Exploring University Students' Mental Constructions of the Limit Concept in Relation to Sequences and Series.

BY

Chagwiza Conilius Jaison (213574316)

A full dissertation for the Degree of

DOCTOR OF PHILOSOPHY (Mathematics Education)

In the School of Science, Mathematics and Technology Education (Edgewood Campus)

Faculty of Education

University of Kwa-Zulu Natal

Supervisors: 1. Prof Aneshkumar Maharaj

2. Prof Deonarain Brijlall

ABSTRACT

The present thesis refers to some first semester calculus 1 university students' mental constructions of the limit concept in relation to sequences and series. A plethora of research on the limit concept is available and suggests that the concept is on record of being difficult for students to learn and comprehend. However, in Zimbabwe, there is inadequate research on mental constructions made by students of the limit concept in relation to sequences and series. This research aims at filling this gap in the literature. This study utilized the Action-Process-Object-Schema (APOS) theory in exploring conceptual appreciative displayed by students when dealing with limits of sequences and series. The study proposes the genetic decompositions on how students might construct the mental constructions in learning the sequences and series through the use of Activities-Classroom discussions –Exercises (ACE).

Collection of data was done by the use of a methodology that used practical teaching. All the thirty students who took calculus 1 accepted to participate in this study and answered the limit test questions. The students' written responses were analyzed using APOS theory. Ten students were selected for interviews through purposive sampling. Two declined to take part leaving eight to take part in the process. The APOS theory was used to analyze the interview results. The revision of preliminary genetic decomposition was done basing on the analyzed data.

The instructional method employed, facilitated the appreciation of the limit concept in relation to sequences and series by the students. Nearly all students showed that they operated at the Action level, a good number showed that they operated at least at the Process level and more than half of the students showed that they operated at the Object level. Three out eight interviewed students indicated that the managed to operate at the Schema level on some of the test questions. However, there is need for the establishment of a conceptual basis that promotes and allows the construction of the limit concept schema in relation to sequences and series. Furthermore, interviewed students' responses paralleled the chronological improvement of the limit concept as reported in literature. Historical analysis of the development of concepts needs to be reflected upon when preparing and designing instruction. This would help the lecturer to foresee the challenges that lay ahead and address students' difficulties during the learning process.

The implementation of APOS Theory is recommended for the learning of other mathematical aspects, which cause difficulties in students' learning. Moreover, other constructivist learning methods can be fused together with the APOS Theory to obtain improved results on students' performance in mathematics.

DECLARATION OF OWN WORK

The researcher (Chagwiza Conilius Jaison) declares that all information presented in this document is his own original work. I am the sole author of this document I present, it is in accordance with the institution's academic rules and ethical conduct of research. The researcher has fully cited and referenced all the materials and results used in this research that are not original to this research. The reproduction and publication of this thesis has no intention of over stepping any third party's privileges. This thesis has I have not been presented before it in its entirety or in part or submitted it for the purpose of attaining any qualification.

Signed ……………………… Date 17/01/2019

The supervisors of this thesis agree that the candidate can submit the thesis.

1. Signed……………………………………Date: 15/01/2019
2. Signed……………………………………Date: 16/01/2019

ACKNOWLEDGEMENTS

Glory be to God almighty for taking me through this PhD. He provided the wisdom, strength, good health and bravery for to battle it out notwithstanding all the hardships that came my way.

I am very grateful to outstanding educators and mentors Prof. A. Maharaj and Professor D. Brijlall for their untiring inspiration, mentorship and guidance throughout my research. I thank them for their dedication, commitment and encouragement. I consider myself very lucky to have had such wonderful people to walk with me through such a long journey.

I express my wholehearted gratefulness to Dr Chipo Shonhiwa a Physics lecturer at Bindura University for her support throughout the writing of this document. She was very instrumental on the designing and drawing of diagrams. She freely offered her time and skills throughout my thesis writing. A debt of gratitude is owed to the students and staff of Bindura University of Science Education for their support.

I wish to register my profound appreciation and sincere thanks to all people who helped and advised me in carrying out this research. The diversity of experience and academic support of the 2013-2017 (Zimbabwe Chapter) Southern African Association for Research in Mathematics, Science and Technology Education, who broadened my perspective on research and challenged many of my long held ideas, is gratefully acknowledged.

I would like to thank Dr. J. Sibanda (Senior Lecturer: English) School of Education of SOLPLAATJE UNIVERSITY for editing my thesis.

LIST OF FIGURES USED IN THE STUDY

LIST OF TABLES USED IN THIS STUDY

TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION

1.1 Introduction

This research study investigated the students' understanding of the limit concept in relation to sequences and series. Proposed in this research is the exploration of the process of finding limits in relation to sequences and series and students' mental constructions of this concept in a university context. The traditional way of teaching 'limits' is usually teaching the concept as a mathematical rule with its formulae. It is a fact that the limit concept was generated over centuries and students cannot understand it through focusing on formulae and rules. Questions like where it comes from, the necessity of its existence, or what it represents and many other conceptual questions, remain unanswered in mathematical text books.

There are many researchers who have shown that students have problems in gaining a deep and correct idea about the limit concept (Bagni, 2005; Tall, 1990; Williams, 1991). The researcher's idea for improved knowledge acquisition of the limit concept in relation to sequences and series was by discovering the concept with students through controlled and oriented discussions. The limit notion presented a conceivable parallelism amid history and perceptive development. Bagni (2005) argues that it would entail an unambiguous theory of knowledge to completely spell out an association of students' knowledge growth and the historical improvement of the concept. The researcher agrees with Bagni on the incompleteness of the theory on knowledge growth and historical development of concepts. However, the theory provides explanations as to why some steps in the history of a concept parallel some steps observed in student learning. Furthermore, educational researchers have noticed parallels between student conceptions and historical conceptions of concepts in calculus (Lakoff & Nunez, 2000; Cornu, 1991). The method of recapitulating historical steps benefits students learning calculus concepts, although the approach has limitations. This calls for class controlled discussion so that students gain profound accepting of the concepts taught. The causes of complications in students' understanding of the limit concept could be partly the language used with respect to limits. The language includes the use of terms like "tends to", "limit approaches", and "converges" Monaghan (1991) cited in Bagni (2005), acknowledges that these words do not convey the same meaning as regard their common place meanings, and students do not readily recognize that such terminologies have the same mathematical meaning. Students' difficulties are also partly influenced by the usual talk about limits, which conjures up the motion notion which is dynamic in nature. The dynamic characteristic of the limit provokes the idea of motion, which is never ending, and which hinders students from developing full understanding of the limit notion.

1.2 Background of the Study

Mathematics is one of the subjects that can be realistic to solve actual lifetime problems hence the call for a teaching approach that can augment a profound understanding of the conceptions that underpin the limit concept in relation to sequences and series. The learning of mathematics is a complex and dynamic process. The goal of teaching and learning is to help students understand the information presented to them or information they discover themselves. The mathematics teaching process needs to give particular attention to instructional steps to ensure that students can grasp and communicate or explain to others what they have understood (Idris, 2006). An unfathomable cognitive understanding is achievable using a cautiously designed teaching method allowing student's reflection on the concepts in many diverse ways (Haripersad, 2011). In higher education, deep learning emanates from an effective form of teaching (active learning instruction). Shimazoe and Aldrich (2010) noted that active learning has benefits for students, which are: a) realizing better grades, b) developing deep learning of materials, c) promoting positive attitudes towards autonomous learning, and d) acquiring social skills. The mathematics lecturer must always aim to help students understand concepts and not regurgitating of facts or merely procedures application for solutions (final answer of the process) (Idris, 2009).

Those concerned with mathematics education the world over have set determined goals for schools and universities, lecturers of mathematics (Frid & Sparrow, 2009). Solving mathematical problems is taken to be a good mental exercise that is critical to taming students' mental capability (Grouws & Cebulla, 2000). Sriraman and English (2010) advocated for students' greater disclosure to problem circumstances, which promotes generation of mathematical ideas. National Council of Teachers of Mathematics (2003) identified problematic resolving, communication of mathematics, mathematics thinking and being able to make mathematical connections as being at the center for reforms of mathematics learning.

Calculus plays a pivotal part in the undergraduate curriculum and the limit concept is one of the key concepts. Calculus is less static and more dynamic; is concerned with change and motion; it deals with quantities that approach other quantities (Stewart, 2015). The limit concept is a central part of the fundamentals of mathematical analysis. Failing understanding it clearly might lead to problems when dealing with concepts such as convergence, continuity and derivatives. The limit concept is connected to many other concepts, e.g. infinity, functions and infinitesimals. If an individual understood the concept of limits, it would be easier to work with the connected concepts. But unfortunately literature has a record that it is difficult for students to make sense of the concept. Furthermore, even many great mathematics researchers have found it hard to accurately handle limits through time. The researcher through analyzing the application of the limit concept found out that it is the basis of calculus-based mathematics and is very important in educational and occupational environments. Here the researcher gives a glimpse of some of the main ideas of calculus by showing how the concept of a limit arises when we attempt to solve a variety of problems.

Differential calculus is concerned with how one quantity changes in relation to another quantity. The central concept of differential calculus is the derivative, which is an outgrowth of the velocities and slopes of tangents. After learning how to calculate derivatives, we use them to solve problems involving rates of change and the approximation of functions, Limits and the Derivative concept. For example, for a function $y = f(x)$ its derivative at x is the function defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ provided that the limit exists. If exists, then we say that f is differentiable at x.

The area and distance problems are used to formulate the idea of a definite integral, which is the basic concept of integral calculus, Limits and the Riemann Sums. The Riemann sum concept originated from infinite series limit (summation) (Stewart, 2015). The integral is used to solve problems concerning volumes, lengths of curves, population predictions, cardiac output, forces on a dam, work, consumer surplus, and baseball, among many others. There is a connection between integral calculus and differential calculus. The Fundamental Theorem of Calculus relates the integral to the derivative. We break up a quantity into a large number of small parts. We next approximate each small part by a quantity of the form and thus approximate by a Riemann sum. Then we take the limit and express as an integral. Finally, we evaluate the integral using the

Fundamental Theorem of Calculus or the Midpoint Rule. Suppose we know that the velocity of an object traveling along a line (think car on a straight highway) is given by a continuous function v(t), where t represents time on the interval [a, b]. How might we determine the net distance the object. When first taking calculus it is easy to confuse the integration process (with its Riemann sums) with simple 'antidifferentiation.' While the First Fundamental Theorem connects these two, they are not the same thing. Most importantly the determination of many quantities can be approximated (interpreted) as Riemann sums and hence evaluated as definite integrals even though it is not obvious at the outset that antidifferention should be involved. The Riemann sum part turns out to be critical. How we interpret definite integrals geometrically: as (net) area under a curve. It can be shown that the net distance travelled over the time interval [a, b] is just the net area under the velocity curve. That's not obvious at first. The key point here is that we were able to use a 'divide and conquer' process to determine the displacement. The steps are as follows:

• We subdivided the quantity into small bits, and we approximate each bit as a product.

• When we reassembled (summed) the bits, we found we had a Riemann sum.

• Once we had a Riemann sum we took a limit as the number of bits got large.

• The limit was a definite integral which we could evaluate easily (if we know an antiderivative) using the Fundamental Theorem of Calculus.

Limits and Taylor Polynomials: Polynomial functions are easy to understand but complicated functions, infinite polynomials, are not obvious. Infinite polynomials are made easier when represented using series: complicated functions are easily represented using Taylor's series. This representation makes some functions properties easy to study such as the asymptotic behavior. Differential equations are made easy with Taylor series. Taylor's series is an essential theoretical tool in computational science and approximation. This paper points out and attempts to illustrate some of the many applications of Taylor's series expansion. If we want to evaluate the definite integral $\int_0^1 sinx^2 dx$ this integrand has no anti-derivative expressible in terms of familial functions. However, we know how to find its Taylor series. $\sin(t) = t - \frac{t^3}{3!} + \frac{t^3}{5!} + \cdots$ If we substitute $t = x^2$ the$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots$ the Tylor series can be integrated: $\int_0^1 sinx^2 dx = \frac{x^3}{3} + \frac{x^7}{7*3!} + \cdots$ This is an alternating series and by adding all the terms, the series converges to 0.31026.

Limits and Differential Equations, perhaps the most important of all the applications of calculus is to differential equations. When physical scientists or social scientists use calculus, more often than not it is to analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying. For example: Let $N(t)$ be the population at time t and Let $N0$ denote the initial population, that is, $N(0) = N0$. Find the solution of the model with initial condition $N(0) = N0$.

Solution: This is a separable differential equation, and its solution is

$\int_0^t \frac{dN(s)}{aN(s)-nN(s)^2} ds = \int_0^t ds = t$

$$\frac{1}{aN - bN^2} = \frac{1}{N(a - bN)} = \frac{A}{N} + \frac{B}{a - bN}$$

To find A and B, observe that $\frac{A}{N} + \frac{B}{a-bN} = \frac{A(a-bN)+BN}{N(a-bN)} = \frac{Aa+(B-bA)N}{N(a-bN)}$.

Therefore, $Aa + (b - bA)N = 1$.

Since this equation is true for all values of N, we see that $Aa = 1$ and $B - bA = 0$.

Consequently, $A = \frac{1}{a}$, $B = \frac{b}{a}$ and

$\int_{N_0}^{N} \frac{ds}{s(a-bs)} = \frac{1}{a}\int_{N_0}^{N}\left(\frac{1}{s} + \frac{b}{a-bs}\right) ds = \frac{1}{a}\left[ln\frac{N}{N_0} + \ln\left|\frac{a-bN_0}{a-bN}\right|\right] = \frac{1}{a} ln \frac{N}{N_0} | \frac{a-bN_0}{a-bN} |$.

Thus, $at = ln \frac{N}{N_0} | \frac{a-bN_0}{a-bN} |$.

It can be verified that $\frac{a-bN_0}{a-bN(t)}$ is always positive for $0 < t < \infty$.

Hence, $at = ln \frac{N}{N_0} \frac{a-bN_0}{a-bN}$

Taking exponentials of both sides of this equation gives

$e^{at} = \frac{N}{N_0} \frac{a-bN_0}{a-bN}$

$N_0(a - bN)e^{at} = (a - bN_0N)$.

Bringing all terms involving N to the left-hand side of this equation, we see that

$[a - bN_0 + bN_0e^{at}]\, N(t) = aN_0e^{at}$.

Limits involved with infinite series summation. The importance of infinite series in calculus stems from Newton's idea of representing functions as sums of infinite series. For instance, in finding areas he often integrated a function by first expressing it as a series and then integrating each term of the series. Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series. Physicists also use series in another way, in studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it. Much information can be obtained by exploring infinite sums of constant terms; however, the eventual objective in analysis is to introduce series that depend on variables. This presents the possibility of representing functions by series. Afterward, the question of how continuity, differentiability, and integrability play a role can be examined. Mathematically, an unending sum, is suggested. We can form sequences of partial sums and then examine the limit. Through the application of the limit concept, infinite series summation is applied in the calculation of area under a curve in a given interval through approximation. This is done through the summation of increasingly thinner rectangular areas.

Limits and Power Series, If the general linear homogeneous (undriven) second-order ODE, $x'' + p(t)x' + q(t)x = 0$, has coefficients p and q that are not both constants, we can write a solution $x(t)$ as a power series: $x(t) = \sum_{n=1}^{\infty} a_n(t)^n$. Much useful information can be deduced about an ODE when its solutions can be expressed as power series. To solve the ODE: $x' + x = 0$, we differentiate the series $x(t) = \sum_{n=1}^{\infty} a_n(t)^n$ term by term and substitute into the given ODE, come up with a recurrence formula, then lastly come up with the solution.

Furthermore, the limit concept is key to the study of modelling situations on growth and decay in Economics, distribution of medicine in Medicine and distribution of pollutants in Biology. All the areas discussed above reveals the importance the limit concept in relation to sequences and series. Despite the importance of the limit concept, literature has on record highlighted students' difficulties with the understanding of the limit concept. This study seeks to explore the understanding of mental constructions made by students as they engage with the limit concept in relation to sequences and series. Hence, for students it is vital for them to understand the limit of sequences and series so that they can solve problems in advanced courses involving them.

The idea of limits emanated from the Greek era (Jaffar & Dindyal, 2011). The Greeks focused intuitively on results and ideas of a limit, and not the mental constructions made by students during and after the learning processes. In this study, the ways first year university students developed the concept of limit were compared to the historical development of the concept. The aim was to find out if the students perceived the notion as mathematicians of the past did as understandings of the concept evolved. A comparison of students' development to historical development of limits Juter (2009) aimed at finding developmental similarities and differences. The historical development as a tool was used to measure with and not something to strive for in modern classrooms. Knowledge of such critical areas can be used to improve the students' opportunities of learning limits of sequences and series. When limits are taught metaphorically or intuitively (Lakoff & Nunez, 2000), they can be thought of as infinitesimals. The infinitesimals are then examples and not the underpinning of the work with limits. Historical examples have facilitated students' learning of other mathematical concepts, for example linear dependence and independence (Radford, 2000). McGinn and Boote (2003) underscored the importance for students to work with different types of tasks to practice both routine skills and problem solving abilities in mathematics. Furthermore, they argued that mathematics history provides a wide range of problems suitable for such practice. Students learning limits of sequences and series can be helped to overcome the shift to abstract thinking. Abstract thinking is essential for students to understand how the definition can be used to solve non-routine problems such as the problem posed to them requiring an understanding of the limit definition. Such work requires historically knowledgeable lecturers who can identify students' potential cognitive obstacles, both historical obstacles and obstacles that are specific for their current students (Katz, Dorier, Bekken & Sierpinska , 2000).

It is a well-accepted fact that in undergraduate calculus, the limit concept plays a foundational role. According to Tall (2011) an excellent foundation for the highest level mathematical analysis has proved to be the limit concept. Its title role in calculus is so acute that Stewart (2015) outlines calculus as a portion dealing with limits in mathematics. Due to the conceptual complexity of the limit concept, students face challenges in developing a firm understanding of the concept in relation to sequences and series. Acquisition of the idea of a limit necessitates students' decoding of its meaning from a comparatively complex symbolic statement. Where limits are first encountered, lecturers and students are faced with the transition difficulty (Jaffor & Dindyal, 2011) which is indispensable, to leap from the routine to the non-routine facets of mathematics. Processes of advanced thinking call for characteristics, generalization, synthesis, abstraction (transition from the concrete to the abstract) and assimilation. Limits unlike algebra, involve a subtle reasoning (Cornu, 1991). The question to ask is will students understand the rest of calculus without a deep understanding of limits? The answer is no. After being taught, the students usually know the techniques of solving problems about limits but not what these concepts mean and represent. It has been noted with concern, that there is a substantial variance between what students actually understand and what lecturers want their students to learn about a concept (Hardy, 2009). The researcher has noted that calculus students he taught display such dissonance between the envisaged and the actual understanding. Normally, calculus students learn an informal conceptualization of limits that focuses on the process, properties and calculating limits. Furthermore, students use different approaches, such as metaphors, to better understand the abstraction of limits of limits of sequences (Patel, McCombs & Zollman, 2014). It is imperative to improve students' understanding of the concept limits in relation to sequences and series conceptually. A number of research papers and policy documents aimed at increasing individual learners' mental understanding and achievement have been developed as perspectives on mathematics education reforms (NCTM, 2000; 2003). These emphasize the learner's competences like multiple representations, communicating mathematically and problem solving.

Researchers have found that computers and communication technologies can be useful in education circles. The use of computers can radically change a mathematical learning environment (Ruthven & Hennessy, 2002). The world is changing very fast and learners should be familiar with the appropriate skills and knowledge, hence the need to be exposed to new methods and technologies of learning (Prensky, 2001). Changes in mathematics teaching, was brought by technological advancement and has opened new opportunities in laboratory work. Thus, a wide range of approaches using technology was attempted in many reform projects. Furthermore, calculus ideas have been supported by dynamic graphics for illustration and symbolic manipulation of computations technological approaches (Tall, 2011). Computational tools like graphic calculators, Maple and Mathematica, are usually applied to provide students with realistic models for concepts (Schibeci, Durrant, Kissane, Miller & MacCallum, 2008). Computer algebra systems can be applied on limits thereby removing the computational drudgery of complex symbolic manipulation (Schibeci et al., 2008). These enable students to visualize trends to help them understand the mathematical concept of limits in relation to sequences and series.

1.3 Motivation for the study

It is known from researcher's experiences that a source of cognitive difficulties for students is the limit concept (Bagni, 2005; Denbel, 2014; Ely, 2010; Hartman, 2008; Maharaj, 2010; Williams, 1991). Many students struggle with limits (Cappetta & Zollman, 2013; McCombs, 2014; Patel, McCombs & Zollman, 2014). These researchers identified specific student difficulties with limits. The researchers (Cappetta & Zollman, 2013; Roh, 2010), found that students have difficulties with understanding limits, involving the infinite processes of limits; the value of limits; and the formal definition of limits. Furthermore, metaphorical reasoning to understanding limits is incorrectly used by students (Dawkins, 2012; Roh, 2010). Additionally, Roh (2010) argues that students struggle with the understanding of limit-based mathematical concepts if they do not understand essential components of the formal limit definition. Additionally, metaphorical reasoning, a way of understanding a situation to resolve disequilibrium students encounters when faced with a new problem situation (Patel et al., 2014), develop without the knowledge of the lecturer, impede their understanding of mathematical concepts. A student who understands a mathematical concept can move between numerical, algebraic, graphical and application representations. However, most students compartmentalize their thinking resulting in them staying with procedures even if such procedures result in illogical or/and contradicting results (McCombs, 2014). Gulcer (2012) argues that even if lecturers move smoothly between limit as a number (end state) and as a process, students seem to focus only on limits as a process. Strong students can view limits as both a value (a static end state) and as a dynamic process (never ending).

Researchers have noted with concern the lack of research in the area of infinite series (Gonzalez-Martin, Nardi & Biza, 2011). The existing literature on series are: infinity and students' understanding of series (Sierpinska, 1987), the way series are introduced to students (Gonzalez-Martin, et al., 2011), and framework development on the determination of students errors on convergence of series (Earls & Demeke, 2016). Thus the documented literature testified to the lack of research on the limit concept in relation to sequences and series. Furthermore, students' difficulties with the understanding of the limit concept are also highlighted.

1.3.1 Personal Motivation

My personal motivation for this study emanated from the time I was studying limits as a high school student and as an undergraduate student, from reading literature on how limits concept developed over centuries, and my experience in teaching the course at undergraduate level for many years. The difficulties students faced when dealing with the limit concept as observed by the researcher through his experience as a lecturer in mathematics at a university since 2008, emanated from the detailed formal theorems. The researcher has been intrigued by the students' poor performance in tests and examinations on questions involving limits in relation to sequences. The researcher explored the mental constructions made by students as they engaged with limits in relation to sequences and series. The failure by students to understand introductory calculus, in particular limit sequences and series, hinders their progress in learning mathematics. This is because learning at higher levels is dependent on appropriate skills and knowledge from lower levels as mathematics is a hierarchical subject. The researcher has noted from own teaching experience that developing understanding of limits is not easy for students, and that traditional ways of teaching do not help students overcome difficulties. Usually, lecturers present sequences and series as separate and unconnected concepts; that is, teaching series without linking them to sequences. Such an approach promotes disconnected and incomplete understanding of limits in relation to sequences and series. The "what" and the "how" of mathematics education, provides a unifying framework which offers a complete picture of the standards for mathematics education (NCTM, 2003).

1.4 Statement of the problem of the study

In summary of '1.3 Motivation of the study' internationally there is a gap in research on students' understanding of limits of sequences and limits of series. To the researcher's knowledge, no study has focused on exploring students' mental constructions of the limit concept in relation to sequences and series. The motivation to embark on this study was trigged by the facts raised by researchers on students' struggle with the limit concept and the researcher's personal experience with the limit of sequences and series.

1.5 Research objectives of the study

This study aimed to explore university students' mental constructions of the limit concept in relation to sequences and series.

This study's objectives were:

1. To determine APOS levels displayed by students when solving limit problems in relation to sequences and series.
2. To determine how the APOS levels displayed by students when solving limit problems in relation to sequences and series relate to the preliminary genetic decomposition.
3. To find how the historical understanding of limit compare with students' mental constructions.
4. To propose a genetic decomposition for mental constructions relevant to the understanding of limits of sequences and series.

1.6 Research questions of the study

This dissertation study tries to explore the APOS levels of the limit concept in relation to sequences and series displayed by first year first semester calculus students at a university in Zimbabwe. The following research question guided this study:

What is the undergraduate students' depth of understanding (APOS levels) of the limit concept in relation to sequences and series at university level? Sub- questions to facilitate answering the main research question were as follows:

1. What APOS levels are displayed by students when solving limit problems in relation to sequences and series?
2. How do the APOS levels displayed by students when solving limit problems in relation to sequences and series relate to the preliminary genetic decomposition?
3. How does the historical understanding of limits compare with the students' mental constructions?

4. How can the mental constructions displayed by students be used to improve the understanding of limits of sequences and series at university level?

1.7 Significance of the study

This study was noteworthy for two core whys and wherefores. Firstly, it has the potential to contribute to the literature on the mental constructions university students display when dealing with limits of sequences and series. It has been noted from literature that many students struggle with limits and also that researchers have noted with concern the lack of research in the area of infinite series (Gonzalez-Martin, Nardi & Biza, 2011). This study aimed to add to the board of research on limits of sequences and series. Furthermore, it suggested an alternative method for the learning of limits in relation to sequences and series in Zimbabwean universities by providing the genetic decomposition to analyze students' mental constructions of limits of sequences and limits of series.

The identification of students' mental constructions was of value in the following ways for the limit concept in relation to sequences and series:

1. The students' thought process was understood which in turn led to a better understanding of learning that took place when they learnt the limit of sequences and series in the calculus course.
2. Better-off learning experiences were offered to students through the use of the ACE learning cycle.

This study provided students with a more interactive and attractive learning environment, through the use of Maple.

Active methods of instruction are supported by the Action, Process, Objects and Schema (APOS) theory (Arnon et al., 2014) and Activity, Classroom discussion and Exercise (ACE) (Arnon et al., 2014) learning cycle. These are regular with constructivist practice that extends customary learning undertakings in a way that stimulates developed thinking process. The (ACE) learning cycle provided students at a Zimbabwean university with an alternative approach to the learning of limits involving sequences and series as a learning trajectory for concept development. This trajectory aimed to help students overcome difficulties in the learning of limits of sequences and series reported in literature, by taking meaningful mathematical actions within a process of mathematical investigation. It integrated mathematical software in a way that enhanced students' mental constructions. The trajectory has been used in many countries with promising results. To this effect, this study has revealed important educational implications to improve the learning of limits in relation to sequences and series in Zimbabwean universities.

1.8 Assumptions of the study

Assumptions are factors that are potentially influential a study for which the researcher does not have hard data, and can't or don't intend to control. The researcher may have anecdotal data related to the assumptions and, if so, it's worth reporting them. It's best if the researcher can discuss a plan to verify the assumptions if possible (thus, taking them from uninformed assumptions to informed opinions).

The assumptions of this study were:

1. The participants will answer the test questions and interview questions in an honest and candid manner.
2. The inclusion criteria of the sample were appropriate and therefore, assumes that the participants will experienced the same or similar phenomenon of the study.
3. The participants will have a sincere interest in taking in the research study and do not have other motives, such as getting better grades in the course.
4. The instruments used in the study will accurately measure the understanding of the students on the limit concept in relation to sequences and series.

1.9 Delimitations of the study

This research study was carried out at a Zimbabwean university. The researcher works in the department of Mathematics and Physics at a Zimbabwean University. Choice of the University did not pose ethical hurdles since the researcher is a lecturer at the university. The targeted context for the study was undergraduate students taking a Calculus course. In particular, the topic on limits in relation to sequences and series was the study area under consideration. The ACE learning cycle was used, it was hoped students would easily grasp the necessary information and skills needed to understand the limit concept in relation to sequence and series. The mental constructions made by the students as they engaged with the limit of sequences and series were explored. Identification of the mental constructions on limits of sequences and series was the intension of this study. The adequate resources; both human and material capital, provided conditions that were conducive for effective teaching and learning. The researcher found such conditions enabling and allowing the conduct of a study that yields reliable findings.

1.10 Concepts and Terminology

The operational definitions of important terms are given in this section.

Exploring	In this study exploring refers to the act of searching for the purpose of discovering mental constructions in understanding the limit concept in relation to sequences and series.
APOS Theory	According to Asiala, Cottrill, Dubinsky and Schwingerndorf (2001), the acronym APOS stands for Action, Process, Objects and Schema, all of which are appropriate mental structures an individual is required to possess to make sense of a given mathematical concept or to learn a concept. Brijlall and Maharaj (2015) define APOS theory as premised on the hypothesis of mathematical knowledge laying in an individual's tendency to deal with perceived mathematical problem situations. This could be constructing mental Actions, Processes, Objects and organizing knowledge in Schema to make sense of the situation and solve problems. Then, the researcher's interpretation of the APOS theory is, principally ideal for recounting how mathematical conceptions may be understood. It is a foundation used to explain how individuals conceptually shape their accepting of scientific notions.
Mathematical understanding	According to Brijlall and Maharaj (2015), mathematical understanding refers to, flexible movement among different mathematical representations of a concept, that is, from graphical to symbolic, then to verbal forms. From the definition, it can be noted that mathematical understanding is not only the ability to perform calculations, but the awareness and ability to make sense of how procedures work. Students need to get the feel of the result

without actually performing all the calculations, for example when they work with variations of a single algorithm, they should see relationships and recognize the same concept presented in various forms.

Calculus The term calculus refers to the study in Mathematics of the behavior of sequences and series in relation to limits. It involves the definitions of limit, of a sequence and of a series.

Sequence A sequence can be thought of as a list of ordered numbers. "The number $a_1$is called the first term, a_2 is the second term, and in general $a_n$is the nth term. We will deal exclusively with infinite sequences and so each term $a_n$will have a successora_{n+1}" (Stewart, 2015, p.714). We notice that for every positive integer n there is a corresponding number a_n, hence a sequence can be defined as a function whose domain is the set of positive integers.

Limit of a sequence A sequence$\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ (Stewart, 2015, p.716). A sequence $\{a_n\}$ converges to a unique finite number L, if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges. If $\lim_{n\to\infty} a_n$ does not exist we say the sequence diverges.

Series Let $\{a_k\}$, $k \geq n_0$ be a sequence (of real numbers). By a **series** (of real numbers) we mean the abstract symbol $\sum_{k=n_0}^{\infty} a_k$. When we say "a series", we are expressing the idea that a sequence (an infinite ordered set) of real numbers a_1, a_2, a_3 etc. (for simplicity we assume now that indexing goes 1, 2, 3,...) and we are summing them up: $a_1 + a_2 + a_3 + \cdots$ (Stewart, 2015). Of course, this addition is infinite, so it never ends and it is not clear what is actually meant by it and whether it can be done at all. We start adding numbers in the series from the left and keep going to the right in it, adding the next and the next number, and if the running totals eventually settle down to some number, it seems natural to declare this number to be the sum of the whole series.

Limit of a series Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$, let $s_n$denote its nth partial sum: $S_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n$. If the sequence $s_n$is convergent and $\lim_{n\to\infty} s_n = S$ exists as a real number, then $\sum a_n$the series is called convergent. If the sequence $s_n$is divergent, then the series is called divergent (Stewart, 2015, p.729). A series is divergent if the limit does not exist or is plus or minus infinity. The limit of its partial sums is the limit of an infinite series. $\{ S_N \}$ is called the sequence of partial sums of an infinite series $\sum a_n$ and is sum-able to S a number or is said to be convergent if the sequence S_N of partial sums converges to S. The infinite series $\sum a_n$ mean the sequence S_N given by $S_N = \sum_{n=0}^{N} a_n$

Genetic decomposition The term genetic decomposition refers to a set of mental structures that students might make during the process of learning a concept and accessing it when needed. A set of mental constructions are a result of a set of teaching intended to help students make the mental constructions and relate them to the desired mathematical concept.

1.11 Thesis outline

This thesis is organized as follows:

Chapter 1: Introduction.

The chapter reports on background of the study, motivation of the study, significance of the study, objectives and research questions of the study, delimitation of the study and concepts and terminology.

Chapter 2: Literature review.

The chapter focused on literature reviewed which covered, learning mathematics, the importance of the limit concept, dynamic and static notion of limit, historical understanding of the limit concept, spontaneous conceptions of limits, helping students overcome misconceptions, the APOS learning theory and the ACE learning cycle, recent studies using the ACE learning cycle and APOS theory and implication of the literature review on this study.

Chapter 3: Theoretical framework.

The chapter explained constructivism and its elements, cooperative learning and its components, construction of mathematical knowledge and understanding mathematical concepts, development of the limit of sequence Schema and limit of series Schema and construction in reflective abstraction.

Chapter 4: Methodology.

Research design and its components were the first to be considered in this chapter. Also considered in this chapter were: participants and sampling procedure, the genetic decompositions, design and implementation of instruction, data collection tools and data analysis strategies, ethics, validity and reliability issues.

Chapter 5: Analysis and discussion of limits of sequences.

The focus for this chapter was data analysis and discussion on: the formal definition of the limit of a sequence, evaluation of limits at infinity, evaluation of recursive limits, proofs of sequence limit and proof of uniqueness of a limit.

Chapter 6: Analysis and discussion of limits of series.

The data analysis and discussion on limits of series was the focus of this chapter: definition of the limit of an infinite series, determination of convergence of a series, the integral test, comparison test and the alternating test.

Chapter 7 Summary of findings, conclusions and recommendations.

In this chapter, the researcher presented a synthesis of foregoing chapters, on the research design and methodology, revisit of data (summary of findings by research sub-questions), modified genetic decomposition, draw conclusion based on findings, limitations, recommendations and themes for further research studies.

CHAPTER 2 LITERATURE REVIEW

2.1 Introduction

Literature and studies relevant to learning and teaching of limits in relation to sequences and series are cited and discussed in this chapter. Also literature on learning co-operatively and computer usage in the mathematics education are cited and discussed as they assist students to make the conceptual structures anticipated by APOS outline. Other recent studies on APOS are summarized later in this chapter.

2.2 Conceptual analysis

Conceptual analysis clarifies meanings and boundaries of the concepts under study to promote understanding and support the research study. This is covered under the sub-headings; learning Mathematics, limit of sequences, limit of series and the importance of the limit concept.

2.2.1 Learning Mathematics

How students learn mathematics has been suggested by many researchers in various frameworks. Through the researcher’s experience and from reading literature, understanding can be defined as composition of knowing the interconnections of the mathematical idea and knowing how and why an algorithm or idea works and when to use it (Usiskin, 2012). Students’ degree of understanding a given mathematical concept can be discovered in situations where they are communicating their idea of that concept. Understanding is also used in the process for assessing students’ learning through their capability to make sense of mathematics. Conversations in small groups gives students the chance to share their ideas about a given concept. Thus, students recognize and strengthen their mathematical schema when they have opportunities to make connections among topics, ideas and perspectives within and outside of mathematics.

In the learning process, students’ understanding is an ongoing process leading to full understanding without reaching the limit (Engelbregcht, Harding & Potgieter, 2010). The authors further argue that the dynamic process of understanding new mathematical knowledge takes place in layers. Students understand deeper the progression from one layer to the other. To gain a deep understanding of a particular concept, students have to repeatedly expose themselves to that concept. The authors see mathematics learning as being made up of two processes, which are; first time exposure and consolidation process. They also believe that working through many problems of a particular concept brings repeated exposure and deeper understanding. Thus, in this study, many examples on limits of sequences and series were solved using various methods, as well as using Maple software.

Full understanding of a concept is achieved when a student can deal effectively with the skills, associated with algorithms, properties, mathematical justifications (proofs), uses, application and representations of the concept (Usiskin, 2012). Each of these aspects can be mastered independently of the others, but attached to a particular concept, these aspects are obviously connected. Understanding can be regarded as the ability to associate these characteristics or connect concepts together and not see them as isolated facts. Understanding may be ascertained by the worth and amount of relations made by the student among these dimensions (Usiskin, 2012). The general appreciation of the sequence limit concept and series limit concept, is the capability to manage successfully the abilities and algorithms related to the limits of sequences and series. Furthermore, one should have the ability to identify and apply sequences or series properties to solve problems involving sequences or series.

Students understand mathematical concepts as a result of the linkage of relations that students create between their mental images connected with their mathematical perception (Harries, Barmby & Huggins, 2007). Thus, mathematics lecturers should be worried with the students' structure of schema or linkages for appreciation of mathematical concepts (Dubinsky, 1991). When Usiskin's dimensions of understanding are properly associated, they construct Dubinsky's linkages or schema for understanding. Lecturers' aim should be on managing students to make these connections. This study defines learning of limits of sequences and series according to APOS theory. The repeated exposure is handy for students operating in the Action stage regarding the understanding of limits in relation to sequences and series.

2.2.2 Limit of sequences

A function is a rule which relates the values of one independent variable to the values of another dependent variable. The value of the second variable is uniquely determined by (i.e. is a function of) the value of the first variable. A sequence is a function whose domain is a set of natural numbers or subset of the natural numbers. Sequences are generated using functions.

A sequence can be described as a special function. 'A sequence$\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$' (Stewart, 2015, p.716). If $\lim_{x\to\infty} f(x) = L$ for a function, then $\lim_{n\to\infty} f(n) = L$ for a sequence. But $\lim_{n\to\infty} f(n) = L \nRightarrow \lim_{x\to\infty} f(x) = L$. For example, $\lim_{n\to\infty} \cos(2\pi n) = 1$ but $\lim_{x\to\infty} \cos(2\pi x)$ does not exist. Sequences can be described as prototypes of discrete objects in mathematics, or natural numbers as the domain of functions. They can be exemplified by unambiguous or recursive formulae, displays, tables or arrow illustrations (Weigand, 2004). The sequence concept can be found in daily lifetime circumstances for example, playing cards sequences, an inordinate length of time and even proverbs like "the punishment follows close on the heels of an evil deed." Also they are found in nearly all mathematical areas. For instance, arrangements of figures, mappings and arithmetical information, scientific processes can be understood of as classifications of precise solitary stages. Fundamentally mathematical objects, sequences have a lengthy history in mathematics. They are developmental tools for other mathematical conceptions like the limit notion and are also implements for mathematization of factual circumstances like development practices (Weigand, 2004).

Through reading literature and dealing with sequences, the researcher has observed with interest, that sequences are interesting objects in themselves like the prime numbers sequence, sequences of Fibonacci and polygonal numbers sequence. The researcher has also noted that theorems dealing with limit of sequences can be put into two categories. The first one deals with ways to combine sequences. Here, numbers and sequences can be added, subtracted, multiplied and divided. Theorems from this category deal with the ways sequences can be written as combinations of several sequences and how the result of the limit can be obtained. The second category of theorems deal with specific sequences and techniques applied to them, for example, $lim_{n\to\infty} \frac{sin(n)}{n}$ requires the squeeze theorem. Computing the limit of a sequence usually involves using theorems from both categories.

The possibility to generate sequences and series is provided by new technology, they can construct representatives, geometric and graphical illustrations, and to transform amid these changed demonstrations (Weigand, 2004). In order to help students, understand sequences and series, the researcher used Maple in the learning sessions. The students through the use of Maple were able to graph sequences and series, explore and interpret given sequence and series graphs.

2.2.3 Limit of series

The researcher considers the most important and interesting illustrations of structures and those that stand up as infinite series, the partial sums. These permit us to clarify functions of exponential and trigonometric in mathematics. Convergent partial sums of a sequence may be defined as a series. Generally, the sum of a convergent series cannot be determined, but an approximation of its sum by numerically computing some partial sums. One of the most challenging topics for students to appreciate and for lecturers to clarify undoubtedly is infinite series (Hartman, 2008). The use of the epsilon – delta proofs for limits is difficult for the students, since the conceptions limits and series are connected. Both of them developed over hundreds of years. We have now an improved standpoint to start from, but it is too much for lecturers to expect students to gainfully appreciate infinite series in two to four weeks that is given to their study. From literature, infinite series development was motivated by the approximation of unknown areas and for the approximation of the value of π (Hartman, 2008). In about 1350, Suiseth indicated that $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3}$ + ….. + $\frac{k}{2^k}$ +… = 2 (Stillwell, 1989). Madhava (1340-1425) used $\frac{\pi}{4}$ = 1 - $\frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ + … to estimate π and $\sqrt{12}$ (1 - $\frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3}$ + …) for π (Joseph, 2000).

Through my experience, infinite series is one of those topics that many students do not find useful in their lives. A number of students cannot appreciate series application outer their calculus class. However, series play a vital part in the field of ordinary differential equations and in the field of partial differential equations. Infinite series is an important aspect of calculus, Riemann sums, sequences and series. The infinite sums and series can be well behaved or badly behaved. For example if one is required to find the sum of $1 + 2 + 3 + 4 + 5$ ..., which is a well behaved sum, one can add the numbers and get an ever increasing sum. The sum of $1 - 1 + 1 - 1 + 1 - 1 \ldots$ represents an example of a badly behaved sum, since for two consecutive terms, the sum is zero. This notion of good and bad behavior requires the idea of a limit. Series, like sequences, can be added, subtracted, multiplied and divided. Furthermore, technology can be used to discover convergence of series and their generation. Technology can be used to create symbolic, numerical and graphical representations; and change among these different illustration methods.

2.2.4 The importance of the limit concept

The limit concept is a fundamental mathematical notion as it is a pre-requisite for later calculus topics (Nagle, 2013). Limit is one of the unifying ideas in undergraduate mathematics. Many mathematical concepts in calculus and other courses depend heavily on the limit concept: derivative at a point as the limit of quotients of increments, the slope of a tangent line as the limit of slopes of secant lines; the definite integral as the limit of Riemann sums; continuity of a function; Taylor series; and the differential in multivariate calculus. The limit concept is applied in approximation methods such as Newton-Raphson to find zeros of functions and Runge-Kutta for approximating solutions to differential equations. The limit concept is a requirement for the acceptance of vital notions in calculus, such as continuity, derivatives and integral conceptions. If students do not understand the concept of pre-requisite conditions, they are bound to have difficulties in understanding things related to the concepts (Duru, 2011). Due to the importance of the limit concept, much research has been conducted by mathematics educators on: students' understanding of the limit concept (Hardy, 2009; Juter, 2009) specific student difficulties with limits (Cappetta & Zollman, 2013; McCombs, 2014; Patel, McCombs & Zollman, 2014); difficulties with understanding limits Cappetta & Zollman, 2013; Roh, 2010); use of metaphorical reasoning on limits by students (Patel et al., 2014). Some researchers have studied the teaching of the limit concept from different theoretical viewpoints, concept image and concept definition (Tall & Vinner, 1981; APOS Theory (Cottril, Dubinsky, Nchols, Schwingendorf, Thomas & Vidakovic, 1996). The purpose of this study was to analyze the mental constructions made by students when learning the limit concept in relation to sequences and series.

2.3 Literature review

A literature review is an evaluation of compiled relevant researched work for a particular topic or issue. Literature review is based on secondary data sources. It is a summary of existing scholarly work about a particular topic (limit of sequences and series) on what other people have already written about the concepts. It is not concerned about discovering new knowledge or information, but reviewed major scholarly books and journal articles in the relevant area. The literature review was carried out under the sub-headings; dynamic notion of limit, static notion of limit, historical understanding of the limit concept, spontaneous conceptions of a limit, helping students overcome misconceptions, the APOS Theory, ACE learning cycle, recent studies using APOS Theory and ACE learning cycle, and recent studies using APOS Theory.

2.4 Dynamic notion of Limit

Tall (1980a) observes that a dynamic conception of the limit of a function is easy to understand and develops naturally for students. The deduction was arrived at by centering on the dichotomy of process and object notions of limit, and students' principal perception of limit as a process and not limit as a stationary object. Thus, even after student's exposure to the formal limit teaching, they remained holding to a dynamic interpretation of limit. The sequence $\lim_{n\to\infty} \frac{n}{\sin(n)} = L$ is regarded to mean "as *n* goes to ∞, *f (n)* goes to *L*", which raises dynamic feeling of motion. But in the prescribed conception of the limit of a function, students deal with intermissions in which *n* and *f (n)* values do not change. This dynamic element in informal notion of limit prevents students from seeing that a sequence or series can reach its limit value. Williams (1991) also noted that dynamic conception of limit prevents the growth of a formal impression of limit.

Mamona- Downs (1990) posits that the getting nearer and nearer of n to a, $f(n)$ nears L does express a feeling of motion or flow. Students had a challenge when answering the question "how close". Some gave tautological answers, others "as close as you can" or infinitesimal. The same author goes on to say that potential infinity is a limiting process for "0. 99…" and must be constructed a never ending process of writing a 9 after 0.9. This declaration is not true even though when dealing with limit it may be understood that it is true that 0. 99… would actually reach 1. An earlier definition by D'Alembert involved the approximation language: "one magnitude is said to be the limit of another magnitude when the second may approach the first within any magnitude, however small, though the first magnitude may never exceed the magnitude it approaches" (Burton, 2007, p.603).

Kabael (2014) carried out a research study on students' formalizing of the limit concept at Anadolu University in Turkey. The results showed that dynamic participants had the dynamic limit images. The students provided a descriptive definition including the dynamic limit notion. Additionally, they reflected the dynamic limit notion and the feeling of motion in their expressions and arguments. It was found that the words like 'approaching points', 'approach' were salient in their expressions and arguments during interviews. Furthermore, the students related the feeling of motion in their dynamic images to the neighborhoods. Students often view the limit as something that cannot be attained and hence they view it only as a process. The everyday-language meanings of words like 'approach' or 'tends to' may reinforce this process (these words suggest that the limit is never reached). Such students failed to realize that $\lim_{x \to a} f(x)$ has a value (if it exists) which results in an object. The dynamic view is an important part of limit understanding, and the teacher should guide students to attain an object view of the limit.

Also Steven (2015) carried out a study on Calculus limits involving infinity: the role of students' dynamic reasoning at Brigham Young University. The results showed that dynamic reasoning used by the students led to good justifications and meaningful interpretations of their answers. On the other hand, results from students who engaged less with dynamic reasoning, revealed that they struggled more and made less rational interpretations of their answers. Furthermore, dynamic reasoning helped students overcome previously documented pitfalls and encouraged covariational reasoning (Steven, 2015).

2.5 Static notion of Limit

The formal conception of a limit relies much on the conceptual understanding of the formal epsilon-delta definition. The epsilon-delta limit definition is based on static terminology which abandons the motion-based terminology of the dynamic interpretation (Nagle, 2013). The main idea of the limit formal definition is that for any small interval or neighborhood chosen around the limit value **L** on the *y*-axis, we can find another neighborhood around *a* on the *x*-axis such that the images of all points in the *x*-axis interval (excluding possibly the image of a itself) are contained in the *y*-axis interval. The formal limit definition is a static notion since it refers to motionless intervals. This advanced calculus conception of limit requires firm foundation on formal static ideas stated in the epsilon-delta definition. There are four stages proposed to build static notions from dynamic notions: (1) dynamic conception only; (2) separate dynamic and static conceptions; (3) dominate dynamic conception integrated with static notions; (4) dominate static conception integrated with dynamic notions (Boester, 2010). Static notion may be gradually added to a dynamic foundation instead of switching from purely dynamic to purely static. There is need for the incorporation of static terminology in introductory instruction, including discussions of closeness through exploring fixed neighborhoods of output values about **L** and corresponding fixed neighborhoods of input values around. Additionally, instruction should motivate the need for static ideas trough real-world examples and problems that require static terminology.

In Boester's (2010) research study, it was found that students were not prepared to adopt static terminology with only two of the eight students advanced to Boester's final stage of a predominantly static limit conception. Furthermore, Boester' (2010) study found that students with a solely dynamic foundation were reluctant to build static notions.

2.6 Historical understanding of the Limit concept

Throughout the history of mankind, infinity has always been a challenging notion. In this thesis, the researcher briefly outlines the chronological development of the conception of infinity leading to limit of infinite functions. This is because "… above all, infinity is the mathematician's realm, for it is in mathematics the concept has its deepest roots, where it has been shaped and reshaped innumerable times, and where it finally celebrated its greatest triumph" (Maor, 1986, p.2).

Mythology has tried to answer the most difficult but basic question of the world's surroundings. The notion of infinity related commonly with cosmology, unboundedness, recurrence and large numbers. Mythology was substituted with explanations about the world's surroundings. The first to introduce the notion of that which has no limit was Thales, who was of the view that the material cause and first element of things was the Infinity (Mancosu, 1996). It was Aristotle who gave the distinction between actual and potential. Contrary to Aristotle's views, Leibniz proposed a really sophisticated and useful view of actual infinity (Mancosu, 1996).

Newton moved a step away from infinitesimals and a step closer to the limit notion, but had amazing comments about limits (Edwards, 1979),

> By the ultimate ratio of evanescent qualities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish… These ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limit towards which the ratios of quantities decreasing without limit do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to till the quantities are diminished in infinitum (p. 225)

The quote above illustrates the role played by motion in Newton's limit concept, and how vague and muddy the concept became as a result. Weierstrass developed the formal algebraic limit definition (Kleiner, 2001). He is credited for the placement of quantifiers, "for every" and "there exists" (Kleiner, 2001).

There are theories that account for the relationship between historical development and cognitive development. The first one is that of embodied cognition, which explains why historical development of concepts and individual development of concepts often parallel each other. This is explained as being due to the metaphorical structure of a concept. Thus, people's conceptions, including mathematical ones, are metaphorically grounded in our bodies and by our physical perception of the world (Lakoff & Nunez, 2000). The second theory is the best and most complete theoretical account of the relationship between historical development and cognitive development, and is provided by Piaget and Garcia (1989). These authors propose that:

> "… [Our] goal is not to set up correspondences between historical and psychogenetic sequences in terms of content, but rather to show that the mechanisms mediating transitions

> from one historical period to the next are analogous to those mediating the transition from one psychogenetic stage to the next." (p. 28)

The transitional mechanisms spoken of here are from analysis of Objects (intra-Objects) to analysis of relations between Objects (inter-Objects), to the construction of a structure of Objects (trans-Objects).

The third and last theory is the Basic Metaphor of Infinity (BMI), which is an example of a metaphor in higher mathematics. The BMI is viewed as "processes that go on indefinitely, are conceptualized as having an end and ultimate result" (Lakoff & Nunez, 2000, p.158). That is, an infinite process inherits the structure of a finite process. (A finite process possess a clear starting point, progression and ending, and an infinite process preserves this structure).

Mathematical truth has to be understood in conjunction with the history that produces it. Hence, mathematics can be fallible as its development was not through a cumulative advancement of rigorous proofs, but through an iterative process of criticism and correction (Lakatos, 1976). The application of this view will help the researcher understand the bizarre developmental concepts like the limit concept.

After seeing proof that 0.99999…=1, a student of Sierpinska observed that algebraically and arithmetically it is all right, but in reality it will not be equal to one but close to one (Sierpinska, 1987). This student admitted that the proof was mathematically correct but was not convinced about its correlation with what happens in reality. The student's intuition was that 0.9999…is not really a static number but an infinite process that cannot finish. In this way, the process is dynamic and the number 1 is never reached. The student had a very strong belief that even after the mathematical proof, the student rejected the proof instead of changing intuition. This is not unusual among calculus students. Research in undergraduate mathematics has repeatedly shown that similar obstacles to concepts of real numbers, functions and limits still holds.

Historical analysis suggests significant parallels between historical conceptions and student conceptions. Sierpinska's student's dynamic conception suggests some parallels with Newton's idea about the limit concept. That is students' conception and historical conception of the limit concept. Educational researchers have noticed parallels between student conception and historical conceptions of calculus (Lakoff & Nunez, 2000; Cornu, 1991). In order to tread our footsteps in those of the first travelers, we need to investigate parallels that exist between student thinking and historical thinking about calculus. In order to investigate the parallels in calculus between cognition and history, we have to determine the connection of students' conception to history.

In Juter's (2009) research study on Limits of functions as they developed through time and as students learn them today, few students displayed a concept image consist of intuitive reasoning and a perception of limits as objects, which indicated that they had reached Tall's second world, at least partially. Additionally, few students were unable to state or even recognize the limit definition through the entire course, but they could solve easy tasks. Their development had not followed the historical development, as they seemed to float between the two first worlds, never entering the third. Furthermore, they regarded infinitesimals as zero at one stage and at the last stage they knew how to deal with infinitesimals. The study did not aim specifically to investigate infinitesimals but these students' work with limits revealed how they perceived infinitesimals. Also, Juter's students showed great confusion about functions' abilities to attain limit values, with most of them treating limits as attainable when they solved tasks but unattainable in theoretical discussions.

A summary of the students' developments revealed they started with a computational approach and some, ordinarily low highfliers, retained this approach for the duration of the semester, while others, ordinarily high highfliers, got improved at describing the theoretical parts of limits during the semester. Almost all students displayed ambiguous views about attainability at some point. The outcome of this comparison of students' developments to the historical development of limits was that high achieving students were more inclined than low achieving students to follow the historical development. This argument was due to the notable fact of their abilities to perform abstraction.

2.7 Spontaneous conceptions of a limit

Spontaneous thoughts can be defined as the output of a broad category of uncontrolled and inaccessible higher order mental processes that arise frequently in everyday life (Morewedge, Giblin & Norton, 2014). It can be noted that people imbue spontaneous thoughts with particular importance. The formal learning of most mathematical thoughts do not start on virgin platforms. When students come to learn limit concept in relation to sequences and series, they come with some understanding about sequences, series and limits. The students already have certain ideas, images and knowledge from daily experiences. Concepts that happen prior to official education are unprompted conceptions (Cornu, 1991). Errors are not only a result of ignorance, but are also caused by previous knowledge which may have been interesting and successful at one level, but manifest as false or unadapted at a new level. During mathematics learning, these spontaneous ideas (previous knowledge which may have been interesting and successful at one level) do not disappear. Instead, they are adapted and modified or mixed with newly acquired knowledge to personal ideas of students.

In the case of the limit of sequences and series, the phrases “approaches”, “tends to”, “converges” and ‘limit’ partake importance for students, before lessons. These phrases have different meanings created by experiences of students’ daily language. Students’ continued reliance on such connotations after the introduction of formal limit definition. Investigations into the use of these phrases have shown that ‘to approach’ means ultimately remaining from the point. ‘Limit’ means impassable but reachable, a point being approached, but unreachable (or reaching it) (Cornu, 1991). Monaghan (1991) discovered the effects of language on the accepting of the limit conception. Monaghan gave students graphs of drawn curves and asked them to choose whether each particular curve tended to 0; had its limit as 0; converged to 0; approached 0. The four phrases are mathematically equivalent but students would accept one of them and disregard the others when dealing with the identical curve. When interviewed the students ‘responses were summarized as:

1. Majority of the students regarded, “limit” as a borderline not to be crossed. In the 0.99…, situation, the limit was viewed as a boundary point that could not be reached. That is 0.9… is the ultimate point of 0.9; 0.99; 0.99…, but border is also 1 to which sequences cannot reach.
2. The majority of the students held a dynamic view of a limit as “approaches” involving movement of one thing towards the other, with the controversy that an approached object is either eventually reached or will never be reached.
3. The use of “tends to” by many respondents indicates distinct predisposition or of wide-ranging trend. “Tends” to, and “approaches” in mathematical situations were viewed as having a similar meaning demonstrating movement of an object never reaching the approached destination.
4. In mathematical settings, students were commonly not sure about the meaning of “converges”.

Basing on the above results, Monaghan (1991) came to the conclusion that the everyday life meanings of these words are a possible source of perceptive conflict to students subjected to the limit instruction. These implications are likely to be retained and held continually, even after instruction.

Norton and Baldwin (2012) carried out a research to determine whether 0.999… really equaled 1. Their research confronted the issue why students resisted to accept the equality. This was the case even after the students had perceived and assumed reasonable opinions for the equality. Four arguments for the equality were raised, (1) relying on the decimal expansion of $\frac{1}{3}$, (2) subtracting of the infinite sequence, (3) generating a contradiction, and (4) defining the decimal expansion with limits.

Norton and Baildwin (2012) used the definition of the limit of a sequence formally as: a sequence S_n converges to the limit S $\lim_{n\to\infty} S_n$ = S if for any $\epsilon > 0$ there exists N such that $|S_{n} {}_{-S}| < \epsilon$ for n > N. Taking S to be the limit of a sequence, this definition amounts to a game of choosing. $\{S_n\}$ for any positive distance, ϵ , you choose, one can find a natural number ‘N’ so that whenever the sequence goes beyond the N th term, then the distance between any of these terms and S is less than ϵ. Using this definition, if the tail of a sequence gets arbitrarily closer to a number, then that number is the limit of that sequence. The decimal 0.99…. can be thought of as the limit of an infinite series, $\frac{9}{10} + \frac{99}{100} + \ldots. = \sum_{k=1}^{\infty} \frac{9}{10^k} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{9}{10^k} = 1$. The $|S_{n} {}_{-S}| < \epsilon$ equality is satisfied for all real values of ϵ that one can think of, there is a natural number N such that 1 is in the range ϵ of $\sum_{k=1}^{\infty} \frac{9}{10^k}$ whenever n > N. The above statement indicates that we are close enough to 1, if when given a ϵ neighborhood prolonging some distance around 1, we always discover a number N so that the numbers at the extreme end of the series are classified in the vicinity. This being the case, the terms of the series and 1 are undistinguishable.

Researchers discovered many reasons for students' rejection of the equality of 1 and 0.999 .... For example, Ely (2010) suggests that the majority of students considered 0.99…. as something in motion instead of a motionless point. Students took the decimal expansion as a restating point that is coming nearer and nearer to 1 without ever attaining the value 1. 0.999…. = 0.9 + 0.09 + 0.009 + …., which leads to infinite series, and the question now is; are students capable of accepting that the aggregate of infinite number of positive real numbers is a real number? The most frequent answer is 'no'.

The researcher through reading literature discovered that the task of finding the sum of infinite series requires coordinated usage of many concepts that are related to the term infinite. These are: 1) the number of terms in the infinite series, 2) the infinite process, 3) the sum of infinite series. The three concepts need to be linked in the student's mind. The usage of these concepts separately causes difficulties.

Eisenmann (2009) identifies problems that can be specified concerning students' understanding of infinite series thus: 1) students' attitude that infinite series cannot be summed up. In regard to this, Eisenmann's (2009) finding revealed that some students argued that it cannot be determined if it goes up to infinite since it has no end after all. The students commented that summing up to the infinite is impossible since something more can be added after all. Furthermore, the students commented that if one adds one more number, it further grows and keeps on growing until the infinite. It was a common idea with the majority of students that the sequences of partial sums of the infinite series of positive terms grow above all limits. Psychogenesis and ontogenesis relation corresponds to Zenon's belief (about 490- 430BC) that the totality of an infinite number of line subdivisions must not be infinite. On $\sum_{k=1}^{\infty} {}^{1}/_{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$, the students argued that the series’ sum never equals one but gets closer and closer to one and never gets there. Furthermore, the students commented that there is always a little bit missing from one, and if we keep on adding, there will always be a little bit missing. Eisenmann (2009) recommended that a suitable procedure for teaching the concept of a sum of the infinite series, starts with a problem of the sum of the infinite series (0.999…. = 1), followed by of a sequence limit, then series sum.

Denbel (2014) carried out a research study on misconceptions of the limit concept held by students in a first year calculus course. The results revealed that a limit is: (a) unreachable, (b) a boundary (c) a moving process and not a non-moving object. The researcher came to the conclusion that the knowledge and understanding of many students rested mainly on detached facts, predictable calculations, and memorized algorithmic procedures which lead to deficient conceptual understanding of limits.

2.8 Misconceptions in the study of the limit concept

Generally, misconceptions manifest through errors, Egodawatte and Stoilescu (2015) pointed out that students’ errors are the symptom of misunderstanding. Misconceptions could be deeply ingrained in the mental map of an individual.

There is documented research literature that testifies to difficulties faced by students when dealing with the limit concept (Corn, 1981; Cotrill et al., 1996; Tall & Vinner, 1981; Williams, 1991). Students’ misconceptions of limits and infinity are a well-documented and researched area in mathematics education literature (e.g., Cory & Garofalo, 2011; Roh, 2008). The difficulties faced were a result of misconceptions held by students. A misconception arises where a concept is misunderstood or misinterpreted which is derived from inaccurate meanings (Ojose, 2015). Hence misconceptions can be defined as perceptions that have different or wrong meaning from that of experts.

Szydlik (2000) found that students had a misconception of the limit, they viewed a limit as a boundary that could not be crossed. Students who held such a misconception, found it difficult to recognize limits of functions that oscillated about a value, but got closer to that value. Furthermore, Szydlik (2000) cautioned that the notions of a limit as a bound and a limit as unreachable were serious misconceptions that contradicted the formal definition of limit.

Przenioslo (2006) found out that students thought that a sequence was not a function, which contradicted textbooks that define a sequence as a function. Furthermore, Przenioslo (2006) noticed that students thought a sequence must be monotonic, the terms of a sequence must be described by an explicit formula and that the difference between sequential terms in the sequence must remain the same always. Roh (2008) found students had a view of limit as cluster points or asymptotes, such students had difficulty determining convergence or divergence of sequences. Additionally, here are some of the misconceptions reported on the limit of a sequence (Flores & Park, 2016; Oehrtman, Swinyard & Martin, 2014):

- The last term in an infinite sequence is the limit.
- A limit is a number which theoretically can be reached.
- A limit is a boundary that cannot be surpassed.
- As a_n gets closer and closer to its limit, ***n*** increases.
- Limit is a boundary that cannot be reached.

Furthermore, these misconceptions can be carried over to related topics in calculus such as infinite series or continuous and differentiable functions. Students held the misconception that the limit of partial sums and the sum of an infinite series were not the same, some students tried to determine the sum the series by first adding all the terms (or as many as one could to determine a pattern) and then took the limit of the partial sums (Martínez-Planell, Gonzalez, DiCristina, & Acevedo, 2012). Martínez-Planell et al. (2012) noticed in general that students relied on properties regarding finite sums, rather than looking at the limit of partial sums. Furthermore, (Nardi & Iannone, 2001) discovered that students had difficulty accepting that the convergence tests can be inconclusive. Students when asked if $\sum_{n=1}^{\infty}\frac{n+2}{n^3-n^2+11}$ converge or diverge, the comparison test was correctly used to obtain a limit of zero. However instead of concluding that the test was inconclusive, the students concluded that the original series does not converge. There is need to find ways to help students minimize or overcome these misconceptions.

Earls (2017) carried out a study on student's misconceptions of sequences and series in second semester calculus. The results of the study revealed that **s**tudents had misconceptions on limits and series when learning the concepts. Earls' students' exam work exhibited that some students got a value of one (1) in the ratio test and concluded that the series converged to one (1). Additionally, the students faced difficulties when determining series convergence test to be used for given problem, the difficulties included identifying a geometric series, and instead used a root test to determine convergence. Furthermore, students had trouble identifying the contrapositive of the nth term test and the logical equivalence of a statement and its contrapositive in geometry. Distinguishing between the limit of a sequence and the sum of a series was also difficult for students, specifically students had difficulty accepting the difference between a sequence and a sequence of partial sums. The results of Earls' study showed that students had problems with limit and series notation and sometimes failed to check satisfaction of the assumptions of a series test before using the test. In the exam data, students neglected to check if the assumptions of the integral test were satisfied, and in the interviews a student failed to test if the assumptions of the direct comparison test were satisfied. Earls' students also had misconceptions about what the conclusions of series tests told them. There were several examples of students who thought that a series test would give a value to which a series converged to rather than just whether the series converged. Furthermore, Earls 'students had a difficult time on how to proceed after they got a numerical value from a series test.

The results of this study contributed to the existing literature on student misconceptions on the limit concept in relation to sequences and series.

2.9 Helping students overcome misconceptions

From reading literature, the researcher noted that the world over, there is a tendency to engage Information Communication Technology (ICT) in the teaching and learning process (Weigand, 2004; Awang & Zakaria, 2012; Yesilyurt, 2010; Fathima, 2013). The lecturer/ instructor and the learner need to gain access to technology to improve learning outcomes. ICT involves the use of computers, computer software, and other devices to convert, store, process, transmit and retrieve information. ICTs are used in this research, as well as an assortment of practices and information of how to bring together materials to come up with preferred product, to solve problems. The solving of problems involves skills, processes, techniques, tools and raw data. Good teaching requires application of technology to represent novel conceptions and develop understanding to the ever changing transactional relationship among the three constituents of understanding, namely; Content, Pedagogy and Technology. This relationship is illustrated in Figure 2.1.

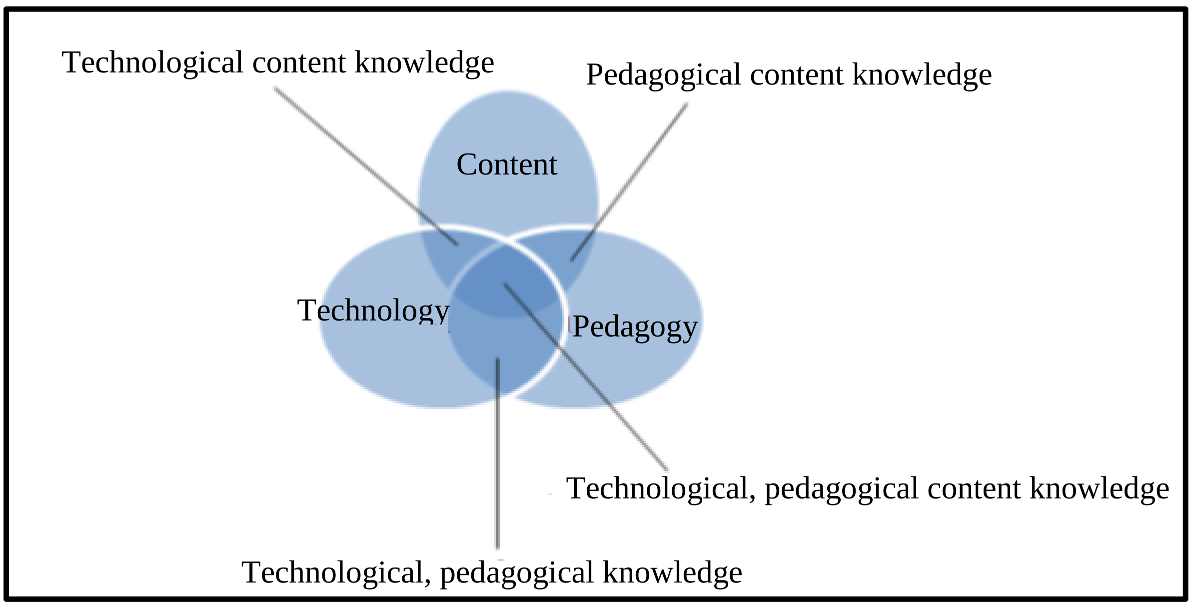


Figure 2.1: The relationship among Technology, Pedagogy and Content.

Technology provides opportunities for doing mathematics by enabling students to work with diagrams animatedly, and encouraging them to picture the geometry as they create their own mental pictures through (Fathima, 2013):

-Observing patterns- the speed of computers and calculators enable students to produce many examples when exploring mathematical problems. This supports their observations of pattern and the making and justifying of generalizations.

> -Seeing connections – the computer enables formulae, tables of numbers and graphs to be linked readily. Changing one representation and seeing changes in the other helps students to understand connections between them.
>
> -Working with dynamic images- students can use computers to manipulate diagrams dynamically. This encourages them to visualize the geometry as they generate their own mental images.
>
> - Exploring data – computers enable students to work with real data, which can be represented in variety of ways. This supports interpretation and analysis.

(p. 53)

From the above, it is apparent that technology allows for more conceptual approach to teaching and learning, and enables students to take note of the relationship between representative algebra and graphical representation, more freely.

In the study of limits, efforts should be made to help students overcome misconceptions brought about by another way of instruction. Monaghan, Sun and Tall (1994) used a computer model for computing values of $s(n)$ for large n to enable learners to embrace the conviction that limit is a process and not an object. They also used a mathematical software called Derive to evaluate limits for students to experience properties of objects produced by a computer. Still, students held misconceptions that limits are never ending processes.

Miller (2011) carried out a study on Technology-enhanced calculus lectures with the main goal of improving students' satisfaction with the course. The secondary goals included improvement of conceptual understanding and students' performance. The lectures were presented using a computer and a projector to improve visibility. A Smart Podium was used for communicating detailed procedures to students. Smart notebook software was prepared before classes, with relevant images, graphs, theorems and equations. The computer was also handy for the instructor to show and annotate other content on the computer such as webpages and calculators. Camtasia was used to record the proceedings of each class session. A table of contents with topics was provided in the video for specific concepts for easy access. The videos offered the students the opportunity for re-watching sections as the attempted written homework. Interactive figures were used to stress the conceptual understanding of topics during lectures. The figures comprised graphs and illustrations that moved as values of parameters changed. The figures were created using GeoGebra or Mathematica.

The results of the student survey were overwhelmingly positive. For the question; “for my next math course, if I was given the choice, I would choose to take the technology- enhanced section”, 87.5% of the first quarter of students and 92.36% of the second quarter chose “agree” or “strongly agree” with 78.17 % of the second quarter strongly agreeing. When asked, “I have a favourable opinion of the mathematics department after this course”, 83.33% of the students agreed or strongly agreed. Students were asked about the use of technology-enhanced calculus lectures, some of their comments were: “There is nothing the chalkboard can do better than the sympodium, especially in terms of understanding; “The sympodium is a great tool and I think every class should use it. “Much” easier to see… the colors are awesome and really help when looking back over the material in my own time”. “I did not do too well in the traditional 151 course, and I attribute much of my success in 152 to the technology features of this course”.

Miller (2011) concluded that students were more satisfied with technology-enhanced lectures than the traditional ones. The instructors enjoyed teaching using the equipment and students’ performance improved through the use of technology. The use of visualization using technology offers an alternative way for appreciating mathematical values and occurrences via appearance and analytic description. It deepens the understanding of the origin of conceptions and enhances the understanding ability of the learner (Mang & Minjun, 2013). Visualization is an instinctive and easy detailed way of demonstrating and handling information. Visual information makes stated facts more tightly-compressed and relationships accessible more vibrantly, intuitive, faster and at ease to communicate than arguments. The visual method is for addressing dynamic and changing problems, regularly troubled with a variable whose “image” is of a trajectory in motion. For example, $|a_{n-\, a_m}| < \epsilon$ depicts the phenomenon that the further the convergent $\{a_n\}$ is, the nearer it is packed about a constant. The imagining technique is the summary and upgrading of numerical-graphical amalgamation process which is often applied to static objects.

In higher mathematics, there is a Basic Metaphor of Infinity (BMI) which authors claim that higher mathematics uses in various and numerous forms. The BMI metaphor states that processes that continue indeterminately are intellectualized as having an ultimate resultant and an end (Lakoff & Nuez, 2000). Thus, an infinite process inherits the structure of a finite process having a beginning, a clear way of progressing, and a final result state. An infinite process also has a beginning, clear way of progressing, and inherits a final resultant state in order to preserve the structure. BMI applies clearly to the limit concept in relation to sequences and series where an unending string of numbers or sums is viewed as having a limit value like a finite sequence or series.

2.10 The APOS learning Theory

The APOS (Action- Process- Objects- Schema) Theory is based on Piaget's principle that an individual student learns Mathematics by applying certain mental mechanisms to build specific mental structures which the student then uses to deal with problems connected or related to the corresponding situation (Piaget, 1972). The APOS theory was used to define students' perspective and outlined students' understanding of the limit concept in relation to sequences and series. Interiorization and encapsulation are the main mechanisms, and Actions, Processes, Objects and Schema (APOS) are the related structures. Piaget (in Pashcos & Farmaki 2006) defined interiorization, as the internal construction process, how the perceived phenomenon is defined; that is, when "a series of actions translate the material into an operating system interiorized" (Piaget, 1980, p. 90). Interiorization enables people to become aware of their actions, to reflect and to combine actions with other actions. Interiorization includes the ability to apply symbols, language, pictures and mental images to construct internal processes as a way of making sense out of perceived phenomena. Actions on objects are interiorized into a system of operations. Encapsulation or adaptation from the process (dynamic) become an object (static), in the sense that, "... the actions or operations into objects are thematized from mind or assimilation" (Piaget, 1985, p. 49). Encapsulation is the ability to conceive a previous process as an object.

Action Level: Actions are transformations of previously constructed cognitive objects that students perceive as external or simply apply recall to answer given problems (Suryadi, 2012). Furthermore, the student arrives at the solution in a step by step manner.

Process Level: when an Action is repeated, then reflection on Action which was done occurs, then subsequently a Process phase is entered into. An individual experiences a Process about a concept if his/her thinking on a mathematical idea is marked by the ability to do reflection towards that mathematical idea (Suryadi, 2012). When an action or actions are repeated, students can reflect upon them in such a way that they do not require external stimuli; they can imagine or carry out them without following the specific order given by the Action level mental structure or by skipping some of them. In such a case, it is considered that the Action has been interiorized into a Process.

Object Level: one experiences an Object conception if one is capable to treat an idea or concept as a cognitive object. This requires the ability to encapsulate the process mental structure (Suryadi, 2012). When students need to apply actions on a process and they are able to conceive it as a whole, they can encapsulate the process and construct a cognitive Object.

Shema Level: Suryadi (2012) stated that a scheme of a certain mathematical material is a collection of Action, Process, Object and other Schema which are related to each other in order to create a framework which is interrelated in the individual's thinking. The structures (action, process, object and schema), are related by general principles in such a way that they generate a coherent framework that the student can use in the solution of problems.
A detailed explanation of the APOS Theory is provided in section 3.3.

2.11 ACE learning cycle.

This study employed the ACE (Activities, Class discussion and Exercises) pedagogical strategy teaching cycle. The ACE learning cycle is a repeated circle of three components: Activities on the computer, Classroom discussion, and Exercises done outside the class (Arnon et al., 2014). The ACE learning cycle's implementation and effectiveness in helping students make mental constructions in the learning of a mathematical concept has been reported in several research studies of the Dubinsky's team (Arnon et al., 2014; Asiala et al., 1996; Weller et al., 2011).

Activities, are the first step of the cycle where students worked cooperatively in groups on tasks designed to help them to make the mental constructions suggested by the genetic decomposition. The tasks focused to promote reflective abstraction rather than to obtain correct answers. This study employed the ACE learning cycle (Activities, Class discussion and Exercises) pedagogical strategy teaching cycle. Activities carried out by students designed in the form of tutorials, were meant to help students make mental constructions.

The second part of the ACE learning cycle, Classroom Discussion, involved all the small groups of students and the lecturer led class discussion. The discussions were on the work students were worked on paper and pen tasks that were built on the lab activities completed in the Activities phase.

Homework exercises, which formed the third part of the cycle, consisted of fairly standard problems designed to reinforce the computer activities and the classroom discussion. The exercises helped to support continued development of the mental constructions suggested by the genetic decomposition (Dubinsky, Weller, & Arnon, 2013). They also help students to apply what they have learned and to consider related mathematical ideas (Dubinsky, Weller, & Arnon, 2013).

2.12 Recent studies using the APOS Theory and ACE learning cycles.

Aydin and Mutlu (2013) used the APOS Theory to examine vocational high school students' understanding of functions at a University in Turkey. There were four types of questions used covering limits on: split-functions, infinity of rational functions, functions undefined at a point, and real functions' continuity. The ACE learning cycle was used, with the activities being carried out using PC tablets. The results revealed that vocational high school students had difficulties in understanding the limits of real functions. The findings advised that many students failed to develop suitable mental arrangements at the Process, Objects and Schema levels.

Maharaj (2014) studied on an APOS analysis on how integration was understood by natural Science students. The ACE learning cycle was employed. The researcher managed to use relevant rules for antiderivatives and definite integrals and their context-based application. The research findings revealed that students found the topic difficult. Students interviewed showed that they could not interpret objects symbolic form of integration. The majority of the students were operating at Action level and failed to develop the suitable mental constructions at the Process, Objects and Schema level.

Voskoglou (2015) used the ACE learning cycle instructional treatment in the study on fuzzy logic. In the study, two groups were represented as fuzzy subsets of the set, graded "A- F"; the experimental group and the control group. The instructional treatment used by Voskoglou (2015) was done in three ACE learning cycle iterations:

Stage one of the ACE learning cycle covered Action level of rational or irrational numbers (an infinite decimal) concept, which is characterized by the attention of determinate decimal approximation. Stage one sought to facilitate interiorization of this Action to a Process. Preloaded decimal expansion packages were used by students to complete activities in the computer laboratory. This helped them develop general descriptions of infinite digit strings. The students also managed to evaluate consecutive infinite decimal calculations of numerous square roots. Students reported and held discussions on group findings from the computer laboratory in the classroom. The class devised a notational system for finite decimals for example 1 < $\sqrt{2}$ < 2, 1.4 < $\sqrt{2}$ < 1.5, 1.41 < $\sqrt{2}$ < 1.42, 1.414 < $\sqrt{2}$ < 1.415 ect then $\sqrt{2}$ can be written as $\sqrt{2}$ = 1.41421…. Student accepted infinite decimal represented symbolically indicating that they could not write all its decimal digits but only the decimal approximation each time. The lecturer gave an explanation of rational numbers (repeating decimal) that can be written as $a.b\bar{c}$ where a is the digit(s) of the integer part, b the decimal fraction before the repeating cycle, and c the restating series of numbers.

The second ACE learning cycle iteration sought to enable the encapsulation of the real number concept to a conceptual object. At one stage, students considered that there were disconnection among different kinds of real numbers fractions, roots, and decimals. The researcher targeted to help students appretiate the sameness of all these numbers and acknowledge alternative forms of writing them. The alternative forms helped students appretiate the set of real numbers as a totallity to which trasnformations can be performed. During the computer activities, students worked on clear and opaque decimal illustrations of real numbers. Examples of transparent decimal representations are $\frac{3}{5} = 0.6$, $\frac{1}{3} = 0.33$ ... since their decimal digits can be foressen. Examples of opaque decimal representations include $\frac{1}{1861} = 0.0005373$ ... which possess a period of 1860 digits. A standard method of converting periodic numbers to fractions is by subtracting, for example, $x = 2.7532323$ ... we can write $10000x = 27532.3232$ ... and $100x = 275.3232..$ subtracting $100x = 275.3232..$ from $10000x = 27532.3232$ ... we get $9900x = 27532 - 275$

which gives $x = \frac{27257}{9900}$. When students reflect on this example, they can conclude that interrupted decimals and fractions are equivalent numbers written in alternative ways.

The last ACE leraning cycle iteration sought to enlist real numbers in general as a way of empowering students to understand irrational numbers. The computer activities at this stage involved the construction of incommensurable lengths like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ excetra. Real numbers can be written in the form of series x = $\sum_{n=0}^{\infty} \frac{k_n}{10^n}$ with k_0 as a positive whole number and k_1 , k_2 .... being natural numbers less than 10. Illustrations of interpolation of irrational numbers and rational numbers between two given integers, aimed to encourage classroom discussion on density of the sets of rational and real numbers.

The research findings showed that the application of the APOS Theory and ACE learning cycle effectively helped students to enlist the real numbers in a powerful cognitive schema including all the basic sets of numbers. The application of the APOS Theory and ACE learning cycle approach enhanced significantly the students' understanding of the real numbers in general and in particular the irrational numbers. The above iterations of the ACE learning cycle were very instrumental in the design and implementation of the ACE learning cycle on the study of the limit of sequences and limit of series.

2.13 Recent studies using APOS Theory

The researcher was motivated by literature on the application of the APOS Theory, mainly from South Africa, to come up with the theoretical framework. Some of these studies are summarized and discussed below.

Maharaj (2010) worked on students' understanding of the concept of a limit of a function using the APOS Theory. His research culminated in the genetic decomposition of a limit concept in calculus. His study focused on how the teaching of a limit of a function could be handled together with the understanding that the APOS Theory analysis of the limit of a function revealed students' understanding. At an Action level when confronted with the limit of a function $\lim_{x \to a} f(x)$ a student merely used values of "x" close to "a" for a variable in f(x), then calculate it with or without seeing the emerging pattern. At the Process level of the function, $\lim_{x \to a} f(x)$, a student constructs a mental process for values of "x" closer to "a", thinking of inputs that produce outputs. An Objects understanding of the function $\lim_{x \to a} f(x)$ emerges as the student sees the sequence as an entirety and can act on mental or written Actions on limits of one-sided functions. The encapsulation of the Process brings about an Object of $\lim_{x \to a} f(x)$, which has the possibility of either existing or not existing. The schema level is characterized by Actions, Processes and Objects' organization and connections into a logical framework. The coherent structure takes into account possible procedures for evaluating $\lim_{x \to a} f(x)$, where "a" could be -∞ or ∞. The above genetic decomposition was used as the basis for analyzing the data. The findings were as follows:

1) The genetic decomposition was adequate.
2) The students found the limit of a function concept difficult to understand, suggesting students' failure to produce appropriate mental constructions suitable for the Process, Objects or Schema levels.
3) More time was needed for the teaching design.
4) Teaching should emphasis on unwritten and graphic methodologies to discover limits of a function, undoing of structures in symbolic form, and forming possible Schema.

Ndlovu and Brijlall (2015) carried out a study on mental constructions made by pre-service teachers on concepts in matrix algebra. Their study was underpinned by APOS Theory and directed by the confidence that understanding mental constructions leads to enhanced instructional methods. The findings of their study revealed concurrence between mental constructions of pre-se rvice teachers and the preliminary genetic decompositions. Majority of the pre-service teachers operated at Action and Process levels and a few operated at Objects level and none at Schema level.

Brijlall and Maharaj (2015) applied APOS Theory discovering pre-service teachers' mental structures when dealing with problems relating to infinite sets. The research study was done with the help of students from a South African university. A qualitative methodology paradigm was employed with five subsections: the case study, participants' background, issues of ethics, validity and reliability, and the Itemized Genetic Decomposition (IGD) on two tasks. The itemized genetic decomposition was introduced by Brijlall and Ndlovu (2013) who defined a specific mathematical task confronting an individual student (IGD) as a genetic decomposition. In this study, the IGDs are of particular interest. The researcher extended this notion for the tasks on the limit concept in relation to sequences and series.

Pre-service teachers were asked to work on two tasks with IGD for each task provided. In order for the students to answer task 1 correctly on base equivalence of two countable infinite sets, the IGD in Figure 2.2 was provided.

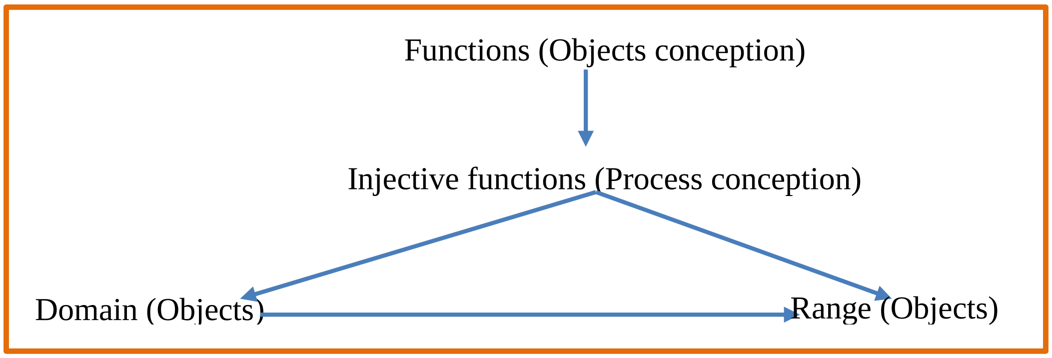


Figure 2.2: IGD for task 1 (Brijlall & Maharaj, 2015)

The question for task 2 was: Is 0.99… = 1 for which the researchers provide IGD Figure 2.3

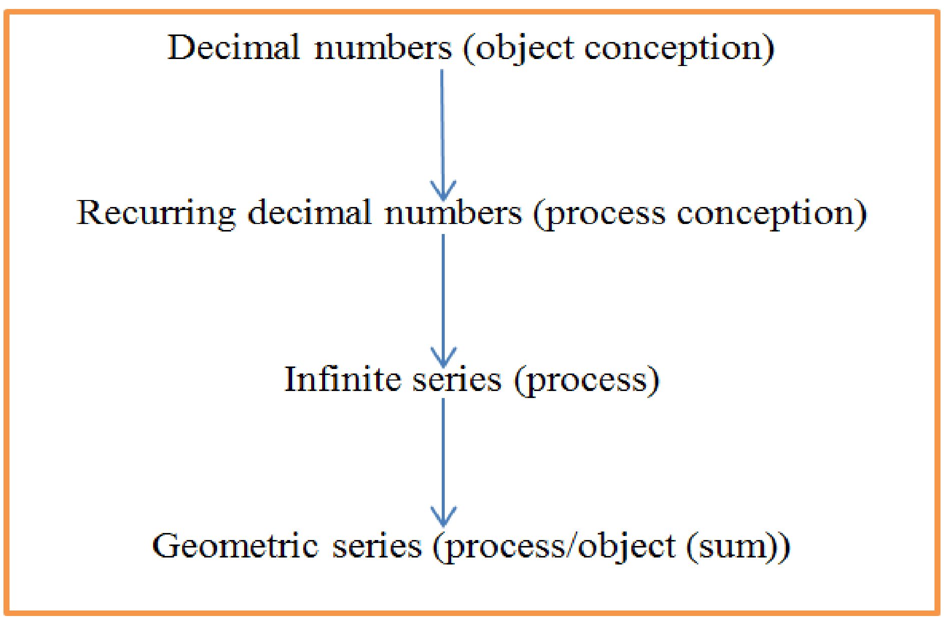


Figure 2.3: IGD for task 2 (Brijlall & Maharaj, 2015)

An alternative way of solving task 2 would be Figure 2.4 which the researchers provided.

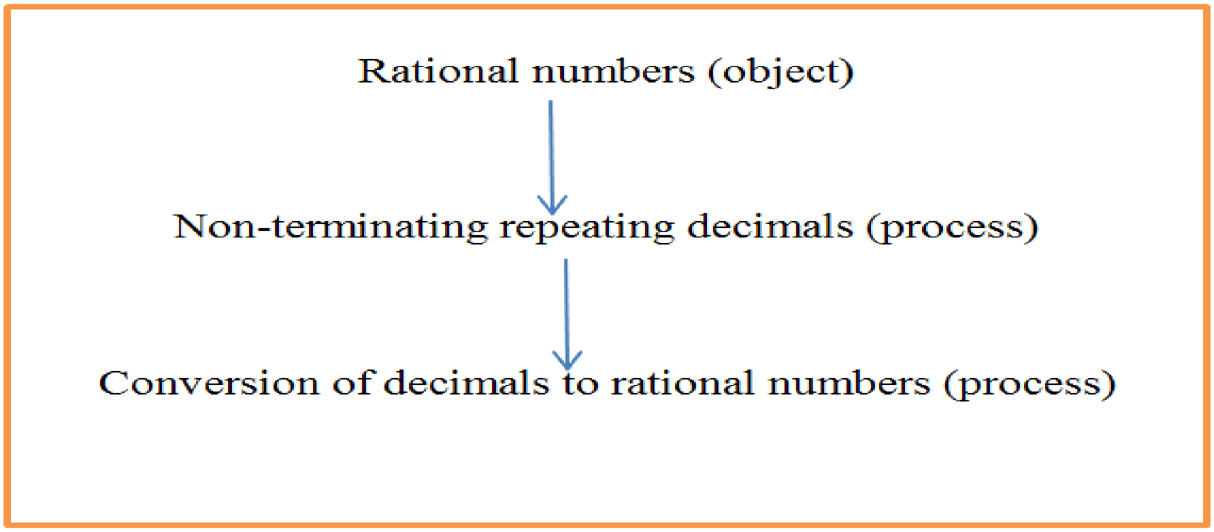


Figure 2.4: An alternative way for IGD for task 2 (Brijlall & Maharaj, 2015)

Brijlall and Maharaj (2015) found the following results. For task 1, $\frac{15}{38}$ of the pre-service teachers gave the answer: the set of positive odd numbers had less elements than the set of natural numbers. Their reason was that all the elements of the set of positive odd numbers are found in the set of natural numbers. This implies that the respondents were operational at the Process level. They believed in a subset having fewer elements than the inherited set. They were not able to notice that positive odd numbers and natural numbers are different objects. Less than half of the students ($\frac{11}{38}$) gave the response that the set natural numbers had fewer elements than the set of odd positive numbers. They could have arrived at this conclusion by considering the magnitudes of the elements of the given sets. This shows that the respondents did not have adequate Process conception of the association rule indicated by IGD for task 1. The pre-service teachers whose conclusion stated, positive odd numbers were equal to natural number basing on injective association from set of natural numbers onto set of positive odd numbers were $\frac{9}{38}$. This gives a possibility of a Process conception of the association rule given by the IGD for task 1. The doubtful responses were captured by $\frac{2}{38}$ of the respondents who gave a contradicting statement by stating that set of positive odd number was larger than set of natural number but their explanation indicated the opposite of what they had said earlier. For the only respond “not larger, smaller or equal to”. It appears no conclusion could be reached about the equivalence of infinite countable sets. This last particular one led to the modification of the IGD as shown in Figure 2.5, to include the notion of infinite countable sets. In their research, Brijlall and Ndlovu (2013) proved that modified itemized genetic decompositions are very useful.

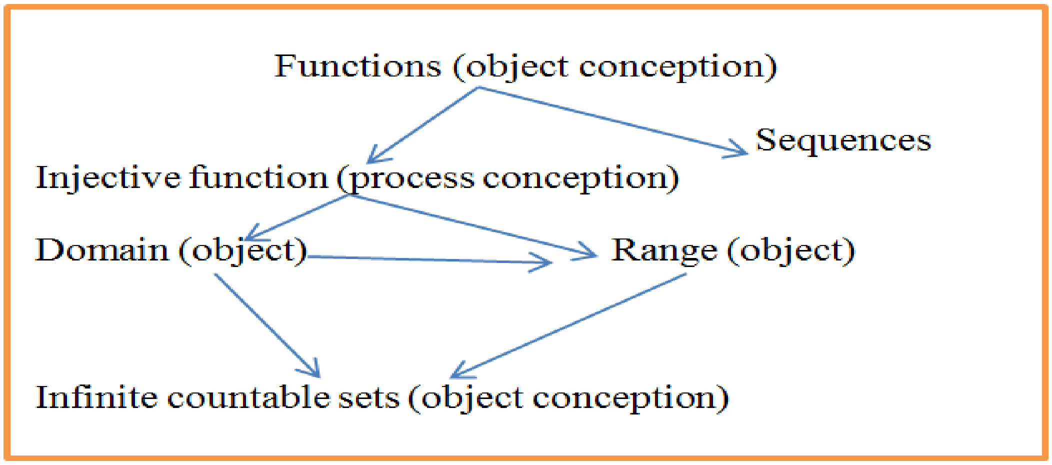


Figure 2.5: A modified IGD for task 1 (Brijlall & Maharaj, 2015)

The findings for task 2 showed that the bulk of the pre-service teachers $\frac{37}{38}$ indicated that $0.99\ldots = 1$ was not valid, $\frac{7}{38}$ responded "yes", to the question is 0.99…=1. Five of them based their response on the fact of rounding off 0.99…. to equal 1. This shows that they made the correct choice but without providing suitable explanations. There were four special cases where one stated that 0.99… is the closet number to one, and so it has to be one. This idea led to the modification of the IGD for task 2. The other responded that two numbers are completely different entities and therefore cannot be equal. The other one argued that, getting the variance between the two numbers gives a non-zero answer. The researchers observed that this last respondent held a Process notion of a repetitive decimal fraction and an Object idea of the number 1. This led to the presence of Object of repetitive decimal fraction in the adjusted IGD for task 2, Figure 2.6.

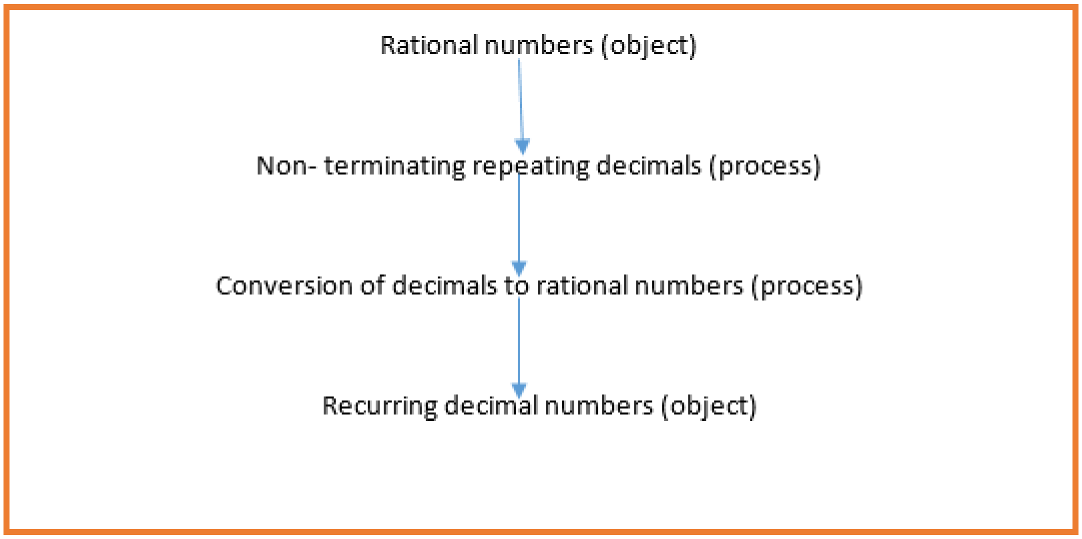


Figure 2.6: A modified IGD for task 2 (Brijlall & Maharaj, 2015)

The researcher first came up with a preliminary genetic decomposition of the limit of sequences and then that of the limit of series and these were later modified in accordance with the responses given by the students, just like in the cases above.

Borji and Voskoglou (2016) studied students' understanding of Polar Coordinates using APOS theory. This research was carried out specifically for the discovery of a genetic decomposition for learning Polar Cooordinates. The students' responces to both the written test and the oral interviews revealed that students' difficulties emanated from imcomplete: previous knowledge of trigonometrical angles and functions, knowledge and usage of the Catersian coordinates and equations, understanding of the polar coordinates and equations and conversion of Cartesian coordinates to polar and vice versa. Based on conclusions made from the written test and the oral interviews, the following genetic decomposition was arrived at:

Stage 1 Prerequisite knowledge

- Drawing angles clockwise and counterclockwise in a plane $\frac{\pi}{4}$ and - $\frac{\pi}{4}$.
- Sketching graphs of trigonometrical functions.
- Cartesian coordinates system linear and quadratic graphs.

Stage 2 Polar coordinates

- Action – identifying a point $P(r, \theta)$

- Process – plotting point $P(r, \theta)$ for positive and negative r and θ.
- Objects – encapsulation of Actions and Processes .
- Schema – Actions, Processes and Objects organization into comprehensible schema which permits students to sketch polar equations.

Step 3 Cartesian coordinates conversion to Polar and vice versa

- Action – conversion of $P(r, \theta)$ to $P(x, y)$.
- Process – understandig of $P(r, \theta)$ and $P(x, y)$ as the same point.
- Objects – conversion of polar equations $f(r, \theta) = 0$ to cartesian equation $g(x, y) = 0$ and vice versa.
- Schema – sketching of polar equation in all quardrants.

Ndlovu and Brijlall (2016) determined the mental constructions displayed by pre-service mathematics teachers on the determinant concept. Activity sheet for thirty-one students supported data collection. Five students were selected for interviews based on their performance on the activity sheet. The belief that the mental constructions made during the learning process could lead to improved instruction guided the study. The results revealed operational level of most pre-service mathematics teachers to be the Action/Process level with a few managing to operate at an Object level. Furthermore, it was revealed that many students had challenges leading to their failure to operate beyond procedural understanding of the determinant concept.

Makonye (2017) carried out a research study to determine pre-service mathematics students' conception of effective interest and nominal rates. The researcher analyzed seventy students' responses doing mathematics at second year university level. Tasks covered work on nomial and effective interest rates. Of the seventy students, twelve were interviewed to ascertain their mathematical thinking when solving tasks. There was variation in the findings, with some students failing due to erratic use of formulae. The researcher discovered some students got the correct solutions due to scrupulus adherence to the formulae. The Action and Process levels marked the end of progress for the majority, with only a few reaching the Object and Schema levels. These showed strong accepting of the association amid nominal and effective interest rate. From these results, the researcher recommended that more time need to be devoted by teachers on these notions.

Kazunga and Bansilal (2017) detected the understanding of matrix operation by Zimbabwean in-service mathematics teachers. Researchers generated their data from written responses of 116 teachers and from these, 13 were selected for interviews. The results of the written responses showed that many students managed to move from matrix addition Action level to Process level, similarily with scalar multiplication and matrix multiplication. The students had challenges with the use of brackets in matrices and distinguishing the operative equal sign denoting equivalence between two objects.

The studies above indicate that APOS Theory is a relevant and effective theory for analyzing mental constructions that students make during the learning of various mathematical concepts. In these studies, pedagogical methods' emphasis is on prompting relevant mental constructs. The researcher has also read many studies exploring the APOS Theory in the understanding of different concepts (Afgani et al., 2017; Bansilal, 2014; Maharaj, 2010). To the researcher's knowledge, no study has focused on analyzing the understanding of mental constructions made by students when learning the limit concept in relation to sequences and series. This study seeks to address this gap.

2.14 Implication of the literature review for this study

Important to mathematics are limits of sequences and series. Limits in relation to sequences and series understanding poses challenges to calculus students. The reports on the reviewed studies concerned with limit conception were examined for confirmation of considerations and discussions about limits in relation to sequences and series. There are few studies covering understanding of limits of sequences and fewer studies on series limits.

2.15 Conclusion

This chapter provided an overview of literature: an understanding of limits of sequences, an understanding of limits of series and the importance placed on instructional design in the teaching of limits, sequences and series. From the discussion, it is evident that to facilitate the understanding of concepts, new knowledge should be incorporated into existing related networks. This chapter concluded by a reflection on the implication of the literature reviewed. The next chapter, chapter three focuses on the theoretical framework informing this study.

CHAPTER 3 THEORETICAL FRAMEWORK

3.1 Introduction

This chapter describes the literature that informed the researcher's theoretical framework. It describes a specific theory that helped the researcher to establish a perspective and a lens through which this study was viewed. This study is immersed in the constructivist paradigm and guided by the APOS theory which "can be thought of as an extension of the last stage of Piaget's theory, the formal operational stage which takes place at about the age of sixteen" (Weyer, 2010, p. 9). A constructivist paradigm is a framework within which theories are built, that essentially stimulates how one sees the world, defines one's perspective and outlines one's understanding of how things are connected. The chapter outlines a discussion on constructivism under the Theoretical Framework for the Study, APOS Theory, and key components in the theoretical framework. Also outlined are: methodological framework, the understanding of a mathematical concept, and construction in reflective abstraction.

3.2 The Theoretical Framework for the Study

The theoretical framework is the structure that can hold or support a research study, it introduces and describes the theory which explains why the research problem under study exists (USC Libraries, 2014). The theoretical framework consisted of concepts (reflective abstraction, cooperative learning, ACE learning cycle), together with their definitions, and existing theory (APOS Theory) that were used for this particular study. The theoretical framework was found through reviewing course readings and pertinent research literature for theories and analytic models that were relevant to the research problem under investigation. The selection of the theoretical framework was considered due to its appropriateness, ease of application, and explanatory power to explore students' understanding of the limit concept in relation to sequences and series.

3.2.1 Constructivism

Constructivism is learner centered; a feature that does not accept rigid loyalty to stages of lecturer input but the stages of lecturer guidance in: a) determining knowledge undertakings by bearing in mind the particular students' needs, b) shifting across sequences of learning episodes through highly structured guidance with varied exploratory activities (Taber, 2011). If constructivism is understood in the above terms, it delivers a firm academic basis for apprising instruction. Constructivism has its basis on how students make sense of their learning experiences in their environment. The object or event; for example, a definition of a limit of a sequence uttered by the lecturer in a classroom and a graphical representation of the limit of a given sequence on the computer, has some inherent meaning which can be identified and added to the store of knowledge about the world, by the student. Constructivism advocates meaningful learning in which the student has to actively construct a meaningful interpretation of what is being seen and heard. In some situations, students in a class may construct similar meaning which may closely reflect the lecturer's intended meaning. However, in other situations, this is far from being the case. From a constructivist viewpoint, both scenarios make sense and most educators of constructivists approaches agree that in many topics, students experience learning difficulties due to the students understanding differently to what was intended, and not a matter of students not understanding the teaching (Taber, 2011). A constructivist model of learning proposes that individual students in a class carries exclusive conceptual and thinking resources to tolerate on the lesson. Kuhn (2012) is one of the 20th century philosophers who is credited with formulating the argument that knowledge is an individual's construction which is relative to the current context (community) working with others, and not the representation of some correspondence to external reality. This led to two major ideas involving constructivism: (1) learning is an active process of knowledge construction and not its acquisition and (2) instruction is not communicating knowledge but a process of supporting knowledge construction.

Constructivism may be taken as a "grand theory" as proposed by (DiSessa & Cobb's, 2004) topology. Constructivism creates a way to test mathematics from the eyes, mind and hands of the student. It was established to satisfy researchers' interest on student's reasoning capabilities, outside humble analytical observation of errors to appreciate the productivity of students' approaches and strategies (Confrey & Kazak, 2006). It is rooted in practice for it addresses two primary concerns of lecturers as indicated by: (Confrey and Kazak 2006)

(1) "Students' weak conceptual understanding with over-developed procedures
(2) Students demonstrated difficulties with recall and transfer to new tasks" (p. 306).

The two above address constructivism by focusing on the strong points and possessions students bring to the tasks and also actively including students in the knowledge acquisition practice. Thus, active involvement and participation of the students is central to the constructivists' theoretical framework. In review of Piaget's work, this research recognizes three major contributions as acknowledged by: (Confrey & Kazak, 2006)

> (1) "The development of an idea determines its meaning, rather than a single statement of a formal definition and set of relationships, known as genetics epistemology.
>
> (2) The Process of moving from Action to operation, to mathematical Objects requires a level of consciousness that is labelled reflective abstraction.
>
> (3) The patterns of thought available for reuse and modification were cast as schemes" (p.309).

The contributions above possess theoretical force which informed this research. It paid attention to how the limit concept in relation to sequence and series' essential notions may be seen as developing, and students could be rich capitals of information.

3.3 APOS Theory

The APOS (Action- Process- Objects- Schema) Theory is based on Piaget's principle that an individual student learns Mathematics by applying certain mental mechanisms to build specific mental structures which the student then uses to deal with problems connected or related to the corresponding situation (Piaget, 1972). APOS Theory is a constructivist theory of how mathematical concepts might be learnt. It is based on the hypothesis about the nature and development of mathematical knowledge (Dubinsky & McDonald, 2001).

When a student can make sense of a given mathematical concept, then the individual possesses appropriate mental structures suggested by APOS Theory (Maharaj, 2010). This theory requires the detection of likely mental structures for a given concept and then suitable learning activities can be designed to support the construction of the structure in the student's mind (Dubinsky & McDonald, 2001). The required construction of a complete concept or mental structure operates through the four stages; Actions, Processes, Objects and Schema (Dubinsky, 1991). The APOS Theory states that the learning of mathematics ought to be based on helping students to use their existing mental structures to develop new ones, more powerful structures, for handling advanced mathematics (Arnon et al., 2014). The APOS Theory hypothesizes that concepts in mathematics are formed as students apply alterations on certain entities to obtain other entities. This transformation is based on the existing mental (or physical) objects. The following descriptions of Action, Process, Objects and Schema are based on what is highlighted by Weller, Arnon, and Dubinsky (2011).

3.3.1 An Action

A mathematical concept is formed first as an Action, which is, directed externally by transformation of an earlier comprehended object (or objects). An Action is an external conception, each step of the transformation needs to be done explicitly and instructed by external guidance. Furthermore, each step controls the next, the steps of the action cannot be imagined and none can be skipped (Arnon et al., 2014). An Action may be defined as any physical or mental transformation of objects to obtain other objects. It is an externally enabled conversion of concrete mathematical objects based on definite algorithm by an individual student. The occurrence of an Action is a reaction to external stimuli to the student. The response may be a single step response like a physical reflex. It may be the recalling of facts from memory; for example, an individual can think of a sequence or series through an explicit expression only, which can be manipulated substituting values in the expression. The response may also be multi-step triggered at each step by what has come before, and not by the individual student's conscious control of the transformation. The results of the various steps' response partially control the individual student (by triggering memory). The individual student may begin to establish conscious control by reflecting upon the Action (Arnon et al., 2014).

The students who act at an Action conception of a limit of a sequence or a series can accomplish the step by step procedure. These procedures can be applied in analytical, tabular or graphic representation context. The student relies deeply on committed memory; for example, the exponents in the given expression or the apparent form of the graph. The student recognizes the difference between a sequence or series and its transformations only in terms of the syntax or the rule that defines the sequence or series. Similarities between a sequence or a series and its transformations are recognized only in terms of some worldwide possessions of the graph. If a student requires an explicit expression to think about a limit of a sequence or a series, he/she is operating at an Action level. Furthermore, if he/she can do little more than substituting values for the variable in the expression, and manipulate it, she/he is also considered to have an Action understanding of the limit of a sequence or a series. For example, a student may think of a sequence as a procedure for assigning elements from one set (i.e. domain) to another (i.e. range) through a specific procedure like in $f(n) = 2n + 3$. In such a situation, a student cannot consider what would happen to an element, such as $n = 2$, without evaluating the sequence of that element.

3.3.2 A Process

A Process is a mental structure that performs some operations as the Action, wholly in the mind (Arnon et al., 2014). A Process performs the same operation as the Action, but wholly in the individual's mind enabling him/her to imagine performing the corresponding operation without having to execute each step explicitly. Students acting on the Process level have the ability to describe changes in the basic sequences or series as the consequences of the application of the transformation, without the need to perform each step of the transformation. An important characteristic of the Process level is that the student is in control of the transformation of an object (or objects). The student is able to look at the graph of the transformed sequence or series and describe the changes that result from the transformation. Such students are able to reverse the Process to identify the sequence on which a set of transformation was applied (from series to sequence). A Process enables the student to imagine the calculation for several values of the sequence or series, and think about these calculations at once. At this level, the student can observe the behavior of the functional values of a variable over an interval without having to evaluate the sequence or series for explicit values of the variable.

A constructed Process can be transformed in several ways. It may be reversed or it may be coordinated with other Processes. For example, the individual can observe the behaviour of the functional values as n varies over the interval, without necessarily evaluating $f\ (n)$ for explicit values of n. A Process is viewed as a transformation of object(s) the individual can control without the need for external stimuli. The individual is able to describe or reflect upon all of the steps in a transformation without necessarily performing them. Individual student's reflections on Actions lead to the interiorization of these Actions into Processes. The individual student may imagine executing the transformation without having to explicitly execute each step. Using our example $f(n) = 2n + 3$, the student can now imagine this sequence as multiplying an element from the domain by 2 and adding 3 to get an element from the range without the need to refer to a specific element. In the case of sequences and series, encapsulation allows the individual to form sets of sequences and series, to define operations on such sets and to equip them with a topology. However, this is not easy or immediate. Encapsulation requires a radical change to the nature of one's conceptualization as it signifies the ability to think of a concept as a mathematical entity to which new higher-level transformation can be applied. The mental process that leads to mental objects through encapsulation may remain available as in many mathematical situations. Such situations may require the individual to de-encapsulate an Object back to the Process that led to it. In Mathematics, it is important for an individual to be able to move back and forth between an Objects and a Process. Thus, Objects can be de-encapsulated to obtain the Process from where they come, for example, from partial sums to sequences. In some instances, the coordination leads to new Processes (partial sums of sequences) and in other cases, the Processes are linked to other constructs to form a Schema. Thus, if the individual becomes aware of the Process as a totality, realizes that transformations can act on that totality, and can actually construct such transformation explicitly, at this stage the individual has encapsulated the Process into a cognitive Object.

3.3.3 An Object

An Object is constructed through the encapsulation of a Process. Encapsulation is achieved when the student becomes aware of the totality of the Process, realizes that transformations can act on it, and is able to construct such transformation (Arnon et al., 2014). At an Object level, students are able to de-encapsulate any possible changes in the original sequence or series Object into the Process involving its construction. They are also able to identify the basic sequences or series on which it is based, and compare different transformed sequences or series in terms of their properties in any representational context. When a mental process can be changed at will by a student, is a target for further manipulation, is encapsulated and forms a detached context, it becomes a formalised Object. If students reflect on all of these processes and are able to think of them as a whole in any representational context, and are able to work flexibly in different representational contexts, then it is considered that the student has encapsulated the Process of applying a transformation of any sequence or series into an Object. Students are said to have an Object conception of transformation if they are able to apply Actions on transformed sequences and series and coordinate their properties in terms of possible changes in the original sequences or series.

Again, considering $f(n) = 2n + 3$, the student can consider translating this sequence by imagining the sequence in its entirety, and moving up 3 rather than moving a series of elements in the range up 3. Mathematical topics often involve many Actions, Processes and Object that need to be organized into a coherent framework which enable the student to make a decision on which mental processes to use when dealing with a mathematical situation; this framework is called a Schema. In sequences or series, it is the Schema structure that is used to see a sequence or series in a given mathematical or real world situation

3.3.4 A Schema

A Schema corresponds to a collection of Actions, Processes, Objects, and other Schema related consciously or unconsciously in the mind of the individual into a coherent structure (Arnon et al., 2014). This structure can be used to find the solution of a problem situation involving a mathematical field. In the case of sequences or series, it is the Schema structure that is used to perceive a sequence or series in a given mathematical or real-world situation. These constructed Schema are applicable in solving limit problems and to get an appreciation of a prescribed definition of limit. Evoking a Schema brings into play the structures previously constructed and the relations among them, that the student would have built up to that point. It can be seen that when students are faced with the same task, different students can use different structures and different relations between them. The establishment of different relations between existing structures by different students makes it possible to identify the level of development of each student's Schema. Students use limit concept in calculus, for sequences and series in compartments. This study is designed to overcome such gaps in future teaching and learning. If one becomes aware of a mental process as a totality and can construct transformations acting on this totality, then he/she has encapsulated the Process into a cognitive Object. A mathematical topic often involves many Actions, Processes and Objects that need to be organized into a coherent framework that enables the individual to decide which mental constructions to use in dealing with a mathematical situation. Such a framework is called a Schema. The APOS theory considers Actions, Processes, Objects and Schemas as an individual's successive mental constructions in learning a mathematical topic and interiorization, encapsulation as the only mental mechanisms needed to build those mental constructions (Arnon et al., 2014). A Schema for a particular mathematical concept is the individual's collection of Actions, Processes, Objects and other

Schema linked by general principles that form a framework in the individual student's mind, which may be brought forward to bear upon a problem situation involving that concept (Asiala et al., 1996). Thematization makes Schema a cognitive Object to which Actions and Processes can be applied. It can be consciously de-thematized to obtain the general Actions, Processes, Objects and other Schema from which the Schema was originally constructed (Clark et al., 1997). The structure of a Schema and its development may be used to explain students' difficulties with aspects of a topic or a concept (Aron et al., 2014). A Schema's existence cannot be separated from its reconstruction and continuous construction. Calculus students may interiorize the Action of taking limit of a sequence or series of various examples using various techniques taught and learnt in calculus courses. Given that the Process is interiorised from relevant Actions, the student may be in a position to reverse it to solve problems in which series are given, and it is desirable to find the sequences that make up the series. An understanding of a certain mathematical concept calls for the successful construction of Schema for that concept. Thus, the chosen activities and instructions given in class induced the mental constructions of the limit concept in relation to sequences and series.

3.4 Some key components in the theoretical framework

The following key components of the theoretical framework are discussed in detail: reflective abstraction, cooperative learning and ACE learning cycle.

3.4.1 Reflective Abstraction

Reflective abstraction as a general structure can be used to define mathematical concepts and their acquirement. Dubinsky (1991) alleged that reflective abstraction as presented by Piaget (Berth & Piaget, 1966) is an influential tool in the study of advanced mathematical thinking that could be used to analyze any mathematical knowledge related to higher education. Reflective abstraction as introduced by Piaget describes the construction of logico-mathematical structures by individuals during cognitive development. Piaget made two important observations about reflective abstraction: firstly, reflective abstraction has no absolute beginning, and is present at the earliest ages in the coordination of the sensori-motor structures (Beth & Piaget, 1966). Secondly, it continues through higher mathematics from antiquity to the present day. Reflective abstraction was drawn from general co-ordinations of actions by the subject internally (Piaget, 1980). He emphasizes that this kind of abstraction leads to a different kind of generalization which is constructive which then results in new syntheses and particular laws to obtain new meaning (Piaget & Garcia,1983). In the current study there were many examples of instances of reflective abstractions: (a) The interiorization, which was the construction of internal processes, which allowed students to make sense out of perceived problem, which was "translating a succession of material actions into a system of interiorized operations" (Piaget, 1980, p. 90). Dubinsky (1991), argued that interiorization permitted the individual to be aware of an action, to reflect on it and to combine it with other actions. (b) The coordination of two or more processes for the construction a new one. (c) The encapsulation of a (dynamic) process into a (static) object. Piaget considered mathematical entities to move from lower level to higher levels. (d) When an individual learns to apply an existing schema to a wider collection of problems, then it is said that the schema has been generalized. Generalisation can also happen when a process is encapsulated to an object. The schema remains the same except that it now has a wider applicability. Students perform several individual actions (not in the context of APOS) in the mind to identify the different sequences and series involved. Furthermore, Piaget (1972) asserted that reflective abstractions in its advanced form leads to the mathematical thinking which separates form or process from the content, and processes are converted, in the mind of the students, to objects of content. The first part of reflective abstractions consists of drawing properties from mental or physical actions at a particular level of thought (Beth & Piaget, 1966). Weyer (2010) stresses that the first occurrence of reflective abstraction should happen as a result of instructional practice. The constructivists, Piaget (1966) and Vygotsky (1986) have insisted that the mental constructions that students use to understand a mathematical concept are made in a social context, with considerable intervention from teachers and fellow students. Hence in this study the lecturer and fellow students were part of the social context helped students to understand sequences and series concepts.

The constructing, recognising and building-with are three characteristics of abstraction identified by (Hershkowitz, Schwarz, & Dreyfus, 2001). An abstraction is a kind of durable change which results in students' ability to recognize new experiences as having the similarities of an already formed concept. The mind chooses, coordinates, amalgamates and registers in memory a collection of mental acts through the process of abstraction. For further explanation on the five kinds of construction in reflective abstraction namely: Interiorisation; coordination, encapsulation, generalization and reversal, see Jojo, Brijlall and Maharaj (2011). When the material or entity has been adequately abstracted and can be re-oriented (re-created) in the absence of perceptual involvement, then the internalization level has been reached. Interiorization level is reached when the material or entity can be disembedded from its original perceptual context and can be freely operated on in imagination, and can be projected into other perceptual material and used in novel situations (Battista, 1999). The most wide-ranging form of abstraction is interiorization. Interiorization leads to the isolation of form (structure), coordination (pattern) and operations (Actions) from experiental things and activities (Seffe & Cobb, 1988). Reflective abstraction focuses on the idea how "Actions and operations become thematised Objects of thought or assimulation" (Piaget, 1985, p. 49). It is the general coordination of Actions with the subject as its source and is completely internal (Piaget, 1980).

Making sense of the limit concept in relation to sequence and series at undergraguate level requires abstraction and reflection. From Piaget's viewpoint, it is the method that " alone supports and animates the immense edifice of logico-mathematical construction" (Piaget, 1980, p. 90). Piaget distinguishes three main kinds of construction in reflective abstraction that matter most to this research study which are :

1. Interiorization, which is the creation of internal processes so as to make sense out of observed phenomena. It is "translating a succession of material Actions into a system of interiorised operations" (Piaget ,1980, p.90). This is supported by Dubinsky (1991) when he argues that one becomes conscious of an Action, then reflects on it and combines it with other Actions through interiorisation.
2. Encapsulation, the construction of new Objects through the coordination or composition of two or more processes. The (dynamic) Process is converted or encapsulated into a (static) objects, that is "… Actions or operations become thematised objects of thought or assimilation" (Piaget, 1985, p.49). Piaget(1972) considered the movement of mathematical objects from one level to another with an operation on such entities as Objects of theory.
3. Generalisation of schmata, when a student is able to apply an existing schemata to a variety and wider collection of phenomena, then the schema would have been generalized.

This can also happen when a Process is encapsulated to an Object. The Schema remains the same but with a broader applicability, which Piaget referred to as reproductive or simplified acclimatization (generalization extension). The mental act of structuring a form for an object or set of objects. Identified spatial components, spatial composites and the establishment of interrelationships between and among components and composites determines an object's nature or shape. Mental representations are non-verbal remembrance of experience like mental versions of circumstances. Mental representations are made up of integrated sets of abstraction that are activated to reason and interpret situations that one is dealing with in thought. The four "agents of change" in the classroom community that initiate reflection, are defined by Cappetta and Zollman (2009) as:

1. Individual: A student spontaneously engages in reflective abstraction.
2. Peer: Classmates challenges or questions each other.
3. Instructor: The instructor challenges or questions the individual student.
4. Curricular: Activities in the curriculum are designed to challenge and question students.

The above four agents of change were employed in the design and eplementation of instruction of of the limit concept in relation to sequnces and series.

3.4.2 Cooperative Learning

Cooperative learning is a student-centered and instructor-facilitated instructional strategy. Small groups of students are responsible for their own learning and that of all group members. Students interact with each other in the same group to acquire and practice the components of a subject matter in order to complete tasks or solve problems. Knowledge is acquired through social negotiation and the evaluation of the feasibility of individual understandings. Cooperative learning is an outgrowth of Constructivism, incorporates the notion that the best learning takes place when students are actively engaged in the learning process with other students to realize a shared goal. Whereas constructivism focuses on personal experience as the foundation for learning new material, cooperative learning utilizes not only the student's own experience to solidify acquired knowledge, nonetheless it also uses the experiences of others. In a constructivist classroom, interactivity through hands-on experiences and practice is at the heart of the curriculum.

A considerable body of investigational research has shown that cooperative learning stimulates higher-level reasoning, generation of new ideas and solutions, group-to individual transfer of learning, achievement, social competence, and cognitive and affective perspective-taking (Mayo, 2010). Cooperative learning is concerned with the learning of students in small groups to help each other learn academic content (Slavin, 2014). In this study, students worked together in small groups of three to make best use of their individual and each other's knowledge of limit of sequences and limit of series. The goal of the learning process was to support students create the required mental creations on limit of sequences and limit of series.

There have been several studies investigating methods of cooperative learning. Noteworthy constructive impression on learner accomplishment has been reported on all cooperative learning methods over competitive and individualistic methods. The consistency and multiplicity of cooperative learning techniques provides a firm confirmation for their usefulness (Johnson & Johnson, 2015). Learning coperatively is among the most fertile and remarkable areas of research, theory and training in education. Multiple factors have necesitated the widespread use of cooperative learning methods. Cooperative learning can be validated by research, is clearly based on theory and it can be operationalised into clear procedures educators can use (Johnson & Johnson, 2015). Research on cooperative learning has been sparked by the diversity and positive outcomes from cooperative efforts. The variety of cooperative learning methods available for the lecturer's use has contributed to its widespread use, ranging from very concrete and prescribed, to very conceptual and flexible methods.

In cooperative learning, the lecturer controls most of what goes on in the class. Cooperative learning is vital, particularly for complex tasks such as abstract and applied learning. From reading literature and the researcher's experience, first as a student and second as a lecturer; limit of sequences and limit of series has proven to be a complex task for students. Hence, cooperative learning was used as an appropriate method to support students create the required mental structures to understand limit of sequences and limit of series.

Cooperative learning has positive effects across all groups and levels, and for high achievers and those who receive special education (Slavin, 2014). Learner–centred instruction is wholly cooperative learning with the priority on fostering facilitative relationships as a foundational and influential practice in education. Highly effective peer -support theories, structures and approaches are what make the field of cooperative learning. In order for students to understand the limit of sequences and limit of series, the researcher used peer- supported structures and approaches to motivate students' discussions and reflection.

There are several common characteristics for successful cooperative learning practices. Johnson and Johnson (2015) propose five components essential for the successful implementation of cooperative learning, thus:

1. Positive interdependence: every member has a unique contribution for the success of cooperative learning since goal achievement depends on other members' attaining their goals. The lecturer has to clearly define the assignment to the group by allocating interconnected goals to each member of the group.
2. Promotive interaction: students have the duty to facilitate each other's learning. This is done through the establishment of information, help, feedback, resources desired, students explaining their thoughts and conveying meaning. The lecturer can encourage the interaction by providing conditions for discussion among group members.
3. Individual accountability: each group member is responsible for contributing to the finished produce of the team. The finished creation is constructed by the efforts of each individual member. The lecturer should provide the opportunity for individual assessement, feedback and monitoring of students' performance during the cooperative learning process.
4. Social skills: in cooperative learning environment, students have the obligation to express their own ways of thinking to others, as well as influence and respect other group members' ways of thinking. The ability to work with others is an crucial component of learning cooperativily.
5. Group processing: the group members have to reflect on their learning process, that is members have to question the usefullness of members' actions, how effective their way to accomplish the group goal was, with the view to change or not. The lecturer can give feedback on the group or/and individual's effectiveness.

The five components essential for the successful implementation of cooperative learning, were taken into account on the design and implementation of instruction for this study. When students worked on limits of sequences and series, they facilitated each other's learning, worked as a team and managed to accomplish the group goal.

There is a consensus among researchers that cooperative learning has positive effects on students' achievement. Biehler and Snowman (1997) reviewed research on cooperative learning and found that,

> Students who learn cooperatively tend to be more highly motivated to learn because of increased self-esteem, the pro-academic attitudes of group mates, appropriate attribution for success and failure, and greater on-task behaviour. They also score highly on tests of achievement and problem solving and tend to get along better with class mates with different racial, ethnic and social class backgrounds (p.421).

The above mentioned factors indicated that cooperative learning has a lot of advantages. Gillies (2007) envisages productive cooperative learning as having the following characteristics: establishment of an education setting that caters for all students; ability to discuss potentials for minor team behaviour; development of communication services that enable discussion in small groups; development of appropriate helping behaviours; choosing tasks for students in small group discussion; and monitoring progress made by students to evaluate outcomes. These components were taken into consideration during the planning of group based activities in the intended sequences and series concepts.

This study established a conducive learning environment for all students by taking into consideration negotiation of small group behaviors, small group discussions, and the development of appropriate culture for students to help each other on given tasks during the learning process. It was easy to monitor students' progress and evaluate the outcome of the limit of sequences and limit of series through cooperative learning.

3.4.3 The ACE Learning Cycle

There are many cooperative learning methods, of which the following five are prominent; Learners Teams-Achievement Division, Teams-Games-Tournament, Jigsaw, Group Investigation, and Activities Class Discussions Exercises (ACE) learning cycle for college Mathematics Education. One or more conceptual structures are made when a student performs the task. A pedagogical approach based on APOS theory, the ACE learning cycle, (A) activities, (C) classroom discussion and (E) exercise, is a useful tool for the development of mental constructions. The hypothesis on learning is that it is a repetitive cycle consisting of ACE (Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996). This study used ACE learning cycle because it offers students opportunities to make mental constructions. The ACE learning cycle was used successfully and yielded good results by (Aydin & Mutlu, 2013; Maharaj, 2014; Voskoglou, 2015).

Mathematical communication ability is part of high-order mathematical thinking ability, and is very important for mathematics and mathematics education. It enables students to share ideas and clarify understanding (National Council of Teachers of Mathematics, 2000). Through communicating mathematics to others in writing or/and orally, students learn to explain and convince others. Listening to explanations of others gives students the opportunity to develop their understanding. Sharing ideas and clarifying mutual understanding occurs when discussions are in progress, in groups and classroom situations. Such activities help students to construct a logical sequence of thought.

In this research study, laptop activities formed the first step of the cycle, where students worked on given tasks using laptops in small groups of threes. The activities were designed to foster the student's development of the appropriate mental structures called for by the APOS theory. During classroom discussion of students' results, the lecturer guided the students to reflect on the laptop activities and their relation to the limit of sequences and limit of series. The homework exercises were standard problems related to the topic being studied. This knowledge was used by students to solve standard problems on limit of sequences and limit of series. This approach's implementation was very effective as students managed to make mental constructions, as reported in the research study by (Weller, Clark, Dubinsky, Loch, McDonald & Merkovsky, 2003). Cooperative education using ACE learning cycle incorporates computer undertakings (in this case laptop activity). Groups of three members meet in a lecture room to study material prepared by the lecturer. This pre-lecture activity was on informal concept before lecture sessions. After the lecture sessions, students meet to work on prepared tasks then one group member presents their findings to the class for discussion. The lecturer clarified certain points, then introduced formal ideas. This was followed by exercises which all group members had to do.

3.5 Linking APOS and ACE learning cycle to the theoretical framework

APOS theory and its applicability to mathematics teaching are based on two psychological assumptions, which are;

(1) the learners' knowledge of mathematics is their "tendency to respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context, and constructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with situations" (Dubinsky, 2001, p. 11).

(2) Students do not learn mathematical concepts directly. "They apply mental structures to make sense of a concept" (Piaget, 1977, p. 178).

These assumptions suggest that "the goal for teaching should consist of strategies to help learners build appropriate mental structures, and to guide them to apply these structures to construct their understanding of mathematical concepts" (Maharaj, 2010, p. 42). If appropriate mental structures are not present, learning the mathematical concept is then almost impossible. In this study, the researcher used the four levels of APOS theory together with an additional mental structure that precedes the Action level called pre-sequence/series (N). The deciding factor in coding the conception of the student was the careful examination of students' responses for their understanding of the test items showing that they may or may not have understood the concept.

In the ACE learning cycle, the activity portion of the cycle, the students worked in groups on tasks that were designed specifically to help them develop the correct cognitive constructions suggested by the constructivist model, that is, to help students to encapsulate Processes into Objects. At the end of the activity, students in groups came together for a class discussion session. The discussion session was there to provide a medium for students to begin the reflective abstraction process. The role of the lecturer was to help the students successfully tie things together. The final stage was an out-of- class exercise assigned to students to work in teams. The exercises were used to help students reinforce their conceptual framework of the mathematical concept under study.

Through the use of the ACE learning cycle, students acquired knowledge and formed meanings based upon their experiences with the representations from Maple. This caused the individual students to develop new outlooks, rethink what were once misunderstandings, and evaluate what was important, ultimately altering their perceptions on the limit of sequences and limit series. The ACE learning cycle promoted social and communication skills by creating a classroom environment that emphasized exchange of ideas. Students learned how to articulate their ideas clearly and exchanged ideas and so learned to "negotiate" with others and evaluated their contributions in a socially acceptable manner. This is essential to success in the real world, since they will always be exposed to a variety of experiences in which they will have to cooperate with others and navigate among the ideas of others. The ACE learning cycle gave both the lecturer and students the opportunity to think of knowledge not as inert factoids to be learned by rote, but as a dynamic, ever changing view of the world we live in and the ability to successfully stretch and explore that view. As the lecturer disseminated information to students; students were not passive recipients of knowledge. The lecturer had a dialogue with students, helped them construct their own knowledge. The lecturer's role was not rooted in authority but was interactive and rooted in negotiation.

3.6 Methodological Framework

A specific framework was used in this study. The framework consists of three research components cycle which influence each other. These were; the theoretical analysis, the design and implementation of instruction, and the data collection and analysis as illustrated in Figure 3.1.

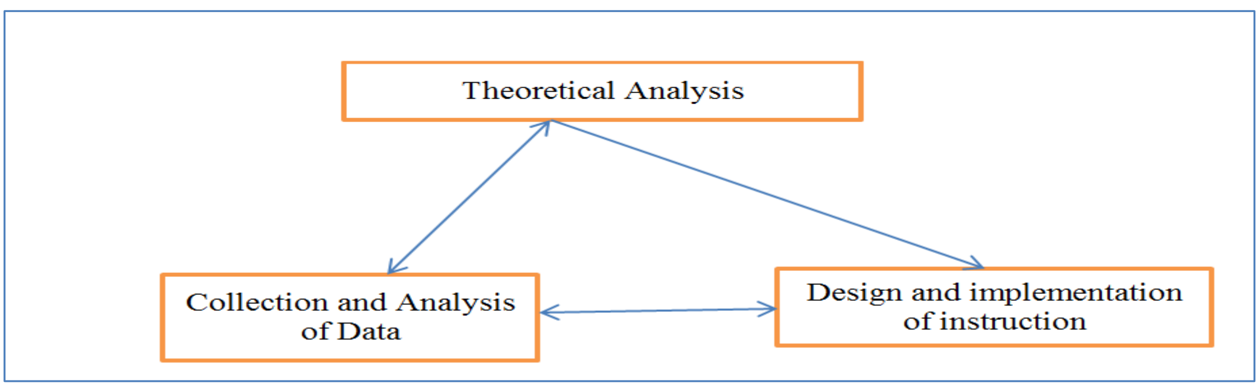


Figure 3.1: Methodological Framework for Research (Asiala et al., 1996)

The methodological framework for this research is made up of three components which are cyclic, and influence each other. The initial step in this approach was to conduct a theoretical analysis of the limit concept in relation to sequences and series relative to specific mental constructions that a learner made in order to develop an understanding of the limit of sequences and limit of series. This analysis resulted in the construction of preliminary genetic decomposition (PGD). This led to a design of instructional treatment that focused on enabling students to make constructions in the PGD of the limit of sequences and limit of series. The last step engaged implementation of instruction through activities intended to foster the mental construction called for by the analysis of the limit of sequences and limit of series. Activities and exercises were designed to help students construct Actions, interiorize them into Processes, encapsulate Processes into Objects, and coordinate two or more Processes to construct new Processes (Asiala et al., 1996). The theoretical analysis led to the preliminary genetic decomposition of the limit concept in relation to sequences and series. The genetic decomposition led to the designing of instructional treatment focusing on getting students make mental constructions in the preliminary genetic decomposition of the limit concept in relation to sequences and series. The final stage engaged implementation of instruction that led to the collection of data which was analysed in the theoretical perspective context.

3.7 Understanding a mathematical concept

The learning of mathematics for understanding involves making connections among ideas related to the concept being learnt. These connections are for the facilitation of the transfer of prior knowledge to novel situations. A mathematical procedure or idea is understood if its mental representation is part of a network of representations (Bransford, Brown & Cocking, 2000). They further postulate that the number and the strength of the connections made with previously acquired mathematics determines the student's degree of understanding a mathematical concept. Thus holders of well-connected ideas can remember and see the connected ideas as part of a large whole where each part shares with other parts within a reciprocal relationship.

The understanding of the concept of a limit concept in relation to sequences and series was explored in relation to the development of Schema relevant to it. In this study, in-class activities were designed to induce students to make desired mental constructions as suggested by the preliminary genetic decomposition. Tasks given helped students to gain exposure and experience in constructing Actions corresponding to the limit concept in relation to sequences and series.

Subsequent activities calling students to reconstruct familiar actions as general processes enabled the students to have the much needed experience. Complicated activities which required students to organize a variety of previously constructed objects, for example, sequences and partial sums of sequences into a schema applied to the limit concept in relation to sequences and series situational problems, were provided. The researcher specifically examined students' attempts to answer given tasks in class, their exercises and tests regarding their understanding of limits, sequences and series.

3.8 Construction in reflective abstraction

Central to APOS theory is reflective abstraction which has two main components: 1) reorganization of existing knowledge structures, and 2) the projection of existing knowledge onto higher thought planes (Dubinsky, 1991). It is a process of construction as outlined by Arnon, et al. (2014).

3.8.1 Interiorization and Processes:

Interiorization is the mechanism which makes conceptual shifts possible, that is, Actions are repeated and reflected on, which make the individual student to move from depending on external cues to internal ones.

> "An Action must be interiorized. As we have said, this means that some internal construction is made relating to the Action. An interiorized Action is a Process. Interiorization permits one to be conscious of an Action, to reflect on it and to combine it with other Actions". (Dubinsky, 1991, p.107)

For the current study, during the phase of interiorization, students were expected to familiarize themselves with Processes and carry them out through mental representations.

3.8.2 Encapsulation and Objects:

At this stage the construction of mathematical appreciative extends from one level to another. Drawn from previous Processes, new forms are built to come up with Objects. The following explanation on objects was given by Dubinsky, Weller, McDonald and Brown (2005a):

> "If one becomes aware of the Process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has encapsulated the Process into a cognitive Objects" (p. 339).

The above implies that the construction of transformations of a process towards and object could be in one's mind (hence the need for interviews to probe this) or it could be explicit in the form of physical evidence, in the form of a written response.

3.8.3 De-encapsulation, Coordination and Reversal of Processes:

A Process that has been encapsulated forms a mental object, when need arises, it can be de-encapsulated back to its underlying Process. Coordination is an indispensable mechanism in Objects construction. Two or more objects can be de-encapsulated, their Processes coordinated, and the coordinated processes encapsulated to form new objects (Arnon et al., 2014). In order to determine the convergence of $\sum_{\mathrm{n}=1}^{\infty} \frac{\mathrm{n}^2+2}{\mathrm{n}^4+5}$, dropped 2 from the numerator and 5 from the denominator since they really did not add anything to the series at the infinity. This results in $\frac{n^2}{n^4} = \frac{1}{n^2}$, which is convergent as a series. If we drop 5 from the denominator, then $\frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4}$. $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^4}$.

These two series converge by the p-series test. Therefore, their sum also converges.

3.9 Conclusion

Constructivism and its contribution to the successful learning of the limit concept in relation to sequences and series was outlined. The cooperative learning and methods, the research framework, implementation, reflective abstraction, and construction in reflective abstraction have been outlined. Students' understanding of the concept under discussion is most likely to be achieved once students are capable of operating at the required mental constructions acquired, and reflecting on explicit structures of these constructions.

The APOS theory provided the theoretical framework for the description of methodology used in chapter 4. The following chapter gives an outline of the research methodology.

CHAPTER 4 METHODOLOGY

4.1 Introduction

The overview provided in chapter 3 of APOS theory gave an insight into the theoretical framework used in this study. This chapter focused on; research design, the participants, genetic decompositions, design and implementation of instruction data collection tools, data analysis strategy, ethical issues and validity and reliability issues.

4.2 Research Design

Durheim (2004), defines research design as a strategic framework for action that serves as a bridge between research questions and the implementation of the research strategy. So a research design is a blueprint that guides the collection and analysis of data of the research project. The research design provided the framework that specified the type of data to be collected, its sources and collection procedures.

4.2.1 Qualitative Research Practice

Denzin and Lincoln (2011) proposed that qualitative research be described as:

> a set of interpretive, material practices that make the world visible. These practices transform the world. They turn the world into a series of representations, including field notes, interviews, conversations, photographs, recordings and memos to self … qualitative researchers study things in their natural settings, attempting to make sense of or interpret phenomena in terms of the meanings people bring to them. (p. 3)

As described by Denzin and Lincoln above, qualitative research is often associated with specific kinds of data, usually involving words or images rather than numbers. Qualitative research is concerned with ‘what’ ‘why’ and ‘how’ questions rather than ‘how many’. It focuses on processes and specific data-generation methods such as semi-structured interviews.

In general terms, qualitative research is scientific research that seeks answers to a question, uses a predefined set of processes to answer the question, collects evidence, comes up with findings that are not determined in advance and are appropriate beyond frontiers of the study. It also provides multifaceted textual images of how people experience a given research issue (Mack, Woodsong, MacQueen, Guest & Namey, 2005). Opportunity for the voice of the participants to be heard was given during the discussions.

The researcher understands qualitative research as research that aims to gain deep understanding of a specific organization to provide an explicit rendition of the structure. That includes broad patterns found among participants and generate data about human groups in social settings. The research findings can often be extended to people in familiar situations as those of the study population. However, rich and multifaceted understanding of a precise phenomenon takes preference over eliciting data that can be generalized to other populations.

Fouche and Delport (2002) outlined the qualities of qualitative research as its ability to:

1. elicit participants' meaning, experience or perceptions about a concept;
2. produce descriptive data;
3. allow diverse responses; and
4. collect data by the use of interviews, observations, pictures and other materials.

The researcher conducted interviews in this study, with the students to elicit their perception and meanings about the limit concept in relation to sequences and series. The data produced from the limit test questions and the interviews was diverse and enabled the researcher to come up with answers to the research questions of this study.

Through reading literature, the researcher characterized qualitative research as having the following attributes:

1. Understanding – pursues to determine people's clarifications.

2. Dynamic – reality varies with variations in people's insights.
3. Inside – reality is what people identify it to be.
4. Holistic – total or complete image is sought.
5. Discovery – theories develop from data collected.
6. Naturalistic – investigations are carried out under unbiased sceneries/settings.

Qualitative research emphasizes on explaining why people think in certain ways. It is good at simplifying and managing data without destroying complexity and context. Major strengths of qualitative research Creswell (2014) are:

1. Open-ended questioning reveals new or unanticipated phenomenon, and raises more issues through broad and open-ended inquiry.
2. It allows researchers to explore the views of homogenous as well as diverse groups of people help unpack these differing perspectives within a community.
3. It can play the important role of suggesting possible relationships, causes, effects, and dynamic processes.
4. It allows people to open up and allows for new evidence that was not even initially considered.
5. It provides a holistic interpretation of the detailed processes.
6. Because of close researcher involvement, an insider's view of the field is gained.
7. Participatory methodologies empower, rather than objectify respondents.

The researcher reflected on the strengths mentioned above during the research process of limit concept in relation to sequences and series. Qualitative research explores a phenomenon that has not been studied before. It is used to understand a phenomenon from the perspective of the actors involved, rather than explaining it from the outside.

Students' interpretation of the questions was revealed through the responses they gave to the written limit test questions and the interview. The researcher also discovered that some students' perceptions changed through probing during the interviews, thereby affording the researcher a bigger picture of what the students had learnt during the learning sessions. In this study, the natural setting gave the students the opportunity to say out how they understood the limit concept in relation to sequences and series. This study took the perspective of describing a set of mental constructions that students could develop after understanding the limits concept in relation to sequences and series.

4.2.2 The interpretive paradigm and how it fits into this study

Interpretivism supports the notion that reality is, not objectively determined but socially constructed, more mental than perceivable, and determined by context (Bryant, 2011). According to interpretivists, reality is an interdependent system that has to be observed holistically and it is an open-ended process. It primarily involves researchers interpreting several elements of the study, and ultimately leads to assimilate human interest into the study. As the primary mode of data collection, interpretivist philosophy applies more naturalistic techniques such as interviews (Dudovskiy, 2016). The use of interviews was implemented in this study.

Interpretivists assumes that the researcher's values are inherent in interviews with the truth being negotiated throughout the interview (Angen, 2000). It should be noted that the nature of the research questions for this study suggests an interpretive research design. Interviews and analysis of existing texts are the main pillars of interpretive approaches (Cohen, Manion & Morrison, 2011) used in this study. These approaches ensured that there was adequate dialogue between the researcher and the respondents. The researcher managed to construct a meaningful reality which enabled him to derive meanings from the research process. This study placed importance on the analysis of students' mathematical thinking in the perspective of the limit concept in relation to sequences and series. It was of importance to find out how students used their past knowledge of limits in general (from A-level mathematics), to construct their knowledge of the sequences limits and of series limits. The interpretive enquiry focused on interpreting participants' perspectives and answers to the enquiry that was practically dependent on the context. The researcher specifically examined students' attempts to answer the given test and their understanding of the limit concept in relation to sequences and series (see Appendix 2).

4.3 Participants and Sampling Procedure

This study aimed at exploring how students understand the limit concept in relation to sequences and series. The students had been taught some calculus concepts at A-level, namely; basic integration, differentiation, and limits of geometrical and arithmetic progressions. The students were first introduced to sequences and series in high school mathematics. However, in high school, formal definition of the limit of a sequence and the definition of a limit of a series are not covered. These are studied at university level. In this study, all the students who were taking the course (30 first year university students) volunteered to participate. There were four mathematics majors and twenty–six statistics and financial mathematics majors who took part in the study under consideration. All the students gave a written consent and allowed the researcher to conduct the study. The researcher used purposive sampling to come up with ten participants (students) for semi-structured interviews which were audio-recorded. The selection of students for interviews was done on the basis of their response to the limit test questions. The determination for the selection for interviews was made on the basis of students who gave correct responses and also those who failed to give correct responses. Five students who gave correct responses to most of the test items and five students who failed to give correct responses to most of the test questions emerged. However, one student who gave correct responses to most of the test questions declined to be interviewed. Furthermore, one student who failed to give correct responses to most test questions also declined to be interviewed. Thus, effectively eight students were interviewed.

4.4 The Preliminary Genetic Decomposition

A genetic decomposition of a concept can be defined as a structured set of mental constructs which can describe how the development of a concept in the mind of an individual might take place (Asiala *et al.,* 1996). It is a hypothetical model that describes the mental Actions, Processes and Objects that must be developed by a student, and the sequence in which the student must develop them, to learn a given mathematical concept. APOS progresses by developing new genetic decompositions, testing them against empirical data pertaining to students' learning, and using them to design pedagogical materials. The theoretical analysis is based initially on the general APOS theory and the researcher's understanding of the mathematical concept in question. After one or more repetitions of the cycle and revisions, it is also based on the fine-grained analyses described above of data obtained from students who are trying to learn or who have learned the concept.

This study engaged the APOS (Action- Process- Objects- Schema) approach in exploring university students' mental constructions of sequence limits and of series limits. The researcher's lens is a preliminary genetic decomposition that can be used to explain developments made by students in their appreciative of the limit concept in relation to sequences and series (Arnon et al. 2014). It was used to explain the discrepancies in performance by students as some performed the tasks correctly, others had difficulties, and yet others failed completely.

The Preliminary Genetic decompositions (PGD) of the limit concept in relation to sequences and series are here presented**.** Limits depended on the types of sequences and series available. The descriptions are displayed as diagrams based on Actions, Processes, Objects and Schema organized with their connections that were detected to be instrumental in the development of mental constructions. The PGD provided in Figure 4.1 illustrate how the limit of a sequence may be presented and the PGD for limit of a series is illustrated in Figure 4.2. The study of mathematics is hierarchical in nature (Dubinsky & McDonald, 2001). The researcher viewed the different levels of mental constructions according to students' proficiency to work with particular content/concepts relating to sequences and series. A similar idea was used by Afgani, Suryadi and Dahlan (2017) when they selected tasks to check for specific APOS Theory levels. This does not mean that students will necessarily function at the indicated APOS Theory levels, but that the researcher chose those problems which he considered required students to have mental structures at a particular level in the context of APOS Theory.

4.4.1 PGD for limit of a sequence.

Pre- Action level: Students at the Pre-Action (N) level, provide no response or totally incorrect response.

Action level: Students at Action level depend on detailed external cues in order to carry out a transformation, one step at a time and lacking any mental image of the overall solution (Asiala, et al., 1997; Dubinsky & McDonald, 2001; Ndlovu & Brijlall, 2015, 2016). The definition of the limit of a sequence was discussed with the students at an Action level. When students were asked for the definition, they were required to, at least, recall and this would be external to the student's imagination. When asked to evaluate the limit of $lim_{n\to\infty} \frac{n^2 + 3n}{2n^2 + 1}$, the researcher expected students to at least come up with the response indicating the use of a step by step process.

Process level: In Process conception, an individual student is expected to have repeated and reflected on an Action, resulting in the interiorization thereof. At this level of conception, students are expected to perform transformations, predict outcomes and even reverse Processes mentally, without external cues (DeVries & Arnon, 2004). At the Process level, students were expected to provide responses with evidence of omission of steps which is an indication that these steps are done in the mind. Furthermore, students were expected to show evidence of having interiorized and coordinated Actions to come up with responses. For example, use the squeeze theorem to evaluate $lim_{n\to\infty} \frac{n!}{n^n}$.

Object level: Object conception is characterized by the students' view of a Process as a totality and is able to apply transformations on that totality, having encapsulated a Process into an Object (Brijlall & Bansilal, 2011). When need arises, the student is able to de-encapsulate Objects in order to access the underlying Processes and Actions. At the Object level, students were expected to show the ability to encapsulate Processes into Objects or choose the correct convergence criteria based on the mathematical structure of the given entity. Furthermore, when students are asked to prove the uniqueness of a convergent a sequence; the students were expected to, at least, choose two sequences that would converge to the same limit, then coordinate procedures to come up with the response.

Schema: A mathematical concept in most cases often involves many actions, processes, and objects that need to be organized and linked into a coherent framework, called a schema. It becomes coherent when it can provide a way of deciding for the individual, when subjected to a particular mathematical situation, whether the schema applies. The coherence might lie in the understanding that to use the $\boldsymbol{\epsilon}$ - $\boldsymbol{N}$ (epsilon – N) definition of the limit of a sequence to prove that $\boldsymbol{lim_{n\to\infty}\frac{1}{3^n}}$ = 0, a student needs to consider all the appropriate steps to be taken together with the justifications and illustrations involved.

The expected APOS level for the limit of a sequence are shown in the PGD in Table 4.1

Table 4.1: PGD of the limit of a sequence

Level	Expected competence
Pre-Action Level	Students at the Pre-Action (N) level, provide no response or totally incorrect response.
Action Level	The individual: gives examples and make representations of sequences; make representations and state the definition of the limit of a sequence; make use of evaluation methods to evaluate of simple sequences e.g. $lim_{n\to\infty}\ \frac{n^2+3n}{2n^2+1}$. Action level depends on detailed external cues in order to carry out transformation, one step at a time or providing definitions from memory.
Process Level	The individual imagines what the limit of sequences will be without carrying out step-by-step procedures. At this level, the individual is able to predict whether sequences cconverge or diverge by looking at its structure; use properties of sequences to evaluate limits of sequences by showing the ability to coordinate Actions to come up with responses e.g. use the squeeze theorem to evaluate $lim_{n\to\infty}\frac{n!}{n^n}$. Students were expected to have repeated and reflected on an Action,

	resulting in the interiorization thereof, perform transformations wholly in the mind, predict outcomes
Object Level	Use of epsilon – N definition to prove or disprove given sequence limits by making use the main components (L, n, N and ε) of the limit definition to prove given sequences. Working with composite sequences e.g. to prove that if a sequence converges, then its limit is unique. Object conception is characterized by viewing a Process as a totality and ability to apply transformations on that totality and encapsulation of a Process into an Object
Schema Level	The individual considers all the appropriate steps to be taken together with the justifications and illustrations involved in a coherent framework. The individual student was expected to show coherence by making use of appropriate schema to use the $\boldsymbol{\epsilon}$ - $\boldsymbol{N}$ (epsilon – N) definition of the limit of a sequence to prove that $\boldsymbol{lim_{n\to\infty}\frac{1}{3^n}} = 0$,

4.4.2 PGD for limit of a series.

Pre- Action level: Students at the Pre-Action (N) level, provide no response or totally incorrect response.

Action level: An action is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and hence is seen by the subject as externally driven. At the Action level, after the definition of the limit of an infinite series had been taught, students were expected to at least recall the definition from memory.

Process level: When individual re-acts on the Action and constructs an internal operation that performs the same transformation then we say that the Action has been interiorized to a Process. Suryadi (2012) furthermore identified that when an Action is repeated, and reflection on Action occurs, then subsequently the individual enters into Process phase. Unlike an Action which can be done through object manipulation or something which is concrete, Process occur internally under the control of an individual who carries it out. Students experience a Process conception, if they are able to do reflection toward that mathematical idea. After the students had beeen taught on the evaluation of convergent series, they were expected to show the ability to choose the correct method, for example, finding the limit of the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. Students were expected to make use of partial fraction, then use a procedure that showed omission of some steps in the response procedure or coordination of different methods to come up with the responses.

Object level: When Actions are performed on a Process, then the individual has encapsulated the Process into a total entity, or an Object. In many mathematical problems, it is necessary to de-encapsulate an Object and work with the Process from which it came. Suryadi (2012) identified that someone is said to have achieved Object conception from a mathematical concept, if the individual is able to treat that concept as a cognitive Object which incorporate the ability carry Actions on that Object, and give interpretation or reason about its properties. Furthermore, that individual should display the capability to decompose an Object to become Process as the properties of Object intended will be used. Students at the Object level were expected to see a dynamic structure (Process) as a static structure to which actions can be applied and to show evidence of the ability to choose the correct convergence criteria. When students were required to evaluate the limit of the series $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$. They were expected to choose the correct convergence criteria, use the correct procedure, then come up with the correct response.

Schema: A mathematical concept in most cases often involves many actions, processes, and objects that need to be organized and linked into a coherent framework, called a schema. It becomes coherent when it can provide a way of deciding for the individual, when subjected to a particular mathematical situation, whether the schema applies. The coherence might lie in the understanding that to use the Integral Test to determine the limit of, $\sum_{\mathrm{n}=2}^{\infty} \frac{1}{\mathrm{nIn\ n}}$, the student requires the ability to verify all the conditions necessary for convergence when using the Integral Test, make illustrations of the given task, and then evaluate the integral to arrive at an appropriate conclusion.

The expected APOS level for the limit of a series are shown in the PGD in Table 4.2

Table 4.2: PGD of the limit of a series

Level	Expected competence
Pre-Action Level	Students at the Pre-Action (N) level, provide no response or totally incorrect response.
Action Level	The individual: gives examples and make representations of series; state the definition of the limit of an infinite series. An action is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and hence is seen by the subject as externally driven. Providing definitions from memory.
Process Level	The individual imagines what the limit of series will be without carrying out step-by-step procedures. At this level, the individual is able to predict whether series cconverge or diverge by looking at its structure; use properties of series to evaluate limits of series by showing the ability to coordinate Actions to come up with responses e.g. to find the limit of the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. Students were expected to have repeated and reflected on an Action, resulting in the interiorization thereof, perform transformations wholly in the mind, and predict outcomes
Object Level	The individual performs evaluation of limits involving convergence Tests e.g. finding the limit of the series $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$. If Actions are performed on a Process, the subject must encapsulate it to become a total entity, or an Object. An object conception is achieved, if a student is capable of treating a limit of a series concept as a cognitive object which comprise the ability to do Action on that Object, and give an explanation or reason about its properties. Object to become process as at the time properties of object intended will be used.
Schema Level	The individual shows the ability to verify all the conditions necessary for convergence when using the Integral Test, make illustrations of the given task, evaluate the integral and give an appropriate conclusion in a coherent framework. The individual was expected to show coherence by making use of appropriate schema to use Integral Test to determine the limit of, $\sum_{\mathrm{n}=2}^{\infty} \frac{1}{\mathrm{nIn\ n}}$.

4.5 Design and Implementation of Instruction

Maple is a computer software or Computer Algebra System (CAS) which is able to solve the problems on limits of sequences and series in numerical and symbolic form. Maple is also able to present symbolic processing and visualization. In general, Maple comprises of the main menu, toolbar, and worksheet. The worksheet is a place to write Maple commands in mathematical calculations. Maple is equipped with "Help" facilities on the main menu. In the main menu, users can select sub menus to see how to work using maple. Maple also provides users with the tools that are very easy to operate contained in the Palettes. Palettes are used to simplify writing in worksheets, and palettes are generally located on the left. Symbol palette is used to write mathematical symbols, expression palettes are used to facilitate writing mathematical expressions such as integrals, Sigma series and root forms. The program allows us to work fluently with students of different computer skills (Samkova, 2012). Teaching Calculus using a variety of computer facilities and equipped with Maple software facilitates lecturers to deliver material quickly, and students can take lessons directly with practice. Learning accompanied by computer practice using Maple facilitated students to understand the concepts being studied, especially the limit concept in relation sequences and series. Determining the limit of sequences and series using Maple is also helpful in strengthening the students' conceptual development.

This study was carried out at a University in Zimbabwe where Science Education and other degree programs are offered. The Mathematics lecturers teach Mathematics courses to Mathematics majors and non-majors. The university has free internet connectivity for students' use in their academic work like any other University in Zimbabwe. Students mainly use their laptops. The university has employed a laboratory technician who helps students with minor software problems. The technician installed maple on the students' laptops for use, before the start of the study.

In this study, the respondents were students taking Calculus 1 during their first semester. The students were introduced to limits, sequences and series of geometric and arithmetic progressions at 'A' level where definitions were not emphasized. At 'A' level they are only expected to carry out simple manipulations involving limits, sequences and series.

4.5.1 Activities

This study employed the ACE learning cycle (Activities, Class discussion and Exercises) pedagogical strategy teaching cycle. The ACE learning cycle is a repeated circle of three components: Activities on the computer, Classroom discussion, and Exercises done outside the class (Arnon et al., 2014). The ACE learning cycle's implementation and effectiveness in helping students make mental constructions in the learning of a mathematical concept has been reported in several research studies of the Dubinsky's team (Arnon et al., 2014; Asiala et al., 1996; Weller et al., 2011).

Activities, are the first step of the cycle where students worked cooperatively in groups on tasks designed to help them to make the mental constructions suggested by the genetic decomposition. The tasks focused to promote reflective abstraction rather than to obtain correct answers. This study employed the ACE learning cycle (Activities, Class discussion and Exercises) pedagogical strategy teaching cycle. Activities carried out by students designed in the form of tutorials, were meant to help students make mental constructions. These activities were carried out in a computer lab. The students worked cooperatively in groups of three on materials prepared by the lecturer. The pre-lecture session's activity introduced the students to the formal lectures. Through working on these activities, the students gained the much needed experience on mathematical issues linked to understanding the limit concept in relation to sequences and series. Early activities provided students with chances for the development of the limit concept, sequences and series. The activities led students to reflect on the Action of limit, sequences and series which led to interiorization of the Actions into Processes, encapsulation of Processes into Objects, and Objects into Schema.

4.5.2 Classroom Discussion

The second part of the ACE learning cycle, Classroom Discussion, involved small groups of three or four students and instructor-led class discussion. Students worked on paper and pen tasks that were built on the lab activities completed in the Activities phase. The class discussions and in-class work gave students an opportunity to reflect on their work, in particular the activities that were done in the lab. The lecturer guided the discussion, and provided definitions, offered explanations, and presented overviews to tie together what the students had said and working on (Weller, Arnon & Dubinsky , 2011). The classroom discussions were held after lab sessions where one group member presented the group's findings to the class for discussion on limits, sequences and series and their application. The lecturer clarified points where need arose and introduced formal ideas. Class discussions considered finite and infinite sequences or series and their limits.

4.5.3 Exercises

Homework exercises, which formed the third part of the cycle, consisted of fairly standard problems designed to reinforce the computer activities and the classroom discussion. The exercises helped to support continued development of the mental constructions suggested by the genetic decomposition (Dubinsky, Weller & Arnon, 2013). They also help students to apply what they have learned and to consider related mathematical ideas (Dubinsky, Weller & Arnon, 2013). Exercises as homework were carried outside classroom management by individual group members at the end of each lesson taught. The exercises reinforced the Activities and Classroom Discussion phases. Exercises were given in form of written questions.

In this study, the students used laptops during the ACE learning cycle for generating and drawing graphs of given sequences and series. The students used the maple software to generate sequences and series, then drew graphs of these sequences and series. Through experience as a lecturer at the university, the researcher was aware that the lecture method used in lecture rooms was not very interactive. It did very little, if any, to help learners in the creation of conceptual structures needed for the limit of series understanding and limit of sequences understanding. Only few students could raise some questions in a lecture due to the short lecture time allocated to each topic in the curriculum. Traditionally, most of the talking is done by the lecturer. However, in this study, interaction among students and the lecturer, and among students themselves, was made possible through the application of the ACE learning cycle. The lecturer would assist when necessary as students worked in the lecture room in small groups of threes, for two hours every week for six weeks.

Sequences are a compulsory component of the calculus 1 for Mathematics major students, Mathematics Education, Computer Science and Statistics and Financial Mathematics. However, series are not part of the course outlines for the above programs. They are only covered for the purpose of this study. The calculus course was taught by one lecturer. The researcher taught students how to use maple to solve the sequences and series problems within the context of the ACE learning cycle. The students met for instruction for two hours per week for six weeks, and studied in the lecture room in groups of three. In the lecture room, students were given programming activities which gave them practice in solving problems on limits of sequences and series using maple. The students also attended four hours of classical (normal) classes per week.

Two hours per week were not enough to complete the activities using maple. Therefore, students continued with the activities at their residents at times. At the beginning of each session, small discussions (of half an hour) about the activities carried out the previous week. After the class, students were given homework covering work done during the week's classical question sets.

The students used their laptops for the activities for this study. This gave the students the opportunity to repeat the activities even after the classroom session. Maple was introduced to students in the first session, and they were required to use a table function of maple to facilitate instinctual overview of limit of a sequences and series. The Maple software was only used during the learning sessions. The examples below are part of the work done in the first session.

> $\lim_{n \to 3} \frac{1}{n};$

$\frac{1}{3}$

>

$\lim_{n \to 0} \frac{1}{n};$

undefined

>

∞

$\lim_{n \to \infty} \ln(n^2);$

> $\lim_{n \to \infty} \frac{1}{n};$

0

> $\lim_{n \to \infty} n^2;$

> $\sum_{n=1}^{3} \frac{1}{n};$

$\frac{11}{6}$

> $\sum_{n=1}^{\infty} \frac{1}{n};$

∞

> $\sum_{n=1}^{\infty} \ln(n^2);$

∞

> $\sum_{n=1}^{\infty} \frac{1}{n};$

∞

> $\sum_{n=1}^{\infty} (n^2);$ ∞

In the second session, characteristics and properties of sequences and series were discussed and later students solved problems using the graphic method. The extracts below give an insight into the work done using maple in the second session.

Sequences

General term > $expr := \frac{1}{3^n}$

$expr := \frac{1}{3^n}$

Index name > n

n

First index value > 0

0

Last index value > 7

7

Members > $seq(expr, n = \mathbf{(3)} .. \mathbf{(4)})$

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}, \frac{1}{2187}$$

Graph > $plot([seq([n, expr], n = \mathbf{(3)} .. \mathbf{(4)})], style = point, symbol = circle, color = red)$

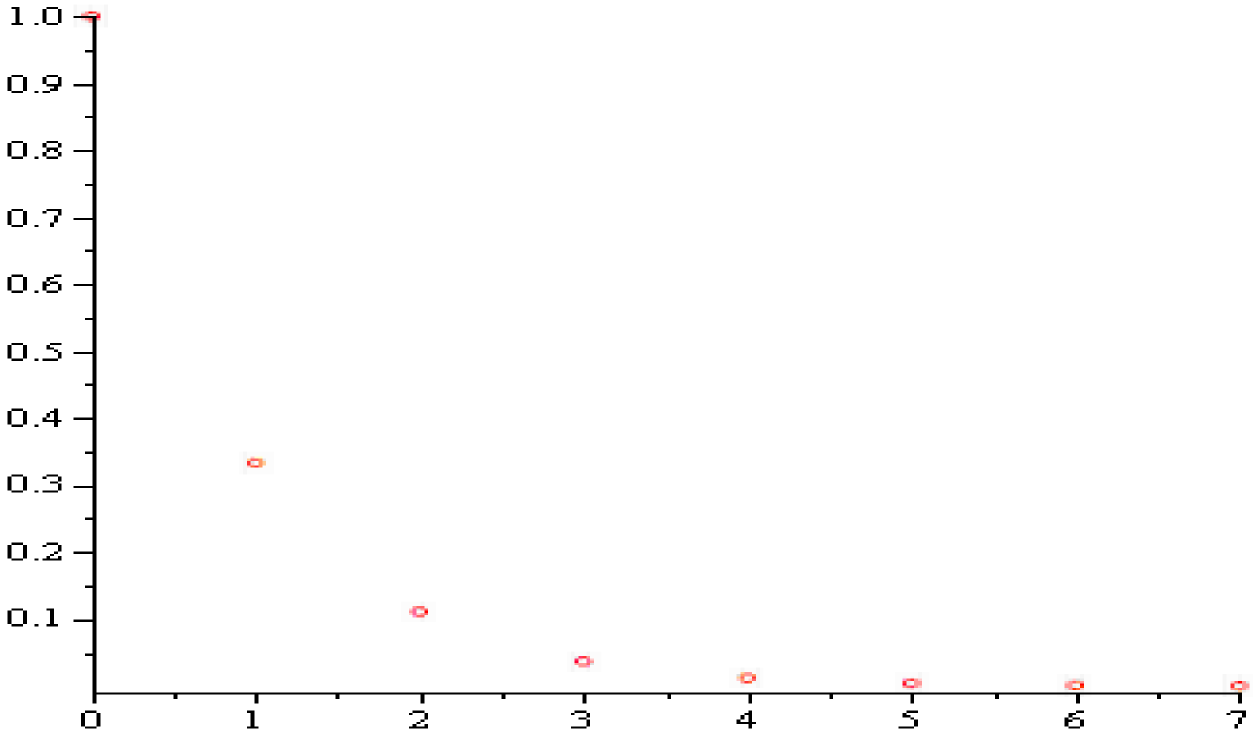


Figure 4.1: Graph of $\frac{1}{3^n}$

> $L := ListTools{:-}PartialSums\left(\left[seq\left(\frac{1}{n}, n = 1..30\right)\right]\right)$:

> *restart*;

> *with*(*plots*);

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*]

$> \ PartialSums\left(\left[seq\left(\frac{1}{n}, n = 1..30\right)\right]\right);$

$$PartialSums\left(\left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}, \frac{1}{21}, \frac{1}{22}, \frac{1}{23}, \frac{1}{24}, \frac{1}{25}, \frac{1}{26}, \frac{1}{27}, \frac{1}{28}, \frac{1}{29}, \frac{1}{30}\right]\right)$$

$> \ L := ListTools:\text{-}PartialSums\left(\left[seq\left(\frac{1}{n}, n = 1..30\right)\right]\right):$

$> \ plots:\text{-}pointplot(L);$

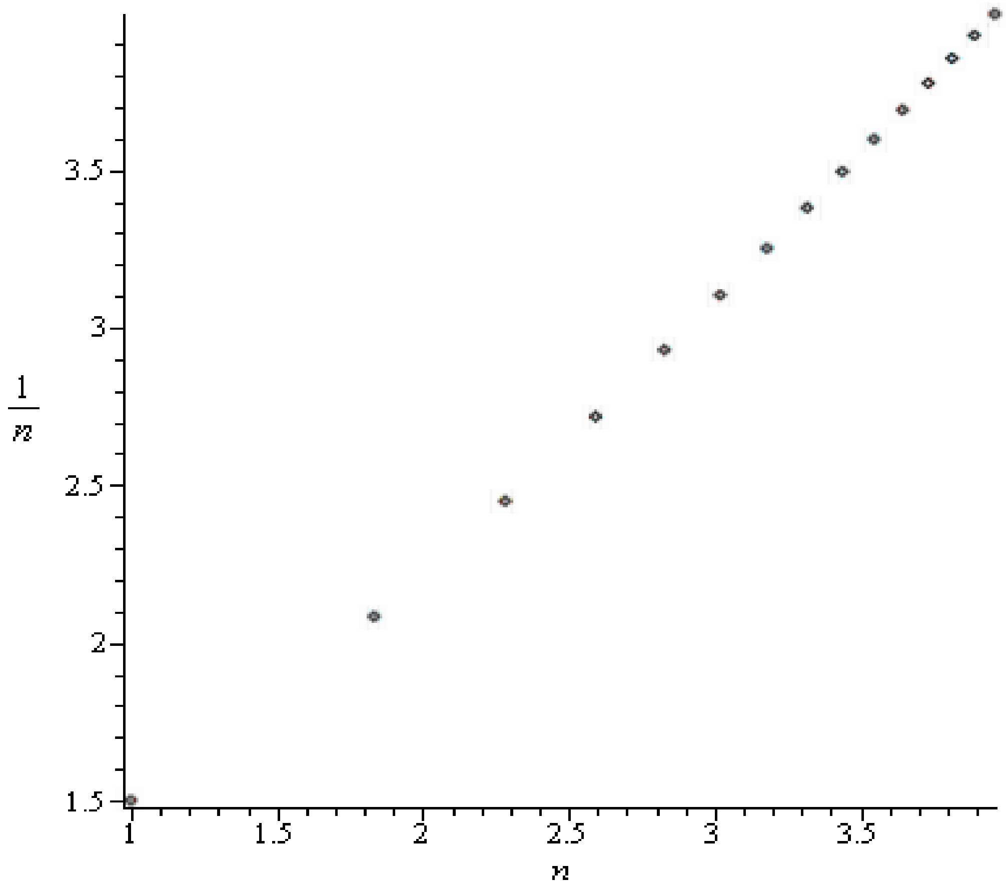


Figure 4.2: Graph for partial sums of a series 1/n from 1 to 30

During the third session, students first repeated what they did in the second session so that they became well acquainted with the concept and required skill of graphing. Also during the third session, students worked on the evaluation of sequences and series using different methods.

In the fourth session, students studied the formal definition (ε – N) of the limit of a sequence and its interpretation. That is, letting a sequence of real numbers (x_n) be a sequence, then the sequence (x_n) converges to limit L, if and only if, for every given positive real number ε, there exists a positive real number N_0 such that for all $n > N_0$ we have $|x_n - L| < \varepsilon$, where$(n \in N)$. What does this mean? $|x_n - L| < \varepsilon$, means the distance between x_n and L is less thanε. That is, the term x_n of the sequence lies between $L - \varepsilon$ and$L + \varepsilon$. This was also represented diagrammatically. Some examples were provided using the definition and illustrated using diagrams.

In the fifth session, students worked on methods of determining whether series converge or diverge. The sixth session was a repeat of session five since this part was not covered during classical lecture times.

After the day classes, the students were given homework to work on for the following week which comprised of a set of questions. Questions were studied in groups and this was followed by individual homework. The researcher prepared homework questions as a way of reinforcing the concepts that were considered and to stimulate students' thinking. Those questions acted as basis for discussion and students' reflections. Drill and practice questions were provided to the students, with some more challenging, while others required reflection on what was learned during class sessions (Appendix 1).

4.6 Data and Data Collection Tools

In this research two data collection tools were employed. These were the limit test questions and a semi-structured interview protocol. Each data collection tool is discussed in detail in the sub-sections that follow.

4.6.1 Limit test questions

The limit test was written traditionally by the students after the learning sessions. The test paper had a set of instructions to be followed as the students took the test. Students were assessed traditionally, hence the use of pen and answer booklet. The limit test given was in paper and pen form and the researcher attached importance to the mental constructions used to answer given questions regardless of them being correct or incorrect. The test was taken after the learning sessions so that students had covered and acquainted themselves with the contents of the test.

The test questions a tool (Appendix 2) in this study, the test questions were used to obtain information for analysis. Areas covered were; evaluation of limits of sequences using different methods, the Epsilon-N definition of a limit of a sequence, the use of the Epsilon-N to prove given limits, and the theorems on limits of sequences. It also covered areas on methods of evaluating limits of series and the determination of convergence and divergence of series. In this research, limit test questions required participants to answer questions that are related to the study with the aim of collecting information about participants' views. All the questions in the test were open-ended. The limit test questions (Appendix 2) were prepared by the researcher. The limit test questions were used to probe how the limits of a sequences and series were understood by the students. This was after the instruction based on APOS theory was done. An explanation of how APOS principles were used to construct the limit test questions, follows. Question 1(a) checked on the stating of the formal definition of the limit of a sequence. This question required an Action conception of stating the formal definition of the limit of a sequence. Furthermore, question 1 (b) checked on the use of different methods to evaluate limits of sequences. In questions 1 (bi) and 1 (b ii) students were given specific limits of sequences and were expected to rely on previous algebraic knowledge to carry out basic manipulations. The questions required students to utilize Action conception of sequence evaluation. Furthermore, questions 1 (b iii) to (b v) required students to utilize Process conceptions of sequence evaluation as students needed to build on their acquired conceptions from questions 1(bi) and1(b ii). Questions 1 (b iii) to (b v), required students to think about how to unpack the concept of transformations which required a Process conception. Question 2 was made up of four items, three checking on application of the formal definition of the limit of a sequence in proving given limits of sequences, and one checking on proof of a sequence theorem. Students were expected to reflect on their Process conception of transformation and unpack the basic tenets thereby building an Object conception of proving the limits of given sequences. Question 3 items covered limits of series with the first item checking on the stating of the formal limit definition of an infinite series. This question required an Action conception of stating the formal definition of the limit of an infinite series. This was followed by three items checking the use of methods to evaluate infinite series. To answer these questions, students were expected to use a Process conception of transformations, assuming they could imagine the transformation while incorporating a different, but related idea to the evaluation of infinite series.

The last three items checked the methods students employ to determine convergence and divergence of series. These questions required students to make use of an Object conception of transformations that needed to be de-encapsulated to analyze its component Processes and repacked to apply the concept to a new situation. The broadening of a Process conception of transformations to consider the concept in a more abstract way, while incorporating other concepts into their developing schema, pushed students towards an Object conception of transformations.

4.6.2 Interviews

After the written test, interviews to substantiate the students' level of understanding of the concept were conducted. Students interviewed were selected on the basis of their responses to pen and paper tasks. Students who gave responses at pre-sequence/series (P) level, Action level, Process level and Object level were selected for the interviews. The focus of the discussion was on the development of students' concept of the limit concept in relation to sequences and series and the APOS framework. The researcher recognized that the students' concept of the limit concept in relation to sequences and series required Action, Process, Object and Schema conceptions.

Interviews are ways to collect data, gain knowledge from individuals, and have participants involved and talk about their views. Kvale (1996, p. 14) regards an interview as "… an interchange of views between two or more people on a topic of mutual interest, sees the centrality of human interaction for knowledge prodiction, and emphasizes the social situatedness of research data." The interviewees are capable of discussing their interpretation and perception with regards to a given situation. Thus, interviews are not only concerned with gaining data about real life, but are part of life. The questions of the interview elicited valid responses through encouraging respondents to give full and accurate replies, while avoiding biases from constructs of disinterest. Gray (2004) gives the following reasons for conducting an interview:

1. The attainment of highly personalized data.
2. The opportunities required for probing.
3. A good return rate which is very important for research study.

Semi-structured interviews are non-starndard and are frequently used in qualitative analysis. This type of interview gave the researcher chances to probe for opinions and views of the individual interviewee. The probing also gave way for the interviewer to explore new paths which were not initially considered in coming up with the interview guide (Gray, 2004). The researcher used a list of key themes, issues, and questions to be covered. One cannot observe thoughts or feelings, hence the need to ask people about points under consideration. The questions were determined beforehand and follow up questions and new questions were asked dependent on responses to questions.

The first question wanted to find out about students' overall understanding of the limit of a sequence. Students gave general comments on the determination of the limit of a sequence at a point. Second and third questions required students to use a particular method to come up with the limit of a given sequence. The fourth required students to give a formal definition of the limit of a sequence, then draw a graph to explain the relationship between epsilon and N. The fifth required students to use the definition of a sequence to prove given limits of sequences. The sixth required students to determine the truth or falsity of given limits of pa particular sequences. The seventh question required students to conclude the convergence or divergence of given series. The use of different methods to find the limits of series that were required to find respective sums of series.

Since participation was voluntary, two students opted not to participate, leaving eight to partake the interviews. The first one declined to be interviewed citing poor performance. She said that she was not at liberty to discuss with someone her response to the written questions. The other one found it difficult to participate due to time and pressure from assignments for other courses. The interviews were conducted with eight students. These students exhibited diverse stages of understanding on the written limit questions. The students who gave correct responses to most of the test items were ST1, ST2, ST12, ST15 and ST28, with ST2 opting not to participate in the interviews. However, ST13, ST14, ST17, ST20 and ST21 failed to give correct responses to most of the test items and ST 21 opted not to participate in the interviews. The purpose of the interview was to get an insight into the students' thinking, and to probe recurring responses that were detected during the analysis of students' written responses. Through the interviews, the researcher developed a deeper understanding of some of the ways of thinking that underpinned students' responses to the questions. Some of the interview questions were shared to all the students and others probed students' topic experiences. The rest of the questions were follow-ups to issues that arose in students' solutions or to what they indicated during the interview.

4.7 Data Analysis Strategies

The current study used two instruments; the test questions (Appendix 2) and the interview procedure (Appendix 3), were used to gather qualitative data. The students were not under time constrains to complete test questions. The researcher gave each student a copy of the test questions, pen and answer booklets. Table 4.3 illustrates the test questions and the related concepts covered.

Table 4.3: Content analysis of test questions

Question	Concepts covered
1 (a)	Formal definition of the limit of a sequence
	Simple sequence evaluation
(b) (i-ii)	
1 (b) (iii-v)	Evaluation of sequences
2a (i) – (iii)	Use of epsilon-N definition to prove given limits
2(b)	Proof of the uniqueness sequence convergence theorem.
3 (a)	Definition of the limit of infinite series.
3b (i) – (iii)	Determination of convergent series that can be represented by a general formula.
3c (i) – (iii)	Selection and use of convergence or divergence tests for series.

The test questions provided the researcher with opportunities to analyze students' written responses, which gave the researcher initial clues about their understanding of limits of sequences and series. The test questions produced qualitative data and the researcher used a coding modified from the work of Asiala, Cottrill, Dubinsky and Schingendorf (1997) to evaluate students' responses as follows:

1. A student awarded a zero (0) pre-sequence/series level N for an irrelevant or empty response.

2. A one (1) was awarded for a response showing a step by step procedure for the solution, (Action level).

3. A two (2) was awarded for a response showing the performing of transformations mentally, prediction of outcomes, (Process level).

4. A three (3) was awarded for a response that showed encapsulation of the Process into a total entity, or an Object, ability carry Actions on that Object, and display the capability to decompose an Object to become Process as the properties of Object intended will be used, (Object level). Furthermore, Wilson and Dubinsky (2013) found out that learners have a pre-function (P) conception of a function if they indicate no or little conception about a function concept. In this study the researcher used four levels of APOS with the a mental structure that precedes Action level named the pre-sequence/series level. Thus in this study where students indicated no or little

conception of the limit concept in relation to sequences and series, the researcher indicated this as (P). In other instances the researcher used the APOS theory. During the marking of students' responses, the researcher looked for, and detected the "None" and different APOS mental constructions as indicated in Table 4.4

Table 4.4: Mental constructions and focused aspects

Mental construction	What was focused on?
pre-sequence/series level N	• No response or totally incorrect response.
Action	• Definition. Correct recall of definition. • Evaluations. Detailed step by step working, with all steps explicitly shown.
Process	• Evidence of some steps omitted; done mentally. Ability to skip steps as well as reverse them.
Object	• Student's ability to choose the correct convergence criteria based on the mathematical structure of the given entity. • Evidence of seeing a dynamic structure (Process) as a static structure to which actions can be applied.
Schema	• Evidence of the collection of Actions, Processes, Objects and other Schemas into a coherent structure: • Determination of truth or falsity of a statements; • Graphical and analytical registers used jointly to determine convergence.

The framework developed by Asiala et al., (1996) was modified so that it would be used to gather data from the interview protocol. The researcher transcribed the data from interviews. The recorded text was segmented into two columns. The first column contained the original recorded manuscript and the second contained the transitory report about what was happening in the first column. After carefully reading each transcribed text, the researcher produced aspects for each individual about their specific mathematical point of view.

Table 4.5: Content analysis of the interview

Mental construction	What was focused on?
pre-sequence/series level N	• No response or totally incorrect response.
Action	• Definition. Correct stating of definition from memory without explanations. • Use of evaluation rule e.g. the largest power wins logic, eyeball etc. without further explanations
Process	• Explanation of evaluations without specific sequences and series. • Ability to draw their action in a verbal manner without doing it. • Ability to explain the relationship between epsilon and N.
Object	• Ability to choose and explain using the correct convergence criteria based on the mathematical structure of the given entity. • Explanation of a static structure to which actions can be applied by use of graphs.
Schema	• Evidence of the collection of Actions, Processes, Objects and other Schemas into a coherent structure to; • Determine the truth or falsity of statements. • Use graphical and analytical registers jointly to determine convergence. • Ability to construct examples of mathematical accord with properties which are possessed by sequences and series.

The interviews accorded the researcher the opportunity to listen to students' explanations and justifications for their responses. Students also had the opportunity to give reasons for their responses which was not apparent in their written responses. The researcher explained the response of each individual in line of Actions, Processes, Objects and schemas. This helped determine whether specific mental constructions were made or not. Based on the students' responses to test questions and interviews, the researcher got the chance to make an informed decision on the instructional options.

4.8 Ethics and Negotiating Access

Written permission to carry out the study at the institution was granted by the human resources of the university on the 7th March 2016 by the then acting registrar (Appendix 5). Ethical clearance was also granted for the study and its number is HSS/0935/016D (Appendix 10). Permission sought from the university authorities allowed the researcher to carry out the research without interfering with the university's day to day running.

For the sake of unrecognizability, the students were coded using tags 'ST1', 'ST2', up to 'ST30' (ST 30 for student number 30). The order did not carry any implication. While enabling the organization of data, the codes ensured that the responses could not be linked in any way in the publication of results, to the original participant.

To ensure that the data acurately reflected students' thinking, a number of measures were taken. Informed consent was given to all partcipants and the researcher read and clarified its contents. Participants were asssured that their responses would firmly be preserved confidentially and that the data so gathered was for the use of the study only. It was communicated the to students that participation was completely voluntary and that one could withdraw his/her services at any stage they wished to do so. The researcher also outlined the nature, purpose and procedure of the study to the participants. Pseudonyms were used to protect the identity of the participants. The researcher clarified on the participants' concerns during the course of the study whenever they arose.

4.9 Validity and Reliability

A pilot study was carried out using 10 first year Masters' students during the December 2016 to January 2017 block. The students were doing a Masters' degree in Mathematics Education and were Advanced Level mathematics teachers. The researcher prepared test questions (Appendix 2). The pilot study aimed to check on:

1. Clarity of the test questions items.
2. Unnecessary and irrelevant items.
3. Time taken to complete the test questions.
4. The effectiveness of the test question items in acquisition of information about students' limit conception in relation to sequences and series.
5. Commonly misunderstood items.

All the test questions were kept after the pilot and none were modified.

The researcher prepared a semi-structured interview protocol (Appendix 3). The pilot was also used by the researcher as a platform for gaining experience in interrogating, probing appropriate follow up questions and analyzing students' response in interviews. Interviewed participants were carefully chosen based on the clarity and explanations of their written responses. The responses were answers to given limit test. The interviews measured the extent of abstraction, critical thinking, insight, conceptualization and imagination levels. Open-ended questions prepared in advance were used to solicit information from each individual participant. The duration of interviews was about an hour. The interviews were conducted at the university over a number of days. The interview schedule aimed to probe further the thinking and reasoning behind student's responses to test questions.

The pilot study aimed to check whether:

1. Question(s) needed rephrasing or not in order to write the specific mental constructions of APOS,
2. questions were ambiguous or not,
3. interview questions were effective in gaining the required evidence about student's limit concept in relation to sequences and series,
4. questions were leading or not.

There were no modifications done to the piloted interview protocol and his was used in the main study.

Reliability as described by Yin (2003) is,

> "… if a later investigator followed the same procedure as described by an earlier investigator and conducted the same case study all over again, the later investigator should arrive at the same findings and conclusion… the goal of reliability is to minimize the errors and biases in the study" (p.37).

Reliability issues were taken care of in this study. Triangulation was carried out by the use of the test questions and the interview protocol to arrive at answers for the research questions:

1. What APOS levels are displayed by students when solving limit problems in relation to sequences and series?
2. How do the APOS levels displayed by students when solving limit problems in relation to sequences and series relate to the preliminary genetic decomposition?
3. How does the historical understanding of limits compare with the students' mental constructions?
4. In what way can the mental constructions displayed by students be used to improve the understanding of limits of sequences and series at university level?

Cohen, Manion and Morrison (2007) define internal validity as "the explanation of a particular event, issue or set of data which a piece of evidence provides can actually be sustained by data" (p.135). Internal validity issues in this research were addressed. The results of responses of students to the test questions and the interview were compared. Attempts were made to support explanations and conclusions drawn by data from test questions and interviews. The researcher made used of APOS theoretical guide line to interpret the student's responses in the interview. Internal validity issues in this research were addressed, the research questions 1 and 2 were addressed by the use of both the test questions and the interview. Furthermore, research question 3 was addressed by the use of the interview and research question 4 was addressed by the modified genetic decomposition.

Triangulation involves the use of multiple referents to draw conclusions about what constitutes the truth. In this study, it was concerned with overcoming intrinsic bias that came from single-observer (Denzin & Lincoln, 2011). Those researchers identified data, investigator, method and theory as the four types of triangulation. The triangulation used in this study involved multiple data collection methods to gather data about the limit of sequences and series. Test questions and interviews were used to come up with a comprehensive appreciation of how the students understood the limit concept in relation to sequences and series. Data triangulation was used, time triangulation, it involved data collection on the phenomenon, from participants, at the same time (answering the limit test questions) and at different times (interviews).

The following criteria for developing trustworthiness are based on the suggestions made by Lincoln and Guba (1985):

1. Credibility was viewed by the researcher as a preponderant goal of qualitative research. It refers to the assurance in the truth of interpretations of the data. The researcher strove to establish assurance in the truthfulness of the findings for each participant, by the use of the informed consent. Thus, the researcher carried out this study in a way that enhanced credibility of the findings through the assurance given by the students' completion of the informed consent form. Furthermore, the researcher took steps that demonstrated trustworthiness to external readers, by having participants' written responses from the limit test questions and interviews on limit concept in relation to sequences and series.

2. Dependability comes second and refers to the constancy (reliability) of data over time and circumstances. A pilot study was carried out to ensure this. Interviews were audio recorded and transcriptions were made as scheduled in (Appendix 3).

3. Confirmability is concerned with the establishment that the data reflected the information given by the participants, and that the interpretation of the data was not the researcher's thoughts. The findings reflected the participant's voice and conditions of inquiry through the checking of participants' response transcriptions to both written and verbal questions. The researcher took the

transcribed data to each participating student for verification. The students confirmed that the transcribed information was correct.

4. Transferability denotes to the magnitude to which the discoveries have applicability to other settings. The researcher provided sufficient descriptive data so that other researchers could evaluate the applicability of the data on sequences and series, to other contexts. The researcher interviewed the students up to a point when no newer information or ideas could be found.

Table 4.6 gives the principles and procedures that provided the trustworthiness of the study on limits in relation to sequences and series.

Table 4.6: Principles and procedures for the establishment of trustworthiness in the study. (Adopted from Lincoln & Guba, 1985), then modified.

Strategy	Criteria	Evidence from
Credibility (internal validity)	Prolonged engagement.	Participants' engagement during lesson sessions, writing of the test and interview on the limit concept in relation to sequences and series.
Dependability (reliability)	Dependability audit	Pilot study, participants' responses to the limit test questions and interview transcripts
Confirmability (Objectivity of the data)	Confirmability audit	Students' checking of transcripts., on responses of verbal questions
Transferability (external validity)	Thick description	Extraction of participants' responses to written questions

and interviews. Verbatim quotes.

4.10 Conclusion

Chapter four covered methodological issues pertaining to this study. The chapter considered the qualitative research paradigm, the methods of data capturing, ethical issues and trustworthiness in line with qualitative research approaches. The next chapter discusses the data presentation, analysis and discussion. Chapter 5 summarizes the findings obtained from analyzed data on the limit of sequences test questions and the interviews.

CHAPTER 5 ANALYSIS AND DISCUSSION OF LIMITS OF SEQUENCES

5.1 Introduction

Chapter 4 highlighted the instructional activities designed to help students make the essential mental constructions that would enable their progression from one conceptual level of APOS theory to the other. This chapter focuses on; students' responses to test questions involving sequences, according to the content analysis given in Table 4.1. The researches that use APOS theory starts with a conjecture of the mental constructions that students need to undergo in order to understand a particular mathematical concept. The conjecture, called a genetic decomposition (GD), was based on the analysis of the mathematical concept itself, and the classroom experience of the researcher. The conjectured GD was then tested by analyzing students' responses to different research instruments. Students gave evidence of doing some expected and unexpected mental constructions, they also showed difficulties to use some of the conjectured constructions.

The focus of this chapter is on the limit test questions 1 and 2 (see Appendix 2). A case analysis of each test question supported by authentic written responses and interview extracts from selected students was done. This was done to provide confirmation of the APOS level at which the students operated, in terms of accepting the limit concept in relation to sequences. Furthermore, the interviews provided evidence of how students' mental constructions of limits compared to historical understanding of limits. The data presentation and analysis was done in line with the genetic decomposition provided in Figure 4.1.

Table 5.1: Complete summary of the categorization of students' mental construction, according to APOS, on each of the test items on limit of sequences.

Test Item	APOS categorization of students' mental constructions				
	None	Action	Process	object	Schema
1 (a)	7	22	0	0	1
1b (i)	2	28	0	0	0
1b (ii)	9	21	0	0	0
1b (iii)	4	0	26	0	0
1b (iv)	6	0	24	0	0
1b (v)	3	0	27	0	0
2a (i)	19	0	0	8	3
2a (ii)	14	0	0	13	3
2a (iii)	12	0	0	15	3
2 (b)	7	0	0	20	3

The questions 1 (a) to 1b (ii) tested students' attainment of the Action level. The results revealed that most of the students attained the Action level, with a few failing to attain the Action level and only one attained the schema level. Also, questions 1b (iii) to 1b (v) tested students' attainment of the Process level and the results showed a similar trend to responses of questions 1 (a) to 1b (ii). Furthermore, responses to questions 2a (i) to 2 (b) showed that the number of students who failed to attain the Object level and Schema level were in a decreasing order from 19 for question 2a (i) to 7 for question 2 (b). The relevant information displayed in Table 5 was extracted for Tables 5.1 to 5.10 to make it easier for the reader to follow the focused discussion on the different sub-questions for test questions 1 and 2.

5.2 Definition of the limit of a sequence: question 1(a), State the ϵ - N definition of the limit of a sequence.

The question addressed the formal definition feature of the Action level in the genetic decomposition. Its intention was to afford insight as to whether students developed an Action notion of the concept of sequence limit definition. There could be a number of ways the definition could be presented but generally, it should have included the description of a sequence in terms of n, N, L and ϵ. The definition could have been stated as follows: let (a_n) be a sequence of real numbers, then the sequence converges to limit L, if and only if, for every given positive real number ϵ, there exists a positive real number N_0, such that for all $n > N_0$ we have $|a_n - L| < \epsilon$, $(n \in N)$ (Stewart, 2015). Any other reasonably correct definition with related quantifiers was acceptable.

5.2.1 Responses of students according to APOS levels

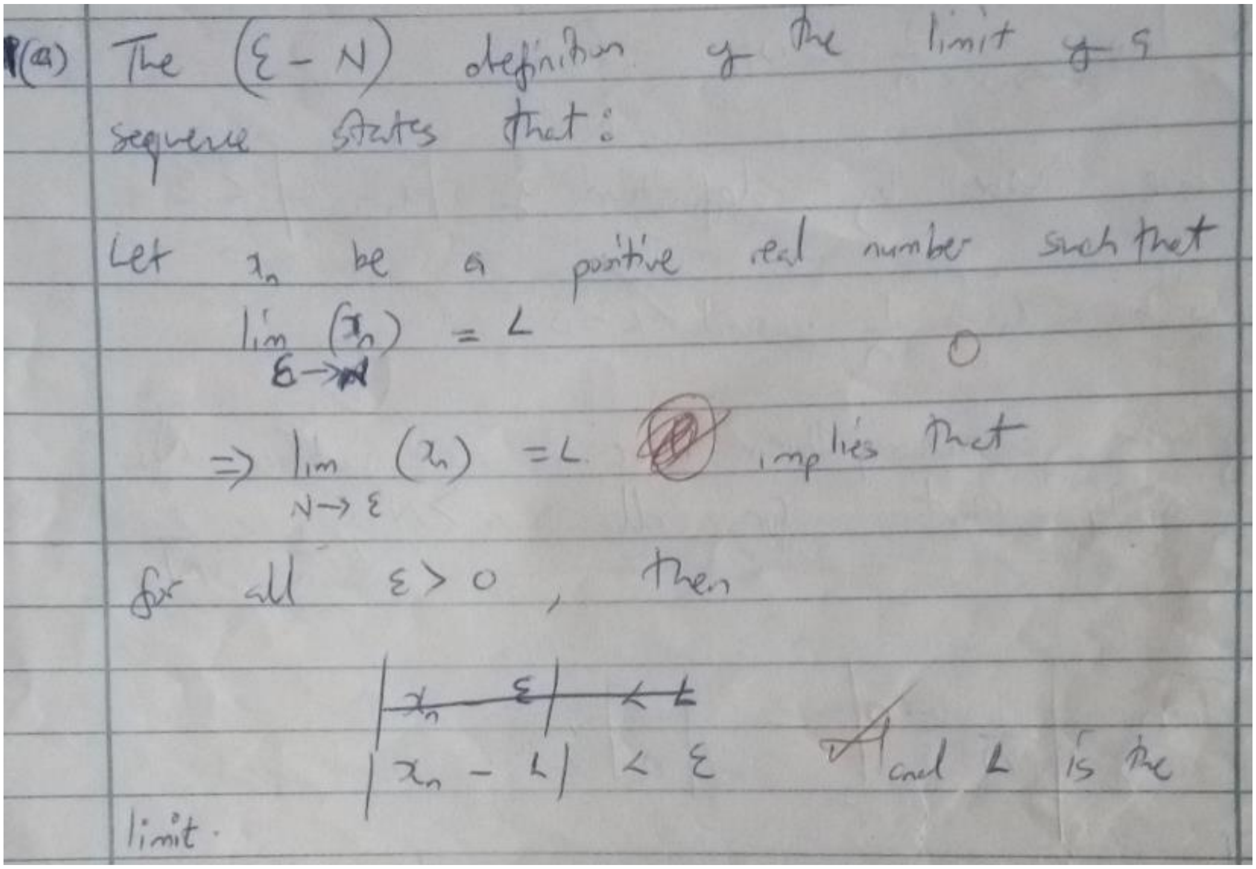
Table 5.2: Frequency of students' responses for question 1 (a) according to APOS level

Frequency of students' responses for question 1(a) according to APOS level.

APOS level	N	Action	Schema
Number of responses	7	22	1

Seven students (23%) gave mathematically incorrect definition in response to question 1(a). Of the seven, five gave the $\epsilon - \delta$ definition of the limit of a function, which most of the times was wrongly stated. That was an indication that the students could have confused a sequence and a function definition. Furthermore, such students may not have understood the demands of the question. This could only be verified through an interview with some of the students. Two out of the seven students gave a totally confused definition of a sequence. This supports findings that students' difficulties in understanding formal limit definitions is due to a lot of much notation and the necessity for this symbolization to be stirred (Fernadez, 2004). In general, students held an unclear notion of the ordinary inequalities involved in the epsilon-delta definition of the limit (Cottrill et al., 1996). Twenty- two (73%) of the students gave responses at the Action level, which was in line with the preliminary genetic decomposition. One student operated at the Schema level of the formal definition of the limit of a sequence.

5.2.2 Responses at N level (totally incorrect response)



1(a) The $(\varepsilon - N)$ definition of the limit of a sequence states that:

Let x_n be a positive real number such that

$$\lim_{\varepsilon \to} (x_n) = L$$

$$\Rightarrow \lim_{N \to \varepsilon} (x_n) = L$$ implies that

for all $\varepsilon > 0$, then

$$\cancel{|x_n - \varepsilon| < L}$$

$$|x_n - L| < \varepsilon$$ and L is the limit.

Figure 5.1: Written response of ST 13 to question 1(a)

ST 13 attempted question 1(a) but failed to give the correct response which showed that he had failed to make the crucial mental constructions to handle such questions. The student wrote $\lim_{N \to \varepsilon}(x_n) = L$ which indicated that he had problems with the correct notation. Furthermore, what he wrote after the confusing limit statement "implies that for all $\varepsilon > 0$, then…" indicated the inability to state the required definition. The definition provided supported the findings that learners had a pre-function conception of a function if they indicate little conception about the function concept (Wilson, 2013). To verify that the student operated at N level, the researcher interviewed him to find out how he understood these concepts and the complications he faced. The interview excerpt that follows reflects the dialogue the researcher had with him.

R: What does the question require you to do?

ST 13: The question requires me to state the $(\varepsilon - N)$ definition of the limit of a sequence.

R: Can you state it?

ST 13: Huuuuu, I cannot correctly state it. I need more time to do that.

R: Can you apply the definition to prove given sequence limits?

ST 13: That too is very difficult for me to do, I think I need a lot of time to be able to do that.

The responses given by ST13 testify to the fact that the student was operating at the N level since he indicated little conception about the formal definition of the limit of a sequence concept. The failed to reach the level expected in the preliminary genetic decomposition of stating the ϵ - N definition of the limit of a sequence.

R: Let a_n be a sequence, how do you determine the limit of a sequence a_n at a point a_0?

ST 13: I will use the number a_0 to come up with the limit of the sequence a_n.

R: What kind of number is this?

ST 13: Figures which are not equal to a_0 but very near to a_0.

R: Which numbers are these?

ST 13: It is not all the time, when it is possible to find the closest number since there is no such resolute values. The numbers will be close to a_0 as much as possible. Suppose $a_0 = 5$, then we can talk of 4.99…, 4.999…, 4.9999… all these numbers are very very close to 5.

R: Okay. Are 5 and 4.9999… infinitely close to each other?

ST 13: Yes. 4.999… is infinitely close to 5 just as 4.999

R: What will you do with these numbers?

ST 13: We will plug into a_n these numbers and notice the resulting values. If the resulting values are pointing (leading) to a certain number, then this number must be the required limit.

For ST 13, infinitely close means very close in a quantifiable manner. This is an indicator and confirmation that he does believe in infinitesimals. His notion is not dynamic but static. This is buttressed by Tall's (2004) analysis of student reasons for accepting that 0.999…= 1, "The difference between them is infinitely small" or "At infinity, it comes so close to 1 it can be considered the same" (Tall & Schwarzenberger, 1978. p 44). The student's notion is also supported by Euler when he remarked that infinitesimal were numbers less than any assignable quantity, and were to be considered as zero (Lolli, 2010).

5.2.3 Responses at Shema level

Figure 5.2 illustrates an example of the response of a student who gave a correct definition of the limit of a sequence.

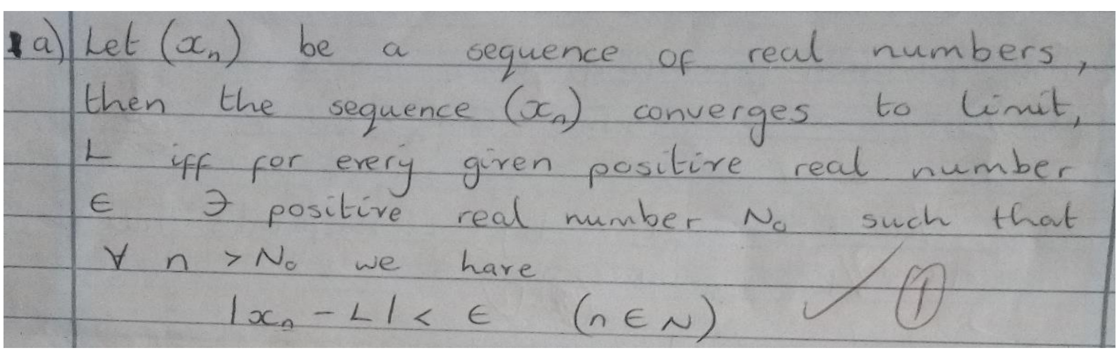

1 a) Let (x_n) be a sequence of real numbers,
then the sequence (x_n) converges to limit,
L iff for every given positive real number
ϵ $\exists$ positive real number N_0 such that
$\forall$ $n > N_0$ we have
$|x_n - L| < \epsilon$ $(n \in N)$ ✓ 1

Figure 5.2: Written response of ST 1 to question 1(a

The response provided by ST 1 indicated that he operated at the Action level of the APOS theory. The student exhibited resilient affinities to remember the definition verbatim (Breidenbach, Dubinsky, Hawks & Nicholas, 1992). The ability to correctly give the definition is an indicator that the student operated at the Action level. To confirm the researcher's claim, an interview was held and the dialogue is captured in the following excerpt:

R: How can you explain an $\varepsilon - x_n$ definition of the limit of a sequence?

ST 1: (The student explains with the aid of a diagram Figure 5.3). The sequence a_n converges to L, if and only if, for every given positive very small number ε,there is a positive number x_n so that

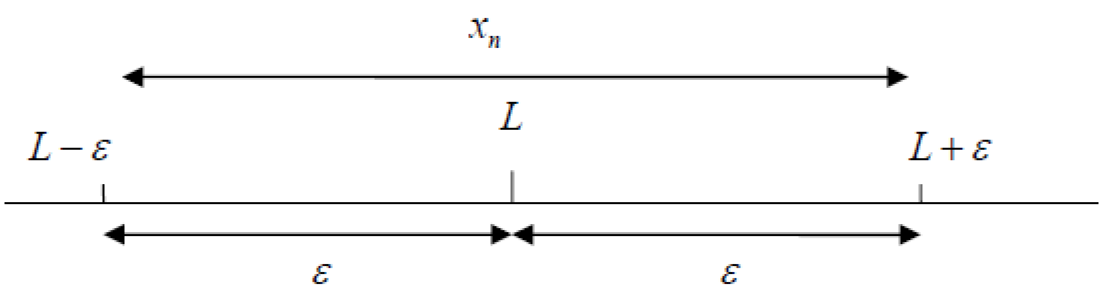


Figure 5.3: Definition of the limit of a sequence

for all $n > N$ we have $|x_n - L| < \varepsilon$. That is, the distance between x_n and L is less than epsilon, and the sequence x_n lies between $L - \varepsilon$ and $L + \varepsilon$ always.

R: What is the relationship between ϵ and x_n?

ST 1: If the sequence is a fraction, x_n is equal to a fraction of epsilon; and if the sequence is not a fraction, then it will be equal to a multiple of epsilon.

R: Well, can you use a graph to explain what this definition means?

ST 1: $|x_n - L| < \varepsilon$ means that the distance between x_n and L is less than ε. Thus, the sequence a_n lies between $L - \varepsilon$ and $L + \varepsilon$. This can be represented in the Figure 5.3.

The written response displayed by ST1 revealed mental constructions at an action level. However, his explanation and graph indicated in Figure 5.3 during the interview demonstrated that he had mental constructions at a Schema level to deal with the application of the definition of limit of a sequence. The explanations and the graph provided by ST 1 showed that he had attained full understanding of the definition of the limit of a sequence. He managed to explain the relationship between Epsilon and x_n, and also to explain his understanding of the definition. ST 1 managed to link graphic and symbolic forms to construct a precise symbolization for the available information in the graph he drew (Wayer, 2010).

5.3 Evaluation of limits at infinity: question 1b (i), $\lim_{n\to\infty} \frac{n^2+3n}{2n^2+1}$.

This question addressed the Action level in the genetic decomposition. Its intention was to provide insight on how the students had acquired an Action conception of the concept of the limit of a sequence evaluation, if any. The students were expected to make use of the different methods of evaluating sequences which they learnt during the classroom activities sessions.

An Action conception in this research is identified by externally directed transformation of previously conceived object(s), which is a step by step evaluation of sequences with each step prompting the next step. The transformation needs to be explicitly performed. For this research, students required an explicit expression of the sequence in order to think of evaluation of a sequence (Arnon et al., 2014). In question 1b (i), the “approach to infinity” and the word “limit” are external cues to the student. The algorithm involved dividing the numerator and the denominator. The step by step procedure and the external cues are expected to enhance the Action conception. An Action conception of the evaluation of simple sequences is necessary for the development of Process conception of sequence evaluation. That is, the student should show evidence of carrying out some steps in the mind and arriving at the answer.

5.3.1 Responses of students according to APOS levels

Table 5.3: Frequency of students’ responses for question 1 b(i) according to APOS level.

APOS level	N	Action
Number of responses	2	28

Two students (7%) operated below the Action level of the APOS theory and failed to attain the expectation of the preliminary genetic decomposition. They failed to provide any response to question 1b (i). Twenty-eight (93%) of the students provided responses to question 1b (i) that indicated that they operated at the Action level. The responses indicated the students' ability to evaluate simple sequences step by step, showing all steps explicitly as the preliminary genetic decomposition expected. The researcher observed that students had the required collection of procedures which they applied to a related context.

5.3.2 Responses at Action level

Figure 5.4 gives the response of student ST 15 who operated at Action level.

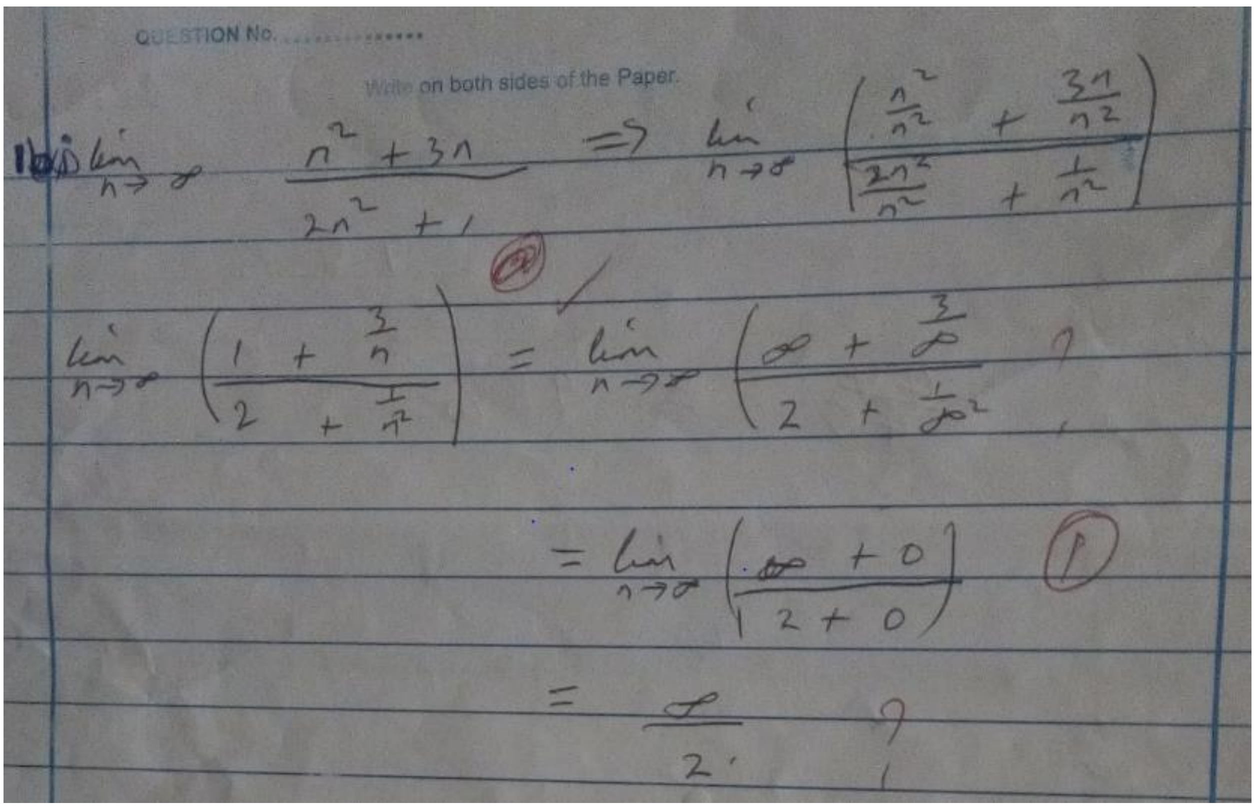


Figure 5.4: Written response of ST 15 to question 1b(i)

ST 15's response is an indication that she operated at the Action level. She demonstrated step by step working from an explicit expression which reflects an Action level of engagement (Dubinsky & McDonald, 2001; Arnon et al., 2014). This student's response was an indication of thinking round the problem in a stage by stage manner, one step at a time. This resonates with Dubinsky's (1991) assertion that the capability to carry out actions step-by-step is indicative of the Action level of the APOS Theory. However, she made a mistake by replacing 1 by infinity when she was replacing n in the denominators. Though ST 15 made a mistake, she could follow procedures for the evaluation of the limit of a sequence.

The substitution of values in a given expression for sequence representation is suggestive that the learner is functioning at the Action level of APOS theory (Dubinsky & Harel, 1992). To authenticate the claim that she operated at the Action level, the researcher interviewed ST 15 on her solution strategy and the reactions are shown in the following interview extracts:

R: Observing at your responses to question 1(bi), $a_n = \frac{n^2+3n}{2n^2+1}$, how did you determine the limit of this sequence as $n \to \infty$?

ST 15: I factor out n^2 from the numerator and then from the denominator first, then simplify the resulting expression.

R: Why did you divide by n^2?

ST 15: We were taught to divide a given expression by the highest power of the denominator.

R: Hoo, how did you proceed?

ST 15: The student wrote and explains, the result is $1+\frac{3}{n}$ for the numerator and $2+\frac{1}{n^2}$ for the denominator. Taking limit of both the numerator and the denominator as $n \to \infty$ give $\frac{3}{n}=0$ and $\frac{1}{n^2}=0$, giving $\frac{1+0}{2+0}$ and this gives the final answer as $\frac{1}{2}$.

R: In your written response, how did you come up with $1=\infty$?

ST 15: Aaaa! it was just a mistake, it should be 1 and not ∞.

Based on the above responses, the researcher concluded that ST 15 could recall what was taught without explanations. ST 15 evaluated the limit of the sequence by placing numbers into an expression but failed to explain why she was dividing by n^2. Remembering what was taught and failing to explain the reason for dividing by n^2 are clear indicators that ST 15 operated at the Action level of the APOS level (Dubinsky & Harel, 1992). The discussion on ST 15's interview responses made the researcher to conclude that she operated at the Action level. The ability to evaluate the limit of a given sequence using a step by step manner indicated that the student had reached the Action level as expected in the preliminary genetic decomposition.

R: Suppose a_n a sequence. How do you come up with the limit of a sequence a_n at a point a_0?

ST15: I plug in a_0 into a_n to find out if it is defined or not. If it is defined, then the resulting value is the limit.

R: So, is a_0 necessary for you to find the limit of a sequence?

ST 15: Yes, apart from plugging in a_0, one can use a number very close to a_0.

R: Is it possible to find a number that is infinitely close to a_0?

ST 15: Yes. very, very close I suppose.

R: Suppose $a_0 = 3$, is it a fair characterization that 2.9999… is infinitely close to 3?

ST 15: It is close. I do not know if it is proper to say infinitely close.

R: Do you have to take just a number or numbers close to a_0?

ST 15: I will take a few numbers to find the resulting pattern from the results, then I will deduce the limit from the resulting values.

R: Can you explain this by use of a specific example?

ST 15 illustrates using $f(n) = \lim_{n \to 5} 2n$.

ST 15: We can use values close to 5, 4.9, 4.99, 4.999. For these values, we get 9.8, 9.98, 9.998 which are coming close to10.

R: How do you comprehend that these are getting close to10?

ST 15: 9.8, 9.98, 9.998, 9.9998 it will keep going like this which are getting close to 10.

Student ST 15's expressions are indicative a never ending process, the dynamic view.

5.4 Evaluation of limits at infinity: question 1b (ii), $\lim_{n\to\infty}(\sqrt{n+2} - \sqrt{n})$.

Question 1b(ii) was also intended to address the Action level in the genetic decomposition. The study showed that some students had reflected and interiorized the Action of evaluation of a sequence into a Process. The researcher observed that written responses revealed students' consideration of all the procedures and their application of these in related context. Students' responses showed that they had constructed the correct concept of sequence evaluation. Some students showed a clear understanding of conjugates ($\lim_{n\to\infty}(\sqrt{n+2} + \sqrt{n})$)($\lim_{n\to\infty}(\sqrt{n+2} - \sqrt{n})$) and how they are applied on evaluating given sequences. Some of the students' responses revealed interiorization of Actions into the Process, showing capabilities of coordinating the relationship between infinity limit laws and sequence evaluation, and of conjugates and sequence evaluation. However, some of the students' responses revealed lack of understanding of the indeterminate forms.

5.4.1 Responses of students according to APOS levels

Table 5.4: Frequency of students' responses for question 1b(ii) according to APOS level.

APOS level	N	Action
Number of responses	9	21

Nine (30%) of the students' responses showed that they operated below the Action level. Twenty-one (70%) of the students gave responses to question 1b(ii) that showed that they operated at the Action level of the APOS theory in line with the preliminary genetic decomposition.

5.4.2 Response at N level (totally incorrect response)

Figure 5.5 illustrates an example of students who operated at N level. They considered infinity as a number, then carried out subtraction failing to realize that this was an indeterminate form which required other methods.

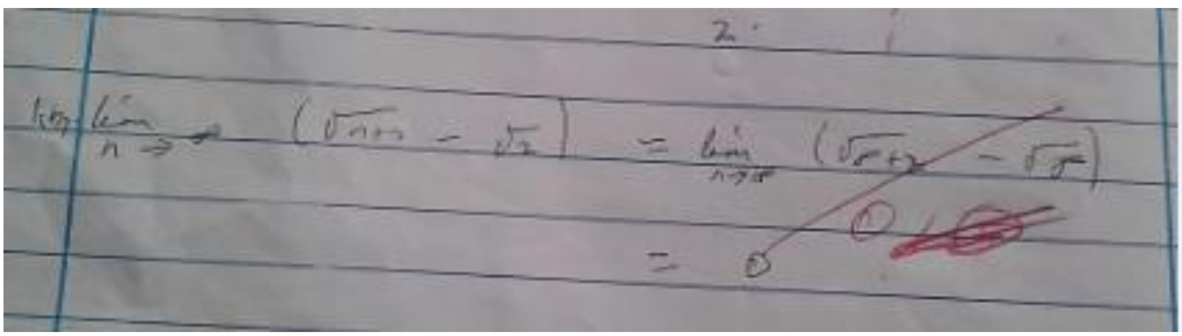


Figure 5.5: Written response of ST 15 to question 1b(ii)

ST 15's written response to question 1b (ii) shows that she had not mastered the indeterminate forms. $\infty - \infty$ is one of the indeterminate forms which calls for alternative methods to be employed to solve the given problem. ST 15 used the procedures unsuccessfully due to lack of understanding. Thus looking at the APOS level, the researcher placed her at the N level. Learners have a pre-function conception of a function if they indicate little or no conception about the function concept (Wilson, 2013). ST 15' written work showed no conception about sequence evaluation involving infinity and indeterminate forms. In question 1b (i), her gap in the understanding of the infinity concept led her to get this question incorrect. She carried over her idea of treating "∞" (infinity) as a constant. To authenticate the researcher's claim that ST 15 was operational at N level, the researcher interviewed her on her solution strategies. The following interview excerpts captured our dialogue:

R: How did you evaluate the limit of the sequence given in question 1 (bii)?

ST 15: I substituted infinity into the expression and write down the outcome of the substitution.

R: After you had substituted, what did you get?

ST 15: Square root of infinity plus 2, minus the square root of infinity which then gives us infinity minus infinity. Because two plus infinity gives us infinity.

R: Do you know cases of indeterminate form?

ST 15: I have heard about them during lectures but I did not fully understand them and how they are applied.

R: Is it possible to evaluate $\sqrt{\infty + \mathbf{2}} - \sqrt{\infty}$ as it is?

ST 15: Yes, because square root of infinity plus two is infinity and square root of infinity is infinity. This leaves us with infinity minus infinity which is zero.

An analysis of this dialogue reveals that ST 15 has not understood the indeterminate forms and the required methods that can be employed to evaluate sequences of this type. Her reasoning indicated that she operated at (P) level, she showed little conception of evaluating sequences that involve indeterminate forms. ST 15 failed to attain the level expected in the preliminary genetic decomposition, the Action level.

5.4.3 Responses at the Action level

There were some students whose responses were as ST 1's in Figure 5.6. Their responses showed that they operated at the Action level of the APOS theory.

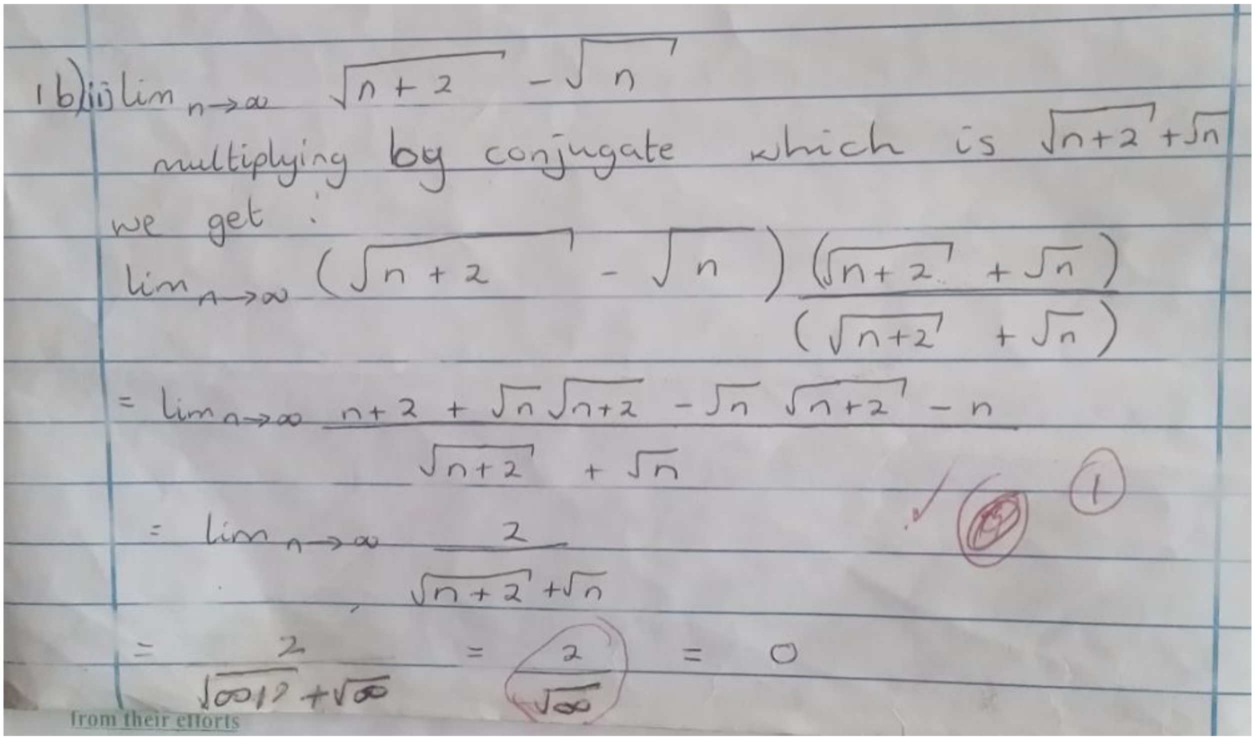


Figure 5.6: Written response of ST 1 to question 1b (ii)

In my analysis, ST 1's written responses indicated a step by step working without omission of any step, indicative of the Action level of the APOS theory. Multiplying with the conjugate and manipulating without explanation on the procedure is a display of the Action level of the APOS theory. The Action is exterior for each step of the transformation, each step was performed explicitly, each step prompted the next step and no step was skipped (Arnon et al., 2014). ST 1's response shows step by step procedure with one step at an interval (Dubinsky & McDonald, 2001). The written response by ST 1's showed that he operated at the Action level of the APOS theory. To verify the assertion that ST 1 operated at the Action level, the researcher interviewed him to explain his written and oral responses:

R: Looking at your response, how did you come up with the final answer?

ST 1: (this time he did not work it out but explained his response method). There are many methods that one can use to solve such questions on limits of sequences. After examining the question, I discovered that multiplying the numerator and the denominator by the conjugate will do the trick.

After carrying out some manipulation this resulted in a 2 as the numerator with the denominator having a power of n, and as n goes to infinite, then the whole expression goes to zero.

R: In your working, you showed all the steps. Is this how you always present your work?

ST 1: In most cases, yes. From lower levels we were encouraged to show all possible working. In this case, the first step led to the next step and then to next again and again until the final answer.

ST 1's interview response revealed that he evaluated the sequence step by step with each step leading to the next step. This confirmed the researcher's claim that he operated at an Action level of the APOS theory and also he managed to meet the expected level in the preliminary genetic decomposition.

R: If we take a_n to be a sequence, how do you determine the limit of a sequence a_n at a point a_0?

ST1: I plug in a_0 into a_n to find out if the sequence is undefined or defined. If it is defined, then the resulting value is the limit of the sequence.

R: So is a_0 necessary for you to find the limit of a sequence?

ST 1: Yes, apart from plugging in a_0 in the given sequence expression I can use a number very close to a_0 to determine the limit of the sequence.

R: Can you find a number that is infinitely close to a_0?

ST 1: Yes. Very, very close to a_0 I suppose.

R: Suppose $a_0 = 3$, is it a fair characterization that 2.9999… is finitely close to 3.

ST 1: It is close; I do not know if I would say infinitely close but it is very close.

R: Do you have to take just a number or numbers close to a_0?

ST 1: I will take into consideration a few numbers to find out the pattern from the results then deduce the limit of the sequence from the resulting values.

R: So can we say the limit of 2.999… is 3?

ST 1: Yes, because the difference between these two numbers is negligible as it is very small and also there is no way 2.999… can exceed 3.

The researcher's dialogue with ST 1 is supported with the findings that the student comments raised an issue pertaining to approximations that one could make as accurately as one desired; and infinitely small errors or differences did not matter (Oehrtman, 2009). Furthermore, an early definition of limit "One magnitude is said to be the limit of another magnitude when the second may approach the first within any magnitude however small, though the first magnitude may never exceed the magnitude it approaches" (Burton, 2007, p. 603), included approximation language. This is the case even though the modern definition attempts to eliminate temporal aspects. But such ideas of approximation language still underlie conceptions of limits.

5.5 Evaluation of limits at infinity: question 1b (iii), $\lim_{n\to\infty}\frac{n}{n+1}$.

Questions 1b (iii) was designed to determine students' understanding of sequence evaluation using the L Hospital's rule. This question aimed to provide understanding about students' conceptual awareness on the method's use as an evaluation tool for the determination of sequence convergence. Furthermore, it was hoped that this question would address the Process conception as indicated in the preliminary genetic decomposition. Processes are interiorized Actions with new Actions leading to the attainment of higher order structures (Arnon et al., 2014). Interiorization results from an individual's ability to carry out an action mentally (some internal creation is made in relation to action). Processes are arrived at through interiorization, the Actions are reiterated and reflected upon, resulting in the individual moving from depending on external prompts, to gaining control over these external cues. This level is categorized by the capacity to carry out steps in the mind devoid of the necessity to accomplish all the steps explicitly.

The students who gave the correct response had understood the question correctly and applied the methods correctly, indicating coordination of schema needed for the evaluation of sequences involving the L Hospital's rule. For the students who provided the incorrect response, it was evident from their work that they had difficulties with notation $n!$ and n^n. Thus, learning a notation call for building some cognitive structures about that notation to upkeep its importance and use (Findell, 2006). From the interview held, there was evidence that some students lacked related knowledge of elementary algebra schema. This negatively impacted the construction of essential mental structures. Radford (1997) contends that absence of background information of mathematical concepts produces difficulties with the learning of algebra. The notion is supported by some of the findings of this research. The notation done at advanced level mathematics was not cognitively constructed. The interviewed student revealed that he had acquired the mental structures as required by the genetic decomposition, at least at the Process level.

5.5.1 Responses of students according to APOS levels

Table 5.5: Frequency of students' responses for question 1b (iii) according to APOS level.

APOS level	N	Process
Number of responses	4	26

Four (13%) of the students gave incorrect responses to question 1b (iii) related to evaluations using the L Hospital's rule, or any other method showing that they had not attained even the Process level of the APOS theory. Twenty-six (87%) operated at the Process level of the APOS theory.

5.5.2 Responses at N level (no response)

ST 20 did not attempt question 1b (iii) which is an indication that he operated at the N level. A learner operated at N level if he/she indicated no conception about the function concept (Wilson, 2013). ST 20 showed no conception of sequence evaluation. The researcher interviewed him to check his understanding of the concepts under discussion and the probable difficulties he faced. The interview excerpts that follows is the dialogue the researcher had with him.

R: You did not attempt question 1b (iii). What challenges did you face in dealing with this question?

ST 20: There are many methods used to evaluate limits of sequences for example, the squeeze theorem, L Hospital's rule, direct evaluation etc. I got confused and could not recall which was which, so I could not proceed to give the response.

R: What else did you find difficult concerning the question?

ST 20: Matching the required method and how it is carried out made me to leave the question without a response.

In the excerpt, ST 20 gave short and precise answer on why he did not attempt the question. The response during the interview was indicative of the student not having understood the techniques of evaluating limits of sequences. The student failed to make the necessary mental constructions needed to evaluate sequences, hence, ST 20 did not manage to operate at the Process level of the APOS theory as expected in the preliminary genetic decomposition. He operated at the N level.

5.5.3 Responses at a Process level

Figure 5.7 represents an example of a student whose response was classified to be operating at an Action level.

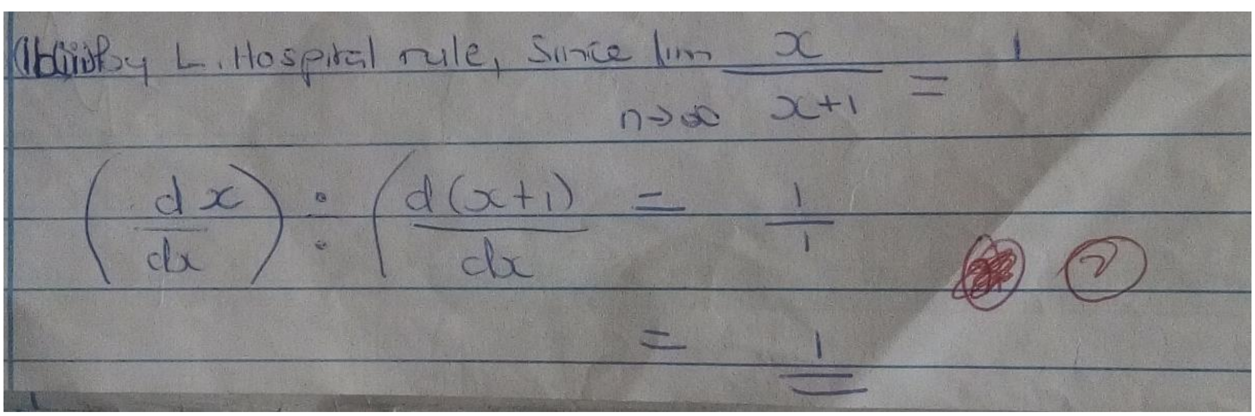


Figure 5.7: Written response for ST 28 to question 1b(iii)

ST 28 used the L Hospital's rule and successfully determined the limit of the given sequence. When an Action is repeated, and reflection on Action occurs, then subsequently the individual enters into Process phase (Suryadi, 2012). ST 28 internally controlled the Action and carried out the sequence evaluation successfully. He experienced a Process conception by showing the ability to do reflection toward that mathematical idea. The researcher's analysis of his written response to question 1b (iii) established that he operated at the Action level of the APOS theory. To confirm the researcher's observation, the student was interviewed so that he could explain his response strategies. This was captured in the following interview excerpts:

R: What condition is required for the use of the L Hospital's rule when evaluating sequence?

ST 28: When the function generating the sequence can be found, for example, $F(x) = \frac{x}{x+1}$ for $f(n) = \frac{n}{n+1}$.

R: Ok, what else is necessary?

ST 28: If you plug in a_0 or ∞ and it gives you any of the indeterminate forms.

R: Explain what you mean by indeterminate form.

ST 28: **(**the student explains verbally) If you plug in and get $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 * \infty$. $\infty - \infty$, 0^0, $\infty^0$and 1^∞.

R: How do you evaluate $\lim_{n\to\infty} \frac{n}{n+1}$ using the L Hospital's rule?

ST 28: If we plug in ∞, we have a situation $\frac{\infty}{\infty}$, and this calls for the L Hospital's rule. If f and g are differentiable and$g' \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. Thus, $\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(x+1)} = \frac{1}{1}$ =1. It then follows that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. That is the limit of the given sequence is 1.

ST 28 was able to give the indeterminate forms and the conditions required for the application of the L Hospital's rule. After the researcher had analyzed his explanation of the response strategy, it was evident that he operated at the Process level of the APOS theory fulfilling the expected level in the preliminary genetic decomposition.

R: Let a_n be a sequence, how do you come up with the limit of a sequence a_n at a point a_0?

ST 28: I will use this number to find the limit of a given sequences.

R: What sort of number is this?

ST 28: These are numbers which are not necessarily one and the same to a_0 but very close to a_0.

R: Which numbers are these?

ST 28: It may not always be possible to find the closest since such numbers cannot be determined. The numbers will be close to a_0 as much as possible. Suppose $a_0 = 2$ then we can talk of 1.99…, 1.999…, 1.9999… as numbers that are close to 2.

R: Okay. Are 2 and 1.9999… infinitely close to each other?

ST 28: Yes, they are infinitely close to each other.

R: What will you do with these numbers?

ST 28: We will plug some of these numbers into a_n and notice the resulting values. If the resulting values are approaching a certain number, then this number is the required limit of the sequence.

The interview conducted with ST 28 presented an argument where he displayed an understanding of infinitesimals. The student's reasoning was that another 9 can always follow the infinity of 9s. ST 28 had a notion that it is possible to insert more to the end of an infinite string (of 9s). A discussion with ST 28 showed that he conceived 1.999… as a static point and not a dynamic one moving towards 2. His conception is in support of the $\varepsilon - N$ sequence limit definition namely; "if the tail of a sequence gets arbitrarily close to a number, then that number is the limit of the sequence" (Norton & Baldwin, 2012, p. 61). Norton and Baldwin gave this explanation when he interpreted Balzano's $\varepsilon - N$ definition for limits of a sequences

5.6 Evaluation of limits at infinity: question 1b (iv), $\lim_{n\to\infty}\frac{n!}{n^n}$.

This question aimed to search students' conceptual knowledge of the squeeze theorem in the evaluation of sequences as indicated in the preliminary genetic decomposition. This question intended to offer insight on students' attainment of the Process conception of sequence evaluation. The responses showed two extremes where most students provided the correct responses and a few provided incorrect responses.

5.6.1 Responses of students according to APOS levels

Table 5.6: Frequency of students' responses for question 1b (iv) according to APOS level.

APOS level	N	Process
Number of responses	6	24

The responses to question 1b (iv) showed that twenty-four (80%) of the students managed to attain the Process level of the APOS theory as expected in the preliminary genetic decomposition. The written response of ST 1 is one example of students' written responses that showed the Process level conception. The students were able to coordinate the procedures and concepts involved in the evaluation process so as to come up with the correct responses. Six (20%) of the students failed to operate even at the Action level of the APOS theory. Some attempted and failed as with the case of ST 17, and three did not attempt the question at all.

5.6.2 Responses at N level (no response)

ST 20 did not attempt question 1b (iv) which showed that he had failed to make the crucial mental constructions to handle such questions. A learner is said to be operating at N level if he/she shows indications of no conception about the function concept (Wilson, 2013). ST 20 showed indications of having no conceptions of squeeze theorem or other appropriate methods to evaluate the given sequence. To verify that the student operated at N level, the researcher interviewed him to find out how he agreed with these concepts and the complications he faced. The interview excerpt that follows reflects the dialogue the researcher had with him.

R: Why did you not attempt question 1b (iv)?

ST 20: Umm, I am used to trigonometrical function when it comes to the use of the squeeze theorem and not such sequences.

R: Ok, what in particular are you used to when it comes to trigonometrical functions?

ST 20: It is easy to draw the graph of the trigonometrical function and get the limits from the graph. For example, for sine and cosine functions, the graphs are between -1 and $+1$ which implies that the limit of a sine or cosine function is either -1 or 1.

R: So what was the problem with 1b (iv)?

ST 20: The $n!$ and n^n were the problem. I failed to find the link between these two so that I could arrive at a reasonable response. That is, I failed to express each term in a form that could lead to respectable response.

R: Well can you use squeeze theorem to evaluate $\lim_{n\to\infty} \frac{\sin(n)}{n}$?

ST 20 correctly used the squeeze theorem to evaluate $\lim_{n\to\infty} \frac{\sin(n)}{n}$ with few explanations on his response strategy.

ST 20 clearly indicated that he worked more with trigonometrical functions and not with problems like question 1b (iv). The responses he provided indicated that he operated at N level. He failed to solve a problem that did not involve trigonometrical function. That showed that he was limited when it came to the application of the squeeze rule which was an indication that the student failed to attain the expected level in the preliminary genetic decomposition.

R: Let a_n be taken as a sequence. How do you find the limit of a sequence a_n at a point a_0?

ST 20: You plug a number close to a_0 but not equal to it.

R: Can you explain using an example?

ST 20: If $a_0 = 4$, then we can use 3.999…

R: Are 3.999… and 4 examples of two infinitely close numbers?

ST 20: Yes they are.

R: Which number is infinitely close to both of these?

ST 20: 3.9999…

R: Why did you write another 9 after 3.999?

ST 20: Its repeating to infinity, shows that there is always going to be one more to infinity. We can continually add 1 to infinity. So you can continuously add another 9 to the infinity of 9s that are coming.

ST 20 conceived 3.9999… dynamically and not as a point that does not move. He interpreted the decimal extension as a representative of a point which was moving nearer and nearer to 4 but not reaching it forever (Tall, 1978). ST 20 showed that there is continually some space outstanding (Zeno's paradox). Such a conception supports well the idea of Aristotle's potential infinity and his refutation of actual infinity, that is, 3.9999… is a progression that does not come to an end. Furthermore, the student conceived 3.9999… dynamically rather than as a static point (Norton, 2012). Such a conception is in support Aristotle's thinking of possible infinity, and his refusal of actual infinity. This is supported by Tall (1978, p.44) who says "0.999…is the nearest you can get to 1 without actually saying it is 1," or "The difference between them is infinitely small".

The decimal expansion produces simply in theory infinite and not truly an infinite string of 9s'.

Figure 5.8 displays the response from a student’s who was believed to be operating at below the Action level.

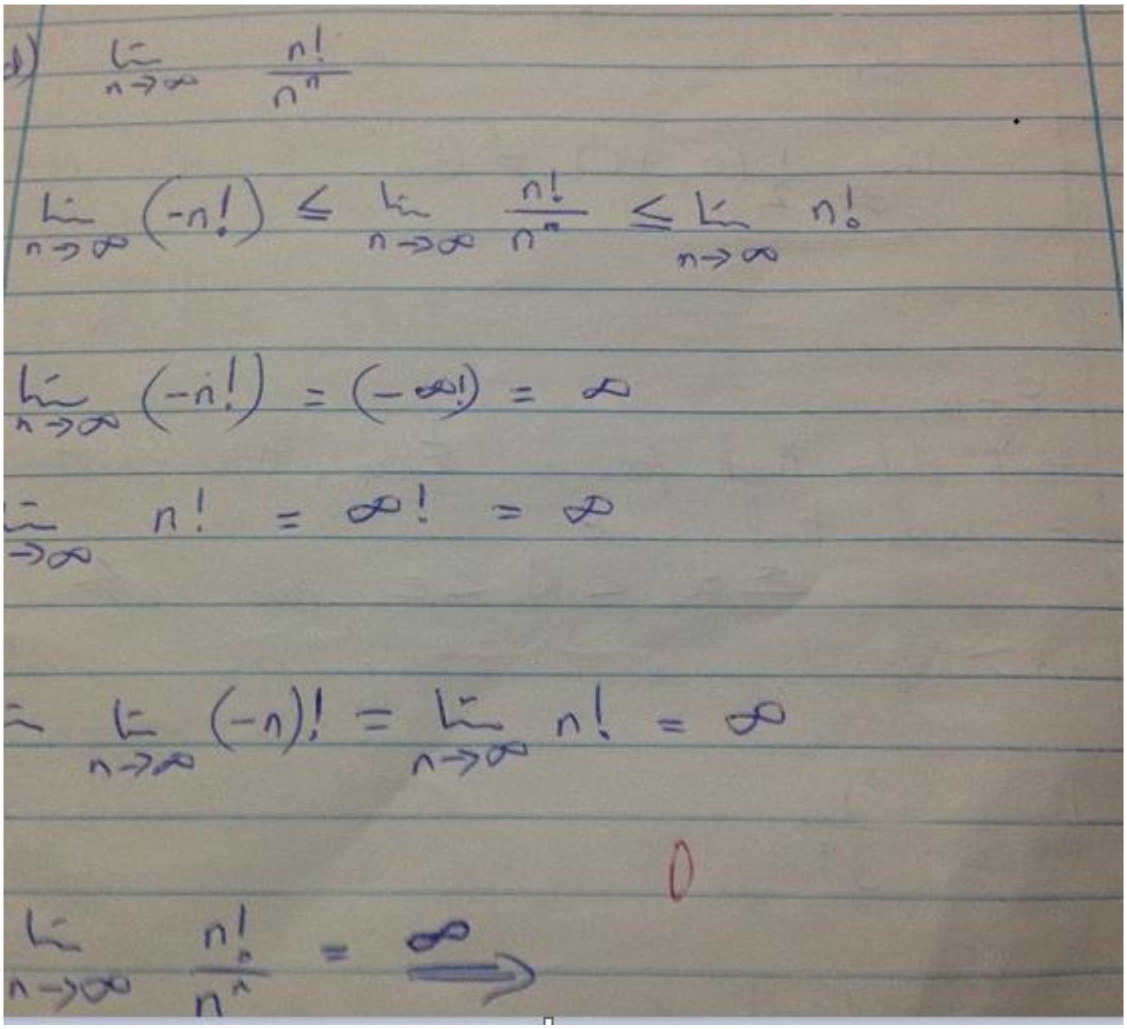


Figure 5.8: Written response of ST 17 to 1b (iv)

ST 17 had an idea of the squeeze theorem, but wrongly applied it. The student failed to express $n!$ and n^n in a way that would allow progress towards the correct response. That is, writing $n!$ and n^n in full then makes the necessary deductions. The way ST 17 presented her work confused her and she ended up giving the incorrect response. This response is indicative of a developing Action level stage of the squeeze theorem conception. The student needed more time and practice to fully develop the squeeze theorem and be able to apply it to solve given problem situations. ST 17’s written response shows the application of the squeeze rule without the precise details. The researcher placed her response at N level. The researcher interviewed her to confirm this claim and the excerpts below captures the resultant dialogue:

R: What are the requirements for the application of the squeeze rule to calculate the limit of sequences?

ST 17: We have $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ that is $\lim_{n\to\infty} a_n$ must equals $\lim_{n\to\infty} c_n$.

R: That is very good. But you provided an incorrect response to question 1(d). What was your challenge?

ST 17: I had challenges with the factorial notation and n^n, I could not find a simpler way of writing it. I got stuck then tried what I wrote as a result.

R: Well, can you use squeeze theorem to evaluate $\lim_{n\to\infty} \frac{\sin(n)}{n}$?

ST 17: Umm, not really I have challenges with the squeeze theorem problems. I need more time to revise and understand the problems involving the squeeze theorem.

The responses to the interview questions provided by ST 17 indicated that she operated at (P) level. She failed to unpack $n!$ and n^n to manageable forms so that she could solve the given problem.

R: Let a_n be a given sequence, what procedure would follow to determine the limit of a sequence a_n at a point a_0?

ST 17: We plug numbers close to a_0 into the expression to find the limit of a sequence.

R: So, if there is some amount of distance between the two, can you give an example of two infinitely close numbers?

ST 17: Infinitely close to each other numbers are 1.001, 1.002, or something to that effect.

R: Okay, so are those two good examples?

ST 17: You can get a lot closer to a number, but there is a ridiculous amount of zeroes in there. There is much less distance between them in reality.

R: Okay. So do you find an infinitely close number to the two numbers?

ST 17: Sure. You can always get a number in between them, I suppose.

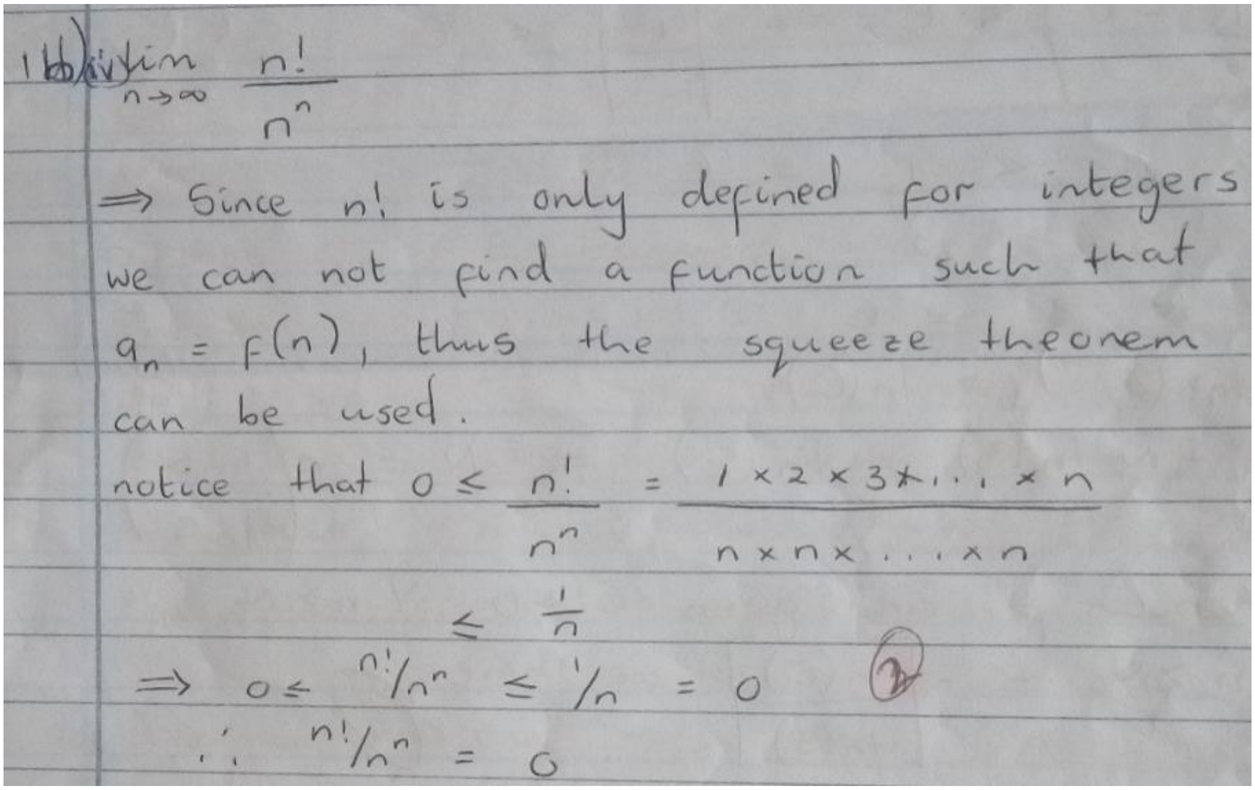
R: Okay. These two numbers [1.001 and 1.002] are infinitely close together. Is that a fair characterization of what you're saying?

ST 17: Yes, they are.

For ST 17, "infinitely" close means "very" close, and in a measurable way. Some distance between 0.1 and 0.001 marks the borderline between "close" and "infinitely close." This ratifies that she believes in infinitesimals. Her notion is static rather than dynamic, and she believes that between numbers, there are infinitely small numbers and distances. Infinitesimal distances are not qualitatively different from finite distances. In particular, a finite number of finite operations can result in an infinitesimal distance.

5.6.3 Responses at a Process level

Figure 5.9 is an example of a student who was believed to be operating at the Process level when responding to question 1b (iv).



1b(iv) $\lim_{n\to\infty} \frac{n!}{n^n}$

$\Rightarrow$ Since $n!$ is only defined for integers we can not find a function such that $a_n = f(n)$, thus the squeeze theorem can be used.

notice that $0 \leq \frac{n!}{n^n} = \frac{1 \times 2 \times 3 \times \dots \times n}{n \times n \times \dots \times n}$

$\leq \frac{1}{n}$

$\Rightarrow 0 \leq n!/n^n \leq 1/n = 0$

$\therefore n!/n^n = 0$

2

Figure 5.9: Written response of ST 1 to question 1b(iv)

ST 1 managed to give the correct response to question 1b (iv), the student gave a justification for the application of the squeeze theorem, then applied it successfully. He managed to write $n!$ and n^n in the appropriate manner to enable progression towards the correct response. This showed that such a student's responses were at Process level of the APOS theory. This conclusion was arrived at after observing the way he dealt with the inequality $\frac{1\times2\times3\times\ldots\times n}{n\times n\times n\ldots\times n}$ which he imagined to be $\leq \frac{1}{n}$. At the Process level of conception, students are expected to perform transformations, predict outcomes and even reverse Processes mentally, without external cues (DeVries & Arnon, 2004). ST 1 performed transformations and predicted the outcome mentally. The researcher interviewed him to check his accepting of the procedure that was correctly followed in the written response so as to ascertain the student's operational level of the APOS theory. The interview excerpts that follows reflects the discussion:

R: You have worked with the squeeze theorem when dealing with the evaluation of functions. Can this theorem be applicable to the evaluation of sequences?

ST 1: Yes, but only when the limit exists. If it is divergent, we cannot conclude.

R: What is your interpretation of the existence of a limit?

ST 1: It works when the sequence is convergent; and when the sequence is divergent, goes to infinity, we cannot conclude.

R: What conditions are necessary for one to use the squeeze rule to determine the limit of a sequence?

ST 1: We have to squeeze the problem in between two other simpler sequences whose limits are easily computable and are equal. We need two sequences with one being greater than or equal to the other. Then our given sequence which we wish determine its limit must be between these two. Which is, for all n the middle sequence must be greater or equal to lower bound, and also the upper bound is greater or equal to the middle sequence.

R: Ok then, how do we apply the squeeze theorem to evaluate $\lim_{n\to\infty} \frac{\sin(n)}{n}$?

ST 1: (the student explains verbally) In this situation, we have $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ and there is always an integer N such that $a_n \leq b_n \leq c_n$ for all $n > N$ then $\lim_{n\to\infty} b_n = L$. Here, we notice that $-1 \leq \sin(n) \leq 1$. For all $n > 1$ we have $\frac{-1}{n}$ $(lower\ bound) \leq$ $\frac{\sin(n)}{n}$ $(middle\ sequence) \leq$ $\frac{1}{n}$ $(upper\ bound)$, from this clearly $\mp\frac{1}{n} \to 0$ as $n \to \infty$. Thus, using the squeeze theorem so must $\frac{\sin(n)}{n}$ $\to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

An analysis of ST 1's responses to the interview questions revealed that he operated at Process level of the APOS theory. His explanations on conditions and application of squeeze theorem indicated conceptual understanding of the squeeze theorem. He showed the ability to link highest and lowest values that can be assigned to the sine function and the given sine sequence. ST 1 performed transformations and predicted outcomes mentally, without external cues (DeVries &

Arnon, 2004). The student's responses showed that he had realized the expected level in the preliminary genetic decomposition.

5.7 Evaluation of recursive limits: question 1b (v), a_1 = 2, and $a_{n+1} = \frac{1}{2}(a_n + 6)$

This question involved a sequence aimed at providing insight on students' knowledge of finding limits of a sequences represented recursively. The students used two methods, where on one hand, some students used the domain and range sequences then after a number of steps the students deduced the required limit. On the other hand, other students used the limit concept, which is- the limit of a sequence is L, which they applied to get a correct response. The students did not start from scratch to construct all steps of the Action. They applied previously constructed Processes for setting the domain and range, then applied the deduction of the limit. Some applied the idea that the limit of a sequence be L and algebraically manipulated the resulting equation to get the required limit.

5.7.1 Responses of students according to APOS levels

Table 5.7: Frequency of students' responses for question 1b (v) according to APOS level.

APOS level	N	Process
Number of responses	3	27

The students who responded correctly were twenty-seven (90%), and they operated at the Process level of the APOS theory in line with the preliminary genetic decomposition. Written responses gave testimony to this level. Some students showed that they could evaluate the sequence in a step by step manner at first, then later they only envisaged the required response (not actually performed the calculations) then came to the conclusion. Furthermore, other students showed that they had mastered the Action level conception of sequence evaluation by displaying their ability to combine repeated Actions involved in the solution methods. Three (10%) of the students did not attempt the question, which made it difficult to draw any meaningful conclusion about their levels of APOS theory. The students who gave the correct responses mainly used the method illustrated in the extract given below.

5.7.2 Responses at a N level (no response)

ST 13 did not attempt question 1b (v) which was an indicator that the student operated at N level. This finding concur with Wilson (2013) who indicated that a learner operated at N level if he/she showed indications of no conception about the function concept. ST 13 showed no conception of evaluating recessive sequences. The researcher interviewed her to ascertain the difficulties she faced and how she understood these concepts. The interview excerpt that follows is a dialogue the researcher had with her.

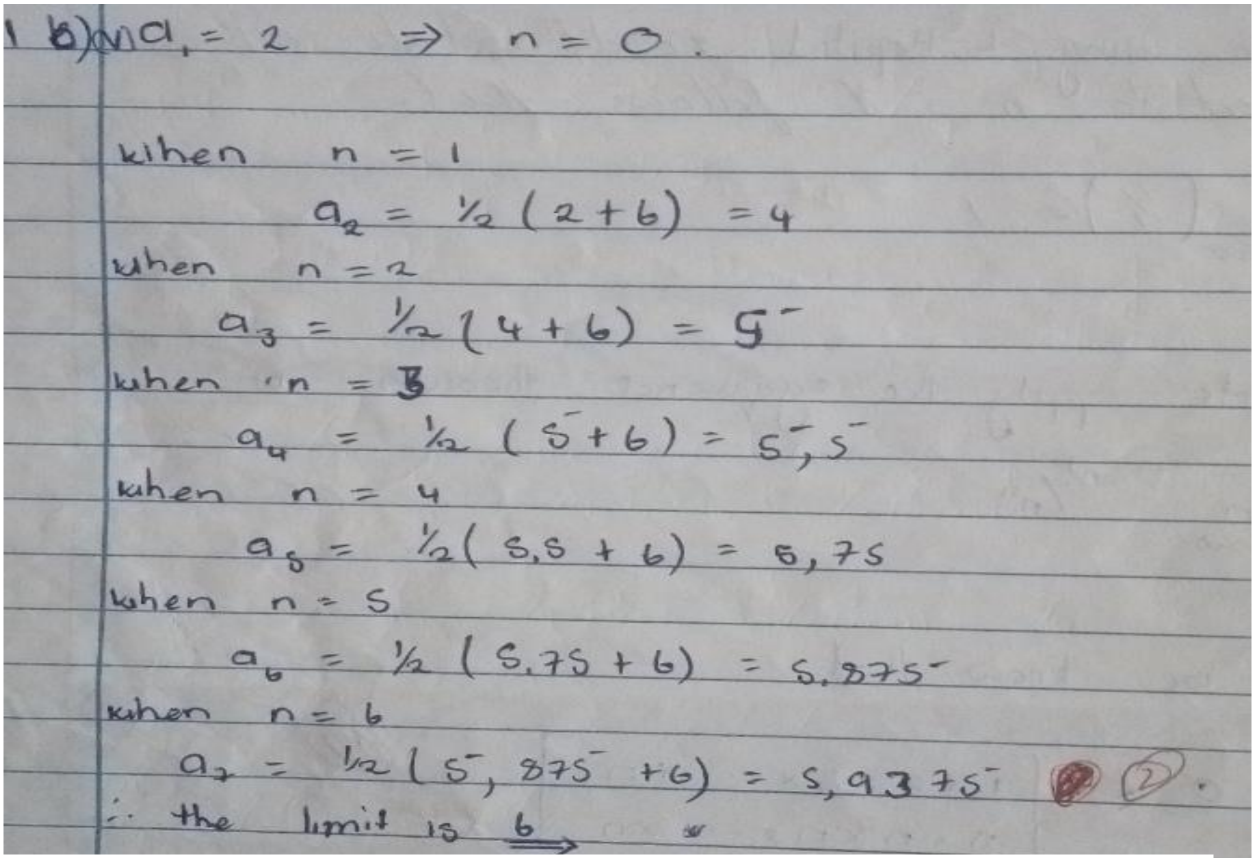
R: You did not attempt question 1b (v). What challenges did you face?

ST 13: Umm, I found the question too difficult to answer. I need more time to work on such type of questions on the evaluation of sequences.

The dialogue with ST 13 was very short and indicative that the student operated below the Process level. She failed to attain the expected level in the preliminary genetic decomposition.

5.7.3 Responses at a Process level

The response provided in Figure 5.10 is typical of the responses of students who operated at Process level conception.



1 b) v) $a_1 = 2 \Rightarrow n = 0$.

when $n = 1$

$a_2 = \frac{1}{2}(2+6) = 4$

when $n = 2$

$a_3 = \frac{1}{2}(4+6) = 5$

when $n = 3$

$a_4 = \frac{1}{2}(5+6) = 5,5$

when $n = 4$

$a_5 = \frac{1}{2}(5,5+6) = 5,75$

when $n = 5$

$a_6 = \frac{1}{2}(5,75+6) = 5,875$

when $n = 6$

$a_7 = \frac{1}{2}(5,875+6) = 5,9375$ 2.

$\therefore$ the limit is 6

Figure 5.10: Written response of ST 12 to question1b (v)

The response of ST 12 was typical of many of the students' responses for this category. The response indicated that ST 12 had done the transformations at Action level repeatedly meaning that the student had interiorized the Action into Process. The student constructed the domain and range sequences. It can be noted that the domain and range sequences were organized through the function. Later after many steps the student deduced the required limit of the given sequence. This shows that ST 12 had not just manipulated numbers, but actually thought at a Process level. The individual student imagined going through the representation processes for the domain and range to demonstrate the Process level. By definition, a Process is a form of consideration that involves imaging of a transformation of mental objects perceived internal and wholly in an individual's control (Dubinsky & Harel, 1992). She showed the ability to control a transformation and her internalization of the procedure by reflecting upon entirely of the stages in the transformation without essentially performing them, which was coming up with final answer 6. The researcher interviewed ST 12 to confirm the claim that her written response indicated that she operated at Process level of the APOS theory. The excerpt below captures our dialogue:

R: How do you explain the limit of a sequence?

ST 12: It is a number that the particular sequence gets closer to and cannot go beyond such a number. The numbers we get when we substitute into the sequence expression approaches a particular number and this number becomes the sequence limit.

R: How do you find the limit of the sequence in question 1b (v)?

ST 12: I substitute values into the sequence expression, then check on the resulting numbers. In this case, the resulting numbers get closer and closer to six and cannot go beyond. Thus, the limit of the sequence is six.

A Process is a way of understanding that involves imaging a transformation of mental objects perceived internal and entirely in an individual's control (Dubinsky & Harel, 1992). ST 12 showed ability to control a transformation and internalize the procedure by reflecting upon the entire steps in the transformation without essentially performing them, which enabled her to arrive at six as the final answer (limit). After consideration of our dialogue during the interview, the researcher placed ST 12 at the Process level of the APOS theory. The student's responses showed that she had had reached the expected level in the preliminary genetic decomposition.

R: If a_n is a sequence, how do you find the limit of a sequence a_n at a point a_0?

ST 12: I will always use numbers to find the limit of a sequence.

R: What type of numbers are these?

ST 12: Numbers which are not necessarily equal to a_0 but very close to a_0.

R: Can you illustrate this?

ST 12: It may not be the case all the time, the closest number since there is no such determined values. The numbers will be close to a_0 as much as possible. Suppose $a_0 = 1$, then we can talk of 0.99…, 0.999…, 0.9999…. Then these numbers are close 1.

R: Okay. Are 1 and 0.9999… infinitely close to each other?

ST 12: You can get very close to 1, but there is a lot of nines in there. 0.999… and 1 are examples of two number infinitely close to each other.

R: Is it possible to square or carry out other operations on infinitely close numbers?

ST 12: Oh yes. It is very possible. If you square such a number a small number, you even get a smaller number. That is any fraction squared, you always get a smaller fraction than you stated off with.

R: Ok, from the work you have covered in line with the limit of a sequence, is this always the case?

ST 12: Umm**,** in some cases you plug in and get infinity or an undefined limit. That is if the denominator becomes zero, the sequence will not be defined.

R: Is 1 equal to 0.9999…?

ST 12: Oh yes. It is possible. If you square such a number, you get a smaller one as I have said before.

ST 12 displayed a coordinated understanding of the infinitesimals which are highly developed and consistent. Her conception shows similarities with Leibniz's understanding (Baron, 1987), with operations of algebra performed on infinitesimal, finite and infinite numbers. ST 12's conception of infinitesimals involved a creation of infinitesimal decimal numbers that can be operated on in the same way as finite decimals. That is, one can find the square of an infinitesimal to get a smaller infinitesimal.

5.8 Proof of a sequence limit

In terms of the APOS level, students who gave the correct response for 2a(i), 2a(ii), 2a(iii) and 2 (b) showed that they had reached at least the Objects level conception of using the formal definition of the limit of a sequence as they were able to encapsulate a Process. The individual became conscious of the totality of the Process and realized the transformations acting on it. These students were able to create such transformations. That is, they coordinated and used many concepts related to sequences, L, n, N, ε and infinity. These transformations were shown both in students' written responses and interviews. Written responses and interviews with ST 12 showed that the student had reached the Object level of proving limits of given sequences. ST 15, ST 1 and ST 28 indicated students' attainment of the Schema level of the application of the formal limit definition of sequences to proving limits of given sequences.

However, some students could not reach the Objects level conception as indicated in the preliminary genetic decomposition through their written and interview responses. The written responses from, and interviews with, ST 14, ST 13 and ST17 indicated a failure by the students to attain the Objects level conception of using the formal definition of the limit of a sequence to prove given sequence limits. This supports the contention made by Ndlovu and Brijlall (2015) that few students operate at the Objects stage. However, in the current study, more than half of the students managed to operate at the Objects stage.

Question 2a (i): Use the $\boldsymbol{\epsilon}$ - $\boldsymbol{N}$ (epsilon – N) definition of the limit of a sequence to prove that $\boldsymbol{lim_{n\to\infty}} \frac{\mathbf{1}}{\mathbf{3^n}}$ = 0.

Question 2a (i) addressed the aspect of proving a given sequence using the formal definition of the limit of a sequence. The question intended to provide understanding of the students on how they had developed the Objects conception of proving a sequence limit using the definition ($\varepsilon - N$) whose denominator was a constant raised to the power n and the sequence limit was zero. Remembering and understanding the required definition was central to responding to the question correctly. Few participants were able to use the $\boldsymbol{\epsilon}$ - $\boldsymbol{N}$ (epsilon – N) definition of the limit of a sequence to prove the test item. More students were able to provide the definition than those who correctly applied the same definition to prove question 2a (i).

5.8.1 Responses of students according to APOS levels

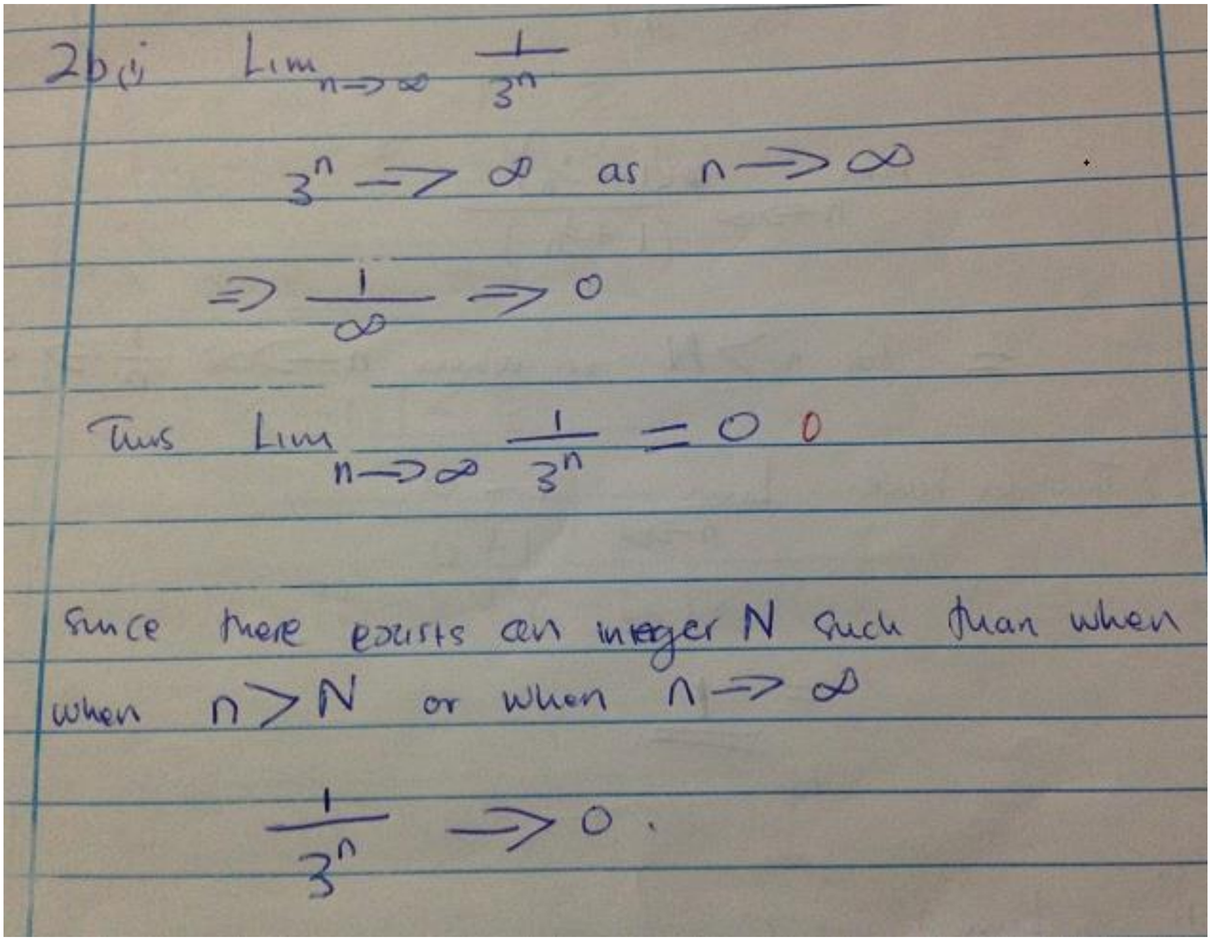
Table 5.8: Frequency of students' responses for question 2a (i) according to APOS level.

APOS level	N	Object	Schema
Number of responses	19	8	3

Nineteen students (63%) managed to provide responses that indicated that they operated below an Object level of the APOS theory. Sixteen students found it difficult to go beyond the first stage of subtracting zero from the fraction in the modulus sign. This is an indication that students used memorization to respond to given questions. Three students did not attempt the question, indicating that such students had not developed the Object level of the APOS theory as expected in the preliminary genetic decomposition. Eight (27%) of the students gave responses that were indicative of the Object level understanding and three (10%) of the students gave response that were indicative of operation at the Schema level.

5.8.2 Responses at N level (totally incorrect response)

Figure 5.11 illustrates an example of a student who operated at N level. There were also some students who showed confusion when they proved limits of sequences.


2b(i) $\lim_{n \to \infty} \frac{1}{3^n}$

$3^n \to \infty$ as $n \to \infty$

$\Rightarrow \frac{1}{\infty} \to 0$

Thus $\lim_{n \to \infty} \frac{1}{3^n} = 0$ 0

Since there exists an integer N such than when when $n > N$ or when $n \to \infty$

$\frac{1}{3^n} \to 0$.

Figure 5.11: Written response of ST 14 to question 2a (i)

The response of ST 14 showed that she was totally confused and gave a response that was not a proof at all. The student failed to correctly write even the first stage of the expected solution. The incorrect response was due to failure to understand the definition and its application. Such students showed lack of the development of up to Object level. Students who gave responses showing little understanding about the function concept were indicative of the pre-function conception of a function (Dubinsky & Wilson, 2013). As for ST 14, she showed very little conception, if any, of the application of the definition of the limit of a sequence in proving limits of given sequences. Moreover, the little understanding students had, this was not very helpful in solving mathematical problems to functions (Breidenbach et al., 1992). ST 14's little understanding of the definition of the limit of a sequence could not help her in proving question 2a (i). In the following excerpt, ST 14 had a vague notion of the formal definition of the limit of a sequence. She could not explain the definition in terms of epsilon and Nas illustrated below:

R: What does question 2a (i) require you to do?

ST 14: I think it required me to show that zero is the limit of the sequence.

R: Is it to show or to prove?

ST 14: Oo yes to prove, but I never understood the formal definition of the limit of a sequence. Hence it is difficult for me to use what I have not understood (the formal definition of the limit of a sequence).

R: Can tell say something about what you can remember?

ST14: I can only remember it deals with N and epsilon, this much I know, but of the finer details uuum I cannot remember.

R: What about epsilon?

ST 14: I understand the difference between $f(n)$ and L is epsilon, um that's all I can remember but cannot proceed from here.

R: We are aware that $\lim_{n\to 2} 5n = 10$. How can one prove this using the formal definition of the limit of a sequence?

ST 14: I cannot prove it. It requires the same understanding as question 2a (i) of which I have already said it is very difficult for me to prove.

The dialogue between the researcher and ST 14 revealed that she had not attained Object level for proving the limit of a sequence in the context of APOS theory. She failed to satisfactorily explain her response strategy and phrases like "that's all I can remember" are indicators of someone trying to work from memory. The student failed to reach the expected Process level in the preliminary genetic decomposition.

R: Let a_n be there for a sequence, how can you determine the limit of a sequence a_n at a point a_0?

ST 14: I plug numbers very close a_0 in the sequence then number that comes out of the plugging in will equals the limit of the sequence.

R: Will that always happen?

ST 14: Uh, well you can get no limit after plugging.

R: Which cases gives you no limit?

ST 14: When we have undefined limit of a sequence, for example when the denominator is zero or infinity divided by infinity.

R: Which numbers close $to\ a_0$ are these? Can you give an example?

ST 14: It may not always be possible to find the closest since there is no such determined values especially in case of fractions. The numbers will be close to a_0 as much as possible. Suppose $a_0 = 10$ then we can talk of 9.99…, 9.999…, 9.9999…

R: Okay. Are 10 and 9.9999… infinitely close to each other?

ST 14: Yes, they are infinitely close to each other.

R: What will you do with these numbers?

ST 14: We will plug them into a_n and notice the resulting values. If the resulting values are pointing towards a certain number, at that point this number is the required limit of the sequence.

The researcher's interview with ST 14 revealed that she arrived at the limit through what can be characterized as sameness by proximity. 10 becomes the limit of 9.999… for students take the difference between them as so infinitely small that it can be considered the same (Tall, 1978). It is interesting that ST 14 believes that there is some immeasurable distance between the two numbers.

5.8.3 Responses at Schema level

Figure 5.12 illustrates an example of a response provided by students who operated at a Schema level. Such students managed to provide the correct response.

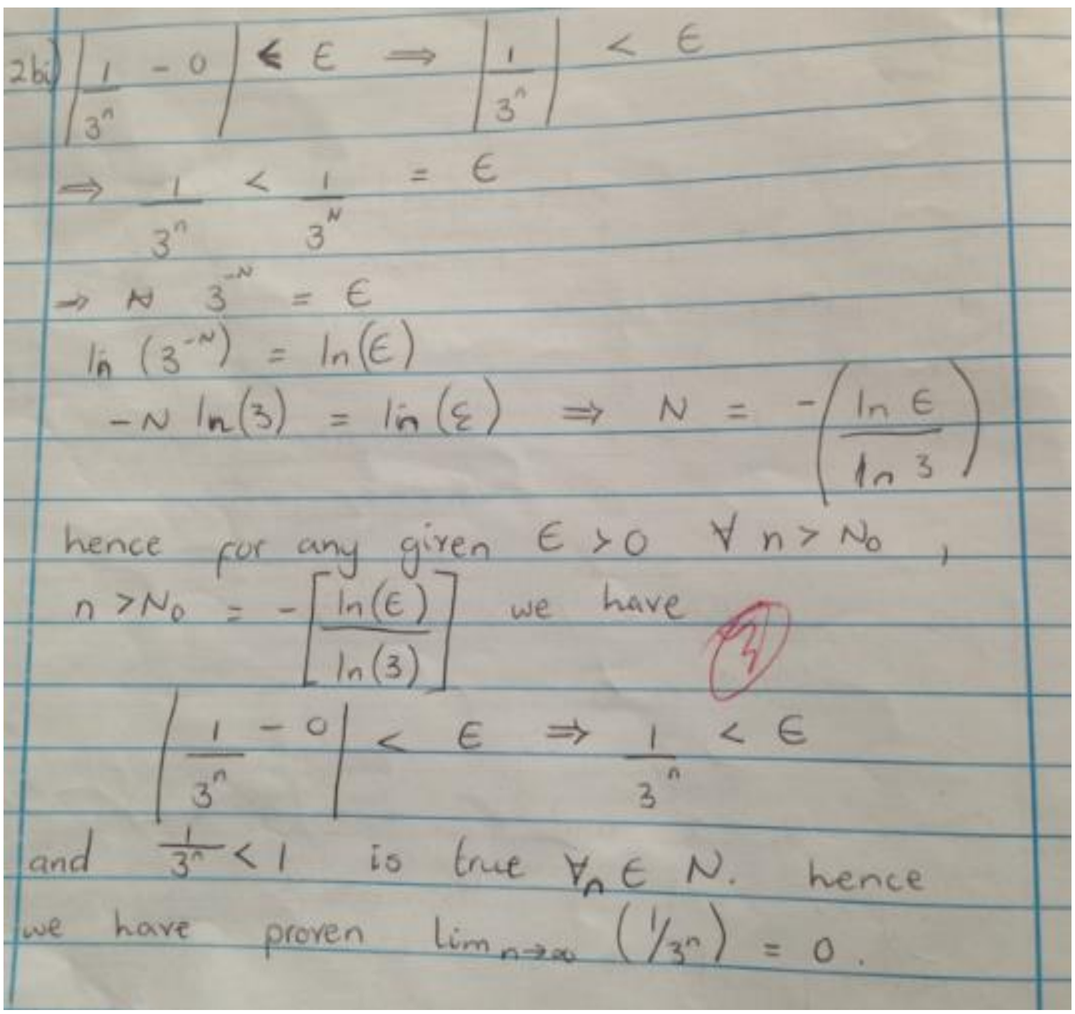


Figure 5.12: Written response of ST 1 to question 2a (i)

ST 1's response in Figure 5.19 is an example of students who operated at Object level of the APOS theory. His response showed that the process is encapsulated into an Object by the recognition of the given question as an object to which actions and processes can be functional (Dubinsky & Harel, 1992). ST 1 saw the process as entirety and understood that transformations can be done on it (Dubinsky & McDonald, 2001). To authenticate the observation that ST 1 operated at the Object level of the APOS theory, he was interviewed on the solution strategies as indicated in the excerpt below.

R: You gave a very good and correct response to question 2a (i). Can you explain your understanding of the solution method you provided?

ST 1: In order to prove this given limit, we need to find a real positive number N so that for all $n > N$ we have $|\frac{1}{3^n} - 0| < \varepsilon$ for any given $\varepsilon > 0$. To find this number N, we use the left hand side so that we have for all $n > N$, $\left|\frac{1}{3^n} - 0\right| = |\frac{1}{3^n}|$ this is equal to $\frac{1}{3^n} < \frac{1}{3^N} = \varepsilon$ (for $n > N$ implies $3^n > 3^N$ which gives $\frac{1}{3^n} < \frac{1}{3^N}$. Now $3^{-N} = \varepsilon$ (by the rules of indices). Then taking natural logarithms, we have $-Nln(3) = \ln(\varepsilon)$. This gives us $N = -[\frac{\ln(\varepsilon)}{\ln(3)}]$. Hence, for any given $\varepsilon > 0$ and for all $n > N = -\left[\frac{\ln(\varepsilon)}{\ln(3)}\right]$, we have $|\frac{1}{3^n} - 0| < \varepsilon$. Thus N is only negative if epsilon is greater than one. That means $\left|\frac{1}{3^n} - 0\right| = \frac{1}{3^n} < \varepsilon$ and $\frac{1}{3^n} < 1$ is true for all $n \in \mathbb{N}$. This completes the proof that $\lim_{n\to\infty} (\frac{1}{3^n}) = 0$. That is, it can be concluded that zero is the limit of the sequence.

R: Let us assume that $\lim_{n\to 2} 5n = 10$. How do you demonstrate this using the formal definition of the limit of a sequence?

ST 1: We draw the graph of $f(n) = 5n$ and here for all epsilon greater than zero we need to find N depending on epsilon. Taking epsilon first, we restrict $f(n)$ in the interval $(10 - \varepsilon, 10 + \varepsilon)$. After this, we now need to find N on the x axis such that the sequence values of x that are taken from the interval $(2 - N, 2 + N)$ must get into $(10 - \varepsilon, 10 + \varepsilon)$. (The student works out and finds the relationship between epsilon and N as $N = \frac{\varepsilon}{5}$). Taking an example, if we choose epsilon as 10, if we need the sequence to be in 0-15 interval, then it should be between 0 and 3. Since we take N as $\frac{\varepsilon}{5}$, we take N as 2

R: Is this true for all epsilon or one epsilon?

ST 1: This is true for all epsilon always.

R: Consider the statement $\lim_{n \to 2} 2n = 5$. Determine the truth or falsity of the statement.

ST 1: We write $|2n - 5| = |2n - 4 - 1|$ This leads to $2|n - 2 - \frac{1}{2}|$. It is known that $-N < n - 2 < N$, then we have $-N - \frac{1}{2} < n - 2 - \frac{1}{2} < N - \frac{1}{2}$. This implies that $|n - 2 - \frac{1}{2}| < N + \frac{1}{2}$. Then showing $2|n - 2 - \frac{1}{2}|$ becomes less than $2(N + \frac{1}{2})$. Now taking $2(N + \frac{1}{2})$ to be epsilon, we have $N = \frac{\varepsilon}{2} - \frac{1}{2}$. But if $\varepsilon = 1$, this leads to $N = 0$, but N must be greater than zero, therefore, it can be concluded that the limit of the given sequence cannot be equal to 5.

ST 1 managed to find epsilon-N relationship in each case. For proving the truth or falsity, he worked out and found $N = \frac{\varepsilon}{2} - \frac{1}{2}$. He took epsilon to be one, that made $N = 0$ which is a contradiction and he concluded that the limit failed to be 5. This appears to be a kind of accommodation of a new condition into epsilon-N argument. Accommodation is not the proper way of coming up with the necessary mental structures to handle a new condition. What is required is reflective abstraction, which is encapsulation of Process conception of formal limit concept into an Object. Nevertheless, the student's opinions made the research to conclude that he was operational at the Schema level of the APOS theory. A Schema for a particular mathematical concept is the individual's collection of Actions, Processes, Objects and other Schema linked by general principles that form a framework in the individual student's mind, which may be brought forward to bear upon a problem situation involving that concept (Asiala et al., 1996). The student's responses testified that he had attained the Schema level fulfilling the expected level in the preliminary genetic decomposition.

5.9 Proof of a sequence limit: question 2a (ii), $\lim_{n\to\infty}\frac{n-1}{n+1} = 1$.

Question 2a (ii) addressed the aspect of a proof of given sequence by the use of the formal definition of the limit of a sequence. The question intended to afford insights as to whether the students had settled for the Objects conception of the proof of a fractional sequence limit using the definition ($\varepsilon - N$) whose limit was one.

5.9.1 Responses of students according to APOS levels

Table 5.9: Frequency of students' responses for question 2a (ii) according to APOS level.

Score	N	Object	Schema
Number of responses	14	13	3

Three (10%) of the students gave responses that indicated that they operated at the Schema level as prescribed by the preliminary genetic decomposition. Thirteen (43%) of the students' responses were indicative of the Object level of the APOS theory as they managed to give the correct response. Their response showed that they had understood the demands of the question and may have attained the Object level conception of proving given limits by the use of the formal limit definition of a sequence. However, fourteen (47%) of the students' responses showed that they operated below the Object level of the APOS theory and failed to attain what was expected in the preliminary genetic decomposition.

5.9.2 Responses at N level (totally incorrect response)

Figure 5.13 shows a response provided by a student who was considered to be operating at N level.

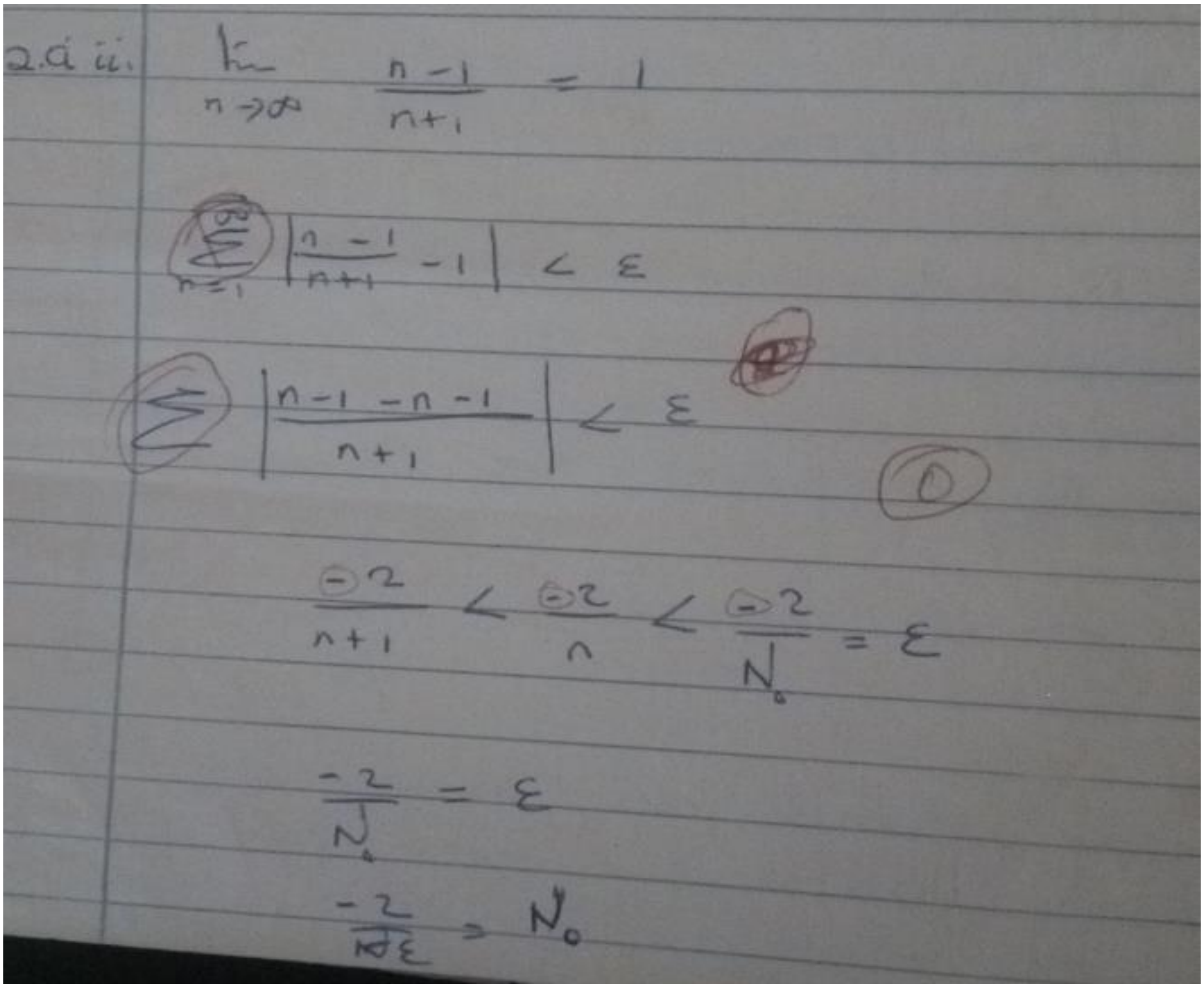


Figure 5.13: Written response of ST 17 to question 2a (ii)

The response provided in Figure 5.12 shows a situation where the student could not apply the modulus function to come up with the required conclusion. This showed that the student had not attained the Action level. The response of ST17 indicated that the student knew the procedure for proving limits of sequences using the definition but messed up along the way. The student was able to find the relationship between N_0 and ε as expected, a procedure with a connection task. However, what the student wrote at the end had nothing to do with what was asked. Thus, the student had not fully developed the Object level of the APOS theory as she failed to carry out the proof successfully.

ST17's responses showed that the student had challenges with the modulus notation. The student did not appreciate the meaning of the negative sign in the modulus notation after the removal of the modulus sign. The student tackled the question poorly due to lack of competency in pre-calculus topics. The incorrect interpretation of the problem was an indication of inadequate conception of modulus operation involving negative numbers. There is need for revising inequalities and the modulus function. That is, if $a < b$ then$\frac{1}{a} < \frac{1}{b}$, and the definition of the modulus function for whole numbers and for fractions, $|-5| = 5$ and$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$. The student needs to be comfortable in applying these inequalities for these are used throughout the proofs of limits of sequences and functions. Furthermore, students need to know properties of the modulus function (sometimes called the distance function) and be comfortable in applying them. This confirms what Ndlovu and Brijlall (2015) pointed out, that lack of related schema, like real number system impact negatively on the construction matrix algebra schema.

An analysis of ST 17's written response made the researcher to place her at N level. This finding concur with Wilson (2013) who indicated that a learner operated at N level if he/she showed little conception about the function concept. ST 17 showed little conception of proving limits of given sequences. However, to confirm the student's understanding of these concepts and the researcher's claim that she at N level, the researcher interviewed her as shown in the interview excerpts that follow:

R: You started quite well when you responded to question 2a (ii), do you understand the meaning of the modulus sign?

ST 17: It means we take what is inside always as positive.

R: Ok, what happens to the negative numbers inside the modulus sign if the modulus sign is removed?

ST 17: The numbers inside the modulus sign remain negative?

R: Why does it remain negative?

ST 17: Ummm, am not sure why. I need more time and reflection on the modulus function and its application.

R: We know that $\lim_{n \to 2} 5n = 10$. How do you prove this using the formal definition of the limit of a sequence?

ST 17: For this one, let me try. (she works out and found the relationship between epsilon and N as $N = \frac{\varepsilon}{5}$).

R: This is good, but is this all that is needed for such proofs?

ST 17: Yaa, I do think so.

R: Can you determine the truth or falsity of $\lim_{n \to 2} 2n = 5$?

ST 17: This is challenging since by examining the expression, one can see that it is false. But I cannot prove this outcome, the understanding of the formal definition limit of a sequence is required.

ST 17 showed a weak schema of the modulus function (distance function) and could not apply the definition of the modulus function properly. That is the modulus function is always zero or positive. When asked to $\lim_{n \to 2} 5n = 10$, she proved well up to finding the relationship of N and epsilon but failed to conclude as expected. On the proof of truth and falsity, she found it too challenging to determine. Thus, she operated at N level and could not attain the Object level as expected in the preliminary genetic decomposition.

5.9.3 Responses at Schema Level

Figure 5.14 illustrates a response that was considered to be of a student who operated at the Schema level. Such students displayed responses that showed an understanding of proving the limit of a sequence.

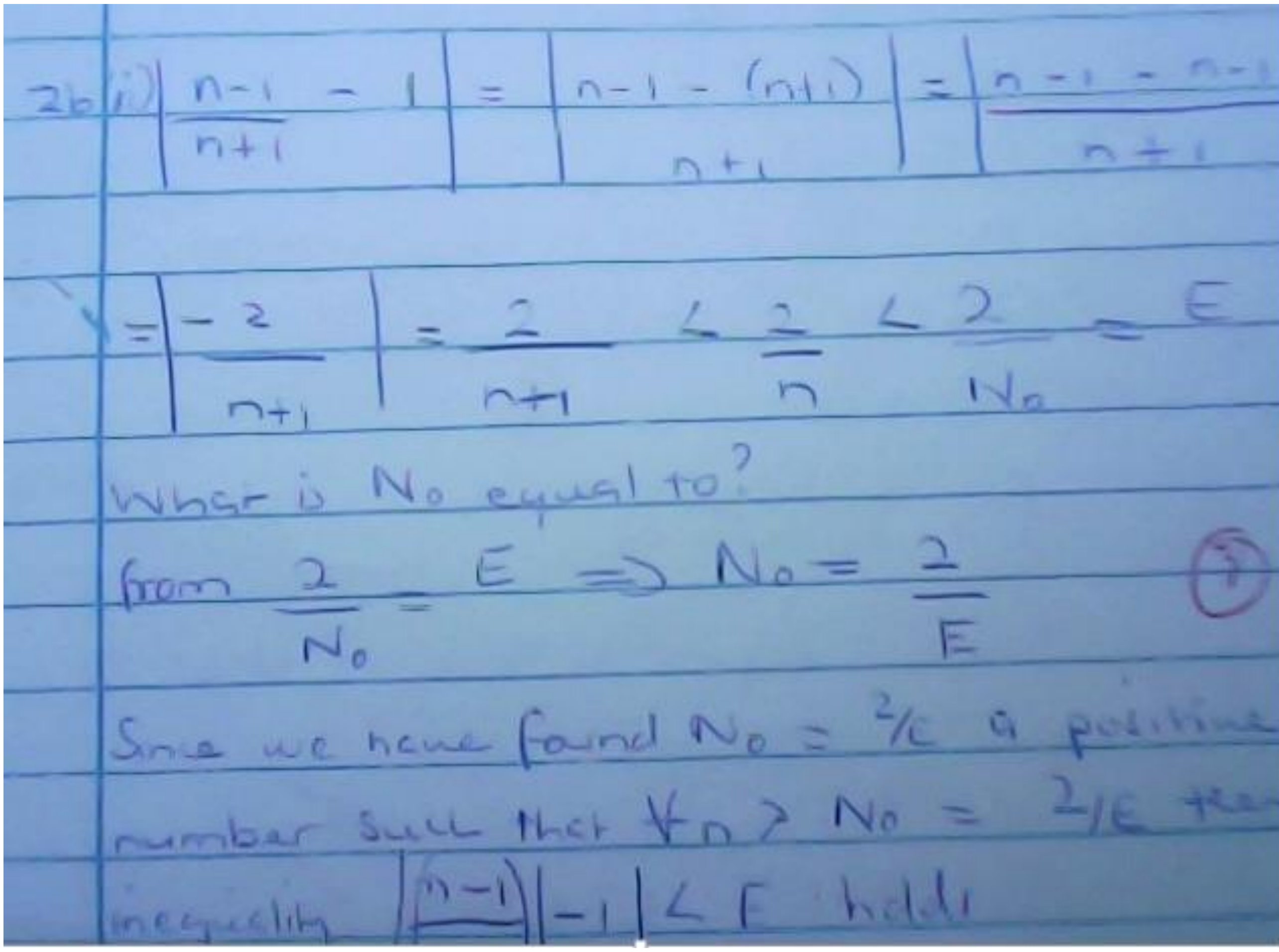


Figure 5.14: Written response of ST 28 for question 2a (ii)

ST 28 shows his ability to encapsulate a Process into an Object. He had seen the process as a one thing and managed to understand that transformations can be performed on it (Dubinsky & McDonald, 2001). The student was able to identify and coordinate the related n, N, ε and L from the formal definition of the limit of a sequence to come up with the correct response. ST 28 showed evidence of seeing a dynamic structure (Process) as a fixed one to which actions can be applied. His response was logically and accurately presented. To confirm the researcher's claim, ST 28 was interviewed and the excerpt that follows captured the resultant dialogue:

R: You gave the correct response to question 2a (ii). Can you explain in detail how you came up with your response?

ST 28: To find N, we use $|x_n - L|$, where x_n is the formula for the sequence and L is the limit of the sequence. Then what is x_n and L equal to? Thus $x_n = \frac{n-1}{n+1}$ and $L = 1$. Given epsilon greater than zero, we have to find a number Nsuch that for all $n > N$ we have $\left|\frac{n-1}{n+1} - 1\right| = \left|\frac{n-1-(n+1)}{n+1}\right|$ this is equal to $\left|\frac{-2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{N} = \varepsilon$ (because $n + 1 > n > N$ implies $\frac{1}{n+1} < \frac{1}{n} < \frac{1}{N}$). Letting $\frac{2}{N} = \varepsilon$ for we want the inequality $\left|\frac{n-1}{n+1} - 1\right| < \varepsilon$. The by transposing $\frac{2}{N} = \varepsilon$ gives $N = \frac{2}{\varepsilon}$. For we have found a positive real number $N = \frac{2}{\varepsilon}$ such that for all $n > N = \frac{2}{\varepsilon}$ the inequality $\left|\frac{n-1}{n+1} - 1\right| < \varepsilon$ holds. This proves that $\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right) = 1$. Hence we have completed the proof as required.

R: It is known that $\lim_{n\to 2} 5n = 10$. How do you substantiate this using the formal definition of the limit of a sequence?

ST 28: Yes. (he correctly proves that $\lim_{n\to 2} 5n = 10$ using the formal definition of the limit of a sequence providing the necessary explanations where necessary).

R: Consider the declaration $\lim_{n\to 2} 2n = 5$. Decide the truth or falsity of the statement.

ST 28: Yaa. I think this proclamation is false. One has to show that limit statement is false by negation. That is, I have to show that$|f(n) - L| \geq \varepsilon$. With negation, one epsilon is enough. I take epsilon say $\frac{1}{2}$. Then for any given N, I have to come up with something so n values have be in the N interval. Let me take $n = 2 - (\frac{N}{3})$. In this case, $f(n) = 2n$, $a = 2$, and $L = 5$. Then we have$\left|2\left(2 - \frac{N}{3}\right) - 5\right| \geq \varepsilon = \frac{1}{2}$. Then we get$|1 + 2\left(\frac{N}{3}\right)|$. This implies that$|1 + 2(\frac{N}{3}) \geq \frac{1}{2}$. This should be true for all N values since $N > 0$. This negation is correct, then the declaration is not true and the limit of $2n$ is not 5 at the point 2.

ST 28 presented an argument why the given statement was false. He displayed a coherent collection of Actions, Processes, Objects and other Schemas which he linked to conclude that the statement was false. This resonated with Dubinsky's (1991) argument that an individual operating at Schema level is able to do the process as a whole by creating linkages among concepts. This is exactly what ST 28 did. The interview with ST 28 showed that the student was able to explain his response strategy clearly and was also able to prove using the definition of the limit of a sequence, for a sequence whose given limit was false. ST 28 gave enough evidence fulfilling the requirements of preliminary genetic decomposition to indicate that he operated at Schema level of the APOS theory for proof of limits of sequences.

5.10 Proof of a sequence limit: question 2a (iii), $\lim_{n\to\infty} \frac{1}{n} = 0$

This question also addressed students' knowledge of the use of the formal definition of the limit of a sequence to prove the limit of a given sequence. This question was provided so as probe students' development of Object conception of the sequence limit proofs.

5.10.1 Responses of students according to APOS levels

Table 5.10: Frequency of students' responses for question 2a (iii) according to APOS level.

APOS level	N	Object	Schema
Number of responses	12	15	3

Three (10%) of the responses provided by the students indicated that they had attained the Schema level for using the formal definition of the limit of a sequence to prove the limit of a given sequence. Fifteen (50%) of the responses showed that students operated at the Object level of the APOS theory. They showed the whole procedure and understood that transformations could be performed on it (Dubinsky & McDonald, 2001). These students' responses showed that they had attained the Object level conception as specified in the preliminary genetic decomposition. Twelve (40%) of the students completely failed to give correct responses to question 2a (iii). Of these, three did not attempt the question at all while nine attempted but failed to give reasonable responses.

5.10.2 Responses at N level (totally incorrect response)

Figure 5.15 shows an example of a student who operated at N level. Such students arrived at a stage where they could not go beyond, revealing that they had challenges to reach the conclusion of proving limits of sequences.

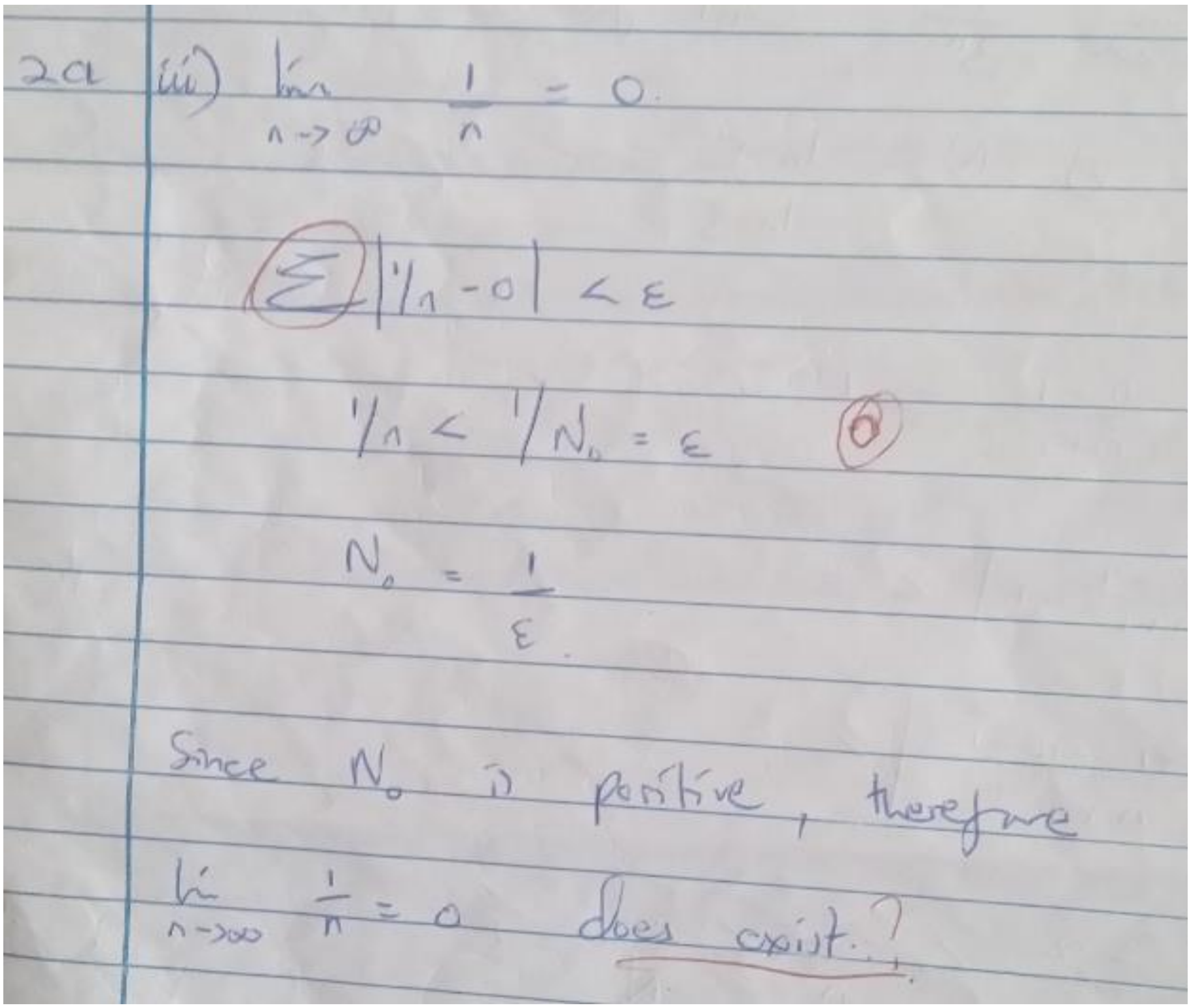


Figure 5.15: Written response of ST 17 to question 2a (iii)

ST 17 provided a response that was indicative of operating at N level. This finding concur with Wilson (2013) who placed a learner whose response showed little conception about the function concept operated at N level. ST 17's response showed little conception of proving the limit of given sequences. The student failed to successfully give a correct response, her solution on procedure lacked the proper links of relationships among n, N, ϵ and L. There is evidence of the student carrying out the proof from memory as no explanations are given for the method, leading to the incorrect response.

To validate the researcher's claim, ST 17 was interviewed on the strategies that she gave in her responses to question 2a (iii). Our dialogue is captured in the following conversation:

R: Can you explain to your response to question 2a (iii)?

ST 17: I only managed to find the value of N_0 and got stuck. I found it difficult to come up to the conclusion as required by the question.

R: What were you asked to do?

ST 17: To prove the given limit, but I have not fully understood the solution procedure. I need more time and practice. The formal definition of the limit of a sequence is difficult to understand. It is even more difficult to apply when proving the limits of sequences.

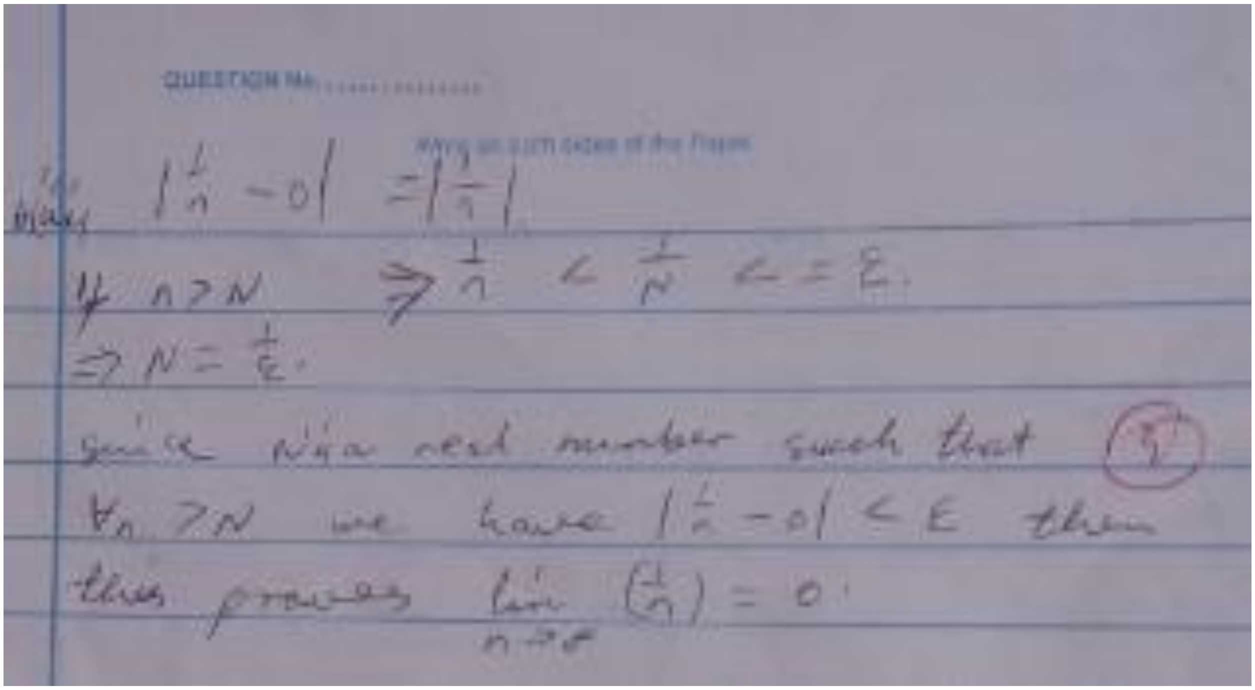
R: What do you mean when you say "since N_0 is positive, therefore $\lim_{n\to\infty} \frac{1}{n} = 0$ does not exist"?

ST 17: Uuuu, I cannot explain this. I just wrote it.

An analysis of the interview dialogue revealed that ST 17 could not proceed after finding the value of N_0, which indicated that she had not made the necessary mental constructions required to deal with such proofs. Her failure to explain and prove the limit made the researcher to place her at N level.

5.10.3 Responses at Schema level

Figure 5.16 is an example of a response by a student who operated at a Shema level of the APOS theory. Such students were able to find the relationship among n, N, ε, indicating that they had understood proofs of limit of sequences.


QUESTION No. ....

Write on both sides of the Paper

$|\frac{1}{n} - 0| = |\frac{1}{n}|$

$\forall\ n > N \Rightarrow \frac{1}{n} < \frac{1}{N} < = \varepsilon.$

$\Rightarrow N = \frac{1}{\varepsilon}.$

since N is a real number such that

$\forall n > N$ we have $|\frac{1}{n} - 0| < \varepsilon$ then

this proves $\lim_{n \to \infty} (\frac{1}{n}) = 0.$

Figure 5.16: Written response of ST 15 to question 2a (iii)

The response provided by ST 15 is indicative of the Object level of the APOS theory. She was able to provide evidence of viewing a dynamic structure as a static one to which actions can be practical. This is an indication that he operated at the Object level of the APOS theory. ST 15 managed to bring out actions resulting in a transformation on a function (in this case a sequence, a special function) (Dubinsky & Harel, 1992). The researcher interviewed the student to check her understanding of the processes that she perused her written response.

R: You gave a very good response to question 2a (iii). Can you explain how you understood the question?

ST 15: To prove that a sequence has a specific limit, there is need to find a positive number N which depends on ε. Using the definition, that is $|x_n - L| < \varepsilon$ where $x_n = \frac{1}{n}$ and $L = 0$, we have to find a number N such that for all $n > N$ the inequality $|x_n - L| < \varepsilon$ holds. We let $\varepsilon > 0$ be given, then there is a positive number Nsuch that for all $n > N$ we have $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right|$ thus, $\frac{1}{n} <$

$\frac{1}{N} = \varepsilon$. Letting $\frac{1}{N} = \varepsilon$ gives us $N = \frac{1}{\varepsilon}$. Then, since we have found a real number N such that for all $n > N$ the inequality $\left|\frac{1}{n} - 0\right| < \varepsilon$ holds, then this proves $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$ as required.

R: We are aware that $\lim_{n\to 2} 5n = 10$. Prove this using the formal definition of the limit of a sequence?

ST 15: Yes. (he correctly proves that $\lim_{n\to 2} 5n = 10$ using the formal definition of the limit of a sequence with explanations where necessary).

R: Considering $\lim_{n\to 2} 2n = 5$. Conclude the truth or falsity of the statement.

ST 15: Yes.

R: Can you explain all this by the use of a graph?

ST 15: Yaa (she draws the graph $f(n) = 2n$ correctly). We choose $\varepsilon = \frac{1}{2}$ and epsilon interval is $(4.5 - 5.5)$. When $n = 2 - \left(\frac{N}{3}\right)$ is chosen, n values can be found in the N interval for all N values. If we take $N = 3$, n takes the value 1 and is in the N interval as required by the given condition.

R: Can you give examples for these N values?

ST 15: Yaa, if one takes $N = 3$, the value of n is $2 - \frac{3}{3} = 1$. $f(1) = 2$. This 2 is not in the interval $(4.5 - 5.5)$, even if we choose $n = 9$. This gives us-1, which again is not in $(4.5 - 5.5)$, also if we take$n = 6$. This results in a zero which also is not in the interval as well.

R: Why did you choose $\varepsilon = \frac{1}{2}$?

ST 15: I can take ε being small but greater than zero. N is greater than zero, $2 - \frac{N}{3}$ is all the time less than 2. The chosen n value will always be at the left of 2, the image so formed less than 4, resulting in the image of n not being near 5. Therefore, the statement cannot be true.

ST 15 showed that she had attained the schema level of proofs of sequence limits. Based on the interview dialogue with ST 15, the researcher concluded that she operated at Schema level of the APOS theory for proving limits of sequences. Dubinsky and Harel (1992) identified a learner at Schema level as one who could show the ability to identify and understand the relationships among Actions, Processes, Objects and other Schemas in connection with functions. ST 15 demonstrated the ability to identify, understand and apply the relationships among Actions, Processes, Objects and other Schemas to prove given limits and even illustrated her responses using diagrams during the interview. Furthermore, she gave convincing explanations on her response strategy that demonstrated that she had met the requirements of the preliminary genetic decomposition of a student operating at Schema level.

5.11 Proof of uniqueness of a limit: question 2(b), Prove that if a sequence converges, then its limit is unique.

The question on the proof of a sequence theorem was intended to provide insight on whether students had developed the Object conception of sequence limit to prove a theorem on uniqueness.

5.11.1 Responses of students according to APOS levels

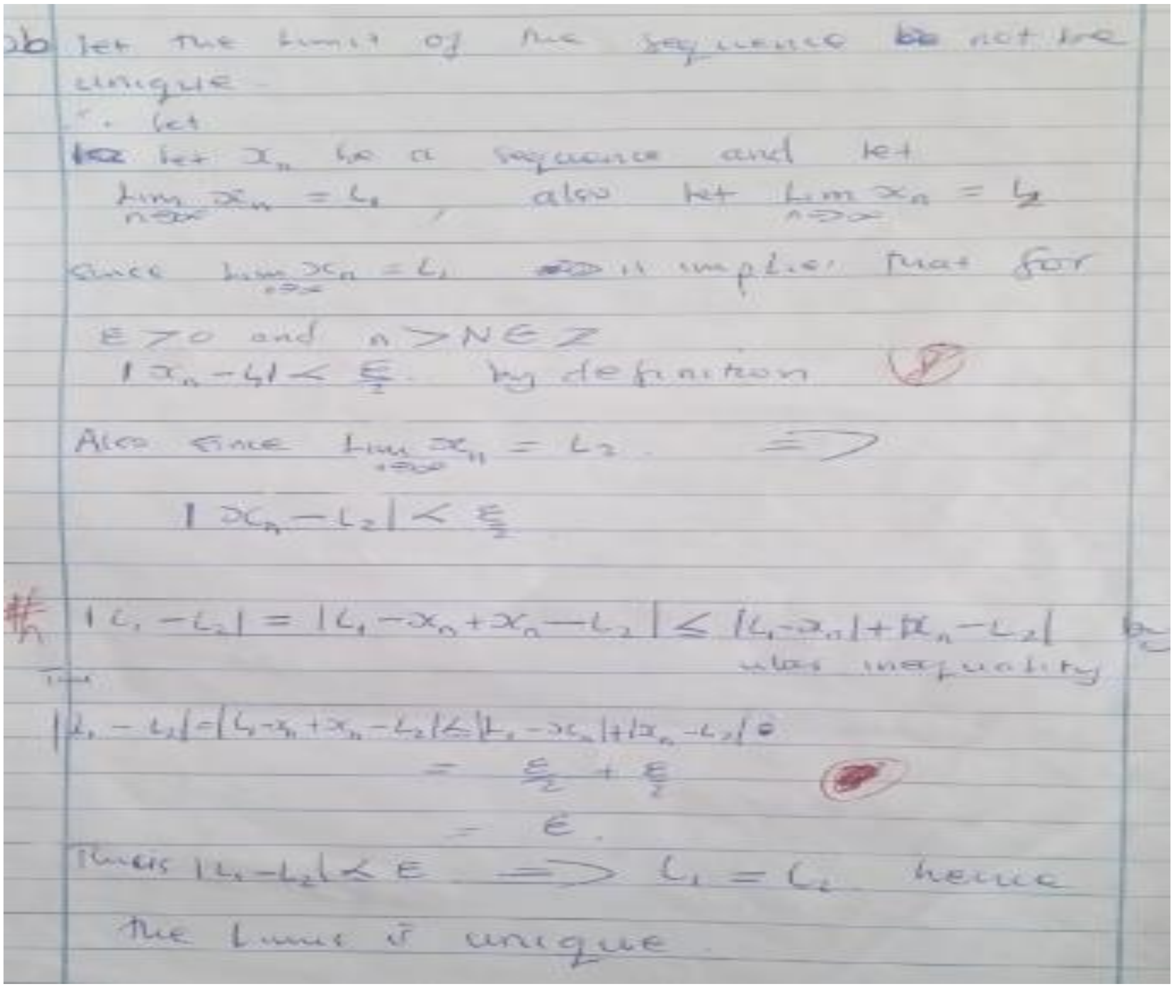
Table 5.11: Frequency of students' responses for question 2(b) according to APOS level.

Score	N	Object	Schema
Number of responses	7	20	3

Three (10%) of the students gave responses that indicated that they operated at the Schema level of the APOS theory. Twenty (67%) of the responses showed that the students operated at the Object level of the APOS theory. Reasonable proofs were provided by the students which was an indication that they had attained the Object level conception in line with the preliminary genetic decomposition. Seven (23%) of the students' responses showed that they had failed to developed to the expectation of the preliminary genetic decomposition, the Object level of the APOS theory. Of the seven, five did not attempt the question and two gave responses that showed they were very confused.

5.11.2 Responses at an N level (totally incorrect response)

Figure 5.17 illustrates a response of a student who operated at the N level. The students showed little ability to work out the proof.


2b let the limit of the sequence ~~be~~ not be unique.

∴ let

~~let~~ let x_n be a sequence and let

$\lim_{n\to\infty} x_n = L_1$, also let $\lim_{n\to\infty} x_n = L_2$

Since $\lim_{n\to\infty} x_n = L_1$ ⇒ it implies that for

$\epsilon > 0$ and $n > N \in Z$

$|x_n - L_1| < \frac{\epsilon}{2}$ by definition

Also since $\lim_{n\to\infty} x_n = L_2$ ⇒

$|x_n - L_2| < \frac{\epsilon}{2}$

$|L_1 - L_2| = |L_1 - x_n + x_n - L_2| \le |L_1 - x_n| + |x_n - L_2|$ by [illegible] inequality

$|L_1 - L_2| = |L_1 - x_n + x_n - L_2| \le |L_1 - x_n| + |x_n - L_2| <$

$= \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$= \epsilon$.

Thus $|L_1 - L_2| < \epsilon \Rightarrow L_1 = L_2$ hence the limit is unique.

Figure 5.17: Written response of ST 14 to question 2b

The response given by ST 14 is indicative of the N level. The response showed a failed recall of the proof done in class (from memorization and unclear understanding). ST 14's response concurred with Wilson (2013) who placed a learner whose response showed little conception about the function concept operated at N level. Her response showed little conception of proving the uniqueness theorem of limits of sequences. The researcher's analysis of ST 14's written response to question 2(b) showed that she operated at the N However, to confirm the student's understanding of these concepts and the researcher's claim that she operated at N level, she was interviewed to explain her response strategies as shown in the following interview excerpt:

R: How did you come up with the statements, let $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$?

ST 14: I recalled this from the work we did some time when dealing with limits of functions.

R: Okay, what do you mean by definition?

ST 14: Umm, it is the definition of the limit of a sequence.

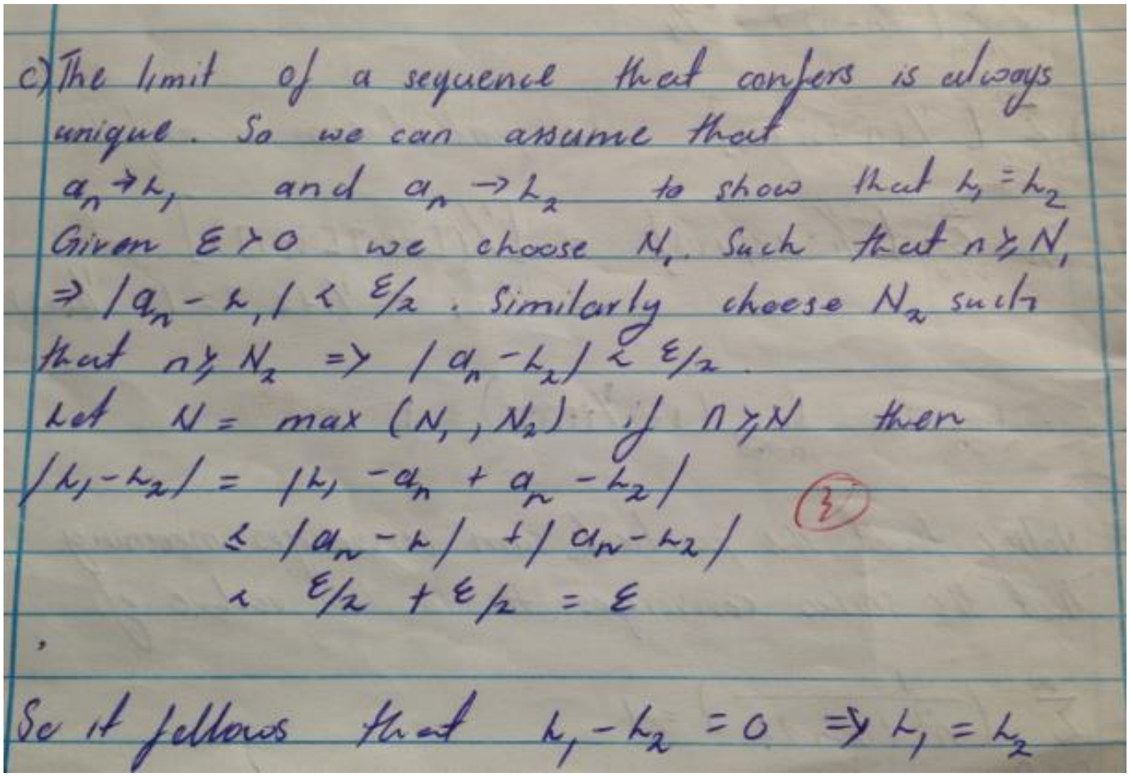
R: What do you understand by the triangular inequality which you talked about in your response?

ST 14: I got this from the book I read and cannot explain it.

An analysis of ST 14's response to the interview questions gives evidence that she operated the N level. Her inability to explain concepts and why particular procedures are indicators of failing to attain the Object level, she just followed a procedure with little or no understanding.

5.11.3 Responses at an Object level

Figure 5.18 shows an example of a student who operated at an Object level. Such students provided responses which showed that they had understood the proof of the uniqueness of the limit of a sequence.



c) The limit of a sequence that confers is always unique. So we can assume that

$a_n \to L_1$ and $a_n \to L_2$ to show that $L_1 = L_2$

Given $\varepsilon > 0$ we choose N_1. Such that $n \geq N_1$

$\Rightarrow |a_n - L_1| < \varepsilon/2$. Similarly choose N_2 such that $n \geq N_2 \Rightarrow |a_n - L_2| < \varepsilon/2$.

Let $N = \max(N_1, N_2)$ if $n \geq N$ then

$|L_1 - L_2| = |L_1 - a_n + a_n - L_2|$

$\leq |a_n - L| + |a_n - L_2|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

3

So it follows that $L_1 - L_2 = 0 \Rightarrow L_1 = L_2$

Figure 5.18: Written response of ST 12 to question 2b

The response provided by ST 12 was quite logical with the necessary explanations accompanied. The excerpt provided evidence indicative of the Object level of the APOS theory. At an Object level, a learner becomes aware of the totality of the Process, realizes that transformations can act on it, and is able to construct such transformation (Arnon et al., 2014). The student constructed an Object through the encapsulation of a Processes, she coordinated two sequences to come up with the correct argument to prove that if a sequence converges, then its limit is unique. To further authenticate the researcher's claim that ST 12 was operated at Object level, she was interviewed on the solution strategies in her written response.

R: Can you state the $\varepsilon - N$ definition of the limit of a sequence?

ST 12: A sequence a_n converges to limit L if, and only if, for every given positive natural number ε, there exists a positive natural number N, such that for all $n > N$, we have $|a_n - L| < \varepsilon$.

R: How can the concept of this definition be used to prove the uniqueness of the limit of a sequence?

ST 12: We note that this limit of a sequence definition goes both ways for we have 'if and only if' statement.

R: What does this mean in relation to the limit of a sequence?

ST 12: It means if $\lim_{n\to\infty} a_n = L$, then for every $\varepsilon > 0$ there is a number N such that for all $n > N$ $|a_n - L| < \varepsilon$. Also going the other way, if for every $\varepsilon > 0$ there is a number N, such that for all $n > N$, $|a_n - L| < \varepsilon$, then $\lim_{n\to\infty} a_n = L$. So when we prove the uniqueness of a limit, we have to establish the inequality $|a_n - L| < \varepsilon$ then we have proved that $\lim_{n\to\infty} a_n = L$.

R: That is good, how do you prove the uniqueness of the limit of a sequence?

ST 12: To prove that if $\lim_{n\to\infty} a_n$ exists, it must be unique, we use the fact that, if n be a number such that for all epsilon greater than zero,$|n| < \varepsilon$, then $x = 0$. We must show that if $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$ then $L_1 = L_2$. By the hypothesis, given any epsilon greater than zero, we can find N such that $|a_n - L_1| < \frac{\varepsilon}{2}$ when $n > N$, $|a_n - L_2| < \frac{\varepsilon}{2}$ when $n > N$. Then $|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |a_n - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. That is $|L_1 - L_2|$ is less than any positive epsilon (however small) and so must be zero. Thus, $L_1 = L_2$. Hence, it can be concluded that the limit of the sequence is unique.

The dialogue between the researcher and ST 12 revealed that she operated at the Object level of the APOS theory. She managed to give convincing explanations for her response strategy linking the definition of the limit of a sequence and the two limits she proposed. Thus, taking into consideration this dialogue, the researcher was convinced that she operated at the Object level.

5.12 Summary of findings on limits of sequences

The response for test question 1 (a) requiring students to display an appreciation of the Action level showed that many students at least 22 managed to attained the Action level, with 7 students failing to attain an Action level. For the failing 7 failed to utilize the first stage of the ACE learning cycle to understand the stating of the definition of the limit of a sequence. This was in contrast to a student who managed to attain a Schema level for the same test question. For the rest of the test questions 1b (i-ii) on the limit of sequences, the students' performance was comparable to other research carried out in the past where most students managed to attain at least the Action level. These are the likes of (Aydin & Mutlu, 2013) who used the ACE learning cycle, with the activities being carried out using PC tablets. Their findings showed that many students developed suitable mental arrangements at the Action level. Furthermore, Maharaj (2014) employed the ACE learning cycle and the research findings revealed that the majority of the students operated at the Action level.

The results of the responses provided by the students to test questions at the Process level highlighted that the results agreed with the past research studies. In this study and the past studies, the ACE learning cycle were employed. The results of this study for the three test questions 1b (iii-v) revealed respectively that 26, 24 and 27students managed to attain the Process level. This was in line with the past studies of Ndlovu and Brijlall (2015) whose findings revealed concurrence between mental constructions of pre-service teachers and their PGD. Majority of the pre-service teachers operated at Action and Process levels. Furthermore, Ndlovu and Brijlall (2016) found out that most pre-service mathematics teachers operated at the Action and Process levels. Also Makonye (2017) carried out a research study and Action and Process levels marked the end of progress for the majority. Lastly Kazunga and Bansilal (2017) detected the understanding of many students was at the Action level and Process level.

The students' response to questions that 2 (a) and 2 (b) revealed that there were students who operated at the Object and Schema levels. The number of students who operated at the Object level for the four test questions were 8, 13, 15 and 20 respectively. The number of students who attained the Object level in the present study was more when compared to those who attained the Object level in past researches. Aydin and Mutlu (2013) found out that many of their students failed to develop suitable mental arrangements at the Object level. Also Maharaj (2014) studied on an APOS analysis on how integration was understood by natural Science students. The majority of the students failed to develop the suitable mental constructions at the Object level. Furthermore, Ndlovu and Brijlall (2016) determined the mental constructions displayed by pre-service mathematics teachers. The results revealed that few pre-service mathematics teachers displayed an Object level. Voskoglou (2015) used the ACE learning cycle instruction and the research findings showed that the application of the APOS Theory and ACE learning cycle approach enhanced significantly the students' understanding of the topic they studied. The current research study indicated that many students managed to operate at the Object level due to the application of the Maple software during the activity stages of the ACE learning cycle.

The number of students who operated at the Schema level for test questions 2 (a) and 2 (b) were three for each test question. This is in line with past research that found very few students, if any, could attain the Schema level. Ndlovu and Brijlall (2015) had none at Schema level, but in this study there were three students. Makonye (2017) found out that few students managed to attain the Schema level.

5.13 Conclusion

Students were prompted through guiding questions to collect sets of linked components of the limit concept in relation to sequences to examine their: knowledge structures, internal representations, how these were coordinated and the mental models they formed. The interviews conducted with the different students on selected sequence test items to clarify some responses presented on the written instrument, helped the students in constructing and reconstructing mental objects. This chapter gave explanations of how students in the first year calculus 1 course constructed concepts of limits of sequences. Written test extracts served to explore the conceptual understanding of these concepts using APOS and the stages on which mental constructs on the limit concept in relation to sequences were made with regards to the Triad mechanism. Chapter 6 discusses the results of students' written limit test and interview responses on the limit concept in relation to series.

6 ANALYSIS AND DISCUSSION OF LIMITS OF SERIES

6.1 Introduction

Chapter five discussed the results the limit concept in relation to sequences. The purpose of this chapter is to discuss the results of the study using the methodology described in chapter four. This chapter focused on students' responses to limit test questions on the limit of series. In this chapter, the research questions were answered, based on the data collected from the series limit test questions and interviews. This chapter focused on the limit test question numbers 3 and 4 (Appendix 2). The test questions granted a case analysis of each test question supported by authentic written responses and interview extracts from selected students. This was done to provide confirmation of the APOS level at which the students operated, in terms of understanding the limit concept in relation to series. The data presentation and analysis was done in line with the genetic decomposition provided in Table 4.2.

Table 6.1: Complete summary of the categorization of students' mental construction, according to APOS, on each of the test items on limit of series.

Test Item	APOS categorization of students' mental constructions				
	None	Action	Process	object	Schema
3 (a)	11	19	0	0	0
3b (i)	5	0	25	0	0
3b (ii)	10	0	20	0	0
3b (iii)	11	0	19	0	0
3c (i)	10	0	0	20	0
3c (ii)	10	0	0	20	0
3c (iii)	10	0	0	20	0

The question 3 (a) tested students' attainment of the Action level. The results revealed that most of the students attained the Action level, with a few failing to attain the Action level. Also, questions 3b (i) to 3b (iii) tested students' attainment of the Process level and the results revealed that most of the students attained the Process level. Furthermore, responses to questions 3c (i) to 3c (iii) showed a constant number of students who managed to attain the Object level. The relevant information displayed in Table 6 was extracted for Tables 6.1 to 6.7 to make it easier for the reader to follow the focused discussion on the different sub-questions for test question 3.

6.2 Question 3 (a), Define the limit of an infinite series?

This question addressed the series definition as predicted by the genetic decomposition at the Action level. It anticipated to provide insight as to whether students had developed an Action conception of the limit definition of a series. Surprisingly a number of students could not state the definition, with some of the students not attempting the question at all.

6.2.1 Responses of students according to APOS levels

Table 6.2: Frequency of students' responses for question 3(a) according to APOS level.

APOS level	N	Action
Number of responses	11	19

Nineteen (63%) of the students gave the correct response to question 3(a) operated at the Action level of the APOS theory. The responses to task 3(a) indicated that eleven (37%) of the students could not attain even the Action level of the APOS theory. Of the eleven, five did not attempt the question at all, while six gave incorrect and confusing responses.

6.2.2 Response at N level (totally incorrect response)

Figure 6.1 is an example of a response of a student who operated at N level. It exemplifies the response of a confused student. It is not clear whether the student was talking of the definition of continuity of a function or the definition of an infinite series.

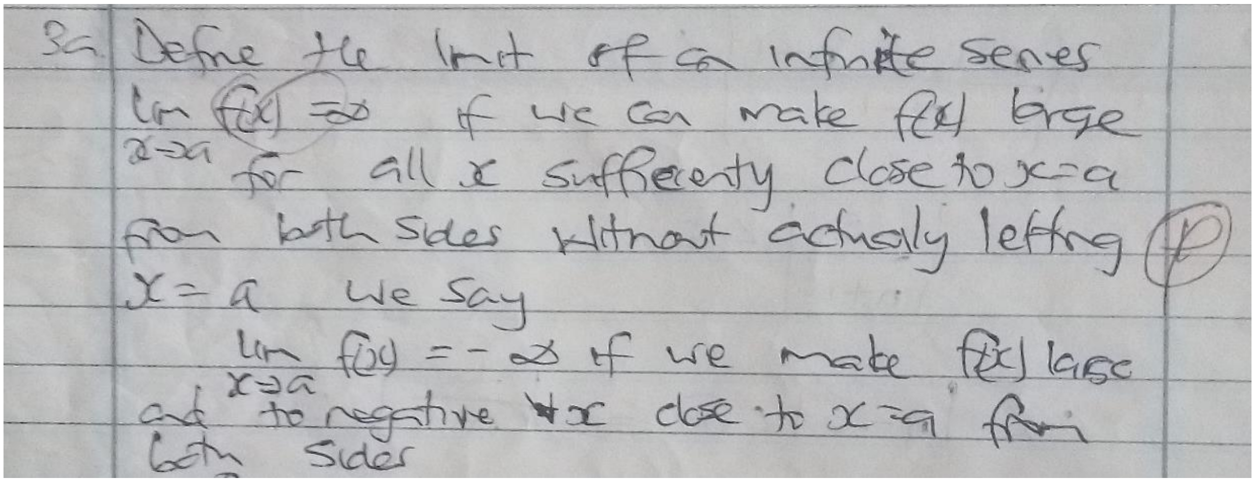

3a Define the limit of an infinite series
$\lim_{x \to a} f(x) = \infty$ if we can make $f(x)$ large
for all x suffiecenty close to $x=a$
from both sides without actualy letting
$x = a$ we say
$\lim_{x \to a} f(x) = -\infty$ if we make $f(x)$ large
and negative $\forall x$ close to $x=a$ from
both sides

Figure 6.1: Written response of ST 20 to question 3a

The response of ST 20 illustrates a completely wrong definition. It appears the student was writing about the definition of the continuity of a function. This response shows a lot of confusion as a number of concepts are mixed up. The idea of a series, though given, accounts for a very small part of the required definition. It was difficult to predict the student's response. ST 20 failed to give the correct response which was an indication that he had failed to attain at Action level of the APOS theory. Existing literature noted that students have difficulty understanding definitions when studying sequences (Roh, 2008). The response provided by ST 20 concurred with Roh's finding. He showed weak understanding of stating the definition of the limit of infinite series. The given response was a mixture of ideas of the limit of a sequence and those for the determination of continuity. To check how he understood these concepts, the researcher interviewed him. The interview excerpt below captures the dialogue between the researcher and the student.

R: Can you explain the definition of the limit of an infinite series you gave?

ST 20: Uuu, by the time I wrote the response, I mixed up issues. I thought of the existence of the limit of a function, yet the question required the definition of an infinite series.

R: Well, but was that the requirement of the question?

ST 20: It required the definition of the limit of an infinite series.

R: Then why did you not give the required response?

ST 20: By that time, I mixed up issues as I said earlier. I only came to understand the demands of the question after I read it again sometime after the test.

R: Well are you now able to give the definition of the limit of an infinite series?

ST 20: Uuu no, I cannot provide the definition accurately at the moment.

R: Anywhere, can a limit be attainable?

ST 20: I do not see if the limit is attainable since we are summing something that is infinite.

ST 20 operated at N level, he failed to correctly state the definition of the limit of an infinite series. Furthermore, his failure to state the definition from memory is an indication that he has not attained the Action level.

6.2.3 Responses at an Action level

The responses provided in Figure 6.2 exemplify an Action level definition of the limit of an infinite series. The student showed the ability to provide the definition without explanations.

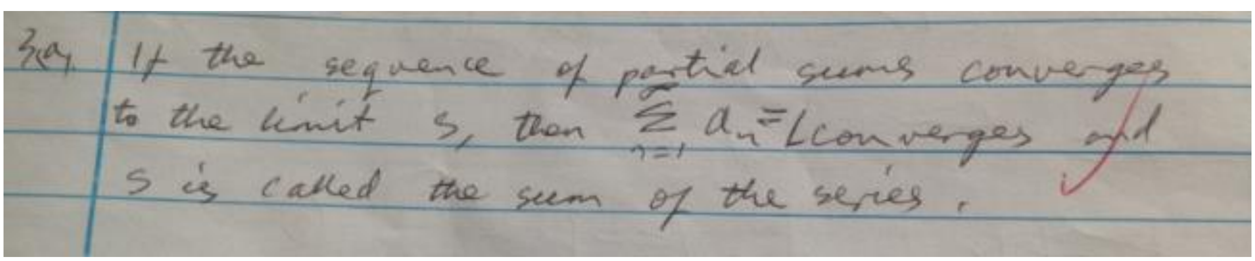


Figure 6.2: Written response of ST 15 to question 3a

ST 15 managed to give a correct response to question 3(a). Her ability to recall the definition correctly is an indication that she operated at the Action level of the APOS theory. She exhibited a strong tendency to recall verbatim, the definition which is an indicator of the Action level (Breidenbach et al., 1992). The researcher analyzed the student's written response and found that she operated at the Action level of the APOS theory. ST 15's understanding of the definition was verified through an interview. The following interview excerpt revealed her understanding of the definition of the limit of an infinite series.

R: You gave the correct response to question 3(a). Can you explain your understanding of this definition?

ST 15: We say an infinite series converges if its sequences of partial sums converge. The infinite series converges to limit L which we get after the summation of terms of a sequence if it is convergent. The series may diverge if its sequence of partial sums diverges, hence does not have a limit.

R: Is the limit L attainable?

ST 15: Why not? It is attainable.

The student was able to give the definition of the limit of an infinite series; an indication that she operated at an Action level of the APOS theory.

6.3 Determination of convergence of a series

Extracts for questions 3b showed that some students mixed up methods for the determination series convergence or divergence with methods of sequence evaluation, by applying theorems and properties of sequences $\{a_n\}$ to $\sum_{n=1}^{\infty} a_n$ or vice versa. A series' convergence is defined in terms of the sequence of the corresponding partial sums $S_n = a_1 + a_2 + \cdots a_n$. Some students did not notice that it is not equivalent to the convergence of the sequence of its terms $\{a_n\}$. Evident in students' work was the use of the series convergent theorem, that is if $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$. The theorem provides the requirement for convergence but fails to guarantee convergence. The two examples $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ illustrate this point. Both series' terms are zero in the limit as n goes to infinity, yet the first converges and the second one diverges. It has to be noted that for a series to converge, the series terms must go to zero in the limit, otherwise the theorem would be violated. Also evident from students' work was the use of the divergence, if $\lim_{n\to\infty} a_n \neq 0$ then

$\sum a_n$ will diverge. The test only states and does not assure that a series is divergent if series terms get to zero in the limit.

These students faced three challenges. Some wanted to use the convergence or divergence theorem without the full understanding of their conditions and procedures needed for their application, and some failed to recognize that use of partial fractions was a very important and necessary technique to solve the given problems. The failure by some students to recognize the use of partial fractions in evaluation of series supports Radford's (1997) contention that limited background information of other mathematical concepts creates complications with the learning of linear algebra. The failure by some students to give the correct responses was a result of difficulties generated by poor understanding and use of partial fractions.

6.3.1 Determination of convergence of series: question 3b (i), $\sum_{n=1}^{\infty} n$

Question 3b (i) addressed the aspect of Process level in the genetic decomposition. It was intended to provide insight as to whether or not students had developed the Process conception of the determination of series convergence.

6.3.2 Responses of students according to APOS levels

Table 6.3: Frequency of students' responses for question 3b (i) according to APOS level.

APOS level	N	Process
Number of responses	5	25

Twenty-five (83%) of the students gave responses that showed they operated at the Process level of the APOS theory. Their responses indicated they understood how to evaluate convergent series problems. Five (17%) of the students' responses showed such students did not attain the Process level of the APOS theory. Four students out of five substituted n by infinity and concluded that the series diverge.

6.3.3 Responses at N level (totally incorrect response)

Figure 6.3 shows an example of a response where the student operated at N level.

The student showed lack of understanding of how to determine whether the series converge or diverge.

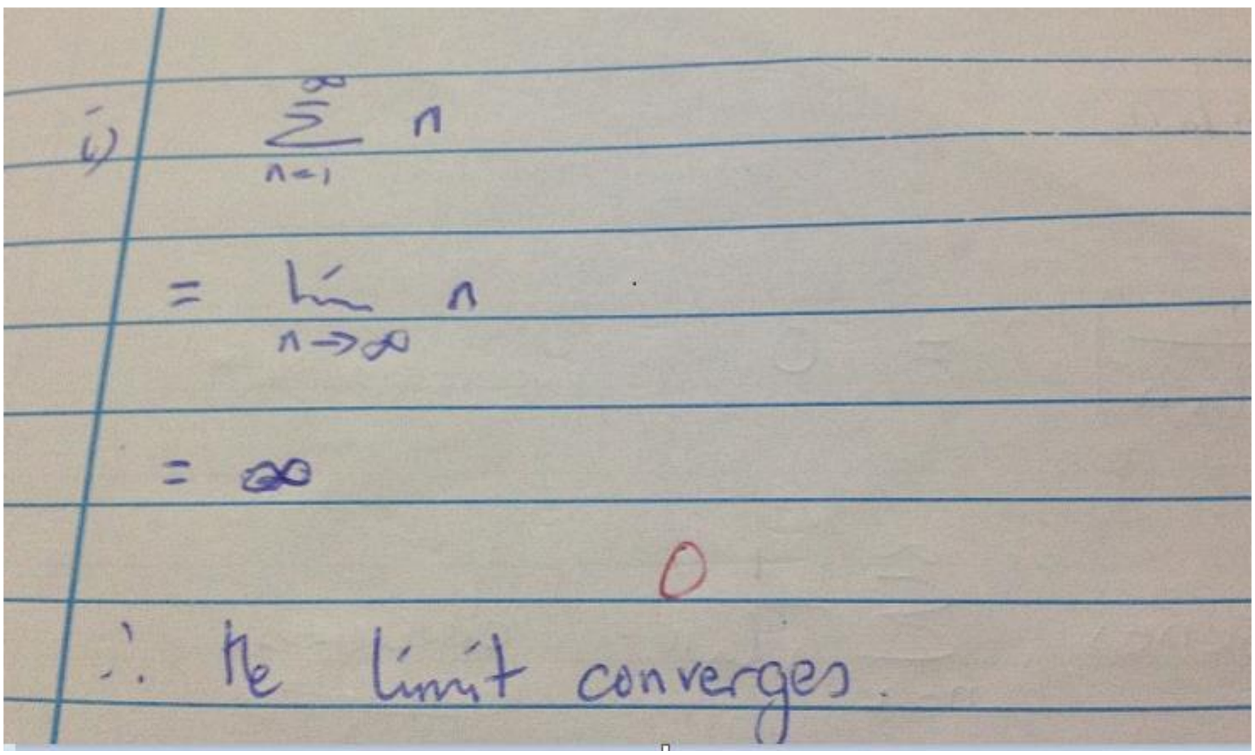


Figure 6.3: Written response of ST 17 to question 3b (i)

The response provided by student ST 17 was incorrect owing to the wrong method of substitution used to determine the convergence of the given series. The student was not able to link the general formula for n terms covered at A-level and the determination of the convergence of a series. Use of the general formula for the sum of n terms was critical for the determination of the divergence and convergence of this series. The response provided by ST 17 agreed with the findings of Earls

(2017) who found that students used and chose inappropriate tests**.** An analysis of students' response showed that if a student failed to express the given series in the form $\frac{n(n+1)}{2}$, then it became impossible for that student to give the correct response. ST 17's written response showed that she was far away from attaining the Process level conception of finding limits of given series. To verify this claim, the researcher interviewed her to triangulate her written and oral responses. An analysis of her written response showed that she operated at the N level. The researcher interviewed her to check how she understood what she was doing. The interview excerpts below show the dialogue held between the researcher and ST 17.

R: How did you determine that the limit of the series $3b(i)$ is convergent?

ST 17: I tried to find the last term. Then discovered that the last term goes to infinite.

R: Did the question ask you to determine the last term or the limit of the series?

ST 17: Uuu no, I was supposed to determine the limit of the series but thought that if I could determine the last term, then the series would be convergent.

R: In your response, you wrote infinity and concluded "the limit diverges." What is the relationship between infinity and limit diverges?

ST 17: Umm no, I have no idea. There is need to find out more about that.

R: Can you find the general formula for the series $\sum_{\boldsymbol{n=1}}^{\infty} \boldsymbol{n}$.

ST 17: Umm, I am not able to do that at the moment. It requires reading more about summation of series, maybe I have to start from advanced level work on series. It is difficult to find a formula for the general term in the sequence of partial sums.

The interview with ST 17 showed that the student could not answer the interview questions. She failed to attain at least the Process level of the APOS theory. Cases of trying to find the last term of the series and the inability to find the relationship between infinity and limit diverges, all pointed to failure by ST 17 to attain the Process level.

6.3.4 Response at a Process level

Figure 6.4 shows an example of a response of a student who operated at the process level. Such students managed to give an expression of the series as a general term, and proceeded to show that the series diverges.

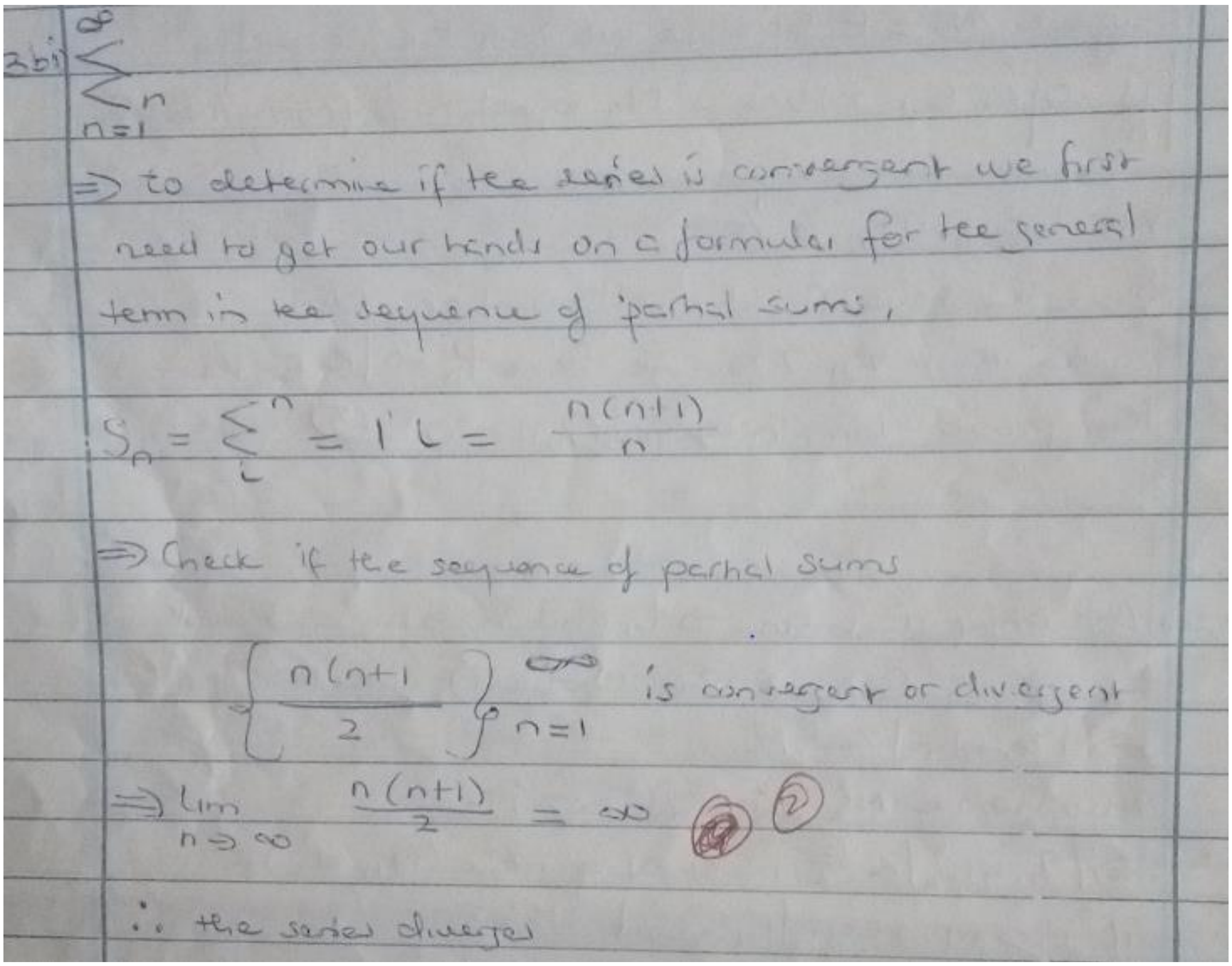


Figure 6.4: Written response of ST 28 to question 3b (i)

The researcher analyzed ST 28’s written response and concluded that he operated at the Process level of the APOS theory. The student managed to give a detailed explanation of his procedure for the solution method. ST 28 noticed the procedure as a whole without need to plug in all specific values (Dubinsky & Harel, 2001). To further authenticate the researcher’s assertion that he operated at the Process level, an interview was held to investigate strategies used.

R: How did you find the response to question 3b (i)?

ST 28: If one adds up numbers that go on and on, then one would never come to a single value. So when I checked on the given question, I saw that was the case so I had to find a formula for the general term in the sequence of partial sums. Then I borrowed the idea of finding the general expression of such a series from advanced level work. That is $\sum_{n=1}^{n} i = \frac{n(n+1)}{2}$ and the limit of the sequence terms is $\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$. Hence the sequence of partial sums diverges to ∞ and it can be concluded that the given series also diverges.

An analysis of the interview responses revealed that the student's ability to draw his action in a verbal manner placed him at the Process level of the APOS theory.

6.4 Determination of convergence of a series: question 3b (ii), $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^2-1}}$

Question 3b (ii) sought to find students' understanding of the determination of the convergence of a series and the value the series converges to. Its intention was to provide insight on students' Process level conception development of determining the convergence of a series and its convergent value.

6.4.1 Responses of students according to APOS levels

Table 6.4: Frequency of students' responses for question 3b (ii) according to APOS level.

APOS level	N	Process
Number of responses	10	20

Twenty (67%) of the students gave the correct responses, which were indicative of the Process level of the APOS theory. These students managed to make use of partial fraction in their responses to this question. Furthermore, ten (33%) of the students' responses showed that they operated below the Action level of the APOS theory. All were confused as to what they should have done to solve the problem. These students failed to utilize partial fractions in coming up with the general formula for the given expression.

6.4.2 Responses at N level (totally incorrect response)

Figure 6.5 illustrates a response of a student who operated at N level. Such students tried to add the resulting series, but without success. There was need to decompose the fraction into partial fractions then apply telescoping series method.

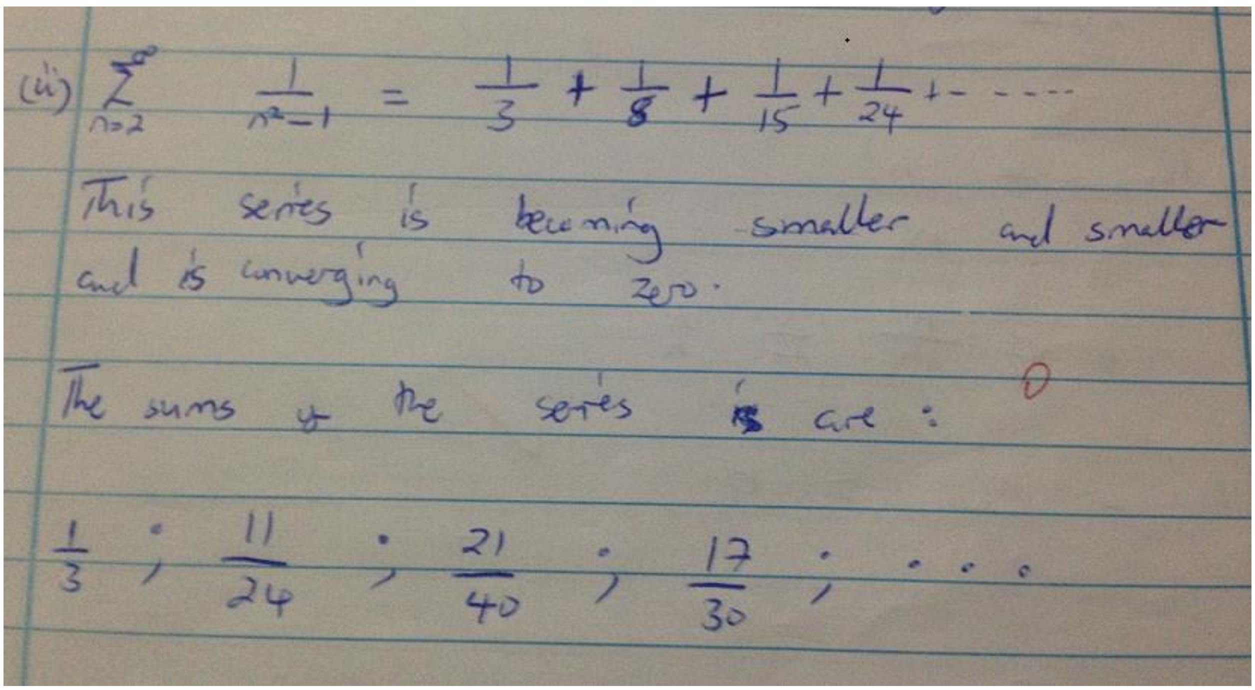


Figure 6.5: Written response of ST 13 to question 3b (ii)

Student ST 13 used a wrong method, substituting values into the given expression. The student chooses an incorrect test method which did not lead to the correct response. Analysis of the above response showed that the limit concept and its application in the determination of convergence or divergence of series was incomplete. The response provided by ST 13 concurred with Earls (2017) who found that students used inappropriate tests to find sums of convergent series**.** The student just listed some sums as indicated in the last sentence, but failed to list all of them. The student could have made use of partial fractions first, then apply the limit laws to come up with the required response of determining the convergence or divergence of a series. An analysis by researcher of ST 13's written response showed that he operated at the Action level of the APOS theory. A verification of the researcher's claim was done through an interview with the student to triangulate his written and oral responses and this is captured in the following excerpts:

R: Can you give the difference between a sequence and a series?

ST 13: A sequence is a list of numbers, usually in an increasing order; and a series is when these numbers are added up and their sum may or may not be found.

R: So how did you determine the convergence of a series given in question 3b (ii)?

ST 13: I substituted the values 2, 3, 4 … in to the given series expression then added the few. It was not possible to add up all the numbers to infinity as the numbers do not come to an end.

R: That is okay. Can you find the general term in the sequence of partial sums of question 3b (ii)?

ST 13: Um, I cannot. This topic on series was difficult for me at advanced level and still it is difficult for me.

ST 13 managed to give the difference between a series and a sequence. However, he failed to find an appropriate method to determine the convergence of the series. The researcher placed him at the N level after taking into consideration the responses he gave to the interview questions.

6.4.3 Responses at a Process level

Figure 6.6 shows the response of a student who operated at the Process level of the APOS theory. Such students showed the ability to omit some steps in their response strategy.

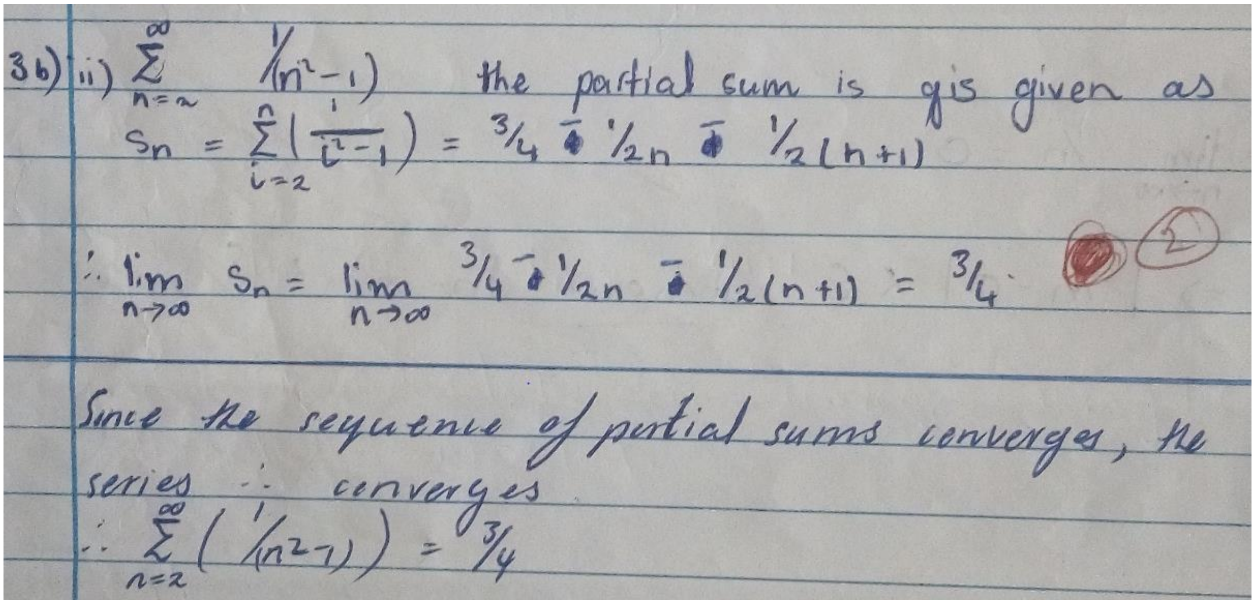


Figure 6.6: Written response of ST 12 to question 3b (ii)

ST 12 managed to give the correct response to question 3b (ii) and an analysis of her response showed that he operated at the Process level of the APOS theory. ST 12's written response shows evidence of omission of some steps in her solution method; which is coming up with the second line in the decomposition of the fraction into partial fractions. She managed to make use of the general formula for n^{th} term in the determination of the convergence of a series. The use of the general formula for the sum of n terms was critical for the determination of the divergence and convergence of this series. She showed the ability to realize a relationship (the given series and the general formula), when it was not stated (Weyer, 2010). Furthermore, to verify the researcher's claim, the researcher interviewed her to check her understanding of the procedure that she correctly followed in her written response. The researcher's dialogue with her is shown in the interview excerpt that follows.

R: You gave the correct response to question 3b (ii), can you take me through your response?

ST 12: I thought of advanced level work where we dealt with partial fraction and then general formula for series summation. So I found the general formula for the given series. I discovered that this results in telescoping series which resulted in $S_n = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2n+2}$ after some terms had canceled out. Then taking limit as $n \to \infty$ gives $\frac{3}{4}$ as the other two terms goes to zero. Thus, the sequence of partial sums converges, so the series does converge to a value of $\frac{3}{4}$.

The dialogue with ST 12 revealed that she operated at Process level of the APOS theory. She showed the ability to explain her response strategy. She explained where she used partial fractions to find the general formula for an expression of the series, and then taking limit as the numbers approached infinity. This indicated that she operated at the Process level.

6.5 Determination of convergence of a series: question 3b (iii), $\sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^2+\ 3n+2}}$

Question 3b (iii) also addressed series determination of convergence to find the value of the series if it converges. The question was made to provide insight on how students had developed the Process conception of the determination of the convergence or divergence of a series, and to find its value if it converges.

6.5.1 Responses of students according to APOS levels

Table 6.5: Frequency of students' responses for question 3b (iii) according to APOS level.

APOS level	N	Process
Number of responses	11	19

Nineteen (63%) of the students gave responses that indicted that they operated at the Process level of the APOS theory. Eleven (37%) of the students' responses showed that they were operating at (P) level. Among the eleven, one did not attempt the question, while some of the students used sequence evaluation methods. The sequence evaluation method mainly used was that of dividing with the highest power of n in the denominator.

6.5.2 Responses at N level (totally incorrect response)

Figure 6.7 shows an example of a student who operated at the N level. Such a student showed inability to determine the convergence or divergence of series.

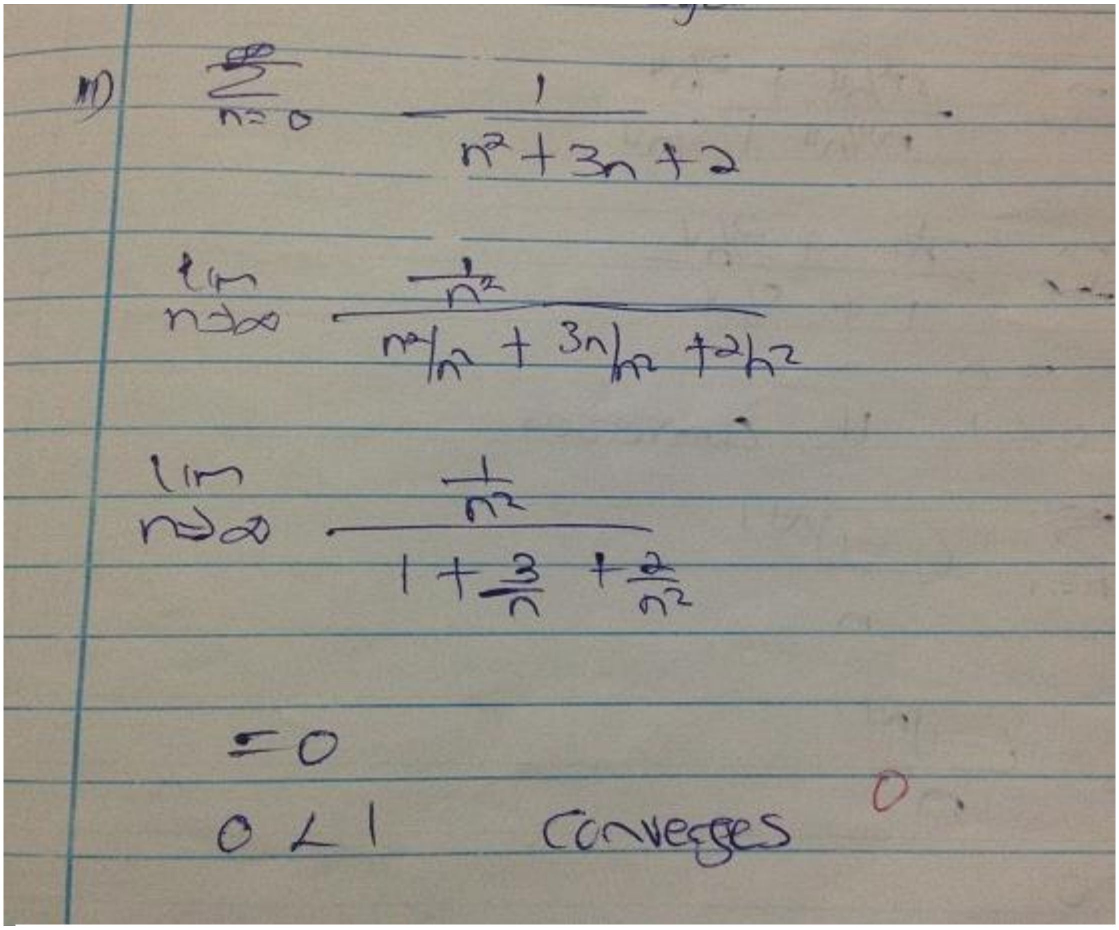


Figure 6.7: Written response of ST 20 to question 3b (iii)

ST 20 provided an incorrect response, and the method used to answer question 3b (iii) was inappropriate and did not address the question. This method is used for evaluating sequences and not determining the convergence or divergence of series. The student confused sequences and series and in this case was inconsistent using summation notation. Also the student seemed unsure of the definition of sequences and series. This student failed to attain a fully developed Process level of the APOS theory. The given response indicated that the student lacked the basic concepts of series evaluation. ST 20's response supported the findings of Earls (2017) who found out that students chose inappropriate tests to determine the sums of convergent series. The researcher placed ST 20's written response at N level after an analysis, as the student used a method meant for the evaluation of sequences. The researcher interviewed him to check on how he understood these concepts and the associated difficulties he faced. The interview excerpt below shows the dialogue held.

R: You responded to question 3b (iii) as though you were answering a sequence problem. Why is this the case?

ST 20: I thought since a series is an addition of numbers of a sequence, then by applying the method of evaluating sequences, I would determine the convergence of the series.

R: During the learning sessions, were the convergence of a series and those of sequences determined using the same methods?

ST 20: Uuu no, we did not use such methods.

R: So why did you use such a method?

ST 20: I got confused during the time I was writing the test so I ended up using the wrong method.

He admitted that he got confused and ended up using the wrong method. This was an indication that he tried to give the written response from memory. An analysis of ST 20's written response and the interview excerpts gave strong indicators that the student operated at N level.

6.5.3 Responses at a Process level

Figure 6.8 illustrates an example of a student who operated at the Process level. Such students showed their ability to express the series in partial fractions, to find out that it results in telescoping series, and that the series converge to one.

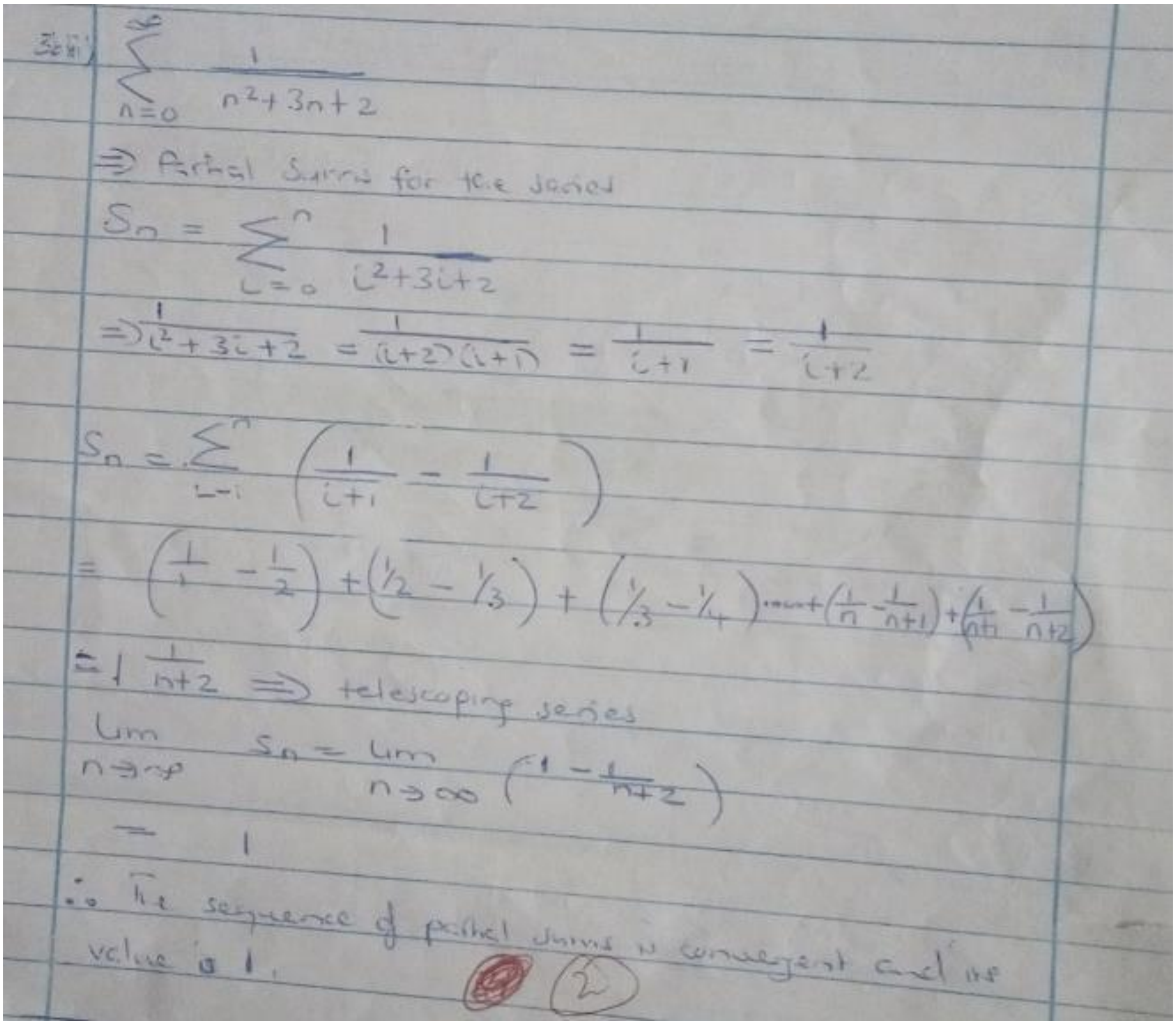


Figure 6.8: Written response of ST 28 to question 3b (iii)

The response given by ST 28 showed that he had attained at least the Process level of the APOS theory. He showed the ability to form and work with telescoping series that led him to successfully come up with the correct response. An analysis of ST 28's response showed evidence of omission of some steps (cancelling out of terms with opposite signs and the finding the limit $\frac{1}{n+2}$ as $n \to \infty$) in the solution method, reflected in thinking of performing a process without actually doing it (Dubinsky & McDonald, 2001). To verify ST 28's understanding of these procedures and confirm whether he operated at the Process level, the researcher interviewed him and the following interview excerpt revealed his understanding of limit related concepts in this question.

R: You responded very well to question 3b (iii), can you explain how you arrived at your final response?

ST 28: Yaa, I had to think of partial fractions first, that is, expressing the given series as a sum of partial fractions. I wrote out the terms of the general partial sum for this series using partial fraction form. This resulted in a telescoping series where successive terms are the ones that canceled out leaving only the first and last terms, then I took the limit of the partial sums whose result is one as $n \to \infty$.

R: Is it always the case that successive terms will cancel out?

ST 28: This is not always the case, some terms cancel with some terms way down the list, for example in the case of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$. (the student used the partial fractions to come up with the general terms which resulted) $\frac{1}{2}[\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right)+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)]$. In this case instead of successive terms cancelling, one term will cancel with another term that is farther down the generated list. The end result this time is that the initial two and the last two terms are left.

R: What are the conditions that a series must satisfy in order for telescoping series method to be applicable to find the value of convergent series?

ST 28: If we can express the series as partial fractions, have the difference of convergent series, and get terms that cancel (some terms must be negative and others positive). If these three conditions are satisfied, then the telescoping series apply.

ST 28 gave detailed explanations on his solution methods during the interview which revealed that the student had understood how to deal with telescoping series. He was able to explain an important issue about telescoping series, which is not all series that can be expressed as partial fractions are not telescoping. A series is telescoping, if terms cancel which is not the case if all terms are positive. His explanations during the interview revealed that he operated at the Process level of the APOS theory.

6.8 Series convergence tests

The convergence test states that if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Students made a common mistake by assuming that if $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ will converge. Yet convergence cannot be guaranteed by the zero result. The case of $\sum_{n=1}^{\infty} \frac{1}{n}$ (diverges) and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (converges) clearly demonstrates that if $\lim_{n\to\infty} a_n = 0$, a convergent situation is not guaranteed. If the series terms happen to go to zero, the series may or may not converge.

The divergence test guarantees divergence if $\lim_{n\to\infty} a_n \neq 0$. The test give assurance on a series' divergent status if the series do not go to zero in the limit. The divergence test should be done if the suggestion is that the series terms may not converge to zero in the limit at a glance.

P – Series: convergent if $p > 1$ and divergent if $p \leq 1$.

Comparison test: only works if a_n and b_n are always positive. If $a_n \leq b_n$ for all n and if $\sum b_n$is convergent, then $\sum a_n$ is convergent. If $a_n \geq b_n$ for all n and if $\sum b_n$is divergent, then $\sum a_n$ is divergent.

Limit Comparison Test: only works if a_n and b_n are always positive. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ where c is finite, then $\sum a_n$ and $\sum b_n$ either both converge or diverge.

Integral Test: if $f(n) = a_n$ for all n and $f(x)$ is continuous, positive and decreasing on $[1, \infty]$, then if $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. Furthermore, if $\int_1^\infty f(x)dx$ diverges then $\sum_{n=1}^{\infty} a_n$is divergent.

Alternating Series test: if $b_{n+1} \leq b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ is convergent.

6.9 Integral Test: question 3c (i), $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Question 3c (i) required students' knowledge of identifying the situation when the integral test is applicable. This question addressed the Object level in the preliminary genetic decomposition. It sorts to provide highlights as to whether the students had developed an Object conception of the determination of series convergence through the application of the integral test. Objects are encapsulated Processes (Dubinsky et al., 2005a). That is, the individual became aware of the process as a totality and realizes that transformations can act on that totality, and such transformations can be constructed in the mind. Encapsulation empowers individuals to give detailed explanations on conditions of series test and how convergence of series is determined.

6.9.1 Responses of students according to APOS levels

Table 6.6: Frequency of students' responses for question 3c (i) according to APOS level.

APOS level	N	Object
Number of responses	10	20

The responses to item 3c (i) were on the extremes, with students either giving the correct response or totally failing to give the correct response. Twenty (67%) students' responses showed that they operated at the Object level of the APOS theory. Ten (33%) students' responses showed that they could not attain even the Action level of the APOS theory. Two students attempted the question but gave totally confused responses.

6.9.2 Responses at N level (no response)

One third of the students did not attempt question 3c (i). This is possibly due to the fact that there are five convergence/divergence tests taught in this module. These students, it seems, could not identify the integral test as the ideal one for this case. ST 13 did not attempt question 3c (i) which is an indication that he operated at the N level. The student could have faced trouble with limit and series notation (Earl, 2017). Specifically, the student did not appear to think of the limit of a series as a limit of partial sums. The researcher interviewed him to check his understanding of the concepts under discussion and the probable difficulties he faced. The interview excerpts that follows is the dialogue the researcher had with him.

R: You did not attempt question 3c (i). What challenges did you face in dealing with this question?

ST 13: There are many methods of solving series limit questions for example, the integral test, ratio test, comparison series test, root test, alternating series test etc. I got confused as to which method was applicable to this particular question.

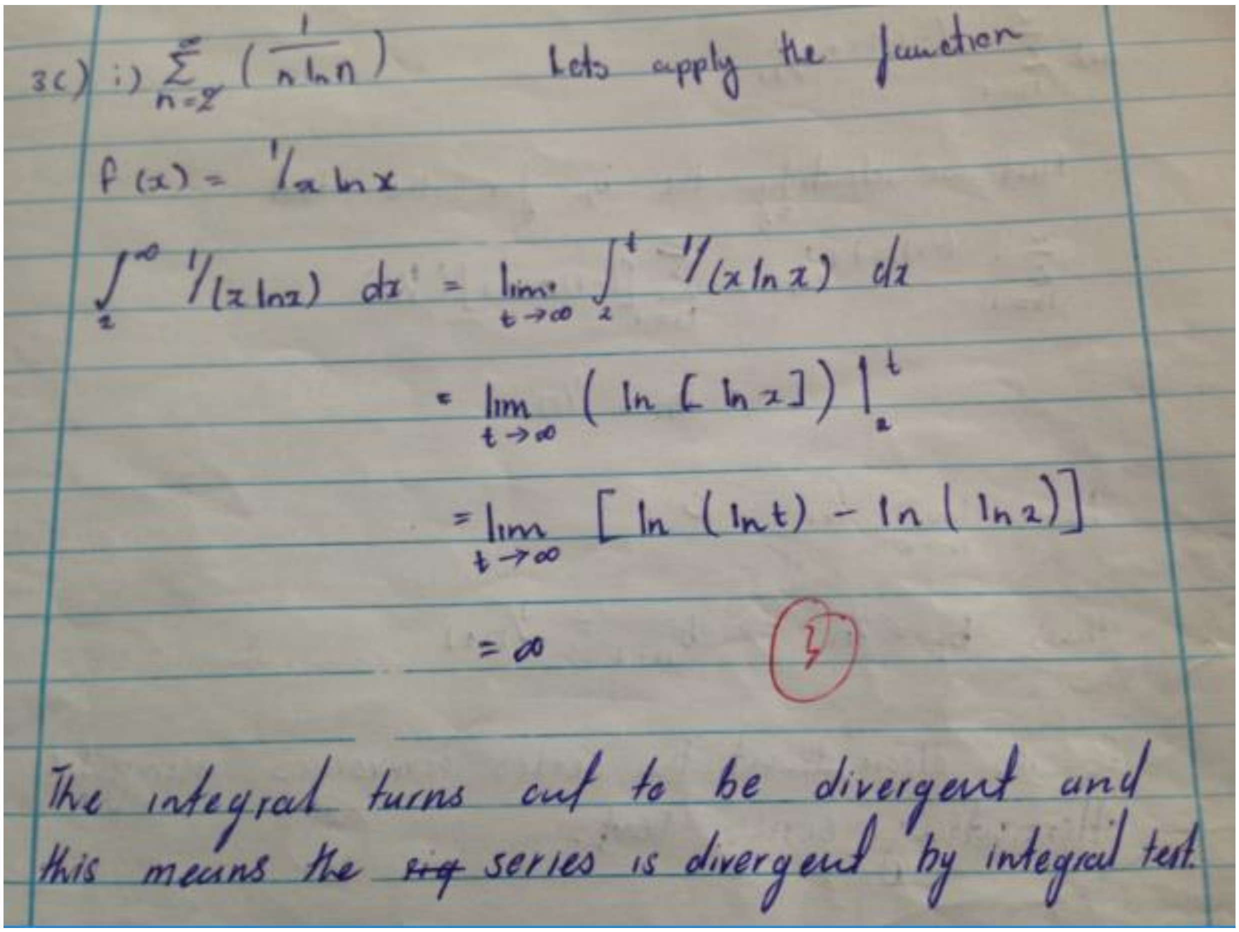
R: Do you understand the criteria for determining which method to use and when to use it?

ST 13: Exactly this where I have a challenges. I need more time to work on such problems.

In the excerpt, ST 13 gave short and precise answer on why he did not attempt the question. The response during the interview was indicative of the student not having understood the techniques of determining the convergence or divergence of series. The student failed to make the necessary mental constructions needed to determine convergence or divergence of a series. Hence, ST 13 did not manage to operate at the Object level of the APOS theory as expected in the preliminary genetic decomposition. He operated at the N level.

6.9.3 Responses at an Object level

Figure 6.9 shows the response of a student who managed to apply the Integral Test to determine whether the series diverges. Such a response reveals the student's ability to make an accurate choice basing on the structure of the series. That revealed that the student operated at an Object level.



3c) i) $\sum_{n=2}^{\infty} \left(\frac{1}{n \ln n}\right)$ Lets apply the function

$f(x) = 1/x \ln x$

$\int_2^\infty 1/(x \ln x)\, dx = \lim_{t\to\infty} \int_2^t 1/(x \ln x)\, dx$

$= \lim_{t\to\infty} (\ln[\ln x]) \Big|_2^t$

$= \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 2)]$

$= \infty$

5

The integral turns out to be divergent and this means the ~~sig~~ series is divergent by integral test.

Figure 6.9: Written response of ST 12 to question 3c (i)

ST 12 managed to give the correct response to question 3c (i) and showed the ability to correctly select the most appropriate test for convergence which he successfully carried out. These abilities are indicators of the Object level conception of the APOS theory. ST 12 checked on the assumptions of a series test, and found that they were satisfied before using this test (Earl, 2017). The student managed to identify the integral test, then successfully demonstrated how the test is carried out, leading to the correct determination that the given series diverges. The researcher's analysis of ST 12's written response revealed that the student operated at the Object level of the APOS theory. To further authenticate the researcher's claim that she operated at the Object level, the researcher carried out an interview with her on the response strategies she gave above. Her responses to the interview are captured in the following interview excerpt:

R: Given $\sum_{n=2}^{\infty} \frac{1}{n\, In(n)}$, how do you determine whether the series diverge or converge.

ST 12: In this case $f(n)$ can be replaced by $f(x)$ and the result is a non-increasing continuous positive function on a given interval. If x is made large, the denominator gets larger and so the function also decreases. Thus, we apply the integral test. (The student performs the integral test correctly with explanations where necessary).

R: What conditions are necessary for the application of the integral test to be carried out?

ST 12: When a function is decreasing, continuous and positive, then the integral function converges if, and only if, the series $f(n)$converges.

R: Can you give situation(s) where the integral test is not applicable.

ST 12: The integral test does not work when series terms are increasing and negative, but when the series is decreasing and positive.

R: Is it possible for one to determine the value of a convergent series using the integral test?

ST 12: My understanding of this series test is that it only determines whether the series convergence or diverges, but does not give the value of the series because we will have replaced a sequence by a function.

ST 12, in responding during the interview, was able to give the conditions necessary to use the integral test. She was also able to perform the integral test correctly. This student indicated that she was operational at Objects level of the APOS theory, in line with the preliminary genetic decomposition. The only aspect lacking was that ST 12 did not check the conditions for the function $f(x) = \frac{1}{xlnx}$ to satisfy the hypothesis of the integral test.

6.10 Comparison Test: question 3c (ii), $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{n^2+2}}{\mathbf{n^4+5}}$

Question 3c (ii) was designed to explore students' understanding of the comparison test to determine the convergence of a series. Its aim was to offer insight as to whether students had developed the series comparison test conception as indicated in the preliminary genetic decomposition. This research study offered researcher a chance to pursue his interested in finding out if students could determine whether or not a given series converges. There are fairly extensive ways of series convergence tests giving various criteria for when a series will converge or diverge.

6.10.1 Responses of students according to APOS levels

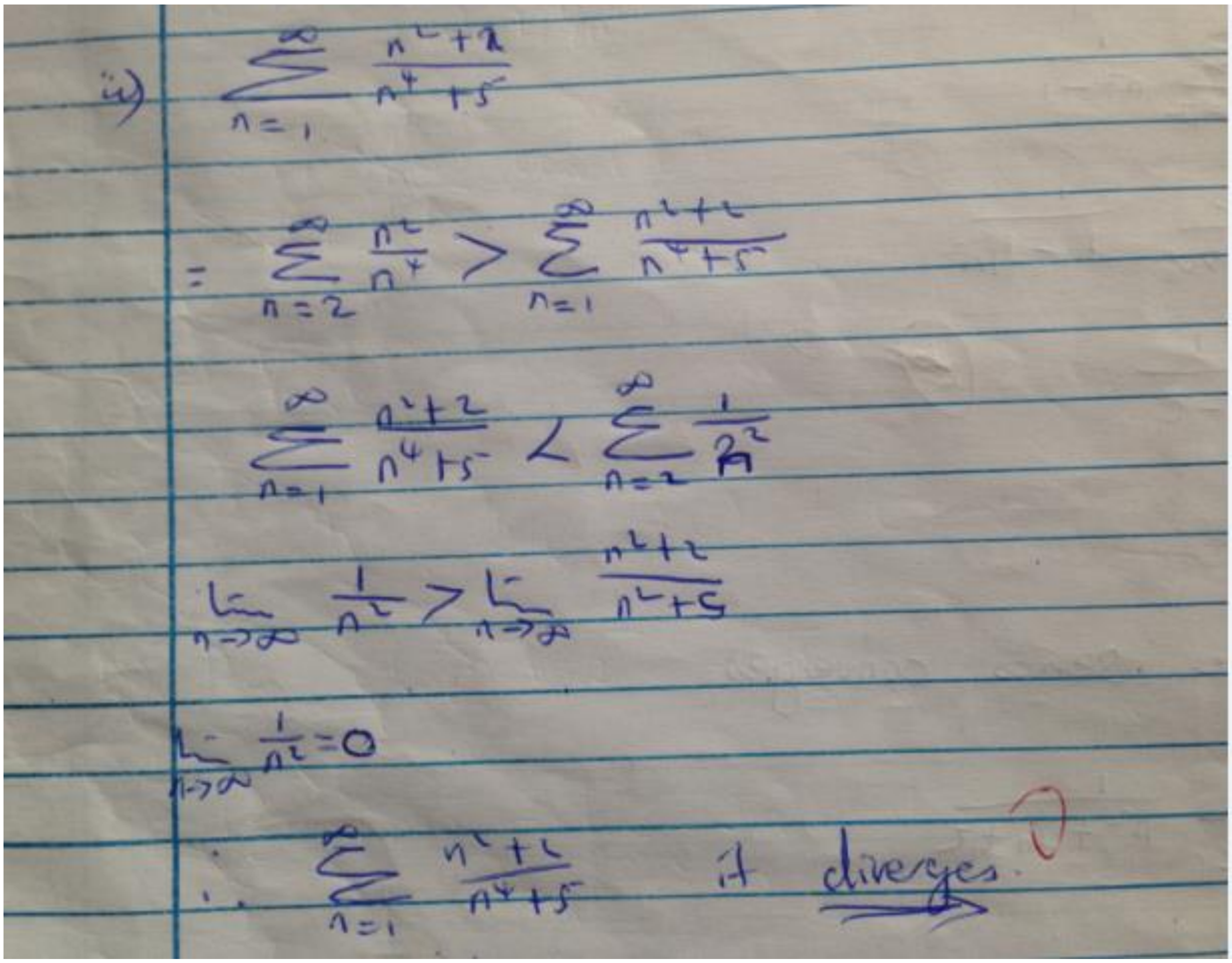
Table 6.7: Frequency of students' responses for question 3c (ii) according to APOS level.

APOS level	N	Object
Number of responses	10	20

The table above indicates that twenty students (67%) of the students gave responses that showed they operated at the Object level of the APOS theory. They managed to coordinate the conditions of the alternating series test and came up with the required solution. This an indication that the students had attained conceptual level as indicated in the preliminary genetic decomposition. Ten (33%) students operated below the Action level of the APOS theory, with four of them not attempting the question at all, while six gave totally confused responses. They displayed their working as though they were evaluating sequences, as they divided the given expression by the highest power of n in the denominator.

6.10.2 Responses at N level (totally incorrect response)

Figure 6.10 shows the response of a student who operated at the N level. Such a student showed that they had not developed an Object level of convergence of a series determination.



ii) $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$

$= \sum_{n=2}^{\infty} \frac{n^2}{n^4} > \sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$

$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5} < \sum_{n=2}^{\infty} \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{1}{n^2} > \lim_{n\to\infty} \frac{n^2+2}{n^2+5}$

$\lim_{n\to\infty} \frac{1}{n^2} = 0$

$\therefore \sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$ it diverges.

Figure 6.10: Written response of ST 17 to question 3c (ii)

The response given by ST 17 to question 3c (ii) is a case where the student tried without success, to apply the comparison test. She failed to come up with the $\sum a_n$ and $\sum b_n$, and to carry out the procedure of the comparison test. The student showed the inability to successfully carry out the comparison test. The student failed to check if the assumptions of the series were satisfied before using the test (Earls, 2019). The student failed to test if the assumptions of the direct comparison test were satisfied. The working displayed showed that she tried to carry it out from memory but failed. ST 17 operated at N level, this was revealed after the researcher had analyzed the student's written response. To confirm the accuracy of the researcher's analysis and the student's understanding of these concepts, the researcher interviewed her. The interview excerpt that follows is a discussion between the researcher and ST 17.

R: Can you tell me anything you know about the comparison test for the determination of the limit of a series?

ST 17: I understand that given two series, say $\sum a_n$ and $\sum b_n$ with $a_n \leq b_n$, then if the bigger is convergent then the smaller is also convergent; and if the smaller is divergent, then also the bigger one is divergent.

R: That is true. Can you explain to me how you came up with your response?

ST 17: I tried to come up with two series and failed to do so. Hence the response I provided.

R: In your written response, why did you write $\lim_{n\to\infty} \frac{1}{n^2} = 0$?

ST 17: I wanted to, um I cannot give the reason for my response.

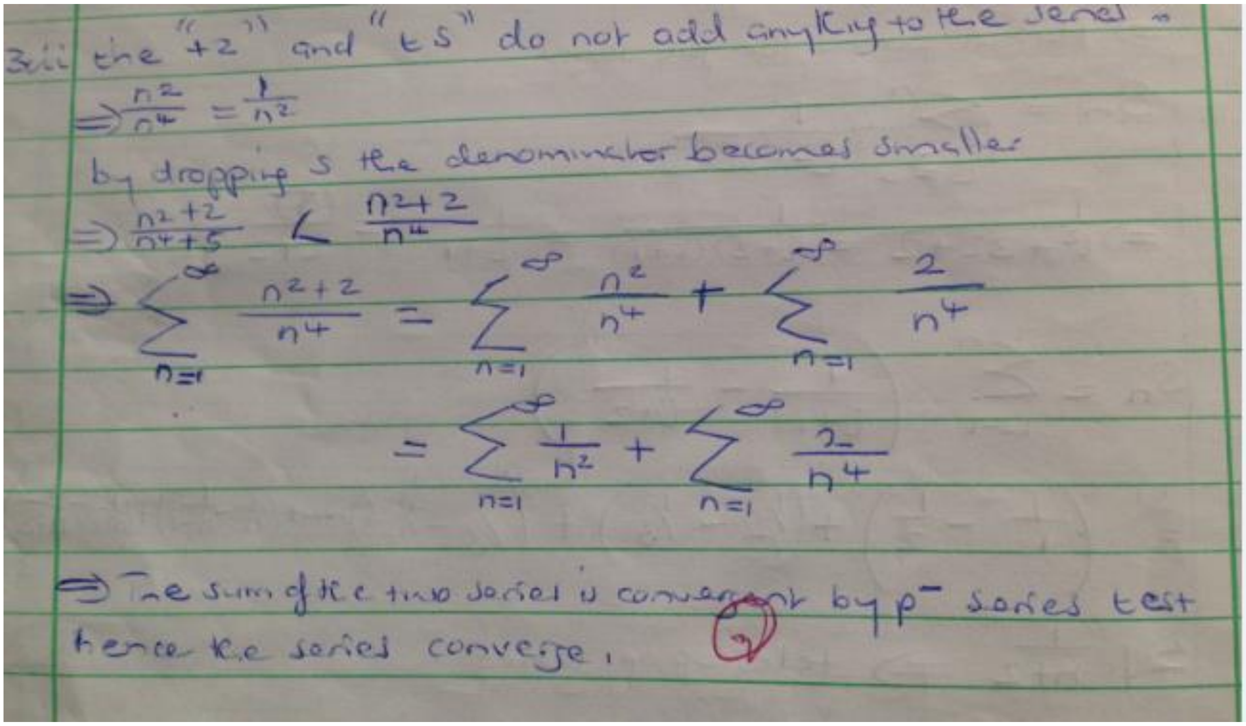
R: Then how did you come up with the conclusion that the series diverges?

(the student did not respond)

An analysis of the interview with ST 17 revealed that she operated at the N level. She failed to answer some of the questions, which showed her inability to make the obligatory mental constructions needed to deal with such questions.

6.10.3 Responses at an Object level

Figure 6.11 is an example of a response of a student who operated at an Object level of the APOS theory. Such students showed the ability to demonstrate strong understanding of the determination of series convergence by p- series test.


3cii the "+2" and "+5" do not add anything to the series ⇒

$\Rightarrow \frac{n^2}{n^4} = \frac{1}{n^2}$

by dropping 5 the denominator becomes smaller

$\Rightarrow \frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2+2}{n^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4} + \sum_{n=1}^{\infty} \frac{2}{n^4}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^4}$$

⇒ The sum of the two series is convergent by p- series test hence the series converge.

Figure 6.11: Written response of ST 28 to question 3c (ii)

ST 28 managed to give the correct response to question 3c (ii) and showed the ability to correctly select the most appropriate test for convergence, which he successfully carried out. These abilities are signs of the Object level conception of the APOS theory. The individual was able to treat that concept as a cognitive Object which incorporate the ability carry Actions on that Object, and give interpretation or reason about its properties (Suryadi, 2012). The student managed to identify that comparison test then successfully demonstrated how the test is carried out, culminating in the correct determination that the given series was convergent. The researcher's analysis of the written response to question 3c (ii) reveled that ST 28 operated at the Object level of the APOS theory. However, in order indorse the student's knowledge of such concepts, the researcher's claim that she operated at the Object level, the interview was conducted revealing an excerpt that follows.

R: What comment, if any, can you give about the partial sums of the infinite series?

ST 28: with the series terms getting to zero in the limit, the limit might converge; but if series terms fail go to zero in the limit, then the series diverges.

R: Can you explain how you came up with the correct response to question 3c (ii).

ST 28: I made use of the terms in the infinite series, infinite process, the sum of infinite series and the links between them. From the problem $\sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$, I dropped 2 from the numerator and 5 from the denominator since they really did not add anything to the series at the infinity. This results in $\frac{n^2}{n^4}=\frac{1}{n^2}$ which is convergent as a series. I then guessed that the original series will converge and I needed to find a larger series which converges. If we drop 5 from the denominator then $\frac{n^2+2}{n^4+5}<\frac{n^2+2}{n^4}$. $\sum_{n=1}^{\infty}\frac{n^2+2}{n^4}=\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{2}{n^4}$. These two series converge by the p-series test. Therefore, their sum also converges. We use the comparison test if series terms are greater than the terms of the original series, and we come to know that the original series converges. Thus, the sum of positive real numbers is a real number.

An analysis of the dialogue with ST 28 reveals a clear explanation about the solution method. The interview with ST 28 revealed that the student had mastered the limit concept and its application to the conception of series convergence or divergence determination. This student was able to make the conceptual structures so-called for by the original genetic decomposition, hence he operated at the Object level.

6.11 Alternating series Test: question 3c (iii), $\sum_{\mathbf{n=1}}^{\infty}\frac{(-\mathbf{1})^{\mathbf{n+1}}}{\mathbf{n}}$

Question 3c (iii) was meant to probe students' knowledge about the determination of series convergence by the application of the alternating series test. This question was intended to access students' conceptual knowledge on methods used to determine series convergence. Furthermore, it was hoped that the question would address the Object conception as specified in the preliminary genetic decomposition. Its intention was to offer students' views on their attainment of an Objects conception of alternating series test for convergence of a series.

6.11.1 Responses of students according to APOS levels

Table 6.8: Frequency of students' responses for question 3c (iii) according to APOS level.

APOS level	N	Object
Number of responses	10	20

Table 6.7 indicates that ten (33%) of the students failed to attain at least the Action level conception applying the alternating series test. Among the students who could not attain the Action level, only three attempted the question. Those who attempted the question showed very confused working, with ST 22 trying to add the resulting sequence, but failing. The other two replaced n by infinity. Twenty (67%) gave responses indicative of operating at the Object level of the application of alternating series test.

6.11.2 Responses at N level (no response)

Again one third of the students did not attempt this question. This could be due to the fact there are five convergence/divergence tests taught in this module. These students, it seemed, could not identify the alternating series test as the ideal one for this case. ST 14 did not attempt question 3c (iii) which was an indicator that the student operated at N level. ST 14's case agrees with Wilson and Dubinsky (2013) who placed learners at pre-function N conception of a function if they indicated no or little conception about a function concept. The researcher interviewed her to ascertain the difficulties she faced and how she understood these concepts. The interview excerpt that follows is a dialogue the researcher had with her.

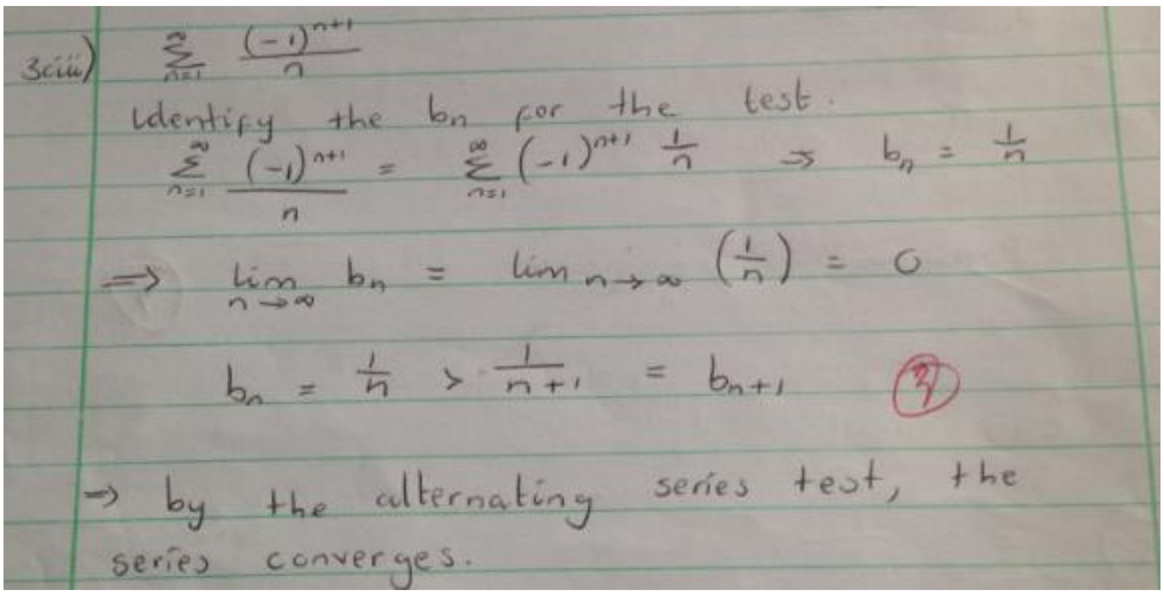
R: You did not attempt question 3c (iii). What challenges did you face?

ST 14: Umm, I found the question too difficult to answer. I need more time to work on such type of questions on the determination of convergence of series.

The excerpt above indicated that ST 14 was not prepared to answer such questions. The researcher's analysis of her responses to question 3c (iii) was also supported by her interview response. She had not made the mental constructions necessary for answering such questions, and operated at N level.

6.11.3 Responses at an Object level

Figure 6.12 is an example of a student who operated at an Object level. Such students showed that they had understood the methods used to determine series convergence.



3c(iii) $\sum_{n=1} \frac{(-1)^{n+1}}{n}$

Identify the b_n for the test.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \Rightarrow b_n = \frac{1}{n}$

$\Rightarrow \lim_{n \to \infty} b_n = \lim_{n \to \infty} \left(\frac{1}{n}\right) = 0$

$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$ 3

$\Rightarrow$ by the alternating series test, the series converges.

Figure 6.12: Written response of ST 1 to question 3c (iii)

In the researcher's analysis of ST 1's written response, the student showed the ability to choose the appropriate convergence criteria based on the mathematical structure of question 3c (iii). The student managed to identify that the alternating series test was the appropriate one, then correctly carried out the solution procedure, and gave a befitting conclusion. A student is said have achieved Object conception from a mathematical concept, if the ability to treat that concept as a cognitive Object, ability to carry Actions on that Object, and give interpretation or reason about its properties (Suryadi, 2012). ST 1's response revealed that he operated at the Object level of the APOS theory. To verify his understanding of these procedures and confirm his operational level, the researcher interviewed him and the following interview excerpt reveals his understanding of limits of series related concepts in this question.

R: You gave the correct response to question 3c (iii). Take me through your response.

ST 1: This is an alternating series, so we have to carry out an alternative series test. The test is applicable if $\sum_{n=0}^{\infty}(-1)^n a_n$is an alternating series with $a_n > a_{n+1} > 0$ for all n and $\lim_{n\to\infty} a_n = 0$, then this series converges. We set $a_n = (-1)^{n+1}b_n$ where $b_n \geq 0$ for values all n. If $\lim_{n\to\infty} b_n = 0$, and $\{b_n\}$ is a decreasing sequence, then series $\sum a_n$ is convergent. This question give $b_n = \frac{1}{n}$. Now I need to run through the two conditions required for the test. That is $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$, and $b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$. The two conditions are met, hence by the alternating series test, the series must converge.

R: Can you explain how an infinite series (sequence of partial sums) has its limit as a real number?

ST 1: If the infinite series converges, then its limit is a real number. One has to view an infinite process as one (a total and complete process) which we can arrive at through the use of a formula or convergence tests.

ST 1 managed to identify an alternating series from the given expression. She was also able to give the required conditions and also identified b_n which is necessary for the determination of convergence or divergence of a given series. This confirmed that ST 1 operated at the Objects level as indicated in the preliminary genetic decomposition.

6.12 Summary of Findings on Limits of series

For the limits of series there was one test question aimed at the Action Level. Students' responses matched some past research studies. Aydin and Mutlu (2013) who used the ACE learning cycle found out that many students developed suitable mental arrangements at the Action level. Likewise, Maharaj (2014) employed the ACE learning cycle and found out that most of the students operated at the Action level. In this study 19 out of 30 students managed to attain the Action level of defining the limit of an infinite series.

In this study and the past studies, the ACE learning cycle were employed just like in the case of the limits of sequences. The results of this study for the three test questions 3b (i-iii) on the limit of series revealed respectively that 25, 20 and 19 students operated at the Process level. The past studies by Ndlovu and Brijlall (2015) revealed concurrence between mental constructions of pre-service teachers and the (PGD) with the majority of the students operating at Process level. Furthermore, Ndlovu and Brijlall (2016) found out that operational level of most students was Process level. Also Makonye (2017) carried out a research study and the Process levels marked the end of progress for the majority of the students. Lastly Kazunga and Bansilal (2017) detected that the understanding of many students was at the Process level.

The students' response to questions that 3c (i-iii) revealed that 20 students operated at the Object level in each case. The number of students who operated the Object level in this study was more as compared to those of past researches. The researchers (Aydin & Mutlu, 2013; Maharaj, 2014; Ndlovu & Brijlall, 2016) found out that many of their students failed to develop suitable mental arrangements at the Object level. It seems that the Object level achievement by many students was indebted to the use of Maple during the activity stages of the ACE learning cycle.

6.13 Conclusion

Students were prompted through guiding questions to collect sets of linked components of the limit concept in relation to series to examine their: knowledge structures, internal representations, how these were coordinated and the mental models they formed. The interviews conducted with the different students on selected sequence test items to clarify some responses presented on the written instrument, helped the students in constructing and reconstructing mental objects. This chapter gave explanations of how students in the first year calculus 1 course constructed concepts of limits of sequences. Written test extracts served to explore the conceptual understanding of these concepts using APOS and the stages on which mental constructs on the limit concept in relation to series were made with regards to the Triad mechanism. Chapter 7 concludes the study by summarizing the findings and emerging themes.

7 SUMMARY OF FINDINGS, CONCLUSIONS AND RECOMMENDATIONS.

7.1 Introduction

In this chapter a summary of the research findings, an abridged discussion of the findings in relation to the literature review and recommendations are given. The research limitations of this study are briefly discussed. Recommendations based on the findings are made, as well as recommendations for future research. This chapter relates the results of the study back to the four sub-research questions stated in the introduction, frames the results in terms of APOS levels of understanding described in the theoretical perspective in chapter three, explains how the results contribute to the existing literature summarized in the literature review, describes changes that need to be made to the preliminary genetic decomposition, and discusses the implications for further research on student understanding of sequences and series. The researcher was able to detect some strengths and weaknesses in students' appreciation of the limit concept in relation to sequences and series. These were detected from students' written and interview responses analysis which showed that most of the students' responses lacked indicators of the schema level.

7.2 Summary of findings by research sub-questions

What is the undergraduate students' depth of understanding (according APOS levels) of the limit concept in relation to sequences and series at university level? For the reader's convenience, the four research sub-questions were:

1. What APOS levels are displayed by students when solving limit problems in relation to sequences and series?
2. How do the APOS levels displayed by students when solving limit problems in relation to sequences and series relate to the preliminary genetic decomposition?
3. How does the historical understanding of limits compare with the students' mental constructions?
4. How can the mental constructions displayed by students be used to improve the understanding of limits of sequences and series at university level?

7.2.1 Research sub-question 1. What APOS levels are displayed by students when solving limit problems in relation to sequences and series?

The results found in this study with regard to students who engaged in dynamic reasoning agreed with the findings by Kabael (2014). The results showed that Kabael's dynamic participants had the dynamic limit images. The students provided a descriptive definition including the dynamic limit notion. Additionally, they reflected on the dynamic limit notion and the feeling of motion in their expressions and arguments. These were also supported by the findings of Steven (2015) who found that students who displayed dynamic reasoning (Process conception) gave good justifications and meaningful interpretations of their answers. On the other hand, results from students who engaged less with dynamic reasoning, revealed that they struggled more and made less coherent interpretations of their answers.

In this study, some students who failed to operate at Action level of stating the correct definition of the limit of a sequence, for example the students ST13 and ST 17, managed to evaluate the limit of some given sequences. It appeared that such students did not believe in the importance of knowing the formal definition of limits to understand a basic course in calculus (Juter, 2006a). They had an intuitive conception that was generally sufficient for them to evaluate sequences at the Action level. ST 1 operated at Schema level, she elaborated on the formal definition during the interview by sketching the graph and explaining the $\varepsilon - N$ relationship by actually referring to the $\varepsilon - N$ terminology in the correct context. Her explanation made the $\varepsilon - N$ limit definition concept applicable and not a collection of words or symbols that were memorized.

Algebra was more problematic than expected. Students failed in several cases of straightforward problem solving due to insufficiently developed algebraic skills, these were the likes of ST 20. Such students operated at the Pre-Action level. Manipulations with symbols in some cases, were done in a manner that indicated that some students the like of ST15, were unable to understand what the symbols represented or what the effects of the operations should be. ST 15's work showed that she "plugged in" infinity to evaluate a limit at infinity, meaning that the student thought evaluating a limit at infinity is the same as "plugging in" infinity. Furthermore, specifically, it appeared the student thought of infinity as a number.

The students, ST 17 and ST 20 who could not connect limit-based concepts back to a formal definition of limit failed to have a deep conceptual definition of limit to utilize in the proving of limits of given sequences. These findings are consistent with several other studies on limit understanding, such as those by Williams (1991) and Roh (2010), which indicated that students had difficulty adopting and using a formal concept of limit. Deep understanding of an advanced mathematical concept entails possessing a formal and operable definition of the concept. In this study less than half of the students struggled to solve limit problems that required the use of the formal definition of the limit of a sequence. The researcher noted that the formal concept definition of limit had a role in limit-based problem solving.

There were notable patterns in the students' thinking on limit. Students who possessed a formal and operable limit definition were successful in solving limit-based calculus problems (Przenioslo, 2006). The students ST 1, ST 12, ST 15 and ST 28 who possessed an operable concept definition of the limit of a sequence could prove limits of given sequences. Those students operated at Process, Object and schema levels, in line with the demands of the given test questions. Thus, Roh (2010) stated that without understanding the essential components of a formal limit definition, students will struggle with understanding limit-based mathematical concepts. Those students did not struggle with questions that required them to prove limits of given sequences which showed that they had understood the essential components of a formal limit definition.

Students were exposed to several series tests they could use to determine divergence, for example the nth term test for divergence, a comparison test, the ratio test, the root test, or the integral test. The students could also determine that a series diverges by showing that the limit of the partial sums does not exist. Though, in practice this rarely happens because in many cases it is difficult to determine a formula for the partial sums. The students who had problems with the determination of whether a series converges or diverges were (33%). For example, conditional series convergence calls for the alternating series test, absolute series converges or if all the terms in the series are positive to begin with, then the comparison test (direct or limit), the ratio test, the root test, or the integral test. Some students failed to check for positivity when using a comparison, root, or ratio tests. Furthermore, those students did not check that the function was non-increasing and continuous when they used the integral test. The lack of a suitable schema deal with selection of an appropriate test to determine whether a series converges or diverges prevented those students from performing up to the expected APOS level in line with the PGD.

In this study more than half (83%) of the students showed that they could not come up with the formula for finding the sum of n terms for question 3b(i). The students who failed to recognize that the use of partial fractions was a very important and necessary technique to solve the given problems questions 3b(ii)-(iii) were (33%), hence, such students operated at (P) level. The failure by some students to recognize the use of partial fractions in evaluation of series supports Radford's (1997) contention that limited background information of other mathematical concepts creates complications with the learning of linear algebra. The failure by some students to give the correct responses was a result of difficulties generated by poor understanding and use of partial fractions. Those students failed to attain the Process level hypothesized by the PGD. The results of this study concurred with Earls (2017) who found that students had difficulty selecting which series test to use when investigating series convergence, why the assumptions of such tests are important, and what conclusions could be drawn. The teaching implication here is that a GD for selection of series tests could be useful to improve the teaching of this concept.

Some students like ST 13 computed the limit of partial sums to show convergence of the telescoping series. Some students, the likes of ST 20 could not distinguish the methods used to evaluate sequences from methods used to determine convergence of series. Furthermore, students like ST 13 showed confusion of series and limit of series. Juter and Grevholm, (2006) noted that students confused functions with limits of functions, which made them give wrong answers to tasks or correct answers with wrong explanations often caused by misconceptions of infinity or the limit definition. Intuitive representations of the concepts, that is concepts not developed enough, in the embodied world (Tall, 2004) caused some of the misconceptions. However, in this study many students operated at Action level and correctly stated the definition of the limit of an infinite series or explained what it essentially meant, which implied that they solved tasks using the fundamental theory behind them.

In APOS context, the findings of this study for question 3c (i-ii) concurred with Ndlovu and Brijlall (2015) who found that the majority of the pre-service teachers operated at Action and Process levels and a few operated at Objects level and none at Schema level. Furthermore, Makonye (2017) carried out a research study to determine pre-service mathematics students' conception of effective interest and nominal rates. The Action and Process levels marked the end of progress for the majority, with only a few reaching the Object and Schema levels. However, in this study more than half of the students managed to operate at the Object level.

The overall responses provided by students showed a pattern of attained APOS levels depending on the type of test question asked. The test question on formal definition of the limit of a sequence and test questions on simple evaluation of sequences had high numbers of students who performed at the Action level. Still at the Action level, results of stating the limit definition of an infinite series showed that nearly half of the students provided incorrect responses. The high incorrect responses could have been a result of not having activities carried out using Maple to support the stating of the limit definition of an infinite series.

Students managed to provide high number of correct responses to test questions on evaluation of sequences at Process level. The correct responses to test questions on determination of convergence of a series were lower compared to those on sequence evaluation. The researcher detected that the failure by students to give correct responses to questions on determination of convergence of series could have resulted from their poor understanding of high school series and application of partial fractions. Those students' Process level mental constructions were not sufficient to deal with questions that required the determination of convergence of a series?

There was a very high number of incorrect responses provided by students at Object level on proofs of given sequence limits. The high incorrect response rate was attributed to students' poor understanding of the relationships among L, n, N and ε and how to link these to proving limits of given sequences. However, there were few students who managed to reach the schema level on the same test question. Test questions at Object level on series tests were correctly responded to by two-thirds of the students. The students who provided the correct responses to questions 3c (i)-(iii) were 67% and 33% of the students who provided incorrect responses in each case. Those students who failed to provide correct responses did not possess the mental constructions necessary for the determination of convergence of series using series tests. However, informal arguments on proving series test could assist students realize the conclusions of series tests. Furthermore, this might help students circumvent thinking that the value of a series test tells them what the series converges to.

There were clear patterns of students' understanding of limits of sequences and series. ST 13, ST 14, ST 17and ST 20 continually operated at Pre-Action level. Those students expressed basic little appropriate knowledge on convergence of sequences and series at the university level. The researcher's reflections on the teaching design indicated that more time needed to be devoted to helping students develop the mental structures at the Process, Object and Schema levels. An analysis using APOS Theory, showed that they had done imperfection "Action', so then they could not progress to perform interiorization of "Action' into "Process". Mental mechanism for proof-structure and conceptual understanding for proving limits of sequences arises when constructing proof through interiorization and coordination, together with encapsulation. Consequently, they required help to learn important lessons that can evoke encapsulation. Therefore, the students needed assistance to refine encapsulation process.

ST 15 at one time failed to attain the Action level of sequence evaluation. This was an indication the student may not have given enough time to develop the required mental constructions needed to deal with the variety of methods used to evaluate limits of sequences, especially the indeterminate forms. However, ST 15's understanding of other areas concerning limits of sequences and series were compatible with the PGD. The teaching implication here is that a PGD should have covered aspects on indeterminate forms to improve the teaching of this section.

Finally, it can be concluded that students ST 1, ST 12, ST 15 and ST 28 displayed all the APOS levels presented in the preliminary genetic decomposition. Furthermore, students who at some time operated at the (P) level failed at each level to attain the level prescribed by the preliminary genetic decomposition.

7.2.2 Research sub-question 2: How do the APOS levels displayed by students when solving limit problems in relation to sequences and series relate to the preliminary genetic decompositions?

There were differences between students who performed in line with the preliminary genetic decomposition and those who could not attain the expected levels of the preliminary genetic decomposition. A formal thinker learning style student, attempts to base his/her work on definitions (Pinto & Tall, 2001). The authors contend that formal thinkers are companionable with the APOS Theory. Concept definition is the starting point for formal learners. Concept images are built by concentrating on rules and processes, and by reflectively routinizing them. This is in line with what was suggested by the preliminary genetic decomposition in this study. Cottrill et al. (1996) consider the limit concept a dynamic concept as a conceptual procedure in APOS terms, and not as solitary process, but fairly as coordinated range and domain process of the function under deliberation. That is what then forms a Schema. A strong dynamic view of sequence limit is indispensable for the formal limit conception understanding. It promotes students' development towards formal understanding.

At the Action level question 1(a) (see appendix 2) required students to state the $\varepsilon - N$ definition of the limit of a sequence of which seven students operated at (P) level and twenty-three students operated at Action level as expected by the PGD. Questions 1b (i) and 1b (ii) tested students' ability to evaluate simple sequences, nearly all students were able to operate at Action level in line with the expectation of the PGD. Question 3(a) required students to define the limit of an infinite series, eleven out of thirty students failed to provide the correct definition thereby failing to reach the level expected by the PGD. The responses of interviewed students revealed that ST 13, ST 15 and ST 20 operated at (P) level which was not in the preliminary genetic decomposition, while ST 1 reached the Schema level. Those findings do not concur with Ndlovu and Brijlall (2015) whose findings revealed concurrence between mental constructions of pre-service teachers and the preliminary genetic decompositions.

The responses of students to test questions at the Process level of the preliminary genetic decomposition revealed that for question 1b (iii) four students gave responses at (P) level and twenty-six gave responses at Process level. It was almost the same results for questions 1b (iv) and 1b (v), six and three students operated at (P) level, and twenty-four and twenty-seven operated at Process level respectively. Most students operated in the Process stage of APOS since they could imagine transformations and perform actions without external stimuli as projected by the PGD. Responses for question 3b (i) had five students, 3b (ii) had ten students and 3b(iii) had eleven students who operated at (P) level, an indication that they had failed to reach the expected Process level in the preliminary genetic decomposition. Thus, most students' responses indicated an assortment of cognitive objects, and their connections and internal processes for using the procedure of determining the convergence of infinite series. When the students' responses were analyzed, it was evident that they could manipulate infinite series to determine convergence. Similarly, findings of Afgani et al. (2017) revealed that the mental constructions made by students on the determination and evaluation of limits of functions in most cases concurred with the preliminary genetic decompositions, and that many students operated at an Action/Process stages.

At the Object and Schema level, questions 2a (i) to 2(b) gave students the opportunity to apply the $\varepsilon - N$ definition of the limit of a sequence to prove limits of given sequences. The proving of given sequence limits indicated that for test item 2a (i) eight students operated at Object level in line with the PGD. Those findings agreed with Ndlovu and Brijlall (2015) that the mental constructions made by pre-service teachers in most cases concurred with the PGD, that is few operated at the Object stage. Most students in this study, only managed to give the first step and failed to proceed; an indication that they had rote learned proving of limit of a given sequence.

The researcher acknowledged the statement by Roh (2010) who stated that without understanding the essential components of a formal limit definition, students were bound to struggle with understanding limit-based mathematical concepts. Thus students who possess a rigorous and operable concept definition of limits were successful in solving limit-based calculus problems (Przenioslo, 2004). In this study, students who worked from memory failed to give a detailed account of steps and explanations taken to successfully arrive at the correct response at an Object level. For test question 2a (ii) thirteen students operated at Object level and three students operated at Schema level. Some of the students' difficulties could be attributed to the structure of the question, that involved a composite sequence with a numerator and a denominator equated to one. The students' difficulties could also be attributed to their difficulties with prerequisite knowledge of algebraic manipulations and the modulus function. The responses to question 2a (iii) revealed that fifteen students operated at Object level and three students operated at Schema level. Lastly for question 2 (b) twenty students operated at Object level and three students operated at Schema level in line with the preliminary genetic decomposition. For strong students, limits can be both a value (a static end state) and as a dynamic process (Zollman et al., 2014). Thus students in this research who operated at the Object and Schema level have exhibited traits of strong students. This research's results were in agreement with Borji1 and Voskoglou (2017) whose study results showed that constructed genetic decomposition was found to be compatible with student data. The conclusions drawn enabled the researchers to make minor revisions to the initial design of PGD.

The questions 3c (i-iii) were designed to test students' ability to apply convergence or divergence series tests on given series problems. The responses revealed that the (same) twenty students (67%) operated at an Object level as expected by the PGD. Those were students who shoed ability to apply the appropriate series test to determine convergence. Interviewed students showed that they possessed the relevant mental constructions to operate at Object level for testing the convergence of given series.

In conclusion, the constructed PGD was nearly adequate and compatible with students ‘responses to written test and interview data. These drawn conclusions enabled the researcher to make minor improvements to the initial design of the PGD.

7.2.3. Research sub-question 3. How does the historical understanding of limits compare with the students' mental constructions?

In this research, first year calculus university students' understanding of the concept of limits were compared to the historical development of the concept. The intention was to find out if the students understood the notion as mathematicians of the past did, as understandings of the concept evolved. The results showed that there were some similarities, for example, the struggle with rigor and attainability. Knowledge of such critical areas can be used to improve the students' opportunities of learning limits of sequence. The students' developments have been compared to the historical development of limits (Juter, 2006a) with the aim to find developmental similarities and differences. The historical development was used as a tool to measure the students' understanding and not something to endeavor for in current classrooms. The outcome of this comparison was that students who performed in line with the PGD were more inclined to follow the historical development than students who could not attain the expected levels of the preliminary genetic decomposition. The researcher argues that this notable fact was due to their abilities to perform abstraction, which is in line with the historical development of the limit concept.

The supposition that the dynamic idea of limit is a crucial condition for accepting a formal limit conception is backed by the fact that it does not deter students' improvement towards formal understanding (Cottrill et al., 1996). The study under consideration strongly supports this assumption. The results obtained in this study revealed that students who appreciated the formal limit concept in relation to sequences and series, had sturdy Process idea of limit or robust dynamic limit view. To find the students' understanding of limits in comparison with historical understanding of limits, the researcher asked the question: How can we determine the limit of a sequence a_n at a point a_0? The question was followed by the prompts: Is a_0 necessary to find the limit of a sequence? Evaluate the limit of a sequence at a_n.

In this research, some students' mental constructions revealed parallels with historical understanding of limits. The conception of infinitesimals has a historical prominence and Leibniz's conception of infinitesimals is well known and documented (Mancosu, 1996). Leibniz acknowledges that infinitesimal quantities really exist and they can be ignored in cases of finite things, but they do not change or disappear in themselves (are static in nature). A way of thinking about infinitesimals was shown in the interview with ST 15. That student tried to avoid the issue of infinity completely. ST 12 showed a highly developed and internally consistent idea of infinitesimal. Thus, some of the interviewed students supported Leibniz's conception. ST 15 and ST 12 gave the correct responses to the questions, but showed the conceptualization of a limit in terms of a dynamic process. This supported the idea that an infinite process can never be completed, which is Aristotle's potential infinity (Moore, 1999). Thus, the unattainable infinite is a form of potential infinity that cannot be understood as a totality (Dubinsky et al, 2005). The interviewed students supported this, since to them n approaches a, $f(n)$ approaches L raised dynamic sensation of motion. In the formal conception, intervals are dealt with, in which n and $f(n)$ values are fixed. This dynamical component prevented students from accepting and understanding that a series or a sequence can attain its limit value. Tall's (1980b) research study supports the observation that the notion of a function is often understood in a dynamic sense as a process that never ends, of getting closer to the limit value and not that actual value.

Interviews with students ST 13 and ST 28 revealed that they supported the ideas of actual infinity. Bolzano in Moore (1999) pointed out that we can use our minds to consider of an endless gathering as being completely devoid of partaking to think of separate components individually. The actual infinity involved the understanding of an infinite process as a totality (Bolzano) and the encapsulation of this totality to a cognitive Object, Cantor in Moore (1999), that is the actual infinity, the attainable form of the infinite. The researcher concluded that historical analysis of the development of the limit concept suggested parallels between historical conceptions and the conceptions held by students who participated in this research study. Some students' conceptions supported Leibniz's infinitesimal. This was predicted from the views displayed by students during interviews. Other students' responses supported Aristotle's conceptions of potential infinity.

Furthermore, some students' responses supported Bolzano and Cantor's conception of actual infinity as revealed by interviews held with the students.

This research's findings showed that there were some similarities between students' interview responses and the historical development of limits. Some of the students struggled with rigor and attainability. The students who performed in line with the PGD were more inclined to the historical development of limits than students who could not attain the expected levels of the PGD.

7.2.4 Research sub-question 4. In what way can the mental constructions displayed by students be used to improve the understanding of limits of sequences and series at university level?

ST 12 and other students had difficulties in stating the epsilon – N definition of the limit of a sequence, and instead, gave the limit of a function definition, which was wrongly stated in most cases. Those students had not understood how to state the definition by linking the relationships among L, n, N and ε. Twenty-three percent (23%) of the students' responses indicated that they struggled to state the definition as required by the PGD. Students need to be exposed to more work on formal definition of sequence.

There were students who showed lack of understanding indeterminate forms. They treated infinity as a constant or as a number which can be added or subtracted. Thus, they subtracted infinity from infinity (an indeterminate form). ST 15 and other students revealed such tendencies in their responses. Thirty percent (30%) of the students exhibited responses that manifested difficulties with the indeterminate form. These students operated at (P) level as they could not provide the correct response. Students need to be exposed to more examples and exercises on the indeterminate forms.

Some students showed that they were comfortable in dealing with sequence evaluation. These included ST 12, ST 1and ST 28. They showed a good understanding of how different sequence structures can be evaluated.

On questions that required proving of limit of sequences, some students had difficulties with inequalities and the modulus function. ST 17 is one of such students, her written and interview responses showed that she struggled with the rules of inequalities and properties of the modulus function. ST 17 had challenges with the interpretation of the modulus notation after she found a negative value inside modulus sign. Students need to have an understanding of rules of inequalities. For example: If $a > b$ where a and b are natural numbers, then $\frac{1}{a} < \frac{1}{b}$, and modulus function properties $|n| = n$ and $|-n| = n$, where $n \in \mathbb{N}$.

There were some students who completely failed to prove the limits of given sequences. ST 14 was one such student. It was difficult to establish from her written response, what she was doing. Students' written responses revealed that they had confusion when it came to the application of the limit of sequence proofs. Most of them got the first step correct but could not proceed. Others provided confused responses. Such students require more time of studying the limit of a sequence definition concept. Furthermore, a learning environment that connects the graphical and numerical representation of the definition of the formal limit of a sequence concept. This could be done by first exploiting such learning activities as the ε-strip (Roh, 2010) or a δ-ε table then $\varepsilon - N$ strips to reinforce the development of the definition concept. the definition concept would then be used to prove limits of given sequences.

There were at least sixteen students who correctly proved the given limit proofs. ST 1, ST 12, ST 15 and ST 28 showed that they were comfortable handling proofs of limits of sequences. It was noted that question 2 (b) had the highest percentage of students who operated at the expected Object level by the PGD. Furthermore, it was noted that nearly in all the questions 2a (i) - 2(b) more than half of the students in each case operated at the expected Object level according to the PGD. The teaching implication here is that a GD for proofs of limits of sequences could be useful to improve the teaching of this section.

Some students found it difficult to define the limit of an infinite series. ST 20 was one such student. He gave a confused definition of the limit of an infinite series. His written response was indicative of one thinking about the limit of a function. For the students who showed difficulties in defining the limit of an infinite series, 37% could not provide the correct definition. Such students need to be exposed to formal representation of the formal limit of infinite series definition. Furthermore, there could be need to look for traces of formal thinking that could occur in graphical, numeric, algebraic, or oral descriptions. Those traces need to be directly connected to the formal infinite series definition so that the students understand the concept of this definition.

Students were challenged in situations when obscure formula were involved in determination of the convergence of series that involved partial fractions and telescoping series. The finding of a suitable formula for the general term in the sequence of partial sums was generally difficult for most students. To come up with the formula for the general term, understanding of advanced level series concepts were required. This involved understanding the sum of an arithmetic progression and partial fractions. Students like ST 13 and ST 17 showed lack of understanding of those concepts. Partial fractions are an integral part in finding the formula for the general term for telescoping series where every term, except the first and last term, cancels out. Questions 3b (i), 3b (ii) and 3b (ii) involved the use of partial fractions and telescoping series which made it difficult for most students to successfully provide the correct responses. From the analysis of those results, there could be need for the revision of arithmetic progression and other concepts related to series covered at advanced level. It was also observed that there could be need to employ you tube video clips on the teaching of telescoping series. Such practice could improve the learning telescoping series as students would be accorded opportunities to reflect and replay the videos to reinforce the learning process.

Failure to choose appropriate tests were seen in both test data and interviews. During interviews, ST 20 used the method of evaluating sequences. The students of Earls (2017) study had misconceptions about how to choose and use an appropriate test for a given limit of series problems. This research’s findings concur with the findings of Earls (2017) that students seemed to rely on the tests they found easiest to work with, rather than tests that were best suited to solve the given problem. Students failed to check that the assumptions for certain series tests were satisfied in both the test data and during interviews. In the test data, students failed to show that they checked the positivity of the ratio test and the three conditions in the alternating series test. During interviews, ST 17 failed to check the positivity condition for the direct comparison test. The evidence suggested that students had misconceptions about checking assumptions when using series tests. Furthermore, it appeared students did not understand the importance of the assumptions in series tests. Some students had problems with understanding properties and conditions necessary for the application of particular series Tests. ST 13, ST 14 and ST 17 had challenges with series test in the determination convergence of series. Series fall into one of the several basic types, recognizing these types helps in the decision making for which tests or strategies will be most useful in determining whether a series is convergent or divergent. Further, there is a need for teaching to focus on checking that the pre-requisite conditions are satisfied before a particular test is applied.

The students who had difficulties with choices of series test to determine convergence of series were 33% for each of the questions 3c (i-iii). Having the understanding of which test to use, why to use it, and the conclusions for particular tests is the knowledge a student with a strong understanding of each test should possess. Each test has its own assumptions, and students need to understand which test is best to use in each circumstance.

The conjectured PGD was tested by analyzing students' responses to two research instruments, the written test questions and the interviews. What typically happened was that students gave evidence of doing some expected and unexpected mental constructions and also showed difficulties in the use of some of the conjectured constructions. This led to the refinement of the PGD to better reflect the constructions students actually made. It is hoped that the modified genetic decomposition would better inform the design and class activities to give students' opportunities to make particular constructions to overcome their difficulties.

7.3 Modified genetic decomposition

Many research studies that employ the APOS Theory have been carried out (Brijlall & Maharaj, 2015; Brijlall & Ndlovu, 2015; Borji & Voskoglou, 2016; Kazunga & Bansilal, 2017). The preliminary genetic decompositions in each case were modified in response to the demands of the analysed data. In this research study, the researcher accepted that there were many mental constructions in response to questions given, that linked to the PGDs. However, some of the responses could not be linked using the models given by the PGDs. The analysis of data from students' written work and interviews indicated that additional schema were required for the conceptual structures to be developed. The researcher proposed modified genetic decompositions (MGDs) as a result of the analysis done. This is the contribution that this thesis makes to the knowledge field of mathematics education especially in rarely considered concepts of limits of sequences and series, at the first year university level. The MGDs appear in Tables 7.1 and 7.2.

Table 7.1: Preliminary and modified genetic decompositions for sequences.

Preliminary genetic decomposition	Modified genetic decomposition
	Pre- requisite concepts: (a) Students conceives rules of inequalities as objects upon which actions can be applied when proving limits of sequences. (b) Students comprehend properties of the modulus function as objects upon which actions can be applied when proving limits
Action level: The individual: gives examples and make representations of sequences; make representations and states the definition of the limit of a sequence; makes use of evaluation methods to evaluate of simple sequences e.g. $lim_{n\to\infty} \frac{n^2+3n}{2n^2+1}$. Action level depends on detailed external cues in order to carry out transformation, one step at a time or providing definitions from memory.	1. The individual can use evaluation methods for limits. 2. The individual can evaluate simple sequences involving indeterminate forms. 3. The individual can involve L, n, N and ε to define the limit of a sequence.
Process level: The individual imagines what the limit of sequences will be without carrying out step-by-step procedures. At this level, the individual is able to predict whether sequences converge or diverge by looking at its structure; use properties of sequences to evaluate limits of sequences by showing the ability to coordinate Actions	The Action is interiorized into a Process when the individual can quickly find the result of the limit of a sequence by imagining without carrying out all steps through:

to come up with responses e.g. use the squeeze theorem to evaluate $lim_{n\to\infty}\frac{n!}{n^n}$. Students were expected to have repeated and reflected on an Action, resulting in the interiorization thereof, perform transformations wholly in the mind, predict outcomes	1. Evaluating indeterminate sequences that require the application of the squeeze rule to evaluate limits of sequences. 2. Evaluating indeterminate sequences that require the application of the L Hospital's rule to evaluate limits of sequences.
Object level: Use of epsilon – N definition to prove or disprove given sequence limits by making use of the main components (L, n, N and ε) of the limit definition to prove given sequences. Working with composite sequences e.g. to prove that if a sequence converges, then its limit is unique. Object conception is characterized by viewing a Process as a totality and ability to apply transformations on that totality and encapsulation of a Process into an Object.	The Process is encapsulated into an Object when the individual is able to prove the results of: 1. Limits of sequences by explaining the relationship among L, n, N and ε using the formal definition of the limit of a sequence. 2. The truth and falsity of given limits of sequences by explaining and giving logical reasoning.
Schema level: Use the collection of Actions, Processes, Objects and other Schemas to determine limits of sequences.	The Schema level is attained when the individual show ability to: 1. Present solutions to given problem involving limit of sequences with evidence of the collection of Actions, Processes, Objects and other Schemas in a coherent structure. 2. Determine limit of sequences using graphical and analytical registers jointly.

Table 7.2: Preliminary and modified genetic decompositions for series

Preliminary genetic decomposition	Modified genetic decomposition
Action level: The individual: gives examples and make representations of series; state the definition of the limit of an infinite series. An action is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and hence is seen by the subject as externally driven. Providing definitions from memory.	The individual can define the limit of an infinite series.
Process level: The individual imagines what the limit of series will be without carrying out step-by-step procedures. At this level, the individual is able to predict whether series cconverge or diverge by looking at its	The Action is interiorized into a Process when the individual can quickly find the result of limit of series by imagining without carrying out all steps through:

structure; use properties of series to evaluate limits of series by showing the ability to coordinate Actions to come up with responses e.g. to find the limit of the series $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$. Students were expected to have repeated and reflected on an Action, resulting in the interiorization thereof, perform transformations wholly in the mind, and predict outcomes.	1. Use partial fractions decomposition to determine limits of series. 2. Show the ability to apply the idea of telescoping series to determine limits of series.
Object level:The individual performs evaluation of limits involving convergence Tests e.g. finding the limit of the series $\sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$. If Actions are performed on a Process, the subject must encapsulate it to become a total entity, or an Object. An object conception is achieved, if a student is capable of treating a limit of a series concept as a cognitive object which comprise the ability to do Action on that Object, and give an explanation or reason about its properties. Object to become process as at the time properties of object intended will be used.	The Process is encapsulated into an Object when the individual is able to: 1. Examine if given expressions satisfy the appropriate hypotheses of the required particular convergence tests. 2. Make the correct choice of series test to be used to determine convergence or divergence of given series. 3. Apply conditions and properties of series tests to determine convergence or divergence of given series.
Schema level: Use the collection of Actions, Processes, Objects and other Schemas to determine limits of sequences.	The Schema level is attained when the individual is able to: 1. Show evidence of the collection of Actions, Processes, Objects and other Schemas in a coherent structure; 2. State and use the conditions of Series Tests and use graphical and analytical registers jointly to determine convergence of series.

7.4 Conclusions

This section highlights the conclusions drawn from the review of literature and the students' written responses to the limit test questions and interviews of this study. In summary, this study synthesized unique contributions to the existing body of knowledge on limits of sequences and series. The study has answered the research objective, to explore university students' mental constructions of the limit concept in relation to sequences and series. Conclusions from the literature study as well as students' written responses to the limit test questions and interviews investigation are provided.

7.4.1 Discussion of Findings

The researcher noted that there were findings supporting the literature surveyed. The literature study was utilized to determine parameters and criteria for all the methods of data collection that were utilized for this study. The literature review sought to explore the APOS levels arrived at by some past researchers and to gain insight into MGD and how it was managed. The following represents the conclusions drawn from the review of literature.

The participating students of the current study had been exposed to educational and digital technology, and more specific computers. Nearly all students' had their own laptop computers, were computer literate and were able to use technology as it posed no threat to them. The essential components of cooperative learning as described in the literature by Mayo (2010) were visible during classroom activities. The students coordinated their efforts with those of their fellow group members. The group members combined their skills and worked together on the activities assigned, each group member was aware of what was expected of him/her. The students acknowledged the obligation to complete their shared work. They displayed proper social skills and inter-group relationships, as they resolved conflict when it arose.

In APOS context, the findings of this study concur with Ndlovu and Brijlall (2016) who found that the operational level of most pre-service mathematics teachers to be the Action level and Process level with a few managing to operate at an Object level. In this research, test questions that tested the Action and Process levels of understanding were usually better performed than those requiring higher levels. Many students gave correct responses to test questions on sequence evaluation.

Finding the difference between the limit of a sequence and the sum of a series was difficult for some students. Specifically, students had difficulty understanding the difference between a sequence and a sequence of partial sums. Existing literature noted that students had difficulty understanding definitions when studying sequences (Roh, 2008). The results of this study contribute to these findings by showing that students specifically had difficulty with proving limits of sequences and determination of series test to determine series convergence. More specifically, students did not appear to think of a series as a limit of partial sums. Furthermore, students had misconceptions about what the conclusions of series tests meant to them. There were examples of students who thought that a series test told them what the series converged to, rather than just whether the series converged and this is similar to the findings of Earls (2017). These examples showed that in addition to the fact that students had a difficult time accepting that series tests can be inconclusive, they had a hard time knowing what to do when they got a numerical value from a series test.

The findings of the current were partially consistent with what is already known. Maharaj (2014) found out that the majority of the students were operating at Action level and failed to develop the suitable mental constructions at the Process, Objects and Schema level; suggesting students' failure to produce appropriate mental constructions suitable for the Process, Objects or Schema levels. Further, Ndlovu and Brijlall (2015) discovered that there was concurrence between mental constructions of pre-service teachers and their PGD. They found that the majority of the pre-service teachers operated at Action and Process levels and a few operated at an Object level and none at Schema level. Ndlovu and Brijlall (2016) determined the mental constructions displayed by pre-service mathematics teachers. Their results revealed that the operational level of most pre-service mathematics teachers to be the Action level and Process level with a few managing to operate at an Object level. Kazunga and Bansilal (2017) detected that the results of the written responses showed that many students managed to attain an Action level and Process level mental constructions.

The study under presentation had findings that differ from the literature and students' written responses to the limit test questions and interviews. In this study it was established that more than 50% of the students managed to attain each of the Action, Process and Object levels and very few reached the Schema level.

The written responses of interviewed students revealed that ST 13, ST 15 and ST 20 continually operated at (P) level which was not in the preliminary genetic decomposition, while ST 1 reached the Schema level. The students who operated at the (P) had difficulties with test questions on sequence evaluation and determination of convergence of series. On average four students gave responses at (P) level on evaluation of sequence and eight students on the determination of series convergence.

Makonye (2017) found out that the Action and Process levels marked the end of progress for the majority of the pre-service teachers, with only a few reaching the Object and Schema levels. From the results, that researcher concluded that more time needed to be devoted by teachers on these notions. Furthermore, Kazunga and Bansilal (2017) detected that the results of the written responses for many students managed to move from matrix addition Action level to Process level. Unlike the results found by Makonye (2017), and Kazunga and Bansilal (2017) there were some students who attained Object and Schema level, for questions (see appendix 2) 2a (i) to 2(b). Those gave students the opportunity to apply the $\varepsilon - N$ definition of the limit of a sequence to prove limits of given sequences. The proving of given sequence limits indicated that for test item 2a (i) nineteen operated at (P) level, eight at Object level and three at Schema level in line with the PGD. Question 2a (ii) had fourteen students who operated at (P) level, like in question 2b (i), most students reacted to external cues but failed to provide exact details on which steps to follow to successfully arrive at the correct response. The responses to question 2a (iii) revealed that twelve students operated at (P) level and three students operated at Schema level. Lastly for question 2 (b) seven students operated at (P) level, twenty at Object level and three students operated at Schema level in line with the PGD. With strong students, limits can be both a value (a static end state) and as a dynamic process (Zollman et al., 2014).

The questions 3c (i-iii) were designed to test students' ability to apply convergence or divergence series tests on given series problems. The responses revealed that the same ten students (33%) failed to operate at the Object level as predicted by the PGD. It was discovered that weak students kept the idea of substituting a finite number of values for *x* to determine the limit of a series. Having the knowledge of which series test to use, why use it, and the conclusions of that series test were the things a student with a strong understanding of each series test would be expected to possess. Each series test had its own assumptions and students needed to understand which test is best to use in each given circumstance.

Lecturers have a very important role in teaching mathematics, they should be aware of the importance of using technology (activities with computers) during the learning process. Furthermore, lecturers should be familiar with mathematical software programs and technologies (e.g., Maple) so that they can design teaching activities that improve students' understanding, and also encourage students to use technologies to enhance their learning. Additionally, lecturers should be aware of the diversity of representations. The use of a variety representations (graphical, algebraic, symbolic and etc.) in the learning of limits of sequences and series and transition between these representations will help students to acquire a deeper understanding of the concepts. By understanding the role of representations in teaching mathematics, teachers and lecturers would have an opportunity to design ACE learning cycle that improve students' mathematical understanding of limits of sequences and series.

Learning limits of sequences and series requires skills from many mathematical areas. Students need to understand formal expositions, perform algebraic manipulations, understand the meanings of quantifiers and absolute values, which students found problematic, and link theory to their everyday problem solving. They also need to find motivation and motives to go through the hard work to make the knowledge meaningful in the context of their concept images (Tall, 2004). The participating students written work and interviews indicated that those students who attained the appropriate schema level, had richer conceptions which enabled them to create many high quality links. These became useful in a variety of situations and gave such students a broader and clearer view of the topic on sequences and series.

Past research studies have indicated that in most cases the Schema level was not attained at all, and very few students managed to attain the object level if ever this level was attained. The attainment of the Process and Object levels by nearly two thirds of the students on test questions involving limits of sequences and series can be attributed to the application of practical Maple activities during the ACE learning cycle. Furthermore, the attainment of the Schema level on proving the limits of given sequences can also be attributed to the ACE learning cycle. The ACE learning cycle instructional treatment by Voskoglou (2015) revealed that application of the APOS Theory and ACE learning cycle enhanced significantly the students' understanding of the real numbers in general and in particular the irrational numbers.

7.5 Limitations of the study

Every study has limitations. Study limitations can exist due to constraints on research design or methodology, and these factors may impact the findings of the study. The limitations for this study were those characteristics of design and methodology that impacted and influenced the interpretation of the findings of this research study. They were the restraints on generalizability, and applications to practice of findings that were the result of unanticipated challenges that emerged during the study. These were, two students opted not to participate in the interviews and the limited literature on limits of sequences and series. However, the impact of these challenges was little since this study was qualitative. Acknowledgement of a study's limitations offered the researcher an opportunity to demonstrate critical thought about the research problem, understood the relevant literature published about it, and correctly assessed the methods chosen for studying the problem. The limitations required a critical, overall appraisal and interpretation of their impact on the research carried out. The researcher answered the question: do these problems with methods and validity eventually matter and, if so, to what extent?

1. The research was a case study and the findings of the study was based on a very small sample; only one class from one university. Therefore, the findings cannot be generalized to other contexts. It is hoped that the findings were informative enough to give useful information on what can be expected in the education of the notion of limits relative to sequences and series in university calculus. The sample size was small and was further reduced by two when some students opted not to participate in the interviews.

2. There were few previous studies in the research area of limits of sequences and series, literature review is an important part of any research, for it helps to identify the scope of research works that other researchers have done so far in an area. Those research findings were used as the foundation by the researcher to be built upon to achieve the research objectives. Citing prior research studies formed the basis of this study's literature review and helped lay a foundation for understanding the research problem that was investigated. However, this was little because of the limited literature on limits of sequences and series.

3. Self-reported data is limited by the fact that it rarely can be independently verified, the researcher had to take what the students said in interviews at face value. However, self-reported data can contain potential sources of bias, these were: (1) discriminatory memory [recollection or not recollection of experiences or events that occurred at some point in the past]; and, (2) telescoping [recalling events that occurred at one time as if they occurred at another time]. The researcher used the written test questions and the interviews, and the students verified the transcriptions made.

The limitations outlined above were acknowledged but they did not detract the significance of the findings.

7.6 Recommendations of the study

The following recommendations are based on the results of this research.

1. The use of genetic decomposition of mathematical concepts to facilitate students' understanding through the development of APOS levels**.** The knowledge of students' mental constructions, and mechanisms to construct them is vital for the designing and sequencing of instruction.
2. The integration of technology such as Maple in the learning of limits and other mathematical concepts. Instructional setting in this study made use of laptops (maple software), which the researcher believed helped students to achieve the results obtained.
3. The use of the ACE learning cycle in the teaching of limit of sequences and series in particular, limits in general and other mathematical concepts. The ACE learning cycle helped students to attain some of the development of the mental constructions suggested

by the genetic decomposition. The ACE learning cycle gave students the opportunity to apply what they had learned and to consider related mathematical ideas.

4. Historical analysis of the development of concepts needs to be reflected upon when preparing and designing instruction. This may help the lecturer to foresee the challenges that lie ahead and help students overcome such difficulties.
5. In this study, cooperative learning environment integrated with technology was utilized. After taking instruction, students showed a good understanding of limit concept. lecturers could use similar strategies in designing instruction about other mathematics concepts.
6. The knowledge of students structures and mechanism to construct these structures are important in both sequencing and designing instruction. This study provided genetic decompositions of the limit of a sequence and a series. These genetic decompositions can be used in designing instruction.

7.7 Further research studies themes

The researcher recommends the following themes for further studies.

1. Mind tools, computer-based tools and learning environments function as intellectual partners with the learner in order to engage and facilitate critical thinking and higher order learning. Research using mind tools can be combined with genetic decomposition suggested by this research to investigate students understanding of limit of sequences or limit of series.
2. The study can be conducted at different universities in different countries with larger sample sizes to increase generalizability of results.
3. Further research can be conducted to explore effect of cooperative learning environment integrated with technology on students′ attitudes toward mathematics.
4. The effectiveness of instruction used in this study can be compared with traditional instruction by using control group.
5. Students performed poorly in proving limits of sequences. A similar study can be carried out using ε-strips for the definition and proving the limit of a sequence.
6. Since some students did poorly on series tests for convergence, a study can be carried out focusing on series tests for convergence.

7. Investigating the historical development of mathematical concepts and their contributions to the design of instruction.

REFERENCES

Afgani, M. W., Suryadi, D., & Dahlan, J. A. (2017). Analysis of Undergraduate Students' Mathematical Understanding Ability of the Limit of Function Based on APOS Theory Perspective. *Journal of Physics: Conference Series 85(1)*, 1-7.

Angen, M. (2000). Evaluating interpretive inquiry: Reviewing the validity debate and opening the dialogue. *qualitative Health Research 10(3)*, 378-395.

Arnon, I., Cottrill,J, Dubinsky, E., Oktac, A., Trigueros, S., & Weller, K. (2014). *A Framework for Research and Curriculum Development in Mathematics Education.* New York: Springer.

Asiala, A., Brown, , A., DeVries, D. J., Dubinsky , E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. *Research in Collegiate Mathematics Education II, CBMS Issues in Mathematics Education, 6,* 1-32.

Asiala, M., Cottrill, J., Dubinsky, E., & Schingendorf, K. E. (1997). The Development of Students' Graphical Understanding of the Derivative. *Journal of Mathematical Behaviour, 16(4)*, 399-431.

Asiala, M., Brown, A., Devries, D. J., Dubinsky , E., Mathews, D., & Thomas, K. (2004). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. *Research in Collegiate Mathematics Education 6*, 1-21.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingerndorf, K. (2001). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior 16(4),* 399-431.

Awang, T. S., & Zakaria, E. (2012). Module for learning integral calculus with Maple: lecturers' views. *The Turkish Online Journal of Educational Technology 11(3)*, 234- 245.

Aydin, S., & Mutlu, C. (2013). Students' Understanding of the concept of Limit of a function in Vocational High School Mathematics. *The Online Journal of Science and Technology 3(1)*, 145-152.

Bagni, C. (2005). Historical roots of limit notion, developments of its representation registers and cognitive development. *Canadian Journal of Science, Mayhematics and Technology Education 5(4),* 453-468.

Bansilal., S. (2014). Using an APOS framework to understand Teachers' responses to questions on the normal distribution. *Statistics Education Research Journal 13(2)*, 42-57.

Baron, M. (1987). *The origins of the Infinitesmal Calculus* . New York: Dover.

Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry based classroom. *Journal for Research in Mathematics Education 30*, 417-448.

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology (W. Ways, Trans.).* Dordrecht, The Netherlands: D. Reidel.

Biehler, R. F., & Snowman, J. (1997). *Psychology applied to teaching (8th Ed.).* New York: Houghton Mifflin Company.

Boester, T. (2010). Testing conceptual frameworks of limit: a classroom-based case study. *Conference on Research in Undergraduate Mathematics Education. 13th Annual Meeting; 2010 February 25–28; Raleigh, NC.* Raleigh, NC.: Google Scholar.

Borji, V., & Voskoglou, M. G. (2016). Applying APOS Theory to Study the Student Understanding of Polar Coordinates. *American Journal of Education Research 4(16),* 1149-1156.

Borji1, V., & Voskoglou, G. M. (2017). Designing an ACE Approach for Teaching the Polar Coordinates. *American Journal of Educational Research, 5(3),* 303-309.

Bransford, J., Brown, A., & Cocking, R. (2000). *How people learn:Brain,mind,experience and school.* Washington, DC: National Academiy Press.

Breidenbach, D. E., Dubinsky, E., Hawks, J., & Nicholas, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics 23(3),* 247-285.

Brijlall, D., & Bansilal, S. (2011). Student teachers' engagement with recontextualised materials: a case of numerical approximation. *US-Chine Education Review – education practice, 5(1),* 691-702.

Brijlall, D., & Maharaj, A. (2015). Exploring Pre-service Teachers Mental constructions when solving problems involving infinite sets. *International Journal for Education Sciences 9 (3)*, 273 -281.

Brijlall, D., & Ndlovu, Z. (2013). Exploring high school learners' mental constructions during the solving of optimization. *South African Journal of Education 33(2),* 157-171.

Bryant, A. (2011). *Leading Issues in Business Research Method.* UK: Academic Publishing International Ltd.

Burton, D. M. (2007). *The history of mathematics:An introduction (6th Ed.).* New York: McGraw Hill.

Cappetta, R., & Zollman, A. (2009). Creating a discourse-rich classroom on the concept of limits in calculus: Initiating shifts in discourse to promote reflective aof of Discourse. Charlotte, NCbstraction. In I. L. Knott, *(Ed.) The Role of Mathematics discouse in Producing Leaders* (pp. 17-39). Charlotte, NC: Information Age Publishing.

Cappetta, R., & Zollman, A. (2013). Agents of Change in Promoting Reflective Abstraction: A Quasi-Experimental, Study on Limits in College Calculus. *Journal of Research in Mathematics Education 2(3),* 343-357.

Clark, J., Cordero, F., Conttrill, J., Czrnocha, B., DeVries, D. J., John, D., . . . Vidakovic, D. (1997). Constructing a schema: The case of the chain rule. *Journal of Mathematical Behavior 16(4),* 345-364.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for research in mathematics education*, 258-277.

Cohen, L., Manion, L., & Morrison, K. (2007). *Research Methods in Education(6 th Ed.).* London: Routledge Falmer.

Cohen, L., Manion, L., & Morrison, K. (2011). *Surveys, longitudinal, cross-sectional and trend studies. Research Methods in Education, (7th Ed.).* Abingdon: Routledge.

Cornu, B. (1991). limits. In D. Tall, *(Ed.) Advanced Mathematical Thinking* (pp. 153-166). Dordrecht: Kluwer Academic Publishers.

Cory, B., & Garofalo, J. (2011). Using dynamic sketches to enhance preservice secondary mathematics teachers'understanding of limits of sequences. *Journal for Research in Mathematics Education, 42(1),* 65-97.

Cottril, J., Dubinsky, E., Nchols, D., Schwingendorf, K., Thomas, k., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematics Behavior 15(2),* 167-192.

Creswell, J. W. (2014). *Qualitative Inquiry & Research Design: Choosing among Five Approaches (4th Ed.).* Thousand Oaks, CA: SAGE.

Dawkins, P. (2012). Metaphor as a possible pathway to more formal understanding of the definition of sequence convergence . *Journal of Mathematics Behavior, 31(3),* 331-343.

Denbel, D. (2014). Students' misconceptions of the limit concept in a first Calculus course. *Journal of Education and practice 5(34),* 24-40.

Denzin, N. K., & Lincoln, Y. S. (2011). *The SAGE Handbook of Qualitative Research (5thEd.).* Thousand Oaks, CA: SAGE Publications.

DeVries.D, & Arnon, I. (2004). Solution--What Does It Mean? Helping Linear Algebra Students Develop the Concept While Improving Research Tools. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (pp. 55–62). RUMEC.

DiSessa, A., & Cobb, A. (2004). Ontological innovation and the role of theory in design experiments. *The Journal of the Learning Sciences 13(1)*, 77-103.

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall, *(Ed.) Davanced Mathematical Thinking* (pp. 95-126). Dordecht: Kluwer Academic Publishers.

Dubinsky, E., Weller, K., & Arnon, I. (2013). Pre-service teachers' understanding of the relation between a fraction or integer and its decimal expansion: The Case of 0.999...and 1. *Canadian Journal of Science, Mathematics, and Technology Education, 13(3)*, 5-28.

Dubinsky, E., & Harel, G. (1992). The Nature of the Process Conception of a Function. In Dubinsky, *(Ed.). The concept of function: Aspects of epistemology and pedagogy, MAA Notes* (pp. 85-106). Washington, DC: The Mathematical Association of America.

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist Theory of learning in Undergraduate Mathematics Education Research. In D. Holton, *(Ed.) The teaching and learning of mathematics at University level: An ICMI study* (pp. 273-280). Dordrecht: Kluwer Academic Publishers.

Dubinsky, E., Weller, K., McDonald, M. A., & Brown, A. (2005a). Some historical issues and paradoxes regarding the concept of infinity: An APOS analysis Part1. *Educational Studies in Mathematics 58(3)*, 335-359.

Dudovskiy, J. (2016). *Purposive sampling. [online] Research Methodology.* Retrieved from net/sampling-in-primary-data-collection/purposive-sampling. 4 Apri 2017: http://research-methodology

Durrheim, K. (2004). Research design. In &. K. In M. Terre Blanche, *Research inpractice: Applied methods for the social sciences* (pp. 29-53). Cape Town: University of Cape Town.

Duru, A. (2011). Pre-service teachers' perception about the concept of limit. *Educational Sciences: Theory and Practice 11(3)*, 1710-1715.

Earls, D. (2017). Student's misconceptions of sequences and series in second semester calculus. Doctoral Dissertations. 156. University of New Hampshire, Durham. Retrieved from https://scholars.unh.edu/dissertation/156.

Earls, D., & Demeke, E. (2016). Does it converge? A look at second semester calculus students' struggles determining convergence of series. *Proceeding for the XIX Annual conference on Research in Undergraduate mathematics Education(RUME)* (pp. 704-710). Pittsburgh: PA.

Edwards, C. H. (1979). *The Historical Development of the Calculus.* New York: Springer-Verlag.

Egodawatte, G., & Stoilescu, D. (2015). Grade 11 Students' Interconnected Use of Conceptual Knowledge, Procedural Skills, and Strategic Competence in Algebra: A Mixed Method Study of Error Analysis. *European Journal of Science and Mathematics Education 3(3),* 289-305.

Eisenmann, P. (2009). The concept of a sum of infinite series as a didactic problem. *Acta Didactica Universistatis Comeniane Mathematics*, 1-8.

Ely, R. (2010). Nonstandard student conceptions about infitesimals. *Journal for Research in Mathematics Education 41(2),* 117-146.

Engelbregcht, J., Harding, A., & Potgieter, M. (2010). Undergraduate students' performance and confidence in procedural and conceptual Mathematics. *University of Pretoria*.

Fathima. (2013). Challenges of ICT in teaching learning process. *International Journal of Engineering and Science 2,* 51-54.

Fernadez, E. (2004). The students' take on the epsilon-delta definition The Students' take on the epsilon-delta definition of a limit. *Problems, Resources, and Issues in Mathematics Undergraduate Studies 14(1),* 43-54.

Findell, B. (2006). Learning and Understanding in Abstract Algebra, Phd Thesis , Unpublshed. Department of Mathematics Education, USA, New Hampshire. *University of of New Hamphshire*.

Flores, A., & Park, J. (2016). Students' guided reinvention of definition of limit of a sequence with interactive technology. *Contemporary Issues in Technology and Teacher Education, 116*, 110-126.

Fouche, C., & Delport, C. S. (2002). Introduction to the research process. In I. A. Vos, *(Ed.). Research at grassroots* (pp. 77-92). Pretoria: Van Shaik Publishers.

Frid, S., & Sparrow, L. (2009). "You just have to take a bit of a risk sometimes": Breaking the "Cycle of Tradition" in primary mathematics. *Mathematics Teacher Education and Development 11*, 36-53.

Gillies, R. (2007). *Cooperative learning: Integrating theory and practice.* Thousand Oacks CA: Sage.

Gonzalez-Martin, A., Nardi, E., & Biza, I. (2011). Conceptually driven and visually rich tasks in texts and teaching practice: the case of infinite series. *International Journal of Mathematical Education in Science and Technology 42 (5)*, 565-589.

Gray, D. (2004). *Doing Research in the real world.* London: SAGE Publications.

Grouws, & Cebulla, K. (2000). *Improving Student Achievement in Mathematics.* Educational Practices Series--4.

Gulcer, B. (2012). Examining the discourse on the limit concept in a beginning-level calculus clssrrom. *Educational Studies in Mathematics, 82(3)*, 439-453.

Hardy, N. (2009). Students' perceptions of institutional practices: the case of limits of functions in college level calculus courses. *Educational Studies in Mathematics 72(3)*, 341-358.

Haripersad, R. (2011). Deep and surface learning of elementary calculus concepts in a blended learning envoronment. *International Journal of Mathematics and Compters in Simulation 5(4)*, 291-298.

Harries, T., Barmby, P., & Higgin, S. (2007). How can we assess mathematical understanding? In H. L. (Ed.). J.H. Woo, *Proceedings of the 31st Conference of the international Group of the Psychology of Mathematics Education 2*, (pp. 41-48). Seoul, South Korea: PME.

Hartman. (2008). *Infinite Series in Calculus.* Ohio: College Board.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in Context: Epistemic Actions. *Journal for Research in Mathematics Education*, 195-222.

Honebein, P. (1996). *Seven goals for the design of constructivist learning environments.In B. Wilson, Constructivist learning environments(pp.17-24).* New Jersey: Educational Technology Publications.

Idris, N. (2006). *Teaching and Learnng Mathemtics: Making Sense and Developing Cognitive Abilities.* Perk: Utusan Publication Sdn. Bhd.

Idris, N. (2009). Enhancing Students' Understanding in Calculus Through Writting. *International Electronic Journal of Mathematics Education 4(1)*, 36-56.

Jaffar, M., & Dindyal, J. (2011). Language-Related Misconceptions in the study of Limits. *Mathematics: Traditions and [New] Practice*, 390-397.

Johnson, D. W., & Johnson , R. T. (2015). Theoretical approaches to cooperative learning. In R. Gillies, *(Ed.). Collaborative learning: developments in research and practice* (pp. 17-4). New York: Nova.

Jojo, Z. M., Brijlall, D., & Maharaj, A. (2011). The reliability of a research instrument used to measure mental constructs in the learning of the chain rule in calculus. *Proceedings of the 16th Annual Congress of the Association of Mathematics Education of South Africa* (pp. 33-349). Johannesburg: AMESA.

Joseph, G. G. (2000). *The Crest of the Peacock: Non-European Roots of Mathematics.* Priceton, NJ: Princeton University Press.

Juter, K. (2009). Limits of functions as they developed through time and as students learn them today. *Mathematical Thinking and Learning 8(4)*, 407-431.

Kabael, T. (2014). Students' formalising process of the limit concept . *Australian Senior Mathematics Journal, 28 (2)* , 23-38.

Katz, V., Dorier, J., Bekken , O., & Sierpinska , A. (2000). The role of historical analysis in predicting and interpreting students' difficulties in mathematics. In J. Fauvel, & J. van Maanen, *(Eds.) History in mathematics education. The ICMI Study* (pp. 149-154). Dordrecht /Boston/London: Kluwer.

Kazunga, C., & Bansilal, S. (2017). Zimbabwean in-service mathematics teachers' understanding of matrix operations. *Journal of Mathematical Behavior 47*, 81-95.

Kleiner, I. (2001). History of the infinitely small and infinitely large in Calculus. *Educational Studies in Mathematics 48(2)*, 137-174.

Kuhn, T. (2012). *The Structure of Scientific Revolutions: 50th Anniversary Edition.* Chicago: University of Chicago Press.

Kvale, D. (1996). *Interviews.* London: SAGE Publications.

Lakatos, I. (1976). *Proof and Reflections.* Cambridge University: Cambridge University Press.

Lakoff, G., & Nuez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry.* Newbury Park, CA: Sage.

Lolli, G. (2010). Infinitesimalls and Infinites in the History of Mathematics: A Brief Survey. *Applied Mathematics and Computation 28(16)*, 7979- 7988.

Mack, N., Woodsong, C., MacQueen, K. M., Guest, G., & Namey. (2005). *Qualitative research methods: a data collectors field guide.* Research Triangle Park, NC: Family Health International.

Maharaj, A. (2010). An APOS analysis of students' understanding of the concept of a limit of a function. *Pythagoras 71*, 41-52.

Maharaj, A. (2014). An APOS Analysis of Natural Science Students' Understanding of Integration. *Journal of Research in Mathematics Education 3(1)*, 54-73.

Makonye, J. (2017). Pre-service mathematics student teachers' conceptions of nominal and effective interest rates. *Pythagoras 38(1)*, 1-23.

Mancosu, P. (1996). *Philosophy of mathematics and mathematical practice in the seventeen century.* New York: Oxford University Press.

Mang, L., & Minjun, C. (2013). Teaching design method and application. *Television Education Research 3*, 16-21.

Maor, E. (1986). *To inifinity and beyond: A cultural history of the infinite.* Chicago: Boston.

Martínez-Planell, R., Gonzalez, A. C., DiCristina, G., & Acevedo, V. (2012). Students' conception of infinite series. *Educational Studies in Mathematics, 81(2)*, 235-249.

Mayo, J. A. (2010). *Constructing undergraduate psychology curricula: Promoting authentic learning and assessment in the teaching of psychology.* Washington, D.C.: American Psychological Association.

McCombs, P. (2014). *Analysis of second semester calculus students' understanding of series and series convergence.* Northern Illinois University: DeKalb, IL.

McGinn, M. K., & Boote, D. N. (2003). A first-person perspective on problem solving in a history of mathematics course. *Mathematical Thinking and Learning, 5(1)*, 71-107.

Miller, E. (2011). Technology- Enhanced Calculus Lectures. *Academic exchange quaterly*.

Monaghan, J. (1991). Problem with the language of limits. *Journal for learning Mathematics 11(3)*, 20-24.

Monaghan, J., Sun, S., & Tall, D. (1994). Construction of the limit concept with a computer algebra system. *Proceedings of PME 18(1)*, 279-286.

Moore, A. W. (1999). *The Infinity, (2 nd Ed.).* London: Routledge and Paul.

Morewedge, C., Giblin, C. E., & Norton, M. I. (2014). The (Percieved) Meaning of Spontaneous Thoughts. *Journal of Experimental Psychology: General, 143(4)*, 1742-1754.

Murphy, E. (1997). *Constructivism: From philosophy to practice.* Retrived June 30,2014 from http://www.cdli.ca/~elmurphy/emurphy/cle.html.

Nagle, C. (2013). Transitioning from introductory calculus to formal limit conceptions. *For the Learning of Mathematics, 33(2)*, 2-10.

Nardi, E., & Iannone, P. (2001). on convergence of a series: the unbearable inconclusiveness of the limitComparison test. *Proceedings of the 25th Conference of the International Group of the Psychology of Mathematics Education (PME).M.van denHeuvel-Panhuizen, ed, 3* (pp. 399-406). The Netherlands: PME Utrecht.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston: VA:NCTM.

National Council of Teachers of Mathematics. (2003). *Programs for Initial Preparation of Mathematics Teachers.* Reston: VA: NCTM.

Ndlovu, Z., & Brijlall, D. (2015). Pre-service Teachers' Mental Constructions of Concepts in Matrix Algebra. *African Journal of Research in Mathematics, Science and Technology Education 19(2)*, 156-171.

Ndlovu, Z., & Brijlall, D. (2016). Pre-service Mathematics Teachers' Mental Constructions of the Determinant Concept. *International Journal of Educational Science 14(1),* 145-156.

Norton, A., & Baldwin, M. (2012). Does 0.999... Really Equal 1? *The Mathematics Educator 21(2)*, 58-67.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education 40(4)*, 396-426.

Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *Journal of Mathematics Behavior, 33*, 131-148.

Ojose, B. (2015). Studeents' misconceptions in mathematics: analysis of remedies and what research says. *Ohio Journal of School Mathematics, 72*, 30-34.

Patel, R., McCombs, P., & Zollman, A. (2014). Metaphor clusters: A study characterizing instructor metaphorical reasoning on limit concept in calculus. *School Science and Mathematics 114(5)*, 236-245.

Piaget, J. (1964). Developing and learning. *Journal of Reseach in Science Teaching 2(3)*, 176-180.

Piaget, J. (1972). *The Principles of Genetic Epistemology,(W. Mays trans.).* London: Routledge& Kegan Paul.

Piaget, J. (1980). *Adaption and Intelligence (S. Eames, trans.).* Chicago (original published 1974): University of Chicago press.

Piaget, J. (1985). *The Equilibration of Cognitive Structures(T. Brown and K. J. THampy,trans.).* Cambridge MA (original published 1975: Harvard University Press.

Piaget, J., & Garcia, R. (1983). *Psychogenese et Hiistoire des Sciences.* Paris: Flammarion.

Piaget, J., & Garcia, R. (1989). *Psychogenesis and the history of science (H. Feider,trans.).* New York: Columbia University Press.

Pinto, M. M., & Tall, D. (2001). Following students' development in a traditional university analysis course. *In PME CONFERENCE 4*, 4-57.

Prensky, M. (2001). Digital natives, digital immigrants part 1. *On the horizon 9(5)*, 1-6.

Przenioslo, M. (2006). Conceptions of a sequence formed in secondary schools. *Journal of Mathematical Education in Science and Technology, 37(7)*, 805-823.

Putman, R., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher 29(1)*, 4-15.

Radford, L. (1997). On psychology, historical epistemology, and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the learning of mathematics 17(1)*, 26-33.

Radford, L. (2000). Historical formation and student understanding of mathematics. In &. J. In J. Fauvel, *(Eds.) History in mathematics education. The ICMI Study* (pp. 143-170). Dordrecht /Boston/London: Kluwer.

Roh, K. H. (2008). Students’ images and their understanding of definitions of the limit of a sequence. . *Educational studies in Mathematics, 69(3)*, 217-233.

Roh, K. H. (2010). An empirical study of students' understanding of logical structure in the definition of limit via the ε-strip activity. *Educational Studies in Mathematics, 73(3)*, 263-279.

Ruthven, K., & Hennessy, S. (2002). A practitioner model of the use of computer-based tools and resources to support mathematics teaching and learning. *Educational studies in mathematics 49(1)*, 47-88.

Samkova.L. (2012). Calculus of one and more variables with Maple. *International Jourmal Mathematics Education Science and Technology, 43(2)*, 230–244.

Schibeci, R. C.-P., Durrant, C., Kissane, B., Miller, E., & MacCallum, J. (2008). Teachers' journeys towards critical use of ICT. *Learning, Media and Technology 33(4)*, 313-327.

Seffe, L. P., & Cobb, , P. (1988). *Construction of arithmetic meanings and strategies.* New York: Springer-Verlang.

Shimazoe, J., & Aldrich, H. (2010). Group can be gratifying: Understanding and overcoming resistance to cooperative learning. *College Teaching 58(2)*, 52-57.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics 18(4),* 371-387.

Slavin, R. E. (2014). Cooperative Learning and Academic Achievement: Why Does Groupwork Work? *Anales De Psicología / Annals of Psychology, 30(3),* 785-791.

Sriraman, B., & English, L. (2010). *Theories of Mathematics Education: Seeking new frontiers (Advances in Mathematics Education).* Berlin: Springer Sience.

Steven, J. R. (2015). Calculus limits involving infinity: the role of students' informal dynamic reasoning. *International Journal of Mathematical Education in Science and Technology, 46(1)*, 105-126.

Stewart, J. (2015). *Single variable calculus: Early transcendentals (8th ed).* Cengage learning.

Stillwell, J. (1989). *Mathematics and Its History.* New York: Springer-Verlag.

Suryadi, D. (2012). *Building a New Culture in Mathematical Thinking.* Bandung: Rizqi Press.

Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education. 31(3)*, 258-276.

Taber, K. S. (2011). Constructivism as educational theory: Contingencecy in learning and optimally guided instruction. In K. S. Taber, *Constructivsm as Educational Theory* (pp. 39-61). University of Cambridge: Nova Science Publisher Inc.

Tall, D. (1999). Reflections on APOS Theory in Elementary and Advanced Mathematical Thinking. In O. Zaslavskey, *(Ed). Proceedings of the 23rd conference of PME* (pp. 111-118). Haifa, Israel.

Tall, D. (2011). Crystalline concepts in long-term mathematical invention and discovery. *For the Learning of Mathematics 31(1)*, 3-8.

Tall, D. O. (1980a). Intuitive infinitesimals in the calculus. *In short communications, Fourth International Congress on Mathematical Education*, p. C5.

Tall, D. O., & Schwarzenberger, R. L. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching, 82(41)*, 44-49.

Tall, D., & Vinner, S. (1981). Concept Image and concept definition in Mathematics with particular reference to Limits and Continuity. *Educational Studies in Mathematics 12(2)*, 151-169.

USC Libraries. (2014). Research Design. USC (University of South Carolina) Libraries: Retrieved August 20, 2015 from www.library.sc.edu.

Usiskin, Z. (2012). What does it mean to understand some mathematics? *International Congress on Mathematics Education.* Seoul: COEX, Korea.

Voskoglou, M. G. (2015). Fuzzy Logic in the APOS/ ACE Instructiona Treatment for Mathematics. *American Journal of Educational Research 3(3)*, 330- 339.

Weigand, H.-G. (2004). Sequences- Basic Elements for Discrete Mathematics. *ZDM 36(3)*, 91-97.

Weller, K., Arnon, I., & Dubinsky , E. (2011). Pre-service teachers' understanding of the relation between a fraction or integer and its decimal expansion: Strength and stability of belief. *Canadian Journal of Science, Mathematics, and Technology Education, 11(2)*, 129-159.

Weyer, R. (2010). APOS theory as a conceptualization for understanding mathematical learning. *Summation: Mathematics and Computer Science Scholarship at Ripon 3*, 9-15.

Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education 22(3)*, 219-236.

Wilson, R., & Dubinsky, E. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior, 32*, 83- 101.

Yesilyurt, M. (2010). Meta analysis of the computer assisted studies in science and mathematics: A sample of Turkey. *The Turkish Online Journal of Educational Technology 9(1)*, 123-131.

APPENDCES

Appendix 1: Work covered during Class sessions

Practice questions:

Session 1

Use Maple to evaluate the following sequences and series.

1. $lim_{n\to 0}\frac{1}{n}$.

2. $lim_{n\to 3}\frac{1}{n}$

3. $lim_{n\to 100}\frac{1}{n}$

4. $lim_{n\to\infty}\frac{1}{n}$

5. $lim_{n\to 1}\ln(n^2)$

6. $lim_{n\to 1000}\ln(n^2)$

7. $lim_{n\to\infty}\ln(n^2)$

8. $lim_{n\to 1}(\frac{2n+5}{n+2})$

9. $lim_{n\to 100}\frac{2n+5}{n+2}$

10. $lim_{n\to\infty}\frac{2n+5}{n+2}$

11. $lim_{n\to 100}\frac{n}{n+1}$

12. $lim_{n\to\infty}\frac{n}{n+1}$

13. $lim_{n\to 5}\frac{n!}{n^n}$

14. $lim_{n\to 100}\frac{n!}{n^n}$

15. $lim_{n\to 100}\frac{n!}{n^n}$

16. $lim_{n\to\infty}\frac{n!}{n^n}$

17. $lim_{n\to 10}\frac{1}{3^n}$

18. $lim_{n\to\infty}\frac{1}{3^n}$

19. $lim_{n\to 100}\frac{n-1}{n+1}$

20. $lim_{n\to\infty}\frac{n-1}{n+1}$

21. $\sum_{n=1}^{3}\frac{1}{n}$.

22. $\sum_{n=1}^{1000}\frac{1}{n}$.

23. $\sum_{n=1}^{\infty}\frac{1}{n}$.

24. $\sum_{n=2}^{100}\frac{1}{n^2-1}$.

25. $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$

26. $\sum_{n=0}^{10}\frac{1}{n^2+3n+2}$

27. $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$

28. $\sum_{n=2}^{10} \frac{1}{nln(n)}$.

29. $\sum_{n=2}^{\infty} \frac{1}{nln(n)}$

30. $\sum_{n=1}^{10} \frac{n^2+3n}{2n^2+1}$.

31. $\sum_{n=1}^{\infty} \frac{n^2+3n}{2n^2+1}$

32. $\sum_{n=1}^{4} \frac{(-1)^{n+1}}{n}$

33. $\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

35. $\sum_{n=1}^{\infty} n^2$

36. $\sum_{n=1}^{\infty} \ln(n^2)$

Session 2

Use Maple to draw the graphs of the following:

1. $lim_{n\to 100} \frac{1}{n}$

2. $lim_{n\to 100} n^2$

3. $lim_{n\to 100} \ln(n^2)$

4. $lim_{n\to 100} \frac{2n+5}{n+2}$

5. $lim_{n\to 100} \frac{n}{n+1}$

6. $lim_{n\to 100} \frac{n!}{n^n}$

7. $lim_{n\to 100} \frac{1}{3^n}$

8. $lim_{n\to 100} \frac{n-1}{n+1}$

9. $\sum_{n=1}^{100} \frac{1}{n}$

10. $\sum_{n=2}^{100} \frac{1}{n^2+1}$

11. $\sum_{n=0}^{100} \frac{1}{2n^2+n+2}$

12. $\sum_{n=2}^{100} \frac{1}{n^2\ln(n)}$

13. $\sum_{n=1}^{100} \frac{n^2+3n}{2n^2+1}$

14. $\sum_{n=1}^{100} \frac{(-1)^n}{n}$

15. $\sum_{n=2}^{100} \sqrt{(n}+2)-\sqrt{n}$

Session 3

Use Maple to draw the following graphs:

1. $lim_{n\to 100} \frac{2n+5}{n+2}$

2. $lim_{n\to 100} \frac{n}{n+1}$

3. $lim_{n\to 100} \frac{n!}{n^n}$

4. $\sum_{n=1}^{100} n$

5. $\sum_{n=1}^{\infty} n^2$

6. $\sum_{n=2}^{100} \frac{1}{n^2+1}$

7. Given a sequence $a_n = \frac{n+3}{n^2+5n+6}$, evaluate $\lim_{n\to\infty} \frac{n+3}{n^2+5n+6}$ by:

   (a) dividing with the highest power of n in the denominator,

   (b) factorizing the denominator,

   (c) using the largest power 'wins' logic, and

   (d) L Hospital's rule.

7. Evaluate:

(a) $\lim_{n\to\infty} (ne^{-n})$,

(a) $\lim_{n\to\infty} \frac{\sin(n)}{n}$, and

(b) $\lim_{n\to\infty} \frac{(-1)^n}{n}$.

8. Determine if the following series converges or diverges. If it converges, find its value.

   (a) $\sum_{n=0}^{\infty}(-1)^n$,

   (b) $\sum_{n=1}^{\infty} \frac{1}{(3)^{n-1}}$

   (c) $\sum_{n=0}^{\infty} \frac{4n^2-n^3}{10+2n^3}$

   (d) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

Session 4

Use the definition of the limit of a sequence to prove the following:

1. $\lim_{n\to\infty} \frac{1}{n^2} = 0$
2. $\lim_{n\to\infty} \frac{3n+1}{4n+2} = \frac{3}{4}$

3. $\lim_{n\to\infty} \frac{2n+7}{n-3} = 2$

4. $\lim_{n\to\infty} \frac{2n+5}{n+2} = 2$

5. The limit of a real convergent sequence is unique.

Session 5 and 6

Determine if the following series is convergent or divergent.

1. $\sum_{n=4}^{\infty} \frac{1}{n^7}$

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

3. $\sum_{n=0}^{\infty} \frac{1}{3^n+n}$

4. $\sum_{n=0}^{\infty} \frac{1}{3^n-n}$

5. $\sum_{n=1}^{\infty} \frac{n}{n^2-cos^2(n)}$

6. $\sum_{n=0}^{\infty} \frac{4n^2+n}{\sqrt[3]{(n^7+n^3)}}$

7. $\sum_{n=0}^{\infty} ne^{-n^2}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$

9. $\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$

10. $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$

11. $\sum_{n=0}^{\infty} \frac{n!}{5^n}$

12. $\sum_{n=1}^{\infty} \frac{9^n}{(-2)^{n+1}n}$

13. $\sum_{n=3}^{\infty} \frac{(-12)^n}{n}$

14. $\sum_{n=0}^{\infty} \frac{n^n}{3^{1+2n}}$

Appendix 2: Learners' test questions

Answer all questions showing all possible working.

1. (a) State the ϵ - N (epsilon – N) definition of the limit of a sequence.

   (b) Evaluate the following limits:

   (i) $lim_{n\to\infty} \frac{n^2 + 3n}{2n^2 + 1}$.

   (ii) $lim_{n\to\infty}(\sqrt{n+2} - \sqrt{n})$.

   (iii) Use the L Hospital's rule to evaluate $lim_{n\to\infty} \frac{n}{n+1}$.

   (iv) Use the squeeze theorem to evaluate $lim_{n\to\infty} \frac{n!}{n^n}$.

   (v) Assume the sequence given recursively by $a_1 = 2$, and $a_{n+1} = \frac{1}{2}(a_n + 6)$ converges, find its limit.

2. (a) Use your definition in 1(a) above to prove the following:

   (i) $lim_{n\to\infty} \frac{1}{3^n} = 0$.

   (ii) $lim_{n\to\infty} \frac{n-1}{n+1} = 1$

   (iii) $lim_{n\to\infty} \frac{1}{n} = 0$

   (b) Prove that if a sequence converges, then its limit is unique.

3. (a) Define the limit of an infinite series?

   (b) Determine if the following series are convergent. If they converge find their sums:

   (i) $\sum_{n=1}^{\infty} n$.

   (ii) $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

   (iii) $\sum_{n=0}^{\infty} \frac{1}{n^2 + 3n+2}$.

(c) Determine if the following series are convergent or divergent:

(i) $\sum_{n=2}^{\infty} \frac{1}{n In\, n}$.

(ii) $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$.

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Appendix 3: Interview Schedule

Hello. My name is Chagwiza Conilius and I am here to discuss with you your understanding of the limit concept in relation to sequences and series. I hope my findings will help students and lecturers in the learning and teaching of limits of sequences and series. Your responses to the questions we discuss is completely confidential and your real name will not appear anywhere on the questionnaire.

1. (a) Can you state the formal definition of the limit of a sequence.

(b) By drawing a graph, can you explain what is meant by the definition?

Prompt: relationship between epsilon, n, N and L. Satisfactory explanation for the relationship.

2. Let a_n be a sequence, how do you determine the limit of a sequence a_n at a point a_0?

Prompt: is a_0 necessary to find the limit of a sequence? Evaluate the limit of a sequence at a_n. Sameness by proximity (the student might think values are the same because of infinitely small difference between values leading to the two values being equal). Infinitesimal difference (there is always a difference no matter how small it might be, the nearest one can get to without actually saying the two are the same).

3. Let a_n be a sequence given by $a_n = \frac{n^2 + 3n}{2n^2 + 1}$. How do you determine the limit of this sequence as $n \to \infty$?

Prompt: use of the methods learnt to handle such situations.

4. How do you determine $lim_{n \to \infty} \frac{sin\, n}{n}$?

5. (a) We know that $lim_{n \to 2}\, 5n = 10$. How do you prove this using the formal definition of the limit of a sequence?

(b) Can you explain what you did in part (a) using a graph?

Prompt: relationship between epsilon and delta.

6. Consider the statement $lim_{n \to 2}\ 2n = 5$. Determine truth or falsity of the statement. How do you prove your response?

Prompt: how to prove by using the formal definition of a limit of a sequence and the manipulations involved. Giving reasons for each step taken.

7. Given the following series, how do you determine whether they converge or diverge? In the case of convergent series, find the respective sums:

(i) $\sum_{n=1}^{\infty} n$.

(ii) $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

(iii) $\sum_{n=0}^{\infty} \frac{1}{n^2+\ 3n+2}$.

Prompt: the use of partial fractions and sum to infinite of real numbers. Telescoping series and their characteristics (cancelation of successive terms and those that cancel farther down the list).

8.Given the following series, how do you determine whether they converge or diverge?

(iv) $\sum_{n=2}^{\infty} \frac{1}{nIn\ n}$.

(vi) $\sum_{n=1}^{\infty} \frac{n^2+\ 2}{n^4+\ 5}$.

(vii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Prompt: the use of the tests for convergence or divergences of series, (n th Term Test, Integral Test, Comparison Test, Limit Comparison Test, Ratio Test, Root Test or Alternating Series Test), their conditions and properties.

Appendix 4: Letter of Consent for Learners

UKZN HUMANITIES AND SOCIAL SCIENCES RESEARCH ETHICS COMMITTEE (HSSREC)

APPLICATION FOR ETHICS APPROVAL

For research with human participants

INFORMED CONSENT RESOURCE TEMPLATE

Note to researchers: Notwithstanding the need for scientific and legal accuracy, every effort should be made to produce a consent document that is as linguistically clear and as simple as possible, without omitting important details as outlined below. Certified translated versions will be required once the original version is approved.

There are specific circumstances where witnessed verbal consent might be acceptable, and circumstances where individual informed consent may be waived by HSSREC.

Information Sheet and Consent to Participate in Research

Date:

Greeting: Dear Undergraduate Calculus Student.

My name is Conilius, J. Chagwiza from Bindura University of Science Education department of Mathematics and Physics. I can be contacted at cjchagwiza@gmail.com and on cell number +263772996601. I am a PhD student at the University of KwaZulu-Natal. My research focuses on the learning of sequences and series in relation to limits in University Calculus by learners.

You are being invited to consider participating in a study that involves exploring undergraduate students' understanding of limits in relation to sequences and series. The aim and purpose of this research is to explore how undergraduate calculus students understand limits in relation to sequences and series as they engage in the learning of these concepts. Purposive sampling will be used to recruit participants. The study is expected to select thirty participants from undergraduate mathematics students who have enrolled for the Calculus 1 at Bindura University of Science Education, first semester 2016. It will involve the following procedures:

1. The researcher (Chagwiza Conilius, J.) will observe what students will be doing in the class since computers will be used in the learning of the concepts.

2. The researcher will monitor the classroom discussions then comment where necessary.

3. The researcher will mark the exercises the given feedback to the participants.

4. Questionnaire on limits in relation to sequences and series shall be answered using pseudonyms.

5. Participants may be asked to take part in the open-ended interview based on how they perform on the written questionnaire. The interviews will be carried out in a private room. The duration of your participation, if you choose to enroll and remain in the study, is expected to be about three months. The study is not funded.

The study is expected to cause no discomforts or stress. Also, no risks are expected. Participation or non-participation will not affect your course grades at all. The study will provide no direct benefits to participants. The participants will gain an understanding of the concept and the skill of using computers and some computer software. After the completion of the study, the researcher shall make a presentation of the results to the participants.

This study has been ethically reviewed and approved by the UKZN Humanities and Social Sciences Research Ethics Committee (approval number_____).

In the event of any problems or concerns/questions, you may contact the researcher at cjchagwiza@gmail.com and on cell number +263772996601, or the UKZN Humanities & Social Sciences Research Ethics Committee, contact details as follows:

HUMANITIES & SOCIAL SCIENCES RESEARCH ETHICS ADMINISTRATION

Research Office, Westville Campus

Govan Mbeki Building

Private Bag X 54001
Durban
4000

KwaZulu-Natal, SOUTH AFRICA

Tel: 27 31 2604557- Fax: 27 31 2604609

Email: HSSREC@ukzn.ac.za

Participation in this research is completely voluntary and has no impact or bearing on evaluation or assessment of the participating or non-participating learner in any study or course during the study period at the university. A participant can withdraw his or her consent at any time by advising the researcher of this intention, without any penalty. Under what circumstances will the researcher terminate the participant from the study?

The participant will not incur any costs and there are no incentives attached to participation in the study.

Data reporting will be by pseudonym, on both the written questionnaire and interviews. The identities of the interviewees will be kept strictly confidential. The answers to the questionnaire and interview will only be used for the purpose of this research and will be destroyed after the publication of this research. Participants are free to review and comment on any parts of the dissertation that represents this research before publication.

CONSENT

I ………… have been informed about the study entitled **Exploring University Students' Mental Constructions of the Limit Concept in Relation to Sequences and Series,** by Conilius, J. Chagwiza.

I understand the purpose and procedures of the study.

I have been given an opportunity to answer questions about the study and have had answers to my satisfaction.

I declare that my participation in this study is entirely voluntary and that I may withdraw at any time without affecting any of the benefits that I usually am entitled to.

If I have any further questions/concerns or queries related to the study I understand that I may contact the researcher at cjchagwiza@gmail.com or +263772996601 or **Bindura University of Science Education, P Bag 1020 , Bindura Zimbabwe.** .

If I have any questions or concerns about my rights as a study participant, or if I am concerned about an aspect of the study or the researchers then I may contact:

HUMANITIES & SOCIAL SCIENCES RESEARCH ETHICS ADMINISTRATION

Research Office, Westville Campus

Govan Mbeki Building

Private Bag X 54001
Durban
4000

KwaZulu-Natal, SOUTH AFRICA

Tel: 27 31 2604557 - Fax: 27 31 2604609

Email: HSSREC@ukzn.ac.za

Additional consent, where applicable

I hereby provide consent to:

Audio-record my interview YES / NO

Video-record my interview YES / NO

Use of my photographs for research purposes YES / NO

If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

	Willing	Not willing
Audio equipment		
Photographic equipment		
Video equipment		

________________________ ________________________

Signature of Participant **Date**

________________________ _________________________

Signature of Witness **Date**

(Where applicable)

__________________ _________________________

Signature of Translator **Date**

(Where applicable)

Appendix 5: Permission to carry out research from Bindura University of Science Education.

REGISTRY DEPARTMENT

P Bag 1020
BINDURA, Zimbabwe

Tel: 0271 – 7531-6, 7621-4
Fax: 263 – 271 – 7534

BINDURA UNIVERSITY OF SCIENCE EDUCATION

HUMAN RESOURCES

7 March 2016

Mr Conilius Chagwiza
University of KwaZulu-Natal
Edgewood Campus
Ashwood 3605,
SOUTH AFRICA

Dear Mr Chagwiza

RE: APPLICATION FOR PERMISSION TO CARRY OUT EDUCATIONAL RESEARCH IN THE UNIVERSITY

Permission to carry out Research on:

EXPLORING UNIVERSITY STUDENTS' MENTAL CONSTRUCTIONS OF THE LIMIT CONCEPT IN RELATION TO SEQUENCES AND SERIES

Bindura University of Science Education has granted you the permission on the following conditions.

a) That in carrying out this research you do not disturb the programmes of the Department.
b) That you avail to the University a copy of your research findings.
c) That the permission can be withdrawn at any time by the Registrar or by any higher officer.

I wish you success in your research work and in your University College studies.

Yours Sincerely

Gmanhando

E Manhando (Mrs)
ACTING REGISTRAR

Appendix 6: Editing Certificate

SOL PLAATJE
UNIVERSITY

Dr. J. Sibanda (Senior Lecturer: English)
School of Education
Private Bag X 5008, Kimberley, 8300
North Campus, Chapel Street, Kimberley
E-mail: Jabulani.Sibanda@spu.ac.za
jbsubsd@gmail.com
Website: www.spu.ac.za
Tel: 27534910142
Cell: 0845282087
23 October 2018

TO WHOM IT MAY CONCERN

I hereby confirm that I have proof read and edited the following PhD Thesis using Windows 'Tracking' System to reflect my comments and suggested corrections for the author(s) to action:

- **Full Name**: Chagwiza Conilius Jaison
- **Student Number**: 213574316
- **Title:** Exploring University Students' Mental Constructions of the Limit Concept in Relation to Sequences and Series.
- **Date:** 23 October 2018

Although the greatest care was taken in the editing of this document, the final responsibility for the product rests with the author.

Sincerely

23.10.2018

SIGNATURE | DATE

Appendix 7: Thesis declaration - plagiarism

I, I Chagwiza, Conilius Jaison, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

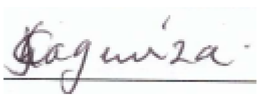
3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

4. This thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:

a. Their words have been re-written but the general information attributed to them has been referenced

b. Where their exact words have been used, then their writing has been placed inside quotation marks, and referenced.

5. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed: 

Appendix 8: Manuscript Forwarded to AJRMSTE for Possible Publication

Inbox x

(African Journal of Research in Mathematics, Science and Technology Education) A revise decision has been made on your submission

AJRMSTE <em@editorialmanager.com>
to me

Nov 27, 2019

Ref.: Ms. No. RMSE-2019-0035R1
University students' mental constructions on the concept limit of a sequence African Journal of Research in Mathematics, Science and Technology Education

Dear Conilius. J Chagwiza,

Reviewers have now commented on your paper. You will see that they are advising that you revise your manuscript. If you are prepared to undertake the work required, I would be pleased to review a revision.

For your guidance, reviewers' comments are appended below.

If you decide to revise the work, please submit a list of changes or a rebuttal against each point which is being raised when you submit the revised manuscript.

Your revision is due by 27 December 2019.

To submit a revision, go to https://www.editorialmanager.com/rmse/ and log in as an Author. You will see a menu item called 'Submission Needing Revision'. You will find your submission record there.

Yours sincerely,

Busisiwe Alant, Ph.D
Editor-in-Chief

APPENDIX 9: TURNITIN REPORT

ORIGINALITY REPORT

13%

SIMILARIT Y INDEX

11%

INT ERNET SOURCES

7%

PUBLICAT IONS

%

ST UDENT PAPERS

PRIMARY SOURCES

etd.aau.edu.et

Internet Source

link.springer.com

Internet Source

etd.lib.metu.edu.tr
Internet Source
repository.up.ac.za
Internet Source
citeseerx.ist.psu.edu
Internet Source
www.csun.edu
Internet Source
www.amesa.org.za
Internet Source

APPENDIX 10: ETHICAL CLEARANCE CERTIFICATE

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

26 October 2016

Mr Conilius Jaison Chagwiza (213574316)
School of Education
Edgewood Campus

Dear Mr Chagwiza,

Protocol reference number: HSS/0953/016D
Project title: Exploring University students' Mental Constructions of the Limit Concept in Relation to Sequences and Series

Full Approval – Expedited Application

In response to your application received on 27 June 2016, the Humanities & Social Sciences Research Ethics Committee **has** considered the abovementioned application and the protocol have been granted **FULL APPROVAL.**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

....
Dr Shamila Naidoo (Deputy Chair)

/ms

Cc Supervisor: Professor Deonarain Brijlall and Dr Aneshkumar Maharaj
Cc Academic Leader Research: Dr SB Khoza
Cc School Administrator: Ms Tyzer Khumalo

Humanities & Social Sciences Research Ethics Committee
Dr Shenuka Singh (Chair)
Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Telephone: +27 (0) 31 260 3587/8350/4557 **Facsimile:** +27 (0) 31 260 4609 **Email:** ximbap@ukzn.ac.za / snymanm@ukzn.ac.za / mohunp@ukzn.ac.za
Website: www.ukzn.ac.za

1910 - 2010
100 YEARS OF ACADEMIC EXCELLENCE

Founding Campuses: Edgewood | Howard College | Medical School | Pietermaritzburg | Westville