ITERATIVE ALGORITHMS FOR SOLUTIONS

OF NONLINEAR EQUATIONS IN BANACH SPACES

by

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Abstract

Analytical methods for finding exact solutions of many nonlinear equations are rare or unknown. Therefore, methods of approximating the solutions of nonlinear equations are of interest where solutions are known to exist. We study nonlinear equations with monotone type mappings, nonexpansive mappings, μ-strictly pseudocontractive mappings and other related problems such as equilibrium problems, variational inequality problems, optimization problems, e.t.c. Constructing iterative algorithms for the approximation of zeros of nonlinear equations and solutions of fixed point problems is an active area of research in Mathematics. It is also observed that most of the existing results on the approximation of solutions of monotone-type mappings have been proved in Hilbert spaces or they are for accretive-type mappings in Banach spaces. Assuming existence, we develop explicit and implicit iterative algorithms for approximating the solutions of nonlinear equations. We propose the concept of generalized Lyapunov functions which are essential in the study of nonlinear analysis and convex analysis that involves important field of monotone mappings from a Banach space into its dual space. We introduce several iterative algorithms for some nonlinear problems which involve monotone type mappings. The strong convergence theorems are obtained in the general Banach spaces for different classes of monotone mappings. These include the class of generalized Φ-strongly monotone mappings, which is the largest (i.e generalized other classes) such that if a solution of an equation $0 \in Ax$ exists, it is necessarily unique, where $x \in D(A)$, the domain of A. The class of monotone type maps is chosen for this study due to its suitability than other maps, such as compact maps, for the study of nonlinear equations. Most operators lack compactness property, it is not always easy to check or verify compactness and it does represent a rather severe restriction on the operators. As an immediate application of our results, we obtain the solutions of generalized convex optimization problems. Our results are of interest to a wide audience due to the monotonicity property of our maps and their applications in other fields such as engineering, Physics, Biology, Chemistry and Economics. We explore the viscosity approximation methods. Numerical examples are used to compare the rate of convergence of implicit midpoint rules, where viscosity is involved with a non-

viscous approximation method. Also, a strong convergence result is established for the implicit midpoint procedures. Under suitable conditions imposed on the control parameters, we show that certain two generalized implicit iterative algorithms will converge to the same fixed points of a nonexpansive mapping. By considering the class of μ-strictly pseudocontractive mappings, we generalize some existing results on viscosity approximation methods of nonexpansive mappings. We propose an iterative algorithm for the class of μ-strictly pseudocontractive mappings and establish its strong convergence to a fixed point of the map, which is also the solution to some variational inequality problems in uniformly smooth Banach spaces. Using generalized contractions, a new iterative algorithm is introduced for the class of nonexpansive mappings. It is shown that the newly introduced sequence converges strongly to a fixed point of the nonexpansive mapping, which is also the solution to some variational inequality problems.

PREFACE

This research focuses on how to develop iterative algorithms to approximate solutions of nonlinear equations. Methods of approximating the solutions of nonlinear equations are of interest where solutions are known to exist. Indeed, approximation method often gives accurate solution which leads to right predictions. Nonlinear or complex equations often defy analytical methods of finding their solutions. They arise in modeling problems, such as maximizing gains in businesses, improving health of individuals, minimizing costs in industries and maximizing the use of resources in the academic institutions.

Business men and women look for ways to optimize profits, individuals think of ways to improve their health, industries search for procedures to minimize cost and academic institutions seek for ways to maximize the use of resources. Suffice it to say that men are always overwhelmed with the thoughts on how to get things done or solve particular problems.

Mathematical tools called ordinary differential and differential algebraic equations can be used to model optimization problems. The solutions to these problems can help the individual to understand the importance of taking basic health tips, for the head of the institutions to value and adopt right strategies, for the managers to welcome new ideas and procedures and for businesses to bring smile on the faces of the concerned people. However, these are often highly complex and dynamic problems, influenced by multiple factors. Sometimes, these cannot be solved by simple means to find their exact solutions.

The best methods to solve such nonlinear equations (otherwise known as complex equations) are the "explicit and implicit iterative procedures". These are the approximation methods which give solutions with least or no computation errors. Many problems which we encounter daily in the world will remain unsolved without application of appropriate mathematical tools. Using suitable iterative algorithms and approximation methods have helped to predict, plan and avert some difficulties.

Assuming existence, we develop explicit and implicit iterative algorithms for approximating the solutions of nonlinear equations. Our results are of interest to a wide audience due to their applications in other fields such as engineering, Physics, Biology, Chemistry and Economics.

DECLARATION

Plagiarism

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2. This dissertation has not been submitted for any degree or examination at any university.

3. This dissertation does not contain other person's data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

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Mathew O. Aibinu, 13 *August*, 2019

Publications based on current work

1. **M. O. Aibinu** and O. T. Mewomo, Algorithm for the generalized Φ-strongly monotone mappings and application to the generalized convex optimization problem, Proyecciones Journal of Mathematics, 38 (1), March (**2019**), 59-82 (Scopus Indexed).

2. **M. O. Aibinu** and O. T. Mewomo, Strong convergence theorems for strongly monotone mappings in Banach spaces, Boletim da Sociedade Paranaense de Matem$\acute{a}$tica, February, (**2018**) DOI:10.5269/bspm.37655 (Scopus Indexed).

3. **M.O. Aibinu**, P. Pillay, J.O. Olaleru and O.T. Mewomo, The implicit midpoint rule of nonexpansive mappings and applications in uniformly smooth Banach spaces, J. Nonlinear Sci. Appl., 11 (**2018**), 1374-1391,
DOI: 10.22436/jnsa.011.12.08.

4. **M. O. Aibinu** and O. T. Mewomo, Algorithm for Zeros of monotone maps in Banach spaces, Proceedings of Southern Africa Mathematical Sciences Association (SAMSA2016) Annual Conference, 21-24 November 2016, University of Pretoria, South Africa, (**2017**), 35-44.

5. **M. O. Aibinu**, On the rate of convergence of viscosity implicit iterative algorithms, **Accepted** to appear in Nonlinear Functional Analysis and Applications, (Scopus Indexed).

6. **M. O. Aibinu** and J. K. Kim, Convergence analysis of viscosity implicit rules of nonexpansive mappings in Banach spaces, **Accepted** to appear in Nonlinear Functional Analysis and Applications, (Scopus Indexed).

Dedication

This thesis is dedicated to the Almighty God.

Contents

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Finally, I am thankful to my siblings and parents for always being by my side to support and encourage. My shout out to my lovely wife and special appreciation to her for being a supportive wife.

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General Introduction

Many phenomena in real life are governed by inherently nonlinear equations. For instance, when chemists model molecules, they are solving Schr*ö*dinger's equation, exploring for oil requires solving Gelfand-Levitan equation and predicting tsunamis means solving Naiver-Stokes equation. Nonlinear equations also represent the problems of minimization of a function, variational inequalities and equilibrium problems. These illustrations drive home the importance of finding the solutions of equations. The researches needed for national and intercontinental development apply mathematical models and principles. These lead to differential and integral equations, which in general are nonlinear. Most nonlinear differential and integral equations cannot be solved analytically. Consequently, we usually resort to iterative methods for finding their solutions. Methods of approximating the solutions of nonlinear equations are of interest where solutions are known to exist.

The contributions of this thesis belong to the general scope of nonlinear functional analysis. This is a scope of Mathematics with increasing amount of study and vast amount of applicability in recent years. This thesis is devoted to present several of the important convergence theorems for solutions of various types of problems associated with nonlinear equations. We provide qualitative and quantitative

descriptions of their solutions.

The existence or construction of solutions of differential and integral equations is often reduced to the problem of finding a fixed point for an operator defined on a subset of a space of functions. Many problems which occur in different areas of mathematics, such as optimization, variational analysis and differential equations, can be modeled by the equation

$$x = Tx,$$

where T is a nonlinear operator defined on a metric space. The solutions to this equation are known as fixed points of T. Fixed point theory (FPT) is one of the most powerful tools of modern Mathematics. FPT includes theorems concerning the existence and properties of fixed points. Also, it blends analysis, topology and geometry. It has numerous application and it has been applied in several fields, such as game theory, engineering, Physics, Economics, Biology, Chemistry, etc. For contraction mapping T, defined on a complete metric space X, that is, for $\alpha \in (0,1)$,

$$d\left(T(x), T(y)\right) \leq \alpha d(x,y) \ \forall \ x, y \in X,$$

it is known by Banach contraction principle that T has a unique fixed point for any $x \in X$. Moreover, the Picard's sequence defined by $\{T^n x\}_{n=1}^{\infty}$, converges strongly to the fixed point of T. Unfortunately, if the contraction mapping T is replaced by a nonexpansive mapping, that is,

$$d\left(T(x), T(y)\right) \leq d(x,y) \ \forall \ x, y \in X,$$

the Banach contraction principle fails. Additional conditions must then be assumed either on T and/or the underlying space to ensure the existence of fixed points. The study of the class of nonexpansive mappings and its generalized form, such as strict pseudocontraction mappings, is one of the major and recent active research areas of nonlinear analysis. On the account of the connection with the geometry of Banach spaces and relevance of the class of nonexpansive mappings in the theory of monotone and accretive operators, considerable attention has been given to it since the sixties.

Fixed point theorems have also been applied to determine the existence of periodic solutions for functional differential equations. In addition to the deep involvement in the theory of differential equations, fixed point theorems have been found to be inevitable in problems such as finding zeros of nonlinear equations and proving surjectivity theorems. Consequently, fixed point theory which is branch of functional analysis has developed into an area of independent research due to its importance and applications in solving real life problems.

Due to the progress in nonlinear functional analysis, it has allowed the study of many nonlinear problems. The concept of monotone operators, introduced in the 1960s, has proved to be very effective in the study of nonlinear problems. The connection between nonlinear analysis and convex analysis has led to the introduction of the important field of monotone operators from a Banach space into its dual space [81]. This is due to the associated problems with the compactness: most operators lack compactness property, it is not always easy to check or verify compactness and it does represent a rather severe restriction on the operators. Monotone mappings extend the properties of compact operators to the infinite-dimensional case. The term 'monotone type' denotes the generalizations of monotone operators. The pseudo-monotone mappings, quasi-monotone mappings and the mappings of type (M) are the examples of monotone type. Monotonicity has provided a more proper tool for solving large classes of nonlinear differential and integral equations.

In this thesis, we develop essential iterative methods for approximating the solutions of nonlinear equations and which have applications in many other areas of mathematics. Basically, the focus of this thesis is on the three important topics:

1. Algorithms for solutions of monotone mappings;

2. Viscosity approximation methods for nonexpansive;

3. Viscosity approximation methods for strict pseudocontraction mappings.

1.1 Motivation for present work

Nonlinear systems and nonlinear phenomena are ubiquitous. Systems such as fluid and plasma mechanics, gas dynamics, elasticity, relativity, chemical reactions, combustion, ecology and biomechanics are governed by inherently nonlinear equations. The facile fact is that nonlinear systems are vastly more difficult to analyze. For this reason, an ever increasing proportion of modern mathematical research is being devoted to their study. In the nonlinear realm, many of the most basic questions remain unanswered: existence and uniqueness of solutions are not guaranteed; explicit formulae are difficult to come by; linear superposition is no longer available; numerical approximations are not always sufficiently accurate; etc. The motivation for this work is discussed under three short sub-headings: monotone mappings, equations of Hammerstein type and viscosity approximation methods.

1.1.1 Monotone mappings

Nonlinear equations have been studied extensively for monotone mappings in Hilbert spaces and accretive mappings in general Banach spaces (see e.g, [32], [75], [89], [26] and references there in). Accretivity can simply be described as the monotonicity from a Banach space into itself. It is known that the dual of a Hilbert space is still a Hilbert space and the normalized duality mapping is an identity in a Hilbert space. Therefore, monotonicity and accretivity coincide in the Hilbert spaces. Monotone mappings were first studied in Hilbert spaces by Zarantonello [110], Minty [75], Kačurovskii [64] and a host of other authors. Interest in monotone mappings stems mainly from their usefulness in numerous applications. For example, consider the following: Let $f : E \to \mathbb{R}$ be a proper and convex function. The subdifferential of f at $x \in E$ is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle \, \forall \ y \in E\},$$

which is an example of monotone mapping (see e.g, [2]). It is known that $0 \in \partial f(x)$ if and only if x is a minimizer of f. Setting $\partial f = A$, it follows that solving the inclusion $0 \in Au$ in this case, is the same as solving for a minimizer of f. Several

existence theorems have been established for the equation $Au = 0$ when A is of the monotone-type (see e.g., Deimling [38]; Pascali and Sburlan [81]).

There have been extensive research efforts on inequalities in Banach spaces and their applications on the iterative methods for solutions of nonlinear equations of the form $Au = 0$. However, it occurs that most of the existing results on the approximation of solutions of monotone-type mappings have been proved in Hilbert spaces or they are for accretive-type mappings in Banach spaces. Unfortunately, as has been rightly observed, many and probably most mathematical objects and models do not naturally live in Hilbert spaces. The remarkable success in approximating the zeros of accretive-type mappings is yet to be carried over to nonlinear equations involving monotone mappings in general Banach spaces. Perhaps, part of the difficulty in extending the existing results on the approximation of solutions of accretive-type mappings to general Banach spaces is that, since the operator A maps E to E^*, the recursion formulas used for accretive-type mappings may no longer make sense under these settings. Take for instance, if x_n is in E, Ax_n is in E^* and any convex combination of x_n and Ax_n may not make sense. Moreover, most of the inequalities used in proving convergence theorems when the operators are of accretive-type involve the normalized duality mappings which also appear in the definition of accretive operators. Certainly, if iterative algorithms can be developed for the approximation of solutions of nonlinear equations with monotone-type mappings in general Banach spaces, these will be a welcome complement and generallization of the existing results in the literature which are available(see e.g, [28], [40]).

1.1.2   Viscosity approximation methods

The set of fixed points of a mapping T will be denoted by $F(T)$. The Viscosity approximation method (VAM) for solving nonlinear operator equations has recently attracted much attention (see [65], [76], [102], [104], [107] and the references therein). In 1996, Attouch [14] considered the viscosity solutions of minimization problems. Following the ideas of Attouch [14], in 2000, Moudafi [76] introduced an explicit viscosity method for nonexpansive mappings. Let $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$, the explicit

viscosity iterative sequence $\{x_n\}_{n=1}^{\infty}$ is defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \in \mathbb{N}, \tag{1.1.1}$$

where f is a contraction on K and the nonexpansive mapping $T : K \to K$ is also defined on K, which is a nonempty closed convex subset of a Hilbert space H. Later in 2004, Xu [102] apply a technique which uses (strict) contractions to regularize a nonexpansive mapping for the purpose of selecting a particular fixed point of the nonexpansive mapping and studied the sequence (1.1.1). Xu [102] showed that under suitable conditions imposed on the parameters, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.1.1) converges strongly to $p \in F(T)$ which also solves the following variational inequality

$$\langle (I - f)p, x - p \rangle \geq 0, \ \forall \ x \in F(T). \tag{1.1.2}$$

Consider the ordinary differential equation

$$x' = f(t), \qquad x(0) = x_0. \tag{1.1.3}$$

The sequence $\{x_n\}_{n=1}^{\infty}$ generated by the the implicit midpoint rule via the recursion

$$\frac{1}{h}(x_{n+1} - x_n) = f\left(\frac{x_n + x_{n+1}}{2}\right), \ n \in \mathbb{N}, \tag{1.1.4}$$

where $h > 0$ is a stepsize and $\mathbb{N}$ is the set of positive integers is efficient in approximating a solution of (1.1.3). The implicit midpoint rule is widely known as a powerful numerical method for solving ordinary differential equations and differential algebraic equations (see [15], [16], [17], [39], [50], [51], [91], [93] and [94] and references therein). Xu et al. [104] recently proposed the concept of implicit midpoint rule

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}, \tag{1.1.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, T and f remain as defined in (1.1.1). Still in a Hilbert space, in 2015, Yao et al. [107] introduced the iterative sequence

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}, \tag{1.1.6}$$

where T and f are as defined in (1.1.1) and $\alpha_n + \beta_n + \gamma_n = 1 \ \forall \ n \in \mathbb{N}$. Ke and Ma [65] introduced generalized viscosity implicit rules which extend the results of Xu et al. [104] and Yao et al. [107]. The generalized viscosity implicit procedures are given by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right), \ n \in \mathbb{N}, \tag{1.1.7}$$

and

$$y_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n T\left(\delta_n y_n + (1 - \delta_n)y_{n+1}\right), \ n \in \mathbb{N}, \tag{1.1.8}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. Replacement of strict contractions in (1.1.8) by the generalized contractions and extension to uniformly smooth Banach spaces was considered by Yan et al. [106]. Under certain conditions on imposed on the parameters, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of the nonexpansive mapping T, which is also the unique solution of the variational inequality

$$\langle (I - f)p, j(x - p) \rangle \geq 0, \ \text{for all } x \in F(T), \tag{1.1.9}$$

where j is a single valued duality mapping. Then, the following questions which arise are of interest to us:

Problem 1.1.1 *Comparing the viscosity implicit iterative schemes (1.1.5) and (1.1.6) with a non-viscosity implicit sequence such as*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}, \tag{1.1.10}$$

where T is a nonexpansive mapping and $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ certify certain conditions, which one has the highest rate of convergence?

Problem 1.1.2 *Analytically, do the sequences (1.1.7) and (1.1.8) always converge to the same fixed point of a nonexpansive mapping?*

Problem 1.1.3 *Can one generalize the results of Ke and Ma [65] to show that the results hold for finite combination of nonexpansive mappings, composition of finite family of nonexpansive mappings and monotone mappings?*

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Problem 1.1.4 *How to extend the results of Ke and Ma [65] and Yan et al. [106] to the more general class of μ-strictly pseudo-contractive mappings?*

Problem 1.1.5 *Does there exist any implicit iterative algorithm which converges strongly to a fixed point of μ-strictly pseudo-contractive mapping in uniformly smooth Banach spaces?*

1.2 Objectives

1. To study the monotonicity of composition of monotone mappings in Banach spaces as the composition need not be monotone but monotonicity provides a broad analytical framework for the study of nonlinear equations.

2. To study the iterative methods for approximating the solutions of nonlinear equations in Banach spaces since there is no known standard method for finding their solutions.

3. To construct coupled explicit iterative algorithms and also try to establish their strong convergence to the unique solution of nonlinear equations in Banach spaces which are more general than the Hilbert spaces, l_p $(1 < p < \infty)$ spaces and 2-uniformly convex spaces which are already existing in the literature.

4. To provide answers to the questions raised in Section 1.1.2.

1.3 Organization of the thesis

The thesis is divided into five chapters as follows:

In chapter 1, a brief historical background of the study is given. The motivations for the study are clearly expressed. The objectives of the study are itemized and finally, we describe the organization of the thesis.

In chapter 2, we introduce some basic concepts and terms that are used in this thesis. We also state some existing results and classical inequalities which are needed

in establishing our results in this work.

Chapter 3 marks the beginning of our contributions and it comprises of four sections. The concept of generalized Lyapunov function is introduced and proof of some essential lemmas are given in section 1. In section 2, we study the convergence of an iterative algorithm for finding the zeros of the class of (p, t)-strongly monotone maps in p-uniformly convex Banach spaces with uniformly Gâteaux differentiable norm. In section 3, we study the class of strongly monotone mappings in uniformly smooth and uniformly convex Banach spaces and prove a strong convergence theorem for an explicit iterative algorithm. In section 4, we establish strong convergence results for the equations within the class of generalized Φ-strongly monotone mappings and apply the results to obtain the solutions of generalized convex optimization problems.

Chapter 4 consists of two sections. In section 1, we use numerical examples to compare the rate of convergence of implicit midpoint rules, where viscosity is involved with a nonviscous method. A strong convergence result is also established for implicit midpoint procedures. In section 2, we establish the conditions under which two generalized implicit iterative algorithms will converge to the same fixed points of a nonexpansive mapping.

Chapter 5 consists of two sections. In section 1, a generalized contraction is applied to introduce an implicit iterative algorithm for the class of μ-strictly pseudocontractive mappings. Moreover, the strong convergence of our implicit iterative algorithm to a fixed point of μ-strictly pseudocontractive mappings is established, which is also the solution to some variational inequality problems in uniformly smooth Banach spaces. In section 2, we introduce a new iterative algorithm based on generalized contractions for nonexpansive mappings. It is also proved that the newly introduced sequence converges strongly to the fixed point of nonexpansive mappings, which is also the solution to some variational inequality problems.

In chapter 6, the results in the thesis are summarized and the contributions to knowledge are discussed. Some areas of future research are also pointed out.

The definitions of essential concepts that are used in this thesis are introduced in this chapter. Some importants results which are used in establishing the main results are also stated.

2.1 Smooth and convex spaces

Definition 2.1.1

(i) Let X and Y be real normed linear spaces and $F : U \subset X \to Y$ be a map with U open and nonempty. The function F is said to have a *Gâteaux differentiable norm* at $x \in U$ if there exists a bounded linear map from X into Y denoted by $F'(x)$ such that for each h in X, we have

$$\lim_{t \to 0} \frac{F(x + th) - F(x)}{t} = \langle F'(x), h \rangle. \tag{2.1.1}$$

We say that F is Gâteaux differentiable if it has a Gâteaux derivative at each x in U and $F'(x)$ is called the *gradient* of F at x.

(ii) Let E be a normed linear space and $S := \{x \in E : \|x\| = 1\}$. E is said to have

a *Gâteaux differentiable norm* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1.2}$$

exists for each $x, y \in S$.

(iii) A Banach space E is said to be *smooth* if for every $x \neq 0$ in E, there is a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$, where E^* denotes the dual of E. E is *Fréchet differentiable* if it is smooth and the limit (2.1.2) is attained uniformly for $y \in S$. Furthermore, E is said to be *uniformly smooth* if it is smooth and the limit (2.1.2) is attained uniformly for each $x, y \in S$.

(iv) The *modulus of convexity* of a Banach space E, $\delta_E : (0, 2] \to [0, 1]$ is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| > \epsilon \right\}.$$

E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Let $p > 1$, then E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. A normed linear space E is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, x \neq y \Rightarrow \frac{\|x + y\|}{2} < 1.$$

Observe that every *p*-uniformly convex space is uniformly convex and every uniformly convex space is reflexive and strictly convex. Also, it is well known that a space E is uniformly smooth if and only if E^* is uniformly convex.

Remark 2.1.2

It is known that a Banach space E is smooth if and only if its norm is Gâteaux differentiable (Alber and Ryazantseva [8], page 7).

2.2 Duality mappings

Definition 2.2.1

In what follows, E will denote a real Banach space.

(i) Let $\nu : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing function such that $\nu(t) \to \infty$ as $t \to \infty$ and $\nu(0) = 0$ for any $t \in [0, \infty)$. Such a function ν is called a *gauge function*.

(ii) A *duality mapping* associated with the guage function ν is a map $J_\nu : E \to 2^{E^*}$ defined by

$$J_\nu(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\nu(\|x\|), \|f\| = \nu(\|x\|)\},$$

where $\langle ., . \rangle$ denotes the duality pairing.

(iii) If the guage function is defined by $\nu(t) = t$, then the corresponding duality mapping is called the *normalized duality mapping*. Therefore, the normalized duality mapping is given by

$$J(x) = \left\{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\right\}.$$

(iv) For $p > 1$, let $\nu(t) = t^{p-1}$ be a gauge function. $J_p : E \to 2^{E^*}$ is called a *generalized duality mapping* from E into 2^{E^*} and is given by

$$J_p(x) = \left\{f \in E^* : \langle x, f \rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\right\}.$$

For $p = 2$, the mapping J_2 is the normalized duality mapping which is simply written as J. In this work, J will denote the normalized duality mapping except where it is specifically stated otherwise.

Remark 2.2.2

In a Hilbert space, the normalized duality mapping is the identity map.

The following results about the generalized duality mappings are well known which are established in [8, 37, 66, 109, 105]. Let E be a Banach space.

(i) E is smooth if and only if J_p is single-valued;

(ii) If E is reflexive, then J_p is onto;

(iii) If E has uniform Gâteaux differentiable norm, then J_p is norm-to-weak* uniformly continuous on bounded sets;

(iv) E is uniformly smooth if and only if J_p is single valued and uniformly continuous on any bounded subset of E;

(v) If E is strictly convex, then J_p is one-to-one, that is, $\forall\ x, y \in E,\ x \neq y \Rightarrow J_p(x) \cap J_p(y) = \emptyset$;

(vi) If E and E^* are strictly convex and reflexive, then J_p^* is the generalized duality map from E^* to E and $J_p^* = J_p^{-1}$;

(vii) E is uniformly smooth and uniformly convex, the generalized duality map J_p^{-1} is uniformly continuous on any bounded subset of E^*;

(viii) If E and E^* are strictly convex and reflexive, for all $x \in E$ and $f \in E^*$, the equalities $J_p J_p^{-1} f = f$ and $J_p^{-1} J_p x = x$ hold.

2.3   Convex functions

Definition 2.3.1

Let E be a Banach space.

(i) A subset K of E is said to be *convex* if for every $x, y \in K$, and $\lambda \in [0, 1]$, we have

$$\lambda x + (1 - \lambda)y \in K.$$

(ii) A function $f : K \to \mathbb{R}$ defined on a convex subset K of E is *convex* if for any $x, y \in K$ and $\lambda \in [0, 1]$, we have

$$f\left(\lambda x + (1 - \lambda)y\right) \leq \lambda f(x) + (1 - \lambda)f(y).$$

If we have strict inequality for all $x \neq y$ in the above definition, the function is said to be *strictly convex*.

(iii) Let $f : E \to \mathbb{R}$ be a convex function. The *subdifferential* of f at $x \in E$ is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle \forall\ y \in E\}.$$

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(iv) A function $f : K \to \mathbb{R}$ is *quasiconvex* if

$$f\left(\lambda x + (1 - \lambda)y\right) \leq \max\left\{f(x), f(y)\right\}, \quad \forall \ x, y \in K \text{ and } \lambda \in [0, 1].$$

Clearly every convex function is quasiconvex but the converse is not always true. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x - 1, & x \leq 1, \\ \ln x, & x > 1. \end{cases}$$

f is quasiconvex but not convex. Certainly, it is concave (Dodos [41]). A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if for each $\alpha \in E^*$ the function $u \mapsto f(u) + \langle \alpha, u \rangle$ is quasiconvex. A classical tool to study lower semicontinuous functions is the Clarke subdifferential.

(v) Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. The *Clarke subdifferential* of f is the operator $\partial f : E \to E^*$ defined for each $u \in E$ by

$$\partial f(u) \ = \ \begin{cases} \left\{u^* \in E^* : \langle u^*, v \rangle \leq f^\uparrow(u; v), \ \forall \ v \in E\right\}, & \text{if } u \in dom f, \\ \emptyset, & \text{if } u \notin dom f, \end{cases}$$

where

$$f^\uparrow(u; v) := \sup_{\epsilon > 0} \inf_{\substack{\gamma > 0 \\ \delta > 0 \\ \lambda > 0}} \sup_{\substack{x \in B_\gamma(u) \\ f(x) \leq f(u) + \delta \\ t \in (0, \lambda)}} \inf_{y \in B_\epsilon(v)} \frac{f(x + ty) - f(x)}{t}$$

is the Rockafellar directional derivative (see e.g., Aussel et al. [11], Clarke [22], pp. 308, Rockafellar [88]). It is known as an axiom of a subdifferential that if f attains a local minimum at u, then $0 \in \partial f(u)$ (see e.g., J. P. Penot [82]).

Recall that a function having a bounded set range is called a bounded function and given a convex function f, if $u \in \textbf{int} \ dom \ f$, then $\partial f(u)$ is nonempty and bounded, where $\textbf{int} \ dom \ f$ denotes the interior of the domain of f.

Lemma 2.3.2 *Aussel et al. [11]. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function on a Banach space E. Then, ∂f is quasimonotone if and only if f is quasiconvex.*

Lemma 2.3.3 *Alber and Ryazantseva [8], p. 17. If a functional ϕ on the open convex set $M \subset dom \ \phi$ has a subdifferential, then ϕ is convex and lower semicontinuous on the set.*

2.4 Nonexpansive mappings

Definition 2.4.1

Let K be a nonempty closed convex subset of a real Banach space E.

(i) A self-mapping $T : K \to K$ is said to be *Lipschitz* if there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\| \;\; \forall \;\; x, y \in K$.

(ii) If $L = 1$, then T is said to be *nonexpansive.*

(iii) A point $x \in K$ is called a *fixed point* of T if $Tx = x$. We shall denote the set of fixed points of T by $F(T)$.

(iv) If E is smooth, $T : K \to E$ is said to be *firmly nonexpansive type* if

$$\langle Tx - Ty, JTx - JTy \rangle \leq \langle Tx - Ty, Jx - Jy \rangle \text{ for all } x, y \in K,$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping (see e.g., Kohsaka and Takahashi [58]).

(v) Let D be a subset of K and let S be a mapping from K to D. Then S is said to be *sunny* if $S(Sx + t(x - Sx)) = Sx$ whenever $Sx + t(x - Sx) \in K$ for $x \in K$ and $t \geq 0$. A mapping S from K into itself is said to be a *retraction* if $S^2 = S$. A set D is said to be a *sunny nonexpansive retract* of K if there exists a sunny nonexpansive retraction from K into D.

(vi) A mapping $T : E \to 2^{E^*}$ is called *J-pseudocontractive* if for every $x, y \in E$,

$$\langle \omega - \eta, \; x - y \rangle \leq \langle \nu - \mu, \; x - y \rangle \text{ for all } \omega \in Tx, \eta \in Ty, \nu \in Jx, \mu \in Jy.$$

(vii) A point $x \in E$ is called a *J-fixed point* of a mapping $T : E \to 2^{E^*}$ if and only if there exists $\omega \in Tx$ such that $\omega \in Jx$.

Remark 2.4.2

It is well known that if E is a smooth Banach space and K is a nonempty closed convex subset of E, then there exists at most one sunny nonexpansive retraction S from E onto K (see e.g., Cioranescu [37], Takahashi [96]).

2.5 Continuous mappings

Definition 2.5.1

Let X and Y be real Banach spaces and let the map $A : X \to Y$.

(i) A is *uniformly continuous* if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\forall\, x, y \in X \text{ with } \|x - y\| < \delta \text{ we have } \|Ax - Ay\| < \epsilon.$$

(ii) Let $\psi(t)$ be a function on the set $\mathbb{R}^+$ of nonnegative real numbers such that:

- ψ is nondecreasing and continuous;

- $\psi(t) = 0$ if and only if $t = 0$.

A is said to be uniformly continuous if it admits the modulus of continuity ψ such that

$$\|A(x) - A(y)\| \leq \psi(\|x - y\|) \; \forall\, x, y \in X.$$

The modulus of continuity ψ has some useful properties which can be found, for instance ([10], pp. 266-269, [80], [60]).

Properties of modulus of continuity

Let X and Y be real Banach spaces and let $A : X \to Y$ be a map which admits the modulus of continuity ψ.

- Modulus of continuity is subadditive: For all real numbers $t_1 \geq 0, t_2 \geq 0$, we have

$$\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2).$$

- Modulus of continuity is monotonically increasing: If $0 \leq t_1 \leq t_2$ holds for some real numbers t_1, t_2, then

$$0 \leq \psi(t_1) \leq \psi(t_2).$$

- Modulus of continuity is continuous: The modulus of continuity $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous on the set positive real numbers, in particular, the limit of ψ at 0 from above is

$$\lim_{t \searrow 0} \psi(t) = 0.$$

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(iii) Let X and Y be linear topological spaces. A mapping $A : X \to 2^Y$ is said to be *upper semicontinuous* if for each point $x_0 \in X$ and arbitrary neighborhood $\vee$ of Ax_0 in Y, there exists a neighborhood U of x_0 such that for all $x \in U$ one has the inclusion: $Ax \subset \vee$.

(iv) A functional f is called *lower semicontinuous* at the point $x_0 \in dom f$ if for any sequence $x_n \in dom f$ such that $x_n \to x_0$ there holds the inequality

$$f(x_0) \leq \liminf_{n \to \infty} f(x_n). \tag{2.5.1}$$

f is called *weakly lower semicontinuous* at x_0 if the inequality (2.5.1) holds with the condition that the convergence of $\{x_n\}_{n=1}^{\infty}$ to x_0 is weak.

Remark 2.5.2

If a map A is uniformly continuous on a bounded set, then A is bounded.

Lemma 2.5.3 *(See, e.g., Chidume and Djitte [30]). Let X and Y be real normed linear spaces and let $A : X \to Y$ be a uniformly continuous map. For arbitrary $r > 0$ and fixed $x^* \in X$, let*

$$B_X(x^*, r) = \{x \in X : \|x - x^*\|_X \leq r\}.$$

Then $A(B(x^*, r))$ is bounded.

2.6 Lyapunov functions

Definition 2.6.1 *Let E be a smooth real Banach space with the dual E^*.*

(i) The Lyapunov function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2, \text{ for all } x, y \in E, \tag{2.6.1}$$

where J is the normalized duality map from E to E^* (introduced by Alber [7]) and has been studied by Kamimura and Takahashi [52] and Reich [87].

(ii) The map $V : E \times E^* \to \mathbb{R}$ is defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \quad \forall \ x \in E, x^* \in E^*.$$

If $E = H$, a real Hilbert space, then Eq.(2.6.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$.

Lemma 2.6.2 *Kamimura and Takahashi [52]. Let E be a smooth uniformly convex real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences from E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

2.7 Monotone type mappings

Definition 2.7.1

Let E be a real Banach space and $A : E \to E$ be a single-valued mapping. J denotes the normalized duality mapping.

(i) A is *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that
$$\langle Ax - Ay, j(x - y) \rangle \geq 0.$$

(ii) A is *m-accretive* if it is accretive and the range of $(I + tA)$ is all of E for some $t > 0$.

(iii) A satisfies the *range condition* if $D(A) \subseteq R(I + tA)$ for all $t > 0$, where $D(A)$ is the domain of A.

(iv) A mapping $T : E \to E$ is said to be a strong pseudocontraction if there exists $k > 0$ such that for all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that
$$\langle (I - T)x - (I - T)y, j(x - y) \rangle \geq k\|x - y\|^2.$$

$T : E \to E$ is said to be pseudocontractive if for each $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that
$$\langle (I - T)x - (I - T)y, j(x - y) \rangle \geq 0.$$

Remark 2.7.2

Pseudocontractive mappings are firmly connected with the class of accretive mappings. A mapping $T : E \to E$ is pseudocontractive if and only if $A := I - T$ is accretive. It is easy to see that the fixed point of pseudocontractive mapping T is the zero of accretive mapping $A := I - T$.

Remark 2.7.3

It known that if A is m-accretive, then A satisfies the range condition (see e.g., Chidume and Djitte [30]).

Definition 2.7.4

Let E be a smooth Banach space.

(i) The multivalued mapping $A : E \to 2^{E^*}$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\langle x^* - y^*, x - y \rangle \geq 0 \quad \forall \ x^* \in Ax, \ y^* \in Ay.$$

The single valued mapping $A : E \to E^*$ is monotone if for each $x, y \in E$, we have

$$\langle x - y, Ax - Ay \rangle \geq 0.$$

(ii) Let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function such that $\phi(0) = 0$. The mapping $A : E \to 2^{E^*}$ is called *ϕ-strongly monotone* if

$$\langle x^* - y^*, x - y \rangle \geq \|x - y\|\phi(\|x - y\|) \quad \forall \ x^* \in Ax, \ y^* \in Ay.$$

If $\phi(t) = kt$, where k is a positive constant, then mapping A is called strongly monotone (Alber and Ryazantseva [8], page 25). That is, there exists a positive constant k such that

$$\langle x^* - y^*, x - y \rangle \geq k\|x - y\|^2 \quad \forall \ x^* \in Ax, \ y^* \in Ay.$$

(iii) (Chidume and Djitte [31] and Chidume and Shehu [36]): Let $p > 1$, $A : E \to E^*$ is said to be *(p, k)-strongly monotone* if there exist a constant $k > 0$ such that for each $x, y \in E$, we have

$$\langle x - y, Ax - Ay \rangle \geq k\|x - y\|^p.$$

Remark 2.7.5

According to definition of Chidume and Djitte [31] and Chidume and Shehu [36], a strongly monotone mapping is referred to as $(2, k)$-strongly monotone mapping.

(iv) Let $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ be a strictly increasing function. $A : E \to 2^{E^*}$ is said to be generalized Φ-*strongly monotone* if

$$\langle x^* - y^*, x - y \rangle \geq \Phi(\|x - y\|) \ \ \forall \ x^* \in Ax, \ \ y^* \in Ay.$$

(v) $A : E \to 2^{E^*}$ is called *maximal monotone* if it is monotone and its graph is not properly contained in the graph of any other monotone mapping. As a result of Rockafellar [90], it follows that A is maximum monotone if it is monotone and the range of $(J + tA)$ is all of E^* for some $t > 0$.

(vi) Let E be a reflexive smooth strictly convex space and $A : E \supseteq D(A) \to 2^{E^*}$ a maximal monotone mapping (or a mapping satisfying the range condition) and let $x \in E$ be fixed. Then for every $t > 0$, there corresponds a unique element $x_t \in D(A)$ such that

$$Jx \in Jx_t + tAx_t. \tag{2.7.1}$$

Therefore, the *resolvent* of A is defined by $J_t^A x = x_t$. In other words, $J_t^A = (J + tA)^{-1}J$ and $A^{-1}0 = F(J_t^A)$ for all $t > 0$, where $F(J_t^A)$ denotes the set of all fixed points of J_t^A. The resolvent J_t^A is a single-valued mapping from E into $D(A)$ and is nonexpansive if E is a Hilbert space (Kohsaka and Takahashi [57]).

Remark 2.7.6

Observe that any maximal monotone mapping satisfies the range condition. The converse is not necessarily true. Hence, the range condition is weaker than maximal monotonicity.

Clearly, the class of strongly monotone mappings is a subclass of ϕ-strongly monotone mappings (by taking $\phi(t) = kt$) and the class of ϕ-strongly monotone

mappings is a subclass of generalized Φ-strongly monotone mappings (by taking $\Phi(t) = t\phi(t)$). It is a well known fact that the class of the generalized Φ-strongly monotone mappings is the largest class of monotone-type mappings such that if a solution of the equation $0 \in Ax$ exists, it is necessarily unique (Chidume et al. [35]). We recall some important generalized monotonicity properties which have been studied for multivalued mappings. Let E be a real locally convex topological vector space and E^* be the dual space. Suppose $K \subseteq E$ is a nonempty subset of E and $A : K \rightarrow 2^{E^*}$ is a multivalued mapping. For each $x, y \in K$, A is said to be respectively pseudomonotone and quasimonotone (see e.g., Karamardian and Schaible [53], Karamardian et al. [54]), if for any $x^* \in A(x)$, $y^* \in A(y)$ the following implications hold:

$$\langle y^*, x - y \rangle \geq 0 \Rightarrow \langle x^*, x - y \rangle \geq 0,$$

and

$$\langle y^*, x - y \rangle > 0 \Rightarrow \langle x^*, x - y \rangle \geq 0. \tag{2.7.2}$$

Also, A is said to be quasimonotone if

$$\min \left\{ \langle x^*, x - y \rangle, \langle y^*, x - y \rangle \right\} \leq 0. \tag{2.7.3}$$

The two definitions of quasimonotonicity coincide (see e.g., Penot and Quang [83]). It is clear that a monotone mapping is pseudomonotone, while a pseudomonotone mapping is quasimonotone. The converse is not necessarily true. In the case of a single-valued linear mapping A, defined on E (where $E := \mathbb{R}^n$), for $\alpha \in E^* \setminus \{0\}$, it is known that if $A + \alpha$ is quasimonotone, then A is monotone (see e.g., Karamardian et al. [54]). This result has been extended by several authors (see, e.g., Hadjisavvas [44], He [48], Isac and Motreanu [61]). In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, monotonicity and accretivity coincide.

2.8   Surjective property of bounded linear functions

Let E and F be two real locally convex topological vector spaces and K a nonempty convex subset of E. Let θ_F denotes the zero vector of F and $T : K \rightarrow L(E, F)$ a set-

valued mapping, where $L(E, F)$ denotes the space of all continuous linear mappings from E into F.

Definition 2.8.1

Recall from Farajzadeh and Plubtieng [43], for x and y in K, $S \subseteq L(E, F)$ is said to have the *surjective property* on $[x, y] = \{x + t(y - x) : t \in [0, 1]\}$ (for short, on x and y) whenever the following equality holds:

$$\langle S, x - y \rangle := \{\langle x^*, x - y \rangle = x^*(x - y) : x^* \in S\} = F,$$

where $\langle x^*, x - y \rangle = x^*(x - y)$ denotes the value of x^* at $(x - y)$.

$S \subseteq L(E, F)$ is said to have the surjective property on K if for every $x \in K$ there exists $y \in K$ such that S has the surjective property on x and y. For $x, y \in K$, consider $x - y$ as a linear functional (denoted by $\widehat{x - y}$) on $L(E, F)$ as follows:

$$\left\langle \widehat{x - y}, f \right\rangle = \langle f, x - y \rangle,$$

where $f \in L(E, F)$. Thus, the surjective property of $S \subseteq L(E, F)$ on x, y implies that the image of S under the linear functional $\widehat{x - y}$ is F. Let S have the surjective property on x, y and $f \in F^* \backslash \{\theta_{F^*}\}$. Then, a set value mapping $foS \subset L(E, \mathbb{R}) = 2^{E^*}$ has the surjective property on x, y. Indeed,

$$\langle foS, x - y \rangle := \{\langle fox^*, \ x - y \rangle : x^* \in S\} = f(F) = \mathbb{R}.$$

Definition 2.8.2

A set of real numbers K is called disconnected if there exist two open subsets of $\mathbb{R}$, say U and V such that

   (i) $K \cap U \cap V = \emptyset$;

  (ii) $K \subseteq U \cup V$;

 (iii) $K \cap U \neq \emptyset$;

 (iv) $K \cap V \neq \emptyset$.

In such a case, we say U and V form a disconnection of K (or we simply say they disconnect K). A set of real numbers K is called *connected* if it is not disconnected. The set of integer $\mathbb{Z}$ is disconnected. Indeed, choose $U = (-\infty, 0)$ and $V = (0.5, \infty)$ respectively.

Lemma 2.8.3 *Farajzadeh et al. [42]. Let E be a real topological vector space, K a nonempty convex subset of E and $A : K \to 2^{E^*}$ a multivalued mapping. Assume $S \subseteq E^*$ is connected and has the surjective property on K. If $A + \alpha$ is quasimonotone for all $\alpha \in S$, then A is monotone on K.*

2.9 Contraction mappings

Definition 2.9.1 *Let (E, d) be a metric space and K a subset of E with $f : K \to K$ a mapping defined on K.*

(i) *f is called a contraction if there exists $c \in [0, 1)$ such that*

$$d(f(x), f(y)) \leq cd(x - y) \text{ for all } x, y \in K.$$

A contraction mapping f will be referred to as c-contraction mapping. Π_K will denote the collection of contraction mapping defined on K.

(ii) *$f : K \to K$ is said to be a Meir-Keeler contraction if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in K$, with $\epsilon \leq d(x, y) < \epsilon + \delta$, we have $d(f(x), f(y)) < \epsilon$.*

(iii) *Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{R}^+$ the set of all positive real numbers. A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an L-function if $\psi(0) = 0$, $\psi(t) > 0$ for all $t > 0$ and for every $s > 0$, there exists $u > s$ such that $\psi(t) \leq s$ for each $t \in [s, u]$.*

(iv) *$f : E \to E$ is called a (ψ, L)-contraction if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an L-function and $d(f(x), f(y)) < \psi(d(x, y))$, for all $x, y \in E$, $x \neq y$.*

The following are the interesting results about Meir-Keeler contractions.

Proposition 2.9.2 *Suzuki [100]. Let E be a Banach space, K a convex subset of E and $f : K \to K$ a Meir-Keeler contraction. Then $\forall \, \epsilon > 0$, there exists $c \in (0, 1)$ such that*

$$\|f(x) - f(y)\| \leq c\|x - y\| \tag{2.9.1}$$

for all $x, y \in K$ with $\|x - y\| \geq \epsilon$.

Proposition 2.9.3 *Lim [69]. Let (E, d) be a metric space and $f : E \to E$ be a mapping. The following assertions are equivalent:*

(i) f is a Meir-Keeler type mapping;

(ii) there exists an L-function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is a (ψ, L)-contraction.

Proposition 2.9.4 *Lim [69]. Let K be a nonempty convex subset of a Banach space E, $T : K \to K$ a nonexpansive mapping and $f : K \to K$ a Meir-Keeler contraction. Then Tf and $fT : K \to K$ are Meir-Keeler contractions.*

Throughout this dissertation, the generalized contraction mappings will refer to Meir-Keeler or (ψ, L)-contractions. It is assumed that the L-function from the definition of (ψ, L)-contraction is continuous, strictly increasing and $\lim_{t \to \infty} \phi(t) = \infty$, where $\phi(t) = t - \psi(t)$ for all $t \in \mathbb{R}^+$. Whenever there is no confusion, $\phi(t)$ and $\psi(t)$ will be written as $\phi \, t$ and $\psi \, t$, respectively.

The results below about contractions and generalized contraction mappings are very essential.

Lemma 2.9.5 *Xu [102]. Let E be a uniformly smooth Banach space, K be a closed convex subset of E, $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $Q \in \Pi_K$. Then the sequence $\{x_t\}$ defined by $x_t = tQ(x_t) + (1 - t)Tx_t$ converges strongly to a point in $F(T)$. If we define a mapping $S : \Pi_K \to F(T)$ by $S(Q) := \lim_{t \to 0} x_t, \, \forall \, Q \in \Pi_K$, then $S(Q)$ solves the following variational inequality:*

$$\langle (I - Q)S(Q), J(S(Q) - p) \rangle \leq 0, \quad \forall \, Q \in \Pi_K.$$

Lemma 2.9.6 *Wong et al. ([99], Lemma 2.12). Let E be a Banach space with a uniformly Gâteaux differentiable norm, let K be a nonempty, closed and convex*

subset of E, let $Q : K \to K$ be a continuous operator, let $T : K \to K$ be a nonexpansive operator and $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Suppose that $\{z_t\}$ is a path in K defined by $z_t = tQ(z_t) + (1-t)Tz_t, t \in (0,1)$ such that $z_t \to z$ as $t \to 0^+$. Then

$$\limsup_{n\to\infty} \langle Q(z) - z, J(x_n - z) \rangle \leq 0.$$

2.10   Rate of convergence

Definition 2.10.1

Berinde [18].   Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be two sequences of real numbers that converge to u and v respectively, and assume that

$$l = \lim_{n\to\infty} \frac{|u_n - u|}{|v_n - v|} \quad \text{exist.} \tag{2.10.1}$$

(i) If $l = 0$, then we say that $\{u_n\}_{n=1}^{\infty}$ converges faster to u than $\{v_n\}_{n=1}^{\infty}$ to v.

(ii) If $0 < l < \infty$ then we say that $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ have the same rate of convergent.

(iii) If $l = \infty$, then we say that $\{v_n\}_{n=1}^{\infty}$ converges faster to v than $\{u_n\}_{n=1}^{\infty}$ to u.

Definition 2.10.2

Berinde [18]. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point p on a normed space X such that the error estimates

$$\|x_n - p\| \leq u_n, \ n \in \mathbb{N} \tag{2.10.2}$$

and

$$\|y_n - p\| \leq v_n, \ n \in \mathbb{N} \tag{2.10.3}$$

are available, where $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are two null sequences of positive numbers (that is, sequences of positive numbers that have zero as their limit). If $\{u_n\}_{n=1}^{\infty}$ converges faster than $\{v_n\}_{n=1}^{\infty}$, then we say that $\{x_n\}_{n=1}^{\infty}$ converges faster to p than $\{y_n\}_{n=1}^{\infty}$.

The following results are well known about the sequences of non-negative real numbers.

Lemma 2.10.3 *Tan and Xu [97]. Let $\{\alpha_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$\alpha_{n+1} \le \alpha_n + \sigma_n, n \ge 0$$

such that $\sum_{n=0}^{\infty} \sigma_n < \infty$. Then $\lim_{n\to\infty} \alpha_n$ exists.

Remark 2.10.4

It is worth stating that if in addition the sequence $\{\alpha_n\}$ has a subsequence that converges to 0, then $\{\alpha_n\}$ converges to 0.

Lemma 2.10.5 *Xu [102]. Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \theta_n)a_n + \gamma_n, \ n \ge 0,$$

where $\{\theta_n\}$ is a sequence in $(0,1)$ and γ_n is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \theta_n = \infty$, *and*

(ii) $\limsup_{n\to\infty} \dfrac{\gamma_n}{\theta_n} \le 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10.6 *Xu [103]. Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relations:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \in \mathbb{N},$$

where

(i) $\{\alpha\}_n \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup \{\sigma\}_n \le 0$;

(iii) $\gamma_n \ge 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10.7 *Suzuki [95]. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be bounded sequences in a Banach space E and $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ with $0 < \liminf\limits_{n\to\infty} t_n \leq \limsup\limits_{n\to\infty} t_n < 1$. Suppose that $u_{n+1} = (1-t_n)u_n + t_n v_n$ for all $n \geq 0$ and $\limsup\limits_{n\to\infty} (\|u_{n+1} - u_n\| - \|v_{n+1} - v_n\|) \leq 0$. Then $\lim\limits_{n\to\infty} \|u_n - v_n\| = 0$.*

2.11 Cauchy-Schwartz's inequality

Let E be a topological real vector space and T a multivalued mapping from E into 2^{E^*}. Cauchy-Schwartz's inequality is given by

$$|\langle x, y^* \rangle| \leq \langle x, x^* \rangle^{\frac{1}{2}} \langle y, y^* \rangle^{\frac{1}{2}}, \tag{2.11.1}$$

for any x and y in $D(T)$ and any choice of $x^* \in Tx$ and $y^* \in Ty$ (Zarantonello [108]). The Cauchy-Schwarz's inequality is also called Cauchy's inequality, Cauchy-Bunyakovsky-Schwarz's inequality or Schwarz's inequality.

Iterative algorithms for monotone type mappings

This chapter focuses on the study of iterative methods for monotone type mappings. New mappings are introduced. We establish strong convergence theorems for monotone type mappings in different spaces such as p-uniformly convex Banach spaces with uniformly Gâteaux differentiable norm and also uniformly smooth and uniformly convex Banach spaces. We shall make use of the following result in this section.

Theorem 3.0.1 *Xu [101]. Let E be a real uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, \infty) \to [0, \infty), \quad g(0) = 0,$$

such that for every $x, y \in B_r(0), j_p(x) \in J_p(x), j_p(y) \in J_p(y)$, the following inequalities hold:

(i) $\|x + y\|^p \geq \|x\|^p + p \langle y, j_p(x) \rangle + g(\|y\|);$

(ii) $\langle x - y, j_p(x) - j_p(y) \rangle \geq g(\|x - y\|).$

3.1 Generalized Lyapunov functions

[1] The concept of generalized Lyapunov function is introduced in this section. We state and give the proof of some lemmas which are useful in establishing our main results.

Definition 3.1.1

Let E be a smooth real Banach space and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(i) We introduce a function $\phi_p : E \times E \to \mathbb{R}$ defined by

$$\phi_p(x, y) = \frac{p}{q} \|x\|^q - p \langle x, J_p y \rangle + \|y\|^p, \text{ for all } x, y \in E,$$

where J_p is a generalized duality map from E to E^*.

(ii) We introduce a function $V_p : E \times E^* \to \mathbb{R}$ defined as

$$V_p(x, x^*) = \frac{p}{q} \|x\|^q - p \langle x, x^* \rangle + \|x^*\|^p \quad \forall \ x \in E, x^* \in E^*.$$

Lemma 3.1.2 *Let E be a smooth uniformly convex real Banach space and $p > 1$ be an arbitrarily real number. For $d > 0$, let $B_d(0) := \{x \in E : \|x\| \leq d\}$. Then for arbitrary $x, y \in B_d(0)$,*

$$\|x - y\|^p \geq \phi_p(x, y) - \frac{p}{q} \|x\|^q, \ \text{where} \ \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Since E is a uniformly convex space, then by Theorem 3.0.1, we have for arbitrary $x, y \in B_d(0)$,

$$\|x + y\|^p \geq \|x\|^p + p \langle y, J_p x \rangle + g(\|y\|).$$

Replacing y by $-y$ gives

$$\|x - y\|^p \geq \|x\|^p - p \langle y, J_p x \rangle + g(\|y\|).$$

---

[1]The results of this section are contents of the following paper

- M. O. Aibinu and O. T. Mewomo [3]

Interchanging x and y, we have

$$
\begin{aligned}
\|x - y\|^p &\geq \|y\|^p - p\langle x, J_p y\rangle + g(\|x\|) \\
&\geq \frac{p}{q}\|x\|^q - p\langle x, J_p y\rangle + \|y\|^p - \frac{p}{q}\|x\|^q + g(\|x\|) \\
&\geq \phi_p(x, y) - \frac{p}{q}\|x\|^q + g(\|x\|) \\
&\geq \phi_p(x, y) - \frac{p}{q}\|x\|^q.
\end{aligned}
$$

∎

Lemma 3.1.3 *Let E be a strictly convex and uniformly smooth real Banach space and $p > 1$. Then*

$$
V_p(x, x^*) + p\langle J_p^{-1} x^* - x, y^*\rangle \leq V_p(x, x^* + y^*) \tag{3.1.1}
$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Proof. Let $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

$$
\begin{aligned}
V_p(x, x^*) &= \frac{p}{q}\|x\|^q - p\langle x, x^*\rangle + \|x^*\|^p, \\
V_p(x, x^* + y^*) &= \frac{p}{q}\|x\|^q - p\langle x, x^* + y^*\rangle + \|x^* + y^*\|^p. \\
V_p(x, x^* + y^*) - V_p(x, x^*) &= -p\langle x, y^*\rangle + \|x^* + y^*\|^p - \|x^*\|^p \\
&\geq p\langle -x, y^*\rangle + \|x^*\|^p + p\langle y^*, J_p^{-1} x^*\rangle + g(\|y^*\|) - \|x^*\|^p \\
&\qquad (\text{ by Theorem 3.0.1, since } E^* \text{ is uniformly convex}) \\
&= p\langle J_p^{-1} x^* - x, y^*\rangle + \|x^*\|^p + g(\|y^*\|) - \|x^*\|^p \\
&\geq p\langle J_p^{-1} x^* - x, y^*\rangle,
\end{aligned}
$$

which implies that

$$
V_p(x, x^*) + p\langle J_p^{-1} x^* - x^*, y^*\rangle \leq V_p(x, x^* + y^*).
$$

∎

Lemma 3.1.4 *Let E be a reflexive strictly convex and smooth real Banach space and $p > 1$. Then*

$$
\phi_p(y, x) - \phi_p(y, z) \geq p\langle z - y, J_p x - J_p z\rangle = p\langle y - z, J_p z - J_p x\rangle \quad \text{for all } x, y, z \in E. \tag{3.1.2}
$$

Proof. Let $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We first show that V_p has a subdifferential on an open subset $M \subset \text{dom } V_p$. For every $h \in E^*$ and $t \in \mathbb{R} \setminus \{0\}$ with a variable y^* and a fixed element y in E, we have,

$$
\begin{aligned}
V_p(y, y^*) &= \frac{p}{q} \|y\|^q - p \langle y, y^* \rangle + \|y^*\|^p, \\
V_p(y, y^* + th) &= \frac{p}{q} \|y\|^q - p \langle y, y^* + th \rangle + \|y^* + th\|^p \\
&\geq \frac{p}{q} \|y\|^q - p \langle y, y^* \rangle - pt \langle y, h \rangle + \|y^*\|^p \\
&\quad + pt \langle J_p^{-1} y^*, h \rangle + g(\|th\|) \\
&= V_p(y, y^*) - pt \langle y, h \rangle + pt \langle J_p^{-1} y^*, h \rangle + g(\|th\|),
\end{aligned}
$$

then $\displaystyle \lim_{t \to 0} \frac{V_p(y, y^* + th) - V_p(y, y^*)}{t} \geq p \langle J_p^{-1} y^* - y, h \rangle.$

Therefore, $\text{grad } V_p(x, y) = p(J_p^{-1} y^* - y)$ and by the Lemma 2.3.3, V_p is convex and lower semicontinuous. Then it follows from the definition of subdifferential that

$$
V_p(y, x^*) - V_p(y, z^*) \geq p \langle J_p^{-1} z^* - y, x^* - z^* \rangle \quad \text{for all } y \in E, x^*, z^* \in E^*.
$$

Since $\phi_p(y, x) = V_p(y, J_p x^*)$, we have

$$
\phi_p(y, x) - \phi_p(y, z) \geq p \langle z - y, J_p x - J_p z \rangle \quad \text{for all } x, y, z \in E.
$$

■

Remark 3.1.5

These remarks follow from Definition 3.1.1:

(i) If E is a smooth reflexive strictly convex space, it is obvious that

$$
V_p(x, x^*) = \phi_p(x, J_p^{-1} x^*) \quad \forall \ x \in E, x^* \in E^*. \tag{3.1.3}
$$

Clearly, for $x \in E, x^* \in E^*$,

$$
\begin{aligned}
\phi_p(x, J_p^{-1} x^*) &= \frac{p}{q} \|x\|^q - p \langle x, J_p \left(J_p^{-1} x^* \right) \rangle + \|J_p^{-1} x^*\|^p \\
&= \frac{p}{q} \|x\|^q - p \langle x, x^* \rangle + \|x^*\|^p \\
&= V_p(x, x^*).
\end{aligned}
$$

(ii) For $p = 2, \phi_2(x, y) = \phi(x, y)$, which is the Definition 2.6.1 (i) of Alber [7], given by

$$\phi(x, y) = \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2, \text{ for all } x, y \in E.$$

Also, it is easy to see from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \text{ for all } x, y \in E. \tag{3.1.4}$$

Indeed,

$$
\begin{aligned}
(\|x\| - \|y\|)^2 &= \|x\|^2 - 2\|x\|\|y\| + \|2\|^2 \\
&\leq \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2 \\
&= \phi(x, y) \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|2\|^2 \\
&= (\|x\| + \|y\|)^2.
\end{aligned}
$$

By similar analysis, interested readers can verify that for each $p \geq 2$,

$$(\|x\| - \|y\|)^p \leq \phi_p(x, y) \leq (\|x\| + \|y\|)^p \text{ for all } x, y \in E. \tag{3.1.5}$$

3.2 Algorithm for zeros of monotone maps in Banach spaces

[2] Let $p > 1, t > 0$, we study the convergence of (p, t)-strongly monotone maps in p-uniformly convex Banach spaces with uniformly Gâteaux differentiable norm. The set of zeros of a mapping A is denoted by $N(A) := \{x \in D(A) : Ax = 0\} = A^{-1}0$. The following result is well known for p-uniformly Banach convex spaces.

Lemma 3.2.1 *Xu [101]: Let $p > 1$ be a fixed real number and E a real Banach space. The following are equivalent:*

(i) E is p-uniformly convex;

(ii) there is a constant $c_1 > 0$ such that for all $x, y \in E$ and $j_p(x) \in J_p(x)$,

$$\|x + y\|^p \geq \|x\|^p + p \langle y, j_p(x) \rangle + c_1 \|y\|^p;$$

(iii) there is a constant $c_2 > 0$ such that

$$\langle x - y, j_p(x) - j_p(y) \rangle \geq c_2 \|x - y\|^p, \forall\ x, y \in X\ \text{and}\ j_p(x) \in J_p(x), j_p(y) \in J_p(y).$$

3.2.1 Background

Let E be a real Banach space and let E^* be the dual space of E. We study the methods of approximating the zeros of a nonlinear equation of the form

$$0 \in Au, \tag{3.2.1}$$

where $u \in E$ and $A : E \to 2^{E^*}$ is a multivalued monotone mapping. This is a general form for problems of minimization of a function, variational inequalities and so on. Assuming existence, for approximating a solution of $Au = 0$, where A is a single valued accretive-type mapping, Browder [19] defined an operator $T : E \to E$ by $T := I - A$, where I is the identity map on E. He called such an operator, pseudo-contractive. It is trivial to observe that the zeros of A correspond to fixed points of T. Chidume [25] defined a sequence of iteration

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n,\ n \in \mathbb{N}$$

where $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ and imposed suitable conditions which made the sequence to converge strongly to the unique fixed point of a Lipschitz strongly pseudo-contractive mapping T in L_p, $2 \leq p < \infty$, spaces. The result of Chidume [25] has been generalized and extended in various directions by numerous authors (see e.g., Censor and Reich [23]; Chidume [26], [30]; Chidume and Bashir [27]; Chidume and Chidume [29]; Chidume and Osilike [56]). Recently, Diop et al [40] introduced an iterative scheme for finding the zeros of a bounded and 2-strongly monotone mapping $A : E \to E^*$ in a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm. A sequence $\{x_n\}_{n=1}^{\infty}$ was defined from an arbitrary $x_1 \in E$ by

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n A x_n), n \in \mathbb{N}, \tag{3.2.2}$$

where J is the normalized duality mapping from E into E^* and $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$. The iteration (3.2.2) was proved to converges strongly to the unique solution of the equation $Ax = 0$ under suitable conditions.

It is our purpose in this section to extend and improve on the existing results in this direction. Let $p > 1$, in a p-uniformly convex real Banach space with uniformly Gâteaux differentiable norm, we shall study the convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ defined from an arbitrary $x_1 \in E$ by

$$x_{n+1} = J_p^{-1}(J_p x_n - \lambda_n A x_n), n \in \mathbb{N}, \tag{3.2.3}$$

where J_p is a generalized duality mapping from E into E^*, $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ and $A : E \to E^*$ is a bounded (p,t)-strongly monotone mapping with $t > 0$. As corollaries, we obtain the results of Diop et al. [40] for $p = 2$ and Chidume et al. [28] for $E := L_p, 1 < p < \infty$ and $\lambda_n = \lambda \ \forall \ n \in \mathbb{N}, \ \lambda \in (0,1)$.

3.2.2 Main result

Theorem 3.2.2 *Let $p > 1$ and E be a p-uniformly convex real Banach space with uniformly Gâteaux differentiable norm. Let $t > 0$ and $A : E \to E^*$ be a bounded (p,t)-strongly monotone mapping such that $A^{-1}0 \neq \emptyset$. Suppose that the inverse duality map J_p^{-1} is Lipschitz continuous. For arbitrary $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined iteratively by (3.2.3) with $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ satisfying the following conditions:*
(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$; (ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$.
Then the sequence $\{x_n\}$ converges strongly to a solution of the equation $Ax = 0$.

Proof. Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in E$ be a solution of the equation $Ax = 0$. There exists $r > 0$ sufficiently large such that:

$$r \geq \max\left\{4\frac{p}{q}\|x\|^q, \phi_p(x_1, x)\right\} \ \text{and} \ \gamma_0 := \min\left\{1, \frac{ptr}{4M_0}\right\}, \tag{3.2.4}$$

where

$$M_0 := pL \sup\left\{\|Ax_n\|^2 : \|x_n\| \leq r^{\frac{1}{p}} + \|x\|\right\},$$

L is a Lipschitz constant of J_p^{-1} and A is a bounded map. We divide the proof into two steps.

Step 1: We prove that $\{x_n\}_{n=1}^{\infty}$ is bounded. It suffices to show that $\phi_p(x, x_n) \leq r, \forall\, n \in \mathbb{N}$. The proof is by induction. By construction, $\phi_p(x, x_1) \leq r$. Assume that $\phi_p(x, x_n) \leq r$ for some $n \in \mathbb{N}$. We show that $\phi_p(x, x_{n+1}) \leq r, \forall\, n \in \mathbb{N}$.

From inequality (3.1.5), we have $\|x_n\| \leq r^{\frac{1}{p}} + \|x\|$. We compute as follow by using the definition of x_{n+1}:

$$
\begin{aligned}
\phi_p(x, x_{n+1}) &= \phi_p\left(x, J_p^{-1}(J_p x_n - \lambda_n A x_n)\right)\\
&= V_p\left(x, J_p x_n - \lambda_n A x_n\right)\\
&\leq V_p(x, J_p x_n) - p\lambda_n \left\langle J_p^{-1}(J_p x_n - \lambda_n A x_n) - x, A x_n\right\rangle\\
&\qquad \text{(by Lemma (3.1.3) with } y^* = \lambda_n A x_n)\\
&= V_p(x, J_p x_n) - p\lambda_n \left\langle J_p^{-1}(J_p x_n - \lambda_n A x_n) - x, A x_n - Ax\right\rangle\\
&\qquad \text{(since } x \in N(A))\\
&= \phi_p(x, x_n) - p\lambda_n \left\langle x_n - x, A x_n - Ax\right\rangle\\
&\qquad - p\lambda_n \left\langle J_p^{-1}(J_p x_n - \lambda_n A x_n) - J_p^{-1}(J_p x_n), A x_n\right\rangle.
\end{aligned}
$$

Using the (p, t)-strongly monotonicity property of A, Schwartz's inequality and Lipschitz property of J_p^{-1}, we obtain

$$
\begin{aligned}
\phi_p(x, x_{n+1}) &\leq \phi_p(x, x_n) - pt\lambda_n \|x_n - x\|^p\\
&\qquad + p\lambda_n \|J_p^{-1}(J_p x_n - \lambda_n A x_n) - J_p^{-1}(J_p x_n)\|\,\|A x_n\|\\
&\leq \phi_p(x, x_n) - pt\lambda_n \|x_n - x\|^p + p\lambda_n^2 L \|A x_n\|^2\\
&\leq \phi_p(x, x_n) - pt\lambda_n \left(\phi_p(x, x_n) - \frac{p}{q}\|x\|^q\right) + \lambda_n^2 M_0\\
&\qquad \text{(using Lemma 3.1.2)}\\
&\leq \phi_p(x, x_n) - pt\lambda_n \phi_p(x, x_n) + pt\lambda_n \left(\frac{p}{q}\|x\|^q\right) + \lambda_n \gamma_0 M_0\\
&\leq r - pt\lambda_n r + pt\lambda_n \frac{r}{4} + pt\lambda_n \frac{r}{4}\\
&= \left(1 - \frac{pt\lambda_n}{2}\right) r\\
&< r.
\end{aligned}
$$

Hence, $\phi_p(x, x_{n+1}) \leq r$. By induction, $\phi_p(x, x_n) \leq r\ \forall\, n \in \mathbb{N}$. Thus, from inequality (3.1.5), $\{x_n\}_{n=1}^{\infty}$ is bounded.

Step 2: We now prove that $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique point $x \in A^{-1}0$. Following the same arguments as in step 1, the boundedness of $\{x_n\}_{n=1}^{\infty}$ and that of A, there exists a positive constant M_0 such that

$$\phi_p(x, x_{n+1}) \leq \phi_p(x, x_n) - pt\lambda_n \|x_n - x\|^p + \lambda_n^2 M_0. \tag{3.2.5}$$

Consequently, $\phi_p(x, x_{n+1}) \leq \phi_p(x, x_n) + \lambda_n^2 M_0$.

By the hypothesis that $\sum_{n=0}^{\infty} \lambda_n^2 < \infty$ and Lemma 2.10.3, we have that $\lim_{n \to \infty} \phi_p(x, x_n)$ exists. From inequality (3.2.5), we have $\sum_{n=0}^{\infty} \lambda_n \|x_n - x\| < \infty$. Using the fact $\sum_{n=0}^{\infty} \lambda_n = \infty$, it follows that $\liminf \|x_n - x\|^p = 0$. Consequently, there exists a subsequence $\{x_{n_k}\}_{n_k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \to x$ as $k \to \infty$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded and J_p is norm-to-weak* uniformly continuous on bounded subset of E, it follows that $\{\phi_p(x, x_n)\}_{n=1}^{\infty}$ has a subsequence that converges to 0. Therefore, by Lemma 2.6.2, $\{\phi_p(x, x_n)\}_{n=1}^{\infty}$ converges strongly to 0. Also, by Lemma 2.6.2, $\|x_n - x\| \to 0$ as $n \to \infty$. ∎

Corollary 3.2.3 *Diop et al. [40]: Let E be a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm and E^* its dual space. Let $A : E \to E^*$ be a bounded and $(2, t)$-strongly monotone mapping such that $A^{-1}0 \neq \emptyset$, where $t \in (0, 1)$. For arbitrary $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined iteratively by:*

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n A x_n), n \in \mathbb{N}, \tag{3.2.6}$$

where J is the normalized duality mapping from E into E^* and $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ is a real sequence satisfying the following conditions:
(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$; (ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$.
Then, there exists $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique solution of the equation $Ax = 0$.

Proof. By taking $p = 2$, the proof follows from Theorem 3.2.2. ∎

3.3 Strong convergence theorems for strongly monotone mappings in Banach spaces

Lemma 3.3.1 *Xu [101]. Let E be a real uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, \infty) \to [0, \infty), \quad g(0) = 0,$$

such that for every $x, y \in B_r(0), j_p(x) \in J_p(x), j_p(y) \in J_p(y)$, the following inequalities hold:

(i) $\|x + y\|^p \geq \|x\|^p + p \langle y, j_p(x) \rangle + g(\|y\|);$

(ii) $\langle x - y, j_p(x) - j_p(y) \rangle \geq g(\|x - y\|).$

3.3.1 Background

Let H be a real Hilbert space and $A : D(A) \subset H \to 2^H$ a maximal monotone mapping. Consider the following problem:

$$\text{find } u \in H \text{ such that } 0 \in Au. \tag{3.3.1}$$

This is a typical way of formulating many problems in nonlinear analysis and optimization. A well-known method for solving (3.3.1) in a Hilbert space is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n \in \mathbb{N},$$

introduced by Martinet [73] and studied further by Rockafellar [90] and a host of other authors. How to extend the monotonicity definition to mappings from a

---

Banach space into its dual was a puzzle in nonlinear functional analysis. Alber [7] introduced a Lyapunov functions which signaled the beginning of the development of new geometric properties in Banach spaces. The Lyapunov function introduced by Alber is suitable for studying iterative methods for approximating solutions of equation $0 \in Au$ where $A : E \to 2^{E^*}$ is of monotone type from a Banach space into its dual (see e.g [5], [28], [77], [111]).

In this section, our purpose is to establish a strong convergence theorem for an iterative scheme for the (p, η)-strongly monotone mappings in uniformly smooth and uniformly convex Banach spaces.

3.3.2 Main result

Theorem 3.3.2 *Let E be a uniformly smooth and uniformly convex real Banach space. Let $p > 1, \eta \in (1, \infty)$, suppose $A : E \to E^*$ is a bounded, (p, η)-strongly monotone mapping such that the range of $(J_p + tA)$ is all of E^* for all $t > 0$ and $A^{-1}(0) \neq \emptyset$. Let $\{\lambda_n\}_{n=1}^\infty \subset (0, 1)$ and $\{\theta_n\}_{n=1}^\infty$ in $(0, \frac{1}{2})$ be real sequences such that,*

(i) $\lim_{n \to \infty} \theta_n = 0$ *and $\{\theta_n\}_{n=1}^\infty$ is decreasing;*

(ii) $\sum_{n=1}^\infty \lambda_n \theta_n = \infty$;

(iii) $\lim_{n \to \infty} ((\theta_{n-1}/\theta_n) - 1) / \lambda_n \theta_n = 0, \; \sum_{n=1}^\infty \lambda_n < \infty \; \forall \, n \in \mathbb{N}.$

For arbitrary $x_1 \in E$, define $\{x_n\}_{n=1}^\infty$ iteratively by:

$$x_{n+1} = J_p^{-1} \left(J_p x_n - \lambda_n \left(A x_n + \theta_n (J_p x_n - J_p x_1) \right) \right), n \in \mathbb{N}, \qquad (3.3.2)$$

where J_p is the generalized duality mapping from E into E^*. Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to a solution of $Ax = 0$.

Proof. The proof is divided into two parts.

Part 1: The sequence $\{x_n\}_{n=1}^\infty$ is shown to be bounded.

Let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in E$ be a solution of the equation $Ax = 0$. It suffices to show that $\phi_p(x, x_n) \leq r, \; \forall \, n \in \mathbb{N}$. From inequality (3.1.5), for real $p > 1$, we have

$\|x_n\| \leq r^{\frac{1}{p}} + \|x\|$. Let $B := \{z \in E : \phi_p(x, z) \leq r\}$. It is known that A is bounded and J_p is uniformly continuous on bounded subsets of E. Define

$$M_0 := \sup\left\{\|Ax_n + \theta_n(J_p x_n - J_p x_1)\| : \theta_n \in (0, \frac{1}{2}), x_n \in B\right\} + 1.$$

Let ψ denotes the modulus of continuity of J_p^{-1}. Then

$$
\begin{aligned}
\|x_n - x_{n+1}\| &= \|x_n - J_p^{-1}(J_p x_n - \lambda_n (Ax_n + \theta_n(J_p x_n - J_p x_1)))\| \\
&= \|J_p^{-1}(J_p x_n) - J_p^{-1}(J_p x_n - \lambda_n (Ax_n + \theta_n(J_p x_n - Jp x_1)))\| \\
&\leq \psi(|\lambda_n| \|Ax_n + \theta_n(J_p x_n - J_p x_1)\|) \\
&\leq \psi(|\lambda_n| M_0) \\
&\leq \psi(\sup\{|\lambda_n| M_0 : \lambda_n \in (0, 1)\}).
\end{aligned}
\tag{3.3.3}
$$

Since A is bounded and the duality mapping J_p is uniformly continuous on bounded subsets of E, the $\sup\{|\lambda_n| M_0\}$ exists and it is a real number different from infinity. Let $M =: \psi(\sup\{|\lambda_n| M_0\})$ and let $r > 0$ be sufficiently large such that:

$$r \geq \max\left\{\phi_p(x, x_1), 4M_0 M, \frac{4p}{q}\|x\|^q\right\}.\tag{3.3.4}$$

The proof is by induction. By construction, $\phi_p(x^*, x_1) \leq r$. Suppose that $\phi_p(x^*, x_n) \leq r$ for some $n \in \mathbb{N}$. We show that $\phi_p(x^*, x_{n+1}) \leq r$. Applying Lemma 3.1.3 with $y^* := \lambda_n (Ax_n + \theta_n(J_p x_n - J_p x_1))$ and by using the definition of x_{n+1}, we compute as follows,

$$
\begin{aligned}
&\phi_p(x, x_{n+1}) \\
&= \phi_p\left(x, J^{-1}(J_p x_n - \lambda_n (Ax_n + \theta_n(J_p x_n - J_p x_1)))\right) \\
&= V_p(x, J_p x_n - \lambda_n (Ax_n + \theta_n(J_p x_n - J_p x_1))) \text{ (By (3.1.3))} \\
&\leq V_p(x, J_p x_n) \\
&\quad -p\lambda_n \left\langle J_p^{-1}(J_p x_n - \lambda_n (Ax_n + \theta_n(J_p x_n - J_p x_1))) - x, Ax_n + \theta_n(J_p x_n - J_p x_1)\right\rangle \\
&= \phi_p(x, x_n) - p\lambda_n \left\langle x_n - x, Ax_n + \theta_n(J_p x_n - J_p x_1)\right\rangle \\
&\quad -p\lambda_n \left\langle J_p^{-1}(J_p x_n - \lambda_n (Ax_n + \theta_n(J_p x_n - J_p x_1))) - x_n, Ax_n + \theta_n(J_p x_n - J_p x_1)\right\rangle.
\end{aligned}
$$

By Schwartz inequality and by applying inequality (3.3.3), we obtain

$$
\begin{aligned}
\phi_p(x, x_{n+1}) \;\leq\;& \phi_p(x, x_n) - p\lambda_n \langle x_n - x, Ax_n + \theta_n(J_p x_n - J_p x_1) \rangle \\
& + p\lambda_n M_0 M \\
\leq\;& \phi_p(x, x_n) - p\lambda_n \langle x_n - x, Ax_n - Ax \rangle \text{ (since } x \in N(A)) \\
& - p\lambda_n \theta_n \langle x_n - x, J_p x_n - J_p x_1 \rangle + p\lambda_n M_0 M.
\end{aligned}
$$

By Lemma 3.1.4, $p \langle x - x_n, J_p x_n - J_p x_1 \rangle \leq \phi_p(x, x_n) - \phi_p(x, x_1)$. Consequently, $p \langle x - x_n, J_p x_n - J_p x_1 \rangle \leq \phi_p(x, x_n)$. Therefore, using (p, η)-strongly monotonicity property of A, we have,

$$
\begin{aligned}
\phi_p(x^*, x_{n+1}) \;\leq\;& \phi_p(x^*, x_n) - p\eta\lambda_n \|x_n - x^*\|^p - p\lambda_n \theta_n \langle x_n - x^*, J_p x_n - J_p x_1 \rangle \\
& + p\lambda_n M_0 M \\
\leq\;& \phi_p(x^*, x_n) - p\lambda_n \|x_n - x^*\|^p + p\lambda_n \theta_n \langle x^* - x_n, J_p x_n - J_p x_1 \rangle \\
& + p\lambda_n M_0 M \\
\leq\;& \phi_p(x^*, x_n) - p\lambda_n \left(\phi_p(x^*, x_n) - \frac{p}{q} \|x^*\|^q \right) \\
& + p\lambda_n \theta_n \phi_p(x^*, x_n) + p\lambda_n M_0 M \\
=\;& (1 - p\lambda_n)\phi_p(x^*, x_n) + p\lambda_n \left(\frac{p}{q} \|x^*\|^q \right) \\
& + p\lambda_n \theta_n \phi_p(x^*, x_n) + p\lambda_n M_0 M \\
\leq\;& (1 - p\lambda_n)r + p\lambda_n \frac{r}{4} + p\lambda_n \frac{r}{2} + \frac{p\lambda_n}{4} r \\
=\;& \left(1 - p\lambda_n + p\lambda_n \frac{1}{4} + p\lambda_n \frac{1}{2} + p\lambda_n \frac{1}{4} \right) r = r.
\end{aligned}
$$

Hence, $\phi_p(x, x_{n+1}) \leq r$. By induction, $\phi_p(x, x_n) \leq r \;\; \forall \;\; n \in \mathbb{N}$. Thus, from inequality (3.1.5), $\{x_n\}_{n=1}^{\infty}$ is bounded.

Part 2: We now show that $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of $Ax = 0$. (p, η)-strongly monotone implies monotone and the range of $(J_p + tA)$ is all of E^* for all $t > 0$. By Kohsaka and Takahashi [57], since E is a reflexive smooth strictly convex space, we obtain for every $t > 0$ and $x^* \in E$, there exists a unique $x_t \in D(A)$, where $D(A)$ is the domain of A such that

$$
J_p x^* = J_p x_t + tA x_t.
$$

Define $J_t^A x^* := x_t$, in other words, define a single-valued mapping $J_t^A : E \to D(A)$ by $J_t^A = (J_p + tA)^{-1} J_p$. Such a J_t^A is called the resolvent of A. Setting $t := \frac{1}{\theta_n}$, for some $x_1 \in D(A)$ and $y_n = (J_p + \frac{1}{\theta_n}A)^{-1} J_p x_1$, we obtain

$$\theta_n(J_p y_n - J_p x_1) + Ay_n = 0, \quad y_n \to x \in N(A). \tag{3.3.5}$$

Observe that the sequence $\{y_n\}_{n=1}^\infty$ is bounded because it is a convergent sequence. Moreover, $\{x_n\}_{n=1}^\infty$ is bounded and hence $\{Ax_n\}_{n=1}^\infty$ is bounded. Following the same arguments as in part 1, we get,

$$
\begin{aligned}
\phi_p(y_n, x_{n+1}) &\leq \phi_p(y_n, x_n) - p\lambda_n \langle x_n - y_n, Ax_n + \theta_n(J_p x_n - J_p x_1)\rangle \\
&\quad + p\lambda_n M_0 M \\
&\leq \phi_p(y_n, x_n) - p\lambda_n \langle x_n - y_n, Ax_n + \theta_n(J_p x_n - J_p x_1)\rangle \\
&\quad + p\lambda_n M_0 M.
\end{aligned}
$$
$$\tag{3.3.6}$$

By the (p, η)-strongly monotonicity property of A and using Theorem 3.0.1 and Eq. (3.3.5), we obtain,

$$
\begin{aligned}
&\langle x_n - y_n, Ax_n + \theta_n(J_p x_n - J_p x_1)\rangle \\
=\ & \langle x_n - y_n, Ax_n + \theta_n(J_p x_n - J_p y_n + J_p y_n - J_p x_1)\rangle \\
=\ & \theta_n \langle x_n - y_n, J_p x_n - J_p y_n\rangle + \langle x_n - y_n, Ax_n + \theta_n(J_p y_n - J_p x_1)\rangle \\
=\ & \theta_n \langle x_n - y_n, J_p x_n - J_p y_n\rangle + \langle x_n - y_n, Ax_n - Ay_n\rangle \\
\geq\ & \theta_n g(\|x_n - y_n\|) + \eta\|x_n - y_n\|^p \text{ (Since } A \text{ is } (p, \eta)\text{-strongly monotone and by Lemma 3.3.1(ii))} \\
\geq\ & \frac{1}{p}\theta_n \phi_p(y_n, x_n).
\end{aligned}
$$

Therefore, the inequality (3.3.6) becomes

$$\phi_p(y_n, x_{n+1}) \leq (1 - \lambda_n\theta_n)\phi_p(y_n, x_n) + p\lambda_n M_0 M. \tag{3.3.7}$$

Observe that by Lemma 3.1.4, we have

$$
\begin{aligned}
\phi_p(y_n, x_n) &\leq \phi_p(y_{n-1}, x_n) - p\langle y_n - x_n, J_p y_{n-1} - J_p y_n\rangle \\
&= \phi_p(y_{n-1}, x_n) + p\langle x_n - y_n, J_p y_{n-1} - J_p y_n\rangle \\
&\leq \phi_p(y_{n-1}, x_n) + \|J_p y_{n-1} - J_p y_n\|\|x_n - y_n\|. \tag{3.3.8}
\end{aligned}
$$

Let $R > 0$ such that $\|x_1\| \le R, \|y_n\| \le R$ for all $n \in \mathbb{N}$. We obtain from Eq.(3.3.5) that

$$J_p y_{n-1} - J_p y_n + \frac{1}{\theta_n}\left(A y_{n-1} - A y_n\right) = \frac{\theta_{n-1} - \theta_n}{\theta_n}\left(J_p x_1 - J_p y_{n-1}\right).$$

By taking the duality pairing of each side of this equation with respect to $y_{n-1} - y_n$ and by the strong monotonicity of A, we have

$$\langle J_p y_{n-1} - J_p y_n, y_{n-1} - y_n \rangle \le \frac{\theta_{n-1} - \theta_n}{\theta_n}\|J_p x_1 - J_p y_{n-1}\|\|y_{n-1} - y_n\|,$$

which gives,

$$\|J_p y_{n-1} - J_p y_n\| \le \left(\frac{\theta_{n-1}}{\theta_n} - 1\right)\|J_p y_{n-1} - J_p x_1\|. \tag{3.3.9}$$

Using (3.3.8) and (3.3.9), the inequality (3.3.7) becomes

$$
\begin{aligned}
\phi_p(y_n, x_{n+1}) \;\le\; & (1 - \lambda_n \theta_n)\phi_p(y_{n-1}, x_n) \\
& + C\left(\frac{\theta_{n-1}}{\theta_n} - 1\right) + p\lambda_n M_0 M,
\end{aligned}
$$

for some constant $C > 0$. By Lemma 2.10.6, $\phi_p(y_{n-1}, x_n) \to 0$ as $n \to \infty$ and using Lemma 2.6.2, we have that $x_n - y_{n-1} \to 0$ as $n \to \infty$. Since $y_n \to x \in N(A)$, we obtain that $x_n \to x$ as $n \to \infty$. ∎

Let $p > 1, \eta \in (1, \infty)$, suppose $A : E \to E^*$ is a bounded, (p, η)-strongly monotone mapping which satisfies the range condition and $A^{-1}(0) \ne \emptyset$.

Corollary 3.3.3 *Let H be a Hilbert space, $p > 1, \eta \in (1, \infty)$ and suppose $A : H \to H$ is a bounded (p, η)-strongly monotone mapping such that $D(A) \subseteq range\,(I + tA)$ for all $t > 0$. For arbitrary $x_1 \in H$, define the sequence $\{x_n\}_{n=1}^\infty$ iteratively by*

$$x_{n+1} := x_n - \lambda_n A x_n - \lambda_n \theta_n(x_n - x_1), \quad n \in \mathbb{N},$$

where $\{\lambda_n\}_{n=1}^\infty \subset (0, 1)$ and $\{\theta_n\}_{n=1}^\infty$ in $(0, \frac{1}{2})$ are real sequences satisfying the conditions:

(i) $\lim\limits_{n \to \infty} \theta_n = 0$ *and* $\{\theta_n\}_{n=1}^\infty$ *is decreasing;*

(ii) $\sum\limits_{n=1}^\infty \lambda_n \theta_n = \infty;$

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(iii) $\lim\limits_{n\to\infty} \left(\left(\theta_{n-1}/\theta_n\right)-1\right)/\lambda_n\theta_n = 0,\ \sum\limits_{n=1}^{\infty}\lambda_n < \infty\ \forall\ n \in \mathbb{N}.$

Suppose that the equation $Ax = 0$ has a solution. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of the equation $Ax = 0$.

Proof.   The result follows from the Theorem 3.3.2 since uniformly smooth and uniformly convex spaces are more general than the Hilbert spaces.   ■

Remark 3.3.4

The generalized Lyapunov functions which we introduced admit the generalized duality mapping. Clearly, our results show the efficacy of the generalized Lyapunov functions which we introduced. Also, our method of proof is constructive and is of independent interest.

Remark 3.3.5

Prototype for our iteration parameters in Theorem 3.3.2 are, $\lambda_n = \frac{1}{(n+1)^a}$ and $\theta_n = \frac{1}{(n+1)^b}$, where $0 < b < a$ and $a + b < 1$.

3.4   Algorithm for the generalized Φ-strongly monotone mappings and application to the generalized convex optimization problems

[4] This section centres on the generalized Φ-strongly monotone mappings, which are the largest class of monotone type mappings. In uniformly smooth and uniformly convex Banach spaces, we study the convergence of a sequence of approximating a solution of a generalized Φ-strongly monotone mapping.

---

[4]The results of this section are contents of the following paper

- M. O. Aibinu and O. T. Mewomo [4]

3.4.1 Background

An important generalized monotonicity property which has been studied for multi-valued mappings is quasimonotonicity. The concept of quasimonotone multivalued mapping broadly generalizes monotone mappings (see e.g., Aussel and Fabian [13], Phelps [84]). Quasimonotone mappings are closely related to the so-called demand functions in mathematical economics (see e.g., Levin [67], Karlin [55] for more details). Classical examples of quasimonotone mappings are the subdifferentials of lower semicontinuous quasiconvex functions. The interest in quasimonotone mapping stems mainly from the fact that the derivative and more generally, the subdifferential of a quasiconvex function is quasimonotone. This is similar to the link between convex functions and monotonicity of their (generalized) derivative (see, Aussel et al. [12], [11] for more details). For a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm, Diop et al. [40] studied the class of strongly monotone mappings and applied their result to the convex minimization problem. Chidume and Idu [33] considered the class of maximal monotone mappings in a uniformly convex and uniformly smooth Banach space and obtained the minimizer of a convex function defined from a Banach space E to $\mathbb{R}$.

Let E be a real Banach space and $A : E \to 2^{E^*}$ be a multivalued mapping. We study the method of finding the zeros of a generalized Φ-strongly monotone mapping A, which satisfies the range condition. It is a well known fact that the class of the generalized Φ-strongly monotone mappings is the largest class of monotone-type mappings such that if a solution of an equation $0 \in Ax$ exists, it is necessarily unique (Chidume et al. [35]). Assuming existence, a sequence is constructed which converges strongly to a solution of the equation $0 \in Ax$. As an immediate application of this result, we apply it to obtain the solutions of generalized convex optimization problems.

3.4.2 Main Results

Theorem 3.4.1 *Let E be a uniformly smooth and uniformly convex real Banach space. Let $A : E \to 2^{E^*}$ be a multivalued mapping which is bounded, a generalized*

Φ-*strongly monotone such that the range of* $(J_p + tA)$ *is all of* E^* *for all* $t > 0$ *and* $A^{-1}(0) \neq \emptyset$. *Let* $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\{\theta_n\}_{n=1}^{\infty}$ *in* $(0, \frac{1}{2})$ *be real sequences such that,*

(i) $\lim\limits_{n \to \infty} \theta_n = 0$ *and* $\{\theta_n\}_{n=1}^{\infty}$ *is decreasing;*

(ii) $\sum\limits_{n=1}^{\infty} \lambda_n \theta_n = \infty$;

(iii) $\lim\limits_{n \to \infty} \left((\theta_{n-1}/\theta_n) - 1 \right) / \lambda_n \theta_n = 0$, $\sum\limits_{n=1}^{\infty} \lambda_n < \infty \ \forall \ n \in \mathbb{N}$.

For arbitrary $x_1 \in E$ *and* $p > 1$, *define* $\{x_n\}_{n=1}^{\infty}$ *iteratively by:*

$$x_{n+1} = J_p^{-1} \left(J_p x_n - \lambda_n \left(\mu_n^x + \theta_n (J_p x_n - J_p x_1) \right) \right), \quad \mu_n^x \in A x_n \quad n \in \mathbb{N}, \qquad (3.4.1)$$

where J_p *is the generalized duality mapping from* E *into* E^*. *Then the sequence* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a point of* $A^{-1}0$.

Proof. The proof is divided into two parts.

Part 1: We prove that $\{x_n\}_{n=1}^{\infty}$ is bounded.

Let $x \in E$ be a solution of $0 \in Ax$. It suffices to show that $\phi_p(x, x_n) \leq r, \forall \ n \in \mathbb{N}$. From inequality (3.1.5), we have $\|x_n\| \leq r^{\frac{1}{p}} + \|x\|$. Let $B := \{z \in E : \phi_p(x, z) \leq r\}$. It is known that A is bounded and J_p is uniformly continuous on bounded subsets of E. Define

$$M_0 := \sup \left\{ \|\mu_n^x + \theta_n (J_p x - J_p x_1)\| : \theta_n \in (0, \frac{1}{2}), x \in B, \mu_n^x \in A x_n \right\} + 1. \quad (3.4.2)$$

Let ψ denotes the modulus of continuity of J_p^{-1}. Then

$$\begin{aligned}
\|x_n - x_{n+1}\| &= \|x_n - J_p^{-1}(J_p x_n - \lambda_n \left(\mu_n^x + \theta_n (J_p x_n - J_p x_1) \right))\| \\
&= \|J_p^{-1}(J_p x_n) - J_p^{-1}(J_p x_n - \lambda_n \left(\mu_n^x + \theta_n (J_p x_n - J p x_1) \right))\| \\
&\leq \psi \left(|\lambda_n| \| \mu_n^x + \theta_n (J_p x_n - J_p x_1) \| \right) \\
&\leq \psi \left(|\lambda_n| M_0 \right) \\
&\leq \psi \left(\sup \{ |\lambda_n| M_0 : \lambda_n \in (0,1) \} \right). \qquad (3.4.3)
\end{aligned}$$

Since A is bounded and the duality mapping J_p is uniformly continuous on bounded subsets of E, the $\sup \{ |\lambda_n| M_0 \}$ exists and it is a real number different from infinity. Let $M =: \psi \left(\sup \{ |\lambda_n| M_0 \} \right)$ and let $r > 0$ be chosen such that

$$\Phi\left(\frac{\delta}{2}\right) \geq r \geq \max\left\{\phi_p(x, x_1), 2M_0M, \; \delta^p + \frac{p}{q}\|x\|^q\right\}$$

and $M \geq \frac{\delta}{2}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and δ is a positive real number. The proof is by induction. By construction, $\phi_p(x, x_1) \leq r$. Suppose that $\phi_p(x, x_n) \leq r$ for some $n \in \mathbb{N}$. We show that $\phi_p(x, x_{n+1}) \leq r$. Suppose this is not the case, then $\phi_p(x, x_{n+1}) > r$. Applying Lemma 3.1.3 with $y^* := \lambda_n \left(\mu_n^x + \theta_n(J_p x_n - J_p x_1)\right)$ and by using the definition of x_{n+1}, we compute as follows,

$$
\begin{aligned}
&\phi_p(x, x_{n+1}) \\
=\; & \phi_p\left(x, J_p^{-1}\left(J_p x_n - \lambda_n\left(\mu_n^x + \theta_n(J_p x_n - J_p x_1)\right)\right)\right) \\
=\; & V_p\left(x, J_p x_n - \lambda_n\left(\mu_n^x + \theta_n(J_p x_n - J_p x_1)\right)\right) \\
\leq\; & V_p(x, J_p x_n) \\
& -p\lambda_n\left\langle J_p^{-1}(J_p x_n - \lambda_n\left(\mu_n^x + \theta_n(J_p x_n - J_p x_1)\right)) - x, \mu_n^x + \theta_n(J_p x_n - J_p x_1)\right\rangle \\
=\; & \phi_p(x, x_n) - p\lambda_n\left\langle x_n - x, \mu_n^x + \theta_n(J_p x_n - J_p x_1)\right\rangle \\
& -p\lambda_n\left\langle J_p^{-1}(J_p x_n - \lambda_n\left(\mu_n^x + \theta_n(J_p x_n - J_p x_1)\right)) - x_n, \mu_n^x + \theta_n(J_p x_n - J_p x_1)\right\rangle.
\end{aligned}
$$

By Schwartz inequality and by applying inequality (3.4.3), we obtain

$$
\begin{aligned}
\phi_p(x, x_{n+1}) \;\leq\;\; & \phi_p(x, x_n) - p\lambda_n\left\langle x_n - x, \mu_n^x + \theta_n(J_p x_n - J_p x_1)\right\rangle \\
& +p\lambda_n M_0 M \\
\leq\;\; & \phi_p(x, x_n) - p\lambda_n\left\langle x_n - x, \mu_n^x - \mu^x\right\rangle \quad (\mu^x \in Ax \;\; \text{since} \; x \in N(A)) \\
& -p\lambda_n\theta_n\left\langle x_n - x, J_p x_n - J_p x_1\right\rangle + p\lambda_n M_0 M.
\end{aligned}
$$

By Lemma 3.1.4, $p\left\langle x - x_n, J_p x_n - J_p x_1\right\rangle \leq \phi_p(x, x_n) - \phi_p(x, x_1)$. Consequently, $p\left\langle x - x_n, J_p x_n - J_p x_1\right\rangle \leq \phi_p(x, x_n)$. Also, since A is generalized Φ-strongly monotone, we have,

$$
\begin{aligned}
\phi_p(x^*, x_{n+1}) \;\leq\;& \phi_p(x, x_n) - p\lambda_n \Phi(\|x_n - x\|) - p\lambda_n \theta_n \langle x_n - x, J_p x_n - J_p x_1 \rangle \\
& + p\lambda_n M_0 M \\
=\;& \phi_p(x, x_n) - p\lambda_n \Phi(\|x_n - x\|) + p\lambda_n \theta_n \langle x - x_n, J_p x_n - J_p x_1 \rangle \\
& + p\lambda_n M_0 M \\
\leq\;& \phi_p(x, x_n) - p\lambda_n \Phi(\|x_n - x\|) + p\lambda_n \theta_n (\phi_p(x, x_n) - \phi_p(x, x_1)) \\
& + p\lambda_n M_0 M. \tag{3.4.4}
\end{aligned}
$$

By the uniform continuity property of J_p^{-1} on bounded sets of E^*, we have

$$
\|x_{n+1} - x_n\| = \|J_p^{-1}(J_p x_{n+1}) - J_p^{-1}(J_p x_n)\| \leq M,
$$

such that

$$
\|x_{n+1} - x\| - \|x_n - x\| \leq M,
$$

which gives

$$
\|x_n - x\| \;\geq\; \|x_{n+1} - x\| - M. \tag{3.4.5}
$$

From Lemma 3.1.2,

$$
\begin{aligned}
\|x_{n+1} - x\|^p \;\geq\;& \phi_p(x, x_{n+1}) - \frac{p}{q}\|x\|^q \\
\geq\;& r - \frac{p}{q}\|x\|^q \\
\geq\;& \left(\delta^p + \frac{p}{q}\|x\|\right) - \frac{p}{q}\|x\|^q \\
\geq\;& \delta^p.
\end{aligned}
$$

So,

$$
\|x_{n+1} - x\| \geq \delta.
$$

Therefore, the inequality (3.4.5) becomes,

$$
\begin{aligned}
\|x_n - x\| \;\geq\;& \delta - M \\
\geq\;& \frac{\delta}{2}.
\end{aligned}
$$

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Thus,

$$\Phi(\|x_n - x\|) \geq \Phi(\frac{\delta}{2}). \tag{3.4.6}$$

Substituting (3.4.6) into (3.4.4) gives

$$
\begin{aligned}
r < \phi_p(x, x_{n+1}) &\leq \phi_p(x, x_n) - p\lambda_n \Phi(\frac{\delta}{2}) + p\lambda_n \theta_n \phi_p(x, x_n) \\
&\quad + p\lambda_n M_0 M \\
&\leq r - p\lambda_n r + p\lambda_n \frac{r}{2} + p\lambda_n \frac{r}{2} \\
&= \left(1 - p\lambda_n + \frac{p\lambda_n}{2} + \frac{p\lambda_n}{2}\right) r = r.
\end{aligned}
$$

a contradiction. Hence, $\phi_p(x, x_{n+1}) \leq r$. By induction, $\phi_p(x, x_n) \leq r \quad \forall \ n \in \mathbb{N}$. Thus, from inequality (3.1.5), $\{x_n\}_{n=1}^\infty$ is bounded.

Part 2: We now show that $\{x_n\}_{n=1}^\infty$ converges strongly to a point of $A^{-1}0$. Recall that A is a generalized Φ-strongly monotone and the range of $(J_p + tA)$ is all of E^* for all $t > 0$. Since E is a reflexive smooth strictly convex space, we obtain for every $t > 0$ and $x^* \in E$, there exists a unique $x_t \in D(A)$, where $D(A)$ is the domain of A such that

$$J_p x^* \in J_p x_t + tA x_t.$$

Define $J_t^A x^* = x_t$, equivalently define a single-valued mapping $J_t^A : E \to D(A)$ by $J_t^A = (J_p + tA)^{-1} J_p$. Such a J_t^A is called the resolvent of A. Setting $t := \frac{1}{\theta_n}$, for some $x_1 \in D(A)$ and $y_n = (J_p + \frac{1}{\theta_n} A)^{-1} J_p x_1$, we obtain

$$\theta_n(J_p y_n - J_p x_1) + \mu_n^y = 0, \quad \mu_n^y \in Ay_n \text{ and } y_n \to x \in N(A). \tag{3.4.7}$$

Observe that the sequence $\{y_n\}_{n=1}^\infty$ is bounded because it is a convergent sequence. Moreover, $\{x_n\}_{n=1}^\infty$ is bounded and hence $\{Ax_n\}_{n=1}^\infty$ is bounded. Following the same arguments as in part 1, we get,

$$
\begin{aligned}
\phi_p(y_n, x_{n+1}) &\leq \phi_p(y_n, x_n) - p\lambda_n \langle x_n - y_n, \mu_n^x + \theta_n(J_p x_n - J_p x_1) \rangle \\
&\quad + p\lambda_n M_0 M. \tag{3.4.8}
\end{aligned}
$$

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By the generalized Φ-strongly monotonicity of A and using Theorem 3.0.1 and Eq. (3.4.7), we obtain,

$$\langle x_n - y_n, \mu_n^x + \theta_n(J_p x_n - J_p x_1)\rangle$$

$$= \langle x_n - y_n, \mu_n^x + \theta_n(J_p x_n - J_p y_n + J_p y_n - J_p x_1)\rangle$$

$$= \theta_n \langle x_n - y_n, J_p x_n - J_p y_n\rangle$$

$$\quad + \langle x_n - y_n, \mu_n^x + \theta_n(J_p y_n - J_p x_1)\rangle$$

$$= \theta_n \langle x_n - y_n, J_p x_n - J_p y_n\rangle + \langle x_n - y_n, \mu_n^x - \mu_n^y\rangle$$

$$\geq \theta_n g(\|x_n - y_n\|) + \Phi(\|x_n - y_n\|) \text{ (Since } A \text{ is } (p,\eta)\text{-strongly monotone and by Lemma 3.3.1(ii))}$$

$$\geq \frac{1}{p}\theta_n \phi_p(y_n, x_n).$$

Therefore, the inequality (3.4.8) becomes

$$\phi_p(y_n, x_{n+1}) \leq (1 - \lambda_n \theta_n)\phi_p(y_n, x_n) + p\lambda_n M_0 M. \qquad (3.4.9)$$

Observe that by Lemma 3.1.4, we have

$$\phi_p(y_n, x_n) \leq \phi_p(y_{n-1}, x_n) - p\langle y_n - x_n, J_p y_{n-1} - J_p y_n\rangle$$

$$= \phi_p(y_{n-1}, x_n) + p\langle x_n - y_n, J_p y_{n-1} - J_p y_n\rangle$$

$$\leq \phi_p(y_{n-1}, x_n) + \|J_p y_{n-1} - J_p y_n\|\|x_n - y_n\|. \qquad (3.4.10)$$

Let $R > 0$ such that $\|x_1\| \leq R, \|y_n\| \leq R$ for all $n \in \mathbb{N}$. We obtain from Eq.(3.4.7) that

$$J_p y_{n-1} - J_p y_n + \frac{1}{\theta_n}\left(\mu_{n-1}^y - \mu_n^y\right) = \frac{\theta_{n-1} - \theta_n}{\theta_n}\left(J_p x_1 - J_p y_{n-1}\right).$$

By taking the duality pairing of each side of this equation with respect to $y_{n-1} - y_n$ and by the generalized Φ-strongly monotonicity of A, we have

$$\langle J_p y_{n-1} - J_p y_n, y_{n-1} - y_n\rangle \leq \frac{\theta_{n-1} - \theta_n}{\theta_n}\|J_p x_1 - J_p y_{n-1}\|\|y_{n-1} - y_n\|,$$

which gives,

$$\|J_p y_{n-1} - J_p y_n\| \leq \left(\frac{\theta_{n-1}}{\theta_n} - 1\right)\|J_p y_{n-1} - J_p x_1\|. \qquad (3.4.11)$$

Using (3.4.10) and (3.4.11), the inequality (3.4.9) becomes

$$\phi_p(y_n, x_{n+1}) \leq (1 - \lambda_n \theta_n)\phi_p(y_{n-1}, x_n) + C\left(\frac{\theta_{n-1}}{\theta_n} - 1\right) + p\lambda_n M_0 M,$$

for some constant $C > 0$. By Lemma 2.10.6, $\phi_p(y_{n-1}, x_n) \to 0$ as $n \to 0$ and using Lemma 2.6.2, we have that $x_n - y_{n-1} \to 0$ as $n \to 0$. Since $y_n \to x \in N(A)$, we obtain that $x_n \to x$. ■

Corollary 3.4.2 *Aibinu and Mewomo [3]. Let E be a uniformly smooth and uniformly convex real Banach space. Let $p > 1, \eta \in (0,1)$ suppose $A : E \to E^*$ is a bounded, (p, η)-strongly monotone mapping such that the range of $(J_p + tA)$ is all of E^* for all $t > 0$ and $A^{-1}(0) \neq \emptyset$. Let $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{\theta_n\}_{n=1}^{\infty}$ in $(0, \frac{1}{2})$ be real sequences such that,*

(i) $\lim\limits_{n\to\infty} \theta_n = 0$ *and* $\{\theta_n\}_{n=1}^{\infty}$ *is decreasing;*

(ii) $\sum\limits_{n=1}^{\infty} \lambda_n \theta_n = \infty;$

(iii) $\lim\limits_{n\to\infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0, \ \sum\limits_{n=1}^{\infty} \lambda_n < \infty \ \forall \ n \in \mathbb{N}.$

For arbitrary $x_1 \in E$, define $\{x_n\}_{n=1}^{\infty}$ iteratively by:

$$x_{n+1} = J_p^{-1} \left(J_p x_n - \lambda_n \left(A x_n + \theta_n (J_p x_n - J_p x_1) \right) \right), n \in \mathbb{N}, \tag{3.4.12}$$

where J_p is the generalized duality mapping from E into E^*. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of $Ax = 0$.

Proof. Take $\Phi(\|x - y\|) := \eta\|x - y\|^p$ in Theorem 3.4.1, then the desired result follows. ■

3.4.3 Application to the generalized convex optimization problem

Generalized Φ-strongly monotone mappings is the largest class of monotone-type mappings such that if a solution of an equation $0 \in Ax$ exists, it is necessarily unique (Chidume et al. [33]). A specific generalized monotononicity property which is quasimonotonicity is used for illustration as it is closely related to the so-called

demand functions in mathematical economics (see e.g., Levin [67], Karlin [55] for more details).

Let E be a real Banach space with the dual E^* and A, a multivalued mapping from E into 2^{E^*}. According to Hassouni [46], for K subset of E, and $\bar{x} \in K$, A satisfies the variational inequality below if and only if

$$\forall \ x \in K, \ \ \langle \mu^x, x - \bar{x} \rangle \geq 0, \ \ \forall \ \mu^x \in Ax. \tag{3.4.13}$$

Consider now the quasiconvex minimization problem

$$\min_{x \in K} f(x), \tag{3.4.14}$$

where $f : E \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and quasiconvex. Let N be a convex open neighborhood of $\bar{x}$. The necessary and sufficient condition to obtain a solution of (3.4.14) is given in the Lemma 3.4.3 below.

Lemma 3.4.3 *Hassouni and Jaddar [47]. If $K = N$ or $K = E$, then following assertions are equivalent:*

(i) $\bar{x}$ is an optimal solution of (3.4.14),

(ii) ∂f satisfies (3.4.13).

Remark 3.4.4

For any single-valued quasimonotone operator ∂f, the operator $h(x) := \{\alpha \partial f(x) : \alpha \geq 0\}$ is also quasimonotone and $Gr(\partial f) \subset Gr(h)$ provided $\partial f \neq 0$, where $Gr(\partial f)$ and $Gr(h)$ denote the graph of ∂f and of h respectively. It follows that for every non-constant smooth quasiconvex function f, the single-valued quasimonotone operator ∂f is not maximal (see e.g., Levin [67]).

Next, we give a useful definition and establish a lemma which is necessary in establishing our main result in this section. Let E be a real locally convex topological vector space, K a nonempty convex subset of E, $A : K \to L(E, \mathbb{R}) = 2^{E^*}$ a multivalued mapping and $S \subseteq 2^{E^*}$.

Definition 3.4.5 *A multivalued mapping* $A : K \to 2^{E^*}$ *is said to have the surjective property if the range of* A *excluding the zero vector (i.e* $R(A) \backslash \{0\}$*) has the surjective property. Indeed, suppose* S *has the surjective property on* K *and* $f \in R(A) \backslash \{0\}$, *then a multivalued mapping* $foS \subset L(E, \mathbb{R}) = 2^{E^*}$ *is said to have the surjective property on* K *provided*

$$\langle foS, x - y \rangle := \{\langle fox^*, x - y \rangle : x^* \in S\} = \mathbb{R}.$$

Lemma 3.4.6 *Let* E *be a uniformly smooth and uniformly convex real Banach space,* K *a nonempty convex subset of* E *and* $A : K \to 2^{E^*}$ *a multivalued mapping. Suppose* $S \subseteq 2^{E^*}$ *is connected and has the surjective property on* K. *Then* A *is monotone and the range of* $(J_p + tA)$ *is all of* E^* *for all* $t > 0$ *if and only if for each* $\alpha \in S$, $A + \alpha$ *is quasimonotone and has the surjective property on* K.

Proof. " $\Rightarrow$ " Suppose A is monotone and $R(J_p + tA) = E^*$ for all $t > 0$. Therefore for each $\alpha \in X^*$, the operator $u \mapsto A(u) + \alpha$ is obviously monotone, hence quasimonotone. Next, suppose for contradiction that $A + \alpha$ has no surjective property, that is $\exists \ x \in K$, a convex subset of E such that $\forall \ y \in K$

$$\langle A + \alpha, x - y \rangle = \{\langle fou^* + \alpha, x - y \rangle, u^*, \alpha \in S\} \neq \mathbb{R}.$$

It follows that for each $u^* \in S$, the range of

$$g(t) := -t \langle fou^*, x - y \rangle - \langle u^*, x - y \rangle$$

is not equal to $\mathbb{R}$. Recall that monotonicity of A gives that

$$\langle x^* - y^*, x - y \rangle \geq 0 \Rightarrow \langle x^*, x - y \rangle \geq \langle y^*, x - y \rangle \quad \forall \ x^* \in Ax, \ y^* \in Ay.$$

Therefore, there exists $t_0 \in \mathbb{R}$ such that

$$\langle x^*, x - y \rangle \geq -t_0 \langle fou^*, x - y \rangle - \langle u^*, x - y \rangle \geq \langle y^*, x - y \rangle.$$

Setting $\alpha := t_0 fou^* + u^*$, we deduce that

$$\langle x^* + \alpha, x - y \rangle \geq 0,$$

while

$$\langle y^* + \alpha, x - y \rangle \leq 0.$$

Thus contradicting the pseudomonotonicity and hence quasimonotonicity of the map $A + \alpha$.

" $\Leftarrow$ " Suppose that $A + \alpha$ is quasimonotone and has the surjective property. We show that A is monotone and the range of $(J + tA)$ is all of E^* for all $t > 0$. By Lemma 2.8.3, A is monotone since $A + \alpha$ is quasimonotone. Next is to show that $R(J_p + tA) = E^*$ for all $t > 0$. Since $A + \alpha$ has the surjective property on K, for every $u^* \in R(J_p + tA)$, the line $L = \{u^* + tfou^* : t \in \mathbb{R}^+\}$ has surjective property on K. But $L \subset E^*$, therefore

$$R(J_p + tA) \subseteq E^*.$$

Also, for a given $u^* \in S$ and each $v^* \in E^*$, define

$$\langle v^*, x - y \rangle = \langle u^* + tfou^*, y - x \rangle$$

for every $x, y \in K$. Therefore, $v^* := u^* + tfou^* \in R(J_p + tA)$. Hence $R(J_p + tA) = E^*$

■

Theorem 3.4.7 *Let K be a nonempty convex subset of a uniformly smooth and uniformly convex real Banach space E and $S \subseteq 2^{E^*}$ is connected and has the surjective property on K. Let $f : K \to \mathbb{R} \cup \{+\infty\}$ be a bounded lower semicontinuous quasiconvex function defined on K with nonempty interior. Suppose for each $\alpha \in S$, $\partial f + \alpha$ is quasimontone and has the surjective property on K with $(\partial f)^{-1} 0 \neq \emptyset$. Let $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ and $\{\theta_n\}_{n=1}^{\infty}$ in $(0, \frac{1}{2})$ be real sequences such that,*

(i) $\lim \theta_n = 0$ *and* $\{\theta_n\}$ *is decreasing;*

(ii) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$;

(iii) $\lim_{n \to \infty} ((\theta_{n-1}/\theta_n) - 1) / \lambda_n \theta_n = 0$, $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then, for arbitrary $x_1 \in E$, the iteration $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = J_p^{-1} \left(J_p x_n - \lambda_n \left((\partial f) x_n + \theta_n (J_p x_n - J_p x_1) \right) \right), n \in \mathbb{N}. \qquad (3.4.15)$$

converges strongly to some $x^* \in (\partial f)^{-1} 0$.

Proof. f is a bounded quasiconvex function, therefore by Lemma 2.3.2, ∂f is a bounded quasimonotone operator. Take ∂f to be A in Lemma 3.4.6. Similar analysis to the proof of Theorem 3.4.1 gives the desire result. ■

Conclusion 3.4.8

Most of the existing results on the approximation of solutions of monotone-type mappings have been proved in Hilbert spaces or they are for accretive-type mappings in Banach spaces. Unfortunately, as has been rightly observed, many and probably most mathematical objects and models do not naturally live in Hilbert spaces. We have considered the class of generalized Φ-strongly monotone mappings in Banach spaces, the class of monotone-type mappings such that if a solution of the equation $0 \in Ax$ exists, it is necessarily unique. Therefore, our results very important results our techniques of proofs are of independent interest.

Viscosity implicit iterative algorithms and applications

[1] In this chapter, we explore the implicit iterative algorithms for approximating the fixed points of nonexpansive mappings in uniformly smooth Banach spaces. Numerical and analystical comparisons are made for various implicit iterative algorithms.

4.1 The implicit midpoint rule of nonexpansive mappings and applications in uniformly smooth Banach spaces

4.1.1 Background

In 1967, Halpern [45] considered an iterative sequence for a nonexpansive mapping T in a Hilbert space. He showed that the conditions $(A_1) \lim_{n \to \infty} \alpha_n = 0$ and $(A_2) \sum_{n=1}^{\infty} \alpha_n = \infty$ are essential for the convergence to a fixed point of T of the se-

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[1]The results of this section are contents of the following paper

- M.O. Aibinu, P. Pillay, J.O. Olaleru and O.T. Mewomo [6]

quence $\{x_n\}$ defined by

$$x_1 \in K, \ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \ n \in \mathbb{N}, \qquad (4.1.1)$$

where $u \in K$ is a given point and $\alpha_n \in [0,1]$. Halpern [45] iteration attracted the attention of many researchers. In 1977, Lions [70] improved on the result of Halpern and showed that for $\{\alpha_n\}$ satisfying the conditions (A_1), (A_2) and (A_3) $\lim\limits_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\alpha_n^2 = 0$, $\{x_n\}$ converges strongly to a fixed point of T in a Hilbert space. In 1992, still in Hilbert spaces and for $\{\alpha_n\}$ satisfying the conditions (A_1), (A_2) and $(A_4) \sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < +\infty$, Wittmann [62] proved a strong convergence theorem for the sequence (4.1.1) to a fixed point of T. By considering various conditions either on $\{\alpha_n\}$ or on the space, there are also several theorems for the strong convergence of Halpern's iteration to a fixed point of T in Banach spaces (see, e.g., [70], [62], [86], [92], [95], [68]). Modifications of Halpern-type iteration have also been studied by many authors [45]. In 2000, Moudafi [76] introduced the concept of a viscosity approximation method for selecting a particular fixed point of a given nonexpansive mapping. He considered an explicit viscosity method for nonexpansive mappings and defined the iterative sequence $\{x_n\}$ by (1.1.1). He showed that the sequence $\{x_n\}$ defined by (1.1.1) converges strongly to a fixed point of T with the conditions that (A_1), (A_2) and $(A_5) \lim\limits_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\alpha_n\alpha_{n-1} = 0$ are satisfied. One of the essential numerical methods for solving ordinary differential and differential algebraic equations is the implicit midpoint rule ([15], [16], [51] and [93]). In 2014, Alghamdi et al. [9] presented a semi-implicit midpoint iteration for nonexpansive mappings in a Hilbert space. They proved a weak convergence theorem for the sequence $\{x_n\}$ defined by (1.1.10). Furthermore, in 2015, Xu et al. [104] defined the viscosity implicit midpoint sequence for a nonexpansive mapping T on K by (1.1.5). Precisely, they proved the following strong convergence theorem.

Theorem 4.1.1 *[104] Let K be a nonempty closed convex subset of a Hilbert space H and $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose $f : K \to K$ is a contraction with coefficient $\alpha \in [0,1)$ and assume that the sequence $\{\alpha_n\}$ satisfies the conditions (A_1), (A_2) and either (A_4) or $\lim\limits_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n-1}} = 1$. Then the sequence $\{x_n\}$ generated by (1.1.5) converges in norm to a fixed point p of T, which*

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is also the unique solution of the variational inequality (1.1.2). That is, p is the unique fixed point of the contraction $P_{F(T)}f$, in other words, $P_{F(T)}f(p) = p$.

Still in a Hilbert space, in 2015, Yao et al. [107] introduced the iterative sequence (1.1.6). They imposed suitable conditions on the parameters and obtained that the sequence $\{x_n\}$ generated by (1.1.6) converges strongly to $p = P_{F(T)}f(p)$. In 2017, Luo et al. [72] extends the result of Xu et al. [104] to a uniformly smooth Banach space. Few among several other works on modified Halpern-type iteration include Qin et al. [85], Wang et al. [98] and the references contained in them. Also, some authors studied modified Halpern-type sequences for various classes of mappings (see e.g., Aibinu and Mewomo [5], [3], Chidume and Mutangandura [34], Hu and Wang [49] and Nandal and Chugh [78]). The following questions are of interest to us:

Problem 4.1.2 *Comparing the three implicit iterative schemes (1.1.5), (1.1.6) and (1.1.10) that are respectively given by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N},$$

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N},$$

and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N},$$

which one has the highest rate of convergence?

Problem 4.1.3 *The main results of Yao et al. [107] which are in Hilbert spaces, can we establish them in general Banach spaces?*

The purpose of this paper is to study the implicit midpoint procedure (1.1.6) in the framework of Banach spaces for approximating a fixed point of nonexpansive mappings. We prove a strong convergence theorem in a uniformly smooth Banach space for the sequence $\{x_n\}$ defined by (1.1.6) and illustrate with a numerical example that it is the most efficient among the three implicit midpoint procedures (1.1.5), (1.1.6) and (1.1.10). Moreover, we obtain the results of Xu et al. [104], Luo et al. [72] and Yao et al. [107] as corollaries.

4.1.2 Main results

Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : K \to K$ a c-contraction. Suppose $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\gamma_n\} \subset (0,1)$ are real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1 \ \forall \ n \in \mathbb{N}$. For arbitrary $x_1 \in K$, we consider the iterative scheme for the sequence $\{x_n\}$ defined by (1.1.6).

Remark 4.1.4 *It is known that the sequence $\{x_n\}$ is well defined [107].*

We first give and prove a lemma which is useful in establishing our main result.

Lemma 4.1.5 *Let E be a uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and suppose $f : K \to K$ is a c-contraction. For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}$ by (1.1.6). Then the sequence $\{x_n\}$ is bounded.*

Proof. We show that the sequence $\{x_n\}$ is bounded.

For $p \in F(T)$,

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \left(f(x_n) - f(p)\right) + \alpha_n \left(f(p) - p\right) \\
&\quad + \beta_n (x_n - p) + \gamma_n \left(T(\frac{x_n + x_{n+1}}{2}) - p\right) \| \\
&\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| \\
&\quad + \beta_n \|x_n - p\| + \gamma_n \|T(\frac{x_n + x_{n+1}}{2}) - p\| \\
&\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| \\
&\quad + \beta_n \|x_n - p\| + \gamma_n \|\frac{x_n + x_{n+1}}{2} - p\| \\
&\leq c\alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| \\
&\quad + \frac{\gamma_n}{2} \|x_n - p\| + \frac{\gamma_n}{2} \|x_{n+1} - p\|.
\end{aligned}
$$

We then have that

$$\left(1 - \frac{\gamma_n}{2}\right)||x_{n+1} - p|| \leq \left(c\alpha_n + \beta_n + \frac{\gamma_n}{2}\right)||x_n - p|| + \alpha_n||f(p) - p||$$

$$\frac{2 - \gamma_n}{2}||x_{n+1} - p|| \leq \frac{2c\alpha_n + 2\beta_n + \gamma_n}{2}||x_n - p|| + \alpha_n||f(p) - p||$$

$$\frac{1 + \alpha_n + \beta_n}{2}||x_{n+1} - p|| \leq \frac{2c\alpha_n + 2\beta_n + 1 - (\alpha_n + \beta_n)}{2}||x_n - p||$$
$$+ \alpha_n||f(p) - p||$$

$$\frac{1 + \alpha_n + \beta_n}{2}||x_{n+1} - p|| \leq \frac{1 + \beta_n + \alpha_n(2c - 1)}{2}||x_n - p|| + \alpha_n||f(p) - p||.$$

Therefore,

$$
\begin{aligned}
||x_{n+1} - p|| &\leq \frac{1 + \beta_n + \alpha_n(2c - 1)}{1 + \alpha_n + \beta_n}||x_n - p|| + \frac{2\alpha_n}{1 + \alpha_n + \beta_n}||f(p) - p|| \\
&= \left(1 - \frac{2\alpha_n(1 - c)}{1 + \alpha_n + \beta_n}\right)||x_n - p|| + \frac{2\alpha_n(1 - c)}{1 + \alpha_n + \beta_n}\frac{1}{1 - c}||f(p) - p|| \\
&\leq \max\left\{||x_n - p||, \frac{1}{1 - c}||f(p) - p||\right\} \\
&\vdots \\
&\leq \max\left\{||x_1 - p||, \frac{1}{1 - c}||f(p) - p||\right\}.
\end{aligned}
$$

This implies that the sequence $\{x_n\}$ is bounded.

Also, for $p \in F(T)$,

$$
\begin{aligned}
||T(\frac{x_n + x_{n+1}}{2})|| &= ||T(\frac{x_n + x_{n+1}}{2}) - p + p|| \\
&\leq ||T(\frac{x_n + x_{n+1}}{2}) - Tp|| + ||p|| \\
&\leq ||\frac{x_n + x_{n+1}}{2} - p|| + ||p|| \\
&\leq \frac{1}{2}(||x_n - p|| + ||x_{n+1} - p||) + ||p|| \\
&\leq \max\left\{||x_1 - p||, \frac{1}{1 - c}||f(p) - p||\right\} + ||p||.
\end{aligned}
$$

Thus $\left\{T\left(\frac{x_n + x_{n+1}}{2}\right)\right\}$ is bounded.

Moreover, we show that $\{f(x_n)\}$ is bounded. For $p \in F(T)$,

$$
\begin{aligned}
||f(x_n)|| &= ||f(x_n) - f(p) + f(p)|| \\
&\leq ||f(x_n) - f(p)|| + ||f(p)|| \\
&\leq c||x_n - p|| + ||f(p)|| \\
&\leq c\max\left\{||x_1 - p||, \frac{1}{1 - c}||f(p) - p||\right\} + ||f(p)||.
\end{aligned}
$$

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■

Theorem 4.1.6 *Let E be a uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : K \to K$ be a c-contraction. Suppose $\{\alpha_n\}$ satisfies*

(A_1) $\displaystyle\lim_{n\to\infty} \alpha_n = 0;$

(A_2) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$

and $\{\beta_n\}$ satisfies

(A_6) $0 < \displaystyle\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and*

(A_7) $\displaystyle\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0.$

For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}$ by (1.1.6). Then as $n \to \infty$, the sequence $\{x_n\}$ converges in norm to a fixed point q of T, where q is the unique solution in $F(T)$ to the variational inequality:

$$\langle (I - f)q, J(x - q) \rangle \geq 0 \ \forall \ x \in F(T).$$

Proof. **Step 1:** The iterative process (1.1.6) is

$$\begin{aligned}
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right) \\
&= \beta_n x_n + (1 - \beta_n)\frac{\alpha_n f(x_n) + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right)}{1 - \beta_n} \\
&= \beta_n x_n + (1 - \beta_n)y_n, \tag{4.1.2}
\end{aligned}$$

where $y_n = \frac{\alpha_n}{1-\beta_n} f(x_n) + \frac{\gamma_n}{1-\beta_n} T(\frac{x_n+x_{n+1}}{2})$, $n \in \mathbb{N}$.

From condition (A_6), we have that

$$0 < \beta_n \leq \beta < 1, \ \text{for some } \beta \in \mathbb{R}^+,$$

where $\mathbb{R}^+$ denotes the set of positive real numbers. Therefore,

$$1 - \beta_n \geq 1 - \beta. \tag{4.1.3}$$

Now f is a c-contraction while $\{x_n\}$ and $\left\{T(\frac{x_n+x_{n+1}}{2})\right\}$ are bounded sequences. These guarantee that $\{y_n\}$ is bounded.

Step 2: We show that $\lim\limits_{n\to\infty} ||y_n - x_n|| = 0$.

We need to first show that $\limsup\limits_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Observe that

$$
\begin{aligned}
y_{n+1} - y_n &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} f(x_{n+1}) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} T(\frac{x_{n+1} + x_{n+2}}{2}) \\
&\quad - \left(\frac{\alpha_n}{1 - \beta_n} f(x_n) + \frac{\gamma_n}{1 - \beta_n} T(\frac{x_n + x_{n+1}}{2}) \right) \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - f(x_n)) + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) f(x_n) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left(T(\frac{x_{n+1} + x_{n+2}}{2}) - T(\frac{x_n + x_{n+1}}{2}) \right) \\
&\quad + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) T(\frac{x_n + x_{n+1}}{2}) \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - f(x_n)) + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) f(x_n) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left(T(\frac{x_{n+1} + x_{n+2}}{2}) - T(\frac{x_n + x_{n+1}}{2}) \right) \\
&\quad + \left(\frac{1 - \alpha_{n+1} - \beta_{n+1}}{1 - \beta_{n+1}} - \frac{1 - \alpha_n - \beta_n}{1 - \beta_n} \right) T(\frac{x_n + x_{n+1}}{2}) \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - f(x_n)) + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) f(x_n) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left(T(\frac{x_{n+1} + x_{n+2}}{2}) - T(\frac{x_n + x_{n+1}}{2}) \right) \\
&\quad + \left(\frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right) T(\frac{x_n + x_{n+1}}{2}) \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - f(x_n)) + \left(\frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right) \left(T(\frac{x_n + x_{n+1}}{2}) - f(x_n) \right) \\
&\quad + \frac{1 - \alpha_{n+1} - \beta_{n+1}}{1 - \beta_{n+1}} \left(T(\frac{x_{n+1} + x_{n+2}}{2}) - T(\frac{x_n + x_{n+1}}{2}) \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
||y_{n+1} - y_n|| &\leq \frac{c\alpha_{n+1}}{1 - \beta_{n+1}} ||x_{n+1} - x_n|| + \left| \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right| ||T(\frac{x_n + x_{n+1}}{2}) - f(x_n)|| \\
&\quad + \frac{1 - \alpha_{n+1} - \beta_{n+1}}{2(1 - \beta_{n+1})} (||x_{n+2} - x_{n+1}|| + ||x_{n+1} - x_n||) . \qquad (4.1.4)
\end{aligned}
$$

We evaluate $||x_{n+2} - x_{n+1}||$.

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\alpha_{n+1}f(x_{n+1}) + \beta_{n+1}x_{n+1} + \gamma_{n+1}T(\frac{x_{n+1}+x_{n+2}}{2}) \\
&\quad - \left(\alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\frac{x_n + x_{n+1}}{2})\right)\| \\
&= \|\alpha_{n+1}\left(f(x_{n+1}) - f(x_n)\right) + (\alpha_{n+1} - \alpha_n)f(x_n) \\
&\quad + \beta_{n+1}(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n)x_n \\
&\quad + \gamma_{n+1}(T(\frac{x_{n+1}+x_{n+2}}{2}) - T(\frac{x_n + x_{n+1}}{2})) \\
&\quad + (\gamma_{n+1} - \gamma_n)T(\frac{x_n + x_{n+1}}{2})\| \\
&= \|\alpha_{n+1}\left(f(x_{n+1}) - f(x_n)\right) + (\alpha_{n+1} - \alpha_n)f(x_n) \\
&\quad + \beta_{n+1}(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n)x_n \\
&\quad + ((\alpha_n - \alpha_{n+1}) + (\beta_n - \beta_{n+1}))T(\frac{x_n + x_{n+1}}{2}) \\
&\quad + (1 - \alpha_{n+1} - \beta_{n+1})(T(\frac{x_{n+1}+x_{n+2}}{2}) - T(\frac{x_n + x_{n+1}}{2}))\| \\
&= \|\alpha_{n+1}\left(f(x_{n+1}) - f(x_n)\right) + (\alpha_n - \alpha_{n+1}) \\
&\quad \times \left(T(\frac{x_n + x_{n+1}}{2}) - f(x_n)\right) + \beta_{n+1}(x_{n+1} - x_n) \\
&\quad + (\beta_{n+1} - \beta_n)\left(x_n - T(\frac{x_n + x_{n+1}}{2})\right) \\
&\quad + (1 - \alpha_{n+1} - \beta_{n+1})(T(\frac{x_{n+1}+x_{n+2}}{2}) - T(\frac{x_n + x_{n+1}}{2}))\| \\
&\leq c\alpha_{n+1}\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}| \\
&\quad \times \left(\|T(\frac{x_n + x_{n+1}}{2})\| + \|f(x_n)\|\right) \\
&\quad + \beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\,\|x_n - T(\frac{x_n + x_{n+1}}{2})\| \\
&\quad + \frac{1 - \alpha_{n+1} - \beta_{n+1}}{2}\left(\|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_n\|\right).
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\left(1 - \frac{1 - \alpha_{n+1} - \beta_{n+1}}{2}\right)\|x_{n+2} - x_{n+1}\| \\
&\leq \left(c\alpha_{n+1} + \beta_{n+1} + \frac{1 - \alpha_{n+1} - \beta_{n+1}}{2}\right)\|x_{n+1} - x_n\| \\
&\quad + |\alpha_n - \alpha_{n+1}|\left(\|T(\frac{x_n + x_{n+1}}{2})\| + \|f(x_n)\|\right) \\
&\quad + |\beta_{n+1} - \beta_n|\,\|x_n - T(\frac{x_n + x_{n+1}}{2})\|.
\end{aligned}
$$

Then,

$$\frac{1 + \alpha_{n+1} + \beta_{n+1}}{2}||x_{n+2} - x_{n+1}|| \leq \frac{1 + \beta_{n+1} + 2c\alpha_{n+1} - \alpha_{n+1}}{2}||x_{n+1} - x_n||$$
$$+ |\alpha_n - \alpha_{n+1}| \left(||T(\frac{x_n + x_{n+1}}{2})|| + ||f(x_n)|| \right)$$
$$+ |\beta_{n+1} - \beta_n| \; ||x_n - T(\frac{x_n + x_{n+1}}{2})||.$$

Let $M_1 = \sup\left\{||T(\frac{x_n + x_{n+1}}{2})|| + ||f(x_n)||\right\}$, $M_2 = \sup\left\{||x_n - T(\frac{x_n + x_{n+1}}{2})||\right\}$ and $M = \max\{M_1, \; M_2\}$. It follows that

$$||x_{n+2} - x_{n+1}|| \leq \frac{1 + \beta_{n+1} + 2c\alpha_{n+1} - \alpha_{n+1}}{1 + \alpha_{n+1} + \beta_{n+1}}||x_{n+1} - x_n||$$
$$+ \frac{2|\beta_{n+1} - \beta_n|}{1 + \alpha_{n+1} + \beta_{n+1}}||x_n - T(\frac{x_n + x_{n+1}}{2})||$$
$$+ \frac{2|\alpha_n - \alpha_{n+1}|}{1 + \alpha_{n+1} + \beta_{n+1}} \left(||T(\frac{x_n + x_{n+1}}{2})|| + ||f(x_n)|| \right)$$
$$\leq \frac{1 + \beta_{n+1} + 2c\alpha_{n+1} - \alpha_{n+1}}{1 + \alpha_{n+1} + \beta_{n+1}}||x_{n+1} - x_n|| + \frac{2M}{1 + \alpha_{n+1} + \beta_{n+1}}$$
$$\times \left(|\alpha_n - \alpha_{n+1}| + |\beta_{n+1} - \beta_n| \right). \qquad (4.1.5)$$

By substituting (4.1.5) into (4.1.4), we get

$$||y_{n+1} - y_n|| \leq [\frac{2c\alpha_{n+1} + 1 - \alpha_{n+1} - \beta_{n+1}}{2(1 - \beta_{n+1})} + \frac{1 - \alpha_{n+1} - \beta_{n+1}}{2(1 - \beta_{n+1})}$$
$$\times \frac{1 + \beta_{n+1} + 2c\alpha_{n+1} - \alpha_{n+1}}{1 + \alpha_{n+1} + \beta_{n+1}}]||x_{n+1} - x_n||$$
$$+ \left| \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right| M + \frac{1 - \alpha_{n+1} - \beta_{n+1}}{2(1 - \beta_{n+1})}$$
$$\times \left(\frac{2M}{1 + \alpha_{n+1} + \beta_{n+1}} (|\alpha_n - \alpha_{n+1}| + |\beta_{n+1} - \beta_n|) \right)$$
$$= [\frac{(1 + \alpha_{n+1} + \beta_{n+1})(2c\alpha_{n+1} + 1 - \alpha_{n+1} - \beta_{n+1})}{2(1 - \beta_{n+1})(1 + \alpha_{n+1} + \beta_{n+1})}$$
$$+ \frac{(1 - \alpha_{n+1} - \beta_{n+1})(1 + \beta_{n+1} + 2c\alpha_{n+1} - \alpha_{n+1})}{2(1 - \beta_{n+1})(1 + \alpha_{n+1} + \beta_{n+1})}]||x_{n+1} - x_n||$$
$$+ \left| \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right| M$$
$$+ \frac{1 - \alpha_{n+1} - \beta_{n+1}}{(1 + \alpha_{n+1} + \beta_{n+1})(1 - \beta_{n+1})} (|\alpha_n - \alpha_{n+1}| + |\beta_{n+1} - \beta_n|) M$$

$$
\begin{aligned}
=\; & [\frac{(1-\alpha_{n+1}-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})+2c\alpha_{n+1}(1+\alpha_{n+1}+\beta_{n+1})}{2(1-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})} \\
& +\frac{(1-\alpha_{n+1}-\beta_{n+1})(1+\beta_{n+1}+2c\alpha_{n+1}-\alpha_{n+1})}{2(1-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})}]||x_{n+1}-x_n|| \\
& +\left|\frac{\alpha_n}{1-\beta_n}-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|M \\
& +\frac{1-\alpha_{n+1}-\beta_{n+1}}{(1+\alpha_{n+1}+\beta_{n+1})(1-\beta_{n+1})}\left(|\alpha_n-\alpha_{n+1}|+|\beta_{n+1}-\beta_n|\right)M \\
=\; & [\frac{(1-\alpha_{n+1}-\beta_{n+1})(2+2\beta_{n+1}+2c\alpha_{n+1})}{2(1-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})} \\
& +\frac{2c\alpha_{n+1}(1+\alpha_{n+1}+\beta_{n+1})}{2(1-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})}]||x_{n+1}-x_n|| \\
& +\left|\frac{\alpha_n}{1-\beta_n}-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|M \\
& +\frac{1-\alpha_{n+1}-\beta_{n+1}}{(1+\alpha_{n+1}+\beta_{n+1})(1-\beta_{n+1})}\left(|\alpha_n-\alpha_{n+1}|+|\beta_{n+1}-\beta_n|\right)M \\
=\; & [\frac{2c\alpha_{n+1}(1-\alpha_{n+1}-\beta_{n+1})+2(1+\beta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}{2(1-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})} \\
& +\frac{2c\alpha_{n+1}(1+\alpha_{n+1}+\beta_{n+1})}{2(1-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})}]||x_{n+1}-x_n|| \\
& +\left|\frac{\alpha_n}{1-\beta_n}-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|M \\
& +\frac{1-\alpha_{n+1}-\beta_{n+1}}{(1+\alpha_{n+1}+\beta_{n+1})(1-\beta_{n+1})}\left(|\alpha_n-\alpha_{n+1}|+|\beta_{n+1}-\beta_n|\right)M \\
=\; & \frac{2c\alpha_{n+1}+(1+\beta_{n+1})(1-\alpha_{n+1}-\beta_{n+1})}{(1-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})}||x_{n+1}-x_n|| \\
& +\left|\frac{\alpha_n}{1-\beta_n}-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|M \\
& +\frac{1-\alpha_{n+1}-\beta_{n+1}}{(1+\alpha_{n+1}+\beta_{n+1})(1-\beta_{n+1})}\left(|\alpha_n-\alpha_{n+1}|+|\beta_{n+1}-\beta_n|\right)M \\
=\; & \left(1-\frac{2\alpha_{n+1}(1-c)}{(1-\beta_{n+1})(1+\alpha_{n+1}+\beta_{n+1})}\right)||x_{n+1}-x_n|| \\
& +\left|\frac{\alpha_n}{1-\beta_n}-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|M+\frac{1-\alpha_{n+1}-\beta_{n+1}}{1+\alpha_{n+1}+\beta_{n+1}}\left(|\alpha_n-\alpha_{n+1}|+|\beta_{n+1}-\beta_n|\right)M \\
<\; & \left(1-\frac{2\alpha_{n+1}(1-c)}{1-\beta_{n+1}}\right)||x_{n+1}-x_n||+\left|\frac{\alpha_n}{1-\beta_n}-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|M \\
& +\frac{1-\alpha_{n+1}-\beta_{n+1}}{(1+\alpha_{n+1}+\beta_{n+1})(1-\beta_{n+1})}\left(|\alpha_n-\alpha_{n+1}|+|\beta_{n+1}-\beta_n|\right)M.
\end{aligned}
$$

Thus,

$$
\limsup_{n\to\infty}\left(||y_{n+1}-y_n||-||x_{n+1}-x_n||\right)\le 0.
$$

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Hence, by Lemma 2.10.7, we have

$$\lim_{n \to \infty} ||y_n - x_n|| = 0.$$

Step 3: We show that $||x_n - Tx_n|| \to 0$ as $n \to \infty$.

We observe from (4.1.2) that

$$
\begin{aligned}
x_{n+1} - x_n &= \beta_n x_n + (1 - \beta_n)y_n - x_n \\
&= (1 - \beta_n)y_n - (1 - \beta_n)x_n \\
&= (1 - \beta_n)(y_n - x_n).
\end{aligned}
$$

Therefore

$$||x_{n+1} - x_n|| \leq (1 - \beta_n)||y_n - x_n|| \to 0 \text{ as } n \to \infty. \qquad (4.1.6)$$

Also, from (1.1.6), we obtain that

$$
\begin{aligned}
||x_n - Tx_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - Tx_n|| \\
&= ||x_n - x_{n+1}|| + \alpha_n||f(x_n) - Tx_n|| + \beta_n||x_n - Tx_n|| \\
&\quad + \gamma_n||T(\frac{x_n + x_{n+1}}{2}) - Tx_n|| \\
&= ||x_n - x_{n+1}|| + \alpha_n||f(x_n) - Tx_n|| + \beta_n||x_n - Tx_n|| \\
&\quad + (1 - \alpha_n - \beta_n)||\frac{x_n + x_{n+1}}{2} - x_n|| \\
&= ||x_n - x_{n+1}|| + \alpha_n||f(x_n) - Tx_n|| + \beta_n||x_n - Tx_n|| \\
&\quad + \frac{(1 - \alpha_n - \beta_n)}{2}||x_n - x_{n+1}||.
\end{aligned}
$$

By (4.1.3), we obtain that

$$
\begin{aligned}
||x_n - Tx_n|| &\leq \frac{3 - \alpha_n - \beta_n}{2(1 - \beta_n)}||x_n - x_{n+1}|| + \frac{\alpha_n}{1 - \beta_n}||f(x_n) - Tx_n|| \\
&\leq \frac{3 - \alpha_n - \beta_n}{2(1 - \beta)}||x_n - x_{n+1}|| \\
&\quad + \frac{\alpha_n}{1 - \beta}||f(x_n) - Tx_n|| \to 0 \text{ as } n \to \infty. \qquad (4.1.7)
\end{aligned}
$$

Step 4: For $t \in (0, 1)$ and $f \in \Pi_K$, define the sequence $\{x_t\}$ by $x_t = tf(x_t) + (1 - t)Tx_t$. By Lemma 2.9.5, as $t \to 0$, x_t strongly converges to a fixed point q of T, which is also a solution to the variational inequality

$$\langle (I - f)q, J(x - q) \rangle \geq 0, \ x \in F(T).$$

Using Lemma 2.9.6 and since $||x_{n+1} - x_n|| \to 0$ and $||x_n - Tx_n|| \to 0$ as $n \to 0$ (by (4.1.6) and (4.1.7) respectively), we get

$$\limsup_{n \to \infty} \langle f(q) - q, J(x_{n+1} - q) \rangle \leq 0. \tag{4.1.8}$$

Step 5: Lastly, we prove that $x_n \to q$.

$$
\begin{aligned}
||x_{n+1} - q||^2 &= \alpha_n \langle f(x_n) - f(q), J(x_{n+1} - q) \rangle \\
&\quad + \alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle + \beta_n \langle x_n - q, J(x_{n+1} - q) \rangle \\
&\quad + (1 - \alpha_n - \beta_n) \left\langle T(\frac{x_n + x_{n+1}}{2}) - q, J(x_{n+1} - q \right\rangle \\
&\leq c\alpha_n ||x_n - q|| \, ||x_{n+1} - q|| + \alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\quad + \beta_n ||x_n - q|| \, ||x_{n+1} - q|| \\
&\quad + \frac{1 - \alpha_n - \beta_n}{2}(||x_n - q|| + ||x_{n+1} - q||)||x_{n+1} - q|| \\
&= c\alpha_n ||x_n - q|| \, ||x_{n+1} - q|| + \alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\quad + \beta_n ||x_n - q|| \, ||x_{n+1} - q|| \\
&\quad + \frac{1 - \alpha_n - \beta_n}{2}(||x_n - q|| \, ||x_{n+1} - p|| + ||x_{n+1} - q||^2) \\
&= \left(c\alpha_n + \beta_n + \frac{1 - \alpha_n - \beta_n}{2} \right) ||x_n - q|| \, ||x_{n+1} - q|| \\
&\quad + \frac{1 - \alpha_n - \beta_n}{2}||x_{n+1} - q||^2 + \alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\leq \frac{1 + \beta_n - (1 - 2c)\alpha_n}{4}(||x_n - q||^2 + ||x_{n+1} - q||^2) \\
&\quad + \frac{1 - \alpha_n - \beta_n}{2}||x_{n+1} - q||^2 + \alpha_n \langle f(p) - q, J(x_{n+1} - q) \rangle \\
&\leq \frac{1 + \beta_n - (1 - 2c)\alpha_n}{4}||x_n - q||^2 + \frac{3 - \beta_n - (3 - 2c)\alpha_n}{4}||x_{n+1} - q||^2 \\
&\quad + \alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
||x_{n+1} - q||^2 &\leq \frac{1 + \beta_n - (1 - 2c)\alpha_n}{1 + \beta_n + \alpha_n(3 - 2c)}||x_n - q||^2 \\
&\quad + \frac{4\alpha_n}{1 + \beta_n + \alpha_n(3 - 2c)} \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&= \left(1 - \frac{4(1 - c)\alpha_n}{1 + \beta_n + \alpha_n(3 - 2c)} \right) ||x_n - q||^2 \\
&\quad + \frac{4\alpha_n}{1 + \beta_n + \alpha_n(3 - 2c)} \langle f(q) - q, J(x_{n+1} - q) \rangle. \tag{4.1.9}
\end{aligned}
$$

By applying Lemma 2.10.6 with $\gamma_n = 0$ to (4.1.8) and (4.1.9), we deduce that $x_n \to q$ as $n \to \infty$. ∎

Corollary 4.1.7 *[107] Let K be a nonempty closed convex subset of a Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $f : K \to K$ be a c-contraction. For given $x_0 \in K$ arbitrarily, let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \geq 0, \qquad (4.1.10)$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\gamma_n\} \subset (0,1)$ are three sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 0$. Assume that $\{\alpha_n\}$ satisfies (A_1) and (A_2) and $\{\beta_n\}$ satisfies

(A_6) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$ *and*

(A_8) $\lim\limits_{n\to\infty} (\beta_{n+1} - \beta_n) = 0.$

Then the sequence $\{x_n\}$ generated by (4.1.10) converges strongly to $p = P_{F(T)}f(p)$.

4.1.3 Application to accretive mappings

Let E be a real Banach space and K be a nonempty closed convex subset of E. The set of zero of an accretive mapping A is denoted by $A^{-1}(0)$, that is $A^{-1}(0) = \{z \in D(A) : A(z) = 0\}$. We denote the resolvent of A by $J_r^A = (I + rA)^{-1}$ for each $r > 0$ ([7], [87]). It is known that if A is m-accretive then $J_r^A : E \to D(A)$ is nonexpansive and $F(J_r^A) = A^{-1}(0)$ for each $r > 0$. Consequently, we can deduce the result below from Theorem 4.1.6.

Corollary 4.1.8 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $f : K \to K$ be a c-contraction. Let $A : K \to K$ be an accretive mapping such that $R(I + rA) = E$ for all $r > 0$ with $A^{-1}(0) \neq \emptyset$. Suppose $\{\alpha_n\}$ satisfies*

(A_1) $\lim\limits_{n\to\infty} \alpha_n = 0;$

(A_2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$

and $\{\beta_n\}$ satisfies

(A_6) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$ *and*

(A_7) $\lim\limits_{n\to\infty} |\beta_{n+1} - \beta_n| = 0.$

For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_r^A \left(\frac{x_n + x_{n+1}}{2} \right), n \in \mathbb{N}. \tag{4.1.11}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\gamma_n\} \subset (0,1)$ are real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1 \ \forall \ n \in \mathbb{N}$. Then as $n \to \infty$, the sequence $\{x_n\}$ converges in norm to $p \in A^{-1}(0)$, where p is the unique solution to the variational inequality:

$$\langle (I - f)p, J(x - p) \rangle \geq 0 \ \forall \ x \in A^{-1}(0).$$

4.1.4 Application to variational inequality problems

Let H be a Hilbert space with inner product $\langle .,. \rangle$. Let K be a nonempty closed convex subset of H and $A : K \to H$ be a nonlinear mapping. The variational inequality problem is finding $x^* \in K$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \text{ for all } x \in K. \tag{4.1.12}$$

We denote the set of all solutions of the variational inequality (4.1.12) by $VI(K, A)$. We shall consider the system of general variational inequalities in Banach spaces recently introduced by Katchang and Kumam [63]. Given two operators $A_1, A_2 : K \to E$, where E is a real Banach space, where K is a nonempty closed convex subset E. The authors considered the problem of finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} \langle \alpha_1 A_1 y^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall \ x \in K, \\ \langle \alpha_2 A_2 y^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall \ x \in K, \end{cases} \tag{4.1.13}$$

where α_1 and α_2 are two positive real numbers and $j(x - x^*) \in J(x - x^*)$. Recall that a nonlinear mapping $A : K \to E$ is called μ-inverse strongly accretive if there exist $j(x - y) \in J(x - y)$ and $\mu > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \mu \|Ax - Ay\|^2, \ \forall \ x, y \in K.$$

We need the two Lemmas below to establish our next result.

Lemma 4.1.9 *[59]. Let K be a nonempty closed convex subset of a real Banach space E and let $\alpha_1, \alpha_2 > 0$ and $A_1, A_2 : K \to E$ be two mappings. Let $G : K \to K$ be defined by*

$$G(x) = S_K[S_K(x - \alpha_2 A_2 x) - \alpha_1 A_1 S_K(x - \alpha_2 A_2 x)], \ \forall \ x \in K,$$

where S_K is a sunny nonexpansive retraction from E onto K. If $I - \alpha_1 A_1$ and $I - \alpha_2 A_2$ are nonexpansive mappings, then G is nonexpansive.

Lemma 4.1.10 *[63] Let K be a nonempty closed convex subset of a real smooth Banach space E. Let S_K be a sunny nonexpansive retraction from E onto K. Let $A_1, A_2 : K \to E$ be two nonlinear mappings. For given $x^*, y^* \in K, (x^*, y^*)$ is a solution of problem (4.1.13) if and only if $x^* = S_K(y^* - \alpha_1 A_1 y^*)$ where $y^* = S_K(x^* - \alpha_2 A_2 x^*)$.*

Corollary 4.1.11 *Let K be a nonempty closed convex subset of a 2-uniformly smooth Banach space E and $f : K \to K$ be a c-contraction. Let $A_1, A_2 : K \to E$ be two possibly nonlinear mappings and G be a mapping defined in Lemma 4.1.9 with $F(G) \neq \emptyset$. Let S_K be a sunny nonexpansive retraction from E onto K. Suppose $\{\alpha_n\}$ satisfies*

(A_1) $\displaystyle\lim_{n\to\infty} \alpha_n = 0;$

(A_2) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$

and $\{\beta_n\}$ satisfies

(A_6) $0 < \displaystyle\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and*

(A_7) $\displaystyle\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0.$

For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \\ y_n = S_K(u_n - \alpha_1 A_1 u_n), \\ u_n = S_K(v_n - \alpha_2 A_1 v_n), \\ v_n = \frac{x_n + x_{n+1}}{2}, \end{cases} \tag{4.1.14}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\gamma_n\} \subset (0,1)$ are real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1 \; \forall \; n \in \mathbb{N}$. Then as $n \to \infty$, the sequence $\{x_n\}$ converges in norm to a fixed point p of G, where p is the unique solution to the variational inequality:

$$\langle (I - f)p, J(x - p) \rangle \geq 0 \; \forall \; x \in F(G).$$

Remark 4.1.12 *Nonlinear mappings that satisfy Theorem 4.1.11 are readily available. Let L be the 2-uniformly smooth constant of a 2-uniformly smooth Banach space and $A_1, A_2 : K \to E$ be μ_1-inverse strongly accretive and μ_2-inverse strongly accretive, respectively. If $0 < \alpha_1 < \frac{\mu_1}{L^2}$ and $0 < \alpha_2 < \frac{\mu_2}{L^2}$, then $I - \alpha_1 A_1$ and $I - \alpha_2 A_2$ are nonexpansive [59].*

4.1.5 Numerical examples

Example 4.1.13 *Let $\mathbb{R}$ be the real line with the Euclidean norm. Let $f, T : \mathbb{R} \to \mathbb{R}$ be maps defined by $f(x) = \frac{1}{4}x$ and $T(x) = 2 - x$ for all $x \in \mathbb{R}$, respectively. It is clear that T is a nonexpansive mapping and $F(T) = \{1\}$. Let $\{z_n\}, \{y_n\}$ and $\{x_n\}$ be the sequences generated by (1.1.10), (1.1.5) and (1.1.6) respectively. We find that $\{z_n\}, \{y_n\}$ and $\{x_n\}$ strongly converge to 1 (by [76], Theorem 4.1.1 of [104] and Theorem 4.1.6, respectively). Take $\alpha_n = \frac{2}{4n+5}$, $n \in \mathbb{N}$ in (1.1.10) and (1.1.5). Notice that the parameters in (1.1.6) are arbitrary sequences satisfying the conditions stated in Theorem 4.1.6. Therefore, the sequence $\{\alpha_n\}$ in (1.1.6) is not necessarily the same as the one in (1.1.10) and (1.1.5). Thus, for the iterative scheme defined by (1.1.6), we choose $\alpha_n = \frac{1}{4n+5}, \beta_n = \frac{n+4}{4n+5}$ and $\gamma_n = \frac{3n}{4n+5}$ for all $n \in \mathbb{N}$. One can rewrite (1.1.10), (1.1.5) and (1.1.6) as follow:*

$$z_{n+1} = \frac{2n+1}{2n+3}z_n + \frac{2}{2n+3}, \tag{4.1.15}$$

$$y_{n+1} = -\frac{4n+2}{12n+13}y_n + \frac{4(4n+3)}{12n+13}, \tag{4.1.16}$$

$$x_{n+1} = \frac{17-2n}{2(11n+10)}x_n + \frac{12n}{11n+10}. \tag{4.1.17}$$

Using Matlab 2015a and by taking $z_1 = y_1 = x_1 = 0$, the results for (4.1.15), (4.1.16) and (4.1.17) are displayed in Table 4.1.1 and Figure 4.1.1. The graphs show that the three algorithms converge to 1 with the iterative algorithm (1.1.6) having the
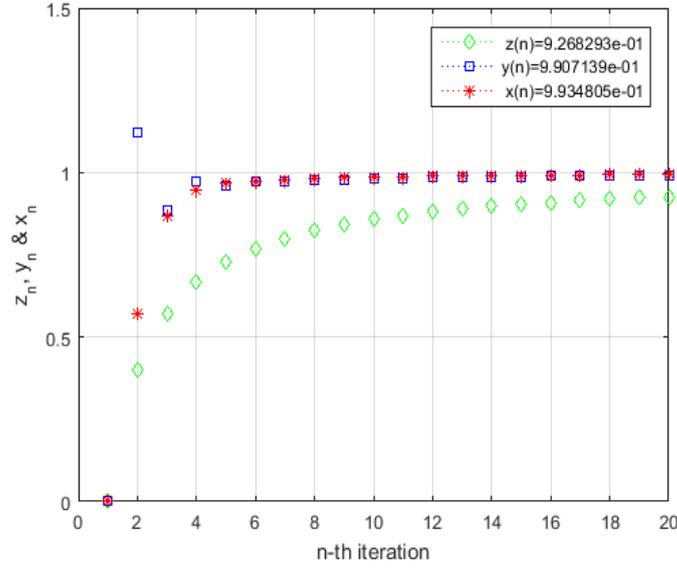
Figure 4.1.1: Comparison of the rates of convergence for the iterative schemes (1.1.10), (1.1.5) and (1.1.6) with different values for α_n.

highest rate of convergence for the viscosity implicit midpoint rule. Therefore, it is the most efficient among the three algorithms.

Remark 4.1.14 *It is worth of mentioning that the efficiency of (1.1.6) depends on the choice of suitable control parameters.*

The next example displays the result where α_n is the same for all the three iterative schemes.

Example 4.1.15 *Let f and T be as defined in Example 4.1.13. Then for the iterative scheme defined by (1.1.6), choose $\alpha_n = \frac{2}{4n+5}, \beta_n = \frac{n+1}{4n+5}$ and $\gamma_n = \frac{3n+2}{4n+5}$ for all $n \in \mathbb{N}$. The equation (4.1.17) then becomes*

$$x_{n+1} = \frac{1-n}{2(11n+12)}x_n + \frac{4(3n+2)}{11n+12}. \tag{4.1.18}$$

The results are presented in Figure 4.1.2 and Table 4.1.2 with the algorithm (1.1.5) having the highest rate of convergence.

The next example compares the convergence rate where where α_n is greater for (1.1.6).

Table 4.1.1: Comparison of the rates of convergence for the iterative schemes (1.1.10), (1.1.5) and (1.1.6) with different values for α_n.

| iteration (n)	z_n e-01	y_n e-01	x_n e-01
1	0	0	0
2	4.000000	11.20000	5.714286
3	5.714286	8.864865	8.660714
4	6.666667	9.712079	9.479859
5	7.272727	9.593157	9.678877
6	7.692308	9.711651	9.751940
7	8.000000	9.735260	9.794472
8	8.235294	9.772600	9.824043
9	8.421053	9.795703	9.846041
10	8.571429	9.816226	9.863092
11	8.695652	9.832470	9.876711
12	8.800000	9.846251	9.887849
13	8.888889	9.857882	9.897130
14	8.965517	9.867896	9.904986
15	9.032258	9.876586	9.911723
16	9.090909	9.884206	9.917565
17	9.142857	9.890939	9.922679
18	9.189189	9.896932	9.927194
19	9.230769	9.902301	9.931210
20	9.268293	9.907139	9.934805

Table 4.1.2: Comparison of the rates of convergence for the iterative schemes (1.1.10), (1.1.5) and (1.1.6) with same value for α_n.

iteration (n)	z_n e-01	y_n e-01	x_n e-01
1	0	0	0
2	4.000000	11.20000	8.695652
3	5.714286	8.864865	9.156010
4	6.666667	9.712079	9.370844
5	7.272727	9.593157	9.497991
6	7.692308	9.711651	9.582210
7	8.000000	9.735260	9.642166
8	8.235294	9.772600	9.687045
9	8.421053	9.795703	9.721907
10	8.571429	9.816226	9.749772
11	8.695652	9.832470	9.772558
12	8.800000	9.846251	9.791537
13	8.888889	9.857882	9.807591
14	8.965517	9.867896	9.821348
15	9.032258	9.876586	9.833268
16	9.090909	9.884206	9.843696
17	9.142857	9.890939	9.852897
18	9.189189	9.896932	9.861074
19	9.230769	9.902301	9.868389
20	9.268293	9.907139	9.874973
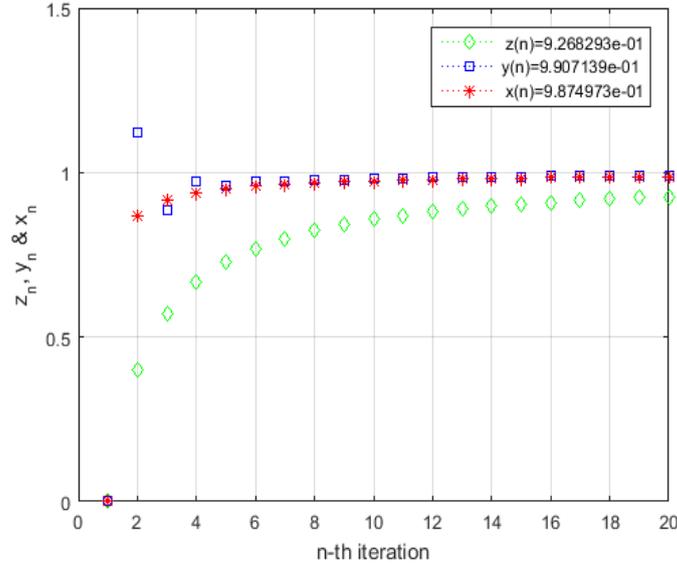
Figure 4.1.2: Comparison of the rates of convergence for the iterative schemes (1.1.10), (1.1.5) and (1.1.6) with same value for α_n.

Example 4.1.16 *Let f and T be as defined in Example 4.1.13. Then for the iterative scheme defined by (1.1.6), choose $\alpha_n = \frac{4}{4n+5}, \beta_n = \frac{n+1}{4n+5}$ and $\gamma_n = \frac{3n}{4n+5}$ for all $n \in \mathbb{N}$. The equation (4.1.17) then becomes*

$$x_{n+1} = \frac{4-n}{2(11n+10)}x_n + \frac{12n}{11n+10}. \tag{4.1.19}$$

The results are presented in Figure 4.1.3 and Table 4.1.3 with the algorithm (1.1.5) having the highest rate of convergence.

Example 4.1.17 *Let $E = \mathbb{R}^2$ with the usual norm and $f, T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x) = \frac{1}{2}x$ and $T(x) = 0$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ respectively. Take $\alpha_n = \frac{4}{4n+5}, \beta_n = \frac{1}{4} - \frac{1}{4n+5}$ and $\gamma_n = \frac{12n+3}{4(4n+5)}$ for all $n \in \mathbb{N}$. Observe that α_n, β_n and γ_n satisfy the conditions of Theorem 4.1.6 and T is nonexpansive. Indeed, for $x, y \in \mathbb{R}^2$*

$$\|Tx - Ty\| = 0 \leq \|x - y\|.$$

Also, it is obvious that $F(T) = \{0\}$. Therefore, $\{x_n\}$ strongly converges to 0. A simple computation shows that (1.1.6) is equivalent to:

$$x_{n+1} = \frac{4n+9}{4(4n+5)}x_n. \tag{4.1.20}$$

75
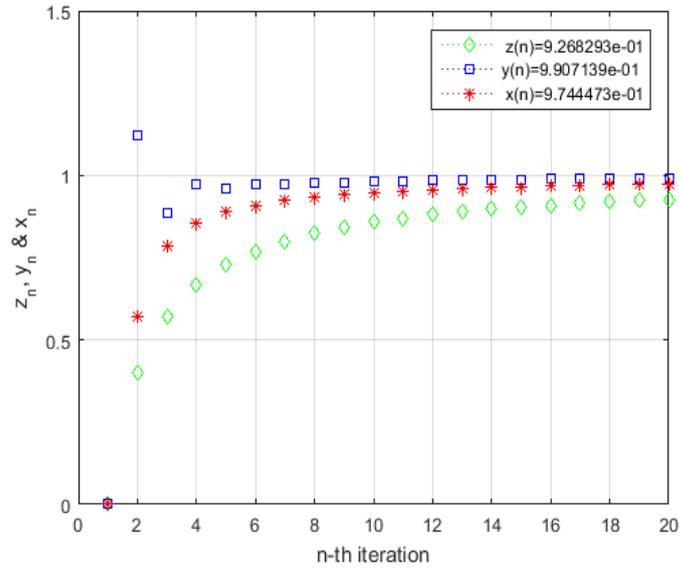
Figure 4.1.3: Comparison of the rates of convergence for the iterative schemes (1.1.10), (1.1.5) and (1.1.6) where α_n is greater for (1.1.6)
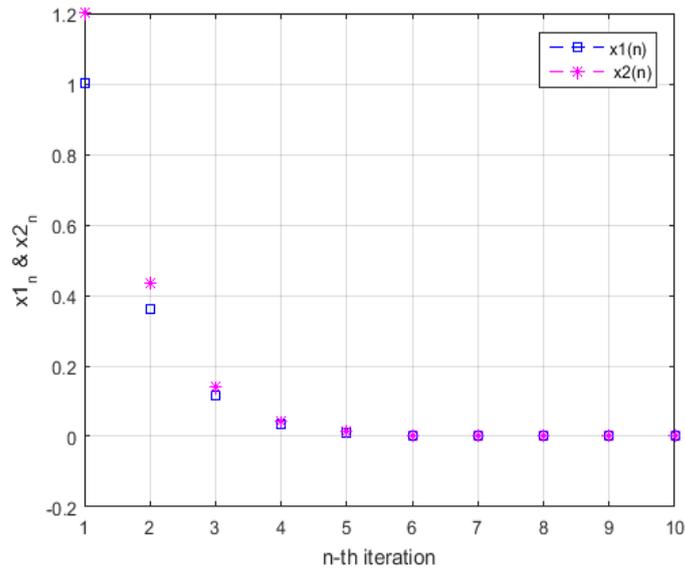
Figure 4.1.4: Two dimensional figure for (4.1.20).

Table 4.1.3: Comparison of the rates of convergence for the iterative schemes (1.1.10), (1.1.5) and (1.1.6) where α_n is greater for (1.1.6)

| iteration (n)	z_n e-01	y_n e-01	x_n e-01
1	0	0	0
2	4.000000	11.20000	5.714286
3	5.714286	8.864865	7.857143
4	6.666667	9.712079	8.554817
5	7.272727	9.593157	8.888889
6	7.692308	9.711651	9.094017
7	8.000000	9.735260	9.234368
8	8.235294	9.772600	9.336746
9	8.421053	9.795703	9.414827
10	8.571429	9.816226	9.476384
11	8.695652	9.832470	9.526181
12	8.800000	9.846251	9.567303
13	8.888889	9.857882	9.601842
14	8.965517	9.867896	9.631264
15	9.032258	9.876586	9.656630
16	9.090909	9.884206	9.678726
17	9.142857	9.890939	9.698147
18	9.189189	9.896932	9.715351
19	9.230769	9.902301	9.730698
20	9.268293	9.907139	9.744473

Table 4.1.4: Values of iteration for (4.1.20).

iteration (n)	$x_1(n)$	$x_2(n)$
1	1.0	1.2
2	3.611111e-01	4.333333e-01
3	1.180556e-01	1.416667e-01
4	3.645833e-02	4.375000e-02
5	1.085069e-02	1.302083e-02
6	3.146701e-03	3.776042e-03
7	8.951823e-04	1.074219e-03
8	2.509223e-04	3.011068e-04
9	6.951226e-05	8.341471e-05
10	1.907349e-05	2.288818e-05

Choosing the initial point for (4.1.20) to be $(1.0, 1.2)$, Table 4.1.4 and Figure 4.1.4 show the results from the Matlab 2015a.

Conclusion 4.1.18 *We have considered the implicit midpoint rule of nonexpansive mappings, using the viscosity approximation method in the framework of Banach spaces. Our method of proof is of independent interest and our result extends the main result of Yao et al. [107] to uniformly Banach spaces. The numerical examples show the application of our work and the efficiency of the algorithm over the existing ones. Moreover, we obtained the results of Xu et al. [104], Yao et al. [107] and Luo et al. [72] as corollaries. It is observed that the iterative scheme (1.1.6) converges faster than (1.1.5) with the following two conditions:*

(i) The value of α_n in (1.1.6) is less than the value of α_n in (1.1.5);

(ii) The sum of values of α_n and γ_n in (1.1.6) is greater than the value of α_n in (1.1.5).

4.2 On the rate of convergence of viscosity implicit iterative algorithms

[2]

4.2.1 Background

In 2000, Moudafi [76] introduced a well-known iterative method known as the viscosity approximation method for approximating fixed points of a nonexpansive mapping. Later in 2004, Xu [102] applied a technique which uses (strict) contractions to regularize a nonexpansive mapping for the purpose of selecting a particular fixed point of the nonexpansive mapping and studied the sequence (1.1.1). Xu [102] showed that under suitable conditions imposed on the parameters, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.1.1), converges strongly in Hilbert spaces to a fixed point p of a nonexpansive mapping T which also solves the following variational inequality (1.1.2). Recently, Xu et al. [104] introduced the implicit midpoint procedure (1.1.5). They proved a strong convergence theorem for the sequence $\{x_n\}_{n=1}^{\infty}$ to a fixed point p of T which also solves the variational inequality (1.1.2) in Hilbert spaces. Yao et al. [107] extended the work of Xu et al. [104] and considered the implicit midpoint sequence (1.1.6). Under certain conditions on the parameters, they obtained that the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.1.6) converges strongly to $p = P_{F(T)}f(p)$. In other words, the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.1.6) converges in norm to a fixed point p of T, which is also the unique solution of the variational inequality (1.1.2).

Luo et al. [72] studied the convergence of the sequence (1.1.5) in uniformly smooth Banach spaces. Furhermore, they used a numerical example to compare the rate of convergence of the sequences (1.1.1) and (1.1.5). Also, numerical methods were used by Aibinu et al. [6] to compare the rate of convergence of the iteration procedures (1.1.10), (1.1.5) and (1.1.6) in uniformly smooth Banach spaces. Ke and Ma [65] chose $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ and generalized the viscosity implicit midpoint

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[2]The results of this section are contents of the following paper

M.O. Aibinu [1]

rules of Xu et al. [104] and Yao et al. [107] to the two viscosity implicit rules (1.1.7) and (1.1.8). It was shown that the sequences generated by (1.1.7) and (1.1.8) converge strongly to a fixed point p of the nonexpansive mapping T, which solves the variational inequality (1.1.2). Extension of the main results of Ke and Ma [65] from Hilbert spaces to uniformly smooth Banach spaces was considered by Yan et al. [106]. Then, the following questions arise naturally:

Question 4.2.1 *Do the sequences (1.1.7) and (1.1.8) which are respectively given by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right), \ n \in \mathbb{N},$$

and

$$y_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n T\left(\delta_n y_n + (1 - \delta_n)y_{n+1}\right), \ n \in \mathbb{N},$$

always converge to the same fixed point of a nonexpansive mapping?

Question 4.2.2 *Do the results of Ke and Ma [65] hold for finite combination of nonexpansive mappings, composition of finite family of nonexpansive mappings and monotone mappings?*

In this section, an affirmative answers are given to those questions raised above. Under suitable conditions imposed on the control parameters, the analytical proof is given to show that the two sequences converge to the same fixed point of a nonexpansive mapping. Moreover, it is shown analytically that the sequence (1.1.8) converges faster than (1.1.7) in approximating a fixed point of a nonexpansive mapping.

4.2.2 Main results

Here, the analytical proof is given to ascertain that the implicit iterative sequences (1.1.7) and (1.1.8) converge to the same fixed point of a nonexpansive mapping.

Theorem 4.2.3 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E, T a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$ and $f : K \to K$, a c-contraction mapping. Given that $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ with*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\displaystyle\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$.

Then (1.1.8) converges in norm to p if and only if (1.1.7) converges in norm to p.

Proof.

We show that (1.1.7) and (1.1.8) converge to the same fixed point of a nonexpansive mapping T.

$$
\begin{aligned}
\|y_{n+1} - x_{n+1}\| &= \|\alpha_n f(y_n) + \beta_n y_n + \gamma_n T(\delta_n y_n + (1-\delta_n)y_{n+1}) \\
&\quad - (\alpha_n f(x_n) + (1-\alpha_n)T(\delta_n x_n + (1-\delta_n)x_{n+1}))\| \\
&= \|\alpha_n(f(y_n) - f(x_n)) + \beta_n(y_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})) \\
&\quad + \gamma_n\left(T(\delta_n y_n + (1-\delta_n)y_{n+1}) - T(\delta_n x_n + (1-\delta_n)x_{n+1})\right)\| \\
&\leq \alpha_n\|f(y_n) - f(x_n)\| + \beta_n\|y_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})\| \\
&\quad + \gamma_n\|T(\delta_n y_n + (1-\delta_n)y_{n+1}) - T(\delta_n x_n + (1-\delta_n)x_{n+1})\| \\
&\leq c\alpha_n\|y_n - x_n\| + \beta_n\|y_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})\| \\
&\quad + \gamma_n\|\delta_n(y_n - x_n) + (1-\delta_n)(y_{n+1} - x_{n+1})\| \\
&\leq c\alpha_n\|y_n - x_n\| + \beta_n\|y_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})\| \\
&\quad + \gamma_n\delta_n\|y_n - x_n\| + \gamma_n(1-\delta_n)\|y_{n+1} - x_{n+1}\| \\
&\leq (c\alpha_n + \gamma_n\delta_n)\|y_n - x_n\| + \beta_n\|y_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})\| \\
&\quad + \gamma_n(1-\delta_n)\|y_{n+1} - x_{n+1}\|.
\end{aligned}
$$

Since $\{y_n\}_{n=1}^{\infty}$ and $\{T(\delta_n x_n + (1-\delta_n)x_{n+1})\}_{n=1}^{\infty}$ are bounded [106], let $M := \sup_n \|y_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})\|$. Then,

$$\begin{aligned}
\|y_{n+1} - x_{n+1}\| &\leq \frac{c\alpha_n + \gamma_n\delta_n}{1 - \gamma_n(1 - \delta_n)}\|y_n - x_n\| + \frac{\beta_n}{1 - \gamma_n(1 - \delta_n)}M \\
&= 1 + \frac{c\alpha_n - (1 - \gamma_n)}{1 - \gamma_n(1 - \delta_n)}\|y_n - x_n\| + \frac{\beta_n}{1 - \gamma_n(1 - \delta_n)}M \\
&= 1 + \frac{-\beta_n - (1 - c)\alpha_n}{1 - \gamma_n(1 - \delta_n)}\|y_n - x_n\| + \frac{\beta_n}{1 - \gamma_n(1 - \delta_n)}M \\
&= \left(1 - \frac{(1 - c)\alpha_n + \beta_n}{1 - \gamma_n(1 - \delta_n)}\right)\|y_n - x_n\| + \frac{\beta_n}{1 - \gamma_n(1 - \delta_n)}M \\
&= \left(1 - \frac{(1 - c)\alpha_n + \beta_n}{1 - \gamma_n(1 - \delta_n)}\right)\|y_n - x_n\| + \frac{\beta_n}{1 - \gamma_n(1 - \delta_n)}M \\
&\leq \left(1 - \frac{(1 - c)\alpha_n}{1 - \gamma_n(1 - \delta_n)}\right)\|y_n - x_n\| + \frac{\beta_n}{1 - \gamma_n(1 - \delta_n)}M \\
&= \left(1 - \frac{(1 - c)\alpha_n}{1 - \gamma_n(1 - \delta_n)}\right)\|y_n - x_n\| + \frac{(1 - c)\alpha_n}{1 - \gamma_n(1 - \delta_n)}\frac{\beta_n}{(1 - c)\alpha_n}M \\
&= (1 - \sigma_n)\|y_n - x_n\| + \frac{\beta_n}{(1 - c)\alpha_n}\sigma_n M, \tag{4.2.1}
\end{aligned}$$

where $\sigma_n = \frac{(1-c)\alpha_n}{1-\gamma_n(1-\delta_n)}$. Notice that $\limsup\limits_{n\to\infty} \frac{\beta_n}{\alpha_n} \leq 0$. Then, we can apply Lemma 2.10.6 with $\gamma_n = 0$ to (4.2.1) in order to deduce that $\|y_n - x_n\| \to 0$ as $n \to \infty$. Furthermore, suppose $\|x_n - p\| \to 0$ as $n \to \infty$, we have that

$$\|y_n - p\| = \|y_n - x_n + x_n - p\| \leq \|y_n - x_n\| + \|x_n - p\| \to 0 \text{ as } n \to \infty.$$

Similarly, suppose $\|y_n - p\| \to 0$ as $n \to \infty$, we have that

$$\|x_n - p\| = \|x_n - y_n + y_n - p\| \leq \|x_n - y_n\| + \|y_n - p\| \to 0 \text{ as } n \to \infty.$$

$\blacksquare$

4.2.3 Applications

The results in this section show an improvement on and generalization of the main results of Xu et al. [104], Yao et al. [107] and Ke and Ma [65]. It will be assumed that the real sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty} \subset [0, 1]$ and $\{\delta_n\}_{n=1}^{\infty} \subset (0, 1)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(iii) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$

(iv) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ for all $n \in \mathbb{N}$.

(I) Finite combination of nonexpansive mappings

The proof of the proposition below is given in Wong et al. [99].

Proposition 4.2.4 *Let K be a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space E and let $\theta_i > 0$ $(i = 1, 2, \ldots, r)$ such that $\sum_{i=1}^{r} \theta_i = 1$. Let $T_1, T_2, \ldots, T_r : K \to K$ be nonexpansive mappings with $\cap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $T = \sum_{i=1}^{r} \theta_i T_i$. Then T is nonexpansive from K into itself and $F(T) = \cap_{i=1}^{r} F(T_i)$.*

Therefore, we have the following result.

Corollary 4.2.5 *Suppose K is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space E, $f : K \to K$ is a c-contraction and let $\theta_i > 0$ $(i = 1, 2, \ldots, r)$ such that $\sum_{i=1}^{r} \theta_i = 1$. Let $T_1, T_2, \ldots, T_r : K \to K$ be nonexpansive mappings with $\cap_{i=1}^{r} F(T_i) \neq \emptyset$. Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ which is defined from an arbitrary $x_1 \in K$ by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{r} \theta_i T_i \left(\delta_n x_n + (1 - \delta_n) x_{n+1} \right), \qquad (4.2.2)$$

converges strongly to a fixed point $p \in \cap_{i=1}^{r} F(T_i)$, which solves the variational inequality

$$\langle (I - f)p, J(x - p) \rangle \geq 0, \ \text{for all } x \in \cap_{i=1}^{r} F(T_i). \qquad (4.2.3)$$

Proof. Define $T := \sum_{i=1}^{r} \theta_i T_i$. It suffices to show that T is a nonexpansive mapping and $\cap_{i=1}^{r} F(T_i) \subseteq F(T)$. This is true by by Proposition 4.2.4. ∎

(II) Composition of finite family of nonexpansive mappings

Corollary 4.2.6 *Suppose K is a nonempty closed convex subset of a uniformly smooth Banach space E and $\{T_t\}_{t=1}^{N}$ a finite family of nonexpansive self-mappings of K such that $F := \cap_{t=1}^{N} F(T_t) \neq \emptyset$. Let $f : K \to K$ be a c-contraction. Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ which is defined from an arbitrary $x_1 \in K$ by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^N T^{N-1} T^{n-2}...T^1 \left(\delta_n x_n + (1 - \delta_n) x_{n+1} \right),$$

converges strongly to a fixed point $p \in F$, which solves the variational inequality

$$\langle (I - f)p, J(x - p) \rangle \geq 0, \ for \ all \ x \in F. \tag{4.2.4}$$

Proof. It is known that a composition of finite family of nonexpansive self-mappings $\{T_t\}_{t=1}^{N}$ on K is nonexpansive with $F(T) \supseteq \cap_{t=1}^{N} F(T_t) \neq \emptyset$. ∎

(III) Monotone mappings

Let E be a real Banach space with the duality pairing $\langle ., . \rangle$ and norm $\|.\|$. The dual of E is denoted by E^*. Let A be a set-valued mapping and denote the domain and range of A by $D(A)$ and $R(A)$, respectively. Monotone mappings have been studied extensively (see, e.g., Bruck [21], Chidume [24], Martinet [74], Reich [87], Rockafellar [89]) due to their role in convex analysis, in nonlinear analysis, in certain partial differential equations and optimization theory. For a maximal monotone mapping $A : D(A) \to 2^{E^*}$ (Kohsaka and Takahashi [57]), one can define the resolvent of A by

$$J_t^A = (J + tA)^{-1} J, \ t > 0. \tag{4.2.5}$$

It is well known that $J_t^A : E \to D(A)$ is nonexpansive, and $F(J_t^A) = A^{-1}(0)$, where $F(J_t)$ denotes the set of fixed points of J_t.

We can then have the following.

Corollary 4.2.7 *Suppose K is a nonempty closed convex subset of a uniformly smooth Banach space E, $f : K \to K$ is a c-contraction and let $\theta_i > 0$ $(i = 1, 2, \ldots, r)$ such that $\sum_{i=1}^{r} \theta_i = 1$. Let $A_i \subset E \times E^*$ be a family of maximal monotone mappings with resolvent $J_t^{A_i}$ for $t > 0$ such that $\cap_{i=1}^{r} A_i^{-1} 0 \neq \emptyset$. Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ which is defined from an arbitrary $x_1 \in K$ by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{r} \theta_i J_t^{A_i} \left(\delta_n x_n + (1 - \delta_n) x_{n+1} \right),$$

converges strongly to a unique solution $p \in \cap_{i=1}^{r} A_i^{-1} 0$, which solves the variational inequality:

find $p \in \cap_{i=1}^{r} A_i^{-1} 0$ such that $\langle (I - f)p, J((x - p) \rangle \geq 0$ for all $x \in \cap_{i=1}^{r} A_i^{-1} 0$.

Proof. Define $T := \sum_{i=1}^{r} \theta_i J_t^{A_i}$. Then T is nonexpansive self-mapping of K and $F(T) \supseteq \cap_{i=1}^{r} F(T_i) \neq \emptyset$. ∎

Implicit iterative procedures based on generalized contractions

We study the implicit iterative procedures which are based on generalized contractions. The implicit iterative procedure is examined for approximating the fixed points of a class of μ-strictly pseudo-contractive mapping. A new implicit iterative procedure based on generalized contractions is also introduced for the class of nonexpansive mappings.

5.1 The implicit iterative algorithms of strictly pseudo-contractive mappings in Banach spaces

5.1.1 Background

Let K be a nonempty, closed and convex subset of a real Banach space E and $f : K \to K$ a contraction. $T : K \to K$ is said to be a μ-strictly pseudo-contractive mapping if there exists a fixed constant $\mu \in (0, 1)$ such that

$$\langle T(u) - T(v), j(u - v) \rangle \leq \|u - v\|^2 - \mu \|(I - T)u - (I - T)v\|^2, \qquad (5.1.1)$$

for some $j(u - v) \in J(u - v)$ and for every $u, v \in K$. For some $j(u - v) \in J(u - v)$ and for every $u, v \in K$, (5.1.1) can be written as

$$\langle (I - T)(u) - (I - T)(v), j(u - v) \rangle \geq \mu \|(I - T)u - (I - T)v\|^2. \qquad (5.1.2)$$

A recent research interest to many authors is the viscosity implicit iterative algorithms for finding a common element of the set of fixed points for nonlinear operators and also the set of solutions of variational inequality problems (see [65], [76], [102], [104], [107] and the references therein). Following the ideas of Attouch [14], in 2000, Moudafi [76] introduced the viscosity approximation method for nonexpansive mapping in Hilbert spaces. Refinements in Hilbert spaces and extensions to Banach spaces were obtained by Xu [102]. Recently, Xu et al. [104] introduced the implicit midpoint procedure (1.1.5). They proved a strong convergence theorem in a Hilbert space for the implicit midpoint sequence (1.1.5) to a fixed point p of a nonexpansive mapping T, which also solves the variational inequality (1.1.2). Yao et al. [107] extended the work of Xu et al. [104] and studied the implicit midpoint sequence (1.1.6). They showed that the implicit midpoint sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.1.6) converges strongly to $p = P_{F(T)} f(p)$ under certain conditions on the parameters, where $F(T)$ is the set of fixed points of a nonexpansive mapping T. In other words, the implicit midpoint sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.1.6) converges in norm to a fixed point p of a nonexpansive mapping T, which is also the unique solution of the variational inequality (1.1.2). Choosing $\{\delta_n\}_{n=1}^{\infty} \subset (0, 1)$, Ke and Ma [65] generalized the viscosity implicit midpoint rules of Xu et al. [104] and Yao et al. [107] to (1.1.7) and (1.1.8) respectively. Yan et al. [106] replaced strict contractions by the generalized contractions and established the main results of Ke and Ma [65] in a uniformly smooth Banach spaces. The sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.1.8) is proved to converge strongly to a fixed point p of a nonexpansive mapping T, which solves the variational inequality (1.1.9). The previous works in this direction generate the following natural questions:

Question 5.1.1 *How to extend the results of Ke and Ma [65] and Yan et al. [106] to the more general class of μ-strictly pseudo-contractive mappings?*

Question 5.1.2 *Does there exist any implicit iterative algorithm which converges*

strongly to fixed points of a μ-strictly pseudo-contractive mapping in uniformly smooth Banach spaces?

Motivated by the previous works, we seek to improve on the existing results in this direction. Precisely, for a nonempty closed convex subset K of a uniformly smooth Banach space E and for real sequences $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ and $\left\{\{\beta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ with $\beta_n^1, \beta_n^3 \neq 0$ such that $\sum_{i=1}^{3} \theta_n^i = 1$ and $\sum_{i=1}^{3} \beta_n^i = 1$, we introduce a new viscosity iterative algorithm of implicit rules from an arbitrary $x_1 \in K$ as follows

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n(\delta_n x_n + (1-\delta_n)x_{n+1}), \qquad (5.1.3)$$

where $S_n x = \beta_n^1 Q(x) + \beta_n^2 x + \beta_n^3 T(x)$, $f : K \to K$ is a generalized contraction, $Q : K \to K$ is a contraction and $T : K \to K$ is a μ-strictly pseudo-contractive mapping. The iterative sequence given by (5.1.3) generalizes the existing schemes and we use the method of Yan et al. [106] to show that it converges strongly to a fixed point p of T, which is also a solution to the variational inequality problem (1.1.9).

5.1.2   Main results

Definition 5.1.3 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $f : K \to K$ a generalized contraction. Let T be a μ-strictly pseudo-contractive mapping defined on K and $Q : K \to K$ a contraction with $F(T) \cap F(Q) \neq \emptyset$. Assume that the real sequences $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ and $\left\{\{\beta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$ with $\beta_n^1, \beta_n^3 \neq 0$ satisfy the following conditions:*

(i) $\sum_{i=1}^{3} \theta_n^i = 1, \ \ \sum_{i=1}^{3} \beta_n^i = 1$

(ii) $\lim_{n\to\infty} \theta_n^1 = 0, \ \ \sum_{n=1}^{\infty} \theta_n^1 = \infty,$

(iii) $\lim_{n\to\infty} |\theta_{n+1}^2 - \theta_n^2| = 0, \ \ 0 < \liminf_{n\to\infty} \theta_n^2 \leq \limsup_{n\to\infty} \theta_n^2 < 1,$

(iv) $\lim_{n\to\infty} |\beta_{n+1}^1 - \beta_n^1| = 0, \ \lim_{n\to\infty} |\beta_{n+1}^3 - \beta_n^3| = 0,$

(v) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ for all $n \in \mathbb{N}$.

We shall study the convergence of the iterative scheme (5.1.3) under the conditions (i)-(v) of Definition 5.1.3.

We show that the scheme is well defined. Firstly, let $c_Q \in [0, 1]$ be the contraction constant of Q, then for all $y, z \in K$,

$$
\begin{aligned}
\|S_n(y) - S_n(z)\|^2 &= \left\|\beta_n^1 Q(y) + \beta_n^2 y + \beta_n^3 T(y) - \beta_n^1 Q(z) - \beta_n^2 z - \beta_n^3 T(z)\right\|^2 \\
&= \left\|\beta_n^1(Q(y) - Q(z)) + \beta_n^2(y - z) + \beta_n^3(T(y) - T(z))\right\|^2 \\
&= \beta_n^1 \langle Q(y) - Q(z), \ J(y - z) \rangle + \beta_n^2 \langle y - z, \ J(y - z) \rangle \\
&\quad + \beta_n^3 \langle T(y) - T(z), \ J(y - z) \rangle \\
&\leq \beta_n^1 \|Q(y) - Q(z)\| \|y - z\| + \beta_n^2 \|y - z\|^2 \\
&\quad + \beta_n^3 \left(\|y - z\|^2 - \mu\|(I - T)y - (I - T)z\|^2\right) \\
&\leq \beta_n^1 c_Q \|y - z\|^2 + \beta_n^2 \|y - z\|^2 \\
&\quad + (1 - \beta_n^1 - \beta_n^2)\left(\|y - z\|^2 - \mu\|(I - T)y - (I - T)z\|^2\right) \\
&\leq \beta_n^1 \|y - z\|^2 + \beta_n^2 \|y - z\|^2 \ (\text{since } c_Q \in [0, 1]) \\
&\quad + (1 - \beta_n^1 - \beta_n^2)\left(\|y - z\|^2 - \mu\|(I - T)y - (I - T)z\|^2\right) \\
&= \|y - z\|^2 - (1 - \beta_n^1 - \beta_n^2)\mu\|(I - T)y - (I - T)z\|^2 \\
&\leq \|y - z\|^2.
\end{aligned}
$$

Next is to show that for all $v \in K$, the mapping defined by

$$
x \mapsto T_v(x) := \ \theta_n^1 f(v) + \theta_n^2 v + \theta_n^3 S_n(\delta_n v + (1 - \delta_n)x) \tag{5.1.4}
$$

for all $x \in K$ is a contraction with a contractive constant $(1 - \epsilon) =: \delta \in (0, 1)$. Clearly, for all $y, z \in K$,

$$\begin{aligned}
\|T_v(y) - T_v(z)\| &= \theta_n^3 \left\| S_n \left(\delta_n v + (1 - \delta_n)y \right) - S_n \left(\delta_n v + (1 - \delta_n)z \right) \right\| \\
&\leq \theta_n^3 \left\| (\delta_n v + (1 - \delta_n)y) - (\delta_n v + (1 - \delta_n)z) \right\| \\
&\leq \theta_n^3 (1 - \delta_n) \| y - z \| \\
&\leq (1 - \delta_n) \| y - z \| \\
&\leq (1 - \epsilon) \| y - z \| \\
&= \delta \| y - z \|. \tag{5.1.5}
\end{aligned}$$

Thus, (5.1.3) is well defined since T_v is a contraction and by Banach contraction principle, T_v has a fixed point. Observe that for each $n \in \mathbb{N}$, $x \in F(T) \cap F(Q) \Rightarrow x \in F(S_n)$. So, $F(T) \cap F(Q) \subset F(S_n) \neq \emptyset$. Indeed, suppose $x \in F(T) \cap F(Q)$, then

$$\begin{aligned}
S_n x &= \beta_n^1 Q(x) + \beta_n^2 x + \beta_n^3 T(x) \\
&= \beta_n^1 x + \beta_n^2 x + \beta_n^3 x \\
&= (\beta_n^1 + \beta_n^2 + \beta_n^3)x \\
&= x.
\end{aligned}$$

Thus, $x \in F(S_n)$.

We give and prove the following lemmas which are useful in establishing our main result.

Lemma 5.1.4 *Let E be a uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a μ-strictly pseudo-contractive mapping and suppose that $f : K \to K$ is a generalized contraction and $Q : K \to K$ is a contraction with $F(T) \cap F(Q) \neq \emptyset$. For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}_{n=1}^{\infty}$ by*

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n(\delta_n x_n + (1 - \delta_n)x_{n+1}), \tag{5.1.6}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded under the conditions (i)-(v) of Definition 5.1.3.

Proof. We show that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $z_n := \delta_n x_n + (1 - \delta_n)x_{n+1}$ and recall that $\phi(t) := t - \psi(t)$ for all $t \in \mathbb{R}^+$. Then for $p \in F(T) \cap F(Q)$,

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n z_n - p\| \\
&= \|\theta_n^1 \left(f(x_n) - f(p)\right) + \theta_n^1 \left(f(p) - p\right) + \theta_n^2 (x_n - p) + \theta_n^3 (S_n z_n - p)\| \\
&\leq \theta_n^1 \|f(x_n) - f(p)\| + \theta_n^1 \|f(p) - p\| + \theta_n^2 \|x_n - p\| + \theta_n^3 \|S_n z_n - p\| \\
&\leq \theta_n^1 \psi \|x_n - p\| + \theta_n^1 \|f(p) - p\| + \theta_n^2 \|x_n - p\| \\
&\quad + \theta_n^3 \left(\delta_n \|x_n - p\| + (1 - \delta_{n+1}) \|x_{n+1} - p\|\right) \\
&\leq \theta_n^1 \psi \|x_n - p\| + \theta_n^1 \|f(p) - p\| + \theta_n^2 \|x_n - p\| \\
&\quad + \theta_n^3 \delta_n \|x_n - p\| + \theta_n^3 (1 - \delta_{n+1}) \|x_{n+1} - p\|.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\left(1 - \theta_n^3 (1 - \delta_n)\right) \|x_{n+1} - p\| &\leq \left(\theta_n^1 \psi + \theta_n^2 + \theta_n^3 \delta_n\right) \|x_n - p\| + \theta_n^1 \|f(p) - p\| \\
&= \left(\theta_n^1 \psi + (1 - \theta_n^1 - \theta_n^3) + \theta_n^3 \delta_n\right) \|x_n - p\| + \theta_n^1 \|f(p) - p\| \\
&= \left(1 - \theta_n^3 (1 - \delta_n) - \theta_n^1 (1 - \psi)\right) \|x_n - p\| + \theta_n^1 \|f(p) - p\| \\
&= \left(1 - \theta_n^3 (1 - \delta_n) - \theta_n^1 \phi\right) \|x_n - p\| + \theta_n^1 \|f(p) - p\|.
\end{aligned}
$$

Observe that $1 - \theta_n^3(1 - \delta_n) > 0$ since $\left\{\{\theta_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3}$, $\{\delta_n\}_{n=1}^{\infty} \subset (0, 1)$. Therefore, we have

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq \frac{1 - \theta_n^3 (1 - \delta_n) - \theta_n^1 \phi}{1 - \theta_n^3 (1 - \delta_n)} \|x_n - p\| \\
&\quad + \frac{\theta_n^1}{1 - \theta_n^3 (1 - \delta_n)} \|f(p) - p\| \qquad\qquad (5.1.7) \\
&= \left(1 - \frac{\theta_n^1 \phi}{1 - \theta_n^3 (1 - \delta_n)}\right) \|x_n - p\| + \frac{\theta_n^1}{1 - \theta_n^3 (1 - \delta_n)} \|f(p) - p\| \\
&= \left(1 - \frac{\theta_n^1 \phi}{1 - \theta_n^3 (1 - \delta_n)}\right) \|x_n - p\| + \frac{\theta_n^1 \phi}{1 - \theta_n^3 (1 - \delta_n)} \cdot \phi^{-1} \|f(p) - p\| \\
&\leq \max\left\{\|x_n - p\|, \ \phi^{-1} \|f(p) - p\|\right\}.
\end{aligned}
$$

Thus, by the induction, we have

$$
\|x_{n+1} - p\| \leq \max\left\{\|x_1 - p\|, \ \phi^{-1} \|f(p) - p\|\right\}.
$$

This implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and hence $\{S_n\left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right)\}_{n=1}^{\infty}$ and $\{f(x_n)\}_{n=1}^{\infty}$ are also bounded.

For $p \in F(T) \cap F(Q)$,

$$
\begin{aligned}
\|S_n\left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right)\| &= \|S_n\left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right) - p + p\| \\
&\leq \|S_n\left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right) - S_n p\| + \|p\| \\
&\leq \|\delta_n x_n + (1 - \delta_n)x_{n+1} - p\| + \|p\| \\
&\leq \delta_n\|x_n - p\| + (1 - \delta_n)\|x_{n+1} - p\| + \|p\| \\
&\leq \max\left\{\|x_1 - p\|,\ \phi^{-1}\|f(p) - p\|\right\} + \|p\| \quad \text{(by induction)}.
\end{aligned}
$$

The boundedness of $\{S_n\}_{n=1}^{\infty}$ implies that Q and T are also bounded siince S_n is defined in term of Q and T. Moreover,

$$
\begin{aligned}
\|f(x_n)\| &= \|f(x_n) - f(p) + f(p)\| \leq \psi\|x_n - p\| + \|f(p)\| \\
&\leq \max\left\{\psi\|x_1 - p\|,\ \psi\phi^{-1}\|f(p) - p\|\right\} + \|f(p)\| \quad \text{(by induction)}.
\end{aligned}
$$

■

Lemma 5.1.5 *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let $Q : K \to K$ be a contraction, $T : K \to K$ a μ-strictly pseudo-contractive mapping and $\{\delta_n\}_{n=1}^{\infty}$ is a real sequences in $(0, 1)$. Define $z_n :=$ $\delta_n x_n + (1 - \delta_n)x_{n+1}$ and let $M_1 = \max\left\{\sup\limits_n \|T(z_n) - z_n\|,\ \sup\limits_n \|Q(z_n) - z_n\|\right\}$. Then*

$$
\begin{aligned}
\|S_{n+1}z_{n+1} - S_n z_n\| &\leq \delta_n\|x_{n+1} - x_n\| + (1 - \delta_{n+1})\|x_{n+2} - x_{n+1}\| \\
&\quad + \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right)M_1 \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Proof. It is known that $\{z_n\}_{n=1}^{\infty}$ is bounded since $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. Notice that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \|\delta_{n+1}x_{n+1} + (1 - \delta_{n+1})x_{n+2} - (\delta_n x_n + (1 - \delta_n)x_{n+1})\| \\
&= \|\delta_{n+1}x_{n+1} + (1 - \delta_{n+1})x_{n+2} - \delta_n x_n - (1 - \delta_n)x_{n+1}\| \\
&= \|(x_{n+2} - x_{n+1}) - \delta_{n+1}(x_{n+2} - x_{n+1}) + \delta_n(x_{n+1} - x_n)\| \\
&= \|\delta_n(x_{n+1} - x_n) + (1 - \delta_{n+1})(x_{n+2} - x_{n+1})\| \\
&\leq \delta_n\|x_{n+1} - x_n\| + (1 - \delta_{n+1})\|x_{n+2} - x_{n+1}\|. \quad\quad (5.1.8)
\end{aligned}
$$

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Then

$$
\begin{aligned}
\|S_{n+1}z_{n+1} - S_n z_n\| &= \|S_{n+1}z_{n+1} - S_{n+1}z_n + S_{n+1}z_n - S_n z_n\| \\
&\leq \|z_{n+1} - z_n\| + \|\beta_{n+1}^1 Q(z_n) + \beta_{n+1}^2 z_n + \beta_{n+1}^3 T(z_n) \\
&\quad - \beta_n^1 Q(z_n) - \beta_n^2 z_n - \beta_n^3 T(z_n)\| \\
&= \|z_{n+1} - z_n\| + \|\beta_{n+1}^1 Q(z_n) + (1 - \beta_{n+1}^1 - \beta_{n+1}^3)z_n + \beta_{n+1}^3 T(z_n) \\
&\quad - \beta_n^1 Q(z_n) - (1 - \beta_n^1 - \beta_n^3)z_n - \beta_n^3 T(z_n)\| \\
&= \|z_{n+1} - z_n\| + \|\beta_{n+1}^1 (Q(z_n) - z_n) + z_n + \beta_{n+1}^3 (T(z_n) - z_n) \\
&\quad - \beta_n^1 (Q(z_n) - z_n) - z_n - \beta_n^3 (T(z_n) - z_n)\| \\
&= \|z_{n+1} - z_n\| + \|(\beta_{n+1}^1 - \beta_n^1)(Q(z_n) - z_n) \\
&\quad + (\beta_{n+1}^3 - \beta_n^3)(T(z_n) - z_n)\| \\
&\leq \delta_n\|x_{n+1} - x_n\| + (1 - \delta_{n+1})\|x_{n+2} - x_{n+1}\| \\
&\quad + \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right)M_1. \tag{5.1.9}
\end{aligned}
$$

∎

Theorem 5.1.6 *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let T be a μ-strictly pseudocontractive self-mapping defined on K while $f : K \to K$ is a generalized contraction and Q is a contraction defined on K with $F(T) \cap F(Q) \neq \emptyset$. Suppose that the conditions $(i) - (v)$ of Definition 5.1.3 are satisfied. Then, for an arbitrary $x_1 \in K$, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by (5.1.3) converges strongly to a fixed point p of T.*

Proof. Observe that one can write the iterative sequence (5.1.3) as:

$$
\begin{aligned}
x_{n+1} &= \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n \left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right) \\
&= \theta_n^2 x_n + (1 - \theta_n^2)\frac{\theta_n^1 f(x_n) + \theta_n^3 S_n \left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right)}{1 - \theta_n^2}.
\end{aligned}
$$

Since $\sum_{i=1}^{3} \theta_n^i = 1$ by condition (i), we have

$$
\begin{aligned}
x_{n+1} &= (1 - \theta_n^1 - \theta_n^3)x_n + (\theta_n^1 + \theta_n^3)\frac{\theta_n^1 f(x_n) + \theta_n^3 S_n \left(\delta_n x_n + (1 - \delta_n)x_{n+1}\right)}{1 - \theta_n^2} \\
&= (1 - \theta_n^1 - \theta_n^3)x_n + (\theta_n^1 + \theta_n^3)w_n, \tag{5.1.10}
\end{aligned}
$$

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where

$$w_n \;:=\; \frac{\theta_n^1 f(x_n) + \theta_n^3 S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)}{1 - \theta_n^2}$$

$$= \; \frac{\theta_n^1}{1-\theta_n^2} f(x_n) + \frac{\theta_n^3}{1-\theta_n^2} S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \qquad (5.1.11)$$

$$= \; \frac{\theta_n^1}{\theta_n^1 + \theta_n^3} f(x_n) + \frac{\theta_n^3}{\theta_n^1 + \theta_n^3} S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right), \; n \in \mathbb{N}.$$

We note that $\{x_n\}_{n=1}^\infty$, $\{f(x_n)\}_{n=1}^\infty$ and $\{T\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\}_{n=1}^\infty$ are bounded sequences. Furthermore, since the $\limsup\limits_{n\to\infty} \theta_n^2 < 1$ by the condition (iii) of Definition 5.1.3, there exists $n_0 \in \mathbb{N}$ and $\eta < 1$ such that

$$1 - \theta_n^2 > 1 - \eta \; \forall \; n \geq n_0. \qquad (5.1.12)$$

The consequence of (5.1.11) and (5.1.12) is that $\{w_n\}_{n=1}^\infty$ is bounded.

Next, we show that $\lim\limits_{n\to\infty} \|w_n - x_n\| = 0$.

We need to first show that $\limsup\limits_{n\to\infty}(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \leq 0$. Observe that

$$w_{n+1} - w_n \;=\; \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3} f(x_{n+1}) + \frac{\theta_{n+1}^3}{\theta_{n+1}^1 + \theta_{n+1}^3} S_{n+1}\left(\delta_{n+1} x_{n+1} + (1-\delta_{n+1})x_{n+2}\right)$$

$$- \left(\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} f(x_n) + \frac{\theta_n^3}{\theta_n^1 + \theta_n^3} S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \right)$$

$$= \; \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\left(f(x_{n+1}) - f(x_n)\right) + \left(\frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3} - \frac{\theta_n^1}{\theta_n^1 + \theta_n^3} \right) f(x_n)$$

$$+ \frac{\theta_{n+1}^3}{\theta_{n+1}^1 + \theta_{n+1}^3}\left(S_{n+1}\left(\delta_{n+1} x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\right)$$

$$+ \left(\frac{\theta_{n+1}^3}{\theta_{n+1}^1 + \theta_{n+1}^3} - \frac{\theta_n^3}{\theta_n^1 + \theta_n^3} \right) S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)$$

$$
\begin{aligned}
= & \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}(f(x_{n+1}) - f(x_n)) + \left(\frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3} - \frac{\theta_n^1}{\theta_n^1 + \theta_n^3}\right) f(x_n) \\
& + \frac{\theta_{n+1}^3}{\theta_{n+1}^1 + \theta_{n+1}^3}\left(S_{n+1}\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\right) \\
& + \left(\frac{\theta_{n+1}^1 + \theta_{n+1}^3 - \theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3} - \frac{\theta_n^1 + \theta_n^3 - \theta_n^1}{\theta_n^1 + \theta_n^3}\right) S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \\
= & \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^k}(f(x_{n+1}) - f(x_n)) + \left(\frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3} - \frac{\theta_n^1}{\theta_n^1 + \theta_n^3}\right) f(x_n) \\
& + \frac{\theta_{n+1}^3}{\theta_{n+1}^1 + \theta_{n+1}^3}\left(S_{n+1}\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\right) \\
& + \left(\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right) S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) \\
= & \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}(f(x_{n+1}) - f(x_n)) \\
& + \left(\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right)\left(S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) - f(x_n)\right) \\
& + \frac{\theta_{n+1}^3}{\theta_{n+1}^1 + \theta_{n+1}^3}\left(S_{n+1}\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|w_{n+1} - w_n\| \leq & \frac{\theta_{n+1}^1 \psi}{\theta_{n+1}^1 + \theta_{n+1}^3}\|x_{n+1} - x_n\| + \left|\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right| \\
& \times \|S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) - f(x_n)\| + \frac{\theta_{n+1}^3}{\theta_{n+1}^1 + \theta_{n+1}^3} \\
& \times \|S_n\left(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}\right) - S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right)\|.
\end{aligned}
$$

Applying Lemma 5.1.5 leads to

$$
\begin{aligned}
\|w_{n+1} - w_n\| \leq & \frac{\theta_{n+1}^1 \psi}{\theta_{n+1}^1 + \theta_{n+1}^3}\|x_{n+1} - x_n\| \\
& + \left|\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right| \|S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) - f(x_n)\| \\
& + \frac{\theta_{n+1}^3}{\theta_{n+1}^1 + \theta_{n+1}^3}\left[\ \delta_n\|x_{n+1} - x_n\| + (1-\delta_{n+1})\|x_{n+2} - x_{n+1}\| \right. \\
& \left. + \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1\ \right]
\end{aligned}
$$

$$= \frac{\theta_{n+1}^1 \psi + \theta_{n+1}^3 \delta_n}{\theta_{n+1}^1 + \theta_{n+1}^3} \|x_{n+1} - x_n\| + \frac{\theta_{n+1}^3 (1 - \delta_{n+1})}{\theta_{n+1}^1 + \theta_{n+1}^3} \|x_{n+2} - x_{n+1}\|$$

$$+ \left| \frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3} \right| \|S_n (\delta_n x_n + (1 - \delta_n)x_{n+1}) - f(x_n)\|$$

$$+ \frac{\theta_{n+1}^1 \psi}{\theta_{n+1}^1 + \theta_{n+1}^3} \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3| \right) M_1. \tag{5.1.13}$$

Next, we need to evaluate $\|x_{n+2} - x_{n+1}\|$. Let $M^1 := \sup_n \{\|x_n - S_n (\delta_n x_n + (1 - \delta_n)x_{n+1})\|\}$, $M^2 := \sup_n \{\|S_n (\delta_n x_n + (1 - \delta_n)x_{n+1}) - f(x_n)\|\}$ and $M_2 =: \max \{M^1, M^2\}$.

$$\begin{aligned}
x_{n+2} - x_{n+1} &= \theta_{n+1}^1 f(x_{n+1}) + \theta_{n+1}^2 x_{n+1} + \theta_{n+1}^3 S_{n+1} (\delta_{n+1} x_{n+1} + (1 - \delta_{n+1})x_{n+2}) \\
&\quad - \left(\theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n (\delta_n x_n + (1 - \delta_n)x_{n+1}) \right) \\
&= \theta_{n+1}^1 (f(x_{n+1}) - f(x_n)) + (\theta_{n+1}^1 - \theta_n^1)f(x_n) + \theta_{n+1}^2 (x_{n+1} - x_n) \\
&\quad + (\theta_{n+1}^2 - \theta_n^2)x_n + \left(\theta_{n+1}^3 - \theta_n^3 \right) S_n (\delta_n x_n + (1 - \delta_n)x_{n+1}) \\
&\quad + \theta_{n+1}^3 \left(S_{n+1} (\delta_{n+1} x_{n+1} + (1 - \delta_{n+1})x_{n+2}) - S_n (\delta_n x_n + (1 - \delta_n)x_{n+1}) \right) \\
&= \theta_{n+1}^1 (f(x_{n+1}) - f(x_n)) + (\theta_{n+1}^1 - \theta_n^1)f(x_n) + \theta_{n+1}^2 (x_{n+1} - x_n) \\
&\quad + (\theta_{n+1}^2 - \theta_n^2)x_n + \left((\theta_n^1 - \theta_{n+1}^1) - (\theta_{n+1}^2 - \theta_n^2) \right) S_n (\delta_n x_n + (1 - \delta_n)x_{n+1}) \\
&\quad + \theta_{n+1}^3 \left(S_{n+1} (\delta_{n+1} x_{n+1} + (1 - \delta_{n+1})x_{n+2}) - S_n (\delta_n x_n + (1 - \delta_n)x_{n+1}) \right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\le \left(\theta_{n+1}^1 \psi + \theta_{n+1}^2 \right) \|x_{n+1} - x_n\| \\
&\quad + |\theta_n^1 - \theta_{n+1}^1| \|S_n (\delta_n x_n + (1 - \delta_n)x_{n+1}) - f(x_n)\| \\
&\quad + |\theta_{n+1}^2 - \theta_n^2| \|x_n - S_n (\delta_n x_n + (1 - \delta_n)x_{n+1})\| \\
&\quad + \theta_{n+1}^3 \|S_{n+1} (\delta_{n+1} x_{n+1} + (1 - \delta_{n+1})x_{n+2}) - S_n (\delta_n x_n + (1 - \delta_n)x_{n+1})\|
\end{aligned}$$

$$\leq \left(\theta_{n+1}^1 \psi + \theta_{n+1}^2\right) \|x_{n+1} - x_n\| + \left(|\theta_n^1 - \theta_{n+1}^1| + |\theta_{n+1}^2 - \theta_n^2|\right) M_2$$

$$+\theta_{n+1}^3 \left[\delta_n \|x_{n+1} - x_n\| + (1 - \delta_{n+1})\|x_{n+2} - x_{n+1}\| \right.$$

$$+ \left.\left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1 \right] \text{ (by Lemma 5.1.5)}$$

$$= \left(\theta_{n+1}^1 \psi + \theta_{n+1}^2 + \theta_{n+1}^3 \delta_n\right) \|x_{n+1} - x_n\|$$

$$+ \left(|\theta_n^1 - \theta_{n+1}^1| + |\theta_{n+1}^2 - \theta_n^2|\right) M_2$$

$$+\theta_{n+1}^3 \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1 \qquad (5.1.14)$$

$$+\theta_{n+1}^3 (1 - \delta_{n+1})\|x_{n+2} - x_{n+1}\|.$$

Let $B_n = \frac{1}{1-\theta_{n+1}^3(1-\delta_{n+1})} \left(|\theta_n^1 - \theta_{n+1}^1| + |\theta_{n+1}^2 - \theta_n^2|\right) M_2 + \theta_{n+1}^3 \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1$, since $1 - \theta_{n+1}^3(1 - \delta_{n+1}) > 0$, we obtain from (5.1.14),

$$\|x_{n+2} - x_{n+1}\| \leq \frac{\theta_{n+1}^1 \psi + \theta_{n+1}^2 + \theta_{n+1}^3 \delta_n}{1 - \theta_{n+1}^3(1 - \delta_{n+1})}\|x_{n+1} - x_n\| + B_n. \qquad (5.1.15)$$

Substituting (5.1.15) into (5.1.13) gives

$$\|w_{n+1} - w_n\| \leq \left[\frac{\theta_{n+1}^1 \psi + \theta_{n+1}^3 \delta_n}{\theta_{n+1}^1 + \theta_{n+1}^3} + \frac{\theta_{n+1}^3(1 - \delta_{n+1})}{\theta_{n+1}^1 + \theta_{n+1}^3} \times \frac{\theta_{n+1}^1 \psi + \theta_{n+1}^2 + \theta_{n+1}^3 \delta_n}{1 - \theta_{n+1}^3(1 - \delta_{n+1})} \right]$$

$$\times \|x_{n+1} - x_n\| + \left|\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right| M_2 + \frac{\theta_{n+1}^3(1 - \delta_{n+1})}{\theta_{n+1}^1 + \theta_{n+1}^3} B_n$$

$$+\frac{\theta_{n+1}^1 \psi}{\theta_{n+1}^1 + \theta_{n+1}^3} \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1$$

$$= \frac{\theta_{n+1}^1 \psi + \theta_{n+1}^3 \delta_n + \theta_{n+1}^3(1 - \delta_{n+1})\theta_{n+1}^2}{[\theta_{n+1}^1 + \theta_{n+1}^3][1 - \theta_{n+1}^3(1 - \delta_{n+1})]}\|x_{n+1} - x_n\|$$

$$+\left|\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right| M_2 + \frac{\theta_{n+1}^3(1 - \delta_{n+1})}{\theta_{n+1}^1 + \theta_{n+1}^3} B_n$$

$$+\frac{\theta_{n+1}^1 \psi}{\theta_{n+1}^1 + \theta_{n+1}^3} \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1$$

$$= \left(1 - \frac{\theta_{n+1}^1(1 - \psi) + \theta_{n+1}^3(\delta_{n+1} - \delta_n)}{[\theta_{n+1}^1 + \theta_{n+1}^3][1 - \theta_{n+1}^3(1 - \delta_{n+1})]}\right)\|x_{n+1} - x_n\|$$

$$+\left|\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right| M_2 + \frac{\theta_{n+1}^3(1 - \delta_{n+1})}{\theta_{n+1}^1 + \theta_{n+1}^3} B_n$$

$$+\frac{\theta_{n+1}^1 \psi}{\theta_{n+1}^1 + \theta_{n+1}^3} \left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1$$

$$
= \left(1 - \frac{\theta_{n+1}^1\phi + \theta_{n+1}^3(\delta_{n+1} - \delta_n)}{[\theta_{n+1}^1 + \theta_{n+1}^3][1 - \theta_{n+1}^3(1 - \delta_{n+1})]}\right)\|x_{n+1} - x_n\|
$$
$$
+ \left|\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right| M_2 + \frac{\theta_{n+1}^3(1 - \delta_{n+1})}{\theta_{n+1}^1 + \theta_{n+1}^3} B_n
$$
$$
+ \frac{\theta_{n+1}^1\psi}{\theta_{n+1}^1 + \theta_{n+1}^3}\left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1
$$
$$
< \left(1 - \frac{\theta_{n+1}^1\phi}{\theta_{n+1}^1 + \theta_{n+1}^3}\right)\|x_{n+1} - x_n\|
$$
$$
+ \left|\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right| M_2 + \frac{\theta_{n+1}^3(1 - \delta_{n+1})}{\theta_{n+1}^1 + \theta_{n+1}^3} B_n
$$
$$
+ \frac{\theta_{n+1}^1\psi}{\theta_{n+1}^1 + \theta_{n+1}^3}\left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1,
$$

since $\theta_{n+1}^1\phi + \theta_{n+1}^3(\delta_{n+1} - \delta_n) > \theta_{n+1}^1\phi$ and $[\theta_{n+1}^1 + \theta_{n+1}^3][1 - \theta_{n+1}^3(1 - \delta_{n+1})] < \theta_{n+1}^1 + \theta_{n+1}^3$. It then follows that

$$
\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \leq -\frac{\theta_{n+1}^1\phi}{\theta_{n+1}^1 + \theta_{n+1}^3}\|x_{n+1} - x_n\|
$$
$$
+ \left|\frac{\theta_n^1}{\theta_n^1 + \theta_n^3} - \frac{\theta_{n+1}^1}{\theta_{n+1}^1 + \theta_{n+1}^3}\right| M_2 + \frac{\theta_{n+1}^3(1 - \delta_{n+1})}{\theta_{n+1}^1 + \theta_{n+1}^3} B_n
$$
$$
+ \frac{\theta_{n+1}^1\psi}{\theta_{n+1}^1 + \theta_{n+1}^3}\left(|\beta_{n+1}^1 - \beta_n^1| + |\beta_{n+1}^3 - \beta_n^3|\right) M_1,
$$

and thus,

$$
\limsup_{n\to\infty}\left(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \qquad (5.1.16)
$$

Invoking Lemma 2.10.7, we have

$$
\lim_{n\to\infty}\|w_n - x_n\| = 0. \qquad (5.1.17)
$$

Obviously from (5.1.10), we can obtain that

$$
\|x_{n+1} - x_n\| = \|(1 - \theta_n^1 - \theta_n^3)x_n + (\theta_n^1 + \theta_n^3)w_n - x_n\|
$$
$$
\leq (\theta_n^1 + \theta_n^3)\|w_n - x_n\| \to 0 \text{ as } n \to \infty. \qquad (5.1.18)
$$

Next, we show that $\lim_{n\to\infty}\|x_n - S_n x_n\| = 0$. From (5.1.3), we can have that

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$$\|x_n - S_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_n x_n\|$$

$$\leq \|x_{n+1} - x_n\| + \theta_n^1 \|f(x_n) - S_n x_n\| + \theta_n^2 \|x_n - S_n x_n\|$$

$$+ \theta_n^3 \|S_n(\delta_n x_n + (1 - \delta_n)x_{n+1}) - S_n x_n\|$$

$$\leq \|x_{n+1} - x_n\| + \theta_n^1 \|f(x_n) - S_n x_n\| + (1 - \theta_n^1 - \theta_n^3)\|x_n - S_n x_n\|$$

$$+ \theta_n^3 \|\delta_n x_n + (1 - \delta_n)x_{n+1} - x_n\|$$

$$\leq \|x_{n+1} - x_n\| + \theta_n^1 \|f(x_n) - S_n x_n\| + (1 - \theta_n^1 - \theta_n^3)\|x_n - S_n x_n\|$$

$$+ \theta_n^3(1 - \delta_n)\|x_{n+1} - x_n\|.$$

$$(\theta_n^1 + \theta_n^3)\|x_n - S_n x_n\| \leq (1 + \theta_n^3(1 - \delta_n))\|x_{n+1} - x_n\| + \theta_n^1 \|f(x_n) - S_n x_n\|$$

$$\|x_n - S_n x_n\| \leq \frac{1 + \theta_n^3(1 - \delta_n)}{\theta_n^1 + \theta_n^3}\|x_{n+1} - x_n\| + \frac{\theta_n^1}{\theta_n^1 + \theta_n^3}\|f(x_n) - S_n x_n\|$$

$$= \frac{1 + \theta_n^3(1 - \delta_n)}{1 - \theta_n^2}\|x_{n+1} - x_n\| + \frac{\theta_n^1}{1 - \theta_n^2}\|f(x_n) - S_n x_n\|$$

$$\leq \frac{1 + \theta_n^3(1 - \delta_n)}{1 - \eta}\|x_{n+1} - x_n\|$$

$$+ \frac{\theta_n^1}{1 - \eta}\|f(x_n) - S_n x_n\| \to 0 \text{ as } n \to \infty, \qquad (5.1.19)$$

by the condition (ii) of Definition 5.1.3 and since $1 - \eta > 0$ (5.1.12). For a unique fixed point $p \in F(T) \cap F(Q)$ of the generalized contraction $P_{F(T) \cap F(Q)} f(p)$ (Proposition 2.9.4), that is, $p = P_{F(T) \cap F(Q)} f(p)$ and since $\lim_{n \to \infty} \|x_n - S_n x_n\| = 0$ (5.1.19), it follows that

$$\limsup_{n \to \infty} \langle f(p) - p, J(x_n - p) \rangle \leq 0.$$

Moreover, since the duality map is continuous and $\|x_{n+1} - x_n\| \to 0$ by (5.1.18), we obtain that,

$$\limsup_{n \to \infty} \langle f(p) - p, \ J(x_{n+1} - p) \rangle = \limsup_{n \to \infty} \langle f(p) - p, \ J(x_{n+1} - x_n + x_n - p) \rangle$$

$$= \limsup_{n \to \infty} \langle f(p) - p, \ J(x_n - p) \rangle \leq 0. \quad (5.1.20)$$

We prove that $x_n \to p \in F(T)$ as $n \to \infty$.

Let us assume that the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge strongly to $p \in F(T)$. Therefore, there exists $\epsilon > 0$ and a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$\|x_{n_j} - p\| \geq \epsilon$, for all $j \in \mathbb{N}$. Thus, for this ϵ, there exists $c \in (0,1)$ such that

$$\|f(x_{n_j}) - f(p)\| \leq c\|x_{n_j} - p\|.$$

$$
\begin{aligned}
||x_{n_{j+1}} - p||^2 &= \theta_{n_j}^1 \left\langle f(x_{n_j}) - f(p), J(x_{n_{j+1}} - p) \right\rangle + \theta_{n_j}^1 \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle \\
&\quad + \theta_{n_j}^2 \left\langle x_{n_j} - p, J(x_{n_{j+1}} - p) \right\rangle \\
&\quad + \theta_{n_j}^3 \left\langle S_n \left(\delta_{n_j} x_{n_j} + (1 - \delta_{n_j}) x_{n_{j+1}} \right) - p, J(x_{n_{j+1}} - p) \right\rangle \\
&\leq c\theta_{n_j}^1 ||x_{n_j} - p||\ ||x_{n_{j+1}} - p|| + \theta_{n_j}^1 \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle \\
&\quad + \theta_{n_j}^2 ||x_{n_j} - p||\ ||x_{n_{j+1}} - p|| \\
&\quad + \left(\theta_{n_j}^3 \delta_{n_j} ||x_{n_j} - p|| + \theta_{n_j}^3 (1 - \delta_{n_j}) ||x_{n_{j+1}} - p|| \right) ||x_{n_{j+1}} - p|| \\
&\leq \left(c\theta_n^1 + \theta_{n_j}^2 \right) ||x_{n_j} - p||\ ||x_{n_{j+1}} - p|| + \theta_{n_j}^1 \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle \\
&\quad + \left(\theta_{n_j}^3 \delta_{n_j} ||x_{n_j} - p|| + \theta_{n_j}^3 (1 - \delta_{n_j}) ||x_{n_{j+1}} - p|| \right) ||x_{n_{j+1}} - p|| \\
&\leq \frac{1}{2} \left(c\theta_{n_j}^1 + \theta_{n_j}^2 + \theta_{n_j}^3 \delta_{n_j} \right) \left(||x_{n_j} - p||^2 + ||x_{n_{j+1}} - p||^2 \right) \\
&\quad + \theta_{n_j}^1 \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle + \theta_n^3 (1 - \delta_{n_j}) ||x_{n_{j+1}} - p||^2
\end{aligned}
$$

$$
\begin{aligned}
2||x_{n_{j+1}} - p||^2 &\leq \left(1 - \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j}) \right) \left(||x_{n_j} - p||^2 + ||x_{n_{j+1}} - p||^2 \right) \\
&\quad + 2\theta_{n_j}^1 \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle + 2\theta_{n_j}^3 (1 - \delta_{n_j}) ||x_{n_{j+1}} - p||^2 \\
&= \left(1 - \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j}) \right) ||x_{n_j} - p||^2 \\
&\quad + \left(1 - \theta_{n_j}^1 (1 - c) + \theta_{n_j}^3 (1 - \delta_{n_j}) \right) ||x_{n_{j+1}} - p||^2 \\
&\quad + 2\theta_{n_j}^1 \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left(1 + \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j}) \right) ||x_{n_{j+1}} - p||^2 \\
&\leq \left(1 - \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j}) \right) ||x_{n_j} - p||^2 \\
&\quad + 2\theta_{n_j}^1 \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
||x_{n_{j+1}} - p||^2 \; &\leq \; \frac{1 - \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j})}{1 + \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j})} ||x_{n_j} - p||^2 \\
&\quad + \frac{2\theta_{n_j}^1}{1 + \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j})} \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle \\
&= \; \left(1 - \frac{2\theta_{n_j}^1 (1 - c)}{1 + \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j})} \right) ||x_{n_j} - p||^2 \\
&\quad + \frac{2\theta_{n_j}^1}{1 + \theta_{n_j}^1 (1 - c) - \theta_{n_j}^3 (1 - \delta_{n_j})} \left\langle f(p) - p, J(x_{n_{j+1}} - p) \right\rangle .
\end{aligned}
$$

$$(5.1.21)$$

By applying Lemma 2.10.6 with $\gamma_n = 0$ to (5.1.21), one can deduce that $x_{n_j} \to p$ as $j \to \infty$. This is a contradiction. Hence, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F(T)$. ∎

5.1.3 Extension to a finite family of strictly pseudo-contractive mappings

The result of Theorem 5.1.6 can be extended to a finite family of μ-strictly pseudo-contractive mappings by using the lemma given below.

Lemma 5.1.7 *[112] Let K be a nonempty convex subset of a real smooth Banach space E and let $\lambda_i > 0$ $(i = 1, 2, ..., N)$ such that $\displaystyle\sum_{i=1}^{N} \lambda_i = 1$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of μ_i-strictly pseudo-contractive mappings and let $T = \displaystyle\sum_{i=1}^{N} \lambda_i T_i$. Then, we have the following:*

(i) $T : K \to K$ is μ-strictly pseudo-contractive mapping with $\mu = \min \{\mu_i : 1 \leq i \leq N\}$.

(ii) If $\cap_{i=1}^{N} F(T_i) \neq \emptyset$ then $F(T) = \cap_{i=1}^{N} F(T_i)$.

The next following result then comes readily.

Theorem 5.1.8 *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N}$ be a finite family of μ_i-strictly pseudo-contractive self-mapping defined on K, Q a contraction defined on K with $\cap_{i=1}^{N} F(T_i) \cap F(Q) \neq \emptyset$*

and $\lambda_i > 0$ $(i = 1, 2, ..., N)$ such that $\sum_{i=1}^{N} \lambda_i = 1$. Let $f : K \to K$ be a generalized contraction and suppose that the conditions $(i) - (v)$ of Definition 5.1.3 are satisfied. Then, for an arbitrary $x_1 \in K$, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n(\delta_n x_n + (1 - \delta_n)x_{n+1}), \qquad (5.1.22)$$

where $S_n x = \beta_n^1 Q(x) + \beta_n^2 x + \beta_n^3 \sum_{i=1}^{N} \lambda_i T_i(x)$, converges strongly to a fixed point $p \in \cap_{i=1}^{N} F(T_i)$ which solves the variational inequality

$$\langle (I - f)p, J(x - p) \rangle \geq 0, \text{ for all } x \in \cap_{i=1}^{N} F(T_i). \qquad (5.1.23)$$

Proof. Define $T = \sum_{i=1}^{N} \lambda_i T_i$, it suffices to show that T is a μ-strictly pseudocontractive mapping with $F(T) = \cap_{i=1}^{N} F(T_i)$. It is known that T satisfies these properties with $\mu = \min \{\mu_i : 1 \leq i \leq N\}$ (Lemma 5.1.7). ■

Remark 5.1.9 *The following result is readily obtained as corollaries of Theorem 5.1.6.*

Corollary 5.1.10 *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let T_i be a μ-strictly $\{T_i\}_{i=1}^{N}$ be a finite family of μ_i-strictly pseudo-contractive self-mapping defined on K, Q a contraction defined on K with $\cap_{i=1}^{N} F(T_i) \cap F(Q) \neq \emptyset$ and $\lambda_i > 0$ $(i = 1, 2, ..., N)$ such that $\sum_{i=1}^{N} \lambda_i = 1$. Let $f : K \to K$ be a generalized contraction and assume that the real sequences $\{\delta_n\}_{n=1}^{\infty} \subset (0, 1)$, $\{\{\theta_n^i\}_{n=1}^{\infty}\}_{i=1}^{3} \subset [0, 1]$ and $\{\alpha_n\} \subset (0, 1)$ satisfy the following conditions:*

(i) $\sum_{i=1}^{3} \theta_n^i = 1$,

(ii) $\lim_{n \to \infty} \theta_n^1 = 0$, $\sum_{n=1}^{\infty} \theta_n^1 = \infty$,

(iii) $\lim_{n \to \infty} |\theta_{n+1}^2 - \theta_n^2| = 0$, $0 < \liminf_{n \to \infty} \theta_n^2 \leq \limsup_{n \to \infty} \theta_n^2 < 1$,

(iv) $\lim_{n \to \infty} |\alpha_{n+1} - \alpha_n| = 0$,

(v) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ *for all $n \in \mathbb{N}$.*

Then, for an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}_{n=1}^{\infty}$ by

$$x_{n+1} = \theta_n^1 f(x_n) + \theta_n^2 x_n + \theta_n^3 S_n(\delta_n x_n + (1 - \delta_n) x_{n+1}), \qquad (5.1.24)$$

where $S_n x = \alpha_n Q(x) + (1 - \alpha_n) \sum_{i=1}^{N} \lambda_i T_i(x)$, converges strongly to a fixed point p of $p \in \cap_{i=1}^{N} F(T_i)$ which solves the variational inequality

$$\langle (I - f)p, J(x - p) \rangle \geq 0, \text{ for all } x \in \cap_{i=1}^{N} F(T_i). \qquad (5.1.25)$$

Proof. Take $\beta_n^2 = 0$ in (5.1.3), then $\alpha_n = \beta_n^1$ and $(1 - \alpha_n) = \beta_n^3$. Also, define $T = \sum_{i=1}^{N} \lambda_i T_i$, it suffices to show that T is a μ-strictly pseudocontractive mapping with $F(T) = \cap_{i=1}^{N} F(T_i)$. It is known that T satisfies these properties with $\mu = \min\{\mu_i : 1 \leq i \leq N\}$ (Lemma 5.1.7). Thus, the desire result follows from Theorem 5.1.6. ∎

5.2 The viscosity implicit iterative algorithms of nonexpansive mappings in Banach spaces

5.2.1 Background

The Viscosity Approximation Method (VAM) for solving nonlinear operator equations has recently attracted much attention. In 1996, Attouch [14] considered the viscosity solutions of minimization problems. In 2000, Moudafi [76] introduced an explicit viscosity method for nonexpansive mappings. The iterative explicit viscosity sequence $\{x_n\}_{n=1}^{\infty}$ is defined by (1.1.1). The sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1.1.1) converges strongly to a fixed point of a nonexpansive mapping T under suitable conditions in Hilbert spaces. Xu et al. [104] recently proposed the concept of the implicit midpoint rule (1.1.5). Under certain conditions, they established that the implicit midpoint sequence (1.1.5) converges to a fixed point p of T which also solves the variational inequality (1.1.2). Ke and Ma [65] introduced generalized viscosity implicit rules which extend the results of Xu et al. [104]. The generalized viscosity implicit procedures are given by (1.1.7) and (1.1.8). Replacement of strict contractions in (1.1.8) by the generalized contractions and extension to uniformly smooth

Banach spaces was considered by Yan et al. [106]. Under certain conditions imposed on the parameters involved, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of the nonexpansive mapping T, which is also the unique solution of the variational inequality (1.1.9).

Inspired by the previous works in this direction, we propose a new implicit iterative algorithm. Precisely, for a nonempty closed convex subset K of a uniformly smooth Banach space E and for real sequences $\left\{ \{\alpha_n^i\}_{n=1}^{\infty} \right\}_{i=1}^3 \subset [0,1]$ and $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ such that $\sum_{i=1}^3 \alpha_n^i = 1$, the strict contraction $f : K \to K$ is replaced by the generalized contraction mapping in (1.1.8) and we propose the implicit iterative scheme, defined from an arbitrary $x_1 \in K$ by

$$x_{n+1} = \alpha_n^1 f(x_n) + \alpha_n^2 x_n + \alpha_n^3 T\left((1-\delta_n)f(x_n) + \delta_n x_{n+1}\right), \qquad (5.2.1)$$

where $T : K \to K$ is a nonexpansive mapping. The technique of Yan et al. [106] has been applied in the analysis.

5.2.2 Main results

Definition 5.2.1 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $f : K \to K$ be a generalized contraction mapping. Let T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$. The real sequences $\left\{ \{\alpha_n^i\}_{n=1}^{\infty} \right\}_{i=1}^3 \subset [0,1]$ and $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ are assumed to satisfy the following conditions:*

(i) $\displaystyle\sum_{i=1}^3 \alpha_n^i = 1$;

(ii) $\displaystyle\lim_{n\to\infty}(1 - \alpha_n^3 \delta_n - \alpha_n^2) = 0, \ \sum_{n=1}^{\infty}(1 - \alpha_n^3 \delta_n - \alpha_n^2) = \infty$;

(iii) $0 < \displaystyle\liminf_{n\to\infty} \alpha_n^2 \leq \limsup_{n\to\infty} \alpha_n^2 < 1$;

(iv) $\displaystyle\lim_{n\to\infty} \alpha_n^3 = 0, \ \sum_{n=1}^{\infty} \alpha_n^3(1 - \delta_n) < \infty$;

(v) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} \leq \delta < 1$ for all $n \in \mathbb{N}$.

We shall study the convergence of the iterative scheme (5.2.1) under the conditions (i)-(v) of Definition 5.2.1 stated above.

First, we show that for all $\omega \in K$, the mapping defined by

$$u \mapsto T_\omega(u) := \alpha_n^1 f(\omega) + \alpha_n^2 \omega + \alpha_n^3 T((1 - \delta_n)f(\omega) + \delta_n u), \qquad (5.2.2)$$

for all $u \in K$, where $\{\{\alpha_n\}_{n=1}^\infty\}_{i=1}^3 \subset [0,1]$, $\{\delta_n\}_{n=1}^\infty \subset (0,1)$, is a contraction with $\delta \in (0,1)$ a contractive constant.

Indeed, for all $u, v \in K$,

$$
\begin{aligned}
\|T_\omega(u) - T_\omega(v)\| &= \alpha_n^3 \|T((1-\delta_n)f(\omega) + \delta_n u) - T((1-\delta_n)f(\omega) + \delta_n v)\| \\
&\leq \alpha_n^3 \|(1-\delta_n)f(\omega) + \delta_n u - (1-\delta_n)f(\omega) - \delta_n v\| \\
&\leq \alpha_n^3 \delta_n \|u - v\| \\
&\leq \delta_n \|u - v\| \\
&\leq \delta \|u - v\|. \qquad (5.2.3)
\end{aligned}
$$

Therefore, T_ω is a contraction. By Banach's contraction mapping principle, T_ω has a fixed point.

We give and prove the following lemmas which are useful in establishing our main result.

Lemma 5.2.2 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $f : K \to K$ a generalized contraction mapping. Let T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$. For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}_{n=1}^\infty$ by (5.2.1). Then the sequence $\{x_n\}_{n=1}^\infty$ is bounded under the conditions (i)-(v) of Definition 5.2.1.*

Proof. We show that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. For $p \in F(T)$,

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n^1 f(x_n) + \alpha_n^2 x_n + \alpha_n^3 T((1 - \delta_n)f(x_n) + \delta_n x_{n+1}) - p\| \\
&\leq \alpha_n^1 \|f(x_n) - p\| + \alpha_n^2 \|x_n - p\| + \alpha_n^3 \|T((1 - \delta_n)f(x_n) + \delta_n x_{n+1}) - p\| \\
&\leq \alpha_n^1 \|f(x_n) - f(p)\| + \alpha_n^1 \|f(p) - p\| + \alpha_n^2 \|x_n - p\| \\
&\quad + \alpha_n^3 \|(1 - \delta_n)f(x_n) + \delta_n x_{n+1} - p\| \\
&= \alpha_n^1 \|f(x_n) - f(p)\| + \alpha_n^1 \|f(p) - p\| + \alpha_n^2 \|x_n - p\| \\
&\quad + \alpha_n^3 \|(1 - \delta_n)(f(x_n) - p) + \delta_n(x_{n+1} - p)\| \\
&\leq \alpha_n^1 \|f(x_n) - f(p)\| + \alpha_n^1 \|f(p) - p\| + \alpha_n^2 \|x_n - p\| \\
&\quad + \alpha_n^3 (1 - \delta_n)\|f(x_n) - f(p)\| + \alpha_n^3 (1 - \delta_n)\|f(p) - p\| \\
&\quad + \alpha_n^3 \delta_n \|x_{n+1} - p\| \\
&\leq \alpha_n^1 \psi \|x_n - p\| + \alpha_n^1 \|f(p) - p\| + \alpha_n^2 \|x_n - p\| + \alpha_n^3 (1 - \delta_n)\psi \|x_n - p\| \\
&\quad + \alpha_n^3 (1 - \delta_n)\|f(p) - p\| + \alpha_n^3 \delta_n \|x_{n+1} - p\| \\
&= \left(\alpha_n^1 \psi + \alpha_n^2 + \alpha_n^3 (1 - \delta_n)\psi\right) \|x_n - p\| \\
&\quad + \left(\alpha_n^1 + \alpha_n^3 (1 - \delta_n)\right) \|f(p) - p\| + \alpha_n^3 \delta_n \|x_{n+1} - p\| \\
&= \left((\alpha_n^1 + \alpha_n^3)\psi + \alpha_n^2 - \alpha_n^3 \delta_n \psi\right) \|x_n - p\| \\
&\quad + \left((\alpha_n^1 + \alpha_n^3) - \alpha_n^3 \delta_n\right) \|f(p) - p\| + \alpha_n^3 \delta_n \|x_{n+1} - p\| \\
&= \left((1 - \alpha_n^2)\psi + \alpha_n^2 - \alpha_n^3 \delta_n \psi\right) \|x_n - p\| \\
&\quad + \left(1 - \alpha_n^2 - \alpha_n^3 \delta_n\right) \|f(p) - p\| + \alpha_n^3 \delta_n \|x_{n+1} - p\| \\
&= \left(\psi + \alpha_n^2 (1 - \psi) - \alpha_n^3 \delta_n \psi\right) \|x_n - p\| \\
&\quad + \left(1 - \alpha_n^2 - \alpha_n^3 \delta_n\right) \|f(p) - p\| + \alpha_n^3 \delta_n \|x_{n+1} - p\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq \frac{\psi + \alpha_n^2 (1 - \psi) - \alpha_n^3 \delta_n \psi}{1 - \alpha_n^3 \delta_n} \|x_n - p\| \\
&\quad + \frac{1 - \alpha_n^2 - \alpha_n^3 \delta_n}{1 - \alpha_n^3 \delta_n} \|f(p) - p\|
\end{aligned}
\tag{5.2.4}
$$

$$
\begin{aligned}
&= \left(1 + \frac{\psi + \alpha_n^2(1-\psi) - \alpha_n^3\delta_n\psi - [1 - \alpha_n^3\delta_n]}{1 - \alpha_n^3\delta_n}\right)\|x_n - p\| \\
&\quad + \frac{1 - \alpha_n^2 - \alpha_n^3\delta_n}{1 - \alpha_n^3\delta_n}\|f(p) - p\| \\
&= \left(1 + \frac{-(1-\psi) + \alpha_n^2(1-\psi) + \alpha_n^3\delta_n(1-\psi)}{1 - \alpha_n^3\delta_n}\right)\|x_n - p\| \\
&\quad + \frac{1 - \alpha_n^2 - \alpha_n^3\delta_n}{1 - \alpha_n^3\delta_n}\|f(p) - p\| \\
&= \left(1 - \frac{(1 - \alpha_n^2 - \alpha_n^3\delta_n)(1-\psi)}{1 - \alpha_n^3\delta_n}\right)\|x_n - p\| \\
&\quad + \frac{1 - \alpha_n^2 - \alpha_n^3\delta_n}{1 - \alpha_n^3\delta_n}\|f(p) - p\| \\
&= \left(1 - \frac{(1 - \alpha_n^2 - \alpha_n^3\delta_n)\phi}{1 - \alpha_n^3\delta_n}\right)\|x_n - p\| \\
&\quad + \frac{(1 - \alpha_n^2 - \alpha_n^3\delta_n)\phi}{1 - \alpha_n^3\delta_n}\,\phi^{-1}\|f(p) - p\| \\
&\leq \max\left\{\|x_n - p\|,\ \phi^{-1}\|f(p) - p\|\right\}.
\end{aligned}
$$

Then by induction, we have

$$
\|x_{n+1} - p\| \leq \max\left\{\|x_1 - p\|,\ \phi^{-1}\|f(p) - p\|\right\}.
$$

For $p \in F(T)$,

$$
\begin{aligned}
\|f(x_n)\| &\leq \|f(x_n) - f(p)\| + \|f(p)\| \\
&\leq \psi\|x_n - p\| + \|f(p)\| \\
&\leq \max\left\{\psi\|x_1 - p\|,\ \psi\phi^{-1}\|f(p) - p\|\right\} + \|f(p)\| \text{ (by induction)}.
\end{aligned}
$$

So, $\{x_n\}_{n=1}^{\infty}$ is bounded. Also,

$$
\begin{aligned}
\|T((1-\delta_n)f(x_n) + \delta_n x_{n+1})\| &= \|T((1-\delta_n)f(x_n) + \delta_n x_{n+1}) - p + p\| \\
&\leq \|T((1-\delta_n)f(x_n) + \delta_n x_{n+1}) - Tp\| + \|p\| \\
&\leq \|(1-\delta_n)f(x_n) + \delta_n x_{n+1} - p\| + \|p\|
\end{aligned}
$$

$$\leq \quad (1-\delta_n)\|f(x_n) - p\| + \delta_n\|x_{n+1} - p\| + ||p||$$

$$\leq \quad (1-\delta_n)\|f(x_n) - f(p)\| + (1-\delta_n)\|f(p) - p\|$$

$$+\delta_n\|x_{n+1} - p\| + ||p||$$

$$\leq \quad (1-\delta_n)\psi\|x_n - p\| + \delta_n\|x_{n+1} - p\|$$

$$+(1-\delta_n)\|f(p) - p\| + ||p||$$

$$\leq \quad (1-\epsilon)\psi\|x_n - p\| + \delta\|x_{n+1} - p\|$$

$$+(1-\epsilon)\|f(p) - p\| + ||p||.$$

Therefore,

$$\|T((1-\delta_n)f(x_n) + \delta_n x_{n+1})\| \quad \leq \quad (1+\delta-\epsilon\psi)\max\left\{\|x_n - p\|,\ \phi^{-1}\|f(p) - p\|\right\}$$

$$+(1-\epsilon)\|f(p) - p\| + ||p|| \text{ (by induction)}.$$

Hence, $\{T((1-\delta_n)f(x_n) + \delta_n x_{n+1})\}_{n=1}^{\infty}$ is bounded. ■

Lemma 5.2.3 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $f : K \to K$ a generalized contraction mapping. Let T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$. Suppose $\{\delta_n\}_{n=1}^{\infty}$ is a real sequences in $(0,1)$ and $\{x_n\}_{n=1}^{\infty} \subset K$. Set $y_n = (1-\delta_n)f(x_n) + \delta_n x_{n+1}$, then*

$$\|Ty_{n+1} - Ty_n\| \quad \leq \quad (1-\delta_{n+1})\psi\|x_{n+1} - x_n\| + (\delta_{n+1} - \delta_n)\|x_{n+1} - f(x_n)\|$$

$$+\delta_{n+1}\|x_{n+2} - x_{n+1}\|. \tag{5.2.5}$$

Proof.

$$\|Ty_{n+1} - Ty_n\| \quad = \quad \|T((1-\delta_{n+1})f(x_{n+1}) + \delta_{n+1}x_{n+2}) - T((1-\delta_n)f(x_n) + \delta_n x_{n+1})\|$$

$$\leq \quad \|(1-\delta_{n+1})f(x_{n+1}) + \delta_{n+1}x_{n+2} - (1-\delta_n)f(x_n) - \delta_n x_{n+1}\|$$

$$= \quad \|(1-\delta_{n+1})f(x_{n+1}) - (1-\delta_{n+1})f(x_n)$$

$$+(1-\delta_{n+1})f(x_n) - (1-\delta_n)f(x_n)$$

$$+\delta_{n+1}x_{n+2} - \delta_{n+1}x_{n+1} + \delta_{n+1}x_{n+1} - \delta_n x_{n+1}\|$$

$$\begin{aligned}
&= \| (1 - \delta_{n+1})(f(x_{n+1}) - f(x_n)) - (\delta_{n+1} - \delta_n)f(x_n) \\
&\quad + \delta_{n+1}(x_{n+2} - x_{n+1}) + (\delta_{n+1} - \delta_n)x_{n+1} \| \\
&= \| (1 - \delta_{n+1})(f(x_{n+1}) - f(x_n)) + (\delta_{n+1} - \delta_n)(x_{n+1} - f(x_n)) \\
&\quad + \delta_{n+1}(x_{n+2} - x_{n+1}) \| \\
&\leq (1 - \delta_{n+1})\|f(x_{n+1}) - f(x_n)\| + (\delta_{n+1} - \delta_n)\|x_{n+1} - f(x_n)\| \\
&\quad + \delta_{n+1}\|x_{n+2} - x_{n+1}\| \\
&\leq (1 - \delta_{n+1})\psi\|x_{n+1} - x_n\| + (\delta_{n+1} - \delta_n)\|x_{n+1} - f(x_n)\| \\
&\quad + \delta_{n+1}\|x_{n+2} - x_{n+1}\|.
\end{aligned}$$

$\blacksquare$

Theorem 5.2.4 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $f : K \to K$ a generalized contraction mapping. Let T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$. Assume that the conditions $(i) - (v)$ of Definition 5.2.1 are satisfied. Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ which is defined from an arbitrary $x_1 \in K$ by (5.2.1), converges strongly to a fixed point p of T.*

Proof. Set $z_n = \frac{x_{n+1} - \alpha_n^2 x_n}{1 - \alpha_n^2}$ and $y_n = (1 - \delta_n)f(x_n) + \delta_n x_{n+1}$, we obtain,

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \alpha_{n+1}^2 x_{n+1}}{1 - \alpha_{n+1}^2} - \frac{x_{n+1} - \alpha_n^2 x_n}{1 - \alpha_n^2} \\
&= \frac{\alpha_{n+1}^1 f(x_{n+1}) + \alpha_{n+1}^3 T(y_{n+1})}{1 - \alpha_{n+1}^2} - \frac{\alpha_n^1 f(x_n) + \alpha_n^3 T(y_n)}{1 - \alpha_n^2} \\
&= \frac{\alpha_{n+1}^1}{1 - \alpha_{n+1}^2}(f(x_{n+1}) - f(x_n)) + \left(\frac{\alpha_{n+1}^1}{1 - \alpha_{n+1}^2} - \frac{\alpha_n^1}{1 - \alpha_n^2} \right) f(x_n) \\
&\quad + \frac{\alpha_{n+1}^3}{1 - \alpha_{n+1}^2}(T(y_{n+1}) - T(y_n)) + \left(\frac{\alpha_{n+1}^3}{1 - \alpha_{n+1}^2} - \frac{\alpha_n^3}{1 - \alpha_n^2} \right) T(y_n)
\end{aligned}$$

$$
= \frac{\alpha_{n+1}^1}{1-\alpha_{n+1}^2}(f(x_{n+1})-f(x_n)) - \left(\frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2} - \frac{\alpha_n^3}{1-\alpha_n^2}\right)f(x_n)
$$
$$
+ \frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2}(T(y_{n+1})-T(y_n)) + \left(\frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2} - \frac{\alpha_n^3}{1-\alpha_n^2}\right)T(y_n)
$$
$$
= \frac{\alpha_{n+1}^1}{1-\alpha_{n+1}^2}(f(x_{n+1})-f(x_n)) + \left(\frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2} - \frac{\alpha_n^3}{1-\alpha_n^2}\right)(T(y_n)-f(x_n))
$$
$$
+ \frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2}(T(y_{n+1})-T(y_n)).
$$

Let $M_1 = \sup_n\{\|T(y_n)-f(x_n)\|\}$, $M_2 = \sup_n\{\|x_{n+1}-f(x_n)\|\}$ and $M = \max\{M_1, M_2\}$. We then have that

$$
\begin{aligned}
\|z_{n+1}-z_n\| \;\leq\; & \frac{\alpha_{n+1}^1}{1-\alpha_{n+1}^2}\|f(x_{n+1})-f(x_n)\| + \left|\frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2} - \frac{\alpha_n^3}{1-\alpha_n^2}\right|\|T(y_n)-f(x_n)\| \\
& + \frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2}\|T(y_{n+1})-T(y_n)\| \\
\leq\; & \frac{\alpha_{n+1}^1}{1-\alpha_{n+1}^2}\psi\|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2} - \frac{\alpha_n^3}{1-\alpha_n^2}\right|\|T(y_n)-f(x_n)\| \\
& + \frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2}\Big[(1-\delta_{n+1})\psi\|x_{n+1}-x_n\| + (\delta_{n+1}-\delta_n)\|x_{n+1}-f(x_n)\| \\
& + \delta_{n+1}\|x_{n+2}-x_{n+1}\|\,\Big] \text{ (by (5.2.5))} \\
=\; & \frac{\alpha_{n+1}^1\psi + \alpha_{n+1}^3(1-\delta_{n+1})\psi}{1-\alpha_{n+1}^2}\|x_{n+1}-x_n\| \\
& + \left(\left|\frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2} - \frac{\alpha_n^3}{1-\alpha_n^2}\right| + \frac{\alpha_{n+1}^3(\delta_{n+1}-\delta_n)}{1-\alpha_{n+1}^2}\right)M \\
& + \frac{\alpha_{n+1}^3\delta_{n+1}}{1-\alpha_{n+1}^2}\|x_{n+2}-x_{n+1}\|. \quad\quad (5.2.6)
\end{aligned}
$$

We now evaluate $\|x_{n+2}-x_{n+1}\|$.

$$
\begin{aligned}
x_{n+2}-x_{n+1} \;=\; & \alpha_{n+1}^1 f(x_{n+1}) + \alpha_{n+1}^2 x_{n+1} + \alpha_{n+1}^3 Ty_{n+1} \\
& - \left(\alpha_n^1 f(x_n) + \alpha_n^2 x_n + \alpha_n^3 Ty_n\right) \\
=\; & \alpha_{n+1}^1(f(x_{n+1})-f(x_n)) + \alpha_{n+1}^2(x_{n+1}-x_n) + \alpha_{n+1}^3(Ty_{n+1}-Ty_n) \\
& + (\alpha_{n+1}^1 - \alpha_n^1)f(x_n) + (\alpha_{n+1}^2 - \alpha_n^2)x_n + (\alpha_{n+1}^3 - \alpha_n^3)Ty_n
\end{aligned}
$$

$$
\begin{aligned}
= \;& \alpha_{n+1}^1(f(x_{n+1}) - f(x_n)) + \alpha_{n+1}^2(x_{n+1} - x_n) + \alpha_{n+1}^3(Ty_{n+1} - Ty_n) \\
& + ((\alpha_n^2 - \alpha_{n+1}^2) + (\alpha_n^3 - \alpha_{n+1}^3))f(x_n) \\
& + (\alpha_{n+1}^2 - \alpha_n^2)x_n + (\alpha_{n+1}^3 - \alpha_n^3)Ty_n \\
= \;& \alpha_{n+1}^1(f(x_{n+1}) - f(x_n)) + \alpha_{n+1}^2(x_{n+1} - x_n) + \alpha_{n+1}^3(Ty_{n+1} - Ty_n) \\
& + (\alpha_{n+1}^2 - \alpha_n^2)(x_n - f(x_n)) + (\alpha_{n+1}^3 - \alpha_n^3)(Ty_n - f(x_n)).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\| \leq \;& \alpha_{n+1}^1\psi\|x_{n+1} - x_n\| + \alpha_{n+1}^2\|x_{n+1} - x_n\| + \alpha_{n+1}^3\|Ty_{n+1} - Ty_n\| \\
& + |\alpha_{n+1}^2 - \alpha_n^2|\|x_n - f(x_n)\| + |\alpha_{n+1}^3 - \alpha_n^3|\|Ty_n - f(x_n)\| \\
\leq \;& \alpha_{n+1}^1\psi\|x_{n+1} - x_n\| + \alpha_{n+1}^2\|x_{n+1} - x_n\| \\
& + \alpha_{n+1}^3[(1 - \delta_{n+1})\psi\|x_{n+1} - x_n\| + (\delta_{n+1} - \delta_n)\|x_{n+1} - f(x_n)\| \\
& + \delta_{n+1}\|x_{n+2} - x_{n+1}\|\,] \text{ (by (5.2.5))} \\
& + |\alpha_{n+1}^2 - \alpha_n^2|\|x_n - f(x_n)\| + |\alpha_{n+1}^3 - \alpha_n^3|\|Ty_n - f(x_n)\| \\
= \;& \left(\alpha_{n+1}^2 + (\alpha_{n+1}^3 + \alpha_{n+1}^1)\psi - \alpha_{n+1}^3\delta_{n+1}\psi\right)\|x_{n+1} - x_n\| \\
& + \alpha_{n+1}^3\delta_{n+1}\|x_{n+2} - x_{n+1}\| \\
& + \left(|\alpha_{n+1}^2 - \alpha_n^2| + |\alpha_{n+1}^3 - \alpha_n^3| + \alpha_{n+1}^3(\delta_{n+1} - \delta_n)\right)M \\
= \;& \left(\alpha_{n+1}^2 + (1 - \alpha_{n+1}^2)\psi - \alpha_{n+1}^3\delta_{n+1}\psi\right)\|x_{n+1} - x_n\| \\
& + \alpha_{n+1}^3\delta_{n+1}\|x_{n+2} - x_{n+1}\| \\
& + \left(|\alpha_{n+1}^2 - \alpha_n^2| + |\alpha_{n+1}^3 - \alpha_n^3| + \alpha_{n+1}^3(\delta_{n+1} - \delta_n)\right)M \\
= \;& \left(\psi + \alpha_{n+1}^2(1 - \psi) - \alpha_{n+1}^3\delta_{n+1}\psi\right)\|x_{n+1} - x_n\| \\
& + \alpha_{n+1}^3\delta_{n+1}\|x_{n+2} - x_{n+1}\| \\
& + \left(|\alpha_{n+1}^2 - \alpha_n^2| + |\alpha_{n+1}^3 - \alpha_n^3| + \alpha_{n+1}^3(\delta_{n+1} - \delta_n)\right)M \\
= \;& \left(\alpha_{n+1}^2(1 - \psi) + (1 - \alpha_{n+1}^3\delta_{n+1})\psi\right)\|x_{n+1} - x_n\| \\
& + \alpha_{n+1}^3\delta_{n+1}\|x_{n+2} - x_{n+1}\| \\
& + \left(|\alpha_{n+1}^2 - \alpha_n^2| + |\alpha_{n+1}^3 - \alpha_n^3| + \alpha_{n+1}^3(\delta_{n+1} - \delta_n)\right)M.
\end{aligned}
$$

Let $d_n = \left(|\alpha_{n+1}^2 - \alpha_n^2| + |\alpha_{n+1}^3 - \alpha_n^3| + \alpha_{n+1}^3(\delta_{n+1} - \delta_n)\right)$. Therefore,

$$\|x_{n+2} - x_{n+1}\| \leq \frac{\alpha_{n+1}^2(1-\psi) + (1-\alpha_{n+1}^3\delta_{n+1})\psi}{1 - \alpha_{n+1}^3\delta_{n+1}}\|x_{n+1} - x_n\|$$
$$+ \frac{d_n M}{1 - \alpha_{n+1}^3\delta_{n+1}}. \tag{5.2.7}$$

Let $S_n = \left|\frac{\alpha_{n+1}^3}{1-\alpha_{n+1}^2} - \frac{\alpha_n^3}{1-\alpha_n^2}\right| + \frac{\alpha_{n+1}^3(\delta_{n+1}-\delta_n)}{1-\alpha_{n+1}^2}$ and substitute (5.2.7) into (5.2.6) to obtain

$$\|z_{n+1} - z_n\| \leq [\frac{\alpha_{n+1}^1\psi + \alpha_{n+1}^3(1-\delta_{n+1})\psi}{1-\alpha_{n+1}^2} + \frac{\alpha_{n+1}^3\delta_{n+1}}{1-\alpha_{n+1}^2}$$
$$\times \frac{\alpha_{n+1}^2(1-\psi) + (1-\alpha_{n+1}^3\delta_{n+1})\psi}{1-\alpha_{n+1}^3\delta_{n+1}}]\|x_{n+1} - x_n\|$$
$$+ S_n M + \frac{\alpha_{n+1}^3\delta_{n+1}}{1-\alpha_{n+1}^2} \times \frac{d_n M}{1-\alpha_{n+1}^3\delta_{n+1}}$$
$$= [\frac{\alpha_{n+1}^1\psi + \alpha_{n+1}^3(1-\delta_{n+1})\psi - \alpha_{n+1}^3\delta_{n+1}(\alpha_{n+1}^1\psi + \alpha_{n+1}^3(1-\delta_{n+1})\psi)}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}$$
$$+ \frac{\alpha_{n+1}^3\delta_{n+1}(\alpha_{n+1}^2(1-\psi) + (1-\alpha_{n+1}^3\delta_{n+1})\psi)}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}]\|x_{n+1} - x_n\|$$
$$+ \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right)M$$
$$= [\frac{\alpha_{n+1}^1\psi + \alpha_{n+1}^3(1-\delta_{n+1})\psi - \alpha_{n+1}^3\delta_{n+1}(\alpha_{n+1}^1\psi + \alpha_{n+1}^3\psi - \alpha_{n+1}^3\delta_{n+1}\psi)}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}$$
$$+ \frac{\alpha_{n+1}^3\delta_{n+1}(\alpha_{n+1}^2 - \alpha_{n+1}^2\psi + \psi - \alpha_{n+1}^3\delta_{n+1}\psi)}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}]\|x_{n+1} - x_n\|$$
$$+ \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right)M$$
$$= [\frac{\alpha_{n+1}^1\psi + \alpha_{n+1}^3(1-\delta_{n+1})\psi - \alpha_{n+1}^3\delta_{n+1}((1-\alpha_{n+1}^2)\psi - \alpha_{n+1}^3\delta_{n+1}\psi)}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}$$
$$+ \frac{\alpha_{n+1}^3\delta_{n+1}(\alpha_{n+1}^2 + (1-\alpha_{n+1}^2)\psi - \alpha_{n+1}^3\delta_{n+1}\psi)}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}]\|x_{n+1} - x_n\|$$
$$+ \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right)M$$
$$= \frac{\alpha_{n+1}^1\psi + \alpha_{n+1}^3(1-\delta_{n+1})\psi + \alpha_{n+1}^3\delta_{n+1}\alpha_{n+1}^2}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\|x_{n+1} - x_n\|$$
$$+ \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right)M$$
$$= \frac{(1-\alpha_{n+1}^2)\psi - \alpha_{n+1}^3\delta_{n+1}\psi + \alpha_{n+1}^3\delta_{n+1}\alpha_{n+1}^2}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\|x_{n+1} - x_n\|$$
$$+ \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right)M$$

$$
\begin{aligned}
= \ & \left(1 - \frac{(1-\alpha_{n+1}^2)(1-\psi) - \alpha_{n+1}^3\delta_{n+1}(1-\psi)}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right) \|x_{n+1} - x_n\| \\
& + \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right) M \\
= \ & \left(1 - \frac{(1-\alpha_{n+1}^2)\phi - \alpha_{n+1}^3\delta_{n+1}\phi}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right) \|x_{n+1} - x_n\| + \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right) M \\
= \ & \left(1 - \frac{(1-\alpha_{n+1}^2 - \alpha_{n+1}^3\delta_{n+1})\phi}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right) \|x_{n+1} - x_n\| + \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right) M \\
\leq \ & \left(1 - \frac{(1-\alpha_{n+1}^2 - \alpha_{n+1}^3\delta_{n+1})\phi}{1-\alpha_{n+1}^2}\right) \|x_{n+1} - x_n\| + \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{[1-\alpha_{n+1}^2][1-\alpha_{n+1}^3\delta_{n+1}]}\right) M.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \ \leq \ & -\frac{(1-\alpha_{n+1}^2 - \alpha_{n+1}^3\delta_{n+1})\phi}{1-\alpha_{n+1}^2}\|x_{n+1} - x_n\| \\
& + \left(S_n + \frac{d_n\alpha_{n+1}^3\delta_{n+1}}{(1-\alpha_{n+1}^2)(1-\alpha_{n+1}^3\delta_{n+1})}\right) M,
\end{aligned}
$$

and thus,

$$
\limsup_{n\to\infty} \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) \leq 0. \tag{5.2.8}
$$

Invoking Lemma 2.10.7, we have

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{5.2.9}
$$

Consequently,

$$
\begin{aligned}
\|x_{n+1} - x_n\| \ &= \ \|(1-\alpha_n^2)z_n + \alpha_n^2 x_n - x_n\| \\
&= \ \|(1-\alpha_n^2)z_n - (1-\alpha_n^2)x_n\| \\
&= \ \|(1-\alpha_n^2)(z_n - x_n)\| \\
&\leq \ (1-\alpha_n^2)\|z_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{5.2.10}
\end{aligned}
$$

Next, we show that $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. From (5.2.1), we obtain that

$$\begin{aligned}
\|x_n - Tx_n\| \;&\leq\; \|x_n - x_{n+1}\| + \|x_{n+1} - T(x_n)\| \\
&\leq\; \|x_{n+1} - x_n\| + \|\alpha_n^1 f(x_n) + \alpha_n^2 x_n + \alpha_n^3 T(y_n) - T(x_n)\| \\
&\leq\; \|x_{n+1} - x_n\| + \alpha_n^1 \|f(x_n) - T(x_n)\| \\
&\quad + \alpha_n^2 \|x_n - T(x_n)\| + \alpha_n^3 \|T(y_n) - T(x_n)\| \\
&\leq\; \|x_{n+1} - x_n\| + \alpha_n^1 \|f(x_n) - T(x_n)\| \\
&\quad + \alpha_n^2 \|x_n - T(x_n)\| + \alpha_n^3 \|y_n - x_n\| \\
&\leq\; \|x_{n+1} - x_n\| + \alpha_n^1 \|f(x_n) - T(x_n)\| + \alpha_n^2 \|x_n - T(x_n)\| \\
&\quad + \alpha_n^3 \|(1 - \delta_n) f(x_n) + \delta_n x_{n+1} - x_n\| \\
&\leq\; \|x_{n+1} - x_n\| + \alpha_n^1 \|f(x_n) - T(x_n)\| + \alpha_n^2 \|x_n - T(x_n)\| \\
&\quad + \alpha_n^3 (1 - \delta_n) \|x_n - f(x_n)\| + \alpha_n^3 \delta_n \|x_{n+1} - x_n\| \\
&=\; (1 + \alpha_n^3 \delta_n) \|x_{n+1} - x_n\| + (\alpha_n^1 + \alpha_n^3(1 - \delta_n)) Q + \alpha_n^2 \|x_n - T(x_n)\| \\
&=\; (1 + \alpha_n^3 \delta_n) \|x_{n+1} - x_n\| + (1 - \alpha_n^3 \delta_n - \alpha_n^2) Q + \alpha_n^2 \|x_n - T(x_n)\|.
\end{aligned}$$

Since $0 < \liminf\limits_{n\to\infty} \alpha_n^2 \leq \limsup\limits_{n\to\infty} \alpha_n^2 < 1$, let $0 < \eta \leq \alpha_n^2 < 1$, then

$$\begin{aligned}
\|x_n - Tx_n\| \;&\leq\; \frac{1 + \alpha_n^3 \delta_n}{1 - \alpha_n^2} \|x_{n+1} - x_n\| + \frac{1 - \alpha_n^3 \delta_n - \alpha_n^2}{1 - \alpha_n^2} Q \\
&\leq\; \frac{1 + \alpha_n^3 \delta_n}{1 - \eta} \|x_{n+1} - x_n\| + \frac{1 - \alpha_n^3 \delta_n - \alpha_n^2}{1 - \eta} Q, \qquad (5.2.11)
\end{aligned}$$

which goes to zero as $n \to \infty$ by (5.2.10) and condition (ii) of Definition 5.2.1.

We claim that

$$\limsup_{n\to\infty} \langle f(p) - p, \; J(x_{n+1} - p)\rangle \leq 0, \qquad (5.2.12)$$

For a unique fixed point $p \in F(T)$ of the generalized contraction $P_{F(T)} f(p)$ (Proposition 2.9.4), that is, $p = P_{F(T)} f(p)$ and since $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ by (5.2.11), it follows that

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle \leq 0.$$

Due to the continuity of the duality map and the fact that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ by (5.2.10), we obtain that,

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(p) - p, \; J(x_{n+1} - p)\rangle \;&=\; \limsup_{n\to\infty} \langle f(p) - p, \; J(x_{n+1} - x_n + x_n - p)\rangle \\
&=\; \limsup_{n\to\infty} \langle f(p) - p, \; J(x_n - p)\rangle \leq 0. \qquad (5.2.13)
\end{aligned}$$

We prove that $x_n \to p \in F(T)$ as $n \to \infty$.

Suppose that the sequence $\{x_n\}_{n=1}^\infty$ does not converge strongly to $p \in F(T)$. Then there exists $\epsilon > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\|x_{n_k} - p\| \geq \epsilon$, for all $k \in \mathbb{N}$. Therefore, for this ϵ, there exists $c \in (0, \frac{1}{2})$ such that

$$\|f(x_{n_k}) - f(p)\| \leq c\|x_{n_k} - p\|.$$

$$
\begin{aligned}
\|x_{n_{k+1}} - p\|^2 &= \alpha_{n_k}^1 \left\langle f(x_{n_k}) - p, J(x_{n_{k+1}} - p) \right\rangle + \alpha_{n_k}^2 \left\langle x_{n_k} - p, J(x_{n_{k+1}} - p) \right\rangle \\
&\quad + \alpha_{n_k}^3 \left\langle T(y_{n_k}) - p, J(x_{n_{k+1}} - p) \right\rangle \\
&= \alpha_{n_k}^1 \left\langle f(x_{n_k}) - f(p), J(x_{n_{k+1}} - p) \right\rangle + \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle \\
&\quad + \alpha_{n_k}^2 \left\langle x_{n_k} - p, J(x_{n_{k+1}} - p) \right\rangle + \alpha_{n_k}^3 \left\langle T(y_{n_k}) - p, J(x_{n_{k+1}} - p) \right\rangle \\
&\leq c\alpha_{n_k}^1 \|x_{n_k} - p\| \, \|x_{n_{k+1}} - p\| + \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle \\
&\quad + \alpha_{n_k}^2 \|x_{n_k} - p\| \, \|x_{n_{k+1}} - p\| \\
&\quad + \alpha_{n_k}^3 \|(1 - \delta_{n_k})f(x_{n_k}) + \delta_{n_k} x_{n_{k+1}} - p\| \, \|x_{n_{k+1}} - p\| \\
&\leq c\alpha_{n_k}^1 \|x_{n_k} - p\| \, \|x_{n_{k+1}} - p\| + \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle \\
&\quad + \alpha_{n_k}^2 \|x_{n_k} - p\| \, \|x_{n_{k+1}} - p\| \\
&\quad + \alpha_{n_k}^3 (1 - \delta_{n_k})\|f(x_{n_k}) - p\| \, \|x_{n_{k+1}} - p\| + \alpha_{n_k}^3 \delta_{n_k} \|x_{n_{k+1}} - p\|^2
\end{aligned}
$$

$$\leq c\alpha_{n_k}^1 \|x_{n_k} - p\| \, \|x_{n_{k+1}} - p\| + \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle$$

$$+ \alpha_{n_k}^2 \|x_{n_k} - p\| \, \|x_{n_{k+1}} - p\| + c\alpha_{n_k}^3 (1 - \delta_{n_k}) \|x_{n_k} - p\| \, \|x_{n_{k+1}} - p\|$$

$$+ \alpha_{n_k}^3 (1 - \delta_{n_k}) \|f(p) - p\| \, \|x_{n_{k+1}} - p\| + \alpha_{n_k}^3 \delta_{n_k} \|x_{n_{k+1}} - p\|^2$$

$$= \left(c\alpha_{n_k}^1 + \alpha_{n_k}^2 + c\alpha_{n_k}^3 (1 - \delta_{n_k}) \right) \|x_{n_k} - p\| \, \|x_{n_{k+1}} - p\|$$

$$+ \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle$$

$$+ \alpha_{n_k}^3 (1 - \delta_{n_k}) \|f(p) - p\| \, \|x_{n_{k+1}} - p\| + \alpha_{n_k}^3 \delta_{n_k} \|x_{n_{k+1}} - p\|^2$$

$$\leq \frac{1}{2} \left(c\alpha_{n_k}^1 + \alpha_{n_k}^2 + c\alpha_{n_k}^3 (1 - \delta_{n_k}) \right) \left(\|x_{n_k} - p\|^2 + \|x_{n_{k+1}} - p\|^2 \right)$$

$$+ \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle + \alpha_{n_k}^3 \delta_{n_k} \|x_{n_{k+1}} - p\|^2$$

$$+ \frac{1}{2} \alpha_{n_k}^3 (1 - \delta_{n_k}) \left(\|f(p) - p\|^2 + \|x_{n_{k+1}} - p\|^2 \right)$$

$$= \frac{1}{2} \left(c(\alpha_{n_k}^1 + \alpha_{n_k}^3 (1 - \delta_{n_k})) + \alpha_{n_k}^2 \right) \|x_{n_k} - p\|^2 + \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle$$

$$+ \frac{1}{2} \left(c(\alpha_{n_k}^1 + \alpha_{n_k}^3 (1 - \delta_{n_k})) + \alpha_{n_k}^2 + 2\alpha_{n_k}^3 \delta_{n_k} + \alpha_{n_k}^3 (1 - \delta_{n_k}) \right) \|x_{n_{k+1}} - p\|^2$$

$$+ \frac{1}{2} \alpha_{n_k}^3 (1 - \delta_{n_k}) \|f(p) - p\|^2$$

$$= \frac{1}{2} \left(c(\alpha_{n_k}^1 + \alpha_{n_k}^3 (1 - \delta_{n_k})) + \alpha_{n_k}^2 \right) \|x_{n_k} - p\|^2 + \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle$$

$$+ \frac{1}{2} \left(c(\alpha_{n_k}^1 + \alpha_{n_k}^3 (1 - \delta_{n_k})) + \alpha_{n_k}^2 + \alpha_{n_k}^3 (1 + \delta_{n_k}) \right) \|x_{n_{k+1}} - p\|^2$$

$$+ \frac{1}{2} \alpha_{n_k}^3 (1 - \delta_{n_k}) \|f(p) - p\|^2$$

$$= \frac{1}{2} \left(c(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}) + \alpha_{n_k}^2 \right) \|x_{n_k} - p\|^2 + \alpha_n^1 \left\langle f(p) - p, J(x_{n_{k+1}} - p) \right\rangle$$

$$+ \frac{1}{2} \left(c(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}) + \alpha_{n_k}^2 + \alpha_{n_k}^3 (1 + \delta_{n_k}) \right) \|x_{n_{k+1}} - p\|^2 \qquad (5.2.14)$$

$$+ \frac{1}{2} \alpha_{n_k}^3 (1 - \delta_{n_k}) \|f(p) - p\|^2.$$

Observe that

$$2 - c(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}) - \alpha_{n_k}^2 - \alpha_{n_k}^3 (1 + \delta_{n_k})$$

$$= 2 - c + c\alpha_{n_k}^2 + c\alpha_{n_k}^3 \delta_{n_k} - \alpha_{n_k}^2 - \alpha_{n_k}^3 - \alpha_{n_k}^3 \delta_{n_k}$$

$$= 2 - c - (1 - c)\alpha_{n_k}^2 - (1 - c)\alpha_{n_k}^3 \delta_{n_k} - \alpha_{n_k}^3$$

$$= 1 - c - (1 - c)\alpha_{n_k}^2 - (1 - c)\alpha_{n_k}^3 \delta_{n_k} + 1 - \alpha_{n_k}^3$$

$$= 1 + (1 - c) \left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k} \right) - \alpha_{n_k}^3 \qquad (5.2.15)$$

and

$$\alpha_{n_k}^1 = 1 - \alpha_{n_k}^2 - \alpha_{n_k}^3$$

$$\leq 1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k} \ (\text{since } \delta_{n_k} \in (0,1)). \qquad (5.2.16)$$

Multiplying (5.2.14) by 2 gives

$$
\begin{aligned}
\|x_{n_k+1} - p\|^2 &\leq \frac{c(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}) + \alpha_{n_k}^2}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3} \|x_{n_k} - p\|^2 \\
&\quad + \frac{\alpha_n^1}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3} \left\langle f(p) - p, J(x_{n_k+1} - p) \right\rangle \\
&\quad + \frac{\alpha_{n_k}^3 (1 - \delta_{n_k})}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3} \|f(p) - p\|^2 \\
&= \left(1 - \frac{(1-2c)(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}) + \alpha_{n_k}^1}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3}\right) \|x_{n_k} - p\|^2 \\
&\quad + \frac{\alpha_{n_k}^1}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3} \left\langle f(p) - p, J(x_{n_k+1} - p) \right\rangle \\
&\quad + \frac{\alpha_{n_k}^3 (1 - \delta_{n_k})}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3} \|f(p) - p\|^2 \\
&\leq \left(1 - \frac{(1-2c)(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k})}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3}\right) \|x_{n_k} - p\|^2 \\
&\quad + \frac{(1-2c)(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k})}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3} \frac{1}{1-2c} \left\langle f(p) - p, J(x_{n_k+1} - p) \right\rangle \\
&\quad + \frac{\alpha_{n_k}^3 (1 - \delta_{n_k})}{1 + (1-c)\left(1 - \alpha_{n_k}^2 - \alpha_{n_k}^3 \delta_{n_k}\right) - \alpha_{n_k}^3} \|f(p) - p\|^2 \ (\text{By } (5.2.16)).
\end{aligned}
$$

Using Lemma 2.10.6, it shows that $x_{n_k} \to p$ as $k \to \infty$. A contradiction, hence, $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F(T)$.

∎

The next result shows that under suitable conditions, the implicit iterative sequences (1.1.8) and (5.2.1) converge to the same fixed point.

Theorem 5.2.5 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $f : K \to K$ a c-contraction mapping with $c \in [0,1)$. Let T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$. Let $\left\{\{\alpha_n^i\}_{n=1}^\infty\right\}_{i=1}^3 \subset [0,1]$ and $\{\delta_n\}_{n=1}^\infty \subset (0,1)$ be real sequences such that $\sum_{i=1}^3 \alpha_n^i = 1$. Given that*

$$\lim_{n \to \infty} \frac{\alpha_n^3}{(1 - \alpha_n^2 - \alpha_n^3 \delta_n)} = 0, \text{ then } \{x_n\}_{n=1}^{\infty} \text{ defined by (5.2.1) converges to } p \text{ if and}$$

only if $\{y_n\}_{n=1}^{\infty}$ defined by (1.1.8) converges to p.

Proof. Notice that (5.2.1) and (1.1.8) are respectively given by

$$x_{n+1} = \alpha_n^1 f(x_n) + \alpha_n^2 x_n + \alpha_n^3 T\left((1 - \delta_n)f(x_n) + \delta_n x_{n+1}\right), \ n \in \mathbb{N},$$

and

$$y_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n T\left(\delta_n y_n + (1 - \delta_n)y_{n+1}\right), \ n \in \mathbb{N}.$$

We first need to show that $||x_n - y_n|| \to 0$, as $n \to \infty$.

$$
\begin{aligned}
||x_{n+1} - y_{n+1}|| &= ||\alpha_n^1 f(x_n) + \alpha_n^2 x_n + \alpha_n^3 T((1 - \delta_n)f(x_n) + \delta_n x_{n+1}) \\
&\quad - \left(\alpha_n^1 f(y_n) + \alpha_n^2 y_n + \alpha_n^3 T(\delta_n y_n + (1 - \delta_n)y_{n+1})\right)|| \\
&= ||\alpha_n^1(f(x_n) - f(y_n)) + \alpha_n^2(x_n - y_n) \\
&\quad + \alpha_n^3\left(T((1 - \delta_n)f(x_n) + \delta_n x_{n+1}) - T(\delta_n y_n + (1 - \delta_n)y_{n+1})\right)|| \\
&\leq \alpha_n^1||f(x_n) - f(y_n)|| + \alpha_n^2||x_n - y_n|| \\
&\quad + \alpha_n^3||T((1 - \delta_n)f(x_n) + \delta_n x_{n+1}) - T(\delta_n y_n + (1 - \delta_n)y_{n+1})|| \\
&\leq \alpha_n^1 c||x_n - y_n|| + \alpha_n^2||x_n - y_n|| \\
&\quad + \alpha_n^3||(1 - \delta_n)(f(x_n) - y_{n+1}) + \delta_n(x_{n+1} - y_n)|| \\
&\leq \alpha_n^1 c||x_n - y_n|| + \alpha_n^2||x_n - y_n|| \\
&\quad + \alpha_n^3(1 - \delta_n)||f(x_n) - f(y_n) + f(y_n) - y_{n+1}|| \\
&\quad + \alpha_n^3 \delta_n||x_{n+1} - y_{n+1} + y_{n+1} - y_n|| \\
&\leq \alpha_n^1 c||x_n - y_n|| + \alpha_n^2||x_n - y_n|| \\
&\quad + \alpha_n^3(1 - \delta_n)c||x_n - y_n|| + \alpha_n^3(1 - \delta_n)||y_{n+1} - f(y_n)|| \\
&\quad + \alpha_n^3 \delta_n||x_{n+1} - y_{n+1}|| + \alpha_n^3 \delta_n||y_{n+1} - y_n|| \\
&= \left(\alpha_n^1 c + \alpha_n^3(1 - \delta_n)c + \alpha_n^2\right)||x_n - y_n|| + \alpha_n^3 \delta_n||x_{n+1} - y_{n+1}|| \\
&\quad + \alpha_n^3(1 - \delta_n)||y_{n+1} - f(y_n)|| + \alpha_n^3 \delta_n||y_{n+1} - y_n||.
\end{aligned}
$$

Since $\{y_n\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are bounded, let

$$M_2 = \max \left\{ \sup_n \|y_{n+1} - f(y_n)\|, \sup_n \|y_{n+1} - y_n\| \right\}. \text{ Then}$$

$$
\begin{aligned}
\|x_{n+1} - y_{n+1}\| &\leq \frac{\alpha_n^1 c + \alpha_n^3(1-\delta_n)c + \alpha_n^2}{1 - \alpha_n^3 \delta_n} \|x_n - y_n\| + \frac{\alpha_n^3}{1 - \alpha_n^3 \delta_n} M_2 \\
&= \left(1 - \frac{(1 - \alpha_n^2 - \alpha_n^3 \delta_n)(1-c)}{1 - \alpha_n^3 \delta_n} \right) \|x_n - y_n\| + \frac{\alpha_n^3}{1 - \alpha_n^3 \delta_n} M_2 \\
&= (1 - \beta_n)\|x_n - y_n\| + \frac{\alpha_n^3}{(1 - \alpha_n^2 - \alpha_n^3 \delta_n)(1-c)} \beta_n M_2, \quad (5.2.17)
\end{aligned}
$$

where $\beta_n = \frac{(1-\alpha_n^2-\alpha_n^3\delta_n)(1-c)}{1-\alpha_n^3\delta_n}$. From the given condition, it follows that $\limsup\limits_{n\to\infty} \frac{\alpha_n^3}{(1-\alpha_n^2-\alpha_n^3\delta_n)} \leq 0$. Apply Lemma 2.10.6 with $\gamma_n = 0$ to (5.2.17) to get that $\|x_n - y_n\| \to 0$, as $n \to \infty$. Next, suppose $\|y_n - p\| \to 0$ as $n \to \infty$. It follows that,

$$\|x_n - p\| = \|x_n - y_n + y_n - p\| \leq \|x_n - y_n\| + \|y_n - p\| \to 0 \text{ as } n \to \infty.$$

Similarly, suppose $\|x_n - p\| \to 0$ as $n \to \infty$. Then,

$$\|y_n - p\| = \|y_n - x_n + x_n - p\| \leq \|y_n - x_n\| + \|x_n - p\| \to 0 \text{ as } n \to \infty.$$

Hence, the implicit iterative sequences (1.1.8) and (5.2.1) converge to the same fixed point under suitable conditions. ∎

5.2.3 Applications

(I) Application to fixed points of λ-strictly pseudo-contractive mappings

Let K be a closed convex subset of a real Banach space E. A mapping $S : K \to K$ is said to be λ-strictly pseudo-contractive mapping if there exists $0 \leq \lambda < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 - \lambda\|(I - S)x - (I - S)y\|^2, \ \forall \ x, y \in K, \qquad (5.2.18)$$

where I denotes the identity operator on K.

Zhou [112] established the following lemma which gives a relationship between λ-strictly pseudo-contractive mappings and nonexpansive mappings.

Lemma 5.2.6 *Let K be a nonempty subset of a 2-uniformly smooth Banach space E. Let $S : K \to K$ be a λ-strictly pseudo-contractive mapping. For $\theta \in (0, 1)$, define*

$$Tx = \theta x + (1 - \theta)Sx \ \forall \ x \in K. \tag{5.2.19}$$

Then, as $\theta \in (0, \frac{\lambda}{L^2}]$, (where L is the 2-uniformly smooth constant of a 2-uniformly smooth Banach space,) $T : K \to K$ is nonexpansive such that $F(T) = F(S)$.

We obtain the following result by using Lemma 5.2.6 and Theorem 5.2.4.

Corollary 5.2.7 *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $f : K \to K$ be a generalized contraction mapping. Let $S : K \to K$ a λ-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Suppose that the conditions $(i) - (v)$ of Definition 5.2.1 are satisfied and T is a mapping from K into itself, defined by $Tx = \alpha x + (1 - \theta)Sx$, $x \in K$, $\theta \in (0, 1)$. Then, for an arbitrary $x_1 \in K$, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by*

$$x_{n+1} = \alpha_n^1 f(x_n) + \alpha_n^2 x_n + \alpha_n^3 T\left((1 - \delta_n)f(x_n) + \delta_n x_{n+1}\right) \ \text{for all } n \in \mathbb{N}, \tag{5.2.20}$$

converges strongly to a fixed point p of S, which solves the variational inequality

$$\langle (I - f)p, J(x - p) \rangle \geq 0, \ \text{for all } x \in F(S). \tag{5.2.21}$$

(II) **Application to solution of α-inverse-strongly monotone mappings**

Let K be a nonempty closed convex subset of a Hilbert space H. The metric projection P_K, is defined from H onto K by

$$P_K x := \arg\min_{y \in K} \|x - y\|^2, \ x \in H \tag{5.2.22}$$

and characterized by

$$P_K(x) := \arg\min_{z \in K} \|x - z\|^2, \ x \in H. \tag{5.2.23}$$

$P_K(x)$ is known as the only point in K that minimizes the objective $\|x - z\|$ over $z \in K$. A mapping A of K into H is called monotone if $\langle Au - Av, u - v \rangle \geq 0$, for all $u, v \in K$. The classical Variational Inequality (VI) problem is to find $u^* \in K$ such that

$$\langle Au^*, u - u^* \rangle \geq 0, \ u \in K, \tag{5.2.24}$$

where A is a (single-valued) monotone operator in Hilbert space H ([20], [71]). In this work, the solution set of (5.2.24) is denoted by $VI(K, A)$. In the context of the variational inequality problem, (5.2.23) implies that

$$u \in VI(K, A) \Leftrightarrow u = P_K(u - \gamma Au), \; \forall \; \gamma > 0. \tag{5.2.25}$$

A is said to be α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2,$$

for all $u, v \in K$. If A is an α-inverse-strongly monotone mapping of K to H, it is known that A is $\frac{1}{\alpha}$-Lipschitz continuous. Also, we have that for all $u, v \in K$ and $\gamma > 0$,

$$
\begin{aligned}
\|(I - \gamma A)u - (I - \gamma A)v\|^2 &= \|(u - v) - (Au - Av)\|^2 \\
&= \|u - v\|^2 - 2\gamma \langle u - v, \; Au - Av \rangle + \gamma^2 \|Au - Av\|^2 \\
&\leq \|u - v\|^2 + \gamma(\gamma - 2\alpha)\|Au - Av\|^2.
\end{aligned}
$$

Therefore, if $\gamma \leq 2\alpha$, then $I - \gamma A$ is a nonexpansive mapping of K into K. Consequently, one can apply Theorem 5.2.4 to deduce the following result:

Corollary 5.2.8 *Let K be a nonempty closed convex subset of a real Hilbert space H and $f : K \to K$ a generalized contractions. Let A be an α-inverse-strongly monotone mapping of K to H with $A^{-1}0 \neq \emptyset$. Assume that the conditions $(i) - (v)$ of Definition 5.2.1 are satisfied. Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ which is defined from an arbitrary $x_1 \in K$ by*

$$x_{n+1} = \alpha_n^1 f(x_n) + \alpha_n^2(x_n) + \alpha_n^3 P_K(I - \gamma A)\left((1 - \delta_n)f(x_n) + \delta_n x_{n+1}\right), n \in \mathbb{N}, \tag{5.2.26}$$

converges strongly to a solution p in $A^{-1}0$, which solves the variational inequality

$$\langle (I - f)p, x - p \rangle \geq 0, \; \text{for all } x \in A^{-1}0. \tag{5.2.27}$$

(III) **Application to Fredholm integral equation in Hilbert spaces**

Consider a Fredholm integral equation of the form

$$x(t) = g(t) + \int_0^1 \Phi(t, s, x(s))ds, \; t \in [0, 1], \tag{5.2.28}$$

where g is a continuous function on $[0,1]$ and $\Phi : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The existence of solutions of (5.2.28) has been studied (see [79] and the references therein). If Φ satisfies the Lipschitz continuity condition

$$|\Phi(t,s,x) - \Phi(t,s,y)| \le |x - y|, \ \ s,t \in [0,1], \ \ x,y \in \mathbb{R}, \tag{5.2.29}$$

then equation (5.2.28) has at least one solution in the Hilbert space $L^2[0,1]$ ([79], Theorem 3.3). Precisely, define a mapping $T : L^2[0,1] \to L^2[0,1]$ by

$$Tx(t) = g(t) + \int_0^1 \Phi(t,s,x(s))ds, \ \ t \in [0,1]. \tag{5.2.30}$$

It is known that T is nonexpansive. Indeed, for $x, y \in L^2[0,1]$

$$
\begin{aligned}
\|Tx - Ty\|^2 &= \int_0^1 |Tx(t) - Ty(t)|^2 dt \\
&= \int_0^1 \left| \int_0^1 \Phi(t,s,x(s)) - \Phi(t,s,y(s))ds \right|^2 dt \\
&\le \int_0^1 \left| \int_0^1 |x(s) - y(s)|ds \right|^2 dt \\
&\le \int_0^1 |x(s) - y(s)|^2 ds = \|x - y\|^2.
\end{aligned}
$$

Thus, finding a solution of integral equation (5.2.28) is reduced to finding a fixed point of the nonexpansive mapping T in the Hilbert space $L^2[0,1]$. Consequently, the following result is obtainable.

Corollary 5.2.9 *Let K be a nonempty closed convex subset of a Hilbert space $L^2[0,1]$, $T : K \to K$, defined by (5.2.30) with $F(T) \neq \emptyset$ and $f : K \to K$ is a generalized contraction. Suppose that the conditions $(i) - (v)$ of Definition 5.2.1 are satisfied. Then, for an arbitrary $x_1 \in K$, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by*

$$x_{n+1} = \alpha_n^1 f(x_n) + \alpha_n^2(x_n) + \alpha_n^3 T\left((1 - \delta_n)f(x_n) + \delta_n x_{n+1}\right), \ n \in \mathbb{N}, \tag{5.2.31}$$

converges strongly to a fixed point p of T, which solves the variational inequality

$$\langle (I - f)p, x - p \rangle \ge 0, \ \text{for all } x \in F(T). \tag{5.2.32}$$

General conclusion, contribution to knowledge and future research

6.1 General conclusion and contribution to knowledge

Introduction of new maps or functions and construction of new algorithms are very essential in Functional Analysis. We proposed the concept of generalized Lyapunov functions in Chapter 3 (Aibinu and Mewomo [3], [5]). The Lyapunov functions given by Alber [7] are obtained from the generalized Lyapunov functions by taking $p = 2$. The generalized Lyapunov functions admit generalized duality mapping. When $p = 2$, the generalized duality mapping becomes the normalized duality mapping and we obtain the definition which was given by Alber [7]. The class of (p,η)-strongly monotone mappings which satisfies the range condition is being considered in Section 3.3, where $p > 1$ and $\eta > 0$. Thus, the results in Section 3.3 extend and generalize the results of Chidume and Idu [33]. Great improvements and expansion have been made to the results of Chidume and Djitte [30]: more general iterative algorithm was considered in Section 3.3 and the results are obtained in uniformly smooth and uniformly convex Banach spaces. The study in Section 3.4 focuses on the class of generalized Φ-strongly monotone mappings in Banach spaces, the class of

monotone-type mappings. Therefore, the results in Section 3.4 extend and improve the existing results on monotone type mappings in the literatures. Reference is also made to the generalized convex optimization problems as an application of the results.

In Section 4.1, the numerical examples display the efficiency of the rate of convergence of implicit midpoint rules, where viscosity is involved over a nonviscous method. We studied the relationship between the existing generalized implicit iterative algorithms and examined the conditions under which they converge to the same fixed points of a nonexpansive mapping. Analytical comparison of the rate of convergence of the existing generalized implicit iterative algorithms is given in Section 4.2. The analytical proof is essential as it is more general and contains numerical examples as corollaries. The study of the class of μ-strictly pseudocontractive mappings in Section 5.1 is very important as it extends and improves the existing results on viscosity approximation methods. Generalized contraction is used in Section 5.2 to introduce a new viscosity iterative algorithm for the class of nonexpansive mappings. The convergence in norm of the newly introduced sequence to a fixed point of a nonexpansive mapping is established. Furthermore, it is shown that it also solves some variational inequality problems in uniformly smooth Banach spaces.

6.2   Recommendation

(i) Can the implicit midpoint rule be applied to approximate a fixed point of non-affine nonexpansive mappings such as $\sin x$? For instance, taking $T(x) = \sin x$ in the implicit midpoint rule (1.1.5), a simplest form of the equation in $\mathbb{R}$ which one would obtain is

$$y = x + \sin(x + y),$$

where y is to be made the subject of the formula in order to get an explicit equation like (4.1.15), (4.1.16) or (4.1.17).

(ii) We do not know if the implicit iteration scheme can be applied in general to

nonexpansive non-affine functions, for example $\cos x$. That is, can one solve for y in

$$y = ax + b\cos(cx + dy)$$

in $\Re$ (the set of real numbers with the usual metric).

Bibliography

[1] M. O. Aibinu, *On the rate of convergence of viscosity implicit iterative algorithms*, **Accepted** to appear in Nonlinear Functional Analysis and Applications, (Scopus Indexed).

[2] M. O. Aibinu and C. E. Chidume, *Existence and uniqueness of solutions of integral equations of Hammerstein type*, M.Sc Thesis, African University of Science and Technology, Abuja, Nigeria, (2013).

[3] M. O. Aibinu and O. T. Mewomo, *Strong convergence theorems for strongly monotone mappings in Banach spaces*, Boletim da Sociedade Paranaense de Matemática, DOI:10.5269/bspm.37655.

[4] M. O. Aibinu and O. T. Mewomo, *Algorithm for the generalized Φ-strongly monotone mappings and application to the generalized convex optimization problem*, Proyecciones Journal of Mathematics, 38 (1), March (2019), 59-82.

[5] M. O. Aibinu and O. T. Mewomo, *Algorithm for Zeros of monotone maps in Banach spaces*, Proceedings of Southern Africa Mathematical Sciences Association (SAMSA2016) Annual Conference, 21-24 November 2016, University of Pretoria, South Africa, (2017), 35-44.

[6] M.O. Aibinu, P. Pillay, J.O. Olaleru and O.T. Mewomo, *The implicit midpoint rule of nonexpansive mappings and applications in uniformly smooth Banach spaces*, J. Nonlinear Sci. Appl., 11 (2018), 1374-1391, DOI: 10.22436/jnsa.011.12.08.

[7] Ya. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, In: Kartsatos AG (ed) Theory and applications of nonlinear operators of accretive and monotone type, Marcel Dekker, New York, (1996), 15-50.

[8] Y. Alber and I. Ryazantseva, *Nonlinear Ill Posed Problems of Monotone Type*, Springer, London (2006).

[9] M. A. Alghamdi, N. Shahzad and H. K. Xu, *The implicit midpoint rule for nonexpansive mappings*, Fixed Point Theory Appl., 2014, (2014), 96.

[10] F. Altomare and M.Campiti, Korovkin-type Approximation Theory and its Applications, Berlin: New York: de Gruyter, (1994).

[11] D. Aussel, J. N. Corvellec and M. Lassonde, *Subdifferential characterization of quasiconvexity and convexity*, J. Convex Anal. 1, (1994), 195-201.

[12] D. Aussel, J. N. Corvellec and M. Lassonde, *Mean value property and subdifferential criteria for lower semicontinuous functions*, Trans. Am. Math. Soc. 347, (1995), 4147-4161.

[13] D. Aussel and M. Fabian, *Single-directional properties of quasi-monotone operators*, Set-Valued Var. Anal., 21, (2013) 617-626, DOI 10.1007/s11228-013-0253-4.

[14] H. Attouch, *Viscosity solutions of minimization problems*, SIAM J. Optim., 6 (1996), 769-806.

[15] W. Auzinger and R. Frank, *Asymptotic error expansions for stiff equations: an analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case*, Numer. Math. 56, (1989), 469-499.

[16] G. Bader and P. Deuflhard, *A semi-implicit mid-point rule for stiff systems of ordinary differential equations*, Numer. Math. 41, (1983), 373-398.

[17] A. Bayreuth, *The implicit midpoint rule applied to discontinuous differential equations*, . Computing 49, 45-62 (1992).

[18] V. Berinde, *Picard iteration converges faster than Mann iteration for a class of quasicontractive operators*, Fixed Point Theory and Applications, 1 (2004), 1-9.

[19] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc., 73, (1967), 875-882.

[20] F.E. Browder, W.V. Petryshyn, *Construction of Fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. 20 (1967) 197-228.

[21] R. E. Bruck Jr., *A strongly convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator U in Hilbert spaces*, J. Math. Anal. Appl., 48 (1974), 114-126. https://doi.org/10.1016/0022-247x(74)90219-4.

[22] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New-York, (1983).

[23] Y. Censor and R. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization, 37, (4), (1996), 323-339.

[24] C. E Chidume, *Geometric properties of Banach spaces and nonlinear iterations*, Lectures Notes in Mathematics, Springer Verlag Series, London, 1965, (2009), ISBN: 978-1-84882-189-7.

[25] C. E. Chidume, *Iterative approximation of fixed points of Lipschitzian strictly pseudo-contractive mappings*, Proc. Amer. Math. Soc., 99, (2), (1987), 283-288.

[26] C. E. Chidume, *An approximation method for monotone Lipschitzian operators in Hilbert-spaces*, J. Aust. Math. Soc. Ser. Pure Math. Stat., 41, (1986), 59-63.

[27] C. E. Chidume and A. Bashir, *Approximation of common fixed points for finite families of nonself asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. App., 326, (2007), 960-973.

[28] C. E. Chidume, A.U. Bello and B. Usman, *Krasnoselskii-type algorithm for zeros of strongly monotone Lipschitz maps in classical Banach spaces*, Springer-Plus, 297, (4), (2015), DOI 10.1186/s40064-015-1044-1.

[29] C. E. Chidume and C. O. Chidume, *A convergence theorem for zeros of generalized Phi-quasi-accretive mappings*, Amer. Math. Soc., 134, (1), (2005), 243-251.

[30] C. E. Chidume and N. Djitte, *Strong convergence theorems for zeros of bounded maximal monotone nonlinear operators*, Abstract and Applied Analysis, (2012), Article ID 681348, 19 pages, DOI:10.1155/2012/681348

[31] C. E. Chidume and N. Djitte, *Approximation of Solutions of Nonlinear Integral Equations of Hammerstein Type*, ISRN Mathematical Analysis, (2012), Article ID 169751, 12 pages.

[32] C. E. Chidume, N. Djitte and M. Sene, *Iterative algorithm for zeros of multivalued accretive operators in certain Banach spaces*, Afr. Math., 26, (2015), 357-368.

[33] C. E. Chidume and K. O. Idu, *Approximation of zeros of bounded maximal monotone mappings, solutions of Hammerstein integral equations and convex minimization problems*, Fixed Point Theory and Applications, 2016, (97), (2016), DOI 10.1186/s13663-016-0582-8.

[34] C. E. Chidume and S. A. Mutangandura, *An example on the Mann iteration method for Lipscitz pseudocontractions*, Proc. Amer. Math. Soc., 129 (8), (2001), 2359-2363.

[35] C. E. Chidume, U. V. Nnyaba and O. M. Romanus, *A new algorithm for variational inequality problems with a generalized phi-strongly monotone*

map over the set of common fixed points of a finite family of quasi-phi-nonexpansive maps, with applications, J. Fixed Point Theory Appl. (2018) 20:29, DOI:10.1007/s11784-018-0502-0

[36] C.E. Chidume and Y. Shehu, *Strong convergence theorem for approximation of solutions of equations of Hammerstein type* Nonlinear Analysis, 75, (2012), 5664-5671.

[37] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic, Dordrecht (1990).

[38] K. Deimling, *Nonlinear functional analysis*, Springer, New York, (1985).

[39] P. Deuflhard, *Recent progress in extrapolation methods for ordinary differential equations*, SIAM Rev. 27(4), (1985), 505-535.omputing 49, (1992), 45-62.

[40] C. Diop, T. M. M. Sow, N. Djitte and C. E. Chidume, *Constructive techniques for zeros of monotone mappings in certain Banach spaces*, SpringerPlus, 383, (4), (2015), DOI 10.1186/s40064-015-1169-2.

[41] P. Dodos, *Generalized gradients ofmonotone type*, Nonlinear Analysis 56 (2004) 201-212.

[42] A.P. Farajzadeh1, A. Karamian1 and S. Plubtieng, *On the translations of quasi-monotone maps and monotonicity*, J. of Ineq. Appl., 1 (192), (2012).

[43] A. P. Farajzadeh and S. Plubtieng *When a vector quasimonotone mapping is a vector monotone mapping*, Optim Lett., 8, (2014), 2127-2134, DOI 10.1007/s11590-013-0716-4.

[44] N. Hadjisavvas, *Translations of quasimonotone maps and monotonicity*, Appl. Math. Lett. 19, (2006), 913-915.

[45] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc., (73), (1967), 957-961.

[46] A. Hassouni, *Quasi-monotone multifunctions; Applications to optimality conditions in quasi-convex programming*, Numer. Funct. Anal. Optim. 13 (3/4), (1992), 267-275.

[47] A. Hassouni and A. Jaddar, *Quasi-convex functions and applications to optimality conditions in nonlinear programming*, Appl. Math. Lett. 14, (2001), 241-244.

[48] Y. He, *A relationship between pseudomonotone and monotone mappings*, Appl. Math. Lett. 17, (2004), 459-461.

[49] L. Hu and J. Wang, *Mann iteration of weak convergence theorems in Banach space*, Acta Mathematicae Applicatae Sinica, 25 (2), (2009), 217-224.

[50] E. Hofer, *A partially implicit method for large stiff systems of ODEs with only few equations introducing small time-constants*, SIAM J. Numer. Anal. 13, (1976), 645-663.

[51] J. Hrabalova and P. Tomasek, *On stability regions of the modified midpoint method for a linear delay differential equation*, Adv. Differ. Equ. 2013, 177, (2013).

[52] S. Kamimura and W. Takahashi, *Strong convergence of proximal-type algorithm in Banach space*, SIAMJ Optim., 13, (3), (2002), 938-945.

[53] S. Karamardian and S. Schaible, *Seven kinds of monotone maps*, J. Optim. Theory Appl. 66, (1990), 37-46.

[54] S. Karamardian, S. Schaible and J.P Crouuzeix, *Characterizations of generalized monotone maps*, J. Optim. Theory Appl. 76, (1993), 399-413.

[55] S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics*, Pergamon Press, London Paris, (1959).

[56] K. Kido, *Strong convergence of resolvents of monotone operators in Banach spaces*, Proc. Am. Math. Soc. 103 (3), (1988), 755-758.

[57] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstract and Applied Analysis 2004:3 (2004) 239Ũ249

[58] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces*, SIAM J. on Optim., 19, (2), (2008), 824-835.

[59] S. Imnang and S. Suantai, *Strong convergence of a viscosity iterative algorithm in Banach spaces with applications*, 10 (52), (2016), 2589-2609.

[60] A. F. Ipatov, Estimation of the error and order of approximation of two variables by Bernstein polynomials Russian, Uc. Zap. Petrozavodsk. Cos. Univ. 4(4). (1955), 3Ũ48 (1957).

[61] G. Isac and D. Motreanu, *Pseudomonotonicity and quasimonotonicity by translations versus monotonicity in Hilbert spaces*, Aust. J. Math. Anal. Appl. 1(1), (2004), 1-8.

[62] J.S. Jung, *Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., (302), (2005) 509-520.

[63] P. Katchang and P. Kumam, *Convergence of iterative algorithm for finding common solution of fixed points and general system of variational inequalities for two accretive operators*, Thai J. Math., 9 (2011), 343-360.

[64] R. I. Kačurovskii, *On monotone operators and convex functionals*, Uspekhi Mat. Nauk., 15, (1960), 213-215.

[65] Y. Ke and C. Ma, *The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces*, Fixed Point Theory Appl., 2015 (2015), 21 pages.

[66] B. T. Kien, *The normalized duality mappings and two related characteristic properties of a uniformly convex Banach space*, Acta Mathematica Vietnamica, 27, (1), (2002), 53-67.

[67] V. L. Levin, *Quasiconvex functions and quasimonotone operators*, J. Conv. Anal., 2 (1/2), (1995), 167-172.

[68] Li-Hua Qiu and Si-Sheng Yao *Strong convergence theorems of nonself nonexpansive mapping in Banach spaces*, Appl. Math. Sci., 5 (56), (2011), 2781-2788.

[69] T. C. Lim, *On characterizations of Meir-Keeler contractive maps*, Nonlinear Anal., 46 (2001), 113-120.

[70] P.L. Lions, *Approximation de points fixes de contractions*, C.R. Acad. Sci. Paris Ser. ACB, (284), (1977) A1357-A1359.

[71] F. Liu, M.Z. Nashed, *egularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal. 6 (1998) 313-344.

[72] P. Luo, G. Cai and Y. Shehu, *The viscosity iterative algorithms for the implicit midpoint rule of nonexpansive mappings in uniformly smooth Banach spaces*, J. Ineq. Appl., 2017, (2017):154, DOI 10.1186/s13660-017-1426-8.

[73] B. Martinet, *Regularisation d́inéquations variationnelles par approximations successives*, 4, (1970), 154-158.

[74] B. Martinet, *Breve communication. Regularization dŠinéquations variationelles par approximations successives*, Revue Francaise DŠinformatique et de Recherche Operationelle, 4 (1970), 154-158. https://doi.org/10.1051/m2an/197004r301541

[75] G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. 29 (1962), 341-346.

[76] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. 241, (2000), 46-55.

[77] A. Moudafi, *Proximal methods for a class of bilevel monotone equilibrium problems*, J. Glob. Optim. 47 (2), (2010), 45-52.

[78] A. Nandal and R. Chugh, *A new strong convergence theorem for Lipschitzian strongly pseudocontractive mappings in a real Banach space*, Int. J. Math. Tren. Tech., 41 (2), (2017), 205-209.

[79] J. J. Nieto and H. K. Xu, *Solvability of nonlinear Volterra and Fredholm equations in weighted spaces. Nonlinear Anal.*, Theory Methods Appl. 24, (1995), 1289-1297.

[80] Forster Otto, Analysis 1, Differential- und Integralrechnung einer Veränderlichen, Vieweg Studium, (1983).

[81] D. Pascali and S. Sburlan, *Nonlinear mappings of monotone type*, Editura Academiae, Bucharest, Romania, (1978).

[82] J.P. Penot, *Are generalized derivatives useful for generalized convex functions?*, University of Pau, (1997), 1-44.

[83] J.P. Penot and P. H. Quang, *Generalized convexity of functions and generalized monotonicityof set-valued maps*, J. Optim. Theory Appl., 92, 2, (1997), 343-356.

[84] R.R. Phelps, *Convex functions, monotone operators, and differentiability*, 2nd edn. In: Lecture Notes in Math, vol. 1364. Springer-Verlag, Berlin, (1993).

[85] X. L. Y. Qin, J. Cho, S. M. Kang and H. Y. Zhou, *Convergence of a modified Halpern-type iteration algorithm for quasi-Φ-nonexpansive mappings*, Applied Mathematics Letters, 22 (7), (2009), 1051-1055.

[86] S. Reich, *Approximating fixed points of nonexpansive mappings*, Panamer. Math. J., 4 (2) (1994) 23-28.

[87] S. Reich, *A weak convergence theorem for the alternating methods with Bergman distance, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. G. Kartsatos (Ed.), Lecture Notes in Pure and Appl. Math., vol. 178, Dekker, New York, (1996), 313-318.

[88] R. T. Rockafellar, *Generalized directional derivatives and subgradients of nonconvex functions*, Canad. J. Math. 32 (1980), 257-280.

[89] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., 14 (1976), 877-898. https://doi.org/10.1137/0314056

[90] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., 149, (1970), 75-88.

[91] C. Schneider, *Analysis of the linearly implicit mid-point rule for differential-algebra equations. Electron*, Trans. Numer. Anal. 1, (1993), 1-10.

[92] N. Shioji and W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc., (125), (1997) 3641-3645.

[93] S. Somalia, *Implicit midpoint rule to the nonlinear degenerate boundary value problems*, Int. J. Comput. Math. 79 (3), (2002), 327-332.

[94] S. Somalia and S. Davulcua, *Implicit midpoint rule and extrapolation to singularly perturbed boundary value problems*, Int. J. Comput. Math. 75 (1), (2000), 117-127.

[95] T. Suzuki, *Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces*, Fixed Point Theory and Applications, 2005, (1), (2005), 103-123.

[96] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama (2000).

[97] H. K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., 178, (1993), 301-308.

[98] Z.M. Wang, Y. F. Su, D. X. Wang and Y. C. Dong, *A modified Halpern-type iteration algorithm for a family of hemi-relatively nonexpansive mappings and systems of equilibrium problems in Banach spaces*, Journal of Computational and Applied Mathematics, 235 (8), (2011), 2364-2371.

[99] N.C. Wong, D.R. Sahu and J.C. Yao, *Solving variational inequalities involving nonexpansive type mappings*, Nonlinear Analysis 69 (2008) 4732-4753.

[100] T. Suzuki, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl. 325, (2007), 342-352.

[101] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., 16(12),(1991), 1127-1138.

[102] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. 298, (2004), 279-291.

[103] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc., 2, (2002), 240-256.

[104] H. K. Xu, M. A. Alghamdi and N. Shahzad, *The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces*, Fixed Point Theory Appl. 2015, 41, (2015), 12 pages.

[105] Z. B. Xu and G. F. Roach, *Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces*, J. Math. Anal. Appl. 157, (1991), 189-210.

[106] Q. Yan, G. Cai, P. Luo, *Strong convergence theorems for the generalized viscosity implicit rules of nonexpansive mappings in uniformly smooth Banach spaces*, J. Nonlinear Sci. Appl. 9 (2016), 4039-4051.

[107] Y. Yao, N. Shahzad and Y. C. Liou, *Modified semi-implicit midpoint rule for nonexpansive mappings*,Fixed Point Theory and Applications (2015) 2015: 166, DOI 10.1186/s13663-015-0414-2.

[108] E. H. Zarantonello, *The meaning of the Cauchy-Schwarz-Buniakovsky inequality*, Proceedings of the American Mathematical Society, 59(1), (1976), 133-137.

[109] C. Zălinescu, *On uniformly convex functions*, J. Math. Anal. Appl. 95, (1983), 344-374.

[110] E. H. Zarantonello, *Solving functional equations by contractive averaging*, Technical Report 160. Madison, Wisconsin: U.S. Army Mathematics Research Center; 1960.

[111] H. Zegeye, *Strong convergence theorems for maximal monotone mappings in Banach spaces*, J. Math. Anal. Appl. 343, (2008), 663-671.

[112] H. Zhou, *Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces*, Nonlinear Anal. 69 (2008), 3160-3173.