$

From the assumptions (4.70) and (4.71), it follows that for $t > 0$, $Av_{1,\varepsilon}(t)$ converges as $\varepsilon \longrightarrow 0$ and that

$$\lim_{\varepsilon \longrightarrow 0} Av_{1,\varepsilon}(t) = \int_0^t AW(t-s)\left(C^{-1}f(s) - C^{-1}f(t)\right) ds. \tag{4.76}$$

The closedness of A then implies that $v_1(t) \in D(A)$ for $t > 0$ and

$$Av_1(t) = \int_0^t AW(t-s)\left(C^{-1}f(s) - C^{-1}f(t)\right) ds. \tag{4.77}$$

To conclude the proof we have to show that $Av_1(t)$ is continuous on $(0, t_0)$. For $0 < \delta < t$, we have

$$\begin{aligned} Av_1(t) &= \int_0^\delta AW(t-s)\,(C^{-1}f(s) - C^{-1}f(t))\, ds \; + \\ &\quad \int_\delta^t AW(t-s)\,(C^{-1}f(s) - C^{-1}f(t))\, ds. \end{aligned} \tag{4.78}$$

For fixed $\delta > 0$ the second integral on the right of (4.78), is a continuous function of t while the first integral is of order δ uniformly in t. Thus, $Av_1(t)$ is continuous and the proof is complete. $\square$

An immediate consequence of Theorem 3.9 is,

Corollary 3.10 Let A be the infinitesimal generator of an analytic semigroup $\{W(t)\}_{t\geq 0}$. If $C^{-1}f \in L_1((0,t_0),X)$ is locally Hölder continuous on $(0,t_0]$ then for every $x \in \operatorname{Im} C$ the initial value problem (4.41) has a unique solution. $\square$

In [25], the author concluded in Section 4.1, with a result, similar to Theorem 3.9, in which the condition on the modulus of continuity of f is replaced by another regularity condition using fractional powers of the generator. A theory of fractional powers of the generator of a C-regularized semigroup has not been established and so we had to conclude with Theorem 3.9.

Chapter 5

A relationship

5.1 Relation between B-bounded semigroups and C-regularized semigroups and similar objects.

Upon a superficial reading of the literature the philosophy and appearance of B-bounded semigroups are similar to C-regularized semigroups and related objects like C-existence and uniqueness families. However, in [8] we show that these objects are quite different, to the extent that the only objects which can be simultaneously C-regularized semigroups and B-bounded semigroups in the same space X are C_0-semigroups. For a relation between C-existence families and B-

bounded semigroups, we begin with noting that, by Theorem 1.5 of Section 3.1 a B-bounded semigroup solves an abstract Cauchy problem in X_B. Since the very concept of a C-existence family is that they provide solutions for the initial values taken from a subspace of the original space X, we are placed in the situation described in Theorem 1.10 of Section 3.1 with $C = \overline{B}^{-1}$. The first result in this direction is the following.

Proposition 1.1 Assume that $A : D(A) \longrightarrow X$ is a closed operator with no eigenvalues in (ω, ∞) for some $\omega \in \mathbb{R}$, and $A \in B - \mathcal{G}(M, \omega, X)$. If B^{-1} is densely defined and $X_B \hookrightarrow X$, then

$$\{W(t)\}_{t\geq 0} = \{\overline{B}^{-1} Y(t) \overline{B}^{-1}\}_{t\geq 0} \tag{5.1}$$

is a mild $\overline{B}^{-1}$-existence family for A. Moreover, if

$$\overline{B}^{-1}(D(A)) \subset D(A), \tag{5.2}$$

then $\{W(t)\}_{t\geq 0}$ is a strong C-existence family for A. If $\overline{B}^{-1} A \subset A\overline{B}^{-1}$ then $\{\overline{B}^{-1} Y(t) \overline{B}^{-1}\}_{t\geq 0}$ is a C-regularized semigroup generated by A.

Proof From Theorem 1.10 of Section 3.1, B is closable and B^{-1} is bounded and we can identify X_B with $D(\overline{B})$ and $\operatorname{Im} \overline{B} = X$. Moreover, then $A = \mathfrak{A}$ restricted to $D_{\overline{B}}(A) = \{x \in D(A) \cap D(\overline{B});\ Ax \in D(\overline{B})\}$ and the B-bounded semigroup is given by $Y(t) = \overline{B}e^{tA}$, where the semigroup acts in $D(\overline{B})$. Then $W(t) = e^{tA}\overline{B}^{-1} = \overline{B}^{-1}Y(t)\overline{B}^{-1}$, $t \geq 0$ is a strong C-existence family for A. Indeed, since $X_B \hookrightarrow X$, $\{W(t)\}_{t\geq 0}$ is a family of bounded operators in X and $t \longrightarrow W(t)x$ is a continuous function for any $x \in X$. Since $\{e^{tA}\}_{t\geq 0}$ is a semigroup in X_B, we have for any $y \in X_B$

$$e^{tA}y = y + A\left(\int_0^t e^{sA}y\, ds\right) \tag{5.3}$$

for $t \geq 0$, and therefore for any $x \in X$ such that $x = \overline{B}y$, we have

$$W(t)x = e^{tA}\overline{B}^{-1}x = \overline{B}^{-1}x + A\left(\int_0^t e^{sA}\overline{B}^{-1}x\, ds\right), \tag{5.4}$$

which is a mild C-existence family identity (note that again due to $X_B \hookrightarrow X$, the integral and A can be considered as X-space operations).

Next note that for a mild C-existence family to be a strong C-existence family it is necessary to leave $D(A)$ invariant; see Definition 1.3 of Chapter 4, so if we have a semigroup acting in a subspace of X which is accessible by an operator C, then we must have $Cx \in D(A)$ whenever $x \in D(A)$ which in our case translates into relation (5.2). If this is the case then using again the fact that $\{e^{tA}\}_{t\geq 0}$ is the semigroup generated by A, we have from $\overline{B}^{-1}x \in D(A)$ that $e^{tA}\overline{B}^{-1}x \in D(A)$ and $\{e^{tA}\overline{B}^{-1}\}_{t\geq 0}$ is a strongly continuous family of operators in $D(A)$ with the graph-norm. Thus A can be moved inside the integral in (5.4) and $\{W(t)\}_{t\geq 0}$ is a strong $\overline{B}^{-1}$-existence family for A.

Finally, if the commutativity property is satisfied (and then (5.2) follows automatically), then for $x \in D(A)$ and $t \geq 0$ we have

$$W(t)Ax = e^{tA}\overline{B}^{-1}Ax = Ae^{tA}\overline{B}^{-1}x = AW(t)x, \tag{5.5}$$

so that by Theorem 3.7 of [15], $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup generated by an extension of A and since $\rho(A) \neq \emptyset$, by Proposition 3.9 of [15], $\{W(t)\}_{t\geq 0}$ is generated by A. $\square$

This proposition suggests that C-evolution families are related to C^{-1}-bounded semigroups rather than C-bounded semigroups. The following theorem shows that the choice is quite limited.

Theorem 1.2 Assume that $A : D(A) \longrightarrow X$ is a closed operator with no eigenvalues in (ω, ∞) for some $\omega \in \mathbb{R}$, B^{-1} is densely defined and $X \hookrightarrow X_B$. Let $\{W(t)\}_{t\geq 0}$ be a mild $\overline{B}^{-1}$-existence family for A. The formula

$$Y(t)x = \overline{B}W(t)\overline{B}\boldsymbol{x}, \tag{5.6}$$

defines a $\overline{B}$-bounded semigroup iff $\overline{B}A\overline{B}^{-1}$ generates a C_0-semigroup in X and $\{W(t)\}_{t\geq 0}$ is exponentially bounded. Then $\overline{BAB^{-1}} = \overline{B}A\overline{B}^{-1}$and $\{Y(t)\}_{t\geq 0}$ is generated by A.

Proof If the equation (5.6) defines a $\overline{B}$-bounded semigroup, then, since $\overline{B}\left(D(\overline{B})\right) = X$, as in the previous theorem

$$W(t)X \subset D(\overline{B}) \tag{5.7}$$

for all $t \geq 0$. Also, since by Theorem 1.5 of Section 3.1, $Y(t) = e^{tK}\overline{B}$ for some K acting in X, $\{\overline{B}W(t)\}_{t\geq 0}$ is a semigroup in X. To identify K, we use the definition of mild existence families to obtain $\int_0^t W(s)xds \in D(A)$, for all $t \geq 0$ and $x \in X$ and

$$W(t)x = \overline{B}^{-1}x + A\left(\int_0^t W(s)xds\right). \tag{5.8}$$

By (5.7), all the terms above are in $D(\overline{B})$ and we have

$$\overline{B}W(t)x = x + \overline{B}A\left(\int_0^t \overline{B}^{-1}\overline{B}W(s)xds\right). \tag{5.9}$$

Since $\overline{B}^{-1}$is bounded, we have $\int_0^t \overline{B}W(s)xds \in D(\overline{B}A\overline{B}^{-1})$ and

$$\overline{B}W(t)x = x + \overline{B}A\overline{B}^{-1}\left(\int_0^t \overline{B}W(s)xds\right). \tag{5.10}$$

By equation (5.6), $t \longrightarrow \overline{B}W(t)x = Y(t)\overline{B}^{-1}x$ and since $\overline{B}^{-1}x \in D(\overline{B})$, this is a continuous function by Definition 1.1 of Section 3.1. Therefore $t \longrightarrow \overline{B}W(t)x$ is a mild solution of the Cauchy problem

$$\partial_t u = \overline{B}A\overline{B}^{-1}u, \qquad u(0) = x. \tag{5.11}$$

Moreover, by Definition 1.1 of Section 3.1,

$$\|\overline{B}W(t)x\|_X = \|Y(t)\overline{B}^{-1}x\|_X \leq Me^{\omega t}\|x\|_X. \tag{5.12}$$

Hence the solutions to (5.11) are exponentially bounded. Since $\overline{B}^{-1}$ is a bounded operator, we obtain also

$$\|W(t)x\|_X = \|\overline{B}^{-1}\overline{B}W(t)x\|_X \leq M'e^{\omega t}\|x\|_X,$$

and hence $\{W(t)\}_{t\geq 0}$ is exponentially bounded.

By Proposition 2.9 of [15], all exponentially bounded mild solutions are unique. Since we have an exponentially bounded mild solution for any $x \in X$, by Theorem 5.5 and (5.16) of [15], the operator $\overline{B}A\overline{B}^{-1}$generates a C_0-semigroup on X. Next we obtain $\overline{B}W(t)x = e^{t\overline{B}A\overline{B}^{-1}}x$ for all $t \geq 0$, $x \in X$ and consequently

$$Y(t)x = e^{t\overline{B}A\overline{B}^{-1}}\overline{B}x \tag{5.13}$$

for all $t \geq 0$, $x \in D(\overline{B})$. Using the semigroup property we obtain

$$\begin{aligned} Y(t)x &= e^{t\overline{B}A\overline{B}^{-1}}\overline{B}x \\ &= \overline{B}x + \int_0^t e^{s\overline{B}A\overline{B}^{-1}}\overline{B}A\overline{B}^{-1}\overline{B}x \qquad (5.14) \\ &= \overline{B}x + \int_0^t Y(s)Axds \end{aligned}$$

and by uniqueness [11], $\{e^{t\overline{B}A\overline{B}^{-1}}\}_{t\geq 0}$ is generated by A and from equation (4.9) of [5], it follows that $\overline{BAB^{-1}} = \overline{B}A\overline{B}^{-1}$. Conversely, if $\overline{B}A\overline{B}^{-1}$generates a C_0-semigroup in X, then repeating the considerations above we obtain that A generates a $\overline{B}$-bounded semigroup, and by Proposition 1.1, $\mathcal{W}(t) = \overline{B}^{-1}Y(t)\overline{B}^{-1}$ defines an exponentially bounded $\overline{B}^{-1}$-existence family for A. Since $\{W(t)\}_{t\geq 0}$ is also an exponentially bounded semigroup, $\{\mathfrak{W}(t)\}_{t\geq 0} = \{W(t) - \mathcal{W}(t)\}_{t\geq 0}$ is also exponentially bounded. However, we have for any $x \in X$, $\mathfrak{W}(t)x = A\left(\int_0^t \mathfrak{W}(s)xds\right)$, i.e. $t \longrightarrow \mathfrak{W}(t)x$ is an exponentially bounded mild solution to the homogenous problem (1.7). By Proposition 2.9 of [15], $\mathfrak{W}(t)x = 0$, and hence $\{W(t)\}_{t\geq 0} = \{\mathcal{W}(t)\}_{t\geq 0}$ and formula (5.6) holds. $\square$

From the proof of the above theorem it follows that the "only if" part can be proved under the weaker assumption that mild solutions of (1.7) in X are unique. Note, that the fact that A generates a semigroup in $X_B = D(\overline{B})$, is not sufficient for that purpose as it gives only uniqueness in a smaller space.

Corollary 1.3 Let the assumptions of the previous theorem be satisfied and let $(W(t))_{t\geq 0}$ be a $\overline{B}^{-1}$-regularized semigroup generated by A. The formula

$$Y(t)x = \overline{B}W(t)\overline{B}x, \tag{5.15}$$

defines a $\overline{B}$-bounded semigroup iff $\{\overline{B}W(t)\}_{t\geq 0}$ is a semigroup in X generated by A.

Proof Since a $\overline{B}^{-1}$-regularized semigroup generated by A is a mild $\overline{B}^{-1}$-existence family for A, Theorem 3.5 of [15], we obtain from Theorem 1.2, that $\{\overline{B}W(t)\}_{t\geq 0}$ is a semigroup generated by $\overline{B}A\overline{B}^{-1}$which, since $\overline{B}^{-1}A \subset A\overline{B}^{-1}$is an extension of A. However, using the definition of the generator we obtain that for $x \in D(\overline{B}A\overline{B}^{-1})$

$$\overline{B}A\overline{B}^{-1}x = \lim_{t\to 0+} \frac{\overline{B}W(t)x - x}{t} = \lim_{t\to 0+} \overline{B}\frac{W(t)x - \overline{B}^{-1}x}{t}. \tag{5.16}$$

Since $\overline{B}^{-1}$is bounded, the existence of the left hand side limit yields the existence of the limit of $W(t)x - \overline{B}^{-1}x$. Thus $x \in D(A)$ and the semigroup $\{\overline{B}W(t)\}_{t\geq 0}$ is generated by A. □

The converse follows as in Theorem 1.2, with the sole difference that we use the uniqueness of solutions of the Cauchy problem (1.7), ensured by Theorem 3.5 of [15]. In [15], the author develops the theory of C-regularized semigroups in the extrapolation spaces (obtained by completion of X with respect to the norm $||C\cdot||$–compare our approach to B-bounded semigroups). This enable a link to be developed between C-regularized semigroups and B-bounded semigroups with a different set of assumptions on B. We have the following theorem.

Theorem 1.4 Let $B : X \longrightarrow X$ be a bounded, injective operator,

$A : D(A) \longrightarrow X$ be a closed operator which generates a B-bounded semigroup $\{Y(t)\}_{t\geq 0}$ and satisfies

$$BA \subset AB. \tag{5.17}$$

Then the extension of A, given by $B^{-1}AB$, generates a B-regularized semigroup $\{W(t)\}_{t\geq 0}$ on X which is given by

$$W(t)x = Y(t)x \tag{5.18}$$

for all $t \geq 0$ and $x \in X$. If $\rho(A) \neq \emptyset$, then $A = B^{-1}AB$.

Proof We check the following points. The operator $\mathfrak{B} : X_B \longrightarrow X$ is a bounded extension of B which satisfies

$$\mathfrak{B}(X_B) = \overline{\operatorname{Im} B} \hookrightarrow X \hookrightarrow X_B \tag{5.19}$$

(the last embedding follows from the construction of the completion and boundedness of B). Moreover, $\mathfrak{A}$ generates a strongly continuous semigroup on X_B. Since $\mathfrak{A}$ is the closure of A in X_B, any $\mathfrak{x} \in D(\mathfrak{A})$ is defined by $\mathfrak{B}\mathfrak{x} = \lim\limits_{n\to\infty} Bx_n$, $x_n \in D(A)$ and

$$\mathfrak{B}\mathfrak{A}\mathfrak{x} = \lim_{n\to\infty} BAx_n = \lim_{n\to\infty} ABx_n, \tag{5.20}$$

where in the last equality we used (5.17). Since $(Bx_n)_{n\in\mathbb{N}}$ converges and, from the above, $(ABx_n)_{n\in\mathbb{N}}$ also converges, by closedness of A we obtain

$$\mathfrak{B}\mathfrak{A}\mathfrak{x} = A\mathfrak{B}\mathfrak{x} = \mathfrak{A}\mathfrak{B}\mathfrak{x} \tag{5.21}$$

as, by the definition, $\mathfrak{A}$ is the closure of A in X_B. Therefore $e^{t\mathfrak{A}}\mathfrak{B} = \mathfrak{B}e^{t\mathfrak{A}}$. Thus all the assumptions of the Proposition 6.4 of [15], are satisfied and $B^{-1}AB$ generates a C-regularized semigroup on X given by

$$W(t)x = \mathfrak{B}e^{t\mathfrak{A}}x \tag{5.22}$$

for all $t \geq 0$ and $x \in X$, which, by Theorem 1.5 of Section 3.1, yields equation (5.18). Note, that condition (5.17) ensures only that $A \subset B^{-1}AB$ as there can be $x \in X \backslash D(A)$ satisfying $Bx \in D(A)$ and $ABx \in \operatorname{Im} B$. The last statement of the theorem follows from Proposition 3.9 of [15]. □

The second set of comparison results stems from the formal similarity of equation (3.2) of Section 3.1, and the formula ii) of Definition 1.3 of Section 4.1, which suggests that a B-bounded semigroup could be a C-existence family with $C = B$ subject to additional conditions. The following theorem shows that again this is possible only for a very restricted class of operators.

Theorem 1.5 Let us assume that $A : X \longrightarrow X$ is a closed operator, such that $[\omega, \infty)$ does not contain its eigenvalues, $B : X \longrightarrow X$ is a bounded operator with range $\operatorname{Im} B$ dense in X and $\{W(t)\}_{t\geq 0}$ is a mild B-existence family for A. Then $\{W(t)\}_{t\geq 0}$ is a B-bounded semigroup $\{Y(t)\}_{t\geq 0}$ satisfying (3.9), generated by some operator $\mathcal{D}$ iff A generates a semigroup in X. In such a case

$$W(t)x = e^{tA}Bx = e^{t\mathfrak{B}\mathfrak{D}\mathfrak{B}^{-1}}Bx = Y(t)x \tag{5.23}$$

for all $t \geq 0$ and $x \in X$, where $\mathfrak{B}$, $\mathfrak{D}$ are the closures of B and $\mathcal{D}$, respectively in X_B.

Proof Let $t \longrightarrow u(t, Bx) = W(t)x$ be a mild solution to (1.7), and $\{W(t)\}_{t\geq 0}$ be a B-bounded semigroup. From the property a) of Definition 1.1 of Section 3.1, we have

$$\|u(t, Bx)\| = \|Y(t)x\| \leq Me^{\omega t}\|Bx\| \leq M'e^{\omega t}\|x\| \tag{5.24}$$

for all $x \in X$. Hence we can use [15, Lemma 2.10], to get

$$(\lambda - B)\int_0^\infty e^{-\lambda t}u(t, Bx)dt = Bx \tag{5.25}$$

for all $x \in X$ and $\lambda > \omega$. On the other hand, from the original version of [11, Lemma 1], we obtain

$$\int_0^\infty e^{-\lambda t}Y(t)xdt = B(\lambda - \mathcal{D})^{-1}x \tag{5.26}$$

for all $x \in X$ and $\lambda > \omega$. Combining (5.25) and (5.26), we have

$$(\lambda - A)^{-1}Bx = B(\lambda - \mathcal{D})^{-1}x \tag{5.27}$$

for all $x \in X$ and $\lambda > \omega$. From Lemma 3.1 of [5], we know that $\lambda - \mathcal{D}$ reduces cosets $X/N(B)$ and therefore equation (5.27) can be written for $\mathfrak{B}$, restricted to X (where $\mathfrak{B}$ is the extension by density of B to the completion X_B of X with respect to the seminorm $\|B \cdot \|$):

$$(\lambda - A)^{-1}\mathfrak{B}x = \mathfrak{B}(\lambda - \mathcal{D})^{-1}x \tag{5.28}$$

for all $x \in X$ and $\lambda > \omega$. Let $\mathfrak{x} \in X_B$ be such $\mathfrak{B}\mathfrak{x} = z \in \operatorname{Im} B$; then from equation (5.28), we have

$$\|(\lambda - A)^{-1}z\| = \|\mathfrak{B}(\lambda - \mathcal{D})^{-1}\mathfrak{B}^{-1}z\| \leq \frac{M}{\lambda - \omega}\|\mathfrak{B}^{-1}z\|_{X_B} \leq \frac{M}{\lambda - \omega}\|z\| \tag{5.29}$$

for all $z \in \operatorname{Im} B$ and $\lambda > \omega$. Since A is closed, $(\lambda - A)^{-1}$ is also closed and, being defined on a dense subspace $\operatorname{Im} B \subset X$ and bounded, it is in fact defined on the whole space X. Therefore $(\lambda - A)^{-1}$is the resolvent of A. Furthermore,

$$\begin{aligned}(\lambda - A)^{-2}z &= (\lambda - A)^{-1}\left((\lambda - A)^{-1}z\right) \\ &= (\lambda - A)^{-1}\left(\mathfrak{B}(\lambda - \mathcal{D})^{-1}\mathfrak{B}^{-1}z\right) \qquad (5.30) \\ &= \mathfrak{B}(\lambda - \mathcal{D})^{-2}\mathfrak{B}^{-1}z\end{aligned}$$

and using (5.29)

$$\|(\lambda - A)^{-2}z\| \leq \frac{M}{(\lambda - \omega)^2}\|z\| \tag{5.31}$$

By induction we see that A satisfies the Hille-Yosida estimates in X. Since $\{Y(t)\}_{t\geq 0}$ satisfies (3.9), $B(D(\mathcal{D}))$ is dense in X. Hence, by equation (5.27), $D(A) = (\lambda - A)^{-1}X \supset (\lambda - A)^{-1}(\operatorname{Im} B)$ is dense in X. Therefore A generates a semigroup in X and from (5.28) we obtain that

$$A = \mathfrak{B}\mathcal{D}\mathfrak{B}^{-1}. \tag{5.32}$$

Thus by Theorem 1.5 of Section 3.1, the B-existence family is given by

$$W(t)x = e^{tA}Bx = e^{t\mathfrak{B}\mathfrak{D}\mathfrak{B}^{-1}}Bx = Y(t)x. \tag{5.33}$$

Conversely, assume that A generates a semigroup in X and define by (5.32)

$$\mathcal{D}\mathfrak{r} = \mathfrak{B}^{-1}A\mathfrak{B}\mathfrak{r} \tag{5.34}$$

for all $\mathfrak{r} \in \mathfrak{B}^{-1}(D(A))$. Since $D(A)$ is dense in X and $\mathfrak{B}$ is an isomorphism, $\mathfrak{B}^{-1}(D(A))$ is dense in X_B. Next, we obtain for any for $\lambda > \omega$ and $x \in X$

$$(\lambda - A)^{-1}x = \mathfrak{B}(\lambda - \mathcal{D})^{-1}\mathfrak{B}^{-1}x. \tag{5.35}$$

Therefore for any $\lambda > \omega$

$$\frac{M}{\lambda - \omega}\,\|x\| \geq \|(\lambda - A)^{-1}x\| = \|\mathfrak{B}(\lambda - \mathcal{D})^{-1}\mathfrak{B}^{-1}x\|, \tag{5.36}$$

which gives for all $\mathfrak{r} \in X_B$

$$\|\mathfrak{B}(\lambda - \mathcal{D})^{-1}\mathfrak{r}\|_{X_B} \leq \frac{M}{\lambda - \omega}\,\|\mathfrak{r}\|_{X_B}\,. \tag{5.37}$$

By induction we obtain all the Hille-Yosida estimates. Thus $\mathcal{D}$ generates a B-bounded semigroup in X. $\square$

Corollary 1.6 If, in the statement of the previous theorem, $\{Y(t)\}_{t\geq 0}$ is generated by an extension of A, then $\{W(t)\}_{t\geq 0}$ is B-regularized semigroup generated by A.

Proof By closedness of A, the equation

$$(\lambda - A)^{-1}z = \mathfrak{B}(\lambda - \mathcal{D})^{-1}\mathfrak{B}^{-1}z, \tag{5.38}$$

originally defined for $z \in \operatorname{Im} B$ is valid (with the same operators) on the whole space X. Therefore $D(A) = \mathfrak{B}(D(\mathcal{D}))$. Moreover, $\mathcal{D}$ is an extension of A, that is, $D(A) \subset D(\mathcal{D})$. Consequently, if $x \in D(A)$ then $\mathcal{D}x = Ax \in X$ and $\mathfrak{B}\mathcal{D}x = BAx$. Also, if $x \in D(A)$, then $Bx = \mathfrak{B}x \in D(A)$. Equation (5.32) can be written as $A\mathfrak{B}x = \mathfrak{B}\mathcal{D}x$, for all $x \in \mathfrak{B}^{-1}(D(A))$, which by the considerations above, is equivalent to $ABx = BAx$, for all $x \in D(A)$. Hence,

$$W(t)Ax = e^{tA}BAx = Ae^{tA}Bx \tag{5.39}$$

for all $x \in D(A)$, and by Theorem 3.7 of [15], $\{W(t)\}_{t\geq 0}$ is a B-semigroup generated by an extension of A. However, since A is the generator of a semigroup, by Proposition 3.9 of [15], $\{W(t)\}_{t\geq 0}$ is generated by A itself. $\square$

These results allow us to prove an interesting observation pertaining to B-bounded semigroups.

Corollary 1.7 If $B : X \longrightarrow X$ is a bounded, one-to-one operator satisfying $BA \subset AB$ and A generates a B-bounded semigroup $\{Y(t)\}_{t\geq 0}$, then the extension of A, $B^{-1}AB$, generates a semigroup in X. If $\rho(A) \neq \emptyset$, then A generates a semigroup in X.

Proof By Theorem 1.4, there is a B-regularized semigroup $\{W(t)\}_{t\geq 0}$ generated by an extension $B^{-1}AB$ of A, such that

$$W(t)x = Y(t)x \tag{5.40}$$

for all $t \geq 0$ and $x \in X$. If $\rho(A) \neq \emptyset$, then $B^{-1}AB = A$ by Proposition 3.9 of [15]. By Theorem 3.5 of [15], this B-regularized semigroup is a mild existence family for $B^{-1}AB$, or A, respectively. From Theorem 1.5, it follows then that $B^{-1}AB$, (or respectively A) generates a semigroup in X. $\square$

Theorem 1.8 Let us assume that $B : X \longrightarrow X$ is a bounded one-to-one operator, image $\operatorname{Im} B$ is dense in X, and let $\{W(t)\}_{t\geq 0}$ be a B-regularized semigroup in X generated by A. Then $\{W(t)\}_{t\geq 0}$ is a B-bounded semigroup iff the semigroup $\{B^{-1}W(t)\}_{t\geq 0}$ extends to a C_0-semigroup on X, generated by A.

Proof By Theorem 3.5 of [15], $\{W(t)\}_{t\geq 0}$ is a strong B-existence family for A. Therefore by Theorem 1.5, A generates a semigroup $\{\exp(tA)\}_{t\geq 0}$ such that

$$W(t)x = e^{tA}Bx \tag{5.41}$$

for all $x \in X$. By Theorem 3.4 of [15], and the definition of B-regularized semigroup, we have

$$BA \subset AB \tag{5.42}$$

and since B is bounded, from the exponential formula for e^{tA} we obtain

$$Be^{tA}x = e^{tA}Bx \tag{5.43}$$

for all $x \in X$. Thus $W(t)x \subset \operatorname{Im} B$ for any $x \in X$, $t \geq 0$ and

$$B^{-1}W(t)x = e^{tA}x \tag{5.44}$$

for all $x \in X$. By (5.42), we have

$$B(\lambda I - A)^{-1} = (\lambda I - A)^{-1}B. \tag{5.45}$$

Indeed, let $y = B(\lambda I - A)^{-1}x$, $x \in X$. Then $B^{-1}y \in D(A)$ and $x = \lambda B^{-1}y - AB^{-1}y$. Equation (5.42) is equivalent to saying that $B^{-1}Ay = AB^{-1}y$ whenever $B^{-1}y \in D(A)$ (and then $y \in D(A)$). This is exactly the condition on y we have above, and therefore $y = (\lambda I - A)^{-1}Bx$. As a result of this, (5.27) can be written as

$$B(\lambda I - A)^{-1}x = B(\lambda I - \mathcal{D})^{-1}x \tag{5.46}$$

for all $x \in X$ and $\lambda > \omega$, and from the invertibility of B we obtain $\mathcal{D} = A$. An application of Corollary 1.6 ends the proof of this part. Conversely, if $\{B^{-1}W(t)\}_{t\geq 0}$ is a C_0-semigroup generated by A, then we define

$$Y(t)x = e^{tA}Bx = Be^{tA}x = W(t)x \tag{5.47}$$

where $\|Y(t)x\| \leq Me^{\omega t}\|Bx\|$ and $t \longrightarrow Y(t)x$ is continuous for any $x \in X$. From the semigroup properties and (5.42), for any $x \in D(A)$, $Bx \in D(A)$ and

$$Y(t)x = Bx + \int\limits_0^t Ae^{sA}Bxds = Bx + \int\limits_0^t e^{sA}BAxds, \tag{5.48}$$

which shows that $\{Y(t)\}_{t\geq 0}$ is the B-bounded semigroup generated by A. $\square$

These questions could be looked at also from the following point of view. Let $\{W(t)\}_{t\geq 0}$ be a B-regularized semigroup generated by A. This means that there are mild solutions $t \to u(t,x)$ originating from $x = By \in D(B)$, that is, there exists an (algebraic) semigroup $\{e^{tA}\}_{t\geq 0}$ such that

$$W(t)y = e^{tA}By \tag{5.49}$$

for all $y \in X$ and $t \geq 0$. This semigroup, however, is not confined to $\operatorname{Im} B$, but in general it could take values in X (more precisely, in some space $Z \hookrightarrow X$).

Only if $x \in \operatorname{Im} B^2$, we obtain $e^{tA}x \in \operatorname{Im} B$. In fact, if $x \in \operatorname{Im} B^2$, then $x = B^2 y$ for $y \in X$. Hence

$$\begin{aligned} e^{tA}x &= B^{-1}W(t)x = B^{-1}W(t)BB^{-1}x = W(t)B^{-1}x \\ &= W(t)By = BW(t)y \in \operatorname{Im} B \quad \text{for any } t \geq 0. \end{aligned}$$

If $\{Y(t)\}_{t\geq 0}$ satisfying (5.15) is a B^{-1}-bounded semigroup, then $\{Y(t)B\}_{t\geq 0}$ is a C_0-semigroup acting in $\operatorname{Im} B$. Unfortunately, requesting only

$$W(t)y \in \operatorname{Im} B \tag{5.50}$$

for all $y \in X$ and $t \geq 0$, or in other words, that $e^{tA} \operatorname{Im} B \subset \operatorname{Im} B$ is not sufficient for $\{W(t)\}_{t\geq 0}$ to be a B^{-1}-bounded semigroup. In such a way $\{e^{tA}\}_{t\geq 0}$ would be a semigroup of bounded operators in $\operatorname{Im} B$ but not necessarily a strongly continuous one. In fact, since each $W(t)$ is a bounded operator and B^{-1} is closed, then $B^{-1}W(t)$ is a closed operator and, being defined on X, it is a bounded operator in X. Thus, for $y = Bx$, $x \in X$,

$$\begin{aligned} \left\|e^{tA}y\right\|_{\operatorname{Im} B} &= \left\|B^{-1}e^{tA}Bx\right\| = \|B^{-1}W(t)x\| \\ &\leq K\,\|x\| = K\,\|B^{-1}y\| = K\,\|y\|_{\operatorname{Im} B} \end{aligned} \tag{5.51}$$

for some constant K.

For $\{e^{tA}\}_{t\geq 0}$ to be strongly continuous on $\operatorname{Im} B$, we see that for any $y = Bx \in \operatorname{Im} B$

$$t \longrightarrow B^{-1}e^{tA}y = B^{-1}W(t)x, \tag{5.52}$$

must be continuous as an X-valued function, that is $t \longrightarrow B^{-1}W(t)$ must be strongly continuous on X.

It follows from Theorem 3.13 of [15], that, provided $W(t)x \in \operatorname{Im} B$, $t \longrightarrow B^{-1}W(t)x$ is strongly continuous iff there is a mild solution of the Cauchy problem for A with the initial value x. Since $x \in X$ is arbitrary then by Corollary 4.11 of [15], A generates a strongly continuous semigroup in X. $\square$

It is interesting and important in some applications (see [11]), to understand what happens when A generates a B-bounded semigroup and at the same time satisfies the Hille-Yosida estimates. We can formulate the following theorem.

Theorem 1.9 Let $\{Y(t)\}_{t\geq 0}$ be a B-bounded semigroup generated by the pair of operators A and B, where $A : D(A) \longrightarrow X$, $B : D(B) \longrightarrow Z$, $D(A)$, $D(B) \subset X$, A satisfies ii) of Definition 1.1 of Section 3.1, and B is a bounded, one-to-one operator. Assume that there exists $M > 0$ such that for all $\lambda > \omega$

and $n \in \mathbb{N}$

$$\|(\lambda - A)^{-n}\| \le \frac{M}{(\lambda - \omega)^n}. \tag{5.53}$$

Then the following hold:

i) The part of A in $\overline{D(A)}$, denoted hereafter by A_0, generates a C_0-semigroup $\{\exp(tA_0)\}_{t\ge 0}$ in $\overline{D(A)}$ and

$$e^{tA_0}x = \left(e^{t\mathfrak{A}} \mid_{\overline{D(A)}}\right) x \tag{5.54}$$

for all $t \ge 0$ and $x \in \overline{D(A)}$, where $\{\exp(t\mathfrak{A})\}_{t\ge 0}$ is the semigroup defining $\{Y(t)\}_{t\ge 0}$. Equivalently, for any $r > \omega$, A generates an $(r - A)^{-1}$-regularized semigroup $\{W(t)\}_{t\ge 0}$ in X given by

$$W(t)x = e^{tA_0}(r - A)^{-1}x \tag{5.55}$$

for all $t \ge 0$ and $x \in X$, and the family $\{e^{-\omega t}W(t)\}_{t\ge 0}$ is bounded and uniformly Lipschitz continuous.

ii) If X has the Radon-Nikodym property, then $\{e^{t\mathfrak{A}} \mid_X\}_{t>0}$ is a strongly continuous exponentially bounded semigroup in X (but not a C_0-semigroup in general-no continuity at $t = 0$), satisfying

$$(e^{t\mathfrak{A}} \mid_X)x = (r - A)e^{tA_0}(r - A)^{-1}x \tag{5.56}$$

for all $t \ge 0$ and $x \in X$.

iii) If X is reflexive or $D(A)$ is dense is X, then A generates a C_0-semigroup in X satisfying

$$e^{tA}x = (e^{t\mathfrak{A}}|_X)x \tag{5.57}$$

for all $t \geq 0$ and $x \in X$.

Proof Without any loss of generality we can assume that $\omega = 0$; then the exponential boundedness should be replaced by the boundedness.

i) The fact that A_0 generates a semigroup in $\overline{D(A)}$ and the corresponding statement for $(r - A)^{-1}$-regularized semigroup follow from Theorem 5.5, 5.10, 5.17 and 17.3 of [15]. To prove equation (5.54), we note that since B is bounded and injective, then $\overline{D(A)} \hookrightarrow X \hookrightarrow X_B$. Using e.g., the exponential formula for the semigroup and the fact that $\mathfrak{A}$ is an extension of A_0 we obtain for $x \in \overline{D(A)}$

$$\begin{aligned} e^{tA_0}x &= \lim_{n\to\infty} \left(I - \tfrac{t}{n}A_0\right)^{-1} x \\ &= \lim_{n\to\infty} \left(I - \tfrac{t}{n}\mathfrak{A}\right)^{-1} x \qquad (5.58) \\ &= e^{t\mathfrak{A}}x = e^{t\mathfrak{A}}|_{\overline{D(A)}}\, x. \end{aligned}$$

ii) Part of the second statement has been proved (see e.g., [2], where it has be proved using the theory of integrated semigroups). For the sake of completeness we provide a proof involving the theory of C-regularized semigroups.

Let us consider the $(r-A)^{-1}$-regularized semigroup given by equation (5.55)

$$W(t)x = e^{tA_0}(r-A)^{-1}x. \tag{5.59}$$

In particular $(e^{tA_0})_{t\geq 0}$ is a family of operators bounded in the X-norm. From [15], Theorem 17.3, $\{W(t)\}_{t\geq 0}$ is bounded and uniformly Lipschitz continuous, and since X has the Radon-Nikodym property, for any $x \in X$, $t \longrightarrow W(t)x$ is differentiable almost everywhere with a measurable and bounded derivative. Next from Proposition 2.7 of [15], if $W'(t_0)x$ exists, then $W(t_0)x \in D(A)$ and $W'(t_0)x = AW(t_0)x$. Since $W'(t)$ exists almost everywhere, for each $t > 0$, there is $t_0 < t$ such that $W'(t_0)x$ exists. Let us consider for any $t > 0$ and $y = (r-A)^{-1}x$

$$\begin{aligned}
\frac{W(t+h)x - W(t)x}{h} &= \frac{e^{(t+h)A_0}y - e^{tA_0}y}{h} \\
&= e^{(t-t_0)A_0}\left(\frac{e^{(t_0+h)A_0}y - e^{t_0A_0}y}{h}\right) \qquad (5.60) \\
&= e^{(t-t_0)A_0}\left(\frac{W(t_0+h)x - W(t_0)x}{h}\right).
\end{aligned}$$

Since the expression in brackets converges in X, the derivative $W'(t)x$ exists by continuity of the semigroup and again by Proposition 2.7 of [15], we obtain

$$W'(t)x = AW(t)x,\ t > 0. \tag{5.61}$$

Hence $AW(t)x$ is well-defined for any $x \in X$ and since $W(t)$ is bounded and A is closed, $\{AW(t)\}_{t\geq 0}$ is a family of bounded operators by the Closed Graph Theorem. In particular we obtain from the Banach-Steinhaus theorem that the family $\{AW(t)\}_{t\geq 0}$ is bounded. Consider now $\{T(t)\}_{t>0} = \{(r-A)W(t)\}_{t>0}$, which is strongly measurable, bounded family of bounded linear operators on X. Moreover,

$$\begin{aligned} T(t)T(s)x &= (r-A)e^{tA_0}(r-A)^{-1}\left((r-A)e^{sA_0}(r-A)^{-1}\right)x \\ &= (r-A)e^{tA_0}e^{sA_0}(r-A)^{-1}x \qquad (5.62) \\ &= T(t+s)x. \end{aligned}$$

Hence $\{T(t)\}_{t\geq 0}$ is a semigroup and by [19, Theorem 10.2.3], it is a strongly continuous semigroup for $t > 0$. Next, from i), we know that $e^{tA_0} = e^{t\mathfrak{A}}\mid_{\overline{D(A)}}$.

Thus we have

$$\begin{aligned} T(t)x &= (r-A)e^{tA_0}(r-A)^{-1}x \\ &= (r-\mathfrak{A})e^{t\mathfrak{A}}(r-\mathfrak{A})^{-1}x \qquad (5.63) \\ &= e^{t\mathfrak{A}}x = \left(e^{t\mathfrak{A}}\,|_X\right)x. \end{aligned}$$

iii) If the space X is reflexive, then any operator satisfying the Hille-Yosida estimates is densely defined. For densely defined operators the statement of this point follows from the standard Hille-Yosida theorem (existence of $\{e^{tA}\}_{t\geq 0}$) and equation (5.58) with A_0 replaced by A. $\square$

Conclusion

The theories of B-bounded semigroups, C-regularized semigroups, C-existence and uniqueness families are generalizations of the theory of strongly continuous semigroups. In this thesis we provide an analysis of the inter-relationship that exists between these theories and show that, despite superficial similarities, the objects they describe are essentially different.

While introducing the concept of B-bounded semigroups we revisit the transport problem from where they originated and in the process we extend the original results of Belleni-Morante (who considered only purely multiplying boundaries; purely absorbing being standard), to cover the mixed, that is, partly multiplying and partly absorbing, case.

Further we present a development of the theory of B-bounded semigroups and demonstrate their usefulness in solving certain types of implicit evolution equations by considering an example of the heat equation with dynamical boundary conditions. In this example we also compare B-bounded semigroups with B-evolutions introduced by N. Sauer.

The main part of the thesis is concluded by a study of the inhomogeneous initial value problem in the context of C-regularized semigroups. We show that the results presented in the monograph "Semigroups of linear operators and applications to partial differential equations" by A. Pazy for strongly continuous semigroups apply, with minor modifications, to the case when the evolution is given only by a C-regularized semigroup. The only gap between these two theories that we could not fill, follows from the fact that the concept of fractional powers of the generator of a C-regularized semigroup is yet to be developed to a satisfactory degree. Thus the results from Pazy that involve the regularity of the data with respect to the domains of the fractional powers of strongly continuous semigroups, have no counterparts in the C-regularized semigroups context.

References

[1] W. Arendt, Resolvent positive operators, Proc. Lond. Math. Soc. 3, 54, 321 – 349 (1987).

[2] W. Arendt, Vector-valued Laplace transforms and Cauchy problems, Israel J. of Math., 59, 3, 327 – 352 (1987).

[3] L. Arlotti, On the B-bounded semigroups as a generalization of the C_0-semigroups, Zeitschrift für Analysis und ihre Anwendungen, 19, 23 – 34 (2000).

[4] L. Arlotti, A new characterization of B-bounded semigroups with application to implicit evolution equations, Abstract and Applied Analysis, 5, 4, 227 – 244 (2000).

[5] J. Banasiak, Generation results for B-bounded semigroups, Annali di Matematica pura et applicata, IV, CLXXV, 307 – 326 (1998).

[6] J. Banasiak, B-bounded semigroups and implicit evolution equations, Abstract and Applied Analysis, 5, 1, 13 – 32 (2001).

[7] J. Banasiak, B-bounded semigroups, existence families and implicit evolution equations, Progress in nonlinear differential equations and their applications, Birkhäuser Verlag Basel/Switzerland, 42, 25 –34 (2000).

[8] J. Banasiak and V. S. Singh, B-bounded semigroups and C-existence families, Taiwanese J. of Math., 6, 1, 105 –125 (2002).

[9] A. Belleni-Morante, A concise guide to semigroups and evolution equations, World Scientific, Singapore (1994).

[10] A. Belleni-Morante and S. Totaro, The successive reflection method in three dimensional particle transport, J. Math. Phys. 37, 6, 2815 – 2823 (1996).

[11] A. Belleni-Morante, B-bounded semigroups and applications, Annali di Matematica pura ed applicata, IV, CLXX, 359 – 376 (1996).

[12] E. B. Davies and M. M. H. Pang, The Cauchy problem and a generalization of the Hille-Yosida theorem, Proc. Lond. Math. Soc. 55, 3, 181 – 208 (1987).

[13] R. deLaubenfels, Holomorphic C-existence families, Tokyo Math. J., 15, 1, 17 – 38 (1992).

[14] R. deLaubenfels, C-semigroups and strongly continuous semigroups, Israel, J. of Mathematics, 81, 227 – 255 (1993).

[15] R. deLaubenfels, Existence families, Functional Calculi and evolution equations, Lecture Notes in Mathematics 1570, Springer Verlag, Berlin (1994).

[16] K.-J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, Springer Verlag, New York (2000).

[17] H. O. Fattorini, The Cauchy problem, Encyclopedia of Mathematics and its applications, Addison-Wesley Publ. Co., 18, U.S.A (1983).

[18] J. A. Goldstein, Semigroups of linear operators and applications, Oxford University Press, New York (1985).

[19] E. Hille and R. S. Phillips, Functional Analysis and semigroups, American Mathematical Society, Colloquium Publications, v. 31, Providence, Rhode Island (1957).

[20] H. Kellerman and M. Hieber, Integrated semigroups, J. of Functional Analysis, 84, 160–180 (1989).

[21] E. Kreyszig, Introductory Functional Analysis with applications, John Wiley and sons, New York (1989).

[22] V. Melnikova and M. A Al'shansky, Well-posedness of the Cauchy problem in a Banach space: regular and degenerate cases, Journal of Mathematical Sciences, 87, 4, 3732 –3779 (1997).

[23] J. R. Mika and J. Banasiak, Singularly perturbed evolution equations with applications to Kinetic theory, World Scientific, Singapore (1995).

[24] F. Neubrander, Integrated semigroups and their applications to the abstract Cauchy problem, Pacific J. of Mathematics, 135, 1, 111 – 155 (1988).

[25] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer Verlag, New York (1983).

[26] G. Da Prato, Semigruppi regolarizzabili, Ricerche di Mat. 15, 223 – 248 (1996).

[27] W. Rudin, Functional Analysis, McGraw-Hill Book Co., New York (1973).

[28] N. Sauer, Linear evolution equations in two Banach spaces, Proc. Royal Soc. Edin., 91A, 387 – 303 (1982).

[29] N. Sauer, Empathy theory and the Laplace transform, in : E. Janas et al. (Eds), Linear operators, Banach Center Publ., Polish Academy of Sci., 38, Warsaw, 325 – 338 (1997).

[30] R. Showalter, Monotone operators in Banach spaces and nonlinear partial differential equations, Mathematical surveys and Monographs, 49, American Mathematical Society, Rhode Island (1997).

[31] G. A. Sviridyuk, On the general theory of operator semigroups, Russian Math. Surveys, 49, 4, 45 – 74 (1994).

[32] N. Tanaka, On the exponentially bounded C-semigroups, Tokyo J. Math., 10, 1, 107 – 117 (1987).

[33] N. Tanaka, Linear evolution equations in Banach spaces, Proc. Lond. Math. Soc. 63, 657 – 672 (1991).

[34] E. Zeidler, Nonlinear Functional Analysis and its applications, v. II/A, Springer Verlag, New York (1990).

[35] A. Belleni-Morante, On some properties of B-quasi bounded semigroups and applications, Quaderno sezione modelli Matematici n° 1/96, Dipartimento di Ingegneria Civile, Universitá di Firenze.

[36] A. Belleni-Morante, Applied Semigroups and Evolution Equations, Clarendon Press, Oxford (1979).