Exploring Grade 11 learners' functional understanding of proof in relation to argumentation in selected high schools

by

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Abstract

Research has established that understanding the functions of proof in mathematics and argumentation ability provide learners with a firm foundation for constructing proofs. Yet, little is known about the extent to which learners appreciate the functions of proof and whether an association between functional understanding of proof and argumentation ability exists. Guided by van Hiele's and Toulmin's theories, this study utilised a sequential explanatory design to randomly select three schools from a cluster grouping of ten Dinaledi high schools in the Pinetown district. Three survey questionnaires, *Learners' Functional Understanding of Proof* (LFUP), self-efficacy scale, and *Argumentation Framework for Euclidean Geometry* (AFEG), were administered to a sample of 135 Grade 11 learners to measure their understanding of the functions of proof and argumentation ability, and to explore the relationship between argumentation ability and functional understanding of proof. Then, *Presh N* (pseudonym)—a female learner who obtained the highest LFUP score despite attending a historically under-resourced township school—was purposively selected from the larger sample. In addition to her responses on the questionnaires, a semistructured interview, and a standard proof-related task served as data sources to explain the origins of her functional understanding of proof. Statistical analyses were conducted on data obtained from questionnaires while pattern matching method was used to analyse the interview data. The analyses revealed that learners held hybrid functional understanding of proof, the quality of their argumentation was poor, the relationship between functional understanding of proof and argumentation ability was weak and statistically significant, and the collectivist culture and the teacher were the two factors which largely accounted for *Presh N*'s informed beliefs about the functions of proof. In addition, although she constructed a deductive proof, she did not perform the inductive segment prior to formally proving the proposition. The recommendation that Euclidean geometry curriculum needs to be revamped for the purpose of making functional understanding of proof and argumentation explicit and assessable content has implications for two constituencies. Instructional practices in high schools and methods modules at higher education institutions need to include these exploratory activities (functional understanding of proof and argumentation) prior to engaging in the final step of formal proof construction. Future research initiatives need to blend close-ended items with open-ended questions to enhance insights into learners' functional understanding of proof. This study not only provides high school teachers and researchers with a single, reliable tool to assess functional understanding of proof but also proposes a model for studying factors affecting functional understanding of proof. Overall, the results of this study are offered as a contribution to the field's growing understanding of learners' activities prior to constructing proofs.

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I, **Benjamin Shongwe**, declare that

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Signed

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Dedication

In loving memory of my late father, Alfred, brother, Vusi, sister, Nomalanga, mother- and sister-in-law, Christina and Portia, respectively, I dedicate this thesis to:

- My mother Thembane, and my children for their moral support and contribution to the reasons for studying.
- My wife Precious for her witty suggestions as well as her courage and keeping the fire burning at home. I cannot ask for a better family.

Publications and Presentations Related to this Research

Publications

Shongwe, B. (In press). The relationship between a learner's conceptions of the functions of proof and behaviour in proof: A case study. *African Journal of Research in Mathematics, Science and Technology Education (AJRMSTE)*.

Shongwe, B., & Mudaly, V. (2017). The development and evaluation of an instrument to measure functional understandings of proof. *PONTE, 73*(12), 87-104.

Presentations

Shongwe, B. (2019). The quality of argumentation in a Euclidean geometry context in selected South African high schools: Validation of a research instrument. *27th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education, 15-17 January 2019, University of KwaZulu-Natal, Durban*, (pp. 98-112). (*Long paper*). ISBN: 978-0-9922269-8-5

Shongwe, B. (2018). The relationship between learners' conceptions of the functions of proof and behaviour in proof: Case of a South African high school learner. *Proceedings of the International Conference on Advanced Teaching Instructions and Learning, 29-30 December 2018, Maseru, Lesotho,* (pp. 48-66). *(Long paper).* ISBN: 978-8-1929580-6-0

List of Acronyms

AFEG	Argumentation Framework in Euclidean Proof
ANOVA	Analysis of variance
LUPF	**Learners'** Functional Understanding of Proof
CAPS	Curriculum and Policy Statement
FET	Further Education & Training (Grades 10-12)
UKZN	University of KwaZulu-Natal
KZNDoE	KwaZulu-Natal Department of Education
KZN	KwaZulu-Natal
LTSM	Learning and Teaching Support Materials
SSPS	Statistical Package for the Social Sciences
STATA	Statistics and data
NCTM	National Council of Teachers of Mathematics
TIMSS	Third International Mathematics and Science Study
US	United States
FAE	Fundamental attribution error
PM	Pattern matching
UK	United Kingdom
ms	millisecond

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Chapter 1

Introduction to the study

> To a very large extent, it seems that the absence, presence, or level of an individual's functional understanding determines that individual's motivation to study and learn mathematics. Without functional understanding, mathematics simply degenerates into a useless, meaningless and arbitrary subject, demotivating the learner from attempting to learn and explore it. The adequate development of functional understanding is therefore an important criterion for evaluating any teaching approach. (de Villiers, 1994, p. 11)

1.0 Introduction to the problem

The purpose of this study was to explore Grade 11 learners' functional understanding of proof and how this understanding related to their argumentation (reasoning and sense making[1]) ability with a view to identify the factors contributing to functional understanding of proof. The definition of proof as an argument that one makes to justify a claim and to convince oneself and others of the claim's veracity (Stylianou, Blanton, & Rotou, 2015) underscores the view that understanding the functions of proof in mathematics and argumentation ability provide learners with a firm foundation for constructing proofs (Bieda, 2010; Stylianou et al., 2015). In addition, Mariotti (2001) shows that proof is more "accessible" to learners if an argumentation activity is developed in the construction of a conjecture.

Attempts to teach proof to high school learners (frequently during short periods of time) have been unsuccessful (Clements & Battista, 1992; Hadas, Hershkowitz, & Schwarz, 2000; Pedemonte, 2007). Given that the 'failure to teach proofs seems to be universal' (Hadas, Hershkowitz, & Schwarz, 2000, p. 128), functional understanding of proof and argumentation, activities Edwards (1997) refers to as the "territory before proof", need to be part of the

[1] The NCTM (2009) foregrounds reasoning and sense making in the learning of high school mathematics and broadly defines "reasoning" as involving the drawing of logical conclusions based on evidence or stated assumptions, and "sense making" as the development of understanding of a situation, context, or concept by connecting it with existing knowledge or previous experience (p. 5).

mathematical activities that precede and support the development of proofs. Along this line, Marrades and Gutiérrez (2000) argue that it is vitally important for both teachers and researchers in the area of proof to know learners' conceptions of functions of mathematical proof in order to understand their attempts to solve proof problems.

The construction of proofs has always been regarded as a defining activity within the mathematics discipline (de Villiers & Heideman, 2014; Lockhart, 2002; Watson, 2008). Also, as Conner, Singletary, Smith, Wagner, and Francisco (2014) put it, '[a]rgumentation, as a precursor to proof, is fundamental to the establishment of mathematical knowledge' (p. 403). Yet, inconsistent with the practices of research mathematicians[2], the focus of high school mathematics has often been on form and established results to pass examinations over the activities that are a precursor to the construction of proofs, for example, understanding the functions of proof and argumentation. Perhaps more importantly, unless learners understand the purpose in studying proofs beyond the goal of preparing for the next mathematics class or test, they are likely to ask the age-old question, "Why do we need to learn this?"

The general motivation for this study came from the need to measure learners' understanding of the *functions of proof in mathematics*[3] since lack thereof contributes to difficulties with learning proofs meaningfully (for example, de Villiers, 1990, Healy & Hoyles, 1998). According to the van Hiele (1986) theory, discussed in some detail in Chapter 3, functional understanding of proof is one of the aspects that determine learners' ability to construct a deductive proof. Mathematical proof performs various functions in mathematics including verification, explanation, communication, discovery, systematisation, and intellectual challenge for the author of the proof. Although these functions are enshrined in the South African policy document,

[2] As Beckmann (2011) puts it, by "mathematicians" is meant individuals in mathematics departments at universities who teach mathematics courses and/or have done research in mathematics. She points out that this descriptions should be viewed as an approximate. However, it must be noted that this definition excludes users of mathematics, such as engineers or physicists, interested in mathematical results but not in the way they are obtained.
[3] Since the phrase "functional understanding of proof in mathematics" needed to be frequently used, it is shortened to "functional understanding of proof", "functions of proof in mathematics" or sometimes "functional understanding" for brevity.

Curriculum and Policy Statement (CAPS) (Department of Basic Education [DBE], 2011), learners' knowledge of these functions is not explicitly assessed and thus not measured.

Rather, it is seemingly assumed that doing proof translates into developing understanding about its functions, an assumption that tends to distort the nature of mathematics. The term "knowledge" is used so frequent in this study such that it is justifiable to define it. Brook and Stainton's (2001) definition of knowledge as a mathematical statement held as true by the mathematics community[4] was adopted in this study on the basis that it is plausible, commonly used, and provided by philosophers (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002).

The present study investigated learners' functional understanding of deductive proof through validation of the current version of the *learners' functional understanding of proof* (hereafter referred to as *LFUP*) scale on the first five functions. The relationship between functional understanding and argumentation was also investigated following Conner's (2007) who suggestion that a relationship between these two constructs exists. Hanna et al. (2009) view proof, argumentation, and justification as parts of a continuum rather than as distinct notions. It is for this reason that I partly drew on the literature related to all three concepts. Further, I explain the reasons why the single learner, *Presh N* (pseudonym) held informed beliefs about the functions of proof in mathematics. Beliefs were classified as informed if they were consistent with those held by the mathematics community.

The investigation reported in this study involved three randomly chosen schools that were part of the Dinaledi Project in the province of KwaZulu-Natal's Pinetown district. Data collection for this study took place in September 2017, a time of the year by which Grade 11 learners were supposedly familiar with the functions of proof and could formulate conjectures and subsequently construct proofs on their own. Briefly, in 2001, the Department of Basic Education (DBE)

[4] The term "mathematics community" is used to refer to mathematicians, mathematics educators (individuals who teach mathematics, mathematics methods course, supervise or coordinate mathematics teaching, and conduct research in mathematics education), and mathematics teachers (individuals who teach within preprimary through Grade 12 (Beckmann, 2011).

launched the *National Strategy for Mathematics, Science and Technology Education* in which learners from 102 historically disadvantaged schools across South Africa were selected for improving mathematics, science and technology (MST) performance.

It is my contention that the scarcity of research in the area of functions of proof (de Villiers, 1990) is a consequence of the conflation of proof with its functions in instructional practices. Bartlo (2013) shares this assumption as she points out that the ‘vast majority of research on the topic implicitly assumes that proof promotes the learning of mathematics without any need for elaboration’ (p. 69). Conflation here refers to these two constructs being assumed to mean the same thing and therefore used interchangeably. Further validation the LFUP scale was helpful in making a clear distinction between constructing proofs and appreciating its functions.

Throughout this study, the term “learning” is used in a broad sense to encompass not only cognitive but also affective (attitudes and beliefs) notions of learners’ mathematical experience (Stylianides & Stylianides, 2018). From a cognitive perspective, the term denotes the social process of appreciating the centrality of proof in mathematics and knowing how to make mathematically acceptable claims and justify them rather than to mean providing answers designed to reflect rehearsed application of procedures and algorithms only. This definition is consistent with Vygotsky’s (1978) sociocultural theory of learning that considers learning as involving scaffolding of a learner by “more knowledgeable others” such as a teachers, parents, older siblings or even peers. This learning theory is compatible with the van Hiele theory of geometric understanding which stresses that each level of thought has its own language and its own interpretation of the same term.

Thus, learners need to clarify and reorganise their ideas using language appropriate for that level (Mason, 1998). However, as already mentioned, inseparable from the cognitive aspect of learning is attitude. For Plotnik (1996), attitude is ‘any belief or opinion that includes a positive or negative evaluation of some target (object, person, or event) and that predisposes us to act in a certain way towards the target’ (p. 540). In this study “value” and “attitude” are treated as distinct terms. In the words of Rokeach (1973), the difference between the two is that, whereas the former refers to ‘a single belief of a very specific kind’, the latter refers to ‘an organization of several beliefs around a specific object or situation’ (p. 18). The meaning of “beliefs” is described in detail in Chapter 2 where it is treated in conjunction with “knowledge” and “understanding”.

It is on the basis of these definitions that searching for a proof is viewed as a social activity involving arguments in the construction and communication of mathematical knowledge; an activity that may lead to discovery of new results thus enabling the systematisation of mathematical propositions[5]. Put differently, argumentation ability and functional understanding of proof serve as a learner’s window into how mathematical knowledge is constructed. According to McMillan and Schumacher (2010), the term “ability” is used interchangeably with “intelligence” and “aptitude”. Depending on the purpose of the definition, mathematical ability is classified as either cognitive or pragmatic (Karsenty, 2014). From a cognitive perspective, O'Donoghue (2009) defines mathematical ability as the capacity to obtain, process, and retain mathematical information. According to Koshy, Ernest, and Casey (2009), mathematical ability refers to learners’ capacity to learn and master new mathematical ideas and skill. In this study, ability is viewed from the perspective of pragmatic as referring to learners’ capacity to perform mathematical argumentation and to effectively solve given mathematical problems (Karsenty, 2014).

In addition, like Dweck (2014), I used this term (that is, ability) to dispel the myths of a fixed mindset, meaning the belief that success in mathematics is instantaneous and that the basic qualities like smartness and talent are innate traits. From the point of view of a fixed mindset, Dweck (2014) goes on to assert that individuals with a fixed mindset have the tendency to devalue effort and as a consequence plateau early. As I saw this definition, this mindset is compatible with an external view of mathematics. I support the idea of a growth mindset which is premised on the

[5] I prefer to make a distinction between “proposition” and “statement”. By proposition and statements here I respectively mean a conjecture whose actual proof is under construction and an axiom, definition, concept or theorem used in the construction of a proof (a meaningful proposition).

notion that ability and talent are malleable and therefore could be developed through dedication and personal effort or hard work. One other positive element of the growth mindset is that it promotes an intellectual culture and a resilience that is essential for proving[6] and other problem solving activities (Blackwell, Trzesniewski, & Dweck, 2007).

It is my premise that focusing on understanding of proof functions harnesses, in Moore's (1994) terms, the typically abrupt introduction to the proof activity for learners. More importantly, it provides learners with a window into the practices of mathematicians. Therefore, functional understanding of proof and argumentation ability are both important aspects of proof competence. De Villiers (1990) argues that learners who understood the functions of proof in mathematics tend to be motivated to do proof meaningfully rather than view proof as just another ritual to be undertaken without meaning. According to Schunk, Pintrich, and Meece (2008) motivation refers to the process whereby goal-directed activity is instigated and sustained. They further point out that motivation is perennially important because it involves goals that provide impetus for and direction to action. As already pointed out, meaningful engagement with proof means that the learning of the proof concept makes sense to learners in such a way that they understand mathematics as a discipline for which proof is central. In this study, as in Usiskin's (2015), I view a concept as an entity that could be analysed in terms of its associated skills, properties, functions, representations and history.

Returning to the concept of proof, Mejía-Ramos and Inglis (2011) lament the fact that school teachers emphasise proof writing prior to ensuring that learners held appropriate understanding of the functions of proof in mathematics. My contention is that functional understanding of proof helps learners to gain an appreciation of subtleties of the practices and arguments employed in the building of mathematical knowledge which in turn motivates learners to do proof meaningfully thus improving their participation and performance in mathematics. Put

[6] Like A.J. Stylianides (2007), I use the term "proving" to describe the activity associated the search for a proof.

more concisely, functional understanding of proof gives learners access to the practices of mathematicians as explained in the next chapter.

By meaningful here I meant that the learning of the proof concept in ways that make sense to learners and help them develop an understanding of mathematics as a discipline for which proof is not only useful but also central. Extensive years of teaching proof to high school learners suggest that learners see little or no value in doing proof. The investigation on functional understanding of proof in the selected high school classrooms was conducted with this concern in mind.

In most high school mathematics classrooms, emphasis is placed on the verification function of proof by using several cases (Herbst, Miyakawa, & Chazan, 2012; Mudaly, 2007). Yet the South African curriculum as presented in *Curriculum and Assessment Statement* (CAPS) insisted that '[i]t must be explained that a single counterexample can disprove a conjecture, but numerous specific examples supporting a conjecture do not constitute a general proof' (Department of Basic Education [DBE], 2011, p. 25). This instruction is directed at teachers. In my experience as both a mathematics teacher and a member of the communities of the mathematical practice in school clusters in various provinces in South Africa, I often withheld the some of the contents of the CAPS document from the learners, especially nonexaminable aspects of the curriculum. I believe that this approach resonates with many other mathematics teachers. For instance, asking a high school mathematics teacher this question, "Are your learners conscious of the Goals, Specific Aims, and Skills stipulated in CAPS?" would draw an emphatic "No" as an answer.

If this belief were correct, helping learners to gain access to the contents of the CAPS document for the further education and training (FET) phase (grades 10–12) may be an important initial step in enabling them to seek adherence to its stipulations and principles. Put slightly differently, learners need access to the CAPS document not only to check that the content is adhered to but also to monitor adherence to its Specific Aims and Skills. Briefly, the CAPS document specified the content area and its accompanying concepts and skills from Grade 10 to

12. In addition, each content area is broken down into several topics: algebra, financial mathematics, trigonometry, probability and statistics, differential calculus, analytical and Euclidean geometries.

The present study focused on Euclidean geometry and measurement in general, and the concepts of proof and argumentation in particular because my observation of the CAPS curriculum was similar to Wu's (1996) conclusion that 'outside of geometry there are essentially no proofs' (p. 228). Research studies have shown that proof is a notoriously difficult concept for learners to learn (de Villiers, 2012; Hanna, et al., 2009; Mudaly, 2007). I posit that one way of making sense of why most learners find doing proof difficult is to capture, with the intention to examine factors affecting the understanding of functions of a mathematical proof. Prior to ending this chapter, it is important to revisit the concept "proof".

In our daily lives we frequently encounter or use the term "proof". Although mathematicians are accustomed to think of "proof" as an unambiguous term (Epstein & Levy, 1995), it has a multiplicity of meanings to the extent that its meaning is still unclear in school mathematics (A. J. Stylianides, 2007). Along this line, Cabassut, Conner, Ersoz, Furinghetti, Jahnke, and Morselli (2012) point out that whereas mathematicians are convinced that, in practice, they know precisely what a proof is, there exist no easy explanations of what proof is that teachers could provide to their learners. The multiple definitions of proof contribute to the difficulty that learners experience in their learning of the concept. According to a widely disseminated definition of proof provided by the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000), proof pertains to the process in which conclusions are derived from axioms in a finite sequence of logical steps.

Further, Tall (1989) points out that the term "proof" means many different things to learners such that interpretation of its meaning may be different from that of the teacher, just as one teacher's interpretation may differ from another's. However, since the term "proof" has been used differently in many situations, in an academic discipline like mathematics education its exact meaning would seem to be important (Reid & Knipping, 2010). Similarly, Epp (2003) points out that mathematical language is required to be unambiguous. CadwalladerOlsker (2011) avers that this difficulty is further compounded by the fact that proof performs several different functions in mathematics and may be written for a specific audience. By way of example, he suggests that a proof written in a research article to verify a theorem is likely to be very different from that written to explain the essential ideas to learners.

In this study, a proof is also viewed as an argument based on accepted truths for or against a mathematical claim (conjecture). The term argument is used to denote a connected sequence of statements generated from the axiomatic method. The term "axiomatic method" means a method of organising a theory (theorem) by beginning 'with the list of undefinable terms and unprovable axioms, including those terms from which the statements of the theory (theorems) should be deduced according to the rules of formal logic' (Demidov, 1980, p. 215). In keeping with de Villiers' (2012) caution, I did not define proof in terms of its verification or any of its multiple functions, to avoid elevating a particular function as more important than the others. For instance, Griffiths' (2000) idea of proof as 'a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion' (p. 2) reflects the systematisation function of proof as it mentions that proof begins with assumptions and logically connecting them to reach a conclusion.

Clearly, the question "What is a mathematical proof?" is difficult to answer despite the extensive literature on proof. However, A.J. Stylianides (2007) provides an apt definition of proof, emphasising argumentation:

> Proof is a mathematical argument, a connected sequence of assertions against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted arguments*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid known to, or within the conceptual reach of, the classroom community;

3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the room community. (p. 291)

Having defined proof in this manner, learners' understanding of the functions of proof was understood at three broad and distinct levels: naïve; hybrid; and informed. That is, learners who understood that proof has no functions other than verification were classified as holding "naïve" views about its functions while those who understand the other functions that proof performs in mathematics were labelled as holding "informed" views and thus assumed as being able to prove propositions. The intermediate level at which the understanding of the functions of proof included both naïve and informed understanding of the functions of proof was labelled as "hybrid". The next section provides evidence from the literature and experience showing that the problem exists and its relevance. In doing so, I also include the philosophical rationale for utilising a mixed methods study.

1.1 Background

In this section I provide a precursor to the literature review by contextualising the study in the literature on functions of proof and argumentation. Though philosophers and the mathematics community including mathematicians and mathematics education researchers continue debating the functions that proof perform in mathematics, recent studies have indicated the existence of consensus on the fundamental functions of proof (Bartlo, 2013; CadwalladerOlsker, 2011; de Villiers, 1990; Herbst, Miyakawa, & Chazan, 2012; Knuth, 2002; Reid & Knipping, 2010). Hence, in this thesis I decided to focus on five functions of proof: verification, explanation, communication, discovery, and systematisation.

As far as I could ascertain, apart from Healy and Hoyles (1998), performing *Google* and *Google Scholar* searches with key search terms such as "functions of proof", "role of proof" or "purpose of proof", no articles reporting empirical studies on learners' functional understanding of proof in literature were located. It was precisely this gap that my study attempted to fill. The ultimate goal of this study was to contribute to the field's growing understanding of the activities that learners need to engage in to prepare for proof construction: functional understanding of proof and argumentation. The implication that emerged from this study was that curriculum designers and teachers have a responsibility to ensuring that learners hold appropriate functional understanding of Euclidean proof and can make engage in argumentation. It must be mentioned that I am not claiming that the functional understanding of proof and argumentation are the only factors affecting learners' performance in geometry education; indeed, there are indeed a plethora of other factors that account for learners' difficulty with proof. This study focused on only functional understanding and argumentation.

The reasons for investigating learners' functional understanding of proof in relation to argumentation within the context of Euclidean geometry is twofold. First, it is in Euclidean geometry where learners first encounter formal proof in the mathematics curriculum. As far back as four decades, Moise (1975) notes that Euclidean geometry component of school mathematics 'seems to be the only mathematical subject that young students can understand and work with in approximately the same way as a mathematician' (p. 477). To this day, the Moise's point has not changed in (at least) South African curriculum contexts. In recent times, support for Moise's point can be found in Tall, Yevdokimov, Koichu, and Whiteley's (2011) assertion that mathematicians often consider the study of Euclidean geometry in school as providing a necessary basis for the formal notion of proof.

Second, Euclidean geometry is tied to an interpretation within physical space which makes it easier to understand for learners, since it is less abstract than other fields (Grigoriadou, 2012). This statement found support in Wu (1996) who says that 'in learning to prove something for the first time, most people find it easier to look at a picture than to close their eyes and think abstractly' (p. 228). Further, functional understanding of proof is important to investigate as it is among other Specific Aims advocated by the South African Department of Basic Education (DBE). There is likely to be no disagreement that this aim, which professed the learning of proofs with a good understanding of why they are important (as stipulated in CAPS), is based on the notions of the van Hiele (1986) theory of geometric thinking. As already mentioned, part of the theory proposes a sequence of stages (level) from the recognition of figures to their description and categorisation, to definitions which furnish the basis for logical deductions, and constructions leading to understanding the functions of axioms, and doing deductive argument (proof).

In order to further elucidate the context to this study, it is necessary to briefly relate the definition of mathematical proof to argumentation. As already mentioned, view proof as a product of an argumentation process in which a learner demonstrates their geometric maturity using a finite sequence of axioms to reach a conclusion. The conclusion states that the proposition has been found to be true. I regard argumentation as an aspect of proof against the background that the first step in proof followed from the given aspects which in argumentation terms constitutes data. Support for this stance about argumentation as a subset of proof is found in Schoenfeld's (1988) definition of proof as 'a coherent chain of argumentation in which one or more conclusions are deduced, in accord with certain well-specified rules of deduction from two sets of givens' (p. 157). By "givens" he refers to the premises or axioms from which deductive arguments proceeded.

As Lakatos (1991) emphasises, the fact that even the axioms on which proofs are based continue to be open to revision by the mathematics community strengthens the interaction between proof and argumentation. The reason why I make this statement is that, for Toulmin (2003), argumentation entails transforming a statement into one that is mutually acceptable following a particular argument pattern. This definition parallels that of Balacheff (1988), who views proof as 'an explanation of a specific form, organized as a succession of statements following specified rules' (p. 148).

1.1.1 Statement of the problem

In all educational research, proof has been found to be a notoriously difficult concept for learners to learn (de Villiers, 1998). Almost three decades ago, de Villiers (1990) suggested that learners' lack of an appreciation of the functions of proof – considered as central in motivating learners to view proof as a meaningful activity – has long been identified as the primary source of their difficulty with proof. Similarly, more than a decade later, an investigation spanning over five countries at different levels of schooling, Ball, Hoyles, Jahnke, and Movshovitz-Hadar (2002) found that learners' difficulty with proof stemmed at least partly from a lack of more refined understanding of the functions of proof in mathematics. The common characteristic of both lamentations is that learning proof without regard to its functions is unfruitful.

Yet scant attention has been given to the extent to which learners appreciated the functions of proof, not even in Euclidean geometry; despite the fact that learning about the functions of proof not only motivates learners to do proof meaningfully, but also helps them to understand how mathematical knowledge develops. Functional understanding of proof is foregrounded by the Specific Aims in the CAPS perhaps on the realisation that other attempts to resolve the problem distorted learners' understanding of the nature of mathematics. According to the curriculum as described in the CAPS document, learning 'proofs without a good understanding of why they are important will leave learners ill-equipped to use their knowledge in later life' (Department of Basic Education [DBE], 2011, p. 8). However, an examination of the CAPS seemed to suggest that little or no instructional time is devoted to functional understanding of proof in mathematics. In this regard, Segal (2000) makes an interesting observation, no less apt today than when it was written:

> It is not clear that there ever is a golden age in which the majority of schoolchildren about to enter higher education understood the role of (especially deductive) proofs. (p. 196)

Personal experience gained from teaching and learning mathematics suggests that lack of appreciation of the functions that proof performs in mathematics invokes rote learning as learners see no value in doing proof. Learning proof this way seems to generate in learners negative attitudes towards mathematics. Aaron (2011) cautions us by pointing out that '[a]s long as students believe that mathematical proof is irrelevant they will not move from an empirical view of proof to a more advanced view of proof' (p. 40). The potential for functional understanding of proof to improve the proving of propositions received a boost by the validation of the LFUP survey instrument designed to measure learners' functional understanding of proof. An instrument is a tool designed for measuring, observing, or documenting quantitative data (Creswell, 2012).

This study is important because learning of mathematics cannot be separated from the need to hold informed functional understanding of its intrinsic means for validation, proof (Balacheff, 2010). For instance, learners, like mathematicians, need to gain insight into why a proposition is true. However, if learners are oblivious to the other functions of proof, it is in part not difficult to see why learner performance in Euclidean proof is poor. In addition, without empirical inquiry into learners' functional understanding of proof, it is difficult to make meaningful recommendations to policymakers and curriculum implementation monitors.

Gaining insights into the character of learners' functional understanding of proof and the factors contributing to learners' persistent belief that empirical arguments (proof by cases) are mathematical proof could encourage further studies by mathematics education researchers. In addition, the insights gained may inform the judgements and decisions of policymakers and curriculum monitors interested in better understanding why learners' performance in Euclidean geometry is poor. More specifically, understanding the impact of collectivist culture provided insights into how future studies may be undertaken to support and thus improve learners' participation in mathematics generally and performance in Euclidean geometry specifically. My contention is that, to motivate learners to do proof meaningfully, it is necessary to capture their functional understanding of proof. The remedy for this problem lay, at least in part, with instructional practices; assessment of functions of proof to be given prominence as they portray the nature of mathematics and the how mathematical knowledge develops. Altogether, the case advanced in this study reflected a desire to make classroom practice akin to that of mathematicians.

As suggested by Driver, Newton, and Osborne (2000), argumentation theory provides a theoretical basis for developing tools to analyse and improve argumentative discourse, either in speech or in writing. Specifically, the CAPS curriculum emphasised the need for learners to be exposed to mathematical experiences that gave them many opportunities to develop their mathematical reasoning[8]. One way in which learners could meet this need is through argumentation which allows them to externalise their thinking (Erduran, Simon, & Osborne, 2004). However, the potential of Toulmin's (2003) argument pattern (TAP) as a tool to measure the quality of arguments in the mathematics classroom has been a neglected component of argumentation discourse analysis (Erduran et al., 2004). In addition, to date, I am not aware of empirical investigations involving Toulmin's model with high school learners from a South African perspective. Apart from attempting to fill this gap, this study contributes to the building of empirical support for the model as it provides quantitative perspectives by measuring the quality of written argumentation in mathematics classrooms.

1.1.2 Overview of the study design

A mixed methods sequential explanatory study is undertaken to obtain answers to the four research questions. The design is sequential in the sense that the quantitative phase follows the qualitative phase for both data collection and analysis. That is, the quantitative phase laid the foundation for identifying the appropriate participant and questions for the qualitative phase of the study. The design is explanatory in that it prioritises quantitative data collection so as to explain the quantitative results. Since the purpose of this study was to quantitatively explore learners' functional understanding of proof with a view to craft a research question to qualitatively explain why *Presh N*'s understanding of the functions of proof was informed, the quantitative phase was prioritised. This study mixing both quantitative and qualitative data within a single study for the purpose of gaining a better understanding of the research problem (Ivankova, Creswell, & Sheldon, 2006). Specifically, the case study phase provided more insight when explaining why she held informed beliefs about the functions of proof in mathematics than a larger sample may afford. The integration of the quantitative and qualitative phases took place at the intermediate stage and the

[8] Hanna (2014) takes reasoning to mean broadly 'the common human ability to make inferences, deductive or otherwise' (p. 405). In this study, I took mathematical reasoning to be the broader term that encompassed both argumentation and proof because both processes entail establishing the "truth" of a proposition.

study design stage. Integration refers to the point in the process of research procedures at which quantitative and qualitative data are brought together (Creswell, Fetters, & Ivankova, 2004).

I adopted Ivankova et al.'s (2006) model to graphically show the connecting points between the quantitative and qualitative phases and to specify the place in the research process where the integration of the findings of both quantitative and qualitative phases occurred. The rationale for choosing this design was that I wanted to collect participants' ideas, as wide as possible, about learners' understanding of the functions of proof in mathematics and their argumentation ability and subsequently conduct an interview to provide in-depth explanation of the reasons why *Presh N* held the beliefs she held about the functions of proof. The explanations were characterised as "in-depth" because, as McMillan and Schumacher (2010) argues, participant's responses were probed. Undertaking a large scale investigation was also intended to provide policymakers with insight from which to base their policies.

1.2 Significance of the study

Proof has been widely studied but little attention has not only been paid to documenting learners' understanding of the functions of proof in mathematics but also to the examination of the relationship between functional understanding of proof and argumentation ability. There is likely to be no contestation to the view that no single explanation accounts for the low scholastic achievement in Euclidean geometry. That view notwithstanding, little systematic[9] investigation of school and curriculum factors and the role they might play in shaping learners' understanding of and competencies in mathematical proof have been conducted (Healy & Hoyles, 2000).

The present study is intended to serve as an attempt to fill this void by building on the work of de Villiers (1990) who asserts that most learners' problems with Euclidean geometry often lie with learners' naïve understanding of the functions of proof in mathematics. Capturing understanding of proof from the perspectives of its functions in Euclidean geometry, aligned with

[9] By systematic is meant "planned, ordered and public" investigation, following rules agreed upon by members of the qualitative research community (Shank, 2002).

argumentation ability, and the factors affecting learners' belief about the function of proof, is significant for a plethora of reasons. The three major contributions of this research are its methodology, the baseline quantitative data gathered on LFUP, and the proposed model for understanding factors influencing learners' functional understanding of proof.

1.2.1 Methodological significance

The vast majority of research has focused on designing intervention programmes (following learners over time and devoting extensive time to data collection) to teach proof relying either on pre-test-post-test designs or qualitative measures thus introducing strong evaluator bias. Specifically, the original development of research methods and instruments is subject to sample size and sampling bias. For instance, the sample size required to provide reliable data is often not statistically determined. In addition, the post-test is usually completed by participants who are still enthusiastic about the experiment and the opportunity to learn somewhat differently; the realities of the environment have not dampened their enthusiasm. Nonetheless, as already mentioned, proof cannot be taught.

The present study advances the argument that only normed and validated methods can provide a scientific basis for addressing the problem from the perspective of focusing on the activities prior to formal proof construction. Hence, the sample for the quantitative phase of this study was randomly selected and the results factor analysed. In addition, rather than focus on one of the two major research paradigms traditionally used in education—positivist paradigm and the constructivist paradigm (McMillan & Schumacher, 2010)—this the study employed a mixed methods design because it was the best way to answer the research questions. A discussion on these paradigms is beyond the scope of this study save to say that they confine the researcher to a particular set of data collection methods or data analysis strategies associated with either of the traditional paradigms (Creswell & Plano Clark, 2011).

1.2.2 Significance in high school Euclidean geometry education

The significance of this study generally lies on the premise that research studies and international assessment bodies often rank Euclidean proof as one of the most difficult topics to teach and learn in mathematics. Thompson, Senk, and Johnson (2012) argue that some of the most persistent proof-related difficulties identified among learners in secondary school and university are a consequence of the confusion about the functions of proof in mathematics. This study will provide clarity by making available an instrument designed with this confusion in mind. First, to date, save for Shongwe and Mudaly's (2017) work, no existing studies have validated the LFUP instrument for measuring learners' functional understanding of proof. In addition, very little (if any) research has been done to characterise learners' functional understanding of proof in Africa, not to mention in South Africa. This instrument is intended to enhance the knowledge base, classroom practice of proof education in the mathematics classroom, and inform research in the area of proof functions. Put another way, this study sought to contribute to a broader knowledge base around understanding difficulties in the learning of proof in Euclidean geometry from the perspectives of activities prior to construction of proof.

Second, this study is one of the few to examine learners' functional understanding of Euclidean proof and the factors that shape this understanding. In particular, it serves as a response to the recommendations of Mariotti (2006) that better insight can be gained from investigating the sources of understanding of proof that are inconsistent with those held by contemporary mathematicians. Usiskin (1980) points out that proofs in Euclidean geometry are different from proofs in other branches of mathematics. The LFUP instrument will be useful in high school mathematics classes as a tool from which instruction in Euclidean proof can be planned given that is has been validated and its reliability established. Reliability means that scores from an instrument are stable (be nearly the same when researchers administer the instrument multiple times at different times) and consistent (when an individual responds to certain items one way, the individual should consistently respond to closely related items in the same way (Creswell, 2012). Validity is the development of sound evidence to demonstrate that the interpretation of scores about the construct that the instrument is supposed to measure matches their use in, for example, statistical analysis to determine if factor structure or scales relate to theory, correlations, and so on (American Educational Research Association/American Psychological Association/National Council on Measurement in Education [AERA/APA/NCME], 2014; Messick, 1980). As Thorndike (2005) points out, this definition shifts the traditional focus on the three-fold types of validity, namely, construct, criterion-referenced, and content validity, to the "evidence" and "use" of the instrument.

1.2.3 Significance in mathematics education monitoring

The Department of Basic Education established the *Dinaledi School Project* in 2001 for the purpose of raising previously disadvantaged high school learners' participation and performance in mathematics and science (Department of Basic Education [DBE], 2009). Part of the budget in the department provides these schools with resources (for example, textbooks and laboratories). The ultimate intention is to improve mathematics and science results and thus increase the availability of key skills required in the economy (Department of Basic Education [DBE], 2009).

In monitoring the performance of these schools, the education officials take note research studies that focus on these schools (Department of Basic Education [DBE], 2009). The fact that this study made findings relating to SA#3 (as mentioned in the next section) in the CAPS document should draw the officials' attention as to whether the stipulations of this aim were achieved. It is reasonable to believe that these officials will have access to this finding given that one of the conditions of approval of this study is that upon its completion, a brief summary of the findings, recommendations, or this thesis in its entirety must be submitted their research office.

1.3 Geometry in South African high schools

The importance of Euclidean geometry education as an integral component of mathematics curriculum was confirmed when it was made compulsory once again in South African high schools in 2011 (Bleeker, Stols, & Van Putten, 2013; Department of Basic Education [DBE], 2011). This reintroduction of proof into the CAPS mathematics curriculum reflected the notion that there is an appreciation of proof as the basis of mathematical knowledge. This notion finds support in Hersh's (1997) claim that proof is an essential tool for promoting mathematical understanding. However, for many learners, proof is just a ritual without meaning (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). This perspective is reinforced when learners are required to write proofs according to a certain scheme or solely with symbols.

In South Africa, as in most countries, the geometry curriculum includes Euclidean proof and analytical geometry. Whereas Euclidean geometry focuses on space and shape using a system of logical deductions, analytical geometry focuses on space and shape using algebra and a Cartesian coordinate system (Department of Basic Education [DBE], 2011; Uploaders, 2013). In this study geometry has been taken to be the mathematics of shape and space, which traditionally incorporates but is not limited to Euclidean geometry. This study focused exclusively on Euclidean geometry on the basis that learner performance in this area has been consistently poor compared to the other geometries just mentioned.

In the South African high school education system, Euclidean geometry is the place where learners should engage in formal deductive reasoning as they do proofs. As previously mentioned, functional understanding of proof, one of the Specific Aims advocated in CAPS for mathematics, is based on van Hiele's (1986) broad theory of geometric thinking. Specifically, Euclidean proof (formal deduction) starts in Grade 10. In this grade, learners are expected to investigate, make conjectures, and prove the properties of the sides, angles, diagonals and areas of quadrilaterals; namely, kite, parallelogram, rectangle, rhombus, square, and trapezium (Department of Basic Education [DBE], 2011). In addition, they are required not only to know that a single counterexample can disprove a conjecture, but also that numerous specific examples supporting a conjecture do not constitute a general proof. Accordingly, very few will contest the notion that Grade 10 instruction is assumed to have had an impact on learners' functional understanding of proof in mathematics. Hence, this study investigated this understanding in Grade 11 learners.

However, the weakness in CAPS is that there appears to be a lack of explicit content on the functions of proof as well as the historical aspects of proof. As I argued earlier, it is precisely this absence of instruction on functional understanding of proof that seem to inhibit learners' ability to construct proofs. By making the functions explicit, the intended curriculum can be realised. Support for this insistence arose out of Idris' (2006) assertion that since functional understanding of proof is a largely conventional concept, its learning cannot take place without explicit instruction. Needless to say, this is not a suggestion that ability to prove is secondary but an attempt to underscore functional understanding as a prerequisite aspect of constructing Euclidean proof.

In primary schools, informal deductive elements are underscored while the formal deductive aspect is delayed until the FET phase. However, the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000) not only underscores inductive proof, it also emphasises the didactic value of deductive proof by noting that all learners must be provided with the opportunity to 'recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof' (p. 56).

Accordingly, Magajna (2011) asserts that the two systems of reasoning in (school) geometry – one based on empirical observation (informal proof) and the other based on deduction (formal proof) – are essential and mutually support each other. In addition, empirical evidence merely gives a sense that something ought to be true (Sundström, 2003). Empirical evidence refers to the testing of a conjecture using numbers after gaining conviction and confidence about the truth of the conjecture (Hanna, 1995). Reference to truth in this project implies contingent truth rather than absolute or infallible truth given that proving is a human activity and humans are prone to making mistakes despite the best efforts to avoid them. However, the problem with viewing proof as a means to make a convincing argument is essentially a return to its everyday usage which engenders semantic contamination (Reid & Knipping, 2010).

Herbst (2002) argues that proof is valuable in mathematics education not only as an opportunity for learners to engage in a process of mathematical reasoning, but more importantly, as a necessary aspect of knowledge construction. This new curriculum at FET phase advocated for teaching that involves not only the "how" of mathematics, but also the "why." For learners, CAPS discouraged the learning of procedures and proofs without a good understanding of why they were important as lack of understanding left them ill-equipped to use their knowledge in later life (Department of Basic Education [DBE], 2011). Reintroducing Euclidean geometry as a compulsory component of mathematics in 2011 seemed to suggest that curriculum designers acknowledged that Euclidean deserves a place in the high school curriculum. According to Adler (2010), this reintroduction was a response to an outcry at universities about the widening gap between school mathematics and tertiary education with a mathematical content. In my view, this development is clear departure from previous perspectives which can reasonably be attributed in large part, to the realisation that:

> An informed view of the role of proof in mathematics leads one to the conclusion that proof should be part of any mathematics curriculum that attempts to reflect mathematics itself. (Hanna, 1995, p. 42)

As Jahnke (2010) points out, the importance attached to proof in the curriculum arose from the perspective that its functions provide a more comprehensive image of the nature of mathematics. Hence, I contend that the pressure on schools to improve pass rates in mathematics examinations encourages the pursuit of rote acquisition of mathematical knowledge thus distorting the nature of mathematics and also undermining some of the Specific Aims in the CAPS document. The Department of Basic Education (DBE) identified eight Specific Aims:

SA♯1: To develop fluency in computation skills without relying on the usage of calculators.

SA♯2: Mathematical modeling is an important focal point of the curriculum. Real life problems should be incorporated into all sections whenever appropriate. Examples used should be realistic and not contrived. Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues whenever possible.

SA♯3: *To provide the opportunity to develop in learners the ability to be methodical, to generalize, make conjectures and try to justify or prove them.*

SA♯4: To be able to understand and work with number system.

SA♯5: *To show Mathematics as a human creation by including the history of Mathematics*.

SA♯6: To promote accessibility of Mathematical content to all learners. It could be achieved by catering for learners with different needs.

SA♯7: *To develop problem-solving and cognitive skills. Teaching should not be limited to "how" but should rather feature the "when" and "why" of problem types. Learning procedures and proofs without a good understanding of why they are important will leave learners ill-equipped to use their knowledge in later life.*

SA♯8: To prepare the learners for further education and training as well as the world of work.

While these are all important aims, only (italised) three of them were relevant for this study; SA♯3, SA#5, and SA♯7. These three aims seem to reflect an internal view of mathematics which emphasises that the processes of mathematics are fallible. In SA♯3, conjecturing and generalising are stressed before engagement in formal proof. In SA#5, the internal view of mathematics is underscored. Functional understanding is the focus of SA♯7, which also stresses the explanatory function of proof in mathematics. Taking SA#7 into account and the fact that a third of the Grade 12 (sometimes loosely known as "matric") second paper examination consisted of Euclidean geometry, making it the component with the highest weighting in the overall assessment of this paper, I think it is reasonable to conclude that the curriculum planners placed value on holding informed functional understanding of proof in mathematics.

Although others may disagree, my opinion is that the South African curriculum assumes that by placing emphasis on making learners understand why a mathematical proposition is true or by merely doing proofs learners will come to understand the functions of proof in mathematics. However, learners' performance in proof is not only evidence that this assumption is unsubstantiated but also a reflection of defective Euclidean geometry instruction. Hence, it would be sensible for future research to examine both preservice and practicing mathematics teachers' perspectives of the functions of proof in mathematics.

While I believe that making Euclidean proof compulsory again is indicative of a willingness by curriculum planners to embrace the functions that proof performs in CAPS mathematics, of concern is the absence of an explicit mentioning of the functional dimensions of proof. This absence can also be detected in the school textbooks that the education authorities recommended for enacting the mathematics curriculum.

1.4 An overview of the theories in this study

The purpose of this brief review of literature was to build the foundation for presenting the research questions by situating the problem in the theories against which the results of the study were interpreted and discussed. The findings of this study were examined through the lenses of van Hiele's (1986) and Toulmin's (2003) theories. The van Hiele (1986) theory is the most comprehensive theory concerning geometry learning and encompasses concepts of functional understanding of proof while the de Villiers (1990) model, adopted as an organising framework for measuring learners' functional understanding of proof, elaborates on the theory. Husband and wife team of Pierre van Hiele and Dina van Hiele-Geldof observed their learners struggling with proof. Their analysis points to lack of appropriate instructional activities to develop geometric maturity based on levels of thinking as learners progress from merely recognising a figure to being able to construct a deductive proof. In addition, given that determining van Hiele levels is a difficult task in that a learner can have different van Hiele levels for different geometric concepts (Mayberry, 1981), I chose to limit the focus of this study to one aspect of Level 4; functional understanding of proof. But, it should be remembered that in terms of the theory this level builds on the previous levels.

The concept of argumentation by Toulmin (2003) is of primary importance in so far as investigating the relationship between learners' functional understanding of proof and their ability to argue mathematically. However, as already mentioned, it is also important in understanding what proof means. For Toulmin, argumentation entails transforming an open statement into one that is mutually acceptable through a “layout of an argument” model. This model consists of six elements: claim, data, warrant, backings, qualifiers, and rebuttals. In Toulmin’s (2003) terminology, I showed how the data (assumptions in the proof) and the warrant (reasons) should be used to justify a claim (conclusion) in written argumentation. In particular, I was interested in characterising the quality of learners’ written arguments in which they substantiated their claims using data.

1.5 Aims and research questions

Effective attempts aimed at developing learners’ informed understanding of the functions of proof in mathematics in learners require a clearer picture of the current status of learners’ understanding of these functions and their argumentation ability. With this background in mind, the aim of this study was to obtain answers to the general question, “How can learners do proof meaningfully?” Specifically, four aims were identified. First to primarily investigate with a view to characterise Grade 11 learners’ functional understanding of proof in mathematics. The corresponding research question was: *What functional understanding of proof do Grade 11 learners hold?* A secondary aim of this study was to investigate how these learners’ functional understanding of proof were related to their argumentation ability. The research question was, *How is the relationship (if any) between learners’ quality of arguments and their functional understanding of proof?* Both these questions required quantitative approach to answer them.

Worth mentioning is that, given the nature of sequential explanatory designs, the results of the quantitative phase were used to craft questions for this qualitative phase of the study to explain and thus enrich the quantitative phase. To explain the factors that influence informed beliefs of the functions of proof. The research question was, *Why does Presh N hold informed beliefs about the functions of proof?* An interest in the overall picture of the findings relating to the nature of interaction among the three central constructs in this study provoked the last research question, *"How is the interaction among the three constructs (that is, functional understanding of proof, argumentation ability, and factors influencing functional understanding?"*

1.6 Delimitation of the study

The scope of the investigation in this study was deliberately narrowed because it is practically impossible to study everything in a single study at once, a phenomenon which Merriam (2009) refers to as delimitation. Specifically, these parameters included, for example, the phenomenon of learners' functional understanding of proof I chose to study, the theoretical frameworks through which I interpreted the results, the research design adopted, the sizes of both quantitative and qualitative phase samples, the geographic location of the research sites, characteristics of the population selected, and the independent variables manipulated in the quantitative phase of the study. Throughout the methods chapter, I provided rationale for almost every choice I made in establishing the parameters.

1.7 Researcher positionality

> The positionality that researchers bring to their work, and the personal experiences through which that positionality is shaped, may influence what researchers bring to research encounters, their choice of processes, and their interpretation of outcomes. (Foote & Bartell, 2011, p. 46)

As is the case with all researchers, my life experiences informed various aspects of this study. In this subsection I provided insights into the paradigm (worldview), hypotheses derived from my experiences with the concept of proof. All these aspects influenced the research process (for example, research questions, sample, methods, interpretations, and so on). For instance, though some township schools tend to achieve 100% pass rate in their Grade 12 examinations, the quality of these passes tend to be weaker than those in previously white schools with similar pass rate. Hence I chose to compare the quality of learners' functional understanding of proof and argumentation between fee-paying as well as no-fee high schools with a focus on mathematics and science (that is, Dinaledi schools).

In this study, the pragmatic paradigm was suitable to serve as a framework with which to describe how this study unfolded. I adopted this paradigm informed by my belief in using "what works" to find answers to research questions. Pragmatism is defined as a paradigm that encompasses both quantitative and qualitative research methods (Johnson & Onwuegbuzie, 2004). However, Creswell and Plano-Clark (2011) use the term "worldview" in this regard. Teddlie and Tashakkori (2010) define a paradigm as 'a worldview together with the philosophical assumptions associated with that point of view' (p. 84). Further, for Teddlie and Tashakkori (2010), pragmatism means typically a worldview associated with mixed methods research as it embraces features associated with both postpositivism and constructivism worldviews and rejects 'the dogmatic either-or choice between constructivism and postpositivism and the search for practical answers to questions that intrigue the investigator' (p. 86).

For the pragmatic paradigm, the view is that the research problem, rather than loyalty to any research paradigm, determine the data collection and analysis methods that are most likely to provide answers to the research questions. In a nutshell, pragmatism as a paradigmatic framework directed the research efforts in this project. In this respect, the use of mixed methods design was not merely a matter of combining qualitative and quantitative methodologies together, but arose from the need for pragmatic response to the research questions at hand. Put another way, my choice of research questions, data collection and analysis methods, and interpretation of findings reflected the underlying pragmatic view of the world.

Consistent with this paradigm, a mixed methods sequential explanatory research design was employed in this study. That is, a quantitative method, which took priority in this study, was followed by a qualitative method as a means to understand why *Presh N* tended to hold informed beliefs about the functions of proof. I agree with Clough and Nutbrown (2012) when they make this observation:

> Since research is carried out by people, it is inevitable that the standpoint of the researcher is a fundamental platform on which enquiry is developed. All social science research is saturated (however disguised) with positionality. (p. 10)

Given the philosophical differences in the structure and knowledge confirmation between quantitative and qualitative approaches (Foss & Ellefsen, 2002), in the next section I provide a personal background for the reader to understand how my experiences with the mathematics discipline might have influenced the results in the qualitative segment of this study. Put another way, it is necessary to disclose to my background to the reader to facilitate their evaluation of the findings of this study.

1.7.1 Early beginnings

I was born in Esilobela township, Carolina, in the Mpumalanga province, but spent my childhood in the Dundonald village situated a few kilometres from the Eswatini border. By township is meant a historically disadvantaged area characterised by, for example, poverty, high crime, antisocial behaviour, shortage of classroom resources that facilitate learning of mathematics, for example, dynamic geometry software (DGS)[11], recreational facilities, and community libraries. After finishing high school in the mid-eighties I received a bursary to study for an integrated teachers' degree with a pure mathematics major and was the first member of my family to attend university. My parents (mother: self-employed; father has since passed on: underground mine worker) separated when I was very young.

Together with my stepbrother and mother who sold mostly second hand clothing items and worked the soil, we lived in a mud house. Compared to my privileged white counterparts, our house had no electricity nor flushing toilets. We relied on public transport as we had no car and attended under-resourced primary and high schools; reflecting realities of apartheid South Africa characterised by an inequitable social and political system. In general, I come from a less privileged background which informs my strong commitment to social justice.

[11] The phrase "dynamic geometry" was originally invented and trademarked by publishers, Key Curriculum Press, to describe the *Geometer's Sketchpad* (Jackiw, 2001).

1.7.2 Interest in mathematics

I was particularly fascinated by mathematics and enjoyed some admiration from friends who I assisted with homework and preparation for examinations. Hence I chose to major in mathematics for my undergraduate degree and particularly enjoyed teaching Euclidean geometry at numerous high schools in the early nineties.

I returned to university in 2010 and obtained an MSc in science education three years later and began teaching Physical Sciences in high schools. Currently, I am a mathematics education lecturer at a public university in South Africa and a proponent and advocate of assuring redress of the past imbalances in the allocation of resources that facilitate mathematics learning and teaching. This fascination with proof may have influenced my thinking about the functions of proof in mathematics as well as the interpretation of the qualitative data with undue bias.

1.8 Organisation of the thesis

In the process of doing this study, I submitted three manuscripts, which are in part based on this thesis to peer-reviewed journals. This was done for two reasons. One was to ensure dissemination of empirical research results arising from this study. Two was to meet the requirement of the School of Humanities at the University of KwaZulu-Natal which stipulated that submission of this thesis must be accompanied by at least one published journal article. The next subsection provides a summary of the focus of the chapters which constitute this thesis.

1.8.1 Chapter 1: Introduction to the study

This introductory chapter broadly encapsulates the notions of functional understanding of proof and argumentation within the context of the South African high school geometry curriculum against the backdrop of reported poor performance of learners in relation to proof. In addition, the discrepancy between actual classroom practice and professed SA in CAPS is underscored. The significance of the study is described within these contexts. The chapter also provides the aims and the resulting research questions that underpin the study. The theoretical bases (van Hiele's and

Toulmin's theories) of this study are then contextualised after which an overview of the research design is explicated. Finally, the delimitations of the study and my positionality as a researcher are described.

1.8.2 Chapter 2: The review of literature

This chapter critically examines literature on the concepts underlying this study and reports on the results of research pertinent to this study: proof functions in mathematics, argumentation, and the factors influencing informed beliefs about the functions proof. The purpose of critically examining and reporting on previous studies is to build the foundation for the present study and thus connect its problem, purpose, and discussions to previous studies. This chapter explores and discusses major concepts and ideas providing conceptual frameworks some of which are permeated by historical and philosophical analyses. The major terms that built the conceptual framework for data analysis purposes include; understanding what mathematics is; functions of proof; mathematical understanding; and, argumentation. The measures for assessing functional understanding of proof – based on previous literature on learner difficulties with proof and its functions, and my own classroom experiences – are discussed.

1.8.3 Chapter 3: The theoretical framework

This chapter presents a brief description of the historical development of the two theories underpinning this study through which data analyses and interpretation of results were undertaken. One is the van Hiele (1986) theory of geometric thinking whose central idea is that learning geometry takes place in discrete levels of thinking and that progress to the next level is a function of instruction. The theory has played a major role in understanding learners' difficulty with geometry. De Villiers' (1990) model provides the concepts for investigating learners' functional understanding of proof in mathematics. Two is Toulmin's (2003) argument pattern (TAP) scheme which was developed for the purpose of explaining how argumentation takes place in the natural contexts of everyday life, especially in law. He suggests that arguments can be understood using six components comprising: claims, data, warrants, backings, qualifiers, and rebuttals. In talking about learning, I draw on the sociocultural theory on the basis that I view mathematics as a human activity in which all learners can participate. Having interrogated these two theories, I construct a conceptual framework to understand the relationships among the concepts.

1.8.4 Chapter 4: Research methodology

First, an overview of methodology and instruments utilised in previous studies on learners' functional understanding of proof is undertaken. A methodological framework that graphically describes the research design is provided. Next, I argue why it is helpful to make a distinction between methods and methodology, terms often treated as synonyms. Then, I provide the rationale for using a mixed-methods sequential explanatory design. Next, instrumentation, data collection measures and procedures as well as analysis procedures are described and justified. Final, issues of rigour are discussed. For the quantitative phase, the data collection instruments, for example, the LFUP and the Argumentation Framework in Euclidean Geometry (AFEG) questionnaires employed to capture and characterise learners' functional understanding of proof and their argumentation ability are also discussed.

Given that the LFUP instrument was already established, its reliability and validity evidence is stipulated. For the qualitative phase, sample task-based questions from the Interview Schedule which were meant to elicit *Presh N*'s beliefs influencing her understanding of the functions of proof in mathematics are provided. The results (analysis, interpretation, and discussed) of this study are presented in independent chapters (5, 6, and 7) using the research questions as an organizing framework. The areas in which the methods were combined are identified and justified. Then, rigour and limitations of the design is discussed.

1.8.5 Chapter 5: Functional understanding of proof in mathematics

The LFUP questionnaire results are analysed (presented and interpreted) to answer the first research question, *What functional understanding of proof do Grade 11 learners hold?* This question sought to understand whether learners held naïve (empiricist), hybrid or informed understanding of the functions of proof in mathematics (verification, explanation, communication, discovery, and systematisation). The SPSS v.24 (2017) software is used to analyse the data. I use descriptive statistics to report on patterns in participants' responses and multivariate techniques to identify factors accounting for variability in LFUP scores and how these vary across participating schools. The results show that learners' functional understanding of proof are of a hybrid nature and inconsistent with those espoused in CAPS and held by contemporary mathematicians.

1.8.6 Chapter 6: The relationship between functional understanding of proof and argumentation ability

This chapter describes the results of the nature of the relationship between learners' functional understanding of proof and their ability to argue, to answer the second quantitative research question; *How is the relationship between learners' functional understanding and argumentation ability?* To this end, statistical techniques are used to describe this relationship. Specifically, SPSS v.24 (2017) is used to analyse the data. The results show that a weak, positive and significant correlation exists between the two constructs. Using Toulmin's theory, the results show that learners' argumentation ability was poor. In addition, multiple regression analysis indicates that the verification function accounts for the largest variability in learners' functional understanding of proof.

1.8.7 Chapter 7: Beliefs about the functions of proof: The case of Presh N

The purpose of this chapter is to answer the third and third research question, *Why does Presh N hold informed beliefs about the functions of proof?* The participant (*Presh N*) is purposively sampled which means that she is selected on the basis that she is an information-rich individual for the most effective use of resources (Patton, 2002). The van Hiele theory is used to examine the findings after the results were analysed with the aid of ATLAS.ti and STATA, a framework for understanding the factors (deductive arguments, semantic contamination, collectivist culture, empirical arguments, teacher, and textbook) influencing understanding of the functions of proof is suggested.

1.8.8 Chapter 8: Exploring the interaction among the three constructs

The purpose of this chapter is to answer the fourth and final research question, *"What is the nature of the interaction among the three constructs (that is, functional understanding of proof, argumentation ability, and factors influencing functional understanding?"* This chapter integrates the results of the quantitative and qualitative phases to discuss the outcomes of the entire study. As indicated at the beginning of this study, both quantitative and qualitative research questions were posed to better understand Grade 11 learners' functional understanding of proof, their argumentation ability, and the factors affecting functional understanding of proof in mathematics. This chapter combines the results from both phases of the study to develop a more robust and meaningful picture of the research problem.

1.8.9 Conclusions

This final chapter concludes the study. In the discussion section, both quantitative and qualitative results were described simultaneously, having kept them independent in previous sections. The two phases are independent in that quantitative and qualitative research questions, data collection, and data analysis are separated for each phase. I take into account results of past empirical investigations in literature concerning learners' functional understanding of proof, argumentation ability, and reasons behind *Presh N*'s informed beliefs about the functions of proof. In the conclusions sections, I consider the overall investigation including the three unique contributions this study makes. One is that this study used a mixed methods design in which participating schools were randomly selected to improve the trustworthiness of the results. Two is that I validated a new measurement scale (LFUP) that allows teachers to gain insights into their learners' understanding of the functions of proof and thus tailor instruction on the meaningful construction of proof. Three is that factors influencing beliefs about the functions of proof were investigated culminating in a suggested model for describing learners' understanding of the functions of proof. In the conclusions section, I provide an overview of the study in relation to the research questions, describe findings and their implications, make recommendations and suggestions that other researchers can consider, acknowledge several limitations, and reflect on the research project as a whole.

1.9 Chapter summary

A brief review of the literature suggested that learners were inclined to hold naïve functional understanding of proof and argue poorly on account of believing that verification is the only function of proof in mathematics. The central argument that ran through this study was that the functional understanding of proof held by learners may not only be distorted and inconsistent with Specific Aims in CAPS and those of the mathematics community, they may also, unfortunately, inhibit learner achievement in Euclidean geometry. An appreciation of the functions of proof convey to learners other important pieces of mathematical knowledge and thus give them a broader picture of the mathematics as a tapestry in which all the concepts and skills are logically interwoven to form a coherent whole. Put another way, holding appropriate understanding of the functions of proof can engender learners' understanding and appreciation of what mathematics entails. Having introduced the study by contextualising the problem, provided the rationale, and pointed out that the results presented in this study are offered as a contribution to the field's growing understanding of the activities that precede the construction of proof (learners' functional understanding of proof and argumentation in mathematics), I now turn attention to the theoretical bases on which I summarise, critique, and relate other primary studies to the present study.

Chapter 2

The review of literature

> A proof demonstrates that a mathematical assertion is true, assuming certain axioms. Yet, mathematicians look beyond establishing truth to seek insight into why; proofs can have explanatory power ... Through the process of proof, mathematicians may discover new results. Proofs communicate mathematical knowledge and situate that knowledge systematically within a framework. (Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012, p. 216)

2.0 Introduction

The aim of this chapter is to conduct a critical summary review of related literature based on three aspects: previous research findings on the research problem in this study; gaining insight into existing knowledge and identification of gaps thereof; and, identifying possible weaknesses in the methodologies of some studies related to functional understanding of proof as well as argumentation. This is done while taking into account two theories underpinning this study: van Hiele theory of geometric thinking and Toulmin's argument theory. The term "theory" denotes a set of assumptions or propositions, together with relevant concepts used to explain and predict behaviour and possible relationships between such variables in a systematic way (Kowalski & Westen, 2011). Prior to conducting the review of literature, a discussion of the major terms and concepts permeating this study are discussed.

2.1 Defining mathematics

I approached the definition of mathematics from two perspectives: school mathematics and mathematical practice by mathematicians. Whereas a mathematician formulates conjectures and develops proofs thereof, this is not the encompassing aim of school mathematics as it offers learners prepackaged content with techniques to be transmitted to learners and regurgitated in tests and examinations (Lockhart, 2002). As Stahl, Çakir, Weimar, Weusijana, and Ou (2010) claim, missing the intellectual mathematical experience may limit learners' lifelong interest in science, engineering, and technology. In his critique of the state of school mathematics as a subject that stunts learners' natural curiosity and love of pattern making, Lockhart (2002) draws attention to the importance of exploration that leads to conjectures and eventual proofs of propositions (process) rather than the meanginless memorisation of proofs (products) prevalent in school mathematics by claiming that:

> If you deny students the opportunity to engage in this activity— to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs— you deny them *mathematics* itself. (p. 5)

In critising high school geometry, which he views as an instrument of the devil, Lockhart (2002) argues that forcing learners to use the rigid two-column proof format in laying out their proofs not only destroys the very essence of what geometric proofs should be but also undermines learners' intuition. For instance, as Lakatos (1991) points out, school mathematics presents theorems by beginning with axioms, lemmas and/or definitions to the conclusion while mathematicians begin with conjectures and construct their proofs as means of analysing the conjecture. Lockhart (2002) further points out that mathematicians enjoy thinking about the simplest possible things, and the simplest possible things are in fact imaginary. Watson (2008) provides a succinct characterisation of the practicies of research mathematicians:

> 'Doing mathematics' is predominantly about empirical exploration, logical deduction, seeking variance and invariance, selecting or devising representations, exemplification, observing extreme cases, conjecturing, seeking relationships, verification, reification, formalisation, locating isomorphisms, reflecting on answers as raw material for further conjecture, comparing argumentations for accuracy, validity, insight, efficiency and power. It is also about reworking to find errors in technical accuracy, and errors in argument, and looking actively for counterexamples and refutations. Mathematics is about creating methods of problem-presentation and solution for particular purposes, tinkering between physical situations and their models, and it also involves proving theorems. (p. 4)

The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) recommends that mathematics education for all learners needs 'to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof' (p. 56). Similarly, in the CAPS document mathematics is defined as 'a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves' (Department of Basic Education [DBE], 2011, p. 8). As already mentioned in the previous chapter, viewing mathematics this way is consistent with the sociocultural theory of learning for which learning is a social process in which learners contribute ideas and critique them. However, most learners in South African mathematics classrooms struggle to achieve the practices embodied in both definitions. My contention is that the emphasis on examinations permeating the education system mitigate against the operational definition (that is, the LFUP scale). In addition, the arguably unintended consequence of participation in this system is that it shapes in learners the development of a distorted image of mathematics. A synthesis of the definitions provided here suggests that for this study, mathematics is seen as human activity concerned with providing proof for the simplest things we observe.

2.2 Understanding in mathematics

The term "understanding" has been invoked several times in the previous sections. I now summarise and make explicit what understanding is and why it is an important concept from the perspective of mathematics. I do so because the term is at the heart of this investigation. Very few will contest the assertion that one of the most important goals of mathematics instruction is that learners should have an "understanding" of the concepts of mathematics. However, various meanings have been ascribed to "understanding" to the extent that defining it is not an easy task.

Some studies use "understanding" with the implicit assumption that there is universal agreement about its meaning. To complicate matters further, Machaba (2016) uses understanding and knowledge interchangeably. In spite of all these difficulties, an explicit attempt to define understanding is made by Sierpinska (1990) who proposes that understanding be regarded as 'an act, but an act involved in a process of interpretation, this interpretation being a developing dialectic between more and more elaborate guesses and validations of these guesses' (p. 26). This understanding is 'acquired through years of watching, listening, and practicing' (Lampert, 1990,

p. 31). The sense I made out of these definitions was that what one thought is "understanding" could in fact turn out to be a myth or a misconception.

To alleviate the multiple meanings ascribed to understanding, Holt (1966) developed a list of seven nonlinear but inexhaustive senses in which the term could be used in education. He suggests that understanding takes place when a learner can do some, at least, of the following about a concept or idea: state it in his or her own words; give examples thereof; recognise it in various guises and circumstances; make connections between it and other facts or ideas; make use of it in various ways; foresee some of its consequences; and state its opposite or converse. However, although the description of features of "understanding" is helpful, the concern with viewing understanding this way is that it does not distinguish between the different types of understanding. I categorised "understanding" into "fundamental" to denote the type of mathematical understanding that is central to arguments permeating this study and "supplementary" to denote the type of mathematical understanding that enhances thinking about "understanding".

2.2.1 Fundamental perspectives on types understanding

As already alluded to, mathematics education research has shown that most learners have serious difficulties with constructing proofs. Attempts to tackle this problem have focused on the widely known and useful distinction between the different types of understanding in mathematics; namely, instrumental, relational, logical, and functional (de Villiers, 1994). Skemp (1976) initially theorised the concept of understanding as either instrumental or relational.

Instrumental understanding refers to the learner's ability to correctly and efficiently manipulate mathematical content by using rules without knowing why these rules work. This understanding is sometimes referred to as computational knowledge, computational skill, computational ability, procedural skill or procedural knowledge (Idris, 2006). In this type of understanding, a learner tends to memorise owing to the isolated nature of many rules. By way of example, instrumental understanding applies to the rule that "we flip and multiply when we divide fractions". Skemp (1976) provides empirical evidence that material learnt relationally is, in a month, remembered seven times better than that which is learnt instrumentally. He adds that without understanding, a learner is mentally lost, anxious, and frustrated in mathematics.

Relational understanding refers to learner's ability to deduce relationships between content and the underlying logic upon which these relationships are based. According to Idris (2006), relational understanding is used interchangeably with conceptual understanding or conceptual knowledge to denote not only knowing facts, rules and procedures, but also knowing why general principles and a network of ideas in mathematics work. According to Schäfer (2010), conceptual understanding relates to acquisition of knowledge that not only revolves around isolated facts but includes an understanding of the different contexts that frame and inform these facts and an understanding of why a particular mathematical idea is important. With this type of understanding, a learner would be able to adjust when a new and different task is introduced. For example, understanding that "the sum of interior angles of a triangle is 180^0", will be useful in proving deductively (informally) that "the angles of a quadrilateral sum up to 360^0".

Although relational understanding provides learners with a broader perspective of the mathematics discipline, the abstract nature of the subject requires further descriptions that go beyond making informal deductions (Idris, 2006). Skemp (1987) improved his theory by including "logical understanding" to instrumental and relational understanding. In mathematics, logical understanding involves a learner's ability to use an appropriate method to perform a task, knowing why the method works, and having mastery of the rhetorical demands of school mathematics in the appropriate context (Tirosh, 1999). By rhetorical demands is meant knowing how mathematical ideas are expressed or written and judged within the mathematics community. In other words, a learner has logical understanding if they are able not only to convince themselves, but being able to convince others when asked to reflect on the logic of the steps in working out a solution to a mathematical exercise or problem.

Resnick and Ford (1981) question the usefulness of Skemp's (1976) dichotomy between instrumental and relational understanding. They point out that, for example, having instrumental understanding without attending to the relational, logical, or functional aspects is counterproductive. I concur with the sentiment in that strict adherence to one theoretical perspective at the expense of others is undesirable in mathematics education. However, as far as I could ascertain, there is no record of Skemp having advocated for this approach. I choose to read Piaget's (1978) argument that instrumental understanding is not understanding at all charitably. Personal experience suggests that instrumental understanding can be useful in building a foundation for relational understanding. If this is taken to be true, instrumental understanding does not seem inferior to relational and logical understanding. However, de Villiers (1994) points out that mathematical understanding cannot be described through these three perspectives of understanding only.

De Villiers (1990) identifies functional understanding to address the affect aspect which is embedded in doing mathematics. For this study it means understanding the role, function, purpose or value of proof in mathematics. He concludes that on the basis of extensive interviews with learners, most of their difficulty with proof seems not to lie so much with poor instrumental proficiency nor inadequate relational understanding as in poor functional understanding of proof (de Villiers, 1994).

2.2.2 Supplementary perspectives on types of understanding

Acknowledging the value of Skemp's (1976) theory, Byers and Herscvics (1977) suggest an extension of understanding that includes "formal understanding" which relates to a learner's ability to express mathematics in conventional forms of notation, and "intuitive understanding" which relates to a learner's perception of a problem with little thought of the solution process. Bell, O'Brien, and Shiu (1980) provide examples of intuitive understanding. In this respect, rather than using a linear sequence of steps in solving $3x + 2 = 8$, a learner spots that 8 is the same as $6 + 2$ and hence $3x$ must equal 6 and so x is in fact 2.

While agreeing with Skemp's dichotomy on mathematical understanding of instrumental and relational understanding, Usiskin (2015) sees understanding of mathematical concepts in five independent and nonsequential (that is, can be learned in isolation from each other and in no particular order) types which he calls dimensions of understanding: skill-algorithm understanding, property-proof understanding, use-application understanding, representation-metaphor understanding, and history-culture understanding. A closer analysis of these forms of understanding revealed that they are an expansion of Skemp's (1976) model that focused on understanding with respect to mathematical concepts only.

He describes skill-algorithm understanding as involving not only mastery of skill to obtain the right answer but also choosing a particular algorithm to obtain the right answer because it is more efficient than others. He suggests that being able to identify the mathematical properties that underlie why a certain method worked in obtaining the correct answer resembled property-proof understanding. As for the use-application dimension, he argues that this relates to individuals who know the uses of algorithms and the mathematical properties associated with a concept. He labels the ability to represent a concept in some way (for example, using manipulatives, pictorial representation or metaphor) representation-metaphor understanding.

In concluding his dimensions of understanding, he convincingly argues that understanding the cultural history of mathematical concepts is very important. For instance, he points out that some mathematical symbols are not the same everywhere; in some places, the fraction a/b is represented by a:b, while in other places the symbol a:b represents a ratio that is mathematically not identical to a fraction. However, Usiskin's (2015) types of mathematical understanding are only limited to mathematical concepts; the problem is that learners may still see no value in learning these concepts.

A little further on Resnick and Ford (1981) emphasise that memorisation of certain facts and procedures is important not so much as an end in itself but as a way to extend the capacity of the working memory by developing automaticity of response and thus free up time to focus on understanding mathematical ideas. While I agree with their view, such a discussion is beyond the scope of this study. That notwithstanding, I am mindful of Tall's (1978) suggestion that any useful classification of mathematical understanding must exhibits the reality that understanding is a dynamic process in the sense that understanding may take place for one week, forgetting and remembering the next. Before turning attention to research studies on the functions of proof, it is important to define what is meant by the term “reality”. Like Berger (1991), I view “'reality” to be a quality appertaining to phenomena that we recognise as being independent of our own volition; things we cannot “wish away”.

2.2.3 Conflation of understanding, knowledge, and belief in mathematics education

Various definitions have been ascribed to “understanding”, “knowledge”, and “beliefs”. Pajares (1992) labels beliefs as a messy construct that travels under alias such as conceptions, perceptions, or understanding. Consistent with a dynamic view of mathematics, knowledge is viewed as contestable facts that are commonly shared among the mathematics community. Mathematical knowledge is indeed contestable given the discovery of non-Euclidean geometries which shattered the view that mathematics provides absolute certainty (Greiffenhagen & Sharrock, 2011). In contrast, beliefs are subsets of knowledge which are consciously held with varying degrees of importance and for which no social consensus regarding their validity is required (Philipp, 2007). In other words, beliefs are ideas, views, assumptions, understanding, conceptions or perceptions, attached to mathematics, proof, and its functions, taken as true by the individual, and not readily amenable to, in Popper’s (1988) term, falsification[12].

In this study, I make explicit the meaning of these terms consistent with Lloyd’s (2002) line of argument to ease communication. Similar to Knuth (2002), she defines understanding as one’s general mental structure encompassing beliefs and knowledge. Beliefs are defined as understanding that are experiential or fantasy in origin and thus disputable while knowledge is defined as understanding which are compatible with consensually held information within the mathematical community. However, beliefs are crucial in that they are thought to influence the application of knowledge in the classroom (Leder, Pehkonen, & Törner, 2002). Therefore, learners’ understanding of proof is a manifestation of their experiences with proof instruction,

[12] The act of deliberately seeking counter-examples to disconfirm a theorem thereby strengthening its truth if it survives such act (Cohen, Manion, & Morrison, 2011).

other social interactions in their environment including simply their own fantasy about proof. In this study, I also used the terms beliefs, understanding, and views interchangeably.

The investigation in this study was meant to determine learners' understanding of the functions of proof which, unlike knowledge, is subject to corrections if inconsistent with accepted mathematical interpretation. Thus, viewing understanding as an ideal to be attained by learners, a model that precisely defines understanding is necessary. De Villiers' (1994) model provides an ideal definition of understanding by taking into account the depth of understanding a learner has experienced. For this reason, I focused only on learners' attainment of functional understanding of proof.

2.2.4 Learners' understanding of the verification function of proof

> [h]aving verified the theorem in several particular cases, we gathered strong inductive evidence for it the inductive phase overcame our initial suspicion and gave us a strong confidence in the theorem. Without such confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is true, you start proving it. (Polya, 1954, pp. 83-84)

Euclidean geometry is the place for learners to "see" the functions of proof in mathematics. However, of all the five functions of proof invoked in this study, studies have shown that the verification function is persistently pervasive. Learners are under the misapprehension that making empirical arguments is justification (proof) for the truth of a proposition; hence, as already pointed out, this function occupies a low status and therefore regarded as being naïve among the functions of proof. But, why is this belief resistant? This is the question that will be answered shortly. Kunimune, Fujita, and Jones (2010) suggest instructional practices to make learners understand: the generality and universality of proof, the roles of figures, and the difference between formal proof and experimental verification. By constructing formal proofs, learners come to understand that the conjectures that they have found to be true in one context are always true. Thus, they will need to understand that proof is required to achieve generality of mathematics propositions.

Indeed, very few will contest that Michael de Villiers has made an outstanding contribution in the field of Euclidean proof. In his book in which he introduces a DGS, *Rethinking Proof with the Geometer's Sketchpad*, he briefly describes activities wherein learners make and verify conjectures using sketches and engage in activities that reflect the various functions of proof at the van Hiele levels lower than 3 (known as informal deduction). However, a serious shortcoming of the van Hiele theory is that it introduces only one function of proof, systematisation, at Level 3. Mudaly's (1999) finding that functions such as verification, explanation, and discovery can be meaningful give support to this criticism. This is why de Villiers (2012) argues that it is far more meaningful to introduce proof within a dynamic geometry context, not as a means to verify, but rather as a means to explain, systematise, and discover prior to engaging in formal proof.

However, the potential risk associated with dynamic geometry is that both learners continue not see deductive proof as the ultimate means of verification (de Villiers, 2006) that provides assurance that there cannot be counter examples to refute a conjecture. Further, Laborde (2000) argues that the opportunity offered by DGS to "see" properties of geometric figures 'so easily might reduce or even kill any need for proof and thus any learning of how to develop a proof' (p. 151). This is in contrast to Chazan's (1993) finding that even extensive use of DGS or measuremnt of examples in geometry classes would not hinder learners' appreciation of mathematical proof. My view is that empiricist (proving propositions by providing specific examples) behaviour persists because of learners' inability to distinguish between inductive and deductive arguments. More broadly, I argue that this behaviour is symptomatic of a lack of functional understanding of proof in mathematics.

Learners are definitely not alone in relying on verificstion. Weber and Mejia-Ramos (2011) point out that the learners' tendency of verifying theorems with examples is akin to how mathematicians gain full confidence that a proof is completely correct. That is, mathematicians do not solely gain confidence by inspecting the logic of the proof line-by-line; they use examples to increase their conviction in, or understanding of, a proof. Weber and Mejia-Ramos (2011) further caution that learners should be aware of the limitations of empirical reasoning and the generality of a deductive proof. It is my view that the behaviour of both mathematicians and learners towards the use of proof as a means to verify the truth of a conjecture is a natural everyday way of gaining evidence by observation.

2.3 Connecting proof with argumentation

The reason for talking about functions of proof alongside argumentation stemmed from A.J. Stylianides' (2007) definition of proof as a mathematical argument, a connected sequence of assertions against a mathematical claim. This definition underscores the fact that in the construction of a formal proof involves both inductive proof and deductive proof. Support for this view s found in Harel and Sowder (1998) who use "proof" to characterise not only deductive proofs but also empirical proofs. Also, in the CAPS document, the definition of mathematics as a social activity in which conjecturing is foregrounded, underscores the notion of argumentation. Further, the communicative function of proof suggests that argumentation is an integral part of proof. In addition, in light of the "didactic contract" of the teacher, proving is a collective process in which the teacher guides their learners in the establishment of the truth of a conjecture; learners and teacher evaluating and critiquing each other's ideas. Brousseau (1997) coined the term "didactic contract" to refer to the teacher's routine instructional obligation. Further support for the link between proof and argumentation is found in Reid and Knipping's (2010) assertion that when mathematics is expressed in a social context it becomes a method of arguing. Thus, proof cannot not be seen as being separate from argumentation.

Hersh (2009) announces that '[i]n fact, "proof" is just "reasoning," but careful, critical reasoning looking closely for gaps and exceptions' (p. 19). Harel and Sowder (2007) seem to use "proof" in the same sense when they assert that 'the term proof often connotes the relatively precise argumentation given by mathematicians' (p. 807). Harel and Sowder's view supports Doeuk's (2009) idea that argumentation and proof have many aspects in common in the sense that the former is often useful to the process of proving. Hence, Pedemonte (2007) considers proof to be a specialised form of argumentation. Hersh's (1993) philosophical work on proof, for whom a proof is an argument, argues that '[i]n mathematical practice, in the real life of living mathematicians, proof is just a convincing argument, as judged by qualified judges' (p. 389).

Lakatos (1991) asserts that a proof follows a zig-zag path beginning with conjecturing then proceeding to refutations (counterexamples) of premises to reach a conclusion which may itself be subject to revision based on the strength of its premises. Certainly, revision does not take place in a vacuum; it invites argumentation. Hersh's (1993) conceptualisation of proof provides an overarching perspective; one from the common mathematical practice and the other from mathematical logic and philosophy of mathematics. From a "working" meaning, proof is a deductive argument (Figure 2—1) that convinces qualified judges while from the "logic" meaning, proof is a sequence of transformations of formal sentences, carried out according to the rules of the predicate calculus. This "logic" definition is adopted in this study on the basis that it is not only consistent with shared understanding and practices in contemporary mathematics but also consistent with what constitutes a mathematical proof at high school geometry level. However, school mathematics curriculum does not always reflect the importance of the relationship between functional understanding of proof and argumentation.

Deductive reasoning process

Axioms

Axiom 1: *(Previously proven conjecture)* — $d + a + e = 180^0$ *(d, a, & e are on a straight line)*

Axiom 2: *(Property of parallel lines)* — $d = b$ *and* $e = c$ *(alternating interior angles are equal)*

Axiom 3: *(Postulate)* — *Substitution of equal values*

Conclusion:

$a + b + c = 180^0$

for <u>any</u> triangle

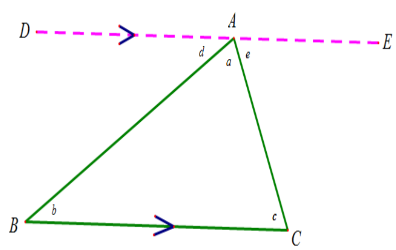


Figure 2—1. An example of a deductive (logical) argument

Kirschner, Shum, and Carr's (2012) definition of argumentation as 'discourse for persuasion, logical proof, and evidence-based belief, and more generally, discussion in which disagreements and reasoning are presented' (p. 2), is problematic. The issue for me lies with their use of "logical proof" in defining argumentation. Toulmin's (2003) reason for theorising about argument is precisely on the basis that logical proof provides limited scope for assessing the effectiveness of an argument generally in everyday discourse and most particularly in law. For instance, an argument could emerge from a mere observation of several cases of a phenomenon and be concluded without requiring the development of a deductive argument. In contrast, as here conceived, a logical proof is based on rules of logic and axioms leading to a true conclusion that applies to all cases of the mathematical objects under investigation; observed or abstracted.

Another way of making this point is to say that relating abstract mathematical objects to rhetorical argumentation, here construed as the process of engaging in domain-specific arguments, is not part of formal logic. In making informal logic a distinct concept from formal deductive logic, Johnson and Blair (2000) define informal logic as a branch of logic with a focus on developing nonformal procedures for the analysis of argumentation in both everyday discourse and domain-specific arguments. In more specific terms, van Eemeren and Grootendorst (2004) see Toulmin's (2003) theory of argumentation as primarily a rhetorical expansion of the syllogism (deductive reasoning) whereas Toulmin (2003) himself emphasises that his theory on argumentation is an effort to make logic 'less of *a priori* subject than it has recently been … more empirical' (pp. 236-237).

2.4 Mathematical inquiry: Experimentation and conjecturing

In this study, mathematical inquiry is defined as an instructional strategy of teaching and learning of mathematical objects through "problems". One important aspect in which mathematical inquiry classrooms differ from conventional classrooms is in the treatment of "problems". The nonhomogeneous nature of schools notwithstanding, instructional practices in most mathematics classrooms are plagued with structured problems, questions and activities that take only a few minutes to respectively answer. Makar (2014) points out that such instructional practices contrast sharply with mathematical inquiries which often involve problems that could take days or even weeks to solve and whose solutions often contain a number of ambiguities.

Various meanings are ascribed to the term “problem”. Some people use it to define mathematics while others use it to refer to routine exercises designed to yield mastery of procedural skills. According to the American policy document *Principles and Standards for School Mathematics*, a problem requires engagement in an activity ‘for which the solution method is not known in advance’ (National Council of Teachers of Mathematics [NCTM], 2000, p. 52). Here, I adopted Makar’s (2014) definition of a problem in the context of mathematics. She claims that in the context of mathematical inquiry a “problem” does not refer to the task as in a textbook, but rather to the larger contextual issue to which there is no readily available procedure for finding the solution. But, what is important in inquiry is that the solution to the problem needs to be strongly underpinned by mathematics. In other words, problems in mathematical inquiry require mathematisation, that is, application of mathematics to an authentic and illstructured contextual problem.

According to Makar (2014), mathematical inquiry is a process of solving illstructured problems – that is, problems whose solutions were typically not “right” or “wrong” but require the learner to justify their conclusion, including the process used to reach it – that significantly relies on mathematics in the solution process. However, most problems in school mathematics are well structured in that they are clearly defined and learners enter the solution process with a limited number of pathways to reach a successful solution. This process involves connecting all four of these elements, purpose-question-evidence-conclusion (Makar, 2014).

Mathematical inquiry, which is very different from discovery learning where learners are expected to “discover” the mathematics they need and the teacher provides little input during investigation, requires high quality scaffolding and expertise from the teacher in knowing how to balance when to step in and when to allow learners to wrestle with challenging ideas. Judging by the features of mathematical inquiry, argumentation aligns closely with mathematical inquiry (Hunter, 2006; Makar, 2014). In this study, the learning activities in which learners conduct investigations, make conjectures, perform measurements and constructions in authentic everyday problems that can be mathematised constitute mathematical inquiry. Makar (2014) points out that in mathematical inquiry, learners are provided with multiple opportunities to use their contextual understanding in building mathematical concepts and structures that underpin the problem and create a need for learning mathematics. Seen in this light, mathematical inquiry need not to be seen as a learning and teaching approach geared towards fulfilling a utilitarian perspective of mathematics but as using real-world problems as a context to the application of mathematical concepts.

In respect of the distinction between proof and proving, the former is an object, a product, and the latter the activity associated the search for a proof (A.J. Stylianides, 2007). However, it is important to mention that proving may involve arguments which ultimately do not lead to proof as defined in the foregone subsection. Also, when referring to nonproof or the colloquial sense of proof, I used the term "empirical argument" to refer to inductive proof or proof by examples or rather put the word proof in inverted commas to refer to its nontechnical meaning. Ideally, proof as a product begins with experimentation involving construction, measurement and observation. This experimentation can either be done by hand or with the aid of DGS. Experimentation is closely associated with, in Felbrich, Kaiser, and Schmotz's (2012) terms, an individualistic culture where the individual participates in the generation of mathematical ideas rather than merely fitting in what authority transmitted. For an elaborated description of the different cultural notions, the reader is referred to Hofstede (1986).

However, experimentation is a result of investigations triggered by the need to prove the truth of a conjecture. Flowing from experimentation is inductive reasoning, usually resulting into some unproved generalisation, a conjecture. On the one hand, a conjecture is a mathematical proposition whose veracity has not been established yet. On the other hand, a generalisation is a proposition that has been accepted as true by a social group (Reid, 2002). This distinction notwithstanding, I used "generalisation" to refer to a conjecture whose truth arose from observation or experimentation with a few or selected cases forming a pattern but lacked a deductive proof to work for all cases exhaustively. The impression I gathered from making a generalisation, as a product that reveals a pattern of mathematical objects was that it is crucial for proof construction. For instance, in an examination of the mathematical practice, Lakatos (1976) points out that conjecturing precedes a proof. Personal experience suggests that such a practice is foreign to school mathematics classrooms where the proving environment begins with the use of axioms and definitions.

According to the CAPS document, it is statutory that investigations be an integral part of instructional practices in mathematics classrooms. The emphasis on investigations reflects two important notions; conjecturing and proving:

> Investigations are set to develop the skills of systematic investigation into special cases with a view to observing general trends, making conjectures and proving them. (Department of Basic Education [DBE], 2011, p. 51)

The emphasis on these activities is a direct reflection of clear evidence that the South African school mathematics curriculum is preparing learners for future success. The emphasis on these activities is a direct reflection of clear evidence that the South African school mathematics curriculum is preparing learners for future success. The focus on investigations further substantiated my claim that the South African society and education system embraced a collectivist culture where lack of success is attributed to a lack of effort on the part of the individual learner. Also, that learners were encouraged to engage in investigations of mathematical objects on their own reflected a mathematics curriculum that promoted a dynamic view of mathematics.

On the basis of the arguments invoked so far, I am inclined to assume that conjecturing from the given diagram implied ability to engage in proof activity. Further, the view that for learners to do proof with understanding, effort must be devoted to ensuring that they develop appropriate understanding of the significance of proof was justified. I further argued that an understanding of the functions of proof facilitated the understanding of the coherent nature of mathematical knowledge. Hence, an informed understanding of the significance of proof must include a consideration of proof in each of these five functions it performs in mathematics (Knuth, 2002).

This probably prompted Balacheff (1991) to suggest that there is a long distance between these functions of proof and their manifestation in school mathematical practices. In particular, most challenging is finding more effective ways of using proof for explanatory purposes (Hanna, 2000). However, this need not to be construed as seeing the explanatory function of proof as more important than the others. Hence, I used de Villiers' (1990) multidimensional framework to organise the discussion of the literature on the functions of proof in mathematics. It is in this framework that investigations and conjecturing, enshrined in the CAPS, aligned well with mathematical inquiry which in turn aligned with argumentation. I now turn to defining what I meant by the term "argumentation" in this study.

2.5 The notion of argumentation

> The skill of developing a logical argument in a geometric setting can focus on a diagram with certain given information. The students are asked to arrive at a conclusion based on the given information. (Hoffer, 1981)

Part of our daily routine as humans involves engaging in arguments and argumentation in attempts to clarify or challenge the rationality of propositions, actions or claims during debate, dialogue, conversation or persuasion. Although in mathematics education there is no shared definition for argumentation (Pedemonte, 2007), for the purpose of this study and in order to place the critical discussion on argumentation in context, it is imperative that I define what I mean by argumentation. But, first, perhaps van Eemeren, Grootendorst, Johnson, Plantin, and Willard's (2013) provide a comprehensive definition of argumentation that did not rely on formal logic (use of premises accepted as true to start an argument as in, for example, Euclidean proof). They define argumentation as:

> a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader to by putting forward a constellation of

> propositions intended to justify (or refute) the standpoint before a rational judge. (van Eemeren et al., p. 5)

According to them, a rational judge is an authority – which could be an existent person or an abstract ideal – to which the assessment of an argument is entrusted. This definition is found not only to be compatible with the practice of mathematicians but also with the CAPS guidelines on handling prescribed tasks. In the latter, emphasis is placed on ensuring that the mathematicians' practices were reflected in high school mathematics as well. That is, learners themselves needed to engage in a line-by-line explanation of the proof and in that process invite argumentation from their peers. Thus, argumentation is indeed an integral part of proving in mathematics.

That notwithstanding, I saw argumentation as the process of linking evidence (information, ground, or datum observed from diagram) to claim (answer to a question in Euclidean proof) where the statements which connected evidence to claims were referred to as the warrants (reasons). My understanding of argumentation is informed by that of Toulmin (2003) who describes it as a process in which substantiated (warranted) claim are made on the basis of data. Typical argumentation in everyday sense involves interactions wherein participants rely on oral or written information to make (1) claims and support them with (2) evidence, both of which can be rebutted (Berland & Reiser, 2008; Toulmin, 2003). So I saw an argument in this thesis, unlike in logic which is a deductive process involving two or more premises resulting in a conclusion, as constituted by data, claim, warrant, and a rebuttal to evaluate the strength of a claim. Thus, in this sense, a logical conclusion is a result of two or more claims. Figures 2—1 and 2—2 illustrate the similarities between deductive proof and argumentation where D = data, C = claim, W = warrant, and R = rebuttal.

I agree with Osborne, Erduran, and Simon's (2004) distinction between "argument" and "argumentation". An argument is regarded as a referent to the claim, data, warrants and backings that form the *content* of an argument and argumentation is viewed as a referent to the *process* of arguing. I argue that since mathematics is viewed as a human activity whose proofs (results) require communication (interactions) among members of the mathematics community, arguments are an integral part of the subject. Figure 2—2 shows how the diagram is used as an argumentation prompt (instrument), that is, to make learners engage in written argumentation.

Argument	**Example**
C: My *statement is* that …	$\hat{e} = \hat{c}$
W: My reason is that …	*Alternating interior angles*
R: Arguments against my idea might be that …	*But, the lines DE and BC are not marked as parallel*

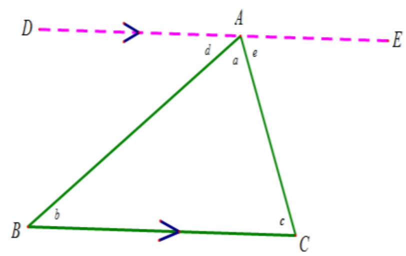


Figure 2—2. An example of an argument in Toulmin's (2003) sense

Argumentation as a social activity is evident in a discourse between two or more interlocutors as they defended their claims and made counterclaims when doing proof. Thus, argumentation in mathematics lessons has become a means to better understand proving processes in class (Reid & Knipping, 2010). Drawing on Lakatos' (1991) perspectives, proof is defined as a product of a process that entails the use of arguments to formulate conjectures that are consistent with evidence whose validity is agreed upon by the mathematics community at a given time. Also, Menezes, Viseu, and Martins (2015) define mathematical proof as a process of argumentation. These perspectives of proof as a particular kind of argument presupposes a relationship between argumentation and proof (Conner, 2007). For instance, Knipping (2003) define argumentation as 'a sequence of utterances in which a claim is put forward and reasons are brought forth with the aim to rationally support this claim' (p. 34).

Further, Aberdein (2012) characterises mathematical proof as an argument. In support of this standpoint, Boero, Garuti, and Mariotti (1996) argue that embedded in the proving process is some continuity – labelled as cognitive unity – which takes place between the construction of a conjecture and the construction of the proof. Before turning to the next section in which I explore functional understanding of proof in mathematics, it is important to end this section by providing a definition of cognitive unity as seen by Boero, Garuti, Mariotti (1996):

> During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ('arguments') produced during the construction of the statement according to a logical chain. (p. 113)

In the classroom, the pursuit of cognitive unity helps learners to connect the two fundamental aspects of reasoning, argumentation and mathematical proof, at the same time. It is precisely for this reason that I claim that argumentation cannot be more than a benefit for the task of constructing a proof.

2.6 Studies on functional understanding of proof

As already alluded to, mathematics education research has shown that most learners have serious difficulties with constructing proofs. Harel and Sowder (1998) locate the cause of learners' difficulty in the logical aspect of proof construction. Thompson, Senk, and Johnson (2012) argue that some of the most persistent proof-related difficulties identified among learners in secondary school and university are a consequence of the confusion about the functions of proof in mathematics. The motivation to conduct this study emanated partly from the premise that to construct proofs in geometry all five functions need to be understood by learners (de Villiers, 1990; Grigoriadou, 2012; Knuth, 2002). Numerous studies show that even South Africa's top mathematics learners perform poorly, on average, compared to their peers in both Africa and the rest of the world (Moloi & Chetty, 2010; Reddy, 2006; Schollar, 2008; Soudien, 2007) notwithstanding the substantial investments in education over the past two decades (Moloi & Chetty, 2010; Reddy, et al., 2012; Taylor, Van der Berg, & Mabogoane, 2013).

Very few can readily disagree with the contention that no single explanation accounts for the low scholastic achievement in Euclidean geometry. However, there is scarcity of empirical evidence on the influence of learners' views on functional understanding of proof. Through this literature review, I extended the existing knowledge on the relationship between functional understanding of proof and argumentation ability. For instance, Knipping (2003) recommends that it would be interesting if the relationship between functions of proof and argumentation structures were examined.

The present study is aimed at advancing research on proof by arguing for the need to capture learners' functional understanding of proof in mathematics as a precursor to doing deductive proof. Most research studies have focused on proof and proving as content of the curriculum to be learnt and taught. For instance, Knuth (2002) investigated teachers' conceptions of proof, Wu (2006) and Chin and Lin (2009) focused on learning how to read and write proofs, Hanna and Barbreau (2008) investigated ways to learn proof, and Harel and Sowder (2007) investigated the teaching of proof. A relatively small number of studies has discussed the functions of proof in mathematics (for example, de Villiers, 1990), Bell, 1976, & Hanna, 2000). Almedia (2000) captured functional understanding of proof through a survey; but, only that of university students. Chin and Lin (2009) conducted an investigation in which they were interested in comparing the performance of Taiwanese learners against United Kingdom (UK) learners in Grades 7–9 proof content.

In contrast, Alibert and Thomas (1991) discusses the relationship between functional understanding of proof largely from a theoretical basis rather than conducting a systematic investigation. They believe that learners' distorted understanding of the functions of proof is a direct consequence of instruction that presents proof as a finished product; an approach that deprives learners of opportunities to be partners in mathematical knowledge construction. In this study, the definition of the term "instruction" is compatible with that of Cohen, Raudenbush, and Ball (2003) who used it to define the interactions among teacher-learners-content in classroom environments.

As far as I could ascertain, only Healy and Hoyles (1998) attempts to capture learners' functional understanding of proof. They conducted a nationwide (England and Wales) survey of 2 459 Grade 10 learners' functional understanding of proof in mathematics and how those learners constructed logical arguments (proof) in algebra and geometry. In particular, they used an open-ended survey questionnaire on which learners were to write about everything they knew of proof and its functions in mathematics. Further, they investigated the influence of statutory instruction on the nature of proof following suggestions that such instruction could contribute to deeper understanding of the notion of proof itself and thus improve its didactic treatment in the classroom. They found that the function of proof as a means to verify was prevalent. Hanna (1995) posit that learning about the functions of proof in mathematics is of primary importance to mathematicians. In the same vein, I contend that the value of understanding the functions of proof in mathematics needs to be reflected in the mathematics classroom itself.

However, because of limited resources, this study only investigated functional understanding of proof and the factors that accounted for the understanding from the perspectives of learners only. In addition, this research project focused on exploring learners' functional understanding of proof in mathematics and, unlike Healy and Hoyles (1998), not on examining learners' competence in distinguishing between deductive and empirical arguments. This must not to be construed as suggesting that such exploration of learners' competence is immaterial. In addition, whilst I acknowledge the influence of a complex set of challenges inherent in geometry education (for example, language of instruction, resources, class sizes, quality of teacher, and so on), I argue that insight into learners' functional understanding of proof has the potential to significantly improve learning of proof. As already alluded to, I take the view that it is improper to expect learners to develop mathematicians' understanding of the functions of proof unless explicit instruction is directed at providing them with opportunities and experiences that reflect mathematicians' practices.

2.7 Studies on argumentation

I begin this section by touching on two prominent theories on argumentation. Perelman and Olbrechts-Tyteca (1982) and Toulmin (2003) are the most influential theorists on argumentation. Perelman Olbrechts-Tyteca (1982) tries to find a description of techniques of argumentation used by people to obtain the approval of others for their opinions. Toulmin (2003), the other influential writer, developed his theory (starting in 1950's) in order to explain how argumentation occurs in the natural process of an everyday argument. He calls his theory "the uses of argument". Ribeiro (2012) points out that Toulmin's model focuses precisely on studying the structure of arguments. In contrast, Perelman and Olbrechts-Tyteca's (1982) model does not seem to give rise to a structure that demonstrates how the components of an argument are related; in fact, they see argumentation as a process opposed to mathematical proof. Of course, as discussed earlier, I see proof and argumentation as inseparable.

As Aberdein (2005) points out, Toulmin's *The Uses of Argument* is arguably the single most influential work in modern argumentation theory. Toulmin's (2003) model explains how the six components of an argument link and also how the argument structure can be employed to analyse arguments. For these reasons, Toulmin's scheme was useful in determining learners' competence in generating arguments to support claims in this study. Further, Toulmin's (2003) theory focuses on argumentation wherein the conclusion, belief or claim is produced by reasoning (justification) as the starting point for the construction of arguments. Hence, its account of argumentation has been found to be insightful on the basis that he focuses on the rhetoric of mathematical practice, arguments (Shapin, 2002). In addition, Toulmin's (2003) layout is intended to encompass all forms of argument, mathematics included (Aberdein, 2005). Taking this brief analysis into account, a "Why?" question calls upon the interlocutor to justify their position which in turn transforms a mere statement into an argument.

Using Toulmin' (2003) argument structure (TAP), Pedemonte (2007) not only describes a proof through argumentation, she also shows that argumentation is useful in the production of a conjecture. Hence, Lakatos (1991) views proof and conjecturing as inseparable. I could only concur with him for the simple reason that conjecturing is an activity undertaken to arrive at a mathematical proof whose validity would eventually be subjected to scrutiny through engaging in the process of argumentation. More specifically, following an analysis of mathematical proofs, Aberdein (2012) concludes that proofs consists of a number of argument structures rather than a single argument structure. In this study, I only focused on requiring participants to make single arguments which by definition, only comprised a claim, data, warrant and a rebuttal, for the purpose of gaining insight into learners' ability to construct an argument.

According to Mariotti (2006) the learner must make sense of this difference between argumentation and proof, without rejecting one for the other. In her characterisation of proof and argumentation, like Pedemonte (2007), she argues that proof is a special case of argumentation, and I agree. She further points out that argumentation, the process of supporting the truth of a particular proposition, introduces learners to the practices of the mathematics discipline.

Some research studies showed that cognitive unity exists between the construction of a conjecture and the construction of a mathematical proof (Boero et al., 1996). In addition, recent researchers suggests that the major goal of teaching mathematics is to develop learners' abilities to establish and defend their own positions while respecting the positions of others (Idris, 2006). Understanding of the functions of proof were found to correlate with the ability to engage in argumentation or proof construction task (Clark & Sampson, 2008; Conner, 2007; Hanna, 2000). For this study, argumentation reflected the communication function of proof in mathematics. Although current research in mathematics education does not offer much insight into the relationship between proof and argumentation, both processes are characterised by being conducted when someone wants to convince (oneself or others) about the truth of a proposition (Pedemonte, 2007). Thus, I viewed proving as an activity that begins with the construction of an argument which is accomplished through argumentation. In this case, Toulmin's model is seen as a powerful tool to characterise the two types of arguments discussed in this study: empirical (informal) and deductive (formal) arguments.

In mathematics education, Krummheuer (1995) started the trend of using Toulmin's scheme of conclusion, data, warrant, and backing by analysing and documenting how learning progresses in a classroom. However, he employed a reduced version of the original scheme, omitting the use of the rebuttals and qualifiers and his study focused on primary mathematics (grade 2). Pedemonte (2007) investigated the structural differences between proof and argumentation. The study took place when 102 high school learners in France and Italy began to learn proof. The learners had prior experience of proof as a means to systematise and knew the theorems necessary to solve the proposed problems. She found that open problems which ask for a conjecture appear to be extremely effective for introducing the learning of proof and that argumentation activities seem to favour the construction of a proofs.

In a comparative study on proving processes in French and German lessons on the theorem of Pythagorean, carried out by Knipping (2003), it was found that proving discourses allow for reflection on underlying functions of proving in class. Conner (2008) study examined the argumentation in one preservice teacher's high school geometry classes and suggested a possible relationship between the observed argumentation and the preservice teacher's understanding of proof. She conducted two semi-structured to infer the teacher's understanding of the concept of proof from her responses. Using Toulmin's scheme, she found that there were difference in the order in which components of an argument were presented. She also found that the teacher's understanding of the functions of proof—as, for example, a means to explain why a statement were true—influenced the support for argumentation in her classroom.

2.8 Perspectives on school type and gender

The exploration in this study involved participants who attended differently resourced schools specifically based on their location and were of at least two genders, male and female. According to Lee and Zuze (2011), the level of school resources make more of a difference in economically developing countries like South Africa than in economically developed countries. A longitudinal study by Healy and Hoyles (1998) found that gender was significantly associated with learners' competence in the construction of proofs. Therefore, the role of school resources and gender on proof functions were reviewed here on the basis that some of the findings could be explained from these perspectives.

2.8.1 School resources

South Africa is a country wracked by rampant inequalities in economic circumstances and educational provision that has resulted in an education system characterised by two different school resource levels (Bertram & Hugo, 2008; Soudien, 2007). In the context of public schools, a minority of resourced and successful schools are found in established urban, middle class areas (Brodie, 2006). In contrast, about 85% of under-resourced schools (Bloch, 2009; Grant, 2014) whose performance was abysmal, are found in townships, rural communities, informal settlement areas of either tin shanties erected by the people themselves or small brick houses, and on farms (Bloch, 2009; Lubben, Sadeck, Scholtz, & Braund, 2010).

Reddy, Prinsloo, Visser, Arends, Winnaar, Rogers, and Mthethwa (2012) point out that where a school is located can have a substantial impact on whether its learners typically are from economically and educationally advantaged home backgrounds and thus able to provide access to important additional resources such as libraries, media centres, or museums. More important for this study, Reddy et al. (2012) found that those schools with resources specifically aimed at supporting mathematics instruction such as providing specialised teachers, computer software, library materials, audiovisual resources, and calculators, tend to perform better.

Very little has changed in terms of resources even under the new democratically elected government, especially in previously disadvantaged schools. This evidence suggests that the promise of equal distribution of resources is yet to materialise (Sedibe, 2011). In this study, I limited school resources to specifically learning and teaching support material (LTSM), particularly DGS which tend to be useful in demonstrating the verification and discovery functions of proof (de Villiers & Heideman, 2014). The DGS has the drag mode that makes it possible for the learner to continuously experiment by varying geometric configurations so as to quickly and easily investigate the veracity of particular conjectures (de Villiers, 1998). I expect no argument from any sensible person contesting that such software is rarely found in previously disadvantaged townships schools. Thus, it is in this context that in this study previously disadvantaged schools were viewed as under-resourced and the others as resourced.

Accordingly, given real differences in schools' resource levels (Soudien, 2007), there is a significant difference in argumentation quality between learners in resourced and under-resourced schools (Lubben et al., 2010). This disparity in Euclidean geometry educational experiences contributed to gaps in learner achievement. Where resources are not equally distributed the inevitable consequence is inequitable access to mathematical knowledge and this did not contribute to the attempts to arrest the notoriously persistent trends of poor learner performance in Euclidean geometry. Thus, assessment on resources showed that learners in under-resourced schools tend to perform poorly notwithstanding attempts by policymakers to redress the conditions under which achievement gaps grew.

2.8.2 Fee-paying and no-fee-paying schools and mathematical proof

South Africa's public schools were previously divided into five categories called "quintiles", according to their poverty rankings based on the assumption that schools in wealthier communities were better able to raise funds and therefore required less financial support from government. According to the South African Government News Agency (2016), the Department of Basic Education (DBE) planned to introduce a two-category system which classified schools as either no-fee paying or fee-paying effectively scrapping the system that divided schools into quintiles. The introduction of the new system is necessary owing to the fact that the quintile system has become difficult to implement as it is based on many different criteria and that in some areas, the question is whether parents could afford to pay or not (South African Governmemt News Agency, 2016).

On the basis of this background, the study investigated functional understanding of proof and argumentation ability in public high schools in and around Durban that were resourced as well as those that were under-resourced. Lubben et al. (2010) refers to a “resourced school” as a school with: up to standard sports fields for a variety of extra mural activities; equipped laboratories; Western European cultural practices; learners’ first language coincides with the teaching and learning language; and, a range of LTSM including a library and internet access. In addition, these schools have the wherewithal to employ and pay, from their coffers, additional teachers (over and above those paid by the government) and thus reduce learner-teacher ratio.

In contrast, the term “under-resourced” school refers to a school that lacks sports fields for a variety of extramural activities, equipped laboratories, library or internet access, is characterised by indigenous cultural practices such as emphasis on respect for the elders and teachers, has large class sizes, and has learners whose mother tongue differed from the medium of instruction. The lack of sporting facilities makes learners turn their playgrounds into rudimentary sports field (Kane-Berman, 2017). I see the former as a referent to "fee-paying" school. The latter, in contrast, is seen as a referent to a "non-fee paying" school. The no-fee policy is a national poverty ranking system which divided all schools into quintiles status in an effort to redistribute resources and improve access to quality education for learners from poor socioeconomic backgrounds. A fee-paying school is a school which charges fees to parents of its learners.

2.8.3 Gender and mathematical proof

Gender is important to consider as a characteristic because, as Hofstede (1986) points out, within certain cultures there is a higher degree of differentiation and inequity between genders than others. The differences among the genders are found in schools; microcosm of society. In simple terms, the differences in gender performance in functional uderstanding of proof and argumentation ability may be assumed to be driven by cultural factors (for example, gender roles). Support for this approach is found in Willingham and Cole's (1997) argument that 'young women [scored] higher than young men on domestic, artistic, writing, social service, and office service vocational interests and young men [scored] higher than young women on business, law, politics, mathematics, science, agriculture, athletics, and mechanical interests' (p. 178). Geary (1998) presents evidence to support the assertion that on average, it appears that women tend to be more interested in careers that involve organic matters, for example, biology and medicine as opposed to men who tend to be more inorganic matters, for example, physics and engineering. In the words of Geary (1999),

> [s]exual selection (male–male competition in particular) has resulted in a greater elaboration of the cognitive and brain systems that support navigation in physical space in men than in women. One feature of these systems is an intuitive understanding of Euclidean geometry. (p. 272)

Interesting arguments have been made on this issue of gender differences in which there seems to be empirical support for the notion that the mathematical domain of Euclidean geometry seems to favour male learners than female learners. In addition, Healy and Hoyles (2000) found that the gender differences in the belief that empirical arguments were proof existed also among learners in algebra. The source of this gender difference is assumed to be social and occupational interests.

2.9 Chapter summary

This chapter provided a critical engagement with the mathematics education research literature by situating this study in terms of previous studies and ideas in literature related to the functions that proof performed in mathematics and argumentation. The case of functional understanding of proof in mathematics and argumentation as essential precursors for motivating learners to construct proof was made. Such understanding includes viewing proof as a means to verify, explain, communicate, discover, and systematise. The interaction of functions of proof and argumentation was discussed by showing that like functional understanding, argumentation is key in doing proof successfully. Literature reviewed here pointed to the fact that school resources and gender differences may influence learners' functional understanding of proof as well as argumentation ability. In the next chapter, I show how I critically engaged with the relevant theories and concepts selected to investigate functional understanding of proof, argumentation, and factors influencing beliefs about the functions of proof.

Chapter 3

The theoretical frameworks

> In simple terms, a theoretical framework involves the presentation of a specific theory, such as systems theory or self-efficacy, and empirical and conceptual work about that theory … a case is built for the importance of the study through a presentation and critique of the concepts, terms, definitions, models, and theories found in a literature base. (Rocco & Plakhotnik, 2009, pp. 125-126)

3.0 Introduction

In the previous chapter I critically examined previous studies and ideas on the functions of proof and argumentation to build the foundation for the theoretical framework with a view to establish the relationships among the various concepts in each one of the two theories underpinning this study. In this chapter, the axiomatic system is briefly discussed. Then, the van Hiele (1986) theory which emphasises the hierarchical nature of geometric thinking as well as Toulmin's (2003) theory which provides a layout of how an argument is structured are critiqued. For both theories, argumentation is an overarching theme; proof is viewed as a specialised form of argumentation. Next a description of the functions of proof is provided followed by an interrogation of the beliefs about the functions of proof (which encompass constructs such as semantic contamination, collectivist culture, empirical arguments, teacher, and textbook). The remainder of this chapter describes the conceptual framework from which the research questions guiding this study are presented.

3.1 Axiomatic geometry in a nutshell

In this section the essential aspects of the axiomatic system with reference to geometry are summarised. The geometry branch of mathematics which Euclid organised into an axiomatic deductive system to study shapes on planes (flat surfaces) is called Euclidean geometry in his honour. Prior to Euclid, geometry was empirical in the sense that it had to do with measurements and constructions given that perception of the world was based on sensorial experiences.

Specifically, answers to practical problems were arrived at by inductive reasoning; a process that involves making generalisations (conjectures) from recognition of patterns as a consequence of observations. However, inductive reasoning is at the heart of science and at the beginning of mathematics (Serra, 1997).

Euclid pioneered the teaching and learning of Euclidean geometry without measurements and constructions (the use protractor, divider, pair of compasses or ruler). Simply put, he was interested in the establishment of the generality of mathematical propositions rather than their verification by sensorial experiences. To that end, he began his study by establishing 10 axioms using the three undefined building blocks of geometry; point, line, segment. These were left undefined because an attempt to define them required the use of words or phrases that themselves needed definition or further clarification. In other words, they are a set of basic assumptions about the primitive objects of mathematics – like points, straight lines, segments, and planes – accepted as truths without proof because they are obvious. These objects are considered primitive in the sense that they cannot be described in terms of simpler concepts. Or better still, these axioms are called postulates because they are self-evident truths that cannot be deduced from others (Serra, 1997).

Euclid's theory is based on 5 axioms, 5 postulates, and 23 definitions. He called the axioms peculiar to geometry "postulates" and those common in both geometry and other domains "common notions" which included concepts such as the transitive, addition, and subtraction properties of equality, the reflexive property; and the notion that the whole is greater than the part. Briefly, nowadays, Euclid's "common notions" and "postulates", are both called "axioms" (Reid & Knipping, 2010). Euclid's book, *Elements*, is a structured presentation of the mathematics of that time. The special historical character of Euclid's definitions is not relevant to the present discussion, so I leave them apart and focus on axioms. The axioms in Figure 3—1 below are stated in modern terms to facilitate understanding.

Axioms	
Common notions	**Postulates**
1. Things which are equal to the same thing are also equal to each other: If $a = b$ *and* $b = c$, *then* $a = c$.	*6. A straight line can be drawn between any two points.*
2. If equals are added to equals, the results are equal: $If\ a_1 = a_2$ *and* $b_1 = b_2$, *then* $a_1 + b_1 = a_2 + b_2$.	*7. Any straight line can be extended indefinitely in a straight line.*
3. If equals are subtracted from equals, the remainders are equal.	*8. A circle can be constructed when a point for its centre and a distance for its radius are given.*
4. Things that coincide with each other are equal to each other, i.e., if a figure (angle or segment) can be moved to fit exactly on top of the other, then it means they are equal, in terms of size.	*9. All right angles are congruent (equal).*
5. The whole is greater than the part	*10. Given a point not on a given line, there exists a unique line through that point parallel to the given line.*

Figure 3—1. The ten Euclid's postulates on which every other proposition is based.

In an attempt to reduce semantic errors and thus ensure that the reader understood his work, he provided 23 definitions of common words, including defining a point as "that which has no dimension", line as "that which has one dimension", and plane as "that which has two dimensions", and so on. These three terms and postulates laid a foundation for a systematic study of geometry. Thus, all proofs of theorems in geometry are based on these axioms. A theorem is defined as a conjecture whose veracity has already been established by a deductive proof.

3.2 The theories underpinning this study

According to Giancoli (2005), theory is a set of concepts offered to explain and order an observed phenomenon. Fox and Bayat (2007) define theory as 'a set of interrelated propositions, concepts and definitions that present a systematic point of view of specifying relationships between variables with a view to predicting and explaining phenomena' (p. 29). Although a theory in the scientific sense is different from a model, in this study the two terms are used interchangeably.

Briefly, the function of theory in this study is to identify the starting point of the research problem and provide the foundation for the conceptual framework and thus establish the focus of the study to make trustworthy findings.

3.2.1 Van Hiele theory of geometric thinking

The van Hiele theory originated from companion dissertations (that is, they worked in a similar area of geometry research but focusing on its different aspects) which were completed simultaneously at the University of Utrecht, The Netherlands, in 1957. Pierre van Hiele devoted his lifetime clarifying, amending, and advancing the theory after Dina died shortly after completing her dissertation. Their work has come to be known as the van Hiele theory, and has helped shape and direct much of the research investigations associated with geometry around the world.

The roots of their theory are found in the theories of Piaget (1978). However, since the van Hiele dissertations and early articles were in Dutch, their findings were not widely disseminated outside Holland until a paper presented in 1957 by Pierre van Hiele to the mathematics education conference brought the theory to the attention of the mathematics education community. The Soviet Union (Russia) educators and psychologists found the paper to be of particular interest and undertook major revisions of their geometry curriculum based on this theory.

Contrary to the claims of Piaget, Inhelder, and Szeminska's (1960) theory, the van Hiele theory suggests that learners progress through levels on the basis of their experiences rather than age, and as such it is imperative that teachers provided experiences and tasks so that learners could develop along this continuum (level 1 to level 4) (Breyfogle & Lynch, 2010). Further, though Piaget et al.'s (1960) theory attempts to explain why learners find geometry difficult, what sets the van Hiele theory apart is its strength in the suggestion of phases of alleviating the problem. Piaget et al.'s (1960) do not go that far. The second strength of the van Hiele theory is that, unlike Piaget el al's theory which applies to several areas of mathematics, it was developed specifically for geometry.

The work of the van Hieles has been presented in Fuys, Geddes, and Tischler's (1984) *English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre M. van Hiele* as part of the research project investigating the van Hiele theory on how people learn geometry. They and subsequent researchers have demonstrated that the van Hiele theory can help improve geometric understanding (Vojkuvkova, 2012; Pusey, 2003) and as such 'has become the most influential factor in the American geometry curriculum' (Van de Walle, Karp, & Bay-Williams, 2010, p. 309), including studies from a South African perspective (for example, Atebe & Schäfer, 2011; de Villiers & Dhlamini, 2013; Luneta, 2015; Siyepu & Mtonjeni, 2014; van der Sandt, 2007; van Niekerk, 2010). That is, the van Hiele has had tremendous influence on geometry education reform in the last half of the twentieth century (Ndlovu, 2013).

The van Hiele theory of learning geometry and de Villiers' (1990) model for the functions of proof influenced my thinking about of the functions of proof in mathematics. The discussion of the theory is modelled around the van Hiele theory's three aspects: characteristics of the levels; properties of the levels; and phases describing steps to help learners progress from one level to the next. This study is confined to the first two aspects; a discussion of the third aspect is beyond the framework of the present study. One remark is worth making, nonetheless. For learners to make progress from one level to the next, the learning process should move through five phases which are not strictly sequential: information, guided orientation, explicitation, free orientation, and integration.

3.2.1.1 General characteristics of the van Hiele theory

The general characteristics of the van Hiele theory are that it is sequential and each level builds on the thinking strategies developed in the previous one. The levels are hierarchical in that advancement to the next level is a function of mastering the thinking strategies of the preceding level(s). Ideas and concepts that are only implied at one level become the objects of study at another level and so become explicit. Each level has its own language and symbols. Therefore, learners working at different levels cannot understand each other's explanations even though they may be describing the same shape or idea. Also important is that teaching needs to match learners' thinking and language. So, if the learner were at different levels, learning cannot take place and as a consequence, progress would be stunted. As already mentioned, progress from one level to the next is more dependent on and can be accelerated by instruction and experiences than age.

3.2.1.2 The five levels of the van Hiele theory

Pierre van Hiele, reporting on the studies that he together with his wife conducted, identified five levels through which learners develop their thinking in geometry. Although originally the van Hieles numbered the levels from 0 to 4, I adopted the American numbering scheme and labelled the levels from 1 to 5. Here I took the liberty to provide an overview of the Levels in the van Hiele (1986, pp. 39-47) theory. In this study, I focused on both general and behavioural terms of the levels but described Level 4 in some detail as it pertained more to this study.

The theory describes the basic Level 1 (visualisation/recognition) as one in which learners recognise shapes on the basis of their physical global, holistic characteristics, like size or position, and therefore formulate their ideas based on visual perception (Usiskin, 1982). At this level, learners need to learn the vocabulary of geometric shapes by comparing the shapes to known prototypes to be able to identify, reproduce, and name a shape as a whole, but not in any orientation (Feza & Webb, 2005). At Level 2 (analysis/description), learners describe the properties of geometric shapes through investigations and practical methods and acquire the appropriate technical terms to make generalisations for classes of figures. Level 3 (informal deduction/ordering) entails learners making sense of definitions although these may be expressed in minimum terms (Lim, 1992). For example, they begin to understand what is meant by the term "proof" in mathematical sense. They also understand the interrelationships between the properties of shapes and see that new results can be obtained by making short chains of deductive arguments based on properties learned from concrete experiences but they may not be able to derive such proofs themselves (Senk, 1989). It is important that at this stage (Level 3) learners are provided with the opportunities to explore, feel and see, build, take shapes apart, and make observations about shapes they created with drawings, models, and computers (Van de Walle, Karp, & Bay-Williams, 2007). These activities involve constructing, visualising, comparing, transforming, and classifying geometric figures.

Level 4 entails understanding of the functions of proof, definitions, axioms, and theorems and making longer chains of deductive arguments (proof) (de Villiers, 1997). The thinking is concerned with logical deduction of new results from axioms, definitions, with theorems and their converses, and the necessary and sufficient conditions in proofs (Crowley, 1987). For instance, on the strength of knowing that, "given parallel lines cut by a transversal, alternate angles are equal and that angles on a straight line are supplementary", a learner can deduce that the interior angles of a triangle are supplementary. As already mentioned, this study was designed to focus on the deductive level which requires learners to be able understand and use the ideas of the Euclidean geometry system. More particularly important, the focus was on one aspect of Level 4 where learners are supposed to understand and hold appropriate understanding of the functions of proof in mathematics. As a consequence, this project sought to explore and understand whether Grade 11 learners' level of geometric thought through measurement of their understanding of the functions of proof. Generally, the van Hiele theory is premised on the understanding that successful construction of proof depends on experiences in thinking at lower levels and specifically an appreciation of its functions in mathematics.

In Level 5 (abstract/rigour), the highest level of the van Hiele theory of development, learners manipulate geometric axioms, definitions, and theorems to compare and establish non-Euclidean geometries. As the label of the level suggests, non-Euclidean geometry is less intuitive and the Euclidean system of axioms that high school learners are accustomed to, are modified. It is worth noting that the first four Levels are the ones mostly pertaining to school geometry and Level 5 is meant for tertiary level geometry courses. Non-Euclidean geometries can also be identified in examples like spherical, elliptical, and hyperbolic geometries. More recently, there has been growing interest in transformation, fractal, turtle, analytical and vector geometries.

3.2.1.3 Critics of the van Hiele theory

I have cited the main ideas emphasised in the theory and illustrated how the main aspects of the theory is related to the research problem. Though I gave an exposition of the theory, to offer a balanced argument, I introduce into the discussion the main proponents and critics of the theory. On the one hand, the van Hiele theory is regarded as one of the best framework known for teaching and learning geometry to date (Wu & Ma, 2006). Whilst van Hiele's model of geometric thinking is of undoubted value in geometric education, I was mindful that there have been some criticism associated with the application of some of its notions of levels and its hierarchical aspect. On the other hand, some studies that have raised questions about some of the characteristics of the theory. For instance, Burger and Shaughnessy (1986) argue that the theory fails to detect the discontinuity between levels and found instead that the levels were dynamic and of a ‘more continuous in nature than their discrete description would lead one to believe’ (p. 45).

Although agreeing with the assertion of the van Hieles that each level has its own language, the study by Fuys, Geddes, and Tischer (1988) also found that learners’ progress was marked by oscillation between levels in different geometric content. Also, Gutierrez, Jaime and Fortuny (1991) found that students can develop more than one level at the same time. As Pegg and Davey (1998) argue, van Hiele’s broad propositions ‘are not as black and white as they are often portrayed to be’ (p. 114). Regarding the levels even van Hiele (1986) himself expresses doubt about the existence or testability of levels higher than the fourth and considered them as of no practical value. I concur with Clements and Battista (1992) suggestion that a pre-recognition level at the lower end of the levels (Level 0) needs to be added to accommodate learners who cannot even identify shapes. Mason (1998) points out that although in terms of the van Hiele theory, a learner cannot achieve one level of understanding without having mastered all the previous levels, research studies in the US and other countries have found that the levels are not sequential as claimed; some mathematically talented learners appear to skip levels, suggesting that they may have developed logical reasoning skills in ways other than through geometry instruction.

For this study, the only drawback related to the learning phases which are meant to move a learner from one level to the next in the van Hiele theory. Although appreciation of the systematisation function of proof is supposed to be achieved at Level 3 (informal deduction/ordering), the need to develop an understanding of the functions of proof is only explicitly introduced at Level 4. I am inclined to suggest that within Level 4, two sublevels, “Functional understanding of proof” as well as “Argumentation” be introduced prior to construction of proof. This view is consistent with de Villiers and Njisane’s (1987) suggestion that the theory needs refinement with regard to the levels at which deduction takes place. In addition, empirical research suggests that functions such as explanation, discovery, and verification can be meaningful to learners at Levels 1 and 2 when introduced gradually. For example, de Villiers (1996) points out that the function of communication pervades geometry education. De Villiers (2004) argues that a prolonged delay renders later introduction of proof as a meaningful activity even more difficult and may also make learners become accustomed to seeing proof as just a means to verify. Despite the criticisms of the theory, it is nonetheless supported by other experts in geometry education as key in understanding learner thinking.

3.2.2 Toulmin's argumentation theory

> For many pupils, proof is just a ritual without meaning. This view is reinforced if they are required to write proofs according to a certain pattern or solely with symbols. Much mathematics teaching in the early grades focuses on arithmetic concepts, calculations, and algorithms, and, then, as they enter secondary school, pupils are suddenly required to understand and write proofs, mostly in geometry. Substantial empirical evidence shows that this curricular pattern is true in many countries. Needed is a culture of argumentation in the mathematics classroom from the primary grades up all the way through college. (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002, p. 907)

Prior to an in-depth explication of Toulmin's theory, I would like to raise and answer one question, which I reckon needs to be taken into account whenever Toulmin's theory is evoked in this thesis: "What is the whole point about argumentation in mathematics?" Perhaps I need to be first clear about what argumentation itself means. Van Eemeren and Grootendors (2004) provide a definition that is consistent with a classroom environments envisaged in most mathematics curriculum reform statements:

> Argumentation is a verbal, social, and rational activity aimed at convincing a reasonable critic of the acceptability of a standpoint by putting forward a constellation of propositions justifying or refuting the proposition expressed in the standpoint. (p. 1)

Further, argumentation theory is the study of arguments that promote informal logic and thus resists deductive logic which is concerned with validity and proof (Aberdein, 2009). In this study, in a word, argumentation is defined as the textual use of one or more elements of TAP and this definition was operationalised as learners linked evidence (given information from diagram) to claim (an assertion in Euclidean geometry) where the statements which connected evidence to claims were referred to as the grounds (axioms, definitions, or theorems).

Schwarz, Hershkowitz, and Prusak (2010) articulate the distinction between argumentation and explanation by stating that '[t]hese two definitions also demarcate argumentative from explanatory activities: in explanatory activities, ideas are clarified, *explained*, but not put into questions' (p. 116). Returning to the question posed earlier, the answer is that argumentation not only helps in the learning of mathematics content, but also provides learners with a window into the practices of mathematicians and by extension into the nature of mathematics. For instance, mathematical practices and content are experienced as learners make their thinking available to others for scrutiny, comparing, and contrasting each other's ideas.

When thinking about argumentation, I drew on Toulmin's (2003) theory. In 1958 he referred to his theory as *"The Uses of Argument"*. Toulmin developed this theory for the purpose of explaining how argumentation takes place in the natural contexts of everyday life and for proofs in mathematics (Aberdein, 2009; Knipping, 2003). Worthy to note is that the model can be used for both developing a theoretical perspective on argument and analysing of argumentation process in classrooms (Simon, 2008). Specifically to note for this section of the study, '[t]he dominant theoretical framework that has been applied to argumentation by educational researchers has been a philosophic model developed by Stephen Toulmin' (Nussbaum, 2011, p. 85). Toulmin (2003) suggests that arguments can be understood using six components comprising: claims, data, warrants which linked data to backings, qualifiers, and rebuttals. This model is still influential in the field of mathematics education (for example, Conner, 2008; Giaquinto, 2005; Krummheuer, 1995; Lampert, Rittenhouse, & Crumbaugh, 1996; Mariotti 2006; NCTM, 2000; Pedemonte, 2007).

Pedemonte (2007) uses TAP as a tool to compare the structures of argumentation and proof and concludes that 'argumentation activity might favour the construction of a proof' (p. 25). However, this need not be construed as suggesting that argumentation can only be associated with proof. The very fact that mathematics is a human activity, it follows that its claims are open to refutations as shown in some of its algebraic generalisations which had survived hundreds of years eventually turning out to be false (de Villiers, 1998). Notably, it is through argumentation within the mathematical community that mathematics often progresses.

Argumentation theory has attracted attention from philosophers, logicians, linguists, law, education, psychology, sociology, political science, and many others (van Eemeren, Grootendorst, Johnson, Plantin, & Willard, 2013). This theory is grounded in conversational, interpersonal communication, but also applies to both group and written communication. In the context of mathematics lessons, the use of TAP has mainly concentrated on the individual learner (for example, Inglis & Mejia-Ramos, 2009; Knipping, 2003; Krummheuer, 1995; Pedemonte, 2007). For the present purpose, the assessment of learners' written argumentation was performed from the perspectives of TAP. The argumentation theory involves arguments that resist deductive formalisation thus emphasising jurisprudential over mathematical approaches to reasoning (van Eemeren, Grootendorst, Johnson, Plantin, & Willard, 2013). In argumentation, interlocutors make claims (conclusions to be evaluated) and defended them, whereas in logic the focus is on conclusions derived from premises. Hence argumentation is also often equated with informal logic.

According to Abdullah and Mohamed (2008), learners' inability to argue make it difficult for them to achieve higher levels of geometric thinking as proposed by the van Hiele model. In addition, argumentation need not be viewed as low level when compared to formal reasoning; they simply represent different tools for appraising human arguments (Aberdein, 2009). As Hanna (2007) suggests, it is widely argued that argumentation and proof need to be central to the practice of learning school mathematics. I was interested in investigating the relationship between learners' functional understanding of proof and their argumentation ability in written form as they make mathematically acceptable rather than logical arguments or everyday talk.

Very few will readily contest the notion that argumentation is a central element in learning to construct proofs. In fact, the centrality of argumentation is exemplified by the communication function of proof which involves the construction of an argument to justify one's claim to the mathematics community using data (evidence). It is through argumentation that controversies resulting from refutations of mathematical propositions or proofs are resolved. More fundamentally, argumentation is a practice that is at the heart of the development of the mathematics discipline. Thus, another answer to the question posed earlier in the section can be summarised by pointing out that although proof is what sets mathematics apart from other sciences, the sense making activities that pave the way for the construction of a deductive proof entail argumentation. In the next section, I describe the constituent components of TAP. Next, I first critiqued the six elements in Toulmin's (2003) argument structure, then consider the criticism levelled against his theory.

3.2.2.1 The six constituent components of Toulmin's argument pattern

Toulmin's argument pattern is a model that decomposes an argument into six constitutive elements and describes the relationships between them: claim, data, warrant, backing, rebuttals, and qualifiers. The elements are further categorised into two triads. The first triad, deemed necessary to make a good argument, comprises claims, data, and warrants. As an example of a mathematical argumentation, Figure 3—2 describes an argument relating for the proposition that "The sum of the interior angles of a triangle equals 180^0".

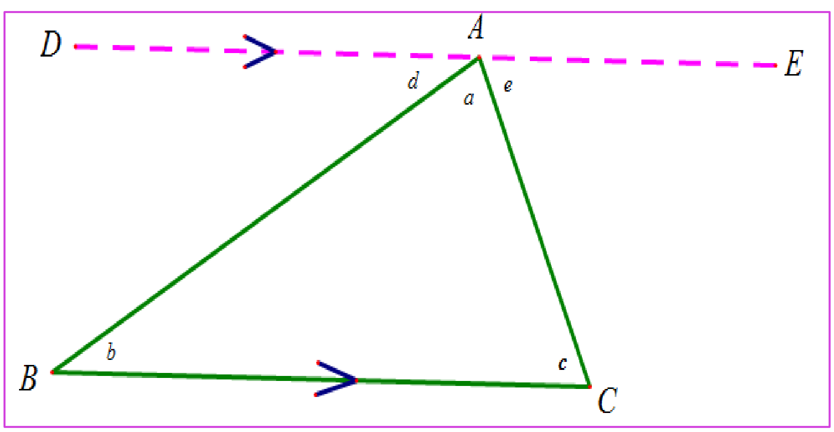


Figure 3—2. The diagram used to make a claim of an argument

The adequacy of Toulmin's model for Euclidean geometry is depicted in Figure 3—3; an illustration of elements of an argument and their relationships in a Euclidean geometry context. In trying to argue their case, a learner's argument structure could take the following form. They make a *claim* (assertion or conclusion) that "angle b = angle d". In this case, a claim is regarded as the point an arguer wants an interlocutor to accept. Asked "How do they know that?", they respond by saying that they observed the given *data* (Figure 3—2 and its labels) and saw that parallel lines were cut by a transversal.

When the arguer began to link the claim with data and mention the theorem that "If parallel lines were cut by a transversal, then alternating angles were equal", that showed that the data *warranted* their claim. In other words, the learner warranted their claim by coordinating evidence (data) with their claim. Thus, the warrant performed a linking function and is typically implicit and therefore often left unstated; hence the dotted ellipse in Figure 3—3. The warrant needs to be a universal proposition and therefore shared among members of the field of Euclidean geometry. When a warrant is unstated, it is the interlocutor's responsibility to recognise the underlying reasoning that led to the claim in light of the data on which the claim is based. In general terms, both the arguer and the interlocutor were engaged in an argumentation process that used perspectives of a mathematics community to which they both belonged. The claims, data, and warrant constitute a primary or basic argument.

Toulmin (2003) adds three more elements of an argument to supplement the primary elements constitute the second triad: *backings*, *rebuttals*, and *qualifiers*. In an argument, the level of confidence with which the claim is made can be indicated; using terms such as "probably", "possibly", "I think" or "perhaps". Also, when the claim is challenged, a warrant, which is the logical connection between the data and the claim, is provided to support it and thus strengthen its validity. In an attempt to provide additional information to support the warrant, a backing is provided. Toulmin (2003) defines backings as the 'other assurances, without which the warrants themselves would possess neither authority nor currency' (p. 96). Put another way, a backing is used to justify why the warrant is a rational assumption.

In this example (Figure 3—3), to support the warrant and answer the question "How do you know that your reason is correct?" The learner could defend the warrant by appealing to a proposition that is commonly shared by indicating that, "The theorem states that alternating angles are equal". Then, the backing is a statement derived by appealing to an axiom, definition, principle, or theorem to support the warrant. However, the claim or warrant could be challenged by a statement that showed exceptional circumstances under which it may not hold, and this sort of statement is referred to as a rebuttal. For example, the warrant provided by the learner could be challenged to show exceptional circumstances under which it did not hold true: "Does your reason that alternating angles are equal work on a sphere?" Then, the response from the learner may be that "My reason applies to plane geometry only".

Rebuttals are necessary to include because they make an argument more nuanced and complete as they demonstrate that the arguer took opposition to his or her claim (or warrant) into account. In addition, rebuttals force the arguer to think beyond their claim as they anticipate potential challenges to their claim or ground. Toulmin (2003) points out that rebuttals not only challenge claims but also warrants by showing exceptional circumstances under which the warranted conclusions were incorrect, in which case the warrants has to be set aside. It is therefore incumbent upon the arguer to anticipate any challenge to the generality of their statements, that is, to leave very little room for a statement that may collapse the structure of their argument.

The elements of Toulmin's (2003) scheme considered in this study are set out and put to use in Figure 3—3. Altogether, the structure of the argument presented here can be summarised as follows: (1) ***Given*** *that BC is parallel to DE (D), and* ***since*** *parallel lines cut by a transversal make alternating angles equal (W),* ***so*** *(Q), angle b is equal to angle d (C)* ***on account of*** *the theorem stating that "If two parallel lines are intersected by a transversal, then alternate interior angles are equal" (B)* ***unless*** *the surface is hyperbolic or spherical (R).* Essentially, these are the elements of Toulmin's model that rationally stand against scrutiny.

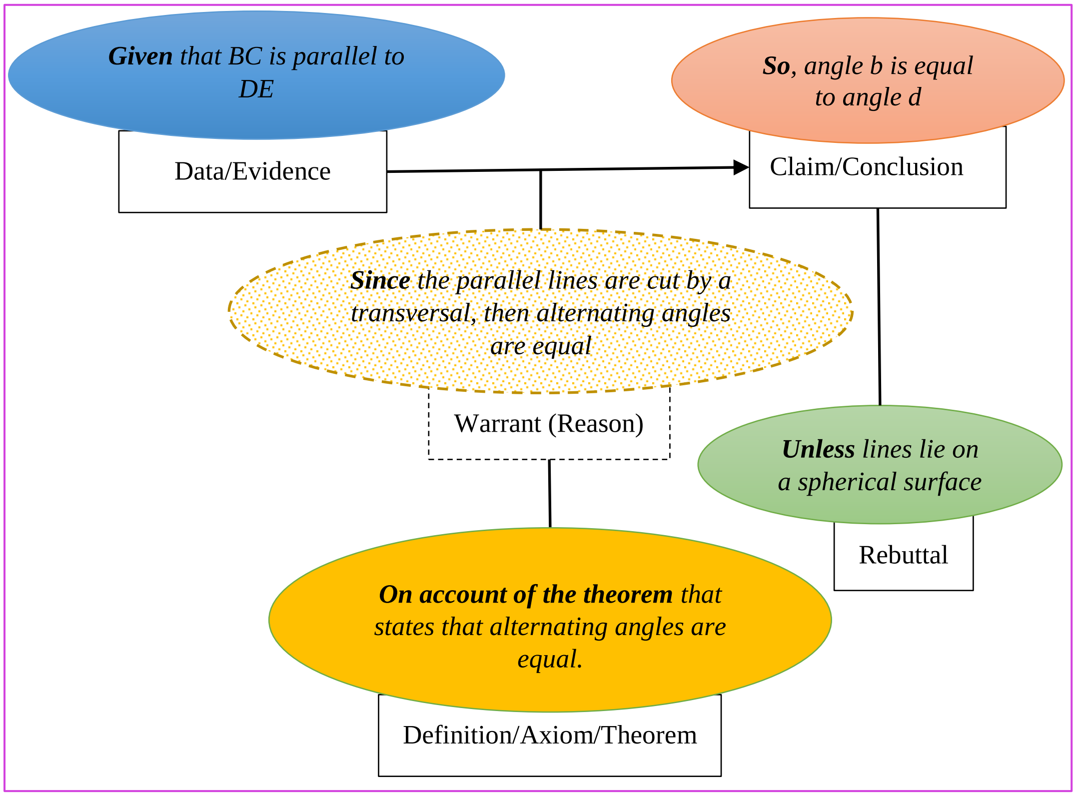


Figure 3—3. An argument in Toulmin's structure (Adapted from Toulmin, 2003, p. 97)

Notwithstanding that qualifiers, backings, and rebuttals are used less often in analysis of argumentation in mathematics education (Inglis, Mejia-Ramos, & Simpson, 2007), only rebuttals in this triad were considered and thus formed part of the analysis process in this study. Returning to the decision to exclude qualifiers, data, and warrants, and include rebuttals, I provide three reasons. First, from a cognitive perspective, unlike counterclaims that introduce new ideas rather than challenge a warranted claim, rebuttals provide learners with opportunities to refine their ideas. Second, argumentations with rebuttals are of better quality than those without given that a rebuttal makes a substantive challenge to the warrant as it refutes its applicability (Osborne et al., 2004). Therefore, challenging a warranted claim engenders learners to consider alternate frameworks that can be construed as undermining their thoughts; this improves the quality of their argument. That is, I included rebuttals as part of argumentation because they provide ground for deciding whether an argumentation is of low or high quality.

The third reason is informed by practical circumstances as they obtained in classroom practice. I did not distinguish among data, warrants, and backings since I would have been naïve to expect learners to begin their claims with adverbs as qualifiers, assumed to be a learner's commitment based on the strength of evidence at their disposal such as "probably", "possibly" or "perhaps", without having received explicit scaffolding on mathematical argumentation. Scaffolding takes place when the teacher guides the learner in extending their knowledge through a series of small steps which they would not be independently capable of undertaking on their own (Cakir, 2008).

As Young-Loveridge, Taylor, and Hawera (2005) argue, learners struggle to appreciate the value of reasoning and attending to the ideas of others. Hence, Mason (1996) emphasises the need to provide suitable instruction as a means to support learners in the acquisition of mathematical knowledge and practices characteristic of the mathematics community. That notwithstanding, I acknowledge the value of these adverbs; they reflect the tentative nature of all knowledge, including mathematical knowledge. As already mentioned, mathematics is a human activity and humans are fallible and that arguments are about uncertainty.

Perhaps more importantly, like Osborne et al. (2004), I found that grouping all data, warrants, and backings as grounds circumvents the difficulty for learners to distinguish among these three elements since they were unlikely to have received instruction on argumentation as a learning strategy. As a consequence, I adapted Toulmin's (2003) argument structure (Figure 3—4) to understand and analyse the quality of learners' argumentation.

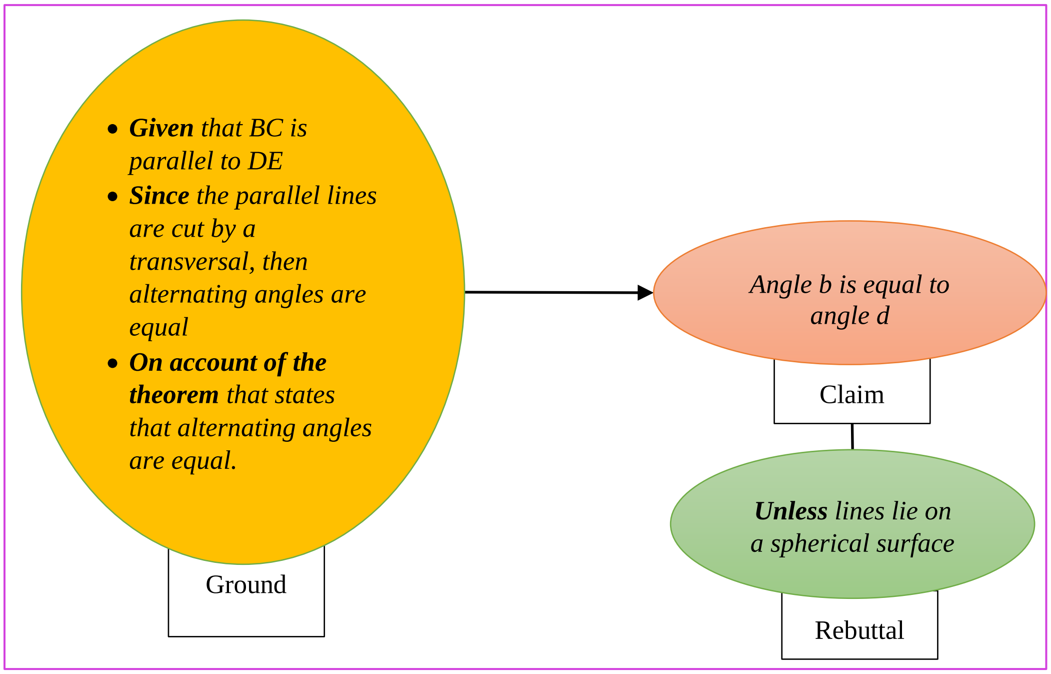


Figure 3—4. Toulmin's adapted model for written argumentation

As Forman, Larreamendy-Joerns, Stein, and Brown (1998) point out, learners see no value in argumentation since for them there is only one correct solution strategy to every problem which the teacher can or should provide in mathematics (this phenomenon is depicted in Figure 3—5). In addition, they also point out that learners hold norms about school mathematics learning which contradict those practiced by mathematicians (for example, expecting that speed and accuracy are more important than relational understanding). Further, Osborne et al. (2004) argue that argumentation is a complex task and therefore learners need guidance and support to construct an effective argument. The situation often gets more complicated because, as Driver et al. (2000) point out, teachers lack the pedagogical skills in organising argumentative discourse within the classroom.

In summary, Toulmin and others treat argumentation in such detail that the only significant modification I could make was to see data, warrants, and backings as "grounds" to circumvent the difficulty in differentiating among them in learners' written argumentation. I believe that Toulmin offers a helpful framework that directs attention to the application of key aspects of argumentation (that is, informal logic). I drew on Toulmin's theory as a means to focus attention on exploring learners' argumentation in response to a specific mathematical task. Learners' written argumentation were mapped onto the adapted Toulmin's argument structure, that is to say the TAP model. In this study, I was only interested in characterising learners' argument rather than requiring them to engage in constructing proofs.

3.2.2.2 Difficulties in the application of Toulmin's framework

Although Toulmin's argument structure has been widely used in educational research on argumentation (Nussbaum, 2008), it is not allowed to pass without critics. Most studies challenged the applicability of Toulmin's model to real-life arguments, mainly on the basis of the ambiguity surrounding the various elements in his argument structure. For instance, Erduran et al. (2004) investigated the application of TAP as a tool for tracing the quantity and quality of argumentation in some topics in science classrooms. They found ambiguity in the characterisation of data, warrants, and backings. I concur and decided to rather consider all reasons (data, warrants, and backings) given in the AFEG instrument as grounds.

Simosi (2003) argues that this criticism is unfair in the sense that when developing his framework, Toulmin was interested in legal argumentation and, consequently, in the different sorts of propositions uttered in the course of a law case. Freeman's (1991) critique of Toulmin's theory relates to the data-warrant-backing distinction; that is, challenging the utility of the scheme. In an attempt to overcome the problem raised by Freeman (1991), Osborne et al.'s (2004) strategy was to collapse Toulmin's (2003) data, warrants, and backings into a single category which they termed "grounds". For this study, this is a fair criticism as I am not aware of any deliberate instruction on argumentation that attempted to make learners aware of the differences among these three elements. However, Freeman (1991) also contests the inclusion of rebuttals as elements of an argument. It is unclear how else could the force of an argument be strengthened if conditions of exceptions were not appreciated.

Another issue is the insistence by researchers on the presence of not only evidence (data) but also rebuttals in argumentation. This idea is rooted in Pollock's (1987) notion of defeasibility.

Defeasibility refers to the insistence that, for an argument to be of high quality, it must not only consist of warranted claims, but must also include rebuttals to show that the argument is successful in withstanding new evidence brought against its warranted claims. However, as found in Clark and Sampson (2008), the problem is that even an integrated argument accommodating multiple rebuttals may have failed to consider some important constraints. In attempting to address this problem, Clark and Sampson (2008) constructed a definition of rebuttals as statements introducing conditions of exception that not only applied to claims, but also to warranted claims. Nonetheless, Toulmin, Rieke, and Janik (1997) regard such criticisms of the theory as an effective way of improving it as they express their views thus:

> Those ideas surviving this critical assessment will be good as scientific ideas. If enough reasons and solid arguments prove their value in a clear fashion, this will mean that their *scientific* basis is also coherent. When the critical assessment fulfills both requirements we can be satisfied: practical argumentation has demonstrated the rational basis of these new ideas. (p. 232)

In light of the difficulties experienced when attempting to differentiate among the various elements of an argument, I used this model as a means of analysing written argumentation. I believe that this approach surpassed the concern just raised in the following way.

First, I provided a frame within which arguments could be made. Second, I grouped data, warrants, and qualifiers into "grounds" in order to resolve the grammatical conflict inherent in the tool. This stance of omitting some elements is in line with Toulmin's (2003) notion that every argument occurs in a context which has its own norms for argumentation. In this study rebuttals were included because they were critical in distinguishing an argument as being either of low or high quality. Rebuttals in this study were generated by the learners themselves through pre-empting the challenges against the claim or ground; that is, taking other conflicting viewpoints into account. Taken as a whole, in spite of the criticisms of TAP, I found the scheme useful by focusing on making it functional in the context of mathematics. In this study I used it with minimal adaptation. The complexity of TAP notwithstanding, attention turns to the exploration of probably another complex matter: learners' functional understanding of proof.

3.3 Learners' functional understanding of proof

> Proof is an important part of mathematics itself, of course, and so we must discuss with our students the function of proof in mathematics, pointing out both its importance and its limitations. (Hanna, 2000, p. 5)

The concept of proof has proven to be one of the most difficult learning thresholds for high school learners of mathematics. Easdown (2012) suggests that this difficulty manifests itself in three ways: appreciating why proofs are important; the tension between verification and understanding; and, proof construction. The present study, which draws on the work of de Villiers (1990) in relation to the functions of proof, investigated the first aspect: learners' appreciation of the functions of proof. This focus on learners' functional understanding of proof was also informed by my personal experience, from having taught high school learners and preservice high school teachers for a number of years.

Often, the terms "role", "function" or "purpose" are conflated in mathematics education literature. For instance, Hanna (2000), seems to use the term "role" when referring to proof in the classroom and "function" for the mathematical practice as opposed to "purpose". CadwalladerOlsker (2011) uses "roles" and "purpose" while de Villiers (1990; 1994; 2004) uses the phrase "role and function" and "purpose" interchangeably. When referring to a specific article, an attempt will be made to use the term used by the author(s) of the article cited. In this study, the term "function" is used to include the phrase "roles and purposes" or the term "role". Apart from the debate over this conflation of terms and phrases, one thing is clear: de Villiers' (1990) list encompasses most of the functions of proof listed by various authors (for example, Bieda, 2010; Hanna, 1995). Like de Villiers (1990), this study framed functional understanding of proof in the classroom within the context of the mathematics scholarship.

De Villiers' (1990) work provided ground for assessing learners' functional understanding of proof in Grade 11 as he argues that lack of understanding of the functions of proof in mathematics impairs learner motivation to seek a proof. He exemplifies this point by reminding us that throughout their schooling years, learners become part of several different mathematics communities and gain a variety of experiences. He points out that in that time, they may have been exposed to and participated in mathematical reasoning involving either informal or formal proving processes. But, he further points out, it is also in that time that they develop a wide variety of understanding about the functions of proof in mathematics, including taking empirical arguments as proof.

Claiming no ranking by importance, I provide an outline of the five dimensions from which the functions of proof in mathematics can be approached. There are theoretical and practical reasons for approaching functional understanding of proof as a multidimensional construct. On the theoretical side, functional understanding of proof is an attitudinal construct and therefore cannot be measured by considering a single dimension as this, in Kern, Waters, Adler, and White's (2015) words, obscures 'potentially valuable information' (p. 263). For instance, the generally nonhomogeneous nature of schools suggests that there is potential variability of the scores on the different dimension based on a school's resources. On the practical side, a multidimensional approach could isolate schools in terms of the various dimensions to provide teachers and departmental officials in charge of curriculum monitoring with specific information. This information could then be used to take practical steps to address those aspects of functional understanding of proof that required attention or strengthening. The next section discusses the five dimensions that entail the functions of proof organised through de Villiers' (1990) multidimensional model. These perspectives were used to gain insights into learners' understanding of the various functions of proof which in turn provided a lens for understanding learners' difficulties with proof.

3.3.1 Proof as a means to verify the truth of a proposition

A proof can be viewed as a tool to establish certainty of a conjecture, that is, verifying (making sure) that a conjecture is true for all cases. In validating the correctness of a mathematical proposition or simply verification, all that is required is to logically connect axioms to arrive at a conclusion regardless of its form or aesthetic appeal (Hanna, 2007). Verification denotes the removal of uncertainty by seeking, in the vocabulary of Harel and Sowder (1998), to "convince" or "persuade" someone or oneself about the validity of a conjecture. Harel (2013) takes this idea of certainty further and claims that the '*need for certainty* is the natural human desire to know whether a conjecture is true—whether it is a fact' (p. 124). Schoenfeld (1994) describes the benefit of reaching certainty so eloquently thus:

> One of the glorious things about proof is that it yields certainty: When you have a proof of something you know it *has to be* true, and why. That feeling of certainty is really powerful, for patterns and trends can be deceptive. All mathematicians have their favorite examples of patterns that look like they ought to hold but fail, or of conjectures that are true for the first N tries but then fail. (p. 26)

This function of proof is most familiar to research mathematicians but regrettably missed by learners as they often complain that it is pointless to prove theorems that "everybody knows" or that have already been proven by other people in the past; a proposition is not a true until it is verified to be so by the construction of a proof (CadwalladerOlsker, 2011). In school mathematics, verification is associated with providing examples as proof that a conjecture is true; nothing more. However, empiricism is only an important process in merely gaining conviction to seek a proof rather than a proving process itself. Empiricism is defined as making an assertion about the truth of a conjecture after verifying several cases (Balacheff, 1988). Therefore, empiricism is defeasible; there are historical examples where counterexamples overturned earlier generalisations. This approach reflects an appreciation of the fact that empiricism and quasi-empirical investigations are unsafe; therefore, a proof provides what is refutably absolute guarantee (de Villiers, 1998).

Another traditional approach in mathematics classrooms is to use some examples and then proceed to introduce deductive proof only as a means to verify that the conjecture being tested with examples is true and thus attain conviction. A conjecture is a proposition that is consistent with data and has not been proven to be either true or false (Uploaders, 2013). The main point here is to note that verification of a mathematical proposition can take two forms: empirical or deductive; empirical by selecting a few cases and deductive by logically connecting a set of axioms to produce a new result. Lakatos (1991) argues that even though proof is regarded as the ultimate authority on the truthfulness of a conjecture, its certainty is vulnerable since the axioms on which it is based continue to be open to revision by the mathematics community. The revision may either have been necessitated by a recognition of human error or inconsistencies in an axiomatic system (Umland & Sriraman, 2014).

Mudaly (1999) argues that research has shown that by engaging in appropriate exploratory activities using DGS learners can gain conviction. This, that is conviction, is the predominant justification method used by learners. Notably, this function of proof is noble on its own in that although the proposition is already undisputed mathematical knowledge, there is value in the learner gaining conviction following the same creative path a mathematician would have taken when discovering that knowledge for the first time (Bartlo, 2013).

De Villiers (1990) argues that if learners see proof only as a means "to make sure" through their own experimentation then they will have little incentive to generate any kind of logical proof. He points out that instead, it is this conviction that propels mathematicians to seek a logical explanation in the form of a formal proof to know why a conjecture must be true. This suggests that it is this role of proof as a means to explain that can motivate learners to seek to generate a proof for a conjecture. Important to consider is that learners need to be aware that the proofs they are learning are new to them but consists of results that are known to be true (Hanna, 1995).

Indeed, given the scientific nature of mathematical knowledge, for each correct conjecture there should be a sequence of logical transformations moving from hypothesis to conclusion (de Villiers, 1990). However, Davis and Hersh (1981) characterise this as a naïve view of mathematics in light of the fact that proof can be fallible. The history of mathematics is littered with instances of "theorems" whose proofs were later found to be false. Hersh's (1979) position that '[w]e do not have absolute certainty in mathematics. We may have virtual certainty, just as in other areas of life. Mathematicians disagree, make mistakes and correct them' (p. 43) captures the tentative nature of proofs. This problem notwithstanding, formal verifications maintain an important and useful function of proof in mathematics (Stylianou et al., 2015). The next section discusses the function of proof as a means to understand why mathematical propositions are true.

3.3.2 Proof as a means to explain why a proposition is true

The explanatory function of proof pertains to the provision of insight into why a proposition is true. This view is supported by Harel (2013) who reserves the term "explain" for the 'mental act one carries out to understand the cause for a conjecture to be true or false' (p. 128). According to Hanna (2000) for a proof to explain, it needs to make use of well-known and well-understood properties of the mathematical objects involved. She suggests that, given that the level of conviction is directly related to understanding, viewing a proof as a means to explain why a conjecture is true deepens existing conviction. Further, she argues that a quest for explanatory power often results in a proof that is economical because it uses only those hypotheses that are absolutely necessary. Bell (1976) asserts that the 'status of a proof would be enhanced if it gives insight as to why the proposition is true as opposed to just confirming that it is true' (p. 6).

Showing how a proposition coheres and connects with the key properties of the concepts involved in the proof provides explanation (Herbst, Miyakawa, & Chazan, 2012). Phrased differently, proof becomes an important tool for presenting mathematical knowledge as a discipline that comprises connected rather than isolated concepts and procedures. I concur with Bartlo's (2013) assertion that understanding the explanatory function of proof helps learners to see the consequences of the concepts. Thus, explanation (or illumination) involves understanding results not from operational construction and measurement of a figure (verification), but from the previously acquired geometrical knowledge.

Traditionally, proof in school mathematics is viewed as a formal way of showing the validity of theorems rather than a way to facilitate conceptual learning of mathematics (Nyaumwe & Buzuzi, 2007). Thus, reading a proof can lead to an understanding of mathematical relationships (Hanna, 1995). Mathematicians often value one proof over another on the basis of its explanatory power (CadwalladerOlsker, 2011). Explanation seems to be of greater importance than verification because when proof is viewed as a means to explain why a proposition is true, substantial improvement in learners' attitude towards proof appears to take place (de Villiers, 1998).

Using DGS, inductive reasoning could result in an explanation why a conjecture is true for *all* figures in different orientations (Figure 3—5). Thus, when a conjecture is supported by intuition, consideration of more and more examples or DGS, the need to understand why it may be true is satisfied by a proof. An example of an inductive argument is provided in the next figure in relation to the proposition that *"The sum of the angles of any triangle is 180°"*. Inductive reasoning not only actively helps in involving learners in the build up to a conjecture, it also helps in the learning of proof with understanding. Here I am reminded of the ancient Chinese proverb that stresses the journey to understanding from behaviourist perspectives: *I hear and I forget; I see and I remember; I do and I understand.* Thus, in addition to creating opportunities for learners to physically engage with the objects of mathematics, inductive reasoning also demands mental activity to enhance understanding.

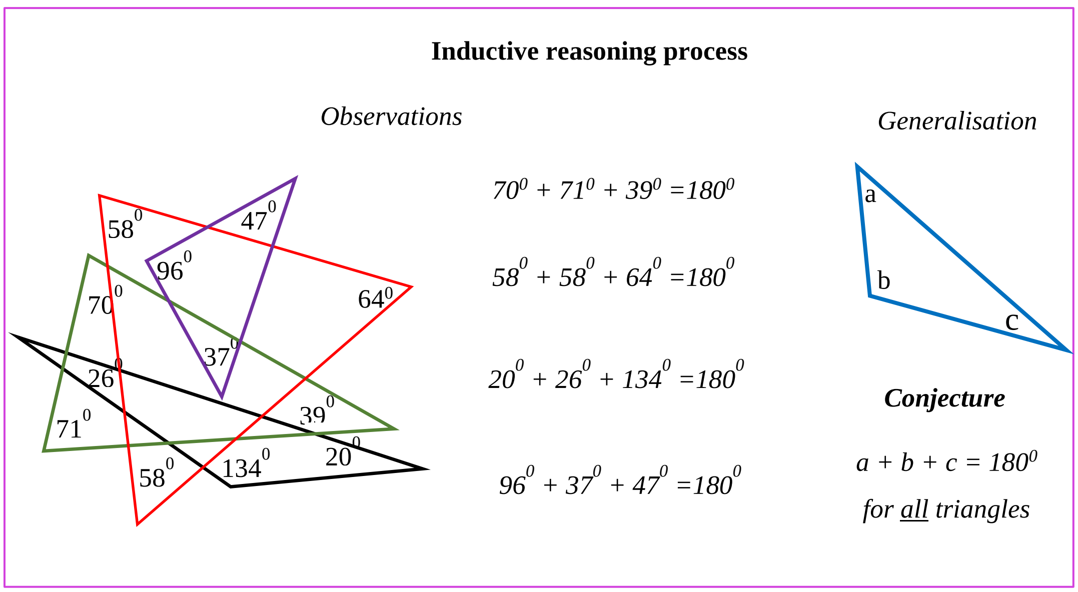


Figure 3—5 Inductive reasoning in which a few cases are observed resulting in a generalisation

Working as Euclid did – by deductive reasoning in which a single figure is used – an explanation why a conjecture is true for any figure can be produced. That is, a deductive argument which holds generally and does not depend on the figure is advanced in order to reject or confirm a conjecture. Using appropriate, previously proven truths (axioms), the conclusion constitutes a mathematical proof. In this study, the conjecture for which a deductive proof has been produced is referred to as a theorem. Having provided an example of inductive reasoning, it is essential to also exemplify deductive reasoning.

Figure 3—6 is an example of a deductive argument that proves the triangle sum conjecture, *"The sum of the angles of any triangle on a plane is 180°".* According to Serra (1997), this conjecture is believed to have been one of those proved by the Greek mathematician, Thales. The proof for this conjecture is made easy by constructing an auxiliary line (an additional line segment or line that is required in proving a conjecture) through one vertex, parallel to the opposite side using the parallel postulate of geometry (through a point not on a given line, exactly one line is parallel to the given line).

Deductive reasoning process

Axioms		*Logical consequences*
Axiom 1: *(Previously proven conjecture)*	*$d + a + e = 180^0$ (d, a, e are on a straight line)*	***Conclusion:***
Axiom 2: *(Property of parallel lines)*	*$d = b$ and $e = c$ (alternating interior angles are equal)*	*$a + b + c = 180^0$* for *any* triangle
Axiom 3: *(Postulate)*	*Substitution of equal values*	

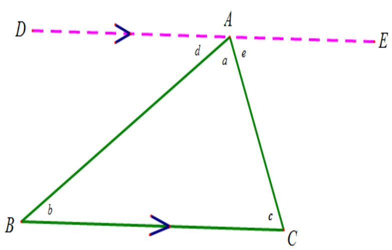


Figure 3—6. Deductive reasoning based on a collection of theorems and a postulate to prove the triangle sum conjecture.

Briefly, proof explains how the propositions are related to each other (Hemmi & Löfwall, 2010). In addition, explaining why a conjecture is true (proving) generally leads to new discoveries (de Villiers, 2002; Hanna, 2000). The next section demonstrates that the social nature of mathematical knowledge construction demands that ideas in a proof must be transmitted to an audience.

3.3.3 Proof as a means to communicate mathematical knowledge

Mathematical proof is an essential tool for the communication of mathematical thinking. One way of communicating a proof of a conjecture is to write it down for publishing to the wider mathematics community. Communication concerns sense making and "transmission" of socially constructed knowledge publicly – both in discourse and in writing – that is acceptable to the mathematics community. That is, this function relates to the communication of the verified and explained mathematical knowledge to others. For instance, although the proof of Fermat′s Last Theorem was initially claimed by Andrew Wiles in a lecture in 1993, it eventually was accepted only after it was communicated in a paper by Taylor and Wiles (Herbst, Miyakawa, & Chazan, 2012). This not only demonstrates that proofs are written and read by human beings (CadwalladerOlsker, 2011) but also that the communication of a proof to a wider audience precedes its acceptance. Thus, a defining characteristic of proof is its public nature. Essentially, mathematicians communicate their results through publishing them in mathematical journals; thus, making them accessible to the public, including the mathematics community. This social aspect of proof facilitates an understanding of what counts as justification in the mathematics community and thus communicates the nature of mathematical argument. In other words, proof is a means to demonstration the standards (criteria) for communication in mathematics.

These standards do not only involve whether the argument has been communicated (using logical chains of deduction) but also whether learners know how to communicate it (for example, essay format or two-column format) (Herbst, Miyakawa, & Chazan, 2012). Accordingly, learners ought to be taught the standards of deductive reasoning so that they can tell when a proof has or has not been established (Hanna, 2000). Proof is the way mathematical results are communicated in the field of mathematics (Bartlo, 2013). Essentially, it is through communication of poofs that mathematicians advance human understanding of mathematics (Hanna, 1995). For instance, as CadwalladerOlsker (2011) points out, proofs can illustrate a new approach or technique which might be just what another mathematician needs to complete their own proof of a different theorem. However, it is important to note that literature provides evidence that, as in other areas of life, mathematicians disagree, make mistakes, and correct them. Thus, similar to other kinds of knowledge, it is through communication or engagement that mathematical knowledge is found to be fallible. Thus, using argumentation, even learners can communicate and debate their ideas about relationships by using a proof. In short, proof is useful in the learning of mathematical language, particularly when learners discover the objects of mathematics for themselves.

3.3.4 Proof as a means to discover a reasonable proposition

> It is exciting and enjoyable to explore, investigate and experiment. I believe that this is what teaching and learning is all about … What you discover - you own, - no matter how often it has been discovered before. When knowledge is discovered in this way, learning becomes enjoyable, exciting, painless and above all, meaningful. (Moodley, 2003, p. 128)

The discovery function of proof relates to the generation of new results (theorems) through proof. Similar to Stylianides (2009), the phrase "new results" is used here to refer to the proof knowledge that learners add to their existing knowledge base as a result of constructing a proof for themselves. Thus, by generation or creation of new mathematics is meant what learners produce as new to them but may have been known to the community of mathematics scholarship. The discovery aspect within the concept of proof involves making conjectures and attempting to provide arguments to justify them. As de Villiers (1997) points out, a proof often leads to new insight which in turn leads to the discovery of new or additional properties of mathematical objects. De Villiers (1998) goes on to point out that proof may lead to the discovery of counterexamples which, as a consequence, necessitates a reconsideration of old proofs and the construction of new ones. He shows how identification of a key idea in the proof of the proposition "If you connect the midpoints of a kite the resulting shape is a rectangle" led to discovery of new results. He further argues that the identification of the key idea that the proof depends on diagonals being perpendicular, results in the realisation that the proposition could then be generalised to any quadrilateral with perpendicular diagonals. De Villiers (2012) extends this notion by asserting that new branches of mathematics often have been invented by producing a deductive proof for a conjecture. Thus, through conjecturing, learners may see and appreciate how new pieces of information are logically deduced by proof and thus gain appropriate understanding of the discovery function of proof. Therefore, conjecturing is the pathway to discovery.

Although it is a rare occurrence, nothing gives greater pleasure to a teacher than when one of their learners produces a conjecture of their own. The conjecture need not be entirely original, but the excitement created in the classroom when something goes "outside" or "beyond" the textbook gives a much more "real" sense of genuine mathematical discovery and invention (de Villiers, 2012). For instance, in studying Euler's rule for polyhedral, Lakatos (cited in Herbst, Miyakawa, and Chazan, 2012) illustrates how proving a naïve conjecture can lead to formulation of a more precise conjecture and its precise concepts (expressed in the naïve conjecture). However, noteworthy is that within mathematics, conjectures continue to be regarded as such, until a deductive proof is provided. This tentative nature of propositions should encourage learners to seek proofs of conjectures so as to put the status of their truth beyond doubt. Doing so adds to the growing volume of mathematical axioms that help to construct new proof by organising these already proven mathematical knowledge into a system.

3.3.5 Proof as a means to systematise mathematical knowledge

Systematisation of mathematical knowledge involves the organisation of results previously thought to be unrelated into a deductive system of axioms, major concepts (definitions) and theorems (de Villiers, 1990). In this case, de Villiers' (1990) central thesis is that proof serves as a means of systematisation in the field of mathematics in that it (1) helps with the identification and weeding out of logical or mathematical inconsistencies elsewhere in that structure, (2) unifies and simplifies mathematical theories, (3) provides a useful global perspective on a topic by exposing the underlying axiomatic structure (4) helps application both within and outside mathematics, and (5) leads to alternate deductive systems that are more elegant or powerful than existing ones. Bartlo (2013) points out that whereas (1), (2), (4), and (5) are addressed in proof literature, the concept of "global perspective" (3) may have been underappreciated in literature.

Knuth (2002) surmises that, based on his experience both as a high school teacher and as a teacher educator, many learners view the many theorems that they are asked to prove as essentially independent of one another rather than as related by the underlying axiomatic system. In brief, I follow Wu's (1996) explanation that an axiomatic system relates to structure built by 'starting with undefined terms and simple propositions (axioms), then deducing complex ones step by step with the use of logic' (p. 230). He points out that this is the best system mankind has ever devised to ascertain the attainment of truth. This systematisation function is important in that it organises individual propositions into a coherent system and exposes the underlying logical relationships between these propositions. Thus, previously disparate results are put together into a unified whole (Weber, 2010) which may lead to uncovering of arguments that may be fallacious, circular or incomplete (de Villiers, 1990). In this case, a proof provides knowledge about how a theorem, definition or proposition relates to the rest of the known geometry or mathematics. That is, proof exposes the underlying logical relationships between propositions in ways no amount of empirical testing could (de Villiers, 2002). Thus, proof serves as a means to organise propositions into a deductive system. A classic example of the use of proof for systematisation is Euclid's *Elements*. He collected many theorems which were first proven by earlier Greek mathematicians and organised them in such a way that they followed from definitions, axioms, and postulates (CadwalladerOlsker, 2011). Proof is therefore an indispensable tool for bringing together various known results into a deductive system of axioms. The next section interrogates the key concepts framing this study.

3.4 Factors influencing functional understanding of proof

> In A-level studies proof … means going through a sequence of symbolic manipulations that many students find hard to follow, only to arrive at a result which they are quite prepared to accept. Why is it necessary to prove something that is known to be true? (Tall, 1989, p. 2)

Learners' efforts to construct proof have been characterised as proceeding from empirical arguments to deductive arguments (de Villiers & Heideman, 2014). A range of factors influences proof construction (Chin & Lin, 2009). Harel and Sowder (2007) provide "comprehensive perspectives on proof" in an effort to understand learners' difficulties with proof and the roots of the difficulties. They outline three categories of factors that influence proof teaching and learning: mathematical and historical-epistemological; cognitive; and, instructional-socio-cultural. This study investigated the instructional-socio-cultural factor. I separated this factor into instructional and sociocultural aspects given that these two factors influence learners' understanding of proof differently. The crucial point in this study, of course, is to understand and explain why *Presh N* held the beliefs she held about the functions of proof in mathematics.

Several scholars made their points in this regard. I addressed each of these ideas, namely, two-column proof ritual, instruction, teacher, textbook, culture, semantic contamination, empirical argument, dynamic geometry software, and language of instruction, in turns. I provided suggestions from literature to address some of the identified sources of distorted functional understanding of proof. For instance, as described by de Villiers (2004), dynamic geometry software could be used to develop learners' understanding of the functions of proof not merely as a means to verify, but also as a means to explain, communicate, discover, and systematise. Thus, the availability of dynamic software has given a new impetus for the learning of Euclidean geometry.

Although the factors are certainly not exhaustive, they are important as I hoped that they will bring structure in future inquiries on the proof phenomenon. This hope made the identification of these factors a key step towards providing insight into the current state of learners' understanding of the functions of proof. In the next section, I provided an overview of what research and practice say about potential sources of *Presh N*'s belief regarding the functions that proof performs in mathematics.

3.4.1 Two-column proof ritual

Learners tend to be convinced that a proof is true and correct based on its form or appearance rather than examining its validity themselves (Harel & Sowder, 2007). That, is, they consider a logically correct deductive argument to be a proof if and only if it is in accordance with a specific mathematical convention; two-column format. Hence, learners tend to believe that this format for geometric proofs is at least as important as its content (Schoenfeld, 1989). Shibli (1932) and Herbst (2002) suggest that the 1913 second edition geometry textbook by Arthur Schultze and Frank Sevenoak is the first to express proofs in two parallel columns of statements and reasons divided by a vertical line. That is, the layout is such that the left hand column consists of statements relating to the current proposition, and the right hand column for references to theorems being assumed as already known and established truths (Bell, 1976). Shibli (1932) argues that the two-column custom has brought improvement in the geometry text. He describes it as not only a work of art and beauty, but also an excellent instrument of instruction in the mathematics classroom.

Knuth, Choppin, and Bieda's (2009) conducted research with 400 learners about their understanding of the functions of proof. None of them mentioned that proof is a means to explain. In addition, they found that if some given diagram consists of facts that are visually obvious to them, learners often see no need to go beyond their observations in proving a proposition as true. Hence, de Villiers (1997) proposes that there need to be less focus on this form of proof in geometry since this can be done with algebra. This presentation of proof (that is, two-column) represents a development in both style and form, from essay type to two-column style. According to Herbst (2002), this format enables both teachers and learners to examine each other's explanation of their deductive written work and facilitates the marking and correction of learners' written work and thus bringing stability to the study of geometry in schools. Hence it has remained the default mode of proving in textbooks and a customary tool of engaging learners in proving in school mathematics (Weiss, Herbst, & Chen, 2009). This format is so prevalent that when proofs are written in narrative format – which uses conversational but logical arguments – learners tend to be unsure of their validity (McCrone & Martin, 2009).

However, if emphasis were that classroom activities must be reflective of the practices of mathematicians, two column proofs distort mathematics because no mathematician has ever worked that way (Lockhart, 2002). Wu (1996) concurs and points out that the format is different from how contemporary mathematicians write proof. Most notably, proofs presented in the two-column format have been found to promote understanding of the function of proof merely as a means to verify the truth of an already known proposition (Ersoz, 2009). In addition, Sowder and Harel (2003) argue that this format influences the development of authoritarian proof schemes wherein a proposition is accepted as true solely on the basis of authority, namely, teacher or textbook.

Apart from the fact that the format is foreign to the mathematical practice, it has ‘brought to the fore the logical aspects of a proof at the expense of the substantive function of proof in knowledge construction’ (Herbst, 2002, p. 307). This statement need not be construed as an indictment on logic in proof; while proof is central to mathematics, logic is central to deductive proof. However, the point is that attention given to the logic of a proof may take away conceptual understanding and limit or distort understanding of the construction of knowledge in the mathematics discipline. I concur with Schoenfeld’ (1988) view that advocates for flexibility in the way a deductive proof can be written given that ‘what matters to the mathematical community is the argument's coherence and correctness’ (p. 11).

Further, Herbst and Brach (2006) found that geometry learners were accustomed to tasks that required proving a proposition presented in given-prove format than in proving a general proposition such as ‘a line through the midpoints of two sides of a triangle is parallel to the third side and half its length’ (p. 84). In my extensive exposure to assessment instruments, I can confirm that such tasks are also found in tests and examinations papers in South African high schools (for example, Department of Basic Education [DBE], 2015). In these assessments, the given-prove format is followed by the two-column format that “guides” learners’ proving activity. In addition, from my experience as a Grade 12 Paper 2 (which included geometry proof questions), there is a rigid memorandum in terms of which learners’ answers had to be marked. Hence, McCrone and Martin (2009) are of the view that geometry learners conceive the function of proof to be the application of recently learned theorems rather than a mathematical process for establishing the truth of propositions. Thus, the approach to proof as merely involving the absorption of what the teacher required reflects distorted understanding of the functions of proof in mathematics.

According to Lortie (2002), the reason why two-column proofs are so prevalent is because of what he terms “apprenticeship of observation”. Lortie (1975) coined this phrase to refer to the phenomenon whereby preservice teachers study for the profession after having spent more than twelve years as learners observing and evaluating the practices of their teachers in action. Specifically, he points out that the average learner spends 13,000 hours in direct contact with classroom teachers by the time he or she finishes high school (Lortie, 2002). Chazan (1993) argues that simply presenting a two-column proof along with a diagram not only obscures the generality of the proposition but also gives learners no indication that the argument presented is not an argument for a single case. To a limited extent, Wu (1996) provides a reconciliatory argument to the use of two-column proof scheme. He attributes the criticism of two-column proof ritual to its abuse by previous generations. While seen as giving learners a distorted view of the functions of proof, the two-column scheme is an admirable educational tool and he advises that the format is only supposed to be used to introduce proof for at most approximately a month. This, he asserts, allows learners to make a smooth transition to writing a proof in a narrative format as it happens in the mathematical practice. Although I think that this assertion is sensible, it does not seem to have been taken serious by curriculum designers and curriculum delivery monitors; not so even by the CAPS designers. Next, I turn to instruction factor.

3.4.2 Instruction

At this stage, it is sobering to define the term "instruction" as referring to at least three distinct categories of activities—what teachers do with the concept of proof, what learners do to learn it, and the pedagogical approach that teachers employ to mediate this concept. Although I believe that there is no one best teaching method in mathematics, van Hiele (1999) argues that learners' instructional experiences, depending on the instructional method, could either foster or impede the development of learners' geometric thought. Specifically, Atebe and Schäfer (2011) found that knowledge transmission methods offers learners scant opportunities to learn geometry.

Van Putten, Howie, and Stols (2010) undertook a study involving preservice teachers enrolled for mathematics methods' courses at the University of Pretoria, South Africa, who all passed Grade 12 mathematics at 50% or more when Euclidean geometry is compulsory in the FET phase. These presevice teachers followed a six-month Euclidean geometry module. They found that preservice teachers expressed the confusion and frustration arising from being taught by teachers, at school level, who did not appear to have either mastered the subject or developed a positive attitude towards it. Nonetheless, it is expected that having successfully completed the mathematics course they would have attained Level 4 of the van Hiele theory (van Putten, Howie, & Stols, 2010). However, despite that this group of preservice teachers studied Euclidean geometry at school, they did not even attain 50% on Level 1 and that attainment of Levels 2 and 3 is even rarer in the test results.

Once in the teaching field setting, very few will contest the conclusion that these teachers will either avoid teaching Euclidean proof or encourage memorisation of proofs rather engage learners in what Edwards (1997) refers to as "exploring the territory before proof" (that is, exploration, conjecturing, and argumentation). Giving learners predetermined propositions to prove reinforces the predisposition that the proposition must be true; so they merely need to verify. Sadly, this is bound to be of little real interest to the learners – the undesirable consequence of such instructional practice is the development of negative attitudes towards Euclidean geometry in particular and mathematics in general.

Generally, presentation of proof to learners as text and diagrams to memorise left them incapacitated to appreciate its value in mathematics. This perspective need not be construed as proposing that memorisation, achieved through drill and practice, is frowned upon in my view of learning. I view memorisation of important concepts of mathematics as a practice that allows learners to acquire basic skills that enabled fast, accurate, and effortless processing of information which frees up working memory for more complex aspects of proof. In this study, in contrast to Western perspectives on learning, memorisation is viewed as the route to understanding. My view of learning draws on perspectives of learning from both Western and Asian cultures. This view rests on the premise that learning is the acquisition of knowledge through primarily the teacher or text and also through learner's own effort. From this view, learning does not preclude memorisation with a view to gain understanding.

I concur with Tavakol and Dennick (2011) who view memorisation not as an end in itself but as a path to understanding. Thus, the memorisation of propositions—not their proofs—is helpful because it engenders understanding. It is in this light that I take the position that instruction designed in terms of van Hiele's five sequential phases of learning (inquiry/information, directed orientation, explicitation, free orientation, and integration) promote learners' acquisition of Level 4 partly through memorisation. The theory suggests that learners have attained Level 4 if they understood the functions of deduction (why proof is important in mathematics) and the roles of postulates, and theorems such that proofs could be done meaningfully.

Van Hiele proposes characteristics of these levels which, like Usiskin (1982), I labelled: fixed sequence; adjacency; distinction; separation; and, attainment. The last characteristic emanates from van Hiele's suggestion that cognitive development in geometry can be accelerated by instruction. They provide detailed explanations of how instruction can move a learner from one level to the next. However, it is not the intent of this study to examine these phases – Hoffer (1981) provides a detailed account. That said, I argue that learners' understanding of proof must be a feature of the information phase because it is in this phase that learners can be acquainted with the significance or importance of proof by emphasising its functions in mathematics. Instruction designed along such van Hiele lines would not only improve learners' ability to write formal deductive proofs but also help to develop learners' understanding of the function of proof in mathematics and thus provide them with insight into the activities of mathematicians. For learners to see the functions of proof and to experience the work of mathematicians, they must see how mathematicians use proof as a way of thinking, exploring, and of coming to understand (Schoenfeld, 1994).

A controversial issue in the field of mathematics education is whether classroom instruction should promote more instrumental (traditional or knowledge transmision method) or relational understanding (reform-based methods). It is my view that certain topics in mathematics need to be taught more effectively with one method or another – teaching methods were guided by context. Put another way, teaching should not be exclusively "instrumental" or "relational". Consistent with this view is the National Mathematics Advisory Panel's (2008) instruction that the widely held belief among teachers in high schools that one method is better than another is not supported by research. However, whichever method is used in the classroom, the argument in this project is that learners need to be exposed to experiences that help learners to develop appropriate conceptions of the functions of proof in mathematics. Thus, I am of the view that teaching for functional understanding of proof needs to be an integral part of whichever method of teaching Euclidean geometry.

In the mathematical practice, proving is a process of learning and *discovering* new mathematics. First, a conjecture would be made based on observation of a number of cases. If available, dynamic geometry software could be an aid in hastening conjecturing. Next, an attempt would be made to *explain* the conjecture through proof. But, this is not the end of the story in the mathematician's work on proof. The created proof needs to be *communicated* to other mathematicians before final acceptance of the conjecture as a mathematical truth. For learners to experience these functions of proof, they must make and test their own conjectures. Of course, the created conjectures and proofs would not necessarily be new mathematics; but, to the learners they would be.

3.4.3 Teacher

Research studies have shown that the failure to teach proof to learners appears to be universal (Balacheff, 1991; Chazan, 1993). An investigation by Senk (1985) found that only 30% of the learners who were taught proof in the US achieved 75% success in proving. However, it was found that even these "successful" learners were not always aware of its functions. It is therefore not surprising that teachers tend to encourage memorisation of proof. These learners reproduce the ready-made proofs because the teacher demands them for passing tests and examinations. Thus, rather than introducing learners to the practices of mathematicians and enable learners to experience the construction of mathematical knowledge themselves through experimentation, measurement, and conjecturing, the teacher reinforces the notion that the sole function of proof is verification. So, the teacher has so much power that the other four goals of teaching proof are (perhaps) deliberately overlooked.

The claim that functional understanding of proof is influenced by the status that teachers occupied in society can be substantiated by personal experience. Indeed teachers seem to wield more power over their learners than parents do. On one particular day, my seven-year old son came home from school and changed into her civil clothes as per usual. He is always adamant that his shirt should be buttoned up to the last button and would come back home in the same state regardless of how hot it has been on any school day. No matter how my wife and I would plead with him to at least loosen the collar button, he would not budge. One day he came back with the collar button lose. Enquiring why he answered it is because "Mrs. P Reddy (teacher's pseudonym) asked me to unbutton it".

This phenomenon of seeing teachers' ideas as uncontestable is further depicted in the cartoon (Figure 3—7) featuring the eponymous pair of boys *Max* and *Moritz* (Busch, 1962). Like Coll, France, and Taylor (2005), and Stephenson and Warwick (2002), I view cartoons as visual tools which combined exaggeratedly drawn characters with dialogues to either depict misconceptions or to stimulate alternative views with minimal use of written language.

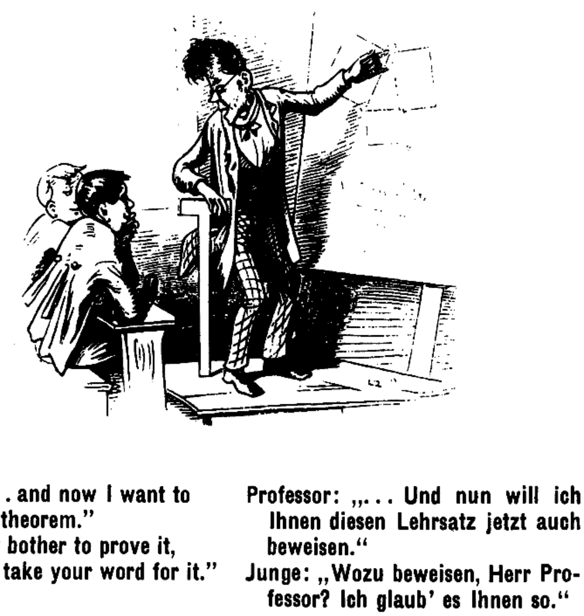


Figure 3—7. Busch's (1962) cartoon illustrating the power teachers wield in the classroom.

Although I am also an educator, the fact that he saw me differently (a mere parent) suggested that teachers were viewed as experts in various things and thus wielded more power over learners than did parents. In fact, Inglis and Mejia-Ramos (2009) point out that appeal to expert opinion is prevalent in everyday situations. Further, Harel and Sowder (1998) found that a teacher presents a proof to convince learners of the truth of a mathematical proposition rather than allowing learners to investigate its truth themselves. They noted during their teaching experiments that most learners' questions were about "how" rather than "why." Such classrooms deprive learners of the insights into understanding that it is not the voice of the teacher that decided on the truth of a mathematical proposition but the mathematics itself. Also, teachers' choice of task and their questions and comments during class affect the development of learners' functional understanding of proof in mathematics (Peressini, Borko, Romagnano, Knuth, & Willis, 2004). Thus, teachers have a more central role in shaping learners' functional understanding of proof than any other curriculum material as they assign tasks to learners to fulfill their didactic contract in the classroom (Tarr, Chávez, Reys, & Reys, 2006).

According to CadwalladerOlsker (2011), there are two suggestions as to why learners do not question or exhibit curiosity about why a proposition is true. The first suggestion is that learners believe that the teacher knew what they were doing. The second suggestion is that learners lack the intellectual curiosity about why a proposition is true because the curriculum emphasises the truth rather than the reasons for the truth (Harel & Sowder, 1998). In this respect, given learners' lack of experiences with the concept of formal proof in the classroom, reliance on the teacher becomes natural. They assume that if a theorem is verified by an authority, there is no value in reading and understanding it for themselves. Harel and Sowder (1998) found that the first and most common manifestation of this behaviour is when learners insisted on being told the procedure to work on a proof rather than participate in its construction. Thus, they saw themselves as mere consumers of mathematical knowledge. As a result, they do not believe that they have the ability, or right, to prove a conjecture (Plotz, Froneman, & Nieuwoudt, 2012).

Thus, it is reasonable to suggest that opportunities that enable learners to make observations, conjectures, and construct deductive proof are to be encouraged in Euclidean geometry classes. Learners needed to become accustomed to the expectation of explaining why an observation is true. The function of proof as explanation is supported in the CAPS document which advocates that while teachers address the "how" part in proof activities, emphasis must be placed on the "why" part, as well. In light of these propositions, making conjectures needs to be the essential feature in proving tasks in order to allow learners to experience mathematics like mathematicians.

3.4.4 Textbook

The design of the curriculum in any particular country influence and shape learners' functional understanding of proof (Healy & Hoyles, 1999). Learners also experience the curriculum through the textbooks. Stylianides (2009) suggests that mathematics textbooks can play an important role in providing learners with opportunities to engage in the proof concept. In fact, given that teachers were hesitant to create their own teaching materials, the reliance on existing textbooks is pervasive in schools (Makgato & Ramaligela, 2012). Yet, little is known about what is in learners' mathematics textbooks in terms of the functions of proof in mathematics (Kajander & Lovric, 2009). That is, while textbooks enable learners to do proofs, they have not engaged into detail about what proof is or about its functions in mathematics.

In most instances, both the teacher and learner source activities such as exercises, assignments, homework, or tests directly from the textbook. Thus, textbooks play a prominent role in the development of learners' functional understanding of proof. However, most textbooks do not develop the concept of proof adequately but instead delve into proof from the beginning of the deductive proof education. Several new textbooks that proclaim to be CAPS-compliant and promoting "discovery". However, upon closer examination of the tasks in these textbooks, I found that they are of the "prove that" type and thus evoked the sense of seeing proof as merely verification of the truth of propositions by testing several cases. More specifically, there is little evidence to suggest that key curriculum materials such as textbooks reflected the emphasis placed on explanatory power of proof in school mathematics. I am of the view that this observation could have triggered de Villiers' (1998) suggestion that the word "explain" rather than "prove" should be used to emphasise the explanatory function as the intended function of proof.

However, an exception is the textbook entitled *Everything Mathematics: Grade 11 (Version 1 CAPS)* that seems to create activities that trigger conjecturing in learners to reflect the ideals professed in the CAPS. In this textbook, conjecturing is 'thought of as the mathematicians [sic] way of saying "I believe that this is true, but I have no proof yet"' (Uploaders, 2013, p. 102). As already mentioned, conjecturing is ideal as it tends to reveal the functions of proof in mathematics as learners verify their claims, explain their ideas, argue (communication and discovery functions), and connect various theorems (systematisation). Nevertheless, in general, textbooks seem to promote the verification function of proof by using the verb “prove” rather than “explain”, an approach that does not seem to embody one of the Specific Aims advocated in CAPS. According to Hadas et al.’s (2000) analysis of teaching materials, these five functions of proof highlighted in this study often remain hidden in textbooks.

In South Africa, the education department planned to achieve "universal coverage", the official term for providing a textbook to each learner in every subject so that the intended curriculum as specified in the CAPS document could be experienced by learners through the recommended textbook (Motshekga, 2015). Thus, the understanding of the functions of proof that learners develop are mediated by the textbooks as they study proof, answer geometry homework questions or proof projects. Fujita, Jones, and Kunimune (2009) performed an analysis of textbooks commonly used for teaching about proof in geometry in Japanese secondary schools. They found that though deductive reasoning is prominent, proof and proving omitted to illustrate convincingly the difference between inductive arguments and deductive arguments. They also argue that an improvement in textbook design needs to involve providing effective instructional activities so that learners can gain fuller appreciation of the generality of a proof in mathematics. Generality of proof refers to the fact that a proof includes an entire class of mathematical objects or situations without exception (Harel & Sowder, 2007; Ottens, Gilbertson, Males, & Clark, 2014).

To engage in the practice of mathematicians, textbooks need to expose learners to opportunities designed to show that, as Lakatos (1991) argues, patterns can forerun the generation of conjectures, which in turn can give rise to the development of proofs. Therefore, treating proof in school mathematics in isolation from the functions that it perform in mathematics not only leads to distorted functional understanding of proof, but also deprives learners the opportunity to experience proof like mathematicians. In addition, distorted understanding of the functions of proof, once formed, will shape learners’ behaviour in ways that have detrimental consequences in subsequent proving effort and performance (Schoenfeld, 1992). Since there is a general acknowledgement that textbooks remain a fundamental learning resource for learners, I suggest that more attention needs to be paid to the ways in which textbooks present the concept of proof in Euclidean geometry. However, the attainment of this ideal is difficult given that most textbook developers write for profit-making purposes rather than for promoting specific pedagogical approaches for a given school subject (Stray, 1994).

3.4.5 Culture in the South African society

As a human activity, learning mathematics is influenced by the culture of the society in which it is practiced. That understanding is influenced by culture in different societies has long been a proposition entertained by many social scientists (for example, Hofstede, 1986; Triandis, 1994). However, there is no consensus even among social scientists on the meaning of the term *culture* (Felbrich, Kaiser, & Schmotz, 2012). For instance, House, Hanges, Javidan, Dorfman, and Gupta (2004) define culture as ‘shared motives, values, beliefs, identities, and interpretations or meanings of significant events that result from common experiences of members of collectives that are transmitted across generations’ (p. 15). Hofstede (1980) define culture as ‘the collective programming of the mind which distinguishes members of one human group from another’ (p. 25). However, I found Schein’s (2004) definition of culture more comprehensive and relevant for this study. He defines culture as:

> A pattern of shared basic assumptions that the group learned as it solved its problems of external adaptation and internal integration, that has worked well enough to be considered valid, and, therefore, to be taught to new members as the correct way to perceive, think and feel in relation to those problems (p. 17).

In considering the culture that is prevalent in South Africa, I drew on two of Hofstede’s (1986) basic four-dimensional model of cultural differences among societies of over 50 countries: collectivism–individualism; power distance; masculinity-femininity; and, uncertainty avoidance. Hofstede (2011) defines dimension as ‘an aspect of a culture that can be measured relative to other cultures’ (p. 7). For a more complete review of all the dimensions the reader is referred to Hofstede (1980). I found his collectivism–individualism and the power distance dimensions with reference to teacher-learner interaction useful in explaining how culture affected *Presh N*’s beliefs about the functions of proof in mathematics.

On the one hand, an individualistic culture is that characterised by practices that tend to view lack of success in mathematics as a consequence of a misfit between learning environment and learner; for example, too demanding tasks. In addition, learners from individualistic countries tend to conceive of proving as a process that entails engaging in social interactions (for example, conjecturing, investigations, measuring or experimentation) in the classroom, and thus taking a dynamic view of mathematics (Felbrich, Kaiser, & Schmotz, 2012). The CAPS supports this view by discouraging silence in the geometry classrooms. Engaging in argumentation creates opportunities for learners to gain an appreciation of the functions of proof. As a consequence, learners' ability to refute a claim on mathematical grounds rather than by appealing to the authority of the teacher or the textbook to resolve disagreements at all times, is valued.

On the other hand, a collectivistic culture is found in societies that attribute failure to individual characteristics of the learner; for example, a lack of effort. In collectivist cultures, where conformity and obedience to group norms are important attributes, it is believed that a learner's behaviour is a consequence of his or her adherence to group expectations. As a consequence, collectivist cultures tends to be less tolerance for deviation from the norm (Lawson, 2015). In addition, teachers expect learners to be proficient in the application of rules and formulae in assessments such as final examinations, a practice that predisposed them to endorse of a static view on mathematics (Felbrich, Kaiser, & Schmotz, 2012). Also, from the perspectives of the power distance dimension, in collectivistic cultures, parents teach their children obedience where older people are both respected and feared leading to teacher-dominated classrooms (Hofstede, 2011).

Hofstede's (1986) hypothesises that a country's culture influence the preferred modes of learning. Felbrich, Kaiser, and Schmotz (2012) examined the results of primary mathematics school teachers in 15 countries to determine their views about the nature of mathematics. This investigation included South Africa's neighbouring country, Botswana, which he classified as a collectivistic country. By extension, very few will contest the view that as a Southern African country with a relatively large African majority, South Africa is collectivistically oriented in its culture. Support for this claim came from the fact that the education system in South Africa places a high premium on examinations. The annual practice of releasing Grade 12 examinations results in a ceremonial style followed by the publishing of names of those learners who passed attested to my claim. The (perhaps) unintended consequence of this practice is that learners who transgress the norm by failing develop feelings of shame, some to the extent of attempting or committing suicide. Further, if my reasoning is accepted, learners in the South African education system – as pointed out by Felbrich, Kaiser, and Schmotz (2012) – seem to engage in the learning process because of an obligation towards their teachers and families who in turn were obliged to grant them the necessary support. Having outlined the practices prevalent in the South African education system, I think that it is reasonable to suggest that South Africa is an archetype of a country that projects a collectivist culture.

3.4.6 Semantic contamination

Pimm (1987) introduces the notion of "semantic contamination" to refer to the interference of natural language in the learning of mathematics. In this study, I treated the term semantic contamination as referring to the notion of associating arguments outside mathematical objects with mathematical proof. I think it is reasonable to suggest that most people are familiar with a commentary after a football match that could be stated along this line *"Christiano Ronaldo has proven once again that he is the best footballer in the world by winning the World Player of the Year contest"*. In this case, the use of "proven" is meant to suggest that he was put on "trial" and found to have passed the "test" (Reid & Knipping, 2010).

Like de Villiers (1998), I equally argue that the different meanings attached to the word proof lead to misunderstanding in the mathematics classroom. In everyday life, people think of "proof" to be synonymous with conviction. As Gopnik, Glymour, Sobel, Schulz, Kushnir, & Danks (2004) note, learners are so adept at proving in nonmathematical contexts, yet they are so poor at proving in Euclidean geometry. Bretscher (2003) concurs and eloquently summarises this phenomenon clearer as she points out that "proof" in everyday life tends to take the form of evidence used to back up a claim. In addition, in an examination of the mathematics classroom from a linguistic point of view, Mejía-Ramos and Inglis (2011) found that the technical meaning of the two main linguistic ways of representing the concept of proof, "proof" and "prove" are not distinguished from their everyday life use in natural language such as ordinary English; they evoke different meanings in different people.

While acknowledging that proving is a complex and cognitively demanding process, Carpenter, Franke, and Levi (2003) argue that semantic contamination is one of the sources of naïve (verificationist) understanding of the function of proof. As already stated, in mathematics, the term "proof" refers to a product of a sequence of logical arguments resulting from a conjecture. Clearly, this technical notion of proof is distinct from its everyday meaning. As a consequence, it is reasonable to suggest that one of the reasons learners experience difficulties with the concept of proof is that the learner and their teacher could be using the terms "proof" and "prove" from different points of view. I share Tall's (1989) belief that an adequate attention to the concept of "mathematical proof" is rarely, if ever, satisfactorily considered in the classroom. In an attempt to provide a guidance on this issue, Epp (2003) stresses that mathematical language needs to be unambiguous – which in my view is a consequence of the nature of language – with each grammatical construct having exactly one meaning. To alleviate the confusion caused by "prove", de Villiers (1998) suggests an interesting introductory statement for the teacher following verification process:

> We now know this result to be true from our extensive experimental investigation. Let us however now see if we can EXPLAIN WHY it is true in terms of other well-known geometric results, in other words, how it is a logical consequence of these other results. (p. 388)

Clearly, everyday usage of "proof", "prove", and "proving" differ considerably from its technical meaning in the mathematics community. Thus, there is a need to "sanitise" the meaning of these concepts, particularly "proof" to indicate that it refers to deductive arguments, and deductive arguments alone. The influence of natural language on the learning of proof underscores the point that learning also needs to involve acquisition of the mathematics register to enable learners to expand their cognitive model of the everyday life terms that assume different meaning in mathematics.

3.4.7 Empirical argument

As mentioned previously, an *empirical argument* is an argument that purports to show the truth of a mathematical claim by considering a few selected cases. Empirical arguments relied on either evidence from examples (sometimes just a single case) or direct measurements of quantities and numerical computations or perceptions to justify the generality of a proposition (Harel & Sowder, 2007). Appreciating the functions of proof is made especially more difficult 'when these proofs are of a visually obvious character or can easily be established empirically' (Gonobolin, 1975, p. 61). However, it needed to be mentioned that this belief persisted despite Popper's (1988) attempts to demystify it by pointing out that 'no rule can ever guarantee that a generalisation inferred from true observations however repeated is true' (p. 25).

I take exception to Harel's (2013) argument that seems to find fault in his learners when they suggest that conviction of the truth of a mathematical proposition is based on empirical evidence rather than on deductive proof. Specifically, he claims that 'students viewed their actions of verifying an assertion in a finite number of cases as sufficient for removing their doubts about the truth of the assertion' (p. 125). Considering this claim in light of his other conclusion that '[a] person is said to have proved an assertion if the person has produced an argument that convinced him or her that the assertion is true. Such an argument is called *proof'* (p. 124). I am ultimately convinced that his argument is false and his learners were correct. In support of my argument, de Villiers (1998) concludes that conviction is 'probably far more frequently a prerequisite for the finding of a proof' (p. 375). Similarly, Bell (1976) stated that 'Conviction is normally reached by quite other means than that of following a logical proof' (p. 24). A similar sentiment is echoed by Schoenfeld (1994) who points out that mathematicians try to produce a proof of a conjecture to show that it works once they suspect that it is true. De Villiers (1998) goes on to clarify why such an observation is flawed:

> For what other, weird and obscure reasons, would we then sometimes spend months or years to prove certain conjectures, if we weren't already convinced of their truth? (p. 18)

Stylianides (2009) argues that learners' engagement in empirical arguments is likely to reinforce the common conception that an empirical argument can be used to "prove" the generality of a proposition. This notion of proof as something constituted by empirical arguments is further perpetuated by the fact that 'all real-life proofs are to some degree informal' (Hersh, 1993, p. 391). According to CadwalladerOlsker (2011), primary school mathematics also contributes to the treatment of empirical arguments as proof. She points out that in primary mathematics, the weight of several examples might be enough to "prove" that the sum of two even numbers is always even. He hypothesises that when these same learners engage in high school geometry, they may try to use similar empirical evidence to prove propositions.

Proof is something quite distinct and as such evidence alone may support a conjecture but would not be sufficient to be constitute a proof (Bretscher, 2003). However, learners develop conflicting understanding within the sciences. For instance, whereas deductive proof is the focus in the mathematics classroom, outside Euclidean proof space, including in the physical and life sciences classrooms, learners freely make generalisations based on a limited number of experiences. Thus, the inability to understand the epistemological distinction between proving in mathematics and proving in science contributes to learners' weak appreciation of the power of deductive proof. Hence, learners often require further empirical evidence even after having proved a proposition in Euclidean geometry (Conner & Kittleson, 2009).

Learners' empirically-based responses in deductive proof tasks indicate a weakness or lack of understanding of the functions of proof in mathematics (Stanovich, 2005). However, Mariotti (2006) remarks that an experimental investigation or a task that require learners to prove the validity of a given proposition do not seem to be as effective in triggering the production of arguments and justifications when compared to the task requiring the production of a conjecture. In an attempt to suggest an approach that bridges the gap between empirical arguments and deductive arguments, Stylianides (2009) describes how learners could follow a mathematician's practice that culminated in a deductive proof. He suggests an activity that involves exploring mathematical relationships to identify and arrange significant facts into meaningful patterns and structure, and using these to formulate conjectures. Then, the conjectures are to be tested against new evidence leading to their revision to formulate new conjectures that are consistent with the evidence, and providing empirical arguments to verify the viability of the conjectures. However, I think that he overlooked the inherent difficulties that arise from implementation of this approach in the classroom. From my point of view, there is a hard wall between actual classroom practice of mathematics and the practice of mathematicians and to break this wall requires a concerted effort from a variety of stakeholders, most of all from politicians. In addition, to produce a conjecture is a task that does not fit the didactical contract[13] in school mathematics wherein propositions are presented and illustrated by the teacher, absorbed and applied afterwards by the learner in tests and examinations (Douek, 2009).

Empirical arguments, frequently the only type of proof comprehensible to learners, may be mathematically valid for establishing refutation by counterexample but invalid if few cases were used for a proof (Hanna, et al., 2009). Hence, “proof” through providing empirical evidence rather than through validation, though prevalent, only limits learners’ understanding of the functions of proof to that of verification or justification. The term “validation” is used to refer to the construction of reasons to accept a specific proposition, within an accepted framework shaped by accepted rules and other previously accepted propositions (Balacheff, 2010).

That empirical explorations provide limited insights into the functions of proof is based on the premise that they provide inconclusive evidence by verifying truth of propositions only for a proper subset of all the cases covered by a deductive proof (Stylianides & Stylianides, 2009). That is, unlike deductive proof which combines logical propositions, there may be an exception or counterexample that negates a conjecture; in actual fact, empirical arguments lead to conjectures because it is virtually impossible to consider every case. Having said that, Harel and Sowder (1998), Healy and Hoyles (2000), Knuth (2002), and Knuth et al. (2009) demonstrated that naïve empiricism is widespread and pervasive way of reasoning among high school learners of mathematics. Balacheff (1988) uses the phrase “naïve empiricism” to describe the practice of

[13] Brousseau (1997) refers to the teacher’s routine instructional obligation as a didactical contract.

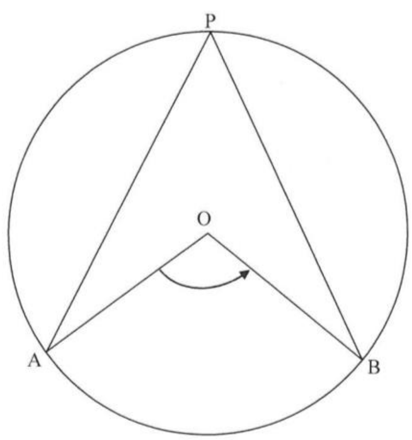
asserting the truth of a proposition after verifying several cases; that is, using empirical arguments as mathematical proof.

My own personal experience as a Grade 12 examination marker also attest to this argument. I suggest that naïve empiricism is encouraged in examinations. For instance, in the South African context, the high school question papers is often sequenced such that immediately after proving a theorem, subsequent questions require learners to verify the validity of that theorem empirically by considering specific cases. This instance is exemplified in Figure 3—8. The sample question is an adjusted version of the National Senior Certificate[14] (NSC) examination (mathematics paper 2) prepared and written in the February/March supplementary examinations period. The question reflects a practice which is akin to seeking conviction about the truth of a proposition by considering particular cases. This practice predominates mathematics classrooms in high schools. For instance, Schoenfeld (1989) and Fischbein and Kedem (1982) found that learners tend to seek conviction by empirical means although they had just performed a deductive proof of a conjecture. This behaviour reflects learners' failure to appreciate that proof provides a firm intellectual foundation which meant that they did not have to appeal to outside experience.

[14] In the context of the South African education system, the National Senior Certificate (NSC), commonly referred to as "matric", is a national, standardised examination, which represents the final exit qualification at the end of high school (Grade 12).

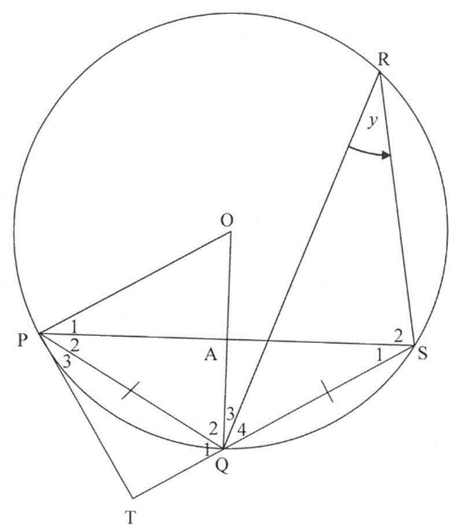
QUESTION 10

10.1 In the diagram, 0 is the centre of the circle and P is a point on the circumference of the circle. Arc AB subtends ∠AOB at the centre of the circle and ∠APB at the circumference of the circle.



Use the diagram to prove the theorem that states that ∠AOB = 2∠APB.

10.2 In the diagram, O is the centre of the circle and P, Q, S and R are points on the circle. PQ = QS and ∠QRS = y. The tangent at P meets SQ produced at T. OQ intersects PS at A.



10.2.1 Give a reason why $\angle P_2 = y$.

10.2.2 Prove that PQ bisects ∠TPS .

10.2.3 Determine ∠POQ in terms of y.

Figure 3—8. An example of examination questions sequenced to written in the February/March supplementary examinations (pp. 12-13)

This phenomenon can be traced back to instances where learners are generally provided with "riders" as extension of classroom instruction to provide practice and thus, at least unwittingly, consolidate conviction in the truth of theorems. Learners need to understand that an empirical argument merely confirms the validity of a conjecture and serves as a spark to understanding why the conjecture is true. This is not to be construed as an indictment on empiricism; it must be stressed that empirical evidence is a necessary but not an altogether sufficient step towards the development of a proof. The point is, one other way in which the seeing proof solely as a means to verify develops is through believing that empirical arguments are mathematical proof.

3.4.8 Dynamic geometry software

> [I]f a student does become convinced … by just observing the computer screen without once measuring the angles himself or herself with a protractor, then I would be inclined to cite this as an example of a catastrophic failure in our education of the young. (Wu, 1996, p. 230)

Mathematics is not immune to the effect of rapidly developing technology in society. De Villiers (2012) points out that DGS provide a wealth of opportunities to develop learners' understanding of functions of proof other than just the traditional function of verification. He asserts that the most powerful ways to promote appropriate functional understanding of proof entails providing learners with an environment to make conjectures by themselves and encouraging their systematical exploration. Mariotti (2007) concurs. Also, in an attempt to mitigate the effect of empirical arguments on proof, Harel and Sowder (2007) point out that learners need to be given opportunities to engage in conjecturing. Like Stylianides (2009), they assert that conjectures are important because they are propositions that call for further examination, including verification which is a precursor to a proof. However, as I indicated earlier, creating such environments in the context of an education system that placed a premium on examinations is a challenge that could not be addressed unless there is political will to do so. Nonetheless the suggestion of conjecturing ventilated here might be useful at a theoretical level.

De Villiers (2012) and Mariotti (2007) point out that the introduction of technology in the form of experimentation with computers and calculators has the potential to promote this ideal through infinite inductive trials using DGS. Thus, with the advent of DGS packages, the efforts to address the problem of transitioning from empirical experiments to deductive reasoning might be enhanced (Harel & Sowder, 2007; Kondratieva, 2011). However, there are various DGS material, but only a few can handle proofs: *GeoGebra, Cabri Geometer, Cinderella, and Geometrix* (the list is not exhaustive though). For instance, Narboux (2004) points out that *Coq* is one DGS found to be a proof assistant on the ground that it allows the learner to conjecture using a base of known lemmas leading to the development and validation of the proof. A proof assistant refers to a software that can be used by a learner in the classroom to interactively build a formal proof such that if it is accepted by the *Coq* proof assistant, the teacher can have a very high level of confidence in the learner's proof (Narboux, 2004).

De Villiers (2012) suggests that it is far more meaningful to introduce proof within a dynamic geometry context not as a means to verify, but rather as a means to explain, systematise, and discover before engaging deductive proof. This assertion is supported by Hanna (2000) who claims that DGS has the potential to encourage both exploration and proof, because it makes hypothesising and testing conjectures easy. However, de Villiers (2012) cautions that while DGS is beneficial to proving activities because it provides accuracy, immediate visual feedback, and the ability to check many cases in a short space of time, it tends to make learners see less need for a deductive proof. Also, Laborde (2000) found that the opportunity offered by DGS to "see" properties of geometric figures 'so easily might reduce or even kill any need for proof and thus any learning of how to develop a proof' (p. 151). Similarly, Magajna (2011) argues that although they promote conceptual understanding, DGSs by their nature favour the empirical aspect of geometry. With respect to measurement, practices that emphasise the importance of quasi-empirical testing, that is, "accurate" construction of some examples, fail to motivate learners to search for a proof (de Villiers, 1998). In my view, increasing learners' access to DGS is one aspect that needs attention to assist in the learning of proof.

Motivated by a desire to create experiences that encourage the discovery of theorems by the zig-zag method rather than merely expecting his learners to accept proofs on blind faith and to improve learners' attitude towards mathematics and proof, Hogan (1999) developed a paper-and-pencil Euclidean geometry unit for deductive reasoning in a Grade 8 classroom. He found that it is feasible to create an environment for learners to experience deductive reasoning while following the prescribed curriculum. Although the activities could be done in pencil and paper contexts, for schools with access to technological equipment such as tablets, laptops, smartboards, and desktop computers, *Sketchpad* could be used to economically make our learners gain insights into the nature of mathematics and thus mitigate time constraints embedded in the curriculum. In addition, Gillis (2005) found that learners who use DGS made more relevant conjectures, fewer false conjectures, and the level of conviction in their conjectures is higher when compared to those working in a static geometry environment. De Villiers (2012) developed and evaluated, by analysing actual mathematical practice, *Sketchpad* activities to assist instruction to focus on the functions of proof where the traditional verification function is not always the sole focus of Euclidean geometry proof instruction.

The sentiments by de Villiers (2012) and Magajna (2011) demonstrate that inductive arguments (verification using limited cases) have a meaningful role to play in the proving process, particularly in formulating conjectures. However, the problem arises when learners use inductive arguments as proof of proposition that in fact require deductive arguments. Hence, Stylianides (2009) suggests that empirical arguments need to be avoided precisely for this problem. Nonetheless, I disagree with Stylianides' (2009) notion on the basis that the verification function of proof has utility if it were used as a precursor to conjecturing. I am mindful of the fact that to act on this suggestion is indeed difficult, but at the same time I think that identification of the factors influencing learners' functional understanding of proof is one good step towards a solution.

3.4.9 Language of instruction different from home language

Children and adults use language as a means to learn to organise their experiences and thoughts. Personal experiences in the mathematics classroom suggests that language is a key factor for conceptual understanding of proof and proving. Specifically, in the South African context, Feza and Webb (2005) found that language proficiency is a barrier to the attainment of understanding geometry in learners whose home language is not English. Schäfer (2010) suggests that a mathematics register in the indigenous African languages is needed for effective teaching and learning of mathematical concepts to happen. Poor performance of South African learners in the Trends in Mathematics and Science Study (TIMMS) is largely ascribed to the problem that the majority of African learners in townships and rural schools in socioeconomically and educationally disadvantaged areas (over 80% of all learners) study science and mathematics through English which is their second or even third or fourth language (Probyn, 2006; Schäfer, 2010). The problem is further intensified by the fact that a mathematical register in any of these languages is absent.

Hence, none of these languages is used as a language of learning and instruction (LoLT) in intermediate, senior, and FET classes.

A mathematics register entails terms, words or expressions for communicating about mathematics. LoLT refers to the language medium wherein learning and teaching including assessment take place while indigenous African language refers to the other “official” languages namely, IsiNdebele, isiXhosa, Sepedi, Sesotho, Siswati, Tshivenda and Xitsonga. However, two other official languages English and Afrikaans are dominant mediums of instruction in the school system as a whole (Department of Basic Education [DBE], 2011). They are languages of a small but powerful elite section of the population who argue that teaching and learning in English gives learners and teachers greater access to scientific and technological knowledge (Halai & Clarkson, 2016). I am mindful of the fact that this so-called official list is not exhaustive in that it left out languages of the other indigenous peoples, the Kois and the Sans. However, the vast majority of learners chose to learn in English rather than Afrikaans (which evolved from the Dutch spoken by early Cape settlers).

Returning to the discussion on language influence in learning the functions of proof in mathematics, Taylor and von Fintel (2016) point out that the extent to which language factors contribute to low scholastic performance is unclear given that language disadvantages were so strongly correlated with other confounding factors such as historical disadvantages, deficiencies in the curriculum, resources, teaching approaches, socioeconomic status, geography, and the quality of both school management and teachers. They asserted that clear empirical evidence about the role of second language instruction on learning outcomes is insufficient for African countries including South Africa. Also, to assess learners’ difficulties with understanding the functions of proof could be semantic rather than conceptual.

In summary, although in my view developing mathematics registers in indigenous languages could go a long way in remedying the problem, I am also mindful of the fact that other factors compound the problem further. Therefore, discussion of challenges associated with developing mathematical registers in indigenous languages, complexities of learning and teaching mathematics in linguistically diverse classrooms, and the influence of LoLT in understanding of proof is beyond the framework of this study and thus are not discussed further here. More specifically, on the basis of the contradictory findings highlighted here I decided not to investigate with a view to explain language as a factor that accounted for learners' functional understanding of proof.

3.5 The conceptual framework for this study

3.5.1 Conceptual framework in a theoretical framework

In this study, part of the critical engagement of previous literature on functions of proof and the concept of argumentation was meant to explore a theoretical framework (the two theories and various concepts) in order to develop a conceptual framework. In deciding to present both the theoretical and conceptual frameworks for this study, I was guided by Kumar's (2005) view that a conceptual framework stems from a theoretical framework and concentrated, usually, on one aspect of that theoretical framework which formed the basis of a research problem. However, according to Miles and Huberman (1994), it seems that no consensus exists in literature on the difference between a theoretical framework and a conceptual framework (for example, Leshem & Trafford, 2007; MacMillan & Schumacher, 2010; Maxwell, 2013; Miles & Huberman, 1994; Sinclair, 2007).

The lack of consensus notwithstanding, Rocco and Plakhotnik (2009) makes a distinction between a theoretical framework and conceptual framework and argues that using these terms interchangeably in research causes confusion. Imenda (2014) and Miles and Huberman (1994) support this stance and venture to make a distinction between these two constructs. Imenda (2014) defines a theoretical framework as 'the application of a theory, or a set of concepts drawn from one and the same theory, to offer an explanation of an event, or shed some light on a particular phenomenon or research problem' (p. 189). Miles and Huberman define a conceptual framework as a visual or written product, one that 'explains, either graphically or in narrative form, the main things to be studied—the key factors, concepts, or variables – and the presumed relationships among them' (p. 18). Kitchel and Ball (2014) go further to point out that though a conceptual framework or model indicates that a relationship exists, it lacks the rationale behind the relationship. In this study, whereas a theoretical framework is overarching and has its genesis in theories and constructs that have already been tested and thus generally accepted in literature, a conceptual framework emerges from a theoretical framework and is more specific in identifying relationships among concepts from the theories already identified in this study.

However, in view of the design of this study, there is a potential stumbling block suggested by Ngulube, Mathipa, and Gumbo (2014). These researchers point out that the use of research frameworks is not yet fully developed in mixed methods studies. In contrast, Grant and Osanloo (2014) believe that both theoretical and conceptual frameworks can be used in mixed methods designs. Taking the latter advice, I went ahead and constructed a conceptual framework on the basis of two theoretical frameworks. These frameworks are manifested to some degree in the methodology, arguments about what might happen, research questions, data collection and analysis, and synthesis of the findings (Bernard, 2013; Silverman, 2013; Royse, 2008).

3.5.2 Connecting theories, research problem and questions

Whereas a theoretical framework was used to ground the study in the van Hiele and Toulmin's theories, the purpose of constructing a conceptual framework was to diagrammatically connect the relevant concepts guiding this study in the theories and constructs underpinning this study. It is worth noting that this conceptual framework which situates the study in relevant literature and illustrates the network of relationships among the concepts with a figure, was constructed rather than found ready-made in literature waiting to be utilised. I schematically present the multidimensional conceptual framework specific to this study in Figure 3—9. For example, the thicker arrows represent the notion that the collectivist culture is hypothesised as stronger than the textbook or teacher factors at accounting for the reasons why the learner held the beliefs she held about the functions of proof. In addition, the conceptual framework addressed both the qualitative and quantitative strands of this study and suggested interactions and relationships among the variables embedded in the problem investigated in this study.

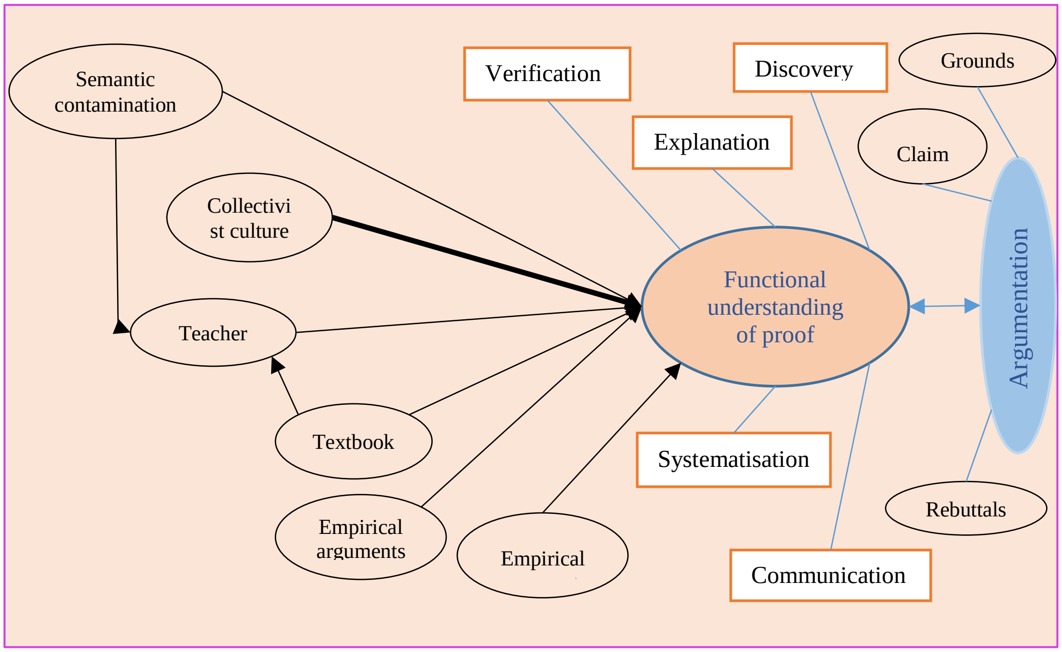


Figure 3—9. The multidimensional framework of this study

According to Silverman (2013), concepts are essential in a research problem and need to be described as clearly specified ideas deriving from a particular theory. Thus, this section focuses on demonstrating the connection among the theories and constructs underpinning this study and their role in contributing to finding answers to the three problems highlighted in this study. The first problem relates to the contention that learners encounter difficulties with proof, yet very little insight on their functional understanding of proof has been provided despite acknowledgement in literature that this understanding makes the learning of proof a meaningful activity. The second problem is that very little is known about whether an association between learners' argumentation quality and functional understanding of proof despite the recognition that argumentation is embedded in proof. Third and final is that while knowing learners' functional understanding of proof is important, more important is knowing the factors that either promote or inhibit the acquisition of informed functional understanding of proof.

Guided by the theoretical and empirical literature previously reviewed and the presented multidimensional framework of the study, the following main research question has been established to frame the design of the study, data collection, and data analyses: "How can meaningful learning of proof construction take place?" This question was broken into four sub-questions:

3.5.3 Quantitative phase questions

3.5.3.1 What functional understanding of proof do Grade 11 learners hold?

3.5.3.2 How is the relationship (if any) between learners' quality of argumentation and their functional understanding of proof?

3.5.4 Qualitative phase questions

3.5.4.1 Why does Presh N hold informed beliefs about the functions of proof?

3.5.4.2 How is the interaction among the three constructs (that is, functional understanding of proof, argumentation ability, and factors influencing functional understanding?

For the purpose of this study, factors such as two-column proof, ability, proof-type, language, teaching methods, and DGS were not part of the investigation; some required examination of learners' written work and others required observation of experimentation and conjecturing, aspects that were beyond the scope of this study. Thus, I would have been unreasonable to posit that current instructional practices were not advocating the aims in CAPS without empirical evidence collected from classroom observations. That said, data were collected and analysed and inconsistencies were explained in light of the theoretical framework.

This study was guided by the view that a theoretical framework dwells on established and tested theories that underpin the findings of numerous investigations on how variables in a phenomenon are interrelated while a conceptual framework is viewed a model that indicate or describe the relationships among specific variables identified in the study. One way of making sense of this differences is to consider the scale on which the frameworks differed in this study. By way of example, the broader framework that provided direction is the van Hiele theory, learners at high schools were expected to understand the functions of proof in mathematics. In brief, theoretical perspectives of the theory informed the conceptual framework. Therefore, the conceptual framework contains my idea on how the research problem identified in the theoretical framework was explored. This is what differentiates a conceptual framework from a theoretical framework.

3.6 Chapter summary

The purpose of this chapter was to present the theoretical and conceptual frameworks guiding this study. I highlighted how the terms theoretical framework and conceptual framework, used interchangeably in some instances, are viewed as different in this study. I showed how the van Hiele and Toulmin's theories together with the relevant concepts underpinning the investigation on learners' functional understanding of proof and argumentation ability in mathematics were connected by providing a conceptual framework. Literature reviewed here pointed to the fact that a variety of factors may work in tandem to produce a litany of understanding of proof functions. The key idea in this chapter was that for learners to begin to improve their performance in Euclidean geometry, efforts must not be spared to capture the concepts that foreground the development of proof: functional understanding and argumentation. The next chapter describes the research design, methods, and methodology, including the various samples of schools and learners and the justification for the instruments used to collect and analyse data.

Chapter 4

Research methodology

4.0 Introduction

The previous chapter discussed theoretical literature on functions of proof and argumentation and presented a conceptual framework to depict the relationships among the different variables considered as useful background for the research investigation that is described in the next chapters. The purpose of this chapter is describe in detail how the research questions posed in the previous chapter and introduced in Chapter 1 was answered. The outline of this chapter is as follows: methodological framework, distinction between methods and methodology, methodological approaches in previous studies on functional understanding of proof, methodological framework, paradigm, design, strategies, sampling procedure, research instruments and interview schedule, data collection procedures, data analysis procedures, data integration, and rigour and limitations.

4.1 Methods versus methodology

Traditionally, there are three research methodologies: quantitative, qualitative, and mixed-methods (Cohen, Manion, & Morrison, 2011). In this study I describe the mixed methods as it relates to issues of design and strategy used, the research instruments used for sampling the data, procedures used to collect data. This also entails probing into how the data collected were analysed to answer the research questions. Although oftentimes, the terms method and methodology are used interchangeably in literature, I argue that there is value in making clear the distinction between these terms. In both terms, the root word is "method" but in methodology, the Greek suffix *logos* is added to mean "reason". As Clough and Nutbrown (2012) note, the value of not viewing methods and methodology as synonyms lies in the fact that one will be able to provide a justification to questions such as "Why carry out a questionnaire survey?", "Why semistructured interviews of 1 rather than 200 participants?" than merely list data collection and analysis techniques. According to Clough and Nutbrown (2012), methods are the ingredients of research while methodology relates to the theoretical reasons marshaled to justify using a particular recipe.

According to Guba and Lincoln (1994), methodology is a systematic analysis of methods and principles employed to obtain data in order to find answers to a problem. To clarify further, methodology involves a consideration of the principles such as ethical issues in data collection and analysis methods. However, I endorse the distinction made by Bogdan and Biklen (2007) who construe "methodology" as the general logic and theoretical perspective of a study, whereas "methods" only refers to specific strategies, procedures, and techniques of analysing and interpreting data. In light of this distinction, Long (2014) points out that methodology is a significant component of research not only because it embodies philosophical assumptions, but also because it guides the selection of research methods.

In this study, methods are the sampling procedures, data collection and analyses techniques such as cluster and purposive sampling, the LFUP and AFEG survey questionnaires, the semistructured interview, and the proof-related task. On the other hand, the methodology denotes the strategies I used to plan these data collection techniques (methods) in terms of how I combined them so as to best attain answers to the research questions. In an admittedly simplistic sense, methods are techniques while methodology is the rationale for the choice of these techniques out of various others, including mentioning their limitations.

4.2 Methodological approaches and methods in studies on learners' functional understanding of proof

In this section I review methodological approaches and instruments used to investigate functional understanding of proof in previous studies. In setting the stage, four examples of studies sufficed in view of the limited space available for this project. However, each of the contentions provided in this section needs not necessarily 'disregard the fact that research methodologies are merely tools that are designed to aid our understanding of the world' (Onwuegbuzie & Leech, 2005, p. 377).

Earlier studies on functional understanding of proof employed primarily quantitative research. One such study was carried out nationwide by Healy and Hoyles (1998). They surveyed 14-15 year-olds in order to, among other objectives, ascertain these learners' functional understanding of proof, in England and Wales. However, though the study was wide (2 459 participants), it lacked depth. In my view, this is an example that demonstrates that qualitative or quantitative research alone is insufficient to better understand a problem (Creswell, 2013). Hence, from a methodological point of view, mixed method research hold the potential for methodologically sound investigations of functional understanding of proof. Therefore, there is no point in the thinking that '[t]he one precludes the other just as surely as belief in a round world precludes belief in a flat one' (Guba, 1987, p. 31).

Whereas the use of qualitative methods in educational research has been regarded as a valuable approach in studying human beings' behaviour and thinking and the experiences they encounter, arguments against the use of quantitative approaches to study human thinking abound (McMillan & Schumacher, 2010). One such argument is that findings emanating from these studies have not led to significant advances in theoretical and applied knowledge within education because statistics cannot explain social behaviour and thinking that must be measured indirectly (Rennie, 1998). However, the argument that quantitative research cannot explain phenomena is not entirely correct.

As Muijs (2004) points out, a well-designed quantitative study will not only investigate what happened but will also provide an explanation of why it happened. For instance, in this study, regression analysis whose correlation coefficients were calculated and equated to validity coefficients, was not only employed to validate the LFUP instrument but also to explain the concept of functions of proof more widely. For this study, learners' behaviour and thinking were measured indirectly because, as Muijs (2004) points out, relatively few phenomena (for example, attitudes and beliefs) in education actually occur in quantitative form. Policymakers often require data that determines the causes of problems, one of the things quantitative approaches such as experimental designs are suitable to determine (McMillan & Schumacher, 2010).

One other recent study that utilised quantitative methods to partly investigate change in learners' functional understanding of proof which provided insightful reflections on the proof phenomenon was reported by Grigoriadou (2012). In particular, she used an experiment (pre-test-post-test comparison) to plan an intervention based on the van Hiele test with a sample of 20 learners in a Greek high school. She found that placing emphasis on making clear to learners the distinction between empirical and deductive arguments at the beginning of the lessons cycle, can help them to understand better the concept of mathematical proof and to produce proofs. Perhaps the most interesting methodological process is found in Atebe (2008). Using both purposive and stratified sampling techniques, he undertook a study to explore and explicate the van Hiele levels of selected high school learners in Nigerian and South African schools. He used interview method to identify the levels of geometric thinking as pen and paper tests cannot provide sufficient information about their levels. To conclude this section, apart from Atebe (2008) and Healy and Hoyles (1998), it is difficult to locate studies on functional understanding of proof which employed an approach other than quantitative or mixed methods research.

The summary of these studies indicated that both dated and recent research on functional understanding of proof seemed to focus more on quantitative than on qualitative methodology. A benefit of engaging in quantitative research is that it seems to attract the attention of policymakers because of the potential of not only obtaining statistically significant results but also those that are of practical significance. This attention may lead to implementation of recommendations emerging from these studies. The challenge therefore is on those mathematics education researchers working within the qualitative framework to explicate solutions to the issues raised here. The methodological position adopted in this study is informed by the belief that research questions should determine whether in a single study quantitative and/or qualitative methods are suitable to provide an understanding of the world.

4.3 Methodological framework

In educational research different methods or sets of methods are utilised to perform specific functions at different stages of the process research process (McMillan & Schumacher, 2010).

However, finding the appropriate sequence of methods requires delicate design steps; methods are combined with each other and designed to perform this function optimally. This overall arrangement is referred to as the methodological framework. According to Guba and Lincoln (1994), a methodological framework is helpful in sequencing methods. They define a methodological framework as a distinctive summary of the approach to the research in such a way that the purpose of the research, data collection procedures, data analyses, and the relationships between the data can be understood.

The framework in Figure 4—1 depicts the three constructs on which data were collected; learners' functional understanding of proof, learners' argumentation ability, and factors affecting functional understanding of proof. The overall interest was to explore the interactions among these three constructs. Worthy to mention is that the examination of learners' functional understanding of proof, their argumentation ability, as well as the interaction among the three were inherently exploratory given that the nature of these three constructs was relatively unknown at the beginning of the study. The study was also correlational in its exploration of the relationship between learners' functional understanding of proof and their argumentation ability.

As already mentioned, the analysis of the interaction of the findings was exploratory because the nature of this interaction was unknown at the start of this investigation. Also, the analysis of the findings was interpretive given that the purpose of the analysis of this interaction was to explore the meaning of the findings (Caracelli & Greene, 1993). Thus, the analysis of the interaction of the findings involves making a personal assessment as to a description that fits the situation (Creswell, 2014). The personal nature of this assessment means that I brought my own perspective to the interpretation of the three findings in this study. Therefore, the interpretation that I made of the interaction will most probably differ from the interpretation that the reader makes (Creswell, 2014).

In the exploration of learners' functional understanding of proof and their argumentation ability, the LFUP scale and the AFEG questionnaire were respectively utilised. The factors influencing *Presh N*'s functional understanding of proof were explored using an Interview Schedule and a proof-related task.

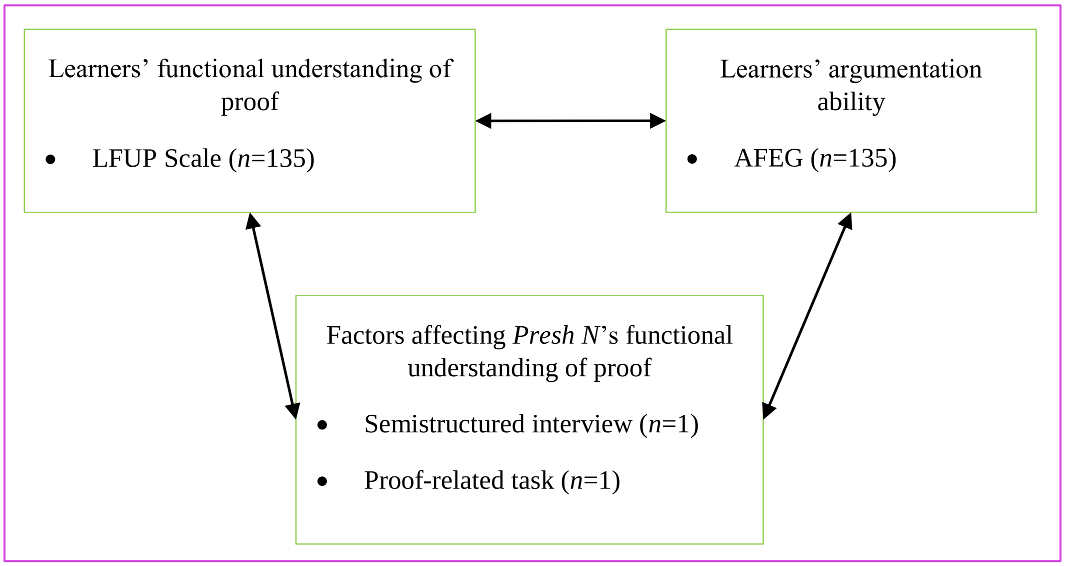


Figure 4—1. The methodological framework

4.4 Research paradigm

I believe philosophically that the best approach to educational research is a *pragmatic* one because, as Tashakkori and Teddlie (2003) point out, this approach is underpinned by the notion that the value of research lies in its effectiveness in finding solutions to research problems utilising multiple methods rather than searching for some "truth". Specifically, in this mixed methods study, the pragmatist paradigm is based on the premise of utilising procedures that "work" (Howe, 1988) in finding answers to the research questions. Subscribing to the pragmatic paradigm means that I moved between the positivistic and constructivist ways of viewing the world. Pragmatism is defined as a 'deconstructive paradigm that advocates the use of mixed methods in research, sidesteps the contentious issues of truth and reality' (Feilzer, 2010, p. 3).

4.5 Research design

The present study followed a mixed methods sequential explanatory design in which the quantitative phase was used to select the single interview participant, *Presh N*. In McMillan and Schumacher's (2010) words, '[a] research design describes the procedures for conducting the study, including when, from whom, and under what conditions the data will be obtained' (p. 20). The main purpose of a research design is to help to avoid instances in which the evidence does not address the initial research questions (Yin, 2014). According to Creswell (2014), a design is explanatory in the sense that the initial quantitative data results are explained further with the qualitative data and sequential in the sense that the initial quantitative phase is followed by the qualitative phase. Creswell and Plano Clark (2011) categorise this design as "QUANT-qual" in which the quantitative component of a study is not only conducted first but is also dominant while the qualitative findings are considered secondary. Various terms are used to refer to this design. Creswell (2014) calls it "explanatory sequential design" and Creswell and Plano Clark (2011) refer to it as a "two-phase model". However, whichever term is used, it remains a mixed methods design which consists of first collecting and analysing quantitative data and then collecting and analysing qualitative data to help explain or elaborate on the quantitative results within a single study.

In this study I assigned a greater weight to the quantitative component given that the priority in this sequential design was to quantitatively explore learners' functional understanding of proof data in order to gain insights into factors that accounted for this understanding (Guba & Lincoln, 1989; Tobin & Fraser, 1991). The reason for adopting this approach is that whereas quantitative data and the subsequent analyses thereof provide general insight into learners' functional understanding of proof and their argumentation ability, qualitative data and analyses thereof refine and explain statistical results by exploring *Presh N*'s views in more depth to explain the factors influencing her beliefs about the functions of proof (Creswell, 2014).

Thus, in this mixed methods sequential explanatory design, quantitative and data collection and analysis (presentation of results, and discussion) were implemented in two distinct phases: collecting QUANTITATIVE data first and looking for an extreme case to follow up in qualitative phase (Caracelli & Greene, 1993). The uppercase letters indicate a priority for quantitative data (McMillan & Schumacher, 2010). For this study, in the quantitative phase, numeric data was first collected and analysed (presented, interpreted, and discussed) and in the qualitative textual data was collected and analysed (presented, interpreted, and discussed) to explain the quantitative results obtained in the first phase. In short, a sequential explanatory design involves beginning with a quantitative design and following with a qualitative design.

Space precluded a detailed consideration of advantages and disadvantages of mixed methods. That notwithstanding, the benefits of using a mixed methods design in this research study were worth noting. First is that quantitative survey findings can be followed up and explained by conducting semistructured task-based interviews with a subsample of those surveyed to gain an understanding of the findings (Doyle, Brady, & Byrne, 2009) obtained in LFUP scale and the Argumentation Frame in Euclidean Geometry (AFEG). Second is that mixed methods can help to increase confidence in findings and provide more evidence while offsetting possible shortcomings from using either a quantitative or qualitative method (Creswell & Plano Clark, 2011). The second benefit is that mixed methods research often has greater impact, because figures can be very persuasive to policymakers whereas stories are more easily remembered and repeated by them for illustrative purposes (Gorard & Taylor, 2004).

I provide a brief and general review background relating to the field of mixed methods research. Creswell (2014) suggests that mixed methods approach can be seen as a new methodology originating around the late 1980s and early 1990s based on work from individuals in diverse fields such as evaluation, education, management, sociology, and health sciences. Teddlie and Tashakkori (2010) chronicle the developments in this field to present a comprehensive snapshot covering the past decade. The original value for mixed methods reside in the notion that both quantitative and qualitative approaches have inherent bias and weaknesses, and the collection of both quantitative and qualitative data neutralises the weaknesses of each form of data (Creswell, 2014). In an attempt to carve a language unique to the field of mixed methods research, a variety of terms are used to refer to this approach. For instance, Teddlie and Tashakkori (2010) suggest the terms frequently used to refer to mixed methods research: integrated or integrative, synthesis, quantitative and qualitative methods, multimethod, and mixed methodology. However, there seems to be consensus around "mixed methods research" as the *de facto* term in this field (Teddlie & Tashakkori, 2010).

Drawing on the perspectives of Creswell (2014), generally, a mixed methods approach enables researchers to minimise the limitations of both quantitative and qualitative approaches. For this study, choosing mixed methods as an approach to research was a useful strategy since it provided better insight into the research problem. In addition, though mixed-methods research studies are more expensive than a single method approach—in terms of time, money, and energy—they improve the validity and reliability of the resulting data (Abowitz & Toole, 2010).

4.6 Survey design and case study design

The survey study in the quantitative phase was crosssectional rather than longitudinal. The decision to use a cross-sectional design related the fact that the problem was not, for example, related to examining the development of the proof concept in Euclidean proof classes in high schools over time and devoting extensive time to data collection. Rather, the problem required a crosssectional design in which learners' functional understanding of proof and their argumentation ability, and the factors influencing functional understanding were examined at one point in time because of time constraints. Thus, I investigated the three variables, that is, functional understanding of proof, argumentation ability, and factors accounting for informed beliefs about the functions of proof, at a specific point in time (Cooper & Schindler, 2014).

I also considered data collection procedures which relates to determining whether survey data collection needed to be based on questionnaires that were self-administered, mailed or electronic and whether the interviews needed to be based on individual, focus group, or telephone. In this study, both questionnaires were self-administered because (1) mailing was going to be time consuming and (2) electronic administration was going to be hampered by learners' inequitable access to computers or appropriate cellular phones. Individual semistructured interview was preferred over focus group interviews or telephone interviews because I was interested in investigating the case of *Presh N*'s beliefs that influenced her understanding of the functions of proof in mathematics in depth utilising multiple sources of data. In addition, individual interview provided opportunities to observe nonverbal cues when the participant was responding to questions and probes which the other methods may not afford.

A graphical representation of the mixed methods model adopted in this study is presented in Figure 4—2 to facilitate the visualisation of the sequence of data collection, sampling techniques, priority of the quantitative phase, and the mixing points of the two approaches in this single study (Ivankova et al., 2006). I adopted Ivankova et al.'s (2006) graphical modelling tool to: portray the sequence of the research activities, show that the quantitative phase was prioritised by capitalising the term "QUANT", show the connecting points between the two phases, specify data collection and analysis procedures, and list expected products from each of the stages of the study. For instance, the figure shows that the quantitative and qualitative approaches were mixed at the sampling stage as well as at discussion of the results stage of the research process.

Phase	Procedure	Product
QUANTITATIVE Data Collection	• Cluster sampling (*n* = 135) • Crosssectional self-administered surveys: LFUP & AFEG	• Numeric data
QUANTITATIVE Data Analysis	• Data screening (univariate, multivariate)	• Normality, linearity, missing data, homoscedasticity, multivariate outliers
	• Principal Axis Factor analysis, frequencies, correlations, ANOVA, multiple regression (with SPSS v.24)	• Factor loadings • Descriptive & Inferential statistics • Comparison of groups
Connecting Quantitative & Qualitative Phases	• Purposefully selecting 1 participant from sample based on highest score	• Case (*n* =1)
	• Developing interview questions	• Interview Schedule
Quantitative Data Collection	• Semistructured interview • Proof-related task	• Text (transcript & document)
Qualitative Data Analysis	• Coding and pattern matching analysis • Cross-thematic analysis & Interpretation (with ATLAS.ti v.8)	• Conceptual framework
Integrating Quantitative & Qualitative Results	• Exploration of the intersection between quantitative and qualitative results	• Discussion • Recommendations • Implications • Future research

Figure 4—2. A visual model of sequential explanatory design procedures employed in this study adapted from Ivankova et al. (2006, p. 16)

4.7 Research strategies

The sequential explanatory method adopted in the present study led to the choice of two strategies: survey and a case study. The "why" research question, capturing the interest in explaining the case of an individual learner, *Presh N*, led to the adoption of a case study as another strategy in this thesis. This strategy was used to gain deeper insights into the reasons why *Presh N* held the functional understanding of proof she held. Findings made were intended to form the basis for the development of a hypothesis that can be tested in future research using other methods. The collection and analysis of LFUP data was in part informed by the work of Liang, Chen, Chen, Kaya, Adams, Macklin, and Ebenezer's (2006). They classified preservice teachers' views on the nature of science in the US, China, and Turkey. This classification builds on the current national and international science education standards documents and existing literature in science education.

4.8 Instrumentation

This section describes the data collection measures utilised in this study. The quantitative phase of the study involved the use of two survey instruments—LFUP and AFEG—to collect data from learners at three schools. The rationale behind the use of surveys is that they are regarded as an inexpensive data gathering technique and often the only feasible strategy to reach a large enough number of respondents to allow for statistical analysis of data (McMillan & Schumacher, 2010). This phase was intended to explore learners' understanding of the functions of Euclidean proof and establish if a relationship exists between this understanding and argumentation ability on a larger scale. Thus, self-administered (LFUP) and open-response argumentation (AFEG) questionnaires were used because the sample was large. By large in this case is meant that the sample size of 135 participants was adequate for performing parametric statistical tests and thus ensuring trustworthy findings (Marshall, Cardon, Poddar, & Fontenot, 2013).

In addition, questionnaires are also low cost and target respondents who can read, write, and answer anonymously and confidentially. By anonymity is meant that participants and their schools could not be identified – by myself or any other person – from the information they provided while confidentiality means that in situations in which I knew who provided the information (for example, in the qualitative phase appertaining the single case study) I was not going to make the connection known publicly (Cohen, Manion, & Morrison, 2011). The principal means by which anonymity was guaranteed was through the use of codes on the questionnaires for identifying participants as well as through the use of password-protected files (Frankfort-Nachmias & Nachmias, 1992).

In the qualitative phase, unlike the quantitative phase whose data collection was limited to one source, that is, crosssectional surveys, data collection consisted of two sources, a semistructured interview and a proof-related task. The data collection tools were the semistructured Interview Schedule and *Presh N*'s proof-related task. Semistructured interview questions, which fall between the completely structured interviews and completely unstructured interviews, constituted the Interview Schedule. An interview schedule contains a set of predetermined core questions to ensure that the same areas were covered with each participant (Ottens, Gilbertson, Males, & Clark, 2014). Standardisation of questions in the form of core questions in the Interview Schedule increases the trustworthiness of data (Creswell, 2014).

A semistructured interview was preferred because it allows adaptation of the formulation of the questions and deviation from the predetermined questions, including the terminology and rephrasing of questions, to fit the background and educational level of respondent (Creswell, 2014). Further, the questions are more likely to receive valid answers than when asked face to face (McIntyre, 2005). Also, the questions place no limitation on how the respondent answered. In addition, a semistructured interview allows participant to delve into detail through probing of responses as and when the need for clarification or explanation arises. The provision of detailed information and probing allows the drawing of conclusions that are trustworthy. In addition, participant can ask for clarification during the interview. However, the interpersonal nature of an interview tends to attract socially desirable responses which offer partial and incomplete understanding of a participant's perspectives (Yin, 2014). In this study, this situation was militated against by appealing to *Presh N* to provide honest responses.

Respondents in this project included learners whose first language was not English. This included learners in township schools who had limited exposure to English outside the classroom and yet the language of learning and teaching mathematics remained English. The use of English and the absence of a mathematics register in the South African indigenous languages demonstrates that language use is inherently political (Setati, 2008). English acquired official recognition in the classroom because it is invariably the language of the small but powerful elite section of the population as well as being used science and technology (Halai & Clarkson, 2016). Discussion of this situation is beyond the scope of this study and thus is not discussed further here.

However, worth mentioning here is the utility of computer software in the analyses of both quantitative and qualitative data. Computer data analysis programs, SPSS and ATLAS.ti, assisted in analysing survey and interview data, respectively. Creswell (2014) makes the point that these programs do not only help during data analysis but also that they are an efficient means for storing and locating qualitative data.

4.8.1 The Learners' Functional Understanding of Proof (LFUP) scale

4.8.1.1 Previous version of the LFUP scale

Shongwe and Mudaly (2017) undertook a methodological study whose purpose was twofold: to develop an objective instrument to measure Grade 11 learners' functional understanding of proof in mathematics. At the time of its development, the instrument was referred to as the *Functional Understandings of Proof Scale* (FUPS). They conducted an exploratory study in two stages: (1) theoretical development of subscales and items and (2) field-tested the instrument and determined its psychometric properties by randomly surveying two groups of participants: 37 mathematics participants and 37 mathematical literacy participants. Mathematical literacy is an FET phase subject that applies mathematical concepts to everyday situations; for example, calculating income tax transfer fees, legal fees, and bond repayment, reading and interpreting statistics in newspaper articles (Clark, 2012).

Initially, a 31-item questionnaire was developed. A panel of experts evaluated content validity and the known-groups method was adopted to assess construct validity. For reliability, internal consistency and item-total correlations were assessed. The instrument received an overall reliability coefficient of .886. In the final analysis, the scale consisted of 25 Likert items. Having given a background to the previous version of the measurement instrument, I now describe the new version (LFUP). In this study it is referred to as learners' functional understanding of proof (LFUP) scale.

4.8.1.2 The LFUP questionnaire for the present study

The validation of the LFUP scale in this study was an effort to provide teachers and educators with an instrument to measure learners' functional understanding of proof and thus inform classroom practice. A curriculum geared towards reflecting the mathematics discipline needs to incorporate the functions of proof in mathematics. As already mentioned, the FET phase mathematics CAPS curriculum stipulated Specific Aims, one of which is understanding that the learning of proof without grasping why it is important, leaves learners ill-equipped to use their knowledge later in their lives. However, effective endeavours aimed at developing learners' informed views of the functions of mathematics require a clearer picture of the current baseline views of these functions: verification, explanation, communication, discovery, and systematisation.

Efforts to evaluate the LFUP scale in this study were guided by the evidence-centred assessment design (ECD) framework. This design framework is based on the principles of evidentiary reasoning embedded in advances in cognitive psychology on how learners gain and use knowledge (Mislevy, Almond, & Lukas, 2003). As Mislevy et al. (2003) put it, 'designing assessment products in such a framework ensures that the way in which evidence is gathered and interpreted is consistent with the underlying knowledge and purposes the assessment is intended to address' (p. 2). This is important in order to provide teachers and teacher educators with information from which accurate instructional decisions can be taken.

In conjunction with Kane's (2004) work, the *Standards for Educational and Psychological Testing*[15] was the basis on which the validity and reliability of the LFUP instrument were framed. The standards are intended to promote sound and ethical use of tests and to provide a basis for evaluating the quality of testing practices. Hill, Ball, and Schilling (2008) have indeed found the standards appropriate and express their belief that for any measurement development effort, data obtained from pilot testing of study items must be analysed to assess whether the instrument meets several measurement-related criteria for it to yield trustworthy results.

Following a trawl of the literature around the concept of proof functions, the structure of LFUP scale has also been modelled on those that were used by Almedia (2000), Ruthven and Coe (1994), and Schoenfeld (1989). These instruments consisted of items that participants typically check-marked on Likert scales ranging from "very true" to "not at all true" and from "strongly agree" to "strongly disagree". The questionnaires contained items such as, "Proof is essential in pure mathematics" or "The key thing is to get the statements and reasons in proper form".

However, some aspects of these questionnaires were found unsuitable for this study for two main reasons. The first is that, unlike in this study, the exploration of proof was not limited to Euclidean proof only. The second is that in this study the key focus area was on exploring functional understanding of Euclidean proof rather than exploring the value of proof in other areas of mathematics. Therefore, the questionnaires were not entirely aligned with the objective of this study.

In this study, quantitative data was collected through administration of a five-point Likert scale questionnaire (LFUP scale in Appendix B1) for analysis in order to answer research question, "*What functional understanding of proof do Grade 11 learners hold?"* The first section of the LFUP questionnaire contains items for gathering demographic data: gender, class name, and home language (Table 4—1). Taking into account Kumar's (2005) guidelines for formulating questions,

[15] For a detailed discussion of these standards, the reader is directed to the manual published jointly by the American Educational Research Association, the American Psychological Association, and the National Council on Measurement in Education (2014).

every effort was made to ensure that simple and everyday language in the questionnaire was used for two reasons. First, English was not the home language of most of the participants. Second, there was no time allocated for explaining the questions to the participants. The purpose of collecting demographics was to be able to adequately describe the sample. Ensuring that language used was appropriate because misunderstanding of the questions by participants would have resulted in irrelevant responses.

Table 4—1. The structure of LFUP questionnaire

Category	**Description**	**Number of items**
Demographics	Code; Gender; Class; Home Language	4
Verification function	Five-point Likert scale assessing understanding of proof as a means to verify	3
Explanation function	Five-point Likert scale assessing understanding of proof as a means to explain	5
Communication function	Five-point Likert scale assessing understanding of proof as a means to communicate	5
Discovery function	Five-point Likert scale assessing understanding of proof as a means to discover/invent	5
Systematisation function	Five-point Likert scale assessing understanding of proof as a means to systematise	7

The second section of the LFUP questionnaire has 25 Likert scale items that range from 1 (“Strongly disagree”) to 5 (“Strongly agree”). The scores on the LFUP scale were treated as interval level scale which was amenable to parametric statistical analyses. There are five dimensions (factors) in the LFUP questionnaire, organised as follows: (1) verification; (2) explanation; (3) communication; (4) discovery; and (5) systematisation. A sample of the explanation function and its associated items is shown in Table 4—2.

Table 4—2. An extract showing items of the Explanation scale on the LFUP instrument (n = 135)

	Item	SD	D	N	A	SA
T4	A proof explains what a maths proposition means.	1	2	3	4	5
T5	A proof hides how a conclusion that a certain maths proposition is true is reached.	1	2	3	4	5
T6	Proof shows that maths is made of connected concepts and procedures.	1	2	3	4	5
T7	When I do a proof, I get a better understanding of mathematical thinking.	1	2	3	4	5
T8	Proving make me understand how I proceeded from the given propositions to the conclusion.	1	2	3	4	5

SD = strongly disagree; D = disagree; N = neutral; A = agree; SA = strongly agree

4.8.2 The AFEG questionnaire

Very few will contest the suggestion that the lack of deliberate instructional practices that explicitly focused on enculturating learners into argumentation is not peculiar to the school mathematics curriculum. This suggestion found support in Driver et al.'s (2000) observation that 'the major barrier to developing young people's skills of argumentation in science is the lack of an opportunity offered for such activities within current pedagogical practices' (p. 308). Hence, it is reasonable to suggest that argumentation itself needs to be taught explicitly to mathematics learners. I chose to conduct this investigation in Euclidean geometry; it is where learners are generally inclined to provide grounds for their claims.

Given that constructing a good argument is not a simple task in the sense that learners require guidance and support to appreciate what constitutes an effective argument (Osborne et al., 2004), I employed Wray and Lewis' (1997) notion of "writing frames" which are meant to support the process of argumentation. They provide vital support and clues as to what is needed in the absence of implicit instruction on argumentation as a learning and teaching tool. In this AFEG instrument, I was only interested in characterising learners' claims rather than requiring them to engage in constructing a formal proof. It is important to note that Euclidean proofs are constituted by a sequence of logical steps, from the initial premises, through one intermediate result after another to the eventual conclusion (Aberdein, 2005). In contrast, argumentation plays the role of developing inductive arguments to eventually become convinced enough of the truth of a claim to seek its proof.

However, in the process of making arguments, a short chain of deductive statement can be made which according to the van Hiele model, is characterised as informal deduction. In other words, as already mentioned, argumentation in this study complements the appreciation of the functions of proof to constitute the "territory before proof". Of course AFEG required participants to use premises (previously learnt theorems) to make the claims. But, they could only use a maximum of two premises which are insufficient to complete a proof; they do not necessarily reach the conclusion. In addition to writing frames, notions of Toulmin's (2003) helpful framework that directs attention to the application of key aspects of argumentation in informal logic, were used (Figure 4—3). I thus drew on Toulmin's theory as a means to develop and adapt theory to this geometric task that constituted the AFEG instrument.

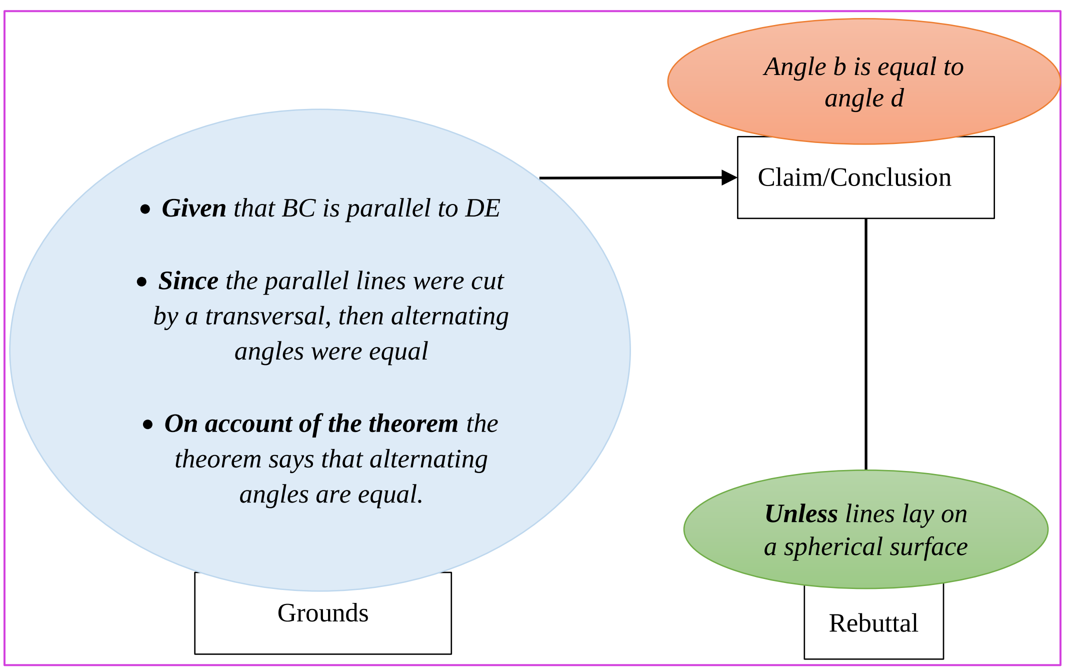
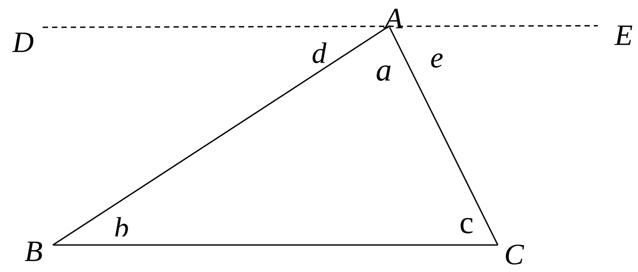


Figure 4—3. The TAP adapted for this study

The modified TAP includes rebuttals. The rationale for including rebuttals is that the AFEG questionnaire required participants to make use of statements which reflect their prior experiences and/or statements they guess to be true, but were not really sure. In this regard, asking for a rebuttal invites thinking about counterexamples. In addition, that a learner is able to think and argue mathematically, a phenomenon Jahnke (2008) refers to as having a "mathematically educated mind", is shown by the presence of a rebuttal in argumentation. Selden and Selden (1998) point out that learners quite often fail to see a single counterexample as disproving a conjecture because they perceive that counterexample as "the only one that exists", rather than seeing it as generic. Put another way, the formulation of counterexamples helps in challenging learners' belief that a particular counterexample is just an exception to the rule at hand and that no other "pathological" cases exist. Therefore, rebuttals not only trigger reorganisation of ideas but also enable learners to develop an understanding of the status of counterexamples in the construction of proofs.

The written argumentation frame (Figure 4—4) is based on making a claim using any two premises to show that "the sum of the interior angles of a triangle sum up to 180^0". The diagram includes a line drawn from a vertex parallel to the opposite side where a participant has to make a claim considering either that (1) corresponding angles or (2) alternate angles of parallel lines. Given that this argumentation task was based on Euclidean geometry, data were regarded as claims that use elements of the figure. For example, a claim or conclusion such as "angle a = angle b" comprises data (*a* and *b*) from the figure which is the foundation for the claim, and the geometric statement, which together complete a claim. Again, apart from the fact that Euclidean geometry is the only context for constructing deductive proof in the CAPS curriculum, I conducted the investigations in the context of Euclidean geometry to capitalise on its visual appeal because, as Wu (1996) puts it, 'almost all of its theorems can be pictorially confirmed' (p. 228). I hope these reasons countered any contestation that I tend to perpetuate the compartmentalisation of mathematics.



My statement is that ……………………………………………………………………………
………………………………………………………………………………………………
My reason is that …………………………………………………………………...............
………………………………………………………………………………………………
Arguments against my idea might be that …………………………………..................
………………………………………………………………………………………………

Figure 4—4. The argumentation frame in Euclidean geometry

As can be seen in the figures above, I used Toulmin's (2003) argument structure and adapted Osborne et al.'s (2004) tool to develop an argumentation framework as a data collection instrument. In light of the fact that English was the second or third language for most of the participants, basic everyday English language was used to create this tool. This approach was also used to improve the reliability of findings. Next I turn attention to the qualitative data collection tool, the Interview Schedule.

4.8.3 The Interview Schedule

The questions in the Interview Schedule was grounded in the literature on functional understanding of proof and the factors influencing this understanding. Because the goal of the second, qualitative phase was to answer the third research question by explain why *Presh N* held the beliefs she held about the functions of proof, seven semistructured questions in the Interview Schedule explored the role of these five factors (semantic contamination, collectivist culture, teacher, textbook, and empirical arguments).

The Interview Schedule was pilot tested on one participant (not *Presh N*), purposefully selected from those who had completed the survey in the first, quantitative phase of the study. On the basis of the results obtained from the pilot interview analysis, the order of the Interview Schedule questions was slightly revised and additional probing questions were developed. For example, the question, "What do you think is your role in the proofs in textbooks?" was regarded as too broad and consequently revised to read "If the textbook has verified the truth of a theorem, what do you think is your role, next?"

The schedule comprised the participant's demographic information, interviewer's, transcriber's, and data capturer's names, time, date, and questions and probes. The participant's responses were probed with the aim of encouraging elaboration on her responses so that a better understanding of the thinking behind her ideas could be gained and also to seek clarity. These questions were organised around the following themes:

1. Establishing whether Euclidean proof was covered in Grade 11, the second term as scheduled in CAPS.
2. Obtaining insights into learner's definition of proof whether it is in terms of a particular role of proof. Also this question will elicit the role of *semantic contamination*.
3. Understanding learners' views about whether understanding proof is innate or takes effort and practice. This will help see if they resort to memorisation.
4. Understanding whether the learner appreciate the need to read and understand a theorem for themselves rather than rely on the authority of the *teache*r.
5. Understanding the extent to which the *textbook* influences her belief about proving.
6. Checking if learner conceives of an *empirical argument* as a means to convince herself that a proposition holds true and/or regards *deductive arguments* as a means to explain and/or communicate to others why the proposition is always true.
7. Eliciting whether the *type of proof* presented to a learner influences their understanding of the functions of proof.
8. Determining what the learner attributes lack of success to and how they think proof learning can best take place in the classroom to elicit the influence of *collectivist culture*.

The semistructured interview also contained task-based interview, a particular form of clinical interview, as the secondary data gathering strategy for this study. According to Maher and Sigley (2014), this type of interview can be traced to Piaget in the early 1960s who pioneered clinical interviews in his quest to gain deeper understanding of children's development. I follow the definition of task-based interviews as ‘interviews in which a subject or group of subjects talk while working on a mathematical task or set of tasks (Maher & Sigley, 2014, p. 579). To Maher and Sigley (2014), task-based interviews make provision for more open-ended questions requiring qualitative analysis wherein participants interact not only with the interviewer but also with the task environment.

The choice of the strategy was informed by the suggestion that task-based interviews provide the best context for assessing and probing for the roots of participants’ beliefs (Hurst, 2008). Thus, the interview was designed to elicit participant’s perspectives on the functions of proof through integrating ordinary text with a cartoon. The reason for using a cartoon in the Interview Schedule was that, as Stephenson and Warwick (2002) point out, they allow for the disassociation of the ideas from those of particular participants so that it is not they who may be proved to be “wrong” but, the cartoon character.

The questions that incorporated diagrams (Figure 4—5) were presented on paper for respondents to use at any time. A cartoon depicting a learner reaching a conclusion about “the sum of angles in a triangle” on the basis of construction and measurement, was used in the Interview Schedule. Ibrahim, Buffler, and Lubben (2009) suggest that the use of real-life figures and names can lead to prejudice towards the making of a decision. As a consequence, the cartoon was used not only because it did not refer to gender, race or culture but also to improve construct validity of the responses (Ibrahim et al., 2009). Generous wait time (a definite pause between asking a question and requiring answers from respondents) of 5 seconds was allowed.

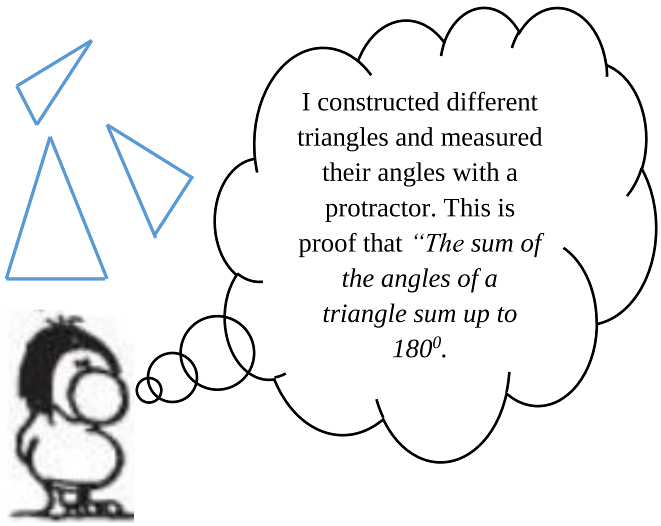
Main questions	**Probes (Follow up questions)**
Please, consider the following cartoon and its proposition.  	Given that the proposition has worked in every case that the teacher has tried so far, how can we be sure that the method always works? Why do you think so?

Figure 4—5. Sample main question and possible probes in the Interview Schedule

Qualitative data were collected through semistructured task-based interview to understand the factors influencing understanding of Euclidean proof in mathematics and thus answer research question, *Why does Presh N hold informed beliefs about the functions of proof?* This interview was considered as the best method to gain insights into *Presh N*'s reasons for holding the understanding of functions of proof she did. The questions in the Interview Schedule (Appendix B4) were organised according to the theoretical perspectives on factors influencing proof understanding. The first question posed was intended to understand whether *Presh N* has had experiences of proof, at all, in the present and previous mathematics classes. Thus, the Interview Schedule included questions such as: "Since this term began, have you done a proof?" and "Tell me about one theorem you just did in class". These questions and their accompanying probes were important in the context of the South African classroom given findings that teachers tend to leave out some topics citing time constraint as a reason (Mji & Makgato, 2006).

Probing as a technique in interviews enables delving more deeply into learners' hidden interpretations of their functional understanding of proof and thus developing deeper insight into how and why they conceive of proof as they do. For example, Figure 4—4 is a sample core question aimed at elicitation of the effect of empirical argument on functional understanding of proof. This question followed the notion that learners inappropriately use $30^0 + 150^0 = 180^0$ to "prove" that the angles on a straight line are supplementary. Also, Kunimune, Fujita, and Jones (2009), report that even learners who can construct deductive proof do not understand why such proofs are necessary in geometry. It is reasonable to suggest that such learners lack an appreciation of the function that proof performs in mathematics.

Another source of qualitative data that was useful in the analysis stage was field notes. These are notes I created during fieldwork to recall and record the behaviours, events, and other features to supplement interview data. A better understanding of what is said in an interview comes from its context, including a range of cues that are simply not captured on the audiorecorder. In addition, the notes were used to reflect on the research settings, difficulties encountered, procedures followed, distractions and nonverbal cues during interview, and as a backup in the event that recording equipment malfunctioned. The benefit that accrues with keeping field notes is that an interrater can use it to understand the coding followed in this study (Saunders, Lewis, & Thornhill, 2012). As with interviews, field notes were transcribed for analysis purposes. Further, the Interview Schedule consisted of spaces between the questions to write these notes.

4.8.4 Proof-related task

The participant was asked to prove the proposition that "the sum of the interior angles of a triangle is equal to 180 degrees". The purpose of examining her proof-related task was twofold: one was to further gain insight into the factors affecting her functional understanding of proof. Two was to validate the information obtained (1) in the semistructured interview and (2) on the survey questionnaires administered during the first, quantitative phase. In particular, as discussed in the literature review, this information was important to elicit given that learners tend to memorise proofs if they see no use for learning to prove them for themselves.

There was no diagram provided and thus she had to construct it from her own experiences. However, not providing the diagram alone was insufficient; the proof was expected to not only show deductive arguments but also empirical arguments to give an indication that she was asked to prove a proposition rather than a theorem as per typical classroom tasks. Thus, the task was an opportunity to gain insight into her ability to engage in the “territory before proof” and possibly explain whether deductive and/or empirical arguments influence her functional understanding of proof.

4.9 Data collection procedures

In this section I describe the steps taken to conduct the investigation reported in this study. This description includes: how access to the schools was gained, how sampling was carried out, and what instructions were given to participants. However, ethical clearance was the first huddle to cross to the findings: ethics committees at the University of KwaZulu-Natal (UKZN) and the KwaZulu-Natal Department of Education (KZNDoE) needed to be content that the research aims and methodologies will be reached through ways that protected the dignity, rights and safety of the participants, and that the research design was ethically sound and likely to render meaningful results. Perhaps more importantly from my perspective as an early-career researcher is that obtaining ethical approval of this study also helped to increase the trustworthiness of the findings. This approval is important to note for classroom teachers and mathematics education researchers who are likely to make decisions based on the results emanating from this study.

4.9.1 Ethical clearance and research permission

The need for ethical approval arises as the result of the fact that the data being sought and the means being used to obtain them may be contentious. Aspects of ethics considered in this study were anonymity, confidentiality, informed consent, protection from economic harm, and reciprocity. This research study sought and obtained clearance from UKZN Humanities and Social Sciences Research Ethics Committee and KZNDoE, respectively. Specifically, the initial application for ethical clearance was submitted to and approved by the UKZN committee, in Protocol Reference number HSS/0437/016M of 9 February 2017 (Appendix A1).

Then, this was followed by the permission granted by the KZNDoE, in Ref 2/4/8/1126 of 14 December 2016 (Appendix A2). These applications were important to do since data collection techniques employed in this study had ethical dimensions by virtue of the fact that they involved humans and the issuance of the clearance certificates was evidence that the study conformed to ethics requirements. Initially, the proposal of this project contained targeted participants, outlined data collection procedures, instrumentation, data protection measures, informed consent forms, and gatekeepers' (KZNDoE) research permission letter. These two institutions did not raise any ethical issues or risk that could be associated with learners participating in this study.

However, it is not always easy to ensure confidentiality for the qualitative phase of the study especially from those who are familiar with the contexts of Dinaledi schools in which the study was conducted (Miles & Huberman, 1994). For example, I needed to identify one learner whose functional understanding was extreme; she obtained the highest LFUP score among learners holding hybrid beliefs about the functions of proof despite attending a township school. There was no way I could not be able to identify her.

That notwithstanding, every effort was made to ensure confidentiality and protection of the identities of the participants by using pseudonyms and withholding any other identifying characteristics for the schools and learners in presentations, journal publications and other public information dissemination platforms. Confidentiality refers to ensuring that only the researcher has access to all data gathered and participants' names and that participants know in advance to whom the data will be divulged (McMillan & Schumacher, 2010). In order to ensure confidentiality, I made sure that the data in the computer were password-protected.

Anonymity refers to the act of keeping individuals nameless in relation to their participation in an investigation. I ensured anonymity by using codes so that the names of participants and their schools could not be matched to data. For instance, the code S2CAL15 denoted school (S) number two 2, class (C) A, learner (L) number fifteen (15) in the class register. I used the classroom registers and allocated these codes to identify the school and the learners for the qualitative phase of the study. Allocating codes on LFUP and AFEG questionnaires assisted in the control and tracking of the questionnaires returned and facilitated the analysis of data. Coding questionnaires this way was helpful in that the participant could later be identified for conducting semistructured interview in the qualitative phase of the study. The participants were informed that the data collected during this study were to be reported on in this thesis and possibly in journal articles and conference proceedings on the basis that functional understanding of proof help in constructing proof meaningfully and thereby improve learner performance in Euclidean geometry.

4.9.2 Sampling

In mixed methods research studies, Bronstein and Kovacs (2013) identify three types of samples: *single sample* where the same sample is utilised for both quantitative and qualitative segments of the research; *single sample with subset* in which data from the quantitative component of the study is used to qualitatively investigate another phenomenon; and, *more than one sample* which describes a mixed methods study that uses one sample for a quantitative component and seeks additional information from a subset of a different sample. In this sequential explanatory study mixed methods, I adopted a *single sample with a subset* approach in which the single participant in the subsequent qualitative component was drawn from the same larger sample after the completion of the quantitative phase of the study. Specifically, in line with the logic of sequential explanatory designs in which the quantitative component is dominant, after administering surveys to one hundred and thirty five (135) learners at selected Dinaledi schools, I invited an extreme case (one learner) to participate in a semistructured interview based on the survey results.

4.9.2.1 Schools

I selected a sample of Dinaledi schools to administer two survey questionnaires in order to answer the first two quantitative research questions. As Wagner, Kawulich, and Garner (2014) suggest, I randomly surveyed three schools from a population of ten Dinaledi schools (Motshekga, 2015) in the Pinetown school district in KZN, South Africa to accommodate the limited resources available for this study. In cluster sampling, convenient and naturally occurring groups are randomly selected which is followed by a selection of individuals in the groups (McMillan & Schumacher, 2010). This sampling method ensured that the fundamental premises of probability sampling, namely, that every of the Dinaledi schools must have an equal chance of being included in the sample, was not violated.

In the pursuit of increasing the participation and performance in Mathematics and Physical Sciences of historically disadvantaged learners, the Department of Basic Education (DBE) established the *Dinaledi School Project*, in 2001 (Department of Basic Education [DBE], 2009). The initiative involved selecting certain secondary schools for Dinaledi status that demonstrated their potential for increasing learner participation and performance in mathematics and science (Department of Basic Education [DBE], 2009). These schools were provided with resources (for example, textbooks and laboratories) and other related resources to improve the teaching and learning of mathematics and science. The ultimate intention was to improve mathematics and science results and thus increase the availability of key skills required in the South African economy (Department of Basic Education [DBE], 2009). The rationale for selecting Dinaledi schools for the investigation was that these schools were monitored by a team that included senior education department officials and individuals with an interest in educational research.

However, only three Dinaledi schools were sampled for the main study and the another one accordingly served as a prelude to the main study (Cohen, Manion, & Morrison, 2011). These were public schools with two of them located in a township and the other two in a suburban area. On the one hand, a township is a residential area previously designated for blacks[16] and characterised by poor socioeconomic conditions whose schools lack resources (for example, qualified mathematics and science teachers, science and computer laboratories, and sports fields). On the other hand, a suburban school has adequate facilities, teachers and educational opportunities for learners.

[16] The use of race as a form of classification and nomenclature in South Africa is still widespread in the academic literature with the four largest race groups being Black (African), Indian, Coloured (mixed-race) and White. This serves a functional (rather than normative) purpose and any other attempt to refer to these population groups would be cumbersome, impractical or inaccurate (Spaull, 2013, p. 437).

4.9.2.2 Learners

Mathematics learners attending in three schools were subsequently sampled. The sample comprised a total of 135 culturally and linguistically diverse and inclusive Grade 11 mathematics learners (seventy eight female and fifty seven male with an average age of 17.4 years and 17.8 years, respectively). The ages of this group ranged from 15 to 18 years. Although learners were informed of the right of their parents to refuse them participation, all of them participated in the study. In each of the three schools, all the learners were studying mathematics, physical sciences, life orientation and at least four other subjects, including two compulsory official South African languages at first- and second-language level.

Survey data was collected, presented, analysed, and discussed to inform both sampling and the development of the Interview Schedule for the subsequent qualitative phase. The size of the sample needed to be large due to the extent of the heterogeneous nature of the population of Grade 11 learners in Dinaledi schools. That is, a bigger sample was required to draw reasonably accurate inferences in light of variation in characteristics of Grade 11 learners in every respect; namely, language, resources, and gender. A summary of the participants across the Dinaledi schools is shown in Table 4—3. By the time of the research, the sampled learners had finished the prescribed Euclidean geometry.

The choice of a single case was guided by two key considerations: appropriateness and adequacy (Morse & Field, 1995). According to Morse and Field (1995), the former implies the identification of participants who can best inform the study, and the latter relates to adequate sampling of participants so as to address the research questions and develop a full description of the phenomenon being studied. Because I was interested in examining a "successful" participant where successful meant holding informed functional understanding of proof as judged by their high LFUP score, the extreme case sampling strategy was used. The term "case" refers to the single participant who took part in the semistructured interview.

Table 4—3. Summary of demographic characteristics of the three schools and participants.

School code	Gender		Home Language		School Location	Total
	Female (54.1%)	Male (45.9%)	IsiZulu	English		
A	22	16	10	28	Suburban	38
B	29	21	46	4	Township	50
C	27	20	36	11	Township	47

In this mixed method design, the qualitative component was subsumed within a primarily quantitative project. The qualitative phase of this study relied exclusively on purposive sampling because I needed that participant whose information was likely to give deeper insight into the factors affecting functional understanding of proof in mathematics. Purposive sampling refers to a sampling technique for the identification and selection information-rich individuals for the most effective use of resources (Patton, 2002). This sampling design involved the selection of a deviant participant for the purpose of learning from an unusual manifestation of functional understanding of proof. Following Guest, Greg, Arwen, Johnson, and Laura's (2006) evidence-based recommendations regarding nonprobabilistic sample sizes for interviews, a semistructured task-based interview was conducted with a single participant, *Presh N*, judged to be holding an informed belief about the functions of proof. The single case study was adopted on the rationale that the depth of data collected is more important than recruiting large samples (McMillan & Schumacher, 2010).

4.9.3 Administration of questionnaires

The principals of the schools received letters of transmittals explaining the purpose of the research and requesting their Grade 11 learners taking mathematics to participate voluntarily. Accompanying those letters was the permission letter from KZNDoE. Further, participants were informed beforehand of the purpose of the research and how the data would be collected and protected. Specifically, the participants received three documents. The first was an informed consent form which also described the nature of the project as approved by the two institutions already mentioned, assuring them of anonymity and confidentiality. Informed consent was sought simultaneously at three points in the study: administration of the two questionnaires; use of the audiorecordings; and for other academic purposes including conference presentations. The second were the LFUP and AFEG diagnostic tools. The LFUP questionnaire collected demographic questions to help respondents get started comfortably while instructions were designed to induce motivation for carefully considered responses. Flipping the LFUP questionnaire would take participants to the AFEG questionnaire.

The third was the information sheet which outlined the purpose of the research, the nature of participation, and how data might be used. In case the questionnaires or interviews evoked emotions, details about relevant counseling services were provided on the information sheet. Also, to improve the accuracy of responses, the questionnaire was of adequate length to avoid taxing participants' concentration while ensuring that items were unambiguously phrased.

In addition, the rows were shaded lightly and alternately to provide a visual cue to help participants reliably match each item with its options. As already mentioned, on each questionnaire, a code representing the school number, teacher number, and the learner number was indicated. Approximately 30 minutes were suggested for the completion of each questionnaire.

4.9.4 Conducting a semistructured interview

A single case whose functional understanding of proof in the LFUP questionnaire were characterised as "informed" was purposively selected. The reason for interviewing a single case was primarily financial: I received no funding beyond those very limited and personally generated funds which were exhausted in the quantitative phase of the project. Secondarily, as Yin (2014) argues, case studies need not always include direct, detailed observations as a source of evidence. In particular, I adopted a single case study to address the research question by providing a detailed description of the participant's experiences gained from her interactions with the notion of proof. In addition, thinking of a single case study as analogous to an experiment, Yin (2014) maintains that single case studies are relevant for the purpose of analysing cases that may be extreme; in this study, *Presh N* was selected as such a case. According to Nock, Michel, and Photos (2007), single case studies refer to those studies in which the phenomena of interest was studied using a single subject who agreed to participate.

The participant was purposefully selected on the basis of her high LFUP score with the main criterion being that she was a deviant case that could provide insight into the reasons why she held informed understanding of the functions of proof in mathematics. That is, although it was helpful to know the factors influencing learners' hybrid functional understanding of proof, it was indeed more helpful to identify and explain why a learner in a historically disadvantaged school, having scored highest, held informed functional understanding of proof. I hoped that this extreme case will provide insights into ways to help learners move to informed functional understanding of proof despite attending township schools. As already mentioned, township schools were characterised by poor provision of quality education. Therefore, I contend that it was in this regard that *Presh N* was an exceptional case. Donaldson, Ching, and Tan (2013) elaborate by mentioning that studying exceptions – those cases that beat the odds – can bring hope to apparently hopeless situations. This use of a single case in this study also gained significance in light of Feagin, Orum, and Sjoberg's (1991) suggestion that while advocates of multiple case studies argued for replication, using more than one case may dilute the importance and meaning that may be derived from the single case.

The purpose of the interview was explained to the learner; to understand the factors influencing their functional understanding of proof. The learner was reminded that the interview was to be audiorecorded and she was assured of confidentiality and anonymity. The interview took place in a classroom after school hours. In an attempt to create a comfortable environment that informalised the atmosphere, conversations about the interviewee's interest in sport, future career prospects, favourite music genre and artist, and so on, took place. On the average the interview was scheduled to take 30 minutes.

As already mentioned, a semistructured interview schedule that consisted of two parts was designed to explain why *Presh N* held informed functional understanding of proof. In the first part, the purpose of the interview was explained out of respect for the participant, as a way to establish rapport, and focus attention and thus elicit more thoughtful responses. Semistructured questions with funneling were derived primarily from literature and personal past experiences with the concept of proof. A funneling technique involves initially asking a general question and then probing with more specific questions (McMillan & Schumacher, 2010). For example, one of the initial questions in the Interview Schedule (2A(i)), "What, in your view, is proof in mathematics?" was followed up with a specific question, "Please, can you explain what a theorem is?"

4.9.5 Proof-related task

I followed *Presh N*'s into the first term of her Grade 12 year (2018). I requested her activities book to investigate performance in a Euclidean geometry baseline assessment by her teacher. The purpose of investigating this work was to triangulate her interview data with documentary and survey evidence. *Presh N*'s written work was produced in response to routine homework assignments to be judged right or wrong by teachers. Specifically, learners were asked to prove the proposition that "The sum of the interior angles of a triangle sum up to 180 degrees". Put another way, learners were required to communicate their thinking in written form so that it could be evaluated by the teacher.

Learners' ability to communicate their mathematical ideas has been the focus of many mathematics curriculum worldwide (for example, NCTM, 2000, 2009; Department of Basic Education [DBE], 2011). For instance, the South African curriculum (2011) advocated for tasks that assessed mathematical processes such as 'communicating mathematical ideas' (p. 43) while the NCTM (2000) points out that 'reflection and communication are intertwined processes in mathematics (p. 61). Apart from this prof related task being an important source of information of learners' level of achievement, it also presents them with an opportunity to systematise the various concepts, axioms, definitions, and theorems into a coherent whole. Seeing mathematics as a coherent whole helps learners to appreciate that mathematics is a tapestry in which all the concepts and skills are logically interwoven to form a single piece.

Thus, the proof-related task was for the learner herself and for inspection by others (for example, peers, teachers, researchers, education authorities, or parent/guardian) and therefore inherently communicative and social in nature. I did not probe the ideas that went into the task. The reason is that learners are not necessarily consistent in their responses to questions requiring recall of ideas in interviews (McMillan & Schumacher, 2010).

4.9.6 Reciprocity

Research participants sacrifice their time to volunteer ideas about a phenomenon under investigation. I thought it was reasonable to reciprocate this generosity. Creswell (2014) affirms the need for researcher's sensitivity to reciprocity or giving back to the participants. He defines reciprocity as something that is returned to participants of a study in exchange for the information collected from them. However, Creswell and Plano Clark (2011) advise that reciprocity should be done within the constraints of research and personal ethics and within the framework of maintaining the researcher's role as an investigator. Hammel, Carpenter, and Dyck (2000) point out that reciprocity 'implies give and take, a mutual negotiation of meaning and power in the research process' (p. 116).

In this study, reciprocity took two forms. First, to express recognition and gratitude to participants for volunteering to participate and share their ideas and experiences about the functions of proof and argumentation, I provided the principal with a summary of the research results to distribute to participants and their respective teachers. Second, in light of limited resources, I relied on McMillan and Schumacher's (2010) point that taking part in a study flatters participants because they are able to express themselves in ways ordinary life rarely affords them; they have someone capturing their beliefs and listening with interest to their experiences. It is in this light that I disagree with Cohen, Manion, and Morrison's (2011) assertion that those participants who agreed to help were doing me a favour.

To avoid any economic harm or burden on the participants, I reimbursed five (5) of them who incurred expenses. For instance, these learners missed their prearranged common transport to their respective homes because the questionnaires were administered after school hours. This caused their participation to stretch beyond normal hours. In addition, they were provided with some refreshments for having given up their time to assist in the study.

4.10 Data analysis procedures

In this study, data analysis entailed separate presentation, interpretation, and discussion of quantitative and qualitative results (findings). In particular, the purpose of analysis was to present and interpret data (attaching meaning to the data) in order to identify patterns, relationships and trends and relate the findings to previous research studies. Marshall and Rossman (1999) define data analysis as 'the process of bringing order, structure and meaning to the mass of collected data' (p. 150). Hitchcock and Hughes (2002) take this notion one step further as they see data analysis as the 'ways in which the researcher moves from a description of what is the case to an explanation of why the case is the case' (p. 295).

In undertaking the analysis process, it is important to note that there were two stages in the design of this study. In the quantitative stage, LFUP questionnaire which consisted of five subscales (verification, explanation, communication, discovery, and systematisation) and the AFEG questionnaire were administered to Grade 11 learners. This stage formed the spine of this research project; the relevance of this study stood and fell on the results of this quantitative phase. In line with McMillan and Schumacher's (2010) suggestion, given the logic of sequential designs, it was 'best to present and interpret results from the first analysis before reporting the second set of data' (p. 406). That is, analysis entailed presentation, interpretation, and discussion of findings for each of the four research questions. However, in Chapter 8, the relationship among the three results concerning the three constructs (functional understanding, argumentation, and factors affecting functional understanding of proof) is explored using an interpretational analysis (Guba & Lincoln, 1994).

Analyses of quantitative data emanating from the administration of the LFUP scale were conducted to determine the validity and reliability of the LFUP scale. The quantitative data analysis program, SPSS v.24 (2017), was used for analysis of demographic and Likert scale data emanating from the first two (quantitative) research questions. Data of items of the five dimensions (functions) in the LFUP scale were subjected to principal axial factoring (PAF). In the analysis of the Likert items, learners' understanding of proof was classified as naïve if the average response score was less than 2.5, hybrid if the score was between 2.5 and 3.5 inclusive, and informed if the score was greater than 3.5 (Liang, et al., 2009). Although the LFUP instrument was designed to be considered as a series of items that when combined measure a learner's functional understanding of proof, I began by determining the mode, median, and frequencies of the individual items that make up the scale to obtain a better understanding of the meaning of data (Boone & Boone, 2012). Next, I examined the data through both univariate and multivariate statistical methods.

In the qualitative stage, an interpretative approach was adopted because the aim of the research question was to elicit participant's meanings of their experience with the concept of proof. For that reason, I paid particular attention to their utterances, acting, and facial expressions. A software program designed primarily for qualitative analysis (data storage, coding, and theme development), ATLAS.ti Version 8, was employed after transcription of raw data. The benefit of using computer software to code qualitative data is that it (a) reduces analysis time, (b) cut out drudgery by facilitating associations and links within data, (c) helps in displaying data more easily (Miles & Huberman, 1994), and (d) results in typically more complex and more detailed analysis than manual thematic sorting (McMillan & Schumacher, 2010). The combination of manual and computer-assisted analysis often leads to greater insight in itself, with just a few clicks (Bazeley, 2009). Thus, I transcribed audiorecords into text which were reduced, displayed and used to draw and verify conclusions (Miles & Huberman, 1994).

4.10.1 The LFUP questionnaire analysis

The LFUP questionnaire was designed such that learners' understanding of the functions of proof was represented by numbers for quantitative analysis. Although numbers were assigned to learners' demographic data (for example, gender, home language, and, grade class), they were merely labels to indicate the differences between these categories of learners. Thus, they required numeric measures of analysis. There exists no consensus amongst scholars as to whether Likert data should be analysed with parametric statistics such as the *t*-test for dependent means or nonparametric statistics such as the Wilcoxon Signed Ranks test (Carifio & Perla, 2008). In this study, the Likert scale was as treated as eliciting interval data and therefore amenable to parametric statistical measures.

On the one hand, if the Likert items were treated as individual items, the data were to be analysed as ordinal; therefore nonparametric measures applied. On the other hand, when multiple Likert items were summed together to describe an attribute (and therefore data considered to be measured on interval scale) parametric measures were appropriate. Another reason for using parametric statistical measures in the LFUP scale was the assumption that, as in psychology research, distributions in education research often approximate a normal curve (Aron, Aron, & Coups, 2014). In addition, the sample was regarded as normally distributed because of the large number of learners who participated in the investigation. Further, 'the Likert scale ("strongly agree" to "strongly disagree") illustrates a scale with theoretically equal intervals among responses' (Creswell, 2012, p. 167).

According to Clason and Dormondy (1994), numbers in Likert scales presumed the existence of underlying continuous variables. Thus, the Likert-type interval scales on LFUP were treated as ratio scale (Austin, 2007). Thus, the five-point LFUP questionnaire responses with five subscales (factors or dimensions) of three to seven items each were be treated as Likert scales where: *1=strongly disagree; 2=disagree; 3=undecided; 4=agree; and, 5=strongly agree*. The "undecided" option was included on the basis that a respondent may truly hold no particular view about an item and if this option is absent, they may choose to respond to the question thus introducing bias in the data. Positively worded items signified agreement with the mathematical community and negatively worded items represented disagreement. Thus, the scoring of the LFUP scale was conducted according to the way in which the response reasonably reflected views in the mathematical community. Also, two items with "*Leave this item blank*" were added to the LFUP instrument to check on participants' attentiveness while completing the questionnaire (Schommer-Aikins, Duell, & Barker, 2003). As already mentioned, LFUP was linked to the five-factor model (verification, explanation, communication, discovery, and systematisation) whose items were derived from research literature about proof functions.

The data were screened to test the presence of outliers and also assessed for linearity, normality and homoscedasticity through scatter plot matrix and boxplot. If outliers were found, the case(s) associated with them were eliminated if they only accounted for less than 5% of the total sample. If elimination were inappropriate, I minimised their effect through data transformation techniques such as square root transformation or logarithmic transformation. Whether the data approximated a normal distribution was verified by using three tests: skewness and kurtosis z-value (ratio with standard error) which must lie between – 1.96 and +1.96 if data distribution is normal and also used the Shapiro-Wilk test for $p > .05$ (Wilson & MacLean, 2011).

The research by Shongwe and Mudaly (2017) was useful in determining and assessing the degree to which the LFUP instrument is unidimensional. Unidimensionality reflects that a scale taps a single composite construct (Streiner, 2003). Having obtained a factor structure that confirmed homogeneity – the existence of unidimensionality in the sample of items – I then proceeded to determine Cronbach's alpha coefficient. The internal consistency reliability, Cronbach alpha, was calculated to determine the degree to which each item on the LFUP scale measured the same construct. Alpha is the mean inter-item correlation measuring internal reliability; determining how closely related a set of items measure the same construct when they are considered as a group. However, since there were five subscales in the LFUP questionnaire, the internal consistency was tested on each subscale rather than on the whole instrument only.

The rationale for determining alpha is that it is the only measure of reliability that can be determined with much less effort because it does not require test-retest (Streiner, 2003; Tavakol & Dennick, 2011). Test–retest reliability involves the administration of a measure to the same group a second time and comparing the two scores (Kline, 2011). In the final analysis, the data were subjected to PAF to test the key assumption that there is one unique factor for each item which affects that item but does not affect any other items. In the factorial ANOVA where the means of three groups of learners (gender and resources), the homogeneity of variances (equal amount of variability of the scores of three groups of schools) assumption could not be assumed because the p-value associated with Levene's statistic was lower than .05. However, I proceeded to perform independent factorial ANOVA on the groups because it turns out that in practice the test gives almost accurate results even when there are fairly large differences in the population variances, particularly when there are equal or near numbers of scores in the groups (Aron et al., 2014, p. 321).

4.10.2 The analysis of the AFEG questionnaire

Current interest in mathematical argumentation underscored the need to develop schemes of analysing argumentation in mathematics classrooms (Aberdein, 2009, p. 1). Although several frameworks focusing on argumentation have been suggested (Aberdein, 2008; Inglis, Mejia-Ramos, & Simpson, 2007; Krummheuer, 1995; Pedemonte, 2007; Yackel, 2001; Zohar & Nemet, 2002), only a few (for example, Kuhn, 1991; Johnson, 1992; Means & Voss, 1996; Zohar & Nemet, 2002) focused on characterising the conceptual quality of learners' argumentation in Euclidean geometry. Thus, further research is encouraged in this direction. I chose to focus on analysing individual learner's rather than group's ability to argue since ultimately individuals will leave the group and enter into new situations that require their own reasoning (Greeno, Smith, & Moore, 1993).

As already mentioned, this study adopted Osborne et al.'s (2004) modified TAP for argumentation that considered the "first order elements of an argument"; claims, grounds, and rebuttals. The second order elements were the components of grounds: data, warrants and backings. However, worth noting is that the analysis of participants' argumentation ability was underpinned by Toulmin's modified argumentation scheme which was interpret to include both informal reasoning and formal proof. Drawing on the ideas of Wray and Lewis (1997), I used a "writing frame" (or sentence starters) to provide support and clues for participants to structure a written argument in a coherent manner (Osborne et al., 2004). According to Wray and Lewis (1998), a writing frame consists of a skeleton outline with different key phrases to scaffold learners' argument so that it is presented as a coherent structure. As pointed out by Osborne et al. (2004), the frame contained a set of stems which provided the prompts necessary to initiate the construction of a written argument: "My statement is that ... (C)", "My reason for this claim is that ... (G)", and "Arguments against my idea might be that ... (R)". The analysis of AFEG was conducted by defining quality in terms of a set of 4 levels (0 – 3) of argumentation (Table 4—4). Learners' attempts to construct an argument in Toulmin's (2003) sense were coded according to the criteria described in Table 4—4 and scored on a scale from 0 to 3.

The analytical framework comprised hierarchical levels in that advancement to the next level was seen as having provided arguments of better quality and therefore increasing in complexity. As Osborne et al. (2004) point out, an improvement in the level of argumentation is judged by the presence of the components of TAP. For instance, a score of 0 indicated that the learner demonstrated little knowledge of the subject matter; their claim was labelled as incoherent or idiosyncratic, that is, it did not make mathematical sense since 'a bare conclusion, without any data produced in its support, is no argument' (Toulmin, 2003, p. 98). That is, uncodifiable, nonargument or idiosyncratic claims were awarded a zero thus representing a low quality of argumentation.

Table 4—4 Analytical framework for assessing the quality of argumentation in AFEG

Argument	**Example**	**Description**	**Score (Quality)**	**Level/Code**
My statement is that …	None/ AD = 180^0	0- No reply/uncodifiable/nonargument	0 (Low)	0
My *statement is* that …	e = c	C (Using data to make a claim)	1 (Low)	1
My reason is that …	*Alternating interior angles*	C+G (Providing reason for claim)	2 (Adequate)	2
Arguments against my idea might be that …	*But, the lines DE and BC are not marked as parallel*	C+G+R (Supplying a rebuttal)	3 (High)	3

C = Claim (assertion or conclusion based on the figure provided); G = Grounds (reasons, or backings used to make a claim); R = Rebuttal (statement that either contradicts the data, warrant of an argument)

Therefore, claims which did not use data in the diagram were not recognised as meriting any significance. However, a score of 1 meant that at least the learner was only able to make a claim, without providing the necessary ground for it. This simplest argument was judged to be at Level 1 (low). Although Osborne et al. (2004) did not use the term “adequate”, I introduced it here to distinguish between low and high quality argumentation. In addition, they seemed to acknowledge that there is a weak, stronger and strongest argument. As a consequence, an argument was judged to be at Level 2 (adequate) if it improved in quality by not only consisting of a claim but also ground to support the claim. However, although providing a ground is important as it can invite a refutation of a claim, this argument was still considered not to be of high quality.

In contrast, arguments were deemed to be at Level 3 (high) when they were accompanied by rebuttals which sought to expose the limitations of a claim. To place emphasis on the importance of rebuttals, Driver et al. (2000) assert that ‘even arguments constructed by an individual are put together by thinking of cases that the arguments have to contest’ (p. 291). As Osborne et al. (2004) point out, arguments with rebuttals were of better quality since those without rebuttals because the latter had potential to continue forever with no evaluation of the quality of the substance of the argument. Thus, rebuttals were regarded as an essential element of arguments of better quality and demonstrated a higher level argumentation ability. Osborne et al. (2004), using TAP, suggest that the quality of arguments is a function of the number of rebuttals to the claim made. More importantly, the consideration of rebuttals promotes the learning of content, especially the understanding of concepts and principles (Nussbaum, 2008). Thus, by considering the three argument elements in Toulmin’s (2003) model, the following three forms of arguments were defined in this study: claim only, claim, claim with ground, and claim with ground and rebuttal.

Apart from nonargumentative statements, the coding scheme was primarily guided by a differentiation between claims, data, warrants, and rebuttals. This approach was successful in circumventing the ambiguity of some of the elements in TAP. The use of quantittative measures to understand argumentation as evidenced in learners' TAP profiles created an avenue for understanding where emphasis was needed to improve the quality of argumentation and by extension, learners' functional understanding of proof. In the next sections, I illustrate how semistructured interview data were analysed.

4.10.3 Pattern-matching analysis

> … while you may commence with either an inductive or deductive approach, in practice your research is likely to combine elements of both. (Saunders, Lewis, & Thornhill, 2012, p. 549)

According to Braun and Clarke (2007), themes within data can be identified in one of two primary ways in qualitative data analysis: an inductive approach in which themes are generated from the data, and theoretical or deductive approach in which data are collected specifically for the research. Thematic analysis is useful at many levels. Pattern-matching (hereinafter referred to as PM) is the thematic analytic method chosen on the basis of the notion that 'researchers cannot free themselves of their theoretical and epistemological commitments, and data are not coded in an epistemological vacuum' (Braun & Clarke, 2007, p. 12). In addition, reporting of themes is an aspect of basic procedures of several other qualitative methodologies by novice researchers like myself (Wilson & MacLean, 2011). Another benefit of thematic analysis is its flexiblility in the sense that it is not tied to specific theories, and is amenable to computerisation (Braun & Clarke, 2007; Cooper & Schindler, 2014; McMillan & Schumacher, 2010).

The essence of this qualitative component of the study is that it downplays statistical techniques and their data collecting methods (Silverman, 2013). In addition, I also sought to use the *deductive qualitative analysis* process to, whenever possible, revise existing concepts (factors) as a consequence of new dimensions to the phenomenon including those that I did not anticipate. More pointedly put, the PM analytic tool was used to pit theoretically-derived pattern against an empirically-based pattern in this qualitative segment of the present study (Yin, 2014). A pattern is described as any nonrandom arrangement of objects or entities that are at least potentially describable (Trochim, 1989). The arrangement formed a pattern to show the relationship among categories (McMillan & Schumacher, 2010).

According to Saunders, Lewis, and Thornhill (2012), the PM technique involves identifying and comparing predicted or hypothesised patterns developed through the conceptual framework with the specific field against patterns evident in the data. In PM techniques, the patterns discerned in the data may match those that were predicted in the conceptual framework. In this case, an explanation would have been found and threats to validity discounted (Yin, 2014). In contrast, if the theoretically-based and the empirically-found patterns do not match, alternative explanations need to be considered. The hypothesised pattern here was: factors in and out of the classroom have an impact on learners' functional understanding of proof in mathematics. Having already developed the conceptual framework in Chapter 3, I was ready to use the framework as a tool in analysing semistructured interview data of a participant (n = 1) to answer the qualitative research question, *Why does Presh N hold informed beliefs about the functions of proof?*

4.10.3.1 Interview data transcription

The first stage in the qualitative analysis was verbatim transcription of the interview data. During the transcription process, which took several weeks to do, I did not only present the verbatim statements but also nonverbal and paralinguistic communication. The analysis involved listening to the entire audio several times and reading the transcription many times to provide a context for the categories specified in the conceptual framework (Cohen, Manion, & Morrison, 2011).

The transcript consisted of notations that explained the participant's body language, for example, facial expressions, gestures, gazes, sighs, breathing rhythms, voice intonation, and pauses. The transcription was more complete if I accounted for these visual cues because, as Fielding and Thomas (2008) note, 'we communicate by body language as well as speech' (p. 253). Furthermore, visual cues were important to note because participant's responses are more trusted if their body language is congruent with their verbal utterances (McMillan & Schumacher, 2010).

By drawing upon the Jefferson (2004) system of transcription notation which features conventions for intonation, changes in volume, intake and exhalation of breath, pauses and their duration, capital letters for volume, I transcribed the interview to a sophisticated level of detail. She (Jefferson) argues that if she were asked for the reason for including all the "stuff" in the transcript, her interesting response would be, 'Well, as they say, because it's there' (Jefferson, 2004, p. 15). On a more serious note, as Irvine, Drew, and Sainsbury (2013) aver, using this system for the transcription of interview data allows for a close examination of precisely what took place – what was said and also the way in which it was said. Appendix C1 is a glossary of the symbols used in the transcript for this study (Appendix C2). I used the numbers in each line of the transcript to facilitate reference to specific points in the interactive in analyses (Hepburn & Bolden, 2013).

As I read the transcript, coding and reevaluating the development of the coding scheme, coding took place. Specifically, all the textual data in the transcript were entered verbatim into ATLAS.ti software for further qualitative data analysis. The decision to incorporate software was primarily based on the reason that ATLAS.ti does not merely speed up the process of grouping data according to categories and retrieving coded themes (Wong, 2008), it also has an attractive search facility that enables interrogation of the data and thus adds rigour to the study (Ozkan, 2004). In general, integrating computer software in this analysis finds support in Welsh's (2002) assertion that 'in order to achieve the best results it is important that researchers do not reify either electronic or manual methods and instead combine the best features of each' (p. 9). Thus, the most compelling reason for using the software is that it provides a quick and simple way of counting who said what and when and in turn, provides a reliable general picture of the data (Wong, 2008). The possible drawback, that using computer software for qualitative data analysis may distance the researcher from the data (McMillan & Schumacher, 2010; Morrison & Moir, 1998), was mitigated by integration of ATLAS.ti with my own analysis.

While I remained the main tool for analysis, all transcribed data, that is, transcript and field notes were converted from word format (.doc extension) into a rich text file format (.rtf extension) in order to use ATLAS.ti's text and visual coding features. Then, I began to attach *in vivo* codes to the text units while placing references into the hierarchical indexing system. The codes are referred to as *in vivo* codes because they retain the respondents' words (Noble & Smith, 2014). Thus, coding involved identifying a paragraph in the data that exemplified a particular category (Wong, 2008). For instance, coding was done through selecting a text about everyday meaning of proof and coding it at the node "semantic contamination". Thus, I ran a text search query to find other such references. From the ATLAS.ti perspective, nodes are categories. These ideas are represented in Figure 4—6.

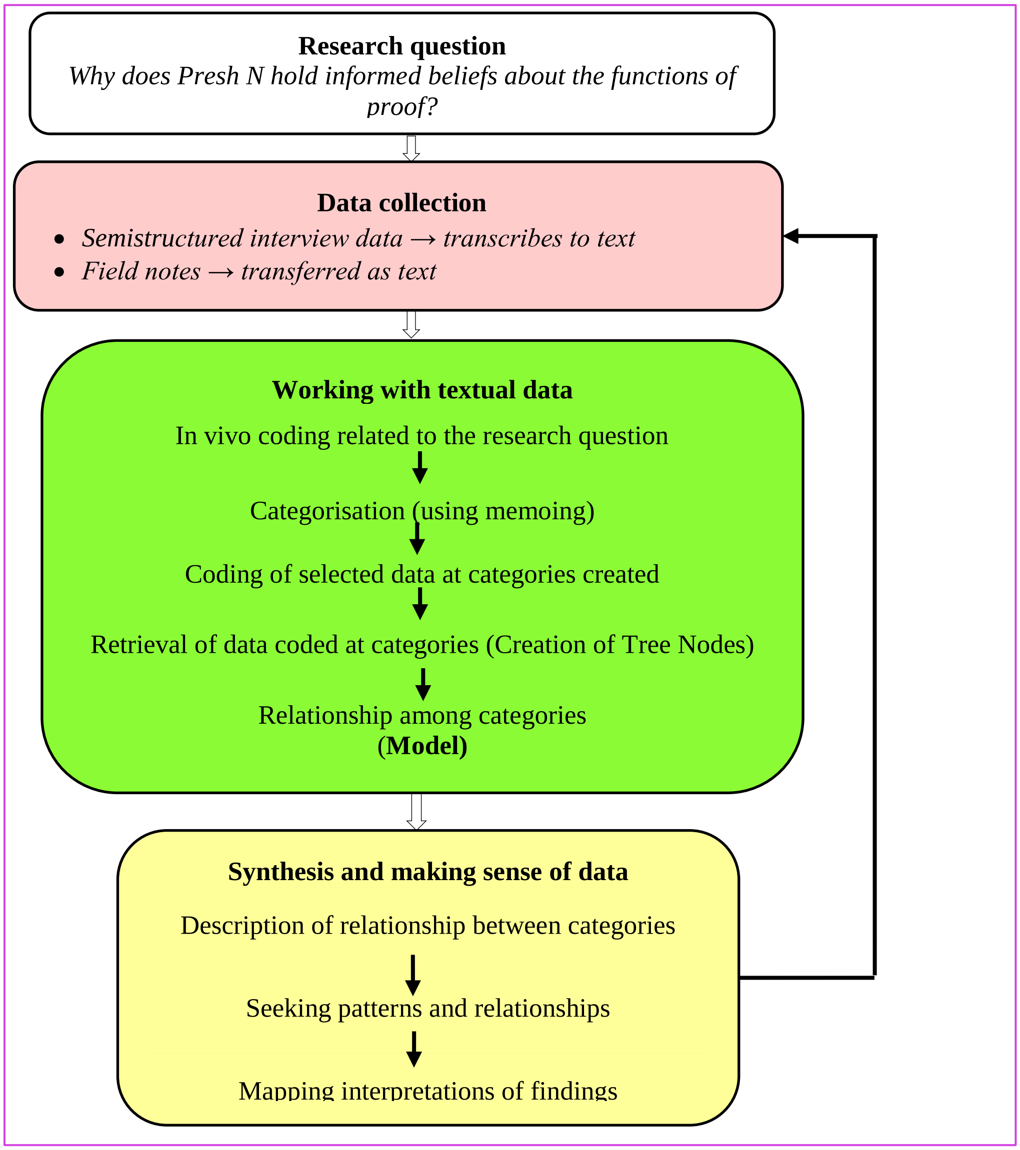


Figure 4—6. Flowchart of the basic steps of data analysis, adapted from Wong (2008)

Also, I attached memos to these text segments to record the ideas, insights, interpretations or understanding that may arise from the data. Then, I displayed tree nodes to see how the participant talked about, for example, “semantic contamination”. Notably, tree nodes are categories organised hierarchically into trees. Thus, I used single item search to ensure that every mention of the word “textbook”, for example, was coded under the “factors” tree node.

The final stage involved recording of insights gained into a memo from the display. This memo contained my commentary on text from the document to use in the interpretation stage of the project. Each node on the tree accommodated similar data and allowed storage of the comments I made. Then, I searched the indexing system to retrieve data according to themes identified in literature. The text was rechecked for the occurrence of these categories to seek patterns so as to determine relationships. I explained the relationships between the categories to seek patterns to interpret the data from the standpoint of participant’s perspectives, in their own voice (McMillan & Schumacher, 2010). As I transcribed the interview verbatim, I demonstrated that the analysis is a nonlinear but recursive process involving a search for themes to categorise.

4.10.3.2 Analysis of interviews data

Having adopted pattern-mathing techniques of data analysis, I was in a position to formulate data collection questions through my conceptual framework. In other words, the factors in the conceptual framework constituted the initial set of predetermined categories from which interview questions emerged. I sought meaning of the data by going beyond the face value of participant’s utterances in order to uncover and analyse hidden meaning of the text. In this regard, ATLAS.ti version 8, a computer software tool for indexing data, aided analysis of interview data. The key area in which this software assisted was with regard to being able to visualise complex relations between categories.

Initially, I coded the text by identifying a unit of analysis (for example, word, phrase, sentence or a group of sentences) and highlighting within the text interesting or salient features of the data without focusing on attempting to answer the research question at this point. The labels allocated to the codes described the explicit or surface meaning of the unit of analysis. Consistent with the approach of thematic analysis in which data are coded to fit into a preexisting coding frame, the purpose of the qualitative phase was to provide a less detailed description of the data overall, and a more richer analysis of some aspects of the data (Braun & Clarke, 2007). The need to provide such a description was driven by what I as researcher and teacher educator have experienced as a current gap: the absence of South African learners' views on functional understanding of proof. This purpose was realised through consideration of the frequency of occurrence and defining each theme sufficiently so that it is clear what it represents. The existence and prevalence of themes is not represented by quantitative methods but by the use of the qualitative phrase "*Most issues raised by Presh N indicated that …*"

All text in the transcript were coded and collated without trying to answer the research question (Kawulich & Holland, 2012). Thus, a long list of the different codes identified across the entire dataset was produced. In the end, the result of this coding process was a coding scheme that listed all the codes thus identified in this way providing an overall insight into the codes. For instance, the inteview subquestions and probes used to obtain answers to code around in order to obtain answers to the overriding research question included "What are the functions you believe proof performs in mathematics?", "Why do you think so?", respectively. Finally, I considered how these themes interrelated and divided into different levels. A detailed analysis for each individual theme was conducted to identify the story that each theme told and how the theme related to others in answering the qualitative research question (Braun & Clarke, 2007). In other words, this analysis phase essentially involved sorting the different codes into potential themes, considering how different codes may combine to form an overarching theme. I selected compelling extract examples to demonstrate the prevalence of the themes.

The coding of participants' answers to questions "What, in your view, is proof in mathematics?" or "Do you think that proving propositions in mathematics is necessary?" involved checking whether the responses demonstrated (1) the definition of mathematical proof as a product of an argumentation process based on a finite sequence of axioms to reach a conclusion, (2) an appreciation of the generality of proof, (3) de Villiers' (1990) categorisation of the functions of proof: truth (verification), explanation, communication, discovery, and systematisation. Further, in response to the question, “How do you learn your theorems?”, the participant may respond, “Our teacher starts by giving us the theorem and follows with problems where we apply the theorems”. This statement would then be coded and categorised as “Teacher influence”.

I performed constant comparison method in which each new piece of data is compared with the previously coded ones thus giving rise to new codes if none of the previous ones provided adequate description (Kawulich & Holland, 2012). These codes were kept brief and succinct. The next step involved grouping these codes into exhaustive and mutually exclusive categories and that their labels not only reflected the purpose of this study but were germane to the research question (Cohen, Manion, & Morrison, 2011). These categories were mutually exclusive in that no data were assigned to more than one category. An example of the coding and categorising process is shown in Figure 4—9.

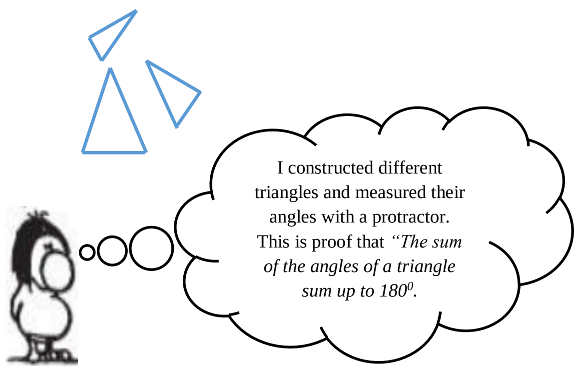
Interview question	**Aim of interview question**	**Data extract**	**Code**	**Category**
(a) What, in your view, is proof?	To obtain insights into whether learner's definition of proof is in terms of a particular role of proof; verification.	"It's like something that convinces you hmmm [inaudible] yes hmmm. It's like it helps you to see that what you are testing is hmmm [hesitation] true. (Thandi B07a)	Proof convinces	Semantic contamination
 (b) Do you agree with the learner's thinking that finding the same answer after trying many cases proves? (c) How can the learner be sure that the statement that *"The sum of the angles of a triangle sum up to 180^0"* always works?	To check if learners conceive of empirical argument as proof.			Empirical argument

Figure 4—7. Sample analysis process of semistructured questions in Interview Schedule

As the participant noted astutely, “All statements that require proof must be proven in ways that show evidence”. Although these descriptions point to a linear analytical process, the analysis involved a back-and-forth movement between the whole and parts of the text. Any text that could not be categorised with the initial coding scheme was allocated a new code.

4.10.3.3 Analysis of proof-related task

I drew on the van Hiele theory of geometric thinking to analyse *Presh N*’s written homework assignment because written work, as is with most work produced by learners, originates from the premise that learners possess mathematical skills and knowledge. As already mentioned in Chapter 3, this theory is underpinned by the assumption that learning geometry takes place in discrete levels of thinking and that progress to the next level is a function of instruction rather than age. The theory has played a major role in understanding learners’ difficulty with geometry in Russian mathematics education. In addition, this theory involved the way in which language is used with the underlying assumption that language can give insights into the participant’s knowledge.

The trustworthiness of the analysis depended on the reliability of the interpretations which in turn was dependent on the researcher’s knowledge of mathematics. Although learning mathematics entails developing mathematical ways of communicating mathematical knowledge, *Presh N*’s work was not judged in terms of whether it was presented in the two-column scheme of narrative form. But, as Morgan (2014) points out, whenever skill and knowledge are assessed in mathematics, reliability and validity involved the true and accurate portrayal of learners’ attributes (skill and knowledge) embodied in the task given the discipline’s focus on right or wrong answers.

According to Teledahl (2016), learners experience problems with vocabulary and symbolism than they do with mathematical content itself. Therefore I focused on scoring *Presh N*’s mathematical knowledge (with content and technical language as indicators of understanding) of definitions, axioms, and theorems in Euclidean geometry as well as her skill to represent this knowledge in accurate mathematical notations. I adopted this approach because language and content in mathematics are inseparable (Barwell, 2005b).

In other words, the analysis was based on the mathematical register and accepted ways of proving a proposition. A mathematical register documents a precisely defined mathematical vocabulary – possibly more than any other discipline – with not only words that are specific for mathematical communication but also everyday words which are taken to mean something specific in a mathematical context (Teledahl, 2016, p. 52). In scoring the task, two researchers independently analysed the participant's written task.

4.11 Connecting the phases and mixing the results

The mixed-methods sequential explanatory study reported in this study took place in two stages (quantitative and qualitative). These phases were connected at the intermediate point and the results were integrated at the design point in the research process. The intermediate point in the research process is when the results of the data analysis in the first phase of the study inform or guide the data collection in the second phase (Ivankova et al., 2006). The quantitative and qualitative phases were connected at the intermediate stage while selecting *Presh N*, the only participant for the qualitative case study, from the sample of learners who took part in the survey in the first, quantitative phase. This selection was informed by her obtaining the highest LFUP score despite attending an under-resourced township school.

The results of both quantitative and qualitative phases were mixed at the study design point. Here, both quantitative and qualitative research questions were used to integrate the results from the two phases during the exploration of the relationship among the three constructs (Chapter 8) that formed the backbone of this study. These constructs are: functional understanding of proof, argumentation, and factors influencing understanding of the functions of proof.

4.12 Consideration of rigour and limitations

In Chapter 1, I outlined the experiences, assumptions, and positions I brought into this study so that its findings could be viewed with these frameworks in mind. For example, I investigated fee-paying and no-fee schools against the background that I believe that resources contribute to differences in learner success in mathematics in general and in Euclidean geometry in particular.

That said, efforts were not spared to ensure that the findings in this study were rooted in evidence and 'worth paying attention to' (Lincoln & Guba, 1985, p. 290). In the course of the research process, I used the terms "reliability" and "validity" to describe attempts to ensure that the results in quantitative segment of the study were rigorously achieved. However, in considering the quality of the findings of the qualitative segment of the study, I adopted Golafshani's (2003) approach in using the term "trustworthiness" to encompass both quantitative notions of reliability and validity.

The term "trustworthiness" is used in this context to refer to the conceptual soundness from which the value of inferences made in this qualitative segment of the study must be judged (Marshall & Rossman, 1995). The reason for this decision is that these two terms, reliability and validity, were problematic for qualitative research (Guba & Lincoln, 1994). For instance, Cohen, Manion, and Morrison (2011) point out that concerns of replicability or uniformity were meaningless given that different researchers studying a single setting may come up with very different findings because reality is multilayered from qualitative research perspectives.

4.12.1 **Evaluation of the quality of quantitative findings**

4.12.1.1 Rigour for the LFUP instrument

The purpose of performing factor analysis was to validate the factor structure that is proposed in Shongwe and Mudaly's (2017) study. When the quality of each of the items for each scale was evaluated, the validity and reliability of the instrument were determined. In quantitative research, reliability refers to the degree to which an instrument yields consistent findings while validity refers to the degree to which an instrument accurately measures what it is intended to measure. Reliability of the LFUP scale was determined through measuring internal consistency of subscales and the global scale through Cronbach's alpha.

However, since "understanding of the functions of proof" is a latent variable and therefore not directly observable, the length of the LFUP scale was adequate, and also given the sufficiently large sample anticipated in this study, an alpha of .50 or above was tolerated (Kline, 2011). The item-total statistics helped in diagnosing if there were problems with the items; for instance, if there is an item that needed to be reverse-coded because it was negative, or deleted to improve reliability, or if there is a negative correlation. Also, high positive correlations were an indication of reliability of the LFUP questionnaire. By item-total is meant the correlation between each item and the overall score of the scale used as an indication of the internal consistency or homogeneity of the scale, suggesting how far each item contributes to the overall theme being measured (McDowell, 2006).

If the findings of this research study were to be helpful, determining the reliability and validity of the LFUP instrument was needed to demonstrate and communicate the rigour of this research process and the trustworthiness of research findings (Roberts, Priest, & Traynor, 2006). Validity of the LFUP scale was established through consideration of three fundamental elements: content validity, criterion-related validity, and construct validity (Long & Johnson, 2000; Saunders, Lewis, & Thornhill, 2012). In addition, face validity of the LFUP scale was also established. Whereas face validity refers to the indication whether, at face value, the questionnaire appears to be assessing the desired qualities, content validity refers to making a judgement as to whether an instrument seems to adequately sample all the relevant or important content or domains (McMillan & Schumacher, 2010). Content validity, which refers to whether or not the content of the items the LFUP questionnaire is appropriate to measure learners' functional understanding of proof is determined through the use of theory on functional understanding of proof.

To establish face and validity, the participants were asked to comment on how the instrument looked to them. However, it is important to obtain expert comments on content validity. To that end, five mathematics teacher educators were asked to judge the content of the instrument. Criterion validity which, like content validity, depends on theory (Muijs, 2004), was determined by considering argumentation ability as being theoretically related to and a predictor of functional understanding of proof. Specifically, participants' scores on the LFUP questionnaire were expected to be related to those they obtained in the AFEG questionnaire. In addition, the theory on argumentation led to the expectation that learners whose self-efficacy levels were high would hold informed functional understanding of proof than those that struggle to appreciate the functions of proof. More than three decades ago, Bandura (1977) theorised that a potent influence on learner behaviour is the beliefs that they hold about their capabilities. Briefly, learners are more likely to have an incentive to learn if they believe that they can succeed in performing a task; they make effort and persist in the face of difficulties. The scale on self-efficacy (Appendix B2) was important in improving the validity of the results.

In sum, establishing criterion validity required knowledge of theory relating to functional understanding of proof so that I could decide which independent variable can be used as a predictor variable. To do this, I needed first of all to collect data on those factors (functions of proof) from the same respondents to whom the LFUP instrument was administered, and secondly to statistically measure relationships among factors using multiple regression, specifically correlation coefficients.

4.12.1.2 Rigour for the AFEG instrument

The participants were required to complete a written argumentation questionnaire consistent with the *Principles and Standards for School Mathematics'* (National Council of Teachers of Mathematics [NCTM], 2000) call for learners to develop mathematical argument 'in written forms that would be acceptable to professional mathematicians' (p. 58). The rationale behind the use of writing frame was that they seem to help in improving the quality of learners' arguments as they present their responses in a structured written form (Sepeng, 2013). The task was deemed appropriate for Grade 11 learners since, at Level 3 of the van Hiele model, they should have begun making informal arguments to justify their conclusions.

A distinguishing feature of this task was that learners had to depend on their observation of the data to make a claim; this process reflected the inductive nature of argumentation. Working inductively could help learners to appreciate the genesis of the objects of mathematics. It is important to note that the examination of learners' geometric knowledge inherent in the task is measured elsewhere (Shongwe, 2019). Osborne's et al.'s (2004) argumentation frame employed in this study has been used in many countries including South Africa (for example, Lubben, Sadeck, Scholtz, and Braund, 2010). As a consequence, the AFEG instrument was deemed valid and reliable.

4.12.2 Evaluation of the quality of naturalistic inquiry findings

The case study approach adopted for the qualitative phase of this study is regarded as, in Lincoln and Guba's (1985) terms, a naturalistic inquiry on the basis that I sought to explain why *Presh N* held the beliefs she held about functional understanding of proof from the perspectives of the participant in her natural setting (the classroom environment where she spent most of her time). Irvine, Drew, and Sainsbury (2013) further provide useful insights into how interviewing in natural setting helps to avoid loss of nonverbal data. These authors point out that interviewing the participant in their natural setting not only facilitates the development and maintenance of a rapport but also provides the opportunity to observe cues such as intonations, facial expressions, levels of interest and attention, and body language during the interview. These nonverbal cues were noted in a reflective journal and used as additional data entered into the interview transcript.

As a consequence of conducting a case study in its natural setting, this rendered the research study not immune to the need to demonstrate and communicate the extent to which research findings were trustworthy (Roberts, Priest, & Traynor, 2006). The trustworthiness of qualitative findings directly relates to the methodological and analytical processes (Daytner, 2006). The following techniques served as safeguards for accomplishing trustworthiness of inferences: cross-checking in methodological triangulation, maintaining an audit trail, and member checking (Bowen, 2009).

Methodological triangulation, defined as a 'method of cross-checking data from multiple sources to search for regularities in the research data' (O'Donoghue & Punch, 2003, p. 78) was helpful in several ways. For instance, I relied on data gathered through semistructured interview, survey data, and document (proof-related) analysis for clues of corroboration and forming themes or categories. Further, it is through triangulation that I attempted to reduce the effect of researcher bias and misrepresentation of views by participant (Cohen, Manion, & Morrison, 2011; Gunawan, 2015). In an attempt to further improve the trustworthiness of coding of the data, I used the principle of multiple coding in which the entire interview transcript was sent to an independent researcher, a fellow doctoral student, to cross check the coding and interpretation of the data to overcome researcher bias. The rationale for cross checking was the interest to gain insight into competing interpretations of the data rather than reaching some degree of concordance. Thus, the reduction of bias and misrepresentation enhanced the understanding of reasons why *Presh N* held informed beliefs about the functions of proof. However, Patton (2002) cautions that inconsistencies arising from triangulation of data need not be seen as weakening the evidence, but as an opportunity to uncover deeper insight into the relationship between the data and the phenomenon under study.

The audit trail involved keeping records on research decisions taken in relation to data collection (including sampling procedure), recording, and analysis to enable inspection of these decisions and/or subsequent findings by others (or "auditors") (Bowen, 2009). In short, the audit trail made the qualitative research process of this study 'visible for all to see' (Bowen, 2009, p. 308). Similar to Guba and Lincoln (1989), I regard member checks as the single most critical technique for establishing trustworthiness of findings because the purpose of this qualitative segment of the study was to explain the reasons why *Presh N* held informed functional understanding of proof from her own perspectives. Member checking, which entails the review and critique of the accuracy of transcript and subsequent interpretations (Lincoln & Guba, 1985), took place in two steps.

Prior to the end of the research process, I provided *Presh N* with a copy of the interview transcript to correct possible errors and provide additional information. Once she was satisfied with the edition and subsequent accuracy of contents of the transcript, I then provided her with a draft copy of interpretations to reflect on the accuracy thereof. Further, the qualitative thematic analytical method adopted in this study, the technical accuracy provided by the audiorecords' verbatim transcription, and applying the rules built into the computerised qualitative data analysis software package, ATLAS.ti, added to the findings' trustworthiness (Roberts, Priest, & Traynor, 2006).

As already mentioned, thematic analysis involves the identification of codes, grouping of these codes (themes) was done for the purpose of describing and interpreting the relationship among categories. ATLAS.ti assisted in the generation of these relationships. The themes generated from the data formed part of the conclusions for this study. As already mentioned, ATLAS.ti aided rather than substituted my responsibility to code, describe, and interpret relationships among categories. Although the analysis was deductive in that it entailed fitting the data into predetermined categories, the participant's perspectives enabled the weighting of each theme. To that end, frequency counts were used to merely to understand which of the factors had a stronger influence on learners' understanding of proof.

The need to weight themes is motivated by Harel and Sowder's (1998) major finding that learners do not appreciate the functions of proof because instructional practices impose deductive proof which is utterly extraneous to the empirical evidence that is used as proof in everyday life. Seen in this light, instructional factors seemed to influence learners' understanding stronger than sociocultural factors. The description of the data analysis procedures for the qualitative phase should provide the reader with a comprehensive account of the context, participant, and research design so that he or she could make their own determinations about whether this work is transferable to their context.

4.12.3 Methodological limitations

The need to maintain rigour in research is critical. However, this need must be balanced with the need to conduct a study in populations where inherent barriers exist relative to key issues (Cohen, Manion, & Morrison, 2011). The limitations of a study are the methodological characteristics that set constraints on application and interpretation of the utility of its findings (Thomas, Nelson, & Silverman, 2005). Consistent with this perspective, the findings of this study were interpreted in the context of these limitations: literacy, assessment techniques, and the practical relevance of the research questions (Crosby, Salazar, DiClemente, & Lang, 2010).

The design of this study included schools characterised by disparities in resources. Some were categorised as resourced and some under-resourced. Often, under-resourced schools were affected by issues such as language proficiency and dysfunctionality which could be obstacles not only in trying to achieve rigour but also in constructing a study designed to understand and influence classroom practice. Although racial segregation has been abolished for over two decades, schools which previously served predominantly white learners under the apartheid system of government remain functional, while those which served predominantly black learners remain dysfunctional and often unable to impart the necessary mathematics and literacy skills (Spaull, 2013).

By dysfunctional schools is meant schools which are characterised by severe underperformance, high grade repetition, high dropout rate, and high teacher absenteeism (Fleisch, 2008; Taylor, Muller, & Vinjevold, 2003). According to Spaull (2013), whereas many of these factors are ascribed to the socioeconomic disadvantage of the learners they serve, there is also an undeniable impact of more intangible elements such as ill-discipline, inefficient management, and low cognitive demand; all legacies of the apartheid system. In contrast, functional schools are those characterised by learning environments which encourage the creation of opportunities for learners to investigate, explore, formulate conjectures, and perhaps use DGS available from computer laboratories.

As mentioned in Chapter 3, for most learners, English is a second, third or even fourth language. The language competency of the participants was evaluated and found literate to cope with the text of the data collection instruments and interview schedule through the use of pilot studies for both questionnaires and interview. Second, the study is limited to descriptions of functional understanding of proof in relation to argumentation and explanations of the factors affecting high school learners' functional understanding of proof. In this way, one-time surveys and a one-time interview with a single participant may not provide a complete picture of Grade 11 learners' functional understanding of proof and their argumentation ability. If this investigation was designed for resourced, functional schools with learners whose parents are classified as middle class, and well-educated, such a study would pose an entirely different set of challenges.

This study was not affected by attrition, sampling, and sample size because it was crosssectional, probabilistic, and met the thresholds for multivariate analysis. The probability sampling was used because of its advantage to infer the results of this study to the larger population of Dinaledi schools. As already mentioned, it was not the goal of this study to achieve generalisability. Thus, the results were not intended to apply to populations other than the sample population.

Thus, judgement on the utility of the findings in this study must be based on the balance between methodological rigour and the inherent limitations imposed by investigating schools with varying degrees of disparities. Key for this study is its ability to address gaps in the empirical literature; gaps which are valuable for informing Euclidean education policy and classroom practice. Most importantly, this study is firmly grounded in its research questions with direct relevance to classroom practice.

4.13 Chapter summary

In this chapter I distinguished between methods and methodology, reviewed past methodologies in relation to functional understanding of proof, provided a methodological framework, described the methods and justified the suitability of the mixed-methods sequential explanatory design for this study, described the points at which the two phases were connected and the results mixed, and considered issues of rigour for this study. The adoption of rigorous approaches in the research design of the study were robust efforts aimed at minimising systematic errors and thus produce trustworthy knowledge. In the next three chapters I engage in data analyses (presentation and interpretations of results, and discussions of findings in relation to previous studies). The focus now turns to the quantitative analysis of the LFUP data.

Chapter 5

Functional understanding of proof in mathematics

5.0 Introduction

In the previous chapter, I framed the study within the mixed-methods sequential explanatory design, provided a diagram of the procedures to identify the priority, connection, and mixing of phases and results within this single study, and described the rigorous quantitative procedures (including the systematic selection of a sample of three Dinaledi schools) and persuasive qualitative procedures followed in this study. In this chapter the first research question of the study, "*What functional understanding of proof do Grade 11 learners hold?*" is answered following the analysis of survey data obtained from the administration of the 5-point Likert scale to 135 Grade 11 learners. The focus of the analysis is twofold. First, it is to validate the LFUP instrument as it is at its infancy. Second, it is to explore and characterise learners' functional understanding of proof as either "informed", "hybrid", or "naïve" and determine which of these factors accounted for functional understanding of proof: verification; explanation; communication; discovery, and systematisation.

The participants had studied the concept of proof including construction of proof before their functional understanding of proof were the subject of analyses. That is, the LFUP questionnaire is employed to measure this understanding to determine if learners were able to appreciate functions of proof other than verification. Learners in Grade 11 are not only expected to construct proofs but also to understand the functions of proof in mathematics. Therefore, they should be at Level 4 of the van Hiele theory. A review of the literature revealed that proof as a means to verify mathematical statements is pervasive.

5.1 Summary of LFUP questionnaire results

Initially, I needed to employ principal axis factor analysis to empirically examine the five–factor LFUP model. In addition, Shongwe and Mudaly (2017) used a *t*-test to validate the instrument. In this project, the data were submitted to factor analyses techniques. The KMO and Bartlett's test respectively determined that the sample size was adequate for performing factor analyses and that there were sufficient items for each factor. Factor analyses results showed that the instrument was valid and reliable. The reason for performing factor analyses was that the sample in this study was different from that employed in the previous study. Therefore, there was a need to validate the LFUP instrument utilising statistical analyses.

5.2 Preliminary analyses for multivariate statistical tests

The purpose of conducting preliminary analyses was to screen the data to establish whether the requirements and assumptions for conducting multivariate statistical analyses were met. For example, the dataset was assessed for missing data, sample size adequacy, linearity, univariate outliers, multivariate normality, and collinearity. Although, ideally, multivariate normality is a requirement for performing factor analysis, deviations are not usually detrimental to the interpretation of the results (Tabachnick & Fidell, 2013). In short, it was necessary to determine whether the sample size was sufficiently large for performing factor analysis and that the items correlated sufficiently. The items included in the LFUP questionnaire were not only derived from theory, literature on the concept of proof (for example, what proof is, why proof is important, mathematicians' practices, and so on), experts in the field, but also from my own hunches about the concept of proof gained as a high school mathematics teacher and teacher educator. Thus, I needed to verify whether the factors carry some conceptual meaning that could be attached to the name that described them. In the next paragraphs, the correlation matrix, communalities, and factor loadings results for each dependent variable (function of proof) are analysed.

The correlation matrix provided the first insight into the appropriateness of the data for factor analysis. Initially, the correlation matrix was examined to gain insight into how each of the 25 items was associated with each of the other 24. In line with Nunnally and Bernstein's (1994) guidelines, correlations were acceptable if they exceeded .30. Some of the correlations were higher than $\pm$.60 (Table 5—1). Whereas relatively high correlations indicate that two items are associated and will probably be grouped together by the factor analysis, items with low correlations indicate that they will not have high loadings on the same factor. In other words, the correlation matrix provides a window into cluster of items that could well be the manifestation of the same underlying factor.

Then, the correlation matrix was further examined for multicollinearity in the items. Although the items had to be intercorrelated, the correlations should not be high because multicollinearity makes the determination of the unique contribution of the items to a factor difficult (Field, 2009). In this study, the value of the determinant of the correlation matrix was 5.25×10^{-10}. According to Field's (2009) threshold of 1.00×10^{-5}, this value is very close to zero which suggested that collinearity was high. This high determinant notwithstanding, it was not zero which would have suggested that a factor analytic solution cannot be obtained. As a consequence, I proceeded with the analysis because of the confidence that none of the items on the LFUP scale could be understood as a linear combination of some set of other items. However, the final decision of whether or not to continue with the analysis of the results was primarily based on the KMO statistic (Mooi, Sarstedt, & Mooi-Reci, 2018).

Table 5—1. Correlation matrix of the LFUP scale

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12	T13	T14	T15	T16	T17	T18	T19	T20	T21	T22	T23	T24	T25
T1	1.000																								
T2	.469	1.000																							
T3	.484	.427	1.000																						
T4	.655	.517	.635	1.000																					
T5	.480	.283	.406	.549	1.000																				
T6	.532	.457	.521	.570	.443	1.000																			
T7	.448	.292	.379	.497	.375	.413	1.000																		
T8	.504	.229	.464	.631	.368	.410	.513	1.000																	
T9	.612	.402	.532	.701	.553	.568	.487	.524	1.000																
T10	.610	.455	.603	.733	.434	.511	.452	.552	.732	1.000															
T11	.489	.397	.453	.547	.331	.453	.343	.354	.454	.563	1.000														
T12	.585	.354	.545	.710	.464	.513	.560	.537	.736	.713	.449	1.000													
T13	.421	.265	.382	.471	.268	.382	.523	.405	.492	.579	.431	.586	1.000												
T14	.605	.413	.562	.689	.454	.578	.428	.575	.670	.763	.609	.714	.595	1.000											
T15	.630	.423	.567	.717	.387	.557	.435	.593	.575	.704	.571	.622	.591	.765	1.000										
T16	.352	.284	.321	.421	.139	.344	.180	.275	.458	.468	.270	.488	.247	.495	.499	1.000									
T17	.435	.189	.313	.399	.264	.305	.428	.377	.398	.503	.445	.432	.651	.428	.554	.160	1.000								
T18	.359	.269	.323	.386	.102	.324	-.022	.214	.306	.534	.385	.376	.307	.544	.518	.514	.265	1.000							
T19	.693	.519	.571	.780	.532	.652	.563	.579	.728	.744	.594	.709	.445	.751	.705	.484	.362	.458	1.000						
T20	.561	.402	.541	.713	.531	.461	.547	.454	.708	.646	.586	.629	.531	.662	.644	.275	.366	.195	.711	1.000					
T21	.559	.400	.526	.657	.428	.445	.347	.433	.555	.677	.520	.471	.442	.612	.635	.212	.426	.382	.648	.616	1.000				
T22	.474	.308	.437	.562	.335	.439	.453	.481	.433	.502	.560	.451	.535	.574	.523	.225	.445	.286	.546	.550	.490	1.000			
T23	.717	.525	.622	.802	.479	.628	.543	.606	.745	.803	.672	.708	.621	.836	.838	.478	.546	.461	.850	.761	.675	.620	1.000		
T24	.509	.414	.501	.539	.516	.492	.470	.335	.624	.573	.539	.569	.440	.544	.406	.262	.262	.215	.571	.601	.388	.434	.570	1.000	
T25	.524	.410	.497	.668	.357	.401	.389	.461	.584	.669	.396	.680	.448	.645	.629	.396	.327	.452	.715	.563	.574	.447	.675	.417	1.000

Determinant = 5.25E-010

The factorability of the sample was further assessed. The Kaiser-Meyer-Olkin (KMO) measure of sample adequacy supported the appropriateness of the sample size for factor analyses. Specifically, the KMO measure of sampling adequacy was .944 (Table 5—2) which, according to Mooi, Sarstedt, and Mooi-Reci's (2018) set of distinctively labelled values, was marvelous because it was above the threshold level of .50. Put another way, the KMO measure indicated that there were sufficient items for each factor. The Bartlett's test of sphericity was performed to examine the hypothesis that the correlation matrix is an identity matrix (where all diagonal values were 1's and all off-diagonal values were 0's). Bartlett's test of sphericity was found to be significant, suggesting that the relationship among the variables was strong implying that the correlation matrix was significantly different from an identity matrix $(\chi^2(300) = 2.277 \times 10^3), p < .0001)$ (Table 5—2). This nonsignificant result suggested that the items were unrelated enough to perform meaningful factor analyses. In addition, the sample (*n* = 135) satisfied Bryant and Yarnold (1995) subject-to-variables minimum ratio of 5 participants for each item on the LFUP instrument.

Table 5—2 Sample adequacy and identity matrix output

Kaiser-Meyer-Olkin Measure of Sampling Adequacy.		.944
Bartlett's Test of Sphericity	Approx. Chi-Square	2.277×10^3
	df	300
	Sig.	.000

In this study, outliers, normality, and linearity were verified through inspecting the normal probability plot of the regression standardised residuals as well as the residual scatterplot. For instance, in Figure 5—1, the points were close to the line on the plot which suggested that the points lay close to the line on the plot and in a reasonably straight line from bottom left to top right. This behaviour of the line implied that the range of residuals were close to zero.

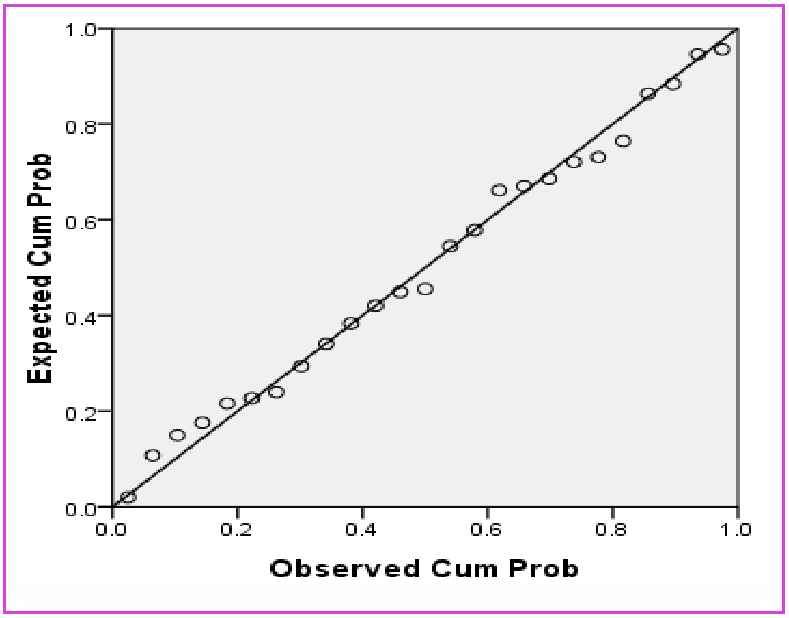


Figure 5—1 Normal P-P of regression standardised residual

In the scatterplot, there were scores predicted by the regression equation in standardised form and the residuals in standardised form. The scatterplot is used to visualise the strength of an association between variables. Here I considered the nondescript cloud of points which showed that most scores were concentrated in the centre along the zero point. In other words, the points were found to be funnel-shaped rather than curved. If they were curved, it would have suggested nonlinearity. Nonetheless, since they were funnel-shaped, the data suggested heteroscedasticity. Residual refers to the difference between an actual score and the score that would be predicted from the regression equation (Wilson & MacLean, 2011).

The scatterplot is also used to detect outliers because their presence can alter the factor solution (Tabachnick & Fidell, 2013). According to Cohen, Manion, and Morrison (2011), outliers are data values with a standardised residual that lay outside the range of – 3.3 and 3.3. Since no such values were found (Figure 5—2), it was clear that the data contained no outliers and this suggested once again that there was no deviation from the assumptions and requirements (for example, normality, linearity, collinearity, and independence of residuals) of multivariate statistics. In addition, the scatterplot showed that the residuals were approximately rectangularly distributed and most of the scores concentrated in the centre, around the zero point.

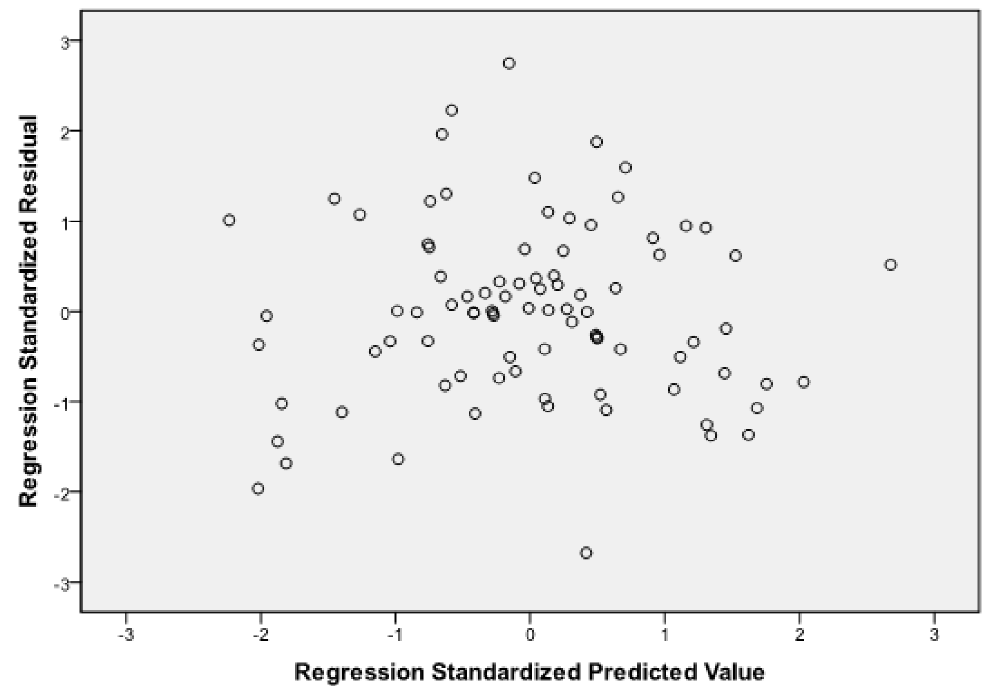


Figure 5—2. Scatterplot of the standardised residual analysis

Having conducted the preliminary analyses in which the KMO statistic indicated that there were enough items for all the factors, the Bartlett's test of sphericity showed that the items were correlated highly enough to provide a reasonable basis for factor analysis, and the scatterplot showed that the assumption of linearity was met, I proceeded to perform the principal axis factoring (PAF) method, a type of exploratory factor analysis (EFA). The PAF is a multivariate statistical method for identifying structure by determining interrelationships between variables (items) to find a smaller number of unifying variables called factors (Mooi, Sarstedt, & Mooi-Reci, 2018). This most widely used form of analysis (Cohen, Manion, & Morrison, 2011), the PAF, is constituted by multiple observable variables (items) with the restriction that each be uncorrelated with other components (Briggs & Cheek, 1986). Put slightly more specifically, for the LFUP scale, each subscale consisted of items that correlated more highly among themselves than they correlated with items not included in that subscale. In this study, the PAF was used for exploratory purposes because the factor structure as proposed in the LFUP scale was hypothetical and therefore needed to be tested.

5.3 Principal Axial Factoring (PAF) analysis

According to Tredoux, Pretorius, and Steele (2006), factor analysis is a statistical technique ‘used to identify relatively small number of factors in order to represent the relationship among sets of interrelated variables’ (p. 248). There are two main methods of factor analysis: principal axis factor (PAF) analysis and principal component analysis (CPA). One key assumption of PAF is that there is one unique factor for each item which affects that item but does not affect any other items (Field, 2009).

The PAF is preferred over the CPA because it takes into account the measurement errors (the variance not attributable to the factor which an observed variable represents) and thus its results are more reliable. In addition, the PAF was more appropriate for this study than PCA in that I postulated that there were five factors underlying the items measured (Cohen, Manion, & Morrison, 2011). Specifically, it was hypothesised that these five factors underlie learners’ functional understanding of proof: verification, explanation, communication, discovery, and systematisation (Figure 5—3). I was interested in gaining insight into whether the items that were considered as indexing each of the five factors actually do cluster together to describe functional understanding of proof in mathematics. Put differently, I wanted to determine empirically whether participants’ responses to, for example, the items characterising the verification function (factor), were more similar to each other than their responses to the items characterising the explanation function (factor). Conducting PAF assisted not only in answering the question “Are there five factors underlying the functional understanding of proof items?” but also in giving support for the construct validity of the LFUP scale in this sample.

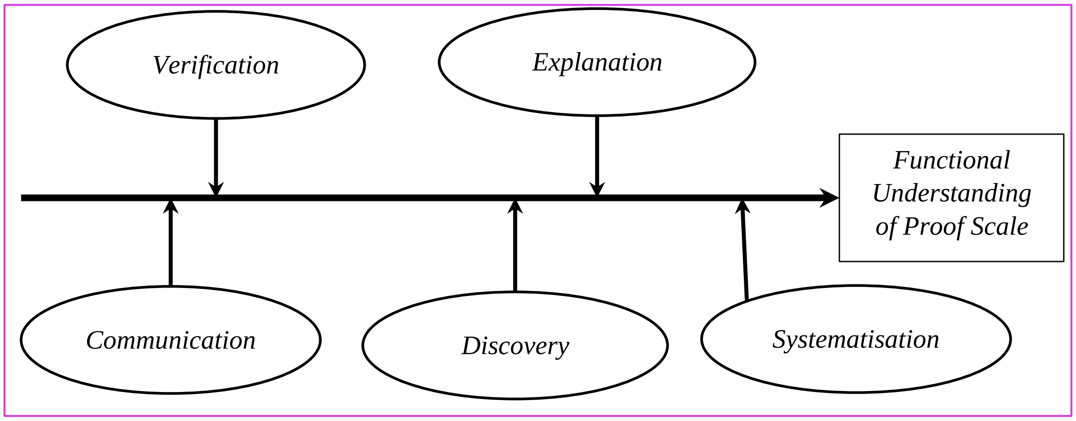


Figure 5—3. The independent variables of the LFUP dataset against the dependent variable.

Although theories and literature provided some ideas about the structure the data, the analysis was exploratory in nature because there were no specific predictions about the magnitude of the association of each item to each factor. Noteworthy in this regard is that eigenvalues were not used to examine the number of factors to retain for interpretation. The logic behind the application of Kaiser's (1974) eigenvalue threshold of greater than 1 for interpretation of results stems from the idea that this point divides the important or "major" factors from the minor or "trivial" factors. Unfortunately, this definition of where the drop occurs is rather vague and thus may encourage the making of arbitrary decisions. For example, it does not make sense to retain a factor with an eigenvalue of 1.01 and then discard a factor with an eigenvalue of .99 (Ledesma & Valero-Mora, 2007). In addition, the eigenvalue method has a tendency to overestimate the number of factors to be retained (Zwick & Velicer, 1986).

In this study, the alternative criterion was to set a predetermined level of cumulative variance and to proceed with the factoring process until Hair, Black, Babin, and Anderson's (2014) minimal threshold value of 60% cumulative variance was reached. As a consequence, applying the cumulative variance criterion as shown in Table 5—3, the five-factor solution captured more than two thirds (71.83%) of the total variance. More specifically, the verification factor accounted for 53.27% of the variance, the explanation factor accounted for 6.17%, the communication factor accounted for 5.19%, the discovery factor accounted for 3.94%, and the systematisation factor accounted for 3.29% of the variance. Important to bear in mind is that the process of deciding on the factor structure was driven by *a priori* theoretical framework which proposed a five-factor solution; it overcame some of the deficiencies inherent in thresholds.

Table 5—3. The ratio of the variance accounted for by each factor to the variations in the dataset

	Initial Eigenvalues			Rotation Sums of Squared Loadings		
Factor	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	13.316	53.266	53.266	4.889	19.557	19.557
2	1.534	6.136	59.402	3.044	12.177	31.734
3	1.298	5.191	64.593	2.998	11.990	43.724
4	.986	3.944	68.537	2.985	11.940	55.665
5	.823	3.290	71.827	2.299	9.195	64.860

Extraction Method: Principal Axis Factoring.

Factor loadings were examined to test the hypothesis that a relationship between items and their underlying factors exists. The term "factor loading" refers to the measure of the contribution that each item makes to the factor in question thus illustrating the correlations between items and factors (Cohen, Manion, & Morrison, 2011). As can be seen in Table 5—4, all factor loadings were greater than Stevens' (1992) threshold of .40. Loadings less than .40 were suppressed to make examination of cross-loadings easy. Cross-loading takes place when one item with coefficients greater than .40 loads on more than one factor. In particular, the first four items that related to verification function of proof loaded onto one factor labelled "verify". The next five items loaded onto the explain factor. The loadings of the items that clustered on the "communicate" factor ranged from .478 to .791. Noteworthy is that item T19 ("Proof shows the lack of connections between theorems and new results") loaded highest onto the "discover" factor.

A continuation of factor loadings is the communality of an item which is defined as the measure of the variance in each item accounted for by all other items (Kline, 2011). Put differently, a communality is a measure of the extent to which an item correlates with all other items on the scale. MacCallum, Widaman, Zhang, and Hong's (1999) suggestion that for sample sizes between 100 and 200, the threshold value of communalities is .50 was adopted. The communalities were all either at or above .50, further confirming that each item shared some common variance with other items. Given these overall indicators, factor analysis was deemed to be suitable with all 25 items. In other words, the dataset obtained from the administration of the LFUP questionnaire to Grade 11 learners was suitable for multivariate statistical analysis, including the validation of the LFUP instrument.

Table 5—4. Factor loadings from principal axis factor analysis with Varimax rotation for a 5-factor solution for LFUP scale (n =135)

	Factor loading					
Item	Verify	Explain	Communicate	Discover	Systematise	Communality
T1	.634					.596
T2	.616					.499
T3	.603					.510
T4		.746				.789
T5		.501				.480
T6		.549				.558
T7		.529				.605
T8		.596				.585
T9			.726			.765
T10			.796			.785
T11			.478			.602
T12			.754			.758
T13			.565			.664
T14				.752		.801
T15				.791		.809
T16				.484		.511
T17				.363		.601
T18				.555		.602
T19					.812	.855
T20					.730	.763
T21					.606	.637
T22					.509	.530
T23					.846	.904
T24					.480	.609
T25					.608	.652
Eigenvalues	4.89	3.04	3.00	2.99	2.30	
% variance	19.57	12.18	11.99	11.94	9.20	
No. of items	3	5	5	5	7	

Note: Loadings<.40 are omitted to aid interpretation

These five factors (subscales) accounted for 78.6% of the total variability (in all of the items together), and were accepted as summarising the data. The ideal solution is when each item loads on (correlated with) only one factor (Kline, 2011). The factors were then rotated to achieve a more interpretable structure. Generally, verimax rotation is preferred over oblique rotation method because it makes the interpretation of results easier (Wilson & MacLean, 2011). In this study, the verimax orthogonal (uncorrelated) rotation method was performed on the basis of the belief that the items were uncorrelated and consequently that the factors were uncorrelated, as well. To gain a better indication of which items loaded on the various factors and thus facilitate interpretation of results, only factor loadings equal to or greater than $\pm$ .40 were interpreted. For example, the items that loaded onto factor 1 related to the verification function of proof and were thus labelled as “verification”, while those that loaded onto factor 2 related to the explanation function of proof and were thus labelled “explanation”, and so on.

In summary, 25 items were subjected to principal axis factoring to assess the dimensionality of the data. These factors were orthogonally rotated using verimax rotation. The KMO measure was high and the Bartlett’s test of sphericity reached statistical significance indicating the correlations were sufficiently factorable. The five hypothesised factors explained a sufficiently large proportion of the variance. This was decided based on cumulative variance and the theory underpinning this analysis. The next section examines the validity and reliability of the LFUP scale.

5.4 Psychometric evaluation of the LFUP instrument

This section is concerned with assessing the validity and reliability of the LFUP instrument. Validity of the LFUP scale was established through consideration of two fundamental elements of validity: construct and criterion (Long & Johnson, 2000; Saunders, Lewis, & Thornhill, 2012). Face and content validity were determined through interviews with panel of experts and participants by checking the instrument’s attributes such as; ease of use, clarity, and readability. Construct validation of the LFUP instrument was done through factor analysis. As already mentioned in the methods and methodology section, the PAF is the standard statistical technique for evaluating construct validity. Criterion validity for the LFUP scale was ascertained through multiple regression analysis (Miller, Meier, Muehlenkamp, & Weatherly, 2009). Regression analysis explains how certain measures predict an outcome of another event by measuring how the predictive variables agree or disagree when they are combined together. Several researchers have used standard multiple regression analysis as a method to validate research instruments (Meyer, Meyer, Knabb, Connell, & Avery, 2013).

5.4.1 Reliability of the LFUP instrument

Although Shongwe and Mudaly (2017) provided evidence that the LFUP instrument is reliable, I needed to administer it anew in order to verify their claim with a new sample. The reason for this approach flowed out of the fact that 'a scale that may have excellent reliability with one group may have only marginal reliability in another' (Streiner, 2003, p. 101). In addition, doing so was going to further strengthen the validity of the LFUP instrument. The data for each of the five factors were analysed for internal consistency using SPSS v.24 (2017). By reliability here I meant the consistency with which the LFUP instrument yields almost the same scores every time it is used to measure Grade 11 learners' functional understanding of proof. To exemplify this idea, a bathroom scale is reliable if it reads, with an acceptable amount of error, the same mass every time I use it under the same health conditions.

The alpha reliability coefficients of the LFUP scale are shown in Table 5—5. The overall alpha coefficient of the 25-item LFUP scale was .961 which, according to Nunnally and Bernstein (1994), indicated that the LFUP instrument had a high reliability. A reliability coefficient of .70 or higher is suggested (Cronbach, Rajaratnam, & Gleser, 2011). For instance, Shinar, Gross, and Bronstein, et al. (1987) obtained a Cronbach's alpha of .98 for 18 patients. Hence scales with reliabilities of .90 or above are sufficient for individual applications (Browne & Cudeck, 1992). Also, because alpha varies with the number of items in a scale (McDowell, 2006), for this study, once validity was achieved (as shown in the next section), this instrument was considered reliable (Khalid, 2013). Similar to Shongwe and Mudaly (2017), the factors as well as the items were retained for interpretation. This result showed the validity of their model which in turn improves the quality of the LFUP scale.

Table 5—5. Internal consistency measure of the LFUP instrument

Cronbach's Alpha	Cronbach's Alpha Based on Standardised Items	No. of Items
.962	.961	25

The alpha coefficients for each of the different subscales ranged from .719 – .910, similarly indicating a high reliability. Shongwe and Mudaly (2017) reported a Cronbach's alpha of .87 in their research on the LFUP results which indicated adequate reliability. Nunnally and Bernstein (1994) suggest .70 as a cutoff such that alpha values below it indicate poor reliability and poor predictive validity of an instrument. However, an alpha greater than .90 would indicate item redundancy wherein some of the items have been rephrased resulting in asking the same question in many different ways (McCrae, Kurtz, & Yamagata, 2011). That said, I posit that the LFUP scale achieved a generally acceptable level of internal consistency judging by the Cronbach alpha value among the five subscales (Table 5—6). In this study, given that there is no sacred acceptable or unacceptable level of alpha, a level as low as .50 may still be useful if a questionnaire is of adequate length (in terms of items not cases) and its dimensionality or construct validity is established (Schmitt, 1996).

Table 5—6. Internal consistency results for the 5-factor LFUP instrument

Factor	Item	Example	α
Verification	T1; T2; T3	Proof makes sure statement is true.	.72
Explanation	T4; T5; T6; T7; T18	Proof makes me gain insight into mathematical thinking.	.82
Communication	T9; T10; T11; T12; T13	Proof communicates maths results even among learners.	.87
Discovery	T14; T215; T16; T17; T18	Analysis of proof may lead to invention of new results.	.81
Systematisation	T19; T20; T21; T22; T23; T24; T25	Proof brings together and connects previous maths results.	.91

The reliability of the LFUP scale was further assessed by determining the item-total statistics. An examination of the item-total correlations (Table 5—7) indicated that all items in each dimension contributed to the consistency of scores with item-total correlations higher than .64 thus exceeding the accepted cutoff value of .30. Put another way, these values indicated that each item related to the overall scale (Nunnally & Bernstein, 1994). The item-total statistics also lists Cronbach's alpha values when a particular scale item was deleted from the instrument. For example, as the first item has a value of r = .960, the Cronbach's alpha of this scale would change from .962 to .960 if that item were removed. Deleting any item would make negligible difference in the Cronbach's alpha value. As a consequence, these items were not removed. The first two columns, "Scale mean if item deleted" and "Scale variance if item deleted" were included for consistency of reporting only. The third column shows item-total values, indicating the correlation between a particular item and the sum of the rest of the items, which shows consistency.

Thus, item-total correlations here were by far greater than zero and therefore high, thus indicating that the each of the items was consistently measuring learners' functional understanding of proof. As a result, no items necessitated a deletion from the scale. All the inter-item correlations were within Cohen's (1988) classification that correlations greater than .50 were large, .30–.50 moderate, and .10–.29 small. Thus, items for the LFUP questionnaire were not only selected on the basis of alpha coefficients, but also on the basis that their item-total and inter-item correlations were relatively within respective cutoff values. These two measures (internal consistency and item-total correlations were used to reach the conclusion that the LFUP was reliable.

Table 5—7. LFUP item-total statistics

Item	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Cronbach's Alpha if Item Deleted
T1	70.4370	510.084	.738	.960
T2	70.9778	525.992	.525	.962
T3	70.4000	517.570	.673	.961
T4	70.6370	498.845	.852	.959
T5	70.9481	530.139	.550	.962
T6	70.7556	522.425	.658	.961
T7	71.9259	530.278	.584	.962
T8	70.4593	521.011	.630	.961
T9	70.6074	518.972	.790	.960
T10	71.1556	502.610	.844	.959
T11	71.5407	515.205	.659	.961
T12	70.5481	516.264	.789	.960
T13	70.8741	526.409	.635	.961
T14	70.4667	510.221	.844	.959
T15	71.1333	497.893	.818	.959
T16	70.4148	543.259	.472	.962
T17	71.0074	530.918	.531	.962
T18	70.4370	543.024	.467	.962
T19	71.0444	498.685	.872	.959
T20	71.3333	508.388	.776	.960
T21	70.8593	526.823	.709	.961
T22	70.8889	526.622	.643	.961
T23	70.9111	492.753	.927	.958
T24	71.7111	522.102	.647	.961
T25	71.3259	509.594	.713	.961

5.4.2 Validity of the LFUP instrument

It is important to note that in the analysis, all the LFUP items were coded so that a higher score represented informed functional understanding of proof consistent with those held by contemporary mathematicians. In terms of the LFUP instruments, validity refers to the appropriateness of conclusions drawn from the use of the LFUP instrument with Grade 11 learners at a particular point in time. One way that the validity of this LFUP instrument was demonstrated was to show that the AFEG instrument produced results that are similar to it. The argument here is that argumentation ability promotes the communication function of proof which may lead to discovery of new results enabling the systematisation of statements.

First, in establishing construct validity, the correlation between two independent variables—intended and proxy—is determined. An intended independent variable identifies correlations between multiple independent variables. A proxy independent variable is a variable used on the basis of the belief that it correlates with the variable of interest. In this study, correlation was assumed between holding functional understanding of proof that is consistent with those of contemporary mathematicians (proxy independent variable) and getting a high score on the FUPI instrument (intended independent variable). Second, the determination of a correlation coefficient between an independent (predictor) variable and the dependent (criterion) variable resulted in establishing criterion validity of the FUPI instrument.

In this study, each subscale representing a function of proof in mathematics is an independent variable and the total score of the instrument is the dependent variable. Specifically, AVC = average of the verification construct; ACE = average of the explanation construct; ACC = average of the communication construct; ACD = average of the discovery construct; and, ACV = average of the systematisation construct. The average of the construct with the highest value was 5. On the other hand, the total score of the LFUP instrument is the dependent (criterion) variable. A total score or index is the aggregate score that summarises a learner's measure of functional understanding in a subscale. This score could also be used in other statistical analyses such as regression or ANOVA. The sum of construct was based out of a possible total score of 25. The correlation coefficient between them is called validity coefficient. Standard multiple regression analysis was used to perform the two validity processes (Table 5—8). All the correlations of five subscales with the total score were highly significant at the .01 level. Tabachnick and Fidell (2013) suggest that for a variable to be significant it should have a correlation in the range of .30 and .70.

Table 5—8 Multiple regression showing correlations of LFUP subscales (n=135)

		ACV	ACE	ACC	ACD	ACS	Total score
ACV	Pearson Correlation	1					
	Sig. (2-tailed)						
ACE	Pearson Correlation	.589**	1				
	Sig. (2-tailed)	.000					
ACC	Pearson Correlation	.595**	.842**	1			
	Sig. (2-tailed)	.000	.000				
ACD	Pearson Correlation	.439**	.639**	.785**	1		
	Sig. (2-tailed)	.000	.000	.000			
ACS	Pearson Correlation	.614**	.874**	.914**	.741**	1	
	Sig. (2-tailed)	.000	.000	.000	.000		
Total score	Pearson Correlation	.697**	.914**	.956**	.818**	.966**	1
	Sig. (2-tailed)	.000	.000	.000	.000	.000	

**. Correlation is significant at the 0.01 level (2-tailed).

In determining criterion validity of the LFUP questionnaire, the extent to which the Likert items of the five functions predicted learners' functional understanding of proof in mathematics was assessed. Wilson and MacLean (2011) suggest that correlation coefficients between the predictor variables should not be greater than .8; hence, the five independent variables were retained for the LFUP instrument.

Continuing with multiple regression analysis, collinearity diagnostics were performed on the five subscales that had significant correlation with the dependent variable (that is, Total Score). This was done to check if the subscales (predictor variables) were highly correlated with each other (Wilson & MacLean, 2011). This phenomenon is referred to as multicollinearity. In addition, multicollinearity made it difficult to identify which of these predictors were important in influencing informed understanding of proof. To understand the role of this aspect, tolerance and variance inflation factor (VIF) were determined (Table 5—9). Whereas tolerance is seen as an indicator of how much of the variability of the specified dependent variable is not explained by the independent variables in the model, VIF is the inverse of the tolerance value.

According to Muijs (2004) and Pallant (2013), a tolerance value of less than .10 and a VIF value above 10 indicates that the correlation of a variable with other variables is high thus suggesting the undesirable multicollinearity. On the basis of these suggestions, my model indicated an absence of multicollinearity. In this regard, the tolerance values of between .122 and .605 were consistent with the multicollinearity results obtained earlier for Pearson's correlation coefficients between these five subscales and the total score. Specifically, these factors correlated significantly among themselves as suggested by Tabachnick and Fidell (2013). They suggest that, for a variable to be significant, it should have correlations of between .30 and .70. These results were an indication that multicollinearity was not detected in the dataset.

Table 5—9 Collinearity coefficients

		Unstandardised Coefficients		Standardised Coefficients			Collinearity Statistics	
Model		B	Std. Error	Beta	t	Sig.	Tolerance	VIF
1	(Constant)	-.315	.495		-.637	.525		
	ACV	5.982	.159	.139	37.693	.000	.605	1.653
	ACE	6.020	.162	.227	37.154	.000	.219	4.568
	ACC	6.033	.199	.242	30.379	.000	.129	7.734
	ACD	6.194	.163	.178	37.982	.000	.374	2.676
	ACS	6.775	.169	.329	40.010	.000	.122	8.219

a. Dependent Variable: Total score

5.5 Analysis and characterisation of learners' functional understanding of proof

I have argued that developing informed beliefs about the functions of proof is important for learners to gain insights into the nature of mathematics and how mathematical knowledge is developed. The five elements constituting the LFUP instrument were intended to serve as a frame of reference on gathering learners' understanding of the functions of proof; a tool that captures the essence of the central aspect of mathematical knowledge development; proof. To remind the reader, the LFUP instrument is the tool to evaluate learners' functional understanding of proof with a focus on de Villiers' (1990) model of five themes: verification, explanation, communication, discovery, and systematisation. Each theme consists of no less than three Likert items, involving both the most common narrow (naïve) views and informed views.

The next section focuses on characterisation of learners' functional understanding of proof as either informed, hybrid, or naïve, and which of these functions accounted for informed understanding of the functions of proof. To this end, I used Liang, Chen, Chen, Kaya, Adams, Macklin, and Ebenezer's (2009) scoring system for gaining a fuller understanding of learners' understanding of individual functions of proof. I began by determining the mode, median, and frequencies of the individual items that make up the scale to obtain a better understanding of the meaning of data and next examined the data through multivariate statistical methods.

5.5.1 Overall trends in functional understanding of proof among learners

In the analysis of the Likert items, I was keen to understand the trends within particular groups of learners using a scoring system clarified as follows. I modified Almeida's (2000) coding convention for the LFUP scale overall responses. Learners' beliefs were classified as *unencultured* if they represented misconceptions about the functions of proof. These uncultured views were represented by the ten negatively-phrased items in the LFUP questionnaire. Learners with a *poorly encultured* classification were those who seemed to believe that proof has just one function, most probably verification. However, both these beliefs were further classified as *naïve*. Further, learners' beliefs about the functions of proof were classified as *highly encultured* if they were partially consistent with those held by contemporary mathematicians. Learners whose views demonstrated the nature of mathematics as exemplified in almost all five functions of proof were deemed to hold *extremely encultured* beliefs about the functions of proof. However, both highly encultured and extremely encultured views were further classified as *informed*.

A five-tiered grading scale was used to assess learners' functional understanding of proof. Mean responses were interpreted according to the following categories of views about the functions of proof: 0–<1.5 (unencultured); 1.5–<2.5 (poorly encultured); 2.5–< 3.5 (hybrid); 3.5–<4.5 (moderately encultured); 4.5– ≤ 5 (extremely encultured). The normative map in Table 5—10 provides a summary of learners' responses to Likert items from all the three schools.

It is important to note that understanding of the functions of proof was judged to be hybrid if contradictions in the responses on the LFUP questionnaire were evident. These contradictory results reflected the fragmented and inconsistent nature of learners' functional understanding of proof and were compatible with a plethora of studies. For instance, Healy and Hoyles (1998) conducted a study of mathematics classes in high schools across England and Wales to investigate, among other variables, factors shaping learners' understanding of proof. They sought to explain these understanding by reference to a landscape of Level 1 variables (learner factors such as individual competency in proof) as well as Level 2 variables (class, school, curriculum such as hours dedicated to mathematics per week, and teacher factors), using statistical and interview methods. Their findings were that learners' understanding of proof were shaped by their functional understanding, gender, and curriculum as they learned about proving in investigations where they informally tested and checked empirical examples.

According to CadwalladerOlsker (2011), the inconsistency is a function of the sociomathematical norm, a term Yackel and Cobb (1996) coined to denote the mathematical practices and standards developed by a mathematics community. They point out that these norms are generally influenced by the textbook, teacher, beliefs and other subtle factors. The results also showed that the majority (45%) of participants' functional understanding could be described as hybrid in the sense that they demonstrated poorly encultured (naïve) beliefs together with highly encultured (informed). I think that the fact that, overall, only 15% of the participants responded "undecided" to the various items, was a function of the pleas repeatedly made to respondents to see their first thoughts as the best. In addition, the "*Leave this item blank*" phrase added to the LFUP instrument helped to check on participants' attentiveness while completing the questionnaire. Therefore, it is reasonable to suggest that the LFUP questionnaire was able to capture learners' ideas about the functions of proof.

In the analysis, I found that only 20% of the participants held informed beliefs about the functions of proof. However, of these participants, none was found to show extremely encultured beliefs about the functions of proof in mathematics. This seemed to suggest that most of the learners understood only few functions of proof that were consistent with those held by contemporary mathematicians. A closer analysis of the data suggested that these few views were in fact those of proof as a means to verify. Put another way, I found that most learners were likely to understand the function of proof as that of verifying the truth of a mathematical conjecture through providing evidence from several examples.

Table 5—10. The normative map based on learners' LFUP mean scores

Classification	General explanation	Mean score range		Count	*f* (%)	*c f*
		From	**To**			
Unencultured	Naïve	0	<1.5	0	0	0
Poorly encultured	Naïve	1.5	<2.5	47	35	35
Moderately encultured	Hybrid	2.5	<3.5	61	45	80
Highly encultured	Informed	3.5	<4.5	27	20	100
Extremely encultured	Informed	4.5	≤ 5	0	0	100
			Total	**135**	**100**	

The reason for saying this was found Table 5—11 where most learners showed agreements with the items that "A proof is useful in making sure that a mathematical statement is true" (64%) and that "Confidence about the truth of a statement motivates me to find its proof" (60%). Therefore, overall, informed beliefs about the functions of proof came from the verification function as it had the highest percentage among the five themes. These results corresponded with empirical findings by Grigoriadou's (2012) who was able to show that learners generally find it easy to appreciate the verification function of proof. She found a negligibly small change in this regard after an intervention programme was conducted.

A similar result was found by Healy and Hoyles (1998). Using an open-ended questionnaire, they found that learners made references to verification, explanation, communication and discovery; no mention was made of the systematisation function of proof. In addition, they found that only a substantial minority ascribed explanations to a function of proof.

Consistent with learners whose learning of proof is characterised by memorisation of proofs for reproducing them in examinations rather than explorations to produce conjectures, only 6% of the portion of participants naively disagreed with the notion that the "Proving prevents me from possibly inventing things about geometry." This result was interesting in that it reflected the dominance of teacher and textbook as the arbiters of mathematical knowledge. Taking into account Usiskin's (1982) stance that 'the student at level *n* satisfies the criterion not only at that level but at all preceding levels' (p. 25), I classified these learners as having not yet mastered Level 4 of the Hiele (1986) theory of geometric thinking.

One encouragingly notable result was that of the nine items phrased in the naïve beliefs sense, only 31% of the participants agreed with them. These statements were, for example, "A proof hides how a conclusion that a certain mathematical statement is true is reached", "Doing a proof shows me how maths is made of isolated concepts and procedures", "Proof restricts the learning of argument standards", and so on. The result showed that the participants appreciated the notion of proof as a means to explain, systematise, and communicate mathematical knowledge. However, this result was in disagreement with the 6% of the participants who naively disagreed with the notion that the "Analysis of proof may lead to invention of new results"

Table 5—11. Sample of distribution of responses across verification and explanation themes (n = 135)

	Verification			Explanation				
	T1	**T2**	**T3**	**T4**	**T5**	**T6**	**T7**	**T8**
	Proof_makes_sure	Some_proposition_true_even_if_no_proof	Confidence_motivates_need_for_proof	Proof_explains_what_statement_means	Proof_hides_how_conclusion_is_reached	Proof_shows_maths_connected	Doing_proof_shows_maths_isolated	Proof_proceeded_from_giveb_to_conclusion
Strongly disagree	25	30	19	38	6	9	61	9
Disagree	15	30	15	6	63	48	40	39
Undecided	9	15	20	11	18	17	20	11
Agree	49	45	46	47	32	41	9	40
Strongly agree	37	15	35	33	16	20	5	36
Total	**135**	**135**	**135**	**135**	**135**	**135**	**135**	**135**

5.5.2 Trends in responses among the five themes across all three schools

Learners' beliefs were classified as naïve if none of the responses in each theme (Table 5—12) received a score greater than 3 and classified as informed if all responses received a score greater than 3 within each theme. The scoring of LFUP responses were done as follows: *strongly agree = 1, agree = 2, not decided = 3, disagree = 4, and strongly disagree = 5*. Nine of the negatively worded items representing naïve beliefs about the functions of proof were reverse scored. These items are: T5, T11, T13, T15, T17, T19, T21, T23, and T25.

Overall, only 15% of the participants responded "undecided" to this statement, "When I do proof, I understand the mathematical thinking". This level of uncertainty is the highest among all of the 25 Likert scale items. This suggested that this item should either be modified or removed in future studies.

Table 5—12. Overall LFUP responses by theme

LFUP subscales (theme)	Items (total number)	Naïve views (%)	Informed views (%)
Verification	T1; T2; T3 (3)	2	35
Explanation	T4; T5; T6; T7; T8 (5)	0	40
Communication	T9; T10; T11; T12; T13 (5)	90	0
Discovery	T14; T15; T16; T17; T18 (5)	5	21
Systematisation	T19; T20; T21; T22; T23; T324; T25 (7)	48	15

5.5.3 Patterns of the participants' responses across the schools by theme

I examined the overall patterns of the learners' responses among the five themes across all three schools. Table 5—13 shows the mean, standard deviation, one-way analysis of variance (ANOVA), and post hoc comparisons for the responses by theme and by school. The descriptive statistics revealed that the learners across the three schools scored higher in the verification theme and lower in the discovery theme. The ANOVA results indicated statistically significant differences ($p < .005$) among the three samples for each LFUP theme. Post hoc multiple comparisons (*Tukey*) were performed to determine statistically significant differences between all possible pairs. According to the analysis across all the three schools, School A sample not only scored highest on all of the five themes, but also demonstrated more informed views on communication, discovery, and systematisation tenets of the LFUP than the School B and School C samples. In contrast, School C sample held more naïve to hybrid beliefs about the five tenets of proof. In the following subsections I provide an analysis of the patterns for each of the tenets (subscales or themes) across the samples in more details.

5.5.3.1 Verification tenet

The learners' total scores in School A on the verification tenet were significantly higher ($p < .05$) than those of School B and School C, whereas no statistically significant differences were found between the School A and School B samples. Generally, most learners in the three samples demonstrated naïve beliefs. For instance, an examination of individual items revealed that 64% of all the participants across the three samples agreed that the function of proof in mathematics is to make sure that a mathematical statement were true. In other words, they recognised that proof created for them the opportunity to use examples as means to gain conviction about the truth of a conjecture. However, as Healy and Hoyles (1998) argue, this function of proof has a "low status" and is considered as naïve.

Two reasons led to considering the verification function as naïve. First, verification is associated with inductive arguments in the sense that it is used to gain conviction by checking the truth of a mathematical statement using several particular cases (Polya, 1954); consideration of empirical evidence as mathematical proof. Second, the notion that "proof makes sure" hides the fact that, as Lakatos (1991) correctly argues, proof does not guarantee absolute certainty in mathematical research. It is for these three reasons that verification is relegated to empirical arguments. However, situations which used verification for conviction about the truth of a mathematical statement provided the motivation to find a proof (de Villiers, 1998). In essence, any statement whose "truth" is generated through such situations has limitations and therefore cannot be accepted as proof.

5.5.3.2 Explanation tenet

A comparison of the LFUP results showed significant differences among means of the three samples. Overall, the mean scores in School A on this tenet were significantly higher ($p < .05$) than those of School B and School C participants. When individual items were assessed across the schools, an overwhelming majority (75%) of the participants disagreed with the belief that doing a proof shows how mathematics consists of isolated concepts and procedures. However, the scores of School B and School C samples were more widely distributed about the mean than those of School A thus implying more varied views about the explanatory function of proof. The overall results (approximately 60%) suggested that some learners needed to be convinced themselves by obtaining the reasons why a statement is true.

5.5.3.3 Communication tenet

Overall, the mean scores in School A on this tenet were significantly higher than those of School B and School C participants. An examination of the participants' responses across the samples appeared to show that they were more likely to believe that proof can be used to debate the correctness of mathematical ideas (57%) and that proof encouraged the learning of argument standards (62%). However, in a study by Healy and Hoyles (1998), it was found that all the learners surveyed had little or no idea of this commutative function of proof in mathematics. I attributed the propensity for learners to view proof as a means to transmit mathematical knowledge to the design of the LFUP instrument; they may have been steered towards this view by the relevant items provided in the LFUP instrument.

Table 5—13. Means, standard deviations, and one-way analyses of variance (ANOVA) by theme and by school

Theme	School A (*n = 38)*		School B (*n = 50)*		School C (*n = 47)*		*ANOVA*	Post hoc
	M	*SD*	*M*	*SD*	*M*	*SD*	$F(2, 132)$	
Verification	3.0000	.36964	2.8900	.52985	2.6277	.58278	6.138*	School B, School C < School A
Explanation	3.2500	.38114	2.7767	.88102	2.1277	.79720	24.667*	School B, School C < School A
Communication	3.5044	.43060	2.9033	.83237	2.2199	.89220	29.873*	School B, School C < School A
Discovery	3.6096	.55653	3.3533	.67397	2.9539	.56327	12.829*	School B, School C < School A
Systematisation	3.4662	.39674	2.8629	1.06864	1.9453	1.06553	29.165*	School B, School C < School A

*$p<.005$

5.5.3.4 Discovery tenet

There was a statistically significantly difference among the mean scores of the three samples on the discovery tenet. The means of School A were significantly higher ($p < .05$) than those of School B and School C samples. However, no statistically significant difference was found between the School A and School B samples. In general, the percentage of hybrid views on this tenet was the highest among the five themes. For instance, respectively 48% and 47% of the participants could not decide whether an analysis of proof could lead to invention of new results and that proof may reveal completely new areas for investigation scales across the three samples. These results were compatible with Healy and Hoyles' (1998) who found that all the learners surveyed had little or no idea of the discovery function of proof in mathematics. I attributed the lack of propensity for learners to view proof as a means to discover new mathematical knowledge to the lack of instructional practices and experiences with the concept of proof steered towards this aspect of proof.

5.5.3.5 Systematisation tenet

According to the results from post hoc multiple comparisons, the School C sample scored the lowest, while the School A sample scored the highest on the systematisation tenet. An examination of the participants' responses to the individual items revealed that about 66% of the School A sample believed that a proof shows that the truth of a theorem is independent of previously proven theorems. However, only about 29–36% of School B and School C participants held similar views. Further examination of the results revealed that School A participants were less likely to agree that proving involved reasoning and argumentation that is different from the rest of mathematics (32%), in comparison to School B and School C participants (51–72%). This finding contradicted Healy and Hoyles' (1998) results which indicated that only 1% of learners made reference to proof as a means to systematise mathematical knowledge.

5.5.4 Variability of learners' functional understanding of proof

To refresh the reader's memory, in Chapter 4, the decision on whether learners hold informed, naïve or hybrid beliefs about the functions of proof was a function of the range within which the mean score in the LFUP questionnaire fell. By determining the mean score, I wanted to find the value that represented all the various scores obtained. The result in Table 5—14 suggests that, on average, learners' functional understanding of proof was characterised as naïve (M = 2.866). In terms of the definition of naïve beliefs in this study, this result suggested that Grade 11 learners at Dinaledi schools in the Pinetown school district learners held verificationist or empiricist beliefs about proof.

According to van Hiele levels of geometric thinking, these learners were at Level 3 in light of the fact that they believed that the sole function of proof is verification of mathematical statements. Put slightly particularly, these learners saw reaching a conclusion based on the consideration of a few cases of mathematical objects as constituting mathematical proof. The Specific Aims served as a signpost to the commitment to ensure that learners gained insights into the nature of mathematics by making them understand the functions of proof in mathematics. However, this result suggested that there is a discrepancy between what CAPS espoused and the actual instructional practices.

The standard deviation $SD = .74$ suggested that the scores were concentrated about the mean because they were spread out, on the average, about .74 above and below the mean. However, this result presented a rather bleak picture of functional understanding of proof among South African Grade 11 learners in Dinaledi schools; approximately about 86 % of these learners believe that proof has functions other than verification.

Table 5—14. Descriptive statistical analysis of LFUP scale

	N	Minimum	Maximum	Sum	Mean	Std. Deviation
LFUP score	135	1.74	3.94	386.92	2.8661	.74048
Valid N (listwise)	135					

Tolerance and variance inflation factor (VIF) of the LFUP data were determined. Whereas tolerance is an indicator of how much of the variability of the specified dependent variable is not explained by the independent variables in the model, VIF is the inverse of the tolerance value. The beta values under Standardised Coefficients (Table 5—15) helped in identifying the extent to which knowledge of a particular functions of proof contributed to the holding of informed functional understanding of proof. By "standardised" here is meant that the coefficients were converted into a standard format thus allowing direct comparison (Wilson & MacLean, 2011). In interpreting Cohen's *d*, Muijs (2004, p. 194) suggests that values .00 – < .10 indicated a weak effect, .10 – < .30 indicated modest effect, .30 – .50 indicated medium effect and those greater than .50 represented a strong effect.

Table 5—15 shows that factor ASC (systematisation) had the largest contribution in accounting for the variability of scores in the dependent variable (β = .329), it meant that the systematisation function made the strongest modest and unique contribution in explaining learners' functional understanding of proof functions in mathematics. Accordingly, knowing that a proof systematises contributes 32.9% to understanding the functions of proof in mathematics. Put another way, knowing that proof systematises could be used to predict and thus characterise learners' functional understanding as either naïve or informed. Understanding of proof as means to explain and communicate were two other independent variables that made a modest contribution of respectively β = .242 and β = .227 in explaining the dependent variable.

However, the beta value for ASV (verification) was slightly lower indicating that it made a less contribution to holding informed views about the functions of proof. Nonetheless, all five factors made statistically significant and unique contributions to predicting the character of learners' functional understanding of proof since their significant values were all less than .000, stronger than .001, where the three zeros after the decimal indicate that this *p* value is never exactly zero. However, a *p* value of this kind is interpreted as less than .001.

Table 5—15. Standardised coefficients

Model		Unstandardised Coefficients		Standardised Coefficients			Collinearity Statistics	
		B	Std. Error	Beta	t	Sig.	Tolerance	VIF
1	(Constant)	-.315	.495		-.637	.525		
	ACV	5.982	.159	.139	37.693	.000	.605	1.653
	ACE	6.020	.162	.227	37.154	.000	.219	4.568
	ACC	6.033	.199	.242	30.379	.000	.129	7.734
	ACD	6.194	.163	.178	37.982	.000	.374	2.676
	ACS	6.775	.169	.329	40.010	.000	.122	8.219

a. Dependent Variable: Total score

Different learners experience the concept of proof, including understanding why proof is important in mathematics, differently. Put another way, the variation in learners' functional understanding of proof partially explains the variation in LFUP scores. The variability in proof experience learners' functional understanding of proof could be explained by scores on the five subscales of the FUPI questionnaire. In Table 5—16, the results summarising the model showed that, in both *R* Square and Adjusted *R* Square values, 99 % of the variance in LFUP scores could be explained by considering all the five subscales. This implied that learners' functional understanding of proof could be predicted by their scores on the LFUP questionnaire. This further strengthened the LFUP tool's utility in measuring functional understanding of proof in mathematics. In this section, reference to "model" denotes the way in which my analysis explains the data (Wilson & MacLean, 2011).

Table 5—16. Regression output showing model summary

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	.999[a]	.999	.999	.757

a. Predictors: (Constant), ACS, ACV, ACD, ACE, ACC

Although obtaining statistically significant results is unquestionably a noble step towards making a contribution to the literature in the mathematics education field, the complete meaning of statistical significance has to be kept in perspective (Salkind, 2012). As Cohen, Manion, and Morrison (2011) notes, 'statistical significance is not the same as educational significance' (p. 520). Thus, I needed to check if the statistical significance obtained would provide sufficient rationale for policymakers to undertake programs aimed at highlighting the importance of functional understanding of proof in improving learners' proof work.

5.5.5 Comparison based on school and gender

I was keen to understand the relation of school (at three levels: A versus B versus C) and gender (at two levels: male versus female) to functional understanding of proof in the selected Dinaledi schools. Put differently, I wanted to see whether my expectation that learners at suburban schools, in view of being resourced, held informed understanding of the functions of proof than those in township schools. Also, I was eager to see whether female learners held informed functional understanding of proof than their male counterparts. Most importantly, I wanted to understand if the effect of gender on functional understanding of proof would be different according to school location. To this end, a 3 (school) ×2 (gender) between-groups analysis of variance (ANOVA) was conducted. The factorial analysis also provided an opportunity to compare the learners' functional understanding of proof per school.

In this analysis, participants were grouped according to their schools (School A: 38 learners; School B: 50 learners; and, School C: 47 learners). It must be mentioned that I proceeded to perform factorial analysis of the LFUP data with caution as one of the assumptions of ANOVA, namely, homogeneity of variances (equal amount of variability of the scores of three groups of schools) could not be assumed because the *p*-value associated with Levene's statistic ($F = 10.71$) was lower than .05 (Table 5—17) (Wilson & MacLean, 2011). Put another way, given that the Levene's test result provided no support for the assumption that the population variances across the three subsamples were the same. Proceeding with this analysis was dependent on the provision that the violation of this assumption is reported in the limitations section of the study. However, I proceeded to perform independent factorial ANOVA on the groups because the test tends to give almost accurate results, particularly when there are equal or near numbers of scores in the groups (Aron et al., 2014, p. 321).

Table 5—17. Levene's test for homogeneity of variance across the three schools

Dependent Variable: LFUP score

F	df1	df2	Sig.
10.710	5	129	.000

Tests the null hypothesis that the error variance of the dependent variable is equal across groups.

a. Design: Intercept + School + Gender + School * Gender

In Table 5—18, the means and standard deviation for each factor, marginal means, and the number of participants were provided. School A, located in the suburb, with relatively the best facilities and educational opportunities for its learners, scored highest (M =3.37). Males in the same school again recorded the highest scores in functional understanding of proof (M = 3.40).

Table 5—18. Mean level of functional understanding of proof by gender and school

School	Gender	Mean	Std. Deviation	*n*
School A	Female	3.339	.245	18
	Male	3.401	.266	20
	Total	3.372	.257	38
School B	Female	2.876	.731	34
	Male	3.125	.664	16
	Total	2.956	.713	50
School C	Female	2.202	.637	26
	Male	2.559	.789	21
	Total	2.362	.723	47
Total	Female	2.758	.751	78
	Male	3.014	.706	57
	Total	2.866	.740	135

As a way to further clarify and elaborate on these results, the scores in Table 5—18 were transformed into a bar graph (Figure 5—4). The results showed that learners in suburban schools tend to have higher levels of functional understanding of proof than those in township schools. Interestingly, the difference was especially large among township schools. After performing an analysis of the 2011 TIMMS data, Visser, Juan, and Feza (2015) suggested that it was not only socioeconomic factors of schools that impacted learners' mathematics performance, but also home resources such as parents' level of education.

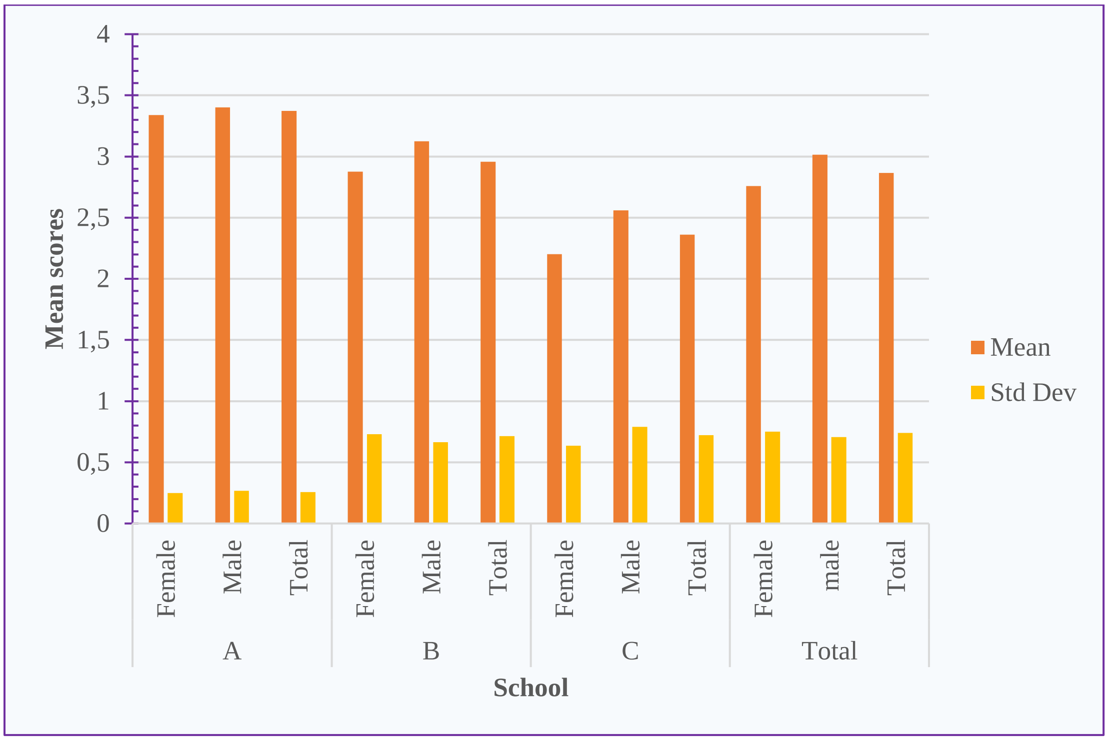


Figure 5—4. Variation of learners' gender scores on LFUP across schools

As can be seen in Table 5—19, the 3×2 analysis of variance (ANOVA) showed a highly statistically significant main effect of school on learners' functional understanding of proof, $[F(2) = 27.955, p < .001, \eta_p^2 = .302]$. Wilson and MacLean (2011) define a main effect as the overall effect of an independent variable, ignoring the effect of any other independent variable, on the dependent variable. Further analysis showed a statistically significant main effect of gender on learners' functional understanding of proof, $[F(1) = 4.134, p < .05, \eta_p^2 = .031]$. This meant that the fact that a learner were male or female affected their understanding of the functions of proof in mathematics. However, this difference in the level of functional understanding of proof among the two groups was small. In addition, there was no statistically significant interaction between school×gender, $[F(2) = .598, p = 552, \eta_p^2 = .009]$. The implication of this result is that school type made no difference in learners' functional understanding of proof for both male and female learners.

In attempting to influence learners' mathematical performance, policymakers need not only know what factors are currently influencing learner performance, they also need to know their practical significance in order to effect changes in the curriculum policy (Visser, Juan, & Feza, 2015). Analysing the results for effect sizes, shown as values of partial eta squared (η_p^2), it was found that the effect for gender was very small even though this factor was statistically significant thus suggesting that the actual differences in the mean values were very small. Specifically, gender's account for the variability in LFUP scores was negligibly small (3.1 %).

Table 5—19. The results of the 2-way ANOVA on LFUP and its factors

Dependent Variable: LFUP score

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	24.274[a]	5	4.855	12.729	.000	.330
Intercept	1080.435	1	1080.435	2.833E3	.000	.956
School	21.324	2	10.662	27.955	.000	.302
Gender	1.577	1	1.577	4.134	.044	.031
School * Gender	.456	2	.228	.598	.552	.009
Error	49.201	129	.381			
Total	1182.408	135				
Corrected Total	73.474	134				

a. Predictors: (Constant), ACS, ACV, ACD, ACE, ACC

In contrast, school location accounted for about a third; 30 % of the variance in the LFUP scores, which, according to Cohen's (1988) criterion, is of moderate practical significance. A similar finding was made by Mbugua, Kibet, Muthaa, and Nkonke (2012) who found that school factors such as overcrowding and insufficient teaching materials (associated with township schools in South Africa) impacted on learners' scholastic performance.

However, since these were results of Dinaledi schools that were selected by the DBE to receive additional equipment and support for effective teaching and learning of mathematics, I attributed this effect to a confounding variables; parents' educational status and their support of and involvement in their children's school matters. Support for this view was found in Desarrollo (2007) whose assembled data provided evidence of a positive relation between the degree to which family members were actively involved in a child's education (for example, participating in activities meant to enhance learning: reading with the child, encouraging watching television channels with educational content, helping with homework, and so on) and scholastic attainment.

In contrast, some studies found a relatively weak relationship between school resources and learner performance (for example, Burtless, 1996; Hanushek, 1997). In spite of the contradictory findings in literature analysing relations between school resources and scholastic performance, I chose not to include this aspect of research. Doing so was appropriate in that this relationship fell outside the scope of this study. Therefore, further research into the relationship between school resources and functional understanding of proof while controlling other variables such as class size, teacher, learner attitude, family involvement, and so on, within the Dinaledi group of schools is needed.

For the purpose of further clarification, the set of pie charts (Figures 5—5 (a) – (c)) displayed the proportion of the variance that was attributed to each effect: school, gender, and school*gender interaction. The two significant effects, school and gender, respectively accounted for 30% and 3% of the variability in the LFUP scores. As shown in the chart, more than two-thirds of the variations in LFUP scores was accounted for by the sum of squares for error (SSE) in each case. The sum of squares for error is a measure of the total deviation of the observed values from the regression line. Given these SSE values, the model had a large error components and therefore poorly predicted the LFUP scores.

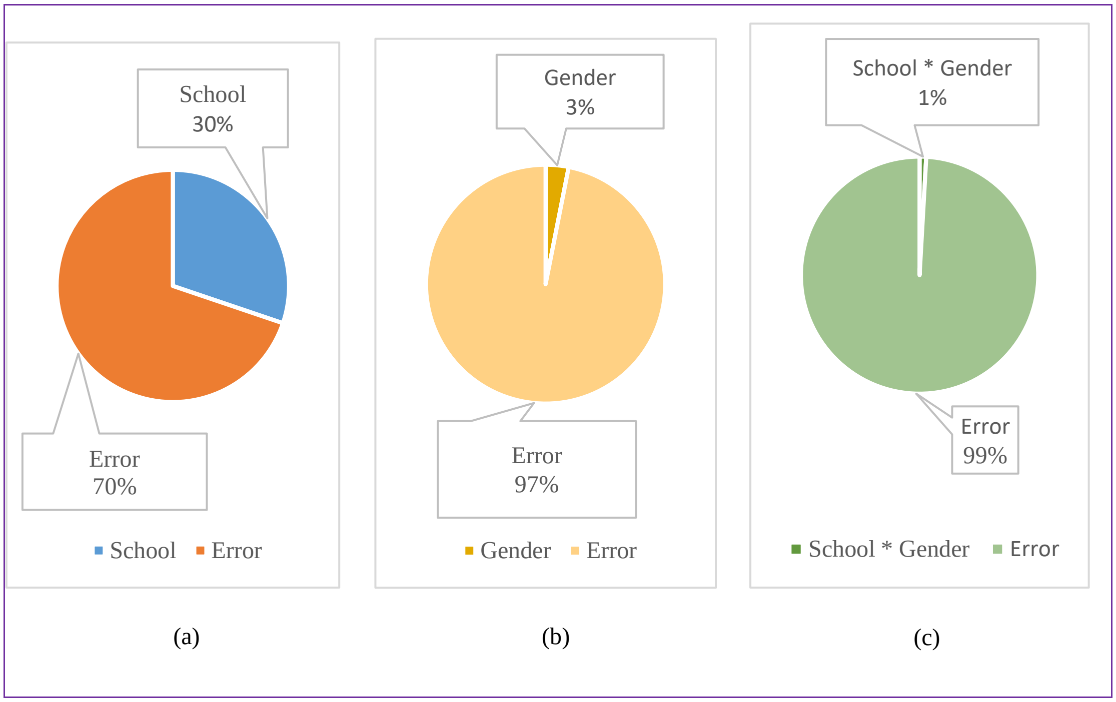


Figure 5—5. Relative effect sizes for the school, gender, and school-by-gender interaction

The profile plots line graph in Figure 5—6 also illustrated these results. Ignoring the effect of gender, the School A line was consistently higher than those of schools B and C. Ignoring the effect of school, the male line was consistently higher than the female line. These results suggested a main effect of gender and school. The fact that the lines did not touch each other was evidence that there was no interaction. However, as literature cited in Chapter 2 suggested, it seemed that these results supported Healy and Hoyles' (1998) findings that gender plays a role in the learning of proof.

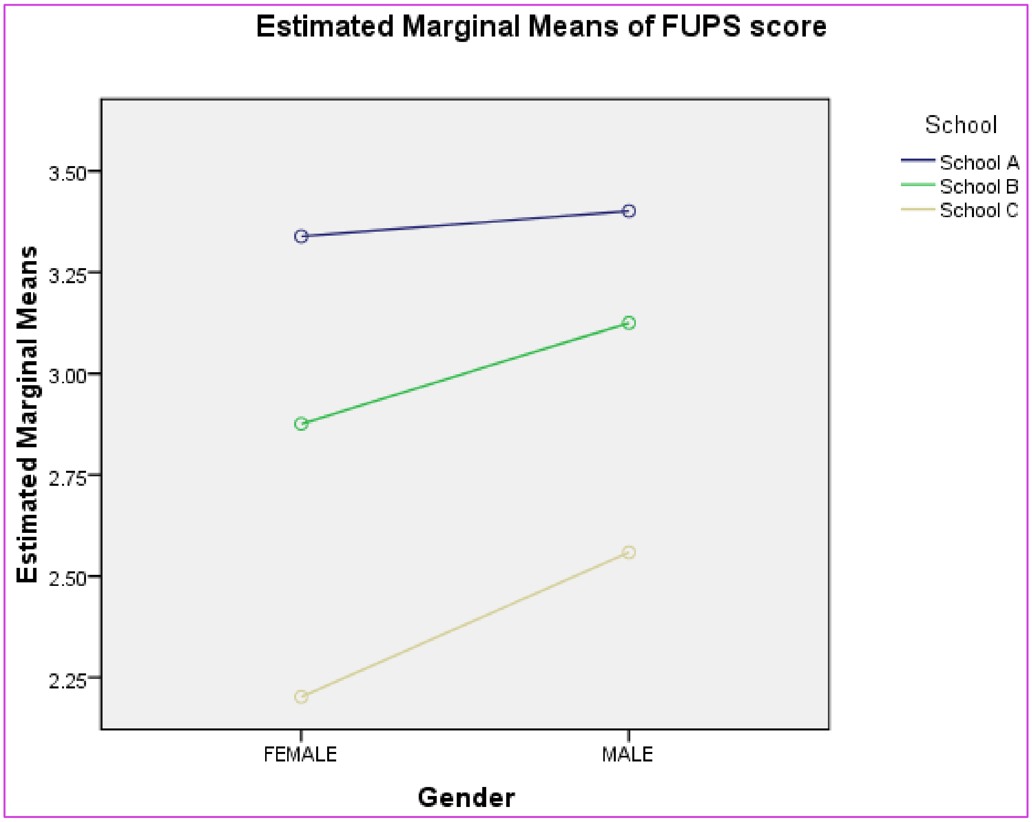


Figure 5—6. Line graph for school type and gender on LFUP scores

This investigation revealed that the overall mean of School A (*M* = 3.370) was higher than schools B (*M* = 3.001) and C (*M* = 2.381) as can be seen in Table 5—20. Therefore, learners in School A held, on average, the highest understanding of the functions of proof than those learners in the other two schools. This finding is interesting in the sense that School A was located in the suburb and as indicated in Chapter 4, was resourced compared to the other two schools which were under-resourced. It is therefore reasonable to conclude that it seemed that school resources may be related to learners' understanding the functions of proof in mathematics across all genders. Also noteworthy was that the overall (marginal) mean for male learners (*M* = 3.029, *SD* = .257) was higher compared to the mean of their female counterparts thus indicating that male learners seemed to hold higher functional understanding of proof than female learners across all schools.

Table 5—20. The mean values for of the functions of proof across schools and gender

Dependent Variable: LFUP score

				95% Confidence Interval	
		Mean	**Std. Error**	**Lower Bound**	**Upper Bound**
School	A	3.370	.100	3.172	3.569
	B	3.001	.094	2.815	3.186
	C	2.381	.091	2.201	2.560
Gender	Female	2.806	.072	2.663	2.949
	Male	3.029	.082	2.866	3.192

But, were these differences in the means of the statistically significant? To answer this question, a Post-hoc Pairwise Comparison Test for this school factor as the main effect was performed. As can be seen in Table 5—21, the Post-hoc comparisons using the *Tukey* test indicated that the mean scores among all three schools differed significantly (p<.001).

Table 5—21. Post-hoc Pairwise Comparison

					95% Confidence Interval	
(I) School	(J) School	Mean Difference (I-J)	Std. Error	Sig.	Lower Bound	Upper Bound
School A	School B	.4160*	.13291	.006	.1009	.7312
	School C	1.0102*	.13473	.000	.6907	1.3296
School B	School A	-.4160*	.13291	.006	-.7312	-.1009
	School C	.5942*	.12547	.000	.2967	.8917
School C	School A	-1.0102*	.13473	.000	-1.3296	-.6907
	School B	-.5942*	.12547	.000	-.8917	-.2967

Based on observed means.
The error term is Mean Square (Error) = .381.
*. The mean difference is significant at the .05 level.

I was motivated to examine the data for gender differences in functional understanding of proof following Forgasız's (2005) argument that it is significantly important to include gender as a variable in research analysis even if it is not the main focus of a study. I hypothesised that there might be surprising results given the ongoing efforts to offset the under-representation of females in the mathematics, science, and technology fields. In my view, the gender differences might be accounted for by cultural influences rather than they being innate. In a study conducted by Blackwell, Trzesniewski, and Dweck (2007), it was found that gender differences in mathematics performance only existed among learners who held fixed rather than growth mindset about mathematical knowledge. By mindset, according to Dweck (2014), is meant assumptions and expectations individuals have for themselves and others that guided their behaviour and influenced responses to daily events.

Research studies on the role of gender in mathematical achievement were varied. Hyde, Fennema, and Lamon (1990) found that males and females have different geometrical skills and knowledge. In contrast, according to the factorial analysis performed in this study, there was a statistically significant difference between male and female learners as also confirmed by the consistently small differences in the means. These results were inconsistent with Halat's (2008) who quantitatively investigated gender related differences in the acquisition of the van Hiele levels and motivation in learning geometry and found that gender was not a factor in learning geometry. In addition, an investigation of gender differences among 145 high school geometry learners by Battista (1990) found no significant evidence of gender differences in geometry proof. In light of the conflicting research results both from this study and those cited here, little suggests that this issue of gender and performance has been laid to rest.

5.6 Chapter summary

This chapter validated the five-factor solution of the LFUP questionnaire in order to ascertain its suitability to use in characterising Grade 11 learners' functional understanding of proof in a few Dinaledi schools. The psychometric results were that the instrument satisfied the psychometric properties with factor analysis providing evidence for the maintenance of the factors as designed. As a consequence, survey data on LFUP questionnaires were subjected to variance tests in order to characterise learners' functional understanding as either naïve, hybrid or informed. It was found that learners held naïve functional understanding of proof. In addition, school location contributed to functional understanding of proof across gender. In particular, learners in suburban schools seemed to hold higher understanding of the functions of proof than learners in township schools. Again, gender seemed to contribute to functional understanding of proof; male learners held higher understanding of the functions of proof than their female counterparts.

Overall, these analyses indicated that five distinct factors were underlying learners' responses to the LFUP scale and that these factors were moderately internally consistent. Validation of the scale suggested that the original five-factor structure can be retained. The next chapter presents results and conducts analyses of the relationship between functional understanding of proof in mathematics and argumentation ability.

Chapter 6

The relationship between functional understanding of proof and argumentation ability

6.0 Introduction

In the previous chapter I presented and analysed learners' functional understanding of proof in mathematics. Apart from finding that the verification function predominated learners' understanding of functions of proof, I also found that generally learners' beliefs about the functions of proof were rather of a naïve nature. These results notwithstanding, mathematicians cannot contest the view that doing proof entails making arguments that require justification for each claim made. In terms of this view, understanding the function of proof as a means to communicate mathematical knowledge and as a means to discover new results involves engaging in argumentation. Hence, it is against this background that this chapter reports on the investigation of the nature of the relationship between functional understanding of proof and argumentation quality. In addition, I deemed it important to understand the relationship between LFUP and argumentation ability as determined by the utility of the AFEG questionnaire (Appendix B3) not only from theoretical perspectives but also empirically by posing this question, *How is the relationship (if any) between learners' quality of arguments and their functional understanding of proof?* Data analysis was performed with the assistance of three software packages: SPSS, ATLAS.ti, and STATA.

The CAPS advocated for tasks that provide learners with opportunities to investigate, make conjectures, and justify or prove them. Yet research in mathematics education (for example, Driver et al., 2000) indicates that teachers lack the pedagogical skills in orchestrating argumentative discourse within the classroom and therefore by extension, learners faced similar difficulties. However, little is known about the learners' argumentation ability as they engage in proof and learning about its functions in mathematics. Perhaps more importantly, the relationship between learners' functional understanding of proof and their ability to argue has not yet been explored.

According to Lockhart (2002), geometry provided a rich context for the development of argument, including making conjectures and validating them. For this reason, I was interested in determining whether learners' understanding of the functions of proof was associated with their argumentation ability. This chapter sought to answer this question by presenting and analysing the questionnaire results of selected Dinaledi high participants.

In Grade 11, learners were expected to accept results established in earlier grades as axioms to investigate and prove theorems of the geometry of circles (Department of Basic Education [DBE], 2011). However, it is only upon completion of Grade 12 that learners were expected to exhibit behaviours that were similar to the van Hiele theory's Level 4; understanding the functions of definitions, axioms, proof and constructing formal proofs. Alex and Mammen (2014) suggest that the CAPS document implied that all learners across the grades in the FET phase were to perform fully at Level 4 of the van Hiele theory in Euclidean geometry. From anecdotal evidence, I found that at Grade 10 learners merely used theorems not to prove but to solve riders which implied that that they were not as yet ready to make conjectures and prove them deductively. However, what is clear in the CAPS is that learners taking mathematics at Grade 12 were expected to have experienced proof in Grade 11.

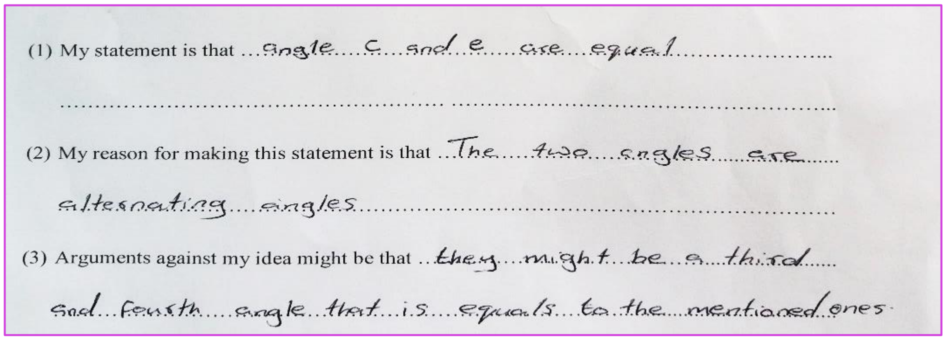
6.1 Summary of the AFEG and LFUP interaction

Analysis of data on the AFEG questionnaire not only focused on how the learners constructed their arguments, but also on whether they made mathematically sound arguments in line with the content in CAPS. Learners' response to the LFUP questionnaire seemed to suggest that the communication function of proof weakly contributed to the holding of informed functional understanding of proof. Given that there was a significant correlation found between functional understanding of proof and argumentation ability, I sought to determine the practical significance of this association as well as whether gender was a factor influencing the relationship between functional understanding and argumentation ability. To that end, I focused on the effect size and partial correlations. The relatively small degree of correlation could not be ignored as paying attention to this relationship in instructional practices may result in practical consequences and bring about the desired change in learner performance.

Second, I considered the Pearson's product-moment correlation coefficient between predictor (independent) variables and criterion (dependent variable). That is, I investigated whether a correlation exists between functional understanding of proof and argumentation ability. To do this, learners' scores on the LFUP instrument were correlated with scores on the AFEG instrument. A statistically significant relationship was found. Third, I checked for multicollinearity to understand whether functional understanding of proof items ar were highly correlated with each other with $r = .80$ or higher (Tabachnick & Fidell, 2013). The importance of checking high correlations is that it becomes difficult to interpret which of the variables is the most important in predicting the dependent variable (Field, 2009). The next section provides a background to the scores obtained from the AFEG questionnaire.

6.2 Qualitative analysis of AFEG scores

I present two sample learners' written argumentation frames: Learner A and Learner B. The sample frames provide an example of application of the coding system adopted in this study. For instance, Learner A's (Figure 6—1) argumentation was judged to be of low quality given that the statement provided by the participant as a rebuttal did not constitute one. Thus, Learner A provided a statement that could not be categorised as a condition under which his claim or ground cannot hold. He suggested that an argument against his claim that "angle c and e are equal" may be that "they might be a third and fourth angle that is equals to the mentioned ones". This statement seems to point to the learners' inability to understand the question; perhaps another example of language interference with learning, a phenomenon common among many in the sample for whom English is not their home language (Setati, 2008).



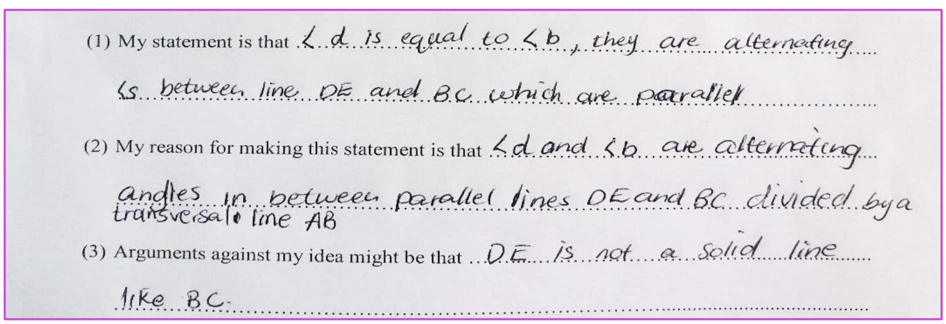
(1) My statement is that ...angle c and e are equal.........

(2) My reason for making this statement is that ..The two angles are alternating angles.........

(3) Arguments against my idea might be that ..they might be a third and fourth angle that is equals to the mentioned ones.

Figure 6—1. Learner A's argument

In contrast, Learner B's (Figure 6—2) frame represented a high quality argumentation. In her rebuttal, she indicated that the claim would not hold if "DE is not a solid line like BC". Indeed this naïve observation might arise particularly from learners who demonstrated lack of understanding that the auxiliary line represents a construction for the purpose of proving.



(1) My statement is that ∠d is equal to ∠b, they are alternating ∠s between line DE and BC which are parallel.........

(2) My reason for making this statement is that ∠d and ∠b are alternating angles in between parallel lines DE and BC divided by a transversal line AB

(3) Arguments against my idea might be that ..DE is not a solid line like BC.........

Figure 6—2. Learner B's argument

First, all the participants' argumentation frameworks were analysed to determine the nature of argumentation. To this end, Osborne et al.'s (2004) analytical tool to code participants' responses in the AFEG instrument. In Figure 6—3, a summary of all the coded data as constructed by 135 participants is provided. The various instances in which elements of TAP were used are indicated as C, C+G, and C+CG+R respectively indicating that there was a claim, claim with ground, and claim which not only included a ground but a rebuttal as well.

Although the data was analysed by two researchers, Cohen's (1968) kappa coefficient (κ) was also used to determine the reliability of the coding scheme. In addition, this coefficient was appropriate to use on the basis that I adopted a multicategory rubric comprising an ordinal scale in which responses were classified into 1 of 5 types of categories. Cohen's interrater agreements (κ) were calculated for each of the five responses using STATA (a syllabic abbreviation of the words *statistics* and *data*), a statistical software that enables analysis, management, and graphical visualisation of data. The very few unanticipated responses received were fitted into the rubric such that the following kappa (κ) coefficients were obtained: content = .95 and argumentation = .97. As Altman (1991) suggests, these values indicated very good agreement between raters. The salient elements of argumentation were counted and the results are shown in Figure 6—3, according to the schools. The notable feature of these data was that rebuttals were few across all the schools, thus reinforcing the argument that argumentation needs to be explicitly taught to learners.

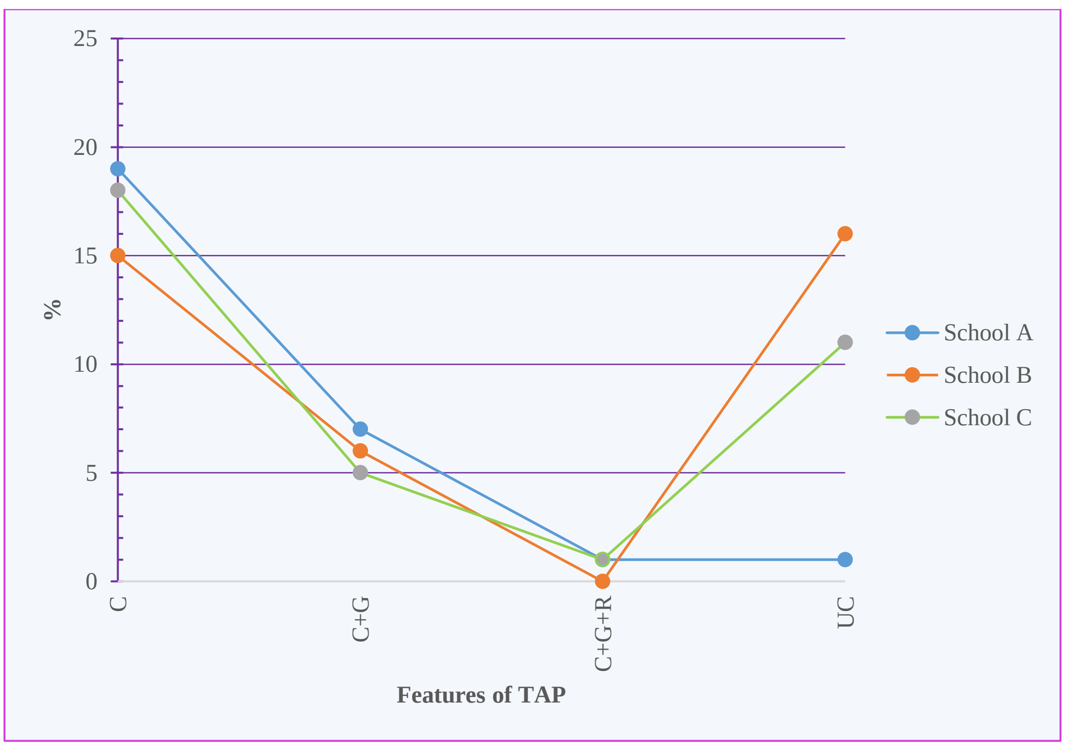


Figure 6—3. Overall distribution of salient features of argumentation across schools

The analysis of the learners' writing frames revealed several noteworthy findings. First, the majority of arguments emerging from the data was at a low level (70%). Second, though only a small minority, 18% of these arguments (claims, claims + grounds) included claims that were substantiated. Third, particularly discouraging was that only 2% of arguments developed by learners were characterised as being of high quality because they consisted of rebuttals. These findings provided deeper insights into learners' difficulties with constructing and sustaining a mathematical argument. The other notable feature of these results was that learners in School A provided the least number of arguments developed by its learners that could not be classified.

6.3 Exploration of a relationship between functional understanding of proof and argumentation ability

6.3.1 Preliminary analysis of AFEG data for multivariate analysis

In performing multivariate statistical analyses of the data, it was necessary to determine whether assumptions inherent in the analyses, such as normality, outliers, homoscedasticity, and linearity, were not violated. The relatively normal distribution curve of the histogram (Figure 6—4) provided evidence of normality.

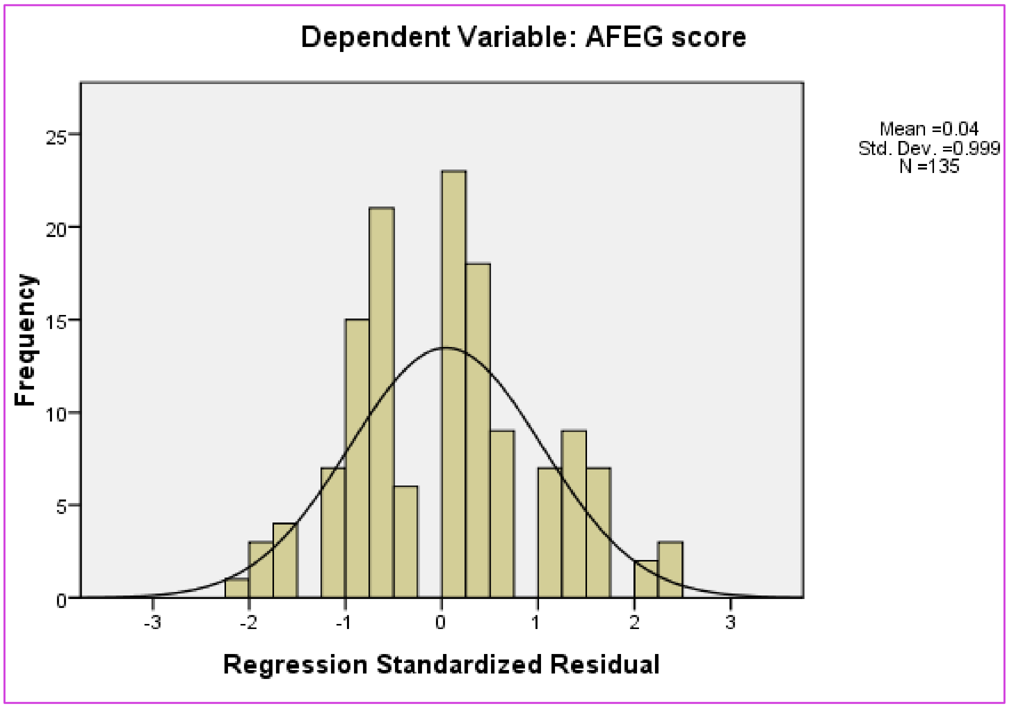


Figure 6—4. The histogram for homoscedasticity

The Normal Probability plot (Figure 6—5) provided sufficient evidence of homoscedasticity. For instance, the points on the P-P plot were close to the line to show linearity thus suggesting that there were no outliers. Therefore, the multivariate statistical assumptions and requirements were not violated and thus analysis was performed on the dataset.

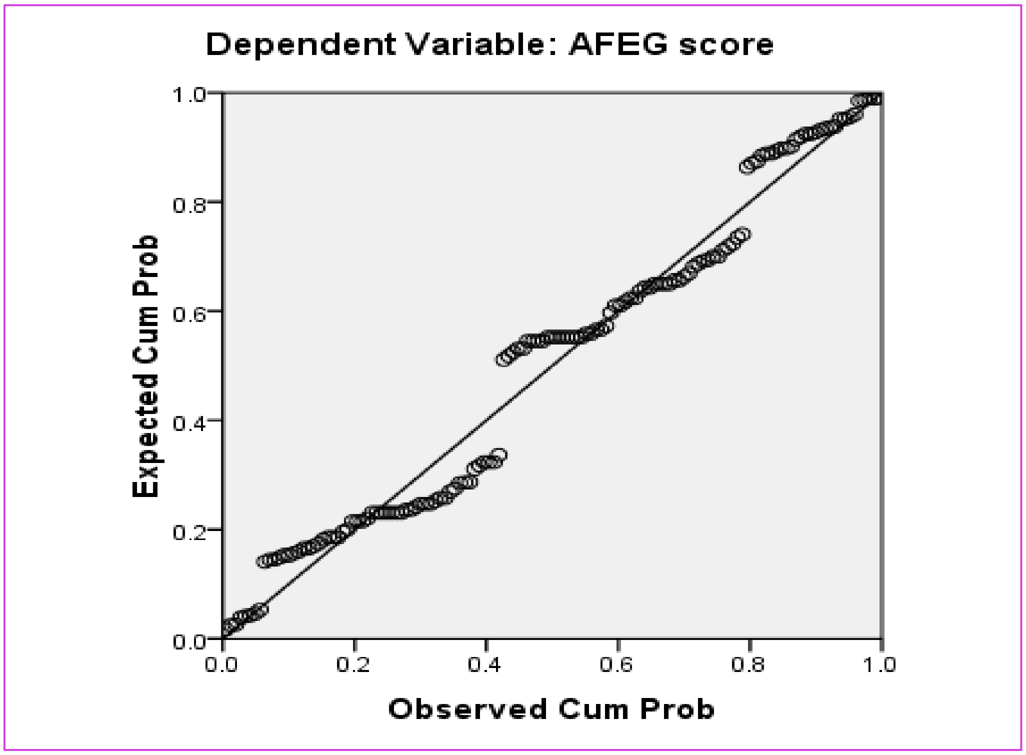


Figure 6—5. The Normal P_P plot for homoscedasticity

6.3.2 Results of the relationship between functional understanding of proof and argumentation ability

The overall mean score of the AFEG questionnaire showed that learners argued poorly (Table 6—1). A research study conducted by Means and Voss (1996) found similar results. In fact, their results showed that learners did not know how to construct an appropriate argument. A detailed presentation and analysis of the AFEG data was done elsewhere (Shongwe, 2019).

Table 6—1. The means of LFUP and AFEG instruments

	Mean	Std. Deviation	*n*
LFUP score	2.866	.740	135
AFEG score	1.415	.900	135

A correlation can be either positive or negative. The intersection of the row LFUP score and the column AFEG score showed that the correlation between functional understanding of proof and argumentation ability was $r = .225$ (Table 6—2). The footnote indicates that the two asteriks after .225 shows that the relationship was statistically significant at p < .01. Thus, the results from a Pearson-product moment correlation coefficient shows that a statistically significant correlation existed between learners' functional understanding of proof and their argumentation ability ($r = .225,\ p < .01$). In this case, the positive correlation indicated that learners who scored above (or below) the mean on functional understanding of proof tended to score similarly above (or below) the mean on argumentation ability. These results suggested that those learners who held hybrid beliefs tended to have the weakest ability to argue in mathematical proofs and those who held informed beliefs tended to have a high quality of argumentation ability. However, a correlation coefficient within the range of .20 and .35 showed only very weak relationships (Cohen, Manion, & Morrison, 2011) between learners' functional understanding of proof and their argumentation ability although it was statistically significant.

Table 6—2. The interactions correlation coefficient

		LFUP score	AFEG score
LFUP score	Pearson Correlation	1	.225**
	Sig. (2-tailed)		.009
	N	135	135
AFEG score	Pearson Correlation	.225**	1
	Sig. (2-tailed)	.009	
	N	135	135

**. Correlation is significant at the 0.01 level (2-tailed).

For this study, a correlation at this level suggested that learners' functional understanding of proof have limited meaning in predicting argumentation ability. Correlations as weak as this study's, used singly, are of little use for individual learner's prediction because they yield only a few more correct predictions than could be accomplished by guessing (Cohen, Manion, & Morrison, 2011). This weak relationship between learners' functional understanding of proof and their argumentation ability could be accounted for by some factors influencing both variables. Learners' response to the LFUP questionnaire seemed to suggest that the communication function of proof weakly contributed to the holding of informed functional understanding of proof.

However, what was also important after a significant relationship was established was estimating the strength of this relationship. In this analysis of results, multiple regression provided a description of the understanding of proof variables that can predict participants' argumentation ability. This analysis was necessary to gain insight into which of understanding of the functions of proof supported argumentation ability. The model (the way in which the analysis was conducted to explain the data) in Table 6—3 provides the *R* square to indicate how much of the variability in argumentation ability is explained by functional understanding of proof. In this the regression model, the adjusted *R* square value of .063 was reported rather than the Adjusted *R* square because the model considered only one independent variable (LFUP scores).

The multiple correlation coefficient between argumentation scores and all the functions of proof variables combined, *R*, was computed. Then, the coefficient of determination (R^2) which is the square of the Pearson product moment correlation coefficient, was used to express the proportion of variability in argumentation that can be accounted for by particular understanding of proof. According to Muijs' (2004) criteria, this model is of poor fit as it meant that only as low as 6.3 % of the variance in the argumentations scores were explained by their functional understanding of proof. The closer R^2 is to 1, the greater is the proportion of the total variation in the argumentation scores that is explained (or accounted for) by learners' understanding of proof functions.

Table 6—3. A summary of the R, R square and adjusted R square in analysis of LFUP and AFEG

					Change Statistics				
Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	R Square Change	F Change	df1	df2	Sig. F Change
1	.252[a]	.063	.056	.87517	.063	9.011	1	133	.003

a. Predictors: (Constant), T15

b. Dependent Variable: AFEG score

It was necessary to determine the effects of the five explanatory factors (independent variables) on learners' argumentation ability (dependent). Multiple regression was run to tease out which of the understanding variables were most closely associated with participants' argumentation ability. Beta weighting (β) in multiple regression was used to predict by how many standard deviation units the values of learners' argumentation ability changed for each unit change in the standard deviation of functional understanding of proof. As Cohen, Manion, and Morrison (2011) point out, the effect size of the predictor variables is given by the Beta weightings in regression analysis. Important to note is that the Beta weightings for the five factors were calculated relative to each other rather than independent of each other (Cohen, Manion, & Morrison, 2011). The beta values in Table 6—4 provide interesting information about some of these factors with regard to their relative effects on argumentation ability.

First, whereas knowing that proof explains had the strongest positive and statistically significant effect on argumentation ability where $\beta = .502$ and the level of significance, $p = .006$, knowing both that proof is a means to verify and discover had nonsignificant impact on argumentation ability. Second, whereas both knowing that proof is a means to systematise and communicate mathematical ideas yielded nonsignificant results, the former had a weakest negative effect ($\beta = -.074$) and the latter the strongest negative effect ($\beta = – .327$). Third, only knowing that proof systematises had a statistically nonsignificant result at .174 ($p > .005$) effect on argumentation ability. This made sense in that knowing that proof systematises had little to do with argumentation ability. The interesting conclusion here was that only having an understanding that proof as a means to explain can be used to predict learners' argumentation ability.

Table 6—4. The beta coefficient in regression analysis

Model		Unstandardised Coefficients		Standardised Coefficients		
		B	Std. Error	Beta	t	Sig.
1	(Constant)	.001	.569		.001	.999
	ACV	.140	.182	.083	.770	.443
	ACE	.524	.186	.502	2.814	.006
	ACC	-.073	.228	-.074	-.318	.751
	ACD	.164	.187	.119	.875	.383
	ACS	-.266	.195	-.327	-1.368	.174

a. Dependent Variable: AFEG score

Attempts to interpret the correlation between functional understanding of proof and argumentation ability were hampered by the possible existence of a third variable that may influence the relationship between the two variables. I used partial correlations technique to statistically control or nullify the effects of gender (Wilson & MacLean, 2011), as the third or secondary variable, on the relationship between the primary variables; namely, functional understanding and argumentation ability. In other words, I determined the association between the primary variables by nullifying the effects of gender and thus assuming that the learners were of the same gender. The partialling out of gender was informed by research which suggests, including the findings in Chapter 5, that learner performance in mathematics tends to be a function of gender.

Since the zero-order correlations have already been analysed above, I considered the section with the partial correlations in Table 6—5. In the previous section, the significant relationship between functional understanding of proof and gender seemed to suggest that gender has influence in explaining the understanding-argumentation association. However, the partial correlations section shows that controlling for gender further weakens the strength of the significant relationship between functional understanding of proof and argumentation ability (r = .214, p = .013). Clearly, gender was one secondary variable that seemed to influence the relationship between the two primary variables.

Table 6—5. Assessing the influence of gender in the primary variables.

Control Variables			LFUP score	AFEG score	Gender
-none-[a]	LFUP score	Correlation	1.000	.225	.171
		Significance (2-tailed)	.	.009	.047
		df	0	133	133
	AFEG score	Correlation	.225	1.000	.089
		Significance (2-tailed)	.009	.	.302
		df	133	0	133
	Gender	Correlation	.171	.089	1.000
		Significance (2-tailed)	.047	.302	.
		df	133	133	0
Gender	LFUP score	Correlation	1.000	.214	
		Significance (2-tailed)	.	.013	
		df	0	132	
	AFEG score	Correlation	.214	1.000	
		Significance (2-tailed)	.013	.	
		df	132	0	

a. Cells contain zero-order (Pearson) correlations.

6.4 Chapter summary

Preliminary assessment of data from the LFUP and AFEG instruments allowed for conducting multivariate data analysis and interpretation. This data were asubmitted to SPSS v.24 (2017) using multiple correlation and regression. There was a relatively weak but statistically significant association between learners' functional understanding of proof and their argumentation ability. Gender, a secondary variable, was found to influence the relationship between functional understanding of proof and argumentation ability. In addition, there was no suggestion of cause-and-effect from this relationship. Whereas, relative to each other, the explanatory function of proof exerted the greatest and statistically significant influence on learners' argumentation ability, the communication function of proof exerted the smallest and statistically insignificant influence on argumentation ability. In other words, the explanatory function of proof was found to be the factor which best predicted learners' success in argumentation ability.

This result was interesting as it suggested that learners appreciate that functional understanding of proof creates an opportunity for them to engage in argumentation. But equally, it is valid to argue that being able to engage in argumentation is encouraged by appreciating the explanatory function of proof. When “partialling” out was conducted, “gender” was found to be a factor that decreased the correlation between functional understanding of proof and argumentation ability. I presented, analysed and interpreted the findings by making comparisons between these results and various other studies. However, the LFUP questionnaire consisted of closed attitude items which restricted participants to certain responses. Thus, I could not detect the reasons why they made the choices they made in the LFUP. Therefore, in the next chapter I seek to investigate one learner’s sources of the beliefs she held about the functions of proof.

Chapter 7

Beliefs about the functions of proof in mathematics: The case of *Presh N*

7.0 Introduction

It has been established in Chapter 5 that Grade 11 learners surveyed in this study hold hybrid functional understanding of proof. In the previous chapter a statistically significant but weak association between functional understanding of proof and learners' argumentation ability was found. The primary focus of the current chapter is to report on the qualitative phase of this study: the case of *Presh N* (one of the participants in the survey) to explain the reasons why she held informed functional understanding of proof in the context of six factors: semantic contamination; teacher; collectivist culture; textbook; empirical arguments; and deductive arguments. I present and analyse semistructured interview results to explain why *Presh N* tended to hold informed understanding of the functions of proof in mathematics.

The single most important contribution made by this qualitative segment of the study pertains to fact that it presents a model to understand factors influencing functional understanding of proof. In addition, the model will stimulate future research and inform practice with regard to paying attention to these sources of beliefs about the functions of proof as they may either hinder or promote the learning of deductive proof. Important to note is that *Presh N*'s case was used to suggest a model for understanding learners' beliefs about the functions of proof in mathematics.

Having obtained the general understanding that most learners hold hybrid beliefs about the functions of proof, I proceeded to identify a participant whose LFUP score was the highest in the range of moderately encultured group of participants to represent an extreme case. Utilising a single case study provided more insight when explaining why she held informed beliefs about the functions of proof in mathematics than a larger study may afford. In addition, given that factors influencing learners' functional understanding of proof are insufficiently understood in research, this qualitative phase of the study is uniquely positioned to investigate this phenomenon in depth.

Conducting an in-depth interview accorded the privilege of getting a glimpse into the participant's experiences in proof education besides being a stranger to her. *Presh N* was a deviant case in the sense that she scored highest (M = 3.94) in the LFUP scale besides attending a township school; it was interesting to explain and illuminate the reasons for this deviant score. A semistructured interview was not only appropriate for this segment of the study, in light of limited time and financial resources, but also because it required adherence to an Interview Schedule. There were very few silent moments during the interview, the longest being a gap of 1,700 milliseconds; however, it seemed that that was one of the moments in which the participant reformulated her response. However, Clayman (2002) cautions that any delay such as this 'may be interpreted as the first move toward some form of disagreement/rejection' (p. 235).

As already mentioned, learners' beliefs about the functions of proof stem from various sources one of which is that, in the classroom, the technical meaning of the term "proof" tends to be conceived of as similar to meaning it takes in everyday talk; evidence. Put another way, the term "proof" is a spontaneous concepts that learners acquire through their interactions within their everyday environment which often refers to providing evidence for a claim. More specifically, I was interested in understanding the influence of these factors—semantic contamination, teacher, textbook, culture, deductive and empirical arguments—on her enacted practice (proving a proposition). To this end, I needed to document empirical evidence of her views on the meaning of proof in mathematics by asking questions such as, "What does the term *proof* mean to you?" In doing so, I intended to capture, analyse, and interpret the participant's experiences with the proof phenomenon in her proof-related task.

The rest of the chapter is organised as follows. First, the case is described followed by a summary of the results obtained from both interview with *Presh N* and her proof-related task. Then, analyses of her responses obtained from the survey, interview, and proof-related task are undertaken. I use her survey data, excerpts from the interview, and perspectives from her proof-related task to illustrate the influence of these factors, highlighting the main and minor factors that seemed to influence her beliefs about the functions of proof. In other words, *Presh N*'s responses from multiple methods formed the basis of the results of this qualitative phase of the study. The analyses end with the presentation of a tentative model for understanding the factors affecting functional understanding of proof. The interrelationships among the factors in this model are justified by references to existing empirical evidence. The last section is a conclusion summarising the chapter.

7.1 The case

The employment of statistical analyses in the quantitative segment of this study showed how widespread hybrid functional understanding of proof was among learners, how these understanding correlated with learners' argumentation ability, and the variability of the functions of proof across the different groups of learners (that is, gender and resourcefulness of schools). The only participant in this case study, *Presh N* was an extreme case in that she obtained the highest LFUP score despite attending a township school which was, at the time of the study, historically under-resourced. *Presh N* was a 17-year old female high school learner at the time of the study. She spoke IsiZulu and indicated that in her home English was a second language. I adopted a case study design to explain the sources of *Presh N*'s informed functional understanding of proof with some degree of thoroughness. Put another way, a case study was preferred as a strategy to provide answers to the "why?" question that I posed as the third research question (McMillan & Schumacher, 2010; Yin, 2014). *Presh N'*s beliefs about the functions of proof were influenced by her experiences with the concept of proof. The experiences were investigated from these perspectives: semantic contamination, teacher, collectivist culture, textbook, empirical arguments, and deductive arguments.

7.2 Summary of interview and proof-related task results

In this section, I first provide a summary of factors facilitating *Presh N*'s understanding of the functions of proof in mathematics from an analysis of her own voice expressed both orally and verbally. To refresh the reader's memory, *Presh N* was purposively selected on the basis that she scored highest in the LFUP questionnaire. A pattern-matching analysis of interview transcript showed that the “collectivist culture” and “teacher” factors had the greatest influence on her functional understanding of proof. The van Hiele theory supported by Edwards’ (1997) notion of “territory before proof” were used to analyse *Presh N*’s written work. Her written response to the proof-related task showed that although she achieved Level 4 of the van Hiele theory of geometric thinking, her work reflected either a misconception of the term “proposition” or lack of experiences with activities on conjecturing which not only involves deductive reasoning but also inductive reasoning.

Although the findings in this qualitative phase were in themselves not generalisable, ‘they can easily become such if carried out in some numbers, so that judgements of their typicality can justifiably be made’ (Giddens, 1984, p. 328). To understand how typical or atypical *Presh N*’s experiences were, as Fraenkel, Wallen, and Hyun (2012) suggest, replication across individuals rather than groups would be needed to make results worthy of generalisation. However, given limited time inherent in investigations such as the present, the findings were intended to lay the foundation for the formulation of hypotheses in subsequent research that document functional understanding of proof in mathematics.

In any event, the intention in this segment of the treatise is not to generalise the findings but to describe and explain the case of a learner found to be the bearer of ideas necessary to understand why some learners hold the beliefs they hold about the functions of proof. Thus, this phase of the study focused on analysing factors influencing hybrid functional understanding of proof from the perspectives of a high school learner. Put another way, in this qualitative component of the present study I attempted to recount the case of *Presh N*, the learner of interest in this study, in relation to the factors influencing her understanding of the functions of proof.

7.3 Analysis of interview data

Initially, two participants, *Linda* and *Presh N*, whose scores were extreme were selected on the basis that they respectively obtained the highest and lowest scores among participants found to be holding hybrid beliefs in the LFUP results. The participants were read each LFUP statement and their response and were asked to explain why they had given the response they gave. There were few instances where it was necessary to clarify or interpret an item. These participants were free to change their responses. As a consequence, responses were rescored; and it was found that *Linda*'s new score did not fit the extreme case categorisation as planned. For this reason, he took no further part in the study. Hence the sole interviewee was only *Presh N*.

The analysis process not only entailed verbatim transcription of interview data but also noting nonverbal and paralinguistic communication in field notes in the interview schedule. The analysis process involved the use of the two principal modes in ATLAS.ti: data level and conceptual level. The data level involved reading the quotations and subsequently assigning codes to selected segments, writing memos and comments that contained my interpretations for each quotation. Memos and comments are methods used to record one's ideas and observations about codes, and quotations (Friese, 2011). In ATLAS.ti, coding is viewed as the procedure of associating code words with quotations. The code may contain more than a single word, but should be concise (Friese, 2011). The coding decision was mainly guided by the conceptual framework. The conceptual level focused on constructing concepts and theories based on relationships between codes, data segments, and memos.

The benefit of this conceptual process is that it helps to uncover other relations in the data that were not previously conceived. Then, codes were assigned to families (categories). Groundedness, the number of quotations associated with a code, determine the strength of the influence (Friese, 2011). For example, in **Semantic contamination*, the number 8 meant that the code **Semantic contamination* has been used for coding fifteen times. Also, density, the number of other codes to which a code is linked, was 2. Analysis began with assigning manually created codes to segments and writing memos that contained my comments or thinking about the data using ATLAS.ti software. I built networks from the codes I had created which together with memos, formed the framework for testing theory. The auto coding facility in ATLAS.ti was used to scan the transcript and automatically assign the predetermined codes. The codes were drawn from the conceptual framework generated from literature review. The network views facility helped in the conceptualisation of the relationships between codes, segments, and memos with the aid of a diagram.

This analysis process also included the use of excerpts to support interpretation (Cohen, Manion, & Morrison, 2011). By interpretation of data is meant identifying its meaning (Brenner, Brown, & Canter, 1985). The data presented here were collected through an audiorecording of the interview with the participant, *Presh N*. Then, it was transcribed and analysed through coding and interpreting. Data resulting from the interview is labelled as excerpts (the units of analysis for this phase of the study) which were associated with the relevant factor influencing *Presh N*'s informed functional understanding of proof. Additional description followed this label to provide more information as to what these excerpts represented. Put differently, in the analysis of the results (that is, *Presh N*'s voice), I provided the most compelling evidence through excerpts to support the analysis.

The analysis began with coding of data segments guided by predetermined categories derived from hunches and literature. The final step was patterning (finding a relationship among categories). Whereas a data segment includes a word, sentence containing a single idea, episode, or piece of relevant information, a code is the abstract term (for example, verification, explanation, communication, and so on) for describing a segment (McMillan & Schumacher, 2010). Thus, the coding process is the analytical translation of responses and participant's information into categories (Miles & Huberman, 1994). In interpreting the results presented here, I was guided by my conceptual and analytical frameworks. As discussed in Chapter 2, the conceptual model provides six categories of factors impacting functional understanding of proof. Seeking patterns was facilitated by ordering categories into "major" and "minor." I present excerpts (coded segments that consisted of a quotation and a linked code) from interview with *Presh N*, interspersed with my interpretation, to capture the essence of her perspectives. A category was labelled major if it had the most "contradicts" and "associated with" relationships. Otherwise, it was a "minor" category.

I found that *Presh N* maintained prior empiricist beliefs about the functions of proof despite appreciating the generality of a deductive proof. Based on the overall impression gained from *Presh N*'s perspectives, I think that learners can be made to modify their beliefs about the functions of proof if they experience cognitive conflict between their beliefs and mathematical knowledge. For *Presh N* and most other learners, theorems are merely mathematical facts to be learned and reproduced in tests and examinations. There was evidence that *Presh N* indeed seemed to hold informed views about the functions of proof beyond verification. When asked to mention the functions that proof performs in mathematics, she said "*When you are proving you... there is a communication*". Her view is consistent with Schoenfeld's (1994) assertion that proof can be seen as a way of communicating with others the ideas resulting from sound thinking. By mathematical thinking he meant the ability to do or use mathematics. *Presh N* also regarded proof as making sure (except that for her, it meant relying on the authority of textbook or teacher) and making her understand mathematical thinking.

The reason I regarded *Presh N* as a learner with informed views about the functions of proof was that even a mathematician, Schoenfeld (1994), thought of the functions of proof in three ways: ensuring certainty (established through verification); a way of communicating ideas with others; and a way of coming to understand (explanation). *Presh N*'s textbook, as is the case with many others, was designed for practice of techniques within a particular timescale to pass examinations rather than to reflect the nature of mathematics. Yet, she expressed views which seemed to reflect a conceptualisation of mathematics as a body of knowledge consisting of connected collection of ideas consistent with viewing mathematics as a dynamic, growing, and changing discipline while at the same time seeing it as body of knowledge meant to be learnt by rote memorisation as transmitted by her teacher. By rote memorisation is meant learning which did not make connections with prior knowledge and is soon forgotten once deliberate attempts to remember it stopped (Wray & Lewis, 1995). In the next subsections, the results are presented and interpreted around the six factors explored in Chapter 2 in this order: semantic contamination; teacher; collectivist culture; textbook; empirical arguments; and, deductive arguments.

7.3.1 Theme 1: Semantic contamination

I presented *Presh N* with a "documentary proof" from the bank. When presented with this letter from the bank certifying the veracity of the account holder's details, *Presh N* indicated that the role of the bank stamp was to verify the details as provided on the "confirmatory letter". She further argued that the stamp as empirical evidence was similar to a proof as understood in the mathematical sense thus giving credence to 'the fact that outside of mathematics, proof can be indistinguishable from evidence' (Healy & Hoyles, 2000, p. 396). Her understanding of proof in this case resembled an everyday rather than a mathematical usage of the term "proof" as convincing through a logical sequence of finite steps. She reached this conclusion based on the fact that both instances arrived at a conclusion through taking some steps:

> *This proof I can say they are almost similar because there's details that aah... the … the client or customer that went to … to … to request for… for the statement can agree to what they see on the statement by agreeing that this is their information so when the bank aah place the stamp on this statement they … they verified with the customer that this is the is their information and the customer agreed and they … they also took steps probably by aah checking the … the clients ID number and this is what they came up with so. Yaah, even though there's not a lot of similarities but they took certain steps to come up with the final aah statement.*

In emphasising her point, she pointed out that "*I see the name of the bank, the branch, the date and that it's the proof*". This seemed to be the effect of everyday ways of verifying the "truthfulness" of claims. Here proof refers to providing visible evidence to convince someone of the correctness of the banking details (Reid & Knipping, 2010). However, she seemed to contradict herself in this regard. For instance, when asked "How can the learner be sure that the statement that *"The sum of the angles of a triangle sum up to* 180^0*"* always works?" she responded:

> *The learner can use aahmm. can use a theorem that talks about parallel lines and a triangle drawn between parallel lines a sketch yah... a learner can use a sketch which has parallel lines and in between the parallel lines there's a triangle and use all the theorems that they have learnt to… to work it out and see using the angles of that triangle talk about it using theorem.*

In the excerpt above, she seemed to indicate that deductive arguments are mathematical proof, yet she equated everyday "proving" to using axioms to logically arrive at a conclusion. She held both mathematically adequate views as well as those that are inconsistent with those held by the mathematics community. These beliefs coexisted in one individual and seemed to be used interchangeably. In this case the learner was unable to evoke each belief on the basis of the context. Put another way, they were unable to appreciate "proof" as an ambiguous term that the learner thought to be consistent with conventional mathematical knowledge. Instructional practices need to place emphasis on the difference between the everday use of the term "proof" and its technical use in mathematics. The following excerpt seemed to highlight the unhelpful consequence of believing that the term can be used unselectively:

> *This proof [letter of confirmation] I can say they are almost similar because there's details that aah... the ... the client or customer that went to ... to… to request for… for the statement can agree to what they see on the statement by agreeing that this is their information so when the bank aah place the stamp on this statement they ... they verified with the customer that this is the is their information.*

It is in light of *Presh N*'s crystalised perspective that I think very few will contest the notion that the inability to distinguish "proof" from its everday sense seemed to hinder understanding of proof and its functions in the field of mathematics. This perspective could be considered as indicative of the lack of mathematical knowledge and therefore it is incumbent upon instruction to create opportunities aimed at addressing it. As Lakatos (1991) points out, through conjectures and refutations, practices which rely on linguistic knowledge, conventions and rules, mathematical knowledge could be developed. So, regardless of the authority, everyday proof does not become proof in a mathematical sense. It is reasonable to suggest that the learner's simplicstic attitude to the term "proof" required deliberate instruction as it is inappropriately dominant.

She seemed to hold "fixed mindset" notions which, as Boaler (2013) asserts, are not only inconsistent with recent research about the plasticity of the brain, but also harmful to all learners. However, the results here showed that indeed it is difficult for learners to abandon completely notions of empirical evidence being mathematical proof. Heinze and Reiss (2003) conducted an interview study of eleven Grade 8 learners from a sample of 700 learners chosen according to their achievement in a written test on geometry items. They found that learners had difficulties to bridge the gap between empirical argumentation and deductive argumentation. Chazan (1993) found similar views of proof when he interviewed seventeen learner, nine females and eight males that lasted for one hour. Similarly, the conclusion I could reach from *Presh N*'s perspectives was that for as long as steps were provided to reach a conclusion, everyday arguments were to be considered as similar to mathematical proof. However, those arguments that rely on empirical evidence within the realm of mathematics do not constitute mathematical proof.

7.3.2 Theme 2: Teacher

Presh N's thoughts about the fucntions of proof were surprisingly in contrast with research findings. She indeed presented a fascinating case. She highlithed the dominant role her teacher played in the classroom. Referring to her teacher, when probed about her role in proving, she said that *"They've been proving for quite some time more than us because are just learning these things they've been exposed to these problems for a while more than us"*. She seemed to set the tone of how she perceived her teacher; as a transmitter of unchanging knowledge by suggesting that she viewed her role as that of absorbing what the teacher said:

> *My role is to … is to go over it again aah maybe try to find out how other learners eeh …. how did the other learners find the … the proof, how… how did they managed to solve it without the teacher just basically going through it again and maybe asking maybe more questions from the teacher …*

This modest view is consistent with learners in countries with a collectivist culture. In a collectivist culture, the teacher commands a highly respected position; their authority in the classroom is seldom challenged. The teacher, by virtue of their legislated position, dominates classroom spoken and written discourse (Morton, 2012; Hayes & Matusov, 2005). Seemingly, she experienced limited, if any, opportunities to engage with the proof content. While *Presh N* was correct in profiling the teacher as an expert by virtue of having had more education in mathematics (Lampert, 1990), she did not paint a picture of a teacher that used this knowledge to enculturate this or her learners into the practices of mathematicians. But, rather, the teacher's role seemed to be the transmitter of mathematical knowledge.

According to Anderson (1989), teachers who spent more time interacting with learners about content create better opportunities for learning. Teachers' choice of task and their accompanying questions and comments during classroom affected the development of learners' functional understanding of proof in mathematics (Peressini, Borko, Romagnano, Knuth, & Willis, 2004). *Presh N*'s perspective seemed to suggest that her experiences with proof was characterised by the teacher providing answers. That seemed to suggest lack of conjecturing and reliance on teacher. Further evidence of the importance accorded to the authority figure, the teacher, *Presh N* argued that she sought thinking strategies about learning to do a proof from her teacher (and peers). Hadas and Hershkowitz (1998) suggest that finding surprising results and the impossibility of checking al1 cases inductively lead to a need for a deductive proof. Schoenfeld (1994) adds that the need for a proof is not only powerful but also necessary because patterns and trends can be deceptive. But, in *Presh N*'s case, the need for proof comes from the teacher whose didactic contract is to fulfil the aspirations of a curriculum designed to make learners reproduce rather than take the zig-zag path as highlighted by Lakatos (1991).

It is reasonable to assume that she viewed her teacher as "all-knowing" including knowing theories of how learning takes place. This result was consistent with Inglis and Mejia-Ramos' (2009) suggestion that learners perceived learning as a process of transferring information from authority figures to them. In response, to succeed in making mathematics classrooms sites of conjecturing rather than places of transmission of knowledge, I endorse Cobb and Yackel's (1996) suggestion that reliance on the teacher who learners regard as having privileged access to knowledge should be declared unacceptable.

However, it would be naïve to apportion blame to the teacher. As is the case in South African schols, all the Dinaledi schools that participated in this study relied upon the concepts in CAPS to pace and structure studied in each term of the school year. Therefore, teachers themselves had little to no authority to change this scheme and by extension, learners cannot deviate from what the teacher has set out to achieve on a particular lesson. As Watson (2008) argues, theories from cognitive psychology whose methods of instruction were unhelpful for inducting learners into mathematical practices (for example, empirical exploration, conjecturing, refutations, or argumentation) since they are less concerned with the practices of mathematics community. She further pointed out, these theories are concerned with seeking the fastest and most productive ways to teach learners how to find answers to broadly isomorphic problems. A similar perspective is echoed by Hersh (1997) who suggests that the school mathematics denied learners opportunities to experience the wonders of mathematics by offering them what Boaler (2010) refers to as an impoverished version of the subject in which mathematicians engaged.

Therefore, it is fair to suggest that school mathematics and the mathematics as practiced by mathematicians were different for many reasons one of which is that, as Boaler (2010) points out, mathematicians work on long and complicated problems that involve combining many different areas of mathematics which encouraged persistence. In contrast, she points out that school mathematics is filled with hours of "problems" that involved the repetition of isolated procedures and rules.

I share Harel's (2008) view that the goal of instruction must unequivocally be to gradually refine learners' proving activities towards those practiced by the mathematicians of today. For instance, Mudaly (2004) investigated, among other ideas, whether learners developed a better understanding of geometric concepts when using *Sketchpad*. The results showed that South African learners were able to discover some geometric facts and ideas through conjecturing and refuting their own conjectures with *Sketchpad*. This finding led to his recommendation, with which I concur, the possibility of integrating DGS into the South African mathematics curriculum to facilitating the learning of deductive proof. Hence, very few will contest the suggestion that his findings demonstrated that despite the constraints imposed by the need to prepare learners for tests and examinations, environments in which learners conjecture and refute can be created in our mathematics classrooms.

While acknowledging that making judgements about educational aims is a difficult task, the stance taken here is that the Specific Aims in CAPS are unrealistic in that the concepts it embodies were not reliably assessed in examinations. For instance, in the Grade 12 final examinations, candidates are required to prove theorems without bothering about the meaning of the term proof, at least. They are expected to do proof when they actually do not have the appropriate mathematical understanding of the term "proof" let alone appreciating the zig-zag between inductive observation and deductive generalisation, and experiencing mathematical work that point to the tentative nature of proof and by extension, mathematical knowledge. Small wonder that most of them perform poorly in Euclidean geometry than they do in algebra where they hardly prove anything. All of these contribute to the understanding of the concept of proof in mathematics. However, more specifically, the lack of consensus in mathematics education about the meaning of the term "proof" (Balacheff, 2008) does not make the situation better.

The point I was trying to make here is that instructional practices do not engage in functional understanding of proof and that the education officials responsible for curriculum delivery were unaware of this problem. The intention of shining the spotlight on functional understanding of proof in mathematics forms part of attempts 'to bring the practice of knowing mathematics in school closer to what it means to know mathematics within the discipline' (Lampert, 1990).

7.3.3 Theme 3: Collectivist culture

Of all the factors influencing functional understanding of proof, the impact of a collectivist culture was strongest. Typical of a collectivist perspective on the world, when asked to explain the reason why she was sometimes unable to do proof, she strongly made the point that one has to blame themselves for failing to apply rules learnt in class. In this regard, she said:

> *Ahh... it might be the language or how the question is put or you didn't exhaust all your theorems or yah all your theorems that you've learnt, I just said it is because you are having a problem in understanding the statement what it says or you haven't applied every, you haven't applied ...*

In her explanation as to how she learnt theorems, she said that "*I learn my theorems by obviously going through them reading try and understand follow the the … the rules and then apply them in a problem*." The tendency to view proof as somebody else's mathematics is not peculiar to high school learners. In Almedia's (2000) investigation of undergraduate mathematics students, he found that they 'see proofs as something that is an external activity rather than an internal activity meant to provide insight and understanding' (p. 871). In this study, *Presh N* pointed out that there is very little that learners can do since mathematicians provided all proofs and theirs as learners was merely to memorise rather than attempt to "invent" theorems:

> *I think as students we can .we rely on … on the text books that are written by mathematicians so by following their way of of proving it is almost guaranteed that you are on the right track you you don't just come up with your own.*

However, her beliefs were punctuated with contradictions. Attempts by *Presh N* to present views that were constistent with mathematical knowledge as practiced by the mathematical community were undermined by images of this influence. For instance, she mentioned that:

> *Math, topics in maths are closely related mmh... Nothing is new you can always relate to the previous chapter you did so you can see a pattern forming and you can see lot of relationships things that you can compare or say you can use to solve other problems in different topics. I mean you can you can you can use aah … ahh... For example algebra, algebra is not only used for aah solving x you can use it in word statements you can use it in in geometry, Euclidean geometry so it's not something new that comes up when you do a different topic it's a follow up or a continuous it's just that it's how you use it, it's how apply it.*

Also, she defined mathematical proof as a process invloving conjecturing through inductive reasoning to eventually applying deductive reasoning to complete a proof by saying that "*Proof in mathematics is about coming up with ideas and developing eeh... Formulas by using things like theorems and measurements …"* Further, she was presented with a ficticious learner's working.

When asked to explain whether she agreed with the learner's argument that a theorem is proven, she articulated the mathematical view of proof by saying that the learner's view was incorrect. This result was contrary to Healy and Hoyles' (1998), who found that learners still sought more examples to be sure that a statement were true despite having produced its deductive proof.

> *Not completely because the learner only relied on the protractor to come up with their conclusion but aah... his conclusion is not supported by statements or other theorems that were ... were that they happen to be proven by mathematicians like maybe theorem of Pythagoras or things like that he only relied on the protractor, it's not proof enough ...*

In contrast, the following statement ssuggetsed that she relied on memorisation rather than the conjecturing she indicated above, "*I learn my theorems by obviously going through them reading try and understand follow the ... the ... the rules and then apply them in a problem.*" *Presh N*'s views mirrored Lampert's (1990) argument that doing mathematics means following the rules provided by the teacher and remembering and applying the rules correctly means knowing maths.

There was a glaring contradiction between what *Presh N* expressed as an ideal situation in which mathematics was viewed as composed of interrelationships among ideas and the actual classroom practice of memorising theorems as prescribed and demonstrated by the teacher. This reflects the discrepancy between the Specific Aims in CAPS and the actual learning outcomes. As a consequence, academic performance of learners is bound to be affected. Further, this situation reduced the opportunities for learners to experience practices akin to mathematicians and thus develop the view that mathematical knowledge is not fixed.

7.3.4 Theme 4: Textbook

The effect of the textbook as a source of *Presh N*'s functional understanding of proof were the next prevalent after the collectivist culture. For instance, her response to the question "*How does the textbook help you to do proof?*', was "*Aah the text book eeh it's got almost all the relevant information that I need so that's how it helps me I get most of the information that I need from the text book*". This response seemed to suggest complete reliance on the textbook. However, the textbook she was using (that is, *Siyavula*), as is the case for many other textbooks, merely presented proofs without giving its reader opportunities to get some insights into the functions of proof through conjectures. Stylianides (2009) suggests that mathematics textbooks can play an important role in providing learners with opportunities to engage in proof. This approach in textbooks seemed to promote the verification function of proof and was similar to that in Japan situation and did not seem to embody the Specific Aims advocated in CAPS. She also insisted that textbooks were reliable sources of mathematical knowledge becaue they were written by mathematicains:

> *I think as students we can .we rely on … on the text books that are written by mathematicians so by following their way of … of proving it is almost guaranteed that you are on the right track you … you don't just come up with your own.*

Of concern was her attempt to define a mathematical theorem as something that can be true or false when, in Lakatos' (1991) terms, theorems are seen as not-yet-falsified conjectures. Most importantly, I concurred with Hanna (1995) that mathematical proof only provided 'contingent truth, rather than absolute or infallible truth, in the sense that its validity hinges upon other assumed mathematical truths' (pp. 46-47). She points out that mathematicians, much as they would like to avoid errors, were as prone to making them as anyone else, in proof and elsewhere. As emphasised by Hofstadter (1997), 'any redblooded mathematician would scream murder at me for referring to a "fact" or "theorem" that I had not proved' (p. 10). While she emphatically mentioned that she goes through the textbook four times a week, it seemed that it was not helpful. In terms of the van Hiele levels, she had not mastered the relevant language for deduction as her definition of a theorem is different from the textbook which considered a theorem to be a hypothesis (proposition) that could be shown to be true by accepted mathematical operations and arguments. What she missed was that only once a statement is deductively proven to be true, it is referred to as a theorem. Also, whereas the textbook defined a proof as the process of showing a theorem to be correct, for *Presh N*, a piece of paper stamped "proof" on a bank letterhead was similar to a mathematical proof.

Although there was evidence of participant attempting to understand mathematics as pattern forming, frequent reliance on textbook tended to limit opportunities for engaging in argumentation in proving lessons so that they could support own or critique other's ideas as they attempted to do proof. *Presh N*'s beliefs about the functions of proof did not only seem to lack a coherent framework, they were also inconsistent with those of contemporary mathematicians. She harboured both adequate and inadequate views such as that mathematics is an activity of finding and studying patterns and relationships yet indicating that she dependent on the teacher who has final authority on mathematics. These findings were consistent with those of Schoenfeld (1989) and Hoyles (1997) in showing that learners who were able to do proof held beliefs that empirical evidence is mathematical proof.

7.3.5 Theme 5: Empirical arguments

The review of literature led me to expect that *Presh N* was inclined to confuse empirical arguments with everyday reasoning (Figure 7—1). I was pleasantly surprised to find that this phenomenon was the least prevalent. The finding was surprising in that there was very little that seemed to suggest even for one moment that she had a challenge to distriminate between empirical arguments and deductive proof.

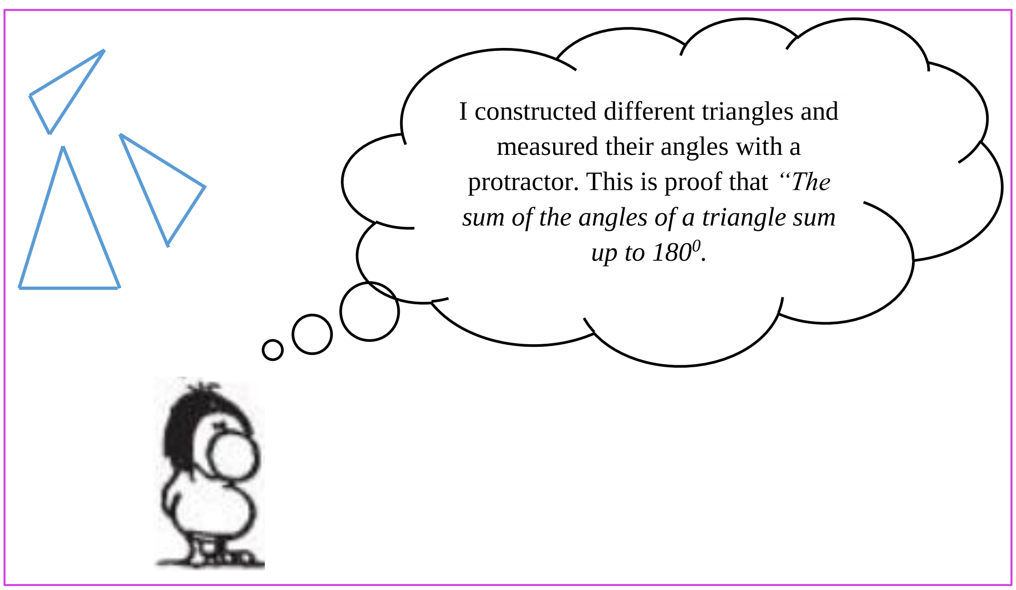


Figure 7—1 Checking Presh N's perspectives on influence of empirical arguments

Her views were reminiscent of Lakatos' (1991) notion of deductively proving as connecting new assertions to a set of previously proven theorems. To illustrate this point, when asked to comment on whether making an observation of several cases yielded a proof, this is what she said:

> *Mmmh...not completely because the learner only relied on the protractor to come up with their conclusion but aah... his conclusion is not supported by statements or other theorems that were … it's not proof enough.*

She seemed to understand that, in proving, it is virtually impractical to investigate or exhaustively measure all cases to which a statement applies – she demonstrated her appreciation of limitations of informal proofs. The response suggests that some learners are aware that checking a few cases is not tantamount to proof; they are aware that checking more varied and/or randomly selected examples does not constitute a proof. This result seemed to contradict findings that learners had difficulty in making a distinction between inductive proof and deductive proof (for example, Chazan, 1993; Schoenfeld, 1994). In addition, this result seemed to undermine Van Asch's (1993) attempt to explain why learners seemed to prefer empirical arguments over deductive arguments:

> It is beyond any doubt that the natural way of learning is an inductive way … Examples and counter-examples play a dominant role in this process … A deductive approach is in fact in conflict with this natural course of things. (p. 312)

Segal's (2000) agrees with this view thus:

> [E]mpirical proof, rather than deductive, are closer to everyday reasoning, in which, for example, a general result may be induced from a set of particular instances or inferred from a set of particular instances or inferred from an archetypical specific instance, ignoring any instances which differ too greatly from the archetype. (p. 196)

Presh N's survey results on the systematisation function were not only interesting but also consistent with her interview views on what it meant to perform a deductive proof. In seven items, she positively answered five of them (71%). Most notably, she correctly disagreed with the statement that "Proving does not require one to decide which axioms may be chosen as true." Generally, most learners struggle to move beyond Level 3 of the van Hiele theory of geometric thinking because they tend to regard empirical arguments as proof even after providing a deductive proof. It was interesting to find a learner who could distinguish between informal proof and formal proof; this result provided hope that doing deductive proof may not be that daunting a task for instruction. That is, learners can be able to appreciate that a deductive proof guarantees safety from counterexamples (Chazan, 1993). It is therefore reasonable to suggest that she seemed to have a good grasp of the systematisation function of proof.

7.3.6 Theme 6: Deductive arguments

Triangulation of data resulted in the emergence of a new factor influencing learners' functional understanding of proof: beliefs about deductive proof. I triangulated *Presh N*'s data from three sources: interview, survey, and proof-related task in search of regularities, clues of corroboration in forming themes, and reduce bias and misrepresentation of views by participant. In doing this, I was mindful of Patton's (2002) caution that inconsistencies arising from triangulation of data need not be seen as weakening the evidence, but as an opportunity to uncover deeper insight into the relationship between the data and the phenomenon under study.

In the interview session, when probed to explain why she thought that proof is a means to verify, this was her response:

> *Because when you are proving you ... bringing together the ideas you have or you know about aah.... in this case geometry things you've learnt from previous grades putting them together things … that are relevant to what you are trying to solve …*

This response was contradictory to Chazan's (1993) finding that learners believe that 'deductive proof is simply evidence' (p. 362). To the contrary, she showed an appreciation of the fact that a deductive 'proof confers universal validity to a statement' (Hadas et al., 2000, p. 128). However, these views were contradictory to what she espoused earlier in the interview. In one moment she seemed to suggest that because in the letter from the bank steps were taken to have "*the name of the bank, the branch, the date and that it's the proof*". She indicated that for her, the two are "proofs" on the basis that "*they also took steps*" to reach a conclusion. She seemed to hold a misconception about the word "step". The steps taken in mathematical proof are different in that previously agreed statements were presumed as true by the mathematical community while in everyday proof the steps followed were not subjected to consensus and therefore could not be regarded as knowledge. Other institutions may decide to follow completely different steps to do the "proof" and still insist that they had provided a "proof". For instance, rather than relying on checking the client's identity (ID) number, biometrics could be used. Thus, the conflation is in the word "steps" as used in the mathematical context and everyday life. She also seemed not oblivious to the fact that even a single case, rather than a few, provides evidence in everyday life.

An examination of her responses to the LFUP questionnaire is fascinating to note. She strongly disagreed with item T2 of the verification function, "Some maths propositions are true even if they have not been verified to be so by proof." *Presh N* seemed to lack some understanding of the history and development of mathematics. Her expression of disagreement with this item was inconsistent with the interview finding that for her, evidence constituted proof. Generally, humans believe many phenomena to be true even in the absence of proofs. For example, there is no proof that the sun will rise from the east the next day yet we believe it will on the basis of previous experiences or evidence. That notwithstanding, it is important to note that this evidence is empirical therefore does not constitute a proof in the mathematical sense. *Presh N*'s response gave credence to Lockhart's (2002) point that mathematics is one of the few school subjects that is taught without reference to its history and philosophical underpinnings.

The history of mathematics has shown that theorems are eventually true on the ground that no counterexamples have emerged. For example, Fermat Last Theorem was referred to as a theorem long before a proof was found; because no one had found a case that served as a counterexample to it, for over three centuries. Another classical example of this phenomenon is the Riemann conjecture, which is a statement about a mathematical curiosity known as the Riemann zeta function. If proven, this function will be used to predict where each prime number will fall on a number line and how many primes exist below a given number. Prime numbers are scattered in an inscrutable pattern across the number line. Hundreds of researcher mathematicians continue to seek a proof for this conjecture; so far none of these proofs have stood up to scrutiny. Once found, the proof of this conjecture will not only illuminate the prime numbers, but will also confirm many mathematical ideas that have been shown to be correct assuming the Riemann conjecture is true (for example, Griffin, Ono, Rolen, & Zagier, 2019). That is, a vast number of additional mathematical propositions have been derived from this conjecture because no counterexample has been found.

Therefore, in Bayesian sense, absence of counterexamples and proofs of corollaries to a proposition strengthens the truth of a proposition. The Bayesian theory is based on Bayes' theorem, which provides a mathematical way of strengthening a hypothesis given new evidence and other rational considerations (Nussbaum, 2011). The following section provides empirical evidence that deductive arguments were associated with *Presh N*'s functional understanding of proof in mathematics. This evidence arose from corroboration of interview results with survey and written homework assignment findings. This evidence created a complete picture (presented as a proposed model) to describe the factors affecting *Presh N*'s beliefs understanding of the functions of proof.

The "discovery" of deductive arguments as a factor influencing *Presh N*'s functional understanding of proof prompted the need to provide an updated analysis. The frequency in which these factors occurred in the interview with *Presh N* is shown in Table 7—1. The categorisation of the factors affecting *Presh N*'s understanding of the functions of proof were scrutinised by a "debriefer" (a fellow doctoral student) for the purpose of uncovering possible taken-for-granted biases, perspectives, and assumptions (Lincoln & Guba, 1985). After discussions with the debriefer, consensus on the categorisation is reached.

In addition, I asked a doctoral researcher at the same institution as I to manually recode *Presh N*'s transcript based on the five predetermined categories: teacher; semantic contamination; empirical arguments; deductive arguments; textbook; and collectivist culture. After a discussions with the doctoral student, consensus on the categorisation was reached. However, the sixth category, "deductive arguments", was not identified by the student. It is reasonable to deduce that the reason for this mismatch is that I primarily employed ATLAS.ti as an analytical tool which, as pointed out earlier, has the capacity to unveil hidden themes. At a more practical level, the student did not have access to the other data collecting instruments (for example, survey and written work data).

Table 7—1. Criteria for the categorisation of factors accounting for Presh N's understanding of the functions of proof

Factor	**Judgement**	**Description**	Frequency
Instructional practices	Naïve	Transmission of facts theorems; valuing of right answers over understanding.	2
	Hybrid	Both memorisation of facts and conjecturing.	0
	Informed	Encouraging conjecturing over memorisation.	0
Semantic contamination	Naïve	Evidence as definitive proof.	1
	Hybrid	Evidence as spontaneous knowledge.	0
	Informed	Evidence is not equivalent to proof.	1
Empirical-deductive arguments	Naïve	Informal proof is proof.	1
	Hybrid	Observation is necessary but insufficient for proof.	0
	Informed	Only deduction provides proof.	1
Collectivist culture	Naïve	Memorisation of proof.	3
	Hybrid	Memorisation of axioms to facilitate proof.	0
	Informed	Knowledge of the functions of proof.	4
Textbook	Naïve	The textbook provides theorems and their proofs.	2
	Hybrid	Riders in textbook provide the basis for construction of proof.	0
	Informed	Textbook need to cover the idea of proof broader by encouraging conjecturing.	0

On the basis of the exemplary excerpts presented above, the model depicting the factors influencing *Presh N*'s informed functional understanding of proof is shown in Figure 7—2.

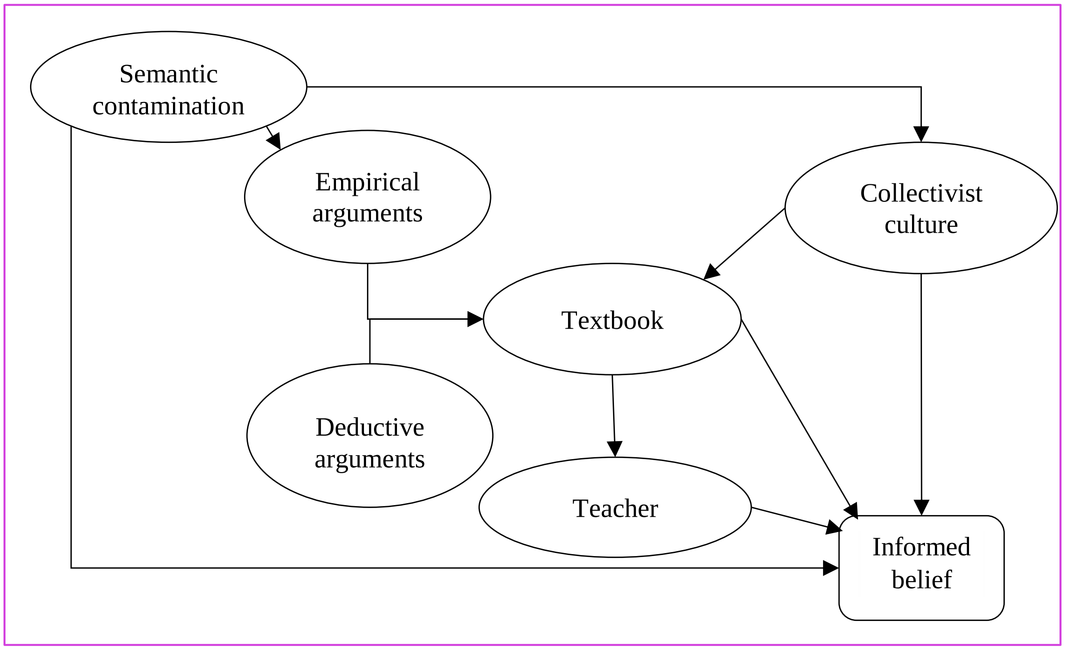


Figure 7—2. Preconceived conceptual framework for Presh N

7.4 Analysis of proof-related task

The van Hiele's (1986) theory of geometric thinking was used to understand and characterise *Presh N*'s work. In particular, her work was analysed in terms of Level 3 (informal deduction) and 4 (formal deduction) of the theory. This approach was informed by the instruction in the task, "Prove the **proposition** that the sum of the interior angles of a triangle is equal to 180 degrees". Working like a mathematician, *Presh N* should be skeptical about this proposition and try to find a counterexample, which will disprove the proposition. It may happen that the proposition is true, so it is not obvious in which direction to go. One counterexample is sufficient to conclude that the proposition is not true, even though there may be many examples in its favour. The key in this task is the value of counterexamples; they would save *Presh N* time and effort. For instance, for a long time mathematicians tried to find a formula that would generate only prime numbers and it was believed that numbers of the form $F_n = 2^{2^n} + 1$, where n is a non-negative integer, are all prime, until Euler found a counterexample which showed that for $n = 5$, F_n is composite.

In addition to van Hiele's theory, to analyse *Presh N*'s work at Level 3, I also drew on Edwards' (1997) notion of "the territory before proof" to describe the kind of thinking that was expected to be in *Presh N*'s work. Having read the question ("prove a proposition"), the next step for her would entail validating the proposition by using inductive reasoning. The inductive reasoning stage would involve checking specific cases (that is, engaging in naïve empiricism) to see if the conjecture holds true under testing and exploration with counterexamples. This stage was to produce mistakes, errors, thrill, joy, even the pain and frustration, all of which would allow her to experience the creative side of mathematics and thus demystify the view that mathematics is just the manipulation of numbers and equations. More importantly, it is at this inductive reasoning stage that one of the persistent misconception in proofs manifests itself as identified by Klymchuk (2010), Mason and Klymchuk (2009): learners at times become convinced that a converse of a theorem can be used as a counterexample to refute a conjecture.

Zaslavsky and Ron (1998) also found that many learners were convinced that single counterexample is not sufficient to refute a false mathematical generalisation. As Edwards (1997) points out, inductive reasoning is a very commonsense and everyday way of thinking. However, more than that, it often forms the basis for building a sense of conviction about the truth of a conjecture (de Villiers, 1990). Edwards (1997) further argues, the validation of a proposition entails using two kinds of reasoning: inductive and deductive reasoning. It is during inductive reasoning that verification by using several cases and formulation of considering counter-examples to disprove a conjecture takes place.

For Level 4 analysis, if no counterexamples could be generated by *Presh N*, then a deductive proof would be constructed. A deductive proof needs to be constructed to show why the generalisation must hold by utilising previously accepted objects (results) of mathematics such as definitions and axioms. According to Level 4 of the van Hiele (1986) theory of geometric thinking, an understanding of the functions of proof is partly indicative of a learner's level of geometric maturity. However, their ability to construct a proof in the Euclidean sense using appropriate language completed the acquisition of this level. Thus, *Presh N*'s written work (Figure 7—3) communicated her ideas in the task designed to gauge her competency in proving a proposition and systematise the system of axioms.

In attempting to the task, the participant was not only proving a proposition but also revealing her thinking. Mention must be made here that the analysis was based on the proof she presented; she was asked to prove a proposition (a conjecture whose actual proof is under construction). Her response to the task was a manifestation of the problem she held with the term "prove". This result was consistent with that of Mejía-Ramos and Inglis (2011) who found that the technical meanings of the two main linguistic ways of representing the concept of proof, "proof" and "prove" are not distinguished from their everyday life use in natural language such as ordinary English; they evoke different meanings in different people. In addition, *Presh N*'s behaviour in this task reflected experiences of having been exposed to teaching routine, prevalent in too many mathematics classrooms, which focused only on the final, polished mathematical product (proof) without showing its evolution. Perhaps the question should have been posed differently and more precisely and thus direct *Presh N*'s attention to the term "proposition". In this regard, de Villiers and Heideman (2014) suggestion was found to be reasonable:

> For example, instead of the usual "Prove that …"it would be pleasing to see the more mathematically authentic version: "Explore whether the following conjecture is true or not. If true, prove it. If false, produce a counter-example." (p. 26)

An analysis of her written work at Level 4 revealed interesting results. She seemed to have partially reached Level 4 of the van Hiele theory of geometric thinking in that she deductively proved that the proposition is indeed true. She accomplished this task by systematising axioms and all previously proven theorems to explain why the proposition is true. According to the proof provided, it seemed that she has also mastered the necessary language for Level 4 as suggested by the van Hiele theory. For example, she used mathematically appropriate language and symbols in her construction of the proof from start to finish; the construction was represented by a broken line segment and the proof, although the proof was presented in the dogmatic two-column format.

However, her proof resembled a memorised piece of work resulting from drill work or a textbook copy in that, rather than beginning some empirical exploration showing how the conjecture arose and then seeking counterexamples for a conjecture as mathematicians do, *Presh N*'s work began with axioms and definitions, what Lakatos (1991) refers to as the standard view of a proof. Similar to the finding of Stylianou et al. (2015), she displayed a disposition towards reasoning in a deductive manner, although her proof was not complete.

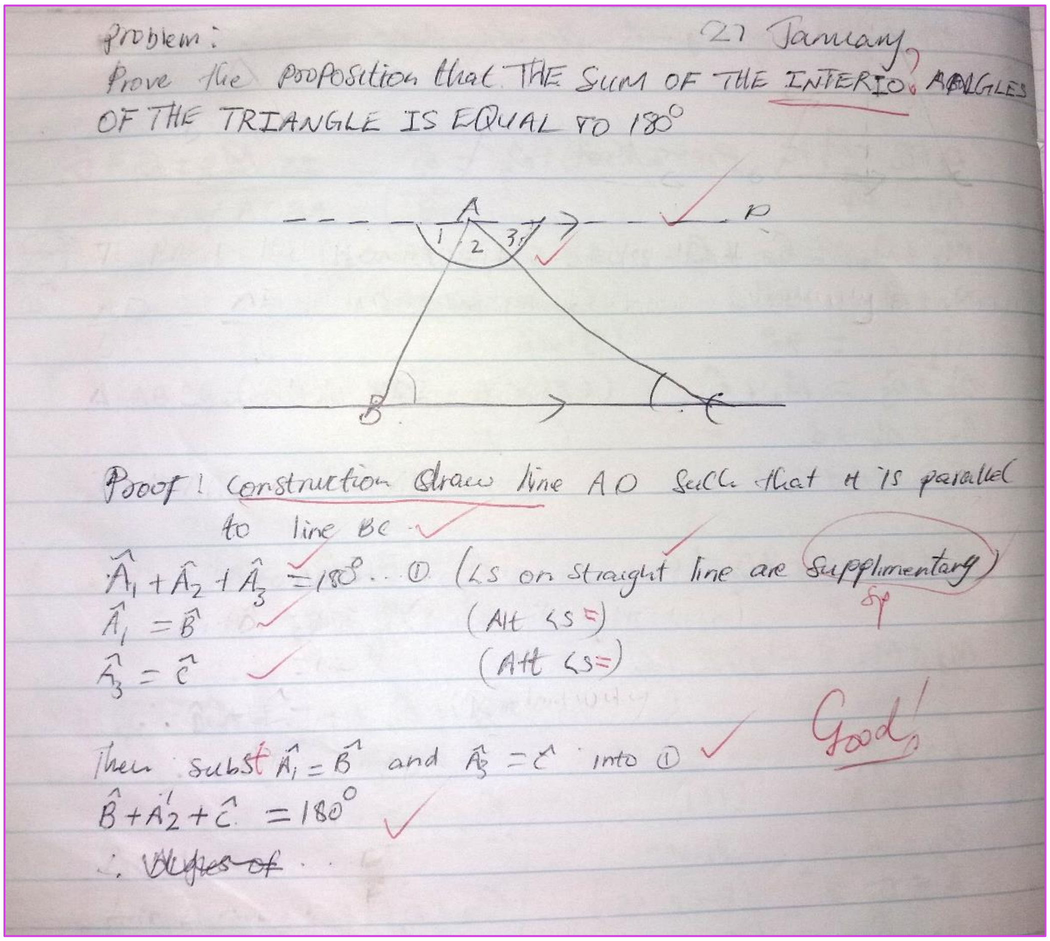


Figure 7—3. Presh N's baseline work on Euclidean geometry

Although I could hardly find fault with her proof, I questioned its origin. As Lakatos (1991) points out, this standard view of proof did not only hide the importance of conjectures and counterexamples, but also distorted the discovery and development of mathematical knowledge and an understanding of the functions of proof. As Lockhart (2002) further argues, standard high school geometry curriculum slowly and painstakingly deflates in learners 'any natural curiosity or intuition about shapes and their patterns by a systematic indoctrination into the stilted language and artificial format of so-called "formal geometric proof" (p. 18). Yet, according to a longitudinal study by Healy and Hoyles (1998), learners were able to formulate conjectures if they begin by using empirical arguments in which they use particular cases to arrive at a generalisation. However, *Presh N*'s achievement of Level 4 of the van Hiele theory was interpreted as an indication that the ability to prove the truth of theorems contributed to holding informed understanding of the functions of proof in mathematics.

This result was the same as for Healy and Hoyles (1998) who found that '[m]ost students appreciate the generality of a valid proof' (p. 3). Worthy to note is that the result was similar Mason's (1998) finding that the levels are not sequential as claimed; *Presh N* seemed to have developed logical reasoning skills having skipped informal deduction mastery. This result seems to confirm the lack of experiences with inductive arguments and a fuller meaning of genesis of a proof. However, this result contradicts van Hiele's (1986) assertion that 'the transition from one level to the following is not a natural process; it takes place under influence of a teaching-learning program' (p. 50). In the next section, a conceptual framework for thinking about factors influencing understanding of the functions of proof is proposed.

7.5 The proposed model

The ultimate goal of the analysis was to determine and describe the relationships among categories by discovering patterns in the data (McMillan & Schumacher, 2010). The interrelationships among the factors in this model were justified by references to existing empirical evidence. It is worth mentioning that I found *Presh N*'s views to be strikingly well thought out; she requested clarity when it is necessary and although openly shared some beliefs which ran contrary to some commonly accepted notions about the functions of mathematical proof. Analysis of the interview transcript gave rise to a free-form model for reporting the findings in the qualitative component of the study (Figure 7—4). In the model, developed form the interview quotations and codes

(Appendix C3), not all arrows will have the same thickness. Bold arrows such as the one pointing from empirical arguments to instructional factors denote the strong influence of one component on another. Data which could be coded into more than one category enabled the identification of linkages and subsequent networks between categories (Cohen, Manion, & Morrison, 2011).

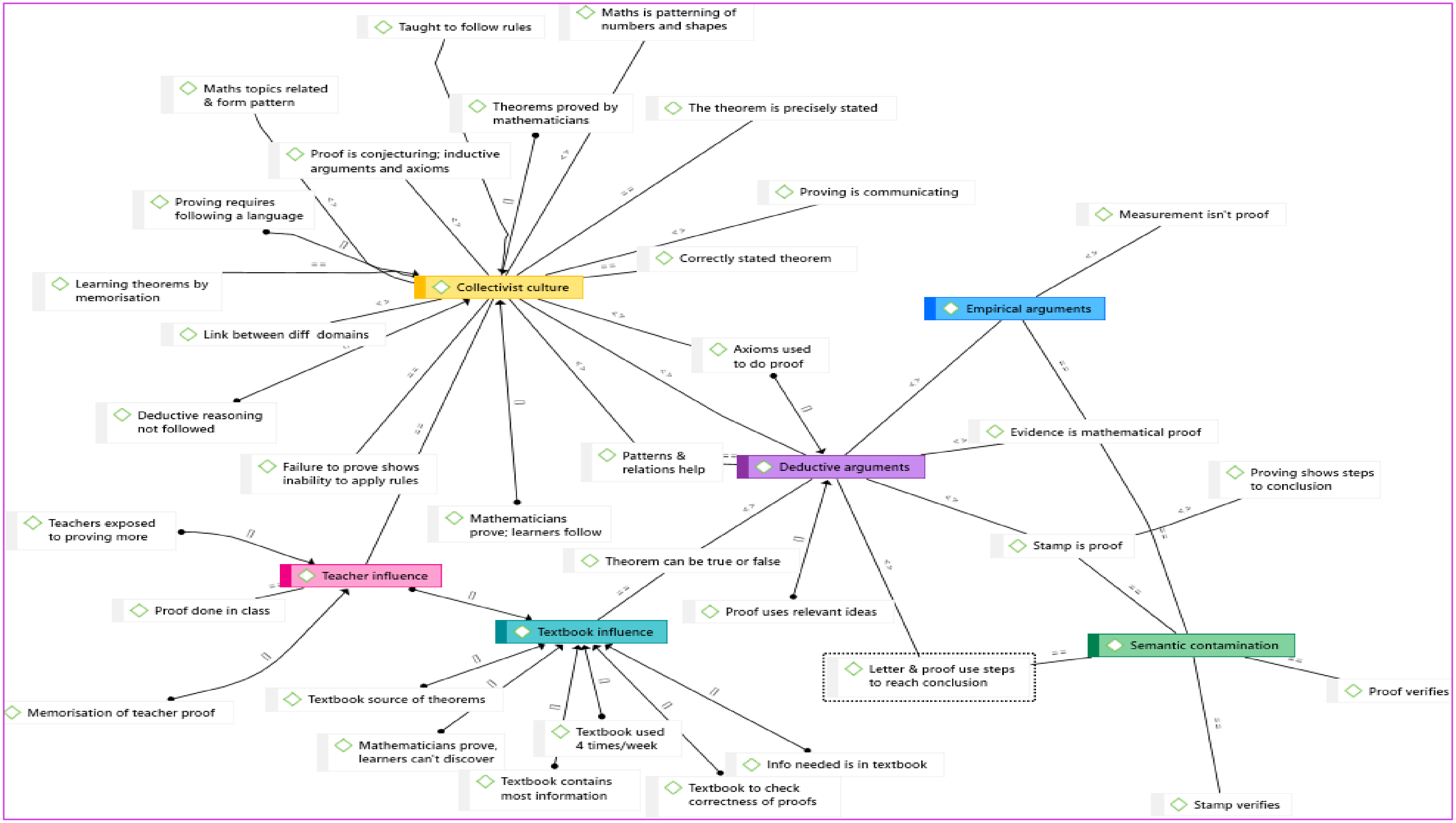


Figure 7—4. The six-theme model on factors shaping functional understanding of proof

As already mentioned, prevalence of a theme was determined by a connection between what the participant stated as important. Thus, the importance of a category depended on its ability to answer the research question. *Presh N* was invited for member checking of the transcript which needed to be subsequently modified after a discussion with them (Yin, 2014). In the write-up stage, key excerpts oriented towards the factors that mediate learners' understanding of proof were extracted for each category and described. Also, this stage entailed looking into the memos created to check for recorded comments that may evolve into important ideas, reviewing of literature to confirm findings, and categorising thus adding to the body of research literature (Corbin & Strauss, 2014). In the next section of this concluding chapter, I drew conculsions by primarily engaging with the research questions and suggesting directions for future research, considering limitations, making recommendations, and providing a brief reflection on the thesis project.

The final version of the model in Figure 7—5 depicts my conception of how the variables in the model related to each other. Given that *Presh N* attended an under-resourced school yet showed some glimpses of informed functional understandings of proof, school resources were seen as having a little role in the acquisition of informed functional understandings. However, this aspect of the model was not directly determined. Hence, a thinnest arrow. In contrast, evidence from the interview suggests that the collectivist culture influenced her functional understandings the greatest; hence the thicker arrow. The thick arrows suggested that those factors seemed to have maximal influence on her appreciation of the functions of proof.

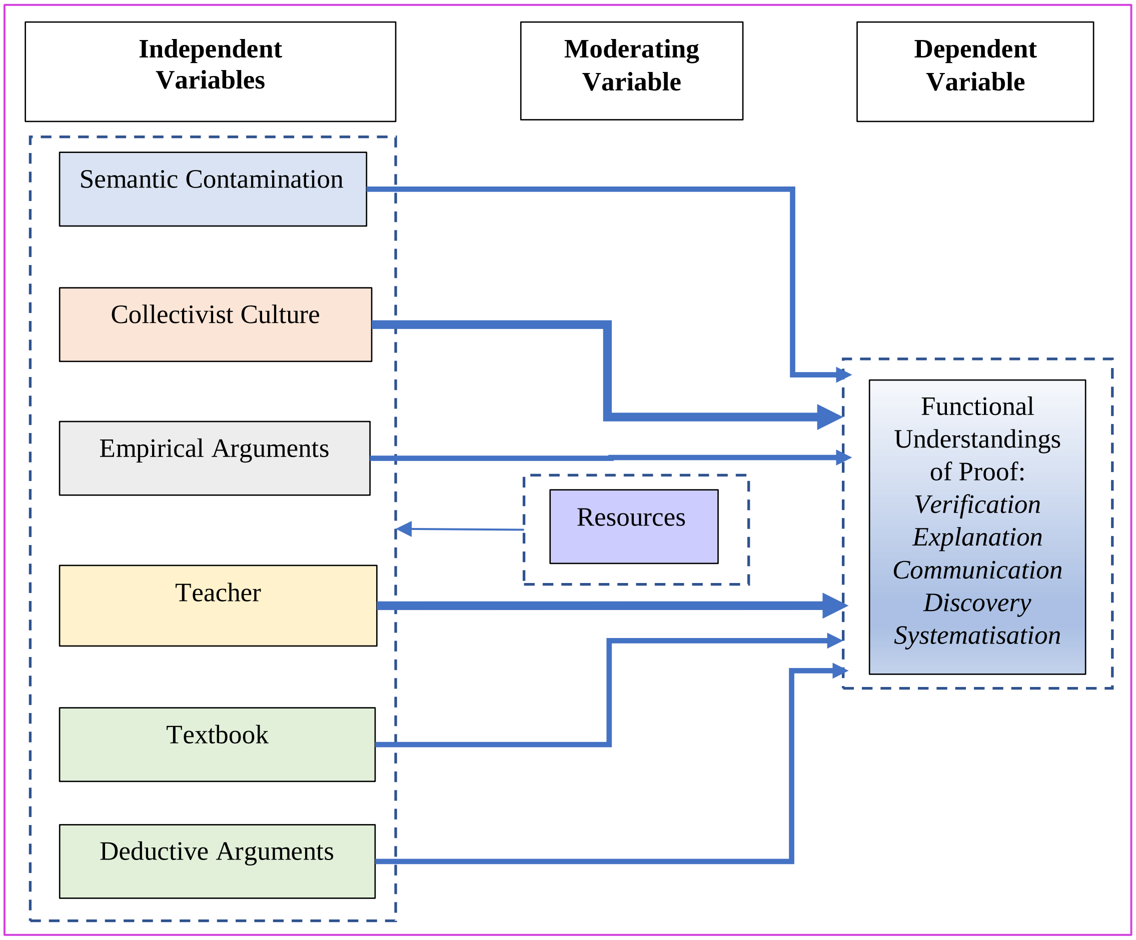


Figure 7—5. A theoretical model should linking variables with theory and concepts discussed in the literature review

In summary, after completing both quantitative and quantitative analyses, I provided an integrated analyses through contrasting and comparing them. I was able to identify and address the dissonances inherent in them. First, that learners could completely fail to make a claim yet others produce rebuttals, though interesting, is a demonstration of the fragmented ability to argue. Second, that on the one hand there is evidence of relational understanding of the mathematical ideas while on the other hand I found treatment of mathematics as a fixed body of knowledge showed the incosistency in learners' functional understanding of proof. That deductive arguments impacted learners' functional understanding of proof not only enriched current theory in the area of systematic investigations on factors influencing hybrid beliefs about the functions of proof, but also showed the fluidity of these factors. I noted how a learner can hold conflicting beliefs about the functions of proof yet argue in ways that were compatible with mathematical knowledge.

The relationship between argumentation and functional understanding of proof could only be described as weak. However, the relationship between functional understanding of proof and factors influencing beliefs about the functions of proof is taken as complex. Similarly, the relationship between argumentation ability and factors influencing beliefs about the functions of proof is deemed to be complex. The relationships were judged as complex because they were charaterised by a repertoire of contradictory stances. Overall, the results suggested an undesirable but coherent behaviour in that in all the three variables, middle ground views dominated. In the final analysis, I argued that the pressure to complete the curriculum and subsequent assesments in school mathematics not only contributed to undesirably incoherent overall results, but also contributed to the distortion of the mathematical practice. However, an investigation of the degree to which school mathematics contributed to the undesirably coherent interaction of the variables went beyond the scope of this primarily exploratory study.

7.6 Chapter summary

The main focus of this chapter was to elicit with a view to explain why *Presh N* held informed beliefs about the functions of proof. The factors influencing beliefs were identified in participant's utterances as represented in excerpts, survey, and task. I matched the themes with excerpts as supporting evidence. I found that the primary factors influencing naïve beliefs were: collectivist culture and teacher. Also, it emerged that the empirical arguments and the textbook weakly influenced functional understanding of proof.

From analysis of this interview data, three key findings stood out about *Presh N*'s underlying perspectives on factors contributing to her beliefs about the functions of proof. First, *Presh N's* views rarely navigated between informed and naïve beliefs; her beliefs about the functions of proof were generally informed. For some moments she articulated ideas that were consistent with mathematical knowledge and for some other moment her beliefs seemed to lean on beliefs that were inconsistent with mathematical knowledge. This observation demystified the notion that learners find it difficult to move from inductive proof to deductive proof. Next, knowing that inductive arguments are not sufficient for a proof did not immunise learners from believing that evidence outside of the mathematics domain is dissimilar to mathematical proof despite the fact that both use some steps to arrive at a conclusion. Last, I found that all the factors identified in literature and consequently explored in this investigation confirmed theory; they seemed to interfere with learners' ability to understand the functions of proof in mathematics. That deductive arguments play a role in functional understanding of proof is a finding that supported the hypothesis that a multitude of other factors influence the learning of functions of proof. Thus, this case study is valuable in that insight gained from it helped in formulating a hypothesis (that is, the model) that can be tested in future research using other methods.

As mentioned in Chapter 3, although a range of factors affected learners' functional understanding of proof, I concentrated upon the the six themes because of being circumscribed by the time constraints imposed on completeing the PhD project. I presented excerpts of the participant's perspectives (oral and written) on her experiences with the proof concept to explain statistical results in more depth. I answered both quantitative and qualitative questions by collecting and analysing data and interpreting results separately. In the next chapter, I integrate the qualitative and quantitative findings into a single discussion in which the interaction among the constructs of functional understanding, argumentation, and factors influencing functional understanding of proof in mathematics is explored from exploratory and interpretive perspectives. These findings are from the two questionnaires, interview, and proof-related task.

Chapter 8

Exploring the interaction among the three constructs

> In the classroom, the teacher and the textbook are the authority, and mathematics is not a subject to be created or explored. In school the truth is given in the teacher's explanations and answer book: there is no zig-zag between conjectures and arguments for their validity, and one could hardly imagine hearing the words *maybe* or *perhaps* in a lesson. (Lampert, 1990, p. 32)

8.0 Introduction

In the three preceding chapters I presented, analysed, and discussed results independently using the first three research questions as an organising framework. The focus of this chapter is on providing an overall picture of the interaction among these three constructs: learners' functional understanding of proof, argumentation ability, and factors influencing functional understanding of proof. In short, this chapter integrates the results of the quantitative and qualitative phases to provide a discussion of the outcomes of the entire study. The purpose of mixing the results is to seek a common point at which the three constructs interacted. Utilising pattern matching technique, the following research question was posed, *"How is the interaction among the three constructs (that is, functional understanding of proof, argumentation ability, and factors influencing functional understanding?"* As already mentioned, the analysis of the findings is exploratory because the nature of this interaction was unknown at the start of this investigation. Also, the analysis of the findings was interpretive given that the focus of the analysis of this interaction was to make sense of the meaning of the findings.

8.1 Background to the findings

The validation of the LFUP instrument facilitated the endeavour to characterise learners' understanding of the functions of proof as either naïve, hybrid or informed. To refresh the reader's memory, in this study, by naïve understanding is meant beliefs about the functions of proof that were characterised by misconceptions. In contrast, by informed understanding is meant beliefs about the functions of proof that were consistent with those held by contemporary mathematicians. A hybrid understanding related to a mix of naïve and informed understanding.

The impetus behind the present study was de Villiers' (1990) insightful link between understanding the functions of proof in mathematics and doing proof meaningfully. The validation of the LFUP instrument was not only to identify the functions of proof that best predict learners' functional understanding of proof and argumentation ability but also to identify the single case, *Presh N*, for the interview process to explain the genesis of her understanding of the functions of proof. The discussion below focuses on the degree to which the research questions that guided this study were answered.

These results reflected Watson's (2008) lamentation that school mathematics is not a subset of the discipline of mathematics. She points out that, among the features that distinguish school mathematics from the mathematics as practised by adult experts is that, for the learners, empirical arguments are privileged over deductive reasoning; seeing proof as empirical argument lingers as a dominant image of the function of proof. The emphasis on the need to ensure that learners' functional understanding of proof are consistent with those of contemporary mathematicians stems from Wu's (2006) assertion that if school mathematics is isolated from mathematics discipline then the former will evolve into 'something that in large part no longer bears any resemblance to mathematics' (p. 1882). Schoenfeld (1994) and Zaslavsky, Nickerson, Stylianides, Kidron, and Winicki-Landman (2012) agree by asserting that proof in school mathematics should be guided by its functions in the mathematics discipline itself so that mathematics learners can gain experience reasoning in the same way as mathematicians do. I believe that it is approaches like this that contribute to the harmonisation of school mathematics and the mathematics practiced by experts in the field.

Although the South African curriculum seeks to promote a "humanist" approach to mathematics where both empirical and logical arguments are emphasised, the pressure to finish the curriculum within specified time period to write examinations constrains these noble ideals which characterise the discipline of mathematics. The lack of efforts to address this dilemma on the part of the education authorities is indicative of what Watson (2008) articulate as a moulding of the discipline of mathematics ‘to fit institutional constraints rather than fit the development of mathematical ideas’ (p. 6). Thus, the inherent time constraint would see to it that the humanistic approach remained just that, and ideal.

8.2 Discussion

The analysis of the findings was interpretive given that the purpose of the analysis of this interaction was to explore the meaning of the findings (Caracelli & Greene, 1993). Thus, the analysis of the findings involved making a personal assessment as to a description that fits the situation that capture the major categories of information (Creswell, 2014). The personal nature of this assessment means that I brought my own perspectives to my interpretation of the three findings in this study. Therefore, the interpretation that I made of these findings will most probably differ from the interpretation that the reader makes. That said, it is important to note that the discussion was undertaken with respect to two different theoretical frameworks underpinning this study: argumentation and van Hiele theories.

In Chapter 2, I identified a plethora of studies pointing to the prevalence of verification or empirical arguments as mathematical proof. However, most of those studies were conducted in Western countries. To explain what influenced *Presh N*’s informed beliefs about the functions of proof, I had to first survey to describe learners’ functional understanding of proof in mathematics. Research studies identified naïve beliefs about the functions of proof as hindering the understanding the functions that proof performs in mathematics and by extension the construction of proof meaningfully (CadwalladerOlsker, 2011; de Villiers, 1990; Hanna, 2000; Healy & Hoyles, 1999). Although the answers to each of the quantitative and qualitative phases were presented and analysed separately, they are integrated in this discussion section. I recap on the research problem as captured in the research questions, relate the findings to previous research including those that motivated the present study, describe how these findings compare and contrast with previous research, and carefully take into account all other possible explanations of the results. The exploration of the interaction among three constructs is organised in terms of these three research questions:

- *What functional understanding of proof do Grade 11 learners hold?*

- *How is the relationship (if any) between learners' quality of arguments and functional understanding of proof?*
- *Why does Presh N hold informed beliefs about the functions of proof?*

8.2.1 Interaction between functional understanding of proof and argumentation

As the results reported in Chapters 5 and 7 showed, learners held hybrid understanding of the functions of proof and that the relationship between learners' functional understadings of proof and their argumentation ability was statistically significant ($r = .225; p < .01$). It was interesting to find that the explanatory function of proof can be used to predict learners' functional understanding and argumentation ability. These conclusions were apparent from learners' responses in the two self-administered questionnaires. In the interview, reported in the previous chapter, it was apparent that the collectivist culture and the teacher profoundly accounted for the informed beliefs about the functions of proof in mathematics that *Presh N* held.

According to the results in Chapter 5, the percentage of participants who demonstrated informed views on all Likert statements within the theme of explanation function theme of was zero. About 47% of the participants chose to "agree/strongly agree" with the statement, “A proof explains what a maths proposition means”. This lack of understanding of the explanatory function of proof was confirmed during the follow-up interview with *Presh N*. When asked to state the various functions that she thought proof performs in mathematics, she responded, “Proof in mathematics is about coming up with ideas and developing formulas by using things like theorems and measurements and hhh, Yeah.” She indeed seemed to have an inclination as to the fact that mathematical practice is a social endeavour, a discipline whose ideas are argued and developed on the social plane.

Analysis of responses on the LFUP instrument revealed that when all five the functions of proof were considered, learners seemed to harbour naïve rather than informed understanding of the functions of proof. Similarly, the WAEC (West African Senior Secondary Examinations Council) (2003) reports that most learners in high school examination in Nigeria proved that a triangle is isosceles by substituting numerical values that two interior angles are equal. Interesting to note is that prior to the report, the council has been recommending the provision of adequate teaching and learning materials to ensure qualitative teaching. Also, this finding is consistent with studies that have been reported over the past ten years in South Africa and beyond. According to Healy and Hoyles' (1998) findings, learners who subscribed to the pervasive notion that the sole function of proof is verification were reliant on memorising proofs. They further found that learners with little or no sense of what proof meant were more likely to choose empirical arguments. Harel and Sowder (1998) describe such understanding of the function of proof as a misconception.

Learners' response to the LFUP questionnaire seemed to suggest that the communication function of proof weakly contributed to the holding of informed functional understanding of proof. This could stem from lack of conjecturing as expressed by *Presh N*'s views that the learning of theorems was dependent on what the teacher told them. This result was not surprising given the Shongwe's (2019) finding that learners argued poorly; communication of mathematical ideas was poorly orchestrated by learners. Most probably because of lack of instruction in and experiences with argumentation. This probable statement is supported by Means and Voss (1996) whose results showed that learners did not know how to construct an appropriate argument. While *Presh N* appreciated the generality of a deductive proof, she also seemed to rely on the authority of her teacher and textbook as arbiters of the truth of mathematical knowledge. CadwalladerOlsker (2011) attributes contradictions in learners' approach to proof to the sociomathematical norms.

The finding that learners seemed to hold hybrid beliefs about the functions of proof is consistent with Schommer-Aikins' (2002) argument that learners' reliance on teachers and textbooks suggested that there is little conjecturing and argumentation in which views were compared, contested, and contrasted in mathematics classrooms. *Presh N* harboured both adequate and inadequate views such as that mathematics is a study of patterns yet indicating that she dependent on the teacher and textbook as final authority on mathematics. That 'the teacher and the textbook are the authorities and mathematics is not a subject to be created or explored' (Lampert, 1990, p. 32) was evident in *Presh N*'s experiences of mathematics in general and proof in particular.

The findings by Chazan (1993), Schoenfeld (1989), and Hoyles (1997) that learners are unable to appreciate that empirical arguments are structurally distinct from deductive arguments and that they seem to prefer empirical arguments over deductive arguments were contradictory to my finding. For instance, an interview with *Presh N* revealed that she understood that proving involved using mathematical language to communicate the relationships or patterns. In addition, her argumentation in AFEG was found to be of high quality. These two results further give merit to the existence of a significant correlation between functional understanding of proof and argumentation quality as found in Chapter 7. She demonstrated very little difficulty in understanding both the notion of proof as well as understanding what the mathematics discipline entailed.

In addition, *Presh N* could make this distinction between empirical arguments and proofs in mathematics clear. Rather, what clouded her thinking is making a distinction between steps in deductive arguments and steps in everyday arguments. It is on the basis of these findings that I suggest that the idea of the dichotomy, empirical arguments vs. deductive arguments, needed rethinking through further investigations. Consistent with the argument in this study, Chazan (1993) found that if instruction highlighted not only differences between measurement of examples and deductive proof but also that measurement of examples has limitations as a method for verifying the truth of geometrical statements, all learners in the urban areas preferred a deductive proof to an argument based on the measurement of examples.

This study was driven by the contention that functional understanding of proof helps learners to gain an appreciation of the subtleties of the practices and arguments employed in the building of mathematical knowledge which in turn motivates them to do proof meaningfully. However, building mathematical knowledge in this way would require recommitment to the Specific Aims enshrined in CAPS and the making of mathematics classrooms sites of argumentation and conjecturing to bring to bear the functions of proof in mathematics. Given these results, it is reasonable to suggest a discrepancy between the Specific Aims in CAPS and the actual learning outcomes about the functions of proof existed. In terms of the van Hiele theory, the results suggested that the Grade 11 learners' geoemtric thinking in these Dinaledi schools were at Level 3. Also, in terms of TAP scheme, the level of the quality of these learners' argumentation quality was low. Further, as part of the discussion here arose from the sociocultural theory of learning, the results suggested that *Presh N* portrayed learning of proof as absorption of others' knowledge rather than a human activity consistent with the SA of CAPS while professing the opposite.

8.2.2 Interaction between argumentation and factors affecting functional understanding

The practice of engaging in argumentation in mathematics is best supported in individualistic cultures where, unlike in collectivist cultures, the "why" question coming from a child is not frown upon but encouraged. The idea that learners' understanding of proof can be composed of different beliefs existing next to each other can be inferred from past studies. In this study, a mathematical statement or simply statement is a sentence in mathematics that consisted of two parts: assumptions or givens and then the conclusion. For instance, when a statement is referred to as a theorem it implies that its proof exists; until then it remains a conjecture. One can only be convinced of the truth of a conjecture. When asked to explain the meaning of the term theorem, *Presh N* responded that *"it is a statement that is true or false*". No evidence indicated that she understood the definition of theorem. I was inclined to believe that here was another example in which semantic contamination was at play and an opportunity for argumentation to be brought in to resolve the contamination.

However, in my view she could be forgiven for thinking this way in view of Fermat's Last theorem, that no integer $n > 2\, and\, x, y, z \neq 0$ satisfies the equation $x^n + y^n = z^n$, which remained a conjecture but was referred to as a theorem for a little over three centuries. This conjecture continued to be referred to as a theorem in the absence of a proof simply on the belief that it is susceptible to logical abstraction and more importantly, no counterexamples were found. Hence it is reasonable to think of mathematics as developing through faith. Again, it is important to provide learners with unambiguous definitions and use of mathematical objects to avoid the contamination of these objects with everyday life talk.

The conclusion I could draw from *Presh N's* classroom experiences, is that conjecturing opportunities were, if any, limited and therefore learners' ideas undervalued. Put another way, *Presh N*'s utterances revealed experiences of an authoritative mathematics classroom context where learners were traditionally reluctant to share their mathematical ideas and answers were sought either from the teacher of textbook. Thus, memorisation and adherence to rules seemed to play a dominant part in her learning experiences. From perspectives of typical instructional practices which are characterised by narrow assessment of competency, it is reasonable to suggest that her case reflected "successful" instruction. But, when a learner tends to believe everyday arguments and mathematical proof are the same on the basis that both require taking of steps to reach a conclusion, something has gone very, very wrong.

Schoenfeld's (1988) case study findings was that "successful" mathematics instruction can foster development of inappropriate perspectives on the nature of mathematics that impeded the acquisition and use of other mathematical knowledge. However, *Presh N*'s high quality of argumentation was indicative of the readiness of learners to engage in mathematical practices that reflected those practices of adult experts. For instructional practices, emphasis needed to be placed on the fact that that both empirical arguments and deductive arguments reach conclusions through following some steps, did not make them the same arguments.

In the South African context, the question is: "How will these Specific Aims and the different functions of proof in mathematics be realised in a mathematics curriculum?" In addition, an important insight gained is that learners' ability to understand that empirical proof is not proof is not going to surpass the understanding of this idea by the teachers themselves. That is, very few will contest the assertion that learners' beliefs were a reflection of instructional practices. If this conclusion is accepted, it is reasonable to suggest that learners are not provided with opportunities to develop in the ability to be methodical, to generalise, make conjectures, and try to justify them as envisaged in the South African mathematics curriculum.

Evidence to support this view was found in *Presh N* who, simply provided a deductive proof rather than begin with inductive arguments in which she would: make a generalisation, search actively for counterexamples, be wrong, be creatively frustrated, and be sufficiently convinced of the truth the proposition to look for its proof. Perhaps it is time that the definition of proof is reviewed so that "proof" is used to explicitly characterise not only deductive proofs but also empirical proofs. Support for this view is found in Mariotti's (2001) words in which she says that '[p]roving consists in providing both logically enchained arguments which are referred to a particular theory, and an argumentation which can remove doubts about the truth of a statement' (p. 30).

Heinze and Reiss (2003) conducted an interview study which served as a qualitative supplement of a large-scale quantitative study on secondary school learners' methodological knowledge involved in constructing proofs. The aim of their research study was to identify both cognitive and noncognitive factors which play a role in proof competencies of learners. They found that most learners appreciated that empirical arguments do not form a proof. This finding is similar to this study's in which *Presh N* explicitly made a distinction between these forms of arguments in proof. However, *Presh N*'s results seemed to be one of few exceptional cases judging by the LFUP results where learners were found to hold hybrid beliefs about the functions of proof.

In conclusion, the discussion of the results have generally showed that the curriculum is in a dilemma in that not only that learners in Grade 11 argued poorly, but also that a coherent understanding of the functions of proof consistent with those held by contemporary mathematicians is attainable yet impractical under the current system. Further, as found in this study, the explanatory function of proof could be used to develop learners' low argumentation ability. The tension between holding both an internal and external view of mathematics while retaining a healthy skepticism for empirical arguments as proof in mathematics reflected the complex interplay of attitude towards proof functions, argumentation, and the curriculum. Having explored the relationship between argumentation and factors influencing functional understanding of proof, I came to the conclusion that this relationship is "complex". The rationale for this judgement emanates from the inconsistences in *Presh N*'s actual behaviour in a proof-related task and her professed beliefs about the functions of proof in mathematics.

8.2.3 Interaction between factors influencing functional understanding of proof and functional understanding of proof

Results of research studies on the roots of the use of verification by examples as proof have been inconsistent. On the one hand, Kunimune, Fujita, and Jones (2010), in various studies they undertook in Japan, consistently found that most learners considered experimental verification as equally valid as a formal proof. They point out that even after intensive instruction in how to proceed with proofs in geometry, learners persistently believed that experimental verifications were enough to demonstrate that geometrical statements were true. They suggested that learners need to be shown the limitations of experimental verification and to make deductive proof meaningful for them. This suggestion needed some qualification in light of the finding in this study. The results were consistent with Schoenfeld (1989) argument that most learners who have had a year of high school geometry are "naïve empiricists".

Despite placing emphasis on proof as verification, explanation and discovery, Grigoriadou (2012) found that most of the learners participating in her study did not know what proof is both before and after an intervention. In contrast, the interview with *Presh N* in this study showed that even without intervention, it is possible for learners to appreciate the meaning of mathematical proof. Healy and Hoyles (1998) finding that learners preferred empirical arguments when they focused on convincing themselves of the truth of a statement. In addition to seeing empirical arguments as proof, learners seem to see the learning of proof as another practical necessity of meeting the teacher's expectations and passing examinations (Almeida, 2000). Indeed, according to the LFUP results, I found that 80% of the participants held naïve beliefs about the functions of proof and none was found to show extremely enculturated beliefs about the functions of proof in mathematics. This seemed to suggest that most of the learners held few of the views about the functions of proof that were consistent with those held by contemporary mathematicians.

Martin and Harel (1989) point out that learners often find the term proof indistinguishable from evidence. In my personal experience, evidence and proof were commonly used interchangeably in everyday talk. Healy and Hoyles (1998) found that, as I have from *Presh N*'s perspectives, learners who seemed to understand the functions of mathematical proof tend to struggle to abandon viewing empirical evidence as mathematical proof. My finding not only supported this point but also signaled the influence of the individualistic culture on learners' beliefs about the functions of proof. From a legal point of view, evidence is used as an attempt to argue that some illegal activity took place.

For example, suppose you arrive home and find your safe open and empty while a man whose pockets are full of bank notes is standing close to it. The open safe and the notes hanging from the pockets of the man are evidence that could be linked in an argument to show that the safe was tempered with and notes taken from it are those in his pockets. Here, proof rested on evidence. In contrast, speaking formally, as already mentioned in previous chapters, proof as a product is not mathematical knowledge derived from evidence gathered through observation of several cases, measurement or experience. Put another way, in proving, evidence is worthless as it is virtually impractical to investigate or measure all cases to which a statement applies. For example, proof for the theorem of Pythagoras can be completed without providing a single measurement of a triangle as evidence.

I concur with Ball et al.'s (2002) suggestion that instructional practices should strive for (1) a more refined perception of the functions of proof in mathematics, (2) a deeper understanding of the gradual processes and complexities involved in learning to prove, and (3) the development, implementation and evaluation of effective teaching strategies using carefully designed learning environments that can foster the development of the ability to prove. I personally would like to see more attention paid to developing curriculum materials that incorporated functional understanding of proof in mathematics as well as argumentation so that learners can gain insights into the nature of mathematics. I believe that the nature of mathematics is best reflected in the appreciation of not only the empiricist function, but also on a range of others that it performed in mathematics. This view found support in Otte's (1994) observation:

> A proof which does nothing but prove in the sense of mere verification must be unsatisfactory. A proof is also expected to generalize, to enrich our intuition, to conquer new objects, on which our mind may subsist. It is expected to renew our entire idea of what mathematics is. (p. 310)

In the final analysis, similar to the previous interaction, it was difficult to characterise the nature of the interaction between the factors influencing functional understanding of proof. Hence this interaction was described as "complex".

8.2.4 The "grand" interaction

The quantitative and qualitative results were mixed to more fully answer the research question, "*How is the interaction among the three constructs (that is, functional understanding of proof, argumentation ability, and factors influencing functional understanding?"* and 'develop a more robust and meaningful picture of the research problem' (Ivankova et al., 2006, p. 14). The grand (overall) result from mixing qualitative and qualitative findings was the common factor among the constructs in this study which could be summarised by the term, undesireable coherence, denoted by UC (Figure 8—1). I use this term to suggest that the constructs reflected "mixed" ideas which were unhelpful for the learner to develop a view of the mathematical practice. The description of the interaction among the constructs as coherent stemmed from the fact that learners' functional understanding of proof was deemed to be inconsistent (hybrid) with those held by research mathematicians, their argumentation quality was fragmented (not high or at least poor, but rather than adequate), and the factors accounting for beliefs varied from hypothesised to unexpectedly contradictory.

"Fluid" in this interaction was a term used to suggest that while six of the factors (for example, semantic contamination, teacher, collectivist culture, textbook, deductive arguments, and emprical arguments) were taken as affecting the development of informed functional understanding of proof, one factor (deductive arguments) was associated with the development of informed functional understanding of proof.

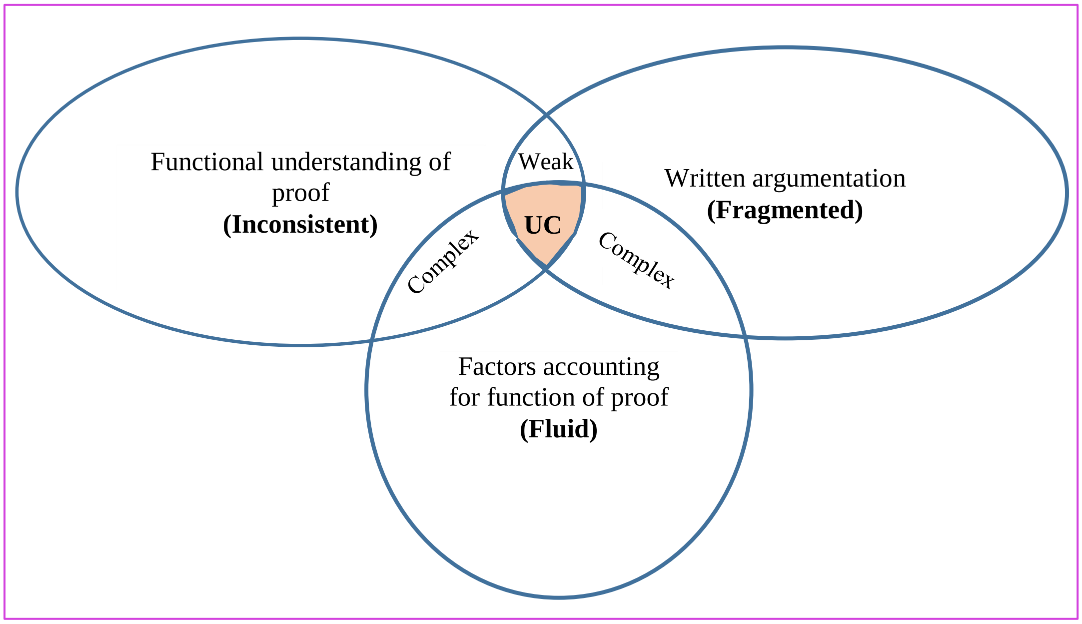


Figure 8—1. The overall description of the interactions between and among the variables investigated in the present study

That the interaction of these three constructs produced undesirably coherent relationship made this interaction not only interesting but also significant for practice. Interesting in the sense that knowing that a deductive proof is the final arbiter in pursuit of mathematical truths contributes to the development of informed functional understanding of proof. Significant in that to date, I am not aware of studies that reported on the interaction of these constructs. I found evidence that bridged the gap in the existing knowledge about the extent to which learners understood the functions of proof in mathematics and in the process validated the LFUP instrument thus strengthening its utility in practice and on research platforms. Also, I showed that learners' argumentation ability was associated with functional understanding of proof. In addition, not only did I find factors influencing learners' beliefs about the functions of proof, but also proposed a framework for thinking about these factors. In the final analysis, it was established that the point at which all the three constructs converged can be described as undesirably coherent (UC).

8.3 Chapter summary

In this chapter, the nature of the interaction between and among learners' functional understanding of proof, argumentation ability, and factors accounting for *Presh N*'s informed beliefs about the functions of proof in mathematics were interrogated. In each of these three constructs, the findings were described along a continuum: from naïve to informed functional understanding of proof (inconsistent); from low to high quality argumentation (fragmented); and, from beliefs that inhibited functional understanding of proof to beliefs that fostered functional understanding of proof (fluid). The interaction between functional understanding and argumentation ability was, though statistically significant, weak. The interaction between functional understanding and factors accounting for beliefs about the functions of proof as well as the interaction between factors accounting for beliefs about the functions of proof and argumentation ability were both described as complex. The analysis of the findings suggested that the common factor in the interaction could be classified as undesirably coherence (UC). This chapter, guided by drawing on the findings across quantitative and qualitative results, is the final and important stage in mixed methods research (Creswell & Plano Clark, 2011). In particular, this chapter entailed the judgements I made about the results/findings (factual information) in relation to the first three research questions.

Chapter 9

Conclusions

9.0 Introduction

In this study I planned to explore Grade 11 learners' functional understanding of proof, explore the relationship between functional understanding of proof and their argumentation ability, explain the sources of *Presh N*'s functional understanding of proof in mathematics and explore the relationship among three constructs (that is, functional understanding of proof, argumentation, and factors influencing functional understanding of proof). To this end, a mixed-methods sequential explanatory study was designed. The purpose of this sequential explanatory study was to systematically determine the factors affecting functional understanding of proof in mathematics. In the quantitative phase of the study, SPSS was used to describe the character of learners' functional understanding of proof in mathematics and explore their argumentation ability. In the qualitative phase of the study, ATLAS.ti and STATA were used to facilitate the pattern-matching analysis of the interview and proof-related data.

Quantitative analysis of data showed that learners held hybrid functional understanding of proof, argued poorly, the collectivist culture and the teacher impacted the gaining of functional understanding of proof, and the interaction among these constructs was described as an undesirable coherence (UC). There are three unique contributions that this study makes in the mathematics education literature. First, this study used a mixed methods design in which participating schools were randomly selected to improve the trustworthiness of the results. Second, this study validated a new measurement scale that allows teachers to gain insights into their learners' understanding of the functions of proof to facilitate the construction of proof meaningfully. Third and final, this study presented a model to understand factors influencing functional understanding of proof in mathematics.

In this this concluding chapter, I provide a summarised overview of the major findings of this study. Then, I highlight the limitations, recommendations, and implications of this study. I end this chapter and the study with reflections on the entire research process undertaken in this study.

9.1 Overview of findings in relation to research questions

The purpose of this mixed-methods sequential explanatory study was to identify factors contributing to functional understanding of proof by obtaining quantitative results from a survey of 135 Grade 11 learners at three Dinaledi schools and then following up with one purposefully selected learner to explore those results in more depth through a qualitative case study analysis. The research process provided an understanding of the participants' experiences and views of the concept of proof, its functions in particular. The overarching conclusion is that engaging in the nature of mathematics is a complex activity requiring time and equipment; resources schools lack due to the practices in school mathematics. This conclusion highlighted the dire inconsistency between actual classroom practice and the SA of CAPS.

9.1.1 Research question: What functional understanding of proof do Grade 11 learners hold?

The LFUP survey findings seemed to reasonably provide evidence that learners hold hybrid understanding of functions of proof in mathematics. This finding suggests that it is incumbent upon the Euclidean geometry teacher to create opportunities that encourage appropriate acquisition of functional understanding of proof. Then, principal axis factor analysis resulted in the validation of the LFUP questionnaire. This analysis provided strong support for a five factor structure for the 25-item LFUP scale, which will serve as a valuable tool for both teachers and researchers intending to capture and characterise high school learners' functional understanding of proof in mathematics. These five dimensions of learners' understanding of the function of proof are: verification; explanation; communication; discovery; and, systematisation.

9.1.2 Research question: How is the relationship (if any) between learners' quality of arguments and functional understanding of proof?

A correlation between learners' functional understanding of proof and argumentation ability was found; weak but statistically significant. In addition, the explanatory function of proof was found to be the factor which best predicted learners' success in argumentation ability.

9.1.3 Research question: Why does Presh N hold informed beliefs about the functions of proof?

The study also found that, consistent with previous studies, *Presh N*'s understanding of the functions of proof was not only influenced by the teacher but also the collectivist culture that permeate the South African education system and society. These results will help in beginning to understand some of the problems that beset Euclidean proof in particular and the education system in general. However, I echo Schoenfeld's (1994) view that '[p]roof is one of the most misunderstood notions of the mathematics curriculum, and I really needed to sort it out. What is it, what roles does it play in mathematics and mathematical thinking …?' (p. 75).

9.1.4 Research question: "How is the interaction among the three constructs (that is, functional understanding of proof, argumentation ability, and factors influencing functional understanding?"

The "grand" interaction among the three constructs underpinning this study was found to be undesirably coherent. Other than the relationship between functional understanding of proof and argumentation ability, the other two relationships were characterised by inconsistencies which led to their description as complex. For instance, *Presh N* disagreed with a statement that conjectures like Riemann's function—which, once proven, will enable us to count prime numbers—are accepted without proof simply because no counterexamples have been found and have been used to prove other mathematical ideas. For instance, at one point she appreciated that only deductive arguments constituted a mathematical proof. However, her behaviour in a proof-related task provided evidence that her understanding of what constitute a theorem and a proposition was flawed.

9.2 Limitations of the study

In this subsection I state the limitations of the study, which Merriam (2009) refers to as the factors that are beyond the researcher's control (for example, time and funding constraints) but threatened the trustworthiness of the findings of a study. This sequential explanatory study is confronted with four shortcomings which necessitated the viewing of its findings with caution. First, although learners' functional understanding of proof is inferred from data collected through Likert scales which inherently do not discriminate unduly on the basis of how articulate participants were (Wilson & McLean, 1994), I suggest the use of open-ended items requiring qualitative analysis to allow probing of responses to provide deeper understanding of the phenomenon (functional understanding of proof). Since the LFUP scale is at its infancy stage, further revision will most likely take place as it is used with more learners and teachers from other populations.

Second, although the choice of the research design adopted in this study is within my control, the inability to infer causality restricted the conclusions drawn in this study. For example, even with the correlational research question investigation the relationship between functional understanding of proof and argumentation ability, this is not possible. Third, the findings in this study could have limited application for Dinaledi schools with access to DGS which could have assisted them in coming to know that, however useful and powerful DGS may be in testing conjectures by dragging points, the conclusions so reached do not constitute mathematical proof. However, I could not make a conclusive finding about DGSs since they were outside the scope of my research problem.

Fourth and final, although ANOVA works even when the spread of the LFUP scores about the mean across the three groups of learners were unequal, the findings emanating from factorial analysis were to be treated with caution. This hurdle arose from the fact that the sample sizes of the three schools that participated in this study were unequal. Stricter observation of homogeneity of variance assumption in the factorial analysis of variance would have made the findings more reliable.

These limitations notwithstanding, this study will serve as a springboard for future research on proof, its functions in mathematics, and the factors influencing understanding of the functions that proof performs in mathematics. I believe that the findings provided important insights into how understanding the nature of mathematics could improve the meaningful construction of proof and learner participation and scholastic achievements in mathematics.

9.3 Recommendations

The present study adopted a sequential explanatory design in which, by definition, the primary emphasis was on quantitatively exploring the concept of functional understanding of proof. Although the semistructured interview provided valuable insights into why *Presh N* held informed beliefs about the functions of proof, making recommendations in this regard is beyond the scope of this study save to draw attention to the suggested conceptual framework to study factors accounting for beliefs about the functions of proof in future research studies. This study makes the following recommendations.

9.3.1 Recommendations for instructional practices

If it were accepted that holding informed functional understanding of proof and being able to engage in argumentation are two of the ways in which to avoid proof remaining a meaningless instructional activity, then there is a need for curriculum monitors as well as beginning, preservice, and inservice teachers to deliberately make functions of proof and argumentation in high school mathematics themes that are assessed in tests and examinations. Why should assessment tools not demand a reflection on the nature of knowledge creation in mathematics? I believe that such tools could be appropriate vehicles for improving the perpetually poor image of mathematics in a society where a mere mention of Euclidean geometry conjures up images akin to, in Popham's (1981) words, 'bubonic plague and the abolition of tenure' (p. 66). Judging by the short time it took to administer and analyse results obtained from LFUP instrument, it is reasonable to propose the use of this instrument for gaining insights into learners' functional understanding of proof prior to instruction on proofs. This perspective is reflected in the assertion that '[t]o plan their instruction, for example, teachers should know about each student's current understanding of what will be taught' (National Research Council [NRC], 1993, p. 82).

Grigoriadou's (2012) investigated how proof was viewed by learners. In her quest to improve learners' appreciation of the concept of mathematical proof and to produce proofs, she suggested that instruction needed to focus on making the distinction between inductive and deductive reasoning early in the lessons cycle. It is hoped that such modification of instructional practices would contribute to the harmonisation of school mathematics and the mathematics as practiced by experts in the field and thus (1) provide learners with a window into the nature and construction of mathematical knowledge and (2) appropriately reflect mathematical practices. By beginning teachers here I meant teachers with less than five years of teaching experience after completing the Bachelor of Education (B Ed) degree.

I believe that the learning and teaching of the functions of proof will be improved only when the curriculum monitoring teams took the Specific Aims in CAPS serious by assessing this aspect of the curriculum in examinations while monitoring instructional practices in Euclidean geometry classes. These results could help policymakers to direct resources to improve environments that contributed to learners' access to opportunities of learning Euclidean geometry meaningfully. Although the insights gained about the research problem in this study were significant, there were questions that I was keen to gain some understanding as a result of new findings that arose in the analysis stage. Thus, this pointed to areas that needed further research.

The CAPS document is conflicted. On the one hand, it sought to advance the interests of the mathematical community. Support for this statement is plentiful in CAPS: the definition of mathematics as a human activity places emphasis on learners appreciating the functions of proof; emphasis that empirical arguments do not constitute proof; and that Euclidean geometry content and assessment take up the largest proportion of the mathematics curriculum, particularly in the FET phase; Euclidean geometry content is modelled on the van Hiele theory of geometric thinking whose principles stipulated the creation of experiences that promoted learner's advancement to the next higher level, a principle that required time to accomplish. Specifically, implementation and monitoring of the first two aspects is sufficient in helping learners to gain insights into the nature of mathematics.

Yet, on the other hand it stipulates time frames by which mathematics content should have been covered which leads to, as Watson (2008) argues, learning of theorems and proofs mechanically as question-spotting activity rather than as mathematical inquiry, answers are expected to be found and problems to be solved, within the confines of a particular timescale. Further evidence of this aspect of the conflict is found in the current examination in Euclidean geometry questions where the word "*Prove that …*" rather than "*Is …?*" is used. As suggested by Furinghetti and Paola (1997), asking learners to "prove" influences them to argumentation rather than conjecturing. These practices can be classified as unmathematics as they constraint engaging in mathematical practices.

Empirical evidence in the form of LFUP results showed that learners hold naïve functional understanding of proof. In addition, the results suggested a statistically significant association between learners' ability to argue and functional understanding of proof. Shongwe's (2019) finding that learners' quality of argumentation is low and that *Presh N*'s sources of mathematical ideas emanate from her teacher and textbook reify the dominant influence of a collectivist culture. In short, the curriculum is in a dilemma and as such, the recommendation emanating from this study is that CAPS requires reexamination of its curricular aims.

I believe that the general finding in this study is clear: the development of learners' appreciation of the significance of functional understanding of proof and argumentation deserve to be given a priority in high school geometry classrooms. I am under no illusion that this is a complex task; it calls for devotion on the part of the teacher. Devotion is required since it is doubtful if there has been a time when functional understanding of proof has been viewed as a measure to capture learners' interest in proof in the South African classrooms of mathematics. This study suggests that qualitative studies are needed to enhance our understanding of the findings as obtained in the quantitative phase. Also, I argue that the significance of TAP as an instrument to quantitatively measure the quality of argumentation has been understudied in mathematics education.

As anticipated, I found that learners held generally naïve beliefs about the functions of proof; that is, there is evidence that most learners viewed the function of proof primarily as systematisation. In addition, I found that their argumentation ability was not only poor, but that the relationship between functional understanding of proof and argumentation ability was also weak. Both these results were influenced largely by the collectivist culture prevailing in the society, that is, depending on the teacher and textbook as sources of knowledge whose authority is not to be questioned.

If learners at the schools such as Dinaledi schools which received vast amount of financial support to improve learner participation in mathematics hold hybrid views about the functions of proof, the question is “How dire is the situation in nonDinaledi schools?” While acknowledging that it would be unfair to assume homogeneity about school practices (Watson, 2008), given the fact that the participating schools were randomly selected, it is reasonable to believe that the situation is no different in schools outside the Dinaledi group.

9.3.2 Suggestions for future research

The results in this study seem to suggest that learners hold a distorted image of mathematics as they have not moved from the descriptive to the theoretical van Hiele level. The recommendation is that instruction needs to heighten the distinction between descriptive level and the deductive level. The findings further suggest that the Specific Aims in CAPS about proof are not necessarily shared by all high school learners. Thus, efforts in the form of intervention programs should be made to impress it upon learners that empirical arguments are merely a prerequisite and therefore do not constitute proof.

In administering both questionnaires, I was confronted with learners in the township classrooms for whom English was their second or third language and struggled to understand some of the question which were, of course, in plain English. This problem was evident particularly in written argumentation (AFEG) questionnaire. Research on how learners are currently using their local language (IsiZulu) in their mathematics classrooms could provide insight into the extent language interfered with argumentation ability.

There is a need to investigate the extent to which the findings in this study apply to other nonDinaledi schools. Although studies on the role of resources in scholastic achievements yielded conflicting results, the situation in other schools require investigation. The investigation is necessary due to the fact that despite increase in budget for education, this has not translated into learner achievement. Learner performance has remained low in mathematics in general and in Euclidean geometry in particular across all schools. In addition, more and more learners seemed to opt or are made to do mathematical literacy rather than mathematics. In short, I chose to refer to it (mathematical literacy) as utilitarian mathematics in that it focused on finding answers to everyday problems which, by definition, not only encourage learners to develop a distorted view of mathematics but also deprived them the opportunities to gain insights into the nature of mathematics.

As already suggested, understanding the concept of proof entails understanding and experiencing all these five functions prior to engagement in the construction of proofs. Thus, it is probably safe to suggest that further research is needed to examine the training of teachers on the concept of proof so as to capacitate them in engaging learners to doing proof meaningfully. At the very least, it represented an attempt to understand the root causes of hybrid beliefs about the functions of proof. In this respect, it is hoped that future studies will further delineate the phenomenon, motivate and inspire the development of innovative intervention programmes aimed at reducing the discrepancy between learners' functional understanding of proof and those of contemporary mathematicians. I recommend that future studies need to conduct longitudinal studies on how learners make a transition from hybrid beliefs to informed beliefs about the functions of proof. Also, each of the factors in the LFUP scale could probably be strengthened through revision (rewriting) items with lower factor loadings and possibly adding new items. In addition, given the contradictory findings in research, future studies could explore the influence of DGS on understanding the functions of proof.

In summary, I believe that with these recommendations and suggestions for further research I have shown the need to balance the amount of attention given to mathematical content with that given to the nature of mathematics. As already emphasised throughout this thesis, learning about the functions of proof in Euclidean geometry and practicing argumentation are the best place to learn about the nature of mathematics. I hope that curriculum delivery monitors will take note of the recommendations in this study and appreciate that they were made in the spirit of giving learners the real deal; giving learners the complete picture of the mathematics discipline.

9.4 Implications of the findings

The recommendation that Euclidean geometry curriculum needed to be revamped for the purpose of making functional understanding of proof and argumentation explicit curriculum content has implications for a variety of constituencies. This then implies further that assessments, which all too often emphasise quick recall of facts, procedures, and memorisation of proofs at the expense of the “territory before proof” include elements of this territory in large-scale, formal assessments. As Marrades and Gutiérrez (2000) put is, ‘[a] complete assessment of students’ justification skills has to take into consideration both products (that is, justifications produced by students) and processes (that is, the ways in which students produce their justifications)’ (p. 88).

In addition, although the qualitative findings of this study are limited to the case of *Presh N*, they provide evidence of the different factors influencing hybrid functional understanding of proof. Whereas the recommendation that learners’ appropriation of the functions of proof is necessary to foster meaningful learning of proof, gaining insight into the factors influencing the understanding is equally important. Thus, this revamp will have implications not only for mathematics teacher education but also for classroom practice. For instance, the teaching of the functions of proof in mathematics is conflated with doing proof in most if not all of teacher education programmes in South Africa owing to the partial alignment of teacher education programmes with school mathematics.

9.5 Reflections

9.5.1 The research process

I have reached the end of what has been a fascinating journey to explore and understand the interaction between and among three variables; learners’ functional understanding of proof; learners’ argumentation ability, and the reasons why a learner held the beliefs she held about the functions of proof. In the conceptions of the study, my single most concern is that participants for whom English is not their home language may find it challenging to engage in written argumentation. However, during the analysis process I pick up very few such instances.

9.5.2 Pattern matching analysis process

I used pattern matching to analyse *Presh N*'s experiences with the concept of proof. I acknowledge that in adopting this theoretical analysis stance, bias could have been inadvertently introduced as I may have ignored some critical aspects of the data or payed too much attention to some specific parts of the data (Javadi & Zarea, 2016). However, consideration of the robust procedures adopted in this study to obtain trustworthy qualitative findings may mitigate the effect of potentially bias procedures.

9.5.3 Personal growth

One of the advantages of this study is that it has contributed to the development of my own knowledge of functions of proof in mathematics. The review of the literature into why learners find proof difficult broadened my knowledge of the functions of proof even more. Also, using ATLAS.ti for coding involves a double learning curve: learning how to make sense and order textual data and learning a new computer programme.

9.6 Chapter summary

I argued that doing proof meaningfully, even if it meant understanding that the primary function of proof is to verify the truth of mathematical statements, is contingent upon holding informed functional understanding of proof. This understanding fulfilled the Specific Aim of CAPS. In fact, I argued that functional understanding of proof and argumentation provide learners with a sense of the nature of mathematics, and believe that such understanding contributes to halting low achievement in Euclidean geometry. I then argued that taking advantage of the relationship between argumentation and functional understanding of proof could contribute to the learning of proof.

I also explained why *Presh N* held informed beliefs about the functions of proof; that is, identifying the roots of beliefs about the functions of proof that were consistent with those of the mathematics community. The fluidity of learners' beliefs about the functions of proof were corroborated by past research studies. Seeking insights into the factors influencing informed beliefs was important for instructional practices to address as such beliefs tend to either hinder or promote gaining of informed beliefs about the functions of proof. I discussed the interactions between and among learners' functional understanding of proof, argumentation ability, and factors accounting for informed beliefs about the functions of proof in mathematics. I acknowledged several limitations inherent in the design of crosssectional studies such as the present study. For each of the research question discussed, I highlighted its implications and made recommendations for (1) actions to be taken to tackle the issues raised and for (2) the pursuit of further research, in the quantitative strand of the study.

The three major contributions of this research are its methodology, the baseline quantitative data gathered on LFUP, and the proposed model for understanding factors influencing learners' functional understanding of proof. Of course, the LFUP result is not a new insight. What is new, however, is another validation of an instrument. This validation enhances the fidelity with which the instrument can be used by both classroom teachers and mathematics education researchers. The large scale investigation was also intended to provide policymakers with insight from which to base their policies.

In a nutshell, I surveyed learners on their functional understanding of proof and the relationship between this phenomenon and argumentation ability. I purposefully selected an extreme case whose reasons for holding informed beliefs about the functions of proof were examined through qualitative methods. Final, I explored the interaction among the three constructs: functional understanding of proof, argumentation, and factors influencing functional understanding of proof. The employment of a mixed-methods sequential explanatory design helped in describing how widespread hybrid understanding of proof was among learners, finding that the relationship between functional understanding of proof and argumentation was weak, *Presh N*'s understanding of the functions of proof was influenced by the teacher and the collectivist culture within which she functioned, and the interaction among the three constructs was undesirably coherent. Overall, the findings presented in this study are offered as a contribution to mathematics education's growing interest in gaining insight into learners' understanding of the functions of proof and argumentation to encourage meaningful learning of proof.

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Appendix A1

Permission letter from UKZN Ethics Committee

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

09 February 2017

Mr Benjamin Shongwe (215081389)
School of Education
Edgewood Campus

Dear Mr Shongwe,

Protocol reference number: HSS/0437/016M
Project title: Grade 11 Mathematics learners' conceptions of Euclidean proof in relation to argumentation in selected high schools

Full Approval – Expedited Application

In response to your application received on 25 April 2016, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and **FULL APPROVAL** was granted for the protocol.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

....
Dr Shenuka Singh (Chair)

/ms

Cc Supervisor: Dr Vimolan Mudaly
Cc Academic Leader Research: Dr SB Khoza
Cc School Administrator: Ms Tyzer Khumalo

Humanities & Social Sciences Research Ethics Committee
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Website: www.ukzn.ac.za

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Appendix A2

Permission letter from KZN Department of Education

education

Department:
Education
PROVINCE OF KWAZULU-NATAL

Enquiries: Phindile Duma Tel: 033 392 1041 Ref.:2/4/8/1128

Mr B Shongwe
43 Everham Rd
Phoenix
4068

Dear Mr Shongwe

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **"GRADE 11 MATHEMATICS LEARNERS' CONCEPTIONS OF EUCLIDEAN PROOF IN RELATION TO ARGUMENTATION IN SELECTED HIGH SCHOOLS"**, in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 09 December 2016 to 31 January 2019.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Connie Kehologile at the contact numbers below.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

Pinetown District

Dr. EV Nzama
Head of Department: Education
Date: 14 December 2016

...Championing Quality Education - Creating and Securing a Brighter Future

KWAZULU-NATAL DEPARTMENT OF EDUCATION
Postal Address: Private Bag X9137 • Pietermaritzburg • 3200 • Republic of South Africa
Physical Address: 247 Burger Street • Anton Lembede Building • Pietermaritzburg • 3201
Tel.: +27 33 392 1004/41 • Fax.: +27 033 392 1203• Email: Kehologile.Connie@kzndoe.gov.za/Phindile.Duma@kzndoe.gov.za •Web:www.kzneducation.gov.za
Facebook: KZNDOE….Twitter: @DBE_KZN….Instagram: kzn_education….Youtube:kzndoe

Appendix B1

Learners' Functional Understanding of Proof (LFUP) Scale

Instructions

- *This questionnaire will **NOT** affect your marks. Please, do not spend a long time on any one question – **your first thoughts are usually your best**.*
- *Each statement is followed by a series of possible responses: Strongly disagree, disagree, undecided, agree or strongly agree.*
- *Put a **tick/circle** over the corresponding response. Please respond to every statement – it's important that you respond to each statement honestly.*
- *All the information will be used for research purposes only. Your responses will be treated **confidentially**. Codes will be used to **protect your identity**.*
- *This survey should take you about **20 minutes** to complete.*

This survey is conducted by Ben Shongwe for his PhD studies at UKZN.

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

Contact Person: Prof. Vimolan Mudaly

E-mail: mudalyv@ukzn.ac.za

Researcher: Mr. Ben Shongwe

E-mail: shongweb@ukzn.ac.za

For any queries please feel free to contact me.

Demographic information

Code: ______________

Please, circle/tick one answer for each of the following.

Personal particulars					
Gender:	*Female*	*Male*	***Class (e.g. 11A)***		
Home language:	*IsiZulu*	*English*	*Afrikaans*	*Other:*	

"1"	***"2"***	***"3"***	***"4"***	***"5"***
Strongly Disagree	*Disagree*	*Undecided*	*Agree*	*Strongly Agree*

Please, circle the number that best reflects your level of agreement with each statement:

[illegible]	A proof is useful in making sure that a mathematical statement is true.	1	2	3	4	5
T2	Some maths propositions are true even if they have not been verified to be so by proof.	1	2	3	4	5
T3	Confidence about the truth of a proposition motivates me to find its proof.	1	2	3	4	5
T4	A proof explains what a maths proposition means.	1	2	3	4	5
T5	A proof hides how a conclusion that a certain maths proposition is true is reached.	1	2	3	4	5
T6	Proof shows that maths is made of connected concepts and procedures.	1	2	3	4	5
T7	When I do a proof, I get a better understanding of mathematical thinking.	1	2	3	4	5
T8	Proving make me understand how I proceeded from the given propositions to the conclusion.	1	2	3	4	5

T9	Proof enables communication of the given propositions, the definitions used, and the theorem to be proven.	1	2	3	4	5
T10	Proof communicates maths results among learners themselves.	1	2	3	4	5
T11	Proof restricts the learning of argument standards.	1	2	3	4	5
T12	Proof can be used to debate the correctness of maths ideas.	1	2	3	4	5
T13	Doing proof limits the learning maths language.	1	2	3	4	5
T14	I like proofs because they give me new insights as they show connections between theorems.	1	2	3	4	5
T15	I do not like proofs and do not see the need for them; I prefer just learning theorems.	1	2	3	4	5
	Leave this item blank	1	2	3	4	5
T16	Analysis of proof may lead to invention of new results.	1	2	3	4	5
T17	Proving prevents me from possibly inventing things about geometry.	1	2	3	4	5
T18	Proof may reveal completely new areas for investigation.	1	2	3	4	5
T19	Proof shows the lack of connections between theorems and new results.	1	2	3	4	5
T20	Proving in maths may lead to an addition of new proposition that can be used in later proofs.	1	2	3	4	5
T21	Proving does **not** require one to decide which axioms may be chosen as true.	1	2	3	4	5
T22	Proving in maths may lead to a replacement of a set of propositions that could be used in later proofs	1	2	3	4	5
T23	Proof does **not** show the existing logical relationships between propositions.	1	2	3	4	5
T24	A proof in maths brings together and connects maths results.	1	2	3	4	5
	Leave this item blank	1	2	3	4	5
T25	Proving involves reasoning and argumentation that is different from the rest of maths.	1	2	3	4	5

Thank you for your valued input and assistance. Please, turn over the page to go to the last part, AFEG.

Appendix B2

Self-efficacy Scale

Instruction

The attached form lists different activities. In the column, rate how confident you are that you can do them as of now. Rate your degree of confidence by recording a number from 0 to 6 using the scale given below:

0	*1*	*2*	*3*	*4*	*5*	*6*
Cannot do at all	*Moderately can do*				*Highly certain can do*	

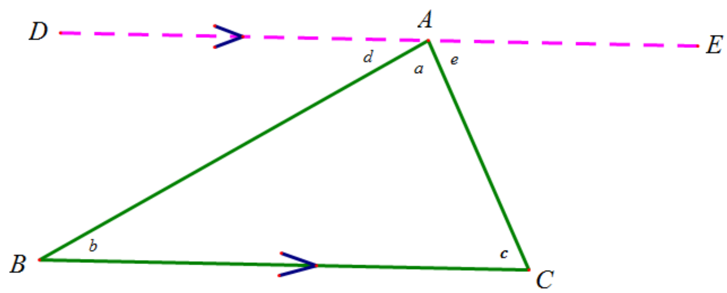
	Confidence (0–7)
Engage in experimentation to seek patterns	__________
Make a conjecture	__________
Verify if the conjecture is true using few cases	__________
Seek counterexamples	__________
Persevere in the face of difficulties	__________
Use previously proven statements	__________
Formally write out and justify each step of your proof	__________
Examine your proof for accuracy and identify any missing steps.	__________

Appendix B3

Argumentation Frame in Euclidean Geometry (AFEG)

I am interested in the claim that you can make about the data in the diagram. This questionnaire is not part of your regular geometry activity and so it will NOT affect your marks. Your name will not be linked to your responses. Please, use the dotted lines to respond to each prompt.

In the diagram below, line DE is parallel to line BC on triangle ABC.



Please, make ANY statement or claim from the diagram and justify it. Please, think carefully as you argue your points using the guide provided below.

(1) My statement is that …………………………………………………...……….

………………………………………………… ………………………………….

(2) My reason for making this statement is that …………………………………...

……………………………………………………………………………………..

(3) Arguments against my idea might be that ……......……………………………..

……………………………………………………………………………………..

End. Thank you.

Appendix B4

Interview Schedule

Code: S1CAL15 **R** = Researcher **P** = Participant (*Presh N*)

Transcriber: Kamlesh M. **Typist:** Zodwa K.

Date: 26 September 2017 **Start:** 14h50 **End:** 15h30

A. Introduction

I am from the University of KwaZulu-Natal (UKZN) conducting interviews to explain the source of your beliefs about the functions of proof. I am meeting with you because you have obtained the highest score in the questionnaire related to the functions of proof.

Thank you for agreeing to be interviewed and audiorecorded by signing the consent form to indicate that you have received information about this study. Audiorecording of this talk will help me in making sure that I do not miss very important information you give and to save time.

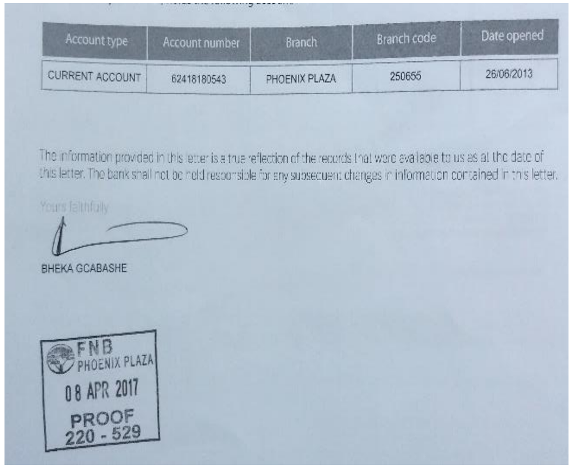
Please, take note that I am interested in your honest thoughts about the functions of proof; not to grade you.

This interview will take approximately 30 minutes. Are you ready to begin?

B. Interview questions

RQ 1.6.2.1	Aim (overall information needed)	Objectives (specific information needed to achieve the aim)	Main questions	Probes (Follow up questions)
Why does Presh N hold informed beliefs about the functions of proof?	1. Confirmation	A. To record informed consent.	(i) Do you consent freely to participate in this audiorecorded interview?	
		B. To establish whether learners have treated Euclidean proof in Grade 10, the second term as scheduled.	(ii) In Grade 10, have you studied proof in geometry?	Tell me about one theorem you just did in class.
	2. Checking *semantic contamination*	A. To obtain insights into whether learner's definition of proof is in terms of everyday meaning.	(i) What, in your view, is proof in mathematics?	Please, can you explain what a theorem is?
			(ii) What are the functions you believe proof performs in mathematics?"	Why do you think so?
	3. Checking *collectivist culture*	A. To understand whether participants hold internal or external views of mathematics.	(i) According to you, what do you think is mathematics?	How do you know this?

RQ 1.6.2.1	Aim (overall information needed)	Objectives (specific information needed to achieve the aim)	Main questions	Probes (Follow up questions)
Why does Presh N hold informed beliefs about the functions of proof?	Checking *collectivist culture* (continued)	B. To be able to see whether learning of proof is through memorisation or investigations.	(i) How do you learn your theorems?	Why do you use this way you have just described?
	Checking *collectivist culture* (continued)	C. To see whether participants attribute failure to context conditions or effort.	(i) What do you think is the reason if you are unable to prove a theorem?	Why do you think so?

<table>
<tr><th>RQ 1.6.2.1</th><th>Aim (overall information needed)</th><th>Objectives (specific information needed to achieve the aim)</th><th>Main questions</th><th>Probes (Follow up questions)</th></tr>
<tr><td rowspan="2">Why does Presh N hold informed beliefs about the functions of proof?</td><td>4. Checking empirical arguments</td><td>A. To understand if proving that something is true in geometry is the same as proving in everyday life (presenting an object as proof).</td><td>Read this paper thoroughly. The stamp states “Proof”.

<table><tr><th>Account type</th><th>Account number</th><th>Branch</th><th>Branch code</th><th>Date opened</th></tr><tr><td>CURRENT ACCOUNT</td><td>62418180543</td><td>PHOENIX PLAZA</td><td>250655</td><td>26/06/2013</td></tr></table>The information provided in this letter is a true reflection of the records that were available to us as at the date of this letter. The bank shall not be held responsible for any subsequent changes in information contained in this letter.
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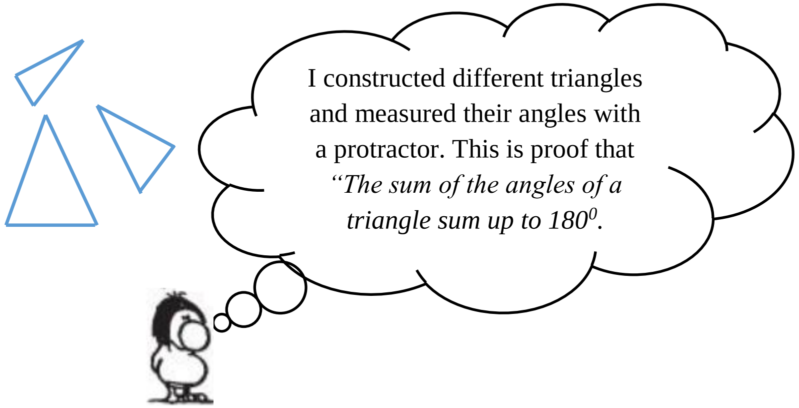
(i) What is the difference between this “proof” and mathematical proof?</td><td>Can you say more?</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
</table>

RQ 1.6.2.1	Aim (overall information needed)	Objectives (specific information needed to achieve the aim)	Main questions	Probes (Follow up questions)
Why does Presh N hold informed beliefs about the functions of proof?	5. Checking *empirical argument*	A. To check if learners conceive of proof as making a number of observations.	Please, consider the following diagrams, cartoon and its statement.  (i) Do you agree with the learner that finding the same answer after trying many cases proves?	Why do you agree?

RQ 1.6.2.1	**Aim (overall information needed)**	**Objectives (specific information needed to achieve the aim)**	**Main questions**	**Probes (Follow up questions)**
Why does Presh N hold informed beliefs about the functions of proof?	Checking *empirical argument (continued)*	A. To check if learners conceive of proof as making a number of observations.	(ii) How can the learner be sure that the statement that *"The sum of the angles of a triangle sum up to* 180^0*"* always works?	How do you know that?
	6. Checking *teacher influence*	A. To understand whether the learner appreciate the need to read and understand a theorem for herself rather than rely on the authority of the *teacher*.	(i) If the teacher has verified the truth of a theorem, what do you think is your role, next?	Why do you think you have to do that?
			(ii) Do you have a mathematics textbook of your own?	How often do you use it?

RQ 1.6.2.1	**Aim (overall information needed)**	**Objectives (specific information needed to achieve the aim)**	**Main questions**	**Probes (Follow up questions)**
Why does Presh N hold informed beliefs about the functions of proof?	Checking *teacher influence* (continued)	B. To understand whether the learner appreciate the need to read and understand a theorem for herself rather than rely on the authority of the *teacher*. (continued)	(iii) If the textbook has verified the truth of a theorem, what do you think is your role, next?	Why do you think you have to do that?
	7. Checking *textbook influence*	A. To check influence of *textbook* in promoting the "prove that" type rather than those that trigger conjecturing by learners.	(i) How does the textbook help you to do proof?	Please, can you elaborate?

C. Closing

- We have come to the end of this interview. Are there any responses you would like to elaborate upon or questions that you would like to ask me about what took place during the interview?
- Thank you very much for the effort you made to participate and the time you spent during the interview.

Appendix C1

Glossary of transcript symbols

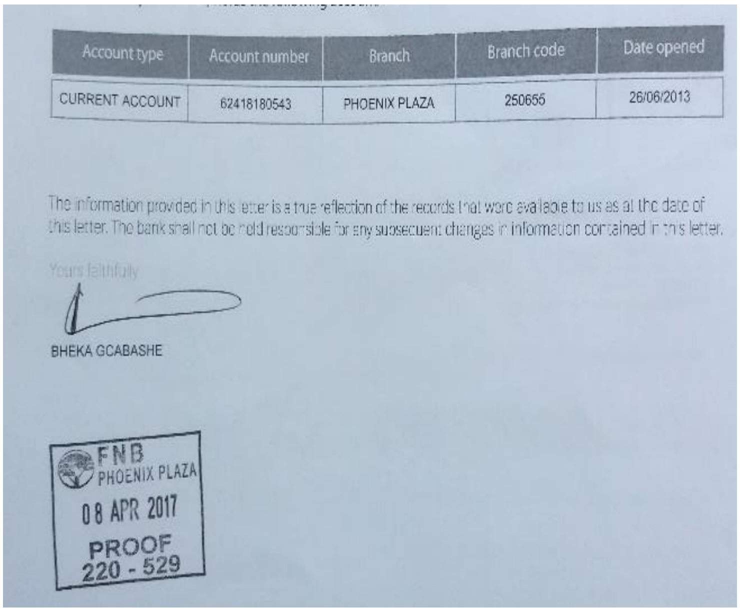
P:/R:	Speaker labels (P: = Participant; R: = Researcher)
(·)	*A dot in parentheses* indicates a brief interval (±a tenth of a second) within or between utterances.
°word°	*Degree signs* bracketing an utterance or utterance-part indicates that the sounds are softer than the surrounding talk.
word	*Asterisk signs* bracketing an utterance or utterance-part indicates that the sounds are harder than the surrounding talk
<u>underline</u>	*Underlining* used to mark words or syllables which are given special emphasis of some kind
CAPS	Words or parts of words spoken loudly marked in *capital letters*
s:::	*Sustained or stretched sound*; the more colons, the longer the sound
( )	*Empty parentheses* indicate that the transcriber is unable to get what is said.
(( ))	*Doubled parentheses* contained researcher's descriptions.
.hhh	A *dot-prefixed* row of 'h's indicates an inbreath. Without the dot, the 'h's indicate an outbreath.
–	*A dash* indicated a cut-off word or sound
=	*Equal signs* indicate no break or gap. *A pair of equal signs*, one at the end of one line and one at the beginning of a next, indicate no break between the two lines.
(1.7)	*Numbers in parentheses* indicated elapsed time by seconds.
↑↓	*Arrows* indicate shifts into especially high or low pitch.
£	*The pound-sterling sign* indicated a certain quality of voice which conveys 'suppressed laughter'

Appendix C2

Presh N's Interview Transcript

R: Do you consent freely to participate in this audiorecorded interview?
P: Yes.
R: In Grade 10, have you studied proof in geometry?
P: Yes.
R: Tell me about one theorem you just did in class.
P: .hhh (1.5) okay that would be the theorem that says if the angles of a quad are (·)=
=supplementary, it is a cyclic quad.
R: What, in your view, is proof in mathematics?
I: Proof in mathematics is about coming up with ideas and developing .hhh (1.5)=
= formulas by using things like theorems and measurements and .hhh (1.5) Yeah.
R: Please, can you explain what a theorem is?
P: A theorem is a statement that can be proven to be true °or not °
R: What are the functions you believe proof performs in mathematics?
P: * Sorry come again*.((clearing her throat))
R: What are the functions you believe proof performs in mathematics?"
P: To <u>verify</u> if a statement is true or a problem is true to (·) to yeah
R: Why do you think so?
P: Because when you are proving you... there is a communication that and a language =
=that you must follow and bringing together the ideas you have or you know =
=about .hhh (2.0) s::: in this case geometry things you've learnt from previous =
=grades putting them together things, that are relevant to what you are trying to =
=solve, the problem you are trying to solve, ( ) something like that.
R: According to you, what do you think is mathematics?
P: Mathematics is about PATTERN, it's a sequence of hhh. No it's a pattern of numbers =
=and shapes how they come about and .hhh (2.0) Yeah.
R: How do you know this?
P: I↑ know this because .hhh (2.0). Math, topics in maths are closely related .hhh (2.0)
=Nothing is new YOU CAN ALWAYS RELATE TO THE PREVIOUS .hhh (2.0)=
=CHAPTER you did so <u>you can see a pattern</u> forming and you can see lot =
=of relationships things that you can compare or say you can use to solve other =
=problems in different topics. I mean you can you can you can use .hhh (1.5)=
=For example algebra, algebra is not only used for .hhh (1.0) <u>solving x you can=</u>
=use it in word statements you can use it in in geometry, Euclidean geometry so it's =
=not something new that comes up when you do a different topic it's a follow up=
=or a continuous it's just that it's how you use it, it's how apply it=
R: How do you learn your theorems?
P: I learn my theorems by obviously going through them reading try and understand =
=follow the the the. £ rules and then apply them in a problem
R: Why do you use this way you have just described?

P: I … I ... That's how we were taught. Sorry, we were taught that, you read, follow the=
= rules then apply them.
R: What do you think is the reason if you are unable to prove a theorem?
(1.5)
P: .hhh (0.5) it might be the language or how the question is put or you didn't exhaust =
=all your theorems or yah all your theorems that you've learnt, I just said it is because=
=you are having a problem in understanding the statement what it says or you =
=haven't applied every, you haven't applied every .hhh (2.0) rule or theorem that you =
=know in the problem.
R: Read this paper thoroughly ((Sheet with bank stamp handed over to participant)). =
=The stamp states "Proof".



Account type	Account number	Branch	Branch code	Date opened
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The information provided in this letter is a true reflection of the records that were available to us as at the date of this letter. The bank shall not be held responsible for any subsequent changes in information contained in this letter.

Yours faithfully

BHEKA GCABASHE

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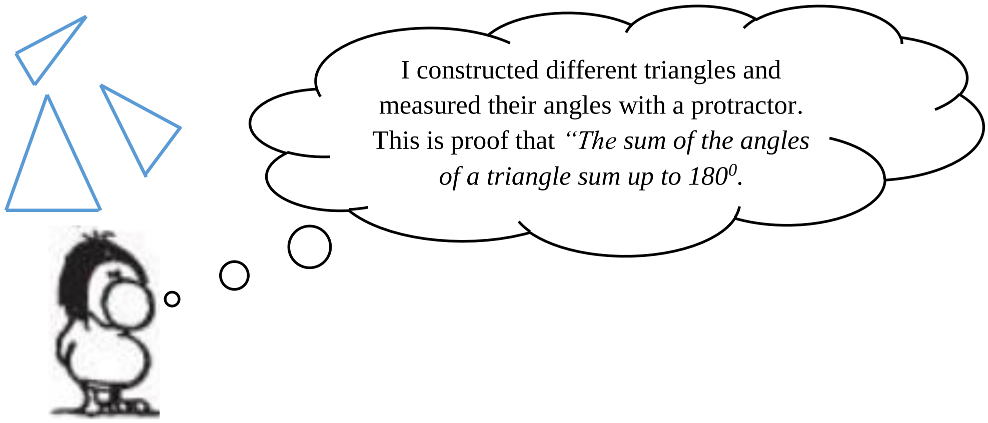
R: What is the difference between this "proof" and mathematical proof?
P: From the stamp?
R: Yes
P: I see the name of the bank, the branch, the date and that it's the proof.
R: Can you say more?
P: .hhh (1.0) in mathematics proof .hhh (1.0) talks about or relates to how you came=
=to a conclusion when solving a problem in this case in Euclidean geometry how=
=how which steps did you take and what those steps were supported by which =
=statement and how did you take those steps to get where to your final answer.
R: Can you say more?
P: This proof I can say they are almost similar because there's details that .hhh (0.5) =
=the the client or customer that went to to to request for for the statement can agree =
=to what they see on the statement by agreeing that this is their information so when =
=the bank aah place the stamp on this statement they they verified with the =
=customer that this is the is their information and the customer agreed and they they =
=also took steps probably by .hhh (0.5) checking the the clients ID number and this is =
=what they came up with so .hhh (1.5), even though there's not a lot of similarities =

= but they took certain steps to come up with the final .hhh (0.5) statement.

R: Please, consider the following diagrams, cartoon and its statement. ((Sheet =
=with bank stamp handed over to participant)). According to the learner, the =



= statement that the angles of a triangle are supplementary has to be proved this way.=
=Tell me what they say.

R: It says I constructed different triangles and measured their angles with a protractor =
=this is proof that the sum of the triangle, the the sum of the angles of a triangle =
=sum up to 180 degrees. Basically the learner is saying .hhh (0.5). They used a =
=protractor to measure the angles of a triangle and all of those triangles they =
=measured they came up to180 degrees.↓

R: Do you agree with the learner that finding the same answer after trying many cases =
=proves?

P: .hhh (2.5) ˚Not completely˚ because the learner only relied on the protractor to come =
=up with their conclusion but .hhh (1.0) his conclusion is not supported by statements =
=or other theorems that were were, that they happen to be proven by mathematicians =
=like maybe theorem of Pythagoras or things like that he only relied on the =
=protractor it's not proof enough, .hhh (0.5).

R: How can the learner be sure that the statement that *"The sum of the angles of a =*
=triangle sum up to 180^0" always works?

P: .hhh (0.5) I think as students we can. We rely on on the textbooks that are written by =
=mathematicians so by following their way of of proving it is almost guaranteed =
=that you are on the right track you you don't just come up with your own. =
=The learner can use .hhh (1.5). Can use a theorem that talks about parallel lines =
=and a triangle drawn between parallel lines a sketch yeah .hhh (0.5) a learner =
=can use a sketch which has parallel lines and in between the parallel lines there's =
=a triangle and use all the theorems that they have learnt to to work it out and see =
=using the angles of that triangle talk about it using theorem.

R: How do you know that?

P: hhh We've done that in class

R: If the teacher has verified the truth of a theorem, what do you think is your role, next?

P: My role is to (2.5) is to go over it again .hhh (0.5) maybe try to find out how =

=other learners .hhh (0.5)how did the other learners find the the proof, how how did =
=they managed to solve it without the teacher just basically going through it again =
=and maybe asking maybe more questions from the teacher it ( ).
R: Why do you think you have to do that?
P: I guess because they .hhh (1.5). They've been proving for quite some time more than =
=us because are just learning these things they've been exposed to these problems =
=for a while more than us.
R: Do you have a mathematics textbook of your own?
P: Yes
R: How often do you use it?
P: Very often.↓ .hhh (1.5) s::: Maybe 4 times a week
R: How does the textbook help you to do proof?
P: To remind myself or to go back and and yeah to remind myself about the theorems or =
=other examples and .hhh (1.5) To check the answers if I'm correct with my =
=problems that I've attempted. .hhh (1.5) Is to continue (2.0) don't stop. =
=hhh (1.5) continue solving more problems more theorems more proofs and.
(2.0)
R: Please, can you elaborate?
P: .hhh (1.5) The textbook .hhh (0.5) it's got almost all the relevant information that =
=I need so that's how it helps me I get most of the information that I need from the =
=textbook.
R: We have come to the end of this interview. Are there any responses you would like to =
=elaborate upon or questions that you would like to ask me about what took place =
=during (·) the interview?
P: No.↓
R: Thank↑ you <u>very much</u> for the effort you made to participate and the time you spent =
=during the interview.

Appendix C3

ATLAS.ti interview quotations and codes

Project: *Presh N* interview

Report created by Shongwe B on 2017/11/30

Quotation Report

All (34) quotations

2:1 the theorem that says if the angles of a quad are supplementary, it is… (224:308)

3 Codes:

● Collectivist culture / ○ Correctly stated theorem / ○ The theorem is precisely stated

---

2:2 Proof in mathematics is about coming up with ideas and developing eeh…… (361:489)

2 Codes:

○ Mathematicians prove, learners can't discover / ○ Proof is conjecturing; inductive arguments and axioms

---

2:3 theorem is a statement that can be proven to be true or not (552:610)

3 Codes:

● Collectivist culture / ○ Proof verifies / ○ Theorem can be true or false

---

2:4 To verify if a statement is true (785:817)

1 Codes:

○ Proof verifies

2:5 when you are proving you... there is a communication (884:936)

1 Codes:

○ Proving is communicating

2:6 language that you must follow (948:977)

2 Codes:

○ Proving is communicating / ○ Proving requires following a language

---

2:7 bringing together the ideas you have or you know about aah.... in this…… (982:1136)

1 Codes:

● Proving is systematising

2:8 that are relevant to what you are trying to solve (1139:1187)

2 Codes:

○ Proof uses relevant ideas / ● Proving is systematising

2:9 Mathematics is about pattern, it's a sequence of ammh... No it's a pat…… (1310:1425

2 Codes:

○ Maths is patterning of numbers and shapes / ● Proving is systematising

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2:10 Math, topics in maths are closely related mmh... Nothing is new you ca…… (1501:1651)

2 Codes:

○ Maths topics related & form pattern / ● Proving is systematising

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2:11 you can see a pattern forming and you can see lot of relationships thi…… (1622:1772)

2 Codes:

○ Patterns & relations help / ● Proving is systematising

2:12 is not only used for aah solving x you can use it in word statements y…… (1857:2131)

2 Codes:

○ Link between diff domains / ● Proving is systematising

2:13 I learn my theorems by obviously going through them reading try and un…… (2172:2308)

2 Codes:

● Collectivist culture / ○ Learning theorems by memorisation

2:14 That's how we were taught. Sorry, we were taught that, you read, follo…… (2375:2471)

3 Codes:

● Collectivist culture / ○ Taught to follow rules / ● Teacher influence

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2:15 Ahh... it might be the language or how the question is put or you didn…… (2549:2892)

2 Codes:

● Collectivist culture / ○ Failure to prove shows inability to apply rules

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2:16 I see the name of the bank, the branch, the date and that it's the pro…… (3053:3125)

4 Codes:

○ Proof verifies / ● Semantic contamination / ○ Stamp is proof / ○ Stamp verifies

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2:17 Aah in mathematics proof aahm talks about or relates to how you came…… (3151:3450)

2 Codes:

● Proving is systematising / ○ Proving shows steps to conclusion

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2:18 This proof I can say they are almost similar because there's details t…… (3476:3718)

4 Codes:

○ Evidence is mathematical proof / ○ Proof verifies / ● Semantic contamination / ○ Stamp verifies

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2:19 when the bank aah place the stamp on this statement they they verified…… (3723:3989)

2 Codes:

○ Proof verifies / ○ Stamp verifies

2:20 yaah, even though there's not a lot of similarities but they took certain…… (3995:4112)

1 Codes:

○ Letter & proof use steps to reach conclusion

2:21 Not completely because the learner only relied on the protractor to co…… (4801:4898)

2 Codes:

● Empirical arguments / ○ Measurement does not imply proof

2:22 his conclusion is not supported by statements or other theorems that w…… (4910:5018)

2 Codes:

○ Deductive reasoning not followed / ● Proving is systematising

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2:23 theorems that were … were , that they happen to be proven by mathematicians…… (4965:5090)

2 Codes:

● Collectivist culture / ○ Theorems proved by mathematicians

2:24 he only relied on the protractor, it's not proof enough (5091:5146)

4 Codes:

● Empirical arguments / ○ Measurement isn't proof / ○ Patterns & relations help / ● Proving is systematising

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2:25 I think as students we can .we rely on … on the text books that are wr…… (5287:5477)

2 Codes:

● Collectivist culture / ○ Mathematicians prove; learners follow

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2:26 you don't just come up with your own (5482:5517)

1 Codes:

○ Mathematicians prove, learners can't discover

2:27 The learner can use aahmm. Can use a theorem that talks about parallel…… (5519:5893)

1 Codes:

○ Axioms used to do proof

2:28 We've done that in class (5931:5954)

3 Codes:

● Collectivist culture / ○ Proof done in class / ● Teacher influence

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2:29 My role is to … is to go over it again aah maybe try to find out how o…… (6052:6340)

2 Codes:

● Collectivist culture / ○ Memorisation of teacher proof

2:30 They've been proving for quite some time more than us because are just…… (6414:6570)

3 Codes:

● Teacher influence / ○ Teachers exposed to proving more / ● Textbook influence

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2:31 Very often. Mmmh...Maybe 4 times a week (6661:6699)

2 Codes:

● Textbook influence / ○ Textbook used 4 times/week

2:32 To remind myself or to go back and and yah to remind myself about the…… (6751:6847)

2 Codes:

● Textbook influence / ○ Textbook source of theorems

2:33 To check the answers if I'm correct with my problems that I've attempt…… (6859:7039)

2 Codes:

● Textbook influence / ○ Textbook to check correctness of proofs

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2:34 the text book eeh it's got almost all the relevant information that I…… (7078:7239)

4 Codes:

○ Info needed is in textbook / ● Teacher influence / ○ Textbook contains most information / ● Textbook influence