AN ANALYSIS OF LEARNERS' ENGAGEMENT IN MATHEMATICAL TASKS

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ABSTRACT

The present project is part of a larger research programme focussed on the analysis of change; one aspect being educational transformation and in particular an emphasis on the explication of the contentless processes (eg. logical operations, reasoning styles, analysis and synthesis) which underlie both learning and teaching at university level. The present project is aimed at an analysis of the teaching-learning dialectic in mathematics courses. This analysis has two major focal points, that is, making explicit the often tacit and mostly inadequate and/or inappropriate rules for engaging in mathematical tasks which the under-prepared learner brings to the teaching-learning situation, and secondly the teaching strategies which may enable these learners to overcome their past (erroneous) knowledge and skills towards the development of effecient, autonomous mathematical problem-solving strategies. In order to remedy inadequate and inappropriate past learning and/or teaching, the present project presents a set of mediational strategies and regulative cues which function both for the benefit of the teacher and the learner in a problematic teaching-learning situation and on the meta and epistemic cognitive levels of information processing. Furthermore, these mediational strategies and regulative cues fall on a kind of interface between contentless processes and the particular content of the teaching-learning dialectic of mathematics in particular, as well as between

the ideal components of any instructional process and the particular needs and demands of under-prepared learners engaged in mathematical tasks.

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CHAPTER 1

INTRODUCTION

1.1 The problem described

By adolescence, the normal person is capable of what Piaget termed formal operations (logico-mathematical thought), or thought which is logical, abstract and flexible (cf. p. 35, section 3.3). Logico-mathematical thought is essential for the engagement in mathematical type and other complex problem solving tasks. Although all normal persons of the same age possess the same mental capacity[1] to make their engagement in age - appropriate problem - solving possible, many learners at university level still fail to engage independently and effectively in mathematical tasks[2].

Mathematics at university is especially a problem for black students. Only a small percentage (less than 10 %) of black matriculants entered first year mathematics courses at the University Natal in 1988 and at most 40% of those who entered managed to pass these mathematics courses.

This state of affairs emphasizes the problem of why " *some students*

[1] Mental capacity in Pascual-Leone's terms refers to the maximum number of 'knowledge units'(schemes) that an individual can simultaneously use while performing a task (cf. section 3.4 on Pascual-Leone's Theory of Constructive Operators, p. 47). In Chomsky's terms capacities refer to both the competencies of a subject which produce performance, and the performance of a subject of knowledge on a task. Competencies refer to underlying cognitive structures which include formal operations (Piaget, cf. section 3.3, p. 35) and executive schemes (Pascual-Leone, cf. section 3.4, p. 48).

[2] Whether or not the learners included in the present project are functioning at the level of formal operations is not addressed here.

at university are unable to engage in typical university tasks successfully-or in a manner which has come to be regarded as appropriate for tertiary education and consistent with the cognitive capacities expected from late adolescence onwards" (cf. Craig, 1988b).

Some reasons for the black learners' poor performance on mathematical tasks and at university in general is provided by Van Den Berg (1978):

(a) In all subjects there is a lack of academic text-books written in the mother tongue as well as literature in general.

(b) The black man is expected to be fluent in one of the official languages of the RSA. However, the danger of learning through the medium of second language lies in the tendency to learn in a parrot-like fashion without much thought or understanding[3]. Mathematical 'story problems' are presented to black learners in english, which makes it difficult for them to extract the pure mathematical meaning of these problems.

(c) The black learners' cultural background does not encourage the black learners' cognitive development of formal operations.

If one compares the cognitive capacities which are expected by late adolescence, and the performance of students on university tasks and the

[3] Orr (1987) shows that learners have trouble understanding mathematics because they are unfamiliar with the basic cultural terms. She focuses on Black English Vernacular's use of different parts of speech that establish semantic relationships (prepositions, pronouns etc.) and demonstrates how fundamental differences between their use in Network Standard English make it difficult for speakers of Black English Vernacular to distinguish between addition and multiplication, between subtraction and division. In section 5.2 (p. 89) we show that this difficulty may also be as a result of overlearned rules.

demands of those tasks, one has to conclude one of the following (Craig, 1988b):

All students have the appropriate cognitive capacities to fulfill the demands of university tasks, but *the learning-teaching situation does not elicit these competencies and/or performances*, and in some cases

some students do not have the appropriate cognitive competencies to meet the demands of tertiary education.

One may, therefore, propose one of the following ways to confront the apparent mis-match between what (some) students bring to the learning-teaching situation, and what the university in general demands in terms of its 'standards of success' (cf. Ibid.):

(1) Change the learning-teaching situation (or aspects of it) so that it will develop or elicit appropriate performances;

(2) Change the university in general to match the skills/knowledge/interests of those students who, at present, fail to meet course and degree requirements;

(3) Only allow those students entry into university whose academic performance match the competence assumed by typical university tasks.

Each of these options has its adherents and critics, but what is common to all three is either an explicit or implicit agreement that *some students do not 'have what it takes' or 'display what makes'* adapting to, benefiting from, and excelling in typical university tasks in general, and mathematical tasks in particular, possible. Rather than enter debates about what will constitute an appropriate response politically, it seems more appropriate to analyze the typical university tasks/performances and the competencies assumed or required or implied by them.

Van Den Berg's analysis, included above, provides some overview of the problems the black learner may experience in formal (tertiary) education. The black learner is under-prepared; not only for school, but also for university and for mathematical tasks in particular. It is, of course, not only black South Africans who are under-prepared for university type tasks: Many other learners who enter university are under-prepared to meet the demands of university tasks/situations. Since black learners are discriminated against in the South African education system and because their general socio-political oppression is so apparent, it seems justifiable to focus primarily on the black learner at this stage.

The present project is part of a larger research programme aimed at an analysis of the *contentless processes*[4] which underlie both learning and teaching at university level. In particular, the present project is focused on the learning-teaching dialectic in mathematics courses.

Van Den Berg (1978) concluded that a solution to the problem of im-

[4]These processes refer to cognitive abilities such as logical operations, reasoning styles, analysis, comparison of parts and the synthesis of parts to a whole (cf. Gellatly, 1987). Piaget's theory (cf. section 3.3 p. 32) and Pascual-Leone's theory (cf. section 3.4 p. 42) are considered contentless theories in the sense of emphasizing logico- mathematical operations (from about puberty onwards) which will be applied to any tasks -mathematics or music, psychology or english, etc.

proving the mathematical competence of black people would have to be sought in the realm of the teaching-learning situation (cf. also Craig, 1988(b)) .

Craig (1988b) suspects that most teaching at university is done on what she calls the *spontaneous model of learning*, that is, the learner is exposed to a flood of content in the hope that 'mind' will develop the necessary contentless processes such as reasoning styles and logical operations spontaneously. Leaving 'mind' to develop 'what it takes' and 'what makes' adaptation to adult learning tasks possible usually suffice, but when 'it' has not developed or is not expressed as in the case of under-prepared students, we may have to attempt something else than follow the 'spontaneous model of learning' (cf. Ibid.).

An analysis of learners' engagement in mathematical tasks will be presented. The aim of this analysis is to direct one's attention to the needs of those learners[5] who come to the university under-prepared to meet the demands of typical university mathematical type tasks. More precisely,

[5] When matriculants are accepted into university mathematics courses it is because they have obtained enough 'points' from the grading of their matric results. This is supposed to ensure that they will be 'competent' to perform university mathematical tasks. But the high failure rate of first year university mathematics courses indicates the unreliability of matriculation results. Since more time than bridging courses are designed for will be necessary to identify and assist those learners who are not yet functioning at the formal operation stage, at this stage of this research it seems more appropriate to concentrate on those under-prepared learners who have reached the stage of formal operations. It will be part of future research to devise an effective ways of determining whether (1) a learner is sufficiently competent in performing formal operations to benefit from mathematical 'bridging' material prior to university entry, (2) a learner needs a reduced first year plus additional support, and (3) a learner is ready to enter the normal first year.

this project will concentrate on reconstructing those, often tacit, inadequate and inappropriate, rules which the under-prepared learner brings to the teaching- learning situation and which he/she mostly applies spontaneously onto unfamiliar mathematical tasks. Reconstructing these tacit rules is done in order to develop the necessary educational 'scaffolding' for the mastering of typical mathematical tasks. The scaffolding will consist of mediational strategies and regulative cues (see section 2.1, p. 16) which function on different levels of cognitive processing and which are meant as 'links' between content and contentless moments in instruction and also between learner and teacher. In general, the regulative cues are meant as 'interruption rules' for the learner to monitor his/her progress on mathematical tasks, and mediational strategies as guides for the teacher in his/her provision of 'other regulation' (Vygotsky, 1978). Both the mediational strategies and regulative cues are aimed at the successful mediation of the task towards the learner's development of efficient, autonomous self-regulation on complex problem solving tasks (see section 3.2 p. 29).

The subjects (N=175) who were used in this project were:

(A) First year science students registered for mathematics 1 at University of Natal (N=150).

(B) Students of the Engineering Bridging Unit at the University of Natal (N=25).

(C) Three learners were used for video recordings: a good student, an average student and a poor student.

1.2 Pilot project

The data base for this pilot project consists of material which was obtained by transcribing video recordings of three learners: (1) A good student, (2) an average student, and (3) a poor student (cf. Appendix C, p. 159).

While the video recordings were being viewed, it became clear that the 'weaker' the learner, the more hints, regulatory cues and other instructional prompts had to be provided by the teacher. In the case of the average and poor student, the teacher's instructional prompts did not seem to become sufficiently internalized by the learner to become self regulation. The good student required less cues from the teacher and tended to develop self-regulatory principles spontaneously. The pilot project, therefore, suggested the internalization of regulatory cues as an important aspect of the learning/teaching of mathematics.

The problems presented to the good and poor student involved integration by parts (e.g. Evaluate the following integral: $\int e^x.xdx$). The epsilon/delta definition of a limit was the subject of study during the interaction of teacher with the average learner (for example, the learner was asked to evaluate the following limit using the ϵ, δ method: $\lim_{x\to 3}(2x+1)$). These problems were chosen because they typically presented difficulties for students.

Poor student

The major difficulty that the poor learner had (and which most under-prepared learners have) was trying to identify which function had to be differentiated and which function had to be integrated (cf. pp. 210, 211). The standard form presented to the learner was:

$$\int (du \times v) = uv - \int (u \times v')$$

The learner was required to apply the rule to the puzzle[6]:
Evaluate $\int (e^x \times x)$.

The learner had to compare the standard form with the puzzle and select which function was to be 'du' and which function was to be 'v'. However, the poor student had great difficulty in *comparing* the two integrals (cf. pp. 210, 211). The nature of the puzzle seemed to be the main cause for the learner's confusion as the 'pure form' had to be extracted out of the puzzle, the pure form of the puzzle being as following:
Evaluate $\int (du \times v)$ where $du = e^x$ and $v = x$.

However, the poor learner could not regard du as a *whole* function which interfered with his comparing of the two integrals (i.e. the integrals on the left hand side of the standard form and the puzzle). Thus what the learner

[6]The word 'puzzle' will be discussed in detail in section 4.2, p. 57. By way of introduction, a puzzle can be defined as a problem where there is only one correct solution and where the solution is guaranteed by using a specific procedure. Central to the present project is differentiating among different *kinds* of puzzles, for example, standard and pure forms. This classification of puzzles is presented in Chapter 4.

needs to know[7] in order to master the given mathematical task (and which the teacher should provide for the learner) is that the original standard form can be expressed in a more acceptable (to under-prepared learners) new form:

$$\int (\square \times \triangle) = (\square)^I \times \triangle - \int (\square)^I \times (\triangle)'$$

where $(\square)^I = \int(\square)dx$ and $(\triangle)' = \frac{d}{dx}\triangle$.

This new form of the standard form contains the integral of the product of two *whole* forms- namely- $\square$ and $\triangle$. The learner can then easily identify the two functions after the $\int$ sign in the puzzle (namely, e^x and x) with these two whole forms in the new form; i.e. $\square = e^x$ and $\triangle = x$. The learner now has to find $(\square)^I$ and $(\triangle)'$ which can be substituted into the right hand side of the new form, etc.

It is important to note that the above mediational strategy (the $\square$ and $\triangle$ form of integration by parts) had already been developed prior to the video recordings (although it was not used while teaching the poor learner) but had come about over a long period of time through trial and error. If the video recordings had been available before these trial and error attempts, viewing of the video recording would have pointed to the learner's difficulty

[7] What the learner needs to know in order to become an efficient, autonomous mathematical problem solver is a major emphasis in the remedial steps proposed in Chapter 5. Such remedial steps involve what is here termed regulative cues and mediational strategies (cf. p. 17) and are aimed at resolving the conflict between the inadequate and inappropriate knowledge and skills under-prepared learners bring to the teaching-learning situation and the demands of mathematical tasks during the first stages of tertiary education.

immediately and the appropriate mediational strategy would have been formulated without the unnecessary 'hit and miss' attempts. This suggests that video recording of students engagement in mathematical tasks may be an important or useful methodology for the necessary task analysis of the learning-teaching situation to be undertaken in all disciplines, mathematics included (cf. Craig, 1985).

Average learner

In viewing the video recording of the average learner it was noted that the learner often 'thought' she understood the teacher (cf. pp. 177, 188, 195, 201) and the teacher often had to *interrupt* her to guide her onto the right path.

The video recordings highlighted an important fact concerning the teaching-learning dialectic involving mathematics (or any object of knowledge). Once the teacher has provided the learner with those rules which are required for a mathematical task, and leaves the learner on his/her own, the learner has to rely on those rules which he/she has internalized to monitor and guide his/her own independent progress during the performance of the given mathematical task. At university it is often assumed that these rules will develop 'spontaneously' while the learner is at university, and we all well know that this does not take place, especially in the case of under-prepared learners in general and black learners in particular.

It is, therefore, important to make explicit those regulatory cues and monitoring rules which are necessary for learners to be competent doers of mathematical tasks.

Good student

In the video recording of the good learner, it was observed that the learner had no difficulty in comparing the integral on the left hand side of the 'standard form' ($\int(du \times v)$) with the integral in the 'puzzle' ($\int(e^x \times x)$) (see p. 165). He immediately saw that $du = e^x$ and $v = x$. It was possible for him to see du as a *whole* and not as two separate functions (d and u). In comparing the good with the poor student we see that the good student had internalized the fact that "when the integral sign and 'd' occur next to each other they 'canceled each other out' so that only the function remains" (e.g. $\int d(e^x) = e^x$) (cf. pp. 163, 165) while the poor learner needed constant reminding about this rule (cf. pp. 203, 207, 212). His preoccupation with trying to compare the standard form with the puzzle hindered the internalization of this rule and, therefore, prevented him from independently regulating his progress.

1.3 Overview of contents

In Chapter 2 the methodology which underlies this research project is discussed and the research paradigm is presented graphically (Fig. 1B, p. 15). This paradigm is an adaption of the methodological framework developed through and for the larger project (cf. Fig. 1A, p. 14).

Expert theories related to cognitive development (eg. Piaget, Pascual-Leone, Vygotsky), adult cognitive processing (eg. Kitchener) and mathematical problem solving (eg. Krutetskii) are reviewed in Chapters 3 and 4. The nature of mathematical problems encountered by learners has a strong influence on the learners' ability to successfully deal with them. It is in Chapter 4 that a possible classification of mathematical problems encountered by learners at first year university level is presented. This classification forms part of the mediational strategies developed in this research project.

Learners' engagement in mathematical tasks is presented in Chapter 5 together with those expert theories which inform the analysis of this task engagement. Juxtaposing the under-prepared learners' tacit rules and the ideal rules for doing mathematical tasks, the emphasis in this analysis falls on the regulative cues and mediational strategies which may bridge the chasm between different realities meeting in the teaching-learning situation.

In the final chapter the results of this project are summarized and discussed in terms of future research plans.

The whole thesis, unless specifically indicated to the contrary in the text, is my own original work.

ORDER C THEORY OF CHANGE

↑ ↑

ORDER B INDIGENOUS THEORY OF CHILDHOOD MEDIATIONAL OPERATORS

↑ ↑ ↑ ↑

ORDER A $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$

1	2	3	4
Ordinary rules for being	Ordinary ways of doing tasks	Ideal ways of doing tasks	Expert rules for being

↑

Engagement in tasks

Fig. 1A

THE STUDY OF THE POSSIBILITY OF CHANGE (CRAIG, 1985)

ORDER C MEDIATIONAL STRATEGIES

↑ ↑

ORDER B TACIT RULES FOR DOING MATHS TASKS IDEAL RULES FOR DOING MATHS TASKS

↑ ↑

ORDINARY RULES FOR BEING EXPERT RULES FOR BEING

↑ ↑

↑

ENGAGEMENT IN MATHS TASKS

↑

ORDER A PRAXIS

Fig. 1B

THE STUDY OF THE DOING OF MATHEMATICS

CHAPTER 2

METHODOLOGY

2.1 Research plan outlined

The research plan for the larger project of which this is a part may be understood in terms of Fig 1A: The study of the possibility of change (p. 14). For the present project, this model has been adapted to the particular emphases in the study of students' engagement in mathematical tasks (see Fig 1B, p. 15)

The research paradigm (cf. Fig. 1B) emphasizes different orders of analysis, as follows:

Order A: Engagement in mathematical tasks

(1) The learners were given written work which was examined for 'typical' errors (cf. Appendix A, p. 129).

(2) The learners were also given a schedule to respond to where the aim was to record their ability to solve a mathematical problem in relation to their conscious awareness of the rules required to solve the problem (cf. Appendix B, p. 143).

Order B: The reconstruction of rules for doing maths

(1) The learners' engagement in mathematical tasks was used as data to reconstruct their often tacit and mostly inadequate and inappropriate rules for doing mathematics.

(2) The author, as a competent doer of mathematics, used his own engagement in the tasks outlined under order A above, as well as expert theories on cognitive development, adult cognitive processing and theories of mathematics, as further data bases from which to reconstruct the ideal rules for doing mathematics. Once these two bodies of 'rules' were reconstructed, the emphasis in analysis shifted to what the learner <u>needs to know</u> in order to master the demands of mathematical tasks. The tacit rules of the under-prepared learner and the ideal rules of the expert, therefore, became vantage points from which to formulate those regulative cues and mediational strategies (cf. Order C below) which may bridge the gap between:

(a) What the under-prepared learner brings to the teaching- learning situation and

(b) What the task demands in terms of its constitution and the rules inherent to it.

Order C: <u>Regulative cues and mediational strategies</u>

(1) The regulative cues are those instructions (verbally given or contained in written materials) which are content bound and meant to prevent the learner form spontaneously, and unconsciously, applying his/her tacit rule onto mathematical tasks. The important point is, these regulative cues, even though they are initially given by a teacher, are aimed at being used by the learner to monitor his/her own progress during problem

solving. These cues are, therefore, presented as guides to the teacher of mathematical tasks in order to provide appropriate other regulation for the under-prepared learner; other regulation which will, through internalization, become self regulation.

(2) The mediational strategies are those content bound moments during the process of instruction which are informed by the particular demands of mathematical tasks on the under-prepared learner as analysed in this project. Included in the present project are the following two mediational stategies:

(a) Providing the necessary regulative cues for self regulation, and

(b) imposing a classification on the different kinds of puzzles encountered to guide the learners' confrontation with stimuli from the teaching-learning situation.

It is important to point out that both the proposed regulative cues and mediational strategies are provisional in the sense that further elaboration of these as well as the development of more of each is envisaged in further empirical work.

2.2 Analysis of data

The tacit rules for doing mathematics, the ideal ways of doing mathematics and the expert formulations of cognitive development, adult cognitive processing and theories of mathematics all confront the researcher as bits and pieces of data : a kind of 'puzzle' consisting of the learning-teaching dialectic, learners, the able and the under-prepared, and teachers engagement in mathematical problem solving which begs analysis and synthesis.

The research paradigm adopted necessitates making explicit the process of analysis in order to underline the rigor and care[1] which is required by a rational reconstruction of the generative mechanisms[2] for overt patterns of actions:

In the process of analysis of an element or body of data, the process proceeds between what may be termed *occasions for surprise*, and their *resolution*, and so on, until *coherence* is achieved. In the process of data analysis undertaken in this present project, the occasions for surprise arose when the mathematics learners either produced incorrect solutions to problems presented to him/her, or failed to produce any solutions at all. Such actions were then analyzed in terms of two frameworks or bodies of knowledge; that is in terms of (1) the learners (tacit) rules for doing mathematics (which were often incorrect, misleading or absent) and (2) the ideal ways of doing mathematics. The data (1) and (2) above obtained in order A (Fig. 1B) may be called 'raw' in the sense of being un- processed; out of this raw

[1] Part of the problem inherent to research such as the larger project and the present part of this larger one is that creative research is necessary to prevent the compilation of *descriptive data* which does not penetrate the obvious.

[2] Generative mechanisms is defined as structures or functions intrinsic to an object or organism which produce that object's or organism's manifest forms (cf. Craig 1985).

data the researcher constructs explanations of the phenomenon of interest (order B), in this case an explanation of that which underlies learners performance on mathematical tasks, i.e. tacit rules, etc. The analyses that are performed on the data at orders B could be called (after Carnap, 1967) 'rational reconstruction'. This may be defined as follows:

> ... It is a description of the essential features of situations in which <u>such</u> an event <u>could</u> occur: it is a story of how something could happen, and, when human actions are concerned, of what is the rationale of its happening that way, not of what did actually take place (Ullmann-Margalit, 1977, p. 1).

<u>Tacit rules for doing mathematics</u>

The tacit rules, and the reasons for them, which the learners used to do the problems incorrectly or not at all were arrived at by scrutinizing: their attempts to do the problems, the nature of the mathematical problems presented to them, the data from the schedule (cf. Appendix B, p. 143) and video recordings (Appendix C, p. 159) as well as using ideas from the body of knowledge from previous research in the area of under-prepared learners, (cf. Kok and Beinhart 1983, Mindry 1984, Craig 1985, Campbell 1985, Kok 1986, and Juckes 1987) which is part of the larger project.

The tacit (often misleading and incorrect) rules for doing mathematics (and used by most under-prepared mathematics learners at university) are often 'inherited' from the rules which have been deliberately taught during informal and formal arithmetic and later mathematics learning/teaching, but are also derived from real life experience. Cultural experiences of under-prepared (especially black African) learners often differ from that of the able mathematics learner (often western and middle class) (cf. Van Den Berg, 1978).

One of the main difficulties of under-prepared learners is their inability to judge the truth of an argument based solely on its logical structure. They rather judge the truth in terms of the content itself- their real world experiences interfere with their judgment (cf. Krutetskii, 1976). In Gellatly(1987) it is stated that for under-prepared learners "real world information overrides purely logical considerations" (p. 37). In mathematics it is crucial to evaluate a mathematical statement in terms of its logical format. Only the information present in the statement must be used in making logical conclusions and real world assumptions will interfere with its understanding.

Many of the tacit rules for doing mathematics adopted by under-prepared learners are derived from their 'rules for being', i.e. the general rules which they abide by in their expression of their own particular humanity (cf. Craig, 1985 & Van Den Berg, 1978). For example, the learner regards the letter x of the alphabet as 'fixed', and find it difficult to regard it as variable when it occurs in a mathematical problem.

The basic arithmetic rules which learners learn at school, namely, addition, subtraction, division, multiplication, finding the common denominator, 'doing the same thing to both sides of the equation' etc., may be regarded as being internalized (via overlearning) to such a degree that they have become part of the mathematics learner's tacit rules for doing mathematics and often prevent the learner from successfully engaging in university type mathematical tasks (see section 5.2, p. 87).

Engagement in activities.

The methodological paradigm in which the larger project and the present one should be located have as their datum line *praxis* (Craig, 1988b). Learners'*goal directed engagement in activities* is, therefore, in each project within the larger one the primary empirical focus. Recording the events and experiences of learners' engagement in tasks (mathematical tasks/problem solving situations in the present case) could be viewed from at least two perspectives:

(1) The learner's own point of view.

(2) From the point of view of an ideal observer (or expert).

The discrepancy between the teacher's world of mathematics and the students understanding of mathematics provides an interactional situation that most clearly highlights the regulatory mechanisms that allow for mathematical problem solving. The magnitude of this discrepancy is important, as a large gap between task and/or teacher's goals/strategies and individual capacities, skills, knowledge may make it exceedingly difficult for an individual to engage meaningfully in any problem solving situations, mathematical tasks included. It is necessary in the case of under-prepared learners to provide that which will *bridge* the gap between what the learner brings to the situation and what the mathematical task/situation entails, i.e. the proposed regulative cues and mediational strategies.

Expert rules for doing mathematics.

Many theorist emphasize the idea of regulatory or generative mechanisms as central to the explanation of manifest phenomena. In terms of the methodological principle established in the present project, it is important to elaborate on the possibility of using these expert formulations of the regulatory mechanisms that operate in cognitive development as one vantage point from which to study the doing of mathematics (cf. Piaget, section 3.3 p. 32 and Pascual-Leone, section 3.4 p. 42).

Expert formulations are the 'rules of being' which the experts have abstracted from their analysis of various aspects of life and formalized into theories. Expert theories which are relevant to this research report include those theories which relate to *the nature of adult cognition* and *the nature of mathematical problems* mentioned above.

Using these expert formulations as one vantage point, among others, from which to analyze the doing of mathematics involves an important analytical principle (cf. Craig, 1985). Were the mathematics learners to become *mathematics experts*, they would formulate 'rules of doing mathematics' similar to those contained in the various theories of mathematical problem solving (cf. Krutetskii, 1976 and Chapter 4, p. 56). In other words, in making explicit both learner's tacit rules, *and* expert rules for doing mathematics, the research is in effect sketching a domain for development -from tacit (often misleading) rules to effective explicit rules, through teaching.

CHAPTER 3

THEORIES WHICH INFORM THE ANALYSIS OF THE DOING OF MATHEMATICS

3.1 Cognition and culture: a cultural relativistic point of view ?

In chapter 1 the role of language and/or culture in determining the black person's ability to engage in university type tasks was emphasized (cf. Van Den Berg 1978). Are there culturally determined variations in basic cognitive structures, and hence in performance ? Cultural relativism implies that there are. Cultural relativism is based on the assumption that cognitive development can be explained as a function of learning alone without any recourse to the notion of a universal aspect of human development common to all people regardless of variations in culture (cf. Campbell, 1985)

A prominent contemporary expression of cultural relativism is the work of the Laboratory of Comparative Human Cognition or LCHC (1982). They assume that all children, regardless of culture, begin their lives with uniform universal cognitive ability. Thereafter their cognitive development is determined predominantly by the learning experiences provided by their particular culture (cf. Ibid.).

This domain-specific theory of learning is similar to the old- fashioned behaviorist S-R approach to cognition. They argue that generalized response tendencies derive not from 'central processor' cognitive structures such as those postulated by Piaget, but from common features shared by the current task and the previous contexts (cf. Campbell, 1985). Context-specific intellectual achievements become the basis for cognitive development. Development is seen by the cultural relativists as the acquisition of increasingly generalized rules that apply to a progressively larger set of specific domains of experience. Since cognitive development is a function of learning, cognition is seen to differ across culture because different cultural conditions pose different kinds of problems.

This view of development as the accumulation of learning experiences is based on three fundamental assumptions about the relationship of mind and culture, which are embedded in the 'mind as a function of culture (M=f(C))' view of the world (cf. Ibid.).

(1) The relation between mind and culture is static and can be examined ahistorically in terms of manifest performances at a given time.

(2) Mind and culture can be conceived of as separate and independent phenomena.

(3) There is an unidirectional, causal relationship between mind and culture, mind being a product of culture.

Implicit in the M=f(C) equation is the assumption that mind and culture can be regarded as separate systems. Mind and culture are assumed to be causally related in an unidirectional way and that they can be operationalized as states rather than processes since human behavior is essentially reactive.

The cultural relativist viewpoint which looks at differences *between* cultures, at *any one given time* is unable to accommodate the everchanging nature of mind in society. In other words, whereas the cultural relativists can explain differences in performances across time and space, this theoretical position does not allow for the explication of that which may generate change or development (cf. Miller, 1984). In a research programme such as the one of which the present project is a part, the major issue addressed is that of change: Can any individual or social (cultural) group familiar with a particular set of tasks given in a certain context adapt to the unfamiliar demands of formal education in general and mathematics in particular ?

According to Vygotsky (1978) and Miller (1984) human activity is not reactive, but should be seen as both responsive to and generative of the world in which it occurs. Neither culture nor mind can be treated as static entities or as pure concepts, functioning or existing independently of each other, neither can one be defined without referring to the other.

The stress on the importance of change points to the inadequacy of the cultural relativists focus on *manifest* performance. Change is the conceptual link between the different explanations of cultural specific skills. "Comparing cultures in terms of manifest performance may be a misguided venture, based on essentially erroneous concepts of mind (IQ, as a measure of intelligence) and culture (genes), and the relationship between them" (Miller, 1984, p. 5, brackets added). Culture and individuals change continuously; adaptation, which for Piaget constituted intelligence is an ongoing *process.* At any time, behavior is only the manifestation of an underlying dynamism of generative mechanisms (cf. Juckes, 1987).

In the establishment of his theory of the mind-culture relation, Vygotsky argues that cognitive processes are often the product of a long history of social transformations which may have become 'fossilized' or mechanized over time. He believes that the best mode of access to behaviors that have become fossilized in the history of culture is to return to their source in the development of the individual and reconstruct them as they are manifested in her/his performance.

The study of the evolution of the structures underlying performance becomes, therefore, a better mode of access to the problem of the mind-culture issue. The study of generative processes underlying performance does not regard the psychological process as fixed, stable objects. The basic task of research becomes a reconstruction of each stage in the development of the process: the process must be turned back to its initial stages (cf. Catán, 1986).

In the developmental paradigm each manifest performance is seen as a moment in the developmental history of the individual. The developmental method prefers to look at particular examples of the unitary mind-culture system in action within a particular situation. It is not a question of how the mind varies as a function of culture, but how the two phenomena simultaneously react to and generate each other in the process of their mutual transformation (cf. Miller, 1984).

The reconstruction of what the learners may intend in the execution of a mathematical task is the crux of this research project. Learners' engagement in mathematical tasks is the 'drama' that this project attempts to understand. This 'drama' is of interest because it may provide the key to the discovery of that which may generate effective, autonomous problem solving skills in general and for successful mastering of mathematical tasks in particular.

3.2 Cognition and culture: an inter-psychological approach

Piaget's main focus is on the 'processing organismic constraints' of the subject and his methods '... minimize learning and maximize conceptual problem solving, while generating a large family of often interrelated genetic-epistemological sequences.' (Pascual-Leone, 1980, p. 2267). Piaget focuses on the universal capacities of individuals to acquire the structures of logico- mathematical thought. These structures (cf. 3.3, p. 32) are conditions internal to the individual.

Vygotsky's focus, by contrast, is on 'Mind in Society', the title of his work (1978). He looks at development not in the context of the epistemic subject, but of the social actor, defined by Craig as follows:

> The social actor is an individual who acts, and is socialized to act, in a particular socio-historical context ... who must meet the demands of a reality which already exists in some form before that individual life can take its course (Craig, 1985, p.13).

For Vygotsky psychological functions appear first *inter- psychologically* or between people - initially the child is regulated from outside by some informed other person, usually the mother. Only later are these functions internalized as *intra-psychological* functions. In Vygotsky's theory culture becomes the steering principle of regulation. Regulation is a social process involving an external mediator. In the teaching of mathematics it is the teacher who provides the 'regulative cues' which the learner needs to solve a given mathematical task, but it is often assumed, incorrectly, that learners come to university equipped with the regulative mechanisms which are necessary before the learner can deal with university mathematics problems. It is therefore necessary to 'lift' the under-prepared learner to a level where he/she is equipped to meet the demands of university mathematical tasks, and this project aims at producing the necessary mediational strategies that will do just that, in the most efficient and effective way.

According to Vygotsky, the interaction between learning and development, or between child and adult, or between teacher and learner, takes place in the *Zone of Proximal Development*. This is the distance between the functions which have already matured and functions which are currently in a state of formation and can be exercised under adult or teacher guidance. The zone of proximal development is thus the interface between mind and culture, in the sense that it is here that mind and culture fuses. The zone of proximal development created when a learner encounters a mathematical problem, is discussed in detail in the section 'Zone of proximal development created for the mathematical task' on page 79.

The external regulation provided by the teacher during the interaction with the learner whilst solving mathematical problems, for example, will be internalized by the learner so that he/she can regulate his/her own engagement in tasks in future -this is the crux of Vygotsky's notion of *internalization*. However the very nature of mathematical learning-teaching and the learning histories of under-prepared learners hinder this process of internalization so that a more thorough analysis of the nature of mathematical problem solving, the existing competence of those engaged in mathematical tasks, and the regulatory or mediational strategies, which could ensure the development of effective autonomous (mathematics) problem skills in learners, is required. In this regard Vygotsky's theory-method is crucial.

3.3 The epistemic subject and cognition

Piaget believed that each person passes through stages of intellectual development that are qualitatively different. These stages are sensorimotor coordinations, pre-operation thought, concrete operations and formal operations. The name of each stage represents the type of thought process that is developing during that stage and is fully accomplished by the end of the stage. For Piaget, operations consist of systematic, coherent processes for manipulating and transforming data. Operations 'concern transformations of reality by means of internalized actions that are grouped into coherent, reversible systems' (Piaget and Inhelder, 1969, p. 93).

One of the main factors which account for an individual's progression through the operational stages of intellectual growth, is logico-mathematical experiences. Logico-mathematical experiences not only involve objects and things in the world but bring about knowledge concerning relationships and coordinations rather about physical properties (cf. Miller, Belkin and Gray, 1982). Piaget used this label because he believed that the formal disciplines of logic and mathematics could be used to describe the thought processes that each person uses, even though the individual could not utilize the definitions or terminology of formal logic or mathematics. For example it is possible for a child to possess arithmetical knowledge, as for instance, to know the names of the numbers and to repeat them in the correct order (count), yet without understanding the significance of the numbers (cf. Van

Den Berg, 1978).

In terms of understanding mathematics, Piaget believes that there are a number of basic mathematical (and scientific) principles which the child must grasp. These underlying basic principles, or concepts, are (cf. Van Den Berg, 1978):

(a) The concept of number.

(b) The concept of space.

(c) The concept of time.

(d) The concepts of length and measurement.

(e) Concepts associated with area and volume.

(f) The concept of substance.

(g) The concept of weight.

Through many experiments Piaget demonstrated that before a child can develop the true concept of number, he should be 'operational' with respect to the following (cf. Van Den Berg, 1978):

(a) Mental representation of a series of actions.

(b) Relational terms. The 'operational child' understands terms of relativity in terms of degrees, space and consanguinity.

(c) Serialization. Basic to the understanding of number is the insight into the cardinal and ordinal properties of numbers.

(d) Reversibility. To Piaget the fundamental skill that underlies all mathematical thinking is the capacity of returning thought to one's starting point.

(e) Conservation. The child must be able to grasp the fact that a certain given mass of matter remains constant although its form may undergo a change.

(f) Class inclusion. Once the child grasps that the number of elements in a given set remains invariant despite changes in their arrangement, and also grasps the fact that when two sets that are equivalent, remain equivalent, no matter what the arrangement of the elements in the respective sets, and when he can furthermore recognize the relationships between groups and sub-groups, then the child can be said to have a understanding of the number concept .

From his experiments Piaget was able to indicate three main stages of cognitive development in the growing child. The first is the non operational stage which is overwhelmingly sensory- motor. The second stage is the pre-operational stage and the third is the operational stage which includes concrete and formal operations.

Children lack the ability to logically manipulate and transform the data they receive during the pre-operational stage.

Concrete operational thought is developed during the elementary school years and overcomes the logical problems of the pre- operational stage. During this stage children can manipulate data using symbols and can solve problems involving mathematical notation. In this stage the child still relies heavily on his observation. The child likes to think with the aid of sketches, for example. During this stage, insight into reversibility, equivalence and conservation break through.

It is during adolescence when the ability to manipulate ideas themselves and to think abstractly, analytically and reflectively is developed. This development of *formal operations* is not as universal as earlier stages (Piaget, 1972) because of cultural influences and personal goals and areas of specialization. However all individuals have the universal capacity to develop such formal operational skills and because of the rapid pace at which societies are changing it is imperative that they possess such skills. This project is concerned with developing appropriate mediational strategies that will assist the under-prepared learner in developing such skills in the most efficient way.

When mathematics learners have reached the stage of formal operations they are expected to have grasped the basic mathematical and scientific principles mentioned above, especially the concept of a number. During the formal operational stage the teen-ager becomes capable of logical thought about what might occur, rather than being restricted to consideration of actual events. A related capacity that also develops during formal operations is the ability to derive a hypothesis based on a set of data and to manipulate the data systematically to test the hypothesis. The teen-ager develops the ability to think in terms of relativity or proportion. The ability to do the intermediate steps of processing required for understanding an analogy develops during this stage. The adolescent becomes capable of judging the truth of an argument based solely on the logical structure rather than on knowledge of the content itself (cf. Gellatly, 1987).

At university level individuals are expected to have developed logico-mathematical thought sufficiently to deal with university type mathematical problems. Although some learners have reached the stage of formal operations, they fail to perform university type mathematical tasks and this project aims at developing mediational strategies that will assist university learners in becoming successful autonomous mathematical problem solvers.

Piaget, instead of regarding mind and culture as separate systems, and then looking at variation in performance between cultures (see section 3.1, p. 25), looks at mind and culture as two aspects of a unitary system, locked in an ongoing process of mutual transformation (cf. Campbell, 1985).

One of Piaget's most important contributions to development psychology, is his insight that this transformation process consists of overcoming constraints which take the form of familiar ways of looking at or understanding the world - in favor of increasingly more sophisticated and unfamiliar ways. For the acquisition of knowledge to take place their must be conflict or non-balance *and* resources for surmounting conflict:

> Non-balance ... produces the driving force of development.

> Without this, knowledge remains static ... It is therefore evident that the real source of progress is to be sought in both the insufficiency responsible for the conflict and improvement expressed in the equilibration. (Piaget, 1977, p13).

In order to examine the overcoming of constraints - or how children proceed to master the unfamiliar - it is necessary to use unfamiliar tasks, as the process of development consists of the child's successful confrontation of increasingly complex unfamiliar tasks. Many of the mathematics problems encountered by first year science learners are not only unfamiliar but are also abstract and embedded within a mass of theory which can be very confusing for the under-prepared learner. The university mathematical tasks provide the necessary conflict for the development of mathematical skills but many under-prepared learners lack the necessary skills required to surmount the conflict. There is therefore the need for 'bridging' or provision of resources to assist the learner in overcoming the conflict.

Piaget's primary concern was the development of logico- mathematical thought discussed above. Piaget sees the development of knowledge as the evolution of increasingly complex psychological structures or mental representations: the emergence of which coincides with developmental stages in the child's life. At each stage there is an extension, reconstruction and surpassing of the structures of the preceding one.

Progress from one stage to the next involves the child's acquisition of increasingly complex psycho-logical structures. The regulatory process which is responsible for the child's transition from stage to stage, constituting cognitive growth, involves the two processes of learning and development, and their integrating principle - the internal regulatory principle which Piaget calls equilibration.

Piaget (cf. Piaget and Inhelder, 1969) outlines four general factors involved in mental development: the first three factors are organic growth, experience and social transmission. The effect of these three factors are integrated by a fourth factor - equilibration which is described as

> A process of equilibrium, in the sense of self- regulation; that is, a series of active compensations on the part of the subject in response to external disturbances and an adjustment that is both retroactive (loop systems or feedbacks) and anticipatory, constituting a permanent system of compensations (Piaget and Inhelder, 1969, p74).

It is the equilibration processes, mediating between maturation on the one hand, and experience and social transmission on the other hand, that engineer the subject's construction of reality.

The concept of equilibration points to the mental structures and functions that generate intelligent behavior, through the successive stages that constitute the development of logico- mathematical thought. For this reason Piaget's theory of equilibration can be interpreted in terms of intrinsic generative mechanisms (cf. Craig, 1985)

Neither Piaget nor the cultural relativists provided a solution to the learning paradox which we shall discuss below.

Cultural relativists believe that a learner's performance while attending to a task is a result of previous learning. Pascual- Leone believes that this is not always the case:

> The subject's production of a given acquired behavior is frequently attributed to previous learning even though (1) the behavior in question has never before been produced by the subject, (2) such a behavior is complex and improbable enough not to have been produced by 'chance' (Pascual- Leone, 1976c, p. 94).

The explanation of cognitive development or mind in terms of culture or learning, as LCHC do, does not resolve the learning paradox: It is not possible to explain a child's spontaneous solution of a problem for the first time in in terms of learning - for the child cannot *know* how to solve the problem unless she/he has *learned* how to do it already (cf. Campbell, 1985).

To avoid this paradox, what Pascual-Leone calls *truly novel performance* must be attributed to some other factor. Truly novel performance is:

> ... behavior which is neither mere transfer of learning or novel integration of pre-existent learned units, nor innately determined (Pascual-Leone, 1976c, p. 94).

Truly novel performance entails the child's overcoming and compelling nature of familiar but misleading cues, in favor of more complex and unfamiliar cues more appropriate to the task at hand. It is this process that constitutes development. In solving (novel) mathematics problems the teacher must provide the learner with mediational strategies- i.e. that which must be taught to 'form' adaptation and help the learner master mathematical tasks.

Equilibration provides the necessity of viewing change in terms of the ongoing resolution of successive contradictions. However, although Piaget's method seems to offer the possibility of a theory of development that adequately accommodates the notion of change, two criticisms have been leveled at him which are of particular importance to the understanding of cognition. The first criticism involves the notion of equilibration as being an incomplete explanation of the process of change in the developing child (cf. Campbell, 1985). Pascual-Leone argues that although Piaget takes important steps towards solving the learning paradox, and the problem of truly novel performance, he does not effect such a solution. Pascual-Leone seeks to 'stand on Piaget's shoulders', using his insights as guidelines for a neo-Piagetian programme better equipped to deal with change (see section 3.4 on Pascual-Leone's theory of constructive operators, p. 42).

The second criticism of Piaget deals with the fact that his theory does not attempt to explain *how* social and cultural factors influence development. He looks at *intrinsic* psychological processes that generate performance, simply taking as given that it occurs in a social context (Craig, 1985). Piaget's focus on intrinsic generative mechanisms is complemented by Vygotsky's notion of interaction between learning and development, incorporating a view on constraints *external* to the individual which govern cognitive development (see section 3.2 on Vygotsky and the zone of proximal development, p. 29). In the teaching-learning of mathematics the extrinsic generative mechanisms can be formulated in terms of mediational strategies -i.e. those rules, cues which the teacher can provide for the learner to help him/her master mathematical tasks.

3.4 Pascual-Leone's theory of constructive operators

Pascual-Leone uses the ideas of Piaget as the foundation stone for his theory of constructive operators (Pascual-Leone 1970, 1976a, 1976b, 1976c, 1976d, 1976e, 1978, 1980). He claims that Piaget fails to provide an adequate psychological theory because Piaget's account of stages and equilibration are valid only at the level of *descriptive structural* theory but fail at the *process structural* level: in other words they are incapable of accounting for the step-by-step temporal unfolding of the subject's behavior (1976c).

Pascual-Leone, by considering Piaget's genetic epistemology, his stages and equilibration as a stepping stone for his own theory, remains within Piaget's framework while elaborating a new psychological approach. His theory of constructive operators (TCO), his neo-Piagetian theory of cognitive development, formulates explicit constructs to account for the step-by-step cognitive growth which is described by Piaget (cf. section 3.3 p. 32).

The TCO is intended as an expansion of Piaget's structuralist framework into a working model of cognitive development which adequately accounts for change, or 'human constructivity' (the organism's ability to synthesize or create truly novel performances using and recombining aspects of past experience, and its ability to permanently modify itself as a result of the new experiences thus achieved). This is clearly of importance to a project such as the present one which attempts to explicate that which will ensure adaptation to the unfamiliar demands of university mathematical tasks.

Pascual-Leone points to Piaget's failure to differentiate between learned habitual cognitive structures and those structures that result from the equilibration processes - truly novel performances for logical-structural tasks, such as conservations, when they are solved for the first time. This first time performance cannot be explained in terms of a learned habitual structure without falling into the snares of the learning paradox (cf. Campbell, 1985).

To resolve this paradox, Pascual-Leone posits situation-free organismic factors or 'constructive operators', which he calls *silent operators*[1]. Through their dynamic effect on schemes[2], these organismic factors, in interaction, account for the developing child's constructivity. These silent operators offer a solution to the stage transition problem in Piaget's theory which is inadequately explicated by the notion of equilibration.

Novel performance is seen in the light of Piaget's description of cognitive growth as an integration of existing learned or innate structures or schemes. Pascual-Leone, on the other hand, speaks of *truly* novel performance which transcends already learned knowledge, and represents a qualitative break

[1]Silent operators are basic organismic factors responsible for the 'choice' among schemes- they are situation-free organismic operators. A silent operator functions as a scheme booster (or a debooster) which increases (or decreases) the activation weight of schemes on which it applies (cf. Pascual-Leone and Goodman, 1979 & p. 51).

[2]Schemes refer to an organized set of reactions that can be transferred from one situation to another by the assimilation of the second to the first (cf. Pascual- Leone, 1978 & p. 47.)

from already learned schemes in the sense that the integration is the result of a higher form of abstraction than the integration underlying novel performances (Craig, 1985).

> In a truly novel performance the integration of habitual schemes occurs serendipitously, without a habitual rule- integration scheme, as the result of hidden interactions among situation-free organismic processes - the silent operators and basic principles (Pascual-Leone and Goodman, 1979, p. 308).

Craig ascribed the possibility of this serendipitous achievement of truly novel performance to the power of the metasubject's[3] intrinsic generative mechanisms '... to achieve greater levels of abstraction than are available in the immediate data from action performed on objects and integration from knowledge thus gained' (1985, p. 77). That is, it is the intrinsic *power* of individuals to achieve truly novel resolution of problems/tasks. However, essential to the present project is the explication of regulatory principles/mediational strategies which could operate on the executive schemes[4] in order to stimulate adaptation to unfamiliarity, i.e. extrinsic generative mechanisms in the sense of Vygotsky's theory. In other words, relying only on the silent operators which develop from the subject's interaction with

[3]The metasubject refers to the subject's psychological organisms-the silent organization of functional structures or 'psychological machinery' underlying the subject's activity.

[4]These are schemes which specify general-purpose plans of action for procedures to accomplish a given task (cf. section 3.4.2, p. 48).

Reality[5], does not give one a handle on deliberately effecting educational change. We can, therefore, by using Vygotsky's focus on extrinsic generative mechanisms (which becomes, through internalization, the intrinsic generative mechanisms) *extend* Pascual-Leone's explication of the functional structures of the metasubject to include the *inter*-psychological development of self-regulation. In moving from the epistemic subject (Piagetian paradigm) to the social actor (Vygotskian paradigm) inter-psychological processes become important in the actual mediation between learner and teacher.

The most important situation-free organismic process, which plays a pivotal role in truly novel performance is the M- operator, the reserve of mental energy which increases quantitatively in power with age. The M-operator refers to a mental energy mechanism that determines the attending to and integration of task-relevant information. The power of M is the same for all individuals of the same age group and is referred to as 'mental capacity'. The growth of M is seen as a maturation process, working in intimate interaction with experience. The developmental growth of M is the 'transition rule' for passing from one Piagetian cognitive stage to the next, in other words a child cannot move to a higher stage until his/her M power has reached a certain level (cf. Campbell, 1985).

[5] Reality is that which is 'out there' and subject of knowledge in interaction with Reality constructs reality or knowledge of Reality.

3.4.1 Principle of bilevel psychological organization

The schemes or subjective operators and the silent operators form two levels of operating or functioning of the metasubject. These levels are strongly hierarchically organized in two, functionally and structurally different interacting systems.

> The first level or *subjective system* is constituted by situation-specific constructs (organismic *schemes*) which apply on the input to categorize and/or modify it: the second-level or *silent system* is constructed by situation free *metaconstructs* (basic *factors* and basic *principles*) which apply on the first level constructs (not on the input) to modify their activation weights (i.e. assimilatory strength) in accordance with organismic requirements (Pascual-Leone and Goodman, 1979, p. 306).

The bilevel organization is therefore a necessary assumption in order to create the possibility for *choice* among schemes. The principle of bilevel psychological organization explains why particular schemes apply, rather than others that are activated in the metasubject in any specific situation and help explain a learner's truly novel performance of a task.

This principle of bilevel psychological organization has powerful implications for the teaching situation - it is not sufficient for the learner to develop content-specific mental structures but it is also necessary for him/her to develop 'contentless' processes such as logic, reasoning styles etc. and it is here that the silent operators play a vital role together with executive schemes which we shall now discuss in detail.

3.4.2 The theory of constructive operators

The theory will be discussed in more detail under the following headings: (1) Schemes, (2) The field of activation, (3) Silent operators.

(1) Schemes

The notion of schemes is taken from Piaget and refers to an organized set of actions which can be transferred from one situation to another. Pascual-Leone characterizes schemes as 'semantic- pragmatic' insofar as each one consists of a bundle of pragmatically relevant blueprints corresponding to expectations, actions, perceptions, beliefs , plans or affects. Structurally all schemes have the same form: if a set of *conditions* is minimally satisfied by the input from the environment or the subject's internal state, the scheme will tend to apply (unless another more dominant scheme prevents its application). When it applies, the set of *effects* (blueprints) which it carries are used by the metasubject to further or modify its ongoing activities.

There are three different kinds of schemes:

(a) Affective schemes: These generate two sorts of effects: physiological reactions and motivational effects.

(b) Cognitive schemes: These schemes include both figurative and operative schemes, which are action schemes which can implement into performance the plan of an executive scheme. Figurative schemes are predicates that have the effect of representing objects and events. Operative schemes have the effect of changing the mental or physical objects they represent.

(c)<u>Executive schemes</u>: These are epistemologically complex and general operative schemes which specify general- purpose plans of action for procedures to accomplish a given task. These procedures are then implemented through the application of specific task relevant figurative and operative schemes which satisfy their plan.

Executive schemes mediate between motives and other cognitive schemes, co-ordinating their combination and temporal sequence to produce a complex goal-directed performance (cf. Campbell, 1985). These executive schemes obviously play a main role in the performance of mathematical tasks as they are responsible for the learner deciding what the best plan of action would be in performing a mathematical task and good executive schemes will therefore be necessary for the learner's successful monitoring of his/her progress throughout the task. These executive schemes are formed by <u>LM</u> learning discussed below and are responsible for the mobilization and allocation of <u>M</u>. Good executive skills will therefore help guide the learner towards the successful solution of the problem. This project involves the development of regulative cues and mediational strategies which may generate the necessary executives to assist the learner in monitoring his/her own progress while performing mathematical tasks.

(2) The field of activation

All schemes are of the same form: they have a releasing component (rc), an effecting component (ec) and a terminal component (tc). The releasing component consists of a set of potential cues or conditions which govern the scheme's activation. When features of an input match at least one condition of the scheme, they *cue* or *release* the scheme. Each condition of the rc causes a 'content activation weight' determined in part on the basis of innate saliency factors (i.e. how salient in a psychophysical sense is the feature matching the condition) and in part on the basis of 'learned' saliency factors (i.e. how important is the condition to the scheme). The local degree of activation of the scheme is given by the sum of a set of weights of satisfied (i.e. activated) conditions . This is the TCO's 'local cue function rule' for scheme activation (Pascual-Leone and Goodman, 1979, p. 308). The effecting component causes the effect or consequence of the scheme, and the terminal component specifies its outcome, should it be realized.

Any performance produced by a subject results from the metasubjective application of schemes. At any particular moment, a set of schemes from the total repertoire of schemes is active, by virtue of the local cue function. This set of schemes is called the initial field of activation. Not all activated schemes actually produce performance - only those which are compatible and dominant in activation strength come to apply. Each one of these dominant schemes shares in the shaping of performance, while other schemes which are weaker and incompatible will be prevented from applying. This law is called the Principle of Schematic Over-determination of Performance or SOP.

The initial degree of activation of schemes is modified by the silent operators, and it is the terminal activation weight of schemes, after silent operators have applied, that determines dominance. Silent operators apply on schemes and, via this application, construct the subject's performance.

Thus at any moment in the performance of a mathematical task the learner may apply a set of schemes which are compatible and dominant in activation strength - these schemes may include mathematics rules which, because they have been 'overlearned', will dominate task specific demands which are supposed to be attended to in order to solve the tasks successfully. The notion of misleading overlearned rules is a definite problem for under-prepared learners and will be addressed in terms of 'interruption cues' or regulative cues which will be necessary to prevent their incorrect application (cf. Chapter 5 p. 87)

(3) Silent operators

Silent operators serve the function of boosting subjective schemes which are appropriate to the situation. The TCO posits two types of learning: C (content) learning, which corresponds to Piaget's notion of empirical experience, and L (logical or structural) learning that corresponds to Piaget's notion of logico-mathematical experience (Pascual- Leone and Goodman, 1978).

Compared to C learning which involves no change in epistemological level, L learning creates 'super- schemes' which reflect structural relations among constituent schemes. It does not replace the constituents, but rather carries information about their interrelationship - which no particular constituent could contain. Their are two types of L learning: L structuring by overlearning (LC learning) and L structuring via M boosting (LM learning).

LC learning occurs through repeated exposure to a situational invariant, i.e. a set of schemes standing in a particular structural relationship to one another. This exposure leads to repeated co-activation of the functionally related schemes. All the schemes involved come to acquire equally high assimilatory strength, and slowly come to assimilate each other, forming an LC structure. LC learning is slow, and results in structures that are functionally interlocked with the schemes that led to their formation.

LC learning is often tacit, taking place latently and without mental effort (i.e. without the application of M boosting to the schemes involved).

In contrast, LM learning takes place when the subject is mentally aroused. A set of schemes is simultaneously and repeatedly boosted by M, and a super-scheme is formed which reflects them all. This learning is rapid, and detached from context, resulting in very generalized structures reflecting trans-situational invariances. *Executive schemes are formed by way of LM learning.*

When under-prepared students perform mathematical tasks they have difficulty in monitoring their own progress (cf. section 1.2, p. 7). This may be because they have weak executive schemes. However they have sufficient M power to boost or create stronger executive schemes. Once again 'interruption cues' which may help improve the learner's performance on mathematical tasks (by, for example, making him/her 'attend to detail') and therefore cause the learner to successfully monitor his/her progress through the task (cf. section 5.2, p. 87).

LM learning is recursive (i.e. the learning process may apply on the super-schemes themselves to create super-schemes of super-schemes, and so on). Its recursive nature, and the combinability of LM and LC structures may lead to the formation of overlearned LM structures called LM/LC structures -automatized mental operations that can be carried out without the intervention of M. In mathematics LM/LC structures are necessary. The student will benefit from 'rote' learning certain mathematical rules so that his/her 'mind' can attend to other, more important aspects of a particular mathematical task. The only problem, as mentioned before, is when these overlearned rules are so heavily weighted that they prevent new and necessary schemes from being applied when the learner meets a task.

There is a developmental ceiling to the quantitative complexity of structures which may be abstracted and schematized via LM learning. This learning is limited by the M power available to the subject at his/her particular developmental stage (cf. Pascual-Leone and Goodman, 1979). This available M power explains why learners of same age exhibit different performances - those whose performances are poor have been subjected mainly to LC learning and have not utilized all their available M power.

The black child comes from an environment where there is little mental stimulation so although they have sufficient M power to master mathematical tasks their culture has not adequately 'trained' them in utilizing it to its maximum potential. In Juckes (1987) it was found that black children's arousal executives were poor which was related to the poor educational environment of the children and it was suggested that performance can be improved by encouraging *active* problem solving.

Vygotsky's idea of an external generative mechanism can therefore be realized in the form of regulatory principles or mediational strategies which will help strengthen the learners executive schemes which are necessary for the learner to successfully monitor his/her progress while performing mathematical tasks. It is therefore part of this project to discover those mediational strategies that will be of the most benefit to under-prepared learners in general and black learners in particular.

M is a limited amount of mental attentional energy that can be used to boost task-relevant schemes that are not sufficiently boosted by other silent operators. The mobilization and allocation of M are carried out by *executive schemes* which carry the subject's representation of the task instructions and the corresponding plans for solving the task. In task situations where schemes are inadequate for task solution, M energy is allocated to boost the activation of the task-relevance schemes, leading to correct performance.

The maximum number of schemes that an individual can simultaneously M boost is called M power (Mp). All individuals of the same age have the same M power which shall be called *mental capacity*. Maximum Mp grows throughout childhood, one unit every year reaching e+7 at adolescence where it remains until and throughout adulthood. The value e represents a constant amount of M energy which is developed during the first two years of life, and later used to boost the task executive.

Learners often find certain mathematics problems 'too difficult' to solve. According to Pascual-Leone's Theory of Constructive Operators a task may be 'too difficult' to solve because more schemes may be required than the maximum number available to the subject via M activation, causing the subject to fail. However in such a case there may be a way out via learning (LM or LC). Schemes may be 'chunked' in such a way that fewer schemes are required to solve a well-known problem than when the problem was first encountered by the learner[6].

These schemes and silent operators presented above are important when analyzing student performance at university level especially when mathematics is involved. This is because of the role that executive schemes play in deciding what rules are needed for the solution of the mathematical task and the importance of silent operators in boosting these executive schemes.

Pascual-Leone's TCO is valuable when trying to solve the problem of why under-prepared learners fail at university mathematical type tasks and will be referred through this research report.

[6]Here one could distinguish between the common sense notions of a 'clever' learner and a 'hard working' learner. The first may rely mainly on LM learning while the second on LC learning. Both learners will eventually solve a given mathematics problem, the latter student taking much longer than the former.

CHAPTER 4

UNDER-PREPARED LEARNERS DOING MATHEMATICAL TASKS AT UNIVERSITY- THE PROBLEM ADDRESSED

4.1 Under-prepared learners and university mathematics

When learners reach university level they are expected to have developed logico-mathematical thought/formal operations (cf. section 3.3, p. 32) sufficiently to handle university type problems. Learners doing mathematics courses are expected to either already have developed the appropriate contentless processes such as logical operations, reasoning styles, analysis, comparison of parts and the synthesis of parts into a whole (cf. Gelatly, 1987), or are expected to develop such processes spontaneously whilst performing mathematical problem solving tasks at university (cf. LC and LM learning, section 3.4, p. 42).

For mathematics lecturers the rules for doing mathematics may have become 'fossilized'. As Vygotsky says:

> ... in psychology we often meet with processes that have already died away, that is , processes that have gone through a very long stage of historical development and have become fossilized. These fossilized forms of behavior are most easily found in the so-called automated or mechanized psychological

> processes which, owing to their ancient origins, are now being repeated for the millionth time and have become mechanized. They have lost their original appearance, and their outer appearance tells us nothing whatsoever about their internal nature. Their automatic character creates great difficulties for psychological analysis (1978, p. 63).

In other words, it is a necessary part of the present research to explicate both the tacit and expert rules for doing mathematics because they have become 'mechanized' or 'automated' and, therefore, are not necessarily available to conscious control by either learner or teacher.

Because of the problems mentioned above it is necessary to analyse learners' engagement in mathematical tasks. This involves focusing one's attention on two important issues which the next two sections deal with - viz.- the nature of mathematical problems and the nature of adult cognition.

4.2 The nature of mathematical problems

Mathematics is not something that exists independently of man and is being gradually discovered by him, but rather mathematics is something that is being *created or developed by man* (cf. Van Den Berg, 1978). It is a widely held belief that mathematics is axiomatic-deductive of nature. The actual development of mathematics as a science is described as follows:

> There are periods of exuberant untidy growth, when existing vital structures rise upon untried assumptions, and loose ends lie about all over the place ... Such periods are followed by pauses for consolidation, when the analysts and systematisers get to work: material is logically ordered, proofs supplied (Van Den Berg 1978, p. 29).

It is therefore important to understand that only after the creative activity has taken place are the new mathematical developments re-arranged in a vigorous structure built up from basic axioms. More importantly, it is only this rigorous structure, the final product, which is presented to the learner. The impression is thus gained that mathematics is axiomatic-deductive of nature, another way in which it is 'fossilized'.

Kitchener (1983) defines mathematical problems as puzzles as opposed to ill-structured problems. It seems appropriate at this stage to point out that puzzles and ill-structured problems describes the types of problems students may encounter at university level, but referring back to the above quote by Van Den Berg, during periods of creativity in mathematics, mathematicians may also be said to confront ill-structured problems which are then converted into a more puzzle like form. In the analysis presented here, we distinguish between different *kinds* of puzzles to assist learners in performing mathematical tasks.

The distinction between puzzles and ill-structured problems used by Kitchener had been made by Churchman (1971). Puzzles and ill- structured problems differ both in their epistemic nature (i.e., in the ways they are knowable) and in the decision making procedure required to solve them.

A puzzle is a well-structured problem. All the elements necessary for a solution are knowable and known, and there is an effective procedure for solving it. Churchman suggests that puzzles, the solution to which may be reduced to a *deductive* algorithm, are characteristic of what he calls the Leibnizian inquiring system (IS). The Lockean IS is distinguished by problems for which single solution may be *inductively* agreed upon via a set of empirical observations and a group of pre-established rules. Although the procedures differ, both assume that all problems are reducible to puzzles which can be solved by the correct application of an algorithm.

Puzzles therefore have two distinguishing characteristics:

(a) there is only one correct final solution and

(b) the solution is guaranteed by using a specific procedure (e.g., following a mathematical formula).

Thus mathematics problems encountered by learners at tertiary undergraduate level are puzzles. However the problems often encountered in the Real world, and in the creating phase of constructing new mathematics problems, rules etc., are of the ill-structured nature ; in ill-structured problems there is not a single, unequivocal solution which can be effectively determined at the present moment by employing a particular decision-making procedure. Such ill-structured problems occur in the social sciences. Such problems are typical of Kantian IS and Dialectial IS. Kantian IS is characterized by problems for which there are two or more complementary conceptualizations or potentially valid solutions. The dilemma is to decide which set of theoretical assumptions best fit the problem and the evidence at hand or how to integrate them into a single solution.

Churchman (1971) defines problems for which solutions are basically antithetical as 'dialectical'. Different and opposing assumptions underlie each side and individuals on opposing sides define the problem in different ways and marshal the same evidence in support of their perspective. A solution or synthesis lies in reframing both or several perspectives into a more general model of the problem, or redefining the problem as one that can be handled by a Kantian or Lockean IS.

In other words, in both types of ill-structured problems evidence, expert opinion, reason and argument can be brought to bear on the issues, but no effective procedure is available which can guarantee a correct or absolute solution.

The distinctions between puzzles and ill-structured problems are important, however, for the development of mediational strategies appropriate to the learning-teaching dialectic of mathematics and science on the one hand, and arts and social science on the other. This is so because the *nature* of the problems confronted will determine the kind of mediational strategies which will be effective in generating adaptation to the unfamiliar. I will again address this problem of the nature of mathematical puzzles at a later stage in this report.

4.2.1 A classification of puzzles

Returning to puzzle type problems I shall postulate that there are two major classes of puzzles in mathematics education (mainly first year university level): *Clear* and *disguised* puzzles. It must be stressed that this distinction/classification is a result of the author's analysis of his own methods of solving mathematical problems. Furthermore, this distinction seems an important aspect for the mathematics learner as the kind or nature of the problem they confront is made explicit (epistemic cognition) (see section 4.3, p. 73).

Figure 2 on page 62 provides a possible classification of puzzles which we shall now discuss in detail. This classification should be made available to learners together with examples of each type of puzzle and the necessary stages required for their conversion to the final, most purest, form. This classification, therefore, forms part of what the learner needs to know in order to master mathematical tasks (cf. p. 82).

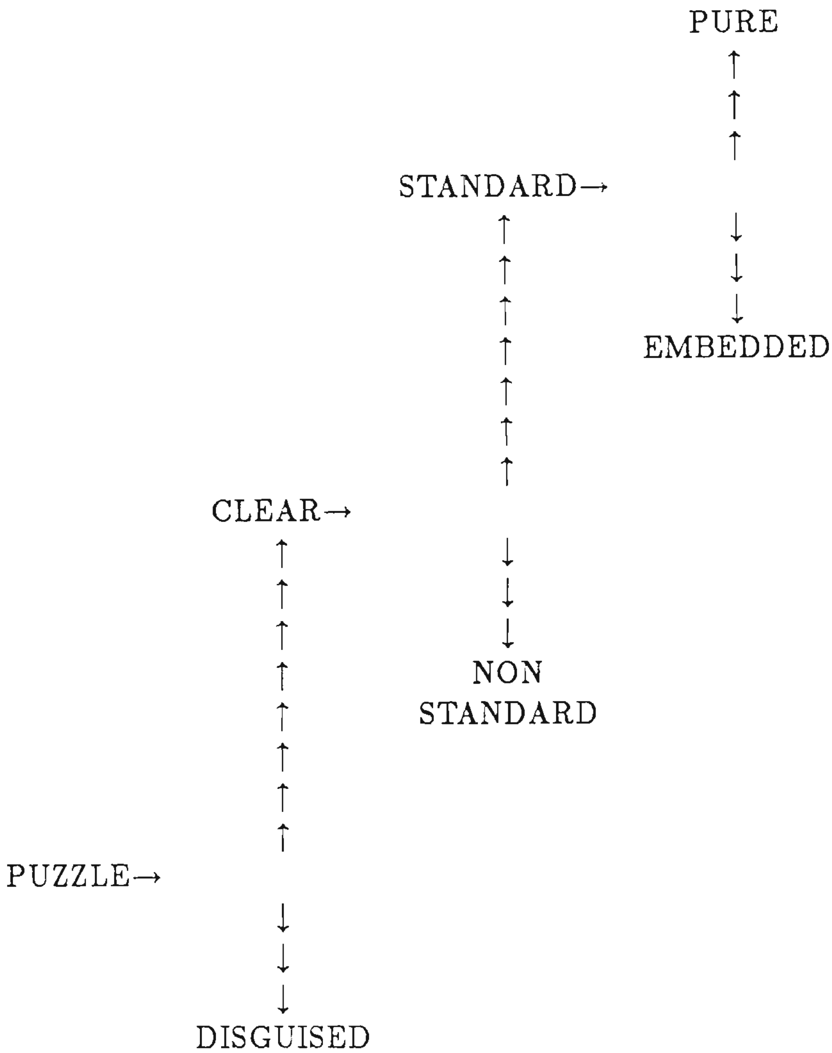

Fig. 2
A POSSIBLE CLASSIFICATION OF PUZZLES

Class 1 (Clear puzzles). These problems are presented almost entirely in *mathematical* symbols/terms and the procedure required to solve them may or may not be explicitly stated. For example

Problem (1a)

Solve the following quadratic equation by completing the square:

$$x^2 - 2x - 3 = 0$$

Problem (1b)

Find the roots of the following quadratic:

$$x^2 - 2x - 3 = 0$$

(Here one can either complete the square or use the formula for solving a quadratic equation).

Problem (1c)

Complete the square:

$$2x^2 - 2x - 3$$

Problem (1d)

If some A's are B's and all B's are C's, are all A's also C?

Problem (1e)

Let $f(x) = \frac{1}{1-x^2} + 1$

Find $f(x + h)$

Problem 1f

Given that

$$(a+b)^2 = a^2 + 2ab + b^2,$$

find

$$(1 + a^2b^3)^2$$

Clear puzzles can be divided into two sub-classes, viz.- those of <u>standard form</u>, and those of <u>non- standard form</u>.

<u>1A. Standard form type puzzles</u>

These puzzles consist of the *pure* type (for example- "evaluate $\frac{d}{dx}x^n$" and examples 1a and 1b above), or of the *embedded* type which consists of a pure form(s)[1] embedded in a collection of symbols and/or functions and/or words which may either be ignored by the learner when he/she tries to solve the puzzle or may hinder the learner's attempt to solve the problem. For example- "evaluate $\frac{d}{dx}(x^3+1)^n$" and examples 1d, 1e and 1f above. Problem 1c is not a clear puzzle of the pure type because the coefficient of x^2 is not 1, nor is it of the embedded type because simplification is necessary to get

[1]A puzzle might have more than one 'pure' form embedded within it. For example: "Evaluate $\frac{d}{dx}(x^2cos^2x)$" expressed in the pure form is: "Evaluate $\frac{d}{dx}(u \times v)$ where $u = x^2$ and $v = cos^2x$". However $\frac{d}{dx}(u \times v)$ has embedded within it the pure form: "Evaluate $\frac{d}{dx}p^3$, where $p = cosx$".

it into a 'pure' form (cf. the non-standard type below).

The embedded type can often be rewritten (*not* simplified mathematically) to reveal its 'pure' nature, i.e. the pure form can be 'extracted' out of the given puzzle. For example. the pure form of problem 1d is "If $A \not\Rightarrow B, B \Rightarrow C$, is it true that $A \Rightarrow C$?". Embedded in problem 1e is that part of the puzzle involving " Find $f(t)$ where $t = x + h$" and from problem 1f we can extract " find $(q + p)^2$ where $q = 1$ and $p = a^2b^3$".

The solving of the pure type standard form puzzles requires mainly <u>LC</u> learning (it is usually this type of puzzle which is first presented to the learner in introducing a new rule/concept) while the embedded type of puzzles clearly require <u>LM</u> learning where executives play a vital role. In solving the puzzles of the embedded type the learner must extract the pure form from the puzzle which involves a particular type of cognition (i.e. epistemic cognition) discussed in the next section.

<u>1B Non-standard form type puzzles</u>

These clear puzzles can be converted to a pure form(s) by appropriate *simplification (i.e. mathematical computation)* so that the most *basic rule* can be applied to obtain its solution. For example the $\frac{1}{x^n}$ in the puzzle: "evaluate $\frac{d}{dx}\frac{1}{x^n}$" can be converted to x^{-n} so that the most basic rule "$\frac{d}{dx}x^m = mx^{m-1}$" can be applied. A more complicated rule (i.e. the quotient rule) can be used to solve this puzzle, but it would take more effort than the procedure mentioned above and a 'clever' learner would opt for the quicker method[2]. Problem 1c can be simplified to $2(x^2 - x - \frac{3}{2})$ so that the most basic rule 'complete the square' can be applied.

Note that the pure standard form of these puzzles cannot just be extracted from the puzzle but must be obtained, either directly or indirectly, via a simplification (i.e. mathematical computational) process. LC learning will be required to initially convert these puzzles into one of the standard form types.

Class 2 (disguised puzzles).

These problems,which contain almost no mathematical symbols and for which the procedure needed to solve the problem is not specifically stated, are masked in *Real life* terms and the learner is expected to convert a given disguised puzzle into a clear one. Underprepared learners have great difficulties in extracting the necessary information required to solve the problem and tend to be trapped by the concrete terms in which the problem is stated (Krutetskii, 1976). For example:

Problem (2a)

One side of the rectangular enclosure is to be made of brick and the remaining sides out of fencing. Given that the fencing is 20m long, what dimensions of the rectangle will give maximum enclosed area ?

[2]'Clever being the common- sense label for those learners who apparently discover short-cut strategies spontaneously. These short-cuts free the mental capacity in the sense of not overloading it with peripheral considerations. cf. also p. 55.

Problem (2b)

A mirror in the shape of a rectangle capped by a semicircle is to be made. Given the perimeter is k meters, what radius of the semicircle will give maximum mirror area?

Problem (2c)

All people over 8 feet tall are Russians. Mrs. Thatcher is over 8 feet tall. What can you deduce about Mrs. Thatcher ?[3]

The above classification of puzzles (cf. Fig. 2, p. 62) has important consequences for the teaching-learning situation. For a given puzzle it is the learner's task to convert the puzzle to the 'pure' standard form(s) which would bring about its solution. For example, the puzzle: "Find the area enclosed by the graphs $y = x^2$ and $y = 4x$" is a clear puzzle of the non-standard form type. To convert this puzzle into a pure standard form type the learner must firstly solve the two given equations simultaneously to determine the limits of integration (they are $x = 0$ and $x = 4$). He/she must then decide which function is the 'greater' of the two functions for $0 \leq x \leq 4$ (clearly $4x \geq x^2$). This acquired information together with the original puzzle becomes a new 'super-puzzle' which in this case is a clear puzzle of the embedded standard form type. The pure standard form

[3]Some responses from the Engineering Bridging Unit are as follows: (1) The statement is incomplete. (2) The statement is incorrect. (3) Mrs. Thatcher is not a Russian. (4) Mrs. Thatcher was caught by the Russians and stretched until she was 8 feet tall.

embedded within this super-puzzle is the puzzle to be solved: "Evaluate the following integral:

$\int_0^4 (4x - x^2)\,dx$"

which the learner should be able to solve without much 'effort'[4].

Clearly it takes much cognitive effort to get the original puzzle into this pure form, however the purpose behind explicating the strategic steps involved in converting a disguised puzzle into a clear puzzle or a clear puzzle of the non-standard/embedded type into a pure standard form type is to enable the learner to have sufficient mental capacity at each step of the way, rather than the mental capacity being flooded at the onset by the disguised/embedded/non-standard form.

The following example illustrates the possible stages that are required in converting a disguised puzzle to a pure standard puzzle. The original puzzle is: "If you have 15 different books and you wish to arrange 5 of them on a book shelf, how many different ways can you do this ?". The clear form of this disguised puzzle is: "How many different ways can you select 5 ordered sub-collections from 15 ?". This new puzzle is a clear puzzle of the embedded standard form type and the embedded pure form is: "Evaluate $15 \times 14 \times 13 \times \ldots \times (15 - 5 + 1)$" which the learner can calculate to obtain the solution of the problem.

[4]'Effort' here indicating the match between strategic steps in problem solving and mental capacity of learner.

Figure 2 on page 62 illustrates the possible choices that one has in deciding what type of puzzle he/she is dealing with and what steps are necessary in converting the existing puzzle into the most workable pure standard form. The maximum number of conversion stages that will be required for any puzzle will be three. This will occur when a disguised puzzle is (1) converted into a non-standard clear puzzle which in turn is (2) converted into a clear puzzle of the standard embedded form type and, finally, this embedded form is then (3) converted into the final pure form. The maximum possible choices which one can make will be 6. One must decide if the puzzle is clear or disguised, and if the puzzle is clear whether it is standard or non-standard, and finally if the puzzle is standard whether it is of the embedded or pure form or not.

When clear puzzles are presented to learners they are viewed as *puzzles*. However when class 2 type puzzles (i.e. puzzles which are disguised by using real world terms) are presented to under-prepared learners they are perhaps regarded as *ill- structured* problems. Here it is important to mention the difficulty under-prepared learners have in distinguishing between 'types' and 'tokens'[5] so that they will certainly have difficulty in distinguishing between the specific and the general in university mathematics (cf. Lyons, 1977 for a more detailed discussion of tokens and types).

[5] The formalisation of the concept of type, is a result of the many instances of tokens -for example, the conceptualization of the type 'fruit' arises from the many different instances 'apples', 'oranges' and 'pears' (cf. Craig 1985, p. 137).

During the stage of formal operations (see section 3.3 p.35) the adolescent is supposed to be capable of judging the truth of an argument based soley on the logical structure rather than on knowledge of the content itself. Problems (1d)[6] and (2c)[7] are similar yet the under-prepared learner more likely get the former correct while allow his/her Real world experiences to affect his/her judgment of the latter. In Krutetskii (1976) it is noted that it is hard for under-prepared learners to distance themselves from the Real world aspect of a problem. Beyond the Real world (concrete) content and form of the problem they do not see its real mathematical meaning. Even word-problems which have additional information in the form of concrete materials make the problems more difficult for most learners (cf. Janvier, 1987, p. 56)

According to Krutetskii (1976) capable learners perceive the mathematical material of a problem *analytically* ('they isolate different elements in its structure, assess them differently, systematize them, determine their hierarchy') and *synthetically*('they combine them into complexes, they seek out mathematical relationships and functional dependencies') (pp. 227-228). Explicating the strategic steps in converting puzzles to the standard pure form attempts to teach under-prepared learners *deliberately* what capable learners do spontaneously. This point is essential to the present project.

[6] If some A's are B's and all B's are C's, are all A's also C's ?

[7] All people over 8 feet tall are Russians. Mrs. Thatcher is over 8 feet tall. What can you deduce about Mrs. Thatcher ?

When under-prepared learners confront a mathematical problem of a new type they are perceiving its separate mathematical elements. 'Going outside' the limits of the perception of one element often means 'losing' it. The able learner firstly perceives each element as part of a whole and secondly he perceives these elements as interrelated and forming an integral structure, as well as the role of each element in this structure. Although problem (1e)[8] is a clear puzzle type problem, under-prepared learners tend to regard the x in $f(x)$ as exactly the same as the x in $f(x+h)$. The arbitrary nature of x conflicts with their real world experience of specific tokens remaining as such. In solving this embedded standard form puzzle the learner must extract or deduce the pure form of the puzzle. Problem (1f)[9] will also create difficulties for under-prepared learners as they will concentrate on the specific elements a and b and lose sight of the problem as a whole. These problems illustrate the difficulty under-prepared learners have in distinguishing between types and tokens and the problems they have in identifying the pure form embedded within the puzzle.

4.2.2 A note on transformations within and translations between mathematical representation systems

In Janvier (1987) five distinct types of representation systems that occur in mathematics teaching are identified which include pictures or diagrams

[8]Let $f(x) = \frac{1}{1-x^2} + 1$. Find $f(x+h)$.

[9]Given that $(a+b)^2 = a^2 + 2ab + b^2$, find $(1 + a^2b^3)^2$.

(static figural models) and written symbols (which involve specialized sentences and phrases such as x+3=7, $A' \bigcup B' = (A \bigcap B)'$ as well as normal english sentences and phrases). Most first year university problems are represented in the form of written symbols. The ability of learners to translate *between* representation sytems and to transform *within* representation systems has a significant influence on both mathematical learning and problem-solving performance. Strengthening or remediating learners' translation and/or transformation abilities facilitates their acquisition and use of mathematical concepts (cf. Janvier, 1987). The classification of puzzles provided above can be used as a transformation model which will assist learners when they engage in university mathematics problem solving. Translating mathematical puzzles represented in terms of written symbols to a system where they are represented in terms of concrete materials may actually make the problem *more difficult* (cf. Ibid.). Therefore it seems more appropriate in this research project to concentrate on transforming puzzles in symbolic form to a more workable (pure standard) form, remaining within the same mathematical representation system.

Having discussed the the nature of mathematical tasks/problems encountered by under-prepared learners in solving mathematical problems we shall now describe the cognitive processes involved when adult learners engage in problem solving.

The aim of the latter is to formulate those mediational strategies which will ensure autonomous and effective problem solving skills in learners of mathematics.

4.3 The nature of adult cognition - A three level model of cognitive processing

The ability of individuals to monitor their own problem solving can be explained by postulating a three-level model of cognitive processing. Each level provides a foundation for the next one but is not subsumed by it. In other words, while the first tier may operate independently of the other two tiers, the reverse is not the case.

At the first level of cognition (level 1), individuals enter into cognitive tasks such as computing, memorizing, reading, perceiving, acquiring language etc.

The second level (level 2), *metacognition*, is defined as the processes which are invoked by the learner to consciously monitor cognitive progress when he/she is engaged in level 1 cognitive tasks. Metacognitive processes include knowledge about cognitive tasks (e.g., how to complete the square), about particular strategies that may be invoked to solve the task (e.g., getting into 'standard' form by making the coefficient of x^2 equal to 1), about when and how the strategy should be applied (e.g., when one is required to solve a quadratic equation) , and about the success or failure of any of these processes.

The third level (level 3), *epistemic cognition*, is characterized as the processes an individual invokes to monitor the epistemic nature of problems and the nature of the task which demand specifiable solutions. It includes the individual's knowledge about the limits of knowing, the certainty of knowing, and the criteria for knowing. It also includes the strategies used to identify and choose between the form of the solution required for different problem types. Most importantly from a mathematical point of view it should include the strategies used to identify those elements of a puzzle type problem that are necessary for its solution and those elements that are superfluous and those elements that are missing. Epistemic cognition involves the use of synthesis and analysis which able students employ in solving mathematical problems (cf. Krutetskii, 1976). Epistemic cognition thus includes the ability to distinguish between clear and disguised puzzles and the ability to deduce the pure standard form of an embedded standard form type puzzle (see Fig. 2, p. 62).

Both metacognition and epistemic cognition require LM learning. LM learning is responsible for the formation of executive schemes (cf. p. 48). Good level 1 cognition is essential when solving mathematical tasks as it allows 'mind' to be 'free' (i.e. little mental energy is required for learning basic skills that are common to all mathematics problems-skills, such as addition, subtraction etc.) to use metacognitive and epistemic cognition skills for effective engagement in mathematics problems/tasks. Hence 'drilling' learners to memorize basic rules may be a necessary part of mathematics teaching as it will provide the learner with more mental energy (little energy is required to recall the rote learned rule) to attend to the more complex mathematical instructions or task demands. However, what rules to drill is a complex problem.

Part of what the under-prepared learner brings to the situation is over-learned misleading and even wrong rules (cf. p. 87). Part of the present project and plans for future research (cf. p. 123) is to identify what to drill learners in; what, in the sense of identifying (through task analysis), are the most important and effective strategic steps and rules.

At tertiary undergraduate level most of the puzzle problems are of the class 1 type (clear puzzles) and some problems may require only level 1 type cognition (for example "find $\frac{d}{dx}x^2$"), while others may require levels 1 and 2 (for example, "find $\frac{d}{dx}\frac{1}{x^2}$"), and others may require all three levels (for example, "find $\frac{d}{dx}(x^2+1)^2$").

Epistemic cognition

Acknowledging epistemic cognition is a vital aspect of learning-teaching mathematics as this level may be responsible for the formation of content-less processes such as synthesis and analysis, etc. where LM learning takes place.

At this third level cognition the teacher must help the student distinguish between those problems which are ill-structured, clear puzzles or disguised puzzles. The teacher must also show the learner which strategic steps to engage in, in order to identify the pure standard form embedded within a puzzle, by emphasizing the difference between a pure standard from clear puzzle and a puzzle where the pure form is embedded within it. The learner must be made aware of the different classes of puzzles (cf. Fig. 2).

As mentioned before problems which are of the disguised puzzle type (class 2 puzzle problems) and problems of the embedded standard form type require epistemic cognitive strategies, to convert the problems into a class 1 type puzzle or pure standard type puzzles, respectively . In Krutetskii (1976) it is stated that in disguised puzzles under-prepared learners at first perceive only disconnected facts; they are 'riveted' to the concrete data from the outset. The learners do not perceive or feel the hidden question in the problem. Masking puzzle problems using real life terms is a great obstacle to such under-prepared learners. The reason for this difficulty may be because the learner's $\underline{M}$ power is being totally taken up by the concrete data and, therefore, the learner cannot attend to the problem as a whole[10].

In a puzzle where integration by parts is required for a solution it is necessary for the learner to write the standard form " $\int du \times v = u \times v -$

[10]Problem (2b) contains the word 'capped' which traps the students into thinking of a hat so they could not even attempt to solve the problem as they could not relate mathematics to the real world (hat).

$\int u \times dv$" in a new form " $\int \square \times \triangle = (\square)^I \times \triangle - \int(\square)^I \times (\triangle)'$" in order to force the learner to attend to contentless forms in order to remove the weight of concrete data (cf. p. 9).

One of the main reasons why disguised puzzles require epistemic cognition is because of the need for the learner to distinguish between those elements which are constant and those elements which vary (eg. Problem (2a))[11]. In problem (2b)[12] the perimeter is fixed while the radius and height of the rectangular part is to vary. The final solution to both problems involves fixed parameters which tends to contradict the fact that the unknowns vary initially. In converting the puzzle (2a) to its pure form the learner must set up the two equations $2x + y = 20$ and $A = x \times y$ (where x is the width of the rectangle and y represents the length of the brick wall). The learners who gave incorrect solutions could not understand how y could be fixed (as the length of the wall) yet vary as a function of x. They wrote down the equation $2x + 2y = 20$ instead of $2x + y = 20$ as they were used to a similar problem where *both* dimensions varied (i.e. where all four sides of the enclosure are made of fencing). This demonstrates how overlearning interfered with their ability to extract the clear puzzle from the disguised puzzle. In problem (2b) the learners (of the Engineering Bridging Unit)

[11]One side of the rectangular enclosure is to be made of brick and the remaining sides out of fencing. Given that the fencing is 20m long what dimensions of the rectangle will give maximum enclosed area ?

[12]A mirror in the shape of a rectangle capped by a semicircle is to be made. Given the perimeter is k meters, what radius of the semicircle will give maximum mirror area ?

did not even attempt the problem as the word capped confused them.

Epistemic cognition is necessary in extracting the pure standard form from an embedded standard form type puzzle, or for deciding what stages are required for its conversion to the pure standard form. Epistemic cognition is also necessary in identifying the clear puzzle within a disguised puzzle.

Metacognition

Capable students, or those students who learn mainly via LM structures, have the ability to monitor their own progress successfully (they have good level 2 type cognition) while less able learners have a problem with deciding what strategic steps are needed for a given puzzle type problem and whether they have applied a chosen strategy correctly or not. Learners have great difficulties in solving clear puzzles of the embedded standard form type, as they cannot easily extract the pure form embedded within the puzzle. The very nature of the problem hinders their ability to solve the problem. The learner brings to the problem solving situation tacit (often incorrect/misleading) rules which tend to override (cf. The SOP principle, p. 50) the appropriate rules needed to solve the problem.

Relating these observations to teaching, we see that with regard to second level cognition, the teacher must provide *cues* or interruption rules so that the learner can monitor his/her own progress and prevent the learner from applying spontaneously his/her incorrect tacit rule. These cues can be used to overcome the confusion of the specific with the arbitrary. For example, when the learner is faced with the puzzle: "If $f(x) = \frac{1}{1+x^2+1}$, find $f(x+h^2)$" the teacher can provide the learner with the metaphor (cue) "If you see $f(x)$ think of $f(blob)$ so that the learner can convert the puzzle into the form: "$f(blob) = \frac{1}{(1+(blob))^2+1}$" which will assist him/her in solving the puzzle[13].

Since most university problems are of the clear puzzle type we shall concentrate on those types in addressing the concept of the zone of proximal development. However there is certainly a problem when it comes to students distinguishing between ill- structured problems, clear and disguised puzzles. We shall address these problems in the next section.

4.4 Zone of proximal development created for the mathematical task

As mentioned in section 1.1, leaving 'mind' to develop 'what it takes'

[13]The ability to understand that x in an equation can represent both a specific number (token) and a 'general' variable (type) requires epistemic cognition, but to assist the under-prepared learner in grasping this distinction *metacognitive* cues such as the metaphor mentioned are provided. The acquired metacognitive skill will in turn lead to the development of efficient, autonomous problem solving skills. It is worth mentioning at this point that discipline specific academics often display a resistance to the use of such metaphors as 'blobs'. This resistance is similar to those advocating a standard English (or any other natural language) over the development of *non*-standard English such as American English, black American English, etc. Important for the present study is, however, that a metaphor such as 'blob' allows the teacher to focus the learner away from a concrete point of information to the type or abstract notion required for problem solving.

and 'what makes' adaptation to solving mathematical problems possible usually suffice, but when 'it' has not developed or is not expressed as in the case of under-prepared students, we may have to attempt something else than follow the 'spontaneous model of learning'.

When a learner engages in a task, that which has already been assimilated by the subject of knowledge is applied to the task. The transaction between the subject and task or situation constitutes the (possible) object of knowledge. This object of knowledge may, therefore, have embedded in it:

(A) What the learner is already familiar with- past experiences, knowledge, skills, etc. - what the learner brings to the situation.

(B) The particular demands of the task, in terms of its own historical development; particulars or demands which may be to a lesser or greater degree at odds with what the learner brings to the situation : What the task has come to mean, regardless of who engages in it. We know that typical university mathematical type tasks and their demands are unfamiliar to most under-prepared students.

(C) What the learner may incorporate into his/her repertoire of knowledge/skills/etc. or what may be 'learned' from the interaction with the task. What may be learned may be, to a lesser or greater degree, at odds with what a teacher intended, or meant, or required by presenting a particular task to students. We know that the way most lecturers define typical university mathematical problems and how they understand these in terms of appropriate responses towards solving them are different from the way under-prepared students define the learning-teaching situation and how they understand the problems; also, merely allowing students to engage in typical mathematical problem solving tasks does not seem to generate the learning of appropriate knowledge/strategies, speedily and effectively enough, in the case of under-prepared learners (cf. Craig, 1988b).

The conflict between what the learner brings to the situation, and what the task demands, creates the intrinsic driving force for acquiring new knowledge about the task and problem solving in general (cf. the acquisition of knowledge, p. 36 ; the learning paradox, p. 39 and equilibration, p. 38).

This model, however, indicates what will happen spontaneously over time. But what we need, if we are to intervene in the slow process of adaptation, is to make explicit those rules and/or strategic steps which capable learners/teachers use *spontaneously*. In those cases where 'time' did not create the necessary learning opportunities (cf. Van Den Berg), it may be necessary to create a 'bridge' between
(1) what the learner brings to the situation, and
(2) the particulars or demands of the task.

In the analysis so far we see that under-prepared learners may come to university equipped with the potential to deal with mathematical tasks but they also bring with them tacit, overlearned often incorrect/misleading rules and, for different reasons, have not learnt to use their mental capacity to its maximum potential. A bridge between (1) and (2) may **consist of what the learner *NEEDS TO KNOW/DO* in order to engage effectively and independently in the task** which translates into Vygotsky's notion of the *ZONE OF PROXIMAL DEVELOPMENT* (cf. section 3.2, p. 29).

This project is concerned with the zone of proximal development mainly from the point of view of the learner- i.e.

(a) what the learner brings to the situation (which includes tacit (incorrect) rules for doing maths (cf. Chapter 5, p. 87), and

(b) what the learner needs in order to master the mathematical task, which may include the following:

(i) the basic arithmetic rules such as addition, multiplication etc.,

(ii) mediational strategies in the form of hints/cues which are necessary to help the learner overcome the overlearned incorrect tacit rules (cf. Chapter 5, p.87),

(iii) a knowledge of the nature of the mathematical problem that the learner is expected to solve (i.e. the learner needs to know the different kinds of mathematical puzzles. These are disguised or clear, non-standard or standard, embedded or pure) and a knowledge of the necessary steps required for the conversion of a given maths puzzle into the most pure form (cf. section 4.2.1, p. 61),

(iv) cognitive skills which include level 1, 2 and 3 cognitive skills discussed in section 4.3, p. 73.

The cognitive skills which the learner needs in the case of typical university mathematical problems may be conceptualized in terms of epistemic and metacognitive operations[14] and because they form an important part of the zone of proximal development further elaboration is necessary:

Epistemic cognition

At this level the learner must be 'taught' or enabled through appropriate mediational strategies how to distinguish between ill-structured problems, clear and disguised puzzles (cf. section 4.2, p. 57). It is here that the learner must develop the strategies to convert disguised puzzles into clear ones - a contentless operation which most teaching-learning situations at university take for granted. The learner must be taught how to pursue goal directed strategic steps towards task completion, that is, how to find those elements of the disguised puzzle which are necessary for its solution and those elements which are superfluous. The learner must also be able

[14]'Operations' because it is both the cognitive capacities of learner and teacher, in transactions, *and* the learning-teaching strategies necessary for successful mastering of mathematics problems which is emphasized. In other words, the explication of mediational strategies are meant to highlight those moments in instruction which a teacher may use to stimulate the development of or expression of executive schemes necessary for adaptation to the unfamiliar demands of complex puzzle tasks.

to identify the pure standard form embedded within a clear puzzle which is not of the pure standard form. He/she must be able to extract the pure form which is entangled in a mass of words/symbols/functions. The role of the teacher in helping learners develop such contentless processes, will be part of future research and it is sufficient at this stage for the learner to be provided with examples of each kind of puzzle and the necessary steps for their conversion to the most purest form.

Conscious monitoring of engagement in a task or meta-cognitive skills

This level involves the learner (consciously) monitoring his/her memorizing, computing, reading, etc. (see section 4.3, p. 73). Such monitoring is required when dealing with clear puzzles especially in deciding whether the puzzle must be simplified into standard form or not and whether a clear puzzle is of the pure standard form or of the embedded standard form type. This conscious monitoring is also necessary when dealing with disguised puzzles in converting them into clear puzzles.

The teacher initially *externally* guides the cognitive operations of the learner by presenting him/her with a set of rules or cues (mainly illustrated by using puzzles of the pure standard form type, for example $\frac{d}{dx}x^n = nx^{n-1}$) which can be used to distinguish between various types of puzzles. This 'other regulation' (cf. Vygotsky, section 3.2, p. 29) must be *internalized*, so that the learner can guide his/her progress (self-regulation). However the very nature of puzzle type problems (the fact that they can be embedded standard from puzzles or non-standard form clear puzzles or disguised puzzles) tends to prevent the learner from applying the appropriate rule (for example, in the puzzle: "Find $\frac{d}{dx}\frac{1}{x^n}$ the learner 'sees' only the pure standard form part x^n below the division line and applies $\frac{d}{dx}$ to this part only and 'does not see' the division sign). Standard form type rules (e.g. $\frac{d}{dx}x^n = nx^{n-1}$ are not sufficient for under-prepared learners. Additional rules/cues are necessary to prevent learners from 'rushing' to apply an overlearnt rule (e.g. the learner must recognize that the form x^n is different from the form $\frac{1}{x^n}$ -the division sign must itself be a cue to interrupt the learner from applying the standard rule).

Remedial programmes tend to concentrate on providing the learner with a large number of typical mathematics problems which learners have difficulties with. This is done with the hope that sufficient *repetition* will help learners overcome these difficulties and master such problems (such a programme was designed by the Ohio State University group -Crosswhite, Demana, Leitzel and Osborne, 1984). A more appropriate approach would be to strengthen the learners' executive schemes, by the use of cues, metaphors etc., which will help monitor and drive their mental engagement during problem solving tasks and help distinguish between the various types of puzzles. This approach will be less time consuming as the 'repetitive' type approach.

A catalogue of typical errors made by learners in performing mathematical tasks at first year university level, together with interruption rules or regulative cues to help prevent them, has been prepared (cf. Chapter 5, p.87 & Appendix A, p. 129) and should be made available to under-prepared students pursuing mathematical courses at university as soon as they begin studying at university. These regulative cues will help individuals check their progress while they perform puzzle type tasks and thus enable them to consciously monitor their engagement in the learning-teaching of mathematics.

The method used to find these errors and regulative cues was by analyzing (i) written work of learners and (ii) video recordings of the teacher guiding learners through mathematical problem tasks (cf. section 1.2, p. 7).

CHAPTER 5

DATA OF LEARNERS' ENGAGEMENT IN MATHEMATICAL TASKS AND THEIR INTERPRETATION

5.1 Data collected

Two classes of data were collected for analysis . The first class (class A) involves a collection of problems (taken from lectures and tutorials) which students have solved incorrectly or have simplified when no simplification was possible (cf. Appendix A, pp. 129-142).

The second class (class B) consists of students responses to a schedule in which mathematics problems, the students solution of these problems, and the rules he/she believes ought be used, has been dis-aggregated (cf. Appendix B, pp. 143-158).

5.2 Class A data

Seven main groups of 'tacit rules' were derived from the first class of data. Each of these groups will be interpreted in terms of the zone of proximal development from the point of view of the learner, i.e.:

(a) That which the learner brings to the task (e.g. tacit rules for doing mathematics).

(b) That which the learner needs to know to master task (e.g. both epistemic and metacognitive operationary cues/rules, the nature of the maths problem (cf. section 4.2.1, p. 61 & section 4.3, p. 73).

With regards to (a) the following comment is necessary:

The learners' *tacit(incorrect) rules* which we shall provide are what we have reconstructed from the different data bases and the learners performances on tasks set and which represent the rules which the learners use to do the mathematical tasks. Given their rules (and if they exist) these would *explain* students performances in mathematics problem solving[1].

Clearly not all mathematical puzzles can be classified according to the classification provided in section 4.2.1 on page 61. For example, those puzzles which require induction for their solution, or puzzles which involve proofs of theorems cannot necessarily be classified according to section 4.2.1. Finally it must be mentioned that this clear/disguised puzzle distinction, discussed in chapter 4, pertains to an 'ideal' way in which one may view certain mathematics problems. However, as the examples in this section will illustrate, ordinary learners/doers of mathematics do not have such a clear perspective of mathematics problems. Our proposal is however, that if the ordinary learner of mathematics were to be given this classification of puzzles, he/she will appreciate the nature of the task demands (epistemic cognition) and therefore the development/expression of appropriate executive schemes.

[1] For a complete explication of the methodological paradigm cf. Craig 85, 88a & 88b.

5.2.1 Tacit rule # 1

(a) *What the learner brings to the situation.*

This tacit rule involves the following (cf. pp. 129-131):

> *Those rules which apply to addition and subtraction of functions also apply to multiplication and division of functions, and vice versa.*

For example, $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ suggests (incorrectly) that $\frac{d}{dx}f(x).g(x) = \frac{d}{dx}f(x).\frac{d}{dx}g(x)$. The learner *distributes* the $\frac{d}{dx}$ over the addition *and* the multiplication.

(b) *What the learner needs to know in order to master the task.*

Students must be made aware of the fact that rules which apply to addition/subtraction do not necessarily apply to multiplication/division and vice versa. The student must check himself/herself whenever he/she encounters $+/-$ or $\times/\div$ and ask himself/herself what rules are allowed to be used.

For example: to distribute a power over a product, the rule to use is e.g.: $(a.b)^3 = a^3b^3$. However $(a+b)^3 \neq a^3 + b^3$. Thus one can distribute powers over multiplication/division but *not* addition/subtraction. Conversely, $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ while $\frac{d}{dx}f(x).g(x) \neq \frac{d}{dx}f(x).\frac{d}{dx}g(x)$. This means that one can distribute a 'derivative' over addition/subtraction but *not* over multiplication/division. Also the fact that $(a+b)^1 = a^1 + b^1$ does not imply that this result will be true for any power other than 1.

The first and third example of rule # 1 (i.e. $(a+b)^{-1} = ?$ and $\sqrt{x^2+1} = ?$) are not puzzles since they cannot even be simplified. The second and fourth examples (i.e. $\frac{d}{dx}[x^2.(x-1)^7]$ and $\frac{d}{dx}\frac{x^2}{x^3+1}$) are clear puzzles of the embedded standard form type (cf. section 4.2.1 p. 61).

5.2.2 Tacit rule # 2

(a) *What the learner brings to the situation.*

This tacit rule for doing mathematics involves (cf. pp. 133, 134):

> *The learner 'forcing' himself/herself to simplify when no simplification is possible.*

For example $log(A+B)$ is simplified (incorrectly) to $log(A) + log(B)$. This might be because $log(A).(B)$ can be written as $log(A)+log(B)$ perhaps suggesting a # 1 tacit rule. Another problem involves the learner 'forcing' himself/herself to simplify $\frac{a}{b+c}$ to $\frac{a}{b} + \frac{a}{c}$.

This tacit rule could also be partially related to the tacit rule #1 since the learner is forcing himself/herself to distribute either powers or derivatives over addition/subtraction or multiplication/division[2].

[2]In general, each of the tacit rules will not function independently nor indicate mutually exclusive domains.

(b) *What the learner needs to know in order to master the task*

The cues/interruption rules the learner needs in order to master such mathematics problems are provided in the tables on pages 133, 134 and below. For example, the learner should emphasize the '+' sign in $log(A+B)$ and realize that one cannot distribute a function (such as log) over addition. Perhaps the learner could regard $log(A+B)$ as a banana where log(...) is the 'skin' of the banana and the inner part $(A+B)$ is the 'flesh' of the banana - one cannot 'cut' the banana at the + sign and create the *sum* of *two* separate bananas since one would have two parts of the same banana. This will also be true for *functions* in general -i.e. $f(A+B) \neq f(A)+f(B)$. However the use of Real world term banana might cause difficulties for some learners (cf. Janvier, 1987) so that more research is required to find the most effective way of overcoming this problem of learners forcing themselves to distribute functions over addition. Note that the problem of learners writing $log(A+B)$ as $log(A)+log(B)$ is similar to the problem of learners (incorrectly) distributing powers over addition (they write $(a+b)^2$ as a^2+b^2) which points to the need to emphasize *functions* <u>and</u> *addition.*

The first two examples of rule # 2 (i.e. $log(M+N)$ and $\frac{a}{b+c}$) are once again not puzzles since they cannot even be simplified. The last example (i.e. $\frac{d}{dx}(x+1)^2 \times cos^2x$) is a clear puzzle of the embedded standard form type. Notice here that the learner first simplified incorrectly (by writing $(x+1)^2 \times cos^2x$ as $[(x+1) \times cosx]^{5/2}$) before applying the rule (differentiation) which illustrates the interwovenness of metacognitive and epistemic operations.

5.2.3 Tacit rule #3

(a) *What the learner brings to the situation.*

The type of problems (cf. p. 135 and below) where this tacit rule is applied illustrates the difficulty under-prepared learners have with the token/type distinction. This tacit rule may be stated as follows:

> *The x which occurs in a problem remains fixed as a* specific x *wherever it occurs throughout the problem, i.e. x is always the same throughout the problem.*

For example, the learner regards the x in $f(x)$ as the same as the x in $f(x+h)$ in the puzzle: " If $f(x) = x^2 + 2x + 1$, find $f(x+h^2)$".

The idea that a variable such as x which occurs in a mathematics problem is a 'dummy' variable is difficult for many learners to grasp.

(b) *What the learner needs to know in order to master the task.*

In order for the learner to overcome the above problem, he/she must replace *each* x in the equation $f(x) = ...$ with a meaningless metaphor for the type x, for example, 'blob' (a symbol such as *(star) could be used for 'blob')[3]. Then what ever is done to * on the left hand side must be done to each * on the right hand side. For example, $f(x) = x^2 + 2x + 1$ becomes

[3] This approach has been found to be very effective in assisting learners overcoming the difficulty concerning the role of the variable x in a puzzle. Part of future research (see p. 123) will be to validate the teaching strategies proposed here experimentally.

$f(*) = (*)^2 + 2(*) + 1$ so that if $f(x + h^2)$ is required then the * in the left and the 'stars' in right hand side each become $(x + h^2)$.

The example of this puzzle as stated above is a clear puzzle of the embedded type and requires epistemic cognition for its conversion to a pure standard form.

5.2.4 Tacit rule # 4

(a) *What the learner brings to the situation*

This tacit rule is a result of overlearning as discussed in (b) below. This rule is as follows (cf. p. 136):

> *The rules you apply to equations can also be applied to non-equations*

For example, the learner adds 'the square of half the coefficient of x' to form a perfect square in the puzzle "complete the square: $x^2 + 3x + 3$" but 'neglects'[4] to compensate by subtracting this amount as he/she is used to dealing with equations where he/she only *adds* to both sides of the equation.

In these type problems the learners have developed 'mechanical' or recipe-like procedural rules for solving certain types of equations and automatically apply them to non-equations. This alerts one again to the problem of *what* is memorized (see pp. 52, 53).

[4]Here 'neglects' refers to the learner not consciously attending to, or applying, the necessary operations.

(b) *What the learner needs to know in order to master the task.*

The appropriate way to overcome such difficulties would be for the learner to ask himself/herself whether the problem at hand has an equal sign or not, and realize that rules used in solving equations cannot necessarily be applied to non-equations. The reason for this (which the learner should be made aware of) is the following: equations consist of two 'sides' (the left and the right) and those rules which can be applied to equations involve 'doing things' to *both* sides which is impossible in the case of non-equations.

For example, when a learner has to complete the square he is required, at a certain stage of the problem, to 'add the square of half the coefficient of x ' (rule 1). Such an operation on an equation is usually linked to another rule (rule 2) ' what you do to one side of the equation you do to the other side'. Since learners perform completing the square on equations rather that on non equations (cf. Matric text book by Dreyer, 1972, pp. 135- 136) this linking results in rule (1) for equations being heavily weighted so that the rule for non-equation ('adding the square of half the coefficient of x and *compensating*') is not as likely to be applied.

The example of this puzzle mentioned above belongs to the first class of puzzles, i.e. clear puzzles, and is of the pure standard form so that mainly metacognitive skills will be required for its solution.

Another example where this tacit rule may be in use is the following puzzle: 'If $f(x) = 3x^2$, find $3f(1) + 2f(2)$'. The incorrect solution was unique to black students. This solution was of the form '$\frac{f(1)}{3} + \frac{f(2)}{2}$' which involved the learners evaluating f(1) and f(2) and then *dividing* the first by 3 and the second by 2. The reason why this problem was included in this section involving tacit rule # 4 is because the learners may be using the 'invert' rule which is applied to equations (for example a/b=c/d becomes b/a=d/c when 'inverting' both sides). However the learners apparantly only inverted 3 and 2 while leaving f(1) and f(2) unchanged. The students themselves could not explain what had caused them to solve the problem in this manner. This problem will be subject to further analysis in future research.

5.2.5 Tacit rule # 5

(a) *What the learner brings to the situation.*

Once the learner internalizes a new rule which is to be applied to a certain function, that function takes on a specific character(token) which the student finds difficult to separate from its generalization (type). The learner tends to see only a fixed 'outer shell' and not a generalized form of the given function.

For example, the learner has a rule to differentiate x^n. He is then expected to differentiate, for example $(x^2 + 1)^n$. The learner tends to focus on the <u>outer</u> $(...)^n$ part of the problem (i.e. the shell) and applies the same rule without consciously monitoring the application against appropriate cues/rules which must be applied to the standard form x^n. We could perhaps state this tacit rule as follows:

> *Those rules which apply to the standard form (y=f(x)) also apply to the general form (y=f(g(x))),* without any further rules required.[5]

A further example of this type of puzzle is " Evaluate $\frac{d}{dx}e^{cosx}$". The learner tends to see only the 'shell' part $(e^{(\cdots)})$ and arrives at the incorrect answer of e^{cosx} since he/she is used to the standard form: $\frac{d}{dx}e^x = e^x$.

(b) *What the learner needs to know in order to master the task.*

The teacher should give the learner a rule which will work for both the standard and general form. For example, to overcome the problem mentioned above the learner can be given the rule $\frac{d}{d*}(blob)^n = n(blob)^{n-1} \times \frac{d}{d*}(blob)$. This rule will work even if $* = blob$ since $\frac{d}{d*}(*) = 1$.

The examples of these puzzles provided above and on pages 137- 140 are clear puzzles of the embedded standard form type. Embedded in the first example (Evaluate $\frac{d}{dx}(x^2+1)^2$) is the puzzle " Evaluate $\frac{d}{dx}(u)^2$ where $u = x^2+1$". The learner is not affected by the fact that the pure standard form has been altered so that the 'tampered' form 'looks like' the standard form and the learner once again applies tacit rules without conscious monitoring.

[5]The standard form $y = f(x) = e^x$ has the general form $y = e^{g(x)}$, i.e., $y = f(g(x))$.

Embedded in the puzzle: "Evaluate $\frac{d}{dx}e^{cosx}$" is the pure form: "Evaluate $\frac{d}{du}e^{u}$ where $u = cosx$".

5.2.6 Tacit rule # 6

(a) *What the learner brings to the situation.*

This tacit rule might be as a result of the basic rule 'if you multiply through by a minus, *all* signs must change' (for example, $-(a - b + 2c) = a + b - 2c$). The learner believes that a change in a sign requires all other signs to be changed (which is true in the case above) regardless of what has brought about the change of that particular sign. This rule could be stated as follows (cf. p. 141):

> *If the power of a number is negative, to make it positive, take the number to the bottom and change* all signs.

For example the learner simplifies the problem $-2(-1)^{-1}$ incorrectly by taking the power term $((-1)^{-1})$ to the 'bottom' and changing *all* signs to obtain: $\frac{2}{(1)^{1}}$.

(b) *What the learner needs to know in order to master the task.*

The learner must be made aware of the fact that taking the number to the 'bottom' results in the sign of only the *power* changing.

The examples provided above when presented to the learner are obviously clear puzzles, but the part to be simplified (for example, the negative power in $(-1)^{-1}$) is entangled within a mass of other numbers and signs which helps contribute to the learners confusion. Hence we could say that these problems are not 'pure' standard form clear puzzles but are of the embedded type. Learners, therefore, need rules or cues for disembedding- e.g. extract only that part of the puzzle which has the negative power attached to it, change only the sign of the power when taking the part to the bottom (or top) and re-embed in the remaining part of the puzzle.

5.2.7 Tacit rule # 7

(a) *What the learner brings to the situation.*

The learner is conditioned to applying a rule to a certain type of function (overlearning), and when a function 'looks like' this particular function the learner spontaneously and unconsciously applies the rule without first simplifying (cf. example on page 142 and below). This tacit rule can be stated as follows:

> *A rule which can be applied to a certain function can be applied to functions which 'looks like' this function even though simplification is necessary for the conversion into the proper form.*

For example the learner is required to solve the puzzle: "Evaluate $\frac{d}{dx}\frac{1}{x^n}$". The learner tends only to see the standard form part $\frac{d}{dx}x^n$ and applies the rule for the standard form without simplifying.

(b) *What the learner needs to know in order to master the task.*

The cues or interruption rules the teacher can provide is provided with the example on page 142 and which we mention below. Here the learner should be made aware of the exact form the standard form takes on-the function to be differentiated is on top (the x^n in the standard form $\frac{d}{dx}x^n$). The problem is presented with the standard form below. When the learner sees the 'one over'(1/) sign in the puzzle " Evaluate $\frac{d}{dx}\frac{1}{x^n}$" he/she must *emphasize* the 'over'(division) part and realize that the puzzle is not in pure standard form. This problem further illustrates how learners tend 'not to see' certain mathematical symbols, as strong 'weights' have been given to other units of the problem as they have been encountered, often as a result of overlearning/overteaching (cf. the SOP principle on p. 50).

The puzzle described above is a clear puzzle of the non-standard form type (see discussion in section 4.2.1, p. 61) and simplification is required to convert the puzzle into a pure form. In this similar example "find $\frac{d}{dx}\frac{1}{x^2}$" the learner tends to see only that part of the problem (x^2) for which he is accustomed to applying the rule (differentiate).

The above problem illustrates the effect that overlearning the wrong rules or recipes[6] has on hindering the learner performing given mathematical tasks. Often the learner will apply the quotient rule to this problem

[6] Future research will involve the analysis of school mathematical tasks/mathematics text books to identify overlearned, misleading or wrong rules.

$(\frac{d}{dx}\frac{1}{x^2})$ and obtain the correct answer. However, although this method is correct, it is not the best method as it is long-winded and it will take much longer to solve the problem. A great amount of M power will be required and little mental power will be left to attend to other, more important, aspects of the problem. The capable learner (cf. p. 66) will spot the short-cut immediately and obtain the solution quickly. However in the present case, the aim is to explicate what the capable learner does spontaneously.

In conclusion it is appropriate to mention that unlearning is a major problem for teachers. There is an implicit idea that an under-prepared student is one without skills and knowledge. However, more important is the fact that they have different skills and knowledge from what the task demands: Skills and knowledge which prevents engaging in appropriate strategies for problem solving.

5.3 Class B data

This data base involved students responses to a schedule (cf. Appendix B, p. 143) in which five mathematics problems were presented to the learners[7]. The students were asked to record the rules he/she believed were necessary to solve the each problem and to actually solve each problem. The learners' responses were graded either 1, 2, 3, or 4 where:

[7] These five problems were selected because they had provided much 'conflict' for mathematics learners, at the university of Natal, in the past. Learners tended to solve these problems in a similar (incorrect) way which is why they were selected for further scrutiny. They represented typical errors.

A grade of 1 implies that the learner has written down the correct rule(s) and has produced the correct solution for a particular problem.
A grade of 2 implies that the learner has written down the wrong rule(s), but has produced the correct solution for the given problem.
A grade of 3 implies that the learner has written down the correct rule(s), but has produced an incorrect solution for the given problem.
A grade of 4 implies that the learner has written down the wrong rule(s), and has produced a wrong solution for the given problem.

The gradings 2 and 3 are primarily of interest because a grading of 2 implies either that the learner obtained the correct solution by accident or that the learner had internalized the necessary rule without being consciously aware of it. A grading of 3 implies either that the learner obtained the incorrect solution by accident or that the learner did not understand the rule or had not internalized the rule to a degree where it could be applied successfully.

The subjects (first year science students registered for mathematics 1 at the university of Natal) were divided into seven main groups:
75% -100% (N=19); 68% - 74% (N=11); 60% -67% (N=10); 50% - 59% (N=18); 40% - 49% (N=17)[8]; 25% - 39% (N=23); 0% - 24% (N=15). The percentages are (average) percentages which the learners have obtained for

[8]Only two science students from this group managed to pass the June mathematics examination.

tests during the first semester of 1988.

The responses to each problem was organized in the following way: The number of 1, 2, 3, and 4 graded responses were recorded alongside each group for each problem. We shall deal with such organization for each of the five problems separately and call such organization tables frequency tables. The analysis of each problem will be similar to that of section 5.2 where aspects of Vygotsky's zone of proximal development was incorporated, that is:

(a) What the learner brings to the situation.

(b) What the learner needs to know in order to master the task.

The statistical analysis performed on the frequency tables for the five problems can be found on pages 154-158, and the method used is based on the Loglinear Model. Each entry under AB refers to the entry in the i'th row and j'th column, respectively, in a frequency table. Such entry will be referred to as the 'i,j' entry (or cell) of the 'matrix' corresponding to a frequency table. Of primary importance is the z-value for an i,j cell which idicates whether there is a significant association between an i'th row (the learners' performance in tests) and a j'th column (the learners' performance according to the given schedule). If $\mid z \mid > 1,96$ for an i,j cell then there is a significant association between row i and column j and one writes $p < 0,05$.

5.3.1 The first problem: Solve for A: $A(A-4) = -4$.

The frequency table corresponding to this problem is as follows:

Groups	# 1	# 2	# 3	# 4
75-100%	18	1	0	0
68-74%	6	4	0	1
60-67%	7	2	0	1
50-59%	11	6	0	1
40-49%	8	3	1	5
25-39%	13	9	1	0
0-24%	8	4	2	1

The the z-values of the 1,1 entry and the 5,4 entry on page 154 indicate that there is a strong correlation between the groups graded responses and their corresponding percentages ($p < 0,05$).

Because of this link between the grading and the percentages obtained in tests, one may conclude that the learners in the range 40-49% (row 5) below average performances in tests, is a result of their failure to solve this first problem and to produce the correct rule necessary for its solution. This problem involves tacit rules which are 'school rules'and those learners who have scored well in the # 1 grading include those who have performed poorly in their tests (0 - 39 % , rows 6 & 7). Therefore one may deduce that those learners who are competent with respect to basic school rules might not necessarily manage university type tasks.

On the other hand the top group (75-100%) performed exceptionally well when solving this puzzle (a high number of # 1's) because the rules necessary to solve the problem may have been internalized to such a degree that they could be produced spontaneously -a characteristic common to able learners.

(a) *What the learner brings to the situation*

Many of the solutions which the learners in the 40 -49 % group produced included the following:

$A = -4$ or $A - 4 = -4$ so that $A = -4$ or $A = -4 + 4 = 0$

The tacit (and incorrect) rule which the learners used was a result of the overlearning of the rule: " If $(x-a)(x-b) = 0$ then either $x = a$ or $x = b$. The learner tends to see only the '2 brackets multiplied together and the = sign' which interferes with the learner's conscious monitoring of the problem -the learner must be taught that there must be '= 0'- the right hand side must be 0 for the rule to apply. The learners tacit rule thus becomes:

$$\textit{If } (x-a)(x-b) = k \textit{ then either } x = a + k \textit{ or } x = b + k$$

(b) *What the learner needs to know in order to master the task*

The teacher must stress the '= 0' part of the rule and emphasize that the rule cannot be applied unless <u>both</u> (1) '2 brackets multiplied together' *and* (2) '= 0' occur in the problem. For example $(x-a)(x-b) = 0$ implies that either $x = a$ or $x = b$.

This problem is a clear puzzle of the non-standard form as simplification is necessary to get it into the pure form:
'$(A-2)(A-2) = 0$'.

5.3.2 The second problem: Evaluate $\frac{d}{dx}[x^2.(1+x^2)^{1/2}]$.

If one looks at the frequency table below corresponding to this second problem, one will notice a deviation in the second column (entry 5,2). The actual z value corresponding to this cell is *almost* significant (z=1,93416, p. 155)

Groups	# 1	# 2	# 3	# 4
75-100%	14	0	3	2
68-74%	6	1	1	3
60-67%	5	0	2	3
50-59%	9	0	6	3
40-49%	7	3	3	4
25-39%	8	0	7	8
0-24%	7	1	3	4

The group in row 5 had a high number of #2 graded responses. This could mean that those learners in this percentage range who responded with a # 2 may have internalized the product rule but could not consciously reproduce it. The majority of the learners who passed their tests knew how to consciously reproduce the rule and how to apply it correctly (they had a high number of #1 graded responses). These results may suggest that teachers should allow learners to have conscious access to the rules which they need to apply to mathematics problems, i.e. they must strengthen their metacognitive operations.

(a) *What the learner brings to the situation*

The tacit rule which the learners used (incorrectly) to solve the problem is the tacit rule # 1 in section 5.2.2, p. 89 (they distributed the $\frac{d}{dx}$ over the product confusing the rule which says $\frac{d}{dx}$ can be distributed over addition).

(b) *What the learner needs to know in order to master the task*

The cues/interruption rules for this problem are provided in section 5.2.2 (the learner must emphasize the $\times$).

This clear puzzle is of the embedded standard form type so that epistemic cognition is required to convert the puzzle into the pure form. The embedded form is: "Evaluate $\frac{d}{dx}u \times v$ where $u = x^2$ and $v = (1 + x^2)^{1/2}$".

5.3.3 The third problem: If $f(x) = 1/(-x^2 + 1)$, find $f(x + h^2)$

This problem once again illustrates the difficulty under-prepared students have with the token/type distinction.

The z-value which is of significance is the 5,1 entry (cf. p. 156). The frequency table is provided below:

Groups	# 1	# 2	# 3	# 4
75-100%	14	2	1	2
68-74%	9	2	0	0
60-67%	6	1	3	0
50-59%	12	6	0	0
40-49%	5	4	4	4
25-39%	9	6	6	2
0-24%	11	3	1	0

The group with percentages between 40 and 49 performed poorly when faced with this problem. Note the striking high occurrence of #1 gradings for the very poor learners (the group 0 - 24%) compared to the group in row 5. This indicates that learners who are able to solve such puzzles may not necessarily be able to master university mathematics. The reason for their high performance could be 'excessive drilling' (i.e. overlearning), during their schooling as this problem is also a school type problem. The group in row 5 also had a relatively high number of #4 responses indicating their difficulty with this type of puzzle, which may suggest that the nature of such puzzles is a result of their poor performance on university type tasks.

(a) *What the learner brings to the situation*

The tacit rule used by learners who fail to solve this problem can be found in section 5.2.3 on page 92 -they were confusing the specific (token) with the general (type), i.e. they regarded the x in $f(x)$ as the same as the x in $f(x+h^2)$ and there answers tended to be of the form $f(x)+h^2$.

(b) *What the learner needs to know in order to master the task*

The teacher strategies to help the learners to overcome the difficulties of this problem can also be found in section 5.2.3 (p. 92) which involves the learner replacing x in the problem with a 'blob'. Thus $f(x) = 1/x^2 + 1$ becomes $f(*) = 1/(*)^2 + 1$ so that the learner must replace *each* '*' with $x + h^2$ in the second equation to obtain the correct solution.

The nature of this puzzle (it is a clear puzzle of the embedded type) is surely the cause of a high percentage of learners (relative to the other groups) in the group 40 - 49 % failing to solve it and know what rule is necessary to solve it. It seems that mainly epistemic cognition is required to solve such problems which contributed to the poor performance by this group. Also the metacognitive cue 'If you see $f(x)$, think of '$f(*)$' will assist learners to successfully monitor their progress while dealing with such puzzles.

5.3.4 The fourth problem: Differentiate: $x.cosa$, where a is a constant.

The overall pattern of the frequency table associated with this problem differs drastically from the frequency tables of the first three problems in that there is a clear overall drop in the number of #1 graded responses.

Groups	# 1	# 2	# 3	# 4
75-100%	6	4	5	4
68-74%	3	2	2	4
60-67%	2	2	1	5
50-59%	5	2	8	3
40-49%	1	1	5	10
25-39%	3	1	6	13
0-24%	2	0	7	6

The nature of the puzzle differs from the first three puzzles - there is an 'a' as well as an 'x' in the puzzle which again highlights the difference between token and type as well as fixed notions of both tokens and type. The z-value corresponding to the cell 4,4 suggets a significant association between the 50-59 % group and their low number (3) of # 4 graded responses (cf. p. 157). Note that column 4 indicates that the group in the range 40 - 49 % performed relatively poorly in solving this problem (a high number of # 4 responses). The reason for this is provided in (a) below.

(a) *What the learner brings to the situation*

When the under-prepared learner solves the problem incorrectly it usually is because he/she regards the 'x' and the 'a' as as being variables, *even though it is explicitly stated that 'a' is a constant.* Most incorrect responses involved learners using the product rule to differentiate $x.cosa$ -they differentiated $cosa$ as if a were a variable. The tacit rule can be stated as follows:

When differentiating a function all *letters of the alphabet, which do not belong to words, are to be regarded as variables* except *the letter k which is* always *a constant*

(b) *What the learner needs to know in order to master the task*

This problem of regarding all letters in a problem as variables when differentiating is common to even the good students. The teacher must emphasize that one does not just differentiate but that one differentiates *with respect to a variable.* If the learner is faced with '$\frac{d}{dx}x.cosa$' the x is the variable while (unless explicitly stated) 'a' is a constant. The problem could have been '$\frac{d}{da}xcosa$' in which case 'a' is the variable. The learner must realize that the letter * occurring in '$\frac{d}{d*}(.....)$' emphasizes that * is the 'main' variable.

Note that the group that 'just passes' (50-59 %) managed not to perform too poorely when solving this problem problem ($p < 0,05$ for the 4,4 cell) which may suggest that the group that 'just doesn't make it' (40=49 %), could pass if they overcame this difficulty of distinguishing between variables and constants.

The nature of this problem is a very interesting one. This clear puzzle is of the embedded pure form type (cf. Fig. 2, p. 62). The pure form which must be extracted from the puzzle is: 'Evaluate $\frac{d}{dx}k.x$ where $k = cosa$'. The standard rule would be to 'take the constant k out and multiply it with the derivative of x^1'. Note that if the learner was given the puzzle in this pure form he/she would have no problem as 'k' is used to represent a constant. But if k is replaced with *cosa* the learner believes that the *cosine function* converts *the constant 'a' into a variable* so that *cosa* is not a constant.

The reason for learners difficulty with this type of puzzle is once again, because of overlearning the wrong rules/recipes. When differentiation is introduced in school or university the basic problems are always presented in the form involving k (for example, $\frac{d}{dx}(k.f(x)) = k.\frac{d}{dx}f(x)$), or involving actual numbers (e.g. $\frac{d}{dx}6x^2 = 6.\frac{d}{dx}x^2$). The student becomes conditioned to $\frac{d}{dx}$ (with emphasis on the x) and constants being either numbers or represented by the number k. Here I believe the best way to overcome these difficulties is to provide many variations of the same type of puzzle- for example: 'Find $\frac{d}{dt}(kt^2.x)$; $\frac{d}{dq}(x^4.p)$; $\frac{d}{dx}(x.e^a)$' etc. The teacher can also provide metaphors like $\frac{d}{d(blob)} function$ implies that 'blob' is the variable.

5.3.5 The fifth problem: Evaluate $\frac{d}{dt}cos^2t^2$.

The pattern of the frequency table related to this problem is similar to that of the pattern of the frequency table for the fourth problem discussed in section 5.3.4, page 109 (i.e. $\frac{d}{dx}xcosa$). There is an overall drop in the number of # 1 graded responses while an increase in the number of # 4 graded responses- the number of #4 graded responses being excessively high for the 'problem' group (the 40 - 49 % group)[9].

Groups	# 1	# 2	# 3	# 4
75-100%	9	2	2	6
68-74%	4	3	3	1
60-67%	3	0	2	5
50-59%	3	2	3	10
40-49%	0	1	1	15
25-39%	1	3	2	18
0-24%	0	1	3	11

There is a significant z-value for the 1,1; 2,4 and 5,4 entries ($p < 0.05$, cf. p. 158). Once again the nature of the puzzle -it is an embedded clear puzzle of the standard type -might be the reason for this trend. The significant z-values for the 1,1 and 2,4 entries are expected -able learners did not perform as poorly as the others while solving this problem.

(a) *What the learner brings to the situation*

There were two main (incorrect) approaches to this puzzle. The first involved the learner rewriting the puzzle as '$\frac{d}{dt}(cost)^2$'. The second approach was to solve the puzzle as follows: '$\frac{d}{dt}cos^2t^2 = -sin^2t^2$. The first approach illustrates the learner 'forcing' the problem to look like a problem he/she can solve- they tend to combine the two squares into one to get only one square. The other approach emphasizes the effect of overlearning- the

[9]This 'problem' group is such because, intuitively one feels that "they are almost there": i.e., almost passing - but this may be a dangerous assumption because they seem, from the analysis, to be the group which is functioning at their maximum. In future research this may become a selection index when further scrutinized empirically.

learner knows that $\frac{d}{dx}cosx = sinx$ so he/she only concentrates on the 'sin' part of the puzzle and ignores the rest of the puzzle.

(b) *What the learner needs to know in order to master the task*

Since this is a clear puzzle of the embedded standard form type the teacher must show the learner how to extract the pure form from the puzzle. The pure form of the puzzle is 'find $\frac{d}{dt}(u)^2$ where $u = cosv$, $v = t^2$. Many learners fail to realize that $cos^n x$ is actually the same as $(cosx)^n$. The appearance of the two squares in the puzzle tended to add to the confusion surrounding the problem. The use of *brackets* is essential in these types of problems- learners often tend to *insert* brackets incorrectly. For example, students often write $cost^2$ as $(cost)^2$ which is incorrect as the absence of brackets in the first expression implies that the square only applies to the t and not to the cosine function. Also there are cases where brackets are ignored - for example, learners often write $(sin(2x))x$ (incorrectly) as $sin2x.x = sin2x^2$. Brackets are powerful and important tools in dealing with mathematics problems and learners must be taught that brackets serve as cues themselves -ways of doing this will be part of future research. Brackets function like interruption rules in the sense that they delineate aspects of parts of the whole which needs to be attended to.

This puzzle is a clear puzzle of the embedded standard form type so that mainly epistemic cognition will be required to convert the puzzle into pure standard form which may be the cause of the overall poor performance by the learners on this problem.

CHAPTER 6

CONCLUSIONS AND FUTURE RESEARCH

6.1 Mediational strategies

This research project was primarily undertaken to develop support material which will assist the learners of first year university mathematics in becoming efficient autonomous mathematics problem solvers. The aim was to provide interruption rules/regulative cues which the learner could use to monitor his/her own progress, as well as epistemic cues regarding the nature of problems, while engaged in mathematics problem solving. Together these metacognition and epistemic cues constitute the mediational strategies presented below.

An important conclusion which was derived from this research project is that under-prepared mathematics learners do not come to university without knowledge and/or ability, but they come to university with knowledge and skills which hinder their ability to engage successfully in autonomous mathematics problem solving. This knowledge is often in the form of incorrect tacit rules which have been overlearned in their schooling years and override the necessary rules required to solve given mathematics problems.

Another important aspect which this research project highlighted is the fact that learners fail on typical maths tasks for different reasons. In section 5.3.3, page 106, it was pointed out that the 40-49% groups' failure in tests can be attributed to their inability to distinguish between types and tokens. However the 25-39% group did not have a difficulty with this distinction so their failure must be attributed to some other factor(s).

The following mediational strategies were developed to assist the teaching of, and the learning of, under-prepared learners when engaged in mathematics problem solving:

(1) *The learner must recognize the class of puzzle which is to be solved and know the transitional stages required for its conversion to pure standard form. (Epistemic cognition).*

For the instructional process to succeed, a focus on the nature of the mathematics problem is important. Teachers must make learners aware of the possible classification of puzzles (cf. Fig. 2, p. 62) by providing the learners with examples of each kind of mathematics puzzle and the steps required for their conversion to a pure standard form (cf. section 4.2.1 p. 61). We provide a summary of different types of puzzles:

(A) **Disguised puzzles** (They have a Real world bias.)

(B) **Clear puzzles** (They consist mainly of mathematical signs and symbols and do not have a Real world bias.)

(B.1) **Non-standard form puzzles** (Mathematical computation is required to get them into standard form.)

(B.2) **Standard form puzzles**(These puzzles are either of the pure or the embedded type.)

(B.2.1) **The embedded standard form puzzle**(The pure form must be extracted from the puzzle -no mathematical computation is involved.)

(B.2.2) **The pure standard form puzzle** (All puzzles must ultimately be converted to this form)

When a learner is faced with a puzzle he/she has to convert the puzzle into its most purest standard form and he/she must therefore be aware of the strategic steps which are required to do the conversion. Some puzzles require *simplification* in order for its conversion into the pure form (for example $\frac{d}{dx}\frac{1}{x^n}$) while others require that the pure form be extracted out of the puzzle (for example $\frac{d}{dx}(x^5+1)^4$).

The actual *form* of a standard form puzzle of the pure type must be emphasized during the instructional process as rules which can be applied to this exact pure standard form will fail if there is any variation of this form as can be seen from the example "Evaluate $\frac{d}{dx}\frac{1}{x^n}$" where the learners applied the rule "$\frac{d}{dx}x^n = nx^{n-1}$" to the denominator not realizing that the puzzle is a quotient.

Although most of university mathematics problems in first year are of the clear type and this research project has concentrated more on these type of puzzles, more research is required into disguised puzzles and this is to be undertaken in future research.

(2) *The learner must attend to the detail of the problem, symbols* $+$, $-$, $\times$ *and* $\div$ *in particular. (Metacognition).*

The teacher must make the learners aware of the fact that rules which apply to the addition and subtraction of functions/numbers do not necessarily apply to division and multiplication of functions/numbers by providing the learners with examples that illustrate this fact. *The learner must regard the symbols* $+$; $-$; $\times$; $\div$ *as <u>interruption cues</u>* so that particular mental attention is required to the rules each symbol demands when any of these symbols occur in the problem. When the learner writes the puzzle down all such symbols occurring in the puzzle must be 'highlighted' or *emphasized* (by using a color different from the rest of the text, for example) to interrupt the learner from applying any incorrect tacit rules.

(3) *The learner must be made aware of the significance of brackets in mathematics puzzles; that is, brackets act as cues themselves -they delineate aspects or parts of the whole which needs attending to. (Metacognition).*

The correct use of brackets in mathematics puzzles is vital for the successful resolution of problems. The given brackets in a mathematics problem serve as *cues* themselves and the learner must understand that their omission may alter the mathematics problem entirely, which could lead to an incorrect solution. Also the insertion of a bracket in an incorrect positions may change the form of the puzzle and result in the problem being incorrectly solved. This problem will be dealt with in a future research project where rules for inserting brackets that have not been provided will be devised. For example the two puzzles :

$$\frac{d}{dx}lnx^2$$

and

$$\frac{d}{dx}x.sinx$$

are equivalent to the following two puzzles with brackets provided

$$\frac{d}{dx}ln(x^2)$$

and

$$\frac{d}{dx}(x.sinx)$$

However learners often insert brackets as follows which lead to incorrect solutions:

$$\frac{d}{dx}(lnx)^2$$

and

$$(\frac{d}{dx}x).sinx$$

.

(4) *The learner must be focused on the role of variables in mathematics puzzles. (Epistemic cognition).*

Many under-prepared learners find it difficult to understand the 'dummy' nature of variables. Letters of the alphabet are used to represent variables and it would be more appropriate if the teacher provided rules where variables are substituted with other symbols such as $\square$ and $\triangle$. Once the rule has been presented in this fashion the the teacher can point out that these symbols can be replaced with letters of the alphabet. This will help the learner understand the 'dummy' property of a variable.

(5) *The learner must must be given guides as to which symbols represent variables and which symbols represent constants. (Metacognition).*

The cause of learners incorrect solutions of mathematics problems is often because they confuse variables with constants. Since x is mainly used to represent a variable, learners find it difficult to regard x as anything other than a variable. In the teaching of calculus learners must be made aware of the fact that if $\frac{d}{d*}$ occurs in a puzzle, *unless otherwise stated*, the symbol that follows the d in the denominator of $\frac{d}{d*}$ (the * in this case) is to be regarded as the variable, while all other symbols (excluding the 'operator' symbols such as $+$, $-$, $\times$ etc.) are to be regarded as being *constant*. Problems encountered in this area provide a vast amount of material for future research.

(6) *The learner must be aware of whether the puzzle he/she is attending to contains an equation or not. (Metacognition).*

Rules which apply to equations do not necessarily apply to non equations. The existence of an equals sign in a puzzle, or the absence thereof, should serve as cue for the learner.

The above mediational strategies (1) to (6) are specifically related to the mathematics learning-teaching situation. As they are presented above, any mathematics teacher would find them obvious, but, because they 'feel' obvious, they are often omitted or not made explicit. This research specifically highlighted them. It is necessary to make explicit that which is taken-for-granted. It is usually that which is obvious to the teacher which is not-so-obvious to the under-prepared learner and which the under-prepared learner needs to know in order to master the mathematical task.

6.2 The main tenets of the claims and evidence presented in this report

(A) All people, from adolescence onwards, are heirs to the potential for logico-mathematical thought (cf. Piaget, 1977; Pascual-Leone and Goodman, 1979). The development of cognitive operations necessary for the kinds of mathematics problems encountered at school and at university, however, seems to depend crucially on appropriate learning opportunities (cf. Gellatly, 1987).

(B) Given differences in the ideological and material conditions of existence, learners from different backgrounds or eco-cultural niches will develop different competencies and skills for engagement in mathematical tasks; these competencies and skills may or may not be suitable for the demands of the mathematical task. (cf. Vygotsky 1978).

(C) Mathematics learners, individually or in groups, have the biological, social, psychological power or adaptive capacities to engage successfully in mathematics problem solving; the degree of successful engagement in mathematical tasks is a function of both the conflict or non-balance provided by the mathematical task and the resources available for surmounting the conflict (cf. Piaget, 1977).

(D) Education, as an institution with educators as those who occupy the 'positioned practices' (Bhaskar, 1979, p. 51) in it, could provide the resources (mediational strategies) to surmount the conflict provided by the mathematical task, thus empowering mathematics learners with the ability to engage successfully in autonomous mathematics problem solving.

(E) This research project and future projects of its kind aim at empowering mathematics learners with the ability to monitor, in the most effective and efficient way, their own progress while performing mathematical tasks. Developing the necessary mediational strategies for learners' successful self-mastering of mathematical tasks seems to be the crux of both addressing the reality of under-prepared mathematics learners at university and creating a basis for the achievement of excellence in and through education (cf. Craig 1988a).

6.3 Future research

Only 8 black pupils were admitted into the Faculty of Science in Durban (11 in Pietermaritzburg) in 1988. There is an increasing demand for black science graduates in both government and private institutions and figures such as above indicate that we do not meet this demand. A need for science graduates are in the following areas: science and mathematics education, operations research, computer programming, chemistry, statistics etc. and the training of more black scientists will obviously help meet this growing need. In order to produce more black graduates a bridging or support unit should be established within the Faculty of Science.

If a programme is developed which is successful in producing more black science graduates from the University of Natal, this programme can be implemented in assisting other under-prepared/disadvantaged learners pursuing science courses.

The present project is part of an envisaged future project where the aim will be to design a support programme in the area of university mathematics which could be part of the support unit in the Faculty of Science, as stated above.

Before such a mathematics programme can be effectively implemented a preliminary period is required to help decide on the structure and content of the mathematics support programme. During this period it will be necessary to recruit a group of black matric pupils, who would be interested in pursuing a degree in science, from schools. This group will then be subject to a period of intense mathematics teaching where (1) it will be determined if a learner has reached the stage of formal operations or not (cf. p. 35) and/or where (2) there will be an assessment of their ability to adapt to and master the set programme. Material from this research report will be necessary in deciding whether a learner will benefit from a support programme or not. From the results of this teaching and testing period two groups of learners will be selected:

(a) The first group will consist of those learners who will enter into a year *bridging course* before they embark on a degree course.

(b) The second group will consist of those learners who will enter into the first year science courses and will receive ongoing assistance from the support unit. (This group will be the smaller).

In designing the support programme, emphasis will be placed on the development of mediational strategies that will assist the learners in becoming efficient autonomous problem solvers. The mediational strategies included in this report, will be subjected to further analysis and also extended by addressing a wider range of mathematics learning-teaching situations. This research project was mainly concerned with what the mathematics *learner* needs to bring to the teaching/learning situation. The role of teacher in the teaching/learning of mathematics will be given more attention in a future research project.

It will be possible to develop further mediational strategies, as well as elaborate on the existing ones developed in this project, while researching the learners engagement in mathematical tasks during the first year of the programme where the learners will be taught/tutored in terms of a pilot programme. The data collected will also help develop the content and structure of a mathematics support programme during the first year and for a bridging year before the first year of a science degree.

Most support programmes concentrate on small groups of pupils. It is the intention of this proposed support programme to eventually provide teaching strategies that will be suited to large-class teaching. In this regard, the development of supplementary materials for use by students during their studies will be an essential and valuable part.

As a result of this research project the following incorrect tacit rules were deduced while investigating learners engagement in mathematical task (cf. section 5.2. p. 86 & 5.3, p. 100)

(1) Those rules which apply to addition and subtraction of functions also apply to multiplication and division of functions and vice versa.

(2) The learner 'forcing' himself/herself to simplify when no simplification is possible.

(3) The x which occurs in a problem remains fixed as

a *specific* x wherever it occurs throughout the problem, i.e. x is always the same throughout the problem.

(4) The rules you apply to equations can also be applied to non-equations.

(5) Those rules which apply to the standard form (y=f(x)) also apply to the general form (y=f(g(x))), *without any further rules required.*

(6) If the power of a number is negative, to make it positive, take the number to the bottom and change *all signs.*

(7) A rule which can be applied to a certain function can be applied to functions which 'looks like' this function even though simplification is required for conversion into the proper form.

(8) If $(x-a)(x-b)=k$ then either $x=a+k$ or $x=b+k$ *for any number k.*

(9) When differentiating a function *all* letters of the alphabet which do not belong to words are to be regarded as variables *except* the letter k which is *always* a constant.

The present project has been successful in delineating the following mediational strategies (which will overcome incorrect and misleading tacit rules):

(1) The learner must attend to the detail of the problem, symbols $+$, $-$, $\times$ and $\div$ in particular.

(2) The learner must be made aware of the significance of brackets in mathematics puzzles; that is, brackets act as cues themselves -they delineate aspects or parts of the whole which needs attending to.

(3) The learner must be focused on the role of variables in mathematical puzzles.

(4) The learner must must be given guides as to which symbols represent variables and which symbols represent constants.

(5) The learner must be aware of whether the puzzle he/she is attending to, contains an equation or not.

Finally, further research will involve the investigation of the following aspects of mathematics teaching-learning:

(1) Disguised puzzles require mainly epistemic cognition for their solu-

tion and it is with these class puzzles that under-prepared learners experience the greatest of difficulty.

(2) The teacher needs strategies to assist learners in developing contentless processes required for solving mathematical tasks (analysis, synthesis etc.).

(3) Rules which have been subjected to excessive drilling must be investigated and it must be decided which rules should be subject to drilling both in school and university.

(4) If it can be determined that teachers are the cause of certain learners' incorrect tacit rules then it will be necessary to devise what the teacher must do in order to prevent these tacit rules from developing in learners.

APPENDIX A

Contents: A catalogue of learners' incorrect tacit rules and possible mediational strategies necessary for their prevention.

Examples of tacit rule # 1

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
$(a+b)^{-1}$ $=?$	Cannot simplify	$a^{-1}+$ b^{-1}	Must distribute -1 over addition Confused with rule $(a.b)^{-1} =$ $a^{-1}.b^{-1}$	$(flesh)^{-1} =$ *banana* Flesh contains $+$. Cannot cut banana at $+$ and get 2 separate bananas
Differentiate the following: $x^2.(x-1)^7$	Using the *product* rule we obtain $2x.(x-1)^7$ $+$ $x^2.7(x-1)^6$	$2x.7(x-1)^6$	The pupil distributes the $\frac{d}{dx}$ over both terms in the product. The pupil is confused with the rule $\frac{d}{dx}(f+g) =$ $\frac{d}{dx}f + \frac{d}{dx}g$	The pupil must emphasise the times $=$'.' in the expression which separates it into *two parts* $1 and 2$ The *product* rule gives $diff(1.2)$ $=$ $diff(1).2$ $+$ $1.diff(2)$

A further example of tacit rule # 1

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
$\sqrt{x^2+1}$	impossible to simplify	$x+1$	The learner is distributing the squareroot over each term. Learner is confused with the rule $\sqrt{a^2 \times b^2}$ $= a \times b$	$\sqrt{x^2+1} =$ *banana* $(x^2+1) =$ *flesh* squareroot function $=$ *skin.* Cannot cut the *banana* at the $+$ to get two separate *bananas*

A further example of tacit rule # 1

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
Find $\frac{d}{dx}$ of $\frac{x^2}{x^3+1}$	The quotient rule must be used.	The learner distributes the $\frac{d}{dx}$ over the top and the bottom to get $\frac{2x}{3x^2}$	The learner is confused with the rule: $(\frac{a}{b})^n = \frac{a^n}{b^n}$ or with the fact that one can distribute $\frac{d}{dx}$ over addition/ subtraction	The division line must be emphasized together with the $\frac{d}{dx}$ so that the learner must know to apply the quotient rule

Examples of tacit rule # 2

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
$Log(M+N)$ $=?$	Cannot simplify	$Log(M)+$ $Log(N)$	Must distribute log over addition	$Log(M+N) =$ *banana* $Log = skin$ $(M+N) =$ *flesh* Cannot cut banana to get two separate bananas
$\frac{a}{b+c}$	cannot simplify	$a/b + a/c$	must break denominator at the + and distribute numerator over separate parts	If denominator contains + treat denominator as a whole and do not break at the +

A further example of tacit rule # 2

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
Evaluate the following: $\frac{d}{dx}[(x+1)^{1/2} \times cos^2 x]$	The product rule must be used	The pupil appears to simplify : $(x+1)^{1/2}.cos^2 x = [(x+1).cosx]^{5/2}$ then the learner applies $\frac{d}{dx}$ to this result	The pupil uses the rule: add the powers.	The pupil must remember the rule: add the tops only if the *bottoms* are the same

Example of tacit rule # **3**

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
If $f(x) = 1/(x+1)$ find $f(x+h)$	$f(x+h) = \frac{1}{x+h+1}$	$f(x+h) = \frac{1}{x+1} + h$	The pupil is confusing the x in $f(x)$ with the x in $f(x+h)$	The Pupil must write $f(blob) = 1/(blob+1)$ i.e. must replace x with *blob* in $f(x) = 1/(x+1)$ Now the pupil can make $blob = x+h$ in both sides of the equation

bf Examples of tacit rule # 4

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
Complete the square: $x^2 - 3x + 8$	$x - 3x +$ $(3/2)^2 + 8$ $-(3/2)^2 =$ $(x - 3/2)^2$ $+8 - (3/2)^2$	$x - 3x + 8 +$ $(3/2)^2 =$ $(x - 3/2)^2$ $+8$	Pupil is used to the standard form: *quadr* $= 0$ and must add to *both sides* of the equation the expression $(3/2)^2$	Pupil must disting- uish between *quadr* $= 0$ and just *quadratic* Pupil must add and compensate in the case where $= 0$ does *not* occur
let $f(x) = 2x^2$ Find $3f(1) - 2f(2)$	$f(1) = 2$ $f(2) = 8$ Thus we want $3.2 - 2.8$ $= -10$	$\frac{2.1^2}{3}$ $-$ $\frac{2.2^2}{2}$ $=$ $2/3 - 8/2$ $=$ $-20/6$	The pupil is confusing rules for *expresssion* and *expres.* $= 0$ The pupil uses the 'flip' rule eg. $\frac{a}{b} = \frac{c}{d}$ implies $\frac{b}{a} = \frac{d}{c}$?	The pupil must distinguish between *expression* and *expres.* $= 0$?

Example 1 of tacit rule # 5

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
Evaluate $\frac{d}{dx}(x^2+1)^2$	$2(x^2+1).2x$	$2(x^2+1)$	$\frac{d}{dx}x^n$ $=$ nx^{n-1} means that $\frac{d}{dx}(expres)^n$ $= n(expres)^{n-1}$	Pupil must rewrite the given problem as $\frac{d}{d*}(blob)^n$ and if $* \neq blob$ then result is $n(blob)^{n-1}$ *times* $\frac{d}{dx}blob$ The rule will still work even if $* = blob$

Example 2 of tacit rule # 5

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
differentiate $sin(x^2)$ i.e. evaluate $\frac{d}{dx}sin(x^2)$	$cos(x^2).2x$	$cos(x^2)$	The pupil is using the rule $\frac{d}{dx}sinx = cosx$	The pupil must rewrite the problem as $\frac{d}{d*}sin(blob) = cos(blob)$ *times* $\frac{d}{d*}blob$ This must be done since $blob \neq *$ The rule will work even if $blob = *$

Example 3 of tacit rule # 5

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
Evaluate $\frac{d}{dx}e^{sinx})$	The chain rule must be used: e^{sinx} *times* $\frac{d}{dx}sinx$ $=$ $e^{sinx}.cosx$	e^{sinx}	The pupil is using the rule $\frac{d}{dx}e^x = e^x$	The pupil must rewrite the problem as $\frac{d}{d*}e^{blob}$ to arrive at e^{blob} *times* $\frac{d}{d*}blob$ since $blob \neq *$ Same rule will work even if $blob = *$

Example 4 of tacit rule # 5

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
Evaluate $\frac{d}{dx} lnx^4$	The chain rule must be used: $\frac{1}{x^4}.4x^3$	$\frac{1}{x^4}$	The pupil is using the rule $\frac{d}{dx} lnx = 1/x$	The pupil must use the following rule: $\frac{d}{d*} ln(blob) = \frac{1}{blob}$ $times$ $\frac{d}{d*} blob$ because $blob \neq *$ The rule will still hold even if $blob = *$

Example of tacit rule # 6

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
Simplify $(-1)^{-1}$	$1/(-1)^1 = -1$	$1/1^1 = 1$	Taking top to the bottom means you must change *all* negatives to positives and positives to negatives	If you take top to bottom or bottom to the top can *only* change the sign of *powers*
Simplify $(1)^{-1}$	$1/(1)^1 = 1$	$1/(-1)^1 = -1$	Same as above	Same as above

Example of tacit rule # 7

Problem	Correct solution	Pupil's solution	Pupil's rule	New rule
Differentiate $\frac{1}{x^2}$	Rewrite the problem as $\frac{d}{dx}x^{-2}$ to obtain $-2x^{-3}$	$\frac{1}{2x}$	The pupil is using the rule $\frac{d}{dx}x^n = nx^{n-1}$ without realizing that x^n is the same as $\frac{x^n}{1}$ where there is a 1 below the division line	The pupil must emphasize the role of the division line in $\frac{1}{x^2}$ and must get the expression into the form where only a 1 is below the division line

APPENDIX B

Contents:

(1) The schedule presented to the learners.

(2) Learners' response to schedule.

(3) Statistical analysis of schedule.

Schedule presented to learners

Problem to be solved	Rules needed to solve problem	Your solution
Solve for A: $A(A-4) = -4$		
Evaluate: $\frac{d}{dx}[x^2.(1+x^2)^{1/2}]$		
Let $f(x) = \frac{1}{-x^2+1}$ Find $f(x+h^2)$		
Differentiate: $x.cosa$, where $a = constant$		
Evaluate: $\frac{d}{dt}cos^2t^2$		

Learners with first class passes (75%- 100%)

% and PROBLEMS	%	A	B	C	D	E
Badenhorst	96	2	1	4	1	1
Chalmers	78	1	1	2	1	4
Davies	82	2	1	1	1	2
Gerber	100	1	1	1	1	3
Govender K	76	1	4	1	2	4
Govender N	90	1	1	2	4	4
Harmer	92	1	4	1	2	4
Holmes	80	1	1	1	2	4
Lorton	76	1	3	1	4	1
Maistry	80	1	1	1	2	1
Moopenar	76	1	3	1	3	1
Nicholson	88	1	1	1	3	4
Njoko	98	1	1	1	3	1
Pincus	88	1	3	1	1	1
Ramus	80	1	1	1	4	1
Rice	78	1	1	1	3	1
Simpson	86	1	1	4	3	1
Stephens	82	1	1	1	1	3
Admund	78	1	1	3	4	2

Learners with upper second class passes (68%-74%).

Binder	74	1	1	1	4	1
Dennehy	72	2	1	2	1	1
Deutschmann	68	1	1	1	3	3
Haley	70	1	4	2	2	1
Henry	72	1	4	1	4	2
Lamp	68	2	4	1	1	4
Lee	74	1	1	1	4	3
Mackay	68	2	2	1	2	1
McCulloch	74	2	1	1	4	2
Nicholson G D	74	4	3	1	1	2
Trofimczyk	68	1	1	1	3	3

Learners with lower second class passes (60%- 67%)

Fitzpatrick	60	1	1	3	1	3
Kattenhorn	62	4	3	2	4	4
Lemmer	66	2	3	3	3	3
Miller	60	2	4	3	4	4
Morton	66	1	1	1	4	4
Naiker	64	1	1	1	2	1
Nair	62	1	4	1	4	4
Rasmussan	66	1	4	1	4	4
Smithday	64	1	1	1	2	1
Theron	66	1	1	1	1	1

Learners with third class passes (50%- 59%)

Aumord	52	1	1	1	3	2
Baxter	50	1	1	2	4	4
Catterall	50	2	1	1	3	4
Dhrampal	56	1	4	1	4	4
Govender R	54	1	1	1	3	4
Hiller	50	1	1	1	3	3
Johnston	52	2	3	1	3	1
Oosthuysen	50	1	3	2	3	1
Patel	50	1	3	1	3	4
Pearton	54	2	1	1	1	4
Pillay	56	2	3	2	2	4
Ramdew	50	1	4	1	2	4
Ramluggan	56	1	3	2	4	4
Reddy A	54	2	4	2	1	4
Roodt	50	1	3	1	1	3
Simjee	52	1	1	1	1	1
Velayudan	54	4	1	1	3	2
Watters	54	2	1	2	1	3

Learners who have failed (40%-49%)

Clark	46	2	1	1	3	4
Ebraham	40	4	3	3	3	4
Frank	44	1	4	2	4	4
Goordeen	44	1	1	1	3	4
Howes	44	1	1	3	3	4
Kader	48	4	4	1	2	4
Kistan	48	1	1	1	4	3
Ljubeko	42	4	1	2	4	4
Marais	44	4	3	3	4	4
Meth	45	2	4	2	4	4
Moolan	46	1	2	4	4	4
Murphy	40	1	1	2	1	4
Niedinger	46	2	2	4	4	2
Padayachee	46	3	1	3	4	4
Ramphal	44	4	3	4	4	4
Reddy V G	42	1	2	4	4	4
Singh T	44	1	4	1	3	4

Learners who have failed (25%-39%)

Balarim	32	3	3	3	4	4
Bayat	32	1	1	1	1	4
Blake	28	2	3	1	3	4
Bodasing	30	1	3	3	3	4
Daly	32	1	4	1	4	4
Doull	36	1	3	1	2	4
Fakir	34	1	1	3	3	3
Fraser	30	2	4	4	4	4
Govender E	36	1	1	1	4	4
Heppel	26	1	1	2	1	2
Hoffmann	26	1	4	3	4	4
Maharaj	26	1	4	1	4	4
Naidoo	34	2	1	2	4	4
Rambhoros	30	2	3	2	4	4
Sader	30	1	4	1	4	4
Sayer	32	2	4	2	4	4
Singh O	32	2	4	2	3	4
Singh S	26	2	1	4	4	4
Sivanker	36	2	4	3	4	4
Tshabalala	36	1	1	3	1	2
Williams	38	1	3	1	3	2
Whitby	28	2	1	2	3	2
Woods	38	1	3	1	4	4

Learners who have failed (0%-24%)

Davis	16	1	1	1	3	4
De Beer	10	4	1	2	4	4
Fincham	6	3	4	1	3	4
Griffin	20	1	3	1	3	4
Jughanath	18	1	3	2	4	4
Moodley	22	1	1	1	3	4
Pierchalski	18	2	1	1	3	4
Poobalan	22	2	4	1	1	3
Poree	24	1	1	1	1	2
Rama	14	1	3	1	4	3
Taberer	22	2	1	2	3	4
Taggart	12	2	4	1	4	4
Thurley	14	1	4	1	4	4
Vainikainen	22	3	2	1	4	4
Vawda	22	1	1	3	3	3

Data collected with regards to problem 1
Solve for A: A(A-4)=-4

Groups	# 1	# 2	# 3	# 4
75-100%	18	1	0	0
68-74%	6	4	0	1
60-67%	7	2	0	1
50-59%	12	5	0	1
40-49%	8	3	1	5
25-39%	13	9	1	0
0-24%	8	4	2	1

Data collected with regards to problem 2
Evaluate $\frac{d}{dx}[x^2.(1+x^2)^{1/2}]$

Groups	# 1	# 2	# 3	# 4
75-100%	14	0	3	2
68-74%	6	1	1	3
60-67%	5	0	2	3
50-59%	9	0	6	3
40-49%	7	3	3	4
25-39%	8	0	7	8
0-24%	7	1	3	4

Data collected with regards to problem 3
If $f(x) = 1/(-x^2+1)$, find $f(x+h^2)$

Groups	# 1	# 2	# 3	# 4
75-100%	14	2	1	2
68-74%	9	2	0	0
60-67%	6	1	3	0
50-59%	12	6	0	0
40-49%	5	4	4	4
25-39%	9	6	6	2
0-24%	11	3	1	0

Data collected with regards to problem 4
Differentiate: $x.cosa$, where a is a constant.

Groups	# 1	# 2	# 3	# 4
75-100%	6	4	5	4
68-74%	3	2	2	4
60-67%	2	2	1	5
50-59%	5	2	8	3
40-49%	1	1	5	10
25-39%	3	1	6	13
0-24%	2	0	7	6

Data collected with regards to problem 5
Evaluated $\frac{d}{dt}cos^2t^2$

Groups	# 1	# 2	# 3	# 4
75-100%	9	2	2	6
68-74%	4	3	3	1
60-67%	3	0	2	5
50-59%	3	2	3	10
40-49%	0	1	1	15
25-39%	1	3	1	18
0-24%	0	1	3	11

Statistical analysis associated with problem 1
Solve for A: A(A-4)=-4

i,j entry	Coeff.	Std.Err.	Z-Value
11	1.04228	0.53066	1.96413
12	-0.77096	0.79877	-0.96519
13	0.02881	1.03698	0.02778
14	-0.30013	1.02560	-0.29264
21	-0.30154	0.49811	-0.60536
22	0.37012	0.54498	0.67915
23	-0.21640	1.00108	-0.21616
24	0.14781	0.78839	0.18748
31	-0.01264	0.49971	-0.02529
32	-0.18827	0.62960	-0.29904
33	-0.08165	1.00611	-0.08115
34	0.28256	0.79476	0.35553
41	0.16254	0.46087	0.35267
42	0.36419	0.52268	0.69678
43	-.44547	0.99810	-0.44632
44	-0.08126	0.78460	-0.10357
51	-0.58950	0.39337	-1.49860
52	-0.49320	0.49827	-0.98984
53	-0.09890	0.75268	-0.13139
54	1.18160	0.50729	2.32925
61	0.13120	0.45128	0.29074
62	0.76650	0.48678	1.57464
63	0.06219	0.79660	0.07807
64	-0.95989	0.98399	-0.97551
71	-0.43235	0.40548	-1.06628
72	-0.04837	0.47768	-0.10126
73	0.75140	0.63058	1.19161
74	-0.27068	0.74346	-0.36409

Statistical analysis associated with problem 2
Evaluate $\frac{d}{dx}[x^2.(1+x^2)^{1/2}]$

i,j entry	Coeff.	Std.Err.	Z-Value
11	0.64143	0.43791	1.46476
12	-0.33775	0.98551	-0.34271
13	0.19101	0.55149	0.34635
14	-0.49469	0.59238	-0.83510
21	0.12157	0.45047	0.26988
22	0.52868	0.77168	0.68511
23	-0.73432	0.72363	-0.01477
24	0.08406	0.52229	0.16094
31	0.08620	0.50719	0.16995
32	-0.01752	0.99752	-0.01756
33	-0.58737	0.75236	-0.78070
34	0.51869	0.53426	0.97085
41	0.15102	0.43364	0.34826
42	-0.54048	0.97857	-0.55232
43	0.68142	0.48060	1.41786
44	-0.29196	0.52701	-0.55399
51	-0.63639	0.39125	-1.62658
52	1.05165	0.54372	1.93416
53	-0.21135	0.47306	-0.44678
54	-0.20391	0.42575	-0.47893
61	-0.25445	0.42716	-0.59567
62	-0.82816	0.97317	-0.85100
63	0.68412	0.45338	1.50298
64	0.40119	0.43909	0.91368
71	-0.10939	0.39985	-0.27357
72	0.14357	0.74896	0.19170
73	-0.02082	0.50099	-0.04155
74	-0.01337	0.45659	-0.02928

Statistical analysis associated with problem **3**
If $f(x) = 1/(-x^2 + 1)$, find $f(x + h^2)$

i,j entry	Coeff.	Std.Err.	Z-Value
11	0.34974	0.38931	0.89834
12	-0.45787	0.57095	-0.80193
13	-0.49896	0.73734	-0.67671
14	0.60709	0.62499	0.97137
21	0.53822	0.52832	1.01875
22	0.17245	0.66285	0.26017
23	-0.56179	1.00707	-0.55785
24	-0.14888	1.01966	-0.14601
31	-0.04053	0.49935	-0.08116
32	-0.69398	0.75388	-0.92055
33	1.05668	0.60774	1.73869
34	-0.32217	0.99504	-0.32378
41	0.47933	0.50307	0.95282
42	0.92449	0.55154	1.67620
43	-0.90837	0.99904	-0.90924
44	-0.49546	1.01172	-0.48971
51	-1.11562	0.38395	-2.90562
52	-0.20046	0.43202	-0.46402
53	0.45159	0.47513	0.95044
54	0.86450	0.50126	1.72465
61	-0.62521	0.33311	-1.87689
62	0.00227	0.39513	0.00575
63	0.65432	0.44187	1.48081
64	-0.03138	0.58293	-0.05383
71	0.41407	0.46789	0.88499
72	0.25310	0.57171	0.44270
73	-0.19347	0.77722	-0.24892
74	-0.47370	0.99334	-0.47688

Statistical analysis associated with problem 4
Differentiate: $x.cosa$, where a is a constant.

i,j entry	Coeff.	Std.Err.	Z-Value
11	0.33770	0.37786	0.89372
12	0.53397	0.44750	1.19321
13	-0.25870	0.38381	-0.67405
14	-0.61296	0.37050	-1.65441
21	0.27598	0.48634	0.56747
22	0.47225	0.56901	0.82995
23	-0.54356	0.53729	-1.01168
24	-0.20467	0.43515	-0.47035
31	0.08939	0.56884	0.15714
32	0.69112	0.59271	1.16604
33	-1.01784	0.70341	-1.44702
34	0.23734	0.44612	0.53200
41	0.38445	0.41223	0.93260
42	0.06989	0.55209	0.12660
43	0.44037	0.36848	1.19511
44	-0.89471	0.44737	-1.99993
51	-0.83283	0.71177	-1.17009
52	-0.23110	0.73098	-0.31615
53	0.36252	0.46209	0.78452
54	0.70141	0.40960	1.71242
61	-0.15858	0.48495	-0.32701
62	-0.65546	0.70781	-0.92605
63	0.27463	0.39972	0.68706
64	0.53942	0.35523	1.51848
71	-0.09610	0.59228	-0.16225
72	-0.88066	0.95615	-0.92105
73	0.74258	0.46462	1.59827
74	0.23418	0.46296	0.50582

Statistical analysis associated with problem 5
Evaluated $\frac{d}{dt}cos^2t^2$

i,j entry	Coeff.	Std.Err.	Z-Value
11	1.16706	0.42398	2.75265
12	-0.12215	0.56121	-0.21765
13	-0.47713	0.54116	-0.88169
14	-0.56778	0.39399	-1.44109
21	0.80407	0.50782	1.58338
22	0.73126	0.53288	1.37228
23	0.37627	0.51171	0.73532
24	-1.91160	0.69381	-2.75522
31	0.73526	0.58094	1.26563
32	-0.84163	0.96387	-0.87318
33	0.18967	0.60928	0.31131
34	-0.08330	0.49549	-0.16811
41	0.11403	0.50855	0.22422
42	-0.07657	0.56077	-0.13654
43	-0.02609	0.48253	-0.05407
44	-0.01138	0.36184	-0.03144
51	-0.88322	0.98938	-0.89270
52	0.02480	0.78615	0.03155
53	-0.33019	0.77196	-0.42772
54	1.18860	0.49399	2.40615
61	-0.85688	0.72422	-1.18316
62	0.45660	0.53171	0.85874
63	-0.30385	0.56580	-0.53702
64	0.70412	0.37793	1.86309
71	-1.08033	0.97477	-1.10829
72	-0.17231	0.76770	-0.22446
73	0.57131	0.57372	0.99580
74	0.68133	0.47329	1.43958

APPENDIX C

Contents: Transcript of video recordings of
(1) A good student.
(2) An average student.
(3) A below average student.
P=Teacher, S=Student

GOOD STUDENT

P: OK we're looking at ddx x to the n.

S:

P: You know what that is ?

S:

P: Now what we want to do is we want to look at reverse differentiation, we want to do this and look at reverse differentiation, and then, the sort of product rule for reverse differentiation. We got the product rule for differentiation, what does the product rule look like ? So we're going to need quite a few results. OK so reverse differentiation of x to the n, we did that today, do you remember what that was ?

S:

P: Plus one, you add and then you divide by the whole power. OK, now what is short hand for that ? Instead of saying reverse differentiation, we're going to say integral x to the n dx. OK now it doesn't, it looks again quite heavy but you've got two symbols, so this just means reverse differentiation. And we get x to the n plus one over n plus one. We say this is the integral of x to the n dx.

S:

P: Will always. It's similar to the ddx, notice, so can you, d will sort of be the reverse of the..., this is like an S, and the over dx is a reverse of the times dx, right So we've got, suppose we've got integral fx dx and then we differentiate that, this whole thing, can you see what will happen. If I say now ddx of this. That will disappear and the d will cancel and you have just fx. And the other away around. Suppose we have ddx fx OK, and we now integral dx, what will happen? cancel, cancel, cancel, cancel and you get ?

S:

P: That will disappear with that.

S:

P: So again they're, like sort of, inverse things of each other, that's not a nice way to say it . When they occurred next to each other they cancel each other out. Let's write that result up. OK so now we've got ddx x to the n equals n x minus one and the new result integral x to the n dx is, but there's only one problem here, can you see where fail ?

S:

P: So n not equal to minus one. Now what we're going to do, let's do some examples. Let's integrate x three dx, what do we get ?

S:

P: Right now if I gave integral three x cubed, how would you handle that one ?

S:

P: What happened to the three?

S:

P: Just look, remember, the rule ddx of a constant, times what happens to that constant ?

S:

P: That's it, so that's wonderful. So all you have to do is that. So generally we can have integral of a constant, is just a constant again, n not equal to minus one. Now we're gonna have to take care of that, because if n equals minus one, it's integral of x to the minus one. What the heck is that ? OK, we want to know what is the integral x to the minus one. Before we do that, there's a special function that when you differentiate you get the same function coming up. Now look, when you differentiate this thing you get something completely different because x square gives you x, x cube gives you three x squared. They're not related, now what mathematicians would like, is a certain function that when you differentiate you get the same thing out again. This function has, you're dealing with, x to the power constant. This involves a constant to the power variable, in other words, if look at two to the power x, the graph of y is two to the x. Have you seen things like that ? If you sketch it, as x becomes very large, it becomes very large. Why ? And if x becomes very large, negative, it's always a positive thing, and it looks like that. But what they do is they choose a specific base, and this base they call the number e. Have you ever heard of the number e ?

S:

P: That's right, you get the, the common logs and the natural logs. So y equal to e to the x. With e you can, e is an irrational number, and it's something like, can't remember, I think it is..

S:

P: Seven yes, it's some constant but the special thing about it is that when you differentiate e to the x you get ?

S:

P: E to the x, and that's one of the magic things. So we've got this result here, if you differentiate e to the x you get. Now what will this be in reverse differentiation and integration ?

S:

P: Don't worry about the constant if you get, in other words if you integrate e to the x. What do you get ?

S:

P: E to the x, so that's one result. Now we can take care of the x to the minus one, you've even mentioned that the natural log, log to the base e, we know what log to the base ten is ? That's the one in matric, so log to the base x we actually write as l n x for shorthand. An amazing thing is when you differentiate this you get one upon x which you can easily prove from the fact that, ln and e are inverses of each other. But we won't worry about it. We've got this result. D d x of l n x is equal to one upon x, now here's your x minus one, so what will the reverse differentiation of x minus one be, the integral ?

S:

P: How would you do that, you've just integrated ? Ddx integrated, the integral will cancel with that, you're left with integral of x to the minus one is ?

S:

P: So we've got quite a few results already but that's all we're going to need.

S:

P: If you want to integrate, you've got to use the log x if it's x to the minus one, otherwise it's always going to be that rule.

S:

P: Ja well, this rule doesn't work on that one.

S:

P: But we can integrate x to the minus one, and we get log x, and of course we've got integral e to the x dx is e to the x. OK, so these results we're going to, I think I'll write them up there, then we can work here. OK, so ddx x to the n, that you know, the integral sometimes we can leave out the dx if you want to, it's not so important, and of course you remember the plus c, but that's not so important for what we're doing. And we've got integral x to the minus one is l n x and we got integral e to the x is ? And differentiate e to the x and differentiate log x you get ?

S:

P: [Mumble] OK, now we can do what we call a reverse differentiation involving the product. Do you remember the product rule for differentiation, ddx of u times v?

S:

P: OK do you want to do some writing? I'm tired of writing.

S:

P: Dd OK That's wonderful, so that's the rule. Now what we want to do is try and get an inter, a rule, a product using integration. If you integrate through it's not going to help you, because you're going to have integral u v, you're going to have two integrals but notice, well look if you integrate with that it will cancel with that and you'll just get uv. So what I'm going to do is I'm going to write this as du dx times v equals that side there which is ddx of uv minus this thing here. Happy there ? now I'm going to integrate through.

S:

P: Either one.

S:

P: That's fine, makes no difference. So let's integrate through, so it will be integral of du dx of v. We should actually put the dx there, so it's going to be confusing, so we leave it out. Is equal to ? So integrate that, so what do we get ? What will the integral du dx times v ?

S:

P: So we're left with ?

P: Ah huh, minus and here we cannot do anything so now this rule says. Given a product, see the product ? It's a weird looking product, it's a derivative of something times v, equals one without any derivative. Differentiate times v minus u and then we differentiate that one, so what we're going to do, given two things we're going to decide which one is going to be du dx and which one is going to be v. And then when we're substituting this side we're going to hope that this will become an easy integral, because if this is difficult then this whole thing is pointless. Because you got an integral giving you an integral, when you've used the ddx rule there was no problem there because you just went and differentiated. So now we're going to need a standard one that we always use, which is e to the x times x. So what you've got to decide, which one is going to be du dx and which one is going to be v. I've given it easy for you as a first time. Then we're going to find out what will u be, that's what we want, we want v once we've decided v,v is always v here, and then we need dv dx. OK so from this what will the du d x be ?

S:

P: Comparing.

S:

P: That's it, so what will u be here ?

S:

P: How would you get u ? Integrate.

S:

P: Integrate mm, so if I integrate this side I'll get u, integrate that side I'll get integral e to the x which is ?

S:

P: So u is e to the x and what will our v be, comparing those ?

S:

P: What will be du dx, because we want du dx there, what will it be ?

S:

P: OK ja, differentiate and differentiate x.

S:

P: Ja dv dx, so it's ddx of x which is ddx of x to the one is one, one minus one.

S:

P: Ja now let's do it and see if it all works. So this is our du dx, this is dv, so the rule says this is u. What is u from there ?

S:

P: That's v.

S:

P: U will be ?

S:

P: And dv dx, ah now notice that this is a decent integral, what is it ?

S:

P: Ah so we've done it, so we've got e to the x, x minus e to the x. Now notice we have integrated a difficult thing and we've got that. OK let me give you one and I'm going to just change it to x squared and see how you handle it. I'm going to help you, I'll leave that on the board. We had integral du dx, can you remember? Times the v was equal u, v minus integral u dv dx. I'm going to make, make it easy, that's e, I'm going to change this to x square so the same thing applies and simplifies. Try it, relax, take your time.

S:

P: OK.

P: That's right, why don't you write, write, why don't you write slowly, put du dx equals. Let's go slowly.

S:

P: A huh.

S:

P: Right.

S:

P: What, what do you need ?

S:

P: Simply substitute.

S:

P: Great, OK plug it in and see what happens.

S:

P: OK so now you've got.

S:

P: Mmm, that's right.

S:

P: We tried to do that, we're not allowed to distribute over a product, so you've just got to take the two out and leave it as two there.

S:

P: Take the two out and you get ? integral e to the x times x. Now the thing is, this an easier integral, than that one. Didn't we start off, that's the first thing I taught you.

S:

P: Now you got to do that thing again.

S:

P: Go through the same process, OK rub this out again and now just evaluate integral e to the x, x , it's the same thing.

S:

P: That's what we're trying to do, that's the only rule that we've got. You can't distribute, ja it's the same as the product rule, you can't distribute, your mind says do that.

S:

P: OK try that.

S:

P: Fine so you've actually got your product rule there, that you now just substitute in there and you complete it.

S:

P: Ja this whole thing comes here, you've completed the problem, there's no more integral here. So integral e to the x, x square is that, and if you've got x to the n , notice you're going to have to carry on.

S:

P: The integral all the time, well what happens, notice that if you had swopped these around it would never have helped you.

S:

P: Cause you would have had x square to integrate, would have given you x cubed. Differentiate e to the x so this one would have been x cubed.

It would have been worse.

S:

P: Well sorry, it would give you a better integral, you've gotta always think back. OK now let's do the log one. OK try the integral of x, OK the way I've written it there. If you decide this was your du, means we've got to integrate log x. Now we don't know what the integral is. We know how to differentiate that, so it might be safer to write x l n x. OK so this will be your du, the one you're going to integrate. That one you're going to differentiate, do you want to try this one ?

S:

P: Umm, all we know that when you differentiate log x you get ?

S:

P: You got to use that.

S:

P: Du is equal to x so what will u be ? So you got to find u and v, v you've got. You must find dv dx, I should actually have put it in this will be du dx.

S:

P: OK so you would put du dx equal to x, it's the first thing. And then you solve, look you want u, OK now how are you going to get u? You put du dx equal to x, do that.

S:

P: So how will you get u from there ?

S:

P: That's right, that's it.

S:

P: That's it, so that's what u is. So we got v, now what other thing must we get ?

S:

P: That's it, no no just this, there's no integral. Ddx of the log x is ?

S:

P: That's it. OK so substitute in and now we're going to hope that integral there will work.

S:

P: That's it.

S:

P: You get an easy integral.

S:

P: Beautiful, [mumbles] take out that half [mumbles] and that whole thing. Brilliant, wonderful, but notice these are all very contrived problems. Ok now let me give you this last one, then you can stop there. We should have used this black. OK I want to integrate, now remember I said we couldn't integrate log x, now we can. Do it, using, can you see how we're going to integrate log x. Can you split that up into two functions, log x, because log times x, so we can't split the log and x up. Can you somehow bring two function there, it's a dirty little trick.

S:

P: Ah huh, ah huh, so this is going to be your du dx, this your v.

S:

P: See ja, you couldn't. If we chose that as our du dx, it would mean you'd have to integrate log x. But that's the question.

S:

P: So it wouldn't help you.

S:

P: Ja so, OK if du dx is one, what would u be ?

S:

P: Ah huh, use your rule, the rule says it's x to the n plus one.

S:

P: Ah huh, that's x so u is equal to, what's u ? And v is log x so it's going to be x ln x minus integral du times. If v is log x what is dv dx ?

S:

P: X lnx, minus integral of one, what is integral of one ? Got it somewhere, ja.

S:

P: So all of these, as long as we can get an easy integral it's done. OK are you happy there ?

S:

P: If, what about the trig ones, we haven't done any of the trig ones. Remember what ddx of sinx is ? So therefore integral of cosx will be ? Sinx and you also had the ddx of cosx is minus sinx, so we've got the integral of sinx is minus cosx. If I gave you integral of x sinx, how we're going to do this one ?

S:

P: OK.

S:

P: That's right.

S:

P: Well we've got it.

S:

P: Both of those we can integrate, but what are we worried about. It's not really this, what are we worried about here. That if we can integrate, now suppose if we chose this thing. Integrate x you get ?

S:

P: X to the one plus one over one plus one which is x square over two. Differentiate sinx you get ?

P: So here we're going to get x square times cos, has that helped you ? You gonna now have x square times cos, does that help you ? So let's swop, make it sinx x, see what happens then.

S:

P: That's right.

S:

P: OK go slowly, write du dx equals sinx.

S:

P: [mumbles]

S:

P: Now check what is the product u times dv dx going to be, u times du dx what you gonna have. Isn't that easy to integrate, there we are, go ahead you'll see. Now if you had chosen those wrong, you would have just carried on and on and on.

S:

P: Ja well, you would have gone through and then you suddenly get that integral and you say, oh shucks! This thing you can't, difficult, then you would swop.

S:

P: [mumbles]

S:

P: Don't need that again, just write it down.

S:

P: Yip.

S:

P: You've now integrated this product, see what I mean ?

S:

P: There's only specific things we can do. There's some things you cannot use. You cannot integrate using this method because it only works if that is easy, OK happy there ?

AVERAGE STUDENT

P: Right, this is what we're going to investigate. So it's meaningless to you now, but the end you're going to have some understanding, mathematical understanding of it, OK. So there's so many things I have to do, before, we can describe this mathematically. Right, so we're going to first need a concept, known as absolute.

S:

P: X, what does this mean ?

S:

P: Right, so if I, start off at, naught and move that way. It's one, two, minus one, minus two.

S:

P: Absolute x what it means, it's really the distance, from naught to the particular x, any x that we choose.

S:

P: So if I take x equal to minus two, what's the distance ?

S:

P: Two, if I take x equal to minus two what's the distance ?

S:

P: Two, so so what minus x does, is just.

S:

P: Make sure it's, positive always, in other words what you get out here will always be positive.

S:

P: So we're going to deal with things like that. But what, we really want to deal with, [I'm going to rub this off], is what does this mean, if I gave x minus c mod, let's say, the number delta. This is a constant, that's greek delta which is a constant.

S:

P: OK, what does this mean, can we draw it down or draw it down or draw another line ? In other words, it say's the distance between x and c must always be less than delta. Now x is the variable.

S:

P: Always.

S:

P: Ja, no no that's ja, these are given.

S:

P: This is a variable thing. So what we have is, you've got, your c over here, you've got your c plus delta, over here you've got your c, minus delta. This says that x always lies between c plus delta and c minus delta.

S:

P: So any, so this means all the x's over here will come from here.

S:

P: You can, get that from there.

S:

P: By, there is a rule that this will be less than delta greater than minus delta, and from that you'll get x is less than c plus delta, and c minus delta.

S:

P: So as long as when you see that it means that x.

S:

P: C, plus delta ....OK so we've got that. Now so now let's go back to the limit. So I'm going to give you a function of x.

S:

P: Call it say, two x plus one and what I want to do, see what, how does this thing behave when x is near the particular number.

S:

P: So, how does fx behave when x is near say, what number would you like it to be ?

S:

P: Three OK, so let's take something near three, give me something very close to three, but not equal to three.

S:

P: Extremely close.

S:

P: OK, so let's take x equal to 2,9 and then if you substitute in there.

S:

P: You're going to have two times 2,9 plus one.

S:

P: And that gives you ?

S:

P: I'm going to rub that out now. OK, two times 2,9 gives you ?

S:

P: Two nines are eighteen, two two's, four, five.

S:

P: No.

S:

P: It's, it's 5,8.

S:

P: Ya.

S:

P: Ja.

S:

P: I thought that was, 6,8, so this, when it's close to three.

S:

P: This thing is close to 6,8. OK which, and you can take something more accurate than that.

S:

P: Let's take three, let's take 3,01, so it gives.

S:

P: Ya.

S:

P: Either side.

S:

P: Smaller or bigger.

S:

P: And what we gonna do, it's going to be twice, so two times three.

S:

P: Plus one, gives you 6,02 plus one.

S:

P: 7,02.

S:

P: So the type of numbers we're getting is, 6,8 and 7,02. So what, number is this thing close to, nice round number ?

S:

P: Seven, so the closer this thing, obviously we can get more closer that we want. We can get 3,0000.

S:

P: And etc.

S:

P: So we now know that when x gets very close to three, this thing is getting very close to seven.

S:

P: And how we're going to write that mathematically.

S:

P: Fx, is very close to seven.

S:

P: Ja, this thing here.

S:

P: The y value.

S:

P: When x is near three, fx is near seven, right. Now instead of saying that all the time you want a shorthand for that.

S:

P: Instead of saying whenever x gets close to three, very close to three, then this fx.

S:

P: Gets, very close to seven.

S:

P: So we want a shorthand, so the shorthand.

S:

P: No late, that comes later.

S:

P: Now we just want a, a shorthand.

S:

P: And we say limit, of this function fx is x, now x is very close to, number, three.

S:

P: Gives you the number.

S:

P: Seven.

S:

P: Well you say that, well you, actually should say the limit of fx as x approaches three, is seven.

S:

P: OK, so this is, but the important thing is. That, in this limit definition we never require that x equal three.

S:

P: It can get very close to three but never equal to three. You'll see the reason why we need that. When we come to, slopes of tangent.

S:

P: Ja.

S:

P: OK.

S:

P: OK.

S:

P: Absolute thing.

S:

P: OK, so we've got absolute x.

S:

P: Right let, let's write this less than delta. What does that mean ? Means x is less than delta, greater than negative delta.

S:

P: But that you can, it's easy to prove that.

S:

P: OK then the other thing is, what do we mean, by this.

S:

P: This x approaches c, is equal to c.

S:

P: So this means, that, limit we can write it down. Limit of fx as x becomes close to c or approaches c doesn't matter what you say, c is l, but the restriction x not equal to c.

S:

P: So the limit of this function.

S:

P: Ya.

S:

P: In other words it gets so close and the important thing, it can approach c.

S:

P: Ja.

S:

P: So now we've got, now what we want to do is to get a mathematical way of describing the limit.

S:

P: So that we can prove things, in other words, that thing I gave you, the two plus one, x approach, equal to seven, how we're going to actually prove that, what limit is it. Knowing that x is getting to three not equal to three.

S:

P: But very, very close.

S:

P: Can we get a mathematical thing for it ? So just remember the x approaches c and it can approach it from both sides. So if that's c there we can go to c from there and there, but we never actually get at c itself.

S:

P: Very close, so let's go to graph of fx.

S:

P: The same, suppose we've got that same thing. Let's make it f, limit fx equal to l. So over here the l refers to the y. ...... and the c refers to x.

S:

P: Approaching c, so let's, there's the graph there.

S:

P: And there's your l.

S:

P: OK, and there's your c, now we don't care what happens so we put a hole there.

S:

P: Cause we don't, care whether x equals c we just want it very much near c, in other words we don't worry what happens at c.

S:

P: ....

S:

P: F of x, so we're not interested in f of c.

S:

P: F of c can be undefined. It can be anything you want, as long as we can get very close to it. Now the actual mathematical way of looking at it.

S:

P: Would say, that when, we've got this set - up here,and when we, we want to choose an interval.

S:

P: Around here.

S:

P: And whatever we choose this, interval, no matter, how close we can get to it, or any thing or any interval here.

S:

P: We want to be able to find an interval around here.

S:

P: So that if we choose a little x.

S:

P: And we go up, it hits the graph. This fx lies in this interval.

S:

P: This is what we want to be able to do. In other words for each, interval we have here, we choose.

S:

P: We must be able to find a corresponding interval. Here around.

S:

P: So that any x that we choose when we go up, except that one, we don't care about that one.

S:

P: Any x will lie in there.

S:

P: So you first, choose, your type of interval that you're going to have.

S:

P: And then you make sure that you can find this interval, so that when x is in that, that will mean, that the fx will lie inside that.

S:

P: OK, so the important thing is that, it doesn't matter, your l over there, you can take this interval here, like that.

S:

P: And make sure you get a corresponding one so that again x .... you can take that one. So this one we can choose anything we want.

S:

P: So that obviously, the obvious thing you're going to choose, is going to be so small, you can actually get almost to l but not equal l.

S:

P: And then there'll always be something, corresponding here. So that if you go up it will hit.

S:

P: And it will go out.

S:

P: Ok, so we've got got that. Now let's try and put that mathematically.

S:

P: So I'm going to re-do that but to, in terms of mathematics. OK so we've got that there is l over there, and there's, the c we're not interested what happens over here.

S:

P: So now we want to be able to set up an interval. So I'm going to make, this distance will always be the same from there to there.

S:

P: So let that distance be epsilon.

S:

P: So therefore this one will be an l plus epsilon, these are positive numbers epsilon.

S:

P: L can be negative or whatever, and this is l minus epsilon. So this little distance here, is a fixed epsilon which is positive.

S:

P: Epsilon is positive so this will make it smaller. So we want to be able to put a little interval around here.

S:

P: No matter what, so we can choose any possible epsilon. And then we, must find.

S:

P: An interval over here.

S:

P: So that when x, is in that interval we get something lying over there.

S:

P: So what we do again, we introduce now a delta distance so this is going to be c plus delta, and this is going to be c minus delta.

S:

P: So what we want, is, for each epsilon here, where l lies inside this interval.

S:

P: We want to be able to find.

S:

P: A delta one , so that when x is inside this delta one that fx will be inside.

S:

P: This here.

S:

P: So.

S:

P: The what's.

S:

P: Umm, tangents.

S:

P: Yes.

S:

P: The, the slope.

S:

P: The gradient.

S:

P: No it's not related to that.

S:

P: No.

S:

P: NO no it's just, it doesn't matter, all we want to do is to be able to get any interval we want, over here.

S:

P: Why do you want to do do that.

S:

P: Because you're trying, look it's like a sort of, you set this thing up.

S:

P: OK you gonna have this, you want to make sure, it doesn't matter how close you get to here.

S:

P: You will always be able to get very close to l.

S:

P: Because you want, you see, you can't, you don't want to get actually right at c itself.

S:

P: We're trying, we're trying, to show, that, that the closer and the closer we get to c, the closer and the closer this thing will always get.

S:

P: To l.

S:

P: OK, so we arb..., so what we want is an arbitrary epsilon or interval here.

S:

P: Doesn't matter what interval we get, as long as we can find this delta interval, then we're guaranteed.

S:

P: That, we can make this thing as small as we want.

S:

P: And there we can always get this l.

S:

P: It is, it is difficult to, it's actually arbitrary this is the thing.

S:

P: If people understand that, it doesn't matter as long as it works for any possible epsilon here.

S:

P: You can find a corresponding delta.

S:

P: Or delta interval.

S:

P: OK so let's write it down. So let's, so we got the for each epsilon can we, we want, delta, so now what we want.

S:

P: Is when x is in the delta interval.

S:

P: OK in other words when x, is in this, this is that, that thing.

S:

P: I said, x is now between c plus delta.

S:

P: Minus delta x not equal to c, to get that.

S:

P: We just do this, because this thing is always greater than equal to naught. When it equals naught then x will equal c.

S:

P: So we put that restriction.

S:

P: So now we now x is not equal to c.

S:

P: You'll see why we need this, later on.

S:

P: OK so we got that, when this is true.

S:

P: So, you pick your epsilon first.

S:

P: Doesn't matter which one you get.

S:

P: As long as you can find a delta.

S:

P: So this is arbitrary, delta, so that when, this is sufficiently small, remember?

S:

P: You've got to get this very close.

S:

P: Then your difference between, so that must imply, that fx minus l must be less than this number epsilon, in other words if fx lies in that epsilon ....

S:

P: OK then, we say, so if we can do this, or we want a delta, if we can find a delta.

S:

P: Then, we've got limit fx, x approaches c will be l.

S:

P: OK [mumbles] do that, we'll go back to that one problem, x approaches three.

S:

P: And see how it works.

S:

P: Remember tha..., the important thing is that for each epsilon, so it doesn't matter what epsilon you choose.

S:

P: And the obvious thing what you would do is that you would get that epsilon that would get you, right on top of l.

S:

P: You see.

S:

P: And that guarantees it because it is arbitrary, so as long as you can know that for every epsilon there will be a delta.

S:

P: Then you know that you will always be, when you're close to c you're going to be on top of l.

S:

P: Because you can choose that epsilon, whenever you want.

S:

P: That's the, the whole thing. So let's try and get this, OK what was that, was two x plus one ?

S:

P: OK and x approached three.

S:

P: And we wanted to make that seven.

S:

P: So let's go through.

S:

P: So we want to now prove this using that definition.

S:

P: So, for each epsilon we must find a delta.

S:

P: Such that, now where x minus three when x is close to three and that just means this, it's the delta interval, remember that always.

S:

P: Was that, and we don't want it equal to three that.

S:

P: We'll describe it later. That must mean that the difference between that and that must be ? Remember this is the fx and that's the l which.

S:

P: Which what, what variable did we use for the fx and l, what, was the umm..

S:

P: No not the, the the, variable, the greek symbol that we used.

S:

P: Good.

S:

P: So therefore we want that to be very close to that.

S:

P: So the distance must be less than ?

S:

P: No.

S:

P: OK go, go back to that thing there when I drew it, there's fx and we had.

S:

P: This is now seven and we had seven plus, what thing did we have here?

S:

P: Ja.

S:

P: Ja, epsilon.

S:

P: And seven minus epsilon.

S:

P: So the distance between this must always be less than, the Greek thing?

S:

P: Epsilon, that's what we want, remember we choo...., for any one, so doesn't matter what epsilon we choose, and obviously you're going to get one. That's going to be so, close to that seven.

S:

P: Then you know that you're going to have to hit that seven.

S:

P: Because any epsilon works.

S:

P: Fx minus this seven, or fx is, two x plus one. This must be, less than epsilon.

S:

P: So now we have got to find that delta.

S:

P: In other words you won't, you've got to get it. When we've got this down obviously the delta's going to be in terms of epsilon because this statement must imply that.

S:

P: So if we can work from this with a particular delta, and eventually get that less than epsilon then it means it doesn't matter what epsilon we get.

S:

P: We'll always be able to get closer, as close as we want to seven, provided that x is, is sufficiently close to three.

S:

P: So what we got to do is always find the delta, remember, this you've got to show, always, you've got to show this.

S:

P: This we must get.

S:

P: Once this is true and it's for any epsilon, then we can obviously choose the epsilon that we want.

S:

P: Which will give you or hit, hit on seven.

S:

P: So the two x plus one hits on seven, so you've got to have to, make that imply that, so now what we do is we start off with this without less than epsilon.

S:

P: And we fiddle with it till we get this thing coming up.

S:

P: And then we choose a corresponding delta. Remember it's always, we've got to find this delta.

S:

P: It's it, it's it's sort of like a cheating thing. Because we're going to start working with this, and make sure we get this less than epsilon. And get that delta, so we can plug that delta in here, substitute the delta for that.

S:

P: Fiddle with it and end up with that, less than epsilon then we're home.

S:

P: Cause we've got that delta so we must always find this delta, we must find this delta. So what we do is we write down this thing.

S:

P: And we write that thing down without the less than epsilon and fiddle till we get this thing looking something like that.

S:

P: And whatever we got on this side becomes that delta automatically.

S:

P: Which will be in terms of epsilon.

S:

P: OK so let's write those two things down. So we gonna have to find the delta, so we've got x minus three. That's less than delta OK.

S:

P: And this side we've got two x plus one minus seven.

S:

P: Right now we're gonna have to fiddle with that until we get x minus three popping up.

S:

P: Over here on this side.

S:

P: And then we can know what we gonna make our delta so this whole thing becomes less than minus epsilon. So how we're gonna fiddle with this to make, make x minus three coming up? So this thing becomes, two x one minus seven is? Minus six.

S:

P: OK now we can take out a two there so you get two into x minus three.

S:

P: Agree, now since two is positive, it doesn't matter with the absolute, you can take it out.

S:

P: It's a positive number, it won't affect.

S:

P: So we've got two absolute minus three, now notice.

S:

P: You've got.

S:

P: That there.

S:

P: And that there, how we want this whole thing to become less than epsilon?

S:

P: So what are we gonna make this thing ? Remember we want to now, replace with something so that we just get an epsilon coming up.

S:

P: So in your mind put an epsilon over there.

S:

P: Right what must this thing be?

S:

P: Solve for this whole thing that we just get, how can you solve all this, using this, how will you get rid of that two. This is, the students have a lot of problems with this because it's so easy, and obvious.

S:

P: But you're trying to think of it too difficult. Remember, we want this whole thing less than epsilon.

S:

P: So this whole thing eventually became like that, so now we must replace this thing with something, and that something will be the delta. What must we make this so that this whole thing becomes epsilon.

S:

P: It's so easy, this is , it's amazing because they all have the same problem and I, I, look you wanna, you want this whole thing to be epsilon.

S:

P: If I gave you two, there, right, I want this whole thing to be an epsilon, what must I make the star, solve the star, how do you get star from there, how do you get rid of this two?

S:

P: D.., how do you make this all alone, how would you?

S:

P: Ja.

S:

P: Then you get epsilon over two.

S:

P: Ah huh.

S:

P: Right, OK, so now this star is the same thing here now.

S:

P: Got it.

S:

P: Ja.

S:

P: No, but what must this whole thing be so that we get this whole thing being epsilon? It must be, what must we put here?

S:

P: Not just epsilon, but?

S:

P: Ja then the two's will cancel and you've got epsilon. So therefore what's your delta going to be?

S:

P: Epsilon over two, if you put epsilon over there going to have epsilon over two there, cancel and you're going to have your epsilon which you want, this thing less than.

S:

P: No, they all have the same.

BELOW AVERAGE STUDENT

P: OK, we have this result, ddx of x to the n. Do you remember that one ? Boring result, what is it ? I'm going to ask you all these things, I'm not going to do everything. X to the n remember? Differentiate it.

S:

P: OK, now what we want, we did reverse differentiation today, we want to reverse differentiate this. Can you remember what this was ?

S:

P: That's it, by the whole thing. OK so reverse differentiate x to the n, we get that. Now we want shorthand for reverse differentiation. We call it integration, and we use, it's like a stretched out S. We say x to the n and then we put a little dx, very similar to that, except the dx is up there and this is reverse differentiated, and which we now know is x to the n plus 1 over n plus 1. So this sort of, is the, cancels with the d. They're like the opposite and the dx is at the bottom and here we put the dx at the top. So where ever this occurs next to the d, it will cancel this one and the dx at the bottom will cancel with the dx at the top. I'll show you now. If you've got, suppose you've got d by dx of fx, OK they now integrate or reverse differentiate so that's this S thing, looks like an S and you put a dx there. Now what will happen ? That will cancel with that and that will cancel with the d, and you will be left with ?

S:

P: Ja fx, and the same if you have ddx and I've got integral fx dx what will happen then ?

S:

P: Remember it's like the inverse, this is like the reverse of differentiation. Inverse is a bad word to use, but the d, these two will always cancel each other and those two, so what are we left with ?

S:

P: Fx.

S:

P: Just to illustrate that, when they occur, they disappear. So let's do a simple example. What's the integral of x squared going to be ?

S:

P: Now notice, sometimes we leave out the dx. OK so I'll put it in for the time being, but people get confused I don't know why, but it just means anti-der- ... or find the reverse derivative of that, or anti-derivative.

S:

P: That's right, the three. If I gave you integral of two x squared what do you get? How would you handle that one ?

S:

P: Very good, that's fine, what did you do there ? Can you tell me what rule did you use, how did you get that ?

S:

P: Ja but but how come that two, didn't that two bother you, why didn't it bother you ?

S:

P: Ja the constant, it's like when we differentiate the constant times x to the n, what happens to the constant ?

S:

P: Doesn't, fall out, it falls out, outside . If it is a constant it goes outside, so this just goes outside when you integrate x square. So we now have integral of a constant times x to the n is any constant times. But is there any n that would make this n go haywire, go totally wrong? Can you think of what type of n would destroy that rule ?

S:

P: That's right, so n can't be equal to negative one, so what the heck is the integral of x to the negative one or one over x ? What will the integral of x be ?

S:

P: Ja so integrate you get ?

S:

P: That's right, and what happens if I give you integral of one dx, what you gonna do now? Remember you gotta have x to the n and one is not of the form x to the n, can you make it of the form x to the n ?

S:

P: You can't it's there, one. Can you try and bring an x in there ?

S:

P: Can, isn't there something to the power naught, anything to the power naught, is always ?

S:

P: Ja, so it's integral of x to the power naught. And now integrate, use the rule, what so you get ?

S:

P: So it's x, so if you integrate any constant or any one, that's always just x. Very similar to when you differentiate a constant, why do we get naught? Because we wrote it's constant as x to the naught and when you differentiate, you bring to naught down and minus one, understand ? Same type of thing. So these are the rules that we've got, we're trying to find that one there. Now there's a certain function that mathematicians want, they like. It's a function that when you differentiate or when you integrate, whatever it is, you get the same thing. Notice here, when you differentiate x to the n you get something completely different. You don't get x to the n coming up, and here you don't, when you integrate you don't get x to the n coming up. So what mathematicians would like is a certain function that when you differentiate it you get the same thing, or converse if you integrate.

S:

P: You get the same thing. Now this function is actually related to this type of things, y equal say, two to the power x, have you seen these ? Can you sketch that graph ? We're dealing with x to, x is the variable bottom with the constant at the top. Now we've got a constant at the bottom and a variable at the top. If you had to sketch this thing, it looks, as x, look. This is always positive since two is positive. Doesn't matter what x is, this will never be a negative number. X can be negative but it will still be positive, that whole number, so it's always above there. And if x becomes very large, it becomes very large and as x becomes very large that way, negative, this goes to naught. But there's a specific number that they choose which we'll call e and e is an irrational number, which is like I think is 2,7. I can't even remember what it is but it's some constant which is near 2,7. And when you differentiate e to the x, you get e to the x. This is the function they've been looking for, and e to the x you can actually write it as, like, one plus x plus x square etc. Don't worry about it, it's just that it's, it's a constant to the power x, and when you differentiate it you get e to the x. In other words, the slope of the tangent to the graph of e to the x is e to the x , and in particular e to the x equal to one will be e. So it's got a, it's a special thing, so you got ddx of e to the x. How would you write this integral using integration reverse differentiation ?

S:

P: Look, if you differentiate the thing you get the same thing, So if you integrate what will you get ?

S:

P: No, it only works with that rule, when you've got x at the bottom, see what we do. Now let's, let's use the reverse differentiation, so we going to integrate there, let's do it. So we're going to integrate both sides, what you use that side of the equation you use the other side. So what do we get now ? What happens with this, remember ?

S:

P: That goes there, that goes there, what are you left with ?

S:

P: So e to the x is, integral of ?

S:

P: So that magical function, when you differentiate you get the same thing, when you integrate you get the same thing.

S:

P: So we got that result there, ddx e to the x, and you've got integral to the x dx is ?

S:

P: Now let's take care of this one. N not equal to minus one. Can you remember the log to the base ten of some number, matric ?

S:

P: OK, given umm, a number equal to base to the power. We convert it to log form. You say that's the same as log of the number, to the base b equals p. If you remember, it's always b cross over p equals eight. Write it in terms of the log rule, it's going to be, log. What's the number going to be ?

S:

P: No.

S:

P: N, eight, what will the base be ?

S:

P: Two, look, it's always base at the bottom, base, that's two and the number, power will be ?

S:

P: Three, and check it. Two to the three gives you eight, but anyway, that's not important, the thing you can express. Why we do this is because it's easier to deal with the powers. What this does, it just plonks everything down, so we deal with powers. Because if you multiply, if I gave him, told him to multiply two to the one hundred times two to the two hundred, it's so much easier just to add those things than to actually multiply out. So that's what it does, but the actual base that one is dealing with, the important base is when we use not ten, but we use that number ... e is a constant. Remember e is like 2,7; e is a, something, can't remember, it's an irrational number. You can not, you can not write it as something over something. It's like pi, know what pi is ? It's twenty two over seven, that's, it's not really because you can't write pi as something over something. So that's an approximation. The same with e, so we want, not log n or log to the x, we want log to the base e of x and shorthand we just write ln x. So when you, see this just means to the base e, so if I write log x to the base e equals y, how would you write that in the other form ? Remember you start with the base, cross over the, it will be e to the.

S:

P: Ja so it's just, we're dealing with, and this is the function that when you differentiate you're going to get one upon x. So when you integrate log, integrate one upon x you get, you get log x. So that I've just given to you that when you differentiate, (let's write on the board). When you differentiate log x, in other words when you find the slope of the tangent to the curve, you can easily prove that, using the fact that e and log are of inverses, you'll get one upon x. So re-write this, using integral, integration, how will we do it ? How will we get rid of the ddx ?

S:

P: Ja integral ddx of log x, (this thing is running out). Dx equals integral of lx dx, what happens here? So the integral of one upon x is log x, so here is the result that we've got. I'll write these up so that you've got them all up there, (this pen must be better). So we've got to have that result, ddx x to the n is n x to the n minus one. We got integral x to the n is x to the n plus one over n plus one. We've also got that, integral of one upon x is ln x and if you differentiate log x you get? There's lots to remember, just don't remember them, cause you can always refer to them when you need them. And we need integral e to the x, what's that ?

S:

P: Ja it's a magical thing and when you differentiate e to the x what do you get ?

S:

P: E to the x, OK now we can do what we're supposed to be doing. Lets go, cause what we're trying to do, remember we got the product rule for differentiation. We want to have the product rule for integration or reverse differentiation, so if you know what the product rule is d... OK now the product rule says d by dx of uv is? Remember we differentiate a product, it is ?

S:

P: No, differentiate two things. Product, differentiate the first, leave plus, remember that thing ?

S:

P: So it's du dx times v plus u times dv dx. Remember that ?

S:

P: If I like, gave you like, x e to the x. When you differentiate, you differentiate the first, leave the second etc. Now we want to try and introduce reverse differentiation. If I integrate this side here I'll just have u times v, and I'm going to have two integrals on that side, which is not going to do us any good. So what we do, I'm going to just solve this. Times v, so leave that that side. So on the other side you're going to have that, and then I'm going to subtract that, so this means that, OK ?

S:

P: I've just fixed that, OK so that definitely equals that, which equals that, agree with that ? Now we're going to integrate through, going to have an integral. We're going to leave out the dx when I integrate, should be there, and when I integrate this what's going to happen ? I put an integral here where the ddx is.

S:

P: Left with ?

S:

P: Uv minus integral, and this is what is called, like a product rule for integration or integration by parts, as we say. So what happens is we're going to integrate a product and you're going to end up with another integral over here. So what we want to do is make sure that we get an integral, which we can do, cause it's hopeless doing this process and getting an integral which is going to be more difficult than what we started off with. So you'll see how it works. I'm going to give you this, this is a famous one, we always start off, x, we want to integrate so what we do, e to the x. I'm going to let du dx, x equal to v so all I'd have to do is I have to find the u. I've got the v, and I must find the dv dx, so we can substitute everything on the side, OK ?

S:

P: OK look, we want to integrate the product of two things, there they are, there. Now I'm going to do, make this the product, there's the product there. So comparing that what do you see, du dx equals what ?

S:

P: And v equals ?

S:

P: So what other things do we need on this side. We need to get u, because we've only got du dx, we've got v haven't we, and what's missing over here, we've got, we haven't got u, got to get u. And we've got to get ?

S:

P: Have we got dv dx? All we've got is that, du dx equals.

S:

P: What ?

S:

P: Comparing those two.

S:

P: You've got to, compare. OK compare this you see the integral, something times something. What is du dx going to be ?

S:

P: Right and what is the x going, what is the v going to be ?

S:

P: X so if we substitute into this formula. E to the x for that, x for that, we must substitute a u that side, a v that side, and a u and a dv dx, so we must get ? How we're going to get a u and how we're going to get a dv dx, can we get dv dx from here ? V equals that, what's dv dx ? Differentiate that side you then, what you do to the one side you do to the other side.

Differentiate x, what is d dx of x, what is x, x to which power ?

S:

P: No, x to the power.

S:

P: One, differentiate it, what do you get ?

S:

P: Differentiate it.

S:

P: Which is ?

S:

P: X to the power zero is ?

S:

P: One, so differentiating that you get one. Now if du dx is equal to e to the x, how we going to get u from here ? How will we get rid of this ddx, what, what, thing will we use. What gets rid of the ddx ?

S:

P: Mmm, so if I integrate this side, I must integrate ?

S:

P: So if the integral of this, will get rid of this, and I'll be left with what over here ?

S:

P: A u, and the integral of e to the x is ? Look over here, where's integral e to the x ?

S:

P: That's fine, so u equals e to the x, and all we have to do now is plug it into here, so what is u ? So you've got u, you've got v and you've got dv dx, substitute , what is u ?

S:

P: What is v ?

S:

P: No.

S:

P: Ah huh, minus integral, and what is u ?

S:

P:Ah huh, dv dx is ?

S:

P: Now do we get something easier that we could integrate? What is integral of e to the x ? We've got it over here.

S:

P: Mmm, so we've created something which is easier. E to the x times x minus e to the x. So when you integrate this thing here, e to the x times x we actually get something out, but it only works provided that this integral is integrable, otherwise it's a meaningless thing. Can I give it again for you to do ? I'm going to give it to you, the same one, go through it cause see I knew there would be some problems, but I have another way of doing this, actually. OK, I want you to do that just like I did , just ask if you're stuck (mumble). Remember all we're going to do, I've now replaced this with that, so now compare, and now all you're going to do is get a u and get dv dx and plug it into that side into the equation. So what you do is you stop, stop there and compare that to that and write the information down, what is du dx ?

S:

P: Start up there, what is du dx going to be ?

S:

P: No no, comparing these two, why have you a problem seeing that ?

S:

P: It's a product.

S:

P: OK it's two things, that times that. You know, that's what? The method that will always work, this doesn't work . See what the other guy did.

S:

P: That's right, OK, and what will v equal ?

S:

P: Now compared to this side, we've got to put things into there to evaluate this, don't we? This is what we are going to work out. What is missing, what things do we need ?

S:

P: How we're going to get the u ?

S:

P: All we've got.

S:

P: That's right, OK do it.

S:

P: Ja, right.

S:

P: But you have, what you do to that side, you must do to that side. Integral, and what's integral e to the x ?

S:

P: That's it, so now write there what is u going to be. Great OK, what other things do we need ?

S:

P: Now we only need? No, you only worry about the things, you gotta only plug things into here.

S:

P: You've got u, have you got v ?

S:

P: Have you got u ?

S:

P: Have you got dv dx ?

S:

P: OK, can we get dv dx from that information ?

S:

P: If v equals x, how would you get dv dx? You d dx both sides.

S:

P: OK you do that, that is ?

S:

P: No no, you just want dv dx, so what's dv dx? Use the rule, where's the rule involving ddx of x to something ?

S:

P: Mmm, what's x to the naught?

S:

P: Ja, x to the naught. OK, so now let's substitute everything in there, put the u.

S:

P: V, ja, actually next time I think we should keep your, your, you wrote v equal x down, see v equals x?

S:

P: X minus that, stays there. Now what is u ?

S:

P: Yes, and dv dx is ?

S:

P: One, OK now looking at that, do we get a simple integral compared to what we started off with ?

S:

P: What is integral e to the x? Times what, is e to the x? What's integral e to the x? OK so now write down.

S:

P: What's, no that, just that one. This is what we want, OK so write that, and write the whole thing down. That's what we want, integral e to the x times x, OK, times x, just copy that thing down and integrate e to the x, what do you get ?

S:

P: Ja, so it's minus e to the x.

S:

P: In other words, we've integrated something which, we can not, we don't know, it's not one of these things given to you. And there, it is using this method which is called integration by parts. Now I'll give you, I'll leave this up here. See this rule, I'll show you later on a nicer way of looking at this rule. OK, I want you to do this, and see, compare again e to the x, instead of x, have x square. Notice that we've got, comparing that, I must now find everything on that side. I've got to get a u, a v, a u and a dv dx, can you do that ? So comparing those two, du dx is going to be ?

S:

P: That's right.

S:

P: OK v is ?

S:

P: Do it, mmm, you write those things down first, everything, you got to put every step in, don't try and skip it. OK now then ask yourself on the right hand side, what things do we have to do, still get.

S:

P: We'll get that now, rather write what u is over here, just leave that.

S:

P: And that gives you ? OK write integral e to the x, what you do to the one side you do to the other. No no no, you're doing this, try here. Integrate that side, you integrate that side, integral e to the x, so this side becomes ?

S:

P: Mmm, and that side becomes ?

S:

P: What else do we need ?

S:

P: Mmm, got a v, but what haven't we got over there ?

S:

P: Right, so v equals x squared, how we're going to get ?

S:

P: OK so differentiate that side, do to that side, so ddx of that side will give you ? That's it, ddx so what you do that side you do to the other side.

the x, to x.

S:

P: Can we do this now . That's, actually what we did at the beginning, we did e to the x times x. OK let's do that one, so now this whole thing goes. All we got to do now is this little thing here which will be. So the minus is outside, but remember the two can come out, that's really minus two integral e to the x times x. OK now we've already done this first thing, we did, you can start all over again so that thing there, this whole thing here is this thing here, same, they, so let's work out integral e to the x times x dx. How do we do it, again you going to go through the same process. We've done it, and I think, remember du dx was that, so it was e to the x times x minus integral x, e to the x which is e to the x. So that will be minus two into that, which is the same as that. We've already done e to the x so I am not going to do it again. But just notice that we have to get something that we can, to do eventually, so if we have x cubed we will have an x squared and you have to do that again. You have an x and eventually get it out. Let's do a log x one.

S:

P: Yes, either you can, it's easy, you can do it, either you can do it or, umm, you going to have to use the same process to eventually to get it down so that you can do it. OK let's do integral of ln x times x is ? OK now if you chose du dx equal to log x, to get u would mean you'd have to, suppose according to the way I've chosen it there, right du dx is log x, so how will you get u from there, what will you do ?

S:

P: And then u will be the integral. What's the integral of log x ? Is it one of the things that we know? Is it integral of log x anywhere here ?

S:

P: Ja, but is there an integral of log x here ?

S:

P: So we can't do it, it's a stupid choice so you'll have to swop, it's not going to help you. See you put x first, notice that all we have to do, if v equals log x, the only thing we got to do is find dv dx. Can we differentiate log x ? Is there a rule that say's differentiate log x ?

S:

P: Surely there's one.

S:

P: Differentiate.

S:

P: So we can, and if du dx equals x can we integrate x ? Can we integrate x to the one ? So therefore that will be the thing. OK do that, try that, the same process. Write that equals that, that, find u, v, dv dx, substitute in. The first thing you write is what du dx is, and v.

S:

P: That's it, and then v.

S:

P: What, what do you want? That's what you got to do. It will always be the same, you'll always have to integrate something.

S:

P: Remember always, this one you going to end up integrating, the one that you choose. So if I had a log x there you cannot integrate, you would not have chosen it, but x you got to integrate and this one you're going to differentiate. It's always the one you're going to integrate and the one you going to differentiate.

S:

P: That's it.

S:

P: So careful before you do that, always do to both sides.

S:

P: See you're too quick,do it, integral, then you start to cancel out. So that side gives you ?

S:

P: That's fine, don't do it yet, the one's there. OK so now go slowly, so what will u be, u equals ... ?

S:

P: NO no no no, what's the rule ? Integral of x to the n, here, what does it say, x to the one it is, x to the, soon as you get rid of it you must do the process. Compare that to that, now.

S:

P: Ah huh, that's it.

S:

P: Two, that's it. So that's u, what else do we need ?

S:

P: OK, get it.

S:

P: No, integrate it. We only want d by dx, so therefore you differentiate it.

S:

P: Ja.

S:

P: Now what you do one side you do the other side.

S:

P: That's it, and then you come here and you look. Do you see ddx of log x ? Look very carefully, each one, until you see it, don't just say it.

S:

P: Ah huh, so carry on, equals, that's it, now got everything. Plug in and see if we get, hope that this is going to be do-able, otherwise we going to be stuffed.

S:

P: That's it.

S:

P: That's it, now on the other side, have we got something that we can work with ? Remember, we're gonna have to evaluate that, that's two. OK can you integrate x squared over x , what is that? Can you simplify it ?

S:

P: Over x, which is, what's x squared over x ?

S:

P: It's there, it's so easy, obviously that's where all your problems lie.

S:

P: OK, in other words it's a half, OK because we got that one, that half there, you can take this out. Write this down again and take out this half and see what you get, just carry on. OK no no, write what you've got. Integral x log x, just write everything down, now take out the half, so minus a half in integral x squared over x, isn't that the same thing ? OK, now what's x squared over x ? Take that to the, what is x, x squared is x times x and the x's cancel, and you're left with ? What is x squared over x ?

S:

P: What is x squared over x ? it's x times x over x. What is it ?

S:

P: Ja, so that's half integral x. Write it down, start here. Equals, that's right, minus half integral of x. What's integral of x to the one, so it's going to be?

S:

P: Ja write, you must always keep your step. You've already did it, you actually did it, you've written the thing down there.

S:

P: Now do that. What is the integral of x ?

S:

P: So, to complete, what is a half ? You see here you going to have a half times that, three times two will give you ?

S:

P: Minus x squared, OK so the whole thing, if you can get something to simplify. If you look at the trig ones, I'll show you what the trig ones look like. If you remember when you differentiated sin, can you remember what we got ?

S:

P: Sinx you get cosx, so when you integrate cosx you get ?

S:

P: Now here we can also do this. If I put an x and a sinx there, if I decide to do this, du dx equals x, which means we integrate x and integral of x is x squared over two. X to the one plus one and then we got to differentiate sinx which gives us cosx. So this side we're going to have an x squared over two times cosx, Is that going to help ? Compare that to that one, gonna integrate, isn't this worse ?

S:

P: So for these you would always swop, it would be much better if these things are trial and error. You have to have sinx, x integral of sinx, well when you differentiate cosx times you get negative sinx, so integral of sinx is negative cosx. So we can integrate sinx and differentiate x, you get ?

S:

P: One, no just one, differentiate x you get x to the naught. So in this rule it will be integral, you see ? It's always going to, u's the integral of sinx, integral of sinx is minus cosx, times v, which is x minus, u is minus cosx. Differentiate x you get one, you get an easy integral which you can do, so these things, as long as we can get an easy integral, they work. OK thanks, Jerry, ... hey ?

S:

P: Thanks Jerry, wasn't so bad, was it ?

S:

P: Well, we'll do this later on and you've already done it.

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