SHAPE INVARIANCE IN SUPERSYMMETRIC QUANTUM MECHANICS

BY

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The whole thesis, unless specifically indicated to the contrary in the text, represents original work by the author, which has not been submitted for any degree or diploma to any other University.

CONTENTS

DIAGRAMS:

ABSTRACT

This thesis presents the results of a study of the use of shape invariance in supersymmetric quantum mechanics. The most original part of the thesis comprises research on (i) the use of shape invariance in the momentum representation (Chapter 5) and on (ii) the shape invariance of certain non-relativistic and relativistic Coulomb problems and of a relativistic oscillator (Chapters 5 and 6). This thesis also contains other contributions, namely (iii) a simple yet complete derivation of the basic results of shape invariance (Chapter 1); (iv) some simple introductory examples on the use of shape invariance (Chapter 2); (v) for all the problems considered it is shown how in the treatment by shape invariance the normalization of bound state eigenfunctions can be achieved (Chapters 1-6); and (vi) in the determination of these eigenfunctions the results are expressed in terms of hypergeometric functions (Chapters 3-6), rather than other special functions. Use of the former functions clarifies the presentation.

CHAPTER 1

INTRODUCTION, SUPERSYMMETRY AND SHAPE INVARIANCE

1.1 INTRODUCTION

In this Chapter a discussion of the origins of supersymmetry and shape invariance in quantum mechanics is presented (Sections 1.2.1 and 1.3.1). The historical outline given in these Sections is by no means complete, although it is hoped that the references quoted will complement the discussion and enable the reader to easily find references to topics that have been omitted.

In Section 1.2.2 the mathematical formalism of supersymmetric quantum mechanics is presented. In this discussion we have emphasized those ideas that are necessary for the introduction of the concept of shape invariance in supersymmetric quantum mechanics. In Section 1.3.2 we introduce the concept of shape invariance and derive the results which give explicit algebraic expressions for the energy eigenvalues (Equation (1.41)) and normalized energy eigenfunctions (Equation (1.46)) of quantum mechanical systems with shape invariant potentials. Our derivation differs from those presented in the literature in that we make use of a construction of two hierarchies of Hamiltonians instead of the single hierarchy used in the standard derivation. This approach simplifies the derivation of the above-mentioned results, and it also provides a clearer insight into the factors required to obtain normalized eigenfunctions. The normalization constants are usually omitted in the literature.

The results obtained in Chapter 1 are applied to three simple quantum mechanical systems in Chapter 2. The first example is the familiar one-dimensional harmonic oscillator (Section 2.1), for which the method of shape invariance is identical to the standard operator method of solving this problem. The second example concerns orbital angular momentum (Section 2.2). We show how shape invariance can be used to obtain the orbital angular momentum quantum numbers and the spherical harmonics. In Section 2.3 we consider the problem of an electron in a uniform magnetic field.

In Chapter 3 we use shape invariance to determine the energy eigenvalues and normalized coordinate-space eigenfunctions of certain one-dimensional systems. Here we consider the one-dimensional Morse potential (Section 3.1), the Rosen-Morse potential (unsymmetric and symmetric cases - Section 3.2) and the first and second Pöschl-Teller potentials (Sections 3.3 and 3.4). The analysis of the first two examples is presented in some detail to clarify the procedure. These problems (and some of the others that are considered in this thesis) have previously been treated by shape invariance in the literature to obtain the energy eigenvalues and the (unnormalized) energy eigenfunctions. The treatment given in this thesis shows how the normalization of bound state eigenfunctions can be performed using shape invariance. We also obtain these eigenfunctions in terms of hypergeometric functions rather than (as is done in the literature) in terms of Jacobi polynomials or other special functions. We believe the use of hypergeometric functions clarifies the presentation and in Chapters 3-6 we have obtained the results in terms of these functions.

In Chapter 4 the results obtained in Chapter 1 are expressed in a form which is suitable for application to the coordinate representation of spherically symmetric problems in an angular momentum basis. We consider the following problems: (i) the three-dimensional isotropic non-relativistic oscillator (Section 4.2), (ii) the non-relativistic Coulomb problem (Section 4.3), (iii) modified oscillator and Coulomb potentials (Section 4.4) and (iv) the Eckart potential (Section 4.5). In Section 4.6 we show how the Hulthén potential can be obtained from the Eckart potential, and in Section 4.7 we extend the results obtained for the one-dimensional Morse and symmetric Rosen-Morse potentials obtained in Chapter 3 to s states of the corresponding three-dimensional potentials.

In Chapter 5 we formulate the method of shape invariance in a manner that is suitable for application to the momentum representation of spherically symmetric systems in an angular momentum basis (Section 5.1). In Section 5.2 we use the formulation of Section 5.1 to obtain the energy eigenvalues and normalized radial momentum-space eigenfunctions of the three-dimensional isotropic non-relativistic oscillator. The non-relativistic Coulomb problem in the momentum representation is discussed in Section 5.3.

In Chapter 6 we consider relativistic problems. A brief introduction to the Klein-Gordon and Dirac equations is presented in Section 6.1. In Section 6.2 we treat the Klein-Gordon and Dirac equations with a Coulomb potential, in a form which is suitable for application to the coordinate representation. We consider a Dirac oscillator in the coordinate representation in Section 6.3, and in the momentum representation in Section 6.4. As a final example we discuss the one-dimensional Klein-Gordon equation with a Coulomb-like potential in the momentum representation (Section 6.5).

We also compare the various types of shift operations that can be constructed for most of the examples considered in Chapters 3-6. Namely, those obtained by the method of shape invariance, the shift operations obtained from the factorization method of Infeld and Hull, and the shift operations for energy. These shift operations are presented in Tables at the end of each Section.

The results presented on the application of shape invariance in the momentum representation (Chapter 5), and on the relativistic Coulomb problems (Chapter 6) have previously been published in the following two papers:

1. de Lange, O.L. and Welter A. (1992). Shape Invariance with Application to the Momentum Representation. *Journal of Physics A : Mathematical and General*, 25, 5753-60.
2. de Lange, O.L. and Welter, A. (1992). Shape Invariance of Coulomb Problems. *American Journal of Physics*, 60, 254-7.

1.2 SUPERSYMMETRY

In this section a historical account of the developments leading to the introduction of supersymmetry in quantum mechanics and the subsequent applications of this theory are given (Section 1.2.1). The relation between the method of supersymmetry in quantum mechanics and other algebraic methods is also traced. A brief review of supersymmetric quantum mechanics is then presented (Section 1.2.2).

1.2.1 **Historical review**

The current interest in supersymmetry stems from work done in the early nineteen seventies by a number of authors [1-3], although the algebra governing supersymmetry had been explored in some mathematical contexts before this [4]. In 1971 Gol'fand and Likhtman [1] discovered the four-dimensional Poincaré superalgebra, and shortly thereafter Volkov and Akulov [2] and Wess and Zumino [3] incorporated supersymmetry in field theories. The current use of supersymmetry is mainly due to the work by Wess and Zumino.

The term supersymmetry is usually associated with two properties.

Firstly, supersymmetry involves transformations of bosons into fermions and vice versa (the Hamiltonian is invariant under such transformations). Secondly, the algebra of supersymmetry is a graded Lie algebra which involves both commutation and anti-commutation relations for the generators of the algebra [5].

An important property of supersymmetry in field theory is that it has led to the reduction of divergences which had plagued certain quantum theories in the past. This success led immediately to the incorporation of supersymmetry in quantum theories of gravity [6], which had run into difficulties due to the non-renormalizability of the gravitational interaction. The study of supersymmetric theories of gravity (supergravity) is still a major topic of interest [7], and it is envisaged that a unified theory of all interactions will be supersymmetric in character.

Whether there is any evidence of supersymmetry in high energy particle physics is still uncertain [8]. However, supersymmetry has been applied successfully in atomic, nuclear, statistical and solid state physics [9]. The degeneracy of the Landau levels for a spin $\frac{1}{2}$ particle in a uniform magnetic field has been recognised to be supersymmetric in character for the special case when the gyromagnetic ratio $g = 2$ [10]. The mathematical properties of the superalgebra have been rigorously analysed by many authors [11]. It has been remarked that supersymmetry is another example of "when the mathematicians in all their wisdom have overlooked a beautiful and most useful structure, and have come to appreciate it only at the demand of physicists" [4].

The development of supersymmetry in quantum mechanics is due to a simple model proposed by Witten [12] to demonstrate a system in which dynamical supersymmetry breaking occurs.* Witten chose the simplest example of supersymmetry in which there are only two generators (see Section 1.2.2). This reduces the problem to one of supersymmetric quantum mechanics.

Since the inception of supersymmetry in quantum mechanics, a considerable amount of work has appeared in the literature on the applications of supersymmetry in quantum mechanics [13], and on the relation between supersymmetry and other algebraic methods in quantum mechanics [15, 19]. This work has provided a deeper insight into these algebraic methods. The method of supersymmetry in quantum mechanics involves the factorization of a given Hamiltonian (or some other operator) into a linear operator and its adjoint (see Section 1.2.2). In terms of differential equations, this is equivalent to replacing a linear second-order differential equation by two linear first-order differential equations. The mathematical groundwork describing the factorization of higher-order differential equations into systems of lower-order equations had already been carried out in the eighteenth and nineteenth centuries [14], but it was not until much later that these connections were pointed out [15].

The idea of using a factorization method to solve certain quantum-mechanical eigenvalue-problems was introduced by Dirac, Weyl and Schrödinger [16]. This work was generalized in a classic paper by Infeld and Hull [17], who obtained six classes of factorization types, and used these to solve Schrödinger's equation for all the known solvable one-dimensional potentials. Infeld and Hull's factorization method was later

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* Broken and unbroken superymmetry is discussed in Section 1.2.2.

shown to be related to the Darboux transformation [18]. The equivalence of supersymmetry in quantum mechanics to the factorization method has now been firmly established [19], and further insight into this equivalence has been given by the introduction of the concept of shape invariant potentials [20] (see Section 1.3).

1.2.2 Supersymmetry in quantum mechanics

In supersymmetric quantum mechanics there are two operators Q_1 and Q_2 which commute with the Hamiltonian H and anti-commute with each other, and whose squares are equal to half the Hamiltonian. That is,

$$[Q_i, H] = 0 \tag{1.1}$$

and

$$\{Q_i, Q_j\} = \delta_{ij} H , \tag{1.2}$$

for i and j equal to 1 or 2.

Consider the operators Q_1 and Q_2, given by

$$Q_1 = \frac{1}{\sqrt{2}} (Q + Q^\dagger) , \qquad Q_2 = \frac{i}{\sqrt{2}} (Q - Q^\dagger) , \tag{1.3}$$

where

$$Q = \begin{bmatrix} 0 & 0 \\ A_+ & 0 \end{bmatrix} , \qquad Q^\dagger = \begin{bmatrix} 0 & A_- \\ 0 & 0 \end{bmatrix} , \tag{1.4}$$

and

$$A_\pm(x) = \pm \frac{i}{\sqrt{2M}} p + W(x) . \tag{1.5}$$

Here p and x are the one-dimensional momentum and position operators of a particle of mass M; they satisfy the canonical commutation relation [21]

$$[x, p] = i\hbar . \tag{1.6}$$

The superpotential, $W(x)$, is a real function of x.

Using (1.3)-(1.5) in (1.2), we find that

$$H_{ss} = \begin{bmatrix} A_-A_+ & 0 \\ 0 & A_+A_- \end{bmatrix} = \begin{bmatrix} H_- & 0 \\ 0 & H_+ \end{bmatrix} . \tag{1.7}$$

Here $H_{ss} = H$ and

$$H_{\pm} = A_{\pm}A_{\mp} = \frac{1}{2M}\, p^2 + V_{\pm}(x) \ , \tag{1.8}$$

with

$$V_{\pm}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2M}} \frac{dW(x)}{dx} \ . \tag{1.9}$$

It is straightforward to show that $Q^2 = (Q^{\dagger})^2 = 0$, and hence that (1.3)-(1.5) satisfy (1.1) and (1.2). The operators $H_{\pm}$ are referred to as supersymmetric partner Hamiltonians and $V_{\pm}(x)$ are supersymmetric partner potentials.

Suppose that $H_{\pm}$ possess discrete spectra, i.e.

$$H_{\pm}|\psi_N^{\pm}\rangle = E_N^{\pm}|\psi_N^{\pm}\rangle \qquad (N = 0, 1, 2, \ \ldots) \ . \tag{1.10}$$

It follows from (1.10), (1.8) and $A_- = (A_+)^{\dagger}$ that

$$E_N^{\pm} \geq 0 \ . \tag{1.11}$$

(This can be seen by premultiplying both sides of (1.10) by the bra $\langle\psi_N^{\pm}|$ and using the non-negativity of a norm.)

Supersymmetry is said to be dynamically broken if the ground state of a quantum mechanical system is degenerate. If supersymmetry is unbroken then there exists a single non-degenerate ground state, determined by an annihilation condition (see equation (1.12)), with zero energy. We assume throughout this work that supersymmetry is unbroken [12]: hence either A_+ or A_- annihilates the ground state. We can, without loss of generality [22], suppose the former and write

$$A_+|\psi_0^-\rangle = 0 \ . \tag{1.12}$$

Premultiplying (1.12) by A_-, and using (1.8) and (1.10), yields

$$E_0^- = 0 \ . \tag{1.13}$$

In the coordinate representation (1.12) corresponds to the first-order differential equation,

$$\left[\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x)\right]\psi_0^-(x) = 0 \tag{1.14}$$

where $\psi_0^-(x) = \langle x|\psi_0^-\rangle$ is the coordinate-space eigenfunction of the ground state. Equation (1.14) has the solution

$$\psi_0^-(x) = C_0 \exp\left[-\frac{\sqrt{2m}}{\hbar}\int^x W(x)dx\right] , \tag{1.15}$$

where C_0 is a constant to be determined so that $\psi_0^-(x)$ is normalized. In practice, if the ground state eigenfunction $\psi_0^-(x)$ is known, one could determine W(x) from (1.15) using the relation

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\ell n\, \psi_0^-(x) . \tag{1.16}$$

Alternatively, one could start with a known potential $V_-(x)$ and solve the Riccati equation (1.9) for the superpotential W(x).

To establish the relationships between the eigenkets of H_+ and H_- we premultiply the eigenvalue equation of H_- (with N replaced by N+1) by A_+ to obtain

$$A_+H_-|\psi_{N+1}^-\rangle = E_{N+1}^- A_+|\psi_{N+1}^-\rangle . \tag{1.17}$$

Use of (1.8) in (1.17) shows that

$$H_+\{A_+|\psi_{N+1}^-\rangle\} = E_{N+1}^-\{A_+|\psi_{N+1}^-\rangle\} . \tag{1.18}$$

Hence $A_+|\psi_{N+1}^-\rangle$ are eigenkets of H_+ corresponding to energy E_{N+1}^-. A similar calculation yields

$$H_-\{A_-|\psi_N^+\rangle\} = E_N^+\{A_-|\psi_N^+\rangle\} , \tag{1.19}$$

and therefore $A_-|\psi_N^+\rangle$ are eigenkets of H_- corresponding to energy E_N^+. From (1.18), (1.19), (1.8) and the condition (1.12) for unbroken supersymmetry, we obtain the following transformations*

$$A_+|\psi_{N+1}^-\rangle = \sqrt{E_{N+1}^-}\,|\psi_N^+\rangle \tag{1.20}$$

and

$$A_-|\psi_N^+\rangle = \sqrt{E_N^+}\,|\psi_{N+1}^-\rangle , \tag{1.21}$$

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* Note that if (1.13) does not hold, then E_{N+1}^- and E_N^+ in (1.20) and (1.21) must be replaced by $E_{N+1}^- - E_0^-$ and $E_N^+ - E_0^-$; see also (1.42) and (1.43).

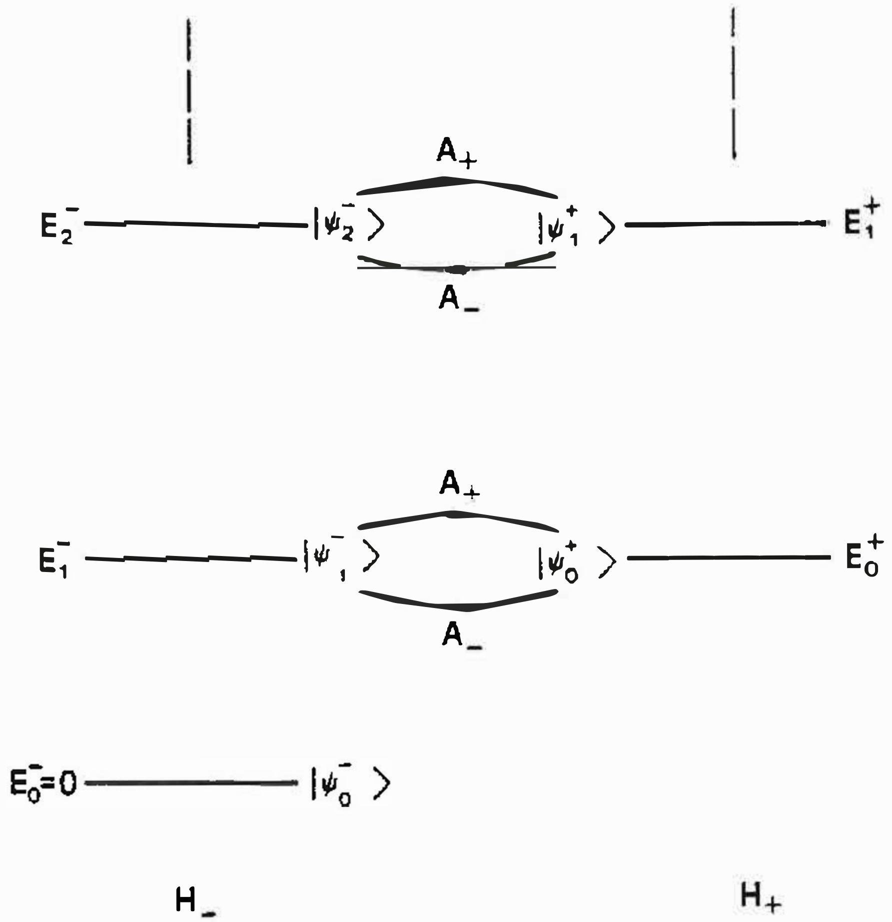

Fig. 1.1 The spectra of the Hamiltonians (1.8) (see Ref. 21 Section 3.8). Examples of the transformations (1.20) and (1.21) are depicted. The kets $|\psi_N^+\rangle$ and $|\psi_{N+1}^-\rangle$ are doubly degenerate for $N = 0,1,2, \ldots$ (see (1.22)).

with

$$E^{+}_{N} = E^{-}_{N+1} \qquad (N = 0,1,2, \ldots) . \tag{1.22}$$

A detailed derivation of Equations (1.20)-(1.22) is provided in Appendix A. The spectra of $H_\pm$ and examples of the transformations (1.20) and (1.21) are depicted in Fig. 1.1. We see that the spectra of H_+ and H_- are identical except that H_+ has no normalizable eigenket with energy zero.*

1.3 SHAPE INVARIANCE IN SUPERSYMMETRIC QUANTUM MECHANICS

Shape invariance was introduced into quantum mechanics in 1983 as a "new type of hidden symmetry" by Gendenshtein [20]. For systems with this invariance property, the eigenvalues and eigenfunctions can be obtained by a simple procedure which is essentially a generalization of the method of shift operators for the one-dimensional harmonic oscillator [20].

In Section 1.3.1 an outline is given of the developments of shape invariance in supersymmetric quantum mechanics since its introduction in 1983. In Section 1.3.2 a derivation is given of the main results which will be used in the remainder of this thesis to obtain the eigenvalues and eigenfunctions of various systems.

1.3.1 Historical Review

Three years after the introduction of the concept of shape invariance in 1983 [20], Dutt *et al.* [23] extended the results derived by Gendenshtein to include calculations of the bound state eigenfunctions from the ground

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* Choosing $A_-|\psi^+_0\rangle = 0$ instead of (1.12) also has the consequence that the spectra of $H_\pm$ are identical, except that in this case it is H_- which has no normalizable eigenket with zero energy.

state using the results obtained by the introduction of shape invariant potentials. Further work on the calculation of eigenvalues and eigenfunctions was carried out by Dabrowska *et al.* [24].

The first formal connection between supersymmetry, shape invariance and solvable potentials was made by Cooper *et al.* [25] who investigated the Natanzon class of potentials and showed that these are, in general, not shape invariant. This work showed for the first time that shape invariance is not a necessary condition for solvability. The equivalence between the shape invariance condition and the factorization condition [17] was shown by Stahlhofen [15,26] and Montemayor and Salem [27].

Besides the calculation of eigenvalues and eigenfunctions, the concept of shape invariant potentials has been used in conjunction with supersymmetry and a factorization method to generate other solvable potentials, which are in general not shape invariant [28]. Shape invariance has also been used to determine scattering amplitudes [29], and to evaluate path integrals for shape invariant potentials [30].

The results of supersymmetry and shape invariance have been derived in an abstract mathematical framework [31], and extended to include applications in the momentum representation [32] (see Chapter 5). Recently it has also been shown how Point Canonical Transformations map twelve types of shape invariant potentials into two potential classes [33].

1.3.2 **Shape invariance**

Gendenshtein showed that if a potential is shape invariant, one can construct a hierarchy of Hamiltonians $H^{(s)}$ $(s = 0,1,2, \ldots)$, each member of which is the supersymmetric partner of the adjacent member in the hierarchy (see below). The method of constructing a hierarchy of Hamiltonians to solve for the eigenvalues and eigenfunctions has also been used by Sukumar [34] without referring explicitly to the concept of shape invariance. The results obtained by Sukumar are the same as those obtained using shape invariance.

If the supersymmetric partner potentials $V_{\pm}(x)$ also depend on a set of parameters α_s (s = 0,1,2, ...), then they are said to be shape invariant if the following relation exists between them:

$$V_+(x,\alpha_s) = V_-(x,\alpha_{s+1}) + R(\alpha_{s+1}) . \tag{1.23}$$

Here α_s denotes a set of parameters with

$$\alpha_{s+1} = f(\alpha_s) , \tag{1.24}$$

and the remainder, $R(\alpha_s)$ in (1.23) is independent of x.

Substituting (1.9) in (1.23), where W(x) is now also a function of α_s, we obtain the condition for shape invariance in terms of the superpotential $W(x,\alpha_s)$:

$$W^2(x,\alpha_s) + \frac{\hbar}{\sqrt{2M}}\frac{dW(x,\alpha_s)}{dx} = W^2(x,\alpha_{s+1}) - \frac{\hbar}{\sqrt{2M}}\frac{dW(x,\alpha_{s+1})}{dx} + R(\alpha_{s+1}). \tag{1.25}$$

It can be shown that equation (1.25) is equivalent to the factorization condition of Infeld and Hull [17,15,26,27].

The usual analysis [20,24] starts with a hierarchy of Hamiltonians

$$H^s = \frac{1}{2M}p^2 + V_-(x,\alpha_s) + \sum_{k=1}^{s} R(\alpha_k) \qquad (s = 1,2, \ldots) , \tag{1.26}$$

$$H^{(0)} = H_- . \tag{1.27}$$

(Note that $H^{(1)} = H_+$. Consequently $H^{(0)}$ and $H^{(1)}$ are supersymmetric partners; one can readily show that $H^{(s)}$ and $H^{(s+1)}$ (s = 1,2, ...) are also supersymmetric partners [24].) In the following derivation we modify* this analysis by considering two hierarchies of Hamiltonians, namely

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* This modification clarifies the presentation, in particular the derivation of the energy-dependent factor $\gamma_N(\alpha_0)$ in (1.46). In the literature this factor is usually omitted [24,27].

$$H_{\pm}^{(s)} = \frac{1}{2M}p^2 + V_{\pm}(x,\alpha_s) + \sum_{k=1}^{s} R(\alpha_k) \qquad (s = 1,2, \dots) \tag{1.28}$$

$$H_{\pm}^{(0)} = H_{\pm} . \tag{1.29}$$

Taking the lower sign in (1.28) and replacing s by s + 1,

$$H_{-}^{(s+1)} = \frac{1}{2M}p^2 + V_{-}(x,\alpha_{s+1}) + \sum_{k=1}^{s+1} R(\alpha_k)$$

$$= \frac{1}{2M}p^2 + V_{+}(x,\alpha_s) + \sum_{k=1}^{s} R(\alpha_k) , \tag{1.30}$$

where the shape invariance condition (1.23) has been used. Comparing (1.28) and (1.30), and using (1.8), we see that $H_{-}^{(s)}$ and $H_{-}^{(s+1)}$ are supersymmetric partner Hamiltonians. Therefore the results obtained from supersymmetry in Section 1.2.2 apply to the eigenvalues and eigenkets of these Hamiltonians. In particular, for discrete spectra, the energy eigenvalues $E_N^{(s)}$ of $H_{-}^{(s)}$ must satisfy (see (1.22) and Fig. 1.1)

$$E_N^{(s)} = E_{N-1}^{(s+1)} . \tag{1.31}$$

Furthermore, the right-hand side of (1.30) is simply $H_{+}^{(s)}$ so that

$$H_{-}^{(s+1)} = H_{+}^{(s)} . \tag{1.32}$$

Consequently if $|\psi_N^{\pm}(\alpha_s)\rangle$ denote the eigenkets of $H_{\pm}^{(s)}$, we can write,

$$|\psi_N^{-}(\alpha_{s+1})\rangle = |\psi_N^{+}(\alpha_s)\rangle . \tag{1.33}$$

The above properties of the spectra and eigenkets of the hierarchies (1.28) are illustrated in Fig. 1.2.

Next we rewrite equations (1.8) and (1.9) to include the dependence on the parameter α_s:

$$A_{\pm}(\alpha_s)A_{\mp}(\alpha_s) = \frac{1}{2M}p^2 + V_{\pm}(\alpha_s) , \tag{1.34}$$

where

$$A_{\pm}(\alpha_s) = \pm \frac{i}{\sqrt{2M}}p + W(\alpha_s) \tag{1.35}$$

and

$$V_{\pm}(\alpha_s) = W^2(\alpha_s) \pm \frac{\hbar}{\sqrt{2M}} \frac{dW(\alpha_s)}{dx} . \tag{1.36}$$

In (1.34)-(1.36) we have simplified the notation by omitting the dependence

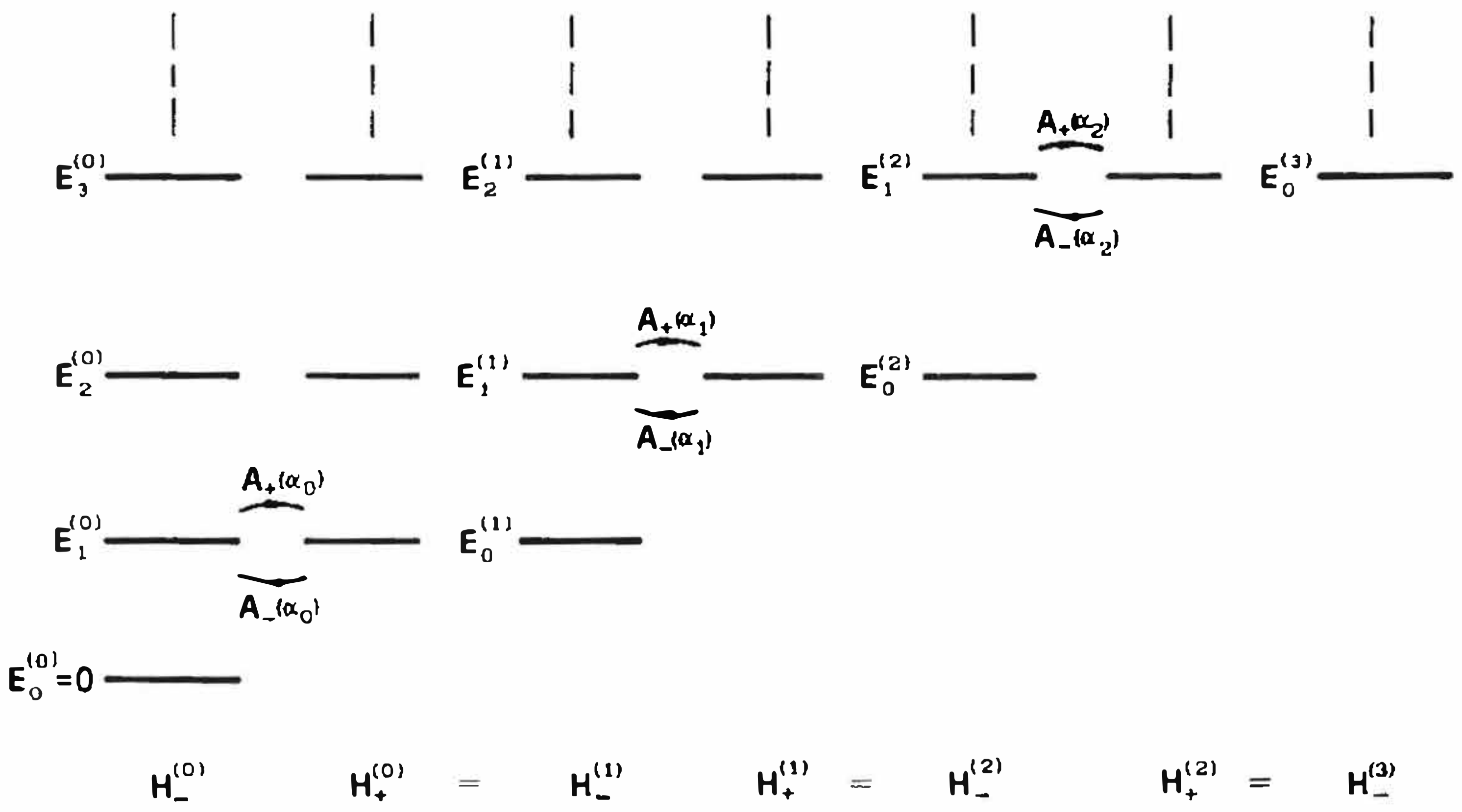

Fig. 1.2 The spectra of the hierarchies of Hamiltonians (1.28). Examples of the transformations (1.42) and (1.43) are depicted.

of $A_\pm$, $V_\pm$ and W on x. From (1.28) and (1.34) we see that

$$H_-^{(s)} = A_-(\alpha_s)A_+(\alpha_s) + \sum_{k=1}^{s} R(\alpha_k) \qquad (s = 1,2, \ldots) \; . \tag{1.37}$$

The annihilation condition (1.12) is

$$A_+(\alpha_s)|\psi_0^-(\alpha_s)\rangle = 0 \qquad (s = 0,1,2, \ldots) \tag{1.38}$$

and the ground-state energy of $H_-^{(0)}$ is (see (1.13))

$$E_0^{(0)} = 0 \; . \tag{1.39}$$

Premultiplying (1.38) by $A_-(\alpha_s)$ and using (1.37) we obtain

$$[H_-^{(s)} - \sum_{k=1}^{s} R(\alpha_k)]|\psi_0^-(\alpha_s)\rangle = 0 \; . \tag{1.40}$$

Letting s = N (= 1,2,...) in (1.40) and noting from (1.31) and Fig. 1.2 that $E_0^{(N)} = E_N^{(0)}$, we obtain from (1.40) an expression for the energy eigenvalues of $H_-^{(0)}$ in terms of the remainder $R(\alpha_k)$, namely

$$E_N^{(0)} = \sum_{k=1}^{N} R(\alpha_k) \qquad (N = 1,2, \ldots) \; . \tag{1.41}$$

Equation (1.41) is a general formula for calculating the energy eigenvalues of shape invariant potentials.

From (1.28) it is obvious that $H_\pm^{(s)}$ are also supersymmetric partner Hamiltonians. Therefore the transformations (1.20) and (1.21) yield

$$A_+(\alpha_s)|\psi_N^-(\alpha_s)\rangle = e^{-i\theta}(E_N^{(s)} - E_s^{(0)})^{\frac{1}{2}}|\psi_{N-1}^+(\alpha_s)\rangle \tag{1.42}$$

and

$$A_-(\alpha_s)|\psi_N^+(\alpha_s)\rangle = e^{i\theta}(E_{N+1}^{(s)} - E_s^{(0)})^{\frac{1}{2}}|\psi_{N+1}^-(\alpha_s)\rangle \; . \tag{1.43}$$

In (1.42) and (1.43) we have included a phase factor $e^{i\theta}$ (see Appendix A). (The energy-dependent factors in (1.42) and (1.43) are similar to those in (1.20) and (1.21), except that they now explicitly include ground state energies $E_s^{(0)} = E_0^{(s)}$ which are simply zero in (1.20) and (1.21).) Examples of the transformations (1.42) and (1.43) are depicted in Fig. 1.2. Using (1.33) in (1.42) and (1.43) we obtain transformations for the kets of $H_-^{(s)}$,

$$A_+(\alpha_s)|\psi_N^-(\alpha_s)\rangle = e^{-i\theta}(E_N^{(s)} - E_s^{(0)})^{\frac{1}{2}}|\psi_{N-1}^-(\alpha_{s+1})\rangle \tag{1.44}$$

$$A_-(\alpha_s)|\psi_N^-(\alpha_{s+1})\rangle = e^{i\theta}(E_{N+1}^{(s)} - E_s^{(0)})^{\frac{1}{2}}|\psi_{N+1}^-(\alpha_s)\rangle \; . \tag{1.45}$$

By repeated application of (1.45), we can generate all the excited states* $|\psi_N(\alpha_0)\rangle$ of H_- from $|\psi_0(\alpha_N)\rangle$. The result is:

$$|\psi_N(\alpha_0)\rangle = e^{-iN\theta}\gamma_N(\alpha_0)A_-(\alpha_0)A_-(\alpha_1)\ldots A_-(\alpha_{N-1})|\psi_0(\alpha_N)\rangle , \tag{1.46}$$

for N = 1,2, ... Here

$$\gamma_N(\alpha_0) = \prod_{k=1}^{N}(E_N^{(0)} - E_{k-1}^{(0)})^{-\frac{1}{2}} , \tag{1.47}$$

and $|\psi_0(\alpha_N)\rangle$ satisfies (1.38) with s=N. The kets $|\psi_N(\alpha_0)\rangle$ in (1.46) are normalized according to

$$\langle\psi_N(\alpha_0)|\psi_N(\alpha_0)\rangle = 1$$

(see also (A.11) and (A.12)).

Equation (1.46) is a general formula for calculating the eigenfunctions of bound states of shape invariant potentials.

We prove (1.46) by induction. Setting N=1 in (1.46) we find that

$$|\psi_1(\alpha_0)\rangle = e^{-i\theta}(E_1^{(0)} - E_0^{(0)})^{-\frac{1}{2}}A_-(\alpha_0)|\psi_0(\alpha_1)\rangle ,$$

which is (1.45) with s=0 and N=0. Assuming (1.46) is true for some N, we must show that it holds for N+1. Let

$$\beta_N = \alpha_{N+1} . \tag{1.48}$$

Then (1.46) with N replaced by N+1 becomes

$$|\psi_{N+1}(\alpha_0)\rangle = e^{-i\theta}\gamma_{N+1}(\alpha_0)A_-(\alpha_0)\{e^{-iN\theta}A_-(\beta_0)A_-(\beta_1)\ldots A_-(\beta_{N-1})|\psi_0(\beta_N)\rangle\}$$

$$= e^{-i\theta}\frac{\gamma_{N+1}(\alpha_0)}{\gamma_N(\beta_0)}A_-(\alpha_0)|\psi_N(\beta_0)\rangle . \tag{1.49}$$

Now

$$\gamma_N(\beta_0) = \gamma_N(\alpha_1) = \prod_{k=1}^{N}(E_N^{(1)} - E_{k-1}^{(1)})^{-\frac{1}{2}} = \prod_{k=1}^{N}(E_{N+1}^{(0)} - E_k^{(0)})^{-\frac{1}{2}} , \tag{1.50}$$

where (1.31) and (1.48) were used. Hence the ratio of the energy-dependent factors in (1.49) is

$$\frac{\gamma_{N+1}(\alpha_0)}{\gamma_N(\beta_0)} = \frac{\prod_{k=1}^{N+1}(E_{N+1}^{(0)} - E_{k-1}^{(0)})^{-\frac{1}{2}}}{\prod_{k=1}^{N}(E_{N+1}^{(0)} - E_k^{(0)})^{-\frac{1}{2}}} = (E_{N+1}^{(0)} - E_0^{(0)})^{-\frac{1}{2}} . \tag{1.51}$$

---

* We simplify the notation by omitting the superscript - on $|\psi_N\rangle$ in (1.44) and (1.45).

Using (1.51) in (1.49), we obtain

$$|\psi_{N+1}(\alpha_0)\rangle = e^{-i\theta}(E^{(0)}_{N+1} - E^{(0)}_0)^{-\frac{1}{2}}A_-(\alpha_0)\,|\psi_N(\alpha_1)\rangle ,$$

which is equation (1.45) with s=0.

The main results of this section are equations (1.41) and (1.46). In the following Chapters we make use of these expressions, and show how they enable one to calculate the energy eigenvalues and normalized eigenfunctions of a Hamiltonian with a shape invariant potential. In Chapter 4 we show how the results obtained in this Section can be extended to treat three-dimensional spherically symmetric systems, and in Chapter 5 we modify these results to treat problems in the momentum representation.

CHAPTER 2

SIMPLE SYSTEMS

This chapter is intended as an introduction to the use of shape invariance: thus the theory presented in Chapter 1 is applied to solve certain simple quantum-mechanical problems. Three systems for which the results are well known are considered. The simplest of these is the one-dimensional harmonic oscillator [1,2]. The standard operator method for the harmonic oscillator closely resembles the method of shape invariance for this problem. In fact, it has been noted that the method of shape invariance is a generalization of the operator method for the harmonic oscillator [3]. In Section 2.2 shape invariance is used to determine the eigenvalues and eigenfunctions for orbital angular momentum, and in Section 2.3 the problem of an electron in a uniform magnetic field is treated by shape invariance.

2.1 ONE-DIMENSIONAL HARMONIC OSCILLATOR

The Hamiltonian of the one-dimensional harmonic oscillator is given by

$$H = \frac{1}{2M}\, p^2 + V(x) \ , \tag{2.1}$$

with

$$V(x) = \tfrac{1}{2} M\omega^2 x^2 \ . \tag{2.2}$$

Here x and p are the position and momentum operators: they satisfy the canonical commutation relation

$$[x, p] = i\hbar \ . \tag{2.3}$$

By inspection of (1.9), it is easy to see that the potential (2.2) can be generated (to within a constant term) from the superpotential

$$W(x) = \sqrt{\tfrac{1}{2}M}\ \omega x \ . \tag{2.4}$$

Furthermore, the potentials generated from the superpotential (2.4) satisfy the shape invariance condition (1.23) without necessitating the introduction of a parameter (see (2.6)). We could therefore set $\alpha_s = 1$ in this problem, however it is customary to let

$$\alpha_s = \omega \ . \tag{2.5}$$

Substituting (2.4) in (1.36) we obtain

$$V_{\pm}(x,\alpha_s) = \tfrac{1}{2}M\omega^2x^2 \pm \tfrac{1}{2}\hbar\omega \ , \tag{2.6}$$

which satisfies the shape invariance condition (1.23) with the remainder

$$R(\alpha_s) = \hbar\omega \ . \tag{2.7}$$

Substituting (2.7) in (1.41) we obtain the energy eigenvalues of $H^{(0)}$ corresponding to the potential $V_-(x,\alpha_0)$:

$$E_N^{(0)} = \sum_{k=1}^{N} \hbar\omega = N\hbar\omega \qquad (N = 1,2, \ldots) \ , \tag{2.8}$$

where $E_0^{(0)} = 0$ (see (1.39)). Comparing $V_-(x,\alpha_0)$ with (2.2), we see that these differ by the constant $-\frac{1}{2}\hbar\omega$, so that

$$H_- = H - \tfrac{1}{2}\hbar\omega \ . \tag{2.9}$$

Hence the energy eigenvalues of the Hamiltonian (2.1) are given by

$$E_N = E_N^{(0)} + \tfrac{1}{2}\hbar\omega = (N + \tfrac{1}{2})\hbar\omega \qquad (N = 0,1,2, \ldots) \ , \tag{2.10}$$

which is the familiar spectrum of the one-dimensional harmonic oscillator. Fig. 2.1 shows the oscillator spectrum compared with the spectra of $H^{(s)}$.

In terms of the superpotential (2.4), the operators $A_{\pm}(x,\alpha_s)$ are

$$A_{\pm}(x,\alpha_s) = \pm \frac{i}{\sqrt{2m}}\, p + \sqrt{\frac{m}{2}}\, \omega x \ . \tag{2.11}$$

It is apparent from (2.11) that $A_{\pm}(x,\alpha_s)$ are identical to the traditional raising and lowering operators $a^{\dagger}$ and a for the one-dimensional harmonic oscillator [2]. The excited states $\psi_N(x)$ are given by (1.46), where the ground state $\psi_0(x)$ can be found by solving equation (1.38). This calculation of $\psi_N(x)$ is identical to the standard operator method for this problem. The procedure will not be repeated here as it is well documented in the literature [2].

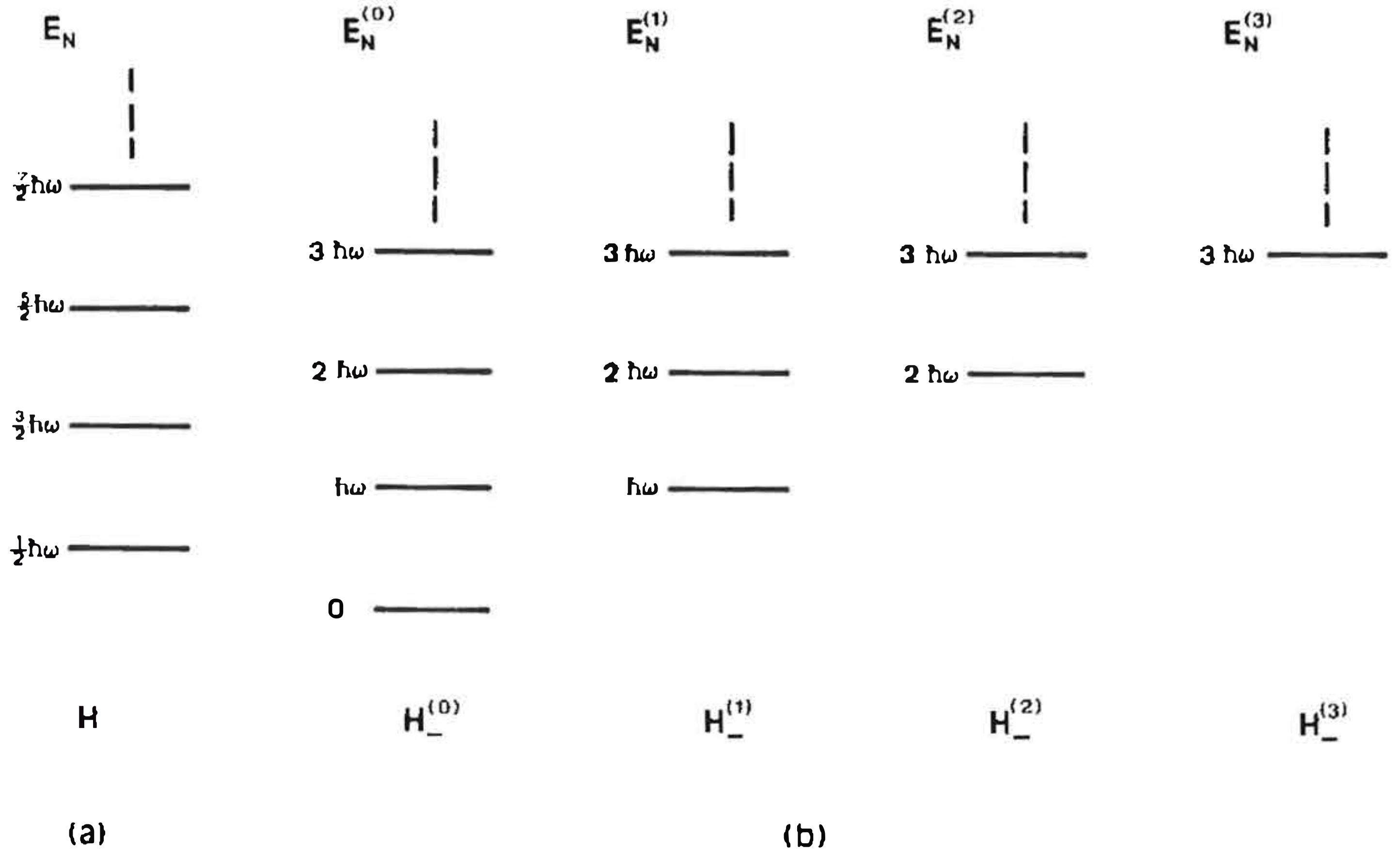

Fig. 2.1 (a) The spectrum of the one-dimensional oscillator Hamiltonian (2.1). The energy eigenvalues E_N are given by (2.10). (b) The spectra of the Hamiltonians $H_-^{(s)}$ where $V_-(x,\alpha_s)$ is given by (2.6). The eigenvalues $E_N^{(0)}$ are given by (2.8).

2.2 ORBITAL ANGULAR MOMENTUM

The orbital angular momentum operator $\underset{\sim}{L}$ is obtained from the definition of classical angular momentum. In classical mechanics, the angular momentum of a particle with position vector $\underset{\sim}{r} = (x,y,z)$ and momentum $\underset{\sim}{p} = (p_x,p_y,p_z)$ is defined by

$$\underset{\sim}{L} = \underset{\sim}{r} \times \underset{\sim}{p} , \tag{2.12}$$

or in tensor notation

$$L_i = \varepsilon_{ijk} r_j p_k \qquad (i = 1,2,3 \text{ or } x,y,z) . \tag{2.13}$$

Here ε_{ijk} is the totally antisymmetric tensor in three dimensions.

The quantum mechanical operator $\underset{\sim}{L}$ is obtained by replacing the position and momentum vectors by the position and momentum operators. The relation (2.13) shows that $\underset{\sim}{L}$ is linear, and using the commutation relations defining the operators $\underset{\sim}{r}$ and $\underset{\sim}{p}$, it can be shown that $\underset{\sim}{L}$ is Hermitian and

$$[L_i, L_j] = i\hbar\varepsilon_{ijk} L_k . \tag{2.14}$$

An operator that commutes with L_i is

$$\underset{\sim}{L}^2 = L_x^2 + L_y^2 + L_z^2 . \tag{2.15}$$

This can be seen by expanding the commutator $[\underset{\sim}{L}^2, L_i]$ using (2.14) and noting that ε_{ijk} is an antisymmetric tensor. Thus $\underset{\sim}{L}^2$ and one of the L_i (traditionally taken to be L_z) can have simultaneous eigenkets:

$$L_z |km\rangle = \hbar m |km\rangle , \tag{2.16}$$

$$\underset{\sim}{L}^2 |km\rangle = \hbar^2 k^2 |km\rangle . \tag{2.17}$$

The algebraic solution for the eigenvalues m and k in (2.16) and (2.17) is well documented in most textbooks on quantum mechanics [4]. In the coordinate representation the kets in (2.16) and (2.17) turn out to be the spherical harmonics.

In what follows we treat this problem using shape invariance with the operators L_z and $\underset{\sim}{L}^2$ expressed in terms of spherical polar coordinates. For this purpose we need to define conjugate momentum operators to the angles θ and ϕ shown in Fig. 2.2. In terms of Cartesian coordinates, the angles θ and ϕ are given by

$$\theta = \cos^{-1} \hat{z} \quad \text{and} \quad \phi = \tan^{-1} y/x , \tag{2.18}$$

and the conjugate momentum operators to (2.18) are [5]

$$p_\phi = L_z = -i\hbar \frac{\partial}{\partial\phi} \tag{2.19}$$

and

$$p_\theta = -i\hbar\left[\frac{\partial}{\partial\theta} + \tfrac{1}{2}\cot\theta\right] . \tag{2.20}$$

Furthermore, $\underset{\sim}{L}^2$ is given in terms of (2.18)-(2.20)by [5]

$$\underset{\sim}{L}^2 = p_\theta^2 + (\sin^{-2}\theta)(p_\phi^2 - \tfrac{1}{4}\hbar^2) - \tfrac{1}{4}\hbar^2 . \tag{2.21}$$

2.2.1 Shape invariance

Substituting (2.21) in (2.17), with p_ϕ replaced by its eigenvalue $\hbar m$, and rearranging we obtain

$$[p_\theta^2 + V(\theta)]\,|km\rangle = \varepsilon\,|km\rangle , \tag{2.22}$$

where

$$V(\theta) = \frac{(m^2 - \frac{1}{4})\hbar^2}{\sin^2\theta} \tag{2.23}$$

and

$$\varepsilon = \hbar^2(k^2 + \tfrac{1}{4}) . \tag{2.24}$$

Potentials of the shape (2.23) can be generated from the super-potential

$$W(\theta,\alpha_s) = -\hbar\alpha_s\cot\theta , \tag{2.25}$$

where

$$\alpha_s = s + m + \tfrac{1}{2} . \tag{2.26}$$

Substituting (2.25) in (1.36), where the differentiation is now with respect to θ, yields the partner potentials

$$V_\pm(\theta,\alpha_s) = \frac{\hbar^2\alpha_s(\alpha_s \pm 1)}{\sin^2\theta} - \hbar^2\alpha_s^2 . \tag{2.27}$$

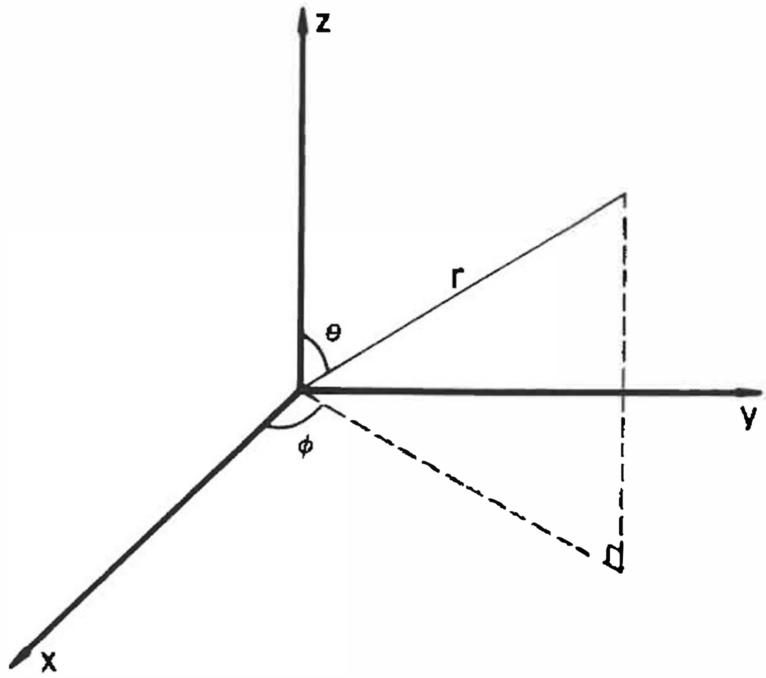

Fig. 2.2 The system of spherical polar coordinates

By inspection it can be seen that $V_-(\theta,\alpha_0)$ is the potential (2.23) minus the constant term

$$\hbar^2\alpha_0^2 = \hbar^2(m + \tfrac{1}{2})^2 \,. \tag{2.28}$$

The partner potentials (2.27) satisfy the shape invariance condition (1.23) with remainder

$$R(\alpha_s) = \hbar^2(\alpha_s^2 - \alpha_{s-1}^2) \,. \tag{2.29}$$

2.2.2 **Eigenvalues**

Substituting (2.29) in (1.41) gives the eigenvalues $E_N^{(0)}$ of the operator on the left-hand side of (2.22) with the potential (2.23) replaced by $V_-(\theta,\alpha_0)$. These are

$$E_N^{(0)} = \sum_{k=1}^{N} \hbar^2(\alpha_s^2 - \alpha_{s-1}^2) = \hbar^2(\alpha_N^2 - \alpha_0^2) \qquad (N = 1,2, \ldots) \,, \tag{2.30}$$

with $E_0^{(0)} = 0$. The eigenvalues ε_N corresponding to the potential (2.23) are therefore equal to $E_N^{(0)}$ plus the constant term (2.28). This yields

$$\varepsilon_N = \hbar^2\alpha_N^2 \qquad (N = 0,1,2, \ldots) \,. \tag{2.31}$$

Substituting (2.31) in (2.24) and using (2.26) with s replaced by N and solving for k^2 we find

$$k^2 = (N + m)(N + m + 1) \,. \tag{2.32}$$

It is customary to write the eigenvalues k^2 in terms of the angular momentum quantum number ℓ = N+m. The kets in (2.16) and (2.17) must be normalizable, and $\underset{\sim}{L}$ must be Hermitian in the Hilbert space of these kets. As is well-known this places the following restrictions on the quantum numbers ℓ and m:

$$\ell = 0,1,2, \ldots \tag{2.33}$$

and

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell \,. \tag{2.34}$$

2.2.3 Coordinate-space eigenfunctions

In the coordinate representation (2.16) and (2.17) are

$$L_z Y_{\ell m}(\theta,\phi) = \hbar m Y_{\ell m}(\theta,\phi) \tag{2.35}$$

and

$$\underset{\sim}{L}^2 Y_{\ell m}(\theta,\phi) = \hbar^2 \ell(\ell+1) Y_{\ell m}(\theta,\phi) \ , \tag{2.36}$$

where we used (2.32) with ℓ = N+m. The orthonormality condition for the $Y_{\ell m}(\theta,\phi)$ is

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{\ell' m'}(\theta,\phi) Y_{\ell m}(\theta,\phi) \sin\theta \, d\theta \, d\phi = \delta_{\ell'\ell}\delta_{m'm} \ . \tag{2.37}$$

The functions $Y_{\ell m}(\theta,\phi)$ that are normalized in this way are known as spherical harmonics.

We separate variables and write

$$Y_{\ell m}(\theta,\phi) = \Theta_{\ell m}(\theta)\Phi_m(\phi) \ . \tag{2.38}$$

From (2.16) with L_z given by (2.19), it is straightforward to find the normalized function $\Phi_m(\phi)$, namely

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \ . \tag{2.39}$$

For N=0, ℓ=m, so that the starting point for determining the function $\Theta_{\ell m}(\theta,\alpha_0)$ is $\Theta_{\ell\ell}(\theta,\alpha_N)$, which must be determined from the annihilation condition (1.38) with s=N. In terms of p_θ and $W(\theta,\alpha_N)$ this yields the first-order differential equation

$$A_+(\theta,\alpha_N)\Theta_{\ell\ell}(\theta,\alpha_N) = \hbar\left[\frac{d}{d\theta} - (\alpha_N - \tfrac{1}{2})\cot\theta\right]\Theta_{\ell\ell}(\theta,\alpha_N) = 0 \ . \tag{2.40}$$

The normalized solution to (2.40) is

$$\Theta_{\ell\ell}(\theta,\alpha_N) = C_0(\alpha_N)\sin^{\alpha_N - \frac{1}{2}}\theta \ , \tag{2.41}$$

where α_N = N + m + $\frac{1}{2}$ (see (2.26)) and

$$C_0(\alpha_N) = \frac{(-1)^{N+m}}{2^{N+m}(N+m)!}\left[\frac{(2N+2m+1)!}{2}\right]^{\frac{1}{2}} \ . \tag{2.42}$$

The phase factor $(-1)^{N+m}$ in (2.42) has been introduced for later convenience.

For this problem, it is easier to determine $\Theta_{\ell m}(\theta,\alpha_0)$ directly from $\Theta_{\ell\ell}(\theta,\alpha_N)$. Operating with $A_-(\theta,\alpha_{N-1})$ on $\Theta_{\ell\ell}(\theta,\alpha_N)$ and setting the phase factor in $A_-(\theta,\alpha_{N-1})$ equal to unity gives

$$A_-(\theta,\alpha_{N-1})\Theta_{\ell\ell}(\theta,\alpha_N) = \hbar C_0(\alpha_N)\,\frac{1}{\sin^{N+m-1}\theta}\,\frac{d}{d(\cos\theta)}\,\sin^{2N+2m}\theta\;, \tag{2.43}$$

where we used the identities

$$2\,\frac{d}{d\theta}\,\sin^{N+m}\theta = \frac{1}{\sin^{N+m}\theta}\,\frac{d}{d\theta}\,\sin^{2N+2m}\theta \tag{2.44}$$

and

$$\frac{d}{d\theta} = -\sin\theta\,\frac{d}{d(\cos\theta)}\;. \tag{2.45}$$

Similarly, operating on (2.43) with $A_-(\theta,\alpha_{N-2})$ yields

$$A_-(\theta,\alpha_{N-2})A_-(\theta,\alpha_{N-1})\Theta_{\ell\ell}(\theta,\alpha_N) = \hbar^2 C_0(\alpha_N)\,\frac{1}{\sin^{N+m-2}\theta}\,\frac{d^2}{d(\cos\theta)^2}\sin^{2N+2m}\theta\;. \tag{2.46}$$

Repeating the above steps N times we obtain

$$A_-(\theta,\alpha_0)\ldots A_-(\theta,\alpha_{N-1})\Theta_{\ell\ell}(\theta,\alpha_N) = \hbar^N C_0(\alpha_N)\,\frac{1}{\sin^m\theta}\,\frac{d^N}{d(\cos\theta)^N}\sin^{2N+2m}\theta\;. \tag{2.47}$$

According to (1.46),

$$\Theta_{\ell m}(\theta,\alpha_0) = \gamma_N(\alpha_0)A_-(\theta,\alpha_0)\ldots A_-(\theta,\alpha_{N-1})\Theta_{\ell\ell}(\theta,\alpha_N)\;, \tag{2.48}$$

where

$$\gamma_N(\alpha_0) = \left[\prod_{k=1}^{N}\hbar^2(\alpha_N^2-\alpha_{k-1}^2)\right]^{-\frac{1}{2}} = \hbar^{-N}\left[\frac{(N+2m)!}{N!\,(2N+2m)!}\right]^{\frac{1}{2}} = \hbar^{-N}\left[\frac{(\ell+m)!}{(\ell-m)!\,(2\ell)!}\right]^{\frac{1}{2}}\;. \tag{2.49}$$

In the last three steps we have used (1.47), (2.30), (2.26) and $\ell = N+m$. Note that in (2.49), $m = \alpha_0 - \frac{1}{2}$ and $\ell = N + \alpha_0 - \frac{1}{2}$. From (2.47), (2.48), (2.49), (2.41) and (2.42) we have

$$\Theta_{\ell m}(\theta,\alpha_0) = \frac{(-1)^\ell}{2^\ell \ell!}\left[\frac{(2\ell+1)(\ell+m)!}{2(\ell-m)!}\right]^{\frac{1}{2}}\frac{1}{\sin^m\theta}\,\frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}\,(\sin\theta)^{2\ell}\;. \tag{2.50}$$

Finally, combining (2.39) and (2.50) we obtain the spherical harmonics

$$Y_{\ell m}(\theta,\phi) = \frac{(-1)^\ell}{2^\ell \ell!}\left[\frac{(2\ell+1)(\ell+m)!}{4\pi(\ell-m)!}\right]^{\frac{1}{2}}\exp(im\phi)(\sin\theta)^{-m}\,\frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}\,(\sin\theta)^{2\ell}\;. \tag{2.51}$$

Equation (2.51) is often written in terms of the associated Legendre function $P_\ell^m(\cos\theta)$. This yields

$$Y_{\ell m}(\theta,\phi) = (-1)^m\left[\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\right]^{\frac{1}{2}}\exp(im\phi)P_\ell^m(\cos\theta) \ , \tag{2.52}$$

where

$$P_\ell^m(\cos\theta) = \frac{(-1)^{\ell+m}(\ell+m)!}{2^\ell\ell!(\ell-m)!}(\sin\theta)^{-m}\frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}(\sin\theta)^{2\ell} \ . \tag{2.53}$$

2.3 ELECTRON IN A UNIFORM MAGNETIC FIELD IN THE BASIS H, p_z, L_z

2.3.1 Introduction

Consider an electron with charge e (<0) and mass M in a uniform magnetic field

$$\underset{\sim}{B} = (0,0,B) \ . \tag{2.54}$$

Here $\underset{\sim}{B} = \underset{\sim}{\nabla}\times\underset{\sim}{A}$, where $\underset{\sim}{A}$ is the vector potential. We choose the gauge $\underset{\sim}{A} = \frac{1}{2}B(-y,x,0)$. We also choose the complete set of commuting observables H, p_z and L_z [2], where p_z and L_z are the z components of the momentum and orbital angular momentum of the particle. We work in cylindrical coordinates ρ, ϕ, z. In terms of these

$$x = \rho\cos\phi \ , \qquad y = \rho\sin\phi \tag{2.55}$$

where $\rho = (x^2+y^2)^{\frac{1}{2}}$. Then

$$p_z = -i\hbar\frac{\partial}{\partial z} \quad \text{and} \quad L_z = -i\hbar\frac{\partial}{\partial\phi} \ . \tag{2.56}$$

For this particle, Schrödinger's equation can be written as [6]

$$H\psi = -\frac{\hbar^2}{2M}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{\partial^2}{\partial z^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+i\frac{M\omega}{\hbar}\frac{\partial}{\partial\phi}-\frac{M^2\omega^2}{4\hbar^2}\rho^2\right]\psi = E\psi \tag{2.57}$$

where

$$\omega = \frac{|e|B}{M} \tag{2.58}$$

is the cyclotron frequency. We can eliminate the linear term in $\frac{\partial}{\partial\rho}$ in

(2.57) by defining the canonical conjugate of ρ, namely*

$$p_\rho = -i\hbar\left[\frac{\partial}{\partial\rho} + \frac{1}{2\rho}\right] . \tag{2.59}$$

In terms of (2.56) and (2.59)

$$H = \frac{1}{2M}\, p_\rho^2 - \frac{\hbar^2}{8M\rho^2} + \frac{p_z^2}{2M} + \frac{\hbar^2}{2M\rho^2} L_z^2 + \tfrac{1}{2}\hbar\omega L_z + \tfrac{1}{8}M\omega^2\rho^2 . \tag{2.60}$$

In (2.60) we replace p_z and L_z with their eigenvalues P_z and m. Making these substitutions and rearranging we obtain the eigenvalue equation

$$\left[\frac{1}{2M}\, p_\rho^2 + V(\rho)\right]\psi = \varepsilon\psi , \tag{2.61}$$

where

$$V(\rho) = \frac{\hbar^2}{2M}(m^2 - \tfrac{1}{4})\frac{1}{\rho^2} + \tfrac{1}{8}M\omega^2\rho^2 \tag{2.62}$$

and

$$\varepsilon = E - \frac{P_z^2}{2M} - \tfrac{1}{2}\hbar\omega m . \tag{2.63}$$

Note that the problem (2.61) is similar to that of an isotropic three-dimensional harmonic oscillator in an angular momentum basis (see Chapter 4).

2.3.2 Shape invariance

Potentials of the shape (2.62) can be generated from the superpotential

$$W(\rho, \alpha_s) = \frac{A}{\rho} + B\rho . \tag{2.64}$$

Here A and B are constants to be determined so that the shape invariance

---

*Obviously, $[p_\rho, \rho] = -i\hbar$. Also, the term $1/2\rho$ in (2.59) makes p_ρ Hermitian: it ensures that

$$\int_0^\infty \psi^* \left[-i\hbar\frac{\partial}{\partial\rho} - \frac{i\hbar}{2\rho}\right]\chi\rho\, d\rho = \int_0^\infty \left[\left[-i\hbar\frac{\partial}{\partial\rho} - \frac{i\hbar}{2\rho}\right]\psi\right]^* \chi\rho\, d\rho .$$

A similar term is required in spherical polar coordinates for the radial momentum operator p_r (see Chapter 4) and in parabolic coordinates [7].

condition (1.23) is satisfied, and so that $V_-(\rho,\alpha_0)$ is the potential (2.62) to within a constant. Substituting (2.64) in (1.36) and comparing $V_-(\rho,\alpha_0)$ with (2.62) we find that

$$A = -\frac{1}{\sqrt{2M}}(|m| + \tfrac{1}{2}) \tag{2.65}$$

and

$$B = \tfrac{1}{2}\sqrt{\tfrac{1}{2}M}\,\omega\ . \tag{2.66}$$

In (2.65) we have written $|m|$ in place of m: this is in anticipation of (2.84), where it emerges that normalizability of the eigenfunctions R_N requires this replacement. To obtain a suitable expression for $W(\rho,\alpha_s)$ we must introduce a parameter in either or both the constants (2.65) and (2.66). Using (2.64)-(2.66) in (1.36) we find that

$$W(\rho,\alpha_s) = -\frac{1}{\sqrt{2M}}\frac{\alpha_s}{\rho} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}M}\,\omega\rho \tag{2.67}$$

where

$$\alpha_s = |m| + s + \tfrac{1}{2} \qquad (s = 0, 1, 2, \ldots)\ . \tag{2.68}$$

From (1.36), the partner potentials corresponding to the superpotential (2.67) are

$$V_\pm(\rho,\alpha_s) = \frac{\hbar^2}{2M}\,\alpha_s(\alpha_s \pm 1)\frac{1}{\rho^2} + \tfrac{1}{8}M\omega^2\rho^2 - \tfrac{1}{2}\hbar\omega(\alpha_s \mp \tfrac{1}{2})\ . \tag{2.69}$$

These satisfy the shape invariance condition (1.23) with remainder

$$R(\alpha_s) = \hbar\omega\ . \tag{2.70}$$

Finally, setting s=0 in (2.69) we find that $V_-(\rho,\alpha_0)$ is the potential (2.62) minus the constant term

$$\tfrac{1}{2}\hbar\omega(|m| + 1)\ . \tag{2.71}$$

2.3.3 Energy Eigenvalues

The eigenvalues of the Hamiltonian with the potential $V_-(\rho,\alpha_0)$ are given by (1.41). Using (2.70) in (1.41) we find

$$E_N^{(0)} = N\hbar\omega \qquad (N = 0, 1, 2, \ldots)\ , \tag{2.72}$$

since $E_0^{(0)} = 0$. The eigenvalues ε in (2.61) differ from $E_N^{(0)}$ by the constant term (2.71). Hence using (2.63), (2.71) and (2.72) we obtain for the eigenvalues E of (2.57)

$$E = \frac{1}{2M} P_z^2 + (N + \tfrac{1}{2}|m| + \tfrac{1}{2}m + \tfrac{1}{2})\hbar\omega \qquad (N = 0,1,2, \ldots) . \tag{2.73}$$

Here N is the radial quantum number, m is the angular momentum quantum number of L_z, and P_z is the eigenvalue of the momentum operator p_z.

2.3.4 Coordinate-space eigenfunctions

Since p_z and L_z are constants of the motion we can write the eigenfunctions in the form

$$\psi_N = \frac{1}{\sqrt{2\pi}} R_N(\rho)\exp\left[\frac{i}{\hbar} P_z z + im\phi\right] . \tag{2.74}$$

We use the method of shape invariance to determine the radial function $R_N(\rho)$. We start with the condition (1.38) with s=N expressed in the coordinate representation. In this representation

$$A_\pm(\rho,\alpha_N) = \pm \frac{i}{\sqrt{2M}} p_\rho + W(\rho,\alpha_N)$$

$$= \pm \frac{\hbar}{\sqrt{2M}}\left[\frac{d}{d\rho} + \frac{1}{2\rho}\right] + W(\rho,\alpha_N) . \tag{2.75}$$

Hence, according to (1.38), the function $R_0(\rho,\alpha_N)$ is the solution to the first-order differential equation

$$\left[\frac{\hbar}{\sqrt{2M}}\left[\frac{\partial}{\partial\rho} + \frac{1}{2\rho}\right] + W(\rho,\alpha_N)\right]R_0(\rho,\alpha_N) = 0 . \tag{2.76}$$

It is easily verified that the normalized solution of (2.76) with $W(\rho,\alpha_N)$ and α_N given by (2.67) and (2.68) is

$$R_0(\rho,\alpha_N) = C_0(\alpha_N)\rho^{|m|+N}\exp(-\rho^2/4a^2) , \tag{2.77}$$

where

$$C_0(\alpha_N) = \frac{1}{a^{|m|+N+1}}\left[2^{|m|+N}(|m|+ N)!\right]^{-\frac{1}{2}} \tag{2.78}$$

and

$$a = \sqrt{\frac{\hbar}{M\omega}} . \tag{2.79}$$

Next we determine $R_1(\rho,\alpha_0)$ by operating with $A_-(\rho,\alpha_0)$ on $R_0(\rho,\alpha_1)$. Premultiplying by $\gamma_1(\alpha_0)$ and setting the phase factor equal to unity we obtain (see (1.46) and (1.47))

$$R_1(\rho,\alpha_0) = C_1(\alpha_0)\rho^{|m|}P_1(\rho^2/2a^2)\exp(-\rho^2/4a^2) , \tag{2.80}$$

where

$$C_1(\alpha_0) = \frac{-1}{a^{|m|+1}|m|!}\left[\frac{(|m|+1)!}{2^{|m|}}\right]^{\frac{1}{2}} \tag{2.81}$$

and the polynomial $P_1(\rho^2/2a^2)$ is given in Table 2.1. Similarly, for $R_2(\rho,\alpha_0)$ we obtain from (1.46)

$$R_2(\rho,\alpha_0) = C_2(\alpha_0)\rho^{|m|}P_2(\rho^2/2a^2)\exp(-\rho^2/4a^2) \tag{2.82}$$

where

$$C_2(\alpha_0) = \frac{1}{a^{|m|+1}|m|!}\left[\frac{(|m|+2)!}{2^{|m|}2!}\right]^{\frac{1}{2}} , \tag{2.83}$$

and $P_2(\rho^2/2a^2)$ is given in Table 2.1.

Equations (2.80) and (2.82) are particular cases of

$$R_N(\rho,\alpha_0) = C_N(\alpha_0)\rho^{|m|}P_N(\rho^2/2a^2)\exp(-\rho^2/4a^2) \tag{2.84}$$

where

$$C_N(\alpha_0) = \frac{(-1)^N}{a^{|m|+1}|m|!}\left[\frac{(|m|+N)!}{2^{|m|}N!}\right]^{\frac{1}{2}} \tag{2.85}$$

and

$$P_N(\rho^2/2a^2) = {}_1F_1(-N;|m|+1;\rho^2/2a^2) . \tag{2.86}$$

Here ${}_1F_1$ denotes the confluent hypergeometric function

$${}_1F_1(-N;B;u) = \sum_{j=0}^{N}\frac{(-1)^j}{j!}\frac{\Gamma(N+1)}{\Gamma(N+1-j)}\frac{\Gamma(B)}{\Gamma(B+j)}u^j . \tag{2.87}$$

To prove that (2.84) holds for all N we use (1.45) with s=0 (see induction proof: Ch 1). In order to obtain $R_N(\rho,\alpha_1)$ we note that $\alpha_0 = |m| + \frac{1}{2}$ and $\alpha_1 = |m| + \frac{3}{2}$, so that $R_N(\rho,\alpha_1)$ can be found simply by replacing $|m|$ by $|m|+1$ in (2.84) and (2.85). Using (2.75), (2.84) and (2.85) we find that (1.45) reduces to

$$\left[1 - \frac{u}{B-1} + \frac{u}{B-1}\frac{d}{du}\right]{}_1F_1(-N;B;u) = {}_1F_1(-[N+1];B-1;u) \ , \qquad (2.88)$$

where $B = |m| + 2$ and $u = \rho^2/2a^2$. The recurrence relation (2.88) is proved in Appendix B.

TABLE 2.1 The constants $C_N(\alpha_0)$ and the polynomials $P_N(u)$ in Equations (2.80) and (2.82) for $N = 0,1,2$. The polynomial $P_0(u)$ is obtained from (2.77) and the constant $C_0(\alpha_0)$ from (2.78).

N	$P_N(u)$	$C_N(\alpha_0)$
0	1	$\frac{1}{a^{\lvert m\rvert+1}\lvert m\rvert!}\left[\frac{\lvert m\rvert!}{2^{\lvert m\rvert}}\right]^{\frac{1}{2}}$
1	$1 - \frac{u}{\lvert m\rvert + 1}$	$\frac{-1}{a^{\lvert m\rvert+1}\lvert m\rvert!}\left[\frac{(\lvert m\rvert+1)!}{2^{\lvert m\rvert}}\right]^{\frac{1}{2}}$
2	$1 - \frac{2u}{\lvert m\rvert + 1} + \frac{u^2}{(\lvert m\rvert + 1)(\lvert m\rvert + 2)}$	$\frac{1}{a^{\lvert m\rvert+1}\lvert m\rvert!}\left[\frac{(\lvert m\rvert+2)!}{2^{\lvert m\rvert}2!}\right]^{\frac{1}{2}}$

CHAPTER 3

ONE-DIMENSIONAL SYSTEMS

In this chapter we consider the application of shape invariance to certain one-dimensional systems. The following potentials are discussed: the Morse potential (Section 3.1), the Rosen-Morse potential (unsymmetric and symmetric cases - Section 3.2) and the first and second Pöschl-Teller potentials (Sections 3.3 and 3.4). The results obtained for the Morse and symmetric Rosen-Morse potentials are extended to the s states of the corresponding three-dimensional potentials in Chapter 4.

The above problems have previously been treated by shape invariance to obtain the energy eigenvalues and the (unnormalized) energy eigenfunctions [1]. The analysis given in this chapter has two purposes: (i) to perform the normalization of the bound-state eigenfunctions using shape invariance, and (ii) to compare the various types of shift operator, namely those given by shape invariance, the Infeld-Hull type operators for changing parameters, and the energy shift operators.

3.1 ONE-DIMENSIONAL MORSE POTENTIAL [2]

The three-dimensional Morse potential was proposed by Morse in 1929 to model the motion of nuclei in diatomic molecules [2]. Morse considered the three-dimensional problem with the angular momentum quantum number ℓ=0 (see Section 4.6). Here we discuss the one-dimensional potential

$$V(x) = D[\exp(-2x/a) - 2\exp(-x/a)] , \tag{3.1}$$

where D and a are positive constants. The potential (3.1) has a minimum value -D at x=0. Also $V(x) \to 0$ as $x \to \infty$ and $V(x) \to \infty$ as $x \to -\infty$. This is shown in Fig. 3.1.

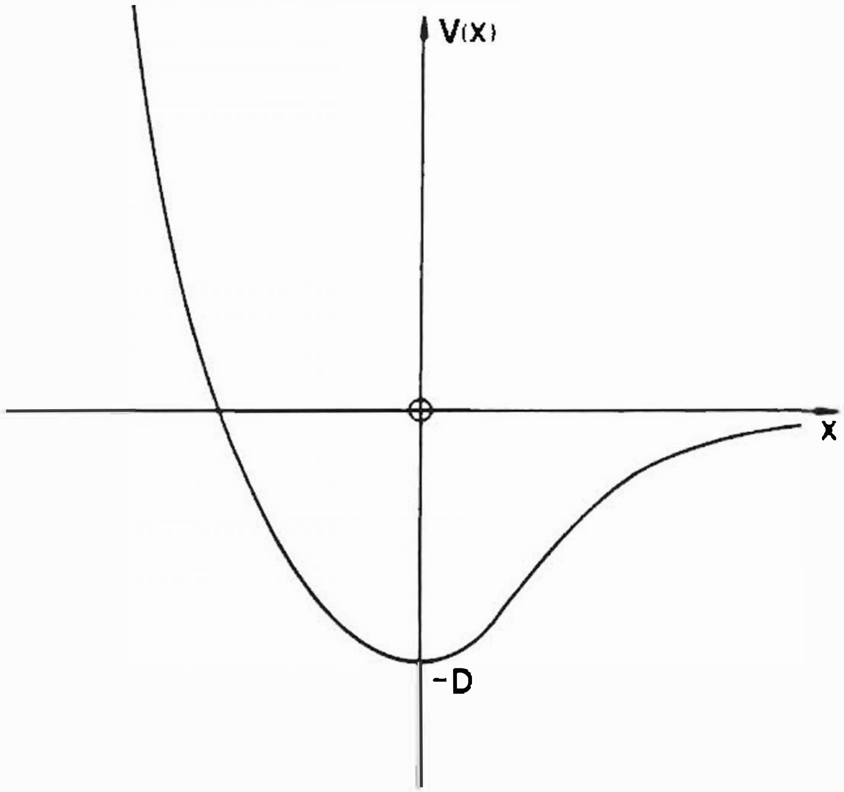

Fig. 3.1 The classical one-dimensional Morse potential (3.1).

Following Ref. 3 we define the dimensionless parameter

$$k = \left[\frac{2Ma^2D}{\hbar^2}\right]^{\frac{1}{2}} - \tfrac{1}{2} , \tag{3.2}$$

and the operators

$$X = \frac{x}{a} - \ell n(2k + 1) , \tag{3.3}$$

$$p_X = \frac{a}{\hbar} p_x . \tag{3.4}$$

In terms of these, the energy eigenvalue equation with the potential (3.1) becomes

$$\left[\frac{1}{2M} p_X^2 + V(X)\right]\psi_N(X) = \frac{a^2}{\hbar^2} E\psi_N(X) , \tag{3.5}$$

where

$$V(X) = \frac{1}{8M}(e^{-2X} - 2(2k + 1)e^{-X}) . \tag{3.6}$$

3.1.1 Shape invariance

By inspection of (1.36) and (3.6) we see that the superpotential associated with the Morse potential must have the form

$$W(X,\alpha_0) = A - Be^{-X} , \tag{3.7}$$

where A and B are constants and α_0 is to be determined. Substituting (3.7) in (1.36) we obtain the supersymmetric partner potentials

$$V_\pm(X,\alpha_0) = B^2e^{-2X} - B\left[2A \mp \frac{1}{\sqrt{2M}}\right]e^{-X} + A^2 . \tag{3.8}$$

Comparing $V_-(X,\alpha_0)$ with the potential (3.6) we see that

$$B = \frac{1}{2\sqrt{2M}} \tag{3.9}$$

and

$$A = \frac{k}{\sqrt{2M}} . \tag{3.10}$$

Next we must make an appropriate choice for the parameter α_s. Consider the simple expression

$$\alpha_s = k - s \qquad (s = 0,1,2, \ldots) . \tag{3.11}$$

From (3.8)-(3.11) we have

$$V_\pm(X,\alpha_s) = \frac{1}{8M}e^{-2X} - \frac{1}{2M}(\alpha_s \mp \tfrac{1}{2})e^{-X} + \frac{\alpha_s^2}{2M} . \tag{3.12}$$

Thus $V_-(X,\alpha_0)$ is the potential (3.6) plus the constant

$$\frac{\alpha_0^2}{2M} = \frac{k^2}{2M} . \tag{3.13}$$

Also, the partner potentials (3.12) satisfy the shape invariance condition (1.23) with remainder

$$R(\alpha_s) = \frac{1}{2M}(\alpha_{s-1}^2 - \alpha_s^2) . \tag{3.14}$$

From (3.7) and (3.9)-(3.11) we see that in terms of α_s the superpotential is

$$W(X,\alpha_s) = \frac{1}{\sqrt{2M}}(\alpha_s - \tfrac{1}{2}e^{-X}) . \tag{3.15}$$

3.1.2 **Energy eigenvalues**

Using (3.14) in (1.41) we obtain the eigenvalues corresponding to the potential $V_-(X,\alpha_0)$

$$E_N^{(0)} = \frac{1}{2M}(\alpha_0^2 - \alpha_N^2) \qquad (N = 0,1,2, \ldots) , \tag{3.16}$$

where we have included N=0 because $E_0^{(0)} = 0$ (see (1.39)). To obtain the eigenvalues corresponding to the potential (3.6) we must subtract the constant (3.13) from (3.16). Hence

$$E = -\frac{\hbar^2}{2Ma^2}(k - N^2) , \tag{3.17}$$

where we have used (3.11) with s=N. Finally, substituting (3.2) in (3.17) and multiplying out, we obtain the energy eigenvalues of the bound states of the one-dimensional Morse potential in the form

$$E = -D\left[1 - 2\left[\frac{\hbar^2}{2Ma^2D}\right]^{\frac{1}{2}}(N + \tfrac{1}{2}) + \frac{h^2}{2Ma^2D}(N + \tfrac{1}{2})^2\right] . \tag{3.18}$$

Here

$$N = 0,1,2, \ldots N_{max} ,$$

where N_{max} is the largest integer less than k. This restriction on N must be placed in order to obtain normalizable eigenkets (see (3.23) and [4]).

3.1.3 Coordinate-space eigenfunctions

The operators (1.35) with the superpotential (3.15) are

$$A_{\pm}(X,a_s) = \frac{1}{\sqrt{2M}}(\pm ip_X + \alpha_s - \tfrac{1}{2}e^{-X}) \tag{3.19}$$

In the coordinate representation, the annihilation condition (1.38) with s=N yields the first-order differential equation

$$\frac{1}{\sqrt{2M}}\left[\frac{d}{dX} + \alpha_N - \tfrac{1}{2}e^{-X}\right]\psi_0(X,\alpha_N) = 0 \ . \tag{3.20}$$

The solution to (3.20) is

$$\psi_0(X,\alpha_N) = C_0(\alpha_N)\exp(-\alpha_N X - \tfrac{1}{2}e^{-X}) \ , \tag{3.21}$$

where $C_0(\alpha_N)$ is a constant to be determined so that

$$\int_{-\infty}^{\infty} |\psi_0(X,\alpha_N)|^2 dX = 1 \ . \tag{3.22}$$

In (3.22) we change the variable to $u = e^{-X}$, and use the integral

$$\int_{0}^{\infty} u^{z-1}e^{-u}du = \Gamma(z) \tag{3.23}$$

where $z = 2k - 2N > 0$. This yields

$$C_0(\alpha_N) = [\Gamma(2k - 2N)]^{-\frac{1}{2}} \ . \tag{3.24}$$

The normalized eigenfunctions $\psi_N(X,\alpha_0)$ can be determined by substituting (3.21) in (1.46). Setting the phase factor in (1.46) equal to unity, we obtain for the first and second excited states

$$\psi_1(X,\alpha_0) = C_1(\alpha_0)\exp[-(k-1)X]\exp(-\tfrac{1}{2}e^{-X})P_1(e^{-X}) \tag{3.25}$$

and

$$\psi_2(X,\alpha_0) = C_2(\alpha_0)\exp[-(k-2)X]\exp(-\tfrac{1}{2}e^{-X})P_2(e^{-X}) \ , \tag{3.26}$$

where the constants $C_N(\alpha_0)$ and the polynomials $P_N(e^{-X})$ are given in Table 3.1. The polynomial $P_0 = 1$ in Table 3.1 is obtained from (3.21), and the constant $C_0(\alpha_0) = [\Gamma(2k)]^{-\frac{1}{2}} = [2k\Gamma(2k+1)/\{\Gamma(2k+1)\}^2]^{\frac{1}{2}}$ from (3.24).

By inspection of $P_N(e^{-X})$ in Table 3.1, we see that

$$P_N(e^{-X}) = {}_1F_1(-N;2k-2N+1;e^{-X}) \tag{3.27}$$

TABLE 3.1 The constants $C_N(\alpha_0)$ and the polynomials $P_N(e^{-X})$ in Equations (3.25) and (3.26) for N = 0,1,2.

N	$C_N(\alpha_0)$	$P_N(e^{-X})$
0	$\left[\frac{2k\Gamma(2k+1)}{\{\Gamma(2k+1)\}^2}\right]^{\frac{1}{2}}$	1
1	$\left[\frac{2(k-1)\Gamma(2k)}{\{\Gamma(2k-1)\}^2}\right]^{\frac{1}{2}}$	$1-\frac{1}{2k-1}e^{-X}$
2	$\left[\frac{2(k-2)\Gamma(2k-1)}{2!\{\Gamma(2k-3)\}^2}\right]^{\frac{1}{2}}$	$1-\frac{2}{2k-3}e^{-X}+\frac{2}{(2k-3)(2k-2)}\frac{e^{-2X}}{2!}$

where ${}_1F_1$ is a confluent hypergeometric function (see (2.87)). Equations (3.25) and (3.26) are particular cases of

$$\psi_N(X,\alpha_0) = C_N(\alpha_0)\exp[-(k-N)X]\exp(-\tfrac{1}{2}e^{-X})\,{}_1F_1(-N;2k-2N+1;e^{-X}) \ , \tag{3.28}$$

where (see Table 3.1)

$$C_N(\alpha_0) = \left[\frac{2(k-N)\Gamma(2k-N+1)}{N!\{\Gamma(2k-2N+1)\}^2}\right]^{\frac{1}{2}} . \tag{3.29}$$

To prove that (3.28) and (3.29) are valid for all non-negative integers N, we must show that the identity

$$A_-(X,\alpha_0)\psi_N(X,\alpha_1) = \sqrt{E_{N+1}}\ \psi_{N+1}(X,\alpha_0) \tag{3.30}$$

holds (see (1.45) with s=0). Using (3.19) and (3.28), with k replaced by k-1 to obtain $\psi_N(X,\alpha_1)$, we find that the left-hand side of (3.30) is

$$\frac{(2k-N-1)C_N(\alpha_1)}{\sqrt{2M}}\exp[-(k-N-1)X]\exp(-\tfrac{1}{2}e^{-X})\phi_{N+1}(X) \ , \tag{3.31}$$

where

$$\phi_{N+1}(X) = \left[1-\frac{e^{-X}}{2k-N-1}-\frac{1}{2k-N-1}\frac{d}{dX}\right]{}_1F_1(-N;2k-2N-1;e^{-X}) \ . \tag{3.32}$$

Setting $u=e^{-X}$ and $B = 2k-2N-1$ in (3.32), and using (B.1), we have

$$\phi_{N+1}(X) = {}_1F_1(-[N+1];2k-2N-1;e^{-X}) \ . \tag{3.33}$$

It is easy to show that the factor independent of X in (3.31) is equal to

$\sqrt{E_{N+1}}\, C_{N+1}(\alpha_0)$. Hence we obtain the right-hand side of (3.30).

In Table 3.2 the operators $A_\pm(X,\alpha_N)$ are compared with the shift operators for a parameter $A_k^\pm$ obtained from the factorization method of Infeld and Hull [5] and the shift operators for energy $B_N^\pm$ [3]. The shift operators $A_k^\pm$ factorize $2MH_k$, where H_k is the Hamiltonian with a Morse potential; the shift operators $B_N^\pm$ factorize an operator related to the Hamiltonian [3]. The shift operations given in Table 3.2 are depicted in Figs. 3.2-3.4. The shift operators obtained from the shape invariance and Infeld-Hull factorization methods transform between eigenkets of different potentials with the same energy, while the energy shift operations transform between eigenkets with different energies belonging to the same potential.

Inspection of Table 3.2 shows that the operators $A_\pm(X,\alpha_s)$ and $A_k^\pm$ are similar in form. This similarity is due to the equivalence of the method of shape invariance and the Infeld-Hull factorization method [6]. Using the method of shape invariance the eigenfunctions are determined from (1.46) (see also Fig. 3.2) and the energy eigenvalues from (1.41). In the Infeld-Hull factorization method, the eigenfunctions are determined by repeated applications of the raising operator A_k^+ to the ground state (see Fig. 3.3), and the energy eigenvalues are determined from the requirement of the non-negativity of the norm [5,7].

3.2 ROSEN-MORSE POTENTIAL [8]

The potential

$$V(x) = -U_0 \operatorname{sech}^2 x/a + U_1 \tanh x/a\ , \tag{3.34}$$

where U_0, U_1 and a are positive constants, was discussed by Rosen and Morse in 1932 to describe the vibrational energies of polyatomic molecules [8]. The potential (3.34) has a minimum if $U_1 < 2U_0$ and tends to $\pm U_1$ as $x \to \pm\infty$ (see Fig. 3.5).

TABLE 3.2 Shift operations for the one-dimensional Morse potential (3.1). The parameter k is given by (3.2). The notation $|\psi_N(X,\alpha_s)\rangle$ is explained in Section 1.3.2; $|Ek\rangle$ and $|Nk\rangle$ are different notations for the same ket. The energy eigenvalues E are given by (3.17).

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1. Shape invariance

$$A_+(X,\alpha_s)|\psi_N(X,\alpha_s) = \frac{1}{\sqrt{2M}}(\alpha_s^2 - \alpha_{N+s}^2)^{\frac{1}{2}}|\psi_{N-1}(X,\alpha_{s+1})\rangle$$

$$A_-(X,\alpha_s)|\psi_N(X,\alpha_{s+1}) = \frac{1}{\sqrt{2M}}(\alpha_s^2 - \alpha_{N+s}^2)^{\frac{1}{2}}|\psi_{N+1}(X,\alpha_s)\rangle$$

where $\alpha_s = k - s$ and $A_\pm(X,\alpha_s) = \frac{1}{\sqrt{2M}}(\pm ip_X - \frac{1}{2}e^{-X} + \alpha_s)$.

2. Shift operators for the parameter k [3,5]

$$A_k^{\pm}|Ek\rangle = [(k + \tfrac{1}{2} \pm \tfrac{1}{2})^2 - n^2]^{\frac{1}{2}}|E,k \pm 1\rangle$$

where $n = k - N$ and $A_k^{\pm} = \pm ip_X + \frac{1}{2}e^{-X} - (k + \frac{1}{2} \pm \frac{1}{2})$

3. Shift operators for energy [3]

$$B_N^{\pm}|Nk\rangle = \left[\frac{(N + \frac{1}{2} \pm \frac{1}{2})(k - N)(2k - N + \frac{1}{2} \mp \frac{1}{2})}{(2k - 2N \mp 1)^2(k - N \mp 1)}\right]^{\frac{1}{2}}|N\pm 1,k\rangle$$

where $B_N^{\pm} = \pm ie^X p_X + (N - k)e^X + (k + \frac{1}{2})/(2k - 2N \mp 1)$

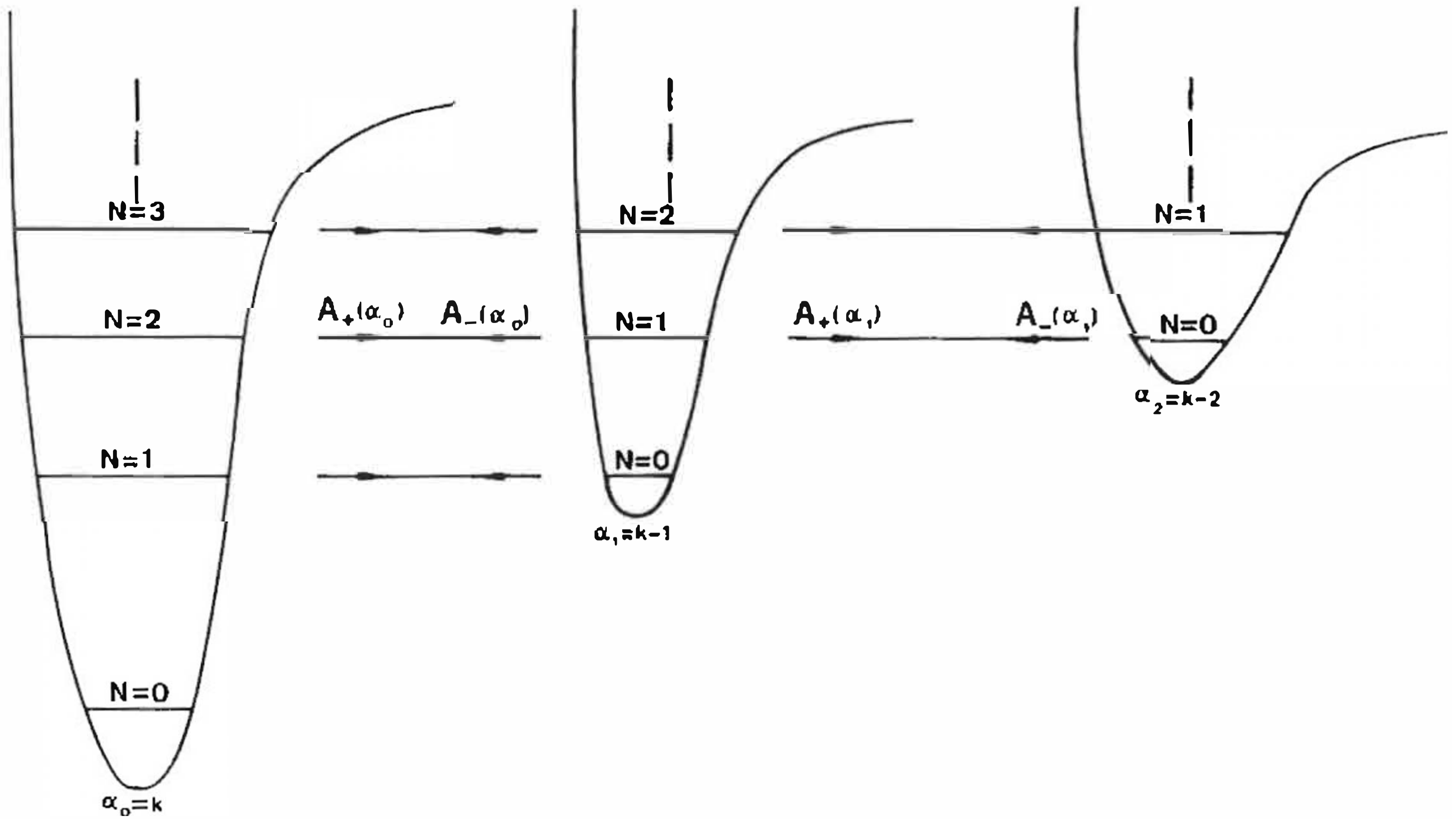

Fig. 3.2 Shift operations obtained by the method of shape invariance for the potentials $V_-(\alpha_s)$ given by (3.12). $A_\pm(\alpha_s)$ are given in Table 3.2, α_s is the parameter (3.11) and N is the vibrational quantum number.

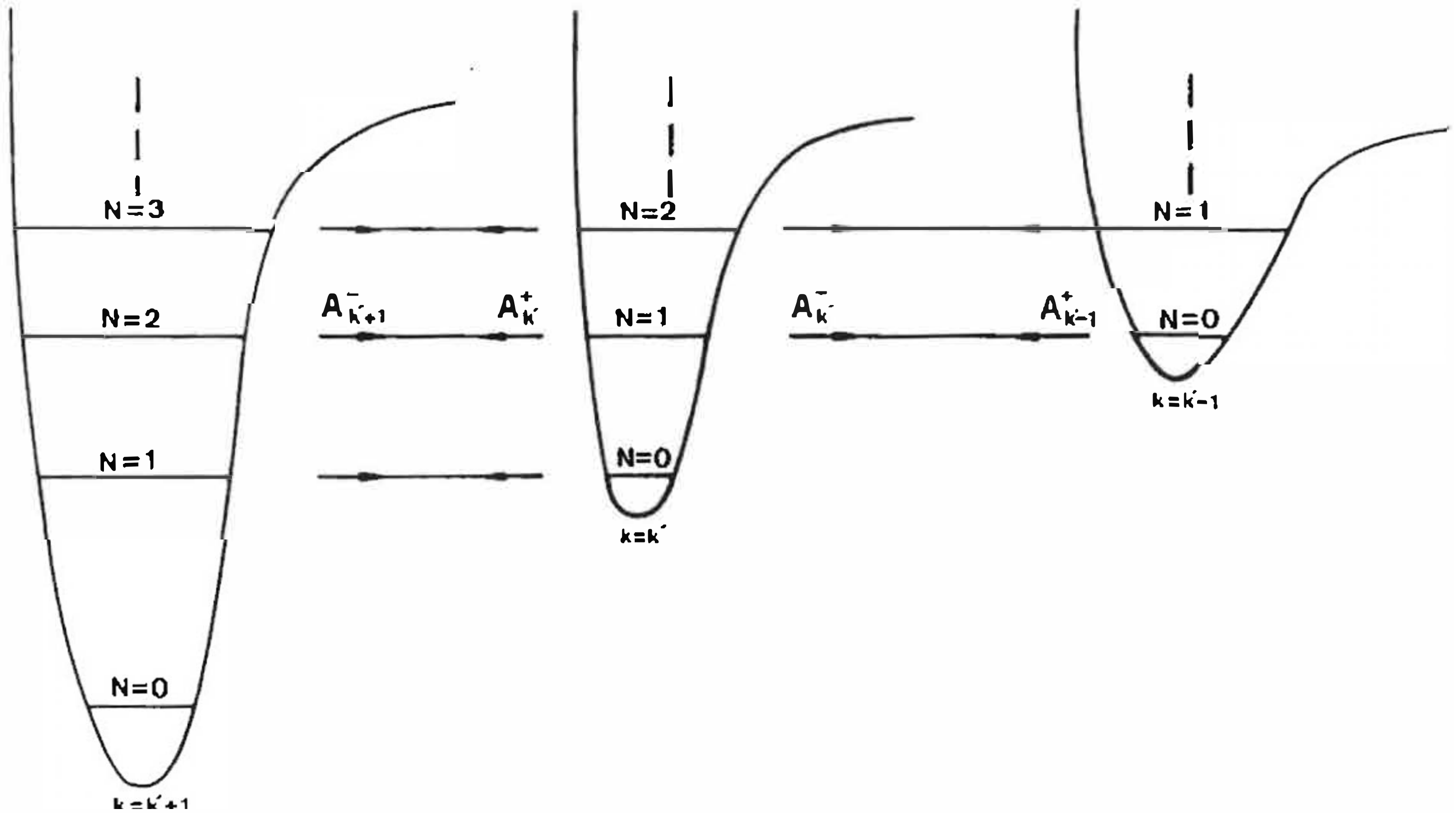

Fig. 3.3 Infeld-Hull type shift operations for Morse potentials [3]: A^+_k and A^-_k are given in Table 3.2, k is the parameter (3.2) and N is the vibrational quantum number.

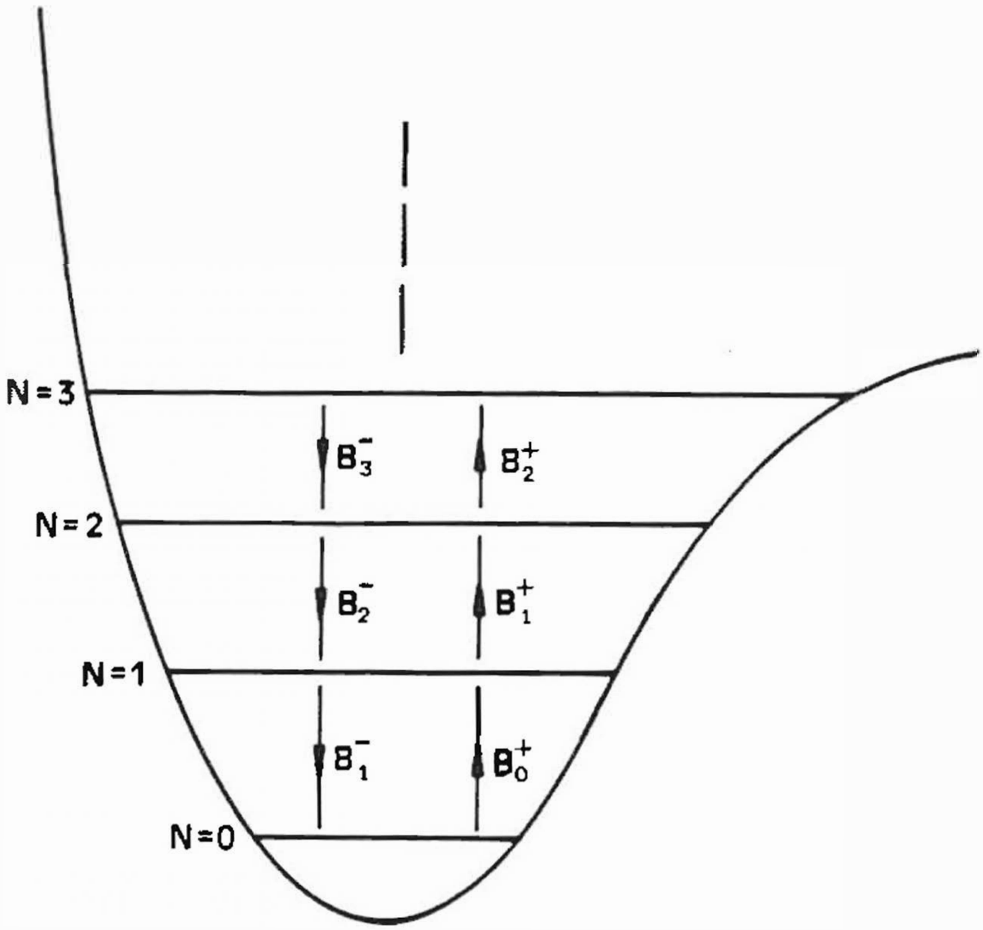

Fig. 3.4 Energy shift operations for the Morse potential. $B_N^{\pm}$ are given in Table 3.2 and N is the vibrational quantum number.

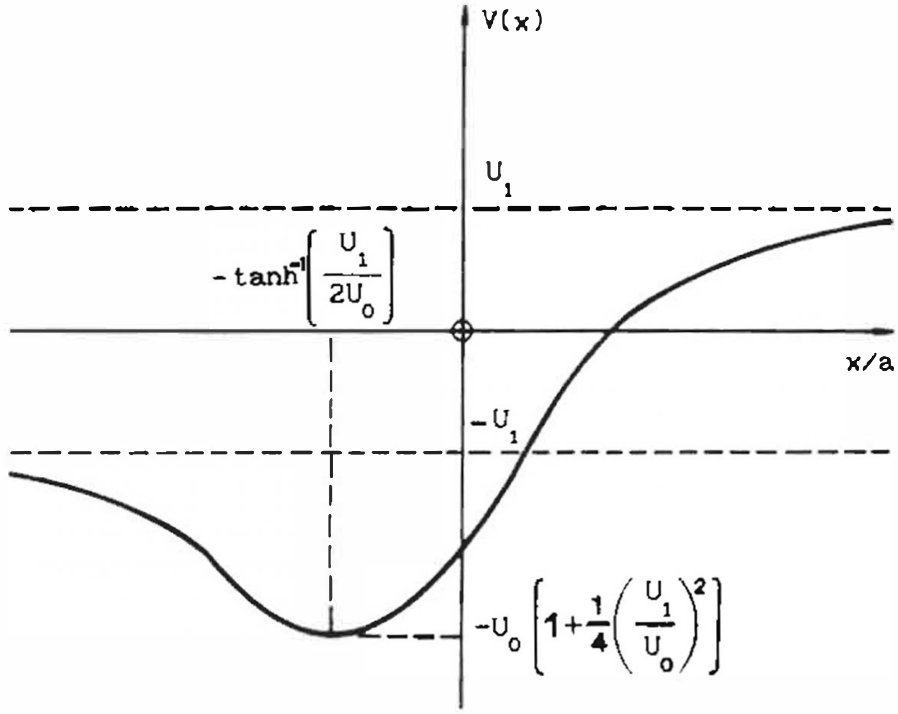

Fig. 3.5 The classical Rosen-Morse potential (3.34).

$$X = \frac{x}{a} \tag{3.35}$$

and

$$p_X = \frac{a}{\hbar} p_x . \tag{3.36}$$

Then

$$[p_X, X] = -i . \tag{3.37}$$

The energy eigenvalue equation with the potential (3.34) can be rewritten

$$\left[\frac{1}{2M} p_X^2 + V(X)\right]\psi = \varepsilon\psi , \tag{3.38}$$

where

$$V(X) = -\frac{a^2U_0}{\hbar^2}\operatorname{sech}^2X + \frac{a^2U_1}{\hbar^2}\tanh X \tag{3.39}$$

and

$$\varepsilon = \frac{a^2}{\hbar^2} E . \tag{3.40}$$

In (3.40), E is the eigenvalue of the Hamiltonian with the potential (3.34).

3.2.1 Shape invariance

Potentials of the shape (3.39) can be generated from the superpotential

$$W(X,\alpha_s) = \frac{1}{\sqrt{2M}}\left(\alpha_s\tanh X + \frac{q}{\alpha_s}\right) , \tag{3.41}$$

where

$$\alpha_s = k - s , \tag{3.42}$$

$$q = Ma^2\hbar^{-2}U_1 , \tag{3.43}$$

and k is a solution to $k(k+1) = 2Ma^2\hbar^{-2}U_0$. That is,

$$k = -\tfrac{1}{2} \pm \tfrac{1}{2}(1 + 8Ma^2\hbar^{-2}U_0)^{\frac{1}{2}} . \tag{3.44}$$

Using (3.41) in (1.36) yields the potentials

$$V_\pm(X,\alpha_s) = -\frac{\alpha_s}{2M}(\alpha_s \mp 1)\operatorname{sech}^2X + \frac{q}{M}\tanh X + \frac{\alpha_s^2}{2M} + \frac{q^2}{2M\alpha_s^2} . \tag{3.45}$$

Thus

$$V_-(X,\alpha_0) = V(X) + \kappa , \tag{3.46}$$

where V(X) is the potential (3.39) and

$$\kappa = \frac{\alpha_0^2}{2M} + \frac{q^2}{2M\alpha_0^2} . \tag{3.47}$$

The partner potentials (3.45) satisfy the shape invariance condition (1.23) with remainder

$$R(\alpha_s) = \frac{1}{2M}(\alpha_{s-1}^2 + q^2/\alpha_{s-1}^2) - \frac{1}{2M}(\alpha_s^2 + q^2/\alpha_s^2) . \tag{3.48}$$

3.2.2 Energy eigenvalues

The eigenvalues corresponding to the potential (3.46) are given by (see (1.41))

$$E_N^{(0)} = \sum_{s=1}^{N} R(\alpha_s)$$

$$= \frac{1}{2M}(\alpha_0^2 + q^2/\alpha_0^2) - \frac{1}{2M}(\alpha_N^2 + q^2/\alpha_N^2) , \tag{3.49}$$

where $E_0^{(0)} = 0$ and $N = 1, 2, \ldots, N_{max}$ with

$$N_{max} < k - (Ma^2\hbar^{-2}U_1)^{\frac{1}{2}} . \tag{3.50}$$

The restriction on N is required to ensure that the eigenfunctions (3.57) are normalizable*. From (3.46), (3.47), (3.40) and (3.49) the energy eigenvalues for the Rosen-Morse potential are given by

$$E = \frac{\hbar^2}{a^2}(E_N^{(0)} - \kappa)$$

$$= -\frac{\hbar^2}{2Ma^2}(k - N)^2 - \frac{Ma^2U_1^2}{2\hbar^2}\frac{1}{(k-N)^2} , \tag{3.51}$$

where $N = 0, 1, 2, \ldots, N_{max}$ and we have used (3.43) and (3.42) with s=N.

3.2.3 Coordinate-space eigenfunctions

The operators (1.35) with the superpotential (3.41) are

$$A_{\pm}(X, \alpha_N) = \frac{1}{\sqrt{2M}}\left[\pm ip_X + \alpha_N \tanh X + \frac{q}{\alpha_N}\right] . \tag{3.52}$$

---

* Normalizability of (3.57) requires $-(k-N) < -q/(k-N) < k-N$. Here $k-N>0$ with k given by the positive solution in (3.44)(see Ref. [3], p.93). Hence normalizability requires $k-N>q^{\frac{1}{2}}$.

The normalized solutions to equation (1.38) with s=N and $A_+(X,\alpha_N)$ given by (3.52) are

$$\psi_0(X,\alpha_N) = C_0(\alpha_N)\cosh^{-\alpha_N}X\ \exp(-qX/\alpha_N)\ , \tag{3.53}$$

where

$$C_0(\alpha_N) = \left[\frac{\Gamma(2\alpha_N)}{2^{2\alpha_N-1}\Gamma(\alpha_N+q/\alpha_N)\Gamma(\alpha_N-q/\alpha_N)}\right]^{\frac{1}{2}} . \tag{3.54}$$

The excited states are obtained by substituting (3.52) and (3.53) in (1.46). The factor γ_N in (1.46) is equal to $\prod_{k=1}^{N}(E_N^{(0)}-E_{k-1}^{(0)})^{-\frac{1}{2}}$, with $E_N^{(0)}$ given by (3.49). The first two excited states are found to be

$$\psi_1(X,\alpha_0) = C_1(\alpha_0)\cosh^{-\alpha_0+1}X\ \exp\left[\frac{-qX}{\alpha_0-1}\right]P_1(\tfrac{1}{2}\tanh X+\tfrac{1}{2}) \tag{3.55}$$

$$\psi_2(X,\alpha_0) = C_2(\alpha_0)\cosh^{-\alpha_0+2}X\ \exp\left[\frac{-qX}{\alpha_0-2}\right]P_2(\tfrac{1}{2}\tanh X+\tfrac{1}{2})\ . \tag{3.56}$$

The constants $C_1(\alpha_0)$ and $C_2(\alpha_0)$, and the polynomials P_1 and P_2, are given in Table 3.3. The polynomial $P_0 = 1$ in Table 3.3 is obtained from (3.53). The constant $C_0(\alpha_0)$ is (see (3.54))

$$\begin{aligned} C_0(\alpha_0) &= \left[\frac{\Gamma(2\alpha_0)}{2^{2\alpha_0-1}\Gamma(\alpha_0+q/\alpha_0)\Gamma(\alpha_0-q/\alpha_0)}\right]^{\frac{1}{2}} \\ &= \left[\frac{(\alpha_0^2-q^2/\alpha_0^2)\Gamma(2\alpha_0+1)\Gamma(\alpha_0+1-q/\alpha_0)}{2^{2\alpha_0}\alpha_0\{\Gamma(\alpha_0+1-q/\alpha_0)\}^2\Gamma(\alpha_0+1+q/\alpha_0)}\right]^{\frac{1}{2}} . \end{aligned}$$

Equations (3.55) and (3.56) are particular cases of

$$\begin{aligned} \psi_N(X,\alpha_0) &= C_N(\alpha_0)\cosh^{-\alpha_0+N}X\ \exp\left[\frac{-qX}{\alpha_0-N}\right] \\ &\quad\times {}_2F_1(-N,2\alpha_0-N+1;\alpha_0-N+1-\frac{q}{\alpha_0-N};\tfrac{1}{2}\tanh X+\tfrac{1}{2})\ , \end{aligned} \tag{3.57}$$

where

$$C_N(\alpha_0) = (-1)^N\left[\frac{[(\alpha_0-N)^2-q^2/(\alpha_0-N)^2]\Gamma(2\alpha_0-N+1)\Gamma[\alpha_0+1-q/(\alpha_0-N)]}{2^{2\alpha_0-2N}N!(\alpha_0-N)\{\Gamma[\alpha_0-N+1-q/(\alpha_0-N)]\}^2\Gamma[\alpha_0+1+q/(\alpha_0-N)]}\right]^{\frac{1}{2}} . \tag{3.58}$$

The validity of (3.57) and (3.58) can be extended to all non-negative integers N by substituting (3.57), (3.58) and (3.52) in (1.45) with s=0. The result can be simplified to yield the recurrence relation

TABLE 3.3 The constants $C_N(\alpha_0)$ and the polynomials $P_N(Y)$ in Equations (3.55) and (3.56) for N = 0,1,2.

N	$C_N(\alpha_0)$	$P_N(Y)$
0	$\left[\dfrac{(\alpha_0^2 - q^2/\alpha_0^2)\Gamma(2\alpha_0 + 1)\Gamma(\alpha_0 + 1 - q/\alpha_0)}{2^{2\alpha_0}\alpha_0\{\Gamma(\alpha_0 + 1 - q/\alpha_0)\}^2\Gamma(\alpha_0 + 1 + q/\alpha_0)}\right]^{\frac{1}{2}}$	1
1	$-\left[\dfrac{[(\alpha_0 - 1)^2 - q^2/(\alpha_0 - 1)^2]\Gamma(2\alpha_0)\Gamma[(\alpha_0 + 1 - q/(\alpha_0 - 1))]}{2^{2\alpha_0 - 2}(\alpha_0 - 1)\{\Gamma[\alpha_0 - q/(\alpha_0 - 1)]\}^2\Gamma[\alpha_0 + 1 + q/(\alpha_0 - 1)]}\right]^{\frac{1}{2}}$	$1 - \dfrac{2\alpha_0}{\alpha_0 - q/(\alpha_0 - 1)}Y$
2	$\left[\dfrac{[(\alpha_0 - 2)^2 - q^2/(\alpha_0 - 2)^2]\Gamma(2\alpha_0 - 1)\Gamma[(\alpha_0 + 1 - q/(\alpha_0 - 2))]}{2^{2\alpha_0 - 4}2(\alpha_0 - 2)\{\Gamma[\alpha_0 - 1 - q/(\alpha_0 - 2)]\}^2\Gamma[\alpha_0 + 1 + q/(\alpha_0 - 2)]}\right]^{\frac{1}{2}}$	$1 - \dfrac{2(2\alpha_0 - 1)}{\alpha_0 - 1 - q/(\alpha_0 - 2)}Y + \dfrac{2\alpha_0(2\alpha_0 - 1)}{[\alpha_0 - 1 - q/(\alpha_0 - 2)][\alpha_0 - q/(\alpha_0 - 2)]}Y^2$

$$\left[1 - \frac{B+N+1}{C+N}\,Y + \frac{B+N+1}{B(C+N)}\,Y(1-Y)\frac{d}{dY}\right]{}_2F_1(-N,B;C;Y) = {}_2F_1(-[N+1],B+1;C;Y)\ , \tag{3.59}$$

where

$$B = 2k - N - 1\ , \tag{3.60}$$

$$C = k - N - \frac{q}{\cancel{K-N-1}} \tag{3.61}$$

and

$$Y = \tfrac{1}{2}(\tanh X + 1)\ . \tag{3.62}$$

Equation (3.59) is proved in Appendix B.

In Table 3.4 we compare the operators $A_\pm(X,\alpha_s)$ with the operators $A_k^\pm$ obtained using the factorization method of Infeld and Hull [5]. In this Table we have not given any shift operators for energy (that is, operators which change N in (3.57)) because, as far as we know, it is not possible to construct such operators for the Rosen-Morse potential. (However, for the symmetric Rosen-Morse potential one can construct shift operators for energy - see Table 3.5 below.)

3.2.4 **Symmetric Rosen-Morse potential [10]**

The potential (3.34) with $U_1 = 0$ is known as the symmetric Rosen-Morse potential. A graph of this potential is shown in Fig. 3.6. From (3.51) we obtain the energy eigenvalues

$$E = \frac{\hbar^2}{2Ma^2}(k - N)^2 \qquad (N = 0,1,2,\ \ldots,\ N_{max})\ , \tag{3.63}$$

where k is given by (3.44) with the positive sign, and from (3.50) $N_{max} < k$.

To obtain the eigenfunctions for the symmetric Rosen-Morse potential we first rewrite (3.52)-(3.54) with q=0. Hence

$$A_\pm(X,\alpha_N) = \frac{1}{\sqrt{2M}}\,(\pm ip_X + \alpha_N \tanh X)\ , \tag{3.64}$$

$$\psi_0(X,\alpha_N) = C_0(\alpha_N)\cosh^{-\alpha_N}X \tag{3.65}$$

and

$$C_0(\alpha_N) = \left[\frac{\Gamma(\alpha_N + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\alpha_N)}\right]^{\frac{1}{2}}\ . \tag{3.66}$$

TABLE 3.4 Shift operations for the Rosen-Morse potential (3.34). The parameter k is given by the positive solution in (3.44), and q is given by (3.43). The energy eigenvalues E are given by (3.51).

---

1. *Shape invariance*

$$A_+(X,\alpha_s)|\psi_N(X,\alpha_s)\rangle = \frac{1}{\sqrt{2M}}[(a_s^2 + q^2/\alpha_s^2) - (a_{N+s}^2 + q^2/\alpha_{N+s}^2)]^{\frac{1}{2}}|\psi_{N-1}(\alpha_{s+1})\rangle$$

$$A_-(X,\alpha_s)|\psi_N(X,\alpha_{s+1})\rangle = \frac{1}{\sqrt{2M}}[(a_s^2 + q^2/\alpha_s^2) - (a_{N+s+1}^2 + q^2/\alpha_{N+s+1}^2)]^{\frac{1}{2}}|\psi_{N+1}(\alpha_s)\rangle$$

where $\alpha_s = k - s$ and $A_\pm(X,\alpha_s) = \dfrac{1}{\sqrt{2M}}(\pm ip_X + \alpha_s \tanh X + q/\alpha_s)$.

2. *Shift operators for the parameter* k [5,9]

$$A_k^\pm|Ek\rangle = [2Ma^2\hbar^{-2}E + q^2/(k + \tfrac{1}{2} \pm \tfrac{1}{2})^2 + (k + \tfrac{1}{2} \pm \tfrac{1}{2})^2]^{\frac{1}{2}}|E, k \pm 1\rangle$$

where $A_k^\pm = \pm ip_X - (k + \frac{1}{2} \pm \frac{1}{2})\tanh X - q/(k + \frac{1}{2} \pm \frac{1}{2})$.

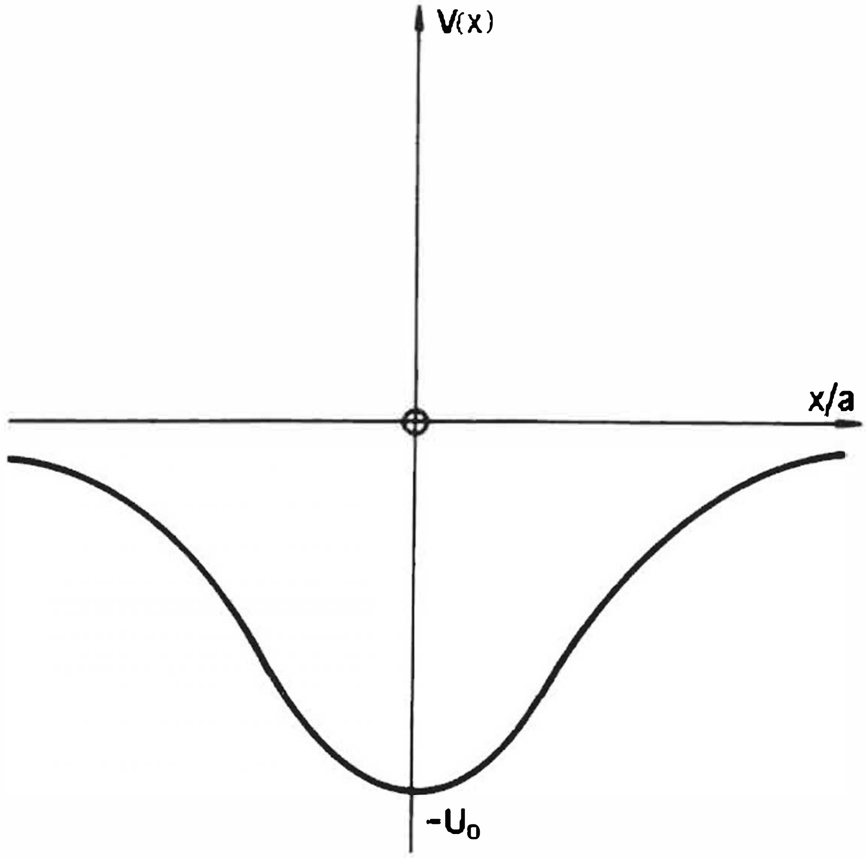

Fig. 3.6 The symmetric Rosen-Morse potential ((3.34) with $U_1 = 0$).

To obtain (3.66) we have used the identity

$$\Gamma(2a) = \frac{2^{2a-1}\Gamma(a)\Gamma(a+\frac{1}{2})}{\Gamma(\frac{1}{2})} . \tag{3.67}$$

Continuing as in Section 3.2.3, we obtain from $\psi_0(X,\alpha_N)$ the first and second excited states $\psi_1(X,\alpha_0)$ and $\psi_2(X,\alpha_0)$, which are odd and even functions of X respectively. In general, $\psi_{2N}(X,\alpha_0)$ is an even function of X, and $\psi_{2N+1}(X,\alpha_0)$ is an odd function of X. General expressions for these eigenfunctions can be obtained using (1.46). We first state the results and then outline the proof. The even eigenfunctions are

$$\psi^{(e)}_{2N}(X,\alpha_0) = C^{(e)}_{2N}(\alpha_0)\cosh^{-\alpha_0}X \; {}_2F_1(-N,N-\alpha_0;\tfrac{1}{2};-\sinh^2X) , \tag{3.68}$$

where

$$C^{(e)}_{2N}(\alpha_0) = \left[\frac{(\alpha_0-2N)\Gamma(N+\frac{1}{2})\Gamma(\alpha_0-N+\frac{1}{2})}{N!\,[\Gamma(\frac{1}{2})]^2\Gamma(\alpha_0-N+1)}\right]^{\frac{1}{2}} . \tag{3.69}$$

The odd eigenfunctions are

$$\psi^{(o)}_{N+1}(X,\alpha_0) = C^{(o)}_{2N+1}(\alpha_0)\cosh^{-\alpha_0}X \sinh X \; {}_2F_1(-N,N+1-\alpha_0;\tfrac{3}{2};-\sinh^2X) , \tag{3.70}$$

where

$$C^{(o)}_{2N+1}(\alpha_0) = \left[\frac{(\alpha_0-2N-1)\Gamma(N+\frac{3}{2})\Gamma(\alpha_0-N+\frac{1}{2})}{N!\,[\Gamma(\frac{1}{2})]^2\Gamma(\alpha_0-N)}\right]^{\frac{1}{2}} . \tag{3.71}$$

To prove (3.68)-(3.71) we first use (1.46) to obtain a transformation which changes N by 2. It is easy to show that (see Appendix C)

$$A_-(\alpha_0)A_-(\alpha_1)|\psi_N(\alpha_2)\rangle = [(E^{(0)}_{N+2}-E^{(0)}_0)(E^{(0)}_{N+2}-E^{(0)}_1)]^{\frac{1}{2}}|\psi_{N+2}(\alpha_0)\rangle . \tag{3.72}$$

For the symmetric Rosen-Morse potential A_- is given by (3.64), and it can be shown that

$$A_-(X,\alpha_0)A_-(X,\alpha_1) = -H^{(2)}+E^{(0)}_2+\frac{1}{2M}\Big[k(k-1)+(k-2)^2-(k-1)(2k-1)\mathrm{sech}^2X -(2k-1)\tanh X\frac{d}{dX}\Big] . \tag{3.73}$$

Here $E^{(0)}_N$ is given by (3.49) with q=0 and

$$H^{(2)} = \frac{1}{2M}p_X^2 + V_-(X,\alpha_2) + E^{(0)}_2 , \tag{3.74}$$

where $V_-(X,\alpha_2)$ can be obtained from (3.45) with q=0. When using (3.73) in (3.72) we can replace $H^{(2)}$ by the energy eigenvalue $E^{(2)}_N = E^{(0)}_{N+2}$ (see

(1.31)). Using (3.73), (3.49) with q=0 and (3.68) in (3.72) and simplifying we obtain the recurrence relation

$$\left[1 - \frac{2(2B-1)(B-1)+2N(2N-2B+3)}{(2N+1)(2B-1)}\, u + \frac{2(2N-2B+3)}{(2N+1)(2B-1)}\, u(u-1)\frac{d}{du}\right]{}_2F_1(-N,B;\tfrac{1}{2};u)$$
$$= {}_2F_1(-[N+1],B-1;\tfrac{1}{2};u)\ , \qquad (3.75)$$

where

$$B = N - k + 2 \qquad (3.76)$$

and

$$u = -\sinh^2 X\ . \qquad (3.77)$$

The proof of (3.75) is given in Appendix B.

We can obtain the odd eigenfunctions (3.70) from (3.68) by using (1.45) with s=0. Upon simplification this yields the recurrence relation

$$\frac{-(2N-2B+1)}{(2N+1)(2B-1)}\left[1 - \frac{2}{(2N-2B+1)}\,(u-1)\,\frac{d}{du}\right]{}_2F_1(-N,B;\tfrac{1}{2};u) = {}_2F_1(-N,B;\tfrac{3}{2};u)\ ,$$
$$(3.78)$$

where

$$B = N - k + 1 \qquad (3.79)$$

and u is given by (3.77). The proof of (3.78) is also given in Appendix B. This completes the proof of (3.68)-(3.71).

In Table 3.5 the operators (3.64) are compared to the shift operators obtained from the factorization method of Infeld and Hull and the shift operators for energy.

3.3 FIRST PÖSCHL-TELLER POTENTIAL [10]

The potentials (3.80) and (3.93) were originally discussed by Pöschl and Teller in 1933 [10]. More recent work on these potentials can be found in Ref. 11. The first Pöschl-Teller potential is

$$V(x) = \frac{\hbar^2}{2Ma^2}\left[\frac{\mu(\mu-1)}{\sin^2 x/a} + \frac{\lambda(\lambda-1)}{\cos^2 x/a}\right]\ , \qquad (3.80)$$

where a is a positive constant and μ and λ are constants greater than one. The potential (3.80) is depicted in the interval $[0,\frac{1}{2}\pi a]$ in Fig. 3.7 for

TABLE 3.5 Shift operations for the one-dimensional symmetric Rosen-Morse potential ((3.34) with $U_1 = 0$). The parameter k is given by (3.44) with the positive sign and the energy eigenvalue E is given by (3.63).

---

1. *Shape invariance*

$$A_+(X,\alpha_s)\,|\psi_N(X,\alpha_s)\rangle = \frac{1}{\sqrt{2M}}(\alpha_s^2 - \alpha_{N+s}^2)^{\frac{1}{2}}\,|\psi_{N-1}(X,\alpha_{s+1})\rangle$$

$$A_-(X,\alpha_s)\,|\psi_N(X,\alpha_{s+1})\rangle = \frac{1}{\sqrt{2M}}(\alpha_s^2 - \alpha_{N+s+1}^2)^{\frac{1}{2}}\,|\psi_{N+1}(X,\alpha_s)\rangle$$

where $\alpha_s = k - s$ and $A_\pm(X,\alpha_s) = \frac{1}{\sqrt{2M}}(\pm ip_X + \alpha_s \tanh X)$.

2. *Shift operators for the parameter* k [5,9]

$$A_k^\pm|Ek\rangle = [(k + \tfrac{1}{2} \pm \tfrac{1}{2})^2 - (k - N)^2]^{\frac{1}{2}}|E,k \pm 1\rangle$$

where $A_k^\pm = \pm ip_X - (k + \frac{1}{2} \pm \frac{1}{2})\tanh X$.

3. *Shift operators for energy* [9]

$$B_N^\pm|Nk\rangle = \left[\frac{(N + \frac{1}{2} \pm \frac{1}{2})(k - N)(2k - N + \frac{1}{2} \mp \frac{1}{2})}{k \mp 1 - N}\right]^{\frac{1}{2}}|N \pm 1,k\rangle$$

where $B_N^\pm = \pm i\cosh X\, p_X - (k - N)\sinh X$.

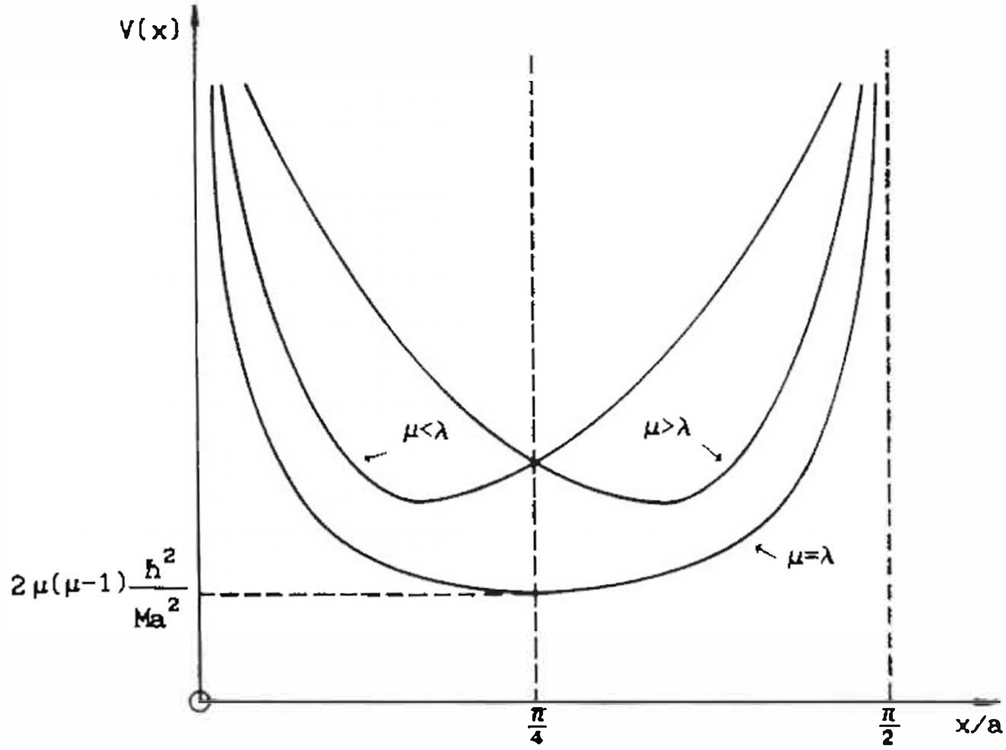

Fig. 3.7 The first Pöschl-Teller potential (3.80).

the three cases $\mu=\lambda$, $\mu<\lambda$ and $\mu>\lambda$.

Using the variables (3.35) and (3.36), the energy eigenvalue equation with the potential (3.80) becomes

$$\left[\frac{1}{2M} p_X^2 + V(X)\right]\psi = \varepsilon\psi \ , \tag{3.81}$$

where

$$V(X) = \frac{1}{2M}\left[\frac{\mu(\mu-1)}{\sin^2 X} + \frac{\lambda(\lambda-1)}{\cos^2 X}\right] \tag{3.82}$$

and

$$\varepsilon = \frac{a^2}{\hbar^2} E \ . \tag{3.83}$$

Potentials of the shape (3.81) can be generated from the super-potential

$$W(X,\alpha_s) = \frac{1}{\sqrt{2M}}\left[(\lambda + \alpha_s)\tan X - (\mu + \alpha_s)\cot X\right] \ , \tag{3.84}$$

where

$$\alpha_s = s \qquad (s = 0,1,2, \ \ldots) \ . \tag{3.85}$$

Equation (3.84) in (1.36) yields

$$V_\pm(X,\alpha_s) = \frac{1}{2M}\left[\frac{(\mu+s)(\mu+s\pm1)}{\sin^2 X} + \frac{(\lambda+s)(\lambda+s\pm1)}{\cos^2 X} - (\mu+\lambda+2s)^2\right] \ ,$$

and hence

$$V_-(X,\alpha_0) = V(X) - \frac{1}{2M}(\mu + \lambda)^2 \ , \tag{3.86}$$

where $V(X)$ is the potential (3.82). It can be shown that the shape invariance condition (1.23) is also satisfied with remainder

$$R(\alpha_s) = \frac{2}{M}(\mu + \lambda + 2s - 1) \ . \tag{3.87}$$

Substituting (3.87) in (1.41) and using (3.82) and (3.86) we find that the energy eigenvalues for the first Pöschl-Teller potential are given by

$$E = \frac{\hbar^2}{2Ma^2}(\mu + \lambda + 2N)^2 \qquad (N = 0,1,2, \ \ldots) \ . \tag{3.88}$$

The eigenfunctions obtained from (1.46) and (1.38) are

$$\psi_N(X,\alpha_0) = C_N(\alpha_0)\,{}_2F_1(-N,\mu+\lambda+N;\mu+\tfrac{1}{2};\sin^2 X)\sin^\mu X \cos^\lambda X \tag{3.89}$$

for $N = 0,1,2, \ \ldots$, where

$$C_N(\alpha_0) = \left[\frac{2(\mu+\lambda+2N)\Gamma(\mu+\lambda+N)\Gamma(\mu+\tfrac{1}{2}+N)}{N!\{\Gamma(\mu+\tfrac{1}{2})\}^2\Gamma(\lambda+\tfrac{1}{2}+N)}\right]^{\frac{1}{2}} \ . \tag{3.90}$$

The validity of (3.89) and (3.90) can be proved for all non-negative integers N by substituting these equations in (1.45) with s=0. This yields the recurrence relation

$$\left[1 - \frac{B-N-1}{C-1}\, u + \frac{u(1-u)}{C-1}\frac{d}{du}\right] {}_2F_1(-N,B;C;u) = {}_2F_1(-[N+1],B-1;C-1;u) \qquad (3.91)$$

where

$$B = \mu + \lambda + N + 2, \qquad C = \mu + \tfrac{3}{2} \quad \text{and} \quad u = \sin^2 X\,. \qquad (3.92)$$

Equation (3.91) is proved in Appendix B. The various shift operations for the first Pöschl-Teller potential are given in Table 3.6.

3.4 SECOND PÖSCHL-TELLER POTENTIAL [10]

The second Pöschl-Teller potential is given by

$$V(x) = \frac{h^2}{2Ma^2}\left[\frac{\mu(\mu-1)}{\sinh^2 x/a} - \frac{\lambda(\lambda+1)}{\cosh^2 x/a}\right], \qquad (3.93)$$

where a is a constant and μ and λ are constants greater than one. The potential (3.93) is depicted in Fig. 3.8 for $\lambda>\mu-1>0$. Using the variables (3.35) and (3.36) and continuing in the same manner as in Section 3.3 we obtain the eigenvalue equation

$$\left[\frac{1}{2M} P_X^2 + V(X)\right]\psi = \varepsilon\psi\,, \qquad (3.94)$$

where

$$V(X) = \frac{1}{2M}\left[\frac{\mu(\mu-1)}{\sinh^2 X} - \frac{\lambda(\lambda+1)}{\cosh^2 X}\right] \qquad (3.95)$$

and

$$\varepsilon = \frac{a^2}{h^2} E\,. \qquad (3.96)$$

Potentials of the shape (3.95) can be generated from the super-potential

$$W(X,\alpha_s) = \frac{1}{\sqrt{2M}}\left[-(\mu - \alpha_s)\coth X + (\lambda - \alpha_s)\tanh X\right], \qquad (3.97)$$

where

$$\alpha_s = s\,. \qquad (3.98)$$

Using (3.97) in (1.36) it is easy to show that

TABLE 3.6 Shift operations for the first Pöschl-Teller potential (3.80). The energy eigenvalues E are given by (3.88).

1. Shape invariance

$$A_+(X,\alpha_s)\,|\psi_N(X,\alpha_s)\rangle = \frac{1}{\sqrt{2M}}[4\alpha_{N+s}(\mu+\lambda+\alpha_{N+s})-4\alpha_s(\mu+\lambda+\alpha_s)]^{\frac{1}{2}}|\psi_{N-1}(X,\alpha_{s+1})\rangle$$

$$A_-(X,\alpha_s)\,|\psi_N(X,\alpha_{s+1})\rangle = \frac{1}{\sqrt{2M}}[4\alpha_{N+s+1}(\mu+\lambda+\alpha_{N+s+1})-4\alpha_s(\mu+\lambda+\alpha_s)]^{\frac{1}{2}}|\psi_{N+1}(X,\alpha_s)\rangle$$

where $\alpha_s = s$ and $A_\pm(X,\alpha_s) = \frac{1}{\sqrt{2M}}(\pm ip_X - (\mu+\alpha_s)\cot X + (\lambda+\alpha_s)\tan X)$.

2. Shift operators for the parameter k [5,12]

$$A_k^\pm|Ek\rangle = 2[(k+N)^2-(k-\tfrac{1}{2}\pm\tfrac{1}{2})^2]^{\frac{1}{2}}|E,k\pm1\rangle$$

where $k = \frac{1}{2}(\lambda+\mu)$ and $A_k^\pm = \pm ip_X - (\mu-\frac{1}{2}\pm\frac{1}{2})\cot X + (\lambda-\frac{1}{2}\pm\frac{1}{2})\tan X$.

3. Shift operators for energy [12]

$$B_N^\pm|Nk\rangle = \left[\frac{(N+\frac{1}{2}\pm\frac{1}{2})(\mu+\lambda+N-\frac{1}{2}\pm\frac{1}{2})(\mu+\lambda+2N)(2\lambda\pm1+2N)(2\mu\pm1+2N)}{(\mu+\lambda\pm2+2N)(\mu+\lambda\pm1+2N)^2}\right]^{\frac{1}{2}}|N\pm1,k\rangle$$

where $B_N^\pm = \pm i\sin X\cos X\,p_X - (\mu+\lambda+2N)\sin^2 X + \frac{1}{2}(\mu+\lambda+2N) + [\mu(\mu-1)-\lambda(\lambda-1)/2(\mu+\lambda+2N\pm1)$.

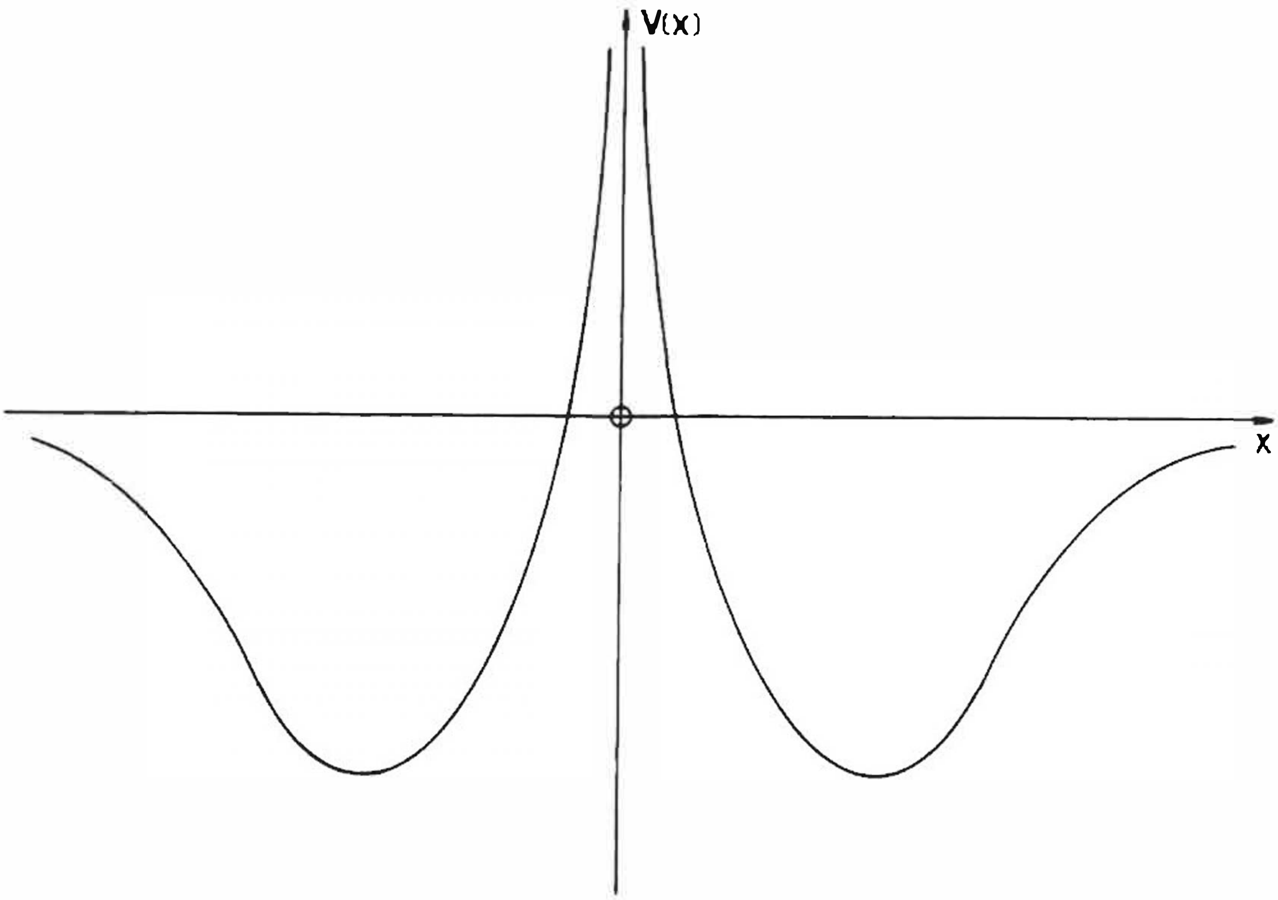

Fig. 3.8 The second Pöschl-Teller potential (3.93) for $\lambda > \mu-1 > 0$.

$$V_{\pm}(X,\alpha_s) = \frac{1}{2M}\left[\frac{(\mu+\alpha_s)(\mu+\alpha_s\pm1)}{\sinh^2X} - \frac{(\lambda-\alpha_s)(\lambda-\alpha_s\mp1)}{\cosh^2X} + (\lambda-\mu-2\alpha_s)^2\right] ,$$

and hence

$$V_-(X,\alpha_0) = V(X) + \frac{1}{2M}(\lambda - \mu)^2 , \qquad (3.99)$$

where $V(X)$ is given by (3.95). Also the shape invariance condition (1.23) is satisfied with remainder

$$R(\alpha_s) = \frac{2}{M}(\lambda - \mu - 2s + 1) . \qquad (3.100)$$

From (1.41), (3.99), (3.98) and (3.96) we obtain for the energy eigenvalues of the second Pöschl-Teller potential

$$E = -\frac{\hbar^2}{2Ma^2}(\lambda - \mu - 2N)^2 \qquad (3.101)$$

for $N = 0,1,2, \ldots, N_{max}$, where N_{max} is the largest integer less than $\frac{1}{2}(\lambda - \mu)$ (see Ref. 3, p.104).

The eigenfunctions obtained from (1.46) and (1.38) are

$$\psi_N(X,\alpha_0) = C_N(\alpha_0)\,{}_2F_1(-N,N+\mu-\lambda;\mu+\tfrac{1}{2};-\sinh^2X)\ \sinh^{\mu}X\ \cosh^{-\lambda}X , \qquad (3.102)$$

where

$$C_N(\alpha_0) = \left[\frac{2(\lambda-\mu-2N)\Gamma(\mu+\frac{1}{2}+N)\Gamma(\lambda+\frac{1}{2}-N)}{N!\{\Gamma(\mu+\frac{1}{2})\}^2\Gamma(\lambda-\mu+1-N)}\right]^{\frac{1}{2}} . \qquad (3.103)$$

The validity of (3.102) and (3.103) can be proved for all non-negative integers N by substituting these equations in (1.45) with s=0. This yields the recurrence relation (3.91) with

$$B = \mu - \lambda + N + 2, \quad C = \mu + \tfrac{3}{2} \quad \text{and} \quad u = -\sinh^2X . \qquad (3.104)$$

The various shift operations for the second Pöschl-Teller potential are given in Table 3.7.

TABLE 3.7 Shift operations for the second Pöschl-Teller potential (3.93). The energy eigenvalues E are given by (3.101).

---

1. Shape invariance

$$A_+(X,\alpha_s)|\psi_N(X,\alpha_s)\rangle = \frac{1}{\sqrt{2M}}[4\alpha_{N+s}(\lambda-\mu-\alpha_{N+s})-4\alpha_s(\lambda-\mu-\alpha_s)]^{\frac{1}{2}}|\psi_{N-1}(X,\alpha_{s+1})\rangle$$

$$A_-(X,\alpha_s)|\psi_N(X,\alpha_{s+1})\rangle = \frac{1}{\sqrt{2M}}[4\alpha_{N+s+1}(\lambda-\mu-\alpha_{N+s+1})-4\alpha_s(\lambda-\mu-\alpha_s)]^{\frac{1}{2}}|\psi_{N+1}(X,\alpha_s)\rangle$$

where $\alpha_s = s$ and $A_\pm(X,\alpha_s) = \frac{1}{\sqrt{2M}}(\pm ip_X - (\mu-\alpha_s)\coth X + (\lambda-\alpha_s)\tanh X)$.

2. Shift operators for the parameter k [5,13]

$$A_k^\pm|Ek\rangle = 2[(k - \tfrac{1}{2} \pm \tfrac{1}{2})^2 - (k - 1 - N)^2]^{\frac{1}{2}}|E,k\pm1\rangle$$

where $k = \frac{1}{2}(\lambda-\mu+2)$ and $A_k^\pm = \pm ip_X + (\mu - \frac{1}{2} \mp \frac{1}{2})\coth X - (\lambda + \frac{1}{2} \pm \frac{1}{2})\tanh X$.

3. Shift operators for energy [13]

$$B_N^\pm|Nk\rangle = \left[\frac{(N+\frac{1}{2}\pm\frac{1}{2})(\lambda-\mu-N+\frac{1}{2}\mp\frac{1}{2})(\lambda-\mu-2N)(2\lambda\mp1-2N)(2\mu\pm1+2N)}{(\lambda-\mu\mp2-2N)(\lambda-\mu\mp1-2N)^2}\right]^{\frac{1}{2}}|N\pm1,k\rangle$$

where $B_N^\pm = \pm i \sinh X\cosh X p_X-(\lambda-\mu-2N)\sinh^2X-\frac{1}{2}(\lambda-\mu-2N)-[\mu(\mu-1)-\lambda(\lambda+1)/2(\lambda-\mu-2N\mp1)$.

CHAPTER 4

SPHERICALLY SYMMETRIC SYSTEMS

In this Chapter we show how shape invariance can be used to treat certain spherically symmetric problems. We first present a brief description of how the results derived in Chapter 1 must be modified for spherically symmetric problems (Section 4.1). In Section 4.2 we discuss the three-dimensional isotropic non-relativistic oscillator. Here we present the analysis in some detail to clarify the difference between the application of shape invariance to one-dimensional and spherically symmetric systems. In Section 4.3 we treat the non-relativistic Coulomb problem in an angular momentum basis; relativistic Coulomb problems are discussed in Chapter 6. In Section 4.4 we show how the oscillator and Coulomb potentials can be modified to include a repulsive inverse-square term. The Eckart potential and the Hulthén potential are discussed in Sections 4.5 and 4.6. In Section 4.7 we extend the results of the one-dimensional Morse and symmetric Rosen-Morse potentials to the corresponding three-dimensional cases. As a final example, we briefly describe how the Manning-Rosen potential can be obtained from the Eckart potential by making certain simple substitutions.

Except for the Hulthén potential, the above problems have previously been treated by shape invariance to obtain the energy eigenvalues and (unnormalized) eigenfunctions [1]. For the Coulomb problem, normalized eigenfunctions have been determined using shape invariance [2]. In this Chapter we perform the normalization of bound-state eigenfunctions for the above systems using shape invariance. At the end of Sections 4.2-4.6 we compare the operators given by shape invariance to the Infeld-Hull type operators that change a parameter, and the shift operators for energy.

4.1 INTRODUCTION

The results of supersymmetry and shape invariance derived in Chapter 1 can easily be extended to treat spherically symmetric systems. This can be achieved by replacing the operators x and p, respectively, by $r = (\underset{\sim}{r}\cdot\underset{\sim}{r})^{\frac{1}{2}}$ and the radial momentum operator [3,4]

$$p_r = \tfrac{1}{2}(\hat{\underset{\sim}{r}}\cdot\underset{\sim}{p} + \underset{\sim}{p}\cdot\hat{\underset{\sim}{r}}) = \frac{1}{r}(\underset{\sim}{r}\cdot\underset{\sim}{p} - i\underset{\sim}{\hbar}) \ . \tag{4.1}$$

These operators satisfy the commutation relation

$$[p_r, r] = -i\hbar \ . \tag{4.2}$$

The results also apply to two non-relativistic spinless particles interacting via a spherically symmetric potential $V(r)$, where $r = |\underset{\sim}{r}_1 - \underset{\sim}{r}_2|$ is the separation of the two particles. The Hamiltonian H_T can be separated into Hamiltonians H_{cm} and H, associated with the motion of the centre of mass and the relative motion of the two particles. We write

$$H_T = H_{cm} + H \ , \tag{4.3}$$

where

$$H_{cm} = \frac{1}{2(m_1 + m_2)} \underset{\sim}{P}^2 \tag{4.4}$$

and

$$H = \frac{1}{2M} \underset{\sim}{p}^2 + V(r) \ . \tag{4.5}$$

Here m_1 and m_2 are the masses of the particles, $\underset{\sim}{P} = \underset{\sim}{p}_1 + \underset{\sim}{p}_2$ is the total momentum, M the reduced mass and $\underset{\sim}{p} = (m_2\underset{\sim}{p}_1 - m_1\underset{\sim}{p}_2)/(m_1 + m_2)$ [3].

It can be shown that $\underset{\sim}{p}^2 = p_r^2 + r^{-2}\underset{\sim}{L}^2$ [3], and hence

$$H = \frac{1}{2M} p_r^2 + \frac{\underset{\sim}{L}^2}{2Mr^2} + V(r) \ .$$

The operators H, $\underset{\sim}{L}^2$ and L_z are a complete set of commuting operators: they act on the kets $|E\ell m\rangle$ of an angular momentum basis. It is a general property of spherically symmetric problems that the energy E does not depend on the quantum number m [5].

For the purpose of operating on the kets $|E\ell m\rangle$ one can consider instead of H the radial Hamiltonian

$$H_\ell = \frac{1}{2M}\, p_r^2 + \frac{\hbar^2 \ell(\ell+1)}{2Mr^2} + V(r) \;, \tag{4.6}$$

which is obtained from H by replacing $\underset{\sim}{L}^2$ with its eigenvalues $\hbar^2\ell(\ell+1)$. In what follows we denote an eigenket of H_ℓ by $|E\ell\rangle$. The corresponding radial coordinate-space eigenfunction is $R_{E\ell}(r) = \langle r|E\ell\rangle$.

For the shape invariance method of Chapter 1, we denote the eigenfunctions for bound states by $R_N(r,\alpha_s)$, where N = 0,1, ... In terms of these, (1.38) and (1.44)-(1.46) are

$$A_+(r,\alpha_s)R_0(r,\alpha_s) = 0 \tag{4.7}$$

$$A_+(r,\alpha_s)R_N(r,\alpha_s) = (E_N^{(s)} - E_s^{(0)})^{\frac{1}{2}} R_{N-1}(r,\alpha_{s+1}) \tag{4.8}$$

$$A_-(r,\alpha_s)R_N(r,\alpha_{s+1}) = (E_{N+1}^{(s)} - E_s^{(0)})^{\frac{1}{2}} R_{N+1}(r,\alpha_s) \tag{4.9}$$

and

$$R_N(r,\alpha_0) = e^{-iN\theta}\gamma_N A_-(r,\alpha_0)A_-(r,\alpha_1)\ldots A_-(r,\alpha_{N-1})R_0(r,\alpha_N) \;. \tag{4.10}$$

Here

$$A_\pm(r,\alpha_s) = \pm\frac{i}{\sqrt{2m}}\, p_r + W(r,\alpha_s) \;, \tag{4.11}$$

and γ_N is given by (1.47). The partner potentials $V_\pm(r,\alpha_s)$ are given in terms of the superpotential $W(r,\alpha_s)$ by (1.36) with x replaced by r.

4.2 THREE-DIMENSIONAL ISOTROPIC NON-RELATIVISTIC OSCILLATOR

The potential of the three-dimensional isotropic non-relativistic oscillator (hereafter referred to as the oscillator) is equal to $\frac{1}{2}M\omega^2 r^2$. The eigenvalue equation for the radial Hamiltonian with this potential can be written

$$H_\ell|E\ell\rangle = \left[\frac{1}{2M}\, p_r^2 + V(r)\right]|E\ell\rangle = E|E\ell\rangle \;,$$

where

$$V(r) = \frac{\hbar^2 \ell(\ell+1)}{2Mr^2} + \tfrac{1}{2}M\omega^2 r^2 . \tag{4.12}$$

Potentials of the shape (4.12) can be generated from the superpotential

$$W(r,\alpha_s) = \sqrt{\tfrac{1}{2}M}\,\omega r - \frac{\hbar(\alpha_s + 1)}{\sqrt{2M}\,r} , \tag{4.13}$$

where

$$\alpha_s = \ell + s \qquad (s = 0, 1, \ldots) . \tag{4.14}$$

Using (4.13) in (1.36) we find that

$$V_{\pm}(r,\alpha_s) = \tfrac{1}{2}M\omega^2 r^2 + \frac{\hbar^2(\alpha_s + 1)(\alpha_s + 1 \pm 1)}{2Mr^2} - \hbar\omega(\alpha_s + 1 \mp \tfrac{1}{2}) . \tag{4.15}$$

Thus

$$V_{-}(r,\alpha_0) = V(r) - \hbar\omega(\ell + \tfrac{3}{2}) , \tag{4.16}$$

where $V(r)$ is the potential (4.12). It can also be shown that the shape invariance condition (1.23) is satisfied with remainder

$$R(\alpha_s) = 2\hbar\omega . \tag{4.17}$$

Substituting (4.17) in (1.41) we obtain the energy eigenvalues of the Hamiltonian with the potential (4.16)

$$E_N^{(0)} = 2N\hbar\omega \qquad (N = 1, 2, \ldots) , \tag{4.18}$$

where $E_0^{(0)} = 0$ (see (1.39)). The energy eigenvalues of the Hamiltonian with the potential (4.12) are given by (4.18) plus the constant $\hbar\omega(\ell + \tfrac{3}{2})$ (see (4.16)). Hence

$$E = (2N + \ell + \tfrac{3}{2})\hbar\omega \qquad (N = 0, 1, 2, \ldots) . \tag{4.19}$$

The coordinate-space eigenfunctions can be obtained by the separation of variables

$$\psi_{E\ell m}(\underset{\sim}{r}) = R_{E\ell}(r) Y_{\ell m}(\hat{\underset{\sim}{r}}) , \tag{4.20}$$

where $R_{E\ell}(r) = \langle r|E\ell\rangle$ is the radial coordinate-space eigenfunction and $Y_{\ell m}(\hat{\underset{\sim}{r}})$ is a spherical harmonic (see Section 2.2). In this representation the radial momentum operator is [3,4]

$$p_r = -\frac{i\hbar}{r}\frac{d}{dr} r . \tag{4.21}$$

We use the method of shape invariance to determine the radial eigenfunctions $R_N(r)$, where N is the radial quantum number.

Using (4.21) and (4.13) we find that the solution to the first-order differential equation (4.7) with s=N is given by

$$R_0(r,\alpha_N) = C_0(\alpha_N) r^{\alpha_N} \exp\left[\frac{-r^2}{2a^2}\right] , \tag{4.22}$$

where $C_0(\alpha_N)$ is a constant to be determined so that

$$\int_0^\infty |R_0(r,\alpha_N)|^2 r^2 dr = 1 . \tag{4.23}$$

In (4.23) we change the variable to $u = r^2/a^2$ and use the integral (3.23) with $z = \alpha_N + \frac{1}{2}$. This yields

$$C_0(\alpha_N) = [\tfrac{1}{2} a^{2\alpha_N+3} \Gamma(\alpha_N + \tfrac{3}{2})]^{-\frac{1}{2}} , \tag{4.24}$$

where

$$a = \sqrt{\frac{\hbar}{M\omega}} . \tag{4.25}$$

We can determine the normalized radial eigenfunctions $R_N(r,\alpha_0)$ from equation (4.10). Setting the phase factor in (4.10) equal to unity we find that

$$R_N(r,\alpha_0) = C_N(\alpha_0) \left[\frac{r^2}{a^2}\right]^{\ell/2} \exp\left[\frac{-r^2}{2a^2}\right] {}_1F_1\left[-N; \ell+\tfrac{3}{2}; \frac{r^2}{a^2}\right] , \tag{4.26}$$

where

$$C_N(\alpha_0) = (-1)^N \left[\frac{2^{\ell-N+2}(2N+2\ell+1)!!}{a^3 \pi^{\frac{1}{2}} N! [(2\ell+1)!!]^2}\right]^{\frac{1}{2}} . \tag{4.27}$$

In equation (4.26) $n!!=n(n-2)\ldots2$(n even), $= n(n-2)\ldots1$(n odd). One can readily verify (4.26) and (4.27) by substituting these expressions in (4.9) with s=0. On simplifying the result one obtains the recurrence relation (2.88) with $B=N+\ell+\frac{5}{2}$ and $u=r^2/a^2$.

The various types of shift operations for the three-dimensional isotropic oscillator are shown in Table 4.1. The shift operators $\mathcal{R}_\ell^\pm(\omega)$ for a parameter shift both ℓ and the energy.

TABLE 4.1 Shift operations for the three-dimensional isotropic non-relativistic oscillator. The energy eigenvalues E are given by (4.19). The kets $|E\ell\rangle$ and $|n\ell\rangle$ are eigenkets of the radial Hamiltonian H_ℓ. The notation $R_N(r,\alpha_s)$ is explained in Section 4.1.

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1. Shape invariance

$$A_+(r,\alpha_s)R_N(r,\alpha_s) = (2N\hbar\omega)^{\frac{1}{2}}R_{N-1}(r,\alpha_{s+1})$$

$$A_-(r,\alpha_s)R_N(r,\alpha_{s+1}) = (2(N+1)\hbar\omega)^{\frac{1}{2}}R_{N+1}(r,\alpha_s)$$

where $\alpha_s = \ell + s$ and $A_\pm(r,\alpha_s) = \dfrac{1}{\sqrt{2M}}\left[\pm ip_r + \sqrt{\tfrac{1}{2}M}\,\omega r - \dfrac{\hbar(\alpha_s+1)}{\sqrt{2M}\,r}\right]$.

2. Shift operators for the parameter ℓ [6]

$$\mathcal{R}^{\pm}_{\ell}(\omega)\,|E\ell\rangle = [2ME + 2M\hbar\omega(\ell + \tfrac{1}{2} \pm 1)]^{\frac{1}{2}}|E\pm\hbar\omega,\ \ell\pm1\rangle$$

where $\mathcal{R}^{\pm}_{\ell}(\omega) = \pm ip_r - M\omega r - \hbar(\ell + \tfrac{1}{2} \pm \tfrac{1}{2})r^{-1}$.

3. Shift operators for energy [7]

$$Q^{\pm}_n|n\ell\rangle = [(n-\ell+1\pm1)(n+\ell+2\pm1)]^{\frac{1}{2}}|n\pm2,\ \ell\rangle$$

where $n = 2N + \ell$ and $Q^{\pm}_n = \pm \frac{i}{\hbar}\, rp_r - \dfrac{r^2}{a^2} + n + \tfrac{3}{2} \pm \tfrac{1}{2}$.

4.3 NON-RELATIVISTIC COULOMB PROBLEM IN AN ANGULAR MOMENTUM BASIS

The energy-eigenvalue equation for an attractive Coulomb potential can be written

$$\left[\frac{1}{2M}p_r^2 + V(r)\right]|E\ell\rangle = E|E\ell\rangle , \tag{4.28}$$

where

$$V(r) = \frac{\hbar^2\ell(\ell+1)}{2Mr^2} - \frac{k}{r} , \tag{4.29}$$

and k is a positive constant. Potentials of the shape (4.29) can be generated from the superpotential

$$W(r,\alpha_s) = -\frac{\hbar(\alpha_s + 1)}{\sqrt{2M}\, r} + \sqrt{\tfrac{1}{2}M}\,\frac{k}{\hbar(\alpha_s + 1)} , \tag{4.30}$$

where α_s is given by (4.14). From (4.30) and (1.36) we obtain

$$V_\pm(r,\alpha_s) = \frac{\hbar^2(\alpha_s + 1)(\alpha_s + 1 \pm 1)}{2Mr^2} - \frac{k}{r} + \frac{Mk^2}{2\hbar^2(\alpha_s + 1)^2} . \tag{4.31}$$

Thus

$$V_-(r,\alpha_0) = V(r) + \frac{\hbar^2}{2Ma^2}\frac{1}{(\alpha_0 + 1)^2} , \tag{4.32}$$

where

$$a = \hbar^2/(Mk) \tag{4.33}$$

and V(r) is given by (4.29). It is easy to show that the shape invariance condition (1.23) is satisfied with remainder

$$R(\alpha_s) = \frac{\hbar^2}{2Ma^2}\left[(\alpha_{s-1} + 1)^{-2} - (\alpha_s + 1)^{-2}\right] . \tag{4.34}$$

From (4.34), (4.14) and (1.41), and subtracting the constant in (4.32), we obtain the energy eigenvalues of the potential (4.29)

$$E = -\frac{\hbar^2}{2Ma^2}(N + \ell + 1)^{-2} \qquad (N = 0,1,\dots) . \tag{4.35}$$

The radial coordinate-space eigenfunctions are obtained in a similar manner to the eigenfunctions of the oscillator in Section 4.2. Using (4.21) and (4.30) in (4.7) with s=N we obtain the radial eigenfunction

$$R_0(r,\alpha_N) = C_0(\alpha_N)\left[\frac{2r}{(\alpha_N + 1)a}\right]^{\alpha_N}\exp\left[\frac{-r}{(\alpha_N + 1)a}\right] , \tag{4.36}$$

where

$$C_0(\alpha_N) = \left[\frac{2}{(\alpha_N + 1)a}\right]^{\frac{3}{2}} [\Gamma(2\alpha_N + 3)]^{-\frac{1}{2}} . \tag{4.37}$$

From (4.10) we obtain the normalized radial coordinate-space eigenfunctions in the form

$$R_N(r,\alpha_0) = C_N(\alpha_0)\left[\frac{2r}{(N+\ell+1)a}\right]^{\ell}\exp\left[\frac{-r}{(N+\ell+1)a}\right]{}_1F_1\left[-N; 2\ell+2; \frac{2r}{(N+\ell+1)a}\right], \tag{4.38}$$

where

$$C_N(\alpha_0) = \left[\frac{2^2\Gamma(2\ell+N+2)}{a^3 N!(N+\ell+1)^4[\Gamma(2\ell+2)]^2}\right]^{\frac{1}{2}} \tag{4.39}$$

and the phase factor in (4.10) has been set equal to unity.

One can readily prove that (4.38) and (4.39) are valid for all non-negative integers N. Using (4.38), (4.39), (4.34) and (1.41) in (4.9) with $s=0$ we find that (4.9) reduces to

$$\left[1 - \frac{N+B-1}{(B-1)(B-2)}\,u + \frac{u}{B-1}\frac{d}{du}\right]{}_1F_1(-N;B;u) = {}_1F_1(-[N+1];B-2;u) , \tag{4.40}$$

where $B = 2\ell+4$ and $u = 2r/[(N+\ell+1)a]$. The proof of (4.40) is given in Appendix B.

The various types of shift operations for the Coulomb potential are summarised in Table 4.2. Note that in Table 4.2 the energy shift operators for the Coulomb potential require the introduction of a scaling operator [7]. The effect of this scaling operator on r and p_r is given by

$$\Delta_n^{\pm}\frac{r}{n} = \frac{r}{n\pm1}\Delta_n^{\pm}$$

$$\Delta_n^{\pm} n p_r = (n\pm1)p_r\Delta_n^{\pm} . \tag{4.41}$$

In the method of shape invariance, the shift (4.41) is automatically achieved by the introduction of the parameter (4.14).

4.4 MODIFIED OSCILLATOR AND COULOMB POTENTIALS

The oscillator and Coulomb potentials of Sections 4.2 and 4.3 can be modified to include a repulsive inverse-square term λr^{-2}, where λ is a positive constant. We can write the potentials (4.12) and (4.29) including

TABLE 4.2 Shift operations for bound states of the Coulomb potential (4.29). The energy eigenvalues E are given by (4.35). The notation for $R_N(r,\alpha_s)$, $|E\ell\rangle$ and $|n\ell\rangle$ is explained in Table 4.1.

---

1. Shape invariance

$$A_+(r,\alpha_s)R_N(r,\alpha_s) = \frac{\hbar}{\sqrt{2M}\,a}\left[(\alpha_s+1)^{-2} - (\alpha_{N+s}+1)^{-2}\right]^{\frac{1}{2}} R_{N-1}(r,\alpha_{s+1})$$

$$A_-(r,\alpha_s)R_N(r,\alpha_{s+1}) = \frac{\hbar}{\sqrt{2M}\,a}\left[(\alpha_s+1)^{-2} - (\alpha_{N+s+1}+1)^{-2}\right]^{\frac{1}{2}} R_{N+1}(r,\alpha_s)$$

where a is given by (4.28), $\alpha_s = \ell+s$ and $A_\pm(r,\alpha_s) = \frac{1}{\sqrt{2M}}\left[\pm ip_r - \frac{\hbar(\alpha_s+1)}{\sqrt{2M}\,r} + \sqrt{\tfrac{1}{2}M}\,\frac{k}{\hbar(\alpha_s+1)}\right]$.

2. Shift operators for the parameter ℓ [6]

$$\mathcal{R}_\ell^\pm |E\ell\rangle = \left[2ME + \hbar^2 a^{-2}(\ell+\tfrac{1}{2}\pm\tfrac{1}{2})^{-2}\right]^{\frac{1}{2}} |E,\ \ell\pm1\rangle$$

where $\mathcal{R}_\ell^\pm = \pm ip_r - \hbar(\ell+\tfrac{1}{2}\pm\tfrac{1}{2})r^{-1} + \hbar a^{-1}(\ell+\tfrac{1}{2}\pm\tfrac{1}{2})^{-1}$.

3. Shift operators for energy [7]

$$Q_n^\pm |n\ell\rangle = \{n^{-1}(n\pm1)[n(n\pm1) - \ell(\ell+1)]\}^{\frac{1}{2}} |n\pm1,\ell\rangle$$

where $n = N+\ell+1$ and $Q_n^\pm = \Delta_n^\pm\left[\pm\frac{i}{\hbar} rp_r - \frac{r}{an} + n\right]$. Here $\Delta_n^\pm$ is defined by (4.41).

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the term λr^{-2} as follows:

$$\tilde{V}_o(r) = \frac{\hbar^2\ell(\ell+1)}{2Mr^2} + \frac{\lambda}{r^2} + \tfrac{1}{2}M\omega^2r^2 = \frac{\hbar^2\tilde{\ell}(\tilde{\ell}+1)}{2Mr^2} + \tfrac{1}{2}M\omega^2r^2 \qquad (4.42)$$

and

$$\tilde{V}_c(r) = \frac{\hbar^2\ell(\ell+1)}{2Mr^2} + \frac{\lambda}{r^2} - \frac{k}{r} = \frac{\hbar^2\tilde{\ell}(\tilde{\ell}+1)}{2Mr^2} - \frac{k}{r}, \qquad (4.43)$$

where

$$\tilde{\ell} = -\tfrac{1}{2} + [(\ell + \tfrac{1}{2})^2 + 2M\lambda\hbar^{-2}]^{\frac{1}{2}} . \qquad (4.44)$$

From (4.42) and (4.43) and Sections 4.2 and 4.3, it is clear that these potentials are shape invariant. The eigenvalues and eigenfunctions of the modified oscillator and Coulomb potentials can therefore be obtained by replacing ℓ with $\tilde{\ell}$ in (4.19), (4.26), (4.27), (4.35), (4.38) and (4.39). For example, for the modified oscillator potential (4.42), the energy eigenvalues are given by (4.19) with ℓ replaced by $\tilde{\ell}$: thus

$$E = \{2N + 1 + [(\ell + \tfrac{1}{2})^2 + 2M\lambda\hbar^{-2}]^{\frac{1}{2}}\}\hbar\omega . \qquad (4.45)$$

Similarly, from (4.35) we obtain

$$E = -\frac{\hbar^2}{2Ma^2}\{N + \tfrac{1}{2} + [(\ell + \tfrac{1}{2})^2 + 2M\lambda\hbar^{-2}]^{\frac{1}{2}}\}^{-2} \qquad (4.46)$$

for the energy eigenvalues of the modified Coulomb potential (4.43). Equations (4.45) and (4.46) show that the degeneracy of the oscillator and Coulomb problems with respect to ℓ is removed by the repulsive inverse-square term in (4.42) and (4.43).

4.5 THE ECKART POTENTIAL

We consider the Eckart potential [1]

$$V(r) = A\,\mathrm{csch}^2 r/a - B \coth r/a , \qquad (4.47)$$

where A, B and a are positive constants. This potential tends to $-B$ as $r \to \infty$, becomes infinite as $r \to 0$ and has a minimum if $B > 2A$ (see Fig. 4.1).

It is convenient to define the dimensionless operators

$$u = r/a \quad \text{and} \quad p_u = \frac{a}{\hbar}\, p_r , \qquad (4.48)$$

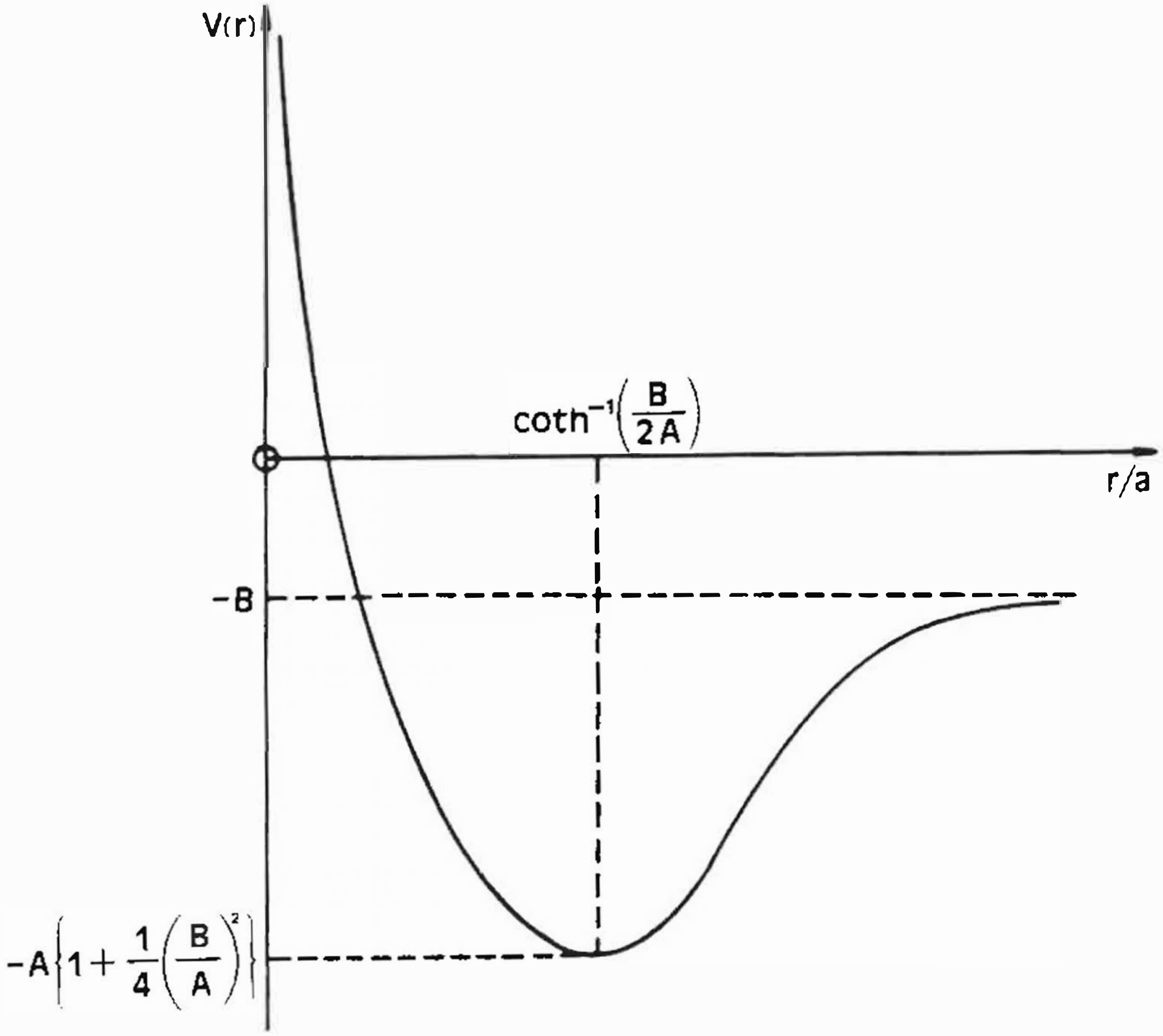

Fig. 4.1 The Eckart potential (4.47).

which satisfy the commutation relation $[p_u, u] = -i$. If we consider s states (ℓ=0), we can write the energy eigenvalue equation with the potential (4.47) in the form

$$\left[\frac{1}{2M} p_u^2 + V(u)\right] R_N(u) = \varepsilon_N R_N(u) , \tag{4.49}$$

where

$$V(u) = \frac{a^2 A}{\hbar^2} \operatorname{csch}^2 u - \frac{a^2 B}{\hbar^2} \coth u \tag{4.50}$$

and

$$\varepsilon_N = \frac{a^2 E}{\hbar^2} . \tag{4.51}$$

Here E denotes the energy eigenvalues of the potential (4.47).

Potentials of the shape (4.50) can be generated from the superpotential

$$W(u,\alpha_s) = \frac{1}{\sqrt{2M}}\left[\alpha_s \coth u - \frac{q}{\alpha_s}\right] , \tag{4.52}$$

where

$$\alpha_s = k - s \qquad (s = 0,1,2,\ldots) \tag{4.53}$$

and

$$q = Ma^2 B\hbar^{-2} . \tag{4.54}$$

Here k is a solution to $k(k + 1) = 2Ma^2 A\hbar^{-2}$. That is

$$k = -\tfrac{1}{2} \pm \tfrac{1}{2}(1 + 8Ma^2 A\hbar^{-2})^{\frac{1}{2}} . \tag{4.55}$$

Which of these two solutions must be used is discussed below (see (4.72)). Using (4.52) in (1.36) we find

$$V_\pm(u,\alpha_s) = \frac{1}{2M} \alpha_s(\alpha_s \mp 1)\operatorname{csch}^2 u - \frac{q}{M} \coth u + \kappa(\alpha_s) , \tag{4.56}$$

where

$$\kappa(\alpha_s) = \frac{1}{2M}(\alpha_s^2 + q^2\alpha_s^{-2}) . \tag{4.57}$$

Hence

$$V_-(u,\alpha_0) = V(u) + \kappa(\alpha_0) , \tag{4.58}$$

where V(u) is the potential (4.47). Also, the shape invariance condition (1.23) is satisfied with remainder

$$R(\alpha_s) = \frac{1}{2M}[(\alpha_{s-1}^2 - \alpha_s^2) + q^2(\alpha_{s-1}^{-2} - \alpha_s^{-2})] \ . \tag{4.59}$$

Substituting (4.59) in (1.41) we obtain the energy eigenvalues corresponding to the potential (4.50)

$$E_N^{(0)} = \kappa(\alpha_0) - \frac{1}{2M}(\alpha_N^2 + q^2\alpha_N^{-2}) \ , \tag{4.60}$$

where $N = 1, 2, \ldots, N_{max}$ and $E_0^{(0)} = 0$. The restriction on N is discussed in Appendix D, where it is shown that

$$N_{max} < k + q^{\frac{1}{2}} \ . \tag{4.61}$$

Using (4.53) and (4.54) in (4.60) and subtracting the constant $\kappa(\alpha_0)$, we find the energy eigenvalues of the Eckart potential (4.47) are given by

$$E = -\frac{\hbar^2}{2Ma^2}(N-k)^2 - \frac{Ma^2B^2}{\hbar^2}\frac{1}{(N-k)^2} \ , \tag{4.62}$$

where $N = 0, 1, 2, \ldots, N_{max}$.

The normalized solution to (4.7) with s=N is

$$R_0(u, \alpha_N) = C_0(\alpha_N)u^{-1}(\sinh u)^{-\alpha_N} \exp(qu/\alpha_N) \ , \tag{4.63}$$

where

$$C_0(\alpha_N) = \left[\frac{2^{1-2\alpha_N}\Gamma(2\lambda - 2\alpha_N + 1)}{a^3\Gamma(2\lambda)\Gamma(1 - 2\alpha_N)}\right]^{\frac{1}{2}} \tag{4.64}$$

and

$$\lambda = -\frac{q - \alpha_N^2}{2\alpha_N} \ . \tag{4.65}$$

It can be shown that the eigenfunctions generated from (4.10) are given by

$$R_N(u, \alpha_0) = C_N(\alpha_0)u^{-1}(\sinh u)^{-k+N}\exp\left[\frac{qu}{k-N}\right]{}_2F_1\left[-N, 2k-N+1; 2\lambda+1; \frac{e^{-u}}{e^{-u} - e^{u}}\right] \ , \tag{4.66}$$

where

$$C_N(\alpha_0) = \left[\frac{2^{2N-2k+1}(\lambda+N-k)\Gamma(2\lambda+N-2k)\Gamma(2\lambda+N+1)}{a^3 N!(N-k)\Gamma(N-2k)2\lambda[\Gamma(2\lambda)]^2}\right]^{\frac{1}{2}} \tag{4.67}$$

and we have set a phase factor equal to unity. In (4.66) and (4.67) it is understood that $\alpha_0 = k$ and $\lambda = -(q-\alpha_0^2)/2\alpha_0$. The validity of (4.66) and (4.67) can be proved for all non-negative integers N by substituting these equations in (4.9) with s=0. This yields the recurrence relation (3.59) with

$$B = 2k-N-1, \quad C = 2\lambda+1 \quad \text{and} \quad Y = \frac{e^{-u}}{e^{-u}-e^{u}}. \tag{4.68}$$

Consider now the behaviour of (4.66) as $u \to 0$. The polynomial ${}_2F_1$ in (4.66) is proportional to u^{-N} for small u, so that $R_N \sim u^{-k-1}$. Thus corresponding to the negative and positive solutions, respectively, of (4.55) we have

$$R_N \sim u^{-\frac{1}{2}+\frac{1}{2}(1 + 8Ma^2A\hbar^{-2})^{\frac{1}{2}}} \tag{4.69}$$

and

$$R_N \sim u^{-\frac{1}{2}-\frac{1}{2}(1 + 8Ma^2A\hbar^{-2})^{\frac{1}{2}}}. \tag{4.70}$$

Now for R_N to be a solution to the Schrödinger equation in Cartesian coordinates, one must have [4,8]

$$uR_N(u) \to 0 \quad \text{as} \quad u \to 0. \tag{4.71}$$

The form (4.69) satisfies (4.71) whereas (4.70) does not. Hence we must take

$$k = -\tfrac{1}{2} - \tfrac{1}{2}(1 + 8Ma^2A\hbar^{-2})^{\frac{1}{2}}. \tag{4.72}$$

This is the expression for k which must be used in (4.62), (4.66) and (4.67).

In Table 4.3 we compare the operators $A_{\pm}(u,\kappa_s)$ with the operators $\mathcal{R}_k^{\pm}$ obtained from the factorization method of Infeld and Hull [9] (see Appendix D). In this Table we have not given any shift operators for energy because, as far as we know, it is not possible to construct such operators for the Eckart potential.

4.6 HULTHÉN POTENTIAL [10]

The Hulthén potential is given by [10]

$$V(r) = -U_0 \frac{1}{e^{r/d}-1}, \tag{4.73}$$

where U_0 and d are positive constants. The potential (4.73) tends to zero

TABLE 4.3 Shift operations for the Eckart potential (4.50). The energy eigenvalues E are given by (4.62), q is given by (4.54) and k is given by (4.72). The kets $|Ek\rangle$ are eigenkets of the radial Hamiltonian with the Eckart potential.

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1. Shape invariance

$$A_+(u,\alpha_s)R_N(u,\alpha_s) = \frac{1}{\sqrt{2M}}\,[(\alpha_s^2 + q^2\alpha_s^{-2}) - (\alpha_{N+s}^2 + q\alpha_{N+s}^{-2})]^{\frac{1}{2}}R_{N-1}(u,\alpha_{s+1})$$

$$A_-(u,\alpha_s)R_N(u,\alpha_{s+1}) = \frac{1}{\sqrt{2M}}\,[(\alpha_s^2 + q^2\alpha_s^{-2}) - (\alpha_{N+s+1}^2 + q\alpha_{N+s+1}^{-2})]^{\frac{1}{2}}R_{N+1}(u,\alpha_s)$$

where $\alpha_s = k - s$ and $A_\pm(u,\alpha_s) = \frac{1}{\sqrt{2M}}(\pm ip_u + \alpha_s \coth u - q\alpha_s^{-1})$.

2. Shift operators for a parameter

$$\mathcal{R}_k^\pm|Ek\rangle = [2MEa^2\hbar^{-2} + q^2(k + \tfrac{1}{2} \pm \tfrac{1}{2})^{-2} + (k + \tfrac{1}{2} \pm \tfrac{1}{2})^2]^{\frac{1}{2}}|E,k\pm 1\rangle$$

where $\mathcal{R}_k^\pm = \pm\, ip_u - (k + \tfrac{1}{2} \pm \tfrac{1}{2})\coth u + q(k + \tfrac{1}{2} \pm \tfrac{1}{2})^{-1}$.

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as $r \to \infty$, and to $-\infty$ as $r \to 0$. This is shown in Fig. 4.2. The Hulthén potential (4.73) can be obtained from the Eckart potential (4.47) by putting

$$A = 0, \quad B = \tfrac{1}{2}U_0 \quad \text{and} \quad a = 2d . \tag{4.74}$$

Using (4.74) in (4.72) yields $k = -1$*. Substituting (4.74) in (4.47) we find

$$\tilde{V}(r) = -U_0\left(\frac{1}{e^{r/d} - 1} + \tfrac{1}{2}\right) . \tag{4.75}$$

Comparing (4.73) and (4.75) it is clear that the energy eigenvalues corresponding to the potential (4.73) can be obtained by adding the constant $\frac{1}{2}U_0$ to the energies in (4.62). Using (4.74) in (4.62), setting $k = -1$ and adding the constant $\frac{1}{2}U_0$ we find

$$E = -\frac{\hbar^2}{8Md^2}\left(\frac{\nu^2 - (N+1)^2}{N+1}\right)^2 \qquad (N = 0, 1, \ldots, N_{max}) \tag{4.76}$$

where $\nu^2 = 2Md^2U_0\hbar^{-2}$ and $N_{max} < \nu - 1$ (see (4.61)).

Using (4.74) and $k = -1$ in (4.66) and (4.67) we obtain the normalized coordinate-space eigenfunctions of the Hulthén potential

$$R_N(u, \alpha_0) = C_N(\alpha_0)u^{-1}(\sinh u)^{N+1}\exp\left(\frac{-\nu^2}{N+1}\, u\right){}_2F_1\left(-N, 2k-N+1; \frac{e^{-u}}{e^{-u} - e^{u}}\right) \tag{4.77}$$

$$= 2^{-N}C_N(\alpha_0)(r/d)^{-1}(1-e^{-r/d})e^{-\lambda r/d}{}_2F_1(-N, N+2\lambda+2;\ 2\lambda+1;\ e^{-r/d}) , \tag{4.78}$$

where

$$2^{-N}C_N(\alpha_0) = d^{-3/2}(-1)^N\frac{\Gamma(2\lambda+N+1)}{(N+1)!\,\Gamma(2\lambda+1)}[(\lambda+N+1)(2\lambda+N+1)2\lambda]^{\frac{1}{2}} . \tag{4.79}$$

To obtain (4.78) from (4.77) we have used the relation [11]

$${}_2F_1(-N, B;\ C;\ z) = (1-z)^N{}_2F_1\left(-N, C-B;\ C;\ \frac{z}{z-1}\right) \tag{4.80}$$

with $z = e^{-2u}$, $B = 2\lambda+N-2k$ and $C = 2\lambda+1$.

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* It is interesting to note that the positive solution in (4.55) gives k=0 when A=0; with this value of k the superpotential (4.52) is undefined for s=0.

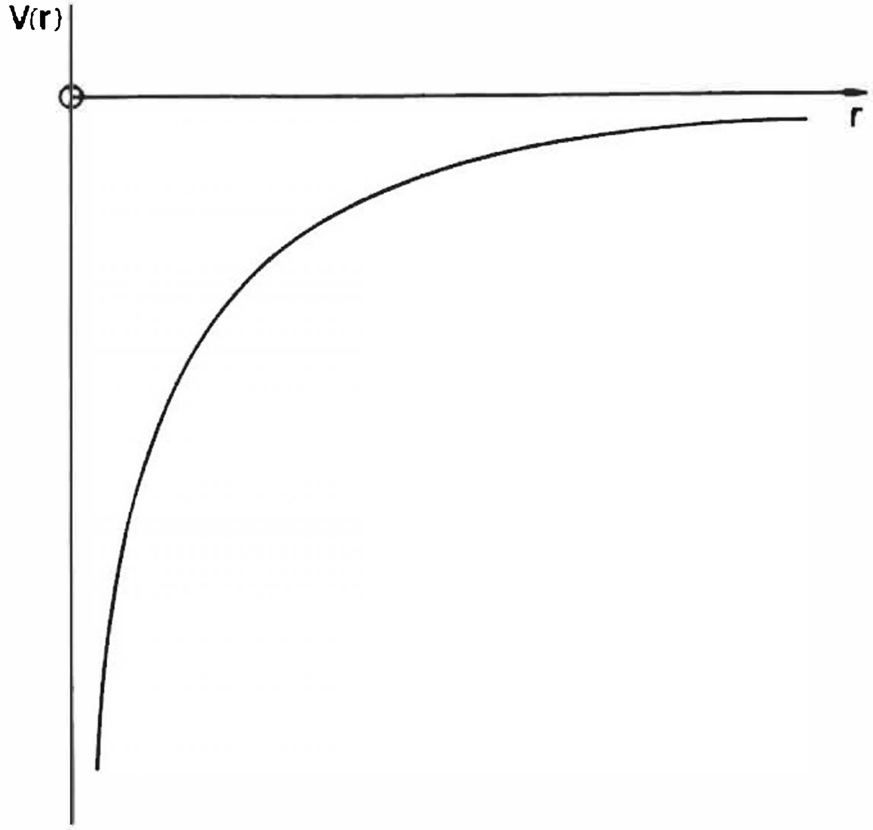

Fig. 4.2 The Hulthén potential (4.73).

In Table 4.4 we compare the various shift operations for the Hulthén potential. The operators $Q_n^{\pm}$ in Table 4.4 are not shift operators for energy, they shift between eigenkets of the same λ belonging to different potentials (different ν) [12]. They factorize an operator related to the Hamiltonian with the Hulthén potential. The operators $A_{\pm}(r,\alpha_s)$ in Table 4.4 are obtained by substituting (4.74) and k = -1 in the corresponding expressions given in Table 4.3 for the Eckart potential. We do not present the operators $\mathcal{R}_k^{\pm}$ obtained using the Infeld-Hull factorization method for the Hamiltonian with the potential (4.73), since the operators $\mathcal{R}_k^{+}$ and $\mathcal{R}_{k+1}^{-}$ are undefined when k = -1 (see the operators $\mathcal{R}_k^{\pm}$ in Table 4.3).

4.7 OTHER SPHERICALLY SYMMETRIC POTENTIALS

For certain one-dimensional problems we can extend the results obtained for the energy eigenvalues and coordinate-space eigenfunctions to give the eigenvalues and eigenfunctions of the corresponding three-dimensional problem. In this Section we show how this can be achieved for the Morse and the symmetric Rosen-Morse potentials.

4.7.1 Three-dimensional Morse potential

Here we consider the s states of the radial Hamiltonian (4.6) with the potential

$$V(r) = D\left[\exp\left(-2\,\frac{r-r_0}{a}\right) - 2\exp\left(-\,\frac{r-r_0}{a}\right)\right] . \tag{4.81}$$

If we let $x = r - r_0$ in (3.1) we obtain the Morse potential (4.81). Here we assume the limit

$$r_0/a \rightarrow \infty , \tag{4.82}$$

TABLE 4.4 Shift operations for the Hulthén potential (4.73). The energy eigenvalues E are given by (4.76) and $\nu = (2Md^2U_o\hbar^{-2})^{\frac{1}{2}}$.

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1. *Shape invariance*

$$A_+(r,\alpha_s)R_N(r,\alpha_s) = \frac{1}{\sqrt{2M}}\left[\left[\frac{\nu^2-\alpha_s^2}{2\alpha_s}\right]^2 - \left[\frac{\nu^2-\alpha_{N+s}^2}{2\alpha_{N+s}}\right]^2\right]^{\frac{1}{2}} R_{N-1}(r,\alpha_{s+1})$$

$$A_-(r,\alpha_s)R_N(r,\alpha_{s+1}) = \frac{1}{\sqrt{2M}}\left[\left[\frac{\nu^2-\alpha_s^2}{2\alpha_s}\right]^2 - \left[\frac{\nu^2-\alpha_{N+s+1}^2}{2\alpha_{N+s+1}}\right]^2\right]^{\frac{1}{2}} R_{N+1}(r,\alpha_s) \,,$$

where $\alpha_s = -(s+1)$ and $A_\pm(r,\alpha_s) = \frac{1}{\sqrt{2M}}\left\{\pm i\,\frac{d}{\hbar}\,p_r + \frac{\alpha_s}{e^{r/d}-1} + \frac{\alpha_s^2-\nu^2}{2\alpha_s}\right\}$.

2. *Shift operators for an operator related to the Hamiltonian with a Hulthén potential* [12]

$$Q_n^{\pm}|n\lambda\rangle = -(n-\lambda\pm1)(2n\pm1)^{-1}[n(n+\lambda)(n+\lambda+1)(n\pm1)^{-1}]^{\frac{1}{2}}|n\pm1,\lambda\rangle \,,$$

where $n = N+\lambda+1$, $\lambda = (-2Md^2\hbar^{-2}E)^{\frac{1}{2}}$ and

$$Q_n^{\pm} = \pm i(1-e^{-r/d})\frac{d}{\hbar}\,p_r + ne^{-r/d} - (2n\pm1)^{-1}[(n-\tfrac{1}{2}\pm\tfrac{1}{2})^2 + (n-\tfrac{1}{2}\pm\tfrac{1}{2}) + \lambda^2] \,.$$

---

which ensures that p_r is Hermitian with respect to the bound states.* In the analysis of Section 3.1 we must therefore make the substitutions

$$X \rightarrow u = (r - r_0)/a - \ln(2k + 1) \tag{4.83}$$

and

$$p_X \rightarrow p_u = \frac{a}{\hbar} p_r \tag{4.84}$$

(see (3.3) and (3.4)). For bound s states the energy eigenvalues are given by (3.18), and from (4.21) and (3.28) we obtain the eigenfunctions of the potential (4.81)

$$R_N(r,\alpha_0) = \frac{1}{r} \psi_N(r,\alpha_0) \ , \tag{4.85}$$

where $\psi_N(r,\alpha_0)$ is given by (3.28) with X replaced by u (see (4.83)). From (4.82) and (4.23) it is apparent that the normalization constants in the functions $\psi_N(r,\alpha_0)$ above are the same as (3.29).

4.7.2 Three-dimensional symmetric Rosen-Morse potential

If we make the substitutions $U_1 = 0$ and $x = r$ in (3.34) we obtain

$$V(r) = -U_0 \operatorname{sech}^2 r/a \ . \tag{4.86}$$

We can extend the analysis for the one-dimensional potential (3.34) with $U_1 = 0$ (Section 3.3) to the three-dimensional case by making the substitutions

$$X \rightarrow u = r/a \tag{4.87}$$

and (4.84).

For bound states of the potential (4.86) we must have $\psi_N(0,\alpha_0) = 0$, where $\psi_N(0,\alpha_0)$ is given by (3.68) or (3.70), and (4.87). From (3.68) we see that the even functions violate the above condition, so that they must be excluded. Hence the coordinate-space eigenfunctions for the bound s

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* The assumption that p_r is Hermitian with respect to the bound states requires that $\psi_N(0) = 0$ in (4.85) [12].

states of the potential (4.86) are given by (4.85) and (3.70) with the substitution (4.87), where we must include a factor $2^{\frac{1}{2}}$ in (3.71) because $0 \le r \le \infty$ whereas $-\infty \le x \le \infty$.

The energy eigenvalues are given by (3.63) where $N = 1,3,\ldots,N_{max}$ and $N_{max} < k$ (see (3.50) with $U_1 = 0$). Here k is given by the positive solution in (3.44).

4.7.3 **Manning-Rosen potential [13]**

Here

$$V(r) = \frac{\hbar^2}{2Ma^2}\left[\frac{\beta(\beta - 1)e^{-2r/a}}{(1 - e^{-r/a})^2} - \frac{\alpha e^{-r/a}}{1 - e^{-r/a}}\right], \tag{4.88}$$

where $\beta(\beta - 1)$ and α are positive constants. We can obtain the potential (4.88) from the Eckart potential (4.47). If we let

$$r \to \tfrac{1}{2}r\ , \tag{4.89}$$

$$A = \frac{\hbar^2}{2Ma^2}\,\frac{\beta(\beta - 1)}{4} \tag{4.90}$$

and

$$B = \frac{\hbar^2}{2Ma^2}\,\tfrac{1}{2}[\alpha + \beta(\beta - 1)] \tag{4.91}$$

in (4.47) we obtain the potential (4.88) minus the constant (4.91). From (4.90) and (4.72) we find

$$k = \tfrac{1}{2}(\beta - 1)\ , \tag{4.92}$$

and from (4.91) and (4.54) we obtain

$$q = \tfrac{1}{4}[\alpha + \beta(\beta - 1)]\ . \tag{4.93}$$

Substituting (4.92) and (4.91) in (4.62) and adding the constant (4.91) we obtain the energy eigenvalues corresponding to the potential (4.88) in the form

$$E = -\frac{\hbar^2}{2Ma^2}\left[\frac{A-\beta-\nu(\nu+2\beta)}{2(\beta+\nu)}\right]^2 , \tag{4.94}$$

where

$$\nu = 2N - 2\beta + 1 \qquad (N = 0,1,\ldots,N_{max}) \tag{4.95}$$

and

$$N_{max} < \tfrac{1}{2}(\beta - 1) + \tfrac{1}{2}[\alpha + \beta(\beta - 1)]^{\frac{1}{2}} ,$$

see (4.61). The coordinate-space eigenfunctions for the potential (4.88) can be obtained by substituting (4.89)-(4.93) in (4.66) and (4.67).

CHAPTER 5

SHAPE INVARIANCE WITH APPLICATION TO THE MOMENTUM REPRESENTATION [1]

In Section 5.1 we formulate the method of shape invariance in a manner that is suitable for application to the momentum representation. Then two examples are discussed: the isotropic non-relativistic oscillator (Section 5.2) and the non-relativistic Coulomb problem (Section 5.3). For these problems we use the formulation of Section 5.1 to determine the energy eigenvalues and normalized momentum-space eigenfunctions for bound states. The content of this Chapter is based on the paper: de Lange, O.L. and Welter, A. (1992). Shape invariance with Application to the Momentum Representation. *Journal of Physics A: Mathematical and General*, 25, 5753-60. A relativistic (Dirac) oscillator and a relativistic Coulomb problem are discussed in Chapter 6.

5.1 SHAPE INVARIANCE IN THE MOMENTUM REPRESENTATION

In this Section we present the method of shape invariance in a form that is suitable for application to the momentum representation of spherically symmetric potentials in an angular momentum basis. The formulation is similar to that in Chapter 4 but with some differences.

Instead of r and p_r we introduce the operators $p = (\underset{\sim}{p}\cdot\underset{\sim}{p})^{\frac{1}{2}}$ and

$$r_p = \tfrac{1}{2}(\hat{\underset{\sim}{p}}\cdot\underset{\sim}{r} + \underset{\sim}{r}\cdot\hat{\underset{\sim}{p}}) = \frac{1}{p}\,(\underset{\sim}{p}\cdot\underset{\sim}{r} + i\hbar)\ , \tag{5.1}$$

which satisfy the commutation relation [2]

$$[r_p, f(p)] = i\hbar\,\frac{df(p)}{dp}\ . \tag{5.2}$$

For the problems considered below, we make use of the identity [3]

$$\underset{\sim}{r}^2 = r_p^2 + p^{-2}\underset{\sim}{L}^2\ , \tag{5.3}$$

to introduce the radial operator (5.1).

Next we consider the operators

$$B_{\pm}(p,\beta_s) = \pm i r_p f(p) + \bar{W}(p,\beta_s) \ , \qquad (5.4)$$

where β_s $(s = 0,1,\ldots)$ is a set of parameters and $\bar{W}(p,\beta_s)$ and $f(p)$ are determined below (Sections 5.2 and 5.3). (The ordering of r_p and $f(p)$ adopted in (5.4) is for later convenience.) We then construct operators which are second order in r_p, namely

$$\Lambda_{\pm}^{(s)} = [r_p f(p)]^2 + \bar{V}_{\pm}(p,\beta_s) + \sum_{k=1}^{s} \bar{R}(\beta_k) \qquad (s = 1,2, \ldots) \ , \qquad (5.5)$$

where

$$\Lambda_{\pm}^{(0)} = B_{\pm}(p,\beta_0) B_{\mp}(p,\beta_0) = [r_p f(p)]^2 + \bar{V}_{\pm}(p,\beta_0) \qquad (5.6)$$

and

$$\bar{V}_{\pm}(p,\beta_s) = \bar{W}^2(p,\beta_s) \mp \hbar f(p) \frac{d\bar{W}(p,\beta_s)}{dp} \ . \qquad (5.7)$$

The operators $\Lambda_{\pm}^{(s)}$ play the role of $H_{\pm}^{(s)}$ in Chapter 1. The partner "potentials" $\bar{V}_{\pm}(p,\beta_s)$ are shape invariant if

$$\bar{V}_{+}(p,\beta_s) = \bar{V}_{-}(p,\beta_{s+1}) + \bar{R}(\beta_{s+1}) \ , \qquad (5.8)$$

where $\bar{R}(\beta_s)$ is independent of p and $\beta_{s+1} = g(\beta_s)$.

Suppose $\Lambda_{\pm}^{(s)}$ possess discrete spectra and consider the eigenvalue equations

$$\Lambda_{\pm}^{(s)} \bar{R}_N(p,\beta_s) = \lambda_N^{(s)} \bar{R}_N(p,\beta_s) \qquad (N = 0,1, \ldots) \ , \qquad (5.9)$$

where $\bar{R}_N(p,\beta_s)$ is a normalized radial momentum-space eigenfunction. By analogy with (1.38) we suppose that

$$B_{+}(p,\beta_s) \bar{R}_0(p,\beta_s) = 0 \ , \qquad (5.10)$$

then

$$\lambda_0^{(0)} = 0 \qquad (5.11)$$

(see (1.39)).

If the shape invariance condition (5.8) is satisfied then the eigenvalues and momentum-space eigenfunctions of $\Lambda_{-}^{(0)}$ are given by formulae which are similar to (1.41) and (1.44)-(1.47):

$$\lambda_N^{(0)} = \sum_{k=1}^{N} R(\beta_k) \ , \tag{5.12}$$

$$B_+(p,\beta_s)\bar{R}_N(p,\beta_s) = e^{-i\phi}(\lambda_N^{(s)} - \lambda_s^{(0)})^{\frac{1}{2}}\bar{R}_{N-1}(p,\beta_{s+1}) \ , \tag{5.13}$$

$$B_-(p,\beta_s)\bar{R}_N(p,\beta_{s+1}) = e^{i\phi}(\lambda_{N+1}^{(s)} - \lambda_s^{(0)})^{\frac{1}{2}}\bar{R}_{N+1}(p,\beta_s) \ , \tag{5.14}$$

$$\bar{R}_N(p,\beta_0) = e^{-iN\phi}\,\bar{\gamma}_N B_-(p,\beta_0)B_-(p,\beta_1)\ldots B_-(p,\beta_{N-1})\bar{R}_0(p,\beta_N) \ , \tag{5.15}$$

$$\bar{\gamma}_N = \prod_{k=1}^{N} (\lambda_N^{(0)} - \lambda_{k-1}^{(0)})^{-\frac{1}{2}} \ , \tag{5.16}$$

for $N = 1, 2, \ldots$. The phase ϕ in (5.13)-(5.15) is a real constant.

For spherically symmetric problems in an angular momentum basis, the momentum-space eigenfunctions can be obtained by the separation of variables [4]

$$\bar{\psi}_{E\ell m}(\underset{\sim}{p}) = \bar{R}_{E\ell}(p)Y_{\ell m}(\hat{\underset{\sim}{p}}) \ ,$$

where $\bar{R}_{E\ell}(p) = \langle p|E\ell\rangle$ is the radial momentum-space eigenfunction and $Y_{\ell m}(\hat{\underset{\sim}{p}}) = \langle\hat{\underset{\sim}{p}}|\ell m\rangle$ is a spherical harmonic (see Section 2.2). (The kets $|E\ell\rangle$ are eigenkets of an appropriate radial operator - see below.) A similar calculation to that presented in Section 2.2 yields the spherical harmonics $Y_{\ell m}(\hat{\underset{\sim}{p}})$, which can be obtained by replacing $\hat{\underset{\sim}{r}}$ by $\hat{\underset{\sim}{p}}$: that is, the angular coordinates θ and ϕ of $\hat{\underset{\sim}{r}}$ in (2.50) are replaced by the angular coordinates θ_p and ϕ_p of $\hat{\underset{\sim}{p}}$. In the following two Sections we denote the radial momentum-space eigenfunctions by $\bar{R}_N(p,\beta_0)$, where N is the radial quantum number.

In Sections 5.2 and 5.3 we give two examples to which the results (5.12)-(5.16) can be applied, namely the isotropic non-relativistic oscillator and the Coulomb problem in an angular momentum basis. Two further applications, to a relativistic oscillator and a relativistic Coulomb problem are discussed in Sections 6.4 and 6.5.

5.2 OSCILLATOR IN AN ANGULAR MOMENTUM BASIS

We start with a radial Hamiltonian for the isotropic non-relativistic oscillator expressed in a form which is quadratic in r_p [2]. Using (5.3) it is easy to show that

$$H_\ell = \tfrac{1}{2}M\omega^2 r_p^2 + \bar{V}(p) \tag{5.17}$$

where

$$\bar{V}(p) = \tfrac{1}{2}M\omega^2\hbar^2\ell(\ell + 1)\frac{1}{p^2} + \frac{1}{2M}p^2 . \tag{5.18}$$

Comparing (5.17) and (5.6), we take $f = \sqrt{\tfrac{1}{2}M}\,\omega$. "Potentials" of the shape (5.18) can be generated from a "superpotential"

$$\bar{W}(p,\beta_s) = \sqrt{\tfrac{1}{2}M}\,\hbar\omega(\beta_s + 1)\frac{1}{p} - \frac{1}{\sqrt{2M}}\,p . \tag{5.19}$$

Substituting (5.19) in (5.7) we find

$$\bar{V}_\pm(p,\beta_s) = \tfrac{1}{2}m\omega^2\hbar^2(\alpha_s + 1)(\alpha_s + 1 \pm 1)\frac{1}{p^2} + \frac{1}{2m}p^2 - \hbar\omega(\alpha_s + 1 \mp \tfrac{1}{2}) . \tag{5.20}$$

If $\beta_0 = \ell$, then

$$\bar{V}_-(p,\beta_0) = \bar{V}(p) - \hbar\omega(\ell + \tfrac{3}{2}) , \tag{5.21}$$

where $\bar{V}(p)$ is the potential (5.18). Also, if

$$\beta_s = \ell + s \tag{5.22}$$

then the potentials (5.20) satisfy the shape invariance condition (5.8) with constant remainder

$$\bar{R} = 2\hbar\omega . \tag{5.23}$$

In (5.6) let $\bar{V}_-(p,\beta_0)$ be given by (5.21): comparing the result with (5.17) we see that

$$A_-^{(0)} = H_\ell - \hbar\omega(\ell + \tfrac{3}{2}) . \tag{5.24}$$

From (5.12), (5.23) and (5.11) we have $\lambda_N^{(0)} = 2N\hbar\omega$, and hence for the eigenvalues of H_ℓ we obtain the familiar result $E = (2N + \ell + \tfrac{3}{2})\hbar\omega$, where $N = 0, 1, \ldots$ is the radial quantum number.

With $\bar{W}(p,\beta_N)$ given by (5.19) with s=N and

$$r_p = \frac{i\hbar}{p}\frac{d}{dp}\,p , \tag{5.25}$$

the normalized solution to (5.10) with s=N is

$$\bar{R}_0(p,\beta_N) = \left[\frac{2^{\beta_N+2}}{p_0^3\pi^{\frac{1}{2}}(2\beta_N+1)!!}\right]^{\frac{1}{2}}\left[\frac{p}{p_0}\right]^{\beta_N}\exp\left[-\frac{p^2}{2p_0^2}\right] . \tag{5.26}$$

Here $p_0 = \sqrt{M\hbar\omega}$ and $n!! = n(n-2)\ldots 2(n \text{ even}), = n(n-2)\ldots 1(n \text{ odd})$. The normalized radial momentum-space eigenfunctions for N = 1,2, ... can be obtained by substituting (5.26) in (5.15). The first few of these are found to be particular cases of

$$\bar{R}_N(p,\beta_0) = e^{-iN\phi}C_N(\alpha_0)\left[\frac{p}{p_0}\right]^{\ell}\exp\left[-\frac{p^2}{2p_0^2}\right]{}_1F_1\left[-N;\ \ell+\tfrac{3}{2};\ \frac{p^2}{p_0^2}\right] , \tag{5.27}$$

where

$$C_N(\alpha_0) = \left[\frac{2^{\ell+2-N}(2\ell+2N+1)!!}{p_0^3\pi^{\frac{1}{2}}N![(2\ell+1)!!]^2}\right]^{\frac{1}{2}} \tag{5.28}$$

and the confluent hypergeometric function ${}_1F_1$ is defined by (2.87). The identifications (5.27) and (5.28) can be extended to all non-negative integers N: we use (5.27) in (5.14) with s=0 to obtain

$$\left[1 - \frac{u}{\ell+\frac{3}{2}} + \frac{u}{\ell+\frac{3}{2}}\frac{d}{du}\right]{}_1F_1(-N;\ \ell+\tfrac{5}{2};\ u) = {}_1F_1(-[N+1];\ \ell+\tfrac{3}{2};\ u) \tag{5.29}$$

$(u = p^2/p_0^2)$, which is the recurrence relation (2.88), with $B = \ell+\frac{5}{2}$.

The radial momentum space eigenfunctions (5.27) can also be obtained from the results in Section 4.2. Comparing the radial Hamiltonian (4.6) with the potential $\frac{1}{2}M\omega r^2$, with (5.17) and (5.18), we see that these transform into each other under the substitutions

$$r \leftrightarrow \frac{1}{M\omega}\, p \tag{5.30}$$

and

$$p_r \leftrightarrow -M\omega r_p \ . \tag{5.31}$$

From (5.30), (5.31), (4.21) and $p_0 = \sqrt{M\hbar\omega}$, it is clear that $r/a \leftrightarrow p/p_0$. Thus (5.27) and (5.28) can be obtained by replacing r and a by p and p_0 respectively in (4.26) and (4.27).

The various types of shift operations for the isotropic non-relativistic oscillator in the momentum representation are given in Table 5.1.

TABLE 5.1 Shift operations for the three-dimensional isotropic non-relativistic oscillator in the momentum representation. The energy eigenvalues E are given by $E = (2N+\ell+\frac{3}{2})\hbar\omega$. The notation $\bar{R}_N(p,\beta_s)$ and $|E\ell\rangle$ is explained in Section 5.1. The kets $|n\ell\rangle$ and $|E\ell\rangle$ denote the same ket.

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1. Shape invariance

$$B_+(p,\beta_s)\bar{R}_N(p,\beta_s) = (2N\hbar\omega)^{\frac{1}{2}}\,\bar{R}_{N-1}(p,\beta_{s+1})$$

$$B_-(p,\beta_s)\bar{R}_N(p,\beta_{s+1}) = [2(N+1)\hbar\omega]^{\frac{1}{2}}\,\bar{R}_{N+1}(p,\beta_s)$$

$$\text{where } \beta_s = \ell+s \text{ and } B_\pm(p,\beta_s) = \sqrt{\tfrac{1}{2}M}\,\omega\left[\pm i r_p - \frac{p}{M\omega} + \hbar(\beta_s + 1)\frac{1}{p}\right] .$$

2. Shift operators for the parameter ℓ [4]

$$P_\ell^\pm(\omega)\,|E\ell\rangle = \pm\,\frac{i}{\omega}\left(\frac{2}{M}[E + \hbar\omega(\ell + \tfrac{1}{2} \pm \tfrac{1}{2})]\right)^{\frac{1}{2}}|E\pm\hbar\omega,\ \ell\pm1\rangle$$

$$\text{where } P_\ell^\pm(\omega) = \mp\, i r_p - \frac{p}{M\omega} - \hbar(\ell + \tfrac{1}{2} \pm \tfrac{1}{2})\frac{1}{p} .$$

3. Shift operators for energy [4]

$$\bar{Q}_n^\pm|n\ell\rangle = [(n-\ell+1\pm1)(n+\ell+2\pm1)]^{\frac{1}{2}}|n\pm2,\ \ell\rangle$$

$$\text{where } n = 2N + \ell \text{ and } \bar{Q}_n^\pm = \mp\,\frac{i}{\hbar}\,p r_p - \frac{p^2}{M\hbar\omega} + n + \tfrac{3}{2} \pm \tfrac{1}{2} .$$

5.3 COULOMB PROBLEM IN AN ANGULAR MOMENTUM BASIS

Our starting point is the radial Hylleraas equation for the Coulomb problem [5,6]

$$\left\{\left[\frac{1}{\hbar} r_p(p^2 - 2ME)\right]^2 + \frac{\ell(\ell+1)(p^2 - 2ME)^2}{p^2}\right\}|E\ell\rangle = 4p_0^2|E\ell\rangle \, , \tag{5.32}$$

where $p_0 = Mk/\hbar$ and k is the constant in (4.29). We consider bound states ($E<0$) and define

$$u = p/\sqrt{-2ME} \, , \qquad r_u = \sqrt{-2ME}\, r_p \, . \tag{5.33}$$

In terms of these (5.32) can be written

$$\bar{H}_\ell|E\ell\rangle = \varepsilon|E\ell\rangle \, , \tag{5.34}$$

where

$$\bar{H}_\ell = \left[\frac{1}{\hbar} r_u(u^2 + 1)\right]^2 + \bar{V}(u) \, , \tag{5.35}$$

$$\bar{V}(u) = \ell(\ell + 1)(u^2 + u^{-2}) \tag{5.36}$$

and

$$\varepsilon = -\frac{2p_0^2}{ME} - 2\ell(\ell + 1) \, . \tag{5.37}$$

In applying the formulae in Section 5.1 we replace p and r_p with u and r_u. Comparing (5.35) and (5.6) we take $f(u) = (u^2 + 1)/h$. With

$$\bar{W}(u,\beta_s) = -(\beta_s + 1)(u - u^{-1}) \tag{5.38}$$

and

$$\beta_s = \ell + s \tag{5.39}$$

in (5.7) we find that

$$\bar{V}_\pm(u,\beta_s) = (\beta_s + 1)(\beta_s + 1 \pm 1)(u^2 + u^{-2}) - 2(\beta_s + 1)(\beta_s + 1 \mp 1) \, . \tag{5.40}$$

For s=0, (5.40) yields

$$\bar{V}_-(u,\beta_0) = \bar{V}(u) - 2(\ell + 1)(\ell + 2) \, , \tag{5.41}$$

where $\bar{V}(u)$ is given by (5.36). One can readily show that the potentials (5.40) satisfy the shape invariance condition (5.8) with remainder

$$\bar{R}(\beta_s) = 8\ell + 8s + 4 \, . \tag{5.42}$$

From (5.12), (5.39) and (5.42) we obtain

$$\lambda_N^{(0)} = 4N^2 + 8\ell N + 8N \ . \tag{5.43}$$

In (5.6) let $\bar{V}_-(u,\beta_0)$ be given by (5.41). Then

$$\Lambda_- = \bar{H}_\ell - 2(\ell + 1)(\ell + 2) \ . \tag{5.44}$$

From (5.37), (5.43) and (5.44) we have

$$4N^2 + 8\ell N + 8N = -\frac{2p_0^2}{ME} - 2\ell(\ell + 1) - 2(\ell + 1)(\ell + 2) \ , \tag{5.45}$$

which yields the Bohr formula

$$E = -\frac{p_0^2}{2M}(N + \ell + 1)^{-2} \ , \tag{5.46}$$

where N = 0, 1, ... is the radial quantum number.

With $\bar{W}(u,\beta_N)$ given by (5.38) with s=N and r_u by (5.25) and (5.33), the normalized solution to (5.10) with s=N is

$$\bar{R}_0(u,\beta_N) = \left[\frac{2^{4\beta_N+6}(\beta_N + 1)^4(\beta_N!)^2}{p_0^3 2\pi(2\beta_N + 1)!}\right]^{\frac{1}{2}} \frac{u^{\beta_N}}{(u^2 + 1)^{\beta_N+2}} \ . \tag{5.47}$$

By substituting (5.47) in (5.15) we generate the normalized radial momentum-space eigenfunctions for N = 1, 2, ... It is straightforward though tedious to show that the first few of these are particular cases of

$$\bar{R}_N(u,\beta_0) = e^{-iN\phi} C_N(\beta_0) \frac{u^\ell}{(u^2 + 1)^{\ell+2}} \, {}_2F_1[-N, 2\ell+2+N;\ \ell+\tfrac{3}{2};\ (u^2+1)^{-1}] \ . \tag{5.48}$$

Here

$$C_N(\beta_0) = \left[\frac{2^{4\ell+6}(\ell+N+1)^4(\ell!)^2(2\ell+N+1)!}{p_0^3 2\pi N![(2\ell+1)!]^2}\right]^{\frac{1}{2}} \ . \tag{5.49}$$

u is given by (5.33) and (5.46), ${}_2F_1$ is a hypergeometric function and $\beta_0 = \ell$. The validity of (5.48) and (5.49) for all non-negative integers N can be proved by using (5.48) and (5.49) in (5.14) with s=N. With $x = (u^2 + 1)^{-1}$ this yields

$$\left[1 - 2x + \frac{2(1 - x)x}{2\ell + 3}\frac{d}{dx}\right] {}_2F_1(-N, 2\ell+4+N;\ \ell+\tfrac{5}{2};\ x) = {}_2F_1(-[N+1], 2\ell+3+N;\ \ell+\tfrac{3}{2};\ x) \ , \tag{5.50}$$

which is the recurrence relation (3.91) with $B = 2\ell+4+N$ and $C = \ell+\frac{5}{2}$.

The various types of shift operations for the Coulomb problem in the momentum-representation are given in Table 5.2.

TABLE 5.2 Shift operations for the Coulomb problem in the momentum representation. The energy eigenvalues E are given by (5.46). The notation $\bar{R}_N(p,\beta_s)$, $|E\ell\rangle$ and $|n\ell\rangle$ is explained in Table 5.1.

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1. Shape invariance

$$B_+(p,\beta_s)\bar{R}_N(p,\beta_s) = \frac{p_0}{\sqrt{2M}}[(\beta_s+1)^{-2}-(\beta_{N+s}+1)^{-2}]^{\frac{1}{2}}\bar{R}_{N-1}(p,\beta_{s+1})$$

$$B_-(p,\beta_s)\bar{R}_N(p,\beta_{s+1}) = \frac{p_0}{\sqrt{2M}}[(\beta_s+1)^{-2}-(\beta_{N+s+1}+1)^{-2}]^{\frac{1}{2}}\bar{R}_{N+1}(p,\beta_s)$$

where u and r_u are given by (5.33), $\beta_s = \ell + s$

and $B_\pm(u,\beta_s) = \pm\frac{i}{\hbar}r_u(u^2+1) - (\beta_s+1)(u-u^{-1})$.

2. Shift operators for the parameter ℓ [6]

$$P^{\pm}_{E\ell}|E\ell\rangle = \mp i2p_0[1 + 2Mp_0^{-2}E(\ell+\tfrac{1}{2}\pm\tfrac{1}{2})^2]^{\frac{1}{2}}|E,\ \ell\pm1\rangle$$

where $P^{\pm}_{E\ell} = \mp i\hbar^{-1}r_p(p^2-2ME) + (\ell+\frac{1}{2}\pm\frac{1}{2})p^{-1}(p^2+2ME)$.

3. Shift operators for energy [6]

$$\bar{Q}^{\pm}_n|n\ell\rangle = -\{n(n\pm1)^{-1}[n(n\pm1) - \ell(\ell+1)]\}^{\frac{1}{2}}|n\pm1,\ \ell\rangle$$

where $n = N + \ell + 1$ and $\bar{Q}^{\pm}_n = \Delta^{\pm}_n\left[\mp\frac{i}{\hbar}pr_p + 2(n\mp1)\dfrac{p_0^2}{n^2p^2+p_0^2} - n\pm2\right]$.

Here the action of $\Delta^{\pm}_n$ on p and r_p is $\Delta^{\pm}_n np = (n\pm1)p\Delta^{\pm}_n$

and $\Delta^{\pm}_n\dfrac{r_p}{n} = \dfrac{r_p}{n\pm1}\Delta^{\pm}_n$ [6] .

CHAPTER 6

RELATIVISTIC PROBLEMS

Certain relativistic problems can be treated by shape invariance and also by the factorization method [1]. In this Chapter we show how shape invariance can be used to obtain the energy eigenvalues and the radial eigenfunctions for the following relativistic systems:

(i) the Klein-Gordon and Dirac equations with a Coulomb potential in the coordinate representation [2] (Section 6.2),

(ii) a Dirac oscillator [3] in the coordinate and momentum representations (Sections 6.3 and 6.4) and

(iii) the one-dimensional Klein-Gordon equation with a Coulomb-like potential in the momentum representation [4] (Section 6.5).

We start by giving a brief introduction to the Klein-Gordon and Dirac equations for a charged particle in an electromagnetic field (Section 6.1).

6.1 INTRODUCTION

Consider a relativistic particle of charge q and rest mass m_0 in an electromagnetic field with vector and scalar potentials $\underset{\sim}{A}(\underset{\sim}{r},t)$ and $\Phi(\underset{\sim}{r},t)$. Let H denote the Hamiltonian of this particle and let

$$p_\mu = (\underset{\sim}{p}, \frac{i}{c} H) , \tag{6.1}$$

$$x_\mu = (\underset{\sim}{r}, ict) , \tag{6.2}$$

(μ = 1,2,3,4). The four-vector potential is

$$A_\mu(x_\nu) = [\underset{\sim}{A}(x_\nu), \frac{i}{c} \Phi(x_\nu)] . \tag{6.3}$$

The relativistic equations for a spin 0 or spin $\frac{1}{2}$ particle can be written in an abstract form [5]. It can be shown that the Klein-Gordon equation is

$$(D_\mu^2 + m_0 c^2)|\psi\rangle = 0 , \tag{6.4}$$

and the Dirac equation is

$$(i\gamma_\mu D_\mu + m_0 c)|\psi\rangle = 0 \ . \tag{6.5}$$

Here

$$D_\mu = p_\mu - qA_\mu \tag{6.6}$$

is the gauge-covariant derivative with $p_4 = i\hbar\partial/\partial t$. Also

$$\gamma_\mu = (i\beta\underset{\sim}{\alpha}, \ \beta) \ , \tag{6.7}$$

where

$$\underset{\sim}{\alpha} = \begin{bmatrix} 0 & \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma} & 0 \end{bmatrix} , \tag{6.8}$$

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.9}$$

and σ_k are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

In (6.4) and (6.5) a repeated Greek index implies summation from 1 to 4.

Consider a stationary state $|\psi(t)\rangle$ of a time independent scalar potential $\Phi(\underset{\sim}{r})$. The time evolution of $|\psi(t)\rangle$ is given by [6]

$$|\psi(t)\rangle = |\psi\rangle \exp(-iEt/\hbar)$$

where E is the energy eigenvalue. Then (6.4) and (6.5) become

$$(c^2\underset{\sim}{p}^2 + m_0^2c^4)|\psi\rangle = (E - q\Phi)^2|\psi\rangle \tag{6.10}$$

and

$$(c\underset{\sim}{\alpha}\cdot\underset{\sim}{p} + \beta m_0c^2)|\psi\rangle = (E - q\Phi)|\psi\rangle \ . \tag{6.11}$$

For the Klein-Gordon equation, the ket $|\psi\rangle$ is a scalar, and for the Dirac equation, $|\psi\rangle$ and $\langle\psi|$ are the column and row vectors (bi-spinors)

$$\begin{bmatrix} |\psi_1\rangle \\ |\psi_2\rangle \\ |\psi_3\rangle \\ |\psi_4\rangle \end{bmatrix} \quad \text{and} \quad (\langle\psi_1|, \ \langle\psi_2|, \ \langle\psi_3|, \ \langle\psi_4|) \ . \tag{6.12}$$

The Dirac ket (6.12) is normalized according to

$$\langle\psi|\psi\rangle = \sum_{\lambda=1}^{4} \langle\psi_\lambda|\psi_\lambda\rangle = 1 \ . \tag{6.13}$$

Next we consider spherically symmetric electrostatic potentials. For the Klein-Gordon equation with such a potential, the operators H^2, $\underset{\sim}{L}^2$ and L_z are a complete set of commuting operators [1]: they act on the kets $|\psi\rangle = |E\ell m\rangle$ of an angular momentum basis (see Section 4.1). For the Dirac equation we denote the spin operator by $\underset{\sim}{S} = \frac{1}{2}\hbar\underset{\sim}{\Sigma}$ and the Dirac operator by

$$K = \beta(\hbar^{-1}\underset{\sim}{\Sigma}\cdot\underset{\sim}{L} + 1) = \begin{bmatrix} \mathcal{K} & 0 \\ 0 & -\mathcal{K} \end{bmatrix} , \tag{6.14}$$

where

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\sigma} & 0 \\ 0 & \underset{\sim}{\sigma} \end{bmatrix} . \tag{6.15}$$

The total angular momentum operator is $\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$. Then H, $\underset{\sim}{J}^2$, $\underset{\sim}{J}_z$ and K are a complete set of commuting operators [7].

The eigenvalue equations for $\underset{\sim}{J}^2$, J_z and K are [7]

$$\underset{\sim}{J}^2|\psi\rangle = \hbar^2 j(j+1)|\psi\rangle \qquad (j = \tfrac{1}{2}, \tfrac{3}{2}, \ldots) , \tag{6.16}$$

$$J_z|\psi\rangle = \hbar m|\psi\rangle \qquad (m = j, -j+1, \ldots, j) \tag{6.17}$$

and

$$K|\psi\rangle = k|\psi\rangle . \tag{6.18}$$

Here k denotes the Dirac quantum number

$$k = -\eta(j + \tfrac{1}{2}) , \tag{6.19}$$

where

$$\eta = (-1)^{j+\ell+\frac{1}{2}} . \tag{6.20}$$

The Dirac ket $|\psi\rangle$ is the bi-spinor

$$|\psi\rangle = \begin{bmatrix} |\phi\rangle \\ |\chi\rangle \end{bmatrix} , \tag{6.21}$$

where

$$|\phi\rangle = \begin{bmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{bmatrix} \quad \text{and} \quad |\chi\rangle = \begin{bmatrix} |\psi_3\rangle \\ |\psi_4\rangle \end{bmatrix} . \tag{6.22}$$

This completes our introductory remarks on the Klein-Gordon and Dirac equations.

6.2 RELATIVISTIC COULOMB PROBLEMS

In this Section we consider the Klein-Gordon and Dirac equations (6.10) and (6.11) for a charged particle in an attractive Coulomb potential

$$q\Phi(r) = -\frac{Zq^2}{r} . \tag{6.23}$$

It is convenient to define the operators

$$u = Er \qquad \text{and} \qquad p_u = E^{-1}p_r . \tag{6.24}$$

These satisfy the commutation relation $[p_u, u] = -i\hbar$. Using the identity $\underset{\sim}{L}^2 = \underset{\sim}{r}^2(\underset{\sim}{p}^2 - p_r^2)$ [8], (6.23) and (6.24) in (6.10) and replacing $\underset{\sim}{L}^2$ by its eigenvalues $\hbar^2\ell(\ell+1)$, we obtain the Klein-Gordon equation in the form

$$\left[\frac{1}{2m_0} p_u^2 + V(u)\right]R(u) = \varepsilon R(u) , \tag{6.25}$$

where

$$V(u) = -\frac{\kappa}{u} + \frac{\hbar^2\zeta(\zeta + \delta)}{2m_0u^2} . \tag{6.26}$$

Here $\kappa = \hbar\gamma/m_0c$, where $\gamma = Zq^2/\hbar c$ is the atomic number times the fine structure constant. The parameters δ and ζ for the Klein-Gordon equation are

$$\delta = 1 , \tag{6.27}$$

$$\zeta = [(\ell + \tfrac{1}{2})^2 - \gamma^2]^{\frac{1}{2}} - \tfrac{1}{2} \tag{6.28}$$

and

$$\varepsilon = -\frac{m_0\kappa^2}{2\hbar^2\lambda^2} , \tag{6.29}$$

where λ is a dimensionless function of energy,

$$\lambda = [\gamma^2E^2/(m_0^2c^4 - E^2)]^{\frac{1}{2}} . \tag{6.30}$$

For bound states ($E < m_0c^2$), λ is real. The eigenfunctions $R(u)$ in (6.25) denote the radial coordinate-space eigenfunctions $\langle r|E\ell\rangle$ (see Chapter 4).

In the Dirac equation (6.11), we can eliminate $\underset{\sim}{p}$ in favour of p_r using the relation [9] $\underset{\sim}{\alpha}\cdot\underset{\sim}{p} = \alpha_r p_r + ir^{-1}\alpha_r(\underset{\sim}{\Sigma}\cdot\underset{\sim}{L} + \hbar)$, where $\alpha_r = \underset{\sim}{\alpha}\cdot\hat{\underset{\sim}{r}}$. From the Dirac equation with the potential (6.23) one can also obtain a second-order equation of the form (6.25). This can be done by first performing a similarity transformation of the Dirac-Coulomb equation and then decoupling the resulting first-order equations [10]. The eigenfunctions of the *transformed* Dirac-Coulomb problem have the form

$$\Psi(\underset{\sim}{r}) = \begin{bmatrix} iR_U(r)V^{\ell}_{jm}(\hat{\underset{\sim}{r}}) \\ R_L(r)\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}V^{\ell}_{jm}(\hat{\underset{\sim}{r}}) \end{bmatrix} . \tag{6.31}$$

Here the angular functions V^{ℓ}_{jm} are the usual spinor spherical harmonics [11]. The radial functions R_U and R_L are solutions to (6.25) with V(u) given by (6.26) and

$$\begin{aligned} \delta &= (-1)^{j+\ell+\frac{1}{2}}, \quad \text{if } R = R_U \\ &= -(-1)^{j+\ell+\frac{1}{2}}, \quad \text{if } R = R_L \end{aligned} , \tag{6.32}$$

$$\zeta = [(j + \tfrac{1}{2})^2 - \gamma^2]^{\frac{1}{2}} \tag{6.33}$$

and $j = \ell \pm \frac{1}{2}$ [12]. The parameters ϵ and λ are given by (6.29) and (6.30). The transformed radial functions must be normalized according to [13]

$$\int_0^{\infty} |R(r)|^2 r^2 dr = \frac{1}{2\sqrt{1 + \gamma^2\lambda^{-2}}} \left[\sqrt{\frac{1 + \gamma^2\zeta^{-2}}{1 + \gamma^2\lambda^{-2}}} \pm 1 \right] , \tag{6.34}$$

where the upper sign applies if R is R_U, and the lower sign if R is R_L. According to (6.34), if $\lambda = \zeta$ then $R_L(r) = 0$.

The nonrelativistic Schrödinger-Coulomb equation is also a particular case of (6.25): it corresponds to the parameters

$$\zeta = \ell , \tag{6.35}$$

$$\lambda = [\gamma^2 m_0 c^2 / 2(m_0 c^2 - E)]^{\frac{1}{2}} , \tag{6.36}$$

$\delta = 1$ and ϵ given by (6.29). Note that for the nonrelativistic equation the rest mass energy is included in the energy E.

The parameters λ, ζ and δ are summarized in Table 6.1.

TABLE 6.1 The parameters λ, ζ and δ in (6.25)

	Klein-Gordon	Dirac	Schrödinger
λ	$[\gamma^2E^2/(m_0^2c^4 - E^2)]^{\frac{1}{2}}$		$[\gamma^2 m_0c^2/2(m_0c^2 - E)]^{\frac{1}{2}}$
ζ	$[(\ell+\frac{1}{2})^2 - \gamma^2]^{\frac{1}{2}} - \frac{1}{2}$	$[(\ell+\frac{1}{2})^2 - \gamma^2]^{\frac{1}{2}}$	ℓ
δ	1	$\delta = \eta$ if $R = R_U$ $= -\eta$ if $R = R_L$ $\eta = (-1)^{j+\ell+\frac{1}{2}}$	1

6.2.1 **Shape invariance**

We now prove the shape invariance of the above Coulomb problems. In applying the formulas of Section 4.1 to (6.25), we must replace r with u and p_r with p_u in (4.7)-(4.11). Potentials of the shape (6.26) can be generated from the superpotential

$$W(u,\alpha_s) = -\frac{\hbar\alpha_s}{\sqrt{2m_0}\,u} + \sqrt{\tfrac{1}{2}m_0}\,\frac{\kappa}{\hbar\alpha_s}\,, \tag{6.37}$$

where

$$\alpha_s = s + \zeta + \tfrac{1}{2}(\delta + 1)\ . \tag{6.38}$$

From (1.36) and (6.37) we have

$$V_\pm(u,\alpha_s) = -\frac{\kappa}{u} + \frac{\hbar^2\alpha_s(\alpha_s \pm 1)}{2m_0u^2} + \frac{m_0\kappa^2}{2\hbar^2\alpha_s^2}\ . \tag{6.39}$$

Thus

$$V_-(u,\alpha_0) = V(u) + \frac{m_0\kappa^2}{2\hbar^2\alpha_0^2}\,, \tag{6.40}$$

where V(u) is the potential (6.26). Also, the partner potentials (6.39) satisfy the shape invariance condition (1.23) with remainder

$$R(\alpha_s) = -\frac{m_0\kappa^2}{2\hbar^2}\left[\frac{1}{\alpha_s^2} - \frac{1}{\alpha_{s-1}^2}\right] . \tag{6.41}$$

6.2.2 Energy eigenvalues

From (1.41), (6.41) and (1.39) we have

$$E_N^{(0)} = -\frac{m_0\kappa^2}{2\hbar^2}\left[\frac{1}{\alpha_N^2} - \frac{1}{\alpha_0^2}\right] \qquad (N = 0,1,2, \ldots) . \tag{6.42}$$

The right-hand side of (6.42) is equal to the eigenvalues ε of (6.25) plus the constant $m_0\kappa^2/(2\hbar^2\alpha_0^2)$ (see (6.40)). Hence, using (6.29), we find that

$$\lambda^2 = \alpha_N^2 \tag{6.43}$$

$$= [N + \zeta + \tfrac{1}{2}(\delta + 1)]^2 \qquad (N = 0,1, \ldots) . \tag{6.44}$$

Note that in the Dirac case, if $j = \ell - \frac{1}{2}$ then R_L does not exist if N=0 (see (6.34), (6.44) and (6.32)). From this and (6.44) it follows that for $j = \ell - \frac{1}{2}$ the values of λ^2 start at $(\zeta+1)^2$.

We can now write down the energy eigenvalues. From (6.44), (6.27), (6.28) and (6.30) we obtain for the positive-energy eigenvalues of bound states of the Klein-Gordon equation

$$E = m_0c^2\{1 + \gamma^2[N + \tfrac{1}{2} + \sqrt{(\ell+\tfrac{1}{2})^2 - \gamma^2}\ ]^{-2}\}^{-\frac{1}{2}} . \tag{6.45}$$

Similarly, for the Dirac equation we find

$$E = m_0c^2\{1 + \gamma^2[N + \sqrt{(j+\tfrac{1}{2})^2 - \gamma^2}\ ]^{-2}\}^{-\frac{1}{2}} , \tag{6.46}$$

where $N \neq 0$ if $j = \ell - \frac{1}{2}$. For the Schrödinger equation, (6.35), (6.36) and (6.44) yield

$$E = m_0c^2[1 - \tfrac{1}{2}\gamma^2(N + \ell + 1)^{-2}] , \tag{6.47}$$

which is the rest-mass energy plus the Bohr value.

In (6.44)-(6.47) the radial quantum number N = 0,1, ... is equal to the number of times that the operator $A_-(r,\alpha_s)$ must be applied to generate $R_N(r,\alpha_0)$ from $R_0(r,\alpha_N)$ (see (4.10) and Table 6.2).

6.2.3 Coordinate-space eigenfunctions

To use (4.10) we must first determine the normalized function $R_0(r,\alpha_N)$. This can be obtained from the first-order differential equation (4.7) with s=N. Using (4.11) (with r and p_r replaced by u and p_u), (6.37) and (6.43) in (4.7) with s=N, and normalizing the solution, we obtain

$$R_0(r,\alpha_N) = (-1)^N a_\lambda^{-\frac{3}{2}}[\Gamma(2\lambda+1)]^{-\frac{1}{2}}\left[\frac{r}{a_\lambda}\right]^{\lambda+1}\exp\left[-\frac{r}{2a_\lambda}\right] . \qquad (6.48)$$

Here $\lambda = \alpha_N = N+q+\frac{1}{2}(\delta+1)$, $a_\lambda = (\hbar/2m_0c\gamma)\lambda$ for the Schrödinger equation, and $a_\lambda = (\hbar/2m_0c\gamma)(\lambda^2+\gamma^2)^{\frac{1}{2}}$ for the Klein-Gordon and Dirac equations. In (6.48) we have for later convenience chosen a phase factor equal to $(-1)^N$.

Substituting (6.48) in (4.10), and using (6.42) and (6.43), we obtain the normalized eigenfunctions of (6.25) in the form

$$R_N(r,\alpha_0) = e^{-iN\theta}a_\lambda^{-\frac{3}{2}}\, C_N(\alpha_0)\left[\frac{r}{a_\lambda}\right]^{\lambda_0-1}\exp\left[-\frac{r}{2a_\lambda}\right]{}_1F_1(-N;\ 2\lambda_0;\ \frac{r}{a_\lambda}) , \qquad (6.49)$$

where $\lambda_0 = \alpha_0 = q + \frac{1}{2}(\delta + 1)$ and

$$C_N(\alpha_0) = \left[\frac{\Gamma(2\lambda_0+N)}{N!\,2(\lambda_0+N)\{\Gamma(2\lambda_0)\}^2}\right]^{\frac{1}{2}} . \qquad (6.50)$$

One can extend the identifications (6.49) and (6.50) to all non-negative integers N by induction. Using (6.49), (6.50) and (6.42) in (4.9) with s=0, we find that (4.9) reduces to (4.40) with $B = 2\lambda_0 + 2$ and $u = r/a_\lambda$.

Equations (6.49) and (6.50) give the normalized radial eigenfunctions for the Schrödinger and Klein-Gordon equations; in these the choice of phase θ in (6.49) is arbitrary. For the Dirac equation there are three additional steps that must be performed to obtain eigenfunctions.

(i) The phase θ is determined by substituting (6.49) in the transformed first-order Dirac equations. For this purpose one can use the limiting form as $r\to 0$, and hence ${}_1F_1$ can be replaced by $1 - r/2\lambda_0 a_\lambda$. This yields $\theta=\pi$ if $j = \ell+\frac{1}{2}$, and $\theta=0$ if $j = \ell-\frac{1}{2}$ [14].

(ii) The functions (6.49) are normalized to unity. Thus the normalizations (6.34) require that an extra factor, equal to the square root of the right-hand side of (6.34), be included in (6.50).

(iii) To obtain eigenfunctions for the original Dirac equation, one must apply the inverse of the similarity transformation used to derive (6.25), (6.26) and (6.29)-(6.31). This procedure has been described in detail in the literature [12,13] and will not be repeated here.

In Table 6.2 we compare the transformations for shape invariance of the above Coulomb problems with the shift operations for energy of these problems. Note that in Table 6.2 the energy shift operators require the scaling operators $\Delta_\lambda^{\pm}$. These are similar to the scaling operators $\Delta_n^{\pm}$ for the non-relativistic Coulomb problem (see Section 4.3). The effect of these operators on r and p_r is given by

$$\Delta_\lambda^{\pm} \frac{r}{a_\lambda} = \frac{r}{a_{\lambda\pm1}} \Delta_n^{\pm}$$

$$\Delta_\lambda^{\pm} a_\lambda p_r = a_{\lambda\pm1} p_r \Delta_\lambda^{\pm} . \tag{6.51}$$

In Table 6.2 we have not given the Infeld-Hull shift operators that change the value of the orbital angular momentum quantum number ℓ; these are discussed in Refs. [16].

6.3 SHAPE INVARIANCE OF A DIRAC OSCILLATOR [17]

In this Section we consider the Dirac equation

$$H|\Psi\rangle = i\hbar \frac{\partial}{\partial t}|\Psi\rangle \tag{6.52}$$

with the Hamiltonian

$$H = c\underset{\sim}{\alpha}\cdot(\underset{\sim}{p} - im_0\omega\beta\underset{\sim}{r}) + m_0^2c\beta , \tag{6.53}$$

where ω is a positive constant, $\underset{\sim}{\alpha}$ is given by (6.8) and β by (6.9). The system described by (6.52) and (6.53) is referred to as a Dirac oscillator [17]. It can be shown that the operators $\underset{\sim}{J}^2$, J_z and K commute with the Hamiltonian (6.53) [17]. The action of these operators on the kets $|\Psi\rangle$ in (6.52) is given by (6.16)-(6.18).

TABLE 6.2 Transformations for bound states of the Klein-Gordon, Dirac and Schrödinger equations with a Coulomb potential. The energy eigenvalues are given by (6.45), (6.46) and (6.47). For the Klein-Gordon and Schrödinger equations $R_N(r,\alpha_0)$ denote the radial functions in (6.31). The kets $|\lambda\zeta\rangle$ denote the upper or lower elements of the transformed Dirac ket in an angular momentum basis [1]. The parameters λ, ζ and δ are given in Table 6.1, a_λ is defined in Section 6.2.3, and u and p_u are given by (6.24). For the Schrödinger and Klein-Gordon equations one can set $\theta=0$; for the Dirac equation $\theta=\pi$ if $j = \ell+\frac{1}{2}$ and $\theta=0$ if $j = \ell-\frac{1}{2}$ [14].

---

1. *Shape invariance*

$$A_+(r,\alpha_s)R_N(r,\alpha_s) = e^{-i\theta}(\tfrac{1}{2}m_0\kappa^2\hbar^{-2})^{\frac{1}{2}}(\alpha_s^{-2}-\alpha_{N+s}^{-2})^{\frac{1}{2}}R_{N-1}(r,\alpha_{s+1})$$

$$A_-(r,\alpha_s)R_N(r,\alpha_{s+1}) = e^{i\theta}(\tfrac{1}{2}m_0\kappa^2\hbar^{-2})^{\frac{1}{2}}(\alpha_s^{-2}-\alpha_{N+s+1}^{-2})^{\frac{1}{2}}R_{N+1}(r,\alpha_s) \; ,$$

where κ is the constant in (6.26), $\alpha_s = s+\zeta+\frac{1}{2}(\delta+1)$ and

$$A_\pm(u,\alpha_s) = \pm\frac{i\hbar}{\sqrt{2m_0}}p_u - \frac{\alpha_s\hbar}{\sqrt{2m_0}\,u} + \sqrt{\tfrac{1}{2}m_0}\,\frac{\kappa}{\hbar\alpha_s} \; .$$

2. *Shift operators for energy* [15]

$$Q_\lambda^\pm|\lambda\zeta\rangle = e^{\pm i\theta}\{\lambda^{-1}(\lambda\pm1)[\lambda(\lambda\pm1) - \zeta(\zeta+\delta)]\}^{\frac{1}{2}}\beta_{\lambda\zeta}^\pm|\lambda\pm1,\zeta\rangle \; ,$$

where $\beta_{\lambda\zeta}^\pm = (\langle\lambda\zeta|\lambda\zeta\rangle / \langle\lambda\pm1,\zeta|\lambda\pm1,\zeta\rangle)^{\frac{1}{2}}$,

$Q_\lambda^\pm = \Delta_\lambda^\pm(\pm i\hbar^{-1}rp_r - \frac{1}{2}ra_\lambda^{-1} + \lambda)$ and $\Delta_\lambda^\pm$ are defined by (6.51) .

The Hamiltonian (6.53) is in 2x2 block form. Therefore the eigenkets in (6.52) can be written as

$$|\Psi\rangle = \begin{bmatrix} i\,|\psi\rangle_U \\ \underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}\,|\psi\rangle_L \end{bmatrix} , \tag{6.54}$$

where the factors i and $\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}$ have been introduced for later convenience. Using the relation $\underset{\sim}{\alpha}\cdot\underset{\sim}{p} = \alpha_r p_r + ir^{-1}\alpha_r(\underset{\sim}{\Sigma}\cdot\underset{\sim}{L} + \hbar)$, where $\alpha_r = \underset{\sim}{\alpha}\cdot\hat{\underset{\sim}{r}}$ [9], we can eliminate $\underset{\sim}{p}$ and $\underset{\sim}{r}$ in (6.53) in favour of the radial operators p_r and r. This yields

$$H = c\alpha_r p_r + i\hbar c r^{-1}\alpha_r \beta K - i m_0 c\omega\alpha_r \beta r + m_0 c^2\beta , \tag{6.55}$$

where we have used (6.14) and $\beta^2 = 1$ (see (6.9)). Using (6.14), (6.15), (6.55) and (6.54) in (6.52) we find for energy eigenkets that

$$\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}} \begin{bmatrix} 0 & cp_r + i\hbar c r^{-1}\mathcal{K} + i m_0 c\omega r \\ cp_r + i\hbar c r^{-1}\mathcal{K} - i m_0 c\omega r & 0 \end{bmatrix} \begin{bmatrix} i\,|\psi\rangle_U \\ \underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}\,|\psi\rangle_L \end{bmatrix}$$
$$= \begin{bmatrix} E - m_0 c^2 & 0 \\ 0 & E + m_0 c^2 \end{bmatrix} \begin{bmatrix} i\,|\psi\rangle_U \\ \underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}\,|\psi\rangle_L \end{bmatrix} . \tag{6.56}$$

From (6.56) it is straightforward to write down the coupled equations

$$(icp_r + \hbar c r^{-1}\mathcal{K} - m_0 c\omega r)\,|\psi\rangle_L = -(E - m_0 c^2)\,|\psi\rangle_U \tag{6.57}$$

and

$$(icp_r - \hbar c r^{-1}\mathcal{K} + m_0 c\omega r)\,|\psi\rangle_U = (E + m_0 c^2)\,|\psi\rangle_L . \tag{6.58}$$

Here we have used $\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}\,\mathcal{K} = -\mathcal{K}\,\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}$, $[p_r, \underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}] = 0$ and $(\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}})^2 = 1$ [18]. It is obvious that $\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}$ commutes with r. Decoupling (6.57) and (6.58) we find

$$\left[\frac{1}{2m_0}\,p_r^2 + \frac{\hbar^2 k(k\mp 1)}{2m_0 r^2} + \tfrac{1}{2}m_0\omega^2 r^2\right] \begin{bmatrix} |\psi\rangle_U \\ |\psi\rangle_L \end{bmatrix} = \hbar\omega[\lambda + (k \pm \tfrac{1}{2})] \begin{bmatrix} |\psi\rangle_U \\ |\psi\rangle_L \end{bmatrix} , \tag{6.59}$$

where

$$\lambda = \frac{E^2 - m_0^2 c^4}{2\hbar\omega m_0 c^2} . \tag{6.60}$$

In terms of $j = \ell \pm \frac{1}{2}$, (6.59) can be written in the convenient form

$$\left[\frac{1}{2m_0}\,p_r^2 + \frac{\hbar^2\zeta(\zeta+1)}{2m_0 r^2} + \tfrac{1}{2}m_0\omega^2 r^2\right] |\psi\rangle = \hbar\omega(\lambda + k + \delta)\,|\psi\rangle . \tag{6.61}$$

Here $|\psi\rangle$ denote the elements $|\psi\rangle_U$ or $|\psi\rangle_L$. The parameters ζ and δ are given by

$$\zeta = \ell ,$$

$$\delta = \tfrac{1}{2} \text{ if } |\psi\rangle = |\psi\rangle_U , \tag{6.62}$$

$$\zeta = \ell - \eta ,$$

$$\delta = -\tfrac{1}{2} \text{ if } |\psi\rangle = |\psi\rangle_L \tag{6.63}$$

and k and η are given by (6.19) and (6.20). The kets (6.54) must be normalized according to (see (6.13))

$$\langle\Psi|\Psi\rangle = {}_U\langle\psi|\psi\rangle_U + {}_L\langle\psi|\psi\rangle_L = 1 \tag{6.64}$$

since $(\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}})^2 = 1$. This requires [3]

$${}_U\langle\psi|\psi\rangle_U = \tfrac{1}{2}(1 + m_0c^2/E) , \tag{6.65}$$

$${}_L\langle\psi|\psi\rangle_L = \tfrac{1}{2}(1 - m_0c^2/E) . \tag{6.66}$$

Because the potential in (6.61) has the same form as (4.12), we can obtain the energy eigenvalues and normalized coordinate-space eigenfunctions from the results obtained in Section 4.2. Replacing ℓ with ζ in (4.19) and equating the right-hand side of (4.19) with $\hbar\omega(\lambda + k + \delta)$ (see (6.61)), we obtain the energy eigenvalues of the Dirac oscillator

$$E = \pm m_0c^2\{1 + 2\hbar\omega(m_0c^2)^{-1}[2N + \ell + 1 + \eta(j + \tfrac{1}{2})]\}^{\frac{1}{2}} , \tag{6.67}$$

for N = 0,1, ... Here we have used (6.60), (6.62), (6.63), (6.19) and (6.20). From (6.65)-(6.67) and (6.20) we see that when N=0 and $j = \ell+\tfrac{1}{2}$, $|\psi\rangle_U$ does not exist for E<0 and $|\psi\rangle_L$ does not exist for E>0.

The coordinate-space eigenfunctions for the Dirac oscillator are given by $\Psi(\underset{\sim}{r},t) = \langle r|\Psi\rangle$, where

$$\Psi(\underset{\sim}{r},t) = \begin{bmatrix} i\ F_{N\ell}(r)V^{\ell}_{jm}(\hat{\underset{\sim}{r}}) \\ \underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}\ G_{N\ell}(r)V^{\ell}_{jm}(\hat{\underset{\sim}{r}}) \end{bmatrix} \exp(-iEt/\hbar) . \tag{6.68}$$

Here

$$F_{N\ell}(r) = M_+C_{N\ell}R_{N\ell} ,$$

$$G_{N\ell}(r) = (\text{sign } E)M_-C_{N'\ell'}R_{N'\ell'} , \tag{6.69}$$

where $R_{N\ell}(r)$ are radial coordinate-space eigenfunctions and $V^{\ell}_{jm}(\hat{\underset{\sim}{r}})$ are the spinor spherical harmonics [11]. In (6.69), $\ell' = \ell - \eta$ and $N' = N-1$ if $j = \ell + \frac{1}{2}$, $N' = N$ if $j = \ell - \frac{1}{2}$ [3]. The constants

$$M_{\pm} = [\tfrac{1}{2}(1 \pm m_0 c^2/E)]^{\frac{1}{2}} \tag{6.70}$$

are required because of the normalizations (6.64)-(6.66). From (4.26) and (4.27) we obtain

$$R_{N\ell}(r,\alpha_0) = e^{-iN\theta} C_{N\ell}(\alpha_0) \left[\frac{r^2}{a^2}\right]^{\ell/2} \exp\left[\frac{-r^2}{2a^2}\right] {}_1F_1\left[-N;\ \ell+\tfrac{3}{2};\ \frac{r^2}{a^2}\right], \tag{6.71}$$

where

$$C_{N\ell}(\alpha_0) = \left[\frac{2^{\ell-N+2}(2N + 2\ell + 1)!!}{a^3 \pi^{\frac{1}{2}} N! [(2\ell + 1)!!]^2}\right]^{\frac{1}{2}}. \tag{6.72}$$

In (4.26) we set the phase factor $e^{-iN\theta}$ equal to unity. For the Dirac oscillator the phase θ can be determined by substituting the limiting form as $r \to 0$ of $F_{N\ell}(r)$ and $G_{N\ell}(r)$ in the coupled first-order Dirac equations (6.57) and (6.58). This yields

$$\theta = \pi \text{ if } j = \ell + \tfrac{1}{2},$$
$$= 0 \text{ if } j = \ell - \tfrac{1}{2}$$

and the factor sign E in (6.69).

The various types of shift operations for the radial eigenfunctions (6.71) of the Dirac oscillator in the coordinate representation can be obtained from the shift operations presented in Table 4.1. Here we must replace ℓ with ζ (see (6.62) and (6.63)) and use the expression (6.67) for the energy eigenvalues E.

6.4 DIRAC OSCILLATOR IN THE MOMENTUM REPRESENTATION

Here we consider equation (6.52) with the Hamiltonian (6.53) expressed in terms of the radial operator r_p (see (5.1)). We can eliminate $\underset{\sim}{r}$ and $\underset{\sim}{p}$ in favour of r_p and p in (6.53) using the relation $\underset{\sim}{\alpha}\cdot\underset{\sim}{r} = \alpha_p r_p - i\hbar^{-1}\alpha_p(\underset{\sim}{\Sigma}\cdot\underset{\sim}{L} + \hbar)$ where $\alpha_p = \underset{\sim}{\alpha}\cdot\hat{\underset{\sim}{p}}$ [19]. This yields for the Hamiltonian

$$H = im_0c\omega\alpha_p r_p\beta - m_0c\hbar\omega p^{-1}\alpha_p\beta K\beta + c\alpha_p p + m_0c^2\beta \ , \qquad (6.74)$$

where we have used (6.14), (6.8), (6.9) and (5.1). From (6.52), (6.54) with $\hat{\underset{\sim}{r}}$ replaced by $\hat{\underset{\sim}{p}}$, and (6.74) we obtain for the upper and lower elements of energy eigenkets the coupled equations

$$(-im_0c\omega r_p + m_0c\hbar\omega p^{-1}K - cp)|\psi\rangle_L = -(E - m_0c^2)i|\psi\rangle_U \ , \qquad (6.75)$$

$$(-im_0c\omega r_p - m_0c\hbar\omega p^{-1}K + cp)i|\psi\rangle_U = (E + m_0c^2)|\psi\rangle_L \ . \qquad (6.76)$$

Comparing (6.75) and (6.76) with (6.57) and (6.58) it is clear that these equations transform into each other, apart from the factor i multiplying $|\psi\rangle_U$ in (6.75) and (6.76), under the substitutions

$$p_r \leftrightarrow -m_0\omega r_p \ , \qquad (6.77)$$

$$r \leftrightarrow (m_0\omega)^{-1}p \ . \qquad (6.78)$$

Thus the radial momentum-space eigenfunctions for the Dirac oscillator can be written down by replacing r/a in (6.71) with p/p_0, where $p_0 = \sqrt{m_0\hbar\omega} = \hbar/a$, and a with p_0 in (6.72). Hence we obtain for the radial eigenfunctions

$$\bar{R}_N(p,\alpha_0) = C_N(\alpha_0)\left[\frac{p^2}{p_0^2}\right]^{\ell/2}\exp\left[\frac{-p^2}{2p_0^2}\right]{}_1F_1\left[-N;\ \ell+\tfrac{3}{2};\ \frac{p^2}{p_0^2}\right] , \qquad (6.79)$$

where

$$C_N(\alpha_0) = (-\eta i)^{\ell}\left[\frac{2^{\ell-N+2}\ (2N + 2\ell + 1)!!}{p_0^3\pi^{\frac{1}{2}}N![(2\ell + 1)!!]^2}\right]^{\frac{1}{2}} . \qquad (6.80)$$

The phase factor $(-\eta i)^{\ell}$ in (6.80) is required because of the factor i multiplying $|\psi\rangle_U$ in (6.75) and (6.76). This factor can also be determined by considering additional factorizations for the Dirac oscillator [20]. The energy eigenvalues are given by (6.67).

The various types of shift operations for the radial eigenfunctions of the Dirac oscillator in the momentum representation can be obtained from the shift operations presented in Table 5.1. Here we must replace ℓ with ζ (see (6.62) and (6.63)) and use the expression (6.67) for the energy eigenvalues E.

6.5 KLEIN-GORDON EQUATION WITH A ONE-DIMENSIONAL "COULOMB" POTENTIAL IN THE MOMENTUM REPRESENTATION [4]

As a final example we consider a spinless relativistic particle with rest mass m_0 moving in the one-dimensional potential

$$V(x) = -\frac{Zq^2}{|x|} . \tag{6.81}$$

If we let $q\Phi = -Zq^2/|x|$ in the Klein-Gordon equation (6.10), replace p_x with $-i\hbar d/dx$ and set $p_y = p_z = 0$, we obtain the eigenvalue equation

$$\left[\frac{d^2}{dx^2} + \frac{2\gamma E}{\hbar c|x|} + \frac{\gamma^2}{x^2}\right]\psi(x) = \lambda^2\psi(x) , \tag{6.82}$$

where

$$\gamma = Zq^2/\hbar c , \tag{6.83}$$

$$\lambda = (m_0c/\hbar)(1 - \varepsilon^2)^{\frac{1}{2}} \tag{6.84}$$

and

$$\varepsilon = E/m_0c^2 . \tag{6.85}$$

It can be shown that the motions to the right and left of the origin are independent: that is, a particle in the right (or left) region will remain there indefinitely [4,21].

In the momentum representation (6.82) becomes [4]

$$\left[(\xi - 1)\xi^2 \frac{d^2}{d\xi^2} + (2 - \nu\xi)\xi \frac{d}{d\xi} - (2 + \gamma^2)\right]\phi^{\pm}(\xi) = 0 . \tag{6.86}$$

Here

$$\xi = 2/(1 \pm i\nu q) , \tag{6.87}$$

$$q = p/(\varepsilon\gamma m_0c) , \tag{6.88}$$

$$\nu = \varepsilon\gamma m_0c/\hbar\lambda , \tag{6.89}$$

and $\phi^{\pm}$ is the Fourier transform

$$\phi^{\pm}(p) = (2\pi\hbar)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx \exp(-ipx/\hbar)\psi^{\pm}(x) .$$

In this tranform $\psi^+(x)$ denotes the solution for a particle confined to $x>0$, and $\psi^-(x)$ the solution for a particle confined to $x<0$.

To treat (6.86) using shape invariance, we must rewrite it in a form that is similar to (5.6). This can be done by making the substitution

$$\phi^{\pm}(\xi) = Cg_{\nu}(\xi)\Omega(\xi) \ , \tag{6.90}$$

where

$$g_{\nu}(\xi) = \xi(\xi - 1)^{\frac{1}{2}(\nu+1)} \ , \tag{6.91}$$

and to simplify the notation we have omitted superscripts ± on g or Ω. Here ν is defined by (6.106) and C is a normalization constant. Using (6.90) and (6.91) in (6.86) we find

$$\left\{[x_{\xi}f(\xi)]^2 + \bar{V}(\xi)\right\}\Omega(\xi) = -\tfrac{1}{4}(\nu + 1)^2\Omega(\xi) \ , \tag{6.92}$$

where

$$x_{\xi} = i\,\frac{d}{d\xi} \ , \tag{6.93}$$

$$f(\xi) = \xi - 1 \tag{6.94}$$

and

$$\bar{V}(\xi) = -\gamma^2/\xi^2 - (\nu - \gamma^2)/\xi \ . \tag{6.95}$$

Equation (6.92) is the desired form which we treat by shape invariance.

Potentials of the shape (6.95) can be generated from the "super-potential"

$$\bar{W}(\xi,\beta_s) = -\beta_s/\xi + (\beta_s^2 + \nu)/2\beta_s \ , \tag{6.96}$$

where

$$\beta_s = k + s \tag{6.97}$$

and k is the solution to $k(k - 1) = -\gamma^2$. That is

$$k = \tfrac{1}{2} \pm (\tfrac{1}{4} - \gamma^2)^{\frac{1}{2}} \ . \tag{6.98}$$

(For regular solutions to (6.86) we must take the upper sign in (6.98) [21].) From (5.7) and (6.96) we find

$$\bar{V}_{\pm}(\xi,\beta_s) = \frac{\beta_s(\beta_s \pm 1)}{\xi^2} - \frac{\beta_s(\beta_s \pm 1) + \nu}{\xi} + \left[\frac{\beta_s^2 + \nu}{2\beta_s}\right]^2 \ . \tag{6.99}$$

(In (5.7) we must set $\hbar=1$ because we are using dimensionless variables.) Using $\beta_0(\beta_0 - 1) = -\gamma^2$ (see (6.97) and (6.98)) in (6.99) it is easily verified that

$$\bar{V}_-(\xi,\beta_0) = \bar{V}(\xi) + \left[\frac{\beta_0^2 + \nu}{2\beta_0}\right]^2 , \tag{6.100}$$

where $\bar{V}(\xi)$ is the potential (6.95). It can also be shown that (6.99) satisfy the shape invariance condition (5.8) with remainder

$$\bar{R}(\beta_s) = \left[\frac{\beta_{s-1}^2 + \nu}{2\beta_{s-1}}\right]^2 - \left[\frac{\beta_s^2 + \nu}{2\beta_s}\right]^2 . \tag{6.101}$$

Consider bound states of (6.86) (that is, $E<m_0c^2$). The eigenvalues

$$\lambda_N^{(0)} = \left[\frac{\beta_0^2 + \nu}{2\beta_0}\right]^2 - \left[\frac{\beta_N^2 + \nu}{2\beta_N}\right]^2 \qquad (N = 0, 1, \ldots) , \tag{6.102}$$

obtained from (6.101) and (5.12), are equal to the eigenvalues $-\frac{1}{4}(\nu+1)^2$ plus the constant $[(\beta_0^2 + \nu)/2\beta_0]^2$ (see (6.92) and (6.100)). Hence

$$\left[\frac{\beta_N^2 + \nu}{2\beta_N}\right]^2 = \tfrac{1}{4}(\nu + 1)^2 . \tag{6.103}$$

Using (6.97) with s=N, (6.89), (6.84) and (6.85) in (6.103) we find

$$E = m_0c^2[1 + \gamma^2(N + k)^{-2}]^{-\frac{1}{2}} , \tag{6.104}$$

where N = 0, 1, ... and k is given by (6.98) with the upper sign.

Finally we use shape invariance to calculate the momentum-space eigenfunctions for bound states $\phi_N^{\pm}(\xi,\beta_0)$. These are determined from (5.10), (5.15) and (5.4) with $\bar{R}_N(p,\beta_s)$ replaced by $\Omega_N(p,\beta_s)$ and with the superpotential given by (6.96). The unnormalized* solution to (5.10) with s=N is

$$\Omega_0(\xi,\beta_N) = \left[\frac{\xi}{\xi - 1}\right]^{\beta_N} (\xi - 1)^{(\beta_N+\nu)/2\beta_N - 1} . \tag{6.105}$$

The momentum-space eigenfunctions $\Omega_N(\xi,\beta_0)$ for N=1,2, ... can be obtained by substituting (6.105) in (5.15). In these calculations care must be taken to use the correct expression, in terms of N, for the ν which appears

---

* The function $g_\nu(\xi)$ introduced in (6.90) does not preserve the normalization of $\phi_N^{\pm}(\xi)$ when treating (6.92) using shape invariance. For this reason we do not follow the normalizations as in the previous examples. The normalization of the functions $\phi_N^{\pm}(\xi)$ is performed in Appendix E.

in (6.105) and (6.96). For positive energy solutions $\nu>0$ (see (6.98)) and hence from (6.103) we obtain

$$\nu_N = k + N . \tag{6.106}$$

The $\Omega_N(p,\beta_s)$ which is used in the shape invariance calculations is associated with the parameter

$$\nu_{N+s} = k + N + s \tag{6.107}$$

that appears in the eigenvalue of (6.92) (see (1.31)). Thus the value of ν to be used in the $\Omega_0(\xi,\beta_N)$ and the $\bar{W}(\xi,\beta_s)$, that are contained in (5.10) (with s=N), (5.15) and (5.4), is given by (6.107) with s=0.

The first few functions obtained from (5.15) are found to be particular cases of

$$\Omega_N(\xi,\beta_0) = \xi^k(\xi - 1)^{-\frac{1}{2}(k+N+1)}\, {}_2F_1(-N,k+1;\ 2k;\ \xi) . \tag{6.108}$$

Here $\beta_0 = k$ (see (6.97)). The identification (6.108) can be extended to all non-negative integers N: we use (6.108) in (5.14) with s=0 to obtain

$$\left[1 - \frac{B(2B-N-1)+2N}{(2B-1)(2B-2)}\,\xi + \frac{\xi(1-\xi)}{2B-1}\frac{d}{d\xi}\right]{}_2F_1(-N,B+1;\ 2B;\ \xi)$$

$$= {}_2F_1(-[N+1],B;\ 2B-2;\ \xi) , \tag{6.109}$$

where B=k+1. (In the calculation leading to (6.109) the ν contained in the superpotential (6.96) is given by $\nu_{N+1} = k+N+1$ - see above.) Equation (6.109) is proved in Appendix B.

From (6.90), (6.91) (with ν = k+N), (E.15) and (6.108) we obtain the normalized momentum-space eigenfunctions for the one-dimensional Klein-Gordon equation with the potential (6.81). For N=0,1, ... we find

$$\phi_N^{\pm}(p,\beta_0) = C_N(\beta_0)\frac{2^{k+1}}{[1\pm i(p/\hbar\lambda)]^{k+1}}\, {}_2F_1\left[-N,k+1;\ 2k;\ \frac{2}{1\pm i(p/\hbar\lambda)}\right] . \tag{6.110}$$

Here

$$C_N(\beta_0) = \frac{1}{\sqrt{\hbar\lambda}}\left[\frac{(2k)^2\Gamma(2k+N)}{4^{2k+1}N!\,(2k+2N)\{\Gamma(k+\frac{1}{2})\}^2}\right]^{\frac{1}{2}} , \tag{6.111}$$

$k = \frac{1}{2}+(\frac{1}{4}-\gamma^2)^{\frac{1}{2}}$ and (see (6.84), (6.85) and (6.104)) $\lambda = [1+\gamma^{-2}(N+k)^{-2}]^{-1}$.

APPENDIX A

We prove that the transformations (1.20) and (1.21) are valid for unbroken supersymmetry. From (1.18) we see that

$$A_+|\psi^-_{N+1}\rangle = C^-_N|\psi^+_{N-\delta}\rangle , \qquad (A.1)$$

and from (1.19),

$$A_-|\psi^+_N\rangle = C^+_N|\psi^-_{N+\delta'}\rangle , \qquad (A.2)$$

where δ' and δ are integers (see (1.10)). Premultiplying (A.1) by A_- we obtain

$$A_-A_+|\psi^-_{N+1}\rangle = C^-_N A_-|\psi^+_{N-\delta}\rangle$$

$$= C^-_N C^+_{N-\delta}|\psi^-_{N-\delta+\delta'}\rangle , \qquad (A.3)$$

where we used (A.2) with N replaced by $N-\delta$. Using (1.8) and (1.10) in (A.3) we find

$$E^-_{N+1}|\psi^-_{N+1}\rangle = C^-_N C^+_{N-\delta}|\psi^-_{N-\delta+\delta'}\rangle , \qquad (A.4)$$

which shows that

$$\delta' - \delta = 1 . \qquad (A.5)$$

If we take N=0, (A.1) yields

$$A_+|\psi^+_1\rangle = C^-_0|\psi^+_{-\delta}\rangle . \qquad (A.6)$$

Because the index labelling the kets is non-negative (see (1.10)), it follows from (A.6) that $\delta \le 0$. Now (A.5) in (A.2) with N=0 yields

$$A_-|\psi^+_0\rangle = C^+_N|\psi^-_{\delta+1}\rangle . \qquad (A.7)$$

Because δ is an integer ≤ 0, we have from (A.7) that δ=0 or -1. Consider the possibility $\delta = -1$ in (A.7): premultiplying (A.7) by A_+, and using (1.8), (1.10) and the condition (1.12) for unbroken supersymmetry we have $E^+_0|\psi^+_0\rangle = 0$, and hence $E^+_0 = 0$. The last result contradicts the property that for unbroken supersymmetry $E^+_N > 0$ (see Fig. 1.1), and thus $\delta \ne -1$.

Therefore δ=0 and from (A.5) we have $\delta' = 1$. Hence (A.1) and (A.2) yield

$$A_+|\psi^-_{N+1}\rangle = C^-_N|\psi^+_N\rangle \tag{A.9}$$

and

$$A_-|\psi^+_N\rangle = C^+_N|\psi^-_{N+1}\rangle \,. \tag{A.10}$$

Taking the norm of (A.9), we find

$$|C^-_N|^2 = \langle\psi^-_{N+1}|A_-A_+|\psi^-_{N+1}\rangle = E^-_{N+1} \,.$$

Hence, choosing a phase factor equal to $e^{i\theta}$, we have

$$C^-_N = \sqrt{E^-_{N+1}}\; e^{i\theta} \,. \tag{A.11}$$

Similarly,

$$C^+_N = \sqrt{E^+_N}\; e^{-i\theta} \,. \tag{A.12}$$

Using $\delta=0$, $\delta'=1$, (A.11) and (A.12) in (A.4) we obtain

$$E^-_{N+1} = E^+_N \,. \tag{A.13}$$

Equations (A.9)-(A.13) yield (1.20)-(1.22).

APPENDIX B

We prove the following recurrence relations for hypergeometric functions:

$$\left[1 - \frac{u}{B+N} + \frac{u}{B+N}\frac{d}{du}\right]{}_1F_1(-N;\ B;\ u) = {}_1F_1(-[N+1];\ B;\ u)\ , \tag{B.1}$$

$$\left[1 - \frac{u}{B-1} + \frac{u}{B-1}\frac{d}{du}\right]{}_1F_1(-N;\ B;\ u) = {}_1F_1(-[N+1];\ B-1;\ u)\ , \tag{B.2}$$

$$\left[1 - \frac{N+B-1}{(B-1)(B-2)}\,u + \frac{u}{B-1}\frac{d}{du}\right]{}_1F_1(-N;\ B;\ u) = {}_1F_1(-[N+1];\ B-2;\ u)\ , \tag{B.3}$$

$$\left[1 - \frac{N+B+1}{N+C}\,u + \frac{N+B+1}{B(N+C)}\,u(1-u)\frac{d}{du}\right]{}_2F_1(-N,B;\ C;\ u) = {}_2F_1(-[N+1];\ B+1;\ C;\ u)\ , \tag{B.4}$$

$$\left[1 - \frac{B-N-1}{C-1}\,u + \frac{u(1-u)}{C-1}\frac{d}{du}\right]{}_2F_1(-N,B;\ C;\ u) = {}_2F_1(-[N+1],B-1;\ C-1;\ u)\ , \tag{B.5}$$

$$\left[1 - \frac{B(2B-N-1)+2N}{(2B-1)(2B-2)}\,u + \frac{u(1-u)}{2B-1}\frac{d}{du}\right]{}_2F_1(-N,B+1;2B;u) = {}_2F_1(-[N+1],B;2B-2;u)\ , \tag{B.6}$$

$$\left[1 - \frac{2(2B-1)(B-1)+2N(2N-2B+3)}{(2N+1)(2B-1)}\,u + \frac{2(2N-2B+3)}{(2N+1)(2B-1)}\,u(u-1)\frac{d}{du}\right]$$
$$\times\ {}_2F_1(-N,B;\ \tfrac{1}{2};\ u) = {}_2F_1(-[N+1],B-1;\ \tfrac{1}{2};\ u)\ , \tag{B.7}$$

$$-\frac{2N-2B+1}{(2N+1)(2B-1)}\left[1 - \frac{2(u-1)}{2N-2B+1}\frac{d}{du}\right]{}_2F_1(-N,B;\ \tfrac{1}{2};\ u) = {}_2F_1(-N,B;\ \tfrac{3}{2};\ u). \tag{B.8}$$

Here N is a non-negative integer. The function

$${}_1F_1(-N;\ B;\ u) = \sum_{j=0}^{N} a_j(N,B)u^j\ , \tag{B.9}$$

where

$$a_j(N,B) = \frac{(-1)^j}{j!}\,\frac{\Gamma(N+1)}{\Gamma(N+1-j)}\,\frac{\Gamma(B)}{\Gamma(B+j)}\ , \tag{B.10}$$

is a confluent hypergeometric function. The function

$${}_2F_1(-N,B;\ C;\ u) = \sum_{j=0}^{N} a_j(N,B,C)u^j\ , \tag{B.11}$$

where

$$a_j(N,B,C) = a_j(N,C)\frac{\Gamma(B+j)}{\Gamma(B)}\ , \tag{B.12}$$

is a hypergeometric function.

The left-hand side of (B.1)-(B.7) has the form

$$\sum_{j=0}^{N+1} b_j u^j \ , \tag{B.13}$$

where $b_0 = 1$. The left-hand side of (B.8) has the form

$$\sum_{j=0}^{N} b_j u^j \tag{B.14}$$

where $b_0 = 1$ (see below).

B.1 PROOF OF (B.1)

Substituting (B.9) and (B.10) on the left-hand side of (B.1) we obtain

$$b_j = \left[1 + \frac{j}{N+B}\right] a_j(N,B) - \frac{1}{N+B}\, a_{j-1}(N,B)$$

if $1 \le j \le N$, and

$$b_{N+1} = \frac{-1}{N+B}\, a_N(N,B) \ .$$

Substituting (B.10) in these equations we find

$$b_j = a_j(N+1,B) \qquad (1 \le j \le N) \ ,$$

$$b_{N+1} = a_{N+1}(N+1,B) \ .$$

Hence (B.13) is equal to the right-hand side of (B.1).

B.2 PROOF OF (B.2)

Here

$$b_j = \left[1 + \frac{j}{B-1}\right] a_j(N,B) - \frac{1}{B-1}\, a_{j-1}(N,B)$$

if $1 \le j \le N$, and

$$b_{N+1} = \frac{-1}{B-1}\, a_N(N,B) \ .$$

Substituting (B.10) in these equations yields

$$b_j = a_j(N+1,B-1) \qquad (1 \le j \le N)$$

$$b_{N+1} = a_{N+1}(N+1,B-1) \ .$$

Hence (B.13) is equal to the right-hand side of (B.2).

B.3 PROOF OF (B.3)

Here

$$b_j = \left[1 + \frac{j}{B-1}\right] a_j(N,B) - \frac{N+B-1}{(B-1)(B-2)} a_{j-1}(N,B)$$

if $1 \le j \le N$, and

$$b_{N+1} = - \frac{N+B-1}{(B-1)(B-2)} a_N(N,B) .$$

Substituting (B.10) in these equations yields

$$b_j = a_j(N+1,B-2) \qquad (1 \le j \le N)$$

$$b_{N+1} = a_{N+1}(N+1,B-2) .$$

Hence (B.13) is equal to the right-hand side of (B.3).

B.4 PROOF OF (B.4)

Here

$$b_j = \left[1 + j\,\frac{N+B+1}{B(N+C)}\right] a_j(N,B,C) - \frac{(N+B+1)(N+B)}{B(N+C)} a_{j-1}(N,B,C)$$

if $1 \le j \le N$, and

$$b_{N+1} = - \frac{(N+B+1)(N+B)}{B(N+C)} a_N(N,B,C) .$$

Substituting (B.12) in these equations yields

$$b_j = a_j(N+1,B+1,C) \qquad (1 \le j \le N)$$

$$b_{N+1} = a_{N+1}(N+1,B+1,C) .$$

Hence (B.13) is equal to the right-hand side of (B.4).

B.5 PROOF OF (B.5)

Here

$$b_j = \left[1 + \frac{j}{C-1}\right] a_j(N,B,C) - \left[\frac{B-N-1+j-1}{C-1}\right] a_{j-1}(N,B,C)$$

if $1 \le j \le N$, and

$$b_{N+1} = - \frac{B-1}{C-1} a_N(N,B,C) .$$

Substituting (B.12) in these equations yields

$$b_j = a_j(N+1,B-1,C-1) \qquad (1 \le j \le N)$$

$$b_{N+1} = a_{N+1}(N+1,B-1,C-1) .$$

Hence (B.13) is equal to the right-hand side of (B.5).

B.6 PROOF OF (B.6)

Here

$$b_j = \left[1 + \frac{j}{2B-1}\right] a_j(N, B+1, 2B) - \left[\frac{B(2B-N-1)+2N}{(2B-2)(2B-1)} + \frac{j-1}{2B-1}\right] a_{j-1}(N, B+1, 2B)$$

if $1 \le j \le N$, and

$$b_{N+1} = -\frac{B(2B+N-1)}{(2B-2)(2B-1)} a_N(N, B+1, 2B) .$$

Substituting (B.12) in these equations yields

$$b_j = a_j(N+1, B, 2B-2) \qquad (1 \le j \le N)$$

$$b_{N+1} = a_{N+1}(N+1, B, 2B-2) .$$

Hence (B.13) is equal to the right-hand side of (B.6).

B.7 PROOF OF (B.7)

Here

$$b_j = \left[1 - \frac{2j(2N-2B+3)}{(2N+1)(2B-1)}\right] a_j(N, B, \tfrac{1}{2}) - \left[\frac{2(2B-1)(B-1)+2(N-j+1)(2N-2B+3)}{(2N+1)(2B-1)}\right] \times a_{j-1}(N, B, \tfrac{1}{2})$$

if $1 \le j \le N$, and

$$b_{N+1} = -\frac{2(B-1)}{2N+1} a_N(N, B, \tfrac{1}{2}) .$$

Substituting (B.12) in these equations yields

$$b_j = a_j(N+1, B-1, \tfrac{1}{2}) \qquad (1 \le j \le N)$$

$$b_{N+1} = a_{N+1}(N+1, B-1, \tfrac{1}{2}) .$$

Hence (B.13) is equal to the right-hand side of (B.7).

B.8 PROOF OF (B.8)

Substituting (B.11) and (B.12) on the left-hand side of (B.8) we obtain

$$b_j = \frac{-(2N-2B+1)}{(2N+1)(2B-1)} \left[\left[1 - \frac{2j}{2N-2B+1}\right] a_j(N, B, \tfrac{1}{2}) + \frac{2(j+1)}{2N-2B+1} a_{j+1}(N, B, \tfrac{1}{2})\right] \qquad (B.15)$$

if $1 \le j \le N-1$, and

$$b_N = \frac{-(2N-2B+1)}{(2N+1)(2B-1)} \left[1 - \frac{2N}{2N-2B+1}\right] a_N(N, B, \tfrac{1}{2}) . \qquad (B.16)$$

Substituting $j=0$ in (B.15), it is easy to check that $b_0 = 1$. Using (B.12) in (B.15) and (B.16) we find

$$b_j = a_j(N, B, \tfrac{3}{2})$$

$$b_N = a_N(N, B, \tfrac{3}{2}) \ .$$

Hence (B.14) is equal to the right-hand side of (B.8).

APPENDIX C

We prove the result

$$A_-(\alpha_0)A_-(\alpha_1)|\psi_N(\alpha_2)\rangle = [(E^{(0)}_{N+2} - E^{(0)}_0)(E^{(0)}_{N+2} - E^{(0)}_1)]^{\frac{1}{2}}|\psi_{N+2}(\alpha_0)\rangle , \qquad (C.1)$$

which is used in Section 3.2.4. Replacing N by N+2 in (1.46) and setting the phase factor $e^{-iN\theta}$ equal to unity, we obtain

$$|\psi_{N+2}(\alpha_0)\rangle = \gamma_{N+2}(\alpha_0)A_-(\alpha_0)A_-(\alpha_1) \;\dots\; A_-(\alpha_{N+1})|\psi_0(\alpha_{N+2})\rangle . \qquad (C.2)$$

Let

$$\beta_N = \alpha_{N+2} , \qquad (C.3)$$

then (C.2) becomes

$$|\psi_{N+2}(\alpha_0)\rangle = \gamma_{N+2}(\alpha_0)A_-(\alpha_0)A_-(\alpha_1)\{A_-(\beta_0) \;\dots\; A_-(\beta_{N-1})|\psi_0(\beta_N)\rangle\}$$

$$= \frac{\gamma_{N+2}(\alpha_0)}{\gamma_N(\beta_0)} A_-(\alpha_0)A_-(\alpha_1)|\psi_N(\beta_0)\rangle , \qquad (C.4)$$

where we have used (1.46) with α_s replaced by β_s.

Now

$$\gamma_N(\beta_0) = \gamma_N(\alpha_2) = \prod_{k=1}^{N}(E^{(2)}_N - E^{(2)}_{k-1})^{-\frac{1}{2}} = \prod_{k=1}^{N}(E^{(0)}_{N+2} - E^{(0)}_{k+1})^{-\frac{1}{2}} , \qquad (C.5)$$

where (1.31) and (C.3) were used. Hence the ratio of the energy-dependent factors in (C.4) is

$$\frac{\gamma_{N+2}(\alpha_0)}{\gamma_N(\beta_0)} = \frac{\prod_{k=1}^{N+2}(E^{(0)}_{N+2} - E^{(0)}_{k-1})^{-\frac{1}{2}}}{\prod_{k=1}^{N}(E^{(0)}_{N+2} - E^{(0)}_{k+1})^{-\frac{1}{2}}} = [(E^{(0)}_{N+2} - E^{(0)}_0)(E^{(0)}_{N+2} - E^{(0)}_1)]^{-\frac{1}{2}} . \qquad (C.6)$$

Using (C.6) in (C.4) we obtain (C.1).

APPENDIX D

We derive the Infeld-Hull type shift operators for the parameter k in the Eckart potential (4.47). We also use the Hellmann-Feynman theorem to determine the condition for bound states of this potential to exist.

Consider the Infeld-Hull factorization type 6 (type F in ref. 17 of Ch. 1) [1] with d = 0 and β replaced by $i\beta$. Then

$$\cot\beta(x + d) \to -i\coth\beta x \,, \tag{D.1}$$

$$\operatorname{cosec}\beta(x + d) \to -i\operatorname{csch}\beta x \tag{D.2}$$

and the potential (3.87) in Ref. 1 becomes

$$V_k(x) = \frac{1}{2M}\,(-2\hbar\beta q'\coth\beta x + \hbar^2\beta^2 k(k - 1)\operatorname{csch}^2\beta x) \,. \tag{D.3}$$

In order to keep the same notation used in Section 4.5, we define

$$H_k = \frac{1}{2M}\,p^2 + V_{k+1} \tag{D.4}$$

and let

$$q' = q\hbar/a \,. \tag{D.5}$$

Thus according to (D.4) we must replace k by k+1 in the operators and coefficients (but not in $|Ek\rangle$) for factorization type 6. Using the operators (4.48), equations (D.1), (D.2) and (D.5), we find that (D.3) with k replaced by k+1 is the potential (4.47) if $\beta = a^{-1}$, and if q and k are given by (4.54) and (4.72) respectively. Hence from (3.88), (3.89) of Ref. 1 and (D.1) we obtain the shift operations

$$\begin{aligned}\mathcal{R}_k^{\pm}|Ek\rangle &= [\pm ip_u - (k+\tfrac{1}{2}\pm\tfrac{1}{2})\coth u + q(k+\tfrac{1}{2}\pm\tfrac{1}{2})^{-1}]\,|Ek\rangle \\ &= [2MEa^2\hbar^{-2} + q^2(k+\tfrac{1}{2}\pm\tfrac{1}{2})^{-2} + (k+\tfrac{1}{2}\pm\tfrac{1}{2})^2]^{\frac{1}{2}}|E,k\pm1\rangle \,.\end{aligned} \tag{D.6}$$

For the Hulthén potential k = -1 (see Section 4.6), which results in the operators $\mathcal{R}_k^{+}$ and $\mathcal{R}_{k+1}^{-}$ being undefined (see Table 4.3). Because the Infeld-Hull factorization method requires the factorizations [1]

$$A_{k+1}^{-}A_k^{+} = 2MH_{k+1} + F_k$$

and

$$A^{+}_{k}A^{-}_{k+1} = 2MH_{k+1} + G_{k} ,$$

we see that it is not possible to use this method to treat the Hulthén potential. Here $\mathcal{R}^{\pm}_{k}$ are operators which are functions of r and p_r whereas $A^{\pm}_{k}$ are functions of x and p_x.

To find the condition for the existence of the bound states, we use the Hellmann-Feynman theorem [2]. Treating A in (4.47) as a continuous variable and using (4.62) and (4.72) in

$$\left\langle \frac{\partial H}{\partial A} \right\rangle = \frac{\partial E}{\partial A} = \frac{\partial E}{\partial k}\frac{\partial k}{\partial A} ,$$

we find

$$\langle \operatorname{csch}^2 u \rangle = \frac{2(k-N)}{2k+1}\left[\left[\frac{Ma^2B}{\hbar^2(k-N)^2}\right]^2 - 1\right] . \tag{D.7}$$

The left-hand side of (D.7) is positive, hence we can determine for which values of k-N bound states exist. Since k<0 (see (4.72)) we find that either

$$-q^{\frac{1}{2}} < k - N < 0 \tag{D.8}$$

or

$$k - N > q^{\frac{1}{2}} . \tag{D.9}$$

Here q and k are given by (4.54) and (4.72). The condition (D.9) implies that $N < k - q^{\frac{1}{2}}$, which is not satisfied if N = 0,1, ... From (D.8) we see that there is a maximum value for N,

$$N_{max} < k + q^{\frac{1}{2}} , \tag{D.10}$$

above which there are no bound states. Because N ≥ 0, (D.10) shows that

$$\tfrac{1}{2}[(1 + 8Ma^2A\hbar^{-2})^{\frac{1}{2}} + 1] < (Ma^2B\hbar^{-2})^{\frac{1}{2}} . \tag{D.11}$$

Hence for fixed M, a and A, there is a minimum value of B, below which no bound states exist.

APPENDIX E

We perform the normalization of the eigenfunctions

$$\phi_N^{\pm}(\xi,\beta_0) = C_N(\beta_0)\xi^{k+1}{}_2F_1(-N,k+1;\ 2k;\ \xi) \qquad \text{(E.1)}$$

for the one-dimensional Klein-Gordon equation with a Coulomb-like potential (Section 6.5). These eigenfunctions must be normalized according to

$$I = \int_{-\infty}^{\infty} |\phi_N^{\pm}(\xi,\beta_0)|^2 dp = 1 \ . \qquad \text{(E.2)}$$

To simplify the notation we denote the coefficients $a_j(N,B,C)$ in the hypergeometric function (B.11) by

$$a_j = a_j(N,B,C) \ . \qquad \text{(E.3)}$$

Using (B.11), (E.1) and (E.3) in (E.2) we obtain

$$I = \left[C_N(\beta_0)\right]^2 \int_{-\infty}^{\infty} \left[\sum_{j=0}^{N} a_j \xi^{j+k+1}\right]\left[\sum_{\ell=0}^{N} a_\ell (\xi^*)^{\ell+k+1}\right] dp$$

$$= \left[C_N(\beta_0)\right]^2 \sum_{j=0}^{N} \sum_{\ell=0}^{N} a_j a_\ell \int_{-\infty}^{\infty} \xi^{j+k+1} (\xi^*)^{\ell+k+1} dp \ . \qquad \text{(E.4)}$$

In what follows it is convenient to change the variable to

$$\xi = \frac{2}{1 \pm iz} \ , \qquad \text{(E.5)}$$

where

$$z = p/\hbar\lambda \qquad \text{(E.6)}$$

(see (6.87)-(6.89)). In terms of (E.5) and (E.6), (E.4) becomes

$$I = \hbar\lambda\left[C_N(\beta_0)\right]^2 \sum_{j=0}^{N} \sum_{\ell=0}^{N} a_j a_\ell\, 2^{j+\ell+2k+2} \int_{-\infty}^{\infty} \frac{dz}{(1\pm iz)^{j+k+1}(1\mp iz)^{\ell+k+1}} \ . \qquad \text{(E.7)}$$

The integral in (E.7) can be evaluated using the relation [1]

$$\int_{-\infty}^{\infty} \frac{dt}{(a+it)^x (b-it)^x} = \frac{2\pi(a+b)^{1-x-y}}{(x+y-1)B(x,y)} \ , \qquad \text{(E.8)}$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function. Using (E.8), (E.3), (B.10) and (B.12) in (E.7) and simplifying, we find

$$I = \hbar\lambda[C_N(\beta_0)]^2 4^{2k} k^{-2} [\Gamma(k+\tfrac{1}{2})]^2 [\Gamma(N+1)]^2 \ S_N \ , \qquad \text{(E.9)}$$

where

$$S_N = \sum_{j=0}^{N} \sum_{\ell=0}^{N} \frac{(-1)^{j+\ell}}{j!\,\ell!} \frac{\Gamma(2k+j+\ell+1)}{\Gamma(N+1-j)\Gamma(N+1-\ell)\Gamma(2k+j)\Gamma(2k+\ell)} \tag{E.10}$$

and we have also used the identity (3.67). The sum over j in (E.10) can be expressed in terms of a hypergeometric function using (B.11), (B.12) and (B.10). Hence

$$S_N = \frac{1}{\Gamma(2k)\Gamma(N+1)} \sum_{\ell=0}^{N} \frac{(-1)^{\ell}}{\ell!} \frac{2k+\ell}{\Gamma(N+1-\ell)}\ {}_2F_1(-N, 2k+\ell+1;\ 2k;\ 1)\ . \tag{E.11}$$

Now

$${}_2F_1(-N, B;\ C;\ 1) = \frac{(C-B)_N}{(C)_N}\ , \tag{E.12}$$

where the Pochhammer symbol $(a)_N = a(a+1)\ \ldots\ (a+N-1)$, $(a)_0 = 1$. (The identity (E.12) is proved below.) Next we use (E.12) in (E.11): the factor $(C-B)_N$ in (E.12) is equal to

$$(2k-2k-\ell-1)_N = (-\ell-1)_N = (-\ell-1)(-\ell)\ldots(-\ell-2+N)\ , \tag{E.13}$$

which is zero except when $\ell = N-1$ and $\ell = N$. Thus (E.11) becomes

$$S_N = \frac{1}{\Gamma(2k)\Gamma(N+1)} \left[\frac{-1}{(N-1)!} \frac{2k+N-1}{\Gamma(2)} N! + \frac{1}{N!} \frac{2k+N}{\Gamma(1)} (N+1)! \right] = \frac{2k+2N}{N!\,\Gamma(2k+N)}\ . \tag{E.14}$$

Thus (E.14), (E.9) and (E.2) yield

$$C_N(\beta_0) = \frac{1}{\sqrt{\hbar\lambda}} \left[\frac{(2k)^2\Gamma(2k+N)}{4^{2k+1}N!\,(2k+2N)[\Gamma(k+\frac{1}{2})]^2} \right]^{\frac{1}{2}}\ . \tag{E.15}$$

This completed the normalization of the eigenfunctions (E.1).

Finally we prove the identity (E.12) by induction. For N=0, ${}_2F_1(0,B;\ C;\ 1) = (C-B)_0/(C)_0 = 1$ since $(C)_0 = 1$. Assuming (E.12) is true for some N, we must show that

$${}_2F_1(-[N+1], B;\ C;\ 1) = \frac{(C-B)_{N+1}}{(C)_{N+1}}\ . \tag{E.16}$$

Consider the left-hand side of (E.16). From (B.11) we have

$${}_2F_1(-[N+1], B;\ C;\ 1) = \sum_{j=0}^{N+1} a_j(N+1, B, C) \tag{E.17}$$

where $a_j(N,B,C)$ is defined in (B.12). Now write

$$a_j(N+1,B,C) = a_j(N,B,C) + b_j(N+1,B,C) \tag{E.18}$$

if $0 \le j \le N$, and

$$a_{N+1}(N+1,B,C) = b_{N+1}(N+1,B,C) . \tag{E.19}$$

Using (B.12) and (B.10) in (E.18) we find that

$$b_j(N+1,B,C) = \frac{j}{N+1}\, a_j(N+1,B,C) ,$$

for $0 \le j \le N+1$. Thus

$$\sum_{j=0}^{N+1} b_j(N+1,B,C) = \frac{1}{N+1}\sum_{j=0}^{N}(j+1)a_{j+1}(N+1,B,C) - \frac{B}{C}\sum_{j=0}^{N} a_j(N,B+1,C+1) , \tag{E.20}$$

where in the last step we have (B.12) and (B.10). Substituting (E.18)-(E.20) in (E.17) and using the induction hypothesis (E.12) we obtain

$$\begin{aligned} {}_2F_1(-[N+1],B;\ C;\ 1) &= \frac{(C-B)_N}{(C)_N} - \frac{B}{C}\,\frac{(C-B)_N}{(C+1)_N} \\ &= \frac{(C-B)_{N+1}}{(C)_{N+1}} , \end{aligned} \tag{E.21}$$

because $(C)_N = (C)_{N+1}/(C+N)$ and $C(C+1)_N = (C)_{N+1}$. This proves (E.16).

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