# EXPLORING MATHEMATICAL ACTIVITIES AND DIALOGUE WITHIN A PRE-SERVICE TEACHERS' CALCULUS MODULE: A CASE STUDY

**By**

**Botshiwe Likwambe**

**Student Number 201512623**

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**DECLARATION**

The work done in this thesis was carried out in the School of Education at the University of Kwa-Zulu Natal under the supervision of Dr J Naidoo.

I declare that this is my original work and has not been submitted in any form for any degree to any tertiary institution. The work used from others has been properly acknowledged.

| | |
|---|---|
| Botshiwe Likwambe | Dr J. Naidoo |
| (Candidate) | (Supervisor) |

**ABSTRACT**

Local and international research findings have shown that high school learners, university students, as well as some of the practicing educators, struggle with calculus. The large numbers of unqualified or under-qualified mathematics educators are a major contributing factor to this problem. Many researchers agree on the fact that profound subject content knowledge is one of the contributing factors to effective teaching. Thus, this study seeks to explore what is counted as mathematics teaching and learning, what is counted as mathematics, as well as the nature of dialogue in a calculus lecture room.

The Mathematics for Teaching framework and the Cognitive Processes framework informed this study, in order to explore what was counted as mathematics teaching and learning in the calculus lecture room. The Mathematical Activities framework and the Legitimising Appeals framework informed this study, in order to explore what was counted as mathematics in the calculus lecture room. The Inquiry Co-operation Model also informed this study, in order to explore the nature of dialogue within the calculus lecture room.

The findings of this study showed that there are various mathematical activities that develop the students' higher order thinking which is required for problem solving. These activities include mathematical activities that promote conjecturing, proving, investigations, the use of multiple representations, the use of symbols, the use of multiple techniques, as well as activities that promote procedural knowledge through conceptual understanding. These activities also keep the students' cognitive demand at a high level. The findings of this study also showed that the types of questions that are asked by the lecturers have a positive impact on the development of the students' high order thinking, as well as in terms of keeping the students' cognitive demand at high levels. The study has also shown that the lecturers exhibited a variety of mathematics for teaching skills and this is done both explicitly and implicitly. It has also been revealed that introducing the rules of anti-differentiation as the reverse of differentiation is an alternative way to introducing the concepts of integral calculus. Based on these findings, it was recommended that students who enrol for the calculus module with low marks in mathematics, ought to use the derivative concept and the rules of differentiation as a foundation to build on the rules of anti-differentiation.

## DEDICATION

This work is dedicated to my hero, my mentor, my role model and my late mom Diniwe Mpofu. You taught me to work hard and I know that you would have been very proud of me and from that I shall continue to draw inspiration.

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# CHAPTER ONE

## INTRODUCTION TO THE STUDY

### 1.1 Introduction

Various researchers including Siyephu (2013) point out that mathematics is perceived to be a subject that opens many doors for students' future studies. Additionally, the mathematical skills that students possess are of great importance in their place of work, as well as within the academia (Lin & Tai, 2015). Thus, students who desire to study courses that rely on calculus such as, engineering, medicine, or advanced mathematics, are expected to have a profound conceptual understanding of mathematical concepts (Ismail et al., 2012). Muzangwa and Chifamba (2012) suggest that the calculus concepts are highly dependent on other mathematical concepts such as algebra and functions. For students to have a profound conceptual understanding of mathematical concepts, they need to be taught effectively (Orhun, 2012), which is the reason why many researchers dedicate their time in search of effective ways to teach mathematics, as well as searching for the qualities that best describe an effective educator. This is confirmed by Mudaly (2016) who points out that the fundamental reason for the pre-service teacher training is to produce effective mathematics educators.

Further to this, calculus is perceived to be important for studying engineering, medicine or advanced mathematics (Tall, 1997). Thus, the profound conceptual understanding of the derivative becomes advantageous to students as they pursue their studies at university level (Kula, 2016). For instance, Feudal (2016) points out that the notion of the derivative has a fundamental function in the study of economics. Thus, students who desire to study economics require a profound understanding of the derivative.

This chapter provides the introduction to the study. The introduction comprises seven sections. Firstly, the study is introduced, followed by the background information to the study. The background is followed by a discussion focusing on the purpose of the study. The discussion on the rationale, addressing the gap in the field, an outline of the study's contribution, the research questions and lastly, an overview of the study, are provided.

## 1.2 Background and purpose of study

Local and international research has shown that a large number of students struggle with concepts in calculus (Firouz, Ismail, Rahman, & Yusof, 2012; Habineza, 2013). In South Africa, this problem is widespread from high school learners to university students, as well as in some of the educators, especially those who are unqualified or under-qualified. This is echoed by Parker (2004) in her seminal work on teacher education and development, when she points out that the field of mathematics teacher education and development is faced with huge challenges which are connected to the education and development of educators who specialise in the teaching of high school mathematics.

Central to her discussion is the point that the education system in South Africa has many educators who are unqualified or under-qualified to teach mathematics and as a result, this has contributed to the "cycle of poverty in mathematics education" (Parker, 2004, p. 122). To add to this "cycle of poverty in mathematics education", learners are deprived of the opportunities to learn mathematics (Stols, 2013). In his study on learners' opportunities to learn, Stols (2013) found that educators spent more time on topics that were mostly procedural and avoided topics that required higher order thinking strategies. This is also supported by Jameel and Ali (2016) who point out that the educators in their study focused on developing learners' procedural knowledge, rather than conceptual knowledge. In addition, due to the limited content knowledge, educators are hindered from selecting, as well as planning and designing good and effective tasks for their classes (Webb & Cox, 2004). As a result, learners are deprived of the opportunities to learn mathematics (Mbugua, Kibet, Muthaa, & Nkonke, 2012) because their educators lack the ability to select or design mathematical tasks that are suitable for promoting the learners' higher order thinking.

Further to this, students' poor performance in mathematics has raised concerns, both locally and internationally (Siyephu, 2013). Siyephu (2013) further agrees with the sentiments by Parker (2004) who argues that the contributing factors are, firstly, the large numbers of unqualified and under-qualified mathematics educators in schools; secondly, the lack of resources in most schools is a major concern. It is a fact that most of the textbooks used in South African schools do not give ample opportunities for learners to make conjectures or investigate and discover formulae.

Additionally, Tshabalala and Ncube (2012) share similar concerns about the poor performances in mathematics, especially at secondary school level. The findings of their study also revealed that the students in rural Zimbabwe perform poorly in mathematics because of inadequate basic knowledge from the lower grades. According to the findings of Tshabalala and Ncube (2012), most rural schools do not have the resources such as textbooks. These schools are also inundated by large numbers of educators who lack good teaching strategies. Similarly, in Kenya, the students' poor performance in mathematics is exacerbated by the shortage of qualified mathematics educators in most rural schools (Gitaari, Nyaga, Muthaa, & Reche, 2013), thus, learners end up being taught by unqualified or under-qualified teachers.

The causes of students' poor performances in mathematics, locally and internationally, are a major concern. deLourdes Mata, Monteiro, and Peixoto's (2012) study indicated that students' poor performance in mathematics is caused by the poor teaching strategies employed by the educators. Moreover, the shortage of qualified mathematics educators causes students to perform poorly in mathematics. Additionally, insufficient subject knowledge by the educators and the lack of resources such as textbooks, also affects the performance of students in mathematics.

Even though this is the case in South Africa, Cavanagh (2008), in his seminal presentation, states that research does not convincingly show which professional credentials demonstrate whether mathematics educators are effective in the classroom. He further points out that defining the qualities of an effective mathematics educator becomes an impossible task. This is in contrast with the views by Hattie (2003; 2013) who argues that providing feedback and the monitoring of learners is one of the qualities of an effective mathematics educator. Similarly, Anthony and Walshaw (2009) describe an effective educator as one who has the following qualities: being able to provide learners with opportunities to make sense of concepts, both individually and in peer groups, being able to host classroom discussions that have a focus towards mathematical argumentation, understanding that the chosen class activities and examples have an impact on how learners see, develop, implement and understand mathematics, being able to carefully choose teaching aids and representations so that they can provide support for learners' thinking and finally, being able to cultivate and use their substantial knowledge and skills to promote learning, as well as to actively respond towards the mathematical needs of all their learners.

In addition, Ansari (2013) describes an effective educator as one who possesses the following qualities. Firstly, one who has a calling for the program of teaching, meaning that such an educator does his or her work of teaching for the of love of teaching. Secondly, one who has the profound subject content and pedagogical knowledge, as well as knowledge of his or her students. Thirdly, personal qualities such as communication skills and passion for the subject they teach. Fourthly, instructional effectiveness, such as being able to use effective teaching strategies in their teaching. The fifth quality is being a good communicator, not only with the learners, but also with other members of the staff. The sixth quality involves the willingness to go the extra mile, which means always doing his/her best to help the learners. Lastly, being a lifelong learner, by always searching for new and better ways to teach their subject, is also considered as one of the qualities of an effective educator.

Cognitive ability, educator personality, classroom management, communication and responsibility, are the intertwined qualities of an effective educator, as pointed out by Hamid, Hassan and Ismail (2012). An effective educator is also one who is able to clearly explain concepts and present them in such a way that learners can easily understand them, as well as using good teaching strategies (Mudaly, 2016).

Calculus builds on some fairly intuitive ideas, which makes it possible to introduce this topic to learners at high school level. At the same time, calculus draws in the much less intuitive limit processes and this constitutes a break away from algebra and geometry (Artigue, 1994; 2001). The limit process is a core component in calculus, but leads to a number of difficulties for learners and students (Artigue, 2001). The seminal work by Tall (1992) confirms this when, in his discussion on the difficulties encountered by students as they study calculus, he mentions that the difficulties met by students include translating real-world problems into calculus formulations and that students prefer methods that involve procedures, rather than conceptual understanding. Additionally, the findings of Zakaria and Salleh (2015) showed that engineering students in their first year of study at university had inadequate calculus background. Their results further revealed that the inadequate calculus background was caused by the insufficient preparation at secondary school level. Thus, effective educators with extensive subject and pedagogical knowledge could ensure that their learners are well prepared for university calculus.

Many educators do not have a strong understanding of the subject matter that they teach (Shúilleabháin, 2013; Ngwenya, 2014). These scholars also mention that insufficient subject matter knowledge amongst educators is widespread. Therefore, if qualified educators are having problems with understanding the concepts in calculus, it is not surprising then, to find that high school learners are also struggling to grasp the concepts in calculus. "One cannot teach what one does not understand well…" (Mogari, 2014, p. 16). It is these high school learners who then enrol for calculus modules at universities, with little or no understanding of the basic concepts of the topic. Thus, the university students' performances are inseparable from their high school performances (Mudaly, 2016). Denebel (2014) also confirms this point by pointing out that most first year students at university have a very weak conception of the concepts in calculus, while they tend to have a better understanding of procedures. The results of the study conducted by Denebel (2014) also showed that the students depended on memorisation and performing routine algorithms. These students' understanding was based on fragmented facts. Skemp (1978) agrees with this by describing instrumental understanding as the mastering of rules or procedures. Skemp (1978) further argues that a student is capable of mastering rules without any knowledge of how the rules or procedures work. Tan and Shahrill (2015) support this by pointing out that students had low conceptual understanding of the integral calculus, while their procedural knowledge was high. Their study shows that most students exhibited instrumental, instead of relational understanding.

Mogari (2014) agrees with this when he points out that the teaching and learning of mathematics in South Africa is examination-based which results in students having to take part in memorisation and rote learning practices. Thus, students resort to memorising and rote learning tactics and sacrifice the conceptual understanding of mathematical concepts in order for them to get through the examination process. However, memorisation and rote learning strategies do not have significant benefits to students' performance. This is supported by Lin and Tsai (2015), whose findings revealed that students who used memorising and rote learning strategies performed poorly, as compared to who used among others, problem solving strategies. Leongson and Limjap (2003), in their seminal work, acknowledge that after observing Filipino students, they were shocked to find that these students did exceptionally well in acquiring knowledge, but struggled in lessons that required higher order thinking skills.

Based on the preceding discussions, the calculus lecture room presented itself as a suitable location for collecting the empirical data for this study. The study is embedded within the interpretivist paradigm using the qualitative approach. Video recording and observations of all calculus lectures were followed by interviews with lecturers. This study was informed firstly by theInquiry Co-operation Model framework. The researcher's motivation for using the Inquiry Communication Model emanates from the desire to explore the communication that the lecturers engage in with the students in the calculus lecture rooms. Secondly, the use of the Mathematical Activities framework, Cognitive Processes framework and the Legitimising Appeals framework was inspired by the researcher's aspiration to explore and explain what was counted as mathematics in the calculus lecture rooms. The research also sought to explore and explain what was counted as mathematics for teaching, elicited by the lecturers, either implicitly or explicitly and this was done by using the Mathematics for Teaching framework as the overarching framework for the study.

### 1.3 The rationale for the study

The rationale for conducting this study is three-fold, thus:

1. Addressing the gap
2. Bringing in new knowledge
3. A follow up from the researcher's Masters Degree.

A more detailed description of the rationale is provided below.

#### 1.3.1 Addressing the gap

There has been a vast amount of research including the research conducted by Siyephu (2013), Tan and Shahrill (2015) and Siyephu (2015) on the teaching and learning of the calculus concepts, both at high school and university levels. Both local and international researchers have done this. Most of these studies have been conducted on either students' or learners' misconceptions of calculus concepts, or students' difficulties in learning calculus. For example, Muzangwa and Chifamba (2012) conducted their research on undergraduate students' errors and misconceptions in calculus, while Makgakga and Makwakwa (2016) explored the Grade 12 learners' difficulties in solving problems in differential calculus. Other studies for example, Adler (2005) and Parker and Adler (2012) have focused on the interaction of educators and learners at high school level, or lecturer and students in a variety of topics which include algebra, probability, functions, sequences and geometry but not calculus. Furthermore, Ndlovu, Amin and Samuel (2017) conducted their research on pre-service teachers' content knowledge of various topics in school mathematics. The study conducted by Hurst, Wallace, and Nixon (2012) was on exploring how the literacy pre-service teachers felt about the social interaction in their lecture room. Very few studies have been conducted on the interactions that take place in the pre-service teachers' lecture room between the lecturer and the pre-service teachers.

Although the study conducted by Habineza (2013) was based on a calculus module, the focus was on the students' concept image of the integral, but not on the mathematical activities or the mathematics for teaching exhibited by the lecturer. The study conducted by Brijlall and Isaac (2011) was based on a calculus module, but the focus was on how the lecturers' subject knowledge influences their reflection in practice. The study conducted by Davis, Adler and Parker (2005) was on the mathematics for teaching exhibited by the lecturer, but it did not include the calculus module. Other studies by Adler (2005), Kazima, Pillay, and Adler (2008) as well as Parker and Adler (2012) were conducted on educators, while teaching their learners on various topics, but these studies did not focus on calculus. More recently Adler et al. (2014) conducted a study in a mathematics course, not particularly about a calculus module, where interviews were conducted with the students. One of their concerns was on mathematics for teaching, which the students had acquired from the course. To bridge these gaps, this study explores what is counted as mathematics, mathematics for teaching, as well as mathematics learning on a calculus module. This study also explores the types of dialogue that take place while teaching a calculus module of the pre-service teachers.

### 1.3.2 Bringing in new knowledge

Several studies including Gitaari et al. 2013 and Jameel and Ali (2016) have shown that when educators lack subject content knowledge, as well as good teaching strategies, the students' performance in mathematics is affected negatively. This study was situated within a calculus lecture room of the pre-service teachers and aims to show what is counted as mathematics for teaching in the calculus lecture room. This contribution may be of benefit in that the lecturers who teach pre-service teachers will be aware of what mathematics for teaching skills can be elicited in the calculus lecture room. Secondly, the mathematical activities that were legitimised in the calculus lecture room may be of benefit in that they do promote the development of a profound subject content knowledge, as will be shown in Chapter Six. Thirdly, this study sought to explore the ways in which lecturers organised their materials and the reasons behind this organisation. Thus, the organisation of the materials by one of the lecturers may be of benefit to the introduction of integral calculus to first year university students. In addition, the findings of this study add a category to the types of questions that may be asked by the lecturer during interactions with the students. Additionally, a category of questions that students may ask is also presented in the findings of this study.

### 1.3.3 A follow up from the researcher's Master's degree

While the researcher was conducting research towards her Master's degree, it was found in her research that, of the in-service educators who had enrolled for a calculus module within the Advanced Certificate in Education (ACE) programme, only a few students developed adequate conceptual understanding of the derivative concept after completing the module (Likwambe & Christiansen, 2008). It is on this basis that this research focuses on the interactions between the lecturers, the students and the teaching and learning materials that were especially developed to teach calculus to pre-service educators. The materials were designed to be conceptually engaging. According to Kilpatrick, Swafford, and Findell (2001), mathematics teaching and learning involves the educator's knowledge, educator's use of mathematical content, educator's attention to the learners, as well as the learners' engagement with the tasks given to them by the educator.

## 1.4 Research questions

As mentioned before, this study was informed by the Inquiry Communication model, The Cognitive Processes framework, the Mathematical Activities framework, as well as the Legitimising Appeals and Mathematics for Teaching frameworks. This was all in an attempt to answer the following questions:

1. What mathematical activities are legitimised in the calculus lecture room?
    - Are the legitimising appeals made to mathematics, mathematics education theories, the textbooks/notes, students' experiences, everyday metaphors, authorities or other?

Thus, this study aims to establish what is counted as mathematics in the calculus lecture room, as well as to what or who the justifications of the activities are made.

2. How and why are the materials organised by the lecturers?

When answering this question, the researcher aims to establish and justify the lecturers' actions.

3. What is the nature of calculus dialogue in the calculus lecture room?

When answering this question, the researcher aims to explain the dialogue with which the lecturers engaged with the students.

## 1.5 The scope of the study

At the university at which this study was conducted, integral calculus is introduced during the first semester of the third year of study. Thus, this study was limited to two groups of third year students and their lecturers from the same institution, but in different years. There were 78 students in the first group, which was taught by Lecturer A and 120 students in the second group, which was taught by Lecturer B.

## 1.6 Terminology used in the study

Some of the terminology and concepts used in this study are explained in detail in Chapter Three, the following are some of them:

- Classroom – The teaching venue at a school.
- Lecture rooms – The teaching venues at a university.
- Learners – The individuals who study at schools.
- Students – The individuals who study at university.
- Pre-service teachers – University students who are studying to become educators.
- Educator – The person who teaches at a school.
- Lecturer – The person who teaches at a university.
- Lecture A1-9 – Lectures taught by Lecturer A.
- Lecture B1-11 – Lectures taught by Lecturer B.

## 1.7 Overview of the study

This study is divided into the following chapters:

**Chapter One: Introduction to the study**

In this chapter, an introduction to the study is given. The background, purpose and rationale of the study are discussed. The reader is introduced to the terminology that is commonly used in this study and key questions are also introduced.

**Chapter Two: Review of literature**

Literature related to this study is reviewed and discussed in this chapter. The literature discussed includes the conceptual learning required by pre-service educators, concept images, lecturers' knowledge necessary for pre-service educators' learning, as well as the learning opportunities – the types of activities and in-service teacher education.

**Chapter Three: Conceptual Framework**

In Chapter Three, the frameworks that informed this study are introduced. Legitimising Appeals, Mathematical Activities, Cognitive Processes and Inquiry Cooperation frameworks, are the four frameworks, with the Mathematics for Teaching as the overarching framework of this study. Chapter Three also explores how each of the components of Mathematics for Teaching framework is illuminated by the components of Cognitive Processes, Mathematical Activities and Inquiry Cooperation frameworks.

**Chapter Four: Design and Methodology**

This chapter gives an outline of the research design of the study. The research paradigms are also explored and the paradigm of this study is identified as the interpretivist. Case studies, research methods and data collection methods are also explored. The chapter concludes with a discussion on the issues of validity and reliability, as well as the ethical issues.

**Chapter Five: Analysis of Data**

This chapter presents the data analysis of this study. The themes that emerged are presented in this chapter, as are also the profiles of the lecturers.

**Chapter Six: Discussion of Findings**

The findings of this study are presented and discussed. These include the types of questions asked by the lecturers, the activities that were legitimised in the calculus lecture room, the communication that took place in the lecture rooms, as well as the mathematics for teaching that was exhibited by the lecturers.

**Chapter Seven: Conclusion and Recommendations**

The research questions are addressed in Chapter Seven and the significance and contributions of this study are presented. In addition, the conclusion of this study is presented in Chapter Seven.

## 1.8 Conclusion

This chapter presented the background of the study, showing how mathematics education is overwhelmed by students' poor performance in mathematics, both locally and internationally. In South Africa, the problem is further exacerbated by the shortage of qualified mathematics educators and the fact that most pre-service teachers have inadequate content knowledge (Mudaly, 2016). In addition, the chapter has presented a discussion on the importance of the need for students to possess a profound understanding of calculus concepts. The rationale and the scope of the study have been presented in this chapter. This chapter is followed by a review of the relevant literature in Chapter Two.

# CHAPTER TWO

## LITERATURE REVIEW

### 2.1 Introduction

This study combines a socio-cultural and a cognitive perspective. Socio-cultural theories give emphasis to the nature of lecture knowledge and the culturally rooted processes by which one becomes part of the lecture room community and thus, participates fully in a community (Silver, Clark, Ghousseini, Charalambous, & Sealy, 2007). This is in agreement with the participationist view on learning, which considers all knowledge to be naturally social. This implies that what we learn is a product of human communication and would not have existed, had we not been part of a community (Sfard, 2005). Learning is also seen as participation in the practice through resources, which include both material and social (Adler, 2005). This is in agreement with the sentiments of Wegner (1998), that learners in a classroom share fundamental goals and knowledge and work together towards achieving their common goal. Thus, learning is seen as a social activity. On the other hand, the cognitive perspective on learning views learning as located within the individual student's head (Adler, 2005). Cognitive theories provide emphasis supporting the fundamental role that individual introspection and cognitive conflict may play, in promoting conceptual development (Silver et al., 2007).

Though it may seem that the two approaches are in conflict, Silver et al. (2007) acknowledge that cognitive and socio-cultural approaches on learning may be constructively perceived as complementary. Thus, it is possible to consider the complexity of the teaching–learning processes of a lecture room from at least two well-defined approaches that are often considered as mutually exclusive. The combination of the two approaches sheds light on the reason why it was possible for the educators in the study conducted by Silver et al. (2007) to have opportunities to learn mathematics in a practice-based professional development, after participating in four activities. The first activity was an individual activity that provoked the educators' thinking and required them to think deeply as they solved the mathematical problem. In the second activity, each educator read an account of a class discussion on how to solve the mathematical problem done in activity one. The third activity was a group discussion on activity one and two and they were given an opportunity to share and learn about issues on pedagogy, as well as on how students learn. The fourth activity gave them an opportunity to consider the second activity in relation to their own teaching.

For students to understand the practice of their mathematics lecture room and the social interactions, they need to take into account the mathematical meaning that they draw from their experiences in that lecture room. Additionally, these same students cannot develop mathematical meaning from their experiences in the lecture room without understanding the patterns of participation in the mathematics lecture room (Christiansen & Chronaki, 2005). The same sentiments are shared by Anthony and Walshaw (2009), whose paper on the characteristics of effective mathematics teaching, discusses the principles of effective pedagogical approaches that promote the learning of diverse students. One of these pedagogic approaches is to provide students with learning opportunities so that they can make sense of concepts, both cooperatively and independently. The researchers point to the fact that sometimes students need time to think quietly on their own, but sometimes they also need to work in pairs or in groups so that they can share ideas. This helps in that the students are motivated to exchange and test ideas, thus, promoting higher levels of thinking.

Learners' performance in mathematics has received much attention locally and internationally. Thus, the first section of this chapter focuses on the factors contributing to learner performace. In addition, students' understanding of concepts and their concept images received much attention by researchers over the past years. Hence, the second part of the chapter focuses on the conceptual learning that pre-service teachers require. This is in line with the seminal work of Hattie (2003), whose review of a substantial number of international research studies indicates that this is one characteristic of expert educators, the other two being monitoring and feedback of learning and challenging learners. The third part of this chapter briefly discusses the literature on students' concept images. This would enlighten the reader on how much has already been researched, with regard to the concept images, as well as give insight on students' concept images, since this research focuses on the teaching and learning materials that have been developed to teach calculus to pre-service teachers.

The fourth part of this chapter discusses the knowledge that the lecturer possesses. This part seeks to enlighten the reader on the types of knowledge that the lecturer possesses, which is necessary for student teachers' learning. The fifth part of this chapter discusses the literature related to the learning opportunities and the types of activities created by the lecturer. Since this research focuses on the interplay between the lecturer, the pre-service teachers and the materials, this section would therefore enlighten the research on the types of activities that have already been researched and how they have created opportunities for students' learning of mathematical concepts. The last section of this chapter discusses the literature on teacher education, to give insights into what goes on in the in-service courses, in terms of the types of practices established, what knowledge is legitimised and how this happens. The literature discussed below focuses on active students who are provided with the opportunity to construct knowledge by making sense of what they are learning, with the lecturer simultaneously helping and guiding the students as they make sense of what they are learning.

## 2.2 Contributing factors to learner performance in mathematics

In most countries, including South Africa, mathematics is a compulsory subject at both primary and secondary school level. This is because mathematical skills and knowledge are required because they are crucial for the scientific and technological development of any community. In addition, the mathematical skills and knowledge are known to be contributing factors to the economic development of any country (Kiwanuka, Van Damme, Van Den Noortgate, Anumendem, & Namusisi, 2015). These sentiments are shared by Zadshir, Abolmaali, and Kiamanesh (2013), who suggest that learners require mathematical skills and knowledge for two reasons. Firstly, for future studies in mathematics and other related subjects and secondly, for the work place, since most industrial and technological positions require a workforce that possesses profound mathematical skills and knowledge. Despite the society's large dependency on mathematical knowledge and skills, learners continue to perfom poorly in mathematics.

Educators, textbooks and learners are the three factors that contribute to learner performance in mathematics (Zadshir et al., 2013). These three factors can affect the learners' performance in mathematics, either positively or negatively. If educators use good teaching strategies, as well as design and use good mathematical activities in their teaching, then the learners' performance is significantly good. However, if educators use poor teaching strategies, the learners' performance is significantly poor (Zadshir et al., 2013). These researchers suggest that learners are seen as the users of mathematical knowledge. Thus, through anxiety, motivation, learning style and attitude, learners as the users of mathematical knowledge are capable of affecting their own performance in mathematics. Zadshir et al. (2013) also suggest that the textbook contributes to the performance of the learners in that if it is a good textbook, the novice educator mostly uses it to make decisions and guide the learners in the right direction. On the other hand, a good textbook might not be used properly by the educator and this could lead to the textbook being a barrier to learners' performance. Additionally, if the educator does not realise that the contents of the textbook do not match the learners' capabilities, the learners' performance is negatively affected.

Furthermore, deLourdes et al. (2012) suggest that educators are a major factor to learner performance in mathematics. They point to the fact that if educators use poor teaching strategies, then the learners perfom poorly in the subject. Moreover, they acknowledge that the poor teaching materials, including the textbooks that are used by the educators, largely impact on the performance of the learners. Kisakali and Kuznetsov (2015) agree with these sentiments by suggesting that in Kenya, learners' performance in mathematics is affected by the lack of qualified educators, as well as the unqualified/untrained educators who lack enthusiasm and use poor teaching strategies in their teaching. In addition, poor learners' performance is caused by the lack of interest from the learners. One can only imagine how difficult it can be, to teach learners who lack motivation and interest in the subject. Thus, if pre-service teachers are exposed to good teaching strategies, they could make a difference in the classroom.

In Nigeria, learners' negative attitude towards learning mathematics contributes to their performing poorly in the subject (Sa'ad, Adamu, & Sadiq, 2014). However, the research findings of Sa'ad et al. (2014) show that when learners were exposed to learning environments that aimed at developing their positive attitude, the learners' performance in mathematics improved significantly. Additionally, their results showed that the lack of qualified educators, the educators' poor teaching strategies, as well as the lack of textbooks, negatively impact on the learners' performance in mathematics. The situation is the same in Kenya, where the learners' attitude, lack of teaching and learning materials, educators' attitude that is caused by the lack of qualifications to teach mathematics, as well as the huge workloads for educators, all contribute to learners' poor performance in mathematics (Karigi & Tumuti, 2015). In view of that, Karigi and Tumuti (2015) also propose that the learners' attitude could be improved if learners are taught by educators who use good and useful teaching strategies and good, as well as useful teaching and learning materials.

In Nepal, educators lack the ability to use learners' prior knowledge as a foundation on which to build new knowledge and this has a negative impact on the learners' performance in mathematics (Acharya, 2017). Additionally, learners have a poor mathematical background and thus, they have no foundation on which to construct new knowledge. This is also the case in rural Zimbabwean schools, where learners leave primary school with poor mathematical background, which then affects their ability to construct new mathematical knowledge (Tshabalala & Ncube, 2012).

The literature reviewed by Hoadley (2012) on the factors affecting South African learners' performance revealed the following factors as fundamentally contributing to learners' poor performance. Firstly, in rural schools, there is a significant lack of learning materials, especially textbooks, secondly, the learners are deprived of opportunities to write and practise what they have learnt. Thirdly, there is a significant lack of classroom interaction and the classroom activities are mostly of low cognitive demand.

On the contrary, Hoadley (2012) also revealed the factors that contributed to learners' improved perfomance and these include the following. Firstly, the educators' ability to be flexible and adjust to their learners' pace and capabilities impacts positively on learners' performance. Secondly, the educators' ability to cover a large amount of content, their ability to design tasks that are of high cognitive demand as well as their ability to assess learners appropriately also impacts positively on the learners' performance.

Thus, pre-sevice teachers need to be equipped with extensive content and pedagogical knowledge, which could alleviate the problem of learners' poor performance. This study seeks to determine what mathematical activities and mathematics for teaching skills are elicited by the lecturers to prepare the pre-service teachers for the classroom situation.

Three factors contributing to learners' performance have been discussed in this section, the educator, the learner and the learning materials, which include the textbooks. Table 2.1 summarises these factors.

**Table 2.1: Factors contributing to learners' performance**

| **Educator** | **Learner** | **Textbooks** |
|---|---|---|
| Unqualified<br>Lack of content knowledge | Poor attitude | Shortage of textbooks |
| Use of poor teaching strategies | Lack of interest and motivation | Content is beyond learners' capabilities |
| Lack of enthusiasm<br>Lack of support from school management | Lack of strong mathematical background | A barrier to learning if not used correctly |
| Large workloads | Lack of parental support | |

This study seeks to shed light into the type of educator/lecturer knowledge, the interaction between the lecturer and the students, with the teaching and learning materials, as well as the lecturer and student interaction that promotes student performance.

**2.3 Conceptual learning required by pre-service teachers**

As suggested by Adler et al. (2005) in their seminal work, there are three types of knowledge which educators need to acquire while enrolled in teacher education programmes. The first type of knowledge discussed is referred to as mathematical knowledge, which is mostly known as content knowledge. Content knowledge is fundamental to the teaching and learning of mathematics (She, Siwatu, Matteson, & Wilhem, 2014). These researchers also point out that content knowledge comprises the knowledge of facts, concepts and how concepts are connected, procedures, as well as the knowledge of organising mathematical concepts. The pre-service students in the study conducted by Van de Merwe and Bekker (2013) felt that content knowledge was important as they took part in their teaching practice.

The second type of knowledge is conceptual knowledge for teaching. This knowledge is described as the relationship between the clarity of the educators' expression of their mathematical objectives of their teaching and the different ways in which they make use of their new practices (Adler et al., 2005). In other words, conceptual knowledge for teaching is the way the educator's mathematical knowledge is adjusted to the challenges of teaching. The third type of knowledge needed by pre-service teachers is mathematical knowledge for teaching, which the pre-service teachers learn in their teaching practice as they learn how to teach and apply in practice for teaching. This is mostly known as pedagogical content knowledge (Adler, 2012).

### 2.3.1 Deep approach to learning

As relevant as *what* the student teachers must learn, is *how* they must engage with that learning. Two types of learning have been identified and called by different names. A deep approach to learning has been described as learning that occurs when one looks beyond the main points by Draper (2013). In other words, one seeks to understand and explore the meaning of the main points. While surface approach to learning is learning that involves focusing on the main points and then memorising them, in the deep approach to learning, the student connects the previous knowledge to new knowledge and is also able to connect concepts from different courses or modules to their day to day experiences and so, memorising is not involved (Draper, 2013). Many mathematics educators suggest that the act of observing relationships and then drawing connections is the key aspect to mathematical practice (She et al., 2014). Additionally, deep learning becomes an advantage to the learning of mathematical concepts, because it enables students to grasp successive concepts with ease (Jao, 2013). Thus, educators should aim at inculcating deep learning in their learners because deep learning becomes helpful in the learners' future studies in mathematics. In contrast, surface learning is characterised by memorising facts and procedures (Hattie & Donoghue, 2016). These researchers also point out that surface learning and deep learning strategies are connected in that a student firstly acquires surface learning, which then develops into deep learning.

#### 2.3.1.1 Transformational reasoning

One important aspect of deep learning which is also central to the nature of mathematics is transformational reasoning, which is described as the mental or physical performance of an operation on an object that enables one to visualise the transformations that the object undergoes (Simon, 1996). Transformational reasoning is being able to consider a dynamic process through which a new state is generated. To illustrate transformational reasoning, which also refers to as being the same as mathematical ability, Simon (1996) discusses the observation of a study on a tenth -grade geometry class, where the learners were asked to explore isosceles triangles. The educator expected her learners to create many examples of isosceles triangles, so that they could realize a pattern and deduce that the base angles of an isosceles triangle are equal. Only one student (Mary) did as the educator expected. Mary showed a different way of reasoning. She did not see the triangle as a static object, but as the result of a dynamic process. She was able to make a representation of an isosceles triangle and justified it by giving an example of two people walking from the ends of one side of the triangle towards each other at equal angles, that they would meet after having walked the same distances. This dynamic process enabled Mary to reason about two ideas that if the base angles are equal, then the legs of the triangle are equal and she ended up connecting the two ideas.

#### 2.3.1.2 Covariational reasoning

Covariational reasoning is another perspective on deep learning. Covariation is when two different quantities are coordinated mentally by an individual, while simultaneously focusing on the way they change in terms of each other (Carlson, 2002). The study conducted by Johnson (2012) shows a learner who used both covariational and transformational reasoning. The learner performed a task in which the area of a square changed as the perimeter changed. As the student performed the task, she predicted that the area would increase at a faster rate than the perimeter. The student synchronised transformational and covariational reasoning to imagine the way in which the area and perimeter of a square increased as the sides of the square increased.

#### 2.3.1.3 Appropriation and the use of technology

Another perspective on deep learning is provided by the notion of appropriation. Moschkkovich (2004) describes appropriation as the ability to take what one produces during an activity that is done in collaboration with others, for one's own use in later activities. Thus, this notion confirms the role of the interaction between the student and the lecturer. Moschkkovich (2004) also found that learners who use appropriation actively participate in the construction of knowledge. In her study, the student was guided to explore functions and had constantly interacted with the tutor as she was being introduced to new meanings and ways of seeing things. The student was then able to appropriate the new ways of seeing lines and equations and was also able to share her knowledge with the others. Moschkkovich (2004) stresses the point that students who use appropriation do not just imitate or replicate what they appropriate, instead, they use appropriated meanings for their own purposes. In a similar way, pre-service teachers must appropriate from the practices in teacher education. Of course, the question remains open as to what exactly we desire them to appropriate and what exactly do they in fact appropriate.

Similarly, Alqahtani and Powell (2016) conducted a study on appropriation using collaborative learning in an online use of the Geogebra, a software that can be used for teaching and learning mathematics. Since Geogebra is a dynamic software, it provides lecturers, educators, students and learners with the opportunities to learn various topics in mathematics, which include geometry, functions and calculus. The educators in the study conducted by Alqahtani and Powell (2016) used Geogebra to answer and perform tasks. They interacted with each other, as well as with the Geogebra thus, interacting with their environment and as a result, this interaction enhanced their knowledge of the use of the Geogebra. As these educators interacted with each other, they also focused deeply and explored beyond the diagrams to find explanations and answers to their tasks. Thus, this study shows that interaction is important in the teaching and learning of mathematics. Mainali and Key's (2012) study on appropriation using collaborative learning in a workshop while using Geogebra software shows that the software provided the educators with learning opportunities as they interacted with it. The educators all agreed that the software would be useful in their teaching and that it would contribute to the development of their learners' conceptual and procedural knowledge. The educators felt that by using Geogebra in their lessons, their learners would be exposed to meaningful learning and this would be due to the interactive nature of Geogebra. In the same way, the study by Daher and Anabousy (2015) supports this by showing that the learners' appropriation of the effect of transformations on functions was exhibited as the learners engaged in the exploration and discovery of the properties of functions, with the aid of Geogebra. The results of Daher and Anabousy's (2015) study also show that there was noteworthy improvement in the learners' conceptual and procedural knowledge of functions. The use of Geogebra enabled the learners to explore, discover and manipulate a variety of functions. Not only did the use of the dynamic software enrich the learners' conceptual and procedural understanding of functions, but also the knowledge of transformations.

Additionally, the findings of the study conducted by Slinas, Quintero and Fernández-Cárdenas (2016) show that the students benefited from the use of technology. They point out that the use of SimCalc, which is software that provides lecturers, educators, students and learners with dynamic and interactive mathematical representations, enabled the students to appropriate the relationship between functions and their derivatives. As these students explored these relationships by means of dynamic visual images, their appropriation of the connection between the functions and their derivative was exhibited through the students' discussions. This also shows that interaction is essential in the teaching and learning of mathematics. Hence, this study seeks to explore the interation that takes place in the calculus module of the pre-service teachers.

### 2.3.2 Self-regulated learning and the use of technology

Syatir et al. (2015) suggest that according to the results of their study, there is a strong relationship between students' high motivation and the formation of self-regulated learning of the students. They also pointed out that students who are highly motivated are capable of acquiring problem- solving skills. Recently, the use of technology has become more popular in classrooms. The Computer Algebra System (CAS) has been identified as the most practical form of technology in calculus courses (Sevimli, 2016), because CAS is capable of performing a variety of calculations involving the derivative and the integral concept, as well as drawing graphs which are three dimensional. The findings by Sevimli's (2016) study show that the more analytically minded students did not prefer to use CAS. On the other hand, the more visually minded students enjoyed and preferred to use CAS. This is supported by Bester and Brand (2013) who point out that one of the benefits of using technology in a mathematics classroom is that it captures the students' attention. Bester and Brand's (2013) research findings show that the students' achievement was noteworthy after the introduction of technology in the lessons of their study. This was because the educators gave the students the opportunities to explore concepts using technology. Furthermore, the use of technology in the mathematics lessons has a positive effect on students' attitudes (Eyyam & Yaratan, 2014). The findings of the study conducted by Eyyam and Yaratan (2014) show that students exhibited a positive attitude towards the use of technology. Thus, the use of technology could be of benefit to pre-service teachers.

### 2.3.3 Problem solving

Govender (2012) proposes that problem solving is an important part of the teaching and learning of mathematics. If one looks at the cognitive levels of the CAPS document, problem-solving questions account for 15% in each of the examination papers at matric level. Therefore, for educators to be able to teach the problem-solving skills to their learners, they need to have profound problem-solving skills themselves. However, the reality is that this is not the case in

South Africa, since a vast number of educators are either unqualified, under-qualified or qualified, but lack confidence (Govender, 2012). Govender’s (2012) study on developing the pre-service teachers’ problem-solving abilities show that it is possible for one to do so. The group of pre-service teachers in his study had little or no problem skills, but after intervention, a significant improvement in their abilities was observed. While in training, pre-service teachers should go through programmes that include insight of school mathematics, as well as programmes that equip them with problem-solving skills, as proposed by Govender (2012).

In addition, the study by Temel (2014) shows that prior to intervention, the pre-service teachers in her study had low levels of critical thinking skills and medium levels of problem-solving skills. After intervention, it was noticeable that both their critical thinking and problem-solving skills had improved significantly. Additionally, the study conducted by Cansory and Türkoğlu (2017) shows that problemsolving and critical thinking skills complement each other. The pre-service teachers in their study appeared to have enough problem-solving skills, which were complemented by low levels of critical thinking skills. These studies thus prove that among others, problem-solving skills are essential to both students and educators. Thus, pre-service teachers are required to engage in problem solving activities to ensure that they are well equipped for the classroom situation. This study seeks to explore the type of mathematical activities that the pre-service teachers engage with in the calculus module.

## 2.4 Concept images

‘Concept image’ refers to the mental pictures and notions that a student has about that concept (Vinner, 1983). These might be in the form of symbols, diagrams, graphs or words. Working within constructivism, Tall and Vinner (1981) describe concept image as the total cognitive structure associated with the concept. A concept image does not have to be consistent or coherent, as it is possible for a student to have compartmentalised concept images. Nor is a concept image necessarily in accordance with the concept definitions that students or learners evoke.

Zandieh (2000) conducted a study in which she analysed the notion of the derivative as it appears in textbooks and the mathematics community as a whole. Her results show that the derivative concept has three layers namely the ratio, limit and function. It is possible to have a

process or a structural concept image on each of these layers and, it is also possible that students may have a pseudo-structural concept image on a particular layer. In addition, Zandieh's (2000) results show that there are various ways in which the derivative concept is usually represented, these being graphically, symbolically, or by velocity, which is in relation to the physical movement and as a general rate of change. Combining the layers and the representations of the derivative, Zandieh (2000) constructed a model for analysing the students' concept images.
The model constructed by Zandieh (2000) has its strengths and weaknesses. Its strength lies in the fact that it focuses on what the students know, not on the discrepancies between the students' concept images and the accepted concept of the derivative, as has been the case with several previous works. For instance, Orhun (2012) found that the students were confusing the graph of the derivative with that of the original function. The students thought the graph of the derivative was the same graph of the function. In addition, Tokgoz (2012) found that students had an incorrect concept image of $h(x) = sinx$, which then resulted in the misconception of the derivative. This also resulted in their increased difficulty in applying the chain rule. Further to this, Siyephu (2015) found that calculus students had errors that were conceptual, procedural and interpretive. The conceptual errors were mainly because of the students' failure to grasp the concepts. The procedural errors emanated from the failure to carry out algorithms, while the interpretive errors were because students were incorrectly interpreting the concepts.

While this makes it possible at a glance to see the extent to which a student's concept image of the derivative is in harmony with the mathematical concept, it does not map any individual images that students may have constructed. Hence, it does have limitations in determining students' concept images when these deviate much from the intended, but it is a good instrument for assessing the impact of teaching. This instrument does not indicate whether or not a student has developed some skills in working with the derivative. This became evident in the research done by Likwambe and Christiansen (2008), when they tried to use Zandieh's (2000) model to analyse interviews with South African educators. In other words, only conceptual knowledge was assessed using Zandieh's (2000) model and not the other strands of proficiency, as proposed by Kilpatrick et al. (2001).

Coming from a background of physicists and engineering, Wagner, Roundy, Dray, Manogue, and Weber (2014) extended Zandieh's (2000) framework by expanding the physical representation with an introduction of measurement, as they perceive the physical representation as a process for measuring the derivative. They also added the idea of thick derivatives, which are small ratios that are practically equivalent to the true derivative (Wagner et al., 2014).

Zandieh's (2000) results show that the development of the students' concept images does not have to follow a certain order. Contrary to this, Likwambe and Christiansen (2008) found that the function layer is less likely to be developed until the other layers have been consolidated. Vincent and Sealy (2016), using Zandieh's (2000) framework, found that the way students define the concept of tangents is strongly influenced by their graphical understanding of the derivative concept. The students exhibited that there was a connection between her knowledge of a tangent with the graphical representation of the derivative. Even though this was the case, the student did not appear to be aware of the connection. Bezuidenhout and Olivier (2000), found that most first-year students at a South African university lacked the suitable conceptions of the integral concept. Serhan (2015) found that students had very little conception of the integral concept, but possessed profound procedural fluency, while Habineza (2013) found that Rwandan students could develop their concept image of the definite integral and their understanding of the Fundamental Theorem of Calculus significantly over the course of a semester.

The study by Desfitri (2015) is one of the few studies that show the participants of the study to have reasonable understanding of the concepts of the derivative and the limit. The participants in Desfitri's (2015) study were in-service educators from various schools and were observed while teaching calculus. The results of the study reveal that the students' understanding of the limit and the derivative concepts is determined by the educators' understanding, as well as by the way the concepts are taught. Panero, Arzarello and Sabena (2016) agree with this by acknowledging that the derivative is a very delicate concept and further argue that its introduction to high school learners is crucial, as well as delicate. For this reason, Panero et al. (2016) suggest that educators pay more attention to the way they introduce the concept to the learners, so that the derivative concept ends up being a resource, rather than a hurdle to the learners' future studies in calculus.

While there have been several studies on students' concept images in calculus, including Tall (1997) and Vincent and Sealy (2016), none of these studies informs us on how the learning situation influences the students' concept images. While Habineza (2013) found that instruction directed at conceptual learning in integration was fairly successful, the processes through which this happened are still fairly opaque. The materials used in this study were developed such that the pre-service teachers could engage conceptually with them. Thus, this study seeks to explore the mathematical activities that promote conceptual learning in integration. Simon's (1996) research on the development of students' concept of area informs us on how the use of the four steps of teaching situations described by Brousseau (1997), helps students develop the concept of area, but it still does not unpack the finer processes through which learning progresses. Artigue (1994) and others engaged with these four steps as a didactical engineering tool, but there is a need to expand this by linking it to the cognitive development of students.

## 2.5 Lecturers' knowledge necessary for student teachers' learning

This section briefly discusses subject matter knowledge, pedagogical content knowledge, as well as pedagogical or didactical knowledge.

Educator's subject and pedagogical knowledge is fundamental in developing the students' mathematical knowledge, as proposed by Bansilal (2012). These sentiments are shared by many researchers, including Krauss and Blum (2012) and Ainley (2012). In South Africa, the Department of Education, together with universities, have been offering in-service courses for educators to upgrade their subject and pedagogical knowledge, for the past 15 years. The Advanced Certificate in Education (ACE) programme is an example of such courses. However, research has shown that not all educators who enrolled for such courses actually improved their knowledge. Likwambe and Christiansen's (2008) study shows that of the five ACE students in their study, only one had deepened their knowledge.

Further to this, the findings of Verbeek (2014) show that the students in the Post-Graduate Certificate of Education (PGCE) programme mostly lacked subject content knowledge. This is said to be due to the nature of the PGCE programme, which is designed based on students having to have acquired content knowledge in their undergraduate degree. Thus, it is therefore assumed that those who enrol for the PGCE programme have profound content knowledge. However, this is not always the case. This is supported by Ngwenya (2014) who found that practising educators had inadequate subject content knowledge. The educators in Ngwenya's (2014) study lacked the conceptual understanding, but had adequate procedural knowledge.

Contrary to this, Bansilal (2012) shows that all four participants in the Master's programme improved their knowledge of teaching. This was because each one of them was involved in their own research and had their own questions that needed to be answered. As each participant engaged with their research, their mathematical knowledge for teaching increased.

Subadi, Khotimah and Suarni (2013) agree with the results of Bansilal's (2012) study. They describe a lesson study as a professional activity that is based in the classroom and aims at developing and empowering the educator. They also point out that a lesson study is also context-based; learner-centred and is owned by the educator. Reporting on the lesson study in their research, Subadi et al. (2013) point out that educators showed improvement in their lessons. This was because these educators were involved in collaborative planning of the lessons, observations and analysing the lessons. Positive points of the lessons were pointed out, areas of improvement were discussed and advice on how to improve was given. The educators found the lesson study very effective and they were more positive about their teaching strategies that were enhanced by participating in the lesson study programme. Additionally, Matanluk, Johari and Matanluk (2013) found that educators and learners had a positive attitude and outlook of the lesson study. This was because the teachers' confidence in their teaching strategies had increased tremendously. The students' performance had also increased noticeably. This is supported by Shúilleabháin (2013) who noted that educators' content knowledge, as well as the educators' pedagogical content knowledge, developed significantly as they took part in collaborative planning and the lesson study programme. The educators had their confidence in their mathematics teaching practice increase noticeably. In addition, Sinclair and Zazkis (2013) suggest that lesson play is vital in developing the pedagogical knowledge of the educators in training. A lesson play is more beneficial to educators in training, because the educators' actual script for instructional interaction is written and acted out by the educators themselves. The findings of Sinclair and Zazkis' (2013) study reveal that this was more beneficial to the educators than simply designing a lesson plan that the educators in the study felt did not allow them to think about some of the fundamental aspects of teaching. These educators felt that lesson play provided them with the opportunity to think about how they would engage in a discussion with learners in the actual teaching and learning environment.

Pedagogical content knowledge and content knowledge are the main essential components of the educators' knowledge that influence the development of the students' progress (Kleickmann et al., 2013). These researchers point out that the pre-service teachers in Germany develop pedagogical content knowledge and content knowledge by going through two phases of learning. Pre-service teachers in Germany are firstly introduced to pedagogical content knowledge and content knowledge at university. This is done through both formal learning during lectures and informal learning during peer learning. The second phase is when the pre-service teachers are in their teaching practice, where they learn through informal learning. Kleickmann et al. (2013) point out that the pre-service teachers are thus exposed to opportunities that enable them to develop profound pedagogical content knowledge, as well as content knowledge.

The notion of lecturer/educator knowledge is increasingly being recognised as a complex phenomenon. Previously, the lecturer/educator was perceived as possessing the understanding of what mathematics educators supposedly knew about mathematics. This is no longer the case, since several studies have revealed that subject matter knowledge alone does not make better teaching; it is necessary, but it is not sufficient. This has been echoed in the works of many researchers, including Ainley and Lutnley (2005; 2007) and Ainley (2012). These studies also show that effective teaching involves much more than an educator being mathematically competent; rather, it involves pedagogical knowledge, subject matter knowledge and pedagogical content knowledge. Yet, despite having been explored widely since Schulman's (1986) original coining of the term, the concept remains elusive (Hoover, Mosvold, & Ball, 2016). Even and Tirosh (1995) claim that pedagogical content knowledge has several sources that include one's own experience, both as a student and as an educator. This is in line with Schulman's (1986) own 'definition' or discussion of the term, which he saw to include the same two elements, the representation of ideas, knowledge of what makes a topic hard or easy, implying knowledge of learners'/students' common conceptions:

> *The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations and demonstrations, in a word, the most useful ways of representing and formulating the subject that makes it comprehensible to other. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons* (Shulman, 1986, p. 9).

Knowledge about students involves knowing how students learn the specific topic, what common pre-conceptions are and, how the two come together. Planned presentation of the subject matter involves one's choices of presenting the subject matter to the students, with the aim of assisting and guiding students to construct their own knowledge in the classroom community. Previously, subject matter knowledge was quantitatively defined by the number of courses one underwent, but over the years, subject matter knowledge has been looked at in a qualitative manner, which includes emphasising the cognitive processes and understanding concepts. Many researchers, including Jadama (2014), Prendergast and O'donoghue (2014), suggest that subject matter knowledge is much more crucial for an educator to be able to take up the responsibility of promoting learning by setting mathematical objectives and creating classroom situations suitable for pursuing and helping students make sense of the subject matter. This means presenting it in a suitable manner, which includes developing activities that lead to discussions, generalisations and conjecturing.

Brijlall and Isaac's (2011) study shows that there is a strong link between content knowledge and classroom practice. This direct link enables lecturers to facilitate learning by guiding students and asking questions that lifted the students' thinking to a higher level, instead of just giving answers whenever students asked for help. Their study also shows that having profound content knowledge enables lecturers to design activities that are at suitable cognitive levels for their students and are able to modify the activities accordingly.

More specifically, Furinghetti (2007) suggests that knowledge for teaching consists of three components: subject matter knowledge, pedagogical knowledge, as well as the educator's beliefs about mathematics and its teaching, which is one of the many components of pedagogical content knowledge. This concurs with Cooney (1994) who noted that the way educators learn mathematics often influences the way they will teach it. Hence, Furinghetti (2007) suggests that teacher education programmes should offer challenging situations that will contribute to the expansion of personal philosophies about mathematics and the teaching of the subject. These challenging situations involve using the history of mathematics to act as a mediator of knowledge for teaching the subject. The main aim of introducing the history of mathematics in teacher education programmes is to make the educators think about the meaning of mathematical objects while they experience the historical moments in which these mathematical objects were created. Moreover, Xenofontos and Papadopoulos (2015) argue that the inclusion of history in mathematical tasks is two-fold. Firstly, because mathematical tasks are perceived to be promoting the history of mathematics as a tool for solving mathematical problems. Secondly, mathematical tasks are seen as promoting the history of mathematics as a goal to achieve high cognitive levels.

Every educator deals with the massive complexity of the classroom situation on a daily basis. Ainley and Lutnley (2005; 2007), as well as Ainley's (2012) research show that apart from subject knowledge and subject-specific pedagogical knowledge, educators also have generalised attentional skills which allow them to draw on what is referred to as attention dependent knowledge. They describe this type of knowledge as a highly contextualised knowledge that is made accessible by paying attention to certain aspects of the classroom situation. The experienced educator is perceived as the one who possesses a large amount of attention skills for attending to cognitive and emotional aspects of the students' activity, which may not be obvious to someone without experience. Ainley and Lutnley (2007) also mention that experienced educators are able to view the classroom situation differently from an inexperienced educator, in that they can use attention dependent knowledge to probe into the learner's answer and end up understanding the learners' reasoning, whereas an inexperienced educator might have thought that the learner was just trying to disrupt the lesson. This attentiveness requires content knowledge. They also mention that this attention dependent knowledge becomes readily available during the course of the lesson, without prior planning on when to use it, because it becomes available in response to students' activities, thus showing that it is also a substantial part of pedagogical content knowledge.  Ainley (2012) confirms that attention dependent knowledge informs teachers' classroom practice.

For educators to be able to represent mathematical concepts as a logical and connected system, they must have  profound content knowledge, as maintained by Anthony and Walshaw (2009). An educator is able to identify his or her students' misconceptions, as well as students' level of understanding of mathematical concepts, if he or she has profound content knowledge. This is in agreement with Kilic (2011), whose study on pre-service teachers showed that they lacked the ability to identify the misconceptions and errors by learners. Thus, these pre-service teachers lacked the ability to identify the conceptual knowledge that the learners needed in order to eradicate the misconceptions and errors. In his study, Kilic (2011) found that when the pre-service teachers had  profound knowledge on a particular topic, it was easy for them to support the reasoning behind mathematical concepts and procedures by using concrete representations or by making connections with other topics. A profound possession of content knowledge also enables an educator to decide on what tasks and resources to use in his or her classroom. The notion of profound subject content knowledge as a necessity has been echoed by many researchers, among others, Kleickmann et al. (2013) and Ngwenya (2014). If an educator does not have adequate subject knowledge, he/she becomes constrained in many ways (Prendergast & O'donoghue, 2014). Firstly, when a student uses a method unknown to the educator, the educator might not be able to identify the student's errors. Secondly, when a student asks a question that is beyond the educator's knowledge, then the educator would be unable to help the student. Thirdly, the educator might not be in a position to identify or anticipate the students' errors (Prendergast & O'donoghue, 2014). Thus, pre-service teachers ought to have extensive subject content knowledge in order to make a difference in the mathematics classroom.

Pedagogical content knowledge is one of the seven groups of educators' knowledge, as proposed by Shulman (1986). The other six being content knowledge, general pedagogical knowledge, curriculum knowledge, knowledge of the learners, knowledge of educational contexts and knowledge of educational ends, values and purposes. Of these seven groups, pedagogical content knowledge has been widely researched, as mentioned in the above paragraph. Depaepe, Verschaffe and Kelchtermans (2013) conducted a web search in three data bases namely ERIC, PsycInfo and Web of Science on pedagogical content knowledge. Their results show that of the 60 articles that they reviewed, while on one hand there are some disagreements amongst the researchers, on the other hand, the researchers concurred on the following: 1. Pedagogical content knowledge links at least two types of knowledge. 2. Pedagogical content knowledge deals with educator knowledge that makes it possible for educators to accomplish the goals in teaching. 3. Pedagogical content knowledge is unique to specific subject content and is the educators' interpretation of specific subject matter. 4. It is an important pre-requisite form of teacher knowledge.

Hill, Ball and Schilling (2008) expanded on Pedagogical Content Knowledge (PCK) and split it into four aspects which include Common Content Knowledge (CCK), Specialised Content Knowledge (SCK), Knowledge of Content Knowledge and how students learn particular content (KCS) and finally, Knowledge of Content and Teaching (KCT). There seems to be a hierarchy in these four aspects. CCK is common content knowledge; the knowledge that teachers use in their daily work. In other words, any mathematics educator has knowledge of school mathematics, just like any other person or professional who has studied school mathematics. SCK is specialised content knowledge and this is special mathematical knowledge that a mathematics educator should possess. An educator with such knowledge is able to explain mathematical concepts using multi-representations of these concepts or explain procedures, why procedures work and why or how concepts are connected. Such an educator is also able to see and accept different methods or procedures presented by students. KCS is knowledge of content knowledge and how students learn particular content. An educator with such knowledge is aware and is able to anticipate errors, mistakes or misconceptions that students are likely to make. With such knowledge, an educator is able to eradicate or correct such mistakes as they arise. KCT is knowledge of content and teaching, including knowledge of the curriculum. Thus, educators with such knowledge are able to select and present, as well as sequence tasks that are appropriate for their particular class.

In concluding this part, it is worth mentioning that the researcher is aware that there are two mutually exclusive views on pedagogical content knowledge, these being, pedagogical content knowledge in practice, as proposed by Adler and Patahuddin (2012) and pedagogical content knowledge separated from practice, as proposed by Krauss and Blum (2012). Since this study is situated in teacher education, the researcher takes the position of pedagogical content knowledge in practice and the implications of this will be the way in which the data have been collected which includes video recordings of the lessons, as well as interviewing the lecturers.

## 2.6 Another dimension to lecturers' knowledge: Technological Pedagogical Content Knowledge

As mentioned in the preceding section, technology plays an important role in the teaching and learning of mathematics. The dynamic nature of software enables educators/lecturers to guide their learners/students to explore and investigate mathematical concepts. As the learners/students engage with the software, they are able to make meaning of what they are learning. For educators/lecturers to be able to design meaningful activities, they need to be in possession of the Technological Pedagogical Content Knowledge (TPCK). In other words, educators/lecturers need to be able to integrate technology into their pedagogical practices (Leendertz et al., 2013). According to Koh, Tsai and Chai (2013), TPCK is the extension of Shulman's (1986) Pedagogical Content Knowledge. In addition, TPCK is a type of knowledge that the educators need in their teaching practices, because it is both transformative and integrative (Koh et al., 2013). A large number of educators have made efforts to apply TPCK in their teaching practices. This shows that the educators deem it necessary to integrate technology in their teaching (Koh et al., 2013). As the teaching profession welcomes a new generation of educators, the educators are dared to use technology in their teaching. The use of technology is meant to improve the teaching and learning of mathematics. Most of the new generation of educators who are entering the profession are already technologically competent and are already comfortable with using technology in their teaching practices (Stewart et al., 2013).

TPCK is the relationship that exists between technology content and pedagogy (Leendertz et al., 2013). The use of technology adds value to teaching and learning and is very much linked to pedagogy as it cannot exist on its own. Thus, TPCK is perceived to co-exist with the following:

- Content Knowledge (CK), which is known as the mathematical knowledge that the educator/lecturer possesses.
- Pedagogical Knowledge (PK), which is known as the ability to select and use suitable teaching strategies.
- Pedagogical Content Knowledge (PCK), which is known as the specialised content knowledge.
- Technological Knowledge (TK), which is known as being able to select and use suitable teching and learning materials which include, textbooks, white boards, smart boards, computers and the internet.
- Technological Content Knowledge (TCK), which is known as the ability to teach mathematics using technology (Koehler, 2012).

Leendertz et al. (2013) show that mathematics educators with TPCK positively contribute to the effective teaching of mathematics. Also, the study conducted by Stewart, Robinson, Antoneko and Mwavita (2013) show that the in-service educators, as well as the pre-service educators, acknowledged the benefits of combining the subject content with technology and teaching strategies. Thus, these educators perceived themselves to be in possession of TPCK. This was because of the educators being easily able to integrate their teaching strategies, as well as their content knowledge, with technology.

Koh et al. (2013), conducted a web search of TPCK on Web of Science, Scopus, ERIC, and EBSCOhost databases and four journal articles were reviewed, which showed that there are two categories into which TPCK can be classified. Firstly, TPCK can be classified as general technology, which is the technological knowledge (TK) dimension. Secondly, it can be classified as subject specific technological knowledge, which is the technological content knowledge (TCK) dimension.

## 2.7 Learning opportunities – types of activities

For effective learning to take place, educators need to allow their learners opportunities to access background knowledge, which can be used as a foundation for building new knowledge (Rosenshine, 2012). Additionally, effective learning takes place when learners are engaging with good tasks (Johnson, Norqvist, Liljekvist, & Lithner, 2014). For a task to be considered appropriate and good, it would have been designed for the following reasons. Firstly, it would have been designed for the development of the learners' conceptual understanding and secondly, for maximising the learners' understanding of mathematical concepts (Chapman, 2013). Webb (2012) showed that when educators engage in planning and designing tasks that promote and provide learners with opportunities to learn, the learners' higher order thinking increased significantly. Additionally, the findings show that such tasks provoke the learners to draw on their higher order thinking skills.

Classrooms are prone to diversity, whereby there may be a mixture of high attaining learners, as well as low attaining learners. This was the case with a classroom in the study reported in the seminal work of Ferguson (2009). The educator designed tasks that were conceptually challenging and kept the learners' cognitive demand at high levels. Despite the fact that there were low attaining learners in that class, the educator did not change the level of the task, but used scaffolding, as well as probing questions to assist the learners. This resulted in the learners developing high level thinking skills, as well as profound understanding of the concepts.

The process of developing and implementing a mathematical activity forms a cycle in that during implementation, if the activity does not work well, the educator/lecturer can rework on the activity and re-implement it in another lesson (Georgius, 2014). Mathematical activities that promote higher order thinking and maintain the cognitive demand at high levels are not easy to develop. Such activities require educators to have profound mathematical knowledge which enables the educator or lecturer to effectively implement the activity. Thus, when an educator does not have profound mathematical knowledge, he/she is confined to frequently using the activities from the text book (Georgius, 2014).

When learners or students are provided with opportunities to engage in classroom activities and tasks that are challenging, their cognitive demand is kept high (Viesu & Oliveira, 2012). Such activities stimulate and allow the learners/students the opportunity to engage in productive classroom dialogue. Hence this study seeks to shed light on the types of mathematical activities that the pre-service teachers are exposed to during their teacher training in the school of education.

Even and Tirosh (1995) acknowledge that the educator's CK,  and knowledge about the students' ways of thinking, are essential in that the educator's decision about whether the students' answer is correct or can be utilised in learning will be based on the educator's content knowledge. The knowledge about the students' ways of thinking helps in developing the students' reaction that can push the students to construct their knowledge and thus, opens an opportunity to learn. Hence, the following part of this literature review focuses on the learning opportunities given to the students. Cooney (1994) suggests that in order for educators to develop the type of mathematical activities that provide the students with learning opportunities, the educators must themselves do these mathematical activities, since the way we learn plays a significant role in the way we teach.

According to Kilpatrick et al. (2001), teaching that promotes the development of mathematical competence over time takes different forms, each with its own potential. They also mention that all forms of teaching can be looked at from the point of how teachers, students and learning materials interact. Kilpatrick et al. (2001) further point out that effective teaching depends on the joint and mutually dependent interaction of the educator, students and the learning materials. They also mention that having high expectations for the students, motivating the students to have value for their activities, allocating sufficient time for the activities, the type of questions asked by the educator, allocating enough time for the students to respond and encouraging the students, all open up many opportunities for the students to learn.

The students have their part to play in all this, which requires taking some level of responsibility for learning and hence, students are expected to engage in mathematical thinking, applications, developing conceptual connections (Johnson et al., 2014) and thus, it may be argued that this is relevant in teacher education practice. The question is to what extent this takes place and how it relates to the knowledge/skills development of the student teachers. In order to learn how to generate conjectures, proofs and definitions, to critique conjectures and look for counter examples, generalise and symbolise, the students need to take part in a practice where such activities are dominant and valued. Hence, this study seeks to explore the types of activities that are legitimised in the pre-service teachers' calculus module.

In any classroom situation, there are various reasons, which would result in an educator having to change tasks. In his study on primary school educators, Olson (2005) observed two educators who changed their tasks during the lessons. The first educator changed her task because she realised after implementing it, that its cognitive demand was low, as the learners in her class were fixated on reproducing an anticipated answer, so she changed the task by elevating it to procedures with connections. The second educator also noticed that the cognitive demand of her task was low for most of the learners, so she let the learners help each other while she maintained the classroom discourse. When the researcher of this study did an action research on teaching trigonometry, she realised that the task that she had in order to elicit learners' prior knowledge needed to be split into manageable bits and spread over a few lessons (Likwambe, 2004). Therefore, we would expect that to be the case in teacher education as well and thus, in this study, the researcher aims to investigate how tasks are changed in teacher education and informed by what, for this to take place. In addition, Brijlall and Isaac (2011) support these sentiments, as their study showed that modifying tasks is necessary for the development of higher thinking skills in students.

Relating this to the observations at school level in a large Gauteng study:

> *We have an important observation about the level of cognitive demand for lessons we saw in South Africa. The observed level was the one implemented by the teacher and not necessarily the level intended… (Carnoy & Chisholm, 2008, pp. 53-54)*

These findings are consistent with results from the TIMSS 1999 video study, as well as the findings by Stein, Smith, Henningsen, and Silver (2000)*:*

*'Mathematical tasks or problems with high level cognitive demands 'are most difficult to implement well, frequently being transformed into less-demanding tasks during instruction' (Stein et al., 2000, p. 4).*

More than the level of cognitive demand alone, this study is interested in the extent to which (a) there is conceptual focus, (b) if the nature of the mathematical activity (see Chapter 3: Theoretical Framework) changes.

In an attempt to avoid tasks that encourage performing routine algorithms, Johnson et al. (2014) designed tasks that required learners to construct their own knowledge. Their research findings showed that such tasks have a significant influence on the students' cognitive efficiency. Their findings also showed that using tasks that require students to struggle with mathematical concepts allows them opportunities to come up with their own solutions as they use higher order thinking.

Thus, pre-service teachers ought to engage with mathematical activities, which are designed to enhance their conceptual understanding of mathematical concepts. Downs and Mamona-Downs (2013) point out those activities that the pre-service teachers take part in, require them to draw on their conceptual and procedural knowledge. Such activities include conjecturing, proving and investigating. Lesseig (2016) and Supratman, Ryan, and Rustina (2016) agree that conjecturing requires students to be deep and critical thinkers. The study conducted by Supratman et al. (2016) showed that the students who were exposed to conjecturing, improved significantly in their thinking skills. Investigations are also essential to the learning of mathematics because they require students to engage in active learning. The findings of Marshman and Brown (2014) showed that the teachers in their study made sense of what they were learning by taking part in investigative activities. Fleron, von Renesse, and Ecke (2014) suggest that proofs are important in the learning of mathematics because they involve logical thinking, which result in students making valid conclusions. The study conducted by Reid (2014) revealed that when students are given opportunities to perform proofs, they develop profound conceptual knowledge of the topics with which they are dealing. Therefore, this study seeks to shed light into the mathematical activities with which the pre-service teachers in the calculus module engage.

## 2.8 In-service teacher education

There are three phases of teacher education (Ogunyinka, Okeke, & Adedoyin, 2015). These researchers point out that the first phase of teacher education is pre-service teacher training, the initial training. The second is the induction phase, whereby newly qualified educators are mentored and given support by experienced educators for the first few years. The third phase is the in-service teacher training, the professional development of already qualified educators. In-service teacher education denotes that educators, who are already qualified continue to develop their professional competences through various teacher development programmes (Naik & Raman, 2013). Thus, educators mostly develop competences in their subject and pedagogical knowledge. This is supported by Bozkurt et al. (2012). These researchers' findings show that firstly, educators acknowledge the importance of enroling in in-service programmes because of the need to keep abreast with the changes in the curriculum. Secondly, educators acknowledge the need for in-service training mostly for professional, as well as personal development.

Apart from improving their subject and pedagogical knowledge, eduactors enrol for in-service programs because professional development ensures the quality of a school, effectiveness of an educator, as well as learner success (Balta, Arslan, & Duru, 2015). Additionally, educators enrol for in-service training so that they develop their subject knowledge, sharpen their teaching skills, as well as become knowledgeable about the developments in technology (Koc, 2016).

The findings of the study conducted by Levi-Keren and Patkin (2016) reveal that educators who enroled for in-service training significantly developed in pedagogical content knowledge. The findings also show that the educators showed vast improvement in their mathematical knowledge, while they also acknowledged that the program empowered them in terms of their understanding of mathematics. Furthermore the educators acknowledged that the program empowered them in their professional practice. The educators pointed out that after their involvement with the program, they were able to apply what they had learnt to their teaching. This is supported by research findings by Balta et al. (2015), which show that educators who enrolled for in-service training significantly improved in their designing of mathematical tasks, classroom management, as well as their pedagogical knowledge.

On the contrary, the findings of the study conducted by Koc (2016) show that the educators who enrolled for in-service training felt that the program failed to meet their needs. On anlysing the course materials, Koc found that the materials which were designed and used in the course did not meet the educators' needs, such as developing their content and pedagogical knowledge. The educators also felt that the activities used in the program did not engage them actively, as well as conceptually. This is supported by the research findings by Muir and Livy (2012), which show that the educators who enrolled in the in-service program had very limited knowledge of their subject. The study also showed that the in-service educators had a variety of misconceptions about many mathematical concepts. Thus, such findings raise many concerns, especially since the educators are already practicing. Thus, in view of this Kidwai et al. (2013) point out that this is a result of the poor quality of the training that the pre-service teachers receive as they initially train to be educators. Also, this is due to the fact that some of the in-service educators do not go through pre-service training and as a result, they possess limited content knowledge. Also, the findings of Ramnarain and Fortus (2013) show that the in-service educators who had enrolled for the Advanced Certificate in Education (ACE) program had insufficient content knowledge, even after completing the program. The findings also show that these educators felt that their Pedagogical Content Knowledge (PCK) was compromised due to the insufficient content knowledge of the new topics that had been introduced to the curriculum.

The preparation of educators in South Africa faces significant challenges, one of which is how the teacher education programme appreciates the notion of mathematics for teaching (Adler & Davis, 2006). In their Quantum project, Davis et al. (2005) draw from Bernstein's performance and social logic competence models. In the performance model, the student can or cannot perform according to the set standard, where as in the social logic competence model, all students are said to be competent and are active, creative, as well as self-regulating. Davis et al.'s (2005) results show that in most cases, the two models co-exist in the teaching practices that they were studying. They conducted studies at three different universities, which they referred to as cases 1, 2 and 3.

In case 1, the in-service educators were to acquire a particular pedagogy, which was to be modelled by their lecturer, which was learning how to teach algebra. The educators in case 1 were to imitate the way their lecturer demonstrated how to teach algebra, although the principles that structured the activity were to be acquired implicitly. Here, the components of teaching were always at hand since this was an activity of teacher education. The meaning of mathematics was profoundly grounded in everyday metaphors. Out of thirty-six (36) evaluative events, four of them specifically appealed to teaching and three of those four were true experiences of the educators in the study, while one appealed to the official curriculum. In this case, no appeal was made to the field of mathematics education. This study seeks to explore whether appeals in the pre-service teachers' calculus module are made to everyday metaphor, curriculum, students experiences or lecturers' authority.

In case 2, although it was not made explicit to the in-service educators as to what counts as knowledge, the practice which was to be acquired by the in-service educators in this study was reflection, where the teachers were to consciously examine their own practices.  The educators in this case were seen as experienced and wellinformed. They were expected to engage with the course materials and in doing so, the values would become clear to the educators because it was presumed that the teachers already possessed these. The course in case 2was aimed at bringing out and strengthening the proficiencies that the educators already had. Unfortunately, the educators in this study did not engage with the materials on their own at home and as a result, the lecturer ended up modelling the expert practice required, without the quality criteria being made explicit.

In case 3, the practice to be modelled was the cross-examination of records of practice with mathematics education as a resource, focusing on mathematics reasoning as a practice. Educators in this course were expected to read three papers before the contact session, which they did, and then watch a video recording during the contact session of a mathematics classroom. During the discussion, the educators were asked to describe how they observed the different strands of mathematics being developed by the teacher in the video. All their sessions were structured in a similar way. The educators in this course were expected to explain and describe, as well as to justify their reasoning on what they observed in the video extracts and what they read in the papers that they were given to read, as well as how they saw themselves in their own practices. In this course, it was made explicit to the in-service educators what counts as knowledge. Thus, this study seeks to explore the components of mathematics for teaching that are exhibited by the lecturers explicitly or implicitly.

An educator's intentions to provoke, identify and then facilitate ideas of proof and various kinds of justification is profoundly important to effective teaching (Adler, 2005). Further to this, this kind of mathematics is not always on what the mathematical preparation of educators focuses (Adler, 2005). Thus, knowing how to ask questions that promote the learners' development of higher order thinking is a fundamental skill that educators need to possess. If educators participate in activities that allow them to engage in cognitive processes that they want their learners to acquire, then they are in a better position to promote higher order thinking in their learners, (Moodley, 2013). The results of Moodley's (2013) study show that the educators who were enrolled for the in-service programme ACE, showed an improvement in the type of tasks they set for their learners after they themselves had taken part in similar activities.

For educators to be able to teach effectively, they need to be confident as they do their work. This is supported by Phin (2014) who pointed out that having enrolled in the in-service training, the educators gained confidence with regard to their content knowledge, as well as their pedagogical knowledge. The educators also indicated that they felt confident with the way they had started planning for their lessons and this resulted in the improvement of their learners' performances. Berg and Huang (2015) support this view because, in their study, the educators showed significant improvement in both their subject and pedagogical knowledge, after having gone through the in-service training.

Similar findings emerged from Ahmad et al.'s (2012) study, even though these researchers also found that the educators in their study lacked reseach skills. If educators are well-equiped in terms of research skills, then they are able to conduct research in their own classrooms on a variety of issues that could emerge as they do their work of teaching. Hine (2013) agrees with the notion of educators conducting their own research in their own classroom, by pointing out that action research is another way for educators to develop their teaching strategies. Action research involves identifying a problem in one's own teaching, then planning and implementing a strategy, observing and reflecting. Additionally, action research provides educators with opportunities to investigate and reflect on their own teaching (Hagevik, Aydeniz, & Rowell, 2012).

## 2.10 Conclusion

In this chapter, the types of conceptual learning that the pre-service teachers need have been discussed. Among these are the notion of a deep approach to learning, transformational reasoning, learning by appropriation, as well as the notion of problem solving. A deep approach to learning has been described by Draper (2013) as the learning that occurs when a student seeks to understand the meaning of the main points. Transformational reasoning occurs when a student is able to perform a mental or physical operation on an object. This results in a student being able to envision the transformation that the object undergoes (Simon, 1996). Moschkkovich (2004), as well as Alqahtani and Powell (2016), agree on the notion of appropriation as having much to do with interaction between the students and the lecturer, or amongst the students. All these require the student to be active, as he or she participates in the learning process. Govender (2012) points out that in order for students to develop profound problem-solving skills, educators must also have profound problem-solving skills themselves.

Concept images of the students, especially the concept images of the topics in calculus, have been widely researched. Some of these studies focused on students' misconceptions of the derivative or the integral concepts, while others focused on the understanding of the derivative or the integral concept (Likwambe & Christiansen, 2008; Habineza, 2013; Serhan, 2015). The pre-service teachers need to engage conceptually with subject content, so that they are able to explain to the learners, so in turn the learners can have profound conceptual understanding of the topics in mathematics.

The different types of knowledge that the lecturer is expected to possess, have also been discussed in this chapter. Some of this knowledge is the same, but just named differently by different researchers. Many researchers, including Bansilal (2012), Kraus and Blum (2012), agree that subject and pedagogical knowledge is crucial in developing the students' mathematical knowledge. The notion of PCK has been widely researched, with some researchers splitting it into categories, in an effort to understand the knowledge that the lecturer needs, in order to be able to develop the students' mathematical understanding. Furthermore, in this chapter, learning opportunities, as well as the reasons why educators or lecturers change tasks during lessons, have been discussed. Cooney (1994) is among the researchers who agree that lecturers need to design activities that provide students with learning opportunities.

Lastly, this chapter discussed the importance of in-service teacher training. Davis et al.'s (2005) Quantum project shows that although what counts as mathematics was justified by appeals made to the student teacher's experiences, as well as the curriculum, most of the justification was profoundly by appeals made to the everyday metaphor. This research seeks to provide insight in to the type of support given by the lecturer to the pre-service teachers, whether it is from the metaphorical or the mathematical domain. The next chapter discusses the conceptual framework that informed this study.

# CHAPTER THREE

## CONCEPTUAL FRAMEWORK

### 3.1 Introduction

The previous chapter deliberated on the literature relevant to this study. This chapter is a description of the conceptual framework underpinning the study. Apart from the cognitive demand level, the study considers the extent to which there is conceptual focus. Essentially, this study seeks to explore the dialogue that takes place in the calculus lecture room, as well as what is legitimised in the calculus lecture room. In this regard, there is need to describe the four different frameworks which capture the different aspects of what is legitimised in the calculus lecture room. Moreover, the frameworks complement each other. It is hoped that the frameworks would assist the researcher with the responses to the following aspects of the calculus lecture room:

1. What is legitimised as mathematics in the calculus lecture room?
2. What is legitimised as mathematics learning in the calculus lecture room?
3. What is legitimised as mathematics teaching in the calculus lecture room?
4. What is the nature of dialogue in the calculus lecture room?

The first question of this study is: What mathematical activities are legitimised by the lecturers? This question is informed by the Legitimising Appeals framework and the Mathematical Activities framework, because the study sought to explore what was counted as mathematics in the calculus lecture room. The second question is: How and why are the tasks from the materials organised by the lecturers? This question is informed by the Cognitive Processes framework, as well as the Mathematics for Teaching framework, because the study seeks to explore what is counted as mathematics learning, as well as mathematics teaching in the calculus lecture room.

The third question is: What is the nature of dialogue in the calculus lecture room? This question is informed by the Inquiry Cooperation Model framework, because the study seeks to explore the communication that the lecturers engage with the students.

Thus, to answer the research questions, the researcher used the Mathematics for Teaching framework, which was developed by Adler et al. (2005), as well as Hill et al. (2008), as the overarching framework. Under the Mathematics for Teaching umbrella, four frameworks: Mathematical Activities which was developed by Niss (2002), Cognitive Processes and types of knowledge, which was developed by Anderson et al. (2001), Legitimising Appeals, which was developed by Davis et al. (2005), as well as the Communication Inquiry Model which was developed by Alro and Skovsmose (2002) are used. As this study draws from a range of conceptual frameworks, all of which are anchored within different perspectives of mathematics education, the following paragraphs briefly discuss how aspects of each framework link to mathematics teaching, as well as to teacher training and their shared assumptions.

Mathematical Activities is based on the understanding of what mathematics is, and thus links with mathematics and education because it allows one to see what mathematical activities are being legitimised within a particular lecture, as proposed by Stein et al. (2000). If a series of questions that lead to conjecturing are being asked by the lecturer, or if switching between representations or symbols is being encouraged, this would imply that the mathematical competences needed by the students are being developed.

The Cognitive Processes and Types of Knowledge framework has a taxonomy table that furnishes educators with a tool that develops common understanding and sensible communication in the classroom. This framework links with mathematics teaching and learning because the explanations or questions that are asked by the lecturer enable the researcher to see what type of learning is legitimised in the calculus lecture (Niss, 2002). By using the taxonomy table, one is able to see whether the dialogue in the lecture room is more focused on procedures or principles.

Legitimising Appeals links with teacher education and mathematics teaching. As the dialogue between lecturer and student transpires within the calculus lecture room, what is counted as mathematics or mathematics teaching is justified. This justification is made by appeals to mathematics, everyday metaphor, lecturer’s authority or experience, students’ experience or curriculum, as indicated by Adler and Davis (2006).

The Inquiry Co-operation Model links with education as this exhibits the dialogue or inquiry processes that take place in the lecture room. If the full cycle of the model is exhibited, then the lecturer is allowing inquiry to take place, thus showing that the students are capable of independently engaging in mathematical thinking.

The Mathematics for Teaching framework enlightens as to what is counted as teaching mathematics, the Mathematical Activities framework enlightens as to what is counted as mathematical activities, while the Cognitive Processes frame work enlightens on whether it is procedural or conceptual knowledge that is legitimised in the calculus lecture room. The Inquiry Co-operation framework enlightens on the nature of dialogue, which takes place in the calculus lecture room. Through communication, the students are likely to acquire what the lecturer intends to legitimise in the calculus lecture room (Parker & Adler, 2012). The lecturer can implicitly or explicitly exhibit what is counted as mathematics for teaching, or what is counted as Mathematical Activities, as well as legitimise procedural or conceptual knowledge. These frameworks thus share the assumption that learning mathematics or the creation of mathematical knowledge is a social activity. This is supported by Msimanga (2016) who points out that communication in any classroom is fundamental to the teaching and learning process.

## 3.2 Mathematics for teaching

The researcher is aware that Mathematics for Teaching is a widely researched phenomenon. For the purposes of this research, the use of the ideas by Adler, Davis, Kazima, Parker and Webb (2005), Kazima, Pillay and Adler (2008), as well as Hill et al. (2008), have been chosen. These researchers focus on lecturer action, rather than lecturer knowledge, as this research study also focuses on the lecturers' actions.

Mathematics for Teaching has been described by Kazima et al. (2008) as specialised mathematical knowledge that educators need to know or already know. In addition, this type of knowledge includes how educators would use it in their teaching so that they are able to deal with a variety of responses from the students. From this description, it is evident that the underlying assumption that underpins the notion of mathematics for teaching is that there are certain aspects in mathematics that are needed to be known by the mathematics educators and they also need to know how to use these aspects in their teaching of the subject (Adler & Davis, 2006). Some of these aspects include unpacking or decompressing of mathematical ideas, as indicated by Ball, Bass and Hill (2005). However, the limitation of using this framework is that this study is located within the calculus lecture room with pre-service teachers, and is not in the mathematics education module, so the lecturer might not explicitly elicit what counts as mathematics for teaching. Although Kazima et al. (2008) focus on school teaching and learning, university lecturing is different but teaching at a university school of education is also similar to school teaching. After reading through articles on Mathematics for Teaching, several components were identified, some of which have also emerged from the data analysis of this study and will be described in the paragraphs that follow.

### 3.2.1 Unpacking mathematical ideas

Being able to break down a concept into manageable bits by students has been named **unpacking** by Hill et al. (2008). In order to be able to unpack mathematical ideas, coupled with the deep understanding of these mathematical ideas, a lecturer must also know how these ideas progress in learning. This unpacking also involves the way in which these mathematical ideas are introduced to the pre-service teachers. Adler et al. (2005) believe that by unpacking the mathematics while teaching various sections of the subject, the lecturer is able to help the pre-service teachers develop profound conceptual understanding of the mathematical concepts and this helps the pre-service teachers to make connections of the concepts with ease. As much as mathematical ideas are unpacked, procedures and symbols may also be unpacked by the lecturer.

Unpacking is exhibited when the lecturer is explaining concepts, procedures, terms or the meaning of symbols. Using the Cognitive Process framework, (see descriptive explanations in paragraphs that follow), would be to enable an understanding of the procedural or conceptual knowledge, as well as remembering these. It would also be reformulating when using the Inquiry Co-operation Model (see descriptive explanations in paragraphs that follow), framework and handling mathematical symbols and formalisms or using tools and aids from the Mathematical Activities framework (see descriptive explanations in paragraphs that follow).

As the lecturer unpacks concepts and processes the meaning of symbols or terms, he at times makes justifications to mathematics, teacher education or students' experiences. This is supported by Adler and Parker (2012), whose research shows the lecturer legitimising content knowledge by appeals made to mathematics, teacher education, curriculum or students' experience.

### 3.2.2 The use of representations

Adler (2005) mentions that mathematics for teaching is exhibited by an educator who is able to work with representations in such a way that they are firstly anticipated and then elicited. This component of mathematics for teaching has been named *the use of representations*, which is exhibited when the lecturer is using a graph or diagram to explain a procedure or concept, or when the lecturer is linking the algebraic form of a function to its graphical form in the explanation. In other words, the use of representations is exhibited when the lecturer uses various forms of representations to teach a concept or procedure. Since explanations are involved, the use of representations is illustrated by reformulating from the Inquiry Co-operation Model framework, understand procedural or conceptual knowledge from the Cognitive Processes framework and representing mathematical entities from the Mathematical Activities framework. When representation is exhibited, the lecturer displays that using various forms of representations when explaining concepts to the pre-service teachers and this is crucial in the teaching of mathematics. This is supported by Akkus and Cakiroglu (2010), who showed that using a variety of representations helps students improve their understanding as they switch between representations. This also deepens the pre-service teachers' understanding of mathematical concepts (Silver, 2015).

### 3.2.3 Mathematical communication

Mathematics for Teaching may be exhibited by a lecturer who uses mathematical language carefully, as highlighted by Adler (2005). Additionally, mathematical communication includes the lecturer being able to put forward mathematical explanations that are clearly understood by the pre-service teachers, as well as explanations that are useful and meaningful to the pre-service teachers. This component of mathematics for teaching has been named *mathematical communication*, which is exhibited when the lecturer reformulates, explains procedures or concepts and when he explains the meaning of symbols and terms. Thus, mathematical communication, as a component of Mathematics for Teaching, is illustrated by getting in contact or reformulation, components of the Inquiry Co-operation Model framework, understand procedural, conceptual or factual knowledge, components of the Cognitive Processes framework, as well as handling mathematical symbols and formalisms and using tools and aids, components of the Mathematical Activities framework. Communication is exhibited by these components because they all involve explaining and as the lecturer explains, he uses mathematical language carefully.

### 3.2.4 Questioning

Asking learners questions that have appropriate levels of mathematical demand and that help them grow in their thinking has been identified as mathematics for teaching by Adler (2005). This component of mathematics for teaching has been named *questioning,* which is exhibited when the lecturer asks questions that promote the development of the pre-service teachers' thinking. This happens when the lecturer is locating, identifying, advocating or challenging his students, as well as when he asks questions that make his students apply or analyse procedural, conceptual or factual knowledge. This also occurs when the lecturer asks the pre-service teachers questions that require them to understand the scope of the problem (thinking mathematically), as well as to lead them to answer or conjecture (reasoning mathematically).

Questioning, a component of mathematics for teaching, is connected to locating, identifying and advocating the components of the Inquiry Co-operation Model framework. When locating or identifying, the lecturer will be checking prior knowledge. In addition, reasoning and thinking mathematically are the components of the Mathematical Activities framework. Apply or analyse procedural or conceptual knowledge, are the components of the Cognitive Processes framework. Thus, the lecturer conveys that in the teaching of mathematics, it is important to ask questions that make the pre-service teachers think about how they perform procedures, as well as how mathematical ideas are connected (Silver, 2015). In addition, the lecturer conveys that checking prior knowledge is crucial in teaching mathematics. This is supported by Mhakure and Jacobs (2016) whose research results show that checking prior knowledge and using it to develop new concepts helps learners to grasp the new concepts.

### 3.2.5 Translating

Adler (2005) identifies mathematics for teaching as being able to translate mathematical ideas from one symbolic system to another or from one representation to another and this component of mathematics for teaching has been named translating, which is exhibited when the lecturer translates symbols or terms from one symbolic form to another, as indicated by Adler (2005). Translating is exhibited when the lecturer is reformulating, when the lecturer is explaining a procedure or a concept that involves terms or symbols, or when the lecturer is explaining the meaning of symbols or shifting between different symbolic forms and showing that they mean the same.

Translating, a component of mathematics for teaching, is also exhibited by handling mathematical symbols and formalisms,which is a component of Mathematical Activities framework. Translation is also exhibited by understanding procedural knowledge, which is a component of the Cognitive Processes framework and reformulation, which is a component of the Inquiry Co-operation Model framework. By exhibiting translation, the lecturer displays that knowing the meaning of symbols, as well as shifting between various symbolic forms, is important in the teaching and learning of mathematics, because this strengthens the students' understanding (Premprayoonk, Loipha, & Inprasitha, 2014).

### 3.2.6 Simplification

When an educator is able to work with definitions appropriate to the class, then this, according to Adler (2005) is referred to as mathematics for teaching. This has been referred to as *simplification*, which is exhibited when the lecturer works with definitions relative to the pre-service teachers in the calculus module (Adler, 2005). This occurs when the lecturer explains the meaning of definitions, as well as the meaning of mathematical symbols. Simplification is illustrated with its connections to understanding conceptual knowledge, which is a component of the Cognitive Processes framework and handling mathematical symbols and formalisms, which is a component of the Mathematical Activities framework. Thus, the lecturer conveys that knowing how ideas are connected, as well as knowing the meaning of mathematical symbols, is important in teaching mathematics (Towers & Proulx, 2013).

### 3.2.7 Perception

Mathematics for teaching was identified by Kazima et al. (2008), when a lecturer exhibited the capability of working with pre-service teacher's ideas. This happens when a pre-service teacher makes a suggestion or when a pre-service teacher puts forward an idea and the lecturer picks up on that idea and works with it and then explains the connection between the pre-service teacher's answer and the method used, or why the method used does, or does not work. In addition, when a lecturer interprets the pre-service teacher's mathematical thinking and reasoning and works with it, he exhibits perception, as noted by Adler and Davis (2006). This component of mathematics for teaching has been referred to as *perception*, which entails being able to work with the pre-service teacher's ideas which is exhibited when the lecturer is reformulating by first repeating what has been said by the pre-service teacher and then carries on to expand the explanation of concepts or procedures. As the lecturer repeats what the pre-service teacher has just said, the lecturer further explains the procedure with emphasis on the meaning of symbols, concepts procedures, etc. Thus, the lecturer conveys that it is essential to know what to do, to know how to substitute using different symbols, as well as to be able to pick up a pre-service teacher's idea and clarify it. Molefe and Brodie's (2010) study confirms this by maintaining that the educator in their study worked with learners' ideas which resulted in strengthening the learners' understanding.

Perception, a component of mathematics for teaching, is reflected by reformulation, which is a component of the Inquiry Co-operation Model framework. It is also reflected by understand procedural knowledge, which is a component of the Cognitive Processes framework and by handling mathematical symbols and formalisms, which is a component of the Mathematical Activities framework.

One of the objetives of this study is to explore the mathematics for teaching that is legitimised in the calculus lecture room of the pre-service teachers. Unpacking, questioning, translation, use of representations, mathematical communication and perception are the components of mathematics for teaching that the lecturers could elicit explicitly or implicitly as discussed in the preceeding sections.

Since mathematics for teaching is the umbrella framework for this study, later in this chapter, a discussion on how the components of the other frameworks link with each other, as well as with mathematics for teaching, is presented.

### 3.3 Mathematical activities

The first question in this research study is: What mathematical activities are legitimised by the lecturers? Thus, to answer this question, the researcher firstly used the categorisation of mathematical activities developed by Niss (2002). These categories are based on the understanding of what mathematics is. Niss (2002) derives from the perspective that is based on the understanding of what mathematics is. Hence, his stance is that there is something constant in the discipline of mathematics over time. Thus, one cannot use the subtopics (for example, calculus, algebra, trigonometry) to define the discipline. The categorisation of mathematical activities was developed by Niss (2002), in order to capture the aspects that we see as essential to mathematics, although one might say that these categories are not themselves constant. It was deemed necessary to use these categorisations of mathematical activities, firstly, because the researcher needed a competency framework to supplement the Mathematics for teaching framework, since it made it possible to distinguish between the different components of mathematics for teaching. Secondly, because Niss' (2002) views, combined with the view that students should engage in mathematical activities in order to learn mathematics, links well with this study, since the materials of the module in this study were developed with the intention that the students engage conceptually with the materials. Hence, one of the objectives of this study is to explore the mathematical activities that are legitimised in the calculus lecture room of the pre-service teachers.

In order for students to carry through any mathematical activity, they require the application of one or several mathematical competences. Therefore, it is necessary to identify competences involved in different mathematical activities. Mathematical competence means being able to comprehend, do, critique and use mathematics in different mathematical situations (Niss, 2002). Mathematical activities are categorised using the following competences. These were adapted from Niss (2002, pp. 7-9).

### 3.3.1 Thinking mathematically

In order to think mathematically, students are required to master mathematical modes of thought, such as:

- Posing questions that are characteristic of mathematics, and knowing the kind of answers
- Understanding and handling the scope and limitations of a given concept
- Extending the scope of a concept by abstracting some of its properties; generalising results to larger classes of objects
- Distinguishing between different kinds of mathematical statements (Niss, 2002, p. 7)

### 3.3.2 Posing and solving mathematical problems

In order to pose and solve mathematical problems, students are required to:

- Identify, pose and specify the different kinds of mathematical problems (pure or applied; open-ended or closed).

- Solve the different kinds of mathematical problems (pure or applied, open-ended or closed), whether posed by others or themselves (Niss, 2002, p. 7)

### 3.3.3 Modelling mathematically

In order to model mathematically, students are required to:

- Analyse foundations and properties of existing models, including assessing their range and validity
- Decode existing models, i.e. translate and interpret model elements in terms of the reality modelled
- Perform active modelling in a given context by:
    - Structuring the field
    - Mathematising
    - Working with (in) the model, including solving the problems, it gives rise to: - validating the model, internally and externally
    - Analysing and criticising the model in itself and possible alternatives
    - Communicating about the model and its results
    - Monitoring and controlling the entire modelling process (Niss, 2002, p. 7)

### 3.3.4 Reasoning mathematically

In order to reason mathematically, students are required to:

- Follow and assess chains of arguments, put forward by others
- Know what a mathematical proof is (not), and how it differs from other kinds of mathematical reasoning
- Uncover the basic ideas in a given line of argument (especially a proof)
- Devise formal and informal mathematical arguments (Niss, 2002, p. 8)

### 3.3.5 Representing mathematical entities

In order to represent mathematical entities, students are required to:

- Understand and utilise (decode, interpret, distinguish between) the different types of representations of mathematical objects and situations

- Understand and utilise the relations between different representations of the same entity, including knowing about their relative strengths and limitations
- Choose and switch between representations (Niss, 2002, p. 8)

### 3.3.6 Handling mathematical symbols and formalisms

In order to handle mathematical symbols and formalisms, students are required to:

- Decode and interpret symbolic and formal mathematical language and understand its relationship to natural language
- Understand the nature and rules of formal mathematical systems
- Translate from natural language to formal/symbolic language
- Handle and manipulate statements and expressions containing symbols and formulae (Niss, 2002, p. 8)

### 3.3.7 Communicating in, with, and about mathematics

In order to communicate in, with, and about mathematics, students are required to:

- Understand others' written, visual or oral texts, in a variety of linguistic registers, about matters having a mathematical content
- Express themselves at different levels of theoretical and technical precision, in oral, visual or written form, about such matters (Niss, 2002, p. 8)

### 3.3.8 Making use of aids and tools (including IT)

In order to make use of aids and tools, students are required to:

- Know the existence and properties of various tools and aids for mathematical activity, and their range and limitations
- Be able to reflectively use such aids and tools (Niss, 2002, p. 9)

## 3.4 Legitimising Appeals

The researcher applied the Legitimising Appeals framework of Adler et al. (2005), to characterise the type of legitimising appeals. Legitimising appeals are important to this study because they are about the extent to which access to principles of the field/discipline is provided to the students by their lecturer. Thus, for example if the students do not get a reason for a particular algorithm, they do not get a sense of what counts as justification within the field/discipline.

Within this framework, evaluative events are analysed. The stance of the framework on evaluative events is partially because of the sociology of knowledge. Bernstein (1996) claims that what is evaluated is learned. In some sense, this relies on the understanding that we are very social individuals, and even our formation of self is determined by how others react to us. By using this framework, the researcher was able to indicate what was counted as mathematics and mathematical activities, as well as mathematics for teaching in the calculus lecture rooms of this study.

Evaluative events refer to teaching-learning sequences focused on the attainment of some or other content, and the purpose of evaluation is to communicate benchmarks for the construction of legitimate texts, whether implicitly or explicitly. On the other hand, an act of evaluation must appeal to some or other authorising ground, to substantiate the selection of the benchmarks. Evaluative events disclose the kind of mathematical and teaching knowledge that may become legitimate, in other words, the kind of mathematical knowledge and teaching knowledge that comes to be privileged. The legitimating appeals are usually spread over appeals from various spheres of influence, especially to mathematics, mathematics education, everyday metaphors, experiences of pre-service teachers/lecturers, aspects of official curriculum documents and some form of authority.

## 3.5 Cognitive processes, types of knowledge

### 3.5.1 The taxonomy table

Perhaps the most wellknown categorisation of cognitive demand in mathematics education comes from Stein et al.’s (2000) rubric with its four categories: memorisation, procedures without connections, procedures with connections and doing mathematics. This was also used in the study conducted by Carnoy and Chisholm (2008) with Grade 6 mathematics educators in Gauteng. However, the researcher has chosen to use the wider framework from Anderson (2005), based on the work by Anderson et al. (2001). As much as this study focuses on the mathematical activities within the calculus lecture room, the intention is not to analyse the tasks, but to look at the lecturers’ actions and thus, this framework has been chosen, instead of Stein et al.’s (2000) rubric.

The second question is: How and why are the tasks from the materials organised by the lecturers? Thus, to answer this question, the researcher intends to adopt the taxonomy table developed by Anderson (2005), which is an extension/revision of Bloom's taxonomy (1956). This framework is based on three aspects, which are: the structure of educational goals, the advances in cognitive psychology and the attempts to categorise educational goals. Thus, the taxonomy table furnishes teachers with a tool that develops common understanding and sensible communication. It also provides a way by which teachers can develop a better understanding of educational objectives, so that they can use this understanding to improve assessments, instruction, etc. The taxonomy table allows the researcher to identify situations where the cognitive demand of the task has been changed.

The main purpose of developing Bloom's taxonomy (1956) was to encourage conformity among teachers, as well as to improve their teaching practices, as indicated by Anderson (2005). Knowledge, comprehension, application, analysis, synthesis and evaluation, are the six categories of Bloom's taxonomy. These categories are arranged from simple to complex and from concrete to abstract. Knowledge and comprehension are said to be simple and concrete, while synthesis and analysis are complex and abstract. These categories are in a hierarchical order, which means that mastering a lower category is a requirement for achieving the next higher category. In contrast, the Anderson taxonomy table is made up of two dimensions, the horizontal dimension, which is known as the cognitive process dimension and the vertical dimension, which is known as the knowledge dimension (Anderson, 2005).

**Table 3.1: The taxonomy table**

| The Cognitive Process Dimension | | | | | | |
|---|---|---|---|---|---|---|
| | Remember | Understand | Apply | Analyse | Evaluate | Create |
| **The Knowledge Dimension** | | | | | | |
| Factual Knowledge | | | | | | |
| Conceptual Knowledge | | | | | | |
| Procedural Knowledge | | | | | | |
| Meta-cognitive Knowledge | | | | | | |

**Adapted from: Anderson (2005, p. 105)**

The Cognitive Processes dimension is an amendment of Bloom's taxonomy (1956), where the categories have been replaced by the terms normally used by the teachers as they speak about their work. In this dimension, hierarchy is not necessarily a key factor. Factual, conceptual, procedural and meta-cognitive knowledge, are the four types of knowledge which make up the knowledge dimension. This dimension allows the taxonomy to be applied to all school subjects, because it is a shift from content knowledge to the types of knowledge.

*Factual knowledge* consists of facts, terms and the basics that the students need so that they familiarise themselves with a subject. A student with *conceptual knowledge* knows the interrelationships among the fundamental concepts of a subject and how they fit in with each other as part of a whole. Such a student has knowledge of classifying, categorising and generalising. *Procedural knowledge* is being familiar with the methods, techniques and skills. It is being knowledgeable about how to do something. Finally, *meta-cognitive knowledge* includes being aware of how one learns and thinks.

The knowledge dimension is most closely related to Bloom's taxonomy (1956), but the dimensions of analyse, synthesize and evaluate have been replaced by the categories analyse and evaluate, and a new category has been added, ‘create’. This framework assumes less hierarchy in the dimensions than Bloom’s taxonomy (1956) did. Anderson et al. (2001) provide a list of sub-categories to the knowledge dimensions, the focus being on the verbs used in text to identify the categories.

According to the developers, the materials used in this study are designed in such a way that the pre-service teachers gain the skills of problem solving, as well as conceptual understanding of the concepts in calculus. Hence, the taxonomy table will be used to identify how the different types of knowledge are developed and whether the types of knowledge change in the process of teaching.

### 3.6.2 The taxonomy table and assessment

Since this study focuses on the interaction between the lecturers, the pre-service teachers and the materials, the taxonomy table and characterisation of assessment will be used in order to examine how the lecturers organise their activities, i.e. how they introduce their lessons and what type of responses they expect from the pre-service teachers. The introductory material, the stem and the responses are the three components that make up the assessment tasks Anderson (2005). The introductory material may be presented in written form, pictorial form or by using real objects. The stem may be presented in the form of an unfinished statement or question, while sometimes it can be presented as a command or instruction. The response may be short where the students provide a short answer, or sometimes they are required to select a response from given options. The response can also be long where the students are required to write text that is more substantial. These three components do not necessarily have to be part of all assessments. Some assessments may not include the introductory components, while others may not include the stem, so the researcher will assess how the lecturers present their introductory materials, stem and the required response.

**Table 3.2: The taxonomy table and assessment**

| **Introductory Material** | **Stem** | **Response** |
|---|---|---|
| Written, Pictorial, Real objects. | Question, Incomplete statement, Instruction. | Short answer, Extended response<br>Supply (fill in blanks), written<br>Select (multiple choice), perform,<br>Match, True-False |

**Adapted from: Anderson (2005, p. 108)**

### 3.7 Communication during the calculus lecture

The qualities of communication in the mathematics classroom situation influence the qualities of mathematics learning, as maintained by Alro and Skovsmose (2002). A dialogue may take place between two or more people, but the number of people taking part in a dialogue does not really matter. What matters is the nature of dialogue taking place, as well as the relationship between the people. Alro and Skovsmose (2002) describe a dialogue as a modest and civil way of collaborating with each other in an equal relationship of mutual understanding. A dialogue may also be described as willingness to suspend one's perceptions at least for a moment and invite opinions, as well as explore them.

The third question is: What is the nature of dialogue in the calculus lecture room? In order for the researcher to be able to answer the third research question, the researcher intends to use the Inquiry Co-operation Model as the conceptual framework. This model was developed by Alro and Skovsmose (2002) and is based on the understanding that particular qualities of communication in the classroom control particular qualities of learning mathematics. In other words, certain qualities of communication may be linked to particular qualities of learning mathematics and that learning is not only in just what is passed on from one person to another, but is also entrenched in the act of communication. Thus, the situations in which people communicate, determines what is learned. The limitation in this study is that the researcher is not able to use this model on the communication between the students, since the data show absence of this. However, the model still fits in well with this study, as the researcher will still examine the communication between the lecturers and the students. By using this model, the researcher will be able to describe the ways in which the lecturers engage in communication with the students. In this model, the most important requirement for communication is active listening. This is mainly because when people are listening, they do not just passively absorb what is being said, but they also take in the information, as well as ask questions and give non-verbal support to the speaker. This is in line with Hewitt (2005), who claims that for communication to take place, both the speaker and the listener must attend to the words so that they can make sense of what is being said.

The Inquiry model is made up of seven components, which are: getting in contact, locating, identifying, advocating, thinking aloud, challenging and evaluating. Alro and Skovsmose (2002) point out that this model should be seen as a characteristic of a communicative cooperation in which some of the components will be explicit, while others will be implicit. They developed their model in relation to communication in school classrooms, but below, the researcher takes the liberty of applying it to the context of pre-service teacher education.

**The Inquiry Co-operation Model**

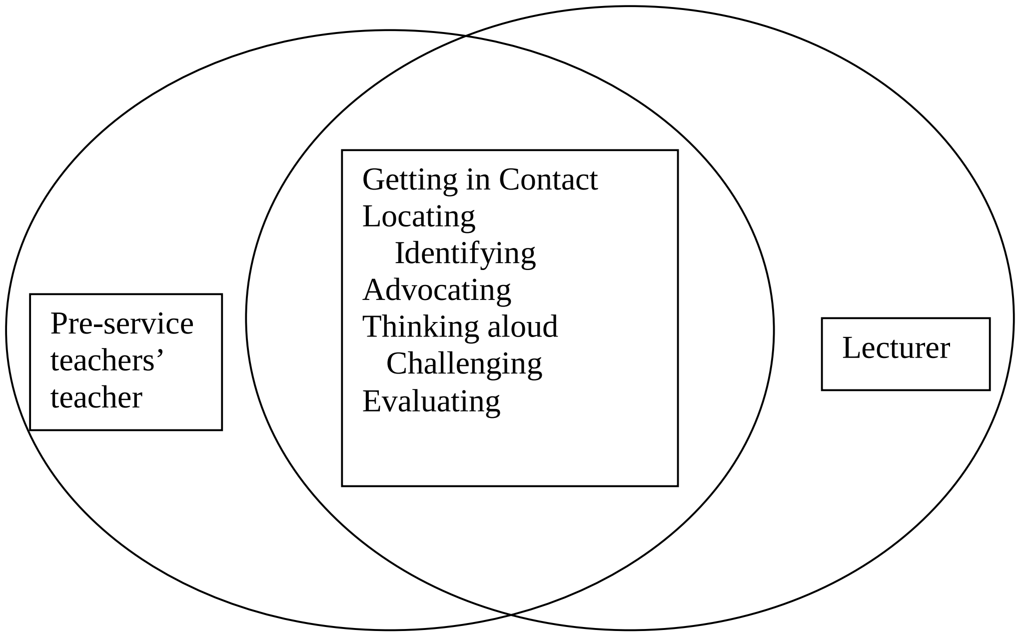


**Figure 3.1: The Inquiry Co-operation Model**

**Adapted from: (Alro & Skovsmose, 2002, p. 72)**

While the model appears to be normative, i.e. prescribes how teaching 'should' take place, it has been successfully used to analyse classroom interaction and its impact on learning (Alro & Skovsmose, 2002), thus, it is in this respect the researcher will use it in this study. Below is the summary of the elements of the model, as described by its creators.

### 3.7.1 Getting in contact and locating

Getting in contact is the first requirement for mutual inquiry, which means making sure both parties are on the same level of understanding, in order to prepare for the cooperation, which is meant to occur between pre-service teachers or pre-service teachers and their lecturers. It also involves paying attention to one another. When this has been established, the lecturer is able to locate the pre-service teachers' perceptions by examining how they understand a particular problem or concept. By doing this, the lecturer will also be finding out what the pre-service teachers do not know, or what they were not aware of, which is done by asking questions. The question then is to what extent the students and lecturer do indeed 'get in contact', and to what extent the students' perceptions are identified in the process?

### 3.7.2 Identifying

When the pre-service teacher expresses his or her perception, this perception may be identified by both the lecturer and the pre-service teacher by using mathematical terms. This process of identification provides a resource for further inquiry. This process can also take the opposite direction, where the pre-service teacher identifies the lecturer's perception.

### 3.7.3 Advocating

Advocating is described as putting forward ideas or points of views as something to be examined. This may result in the pre-service teacher or lecturer re-evaluating their initial perception. Advocating is also described as insinuating arguments for a certain position, but not necessarily having to stick to that position. Advocating can take the form of thinking aloud, because by thinking aloud, perceptions become visible on the surface of communication and as a result, it becomes possible to probe into these perceptions. The communication between students and lecturer or amongst students themselves can be considered in this light, to see to what extent this happens, facilitates the accommodation of useful mathematical or pedagogical content knowledge and is encouraged in the classroom.

### 3.7.4 Reformulating

The lecturer can reinforce the clarification of perceptions by reformulating the pre-service teacher's formulations. This can be done because the lecturer wants to make sure that he understands what the pre-service teacher intends to say. Reformulation can also be done by the pre-service teachers when they are making sure they understand what the lecturer is saying. The lecturer, as well as the pre-service teacher, will reformulate in order to make sure that there is clarity and as a result, avoid misunderstandings.

### 3.7.5 Challenging

Clarification of perceptions serves as a pre-requisite for making a proper challenge. Challenge is described as an attempt to push things in a new direction. When a challenge is being made, the lecturer plays the role of an opponent, as well as the role of a partner and the challenge should be adjusted to the pre-service teachers' conceptions so that they build confidence. Making a challenge can happen either way. The pre-service teacher can also challenge the lecturer.

### 3.7.6 Evaluation

Evaluation can take many forms, which includes correction of mistakes, negative criticism, positive criticism, giving advice, as well as praise. Evaluating the lecturer and the pre-service teachers' perceptions is part of the inquiry process. It also relates back to the evaluative events and the types of legitimising appeals used in the classroom interactions.

For the analysis of the lecturers' reasons for their decisions, the researcher will simply draw on the distinction between content knowledge, pedagogical content knowledge and general pedagogical knowledge. However, the researcher is open to other aspects manifesting themselves in the interaction with the lecturers.

## 3.8 Overview of Conceptual Frameworks

**Table 3.3: Overview of frameworks used in this study**

<table>
<tr><th>Conceptual Framework</th><th colspan="2">Components of the Framework</th></tr>
<tr><td>Mathematics for Teaching</td><td colspan="2">Unpacking<br>Use of Representation<br>Mathematical Communication<br>Questioning<br>Translating<br>Simplification<br>Perception</td></tr>
<tr><td>Cognitive processes</td><td>Cognitive Dimension<br>Apply<br>Analyse<br>Understand<br>Remember<br>Create<br>Evaluate</td><td>Knowledge Dimension<br>Factual Knowledge<br>Conceptual Knowledge<br>Procedural Knowledge<br>Meta-cognitive Knowledge</td></tr>
<tr><td>Mathematical Activities</td><td colspan="2">Thinking mathematically<br>Posing and solving mathematical problems<br>Modelling mathematically<br>Representing mathematical entities<br>Reasoning mathematically<br>Handling mathematical symbols and formalisms<br>Making use of aids and tools</td></tr>
<tr><td>Legitimising Appeals</td><td colspan="2">Appeals made to:<br>Mathematics; Teacher education; Lecturer experience<br>Students experience; Curriculum; Authority<br>Lecturer authority</td></tr>
<tr><td>The Inquiry Co-operation Model</td><td colspan="2">Getting in contact; Locating; Identifying; Advocating<br>Thinking aloud; Challenging; Evaluating</td></tr>
</table>

## 3.9 Connections amongst the different frameworks to each other

In this section of the chapter, an illustration of how the frameworks used in this study are connected to each other, as well as to the components of mathematics for teaching, is presented.

In any lesson, one will find that there is always some form of communication between the lecturer and the pre-service teachers or amongst the pre-service teachers themselves. In this study, the researcher chose to analyse the communication in the teaching and learning of the calculus module, by means of the Inquiry Co-operation Model (ICM) which helped the researcher to see the way in which the legitimising appeals come in at the end of a process of mutual engagement. This is in line with Bernstein's (1996) theory of pedagogic discourse, which maintains that in any classroom, the educator will disseminate benchmarks of what his students are to know in that lesson and will legitimise what his students ought to know. It is in this process that the different representations are evoked and the different mathematical proficiencies and cognitive demands help in the development of conceptual understanding.

This study was informed by the Cognitive Processes (CP), Mathematical Activities (MA), Inquiry Co-operation Model (ICM) and Legitimising Appeals (LA) frameworks, together with the Mathematics for Teaching framework. In this section of the chapter, the researcher presents how the data show the connections of frameworks, as well as how the frameworks are linked to mathematics for teaching. A discussion of how the various elements of the coding co- exist, as well as how they connect with each other follows.

This research aims to assess the communication between the lecturers and the pre-service teachers, as well as amongst the pre-service teachers themselves, the mathematical activities that are legitimised and if cognitive demand is changed during the lesson, as well as the mathematics for teaching that is elicited for the pre-service teachers in the calculus module. In the following paragraphs, a presentation of the two distinct groups of components of the CP, MA, ICM as well as the LA frameworks that emerged from the data analysis in relation to the components of mathematics for teaching, is given.

### 3.9.1 The links amongst the different frameworks

The Mathematics for Teaching, Mathematical Activities and Cognitive Processes frameworks are all linked to the Legitimising Appeals framework in that they all have something that ought to be legitimised. The Legitimising Appeals, Mathematical Activities and Cognitive Processes frameworks are connected in that what is counted as mathematics and mathematics learning is legitimised, as the pre-service teachers engage with the tasks in their lecture. Mathematics for Teaching is connected to Legitimising Appeals in that as the lecturer interacts with the pre-service teachers, as well as the materials, the pre-service teachers learn from the lecturer what is counted as mathematics teaching, because the lecturer exhibits mathematics for teaching, either explicitly or implicitly. Figure 3.2 illustrates these connections.

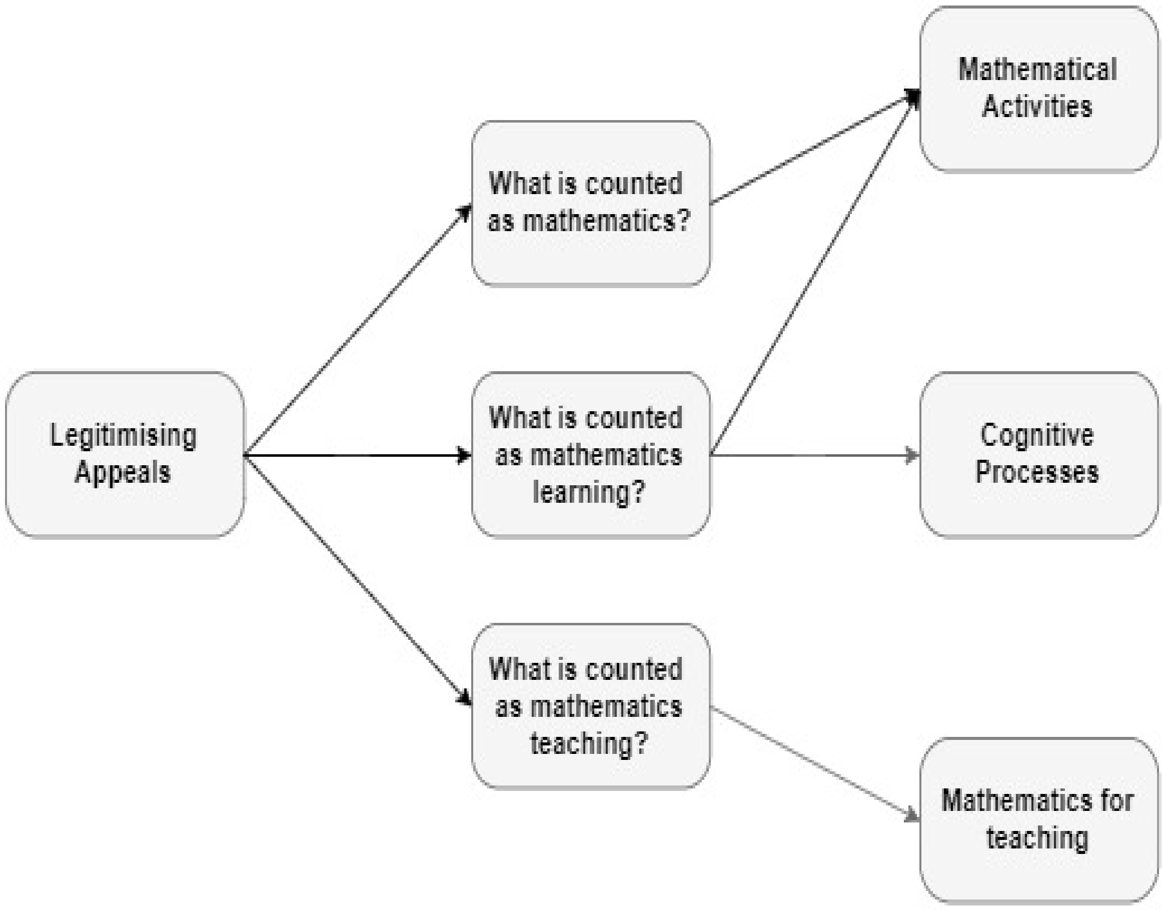


**Figure 3.2: The links amongst the aspects of different frameworks**

In the following paragraphs, a presentation of the two distinct groups of components of the four frame works, that emerged from the data analysis, in relation to the components of mathematics for teaching, is given.

#### 3.9.1.1 The first group

The first group comprises reformulation from the Inquiry Co-operation Model, understand or remember procedural, conceptual and factual knowledge from the Cognitive Processes framework, handling mathematical symbols and formalism, representing mathematical entities and using tools and aids from the Mathematical Activities framework, as well as appeals made to mathematics education, mathematics and students' experience from the Legitimising Appeals framework. All these components are linked to unpacking, communication, translating, representation, perception and simplification from mathematics for teaching because the lecturer does much explaining of procedures, concepts and the meaning of symbols and terms.

##### 3.9.1.1.1 The links between aspects of mathematics for teaching and the other frameworks.

As each one or more of the above components occur, one or more components of mathematics for teaching is exhibited. The lecturer is either *unpacking* an algorithm or concept. When heexplains, the lecturer uses mathematical language carefully, thus exhibiting *mathematical communication*. Sometimes the lecturer uses graphs to explain a concept, thus displaying the use of *representations*. *Simplification* and *translation* are exhibited when the lecturer is explaining the meanings of symbols, as well as definitions. When the lecturer is reformulating, he displays *perception*, when he picks up the pre-service teacher's idea and works with it in his explanations. Figuer 3.3 shows the link between the frameworks and mathematics for teaching.

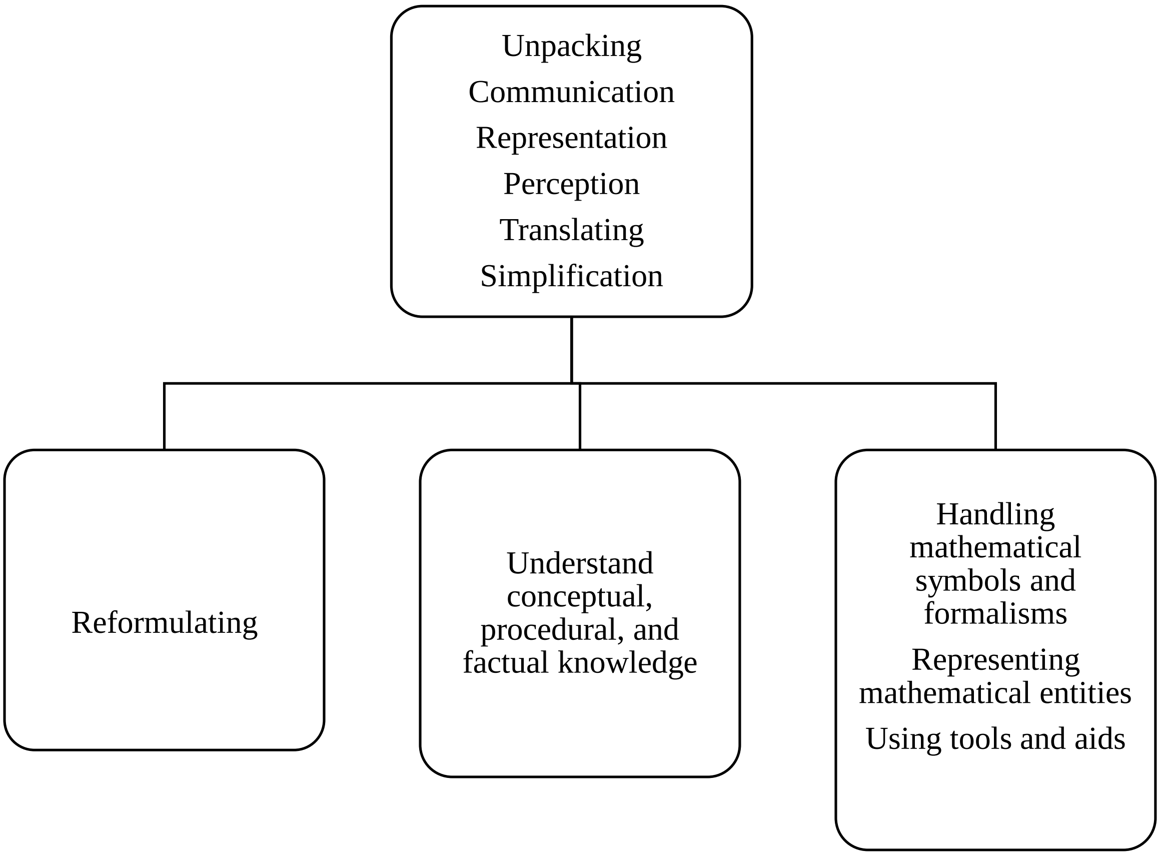


**Figure 3.3: The links between aspects of mathematics for teaching and the other frameworks**

### 3.9.1.2 The Second group

#### 3.9.1.2.1 Links between aspects of different frameworks

The second group comprises locating, identifying, advocating and challenging from the Inquiry Co-operation Model, as well as analysing or applying procedural, conceptual and factual knowledge from the Cognitive Processes, as well as reasoning or thinking mathematically from the Mathematical Activities framework. All these are then linked to *questioning* from mathematics for teaching. In this second group, there is much to do with asking questions which happens when the lecturer is locating, identifying, advocating or challenging, as well as when he asks questions that force his students to apply or analyse their conceptual, procedural or factual knowledge. The lecturer asks a series of questions with the intention of leading his students to an expected answer, by reasoning mathematically. In addition, sometimes he asks questions that the pre-service teachers can only answer if they understand the scope of the problem at hand. All this is linked to the component of mathematics for teaching, which is *questioning.* Questions are asked with the intention of developing the thinking progress of the pre-service teachers, as stated by Silver (2015), that teachers need to ask purposeful questions. In support of this, Walsh (2012) discusses the notion of dialogic teaching, where questions are asked by the lecturers to encourage their students to use their deep thinking skills.

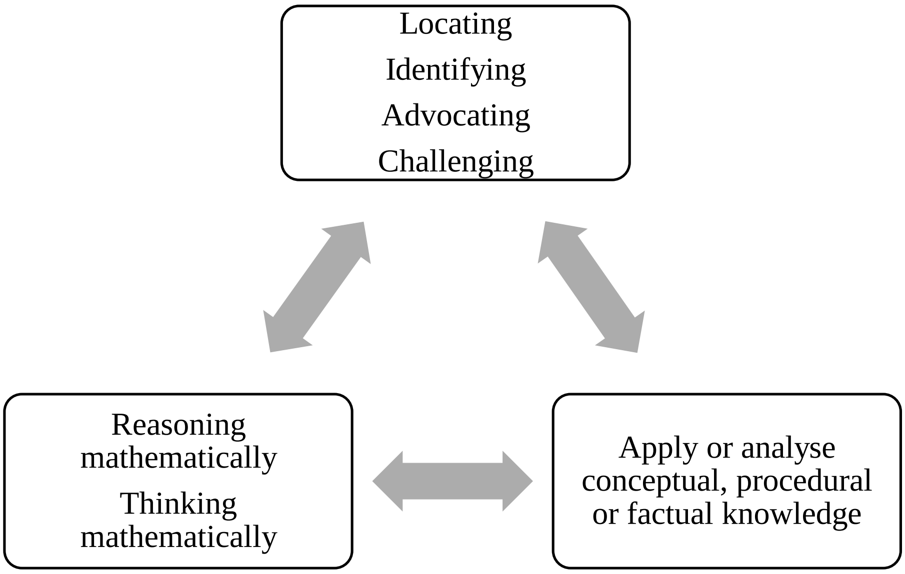


**Figure 3.4: Links between aspects of different frameworks**

**3.9.1.2.2 The links between aspects of mathematics for teaching and the other frameworks**

The elements from the Inquiry Co-operation Model, Cognitive Processes and Mathematical Activities frameworks are exhibited when the lecturer locates and identifies what the pre-service teachers may or may not know, at the same time the lecturer wants them to apply their conceptual understanding and the pre-service teachers can only answer the question if they understand the scope of the problem. This then links with *questioning* from mathematics for teaching, where questions are asked with the intention of developing the pre-service teachers' thinking abilities. Thus, the lecturer conveys that asking questions that make the pre-service teachers think about what they are learning is important in the teaching of mathematics (Olmsted, 2012).

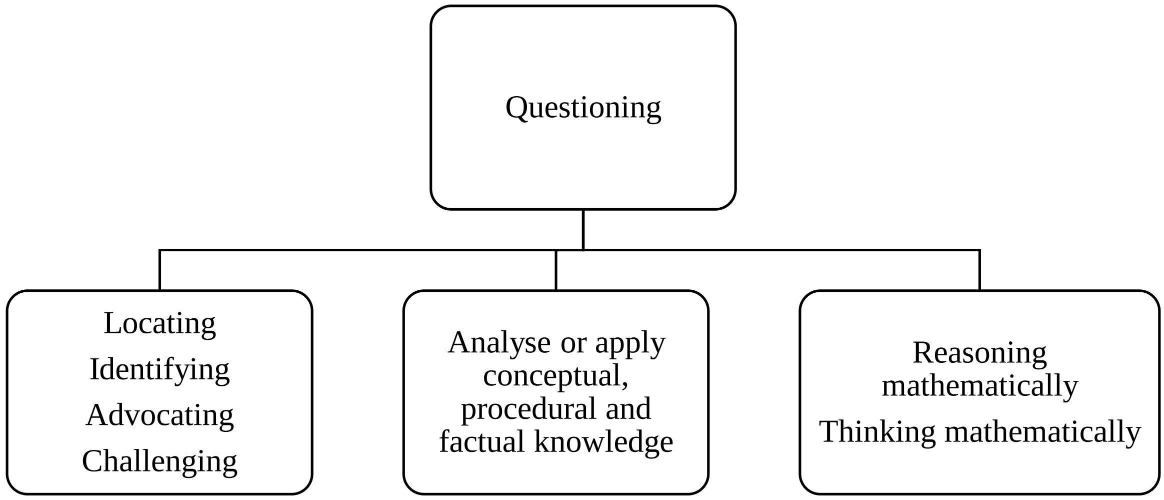


**Figure 3.5: Links between an aspect of mathematics for teaching and other frameworks**

Two types of teaching also emerge, one that involves explanation of concepts, procedures, symbols, terms or basic ideas, while the other involves asking questions that make the pre-service teachers examine their line of thought, apply or analyse their procedural or conceptual understanding, follow a line of argument and come up with conclusions or conjectures. All this is conveyed to the student teachers, either explicitly or implicitly.

## 3.10 Conclusion

In summary, the lecturer explains concepts, procedures, symbols or terms, or asks questions that lead the pre-service teachers to apply or analyse their procedural or conceptual knowledge, or leads them to reason or think mathematically. While the lecturer locates, identifies or advocates, he will be legitimising this with appeals made to mathematics, mathematics education, students' experiences, or to his own authority. This is in line with Bernstein (1996), who proposed that as communication proceeds in any classroom, the eductor will, at different instances, validate to his students, what counts as mathematical knowledge.

In this chapter, the conceptual frameworks have been outlined. The discussion of how mathematics for teaching is illuminated by the different components of the frameworks, has also been presented. In the next chapter, the design and methodology of this study is presented.

# CHAPTER FOUR

## RESEARCH DESIGN AND METHODOLOGY

### 4.1 Introduction

In the preceding chapter, the following frameworks, which informed this study have been explored: Legitimising Appeals, Cognitive Processes, the Inquiry Cooperation Model, Mathematical Activities and Mathematics for Teaching. It has also been shown how these frameworks are linked, as well as how each component of Mathematics for Teaching is illuminated by the components of the other frameworks. This chapter has eleven sections: a discussion of the paradigm of this study follows. The fourth section of this chapter discusses the case study, while the fifth section discusses the research methods, which is then followed by a discussion on the issues of reliability and validity. The last three sections include a discussion on sampling which is then followed by a discussion on ethical issues and finally, the conclusion of this chapter.

#### 4.1.1 Research Questions

This study was guided by the following key questions:

1. What mathematical activities are legitimised in the calculus lecture room?
    - Are the legitimising appeals made to mathematics, mathematics education theories, the textbooks/notes, students' experiences, everyday metaphors, authorities or other?

2. How and why are the materials organised by the lecturers?

3. What is the nature of calculus dialogue in the calculus lecture room?

Table 4.1 shows an overview of the research design of this study.

**Table 4.1: Overview of the research design for this study**

| | | | |
|---|---|---|---|
| **Paradigm of study** | | | Interpretivist |
| **Methodology of study** | | | Qualitative |
| **Strategy of study** | | | Case study |
| **Sampling of study** | | | Convenience sampling |
| **Data collection methods** | | | |
| **Participants** | **Method** | **Instrument** | **Number of video recorded lectures** |
| Two lecturers and their students | Semi-structured interviews<br>Video recording lectures | Interview process<br>Video recorder | 8 from Lecturer A<br><br>10 from Lecturer B |
| **Data analysis** | Coding using Nvivo 10<br>Manual coding<br>Identification of themes | | |

### 4.2 Exploring the use of paradigms

Research studies are conducted in ways that differ from one researcher to another, because researchers' beliefs, as well as the ways in which different researchers view their surroundings, may vary. Thus, the researcher's ontological, epistemological and methodology determine the paradigm for their research. All researchers are guided by rules, principles or standards as they conduct their research and these rules, principles and standards are referred to as a paradigm (Guba & Lincoln, 2005), cited in (Salma, 2015, p. 1).

A paradigm is a collection of assumptions about the type of what is real, the significance of what is to be known and the methods that the researchers intend to employ as they attempt to answer the research questions. In other words, a paradigm is generally the researcher's philosophical viewpoint of the world, as proposed by Creswell (2014). The functions of a paradigm are as follows:

- Outline how the world works, how knowledge is extracted from this world and how one is to think, write and talk about this knowledge.
- Outline the types of questions that need to be asked and the methods and approaches that should be used in answering these questions.
- Choose what is published and what is not published.
- Construct the world of the academic.
- Provide meaning and its significance (Dills & Romiszowski, 1997)

Ontology, epistemology, methodology and methods are the four components that make up a paradigm. Ontology is described by Wahyuni (2012) as the view of how the world is perceived by an individual, which is the position that one takes on the nature of reality. Hence, the assumptions that ontology makes are what reality is made of. Epistemology relates to the type of knowledge that the researcher seeks to know, as well as the relationship between the researcher and that which the researcher seeks to know (Irene, 2014). In addition, knowledge may be constructed, attained and conveyed. Thus, ontology suggests what is, while epistemology suggests what is to know. It is fitting then to say that different paradigms will hold different views in their ontological and epistemological assumptions, since each paradigm is grounded in its own ontological and epistemological assumptions.

Methodology is a strategy that one anticipates with which to approach certain methods, so it is concerned with why the researcher has to collect data, what data must be collected by the researcher, from where the data should be collected, as well as when and how the data will be collected. It is a model or technique that one uses to carry out research within a particular paradigm (Wahyuni, 2012). Essentially, Guba and Lincoln (2005) describe methodology as asking questions of how the researchers will go about finding out what they believe can be known.

Thus the ontological position of this study is grounded in the fact that there are mathematical activities, mathematics for teaching and dialogue that take place in the calculus lecture room of the pre-service teachers. The epistemological position of this study is grounded in an attempt to know what is legitimised as mathematics, mathematics teaching and mathematics learning in the calculus lecture room of the pre-service teachers. The methodological position of this study is to conduct the study in a suitable paradigm and to collect data using suitable methods.

Positivism, post-positivism, critical theory, pragmatism and constructivism, have been identified by Wahyuni (2012) as the major research paradigms. Scotland (2012), on the other hand, identified scientific, interpretive and critical theory as the key research paradigms. However, while all the different paradigms have been acknowledged, the most suitable paradigm for this research study is interpretivism, which is the subject of the following discussion.

### 4.3 The paradigm within which this study was framed: The interpretivist paradigm

Upon reviewing papers on research paradigms, at first it was difficult to distinguish between constructivism and interpretivism. This was because some writers such as Wahyuni (2012) would describe constructivism in the same way that, for example, Creswell (2014) and Scotland (2012), would describe interpretivism. It became clearer when the researcher reviewed the paper by Aliyu et al. (2014), where the difference between the two is clearly stated. The fact is that the two paradigms share the same ontological, epistemological and methodological views. Thus, the difference is that the ontological position of constructivism is more fundamental and thorough, in as much as it is spread across all spheres of truths and reality, while the ontological position of interpretivism is restricted to social truth reality (Aliyu et al., 2014).

Relativism is the ontological position that the interpretivists take. This means that the interpretivist view on reality is subjective and is not the same from individual to individual. Since reality is subjective, rather than objective, interpretivists find it necessary to make sense of the world subjectively through the participants’ views and experiences, which is done through interacting with the research participants (Ponelis, 2015). Hence, in this study actual lectures were video recorded and interviews with the lecturers were conducted. The interpretivist epistemological position is that of subjectivism and is based on real world phenomena, as proposed by Scotland (2012). These realities are constructed because of the interactions between a researcher and the research group. Thus, the methodological position is that of understanding the truth or reality from the researcher's perspective (Creswell, 2014). The methods used include case studies, phenomenology and hermeneutics.

This study is located within the interpretivist paradigm, because the aim of the study is to describe and develop an understanding of the lecturers' and the pre-service teachers' actions. Therefore, there was the need to understand and interpret the dialogue that took place between the lecturers and the students, as well as among the students themselves, in the calculus lecture room and thus be able to answer research question two. By working within the interpretivist paradigm, the researcher was able to see the reality of the mathematical activities in the calculus lecture room through the eyes, activities and experiences of the research participants. The researcher used these experiences and views to construct her own understanding of the teaching of calculus by the lecturers to the pre-service teachers i.e the interactions, activities and dialogue which took place during lectures (Thanh & Thanh, 2015). The researcher was therefore able to answer research question one. Each lecture room had some interactions, activities and dialogue, so, video recordings were taken and analysed, in order to gain insight into the activities that took place during the lectures.

There was also the need to understand and interpret how and why the lecturers organised the activities the way they did and thus answer research question three. According to Guba and Lincoln (2005), the lecturers' intentions would be elicited and shared as they interacted with their students during the lectures, as well as when they responded to the interview questions, though it must be noted that some issues of identity and power always manifest in interviews. The interviews have been analysed qualitatively to gain insight into the lecturers' intentions. Table 4.2 shows the ontological, epistemological and methodological standpoint of this study.

**Table 4.2: Overview of the study's paradigm**

| | |
|---|---|
| Purpose of research | • To understand and interpret the lecturers' actions.<br>• To understand and interpret the dialogue and the mathematical activities in the calculus lecture room.<br>• To understand and interpret what is counted as mathematics and mathematics for teaching in the calculus lecture room. |
| Ontology | • There exist multiple realities in the calculus lecture room.<br>• Dialogue, the lecturers' actions and mathematical activities are the realities of the lecture room of this study that are inclined to change from one lecture to another. |
| Epistemology | • The events that take place during the teaching of calculus are understood through mental processes as the researcher interprets them, which is influenced by the researcher's interaction with the research participants.<br>• The researcher maintains a close link with the research participants. |
| Methodology | • Data were collected by means of video recording of the lectures, as well as interviews with the lecturers. |

**Adapted from: Cantrell (2001, p. 35)**

## 4.4 Exploring the notion of a case study

A case study is a thorough study of a social phenomenon, which is carried out within social boundaries of one social system (Harrison, Birks, Franklin, & Mills, 2017). Carrying out a case study means that one has to monitor or observe the phenomenon over a certain period. During the observation, the researcher focuses on the description and explanation of social processes that unfold in that particular social system. A case study is carried out by means of several data sources, which include interviews and direct observation (Davis, 2011). The researcher may opt to invite the research participants to a discussion, which is meant to clear up any misunderstandings that the researcher might have, which also helps in getting a more solid base for a final report on the research. Case studies are classified according to time frame, as well as according to theory formation (Bennet & George, 2005; Thomas, 2011), cited in (Starman, 2013, p. 33). These have been summed up in Table 4.3 and Table 4.4.

### 4.4.1 Summary of classification of case studies

**Table 4.3***:* **Classification of case studies according to time frame (adapted from: Thomas, 2011, p. 517)**

| Type of case study | Description |
|---|---|
| Singular case study | One case is studied at a time. |
| • Retrospective case study | This is the very simplest study, whereby the researcher looks back at a past phenomenon and collects data, which relate to that past phenomenon or situation and studies it again. |
| • Snapshot case study | The case gets to be studied over a fixed period. This could be a current event, or even a person's life over a week or a month. |
| • Diachronic case study | These are case studies that change over a long time. |
| Multiple case studies | These involve multiple cases which are studied separately and treated as if they were singular, and then compared to other cases. Analysis of each is built on the knowledge of the other case. |
| • Nested case studies | These involve comparison of elements within one case. |
| • Parallel case studies | These are cases, which are studied concurrently as they happen at the same time. |
| • Sequential case studies | It is assumed that what happens in one case will affect what happens in the next case and thus, these cases are studied consecutively. |

**Table 4.4 Classification of case studies according to theory formation (adapted from: Bennet & George, 2005, p. 75)**

| Type of case study | Description |
|---|---|
| Theoretical case studies | These are illustrative case studies that do not contribute directly to theory. |
| Disciplined configurative case studies | These use theory to explain the case. |
| Heuristic case studies | These rely on the usefulness of marginal, deviant or outlier cases to identify new unexpected paths. |
| Theory testing case studies | These studies assess the validity of theories or competing theories. |
| Plausibility probes | These are pilot studies that are used to check if there is need for further studies. |
| Building blocks | These are studies of the same type that contribute to theory when they are put together. |

Case studies have their own advantages and disadvantages. Rose, Spinks and Canhoto (2015) listed the following advantages and disadvantages.

**Advantages of a case study**

Firstly, case studies may be adjusted to a variety of research questions, as well as research environments and the phenomenon can be studied in detail in its usual surroundings. This means that a case study is conducted within the situation where the activity takes place. Thus, when one wants to explore the dialogue and activities in a calculus lecture room, one has to observe these. Secondly, case studies allow for a variety of data collection methods. In addition, the use of a case study allows for both qualitative and quantitative means of analysing data. Thirdly, using qualitative analysis of data allows for detailed accounts of the case, which then helps to explore or describe real life situations (Rose, Spinks, & Canhoto, 2015).

**Disadvantages of a case study**

Firstly, case studies are seen to lack rigour and secondly, they are considered too long and not so easy to conduct, especially if access to the site proves to be problematic. Bias may prove to be unavoidable, since the research findings are dependent on the selection of the case. It is not possible to generalise the research findings if only one case has been used in the study and since it is conducted on real life situations (Rose et al., 2015).

### 4.4.2 The case study explored in this research study

This case study focused on exploring the activities, dialogue and interactions which occurred during the teaching and learning of calculus. As mentioned above, a case study is defined as the collection and presentation of comprehensive facts about a particular participant or a small group of people (Harrison et al., 2017). The case study approach was chosen for this research because there was need to understand and interpret the interplay between the variables in this study, which included the lecturers, the pre-service teachers and the materials. The case study was employed in this study because the researcher was then able to provide a rigorous description of what was legitimised in the calculus lecture room (Ponelis, 2015). This is a multiple case study, since the data used in this study were collected from two different calculus lecture rooms, in two different years. Although this is a multiple case study, it does not fit in the classifications mentioned in Table 4.3. This is because no elements within the same case are being compared. In addition, the studies were not done concurrently or consecutively in order to use the result of one to influence the next study.

The researcher of this study is aware of the weaknesses and limitations of case studies as mentioned in the preceding paragraph. Since quantitative means of analysis have not been used in this study, there is no need to worry about the data not being large enough to meet statistical significance, as data were analysed using qualitative means.

Exploratory, explanatory and descriptive case studies, are the three categories of case studies. Exploratory case study is used when the researcher intends to explore a phenomenon that interests him/her. In other words, the aim is to explore and question what is happening in the case and this is done by engaging with the research participants through questions and interviews. In an explanatory case study, the researcher looks comprehensively through the data so that he/she can explain the phenomenon. This means that the researcher's aim is to explain why the events of the lecture room occur in the way that they do. In addition, the researcher seeks to explain how the events of the lecture room occur in the way they do (Gray, 2013). In a descriptive case study, the researcher's intentions are to describe the phenomenon

(Zainal, 2007). This means that the researcher's aim is to provide a picture of the events that take place in the lecture room, as they occur in their natural setting (Gray, 2013). Thus, the case study of this research was exploratory in that the researcher intended to explore the interactions between the lecturers and their students in the calculus lecture rooms. In addition, the researcher, through interviews with the lecturers, questioned the lecturers' intentions and actions. It was also explanatory because the researcher's intentions were to explain the lecturers' actions, as well as the dialogue that took place in the calculus lecture rooms. Thirdly, it was a descriptive case study because the researcher's intentions were to describe what was counted as mathematics, as well as mathematics for teaching in the calculus lecture rooms. Thus, the researcher's intentions were to paint a picture of the events that took place in the lecture room.

## 4.5 The research methods

Research methods are the different procedures, which are used to help the researcher collect samples and data, as well as interpret the data. Methodology is an overall research strategy that outlines the way in which the research will be undertaken (Rajasekar, Philominathan, & Chinnathambi, 2014). In other words, methodology is the research trajectory, a plan and action behind the methods that the researcher chooses to use in their research. In addition, Creswell (2013) suggests that methodology is an approach which is systematic that guides the researcher to accomplish his/her aim. The qualitative and quantitative methods are the most commonly used research methods. These are ways in which data are collected and analysed. The quantitative methods were developed for natural sciences and are used in the studies of natural phenomena, while the qualitative methods were developed for social sciences and are used in the study of social and cultural phenomena. The difference between the two methods is that, qualitative methods are flexible, while quantitative methods are inflexible.

Researchers who use qualitative research methods do so because they are in search of insight into the research participants and their actions seek to understand the research participants' actions better. Researchers who use quantitative research methods seek to expand or verify existing theories and they do so by collecting facts about the behaviour of the research participants.

### 4.5.1 The qualitative research method

This study adopted the qualitative method, since the aim of the research is to understand and interpret the communication, and activities of the calculus lecture room. Qualitative research methods involve documents analysis, conducting interviews, which could be open-ended, semi-structured or unstructured. Qualitative research methods also involve direct observation of the research participants in their natural settings. Qualitative research methods have eight core characteristics, as shown in Table  4.5.

**Table 4.5: The characteristics of qualitative method**

| **Characteristic** | **Description** |
|---|---|
| Natural setting | A qualitative study takes place in its original context. The researcher may not move participants to another site to conduct the study. |
| Researcher as the key instrument | The researcher takes it upon themselves to collect data either by video recording, taking notes or conducting interviews. |
| Multiple sources of data | Qualitative research is characterised by a variety of data, which is then organised and analysed. |
| Inductive and deductive data analysis | Firstly, data are analysed inductively, where by the researcher looks for patterns (themes) in the data. Secondly, deductive analysis is applied by using data to support the themes. |
| Participants' meaning | The researcher recognises and acknowledges the participants' understanding of the issues under study. |
| Emergent design | The researcher must be flexible because various things such as key questions and methods may change during the research. |
| Reflexivity | The researcher is always reflecting about things such as how their background, personality and many others could have an effect on their study. |
| Holistic account | A broad picture of the issue under study needs to be painted by the researcher from various perspectives. |

**Adapted from: Creswell (2014, p. 234)**

Considering the characteristics listed in Table 4.5, this study locates itself in the qualitative method since the data were collected within the participants' original site. The researcher was the key instrument for collecting data by video recording the lectures, as well as personally conducting the interviews with the lecturers. This led to the researcher ending up with multiple data sources. Inductive analysis of data was used by looking for patterns and themes that emerged from the data and then confirming the themes by supporting them with the data, as well as with literature. The researcher was always aware of, and respected the lecturers' understanding of the mathematical activities, as well as the dialogue in the calculus lecture room. The researcher was aware that her background as a mathematics educator, values and personality, would influence the data during video recordings or during the interviews with the lecturers. Thus, working within the interpretivist paradigm, by using the qualitative research method, the researcher sought to understand and explain what is counted as mathematics, mathematics for teaching, as well as the dialogue in the calculus lecture room.

### 4.5.2 Exploring sampling

A sample is part of a larger group or population of people that serves the purpose of representing the larger group or population (Lynch, 2015), because the sample is a smaller version of the population. Theoretical sampling, convenience sampling and snowball sampling are the most commonly used sampling methods. In theoretical sampling, the researcher chooses a sample for the sake of developing new theories or exploring the existing theories. In snowball sampling, the researcher builds up the sample numbers through connections from the initial participants. Convenience sampling involves choosing a sample based on its accessibility and the willingness of the participants to take part in the research. It enables the researcher to collect data in a quick and effortless manner (Elfil & Negida, 2017). Hence, this study adopted the convenience sampling, since the sample was selected because the site was close to the researcher and this minimised the travelling costs, because video recordings of the lectures were done twice a week. The participants were two lecturers and a combination of second and third year students in each lecture room.

### 4.5.3 Data collection methods

After carefully reviewing the research questions of the study, the need to consider which data collection methods to use, became known. To make sense of the lecturers' intentions, which included why the lecturers changed or arranged the activities the way they did, there was a need to interview them.

#### 4.5.3.1 Types of interviews

Interviews are mostly used in qualitative research and they help the researcher acquire profound understanding of the opinions, experiences and beliefs of the research participants (Castillo-Montonya, 2016). Structured, semi-structured and unstructured interviews are the common types of interviews in research (Wahyuni, 2012).

Within structured interviews, the researcher takes along a list of pre-determined questions to use as a questionnaire. There are no follow up questions since the type of questions in this case are straightforward. In the case of unstructured interviews, while the researcher does not have a list of pre-determined questions, there will be an opening question, after which follow up questions will be asked. Unstructured interviews aim to bring out as much information from the participant as possible. In semi-structured interviews, the researcher takes along a set of key questions and may ask follow up questions, for the participant to elaborate. This set of key questions is regarded as a guide in which the order may change during the interview (Edwards & Holland, 2013). Thus, the interviews of this study were semi- structured, since there was a set of questions that were taken along for the interview. These questions were used as a guide and follow up or probing questions were asked where it was felt that the lecturer needed to elaborate or explain further. The researcher saw the need to use semi-structured interviews because such interviews have structure and the researcher did not want to lose track of the objectives of the study. Semi-structured interviews provided the researcher with a guide of questions to ask the lecturers. Such interviews also allow for probing questions. This means that although the researcher took along a set of questions to ask the lecturers, depending on how the lecturers answered the intended questions, the researcher could ask questions, which were intended for the lecturers to expand or elaborate on their responses.

#### 4.5.3.2 The video recorded data

It was necessary to video record the lectures in order to capture the dialogue in the calculus lecture room, to determine what was counted as mathematics, as well as what was considered as mathematics for teaching. Jewitt (2012) listed the following as the key advantages of collecting data by means of video recordings. Firstly, video -recorded data can be re-opened for later analysis to access things that the researcher would have originally missed. Secondly, videorecorded data can be used to support empirical data, especially when comparing two or more sets of data and thirdly, the researcher is able to return to an instant, as well as to be reminded of an occurrence. In view of what has just been pointed out, the researcher deemed it necessary to video record the lectures. Since this was direct observation of the participants in their natural setting, there was a need to video record the lectures and capture the happenings of the lecture room. The researcher was aware of the fact that the actions of the research participants could be influenced by the fact that the participants are aware that they are being recorded and this may be a disadvantage.

## 4. 6 Ethical issues followed in this study

The importance of ethical issues needs to be considered by researchers, as they embark on their research journey (Hamza, 2014). Ethical issues such as obtaining consent from the research participants, as well as the withdrawal from the research by the participants, are among many issues that need to be considered by researchers.

### 4.6.1 Obtaining informed consent

The lecturers and the students participating in this study were informed about the purpose of the study, as well as the fact that the lectures were to be video recorded. The students and lecturers gave the researcher of this study informed consent to the recording of the lectures, which were deemed to have minimal effect on their studies and interactions. The lecturers also gave informed consent to the interviews, which were conducted after recording the lectures.

### 4.6.2 Withdrawing from the research

The participants of this study were made aware that they had the right to retract from the study if ever they felt uncomfortable participating. Banister (2007) points out that participants may choose to retract, not only during the interviews, but at any time during the research. This means that research participants may withdraw from being video recorded if they so wish at any particular time during the recording of the lectures.

### 4.6.3 Protection of research participants’ identities

The participants of this study were also made aware of the fact that their identities would be kept confidential. The students were given pseudonyms, while the lecturers were referred to as Lecturer A and Lecturer B. The names of the institution in which the study was conducted was also kept confidential.

The study followed all ethical guidelines. Permission was obtained from the Dean of the Faculty before the study started. Ethical approval was obtained from the University's research office before the study commenced.

### 4.7 Issues of reliability and validity

Reliability addresses how accurate the research methods and techniques of a study produce the data, while validity addresses what the researcher initially intended to explain (Cano, 2009). Thus, in qualitative research, the issues of validity are connected to how suitable the tools, and the processes are, as well as the data for the particular research (Leung, 2015). The researcher ensures validity by using a variety of procedures such as group coding of the data, as well as confirming the accuracy of findings with the research participants. While considering the methods of collecting and analysing data, the issues of reliability and validity had also to be considered. To increase validity, the first coding session of lecture A 1 was done by colleagues of the researcher, a fellow student and the researcher. The researcher verified the findings of the study with each participant to ensure reliability and validity.

Using more than one data collection method, data source, theory, as well as two or more researchers, is all means of triangulation, in an effort to increase reliability and validity of a qualitative study (Blandford, 2013). In order to increase reliability, this study employed the use of two data collection methods, the interviews and video recording of the lectures. In addition to the two data collection methods, this study employed the use of three frameworks in an attempt to explain the findings, as well as increase the reliability of the findings. This is important since triangulation assists the researcher in developing confidence in his/her research findings (Yeasimn & Rahman, 2012).

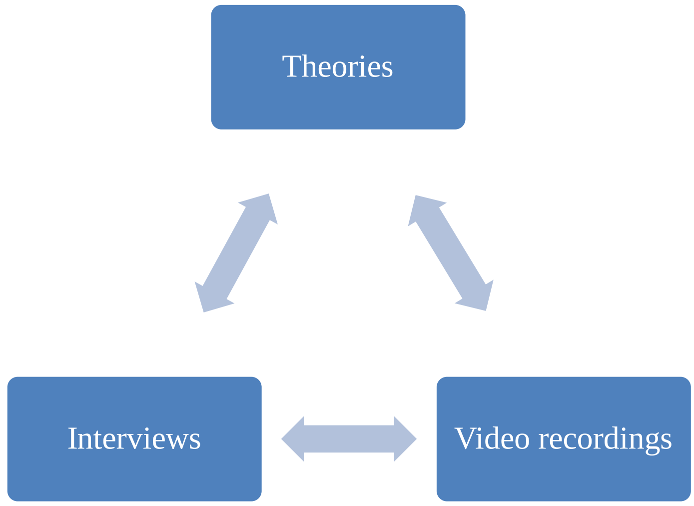


**Figure 4.1: Triangulation used in this study**

The combination of interviews and video recordings as a means of collecting data, coupled with multiple theories to explain the findings of the research, ensured that the issues of reliability and validity were addressed and that the research findings were credible (Blandford, 2013).

**4.8 Limitations of this study**

The data of this study were collected by video recording, which implies that in this research, data might have been affected by the fact that the lecturers and the students were aware of being video recorded, especially in the first lesson and they could have acted differently. There is a possibility that the students might be intimidated by being video recorded. The lecturers who participated in this study were more qualified and more experienced than the researcher, so the researcher felt intimidated thus, ending up also not getting enough information from them. In addition, since there were only two lecturers and two groups of a combination of second year and third year students, this was a limitation, with respect to generalisation.

## 4.9 Overview of this study with respect to the critical research questions

**QUESTION 1**

What mathematical activities (conjecturing, exemplifying, etc.) are legitimised by the lecturers?

Are the legitimising appeals made to mathematics, mathematics education theories, the textbooks/notes, students' experiences, everyday metaphors, authorities or more?

**Table 4.6: Data collection plan to respond to Question 1**

| Questions for developing a data collection plan | Data collection plan |
|---|---|
| What was the research strategy? | Video and audio recordings of the actual lessons. Coding of lecturers' statements in NVivo 10 to identify and characterise legitimising appeals. |
| Why was the data collected? | To identify the mathematical activities that are legitimised in the calculus lecturer room. |
| Who was the source of data? | The interaction between the B.Ed. students in the calculus module and their lecturers. Interviews with the lecturers. |
| How many of the data sources were accessed? | Two classes of B.Ed. students and two lecturers. |
| Where was the data collected? | At a university in South Africa. |
| How often was the data collected? | 20 lectures were recorded. |
| How was the data collected? | Each lecture was video and audio recorded. |
| Why was this the best way of collecting the data? | The video recordings provided the best possible record of the actual events that took place in the lecture rooms. |

**QUESTION 2**

How are the tasks from the materials organised by the lecturer?

Why does the lecturer organise the materials the way he does?

Is conceptual demand changed?

How does his mathematical knowledge inform his decisions and reflection in practice?

How does the knowledge that he is engaged in training teachers inform his decisions and reflection in practice?

**Table 4.7: Data collection plan to respond to Question 2**

| **Questions for developing a data collection plan** | **Data collection plan** |
|---|---|
| What was the research strategy? | Video recordings of the actual lectures. |
| Why was the data being collected? | To identify the way the lecturers organised the activities and why. |
| Who was the source of data? | Lecturers and the students |
| How many of the data sources were accessed? | Two classes of B.Ed. students and two lecturers. |
| Where was the data collected? | At a university in South Africa. |
| How often was the data collected? | Daily throughout the module. 20 lectures were recorded. |
| How was the data collected? | Each lecture was video recorded.<br>Interviews with the lecturers were recorded. |
| Why was this the best way of collecting this data? | The video recordings provided insight into the actual events that took place in the lecture rooms.<br>The interviews with the lecturers provided insight into why the lecturers organised the activities the way they did. |

**QUESTION 3**

What is the nature of the dialogue in the calculus classroom?

How does it influence the access of the students to the practice?

**Table 4.8: Data collection plan to respond to Question 3**

| **Questions for developing a data collection plan** | **Data collection plan** |
|---|---|
| What was the research strategy? | Video and audio recordings of the actual lessons were taken. Characterisation of all interactions according to the inquiry co-operation model. |
| Why was the data being collected? | To identify the nature of dialogue in the calculus lecture room. |
| Who was the source of data? | The B.Ed. students in the calculus module and their lecturers. |
| How many of the data sources were accessed? | Two classes of B.Ed. students and two lecturers. |
| Where was the data collected? | At a university in South Africa. |
| How often was the data collected? | Every day's lecture of the module. |
| How was the data collected? | Each lecture was video and audio recorded in an attempt to capture the nature of dialogue in the calculus lecture room. |
| Why was this the best way of collecting this data? | The video recordings provided the actual events that took place in the lecture room. |

## 4.10 Conclusion

This chapter explored the use of paradigms and identified the interpretivist paradigm as the most applicable for this study. Working within the interpretivist paradigm and using a qualitative case study approach, the researcher sought to explore and understand the interactions of the participants in the calculus lecture room, which included the dialogue and the mathematical activities, as well as the mathematics for teaching elicited by the lecturers. Triangulation has been discussed as the best way to confirm the validity and reliability of the research findings, as well as to check the consistency across other research of the processes and methods employed in the research (Gibbs, 2007). The ethical issues have also been discussed, highlighting the fact that the research participants need to give the researcher informed consent before participating in the research. The next chapter explores the analysis and presents the data gathered through the methods described in this chapter.

# CHAPTER FIVE

## PRESENTATION AND ANALYSIS OF DATA

### 5.1 Introduction

This chapter presents the data, as well as providing a discussion focusing on the analysis of data. The study sought to gain insight into the mathematical activities that are legitimised in the calculus lecture room. Secondly, the study sought to gain insight into the dialogue that takes place in the calculus lecture room. Thirdly the study sought to gain insight into the actions of the lecturers, as well as the mathematics for teaching, that is elicited by the lecturers. Two lecturers and their students were video recorded in eighteen lectures, after which interviews with the lecturers followed. Themes emerged during the coding of both the video recorded data and the interviews with the lecturers which are also presented in this chapter.

### 5.2 The coding of data

The coding process is fundamental to the analysis of qualitative data, because qualitative data are in textual form (Creswell, 2013). Coding of data is the oldest and most popular technique of data analysis (Gläser & Laudel, 2013). Theron (2015) is one of the many researchers who recommends coding as a technique of qualitative data analysis. Coding of data involves grouping data into categories and searching for themes (Creswell, 2013). Part of the data in this study, data from Lecturer A was transcribed and coded using Nvivo 10 while the other part, data from Lecturer B was coded manually. Firstly, video recordings of the actual lessons were recorded and then transcribed. The first coding was done by the researcher of this study and her colleagues, as well as a fellow student. This was done to account for reliability. After this session, the researcher had a few more sessions of coding with a fellow student and this helped with the coding reliability, as the coding was compared. The supervisor of this study also checked the coding, which strengthened the coding reliability.

### 5.3 Analysis of data

Data analysis does not occur in isolation; instead, it occurs concurrently with other parts of the research, such as data collection, coding and the writing up of findings (Creswell, 2013). This is because the researcher might see the need to interview the participants so that they can draw more information from the participants. Data analysis is also a process that involves backwards and forward movements between coding, generating and interpreting themes (Noble & Smith, 2013).

Figure 5.1 shows the steps in data analysis, as identified by Creswell (2014), the first of which was to organise and prepare the data for analysis, which was done by transcribing the data. After transcribing the data, reading through all the transcription was necessary so that the researcher could be familiar with the data. Data were then grouped into categories or clusters by means of coding, which could be done manually or by computer software. Part of coding of this study was done using Nvivo 10, and then the rest of the coding was done manually. The process of coding was then used to generate codes for describing the events in the calculus lecture room, such as the dialogue, the mathematical activities, as well as the lecturers' actions. Coding was also used to generate themes that emerged from the data analysis. These themes were then used to make known the findings of the study and the final step was the interpretation of the findings of the study.

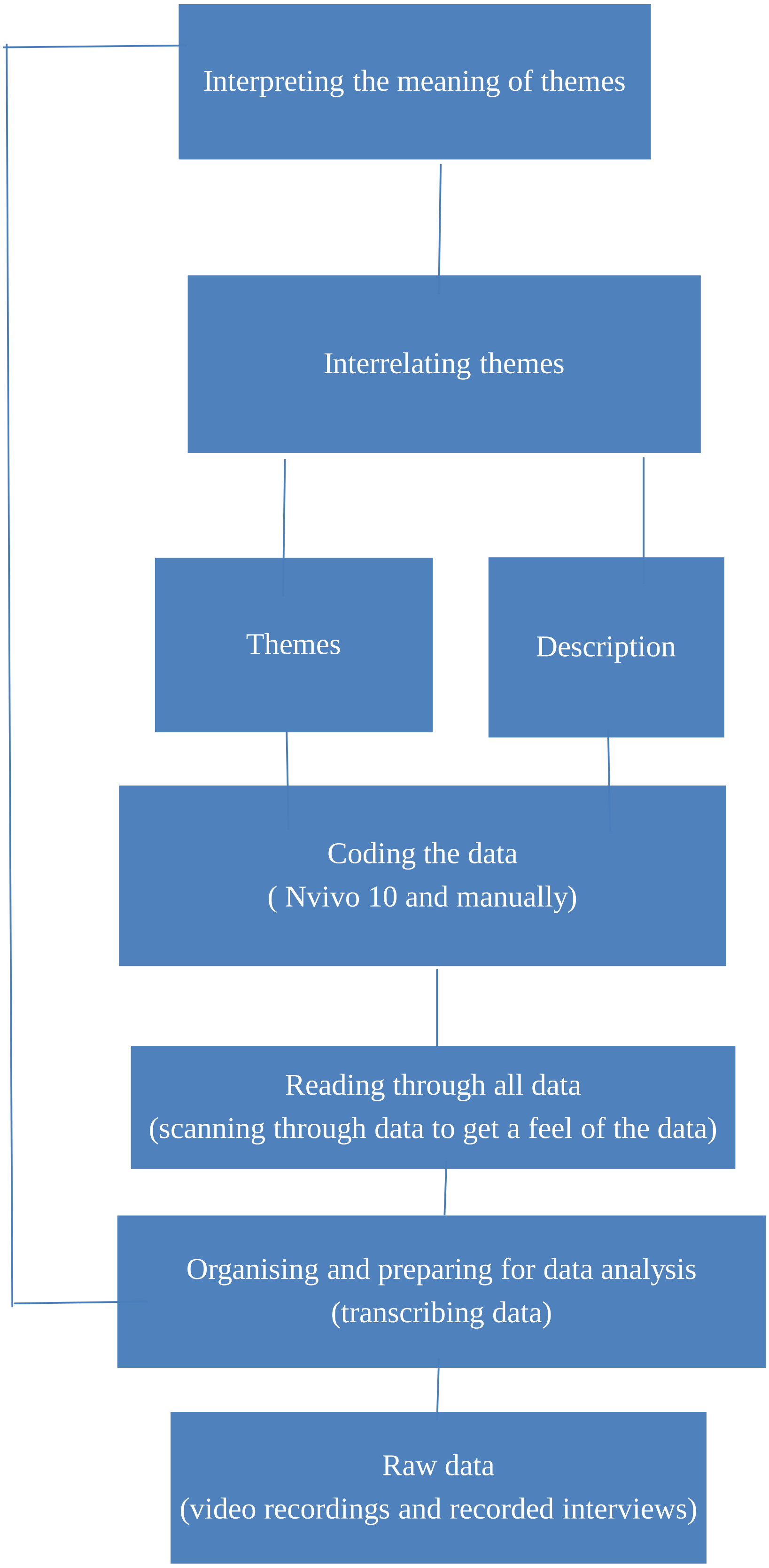


**Figure 5.1: Overview of data analysis process**

**Adapted from: (Creswell, 2014, p. 214)**

## 5.4 The research participants

The research participants of this study were two lecturers. In Lecturer A's lecture room, there were 78 registered students, while Lecturer B had 120 registered students.

### 5.4.1 The profiles of the lecturers

The participating lecturers of this study were highly qualified people who had many years of experience in lecturing pure mathematics. Table 5.1 depicts a summary of the lecturers' profiles.

**Table 5.1: Profiles of the lecturers**

| Lecturer | A | B |
|---|---|---|
| Gender | Male | Male |
| Age group | 50-60 | 50-60 |
| Years of experience | 31 | 29 |
| Qualifications | B. Ed (Hons); B. Sc; B. Sc. (Hons); M.Sc.; PhD | HDE; B.Sc.; B.Sc.(Hons); M.Sc. |
| Use of technology | No | Yes |
| Use of white board | Yes | Yes |
| Number of students | 78 | 120 |

#### 5.4.1.1 Profile of Lecturer A

Lecturer A is currently lecturing mathematics at one of the universities in South Africa. He began by teaching at a high school and then moved on to lecture at a teacher training college and finally at a university. His combined years of teaching and lecturing experience amount to

31 years. Lecturer A is highly qualified, with numerous degrees, which include the Bachelor of Education Honours, Bachelor of Science, Bachelor of Science Honours, Master of Science, as well as a Doctorate in Pure Mathematics. Lecturer A also supervises the Masters and PhD students. His research interests are topology, as well as exploring teaching and learning theories and practice within university mathematics.

While observing the video recordings of Lecturer A, it was noticeable that he is well respected by his students and appeared to have a good relationship with them. There appeared to be a friendly atmosphere in the lecture room of Lecturer A. Lecturer A made frequent use of a whiteboard for his lectures. The whiteboard was used to demonstrate and explain procedures and concepts.

#### 5.4.1.2 Profile of Lecturer B

Lecturer B began by teaching at a high school, after which he moved to lecture at a college of education. From there, he spent a full year working with teachers as a mathematics subject advisor before taking up a position as a senior lecturer at a university in 2002, where he is currently working. His combined teaching and lecturing experience is 29 years. In his teaching career, he has continued to facilitate workshops and seminars aimed at supporting learners and mathematics educators as they improve their mathematics content knowledge, in both the GET and FET band. He has also written a number of learning and teaching support materials for Educational Projects such as Primary Mathematics Project (PMP), Upward Bound Project, Centre for the Advancement of Science and Mathematics Education (CASME), Mdiphi Consultants and Programme to Improve Learning Outcomes (PILO).

Lecturer B is relatively well qualified with the following qualifications: HDE (Mathematics Junior and Senior Secondary), Bachelor of Science (Mathematics and Mathematical Statistics), Bachelor of Science (Hons) (Applied Mathematics) and a Master of Science (General Relativity and Cosmology). Over and above this, he has presented several papers in a number of national conferences and published six peer reviewed journal articles in SAPSE accredited journals to date.

While observing the video recordings of Lecturer B, it was evident that the students respected Lecturer B and that there was a friendly atmosphere in all his lectures. Lecturer B had 120 students in total. He appeared to be comfortable with the use of technology, as he used power point in most of his lectures. He also used the whiteboard to demonstrate procedures, as well as explain concepts.

### 5.4.2 The students

In South Africa, one can qualify as a teacher in two ways. One way is to go through a four-year degree, which is the Bachelor of Education (B.Ed.) degree at a university, which requires one to have English language at level 4 and Life Orientation at level 4 and any other 2 subjects at level 3. Additionally, one's total points have to be between 28 and 48. The levels and points that are currently used in South Africa are discussed in detail in Table 5.2:

**Table 5.2: Calculation of Composite Academic Performance Score (APS) for university entrance**

| NCS Rating (Level of Performance) | NCS Percentage | Points value for calculation at (APS) | Adjusted % for APS Calculation |
|---|---|---|---|
| | | 8 | 90-100 |
| 7 | 80-100 | 7 | 80- 89 |
| 6 | 70-79 | 6 | 70-79 |
| 5 | 60-69 | 5 | 60-69 |
| 4 | 50-59 | 4 | 50-59 |
| 3 | 40-49 | 3 | 40-49 |
| 2 | 30-39 | 2 | 30-39 |
| 1 | 0-29 | 1 | 0-29 |

**Adapted from: The website of the university in this study**

The second way to qualify as a teacher is to study towards a three-year degree, which is followed by a Post-Graduate Certificate in Education (PGCE) (Ungersbock, 2015). The PGCE is a one-year, full-time course that can be studied at a university. In order for one to enrol as a PGCE student, one must have completed a recognised undergraduate degree and must have studied the subject in which one wishes to major.

In both lectures, some of the students were in their second year while others were in their third year of studying. This was because if students were accepted at university with a mark less than

60% in NCS mathematics, i.e. below a level 5 in mathematics, these students would have to complete a foundational module in mathematics in their first year. Level 5 is part of the national codes, which are related to percentages as prescribed by the Department of Basic Education in the National Protocol for Assessment (2011). Table 5.3 illustrates the codes and related percentages.

**Table 5.3: Codes and percentages for recording and reporting in Grades 10-12**

| Rating Code | Achievement Description | Marks % |
|---|---|---|
| 7 | Outstanding Achievement | 80-100 |
| 6 | Meritorious Achievement | 70-79 |
| 5 | Substantial Achievement | 60-69 |
| 4 | Adequate Achievement | 50-59 |
| 3 | Moderate Achievement | 40-49 |
| 2 | Elementary Achievement | 30-39 |
| 1 | Not Achieved | 0-29 |

**Adapted from: The National Protocol for Assessment (2011)**

In their second year, the students would do a pre-calculus module in the first semester and differential calculus in the second semester. In their third year, they would do the integral calculus module in the first semester. Some of the students went to university with a mark of 60% or more, which means that they had a level 5 or more in mathematics and thus, could study pre-calculus in the first semester of their first year. They would then study differential calculus in the second semester of their first year and then integral calculus in the first semester of their second year. Thus, both groups of students in this study had a combination of third year students and second year students. There were 78 students in lecture A1-9 and 120 students in lecture B1-11.

Some of the students in this study were taking the integral calculus module because they were majoring in mathematics so that they could teach in the Further Education and Training Band (FET), which means they would be teaching learners in Grades 10, 11 and 12. The other group of students was studying mathematics only as a learning area. This means that they were studying towards the B.Ed. degree and focusing on the intermediate and senior phases, which means that they would be teaching learners in Grades 4, 5 and 6 in the intermediate phase, as well as those in Grades 7, 8 and 9 in the senior phase. These students would have been taking other subjects as their majors, but taking mathematics as a learning area. This would have been because the B.Ed. programme requires students to register for at least one learning area which would have been either mathematics, Life Sciences or Technology and thus, the integral module would be their final content module.

### 5.4.3 The university in this study

The data for this study were collected from one university. The university has a well- resourced library and several computer laboratories. All lecture rooms are equipped with resources in the form of data projectors, white boards and chalk boards. The university has free Wi-Fi access so that students are able to access the internet easily when they are on campus.

### 5.4.4 The materials in this study

The materials in lecture A1-9 were developed with the intention of engaging students conceptually. The developers of the materials took careful consideration of what was to be taught, since the intentions were to allow the students to engage conceptually with the materials. Lecturer A did not develop the materials used in lecture A1-9.

Lecturer B himself developed the materials used by Lecturer B. As he developed the materials, he took consideration of the fact that he had taught the same class in the previous module Differential Calculus and his intentions were to link the two concepts with ease.

Table 5.4 illustrates a list of topics from lectures of the two data sets in this study.

**Table 5.4: Lecture content of Lecturer A**

| Lecture | Duration | Lecture content |
|---|---|---|
| **One** | **1hr 30 minutes** | Introduction of area under a curve. |
| **Two** | **1hr 30 minutes** | Continuation of area under a curve.<br>Finding the sum of the areas of rectangles under a curve. |
| **Three** | **1hr 30 minutes** | Introducing the Fundamental Theorem of Calculus.<br>Linking to area under a curve. |
| **Four** | **1hr 30 minutes** | Calculating anti-derivatives.<br>Linking to area under a curve. |
| **Five** | **1hr 30 minutes** | Calculating anti-derivatives by using the substitution technique.<br>Calculating anti-derivatives of trigonometric functions. |
| **Six** | **1hr 30 minutes** | Calculating anti-derivatives of log functions. |
| **Seven** | **1hr 30 minutes** | Calculating definite integrals. |
| **Eight** | **1hr 30 minutes** | Working with partial fractions.<br>Integration by parts. |
| **[1]Nine** | **1hr 30 minutes** | Consolidation of module. |

This lecture content plan for lecture A1-9 was designed by Lecturer A and the developers of the materials used in lecture A1-9. The university did not have a master plan for the module, but the module outline, which stated the topics to be covered in the semester.

[1] Although there were 9 lectures recorded from Lecturer A, 8 were transcribed because lecture 9 was a module consolidation lecture.

**Table 5.5: Lecture content of Lecturer B**

| Lecture | Duration | Lecture content |
|---|---|---|
| **One** | **1hr 30 minutes** | Introduction of integral calculus through a process of anti-differentiation.<br>Introducing the properties of indefinite integrals. |
| **Two** | **1hr 30 minutes** | Application of properties in finding indefinite integrals.<br>Introducing useful simplification methods when finding indefinite integrals. |
| **Three** | **1hr 30 minutes** | Application of simplification procedures to a variety of problems.<br>Finding indefinite integrals of powers of functions.<br>Finding indefinite integrals that lead to logarithmic functions and using suitable substitution techniques. |
| **Four** | **1hr 30 minutes** | Solving first order separable ordinary differential equations.<br>Solving initial value problems as an application of indefinite integration. |
| **Five** | **1hr 30 minutes** | Finding integrals of trigonometric functions. |
| **Six** | **1hr 30 minutes** | Using known trigonometric identities to transform the given integrand into something easily integrable. |
| **Seven** | **1hr 30 minutes** | Revisit procedures for resolving an algebraic fraction into partial fractions.<br>Use this idea to introduce integration by partial fractions as an alternative technique for integration.<br>Apply integration by partial fractions appropriately. |
| **Eight** | **1hr 30 minutes** | Introduce integration by parts as a useful technique. |
| **Nine** | **1hr 30 minutes** | Introducing area under a curve.<br>Contextualising definite integration.<br>Confirming: Area (A) = $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$ |
| **Ten** | **1hr 30 minutes** | Continuation of area under a curve. |
| [2]**Eleven** | **1hr 30 minutes** | Consolidation of module. |

[2] Although there were 11 lectures recorded from Lecturer B, 10 were transcribed because lecture 11 was a module consolidation lecture.

This work plan was designed by Lecturer B, who was guided by the module template that specified the broad topics to be covered in the module, the notional hours for the module, as well as the number of weeks in a semester that the module is run. This plan is fluid and changes, depending on the number of weeks available in a semester, which is normally between 12 and 14 in Semester 1. There is no generic lecture plan from the university from which the lecture and assessment plan was designed.

## 5.5 The primary themes emerging from video recordings and interviews of the lectures

The primary themes that emerged from the video recordings of the lectures of this study, as well as from the interviews with the lecturers are presented in Table 5.6.

**Table 5.6: The primary themes**

| | |
|---|---|
| Theme 1<br>Mathematical activities in the calculus lecture room | 1.1 Activities that promote the use of investigations. |
| | 1.2 Activities that promote the use of conjectures. |
| | 1.3 Activities that promote the use of proofs. |
| | 1.4 Activities that promote the use of symbols. |
| | 1.5 Activities that promote the use of multiple representations. |
| | 1.6 Activities that promote the procedural fluency through conceptual understanding. |
| | 1.7 Activities that promote the use of multiple techniques in problem solving. |
| Theme 2<br>Nature of dialogue in the calculus lecture room | 2.1 Dialogue through explanation of concepts, procedures and symbols. |
| | 2.2 Questions to check prior knowledge. |
| | 2.3 Probing and follow up questions. |
| | 2.4 Leading questions. |
| | 2.5 Interrogative questions. |
| | 2.6 Confidence boosting questions (Affirmation questions). |
| Theme 3<br>Organising learning materials for the purpose of engaging students conceptually with the materials. | 3.1 Building on foundation knowledge. |
| | 3.2 Working with prior knowledge. |
| | 3.3 Scaffolding. |

Table 5.6 illustrates the primary themes that emerged during data analysis. Themes 1.1, 1.2, 1.3 and 2.2 to 2.6 involve questioning, hence, they were grouped together to form a new theme called: lecturing through questioning. Many researchers including Chikiwa (2017), Mhakure and Jacobs (2016) and Olmsted (2012), advocate lecturing through questioning because the questions invoke the students' thinking and thus, assists the students in developing mathematically (Jancarik, Jancaricova, & Novotna, 2013). Themes 1.4, 1.6 , 1.7, 2.1 and 3.1 involve explanations, hence they were grouped together to form a new theme called lecturing through explanations. During these lectures, Lecturer A and Lecturer B explained the concepts, procedures and the meanings of different mathematical symbols and notations.

## 5.6 New themes that emerged from primary themes

**Table 5.7: The new themes**

| | |
|---|---|
| Theme 1 | Lecturing through explanations |
| Theme 2 | Lecturing through questioning |

Table 5.7 shows the new themes 1and 2 that emerged from the primary themes in Table 5.6.

## 5.7 Interaction between lecturer and students

While observing the video recordings of both lectures, it was noticeable that dialogue in both lectures was two sided, because the students were also given the opportunity to explain how they arrived at different solutions and specific answers. Both lecturers explained the concepts, procedures and meanings of symbols and notations. Both lecturers asked questions and provided their students with opportunities to ask questions, which were answered by the lecturer or by fellow students.

## 5.8 Interaction amongst students

Students in both lectures were given opportunities to work either in groups or in pairs. Thus, students were given the opportunities to discuss and find solutions to problems. They also had opportunities to explain concepts and procedures to each other. As students worked either in pairs or in groups, they had the opportunities to ask each other questions, at the same time the other students had opportunities to answer these questions.

### 5.9 Interaction with the materials

Students in both lectures had no prescribed textbooks. The students in lecture A1-9 used the materials, which were designed by the organisers of the NRF project, while students in lecture B1-11 used the materials that were designed by Lecturer B. Students in both lectures, interacted with the materials during the lectures and tutorials. For homework, they also used questions from the materials. Students also had access to notes and tutorials via Moodle, a website designed for communications between lecturers and students. Lecturers posted tutorials, extra notes, past exam papers, past tests and answers on the website.

### 5.10 Analysis of the lectures of this study

While 20 lectures, of which 9 were from Lecturer A and 11 from Lecturer B, were video recorded, 18 lectures were transcribed and analysed, 10 of which were from Lecturer A and 8 from Lecturer B. This was because the last lecture from each lecturer was a module consolidation lecture and the researcher found no need to analyse the module consolidation lectures, since it was based on revision of work already covered in the module. Each lecture was one hour thirty minutes long. The students had tutorials, which were also one hour thirty minutes long and served the purpose of consolidating what had been learnt in the lectures. Both lecturers used worksheets and the materials that were designed for the module in their teaching.

### 5.11 Analysis of semi-structured interviews and the legitimising appeals

#### 5.11.1 Why does the lecturer change tasks?

In any mathematics lecture, it is possible for the lecturer to change tasks during planning or during the lesson for various reasons (Georgius, 2014). Some lecturers may change the task because it is of high cognitive demand and they might see the need to lower the level of the task.  Sometimes the task might be of low cognitive demand and thus, the need to raise the task to a higher level (Georgius, 2014). Thus, in most lectures, tasks that are planned for the lectures are not always implemented as initially intended.

The lecturers participating in this study changed tasks during lectures, after observing that their students lacked foundation knowledge. An example is when students in lecture B1-11 were experiencing problems with integrals of trigonometric functions. The lecturer noticed that the students had no problems with integration but the problem was with trigonometry, which the students were expected to have done in high school. Thus, instead of carrying on with finding integrals of trigonometric functions, the lecturer changed his tasks to working with the basic concepts of trigonometry, which included trigonometric identities. Another example is when the lecture was based on integration using partial fractions, the lecturer noticed that the students were struggling with partial fractions and thus, changed his task to working with partial fractions, which the students had already done in their first module. This is supported by Mapolelo and Akinsola (2015), who point out that the lecturers' mathematical knowledge profoundly influences the way they teach mathematics, as well as the way they reflect and make decisions about their teaching.

### 5.11.2 How does the lecturer's mathematical knowledge inform his decisions and his reflection in practice?

Lecturer A viewed content knowledge (mathematical knowledge) as knowledge that was appropriate or fitting for a particular topic that he teaches, in this case integral calculus. He referred to mathematical knowledge as being able to deal with a mathematical topic and the mathematics around that particular mathematical topic. This is what Lecturer A said in this regard:

> **Lecturer A:** *My knowledge of mathematics plays a major role firstly in my planning for the lecture and secondly in the actual lecture. Profound content knowledge is essential in that it is the knowledge that one needs for teaching. You cannot teach what you don't know. If one does not have the content knowledge, one cannot engage conceptually with the students at a high level of thinking. One needs to make sure that the students develop high level of thinking in the subject.*

Lecturer A had an extensive background in pure mathematics and was comfortable teaching integral calculus. He also believed that his mathematical knowledge was more than enough to teach this topic at this level. With the possession of such profound mathematical knowledge, the lecturer was able to reflect and make decisions during the lesson. An example is when he guided the students to calculate the area under a curve by dividing the area into rectangles. He also allowed his students to think about how they were going to do so. This is supported by Jadama (2014), who points out that the possession of deep mathematical knowledge allows lecturers or educators to plan well for their lessons. This also enables a lecturer to be in a position to use a variety of teaching strategies.

Lecturer B mentioned that his deep mathematical background, coupled with his experience of teaching the module, enabled him to reflect and make decisions in his practice. He mentioned that he reflects on the materials that he uses in his lectures. If the materials work well, he reflects on why they work well and why the materials do not work well. This is what Lecturer B said in this regard:

> **Lecturer B:** *Look, it's not only my mathematical knowledge, it's coupled with my experience in teaching this module and understanding what students know and what students battle with. Also with how else to assist them as they battle. You know, the materials that I have, I use the materials, but now and then because I sit down and reflect on what has worked well, and if something has worked well, I say ok, why has it worked well? I will see what has worked well and if there is something that has not worked well I will try and find out what exactly was the problem in this particular instance.*

Both lecturers participating in this study agreed that their strong mathematical knowledge informed their decisions and their reflections on their practices. This is in agreement with many researchers, including Kleickmann et al. (2013), Jadama (2014), Santangata and Yeh (2015), who argued that the teachers' mathematical knowledge greatly affects their decision-making and reflections.

### 5.11.3 How does the lecturer's knowledge of teacher training inform his decisions and his reflection in practice?

The lecturers in this study believed that there is an extensive connection between their mathematical knowledge and their lecture room practice. For example, Lecturer A pointed out that his decisions on which approach to use, as well as what lecture room practice to use, are strongly based on his vast mathematical knowledge. He alluded to the fact that he has a complete global picture of where he is going with a particular topic. This is what Lecturer A said in this regard:

> **Lecturer A:** *There is a strong link between my knowledge of mathematics and my practice in the classroom. When I notice that students are struggling with a concept, I am able to re- explain in a different way so as to make sure the students understand. In other words, I am able to represent concepts in a variety of ways to help students understand. Having a profound knowledge about your subject influences how you teach the subject.*

This is supported by Kleickmann et al. (2013), who point out that teaching experience, coupled with subject knowledge, enable lecturers to engage in thoughtful reflections about their lecture room practices. Lecturers can only have a global picture of where they are going with the topic being taught if they have in depth content knowledge. This helps the lecturers or educators to plan for their lessons well, taking into account what their students are expected to alreadyknow and what they are going to know. It also allows the lecturer to lecture using a variety of lecturing methods (Jadama, 2014).

Lecturer B mentioned that his mathematical knowledge, experience in teaching the module coupled with the understanding of what students know or do not know, as well as what students struggle with, informs his decisions and reflections in practice. He mentioned that it was also the knowledge of how students learn that informed his decisions and reflections in practice. In view of this, he had this to say:

> **Lecturer B:** *As I am saying it is knowledge of how they also learn and what is that they battle with. I mean, we know part of our problem is that when you are teaching calculus, you teach calculus, you teach integration but when you talk about integrals of trig functions, then they need to know their trigonometry. That immediately it is going to say to me, what activities or what is it that I should do in order to satisfy myself that they got the basics before I go on?*

An example is when he changed the task because his knowledge of the subject, as well as the knowledge of how students learn, gave him the reason to decide on changing the task from integrating partial fractions to just working with partial fractions. This is supported by Ramli, Shafie and Tarmizi (2013), who point out that the lecturers' knowledge of how students learn, think or struggle with concepts, allows the lecturers to plan well for their lessons. The findings of Santangata and Yeh (2015) are in agreement with this. In their study, Santangata and Yeh (2015) highlighted the fact that their mathematical knowledge, coupled with the knowledge of how students learn, played a fundamental role in their decision-making. This knowledge also allows the lecturers to make decisions with regard to their classroom practices.

Both lecturers agreed that their mathematical knowledge, their experience of teaching the module, as well as their knowledge of how the students learn, informed their decisions and reflections in practice. This is what they said in this regard:

> **Lecturer B:** *Look, it's not only my mathematical knowledge, it's coupled with my experience in teaching this module and understanding what students know and what students battle with.*
>
> **Lecturer A:** *There is a strong link between my knowledge of mathematics and my practice in the classroom. When I notice that students are struggling with a concept, I am able to re- explain in a different way.*

Many researchers (Ramli, Shafie, & Tarmizi, 2013; Santangata & Yeh, 2015) agree with the fact that the lecturers' mathematical knowledge and experience inform their decision- making and reflections in practice.

**5.12 Conclusion**

This chapter presented a discussion of the analysis of data. The data generated in this study showed that two lecturers and their students were the participants of this study. The steps taken to analyse the data have also been discussed. These steps include transcribing of data, reading through the transcriptions in order to be familiar with the data, coding the data, searching for patterns and themes and finally interpreting the patterns and themes.

Two types of lecturing emerged from the primary themes and these are lecturing through explanations and lecturing through questioning. These themes are discussed in detail in Chapter Six. The next chapter discusses the findings of this study.

# CHAPTER SIX

## DISCUSSION OF FINDINGS

### 6.1 Introduction

This study sought to gain insight into what is counted as mathematics, the dialogue that took place in the calculus lecture room, the lecturers' actions, as well as the mathematics for teaching that is exhibited by the lecturers. Thus, this chapter explores and discusses the findings of this study. There are three sections in this chapter. The first section explores the activities that were legitimised in the lectures. The second section explores the nature of dialogue that took place during the lectures and the third section explores the lecturers' organisation of the materials and the reasons behind the organisation.

Themes that are related to the research questions of this study emerged during the analysis of data as discussed in Chapter Five. The first theme focuses on the mathematical activities that were legitimised during the lectures. The second theme focuses on the dialogue that emerged during the lectures, while the third theme focuses on how the activities were organised by the lecturers in the calculus lecture room. These themes were informed by the research questions of the study. From these primary themes, two themes emerged; lecturing through explanations, as well as lecturing through questioning.

### 6.2 Theme 1: Mathematical activities that were legitimised during the lectures

In every lecture room, there is a variety of mathematical activities in which the students participate. Some of the activities require the students to employ deductive or inductive reasoning (Downs & Mamona-Downs, 2013), while other activities require the students to employ their procedural knowledge. The following section explores the mathematical activities that were legitimised during the teaching and learning of calculus.

#### 6.2.1 Activities that promote the use of investigations

Investigations are designed to engage students in active learning, in which they are given an opportunity to explore a given concept. The inclusion of investigations during the teaching and learning of mathematics is important because investigations not only allow the students to participate in the activities that take place in that lecture room, but also, are fundamental to the development of the students’ critical and creative thinking (Yuliani & Saragih, 2015). In addition, investigations deal with complex thinking processes that encourage the reinforcement of learning, since the students are required to draw on higher abilities. The students are also required to communicate with their lecturer, as well as with each other and thus, investigations are mostly related to inquiry-based learning (Day, 2014).

Lecturer A used investigations in his teaching. He introduced the module with an investigation focusing on the area under the curve, while Lecturer B concluded the module with the concept of the area under the curve. The following is an example of an investigation from Lecturer A, where students were investigating the area under a given curve:

> **Lecturer A:** *Ok, now the area between the graph and the x axis you are going to shade now, over the interval 1 to 7. So, this is what I expect you to do. Firstly. [Draws graph on board.][3] So in pencil, I want you to shade this portion of the graph [shades between 1 and 7 and* students follow instructions].

> **Lecturer A:** *Okay have you shaded the area? Right, so you should have this particular region under the graph shaded [Holds up example by student] Okay. Now, I want you to read on and you are going to do the rest of the stuff. In fact, I want you to go ahead with the work. And you would answer the questions in your books. You can work in pairs if you want to.*

In this investigation, students were required to calculate the area under the given curve between the interval x = 1 and x = 7. There was a series of questions that required them to calculate the sum of the areas of the rectangles in the first diagram where only three rectangles were drawn under the curve. The task focused on the students realising that not all the area under the curve was accounted for in the three rectangles and so the students had to apply higher order and critical thinking (Sanders, 2016), so that all the space under the curve was accounted for. The investigation task of this study required the students to work in pairs.

Lecturer B also used investigations in his teachings. An example is when he also guided his students to investigate the area under a curve. The following transcript illustrates this:

[3]Words in square brackets have been added by the researcher to assist the reader when reading the transcripts.

**Lecturer B:** *We are looking for the area under this curve bounded by the line x = a, x = b and the x-axis. You get That?*

**Students:** *Yes.*

**Lecturer B:** *You can actually show this by shading this area. [shades the area under the curve]. The question is, how do we find the area under this curve? If I asked you to find the area of my table, what would you do?*

**Students:** *Length times breadth.*

**Lecturer B:** *Yes, because you know the formula for area of a rectangle. Now for the area under a curve, this is what Riemann did. He then subdivided that region, the shaded region, by inserting one vertical line whose base lies on the x-axis but it is within the region itself. [draws the vertical line on the shaded region]. Once you do that, you can actually get a rectangle. [draws a rectangle on the shaded region]. But also you can get another rectangle. [draws in another rectangle]. Now I want you to use this idea and try and find the area under this curve.*

By using investigations in their teaching, the lecturers encouraged their students to reason mathematically. This is supported by Marshman and Brown (2014), whose findings show that the educators in their study encouraged their learners to think mathematically. This allowed the eductors to make sense of the mathematics they applied as they took part in investigative tasks. Reasoning mathematically is one of the components of the Mathematical Activities framework of this study. Reasoning mathematically was exhibited through the process of investigating the area under the curve when the students answered the questions that led them to uncover the concept of a definite integral. A definite integral is the exact limit and summation of the areas of rectangles used to find the net area between a function and the x-axis. During the process of the investigation, the lecturers also guided the students by asking leading questions that required them to reason mathematically. The following is an example of the series of questions that were asked by Lecturer A:

**Lecturer A:** *Ok. So, the areas changed. The values of the areas. How did the change take place? What was the change?*

**Student:** *It was a consequence of the x values.*

**Lecturer A:** *Ok, it was a consequence of the x values. But when you compare the change, what do you observe, what makes the change? I mean, by just looking at it, how do you know it changed? You say it got bigger. How did you know that? You* *looked at the values. So, the first area sum was … what was the first one? Why does that happen?*

The preceding example shows a series of questions that the lecturer asked the students, which required them to reason mathematically so that they could uncover the idea behind the change of the areas under the curve and eventually uncover the idea of a definite integral. Thus, by using tasks that promote the use of investigations while introducing new concepts, the lecturer created opportunities for students to comprehend the new concepts (Day, 2014). Yuliani and Saragih (2015) support the idea of using investigations in the teaching of mathematics by pointing out that investigations encourage students to apply high-level thinking. Students in this study appeared to apply high level thinking as they engaged in reasoning mathematically when answering the series of questions that were asked by their lecturers.

### 6.2.2 Activities that promote the use of conjectures

Conjecturing involves identifying patterns and using them to develop new mathematical knowledge, which only occurs when students are given enough opportunities to take part in the conjecturing process (Liu & Chin, 2016). Thus, conjecturing involves inductive reasoning since new knowledge is developed through the observation of patterns. Therefore, teachers need to design tasks that allow students to make conjectures and also allow students to talk about these conjectures, so that they can discover mathematical concepts. Thus, when students take part in activities that promote conjectures, they develop the confidence to share their ideas with others (Rahman, Yusof, Ismail, & Kashefi, 2012). Cañadas, Deulofu, Fgueiras, Reid and Yevdokimov (2007) identified seven stages of conjecturing, while Lin, Yang, Lee, Tabach and Styliandes (2012) identified four principles of conjecturing which were developed with seven stages in mind. Table 6.1 clearly illustrates the seven stages of conjecturing.

**Table 6.1: The seven stages of conjecturing**

| Stage Number | Stage Name |
|---|---|
| 1 | Observing cases |
| 2 | Organising cases |
| 3 | Searching for patterns |
| 4 | Formulating a conjecture |
| 5 | Validating the conjecture |
| 6 | Generalising the conjecture |
| 7 | Justifying the conjecture |

**Adapted from: Canadas et al. (2007)**

Table 6.2 shows the four principles of conjecturing.

**Table 6.2: The four principles of conjecturing**

| Principle number | Principle name |
|---|---|
| Principle 1 | Observation |
| Principle 2 | Construction |
| Principle 3 | Transformation |
| Principle 4 | Reflection |

**Adapted from: Lin et al. (2012, p. 495).**

The four principles of conjecturing guide the seven stages identified by Cañadas et al. (2007) in the following manner. Firstly, when students are observing and organising cases, in stages 1 and 2, they are guided by the principle of observation and construction of new knowledge from their prior knowledge. Secondly, when they are searching for patterns and formulating conjectures, they would be guided by the principles of construction and transformation. The stages of validating, generalising and justifying are guided by the principles of transformation and reflection, because it is possible for students to make and justify a wrong generalisation. Thus, the principle of reflection is important as students transform prior knowledge into new knowledge (Lin et al., 2012).

Both lecturers in this study used conjecturing even though not all the stages in Table 6.1 are exhibited in all 18 lectures of this study. The following is an example where conjecturing was used by Lecturer B:

**Lecturer B:** *From standard 1 or grade 2, you know that* $6 + 3 = ?$

**Students:** *9*

**Lecturer B:** *We can go backwards and say* $9 - 6 = ?$

**Students:** *3*

**Lecturer B:** *We can do a similar thing with multiplication and say* $5 \times 4 = ?$

**Students:** *20*

**Lecturer B:** *But we can start from 20 and say,* $20 \div 4 = ?$

**Students:** *5*

**Lecturer B:** *For those of you who came from Maths 120,* $10^2 = ?$

**Students:** *100*

**Lecturer B***: We can start from 100 and take the logarithm of 100 of base 10 and go back to?*

**Students:** *2*

**Lecturer B:** *Even in grade 10, we can find the product of* $(x - 2)(x + 2)(x - 1)$ *and get?*

**Students:** $x^3 - x^2 - 4x + 4$

**Lecturer B:** *What is the derivatives of the following: 1.* $f(x) = x^5$, *2.* $f(x) = x^5 - \sqrt{7}$

*3.* $f(x) = x^5 + 9$

**Students:** $5x^4$

**Lecturer B:** *Which function did I differentiate to get* $3x^2$*?*

Lecturer A also had activities that promoted the use of conjecturing. The following is an example where conjecturing was used by Lecturer A:

**Lecturer A:** *What is your exponent here? 1.*

**Students:** *1*

**Lecturer A:** *And what does it become?*

**Students:** 2.

**Lecturer A:** *2. 2, if you add 1?*

**Students:** 3

**Lecturer A:** *3 if you add 1?*

**Students:** *4*

**Lecturer A:** *So, in general, what would be the exponent here?*

**Students:** *It's n+1*

The transcripts presented here are part of the conjecturing process in the lectures of Lecturer B and Lecturer A. The students had to answer a series of questions while they took part in activities that required them to make a conjecture about the process of anti-differentiation. In these activities, the students had to observe a pattern, firstly, that anti-differentiation was a reverse process of differentiation and secondly, that the anti-derivative would have a constant ***c***. As students made conjectures about the notion of the anti-derivative, they may have been required to use higher levels of thinking. This is supported by Lesseig (2016), who points out that during the process of conjecturing, students are expected to be deep and divergent thinkers as they seek to understand this mathematical knowledge. Similarly, Supratman et al. (2016) show that when students are given the opportunity to take part in the conjecturing process, they are prepared to use their prior knowledge to construct new knowledge by applying deep thinking.

By using conjecturing, Lecturer A and Lecturer B gave their students an opportunity to reason mathematically, as well as to think mathematically. Reasoning mathematically and thinking mathematically are some of the components of the Mathematical Activities framework used in this study. Reasoning mathematically was exhibited, for example, when Lecturer B’s students were answering a series of questions, which led them to make a conjecture about anti-differentiation being the reverse of differentiation. Thinking mathematically was exhibited when the students had to extend the scope of the concept of differentiation to the concept of anti-differentiation and thereby applying deep and broad thinking. Similarly, in the United Kingdom, teachers are required to give their students opportunities to reason and think mathematically, which is promoted by doing tasks that involve conjecturing (Lesseig, 2016).

### 6.2.3 Activities that promote the use of proofs

Proofs are important in mathematics because they assist in the completion of the cycle of mathematical sense making, as they involve the use of logical and rigorous arguments to make valid conclusions (Fleron et al., 2014). de Villiers (2012) and Seldon and Seldon (2015) agree that logical thinking is one of the fundamental requirements for conducting a proof. As stated by de Villiers (2012), a proof is an argument that shows how one can get an expected result, while using deduction and logic. Thus, deductive reasoning is essential in conducting a proof. Seldon and Seldon (2015) identified two different parts of a proof. The first part is formal-rhetorical, which depends on unpacking the statement or theorem and does not need the deep understanding of the mathematical concepts, but uses logic in the process. The second part is the problem-centred part, which depends on the deep understanding of mathematical concepts.

Both lecturers in this study used proving in their lectures. Among other concepts, both lecturers provided their students with the opportunity to prove The Fundamental Theorem of Calculus (the FTC). The Fundamenatl Theorem of Calculus is a theorem that links the idea of the anti-derivative with the area under a given function. The following is an example from Lecturer A's lecture on proving the Fundamental Theorem of Calculus with his students:

> **Lecturer A:** *Then the fundamental theorem is correct. It's telling us the correct thing. Hey, did any of you come across a strategy, this method of proof, where you have to verify something and you take the left-hand side of that whatever, and it should equal the right-hand side. Did you come across such a thing? Check and compare the two. Take the left-hand side, work on it, then take the right-hand side. Same thing we're doing. We are proving an identity.*

The following is an example from Lecturer B's lecture on the Fundamental Theorem of Calculus:

> **Lecturer B:** *Here is our generalisation,* $\int \left(f(x)\right)^n . f'^{(x)} dx = \frac{1}{n+1}[f(x)]^{n+1} + c, where\ n \neq -1$. *What happens if n = -1? Now do this* $\int \frac{2x}{x^2-4} dx,$ *and see what happens also to check that our generalisation is correct.*

Verifying a statement or a theorem is one of the fundamental purposes of a proof. The students in this study had to verify the Fundamental Theorem of Calculus with the guidance of their lecturers. This may have led them to understand the core concept of the anti-derivative.

Similarly, the students in Reid's (2014) study had to perform proofs and ended up showing the ability to understand the core concepts of the topics with which they were dealing.

As the students engage in the process of conducting a proof, they think mathematically, as well as reason mathematically which is supported by Marshman and Brown (2014), who suggest that encouraging students to think and reason mathematically helps them to make sense of what they are learning. Reasoning and thinking mathematically are some of the components of the Mathematical Activities framework of this study. Reasoning mathematically suggests that the students were able to uncover the basic ideas as they follow a line of argument in proving the Fundamental Theorem of Calculus. Thinking mathematically suggests that the students were able to understand and handle the scope of the Fundamental Theorem of Calculus. This is supported by de Villiers (2012), who points out that allowing students to engage in tasks that promote proving assists them in understanding the concepts, as well as with seeing the importance of generalisations.

### 6.2.4 Activities that promote the use of symbols

In mathematics, there is a vast amount of symbols and notation, which often have different meanings in different situations. Thus, it is essential for students to know the meaning of symbols and notation. This is in line with the point raised by Quinnell and Carter (2013) when they point out that some mathematical concepts may be represented in a variety of ways using a variety of symbols. In addition, Chirume (2012) affirms this by arguing that mathematical symbols are used in many contexts in which their function is to decode or shorten sentences.

Calculus is inundated with symbols and notation and thus, it was not surprising to see that both lecturers in this study used symbols in their teaching. Both lecturers explained the meaning of symbols and notation to their students. The following is an example from Lecturer A explaining the meaning of the symbols and notation:

> **Lecturer A:** *Ok, now how do you read this? We're going to use this kind of a symbol, right? This is* $\int_1^7 \frac{1}{2}x^2 + 2\,dx$ *, first of all, it's called a definite integral. And if you read this as the integral of the function f with respect to x, wrt stands for with respect to -- the variable x from x=1 to x=7. Remember in mathematics we have symbolic notation, so there is (*$\int_1^7 fx$*) is the integral -- or you can say definite*

*integral if you want to the integral of which function? Of the function f. And this (dx) stands for? With respect to x.*

Lecturer A confirmed the point raised by Premprayoonk et al. (2014) that mathematical concepts are sometimes difficult to work with, when using ordinary language and therefore, there is a need for symbols and notation to make it easy when working with mathematical concepts. Thus, students are required to be confident in manipulating symbols because doing mathematics involves working with symbols. Lecturer B also explained the meaning of the integral symbol and notation. The following is an example from Lecturer B:

**Lecturer B:** *The indefinite integral of, f(x), symbolized by* $\int f(x)dx$ *is defined to be* $\int f(x)dx = F(x) + c$ *where c is an arbitrary constant. This is the symbol* $\int$ *it looks like an S an elongated S, it is called an integral sign. f(x) is the function that you are integrating, and is called the integrand.*

Both lecturers in this study confirm the point that was raised by Quinnell and Carter (2013), that it is of fundamental importance that lecturers guide their students on how to use and decode symbolic notation. This is because, in order for students to perform well in mathematics, they are required to handle mathematical symbols with ease. As the lecturers explained the meaning of the symbols, they exhibited handling mathematical symbols, which is one of the components of the Mathematical Activities framework of this study. Since calculus has many symbols and notation, the calculus students are required to handle these symbols with ease.

### 6.2.5 Activities that promote the use of multiple representations

In this study, the use of multi-representations means presenting mathematical concepts in more than one form. For example, a function may be represented both in symbolical and graphical forms. Cope (2015) pointed out that representational modes include manipulatives, real-world situations, spoken symbols, written symbols, tables as well as pictures or graphs. It is also argued that the use of multiple representations enhances the students' understanding of mathematical concepts.

Both lecturers in this study used a variety of representations except for manipulatives. Calculus concepts call for multiple representations in the form of graphs or symbols, whether written or spoken, as well as real- life situations. The following is an example from Lecturer A when he was showing the students why they were getting negative values. Lecturer A, switched from symbolic representation to graphical representation, so that the students could see why they were getting negative values as they were finding the anti-derivative of one particular function:

> **Student:** *We seem to be getting the negatives.*
> **Lecturer A:** *You see that? Not always, but for this particular function, we seem to be getting the negatives. [Draws graph on board]. And if we're looking at graph, there's 2 and 4. If you want to find this area here, where would you integrate, from where to where? If you go from the smallest to the largest, if you go from 2 to 4. And if – but if you looked at the Fundamental Theorem somewhere along the line ya, proving the Fundamental Theorem. You reverse the order of the limits of the same function … What was the negative of the other … So, this can represent the area of x provided … the anti-derivative of an integral is not necessarily the area; it is used to find the area – calculate the area.*

The following is an example from Lecturer B, who used spoken, graphical and symbolical representations:

> **Lecturer B:** *Why are we not writing ln* $(1 + x^2)$ *in absolute value form? It is because* $1 + x^2$ *will always be positive, no matter what value of x. It will always have a minimum value of one, because it turns at* $y = 1$ *[Lecturer B draws the graph on the board].*

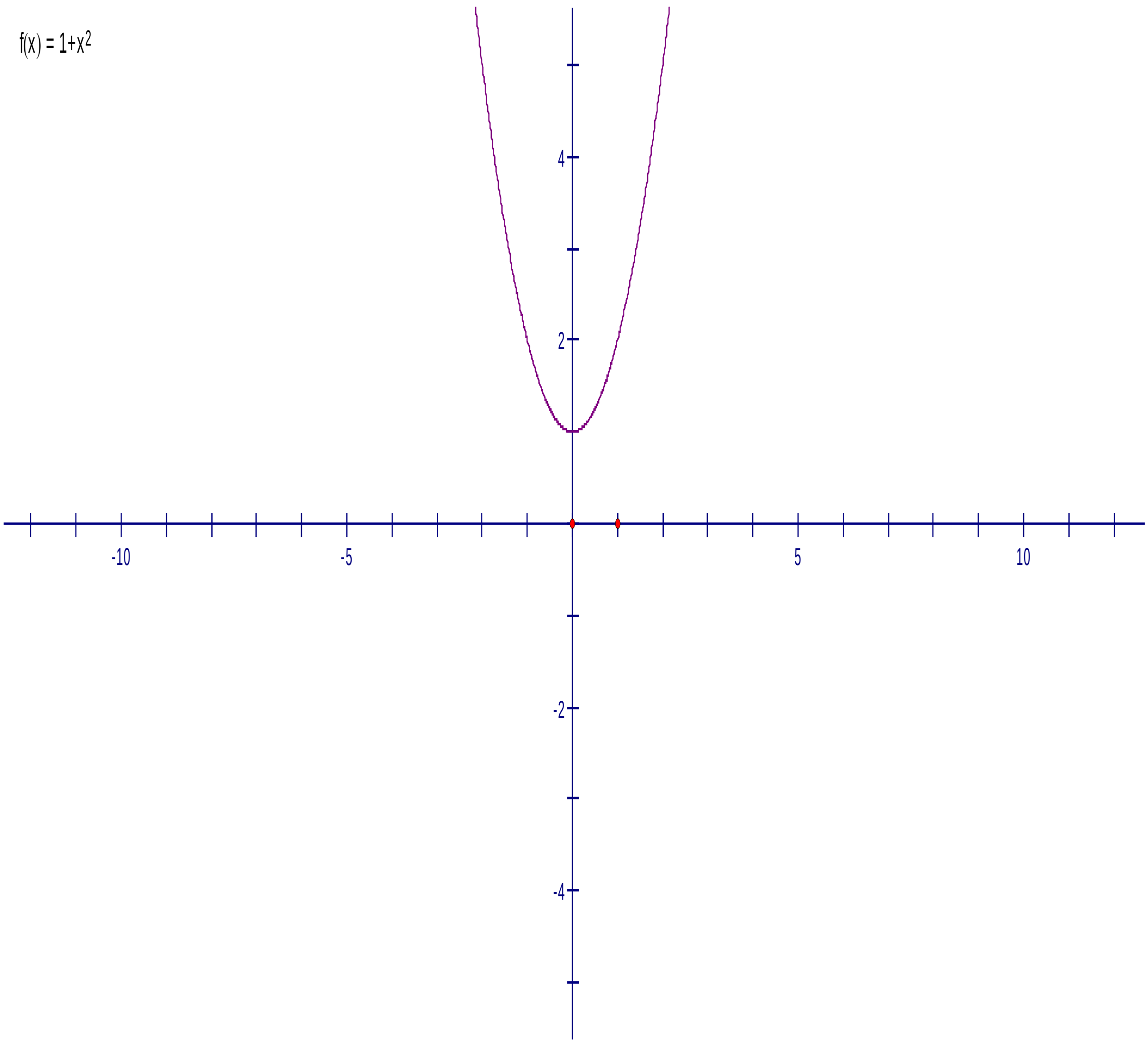


**Figure 6.1: The graphical representation of 1+ $x^2$**

By representing $1 + x^2$, by means of a graph, students may have found it easy to understand why it was not necessary to write ln $(1 + x^2)$ in absolute form. This is in agreement with the research findings by Gulkilik and Arikan (2012), which show that switching between representations enables students to understand mathematical concepts with much ease, while making connection between the mathematical concepts. These researchers point out that the students' concept images are strengthened by the variety of representations.

Representing mathematical entities is one of the components of the Mathematical Activities framework of this study and is exhibited by the use of multiple representations by both lecturers in this study. Representing mathematical entities involves switching between representation, which is supported by Cope (2015), who points out that teaching for understanding means that teachers use multiple representations. Bautista, Cañadas Brizuela and Schliemann (2015) also support the use of multiple representations through their study, in which the participants were shown how the use of multiple representations helped to develop the students’ understanding of mathematical concepts.

Conjecturing, investigations, proving, using symbols and using multiple representations, all have something to do with thinking mathematically, representing mathematical entities, reasoning mathematically, as well as handling mathematical symbols. These were present in most of the 18 lectures of this study. This supports the call by Bailey, Leinwand, Smith, Stein, Surr and Walter (2014), for lecturers and teachers to meaningfully engage students in activities so that they think and reason mathematically. Figure 6.2 shows the occurrence of these components in one of the 18 lectures of this study. Additionally, Figure 6.2 shows that the activities in lecture B 3 required the students to think mathematically because in this lecture, they were finding the indefinite integrals of powers of functions, as well as indefinite integrals that led to logarithmic functions.

All this required the students to think mathematically, which meant that they were required to understand the scope of working with exponential functions, as well as logarithmic functions. This is supported by Hudson, Henderson and Hudson (2015), who argue that students are given the opportunity to think mathematically when they are constantly exposed to activities such as conjecturing, proving and investigations, which involve questioning. The lecturers in this study were also switching between equations of functions and the graphs of functions, hence, representing mathematically was also significant in lecture B 3. There was also a significant amount of symbolic notation that was explained and then used in lecture B 3, thus, handling mathematical symbols featured in lecture B 3. Reasoning mathematically occurred the least because only on a few occasions did the lecturer ask a series of questions that required the students to come up with a conjecture.

Boaler (2016) points out that activities that promote conjecturing, proving, investigations, the use of symbols and the use of multiple representations, are crucial to students as they develop to become successful problem solvers.

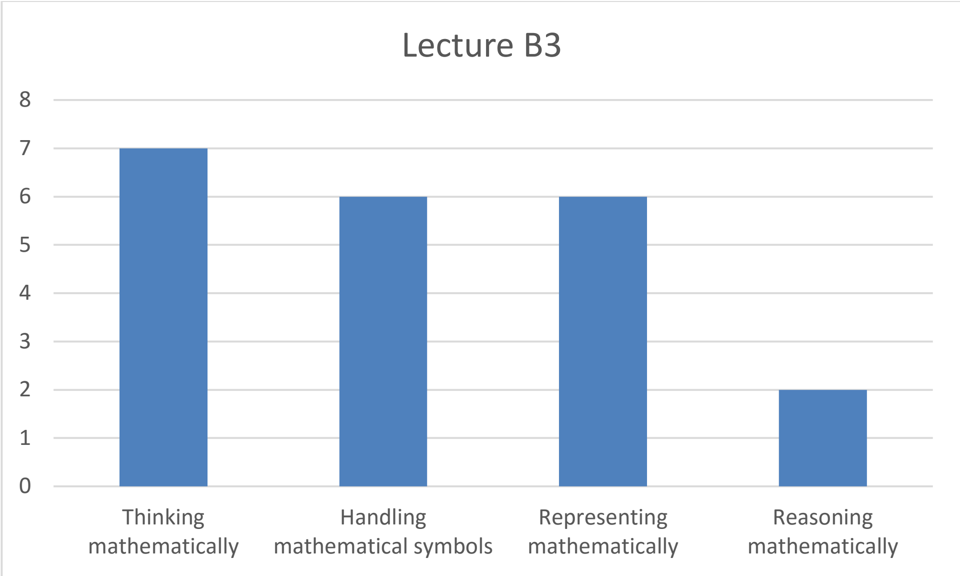


**Figure 6.2: Components of the mathematical activities**

The components of the mathematical activities as shown in Figure 6.2 are supported by Copely (2013) who points out that thinking, reasoning mathematically and representing mathematical concepts is central in mathematics. Thus there is  need for lecturers to focus on improving their students' abilities to represent, reason and think mathematically. This is because such abilities promote higher order thinking, which is increasingly becoming one of the most important skills that students must possess so that they can easily cope with situations that they encounter (Cansory & Türkoğlu, 2017).

### 6.2.6 Activities that promote procedural fluency through conceptual understanding

Procedural fluency is the skill of carrying out procedures flexibly, accurately, effectively and appropriately, while conceptual understanding is the comprehension of mathematical concepts, operations and relationships (Kilpatrick et al., 2001). Many researchers including, Groves (2012) and Bautista (2013) are in agreement with this definition. Students with only procedural fluency find it difficult to cope in answering questions that are not familiar to them. This is evident in the study conducted by Groves (2012), where the results show that there is a very close link between procedural fluency and conceptual understanding.

Both lecturers from this study promoted procedural fluency and conceptual understanding in their lectures. However, there was evidence that both lecturers promoted more procedural fluency than conceptual understanding. This supports the findings by Ally and Christiansen

(2013), which show that in all the lessons that were video recorded, procedural knowledge was common, while conceptual knowledge appeared in half of the lessons. Perhaps this may have been because procedural fluency needed to be developed more frequently (Askew & Venkat, 2012). The following is an example of Lecturer B promoting conceptual understanding of the differential equation:

> **Lecturer B:** *A differential equation is an equation that contains an unknown function, and one or more of its derivatives. Equations such as:*$\frac{dy}{dx} = f(x)$*or*$\frac{dy}{dx} = x^3 - 2x + 1$. *In general, it is written as* $\frac{dy}{dx} = f(x).h(y)$ *for example* $\frac{dy}{dx} = y^{-3}.e^x$*so there are two functions here* $y^{-3}$ *and* $e^x$*…………*

In the preceding example, the lecturer explained the concept of the differential equations before the procedure of solving the differential equations. This is in line with the suggestion by the Kansas College and Career Ready Standards (2013) which highlights that procedural fluency should always come after conceptual understanding. In this lecture, Lecturer B went on and promoted procedural fluency, firstly by demonstrating an example and then by providing students with the opportunity to work on their own. The following example illustrates procedural fluency:

> **Lecturer B:** *What is the order of this equation?*
> **Students:** *1*
> **Lecturer B:** *Why?*
> **Students:** *Because the highest derivative is the first derivative.*
> **Lecturer B:** *We want to try and see if this can be written as f(x).g(y). How can we do that?*
> **Students:** $\left(\frac{x}{1+x^2}\right).y$
> **Lecturer B:** *This is indeed f(x).g(y), we can divide by y and actually multiply by dx and we have*$\frac{dy}{y} = \left(\frac{x}{1+x^2}\right).dx$. *We can now integrate this equation* $\int \frac{dy}{y} = \int \frac{x}{1+x^2}\, dx$*……………*

The preceding example shows that the lecturer promoted procedural fluency, while supporting this with conceptual understanding.

The following is an example of Lecturer A promoting conceptual understanding by explaining the meaning of an inverse of a function.

> **Lecturer A:** *The inverse of a function – if you've got some function f, which can be written in this way: Let's just say y is equal to 3x-1. What will the graph – how would the curve look – what kind of a graph would you get for this?*
> **Students:** *Straight line.*
> **Lecturer A:** *Now the inverse you will represent by f to the minus 1. The rule for the inverse function, we interchange x and y. So, y becomes x and x becomes y. Which really means is that the domain will determine the range of the inverse and the range will determine the domain. So, to obtain this we will say: x is equal to 3y minus 1 – to obtain the inverse – y became x, x became y. To obtain the graph of the inverse, we reflect the function in the line y = x, that is why the x and y values interchange.*

In the preceding example, Lecturer A explained the concept of an inverse of a function, as well as the process of obtaining both the symbolic and graphical representation of an inverse of a function. Thus, Lecturer B promoted procedural fluency, while supporting it with conceptual understanding.

Procedural fluency and conceptual understanding are in line with the Cognitive Processes framework used in this study. They both fall under the knowledge domain of the taxonomy table namely procedural knowledge and conceptual knowledge. In this example, conceptual knowledge was exhibited when the lecturers explained the concept of differential equations. This was reinforced by the questions Lecturer A asked the students on why they knew how it was the equation of order 1. Thus, questions like these required students to apply, as well as analyse their conceptual knowledge. Analyse and apply the components of the Cognitive Processes framework, fall within the cognitive domain of the taxonomy table. Procedural knowledge is exhibited when the lecturer leads the discussion on the process of solving the equation. This supports Van Der Hayden and Alssop's (2014) study, which showed that the Chinese students, in their study, had deep understanding of procedures and concepts because their teachers not only explained the algorithms, but also went on to explain why the algorithms work. This is also affirmed by Smith (2014), who points out that for students to make connections between processes and concepts, teachers need to give them the opportunities to develop their procedural knowledge together with conceptual knowledge.

As stated earlier in this section, procedural and conceptual knowledge were present in all the 18 lectures of this study. Figure 6.3 is an example of an analysis of lecture A 2. This is one of the lectures of this study that exhibited occurrences of procedural and conceptual knowledge. Figure 6.3 shows that there were more instances of procedural knowledge than conceptual understanding in lecture A 2. This was mainly because the lecturer and the students were discussing and explaining the process of finding the sum of the areas of the rectangles under a curve. Students were given opportunities to explain their method and this contributed to more instances of procedural knowledge. These discussions appeared to have been intended to guide the students to understand the concept of the integral. Conceptual and procedural knowledge complement each other (Rittle-Johnson, Schneider, & Star, 2015). These researchers also claim that for students to be mathematically competent, their procedural and conceptual knowledge need to be developed.

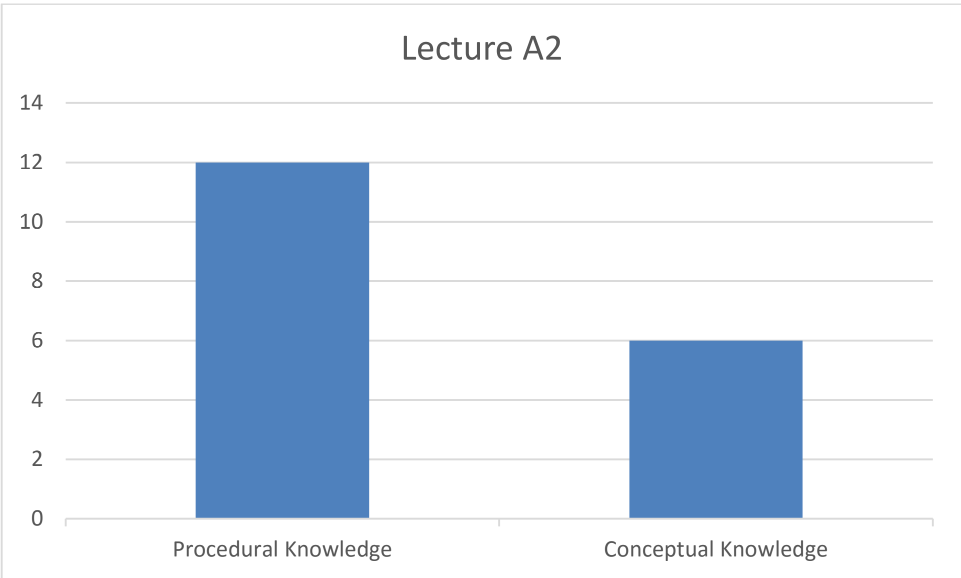


**Figure 6.3: Occurrence of procedural and conceptual knowledge**

### 6.2.7 Activities that promote the use of multiple techniques in problem solving

Integral calculus is grounded in an environment where there are various techniques or ways to solve mathematical problems. Both lecturers in this study exposed their students to the various techniques used in integral calculus, such as substitution techniques, integration by parts and many others. These methods are coherent with the outline of the integral calculus presented by various calculus textbooks, for example, Stewart (1997).

Other than providing their students with opportunities to use a variety of techniques, both lecturers in this study allowed their students to explore their own ways of solving problems. The following is an example where Lecturer A allowed his students to explore their own method:

> **Lecturer A:** *Ok, let's see what he says. It's definitely correct what he has done. So, he says this here – what is this whole thing here equal to? 1 – (x – 2)*$^2$*. Ok? Yes. And he says let's make this the substitution for x minus 2. Let's see what he says then. Let u = x – 2 what is d u /dx? It's one, isn't it?*

Lecturer A provided his students with the opportunity to explore their own ways of solving problems. Similarly, Lecturer B also allowed his students to explore their own ways of solving problems. The following transcript is an example showing two students using two different methods, both different from their lecturer.

> **Lecturer B:** *Now Mandla, did something else. He said, since he needed to integrate* $\int (3x^2 - 7)^5 . 2xdx$ *, then he needed the derivative of the function which is 6x and he created 6 by multiplying 2xdx by* $\frac{3}{3}$*. But Nompilo did it this way,* $2\int (3x^2 - 7)^5 . xdx$ *and then let u =* $(3x^2 - 7)^5$ *and her answer is also correct.*

There were many instances where both lecturers affirmed the methods used by their students. Arikan (2016) suggests that providing students with tasks that have multiple solutions, contributes to the profound development of the students' mathematical understanding. The educators in the study conducted by Arikan (2016) claimed that exposing students to multiple techniques gave them the opportunities to firstly approach a task with divergent pespectives, secondly, strenghten their basic skills, thirdly, develop their creativity in solving problems and fourthly, allow them to see how concepts are connected.

Thus far, the mathematical activities that were legitimised in the lectures during this study have been discussed. These activities include conjecturing, investigations, proving, using multiple representations, using symbols, promoting procedural fluency through conceptual understanding and using multiple techniques in problem solving. These activities were spread across all the 18 lectures that formed part of this study.

### 6.3 Justification of mathematical activities

The lecturers in this study justified the mathematical activities with legitimating appeals (see Chapter Three for a discussion on legitimating appeals) made to mathematics, students' experience, lecturers' own experience, lecturers' own authority, everyday metaphor and teacher education. The seminal work of Davis et al. (2005) shows that legitimising appeals were mainly spread over appeals made to mathematics, mathematics education, everyday metaphors, students' experience, the lecturer and curriculum. In addition, Parker and Adler's (2012) study found that the lecturer made appeals to mathematics, teacher education, curriculum and to his own authority.

#### 6.3.1 Justification with an appeal made to mathematics

Justification with an appeal made to mathematics, refers to instances where the lecturers justify why they are doing what they are doing, with reference to mathematics. The following example illustrates the lecturer justifying the use of symbols with an appeal made to mathematics.

> **Lecturer A:** *We're going to use this kind of a symbol, right? This is first of all, it's called a definite integral. You read this as the integral of the function f with respect to...wrt stands for with respect to the variable x from x=1 to x=7. Right, that's how we read that. Remember in mathematics we have symbolic notation, we have symbols -- mathematical symbols ….*

Justification of the use of symbols was made to mathematics, with the lecturer pointing out that in mathematics, symbols are used. Thus, this implies that the lecturer made a point that appeared to justify why they were using symbols. This is supported by Parker and Adler (2012) who noted that the justification for solving a quadratic equation was made to mathematics.

### 6.3.2 Justification made to students' experience

An appeal made to students' experience refers to instances where the lecturers justify what they are doing, with reference to their experience. In the following example, the lecturer justified the process of anti-differentiation by referring to what the students learnt in the previous module, which covered the differential;

> **Lecturer A:** *What he says is - because last year you subtracted 1, to find that number he's adding 1 to this number. To get back there, it means first of all you've got to add 1 to n-1. If you add 1 there, what do you get? You get n. And if you divide by n, see n-1+1 is what? n-1+1 is n. And if you divide by n wouldn't this n cancel? You'll divide once and you'll get this 1 there. We're reversing the process of the power rule as well………*

The preceding example illustrates justification with an appeal made to students' experience. The students were participating in an activity that promoted conjecturing. The reversing process was justified with an appeal made to their experience, with the concept of differentiation. Davis et al. (2005) showed that justification by the lecturer in their study was made to students' experience.

### 6.3.3 Justification made to the lecturer's own experience

The following is an appeal made to the lecturer's own experience, although he was not saying, "in my experience…" but the fact that he said, "to me ….", makes this an appeal made to his experience.

> **Lecturer A:** *To me, it didn't look like it's a 2 ½, but anyway. I suppose, because you were estimating, so you decided you would even estimate the length. But remember, if you're estimating, we want as best an estimate as possible.*

Thus, the lecturer used his own experience to justify why his estimate was the closest, although he did not explicitly mention that it was because of his experience. This is supported by Adler (2012), who also noted a teacher justifying his lessson preparations to his own experience and that his experience allowed him to watch out for learner misconceptions.

### 6.3.4 Justification with appeal made to the everyday metaphor

An appeal made to the everyday metaphor is when the lecturers justify what they are doing with reference to everyday life. The following is an example of an appeal made to the everyday metaphor.

> **Lecturer B:** *From last year up until Tuesday, we were dealing with differential calculus. But now we are going to do the reverse of that……It's like learning to drive, you learn how to move forward and then learn how to reverse. Now I know reversing is not easy because you don't have eyes at the back but at least we have mirrors which we are going to use. By mirrors I mean all that we know about calculus, everything that we know about the derivatives, those are our mirrors in order for us to go back.*

In this example, the lecturer justified why integral calculus was the reverse of differential calculus by referring to the metaphor of learning to drive a car. This is supported by the findings by Adller and Davis (2006), who noted that the lecturer in their study justified the distributive law with an appeal made to everyday metaphor.

### 6.3.5 Justification made to teacher education

The following is an example of an appeal made to teacher education because the lecturer is making a justification of what the students ought to know. In this case, the lecturer talks about trigonometric identities. The lecturer refers to this section of mathematics because the students will be teaching this section when they go to schools on teaching practice, or when they start their careers as practicing teachers.

> **Lecturer A:** *Right, now these formulae are all schoolwork, you would have to know this, because you're going to teach these things in 2 years' time. In Grade 12, the learners they're being taught this section on compound angles and double angles in trigonometry………*

The preceding example illustrates that the lecturer's justification of students having to know the Grade 12 trigonometric identities was because they would be teaching them to the high school students. Along similar lines, the lecturer in Parker and Adler’s (2012) study justified what he was teaching with appeals made to teacher education.

### 6.3.6 Justification made to the lecturer’s own authority

When the lecturers do not justify why the answer is wrong or why they are doing what they are doing, then the particular instance would be referred to as an appeal made to lecturers’ own authority. In the following example, the lecturer did not give a reason why the answer given by the student was wrong, that is why this is an appeal made to his own authority. When the lecturer said “*No. That's wrong. Your lower sum is wrong.*” He did not say why the answer was wrong;

> **Lecturer A:** *Anybody else? Worked with 4 rectangles?*
> **Student:** *But I got the right area sum at 88.125.*
> **Lecturer A:** *90?*
> **Student:** *88.125.*
> **Lecturer A:** *No. That's wrong. Your lower sum is wrong.*

The preceding example illustrates the justification made to the lecturer’s own authority, since he did not say why the student’s answer was wrong. Parker and Adler’s (2012) study illustrated that the lecturer in their study showed evidence of justification made to his own authority.

## 6.4 Theme 2: The nature of calculus dialogue in the lecture room

Every lecture room is characterised by communication between the lecturer and the students, as well as communication amongst the students. Communication is fundamental to the teaching and learning process (Msimanga, 2016). For maximum and productive communication in the lecture room, lecturers are required to create opportunities for students to be part of the lecture room dialogue, as suggested by Walsh (2012). Dialogue may be in the form of explanations, either by the lecturer or by the students. It can also be in the form of questions and answers. The sections that follow explore the dialogue that was exhibited in the calculus lecture rooms observed in this study.

### 6.4.1 Dialogue through the explanation of concepts, procedures, facts and symbols

Explanations of concepts, procedures, facts and symbols were evident in all the 18 lectures that were observed for this study. Both lecturers explained concepts, facts and the meaning of symbols, while both the lecturers and the students explained procedures. Students also explained the method that they used to answer a particular question. The following example shows a student from Lecturer A explaining the procedure of finding the anti-derivative of a trigonometric function:

> **Student:** *So, you get tan x equal u, and sec squared x dx equal dv. Derivative of tan x is 6x squared so du will equal this. And the anti-derivative of dv will equal tan x. Now we have this form and then we just apply the formula. U is equal to tan x. B is tan x as well. And minus the anti-derivative of b is tan x, sec x squared dx because du is sec squared dx….*

Allowing students to verbalise what they are thinking enhances the development of their mathematical concepts. Additionally, by allowing students to explain and support why their method or answer is correct, assists in developing the students' mathematical understanding (Bansilal, 2012). This also reaffirms that lecturers need to help students develop mathematical language through symbols and notations, as this helps them with making links between concepts. This idea is supported by Quinnell and Carter (2013) who point out that developing students' symbolic language is fundamental to the students' mathematical development.

As the lecturers explained concepts and procedures, understanding of procedural knowledge and understanding conceptual knowledge was enhanced. This is supported by Bautista (2013) who points out that students' mathematical knowledge is linked to their understanding of concepts and procedures. Understanding procedural and conceptual knowledge is a component of the Cognitive Processes framework of this study. Handling mathematical symbols, a component of the Mathematical Activities framework of this study, was exhibited during the explanation of symbols and the following example illustrates this:

> **Lecturer A:** *Suppose f(x) is the function on an interval [a ; b] , that means it's a closed interval, square brackets means it's closed from a to b, inclusive of a and b. This means you are finding the definite integral from a to b.*

Lecturer B also used a similar example to explain the idea of upper and lower limits.

> **Lecturer B:** *I drew this graph earlier to save times. [projects the graph on the screen and shades the area form a = -1 to b = 1]. We say a = -1 is your lower limit and b = 1 is your upper limit. This means you are starting from x = -1 to x = 1.*

### 6.4.2 Dialogue through questioning and answering

Msimanga (2016) pointed out that dialogue through questioning and answering is fundamental to the learning of mathematics. This is because the questions invoke the students' thinking and thus, helps them develop mathematically, through sound ideas and concepts. In all of the 18 lectures observed in this study, there was evidence of dialogue through questioning and answering. Both the participating lecturers and students did this. Lecturers asked questions to verify students' prior knowledge, to verify and interrogate students' conceptual understanding and procedural knowledge. Additionally, lecturers asked questions to guide the students through a particular mathematical problem- solving process. Students asked questions to affirm their understanding of either a concept or a procedure. Similarly, research results by Mhakure and Jacobs (2016), show that asking questions help students to enhance the understanding of the mathematical problem under discussion. These researchers identified a variety of types of questions, of which three types have also been identified in this study: firstly, questions to check prior knowledge, secondly, probing and follow up questions and thirdly, leading questions.

#### 6.4.2.1 Questions to check prior knowledge

Questions to check prior knowledge were asked by both lecturers who participated in this study. This type of questioning is in line with the discussion by Chikiwa (2017), in that it is common practice by lecturers to ask questions that are intended to check prior knowledge before they teach new concepts. The following is an example from Lecturer B's lectures, verifying the students' prior knowledge:

> **Lecturer B:** *What is the abscissa?*
> **Students:** *x coordinate*
> **Lecturer B:** *Yes, only grade 9 learners will tell you that. What is a slope?*
> **Students:** *Gradient.*

The following is an example of Lecturer A checking his students' prior knowledge:

> **Lecturer A:** *Right, there is a graph -it's part of the whole graph, isn't it? But here the domain is restricted to 0 to 7, so that is the portion you see. What type of graph is that?*
> **Students:** *It's a parabola*
> **Lecturer A:** *It's parabolic. Why do you say its parabolic?*
> **Students:** *Because it represents a quadratic equation.*

Although these questions might seem to be rudimentary for third year students, the lecturers appeared to be doing a quick verification, to ensure that the students had the elementary ideas needed for building on to the new concept. Asking such questions is fundamental because these questions require students to recall what they previously learnt, with the aim of using this knowledge as a base for building new knowledge (Elsner, Haines, & Tofade, 2013).

The questions on verifying students' prior knowledge fit in with the Inquiry Communication Model, one of the frameworks used in this study. These questions are examples of locating and identifying. In the example, the lecturer was locating his students' knowledge of the term abscissa, as well as the meaning of gradient. These questions also fall under the Cognitive Processes framework, because the lecturer was asking questions that required the students to remember factual, procedural or conceptual knowledge.

#### 6.4.2.2 Probing and follow up questions

In this study, probing and follow up questions are questions that are either asked by the lecturer to answer a student's question, or when the lecturer begins by asking a question which is then followed by a series of questions until the original question is answered. If a student asks a question, the lecturer does not directly answer the student's question, but asks a series of questions instead, until the student's question is answered by other students or the very student who asked the question. The following is an example of probing or follow up questions, where the lecturer begins by asking a question:

**Lecturer A:** *Anybody else that has a different understanding of what these symbols mean?*

**Student:** *Differentiate with respect to x.*

**Lecturer A:** *Differentiate? With respect to x.?*

**Student:** *Yes*

**Lecturer A:** *With respect to x. How do you know with respect to x? Anybody else? What do others think?*

Lecturer B also asked probing and follow up questions. The following is an example of such questions:

**Lecturer B:** $\int \frac{(4-x)}{(2-\sqrt{x})} dx$, *look there is something you need to do first, what is it?*

**Student:** *Should we rationalise the denominator?*

**Lecturer B:** *Let us see what he is saying. He is saying we must rationalise the denominator. Why do you want to rationalise the denominator?*

**Student:** *I am trying to eliminate the square root so that I can have something that can be a factor of the numerator.*

**Lecturer B:** *Let us see if that will work. So what do you want us to do?*

The appearance of probing or follow up questions supports the results by Hähkiöniemi (2013), in which students were asked probing questions which were intended for them to scrutinise their line of thinking and procedural or conceptual knowledge. These examples also support the findings by Mhakure and Jacobs (2016) in which educators asked similar questions so that they could find out whether the students had understood what they had just taught them.

These types of questions fall under the Inquiry Communication Model framework of this study. By asking such questions, the lecturers appeared to be locating and identifying the students' perceptions about a particular concept or method. Moreover, these types of questions also fall under the Cognitive processes framework of this study. Thus, by answering such questions, the students were required to apply their conceptual or procedural knowledge.

### 6.4.2.3 Leading questions

Leading questions are intended to guide or lead students to a desired procedure or conclusion. These types of questions were evident in most of the lectures observed for this study. The following is an example that demonstrates this idea:

> **Lecturer B:** *….which means* $\int \frac{2x}{x^2-4}\,dx = \int \frac{f\prime(x)}{f(x)}\,dx$, *now which function have we seen before whose derivative is* $\frac{f\prime(x)}{f(x)}$*?*
>
> **Students:** *Natural logarithms.*
>
> **Lecturer B:** *…… then which function did we differentiate?*
>
> **Students:** *It's the natural logarithm of, f(x).……*

Lecturer A also asked leading questions and the following transcript illustrates this:

> **Lecturer A:** *I see that some of you are struggling with finding* $\int \frac{sin^2 a}{1+cos\,a}\,da$ ?. *In this case you would use which identity?*
>
> **Student:** $1 - cos^2 a = sin^2 a$.
>
> **Lecturer A:** *How would this help?*
>
> **Student:** *We can then factorise and simplify.*

The findings of Mhakure and Jacobs (2016) show that the educators in the study asked similar questions in order to help their learners achieve a desired solution, especially if the students were having difficulties with the mathematical problem under discussion In their study, Mhakure and Jacobs (2016) identified such questions as prompting questions. In the given example, Lecturer A and Lecturer B asked leading questions, which guided their students to finding the integral of the given function.

These questions fall under the Cognitive processes framework, because Lecturer B was asking questions that required students to apply their conceptual knowledge of the natural logarithm when finding the integral of a fraction. Lecturer A was asking questions that required the students to use their conceptual knowledge of trigonometric identities. The questions also fall under the Inquiry Co-operation Model framework, because Lecturer B was advocating, since the questions appeared to have been intended to guide the students in the direction of natural logarithms. Lecturer A was guiding his students in the direction of using trigonometric identities. This helps the students think more about what they are learning and thus, enhances their thinking progress (Olmsted, 2012). Finally, these questions fall under the Mathematical Activities framework because the students were required to think mathematically in order for them to answer the questions.

#### 6.4.2.4 Interrogative questions

In this study, interrogative questions were those that required students to interrogate their conceptual understanding or procedural fluency. The following is an example from Lecturer B's lecture. These questions were aimed at allowing the students to interrogate the method employed.

> **Lecturer B:** *Mr Singh here is saying he can write cosecx as* $\frac{1}{sinx}$ *and cotx as* $\frac{cosx}{sinx}$. *Now* $\int cosecx.cotx\,dx = \int \frac{1}{sinx}.\frac{cosx}{sinx}dx$. *Why is he doing that? Why are you doing that sir?*
>
> **Student:** *Because it is easy to differentiate sinx and cosx.*

Lecturer A also asked interrogative questions and the following is an example to illustrate this:

> **Lecturer A:** *Your answer is 2. And I'm asking you why did you respond with 2 as your answer.*
>
> **Student:** *So I look at the width of each rectangle.*
>
> **Lecturer A:** *Each rectangle?*
>
> **Student:** *Yes.*
>
> **Lecturer A:** *How many rectangles have you got?*
>
> **Student:** *I've got 6 rectangles.*
>
> **Lecturer A:** *Why do you have 6 rectangles? What made you choose 6 rectangles? How are 6 rectangles going to help you?*

This supports the results by Olmsted (2012), which show that by asking challenging questions, students are compelled to think about meaningful answers, as well as develop conceptual understanding. Lecturer B asked a question to challenge the students to interrogate their conceptual understanding of trigonometric identities and Lecturer A asked a series of questions to challenge the students to interrogate his procedural knowledge. The notion of asking challenging questions is also supported by Jancarik et al. (2013), whose research findings show that these type of questions give room for students to take part in the inquiry process in the lecture room and also promotes deep thinking, which then leads to students' mathematical development. Thus, through this challenging and interrogative approach, the students were compelled to analyse their conceptual and their procedural knowledge.

This type of questioning falls under the Inquiry Co-operation Model as challenging. When the lecturers asked such questions, students were challenged to think deeper about why they were choosing the particular method. Such questions also fall under the Cognitive Processes framework, as analyse conceptual or analyse procedural knowledge. The students were required to think deeper and analyse their methods of choice. Hudson et al. (2015) support the use of interrogative questions because they challenge students to think mathematically.

#### 6.4.2.5 Confidence boosting questions (Affirmation questions)

The questioning in both lectures was not lecturer-centred. Students also asked questions. Students often ask questions when they are unsure of something and sometimes when the lecturer says something that triggers the students' prior knowledge or when the students want to extend their knowledge (Almeida, 2009). The questions, which students asked and appeared to have been asked with the intention to affirm or boost their confidence on a particular method, concept or solution to a problem, were also identified in this study. The following is an example of these questions:

> **Student:** *Can you get a negative answer? Why?*
> **Lecturer A:** *Did any of you get a negative answer?*
> **Students:** *Yes*
> **Lecturer A:** *The first problem, did you get a negative answer? And you thought you were wrong?*

The student in this example appeared to lack confidence in her answer; hence, she asked the question to confirm her answer. The student appeared to have obtained confirmation of her answer directly from fellow students and indirectly from the lecturer. Supportive lecture room environments allow students to be confident in their own understanding of mathematical concepts and procedures (Bailey et al., 2014).

Getting in contact, locating, identifying, advocating, reformulating and challenging are the components of the Inquiry Co-operation Model that were exhibited through questioning. Getting in contact and reformulating are the two components that did not appear in most lectures, with getting in contact appearing the least. Locating and identifying occurred when the lecturers were checking the students' prior knowledge. Figure 6.4 shows that in Lecture B 4, the lecturer checked prior knowledge quite often; perhaps this was because they were dealing with differential equations for the first time. When the lecturers were advocating, they were asking leading questions and Figure 6.4 shows that there were a few instances where Lecturer B was advocating, in other words, guiding the students to an expected solution. When the lecturers were challenging the students, they were asking interrogative questions that required students to think deeper about their method or their line of thought. The occurrence of the different types of questioning across the 18 lectures of this study is supported by Bailey et al. (2014), who point out that questioning by lecturers enables students to reason, think and communicate mathematically. When the lecturers or students were reformulating, they would be repeating a question or repeating an explanation that had been given by the other, in order to make sure that they had understood what had been said. In lecture B 4, there were four instances as exhibited in Figure 6.4.

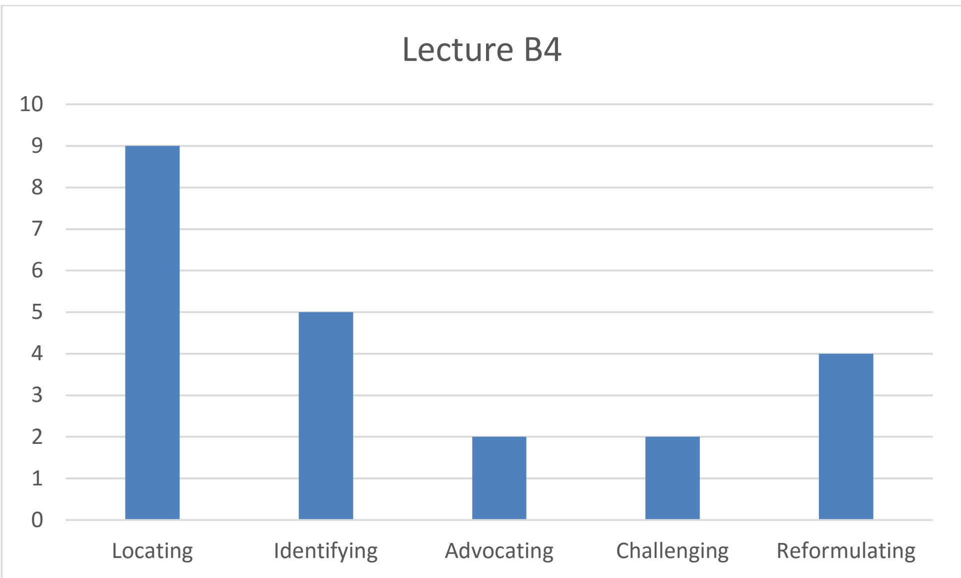


**Figure 6.4: The Components of the Inquiry Co-operation Model**

Apply or remember factual knowledge, remember procedural knowledge and remember conceptual knowledge, are components of the Cognitive Processes framework, which were evident when lecturers asked questions to check prior knowledge. Figure 6.5 shows that the lecturer checked prior knowledge on numerous occasions and this is in line with Figure 6.4 that also shows that the lecturer checked prior knowledge quite often in that lecture. When the lecturer asked leading, probing or follow up questions, students were required to apply their conceptual or procedural knowledge. Thus, Figure 6.5 illustrates more applying procedural knowledge, followed by conceptual knowledge, since they are exhibited by both leading and probing questions. When the lecturer asked interrogative questions, students were required to analyse their conceptual or procedural knowledge, thus, allowing them to think mathematically. This is supported by Hudson et al. (2015), who noted that thinking mathematically can be stimulated by questioning, especially when lecturers ask questions that challenge students to interrogate their conceptual or procedural knowledge. In lecture B 4, there appeared to be no evidence of questions that required students to analyse their procedural knowledge, as shown in Figure 6.5.

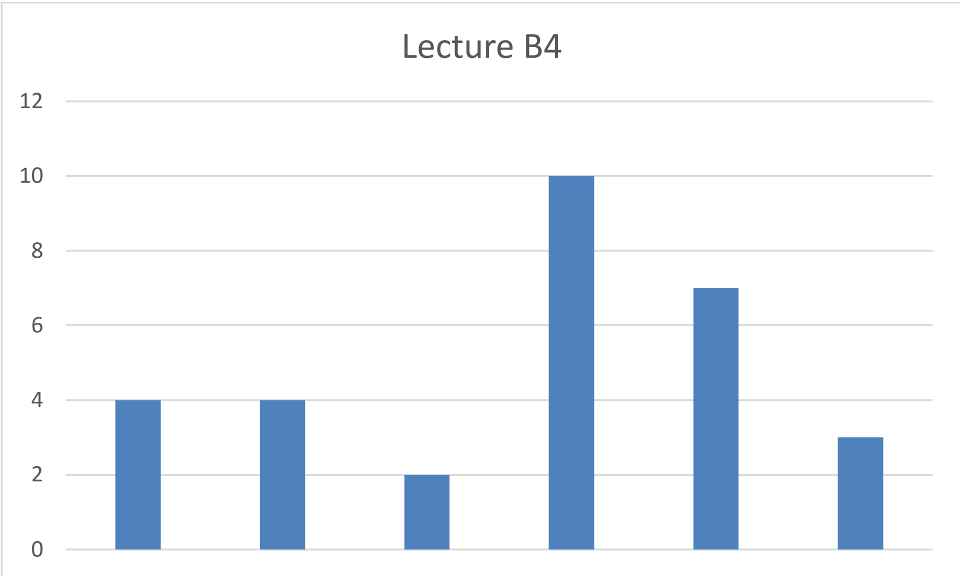


**Figure 6.5: The Components of the Cognitive Processes Framework**

Two types of dialogue took place in both lectures A and B. These were dialogue through explanation of concepts, procedures and symbols, as well as through questioning. The lecturers, as well as the students provided explanations and questioning. Lecturers asked questions to check prior knowledge. They also asked probing questions, leading questions, as well as questions that challenged the students so that they could interrogate the students' line of thought. Thus, two types of lecturing appear to have emerged and these are lecturing through explanations, as well as through questioning.

**Table 6.3: Lecturing through explanations**

| **Components of Mathematics for Teaching** | **Unpacking** | **Use of representations** | **Translation** |
|---|---|---|---|
| **Mathematical Activities Framework** | | Use of representations. | Handling mathematical symbols. |
| **Cognitive Processes Framework** | Understand:<br>Procedural Knowledge<br>Conceptual Knowledge. | | |
| **Inquiry Cooperation Model Framework** | Reformulation. | | |
| **Activities in the calculus lecture room** | | | |
| | Promoting procedural knowledge through understanding conceptual knowledge.<br>Promoting use of multiple techniques. | Use of multiple representations. | Use of symbols. |

### 6.4.3 The components of Mathematics for Teaching

The components of mathematics for teaching that were exhibited by lecturing through explanations were *unpacking*, the *use of representations, perception* and *translation*. These were evident in most of the lectures of this study.

*Unpacking* was exhibited when the lecturers were explaining the concepts or algorithms. For example, when in lecture A 1, Lecturer A was explaining the concept of the left area sum and how to calculate the area.

> **Lecturer A:** *You have to calculate the length. You substituted the x value from the left-hand side - from here - from this rectangle here. So, we're going to call them left rectangles, and the total area is called a left area sum. So, when we're dealing that way, we'll talk about left area sum and right area sum. So, it means, we don't even need a picture in front of us: when I say, 'left area sum' you've got an idea of what we're communicating about, that we're talking about rectangles which are now not necessarily are those rectangles below in this graph it turns out that the left rectangle is below the right. But the point at this stage is that if you talk about left area sum, we're talking about the sum of areas of rectangles whose left-end point meets the curve.*

Another example is when Lecturer B was unpacking the concept of the anti-derivative of a function:

> **Lecturer B:** *So, in each of these cases, the inverse operation takes us back to the original value or function. Now, what is anti-differentiation? Let me define what an anti-derivative is. In simple terms, if you have got G and f and these two functions are such that the derivative of G is in fact f, for all values of x in the domain of f, then we will say that G is the anti-derivative of f(x). If G and f are functions such that if I differentiate G, I get f, for all values of all values of x in the domain of f, then I am saying G is in fact the anti-derivative of f.*

This is supported by Adler and Davis (2006), whose study indicates that unpacking mathematical ideas assists students in developing a profound conceptual understanding of the concepts and it is in this way that students are able to make mathematical connections. In lecture A1, there were 10 instances coded as unpacking, out of 23 instances coded. Of these 10, 4 instances were principled unpacking, while 6 instances were procedural. Thus, Lecturer A, appeared to convey that knowing what to do and how or why a procedure is done, are the central components of mathematics and that it is part of teaching mathematics. In lecture B 1, there were 13 instances coded as unpacking out of 19 coded instances. Of these 6 instances were principled unpacking and 4 were procedural. This also shows that Lecturer B appeared to convey that it is important to know what to do and why a procedure must be done.

The use of *representations* was exhibited when Lecturer A used both the graph and its equation to determine the area of the rectangles under the curve, moving between the two representations as he was explaining concepts or procedures. The following example illustrates this:

> **Lecturer A**: *Right, there is a graph -it's part of the whole graph, isn't it? But here the domain is restricted to 0 to 7, so that is the portion you see. It's parabolic, yes. Right, it's a parabolic graph. Because it represents an equation there, a quadratic equation. To calculate the total area - the area sum - of the three rectangles, we first find the length by substituting into the equation of the graph y =* $\frac{1}{2}x^2 + 2$. *If you take this particular rectangle here., to find the length, you take this x value here and substitute it into this equation…*

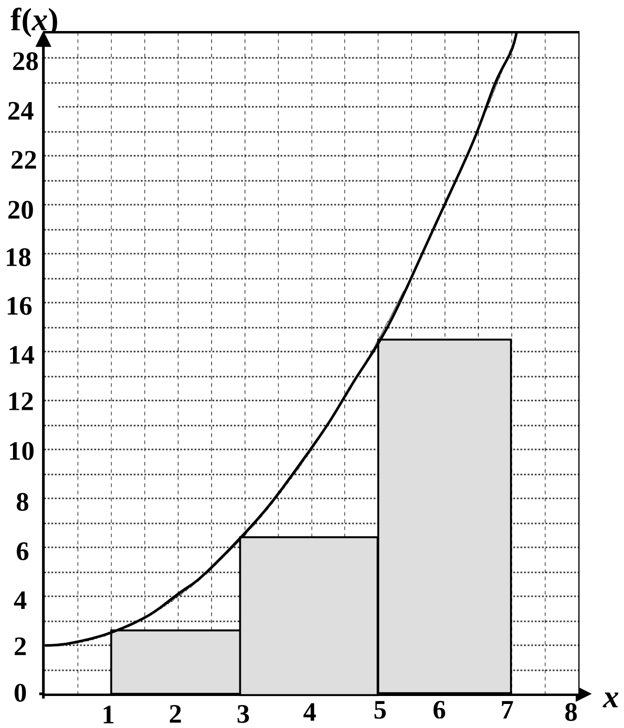


**Figure 6.6: The graph of** $y = \frac{1}{2}x^2 + 2$

Thus, Lecturer A appeared to convey that it is acceptable to use a variety of representations to solve a mathematical problem. This is also supported by Bardini, Bauer, Bichler, Combes and Weigan (2011), who showed that using multiple representations enriched conjectures and strengthened the students' understanding of the concepts as they captured the links between concepts.

Also, the use of representations was exhibited by Lecturer B when he was explaining to his students why they had to take the positive square root:

> **Lecturer B**: *So, then we can say, 1 + tan$^2$x = sec$^2$x and tan$^2$x = sec$^2$x – 1. So tanx* $= \pm\sqrt{sec^2x - 1}$ *, but we take the positive square root. There is a reason for that. If you think of a right-angled triangle, (draws the triangle on white board) we are*

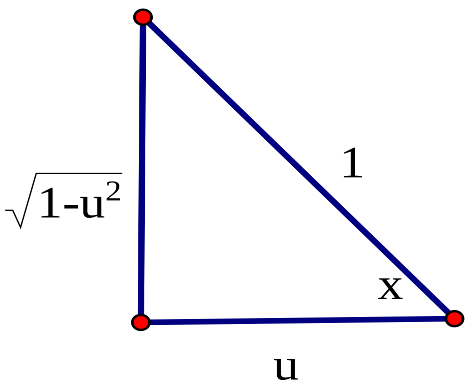


**Figure 6.7: The right-angled triangle**

> *Saying the secant of x is u. So how do you describe the secant of an angle?*
> **Students:** *Hypotenuse over adjacent.*
> **Lecturer B:** *So the sides are u for the hypotenuse and 1 for the adjacent side. How do you find the third side?*
> **Students:** *By using Pythagoras theorem.*
> **Lecturer B:** *Now you have everything you need in the diagram. Now what is tan x?*
> *Students: Opposite divided by adjacent.*

Copely (2013) supports the use of representations by pointing out that it is fundamental to the development of students' thinking and reasoning. Also, the use of representations is supported by Akkus and Cakiroglu (2010), who suggest that switching between different forms of representations of mathematical concepts has a positive impact on the students' understanding of mathematical concepts. Furthermore, Silver (2015) supports the use of representations by pointing out that students who are exposed to various forms of representing mathematical concepts have a deeper understanding of the mathematical concepts.

Reformulating indicates repeating what a student has just said which is done to make sure that the lecturer has the correct understanding of what the student intends to put across. When the lecturer then picks up the idea raised by the student and works with it, he exhibits *perception*. The following is an example of perception by Lecturer A:

> **Student:** *You let x minus 2 to be k of t.*
> **Lecturer A:** *And then?*
> **Student:** *Then of course you substitute….*
> **Lecturer A:** *Ok, let's see what he says. So he says this here, what is this whole thing here equal to?* $1 - (x-2)^2$. *Ok? Is that true? Yes. And he says let's make this the substitution for x minus 2. Let's see what we say then. Let u = x – 2 what is d u /dx? It's one, isn't it? So what would this become?*

In the preceding example, Lecturer A is repeating what the student has just said about letting *x – 2 = k*, thus, he is exhibiting *perception.* This is because he further takes up the student's idea and uses it to explain the process of finding the derivative of a function by using substitution. Thus, Lecturer A appears to convey that following up on a student's idea is acceptable and is part of teaching mathematics.

Lecturer B also exhibited perception a component of the Mathematics for Teaching. The following example illustrates this:

> **Student:** $\frac{1}{2}x^4$ + *c.*
> **Lecturer B:** $\frac{1}{2}x^4$ + c, *that's what was he is saying. Think about it. He is saying* $\frac{1}{2}x^4$ + c. *Where is he getting that from? He is so clever. He is saying the answer is* $\frac{1}{2}x^4$ + c. *Let us check. Let us differentiate* $\frac{1}{2}x^4$ + c. *We get* $2x^3$. *So he was right.*

In the preceding example, the lecturer repeats what the student has just said and then carries on differentiating the expression to check that the student gave the correct answer.

Perception plays an important role in developing the students' mathematical understanding. This is supported by Molefe and Brodie (2010), who suggest that the ability to notice the students' idea and work with it, has a positive impact on the students' mathematical development. In addition, this is supported by the research findings by Kazima et al. (2008), which revealed that the lecturer exhibited perception by working with the ideas put forward by his students, which resulted in strengthening the students' mathematical understanding.

*Translation* was exhibited when the lecturers were explaining the meaning of symbols and notations. The following is an example of Lecturer A explaining the meaning of symbols:

> **Lecturer A** *: Now the capital D of x, can you see here you you have ddx, you would have probably come across a symbol like this as well. ( 4 sec writes on the board ). This also stands for the derivative with respect to x. It's the same thing ; it has the same meaning as* $\frac{dy}{dx}$*. So don't get confused with that thing.*

Lecturer B also exhibited translation of a component of the Mathematics for Teaching framework when he was explaining the meaning of symbols. The following is an example to illustrate this:

> **Lecturer B:** *If you are looking at* $\int dx$ *this means you are looking at* $\int 1.dx$ *but one can be written in terms of x, because 1 is the same as* $x^0$ *so* $\int x^0 dx$.

Thus, the lecturers in this study appeared to convey that concepts, mathematical terms and symbols are important to mathematics, which is supported by justification with appeals made to mathematics. The seminal work of Kamina and Iyer (2009) supports this by pointing out that, mathematical symbols play a critical role in mathematics teaching and learning. Additionally, this supports the point made by Premprayoonk et al. (2014), that the ability to shift between symbols, as well as knowing the meaning of symbols, is fundamental in the teaching and learning of mathematics. This is because such abilities play a major role in developing, as well as strengthening the students' understanding of mathematical concepts.

*Simplification*, a component of Mathematics for Teaching, was exhibited when the lecturers worked with definitions that were appropriate to the calculus community, in this case the calculus module of the pre-service teachers (Adler, 2005). The following is an example of Lecturer A illustrating simplification:

> **Lecturer A:** *I will go through the statement with you. So if you've got a function g(x), which is the interval from a to x. Then g(x) is the anti-derivative of f(x). It's the anti-derivative of f(x). Now what does that mean? That means if I find the derivative of this of g(x), if I find ddx, I'll get f(x) as the answer. So that's what you need to write down somewhere in your notes. So in other words, if I find the integral of the f(t), first of all, and then if I differentiate, I'll come back to the original function f. Right if you find the integral and then differentiate.*

In the preceding example, the lecturer explains the meaning of the anti-derivative of a function and thus exhibits simplification.

Lecturer B also exhibited simplification, for example, when he was explaining that the rule that is applied in differentiating a product of the coefficient and a function is also applied in integration. The following example illustrates this:

> **Lecturer B:** *Then we can conclude that* $\int k.f(x)dx = k\int f(x)dx + c$. *This means that the integral of a coefficient times a function will be equal to the coefficient times the integral of the function. Just like the derivative of a coefficient times a function is equal to the coefficient times the derivative of the function.*

By exhibiting simplification, the lecturers support the suggestion made by Towers and Proulx (2013) that by doing so, the lecturers convey that the ability to know how mathematical concepts are connected is crucial in the teaching and learning of mathematics. Simplification is connected to understanding conceptual knowledge, and is exhibited when the lecturers work with definitions of mathematical concepts.

**Table 6.4: Lecturing through questioning**

| **Component of Mathematics for teaching** | | **Questioning** | |
|---|---|---|---|
| **Frameworks** | | | |
| Inquiry Cooperation Model | Identifying<br>Locating | Advocating | Challenging |
| Mathematical Activities | | Thinking, Reasoning mathematically | |
| Cognitive Processes | Remember:<br>Conceptual,<br>Procedural,<br>Factual knowledge | Apply:<br>Conceptual, Procedural knowledge | Analyse:<br>Conceptual,<br>Procedural knowledge |
| **Types of questions** | Checking prior knowledge | Leading<br>Probing | Interrogative |
| **Activities in the calculus lecture room**<br>Conjecturing<br>Proving<br>Investigations, | | | |

Table 6.4 shows the different types of questions that were asked by the lecturers. Questioning is a component of mathematics for teaching and it was exhibited through the different types of questions that were asked by the lecturers who participated in this study. Thus, the lecturers appeared to convey that asking leading, probing and interrogative questions, as well as asking questions to check the students' prior knowledge, is important in teaching mathematics. This is supported by Olmsted (2012) who points out that asking questions is fundamental to the development of students' thinking, as well as their mathematical development.

There were many instances where the lecturers in this study asked a variety of questions. The following is an example of Lecturer A asking questions:

**Lecturer A:** *Okay, there's a gentleman here says use more rectangles. How is that going to help?*

**Student***: Make them smaller.*

**Lecturer A:** *Yeah, make them smaller: how will that help?*

**Student:** *They would take more of the left space.*

Lecturer B also exhibited questioning, a component of the Mathematics for Teaching framework by asking a variety of questions. The following is an example to illustrate this:

**Lecturer B:** *So when we say let* $f(x) = x^2 - 4$*, what will* $f'(x)$ *be?*

**Students:** *2x.*

**Lecturer:** *So this means* $\int \frac{2x}{x^2 - 4} dx = \int \frac{f'(x)}{f(x)} dx$*. Now which function have we seen before whose derivative is* $\frac{f'(x)}{f(x)}$*?*

**Students:** *Natural logarithms.*

**Lecturer:** *Yes, it's the natural log of f. So we know that if* $\frac{f'(x)}{f(x)}$ *is our derivative, then what function did we differentiate?*

Thus, by exhibiting questioning, a component of Mtahematics for Teaching, the lecturers appear to convey that asking questions that make students think about what they are learning is fundamental in the teaching and learning of mathematics (Silver, 2015).

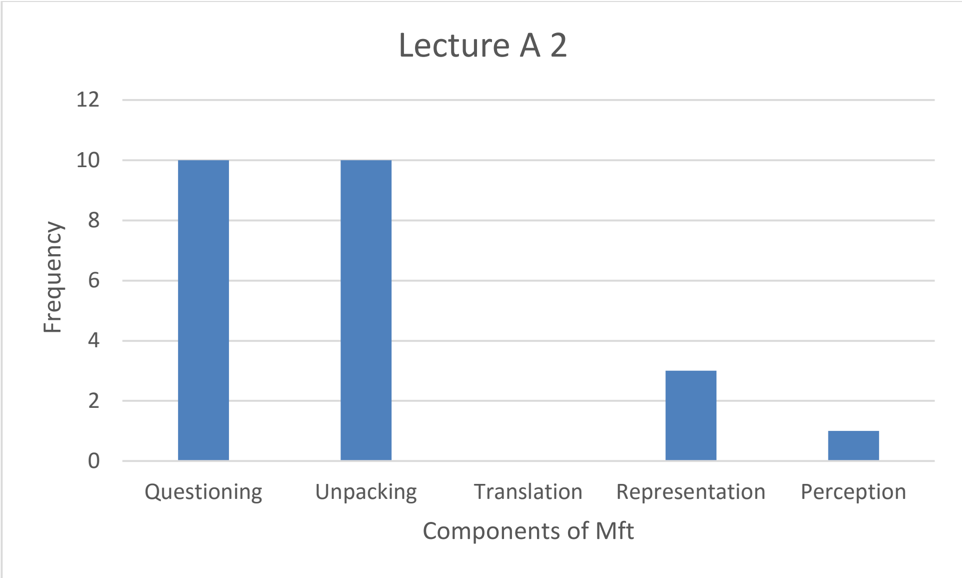


**Figure 6.8: The Components of Mathematics for Teaching as exhibited in lecture A 2**

Figure 6.8 shows the occurrences of the components of mathematics for teaching in one of the lectures of this study. Questioning and unpacking were exhibited in all the lectures that were observed for this study, while translation, use of representations, simplification and perception did not appear in all of the observed lectures.

### 6.5 Theme 3: The organisation of materials by the lecturers

Every institution of learning follows some form of syllabus or module guideline. The university in this study regards the purpose of a syllabus or module outline as three-fold. Firstly, the module outline is that of contractual purpose. Secondly, it serves the purpose of permanent record keeping and thirdly, it serves the purpose of a learning tool (Richmond, 2016). As a contract between the lecturer and the students, the syllabus or module guideline states things like the number of assignments, due dates and the rules that the students are expected to follow. As a permanent record, the syllabus or module guideline provides the outline of the module content, the assessment of the module, the materials and the requirements of the module. As a learning tool, the syllabus provides the students with information such as how to succeed and avoid unnecessary failure, as well as where to obtain counselling.

#### 6.5.1 Working within the existing curriculum outline

Lecturer A did not organise his materials because he was working within a project where the materials had been developed for the purposes of the project. The materials of Lecturer A were organised such that he introduced the module with an investigation of area under the curve. This was then followed by proving the Fundamental Theorem of Calculus, after which the students investigated the rules of anti-differentiation. The materials also included the finding of the anti-derivatives of trigonometric and logarithmic functions. The module was concluded with finding the anti-derivatives of partial fractions. Starting with the area under a curve is coherent with the curriculum and with most university textbooks. Thus, Lecturer A followed the outline of the existing curriculum.

On the other hand, Lecturer B organised his own materials. He introduced the module by first starting with the rules of anti-differentiation, as well as working with the properties of the indefinite integrals. This was then followed by proving the Fundamental Theorem of Calculus. The materials of Lecturer B included finding of the anti-derivatives of trigonometric and logarithmic functions, as well as partial fractions. The module was concluded by finding the area under a curve.

Thus, the lecturers in this study organised their materials with the needs of their students in mind. This is supported by Adler (2012) who points out that as teachers or lecturers organise their materials and do their work of teaching, they utilise a variety of resources and adapt them to suit their students' needs as well as to legitimise what is counted as mathematics.

> **Lecturer B:** *I don't follow a particular text book, I have summaries but within a particular module outline……but there is a plan in terms of what I am doing. The plan is a plan from the module itself. These are the topics that need to be done, so as a person teaching, then I say how am I, then going to put this within the given time frame…..*

Both lecturers in this study followed the module outline which was prescribed by the university.

### 6.5.2 Building on foundation knowledge

Lecturer B arranged his activities by starting with the rules of anti-differentiation because he was building on the students' foundation knowledge. Integral calculus is linked to differential calculus in that it is the reverse of differential calculus. Thus, Lecturer B's intention was for his students to link the two concepts with ease, since he had taught the same group of students in the differential calculus module. Hence, differential calculus acted as a foundation for building on integral calculus.

> **Lecturer B:** *The module itself is a continuation of differential calculus…. The way in which I introduced the integral was through the anti-derivatives, ….. My purpose of doing that was because they could see that they get a family of functions which is why I had to introduce the whole idea of what else can they do, what is this integration concept leading to?*

The idea of using differential calculus as a foundation on which to build the integral concept is supported by Awang and Zakaria (2012), who point out that in Malaysia, students find the integral concept difficult to grasp. Hence, in Malaysian schools, teachers introduce the integral concepts as a reverse of differentiation, because the rules of anti-differentiation are closely related to the rules of differentiation.

### 6.5.3 Using prior knowledge to support the students' understanding

The knowledge and skills that students already possess and take along to the lecture room and make available for the construction of new knowledge is referred to as prior knowledge (Braithwaite & Goldstone, 2015). This is because when students construct new knowledge, they use prior knowledge to make meaning of the new knowledge (Akinsola & Odeyemi, 2014). In this study, both lecturers tapped into their students' prior knowledge as they worked through the tasks. The following is an example from Lecturer B's lecture, where he used his students' prior knowledge of square identities in trigonometry to support their understanding of the integration of trigonometric functions:

> **Lecturer B:** *If u =secx, again we go back to high school, you remember square identities. What are they?*
> **Students:** $sin^2x + cos^2x = 1$
> **Lecturer B:** *…..There is another one* $1 + tan^2x$
> **Students:** $sec^2X$

Lecturer A also tapped into his students' prior knowledge, as they worked with integration by parts. The following example shows the knowledge of working with algebraic fractions being used, to support their understanding of partial fractions:

> **Lecturer A:** *So, you've got 2 over x minus 1 plus 3 over x. Where you've got fractions and then have to simplify – have to add to fractions. How do you go about doing this?*
> **Students:** *Find the lowest common denominator.*

Prior knowledge has a vital role in students' learning of new concepts (Akinsola & Odeyemi, 2014). This is because prior knowledge may be beneficial to the construction of new knowledge, but limited prior knowledge may also be an obstruction to learning new concepts. This is because students with enough accurate prior knowledge may use this existing knowledge to build new knowledge, while students with limited prior knowledge might feel overwhelmed by the construction of new knowledge (Cernusca, Collier, & Ionas, 2012). Once precise and adequate prior knowledge is stimulated, this prior knowledge may support learning. On the other hand, if prior knowledge is not stimulated, or is inadequate, then this impedes learning (Ambrose, 2012). This researcher found that the participants in the study benefitted by being given the opportunity to tap into their prior knowledge. Thus, prior knowledge has a significant influence on how students acquire new knowledge.

### 6.5.4 Using Scaffolding to support the students' understanding

Scaffolding involves providing students with guidance as they perform a task (Casem, 2013). The metaphor of a temporary support structure for a building under construction is interpreted by Bakker, Smit and Wegerif (2015) as the help that students receive, so that they are able to do tasks that they would not have been able to do on their own. Scaffolding is fundamental to tasks that promote conjecturing, investigations or proofs (Bakker et al., 2015), because such types of tasks enable students to develop higher order thinking. Thus, students are given the opportunity to develop higher order thinking if they are taught through scaffolding tasks (Collins, 2014).

There are benefits to scaffolding, which include students becoming more independent as they work individually or in groups while getting assistance and support from their lecturer periodically (Bakker et al., 2015). The following is an example from Lecturer A's lecture, who went around the lecture room assisting the students as they worked individually or in pairs:

**Lecturer A:** *What does decompose mean?*

**Student:** *Break.*

*T: Break up. Right. So, you're going to break this up into partial fractions. If you recall in Grade 10 you should have had problems like these: Simplify:*$\frac{2}{x-1}+\frac{3}{x}$*……how do you go about doing this?*

**Students:** *Find the lowest common denominator.*

As the lecturer went around assisting the students, they benefitted in that there was instant feedback from the lecturer. This is supported by Casem (2013), who illustrated that students showed significant improvement in their performance after learning through scaffolding, because they were given instant support and feedback.

The lecturers participating in this study organised their materials differently. Lecturer A began with an investigation of area under a curve, while Lecturer B began with the rules of anti-differentiation and concluded with the area under a curve. Both lecturers followed the module outline from their institution. Lecturer B had his materials organised that way because he was building on students' foundation knowledge of the derivative concept. Both lecturers used scaffolding in their lecturing and gave their students the opportunities to use prior knowledge to make meaning of new knowledge. What was interesting in this section was realising that there exists a difference between foundation knowledge and prior knowledge. Foundation knowledge includes the ideas and the knowledge that is pertinent to the concept that is being taught and is always accurate (Reynold, 2010). In the same way, prior knowledge may be correct or incorrect knowledge that a student already possesses and takes to the lecture room, which may affect learning positively or negatively (Ambrose, 2012).

### 6.5.5 Exploring the students' cognitive demand

Many researchers including Chinyoka, Denhere and Mambeu (2013) and Bature and Jibrin (2015) attest to the fact that scaffolding tasks promote the development of higher order thinking. As indicated before, both lecturers in this study used scaffolding of tasks in their lectures, thus students were given the opportunity to develop higher order thinking. Hence, cognitive demand appears to have been kept at high levels since the students would have employed higher order thinking as they worked through the tasks. Ferguson's (2009; 2013) findings show that even though one is teaching a mixed class of high and low attaining learners, the cognitive levels of a task can be kept high by scaffolding and asking probing questions.

Cognitive demand appeared to have been kept at high levels since the students were given the opportunities to participate in tasks that promoted investigations, conjecturing, proving and the use of multiple representations. Such tasks require students to be deep thinkers (Quinnell & Carter, 2013), as well as to reason mathematically as they work through the tasks (Boaler, 2016). Akkus and Cakiroglu (2010) also showed that allowing students to engage with tasks that promote the use of multiple representation keeps their cognitive demand high. This is because as the students switch between representations, they are given the opportunity to understand better and thus, avoiding memorising.

Figure 6.9 shows that in lecture B 6, there were more activities that required the students to think, represent and reason mathematically. Thus, the students' cognitive demand was kept at high levels.

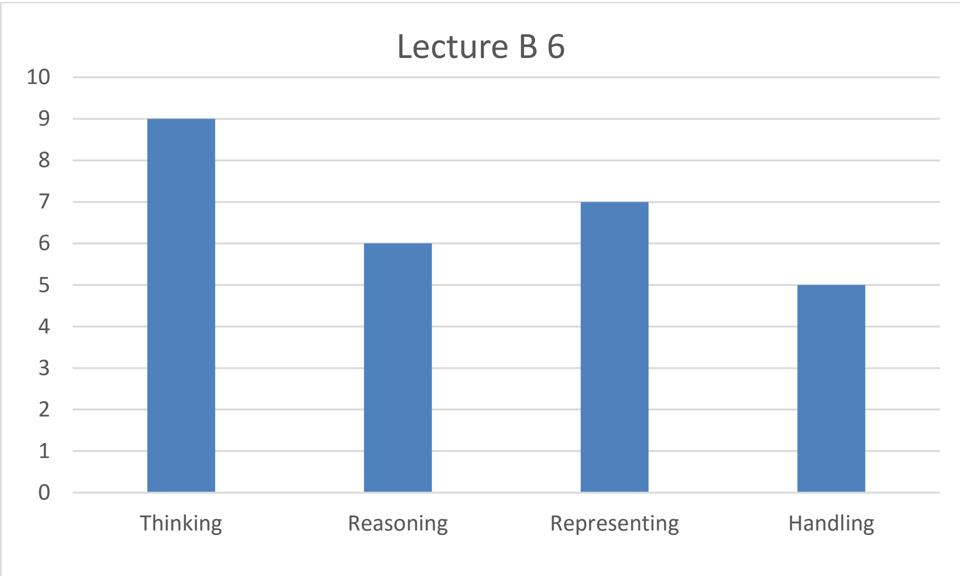


**Figure 6.9: Components of Mathematical Activities Framework**

The lecturers asked questions to check prior knowledge, leading and follow up questions, probing questions, as well as interrogative questions, which challenged the students to think deeply about what they were learning. Such questions, coupled with the tasks that the students took part in during the lectures, enabled them to reason and think mathematically (Bailey et al., 2014), thereby keeping the cognitive demand at high levels. Across all the 18 lectures that were observed during the study, there were few questions, which aimed at checking students' prior knowledge, as compared to leading, follow up, probing, as well as interrogative questions. Apply, or analyse procedural and conceptual knowledge questions fall under leading and follow up questions, probing questions, as well as interrogative questions and such questions require students to use higher order thinking, since they are in the middle to higher level of the taxonomy table (Mathumbu, Braun, & Rauscher, 2014).

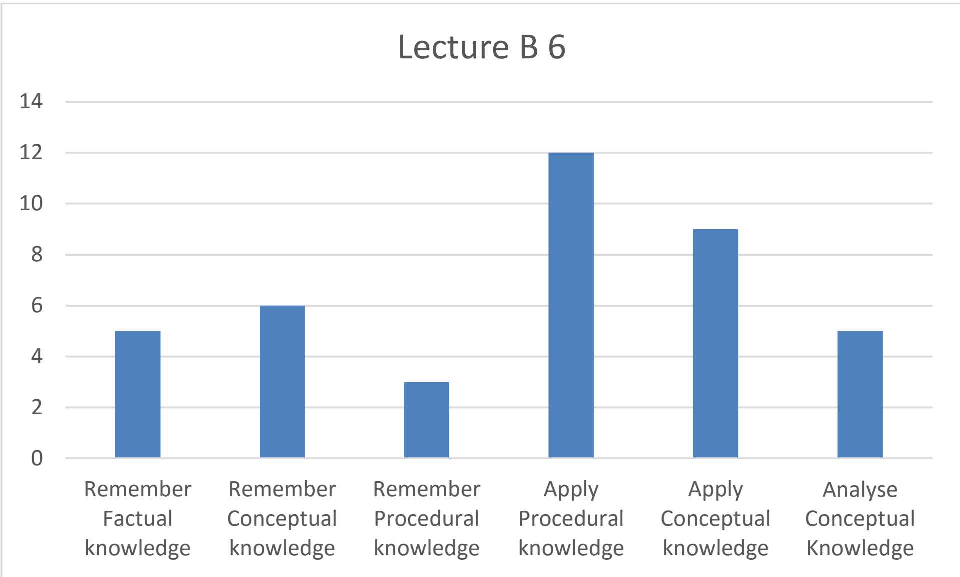


**Figure 6.10: The components of the Cognitive Processes Framework**

Questions that require students to remember factual, conceptual or procedural knowledge fall under questions aimed at checking students' prior knowledge and they are in the lower level of the taxonomy table (Mathumbu et al., 2014). Figure 6.10 illustrates that there were few questions in the lower level of the taxonomy table and more questions in the middle to upper level of the taxonomy table. This implies that the cognitive demand was not reduced, but kept at a high level. Figure 6.9 and 6.10 both show that the students' cognitive demand, in lecture B 4 was kept high. This was the pattern across all the lectures that were observed.

Table 6.5 supports Figures 6.9 and 6.10 by showing that questions that needed the students to apply or anaylse their procedural knowledge or conceptual knowledge were asked most frequently in lecture B 6. This also shows that the cognitive levels were kept at high levels, since these type of questions are in the middle to upper level of the taxonomy table.

**Table 6.5: The Taxonomy Table: Lecture B 6**

| | **The Cognitive Process Dimension** | | | | | |
|---|---|---|---|---|---|---|
| | Remember | Understand | Apply | Analyze | Evaluate | Create |
| **The Knowledge Dimension** | | | | | | |
| Factual Knowledge | | | | | | |
| Conceptual Knowledge | | 3 | 7 | 5 | 1 | |
| Procedural Knowledge | | 2 | 6 | 5 | | |
| Meta-cognitive Knowledge | | | | | | |

### 6.6 Conclusion

This chapter explored the findings of the study and highlighted the themes that emerged during the analysis of data. Two final themes emerged from the primary themes, which were lecturing through explanations, as well as lecturing through questioning. This chapter also explored the reasons behind the lecturers' choice of arranging their tasks and materials in the way that they did. Additionally, this chapter explored what was legitimised during the teaching and learning of the calculus module and these included the mathematics, mathematics learning and mathematics teaching.

The findings of this study suggest that there are two types of lecturing that may occur in a calculus lecture room. Firstly, lecturing by explanations of concepts and processes and secondly lecturing by questioning. To take up on the point raised above, lecturing by explanations involves the unpacking of mathematical concepts, processes and symbols. It also involves representation, translation and perception. Lecturing through questioning involves asking a variety of questions, with the aim of checking prior knowledge, leading and following up questions, probing and interrogative questions. The findings also showed that students might also ask questions, known as the affirmation or confidence boosting questions. It was also revealed that activities that promote investigations, conjecturing, proving, the use of symbols, the use of multiple representations, and the use of multiple techniques and activities that promote procedural fluency through conceptual understanding, were legitimised in the calculus lectures that were observed during this study. The conclusion of the study, as well as the recommendations and significance of the study, are provided in the next chapter.

# CHAPTER SEVEN

## CONCLUSION AND POSSIBILITIES FOR FUTURE RESEARCH

### 7.1 Introduction

The preceding chapter explored the findings of this study. Based on the findings, the themes that emerged during the data analysis were grouped into two major themes, which are lecturing through explanations, as well as lecturing through questioning.

This study aimed at investigating what was legitimised as mathematics, mathematics teaching and learning as well as the dialogue that took place in the calculus lecture room. In that view, the first chapter set the tone for the study by highlighting the pertinent issues relevant to the study and Chapter Two discussed the literature related to this study. Chapter Three outlined the frameworks that informed this study and Chapter Four presented the research design and related issues while Chapter Five presented the data used in this study. Therefore, this chapter presents the conclusion of this study. Additionally, this chapter attempts to provide a response to each of the key questions. The recommendations, possible contributions of this study, as well as the limitations of this study, are also presented in this chapter.

### 7.2 Responding to the research questions

This study has three key questions, which were informed by the conceptual frameworks of this study. The first question addressed the mathematical activities that were legitimised during the teaching and learning of the calculus module. The second question addressed the lecturers' actions, how and why they arranged the materials of the module this way as well as how this affected the students' cognitive demands. The third question addressed the nature of dialogue that took place during the lectures.

#### 7.2.1 The mathematical activities that were legitimised in the calculus lecture room

The study has shown that some activities, which promoted investigations, conjecturing, proving multiple representations, the use of symbols, procedural and conceptual knowledge, as well as multiple techniques, were legitimised in the calculus lecture rooms. Rittle-Johnson, Schneider and Star (2015) support the engagement of activities which aid the development of procedural and conceptual knowledge by pointing out that the two complement each other and that they are fundamental to the learning of mathematics. Premprayoonk et al. (2014) also point out that doing mathematics involves working with mathematical symbols, thus, students are required to manipulate mathematical symbols with confidence and ease. Activities that promote investigations, conjecturing, proving and multiple representations ensure that the students develop as strong problem solvers (Boaler, 2016).

The findings of this study have shown that the mathematical activities that were legitimised in the calculus lecture rooms were justified with appeals made to mathematics students' experiences, lecturer's own experience, everyday metaphor, teacher education and lecturers' own authority. These findings are supported by the research results of Adler and Davis (2006) and Parker and Adler (2012) which showed that justifications were made to mathematics, lecturers' own authority, lecturers' own experience, students' experiences, teacher education as well as everyday metaphor.

### 7.2.2 Organisation of materials and the reasons behind that

The findings of this study showed that the lecturers organised their materials in different ways. Lecturer A began with an investigation of area under a curve, while Lecturer B began with the rules of anti-differentiation. Although this was the case, both lecturers followed the prescribed curriculum.

Lecturer A and the organisers of the project within which he was working designed the materials used by Lecturer A. The way in which the materials were organised was in line with most calculus textbooks. However, Lecturer B organised his own materials, with the intention of linking the rules of anti-differentiation with the rules of differentiation. Lecturer B intended to build on the students' foundation knowledge. Thus, while organising his own materials, Lecturer B employed the notion of the derivative and the rules of differentiation as foundation knowledge to build on the notion of anti-differentiation, since anti-differentiation is the reverse of differentiation. Adler (2012) points out that as teachers go about teaching mathematics, they employ a variety of resources and strategies and adapt them to suit their students' needs. Both lecturers in this study took part in the organisation of their materials and organised them to suit their students' needs. This is supported by Ostova-Namghi (2017), whose study's participants felt that they should be allowed to organise their own materials because they are the ones involved in the implementation process.

Both lecturers tapped into students' prior knowledge, for example, when they were working with integration by parts, they tapped into students' prior knowledge of partial fractions. In addition, when they were working with integration of trigonometric functions, they employed the students' prior knowledge of trigonometric identities.

Both lecturers worked within the existing curriculum of their institution. The way the materials were organised by both lecturers allowed for scaffolding. Students may have benefitted from scaffolding by getting instant feedback from the lecturers and by having the tasks broken into manageable portions (Bature & Jibrin, 2015).

### 7.2.3 The students' cognitive demand

The findings of this study have revealed that the students' cognitive levels appeared to have been kept high. The activities that promote conjecturing, proving and use of multiple representations required the students to employ their deep-thinking skills, thus keeping their cognitive levels high, as suggested by Boaler (2016). While engaging with activities that required them to switch between various representations of mathematical concepts, the students were required to use their higher order thinking and this increased their cognitive demand (Akkus & Cakiroglu, 2010). The findings of this study are supported by Ponte (2005), as cited in Viesu and Oliveira (2012, p. 290) who pointed out that tasks that promote the use of investigations present themselves with higher levels of difficulties, as compared to normal textbook exercises that mostly have lower levels of difficulties. Thus, when students are given the opportunity to engage with tasks or activities that promote the use of investigations, their cognitive demand is raised because such activities are not routine, but require students to be creative and use their deep thinking skills, thus maintaing the students' cognitive demand at high levels.

Additionally, the type of questions that were asked by the lecturers required the students to reason and think mathematically. Bailey et al. (2014) suggest that such questions also encourage the students to draw on to their deep-thinking skills, thus keeping the students' cognitive levels high.

### 7.2.4 The connection between the lecturers' mathematical knowledge and their decisions and reflections in practice

Kleickmann et al. (2013) and  Jadama (2014) highlighted the fact that the lecturers'/educators' mathematical knowledge largely impacts on their reflection and decisions in their practice. Also, the study conducted by Olfos, Goldrine and Estrella (2014) revealed that the educators' content knowledge, as well as the educators' teaching strategies and experience, are closely linked to the learners' performance. Their findings revealed that the educators who had more experience in teaching and were in possession of good teaching strategies, coupled with their profound mathematical knowledge, had a positive impact on the learners' performance in mathematics. This supports the findings of this study, which showed that both lecturers felt that their profound mathematical knowledge influenced the way they planned, the decisions they made, as well as their reflections of their lecture room practice. The findings of this study reveal that the lecturers agreed that their profound mathematical knowledge was also helpful in their planning of their lectures, because they had a comprehensive picture of where they were heading with the topic that they were teaching. The comprehensive picture of the topic that they were teaching also enabled them to be aware of their students' prior knowledge and to link the students' prior knowledge with the new concepts. The lecturers also agreed that their mathematical knowledge, combined with their experience in teaching the module, enabled them to anticipate where the students would encounter problems, thus the lecturers were prepared to help their students when needed to do so. Thus, the lecturers in this study appeared to be in possession of the four categories of Pedagogical Content Knowledge suggested by Hill et al. (2008), which are Common Content Knowledge (CCK), Specialised Content Knowledge (SCK), Knowledge of Content and knowledge of how students learn particular knowledge (KCS), as well as Knowledge of Content and Teaching (KCT).

### 7.2.5 The nature of dialogue in the calculus lecture room

In every lecture, there is some form of dialogue that take place, which is important to the learning of mathematics (Walsh, 2012). Dialogue in both lectures was twofold. Firstly, there was dialogue that involved explanations of concepts, procedures and mathematical symbols. The lecturers and the students did the explanation of concepts, and especially procedures. Usually, the lecturer is the only one expected to do the explanation of concepts and procedures, but in recent times, the students are tasked to explain their thought processes, which include their understanding of the concepts and procedures (Hähkiöniemi, 2013). Engaging students in dialogue is fundamental to their mental development. In addition, having students engage in dialogue contributes to their profound understanding of concepts, as well as their development into deep thinkers (Sedova, Sedlacek, & Svaricek, 2016).

Secondly, there was dialogue that involved questioning and answering. The lecturers, as well as the students did the questioning. The students asked questions to affirm or boost their confidence, while the lecturers asked questions to check prior knowledge, leading or follow up questions, probing questions, as well as interrogative questions. Such questions are important because they help in developing the students' higher order thinking (Elsner et al., 2013).

### 7.3 Mathematics for teaching

The findings of this study have shown that the lecturers exhibited several components of mathematics for teaching. These included unpacking, the use of representations, questioning, translating, simplification and perception. Mathematics for Teaching plays an important role in the teaching and learning of mathematics. Adler (2017) confirms this by pointing out that students who were taught by educators, who had enrolled for an in-service course focusing on developing their mathematics for teaching, outperformed those who were taught by teachers who did not enrol for the course.

Although the lecturers did not explicitly mention the components of mathematics for teaching that they were exhibiting through the lecturers' actions, the pre-service teachers may have learnt the following from their lecturers. Firstly, the fact that concept, procedures and symbols needed to be unpacked. Secondly, the fact that representing concepts in a variety of ways, as well as switching between symbolic notations, was beneficial to the pre-service teachers (Quinnell & Carter, 2013). Thirdly, the fact that the different types of questioning strategies was fundamental to the mathematical development of the students (Bansilal, 2012). Fourthly, the fact that re- explaining and following up on students' ideas, as well as working with the ideas, is fundamental to maintaining coherence in the students' contribution to the group discussions (Towers & Proulx, 2013).

## 7.4 Recommendations

One of the aims of the study was to explore the way in which the lecturers organised the materials and the reasons behind their style of organisation. The findings of this study exhibit that the way in which Lecturer A organised his materials was consistent with most of the calculus textbooks. He began with the investigation of the area under the curve and then moved on to the rules of anti-differentiation.

On the other hand, Lecturer B working within the prescribed curriculum began with the rules of anti-differentiation and concluded the module with the area under the curve. Lecturer B intended his students to use the notion of the derivative as well as the rules of differentiation as foundation knowledge on which to build the notion of anti-differentiation. He did so because anti-differentiation is the reverse of differentiation and so the students could easily make the connection. Based on the statistics for this module, 82% of the students passed the module.

In South Africa, Integral Calculus is taught at high school level in a few private schools that follow the Independent Examination Board (IEB) curriculum. The majority of public high schools in South Africa only teach Differential Calculus following the Curriculum Assessment Policy Statement as prescribed by the Department of Basic Education (2011). Thus, the majority of first-year university students are introduced to the integral concepts for the first time at university level. Many researchers, including Awang and Zakaria (2012) and Siyepu (2013) have shown that most first-year students struggle with calculus concepts. The findings of this study have shown that introducing the integral concept by using the derivative concept and the rules of differentiation as background knowledge, may be beneficial to the students. This is supported by the findings of the study conducted by Awang and Zakaria (2012), which revealed that in Malaysian high schools, educators find it helpful to the students, when they teach the integral concept by using the derivative and the rules of the differentiation as foundation knowledge. Therefore, this study recommends that students who enrol with low marks in mathematics should use the derivative concept and the rules of differentiation as their foundation in order to build on the rules of anti-differentiation.

## 7.5 The limitations of this study

The data collected for this study were collected through video recording of the 18 lectures. Although both lecturers in this study were highly experienced and qualified, the presence of the camera might have affected their lecture presentations in the first few lectures. Similarly, the students' actions and responses might have also been affected by the presence of the camera during the first few lectures. The participants of this study appeared to get used to having the camera in their midst as the days progressed.

Both lecturers in this study were highly qualified and more experienced than the researcher. This might have threatened the researcher during the interviews and resulted in the researcher not getting sufficient information from the interviewees.

This was a small-scale study in which both the lecturers were from the same university. Thus, the findings of this study cannot be generalised to all third-year calculus modules across all universities.

Having not interviewed the students has been a limitation of this study. Had students been interviewed on which components of the mathematics for teaching they observed or learnt from their lecturer, the researcher would have been in a better position to confidently report that the students were well equipped with the components of mathematics for teaching, which were exhibited by the lecturers. In addition, there was no follow up with the students, to see if they employed the same mathematics for teaching that had been exhibited by the lecturers. Again, concerning the taxonomy table, there was no follow up with the students to confirm that they could actually apply, or analyse the conceptual or procedural knowledge.

### 7.6 The significance and contributions of the study

This study sought to explore what was legitimised in the calculus lecture room. This is in agreement with the call made by Hoffman and Mercer (2016) that since the lecture room is a social setting, researchers must attend to what the participants of the lecture room are taking part in. Many studies have been conducted on students' errors, misconceptions and concept image of the concepts in calculus, while other studies have focused on the interaction of lecturers and students in various topics but not calculus in the pre-service teachers'module. Furthermore, many studies have been conducted on mathematical activities in various topics but not in the calculus module of the pre-service teachers. Thus this study addresses a significant challenge in pre-sevice teacher education that of getting pre-service teachers to engage conceptually and cooperatively with mathematical activities in large classes of the calculus module. Hence, this study has combined the socio-cultural and cognitive perspectives to learning. The combination of these two mutually exclusive approaches that seem to be in conflict but complement each other, has shed light into the reason why it is possible for the pre-service teachers in the calculus module to succeed in taking part in the mathematical activities of that module. Therefore, this study provides insight into the interaction of the lecturers and pre-service teachers as well as the mathematical activities with which the pre-service teachers in the calculus module engage.

This study contributes to the field of mathematics education, firstly by identifying the mathematical activities that are legitimised in the pre-service teachers' calculus lecture room. This study has shown that activities such as investigations, conjecturing, proving and the use of multiple representations are fundamental to the learning of integral calculus because they require students to think and reason mathematically. This in turn, helps to develop higher order thinking, which is an important skill that students should possess so that they are able to cope with situations in problem solving (Cansory & Türkoğlu, 2017). Additionally, these activities keep the students' cognitive demand high, since such activities allow students to develop their own mathematical understanding (Liu & Chin, 2016). The knowledge of such activities being promoted in the pre-service teachers' calculus lecture room may be of benefit to the pre-service teacher education because such activities contribute to the development of profound subject knowledge. Tshabalala and Ncube (2012), Mogari (2014), as well as Stols (2013) have shown that the learners' poor performance in mathematics at high school level is attributed to the educators' inadequate subject knowledge. Thus, the findings of this study show that the pre-service teachers' cognitive demand was kept high by participating in such activities. Hence, the students may have developed profound subject knowledge by participating in such activities which in turn might alleviate the problem of educators' inadequate subject knowledge. Thus lecturers could design their pre-service teacher calculus modules and include such mathematical acivities.

Secondly, this study has significantly contributed by showing the mathematics for teaching skills that are exhibited by the lecturers in the calculus lecture room of the pre-service teachers. Thus, the pre-service teachers are exposed to what is counted as teaching mathematics, although they are in the calculus module. The components of mathematics for teaching, which were exhibited by the lecturers are unpacking, questioning, translating, simplification and perception. In view of this, Gitaari et al. (2013) and others point to the fact that learners' poor performance in mathematics is caused by the educators' poor teaching strategies. This study shows that it is possible for pre-service teachers to be exposed and introduced to good teaching strategies, even if they are not in the teaching methods module. Thus, the components of mathematics for teaching that were exhibited by the lecturers may be of benefit to the field of teacher training.

Thirdly, this study sought to explore the ways in which the lecturers organised their materials. Lecturer B began with the rules of anti-differentiation because he wanted the students to link them with the rules of differentiation. This may have benefitted the students because anti-differentiation is the reverse of differentiation, thus, the students could easily see the connection between the two concepts. Siyepu (2013) and Serhan (2015) have shown that students struggle with grasping the concepts in calculus. The researcher has not come across literature focusing on the introduction of integral calculus using the rules of differentiation as background knowledge at university level. In addition, given the fact that many students in South Africa have an inadequate grasp of mathematical concepts from secondary schools, thus, giving the rules of anti-differentiation as an introduction to integral calculus, may be of benefit to many first-year university students.

While exploring the nature of dialogue in the calculus lecture room, two types of teaching emerged. Firstly, the teaching that involves questioning and secondly, the teaching that involves explanations. The teaching that involves questioning may benefit both the lecturers and the educators because the types of questions require students to use higher order thinking, which is one of the important skills needed by students (Cansory & Türkoğlu, 2017). This in turn helps the students to develop a rich understanding of the mathematical concepts (Sedova, Sedlacek, & Svaricek, 2016). The teaching that involves explanation of concepts, procedures or mathematical symbols and notations may be of benefit to lecturers and educators because students or learners can be involved in explaining the concepts or procedures. The findings of this study have shown that students can also take the role of explaining the concepts and procedures. This is supported by Hähkiöniemi (2013) who points out that students or learners can also take the role of explaining concepts or procedures. Allowing students to talk about their understanding of concepts or their procedures is fundamental to the students' mathematical development (Bansilal, 2012).

One of the aims of this study was to explore the nature of dialogue in the calculus lecture room of the pre-service teachers. The findings of the study showed that dialogue in the calculus lecture room could be in the form of explanation of concepts, procedures or symbols. Dialogue can also be in the form of questioning. Hence, this study offers a classification of broad moves by the lecturers who lecture in the pre-service teachers' calculus module to promote a culture of dialogic learning. Previous researchers including Chikiwa (2017), Mhakure and Jacobs (2016) and Olmsted (2012) have shown that lecturers ask questions to check prior knowledge, leading and follow up questions, as well as probing questions. The findings of this study have shown that, added to the mentioned types of questions, lecturers can also ask interrogative questions, which challenge students to interrogate their conceptual or procedural knowledge. These types of questions are sub-categories of dialogue through questioning that was exhibited by the lecturers. Aligning of such types of questions with the ICM, CP and MA frameworks is a contribution to the field of mathematics education and is essential because it signifies that practice fosters a cooperative learning culture that makes inquiry a priority. Thus, interrogative questioning may be an addition to the existing types of questions framework.

Almeida (2009) also indicated that students ask questions, most of which are triggered by what might have been said by their lecturer. This study has shown that students can also ask questions, which are intended to boost their confidence or to affirm their confidence. Hence, this type of question may be an addition to the existing types of questions asked by learners or students. Thus, lecturers or educators ought to be aware of this type of questions so that they can be in a position to give support to their students.

### 7.7 Areas for future research

This study has investigated the dialogue as well as the mathematics, mathematics teaching and learning that were legitimised in the calculus lecture rooms. The findings of the study have shown that there are various mathematical activities that are legitimised in the calculus lecture rooms as well as the mathematics for teaching skills that are exhibited by the lecturers explicitly as well as implicitly. However, the students were not interviewed to explore their perceptions on the mathematics teaching and learning that they were exposed to in the calculus module. In view of this, further studies could investigate this issue from the students' perspectives. Further studies could also follow up on the students in the practice, to see how they apply what they would have learnt in the classroom situation. In addition, the materials that were used in the calculus modules of this study were not analysed. Further studies could explore the materials that are used in the calculus modules to see if they are such that they keep the students' cognitive demand at high levels.

## 7.8 Conclusion

This chapter presented the conclusion of the study by firstly responding to the research questions. Secondly, the limitations of this study are highlighted, especially that the findings could not be generalised since this was a small-scale study with only two lecturers. The significance of the study has been explored and the possibilities for future research have been discussed alongside the recommendations of the study.

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# APPENDICES

## APPENDIX A: Consent Information: Lecturer

Dear Lecturer
Re: Consent for participation in the PhD research project

This letter is to inform you about my PhD research project that involves a case study of the interplay between the lecturer, students and the module materials. The aim of this project is to find out what types of mathematical activities are recognised as relevant as well as what counts as mathematics for teachers, so that lecturers can design their activities for student teachers to benefit more from their modules.

You and your class have been selected as possible research group because I am studying at UKZN and I live in PMB and this is the only B.Ed students group nearest to me. This letter formally invites you as a lecturer in the B.Ed calculus module to participate in the project.

Your participation will involve:
1. Being video recorded as you teach the integral calculus module to your class. The recordings will be strictly used for this research purpose and will be kept confidential.
2. At the end of the module, I will interview you as soon as I begin with my data analysis. The interviews will be taped. The questions will be based on what I will observe from my data analysis. If, however you will feel uncomfortable during the interview, the interview will be stopped immediately. The interview will last at least 30 minutes per person.

The interview will be strictly confidential. You will not be paid for participating in the project. Your real name will not be used.

Yours sincerely
Botshiwe Likwambe
PhD student University of KwaZulu-Natal PMB

**APPENDIX B: Consent form: Lecturer**

I…………………………………………………( *please print your full name)* as the lecturer of the B. Ed Calculus module, I am aware of the data collection process in the research project as listed in the information letter above.

I give consent to being video recorded while I teach the module as well as to being interviewed at the end of the module and having these interviews taped and transcribed.

I am aware that the data collected will be used in a research project focused in finding out what types of mathematical activities are recognised as relevant as well as what counts as mathematics for teachers.

I know that all the information provided and used in the research report will not be connected to me personally and my name will not be used. Full confidentiality will be adhered to and a suitable pseudonym, selected in consultation with me will be used to identify my contribution to the report.

Signed……………………………………..

Date

## APPENDIX C: Consent Information: Students

Dear University of KwaZulu-Natal Edgewood Campus B. Ed Student
Re: Consent for participation in the PhD research project

This letter is to inform you about my PhD research project that involves a case study of the interplay between the lecturer, students and the module materials. The aim of this project is to find out what types of mathematical activities are recognised as relevant as well as what counts as mathematics for teachers, so that lecturers can design their activities for student teachers to benefit more from their modules.

You and your class have been selected as possible research group because I am studying at UKZN and I live in PMB and this is the only B. Ed students group nearest to me. This letter formally invites you as a lecturer in the B. Ed calculus module to participate in the project.

Your participation will involve:
1. Attending your lectures as usual and your lectures being video recorded. These recordings will be strictly used for this research purpose and will be kept confidential.
2. In your first lecture, you will be provided with a consent form that you will be asked to complete and sign.
3. Some of you will be interviewed once towards the end of your module. (I will ask you for your consent when the time comes). The interviews will be taped. The questions will be based on what you will be learning on integral calculus. If, however you will feel uncomfortable during the interview, the interview will be stopped immediately. The interview will last at least 30 minutes per person.
The interview will be strictly confidential. Your decision to participate or not participate will not affect your marks in any way. If you participate, your lecturer will not have access to the recorded interview. You will not be paid for participating in the project. Your real name will not be used.

Yours sincerely
Botshiwe Likwambe
PhD student University of KwaZulu-Natal PMB

**APPENDIX D: Consent form: Student**

I……………………………………………….( *please print your full name)* as a B. Ed student specialising in Mathematics, I am aware of the data collection process in the research project as listed in the information letter above.
I give consent to being video recorded in my Mathematics lectures.

I am aware that the data collected will be used in a research project focused in finding out what types of mathematical activities are recognised as relevant as well as what counts as mathematics for teachers.

I know that all the information provided and used in the research report will not be connected to me personally and my name will not be used. Full confidentiality will be adhered to and a suitable pseudonym, selected in consultation with me will be used to identify my contribution to the report.

Signed……………………………………..
Date

**APPENDIXE: Certificate of Editing**

**PROOF OF EDITING CERTIFICATE**

**TO WHOM IT MAY CONCERN**

**Re: LANGUAGE EDITING**

I, THE UNDERSIGNED, hereby confirm that I have edited the thesis titled **EXPLORING MATHEMATICAL ACTIVITIES AND DIALOGUE WITHIN A PRE-SERVICE TEACHERS' CALCULUS MODULE: A CASE STUDY, by Botshiwe Likwambe,** for the degree of **DOCTOR OF PHILOSOPHY.**

Regards

*H Mapudzi*

Professional EDITORS Guild

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078 086 5605
083 527 1804
fmapudzi@yahoo.co.uk

www.editors.org.za

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APPENDIX F: Turnitin Certificate

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**APPENDIX G: Letter of Editing**

# Angela Bryan & Associates

6 La Vigna
Plantations
47 Shongweni Road
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Date: 23 July 2018

To whom it may concern

This is to certify that the Doctoral Thesis: Exploring Mathematical Activities and Dialogue Within a Pre-service Teachers' Calculus Module: A Case Study written by Botshiwe Likwambe has been edited by me for language.

Please contact me should you require any further information.

Kind Regards

Angela Bryan

angelakirbybryan@gmail.com

0832983312