Aspects of Connectedness in Metric Frames

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Submitted in fulfilment of the academic requirements for the degree of Doctor of Philosophy in Mathematics

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January, 2019

Declaration

I, Cerene Rathilal, affirm that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. The study presented in this thesis was carried out in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban.

3. This thesis has not been submitted for any degree or examination at any other university.

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Signed: Rathilal

C Rathilal

Dedicated

to

Mum and Dad.

Acknowledgements

I wish to express my gratitude to my supervisors Dr P. Pillay and Prof D. Baboolal for their invaluable guidance, immense support and unwavering patience throughout the preparation of this thesis.

I am also grateful to my friend Byron Brassel, who sacrificed his time to read and identify my grammatical errors.

I am especially grateful to the Canon Collins Educational and Legal Aid Trust for their financial assistance and all their moral support during my PhD studies.

Lastly, I wish to thank my parents for all their love and support, and for all the sacrifices they had made so that I could pursue my dreams.

Abstract

Within the context of frames there is a well-established theory of metric frames. This is the frame analog of metric spaces. In this thesis we investigate the extension of the theory around the concepts of connectedness, local connectedness, Property S and uniform local connectedness in the setting of metric frames. Connectedness of sublocales and diameters on sublocales are also studied. A new diameter on a connected, locally connected metric frame is constructed. It is shown that the resulting metric frame is uniformly locally connected and positively answers the question on the existence of a diameter on a connected, locally connected frame possessing Property S, such that the spherical neighbourhood of every point is connected and has Property S. Completion of metric frames and conditions under which a frame is S-metrizable are investigated. It is established that in a locally connected frame, S-metrizability is equivalent to the existence of a countably locally connected and uniformly locally connected Wallman basis. Locally non-separating sublocales of metric frames are studied and is essential in obtaining an equivalent criteria for a non-compact, connected and regular continuous frame to be S-metrizable .

Contents

Chapter 1

Introduction

The origin of the theory of frames (or locales) can be traced back to as early as the 1930's, when Stone [28] and Wallman [29] were the first mathematicians to apply lattice theory to Topology. Since then, interest in the approach to studying topology from a lattice theoretic viewpoint began to grow. The term *frame* was coined during the 1960's, by Dowker and Papert [13], in order to describe a *local lattice* that was being studied at a seminar by Ehresmann in Paris. But it was only until 1972, through Isbell's pioneering paper [15], that the theory of frames came to prominence. This paper opened several important topics for investigation. Since then, more mathematicians have become interested in the area of frames and a richer body of work now exists.

Isbell [15] was the first to consider the notion of a *uniformity* on a frame using a system of covers. Of particular relevance to this thesis, Isbell had defined a frame L as *metrizable* if it admits a countably generated uniformity, using the assumptions from the Uniform Metrization Theorem.

In 1984, Pultr [25] defined a *metric diameter* d on a general frame L, to imitate distance functions in spaces. This was a natural modification of the notion of distance in a generalised setting. Pultr called the pair (L, d) a *diametric frame*, which is now commonly referred to as a *metric frame* (and is the accepted usage).

Sublocales, which are generalised subspaces, are also notably attributed to Isbell [15] and Dowker and Papert [13]. Although, it has since been studied by several authors. The use of a *nucleus* to study sublocales is due to Simmons in 1978. In this thesis we use the nucleus as a tool to study diameters on sublocales, and hence *metric sublocales*.

The *completion* of a uniforms frame was first described by Isbell [15]. In [19], Kříž introduced an alternate and more straightforward description. Later, Banaschewski and Pultr [9] showed that every metric frame has a unique completion. The known theory on Metric frame completions will later be discussed in detail, so that we may study properties of dense metric sublocales.

The study of *compactifications of frames* articulated in terms of strong inclusions is due to Banaschewski [7]. Banaschewski showed that every compactification of a frame induces a strong inclusion on the frame and, vice versa. In this thesis, we will be concerned with the Wallman compactification of a frame. The *Wallman compactification* of a frame was first defined by Johnstone [16]. However, our approach is different from his construction, and instead we will follow the approach of Baboolal [4], where the Wallman Compactification is defined from a Wallman basis.

This thesis is concerned with some of the aspects of connectedness in metric frames. One of the central themes in this thesis, is the concept of *Property S* and *S-metrizability*, which is due to Sierpinski but introduced in frames by Baboolal. For the remainder of this chapter, we shall provide a brief overview of results and provide a summary of the forthcoming chapters.

Chapter 2. This chapter is a survey of known results in the theory of Frames. Definitions, notations and prerequisite theory is provided. Proofs for some of the results will be included. An introduction to frames, sublocales and metric frames is presented.

Chapter 3. Property S, uniform local connectedness and local connectedness are investigated in this chapter. It is shown in Theorem 3.1.8, that under the assumption of compactness, the above three mentioned concepts are equivalent. Diameters on sublocales are also studied with the use of nuclei [20]. Detailed proofs for properties of diameters on sublocales are provided. The main result of this chapter, Theorem 3.3.8, establishes an equivalent criteria for Property S on a metric frame, in the language of metric sublocales.

Chapter 4. In [18], Kelley constructed a metric on a connected, locally connected space which answered a question posed by Whyburn. Whyburn asked whether there exists a metric on a connected space having Property S, such that every open ball of a point will be connected and have Property S. In Definition 4.1.5, we define a new diameter ρ on a connected, locally connected metric frame (L, d). Using a result of Pultr and Picado [20], we obtain a compatible metric diameter $\tilde{\rho}$ from ρ, which is the frame analogue to the construction of Kelley's metric and we show that $\tilde{\rho}$ positively answers Whyburn's question in the point-free setting. The relationship between Kelley's metric and the new constructed metric diameter is investigated in Proposition 4.2.4, and we show that $(L, \tilde{\rho})$ is uniformly locally connected, in Theorem 4.3.8.

Chapter 5. In this chapter we generalise results by Garcia-Maynez [14] and present equivalent characterisations of S-metrizability for metric frames. We discuss metric frame completions and show that every metric frame is a dense metric sublocale of its metric frame completion. It is shown that metric frames are perfect extensions of their uniformly locally connected dense metric sublocales. One of the main results in this chapter, Theorem 5.3.19, states that a connected locally connected metric frame is S-metrizable if and only if it has a perfect locally connected metrizable compactification. In addition, we discuss the Wallman compactification and show that every compact metric frame is a Wallman compactification of each of its dense sublocales. S-metrizability of a locally connected frame M is shown to be equivalent to M having a countable, locally connected and uniformly locally connected Wallman basis, in Proposition 5.4.19. This result is then used to obtain an equivalent characterisation of S-metrizability on a locally connected frame M, in terms of a Wallman basis on M.

Chapter 6. The concept of a locally non-separating remainder is due to Curtis [12]. In Chapter 6, we define a locally non-separating sublocale on a locally connected frame and deduce natural properties of it. Using the properties of locally non-separating remainders, Curtis determined the conditions under which a Peano compactification of a connected space X would exist. We provide a generalisation of Curtis's result under the assumption of M being a regular continuous and connected frame. We show in Theorem 6.3.9 that a non-compact, connected and regular continuous frame M is S-metrizable if and only if M has a Peano compactification $h : L \longrightarrow M$ with locally non-separating remainder $L \setminus h_*(M)$.

Chapter 2

Preliminaries

This chapter is purely introductory in nature, and we provide the preliminary material to the theory of frames and metric frames. Important terminology is introduced, notational convention will be set and basic known results are provided. Proofs for some of the lesser known results will be included. As a general reference to the theory of frames, and for proofs of well-known results that are not provided, we refer the reader to [8], [17] and [20].

2.1 An Introduction to the Theory of Frames

In this section, we provide an introduction to frame theory by stating essential definitions and required results.

2.1.1 Frames

Definition 2.1.1. Let L be a set equipped with a partial order $\leq$. If any two elements $x, y \in L$ have an infimum (a *meet*, written $x \wedge y$) then we call L a *meet-semilattice*, and if any $x, y \in L$ have a supremum (a *join*, written $x \vee y$) then L is a *join-semilattice*. L is a *lattice* if $x \wedge y$ and $x \vee y$ exist for all x, y in L . A lattice is said to be *complete* if every subset F of L has a supremum (written $\bigvee F$) and hence also an infimum (written $\bigwedge F$).

Remark 2.1.2. In a complete lattice, the empty join and the empty meet exist, so there is a bottom and top element, respectively.

Definition 2.1.3. A *frame* L is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{\, x \wedge s |\, s \in S\},$$

for all $x \in L, S \subseteq L$, where $\bigvee S$ denotes $\bigvee \{s \mid s \in S\}$.

Remark 2.1.4. Throughout this thesis we will denote the top element of a frame L by 1_L and the bottom element by 0_L. If no ambiguity is caused then we simply use 0 and 1.

Definition 2.1.5. Let L be a frame and suppose $M \subseteq L$ is closed under finite meets and arbitrary joins in L. Then M is a frame and M is called a subframe of L.

Example 2.1.6. Let X be a topological space, then $\mathcal{O}X = \{U \subseteq X |\ U \text{ is open}\}$ is a frame with partial order $\subseteq$ and intersection as a binary meet. $\mathcal{O}X$ is called the frame of open sets of a topological space.

Definition 2.1.7. Let L be a frame. Any subset J of L is called an *ideal* of L if

1. $0 \in J$,
2. whenever $x, y \in J$, then $x \vee y \in J$,
3. if $x \leq y$ for $y \in J$, then $x \in J$.

Proposition 2.1.8 ([8]). *Let L be a frame and let $\mathcal{I}L = \{J |\ J \text{ is an ideal of } L\}$, then $(\mathcal{I}L, \subseteq)$ is a complete lattice and $\mathcal{I}L$ is a frame.*

2.1.2 Homomorphisms and Adjoints

Definition 2.1.9. Let $h : L \longrightarrow M$ be a map, where L and M are frames. h is called a *frame homomorphism*, if h preserves all finite meets including the top element 1, and all arbitrary joins, including the bottom 0.

Definition 2.1.10. Let $h : L \longrightarrow M$ be a frame homomorphism.

1. h is *dense* if whenever $h(x) = 0_M$ then $x = 0_L$.

2. h is an *onto* frame homomorphism if for every $y \in L$ there is an $x \in M$ such that $h(x) = y$, and h is *one-to-one* if whenever $h(a) = h(b)$, then $a = b$ for $a, b \in L$.

3. h is a frame *isomorphism* if and only if h is onto, one-to-one.

The definition and results that follow are stated in the more general case of complete lattices in Pultr and Picado [20]. Since we work in a frame, we will restate the theory for frames.

Definition 2.1.11. Let $h : M \longrightarrow L$ be a frame homomorphism, then h has a *right adjoint* $h_* : L \longrightarrow M$ satisfying the property that for all $x \in M$ and for all $y \in L$,

$$x \leq h_*(y) \text{ iff } h(x) \leq y.$$

h is called the *left adjoint* of h_*, and h together with h_* are in a *Galois connection*, or are simply said to be *Galois adjoint*.

Fact 2.1.12 ([20]). *Let $h : L \longrightarrow M$ be a frame homomorphism with right adjoint $h_* : M \longrightarrow L$, then*

1. $h \cdot h_* \cdot h = h$ and $h_* \cdot h \cdot h_* = h_*$.

2. h is onto if and only if h_* is one-to-one.

Theorem 2.1.13 ([20]). *If L, M are frames then a frame homomorphism $h : L \longrightarrow M$ is a left adjoint (respectively, right adjoint) if and only if it preserves all suprema (respectively infima).*

The next two well-known results about right adjoints are part of the folklore in the theory.

Fact 2.1.14. *Let $h : L \longrightarrow M$ be an onto frame homomorphism with right adjoint $h_* : M \longrightarrow L$.*

1. For any $c \in M$, $h \cdot h_*(c) = c$.

2. If h is dense then $h_*(0_M) = 0_L$.

Lemma 2.1.15. *Let $h : L \longrightarrow M$ be a frame homomorphism with right adjoint $h_* : M \longrightarrow L$. Let $f : L \longrightarrow A$, and $g : A \longrightarrow M$ be frame homomorphisms such that $h = gf$. Then $h_* = f_* g_*$, where f_* and g_* denote the right adjoint of f and g, respectively.*

2.1.3 Frames and Spaces

Let X be a topological space. From Example 2.1.6, $\mathcal{O}X = \{U \subseteq X|\ U \text{ is open}\}$ is a frame. If $f : X \longrightarrow Y$ is a continuous map from the topological space X to a topological space Y, then we have a frame homomorphism,

$$\begin{aligned} \mathcal{O}(f) :\ \mathcal{O}(Y) &\longrightarrow \mathcal{O}(X), \\ U &\mapsto f^{-1}(U). \end{aligned}$$

Hence we have a contravariant functor, $\mathcal{O} : \mathbf{Top} \longrightarrow \mathbf{Frm}$, where **Top** denotes the category of topological spaces and continuous maps, and **Frm** denotes the category of frames and frame homomorphisms. Now, we also have the contravariant functor,

$$\begin{aligned} \Sigma : \mathbf{Frm} &\longrightarrow \mathbf{Top}, \\ L &\mapsto \Sigma L. \end{aligned}$$

ΣL, called the spectrum of L, is the space of all frame homomorphisms $\psi : L \longrightarrow \underline{2}$, where $\underline{2}$ denotes the two element frame $\{0, 1\}$, and ΣL has open sets
$\Sigma_a = \{\psi \in \Sigma L \mid \psi(a) = 1\}$, for $a \in L$. Thus, $\{\Sigma_a|\ a \in L\}$ is a topology on ΣL. For any frame homomorphism $h : L \longrightarrow M$, we have $\Sigma h : \Sigma M \longrightarrow \Sigma L$ which is defined by composing a frame homomorphism from ΣM with h, that is, $\Sigma h(\psi) = \psi \cdot h$, for $\psi \in \Sigma M$.

Proposition 2.1.16 ([8])**.** *Σ and $\mathcal{O}$ are adjoint on the right with adjunctions*

$$\eta_L : L \longrightarrow \mathcal{O}\Sigma L, \quad \textit{defined by}\ \ \eta_L(a) = \Sigma_a \ \ \textit{for}\ a \in L,$$

and

$$\varepsilon_X : X \longrightarrow \Sigma\mathcal{O}X, \quad \textit{defined by } \ \varepsilon_X(x) = \tilde{x} \ \textit{ for } x \in X,$$

where

$$\tilde{x} : \mathcal{O}X \longrightarrow \underline{2}, \quad \textit{defined by } \ \tilde{x}(U) = \begin{cases} 1 & , \textit{ if } x \in U \\ 0 & , \textit{ if } x \notin U. \end{cases}$$

Definition 2.1.17. A frame L is called *spatial*, if η_L is an isomorphism.

2.1.4 Pseudocomplements and the Heyting Operation

We now recall some theory about pseudocomplements and Heyting algebras from [20], which we re-state in the theory of frames for the purpose of this thesis.

Definition 2.1.18. The *pseudocomplement* of an element a from a frame L, is the largest element b such that $b \wedge a = 0$. The pseudocomplement of a is denoted a^* and is characterized by the following formula

$$a^* = \bigvee \{x \in L \mid a \wedge x = 0\}.$$

Lemma 2.1.19 ([20]). *In any frame L,*

(1) If $a \leq b$ then $b^* \leq a^*$.

(2) $a \leq a^{**}$.

(3) $a^{***} = a^*$.

(4) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

Proposition 2.1.20 ([20]). ***(First De Morgan law)*** *Let L be a frame and $a, b \in L$. The pseudocomplement of $a \vee b$ is given by*

$$(a \vee b)^* = a^* \wedge b^*.$$

Proposition 2.1.21 ([20]). *Let L be a frame, then*

$$\left(\bigvee_{i\in I} a_i\right)^* = \bigwedge_{i\in I} {a_i}^*.$$

Proposition 2.1.22 ([6]). *Let $h : L \longrightarrow M$ be a dense onto frame homomorphism with right adjoint h_*, then*

1. $h(x^*) = h(x)^*$, *for all $x \in L$.*

2. $h_*(a^*) = h_*(a)^*$, *for all $a \in M$.*

Definition 2.1.23. Let L be a frame, with binary operation $\rightarrow$ such that for all a, b, c in L,

$$c \leq a \rightarrow b \quad \text{if and only if} \quad c \wedge a \leq b.$$

The arrow, $\rightarrow$, is called a *Heyting operation* on the frame L.

Proposition 2.1.24 ([20]). *In any frame L,*

(H1) $1 \rightarrow a = a$ *for all* a,

(H2) $a \leq b$ *iff* $a \rightarrow b = 1$,

(H3) $a \leq b \rightarrow a$,

(H4) $(\bigvee_{i\in I} a_i) \rightarrow b = \bigwedge_{i\in I}(a_i \rightarrow b)$,

(H5) $a \rightarrow (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \rightarrow b_i)$,

(H6) $a \leq b$ *implies* $c \rightarrow a \leq c \rightarrow b$,

(H7) $a \wedge b = a \wedge c$ *iff* $a \rightarrow b = a \rightarrow c$,

(H8) $(a \wedge b) \rightarrow c = a \rightarrow (b \rightarrow c)$ *and* $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$,

(H9) $a = (a \vee b) \wedge (b \rightarrow a)$.

2.1.5 Regularity, Compactness and Local Connectedness

Definitions which are extensions of classical properties in topological spaces are now provided. Regularity, compactness, connectedness and local connectedness are the classical properties which we recall next.

Definition 2.1.25. For elements a, b in a frame L, we say that a is *rather below* b, written $a \prec b$, if there exists an element c in L such that $a \wedge c = 0$ and $b \vee c = 1$.

Remark 2.1.26. The condition provided in Definition 2.1.25, for determining if an element a is rather below an element b in a frame L, is equivalent to the condition that $a^* \vee b = 1$, where a^* is the pseudocomplement of $a \in L$.

Lemma 2.1.27 ([20]). *Let L be a frame, then the following hold:*

(1) If $a \prec b$ then $a \leq b$,

(2) $0 \prec a \prec 1$ for every a in L,

(3) If $x \leq a \prec b \leq y$ then $x \prec y$,

(4) If $a \prec b$ then $b^* \prec a^*$,

(5) If $a_i \prec b_i$, for $i = 1, 2$, then $a_1 \vee a_2 \prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec b_1 \wedge b_2$.

Definition 2.1.28. A frame L is said to be *regular* if

$$a = \bigvee \{x \in L \mid x \prec a\}, \text{ for every } a \text{ in } L.$$

Definition 2.1.29. Let L be a frame.

1. A subset $U \subseteq L$ is a *cover* of L, if $\bigvee U = 1$.

2. A cover U of L is a *refinement* of a cover T of L, written $U \leq T$, if for each x in U, there exists y in T such that $x \leq y$.

3. For a cover U of L and any $x \in L$, let $Ux = \bigvee \{a \in U \mid a \wedge x \neq 0\}$.

Definition 2.1.30. A frame L is *compact* if whenever $\bigvee S = 1$ (that is, whenever S is a cover of L), then there exists a finite subset F of S such that $\bigvee F = 1$.

Proposition 2.1.31 ([8]). *Let L be a compact regular frame, then for any $a, b \in L$, $a \prec b$ implies that there exists $c \in L$ such that $a \prec c \prec b$. We say that $\prec$ interpolates in a compact regular frame.*

Definition 2.1.32. Let L be a frame.

1. Any $a \in L$ is said to be *connected*, if whenever $a = b \vee c$ and $b \wedge c = 0$ then we either have $a = 0$ or $b = 0$.

2. L is said to be connected if its top element 1 is connected.

Proposition 2.1.33 ([1]). *Let $h : L \longrightarrow M$ be a dense onto frame homomorphism such that the right adjoint h_* of h preserves disjoint binary joins, then $h(c)$ is connected in M for any connected $c \in L$.*

The next result on connectedness of elements in a frame is well known and part of the folklore.

Lemma 2.1.34. *Let L be a frame and let x be any element in L.*

1. *If $x = \bigvee_{i \in I} c_i$, where c_i is connected in L for each $i \in I$, and $\bigwedge_{i \in I} c_i \neq 0$, then x is connected in L.*

2. *If $x = c_1 \vee c_2 \vee ... \vee c_n$, where c_i is connected for each i, and $c_i \wedge c_{i+1} \neq 0$ for $i = 1, ..., n-1$, then x is connected in L.*

3. *If x is connected then x^{**} is connected in L.*

Definition 2.1.35. Let L be a frame and $B \subseteq L$. B is called a *base* of L if every $x \in L$ can be written as a join of elements from B. In the literature, B is also referred to as a *basis* of L. We will use the terms *base* and *basis* interchangeably.

Definition 2.1.36. A frame L is said to be *locally connected* if there exists a base B of L consisting of connected elements. That is, for every $a \in L$, a can be written as a join of connected elements from L.

2.2 Sublocales

We now consider the notion of generalised subspaces. In this section, the theory of *sublocales* of a frame L, as introduced in [20], is recalled and discussed. Amongst other important concepts, we will recall the theory of open, closed and connected sublocales, and closures of sublocales of a frame. A proof of a new result on the frame distributivity law for sublocales is given in Proposition 2.2.17, and we provide a proof for the generalisation of a result by Chen [11].

Definition 2.2.1. Let S be a subset of a frame L. S is called a *sublocale* if :

1. S is closed under arbitrary meets.
2. For $x \in L$ and $s \in S$, $(x \to s) \in S$.

Remark 2.2.2.

1. A sublocale S is always non-empty, since $1 = \bigwedge \emptyset \in S$. The meet in S is exactly the meet in L, however, in general, the join in S differs from the join in L. We will, therefore, denote the join in a sublocale S, by $\vee_S$, to avoid confusion.
2. We denote the bottom element of S by 0_S, where $0_S = \bigwedge S$.
3. An arbitrary intersection (or meet) of sublocales, is a sublocale.
4. The collection of all sublocales of a frame L, denoted by $\mathcal{S}(L)$, is a complete lattice $(\mathcal{S}(L), \subseteq)$.
5. Let $S_i \in \mathcal{S}(L)$ for $i \in I$. The join of sublocales is defined as follows:

$$\bigvee_{i \in I} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i \in I} S_i\}.$$

   We note that the join of sublocales is indeed a sublocale.

Definition 2.2.3. A *nucleus* on a frame L is a mapping $\nu : L \longrightarrow L$ such that

(N1) $a \leq \nu(a)$ for any $a \in L$,

(N2) $a \leq b \Rightarrow \nu(a) \leq \nu(b)$ for $a, b \in L$,

(N3) $\nu(\nu(a)) = \nu(a)$ for any $a \in L$, (N4) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$ for $a, b \in L$.

Proposition 2.2.4 ([20]). *Let L be a frame. For a sublocale S of L, set*

$$\nu_S(a) = \bigwedge \{s \in S \mid a \leq s\},$$

and for a nucleus $\nu : L \longrightarrow L$, set

$$S_\nu = \nu[L].$$

Then ν_S is a nucleus on the frame L, and the formulas $S \mapsto \nu_S$ and $\nu \mapsto S_\nu$ constitute a one-to-one correspondence between sublocales of L and nuclei of L.

Remark 2.2.5.

1. For a sublocale S of a frame L, the join in S, denoted by $\bigvee_S$, is given by

$$\bigvee_S \{a_i \in S \mid i \in I\} = \nu_S \left(\bigvee_{i \in I} a_i \right).$$

2. We observe that every sublocale S of a frame L, is indeed a frame with meet as in L and join as defined above with bottom element 0_S and top element 1.

Definition 2.2.6. (**Co-frame Property**) A *co-frame* L' is a complete lattice satisfying:

$$x \vee \left(\bigwedge A \right) = \bigwedge \{x \vee a \mid a \in A\},$$

for all $x \in L', A \subseteq L'$.

Theorem 2.2.7 ([20]). *Let L be a frame. $\mathcal{S}(L)$ is a co-frame.*

2.2.1 Open and closed sublocales

In what follows, we examine *open* and *closed* sublocales and recall important consequences of them. For detailed proofs, we refer the reader to Pultr and Picado [20].

Definition 2.2.8. Let L be a frame.

1. The *open sublocale* associated with any $a \in L$ is : $\mathfrak{o}(a) = \{x \in L \mid a \to x = x\}$.

2. The *closed sublocale* associated with any $a \in L$ is : $\mathfrak{c}(a) = \uparrow a = \{x \in L \mid a \leq x\}$.

Remark 2.2.9. Let L be a frame.

1. An alternative formula for the open sublocale of $a \in L$, is $\mathfrak{o}(a) = \{a \to x \mid x \in L\}$.

2. For any $a \in L$, $\mathfrak{o}(a) \cong \downarrow a$, where $\downarrow a = \{x \in L \mid x \leq a\}$.

Fact 2.2.10 ([20])**.** *$\mathfrak{o}(a)$ is a sublocale of L, for $a \in L$.*

Proof.

Take $M \subseteq \mathfrak{o}(a)$. For each $m \in M$, we have that $a \to m = m$. Thus by Proposition 2.1.24, $\bigwedge M = \bigwedge_{m \in M}(a \to m) = a \to \bigwedge M$. Hence $\bigwedge M \in \mathfrak{o}(a)$. For $x \in L$ and $y \in \mathfrak{o}(a)$, we show that $x \to y \in \mathfrak{o}(a)$. Now using *(H8)* of Proposition 2.1.24 and the fact that $a \to y = y$, we obtain that $a \to (x \to y) = x \to (a \to y) = x \to y$. Hence $x \to y \in \mathfrak{o}(a)$.

$\square$

Remark 2.2.11. One can easily establish that if $\mathfrak{o}(x) = \mathfrak{o}(y)$, then $x = y$.

Fact 2.2.12 ([20])**.** *$\uparrow a$ is a sublocale of L, for $a \in L$.*

Proof.

For $M \subseteq \uparrow a$, we have that $a \leq m$ for each $m \in M$. Thus $a \leq \bigwedge M$ and $\bigwedge M \in \uparrow a$. It only remains for us to check that for any $x \in L$ and $y \in \uparrow a$, $x \to y \in \uparrow a$. Now $a \leq y$, therefore $x \to a \leq x \to y$. But by Proposition 2.1.24, $a \leq x \to a$, so we conclude that $a \leq x \to y$. Hence $x \to y \in \uparrow a$.

$\square$

Proposition 2.2.13 ([20])**.** *Let L be a compact frame and S be a closed sublocale of L. Then S is compact.*

Proposition 2.2.14 ([20])**.** *Let L be a frame and $a \in L$. $\mathfrak{o}(a)$ and $\uparrow a$ are complements of each other in $\mathcal{S}(L)$ (that is, $\mathfrak{o}(a) \bigvee \uparrow a = L$ and $\mathfrak{o}(a) \bigcap \uparrow a = \{1\}$).*

Proposition 2.2.15 ([20]). *Let L be a frame.*

1. $\uparrow (\bigvee_{i\in I} a_i) = \bigcap_{i\in I} \uparrow a_i$, *for* $a_i \in L,\ i \in I$.

2. $\uparrow a \vee \uparrow b = \uparrow (a \wedge b)$, *for* $a, b \in L$.

3. $\uparrow 0 = L$ *and* $\uparrow 1 = \{1\}$.

4. $\mathfrak{o}(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} \mathfrak{o}(a_i)$, *for* $a_i \in L,\ i \in I$.

5. $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b)$, *for* $a, b \in L$.

6. $\mathfrak{o}(0) = \{1\}$ *and* $\mathfrak{o}(1) = L$.

Recall that Theorem 2.2.7 states that the collection of all sublocales of a frame L, $\mathcal{S}(L)$, is a co-frame. However, under special conditions for open and closed sublocales we also have that the frame condition is satisfied. This is illustrated in the two results that follow, the first of which is well known and part of the folklore, and the second of which we state and prove.

Proposition 2.2.16. *Let L be a frame, $a \in L$ and S_i be sublocales of L, where $i \in I$.*

1. $\mathfrak{o}(a) \cap \bigvee_{i\in I} S_i = \bigvee_{i\in I}(\mathfrak{o}(a) \cap S_i)$.

2. $\uparrow a\ \cap \bigvee_{i\in I} S_i = \bigvee_{i\in I}(\uparrow a\ \cap S_i)$.

Proposition 2.2.17. *Let L be a frame and S be a sublocale of L. For $a_i \in L$ where $i \in I$, the following holds:*

$$S \cap \bigvee_{i\in I} \mathfrak{o}(a_i) = \bigvee_{i\in I} S \cap \mathfrak{o}(a_i).$$

Proof.

$$\begin{aligned}
S \cap \bigvee_{i \in I} \mathfrak{o}(a_i) &= S \cap \mathfrak{o}(\bigvee_{i \in I} a_i) \quad \text{(by Proposition 2.2.15)} \\
&= \{x \in S \mid x = \left(\bigvee_{i \in I} a_i\right) \to x\} \\
&= \{x \in S \mid x = \bigwedge_{i \in I}(a_i \to x)\} \quad \text{(by Proposition 2.1.24)} \\
&= \bigvee_{i \in I} S \cap \mathfrak{o}(a_i).
\end{aligned}$$

The final equality holds, since $a_i \to x \in S \cap \mathfrak{o}(a_i)$ for each $i \in I$, which implies that $\bigwedge_{i \in I}(a_i \to x) \in \bigvee_{i \in I} S \cap \mathfrak{o}(a_i)$.

□

In [11], Chen proves the following :

If two families of congruences $\{\theta_i | i \in I\}$ and $\{\Delta_{a_i} | i \in I\}$ satisfy the conditions:

(1) $\bigwedge\{\theta_i | i \in I\} = \bigwedge\{\Delta_{a_i} | i \in I\}$,

(2) $\theta_i \vee \theta_j =$ the top element in the congruence frame, where $(i \neq j)$,

(3) $\theta_i \leq \Delta_{a_i}$, for all $i \in I$,

then $\theta_i = \Delta_{a_i}$, for all $i \in I$.

It is well known that for any frame L, the congruences of L are in one-to-one correspondence with the sublocales of L. The next Proposition is a reformulation of Chen's result stated in the context of sublocales.

Proposition 2.2.18. *Let L be a frame. Let $\{S_i \mid i \in I\}$ be a family of sublocales of L and $\{\mathfrak{o}(a_i) \mid a_i \in L, i \in I\}$ be a family of open sublocales of L such that, if*

(1) $\bigvee\{S_i \mid i \in I\} = \bigvee\{\mathfrak{o}(a_i) \mid a_i \in L, i \in I\}$,

(2) $S_i \cap S_j = \{1\}$ for $i, j \in I$ and $i \neq j$,

(3) $\mathfrak{o}(a_i) \subseteq S_i$ for all $i \in I$,

then $S_i = \mathfrak{o}(a_i)$ for each $i \in I$.

Proof.

For any $j \in I$,

$$\begin{aligned}
S_j &= S_j \cap \bigvee_{i \in I} S_i \\
&= S_j \cap \bigvee_{i \in I} \mathfrak{o}(a_i) \quad \text{(by (1) of the hypothesis)} \\
&= S_j \cap \ \mathfrak{o}(\bigvee_{i \in I} a_i) \quad \text{(by Proposition 2.2.15)} \\
&= \bigvee_{i \in I} (S_j \cap \mathfrak{o}(a_i)) \quad \text{(by Proposition 2.2.17)} \\
&= \mathfrak{o}(a_j) \cap S_j
\end{aligned}$$

Therefore we have shown that $S_j \subseteq \mathfrak{o}(a_j)$ for all $j \in I$. By (3) of the hypothesis, $S_i = \mathfrak{o}(a_i)$ for all $i \in I$.

□

2.2.2 Closure of sublocales

In spaces, the closure of $A \subseteq X$, where X is a space, is the least closed set that contains the subset A. In frames, if S is a sublocale, we observe that any sublocale T containing S must contain $\bigwedge S$, and we always have that $S \subseteq \uparrow (\bigwedge S)$. Thus this motivates the following meaningful definition, as found in [20].

Definition 2.2.19. Let S be a sublocale of a frame L. The *closure* of S in L, denoted $\overline{S}$, is defined as

$$\overline{S} = \uparrow \left(\bigwedge S \right).$$

Remark 2.2.20.

1. It can be easily verified that $\overline{S}$ is a sublocale and is in fact the least closed sublocale containing S.

2. Let $S \subseteq T \subseteq L$, where L is a frame and S and T are sublocales of L. Then regarding

T as a frame and S as a sublocale of T, we shall denote the closure of S in T by $\mathbf{Cl}_T(S)$, where $\mathbf{Cl}_T(S) = \uparrow^T (\bigwedge S) = \{x \in T \mid \bigwedge S \leq x\}$.

Definition 2.2.21. Let S be a sublocale of a frame L. S is called a *dense* sublocale of L, if $\overline{S} = L$.

Proposition 2.2.22. *Let S be a sublocale of a frame L, and $a \in L$. If $S \subseteq \uparrow a$, then $\overline{S} \subseteq \uparrow a$.*

Proof.
$\overline{S} = \uparrow (\bigwedge S)$. If $s \in \uparrow a$, then $a \leq s$ for all $s \in S$ and hence $a \leq \bigwedge S$. It follows that $\uparrow (\bigwedge S) \subseteq \uparrow a$, and so $\overline{S} \subseteq \uparrow a$, as required.

□

Proposition 2.2.23 ([20])**.** *Let S and T be sublocales of L, then $\overline{S \vee T} = \overline{S} \vee \overline{T}$.*

Proposition 2.2.24 ([20])**.** *Let L be a frame and $a \in L$. Then $\overline{\mathfrak{o}(a)} = \uparrow a^*$.*

2.2.3 Connected sublocales

We recall that an element a in a frame L is connected, if whenever $a = b \vee c$ and $b \wedge c = 0$, then either $b = 0$ or $c = 0$, and L is connected if the top element $1 \in L$ is connected in L. Thus if S is a sublocale of L, then regarding S as a frame, we obtain the following definition.

Definition 2.2.25. A sublocale S of L is connected, if the top element $1 \in S$ is connected in S. That is, S is connected if and only if whenever $1 = a \vee_S b$, for $a, b \in S$, and $a \wedge b = 0_S$ then either $a = 0_S$ or $b = 0_S$.

Proposition 2.2.26. *Let L be a frame. A sublocale S of L is connected if and only if whenever $S \subseteq \uparrow a \vee \uparrow b$ and $\uparrow a \cap \uparrow b \ \cap S = \{1\}$ then either $\uparrow a \cap S = \{1\}$ or $\uparrow b \cap S = \{1\}$.*

Proposition 2.2.27. *A sublocale S of a frame L is connected if and only if whenever $S \subseteq \mathfrak{o}(a) \vee \mathfrak{o}(b)$ and $S \cap \mathfrak{o}(a) \cap \mathfrak{o}(b) = \{1\}$, then either $S \cap \mathfrak{o}(a) = \{1\}$ or $S \cap \mathfrak{o}(b) = \{1\}$.*

Remark 2.2.28. It is clear that determining the connectedness of a sublocale can be obtained using different criteria, all of which are equivalent. Proposition 2.2.26 which invokes closed sublocales in its criteria, Proposition 2.2.27 which invokes open sublocales in its criteria or the first principles definition which can be used interchangeably to one's convenience.

Proposition 2.2.29 ([20]). *Let S be a sublocale of a frame L. If S is connected, then $\overline{S}$ is connected.*

Proposition 2.2.30. *If S is a connected sublocale of L and T is a sublocale such that $S \subseteq T \subseteq \overline{S}$, then T is connected.*

Proof.

We will show that if $S \subseteq T$, then $\mathbf{Cl}_T(S) = \overline{S} \cap T$. To see this let $\bigwedge S = 0_S$. Then $\overline{S} = \uparrow 0_S$, so $\overline{S} \cap T = \{x \in T \mid 0_S \leq x\} = \uparrow^T 0_S = \mathbf{Cl}_T(S)$. Thus if $T \subseteq \overline{S}$, then we have that $T = \mathbf{Cl}_T(S)$, which is connected by Proposition 2.2.29.

□

Proposition 2.2.31 ([20]). *Let S_i, $i \in J$, be a system of connected sublocales of a frame L. Suppose that for any two $i, j \in J$ there exists $i_1, ..., i_n \in J$ such that $i_1 = i$, $i_n = j$ and for $k = 1, 2, ..., n-1$, $S_{i_k} \cap S_{i_{k+1}} \neq \{1\}$. Then $\bigvee_{i \in J} S_i$ is connected.*

Proposition 2.2.32 ([20]). *Let S_i be connected sublocales of L, for $i \in I$. If $\bigcap_{i \in I} S_i \neq \{1\}$, then $S = \bigvee_{i \in I} S_i$ is connected.*

The next result is part of the folklore for the theory of sublocales. Although well-known, a detailed proof in the literature is not found, and we therefore provide one.

Theorem 2.2.33. *Let L be a frame. $a \in L$ is connected if and only if $\mathfrak{o}(a)$ is connected as a sublocale.*

Proof.

($\Rightarrow$) Suppose a is connected in L, and $\mathfrak{o}(a) \subseteq \uparrow b \vee \uparrow c$ with $\uparrow b \cap \uparrow c \ \cap \mathfrak{o}(a) = \{1\}$. Then $\uparrow b \cap \uparrow c \ \subseteq \uparrow a$ and this is equivalent to $\uparrow (b \vee c) \subseteq \uparrow a$. Thus $a \leq b \vee c$, and it follows that $a = (a \wedge b) \vee (a \wedge c)$. Now $L = \uparrow a \vee \mathfrak{o}(a)$ (by Proposition 2.2.14), and by our assumption we have that $\mathfrak{o}(a) \subseteq \uparrow b \vee \uparrow c$, therefore

$L = \uparrow a \vee \mathfrak{o}(a) \subseteq \uparrow a \vee \uparrow b \vee \uparrow c$. Thus $L = \uparrow a \vee \uparrow b \vee \uparrow c = \uparrow (a \wedge b \wedge c)$, by Proposition 2.2.15. Now $0 \in L$, therefore $a \wedge b \wedge c = 0$. Hence we have a connected in L, with $a = (a \wedge b) \vee (a \wedge c)$ and $(a \wedge b) \wedge (a \wedge c) = a \wedge b \wedge c = 0$. Thus $(a \wedge b) = 0$, say, and then $a = a \wedge c \leq c$, so $\uparrow c \subseteq \uparrow a$. Now $\mathfrak{o}(a) \subseteq \uparrow b \vee \uparrow c \subseteq \uparrow b \vee \uparrow a$, so $\mathfrak{o}(a) \subseteq \uparrow b$, since $\mathfrak{o}(a) \wedge \uparrow a = \{1\}$. Hence $\uparrow b \cap \uparrow c \cap \mathfrak{o}(a) = \{1\}$ and this implies that $\uparrow c \cap \mathfrak{o}(a) = \{1\}$. Thus $\mathfrak{o}(a)$ is connected as a sublocale.

($\Leftarrow$) Now suppose $\mathfrak{o}(a)$ is connected as a sublocale. Assume $a = b \vee c$ and $b \wedge c = 0$. Now $\uparrow a = \uparrow (b \vee c) = \uparrow b \cap \uparrow c$, and $L = \uparrow 0 = \uparrow (b \wedge c) = \uparrow b \vee \uparrow c$. Now $\mathfrak{o}(a) \subseteq \uparrow b \vee \uparrow c$ and $\uparrow b \cap \uparrow c \cap \mathfrak{o}(a) = \uparrow a \cap \mathfrak{o}(a) = \{1\}$. By the connectedness of $\mathfrak{o}(a)$, $\uparrow b \cap \mathfrak{o}(a) = \{1\}$, say. Now $\mathfrak{o}(a) \subseteq \uparrow c$ and we also have that $\uparrow a \subseteq \uparrow c$, so $\uparrow a \vee \mathfrak{o}(a) \subseteq \uparrow c$. That is, we have shown that $L = \uparrow c$, which implies $c = 0$. Hence a is connected in L.

□

2.3 Metric frames

Metrizability was first defined by Isbell [15] and the theory was later developed by Pultr [25]. We now recall the definition of a diameter function which mimics the properties of a diameter in a classical metric space. This defines the required metric structure in the point-free context.

Definition 2.3.1 ([25]). A *diameter* on a frame L is a map $d : L \longrightarrow \mathbb{R}^+$ (the non-negative reals including ∞) such that

(M1) $d(0) = 0$.

(M2) if $a \leq b$ then $d(a) \leq d(b)$.

(M3) if $a \wedge b \neq 0$ then $d(a \vee b) \leq d(a) + d(b)$.

(M4) For each $\varepsilon > 0$, $U^d_\varepsilon = \{u \in L | \ d(u) < \varepsilon\}$ is a cover.

A diameter d is called *compatible* if

(M5) For each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd_d a\}$, where $x \lhd_d a$ means there exists U_ε^d such that $U_\varepsilon^d x = \bigvee\{u \in U_\varepsilon^d \mid u \wedge x \neq 0\} \leq a$.

A diameter d is called a *metric diameter* if

(M6) For each $a \in L$ and $\varepsilon > 0$ there exists $u, v \leq a$, $d(u), d(v) < \varepsilon$ such that $d(a) - \varepsilon < d(u \vee v)$.

Remark 2.3.2. A frame L with a specified compatible metric diameter d is called a *metric frame*. This is denoted by (L, d) and will be adopted as standard notation throughout this thesis.

Note that (M6) is equivalent with the following condition, which is sometimes handier:

(M6)$'$ For every a in L and $\varepsilon > 0$ there are u, v with $u \wedge a \neq 0 \neq v \wedge a$ such that $d(u), d(v) < \varepsilon$, and $d(a) < d(u \vee v) + \varepsilon$.

Definition 2.3.3. A frame L is said to be *metrizable* if there exists a compatible diameter, d on L, such that (L, d) is a metric frame.

Definition 2.3.4. A metric diameter d on a frame L is called *bounded* if there exists $r > 0$ such that for all a in L, $d(a) < r$.

Theorem 2.3.5 ([23]). *If d is a diameter on a frame L, then for $a \in L$, $\tilde{d}$ defined by*

$$\tilde{d}(a) = \inf_{\varepsilon > 0} \sup\{d(u \vee v) |\ u, v \leq a, d(u), d(v) < \varepsilon\}$$

is a metric diameter with the property that $\tilde{d} \leq d$.

Proposition 2.3.6 ([2]). *Let (L, d) be a locally connected metric frame. Then for $a \in L$,*

$$\tilde{d}(a) = \inf_{\varepsilon > 0} \sup\{d(u \vee v) |\ u, v \leq a, u, v \text{ connected and } d(u), d(v) < \varepsilon\}.$$

Lemma 2.3.7 ([2]). *Let d be a metric diameter on a locally connected frame L. Then for all $a \in L$, and for all $\varepsilon > 0$ there exists $u, v \leq a$, $d(u), d(v) < \varepsilon$ and u, v connected such that $d(a) - \varepsilon < d(u \vee v)$.*

Proposition 2.3.8 ([9]). *Let (L, d) be a metric frame, then $d(x) = d(x^{**})$, for all $x \in L$.*

We now state the definition of a uniform frame, and briefly discuss the relationship between uniform frames and metric frames.

Recall that if U and V are covers of L, then $U \leq V$ means that for each $x \in U$, there exists $y \in V$ such that $x \leq y$.

Definition 2.3.9. A *uniformity* on a frame L is a system of covers $\mathcal{U}$ of L satisfying:

1. If $U \in \mathcal{U}$ and $U \leq V$, then $V \in \mathcal{U}$.

2. If $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \wedge V \in \mathcal{U}$, where $U \wedge V = \{a \wedge b \mid a \in U, b \in V\}$.

3. For all $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $\{Vx \mid x \in V\}$ is a refinement of U.

4. For all $a \in L$, $a = \bigvee\{x \in L \mid Ux \leq a, \text{ for some } U \in \mathcal{U}\}$.

A frame L together with a specified uniformity, denoted $\mathcal{U}L$, is called a *uniform frame* and is denoted by the pair $(L, \mathcal{U}L)$. The members of the uniformity $\mathcal{U}L$ are called uniform covers of L.

For any metric frame (L, d), the covers U_ε^d define a uniformity which is countably generated $\big($by taking $\varepsilon = \frac{1}{n}\big)$ on L. In addition, Pultr showed in [24], that a uniform frame $(L, \mathcal{U}L)$ has a metric diameter d such that its given uniformity $\mathcal{U}L$ is the same as the uniformity induced by d. Hence every metric frame is a uniform frame.

Chapter 3

Property S

In this chapter we will investigate well known properties that are inherent in spaces (and their subspaces), within the point-free context. We begin by examining the relationships between Property S, uniform local connectedness and local connectedness in a metric frame. The equivalence of these three properties is later shown under the assumption of compactness. In order to establish these properties on sublocales of metric frames, we first introduce a metric structure on a sublocale as defined in [20]. The main result of this chapter provides equivalent criteria for Property S on a metric frame, formulated in the theory of sublocales.

Certain definitions from frame theory will be recalled here, for the reader's convenience.

3.1 On Property S, uniform local connectedness and local connectedness.

We now show that each of Property S and uniform local connectedness imply local connectedness. These results have previously been investigated by Baboolal [5], in the context of uniform frames. Hence these results are expected and we provide a direct proof within metric frames for the sake of completeness.

Property S is a concept due to Sierpinski ([26]), that was originally defined in metric spaces. We now state the corresponding, well known, definition of Property S for a metric frame.

Definition 3.1.1. Let (L, d) be a metric frame. L is said to have *Property S* if, given any $\varepsilon > 0$, there exists $a_1, a_2, ..., a_n$ such that $\bigvee_{i=1}^{n} a_i = 1$, where a_i is connected and $d(a_i) < \varepsilon$ for each i.

For the purpose of the next result, we recall that a frame L is locally connected if each element in L can be written as a join of connected elements from L.

Proposition 3.1.2. *If (L, d) has Property S, then L is locally connected.*

Proof.

Let (L, d) be a metric frame with Property S and let $a \in L$ be arbitrary. We will show that a is a join of connected elements from L. (L, d) is a metric frame and therefore d must be a compatible diameter. Thus $a = \bigvee\{b \in L |\ b \lhd_d\ a\}$ and so we find that for each b with $b \lhd_d a$, there exists $\varepsilon > 0$ such that $U_\varepsilon^d b \leq a$. Since (L, d) has Property S, there exists connected $y_1, ..., y_n$ such that $\bigvee_{i=1}^{n} y_i = 1$ and $d(y_i) < \varepsilon$ for each $i = 1, 2, ..., n$. Now,

$$\begin{aligned} b = b \wedge 1 = b \wedge \left(\bigvee_{i=1}^{n} y_i \right) \\ = \bigvee_{i=1}^{n} (b \wedge y_i) \\ = \bigvee_{i=1}^{n} \{b \wedge y_i \mid b \wedge y_i \neq 0\}. \end{aligned}$$

Let $c_b = \bigvee\{y_i \mid y_i \wedge b \neq 0\}$, then c_b is connected and $b \leq c_b \leq a$. For each b such that $b \lhd_d a$, we have obtained a c_b, and thus we have that

$$a = \bigvee\{b \in L |\ b \lhd_d\ a\} \leq \bigvee_{b \lhd_d a} c_b \leq a.$$

Hence $a = \bigvee_{b \lhd_d a} c_b = \bigvee_{b \lhd_d a} \bigvee\{y_i \mid y_i \wedge b \neq 0\}$, and as required we have shown that a is a join of connected elements from L.

□

Definition 3.1.3. (L, d) is said to be *uniformly locally connected* (abbreviated ulc) if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(a) < \delta$ then there exists a connected c, $a \leq c$ and $d(c) < \varepsilon$.

Proposition 3.1.4. *If (L, d) is uniformly locally connected then L is locally connected.*

Proof.

Suppose that (L, d) is uniformly locally connected and let $a \in L$ be arbitrary. Since d is a compatible diameter, $a = \bigvee\{b \in L \mid b \lhd_d a\}$ and therefore for each b where $(b \lhd_d a)$ there exists U_ε^d such that $U_\varepsilon^d b \leq a$. Since (L, d) is uniformly locally connected, there exists $\delta > 0$ such that if $d(w) < \delta$ then there exists a connected $c \in L$ with $w \leq c$ and $d(c) < \varepsilon$. Now $U_\delta^d = \{x \in L \mid d(x) < \delta\}$ is a cover of L, thus

$$\begin{aligned} b = b \wedge 1 = b \wedge \bigvee\{x \in L \mid d(x) < \delta\} \\ = \bigvee\{b \wedge x \mid x \in U_\delta^d\} \\ = \bigvee\{b \wedge x \mid x \in U_\delta^d, b \wedge x \neq 0\} \end{aligned}$$

By the uniform local connectedness of (L, d), for each x such that $x \in U_\delta^d$ and $b \wedge x \neq 0$, we have that there exists a connected c_x, $x \leq c_x$ and $d(c_x) < \varepsilon$. Now $b \wedge c_x \neq 0$ and $c_x \in U_\delta^d$, therefore $c_x \leq a$. Furthermore,

$$b = \bigvee\{b \wedge x \mid x \in U_\delta^d, b \wedge x \neq 0\} \leq \bigvee\{x \in L \mid x \in U_\delta^d, b \wedge x \neq 0\} \leq \bigvee\{c_x \mid x \in U_\delta^d, b \wedge x \neq 0\}.$$

Thus $a = \bigvee\{b \in L \mid b \lhd_d a\} \leq \bigvee_{b \lhd_d a} \bigvee\{c_x \mid x \in U_\delta^d, b \wedge x \neq 0\} \leq a$, and therefore a is a join of connected elements from L. Hence L is locally connected, as required.

□

Definition 3.1.5. (L, d) is *totally bounded* if, given $\varepsilon > 0$, there exists a finite cover $\{a_i\}_{i=1}^n$ of L such that $d(a_i) < \varepsilon$ for all i.

Proposition 3.1.6. *Every totally bounded, uniformly locally connected metric frame (L, d) has Property S.*

Proof.

Let $\varepsilon > 0$ be given. Since (L, d) is uniformly locally connected, there exists $\delta > 0$ with the uniformly locally connected property. Now (L, d) is totally bounded, therefore there exists $\{a_1, a_2, ..., a_n\} \subseteq L$ such that $\bigvee_{i=1}^{n} a_i = 1$ and $d(a_i) < \delta$ for $i = 1, 2, ..., n$. By the uniform local connectedness of (L, d), since $d(a_i) < \delta$ for $i = 1, 2, ..., n$, there exists connected $c_i \in L$ such that $a_i \leq c_i$ and $d(c_i) < \varepsilon$ for $i = 1, 2, ..., n$. Now, $\bigvee_{i=1}^{n} a_i \leq \bigvee_{i=1}^{n} c_i$. Hence $\bigvee_{i=1}^{n} c_i = 1$ and it follows that (L, d) has Property S.

□

We now establish a result with the aim to show the equivalence of Property S, uniform local connectedness and local connectedness, in a compact metric frame.
Recall that a frame L is compact if each cover of L has a finite subcover.

Lemma 3.1.7. *If (L, d) is a compact metric frame and $U \subseteq L$ is a cover of L, then there exists $\delta > 0$ such that if $a \in L$ has $d(a) < \delta$ then there exists $u \in U$ such that $a \leq u$.*

Proof.

Suppose that U is a cover of L. d is a compatible diameter, therefore for each $u \in U$, $u = \bigvee\{x \in L |\ x \lhd_d u\}$. Since U is a cover of L,

$$\bigvee\{x \in L |\ x \lhd_d u \text{ for } u \in U\} = 1.$$

By the compactness of L, there exists $x_1, x_2, ..., x_n$, where $x_i \lhd_d u_i$ for some $u_i \in U$ for $i = 1, 2, ..., n$, such that $x_1 \vee x_2 \vee ... \vee x_n = 1$. For $i = 1, 2, ..., n$, there exists $\varepsilon_i > 0$ such that $U^d_{\varepsilon_i} x_i \leq u_i$. Let $\delta = \min\{\varepsilon_i\}_{i=1}^{n}$ and suppose $a \in L$ and $d(a) < \delta$. Then
$a = a \wedge 1 = \bigvee_{i=1}^{n}(a \wedge x_i)$, and $a \wedge x_i \neq 0$ for some i. Now, $d(a) < \delta \leq \varepsilon_i$, so $a \in U^d_{\varepsilon_i}$. Since $a \leq U^d_{\varepsilon_i} x_i \leq u_i$, it follows that $a \leq u_i$, as required.

□

Using Lemma 3.1.7, it is now possible to prove the following result:

Theorem 3.1.8. *Let (L, d) be a compact metric frame. The following are equivalent:*

1. *L is locally connected.*

2. *(L, d) has Property S.*

3. *(L, d) is uniformly locally connected.*

Proof.

$(3) \Rightarrow (2)$: Let $\varepsilon > 0$ and suppose that (L, d) is uniformly locally connected. We will first show that (L, d) is totally bounded. U_ε^d is a cover of L, hence by compactness of L, there exists $x_1, x_2, ..., x_n$ in U_ε^d such that $\bigvee_{i=1}^n x_i = 1$. Since $d(x_i) < \varepsilon$ for $i = 1, 2, ..., n$, we have that (L, d) is totally bounded. Hence by Proposition 3.1.6, (L, d) has Property S.

$(2) \Rightarrow (1)$: Follows from Proposition 3.1.2.

$(1) \Rightarrow (3)$: Suppose L is locally connected and take any $\varepsilon > 0$. Now U_ε^d is a cover of L, therefore $\bigvee U_\varepsilon^d = 1$. By local connectedness of L, each $y \in U_\varepsilon^d$ is a join of connected elements from L. Thus we have that $\bigvee\{x \in L \mid x \text{ is connected and } d(x) < \varepsilon\ \} = 1$. By lemma 3.1.7, there exists $\delta > 0$ such that if $d(a) < \delta$, then $a \leq x$ for some connected $x \in L$ with $d(x) < \varepsilon$. Hence (L, d) is uniformly locally connected.

$\square$

3.2 Diameters on sublocales

In the preliminaries, we had surveyed a considerable amount of theory on sublocales, which has prepared us to discuss the induced metric structure on a sublocale S, of a metric frame L. The metric diameter on a sublocale was first introduced by Pultr [22].

We begin by recalling the following:

1. For any sublocale S of a frame L, there exists a nucleus $\nu_S : L \longrightarrow L$, defined by $\nu_S(a) = \bigwedge\{s \in S \mid a \leq s\}$, for $a \in L$.

2. Every sublocale S of a frame L, is a frame itself. The meet in S is exactly the meet in L and the join in S, denoted $\bigvee_S$, is given by

$$\bigvee_S \ _{\{i \in I\}} a_i = \nu_S(\bigvee_{i \in I} a_i).$$

The bottom element of S, denoted 0_S, is $\bigwedge S$ and the top element of S is the top element of L, written as 1.

Theorem 3.2.1 ([22]). *Let (L, d) be a metric frame and S be a sublocale of L. Define $d_S : S \to \mathbb{R}^+$, by*

$$d_S(b) = \inf\{d(a) \mid a \in L,\ b \leq \nu_S(a)\},$$

for $b \in S$, where ν_S is the nucleus obtained from the sublocale S. Then d_S is a metric diameter on S.

Remark 3.2.2.

1. Given a metric frame (L, d), the above theorem asserts that every sublocale S of L inherits a metric diameter d_S.

2. The pair (S, d_S) is called a metric sublocale of the metric frame (L, d). We observe that (S, d_S) is itself a metric frame.

The following question on the well-definedness of the diameter from Theorem 3.2.1 now arises: Given $S \subseteq T \subseteq L$ where S and T are sublocales of (L, d), is the metric diameter on S inherited from L and the metric diameter on S inherited from T the same?

For the purpose of the next result, which addresses the above question, we now clarify the notation for the inherited metric diameters of sublocales and their corresponding nuclei.

Let $S \subseteq T \subseteq L$, where S and T are sublocales of (L, d). By ν_S and ν_T, we shall denote the nucleus of S on the frame L and the nucleus of T on the frame L, respectively. Regarding

S as a sublocale of L, we have that for $a \in S$,

$$d_S(a) = \inf\{d(b) \mid a \leq \nu_S(b) \text{ for } b \in L\}.$$

We define $d_T(a)$ for any $a \in L$, analogously, when regarding T as a sublocale of L. Thus d_S and d_T, denote the metric diameters of S and T, respectively, inherited from (L, d). Regarding S as a sublocale of T, we define for $a \in S$,

$$\rho_S(a) = \{d_T(b) \mid a \leq \mu_S(b) \text{ for } b \in T\},$$

where $\mu_S : T \longrightarrow T$ is the nucleus of S on the frame T. We note that $\mu_S = \nu_S\big|_T$, where $\nu_S : L \longrightarrow L$ is the nucleus of S on the frame L. So ρ_S, which denotes the metric diameter of S inherited from T, can be restated as follows: For $a \in S$

$$\rho_S(a) = \{d_T(b) \mid a \leq \nu_S(b) \text{ for } b \in T\}.$$

Now that we have defined and discussed the required notation, we are able to state and prove the next result.

Theorem 3.2.3. *Let $S \subseteq T \subseteq (L, d)$, where S and T are sublocales of L, then for any $a \in S$, $d_S(a) = \rho_S(a)$. That is, the diameter of S inherited from L is the same as the diameter of S inherited from T.*

Proof.
Let $a \in S$ be arbitrary. We will first show that $d_S(a) \leq \rho_S(a)$. Take any $d_T(b)$, $b \in T$, where $a \leq \nu_S(b)$. To show that

$$d_S(a) \leq d_T(b) = \inf\{d(c) \mid b \leq \nu_T(c) \text{ for } c \in L\}.$$

We will show that $d_S(a) \leq d(c)$ whenever $b \leq \nu_T(c)$ for $c \in L$, and $a \leq \nu_S(b)$ for $b \in T$.

For this we show that $a \leq \nu_S(c)$. Now,

$$b = \nu_T(b) \leq \nu_T(\nu_T(c)) = \nu_T(c) \leq \nu_S(c).$$

So $a \leq \nu_S(b) \leq \nu_S(\nu_S(c)) = \nu_S(c)$, as required.

To obtain equality, it remains for us to check that $\rho_S(a) \leq d_S(a)$. We will show that $\rho_S(a) \leq d(b)$, whenever $a \leq \nu_S(b)$. Now,

$$\rho_S(a) = \inf\{d_T(c) \mid a \leq \nu_S(c) \text{ for } c \in T\},$$

$a \leq \nu_S(b) \leq \nu_S(\nu_T(b))$ and $\nu_T(b) \in T$, so $\rho_S(a) \leq d_T(\nu_T(b))$. Now,

$$d_T(\nu_T(b)) = \inf\{d(w) \mid \nu_T(b) \leq \nu_T(w)\} \leq d(b),$$

so $\rho_S(a) \leq d(b)$. Hence $\rho_S(a) \leq d_S(a)$, and hence we have equality.

$\square$

Remark 3.2.4. In view of the previous result, throughout the remainder of this thesis, d_S shall denote the inherited metric diameter of S, where S is a sublocale of a metric frame. The pair (S, d_S) shall be called a metric sublocale.

The concept of the *diameter* of a sublocale S, is discussed next. Since every nucleus $j : L \to L$ on a frame L induces a frame homomorphism given by $h : L \to Fix(j)$, where $Fix(j) = \{x \in L \mid j(x) = x\}$, the following definition follows from [20].

Definition 3.2.5. Let (S, d_S) be a metric sublocale of a metric frame (L, d). The *diameter* of S, denoted $\mathrm{diam}(S)$, is defined as follows

$$\mathrm{diam}(S) = d_S(1) = \inf\{d(a) |\ a \in L,\ 1 \leq \nu_S(a)\}.$$

Remark 3.2.6. From Theorem 3.2.3, it follows that for any metric sublocale S, $\mathrm{diam}(S)$ is well-defined.

To conclude this section, we present known results on properties of diameters of sublocales which will be required in later chapters. We will provide detailed proofs of the results, since they are part of the folklore with no literature to reference from.

Proposition 3.2.7. *Let (L, d) be a metric frame and $a \in L$, then $d(a) = diam(\mathfrak{o}(a))$.*

Proof.

Let $a \in L$ be arbitrary, we need to show that

$$d(a) = d_{\mathfrak{o}(a)}(1) = \inf\{d(b) \mid 1 \leq \nu_{\mathfrak{o}(a)}(b),\ \text{for } b \in L\ \}.$$

Now by Proposition 2.1.24, $b \leq x$ implies that $a \to b \leq a \to x$, and $\nu_{\mathfrak{o}(a)}(b) = \bigwedge\{a \to x \in \mathfrak{o}(a) \mid b \leq x\}$, thus we have that $a \to b \leq \nu_{\mathfrak{o}(a)}(b)$. So it follows that,

$$\begin{aligned}
d_{\mathfrak{o}(a)}(1) &= \inf\{d(b) \mid 1 \leq \nu_{\mathfrak{o}(a)}(b),\ \text{for } b \in L\} \\
&= \inf\{d(b) \mid 1 = \nu_{\mathfrak{o}(a)}(b),\ \text{for } b \in L\} \\
&= \inf\{d(b) \mid a \to b \leq 1,\ \text{for } b \in L\} \\
&= \inf\{d(b) \mid a \leq b \text{ or } a \to b < 1,\ \text{for } b \in L\} \\
&= d(a)
\end{aligned}$$

Hence $\text{diam}(\mathfrak{o}(a)) = d(a)$.

$\square$

Proposition 3.2.8. *Let S and T be sublocales of (L, d). If $S \subseteq T$, then $diam(S) \leq diam(T)$.*

Proof.

Let S and T be sublocales of (L, d) and $S \subseteq T \subseteq L$. Since $S \subseteq T$, it follows that for $a \in L$, $\bigwedge\{x \in T \mid a \leq x\} \leq \bigwedge\{x \in S \mid a \leq x\}$. Thus $\nu_T(a) \leq \nu_S(a)$, for $a \in L$. Now, $\{d(a) \mid 1 \leq \nu_T(a),\ a \in L\} \subseteq \{d(a) \mid 1 \leq \nu_S(a),\ a \in L\}$, and taking the infimum on both sides we have that $d_S(1) \leq d_T(1)$. Thus $\text{diam}(S) \leq \text{diam}(T)$, as required.

$\square$

Proposition 3.2.9. *Let S and T be sublocales of (L, d). If $S \cap T \neq \{1\}$, then $diam(S \vee T) \leq diam(S){+}diam(T)$.*

Proof.

If either $\text{diam}(S) = \infty$ or $\text{diam}(T) = \infty$, then we are done. So assume that $\text{diam}(S) < \infty$ and $\text{diam}(T) < \infty$.

Claim 1: $\nu_{S\vee T} = \nu_S \wedge \nu_T$.

For any $x \in L$,

$$\begin{aligned} \nu_{S\vee T}(x) &= \bigwedge\{y \in S \vee T \mid x \leq y\} \\ &= \bigwedge\{s \wedge t \mid x \leq s \wedge t,\ s \in S,\ t \in T\} \\ &= \bigwedge\{s \mid x \leq s, s \in S\} \ \wedge\ \bigwedge\{t \mid x \leq t, t \in T\} \\ &= \nu_S(x) \wedge \nu_T(x). \end{aligned}$$

Thus $\nu_{S\vee T} = \nu_S \wedge \nu_T$, as claimed.

We now show that

$$\begin{aligned} d_{S\vee T}(1) &\leq d_S(1) + d_T(1) \\ &= \inf\{d(b) \mid 1 = \nu_S(b)\} + \inf\{d(c) \mid 1 = \nu_T(c)\}. \end{aligned}$$

Since $S \cap T \neq \{1\}$, there exists $x \neq 1$ such that $x \in S$ and $x \in T$. Fix any b such that $1 = \nu_S(b)$. Then for any c such that $1 = \nu_T(c)$, we have that $1 = \nu_S(b)$. Now,

$$\begin{aligned} d_{S\vee T}(1) &= \inf\{d(r) \mid 1 = \nu_{S\vee T}(r) = \nu_S(r) \wedge \nu_T(r)\} \\ &= \inf\{d(r) \mid 1 = \nu_S(r) \text{ and } 1 = \nu_T(r)\}. \end{aligned}$$

Claim 2: $\nu_S(b) = 1$ and $1 = \nu_T(c)$ implies $b \wedge c \neq 0$.

If $b \wedge c = 0$, then $\nu_S(b \wedge c) = \nu_S(0)$, thus $\nu_S(b) \wedge \nu_S(c) = \nu_S(0)$. This implies that

$\nu_S(c) = \nu_S(0)$. Now $\nu_S(0) \leq \nu_{S\cap T}(0)$, therefore

$$1 = \nu_T(c) \leq \nu_T(\nu_S(c)) \leq \nu_T(\nu_{S\cap T}(0)) = \nu_{S\cap T}(0).$$

Thus $0_{S\cap T} = 1$, but then this means $S \cap T = \{1\}$, which is a contradiction. Hence $b \wedge c \neq 0$, as claimed.

Now $\nu_S(b \vee c) = 1$ and $\nu_T(b \vee c) = 1$, so $\nu_{S\vee T}(b \vee c) = \nu_S(b \vee c) \wedge \nu_T(b \vee c) = 1$. Hence $d_{S\vee T}(1) \leq d(b \vee c) \leq d(b) + d(c)$, since $b \wedge c \neq 0$. So $d_{S\vee T}(1) - d(b) \leq d(c)$. Taking the infimum over the $d(c)'s$, we obtain $d_{S\vee T}(1) - d(b) \leq \inf\{d(c) \mid 1 = \nu_T(c)\} = d_T(1)$. Thus $d_{S\vee T}(1) - d_T(1) \leq d(b)$, and taking the infimum over the $d(b)'s$, we have
$d_{S\vee T}(1) \leq d_S(1) + d_T(1)$. Hence $\mathrm{diam}(S \vee T) \leq \mathrm{diam}(S)+ \mathrm{diam}(T)$.

□

Next, we aim to establish that for any sublocale, S, of a metric frame, the diameter of the sublocale and the diameter of the closure of the sublocale, $\overline{S}$, are the same. We recall that the closure of S is defined as $\overline{S} = \uparrow (\bigwedge S)$, as presented in Definition 2.2.19.

In order to prove that $\mathrm{diam}(S) = \mathrm{diam}(\overline{S})$, we require the next Theorem.

Theorem 3.2.10. *Let S be a sublocale of (L, d). For any $a \in S$, $d_S(a) = d_{\overline{S}}(a)$.*

Proof.

Let $a \in S$ be arbitrary. Since $\nu_{\overline{S}}(b) \leq \nu_S(b)$ for any $b \in L$,

$$d_S(a) = \inf\{d(b) \mid b \in L,\ a \leq \nu_S(b)\} \leq \inf\{d(b) \mid b \in L,\ a \leq \nu_{\overline{S}}(b)\} = d_{\overline{S}}(a).$$

Hence $d_S(a) \leq d_{\overline{S}}(a)$.

It remains to show that $d_{\overline{S}}(a) \leq d_S(a)$, that is, to show that
$d_{\overline{S}}(a) \leq \inf\{d(b) \mid a \leq \nu_S(b), b \in L\}$. Take any $d(b)$, $b \in L$, with $a \leq \nu_S(b)$. We will show that $d_{\overline{S}}(a) \leq d(b)$. Let $\varepsilon > 0$, we show that $d_{\overline{S}}(a) \leq d(b) + 5\varepsilon$. Now

$U_d^\varepsilon = \{x \in L \mid d(x) < \varepsilon\}$ is a cover of L. Let

$$w = \bigvee\{x \in L \mid d(x) < \varepsilon \text{ and } x \wedge b \neq 0\},$$

and let $0_S = \bigwedge S$. We note that $0_S \in S$, since S is a sublocale. Also since $\overline{S} = \uparrow 0_S$, we have that for any $t \in L$,

$$\begin{aligned}
\nu_{\overline{S}}(t) = \bigwedge\{s \in \overline{S} \mid t \leq s\} &= \bigwedge\{s \in \uparrow 0_S \mid t \leq s\} \\
&= \bigwedge(\uparrow 0_S \cap \uparrow t) \\
&= \bigwedge \uparrow (0_S \vee t) \\
&= 0_S \vee t.
\end{aligned}$$

<u>Claim</u>: if $x \in U_d^\varepsilon$, with $x \wedge b = 0$, then $x \wedge a \leq 0_S$.

Let $x \in U_d^\varepsilon$, such that $x \wedge b = 0$.

$$\begin{aligned}
x \wedge b = 0 &\Rightarrow \nu_S(x \wedge b) = \nu_S(0) = 0_S, \\
&\Rightarrow \nu_S(x) \wedge \nu_S(b) = 0_S, \\
&\Rightarrow \nu_S(x) \wedge \nu_S(b) \wedge a = 0_S \wedge a = 0_S \quad (\text{since } a \in S \text{ implies } 0_S \leq a), \\
&\Rightarrow \nu_S(x) \wedge a = 0_S \quad (\text{since } a \leq \nu_S(b)), \\
&\Rightarrow x \wedge a \leq \nu_S(x) \wedge a = 0_S.
\end{aligned}$$

Hence $x \wedge a \leq 0_S$, as claimed.

Now $a \leq 1$, and

$$1 = \bigvee\{x \in U_d^\varepsilon \mid x \wedge b = 0\} \ \vee \ \bigvee\{x \in U_d^\varepsilon \mid x \wedge b \neq 0\}.$$

This implies
$$\begin{aligned} a &= \Big(a \wedge \bigvee\{x \in U_d^\varepsilon \mid x \wedge b = 0\}\Big) \ \vee \ \Big(a \wedge \bigvee\{x \in U_d^\varepsilon \mid x \wedge b \neq 0\}\Big) \\ &= \bigvee\{a \wedge x \mid x \in U_d^\varepsilon, \ x \wedge b = 0\} \ \vee \ (a \wedge w) \\ &\leq \ 0_S \ \vee \ (a \wedge w) \\ &\leq \ 0_S \wedge w \ = \ \nu_{\overline{S}}(w). \end{aligned}$$

Hence $d_{\overline{S}}(a) \leq d(w)$. Since d is a metric diameter, there exists $0 \neq u, v \leq w$, $d(u), d(v) < \varepsilon$ and $d(w) < d(u \vee v) + \varepsilon$. Now $w = \bigvee\{x \in U_d^\varepsilon \mid x \wedge b \neq 0\}$, hence there exists $x_1 \in U_d^\varepsilon$, such that $x_1 \wedge b \neq 0$ and $u \wedge x_1 \neq 0$. Similarly, there exists $x_2 \in U_d^\varepsilon$ such that $x_2 \wedge b \neq 0$ and $v \wedge x_2 \neq 0$. If we consider the element $u \vee x_1 \vee b \vee x_2 \vee v$, then we have that

$$\begin{aligned} d(w) < d(u \vee v) + \varepsilon &\leq d(u \vee x_1 \vee b \vee x_2 \vee v) + \varepsilon \\ &\leq d(u) + d(x_1) + d(b) + d(x_2) + d(v) + \varepsilon \\ &< d(b) + 5\varepsilon \end{aligned}$$

Hence $d_{\overline{S}}(a) \leq d(b) + 5\varepsilon$, and since ε is arbitrary this shows that $d_{\overline{S}}(a) \leq d(b)$, and hence $d_S(a) \leq d_{\overline{S}}(a)$, and so we get equality.

□

Corollary 3.2.11. *Let S be a sublocale of (L, d), then $diam(S) = diam(\overline{S})$.*

Proof.

Follows immediately from Theorem 3.2.10.

□

3.3 Sublocales and Property S

The purpose of this section is to obtain an equivalent criterion of Property S, determined by the sublocales of a metric frame, which is the main result of this chapter. In order to do this, we will discuss the concept of Property S^*, as introduced by Baboolal in [5] for uniform frames and provide a reformulation of Property S^* in the theory of sublocales. All results in this section are geared towards proving the main result.

Proposition 3.3.1. *Let (L, d) be a metric frame and $\varepsilon > 0$, then there exists a collection of open sublocales $\{\mathfrak{o}(a_i)\}_{i \in I}$ each with diameter less than ε and such that $L = \bigvee_{i \in I} \mathfrak{o}(a_i)$.*

Proof.
Let $\varepsilon > 0$ and note that $U_d^\varepsilon = \{x \in L \mid d(x) < \varepsilon\}$ is a cover of L. For each $a \in U_d^\varepsilon$, consider the open sublocale $\mathfrak{o}(a)$, then by Proposition 3.2.7, $\mathrm{diam}(\mathfrak{o}(a)) = d(a) < \varepsilon$. We must now show that $L = \bigvee_{a \in U_d^\varepsilon} \mathfrak{o}(a)$. Since U_d^ε is a cover of L,

$$\bigvee_{a \in U_d^\varepsilon} a = 1,$$

and therefore $\mathfrak{o}(\bigvee_{a \in U_d^\varepsilon} a) = \mathfrak{o}(1)$. By Proposition 2.2.15, we know that $\mathfrak{o}(1) = L$ and $\mathfrak{o}(\bigvee_{a \in U_d^\varepsilon} a) = \bigvee_{a \in U_d^\varepsilon} \mathfrak{o}(a)$. Hence, we have that $L = \bigvee_{a \in U_d^\varepsilon} \mathfrak{o}(a)$, as required.

□

Theorem 3.3.2. *Let (L, d) be a metric frame. (L, d) has Property S if and only if, given any $\varepsilon > 0$ there exists a finite number of open connected sublocales S_i such that $L = \bigvee_{i \in I} S_i$, and diam$(S_i) < \varepsilon$.*

Proof.
($\Rightarrow$) Let $\varepsilon > 0$ and suppose that (L, d) has Property S. Then there exists a finite collection $\{a_1, a_2, ..., a_n\}$, consisting of connected elements such that $\bigvee_{i=1}^n a_i = 1$ and $d(a_i) < \varepsilon$ for $i = 1, 2, ..., n$. Consider $\mathfrak{o}(a_i)$, for each a_i where $i = 1, 2, ..., n$. By Theorem 2.2.33, it follows that each $\mathfrak{o}(a_i)$ is connected as a sublocale, since each a_i is connected in L. Furthermore, by Proposition 3.2.7, $\mathrm{diam}(\mathfrak{o}(a_i)) = d(a_i) < \varepsilon$, for $i = 1, 2, ..., n$. Finally, by invoking

Proposition 2.2.15 and since $\bigvee_{i=1}^n a_i = 1$, we have that $L = \mathfrak{o}(1) = \mathfrak{o}(\bigvee_{i=1}^n a_i) = \bigvee_{i=1}^n \mathfrak{o}(a_i)$. Hence $L = \bigvee_{i=1}^n \mathfrak{o}(a_i)$, as required.

($\Leftarrow$) Let $\varepsilon > 0$. Suppose that there exists a finite collection of open connected sublocales $\{\mathfrak{o}(a_1), \mathfrak{o}(a_2), ..., \mathfrak{o}(a_n)\}$, for $a_1, a_2, ..., a_n \in L$, such that $\text{diam}(\mathfrak{o}(a_i)) < \varepsilon$ and $L = \bigvee_{i=1}^n \mathfrak{o}(a_i)$. By Proposition 2.2.33, each a_i is connected since $\mathfrak{o}(a_i)$, and by Proposition 3.2.7, $d(a_i) = \text{diam}(\mathfrak{o}(a_i)) < \varepsilon$ for $i = 1, 2, ..., n$. Now, by Proposition 2.2.15

$$\mathfrak{o}(1) = L = \bigvee_{i=1}^{n} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i=1}^{n} a_i).$$

Thus $\mathfrak{o}(1) = \mathfrak{o}(\bigvee_{i=1}^n a_i)$ implies that $1 = \bigvee_{i=1}^n a_i$. Hence (L, d) has Property S.

$\square$

Baboolal [5] defined Property S^* for a uniform frame. Since we are working with sublocales, this can be reformulated as follows:

Definition 3.3.3. A uniform frame $(L, \mathcal{U}L)$ has *Property* S^* if, given any $U \in \mathcal{U}L$, there exists a finite number of connected sublocales $C_1, C_2, ..., C_n$ such that $\bigvee_{i=1}^n C_i = L$ and $\{C_i\}_{i=1}^n \leq \{\mathfrak{o}(u) \mid u \in U\}$.

Property S^* was introduced in order to show that for uniform frames, Property S is reflected by dense uniform maps. This was done by showing that Property S^* is reflected by dense uniform maps, and then showing that Property S and Property S^* are in fact equivalent (Proposition 4.21 in [5]). Since metric frames are uniform frames, the following follows from [5].

Proposition 3.3.4. *For metric frames, Property S and Property S^* are equivalent.*

The next result is essential in obtaining an equivalent formulation of Property S^*.

Proposition 3.3.5. *Let S be a sublocale of (L, d). Suppose that diam$(S) < \varepsilon$ and $S \subseteq \bigvee\{\mathfrak{o}(x) \mid \mathfrak{o}(x) \cap S \neq \{1\},\ d(x) < \varepsilon\}$, then*

$$\textit{diam}\left(\bigvee\{\mathfrak{o}(x) \mid \mathfrak{o}(x) \cap S \neq \{1\},\ d(x) < \varepsilon\}\right) < 6\varepsilon.$$

Proof.

Let $z = \bigvee\{x \in L \mid \mathfrak{o}(x) \cap S \neq \{1\},\ d(x) < \varepsilon\}$, then

$$\mathfrak{o}(z) = \bigvee\{\mathfrak{o}(x) \in L \mid \mathfrak{o}(x) \cap S \neq \{1\},\ d(x) < \varepsilon\}.$$

We have to show that $\operatorname{diam}(\mathfrak{o}(z)) < 6\varepsilon$, but by Proposition 3.2.7 it suffices to show that $d(z) < 6\varepsilon$. Since d is a metric diameter, we can find $u, v \leq z$, $d(u), d(v) < \varepsilon$ such that $d(z) - \varepsilon < d(u \vee v)$. Since U^d_ε is a cover of L, there exists x_1 with $d(x_1) < \varepsilon$ such that $u \wedge x_1 \neq 0_L$, and similarly, there exists x_2 with $d(x_2) < \varepsilon$ such that $v \wedge x_2 \neq 0_L$. Then $\mathfrak{o}(u) \cap \mathfrak{o}(x_1) \neq \{1\}$ and $\mathfrak{o}(v) \cap \mathfrak{o}(x_2) \neq \{1\}$. Thus by Proposition 3.2.7,
$d(u \vee v) = diam(\mathfrak{o}(u \vee v))$, and

$$\begin{aligned}
&\quad diam(\mathfrak{o}(u \vee v))\\
&= diam(\mathfrak{o}(u) \vee \mathfrak{o}(v)) \quad \text{(by Proposition 2.2.15)}\\
&\leq diam(\mathfrak{o}(u) \vee \mathfrak{o}(x_1) \vee S \vee \mathfrak{o}(x_2) \vee \mathfrak{o}(v))\\
&\leq diam(\mathfrak{o}(u)) + diam(\mathfrak{o}(x_1)) + diam(S) + diam(\mathfrak{o}(x_2)) + diam(\mathfrak{o}(v)) \text{ (by Proposition 3.2.9)}\\
&= d(u) + d(x_1) + diam(S) + d(x_2) + d(v) \quad \text{(by Proposition 3.2.7)}\\
&< \varepsilon + \varepsilon + \varepsilon + \varepsilon + \varepsilon = 5\varepsilon.
\end{aligned}$$

Hence $d(z) < \varepsilon + d(u \vee v) = 6\varepsilon$.

□

For metric frames, Property S^*, as introduced for uniform frames by Baboolal [5], reduces to the following definition:

Definition 3.3.6. (L, d) has Property S^* if whenever $\varepsilon > 0$ then there exists a finite collection of connected sublocales $\{S_1, ..., S_n\}$ such that $\bigvee_{i=1}^n S_i = L$ and
$\{S_i\}_{i=1}^n \leq \{\mathfrak{o}(x) \mid d(x) < \varepsilon\}$.

Proposition 3.3.7. *(L, d) has Property S^* if and only if given $\varepsilon > 0$, there exists a finite collection of connected sublocales $\{S_1, ..., S_n\}$ such that $\bigvee_{i=1}^n S_i = L$ and $diam(S_i) < \varepsilon$ for*

$i = 1, ..., n$.

Proof.

($\Longrightarrow$) Let $\varepsilon > 0$ be given and suppose that (L, d) has Property S^*. Then there exists connected sublocales $S_1, ..., S_n$ such that $\bigvee_{i=1}^n S_i = L$ and $\{S_i\}_{i=1}^n \leq \{\mathfrak{o}(x) \mid d(x) < \varepsilon\}$. Then for all i, there exists x_i, such that $d(x_i) < \varepsilon$ and $S_i \subseteq \mathfrak{o}(x_i)$. By Proposition 3.2.8, $\text{diam}(S_i) \leq \text{diam}(\mathfrak{o}(x_i)) = d(x_i) < \varepsilon$, as required.

($\Longleftarrow$) Take $\varepsilon > 0$ and let $\{S_1, ..., S_n\}$ be a collection of connected sublocales such that $\bigvee_{i=1}^n S_i = L$ and $\text{diam}(S_i) < \frac{\varepsilon}{6}$ for $i = 1, ..., n$. Now $1 = \bigvee\{x \in L \mid d(x) < \frac{\varepsilon}{6}\}$, so $L = \mathfrak{o}(1) = \bigvee\{\mathfrak{o}(x) \mid d(x) < \frac{\varepsilon}{6}\}$. For any S_i,

$$
\begin{aligned}
S_i = S_i \wedge L = S_i \ \wedge \bigvee\{\mathfrak{o}(x) \mid d(x) < \frac{\varepsilon}{6}\} \\
&= \bigvee\{S_i \cap \mathfrak{o}(x) \mid d(x) < \frac{\varepsilon}{6}\} \quad \text{(by Proposition 2.2.17)} \\
&= \bigvee\{S_i \cap \mathfrak{o}(x) \mid d(x) < \frac{\varepsilon}{6},\ S_i \cap \mathfrak{o}(x) \neq \{1\}\} \\
&\subseteq \bigvee\{\mathfrak{o}(x) \mid d(x) < \frac{\varepsilon}{6},\ S_i \cap \mathfrak{o}(x) \neq \{1\}\} = \mathfrak{o}(z),
\end{aligned}
$$

where $z = \bigvee\{x \mid d(x) < \frac{\varepsilon}{6},\ S_i \cap \mathfrak{o}(x) \neq \{1\}\}$. So $S_i \subseteq \mathfrak{o}(z)$, and by Proposition 3.3.5, $\text{diam}(S_i) \leq \text{diam}(\mathfrak{o}(z)) < 6(\frac{\varepsilon}{6}) = \varepsilon$. Hence $\{S_1, ..., S_n\} \leq \{\mathfrak{o}(y) \mid d(y) < \varepsilon\}$, and hence (L, d) has Property S^*.

□

We now prove the main result.

Theorem 3.3.8. *Let (L, d) be a metric frame. The following are equivalent:*

1. *(L, d) has Property S.*
2. *For $\varepsilon > 0$, $L = \bigvee_{i=1}^n K_i$, where K_i is an open connected sublocale with diam$(K_i) < \varepsilon$, for each $i = 1, ..., n$.*
3. *For $\varepsilon > 0$, $L = \bigvee_{i=1}^n T_i$ where T_i is a closed connected sublocale with diam$(T_i) < \varepsilon$, for each $i = 1, ..., n$.*

4. *For $\varepsilon > 0$, $L = \bigvee_{i=1}^{n} S_i$ where S_i is a connected sublocale with diam$(S_i) < \varepsilon$, for each $i = 1, ..., n$.*

Proof.

(1) $\implies$ (2): Follows from Theorem 3.3.2.

(2) $\implies$ (3): Take $\varepsilon > 0$ and suppose $L = \bigvee_{i=1}^{n} K_i$, where K_i is an open connected sublocale with $\operatorname{diam}(K_i) < \varepsilon$, for each $i = 1, ..., n$. Now for each i, $K_i \subseteq \overline{K_i}$, therefore $L = \bigvee_{i=1}^{n} \overline{K_i}$. Since K_i is connected, it follows from Proposition 2.2.29 that $\overline{K_i}$ is connected for each i. By Corollary 3.2.11, $\operatorname{diam}(\overline{K_i}) = \operatorname{diam}(K_i) < \varepsilon$. Hence (3) follows.

(3) $\implies$ (4): Follows immediately.

(4) $\implies$ (1): Let $\varepsilon > 0$ and suppose $L = \bigvee_{i=1}^{n} S_i$ where S_i is a connected sublocale with $\operatorname{diam}(S_i) < \varepsilon$, for each $i = 1, ..., n$. By Proposition 3.3.7, (L, d) has Property S^* and hence by Proposition 3.3.4, (L, d) has Property S, as required.

□

Chapter 4

A diameter on a connected, locally connected frame

In this chapter we provide the construction of a compatible metric diameter for a connected, locally connected metric frame. The construction of the new diameter is the analogue of a metric defined by Kelley [18].

In [30], Whyburn constructed a metric, ρ^* from ρ, on a connected, locally connected space X, given by

$$\rho^*(x, y) = \inf\{\operatorname{diam}(C) \mid x, y \in C \text{ and } C \subseteq X \text{ is connected}\},$$

for $x, y \in X$ and where $\operatorname{diam}(C) = \sup\{\rho(a, b) \mid a, b \in C\}$.

Whyburn posed further questions regarding properties of his new constructed metric and on the existence of a metric for a connected space having Property S such that the open ball of every point will be connected and inherit Property S. Kelley answered the latter question posed by Whyburn, by defining a new metric on a connected, locally connected space found in [18] .

We also note that Baboolal [2], provided the analogue of Whyburn's construction on a locally connected frame. In [2], he also shows that the new constructed metric frame is uniformly locally connected. In the next section, we will present the analogue of Kelley's construction.

4.1 Construction and properties of the new diameter

In this section we will present the construction of a compatible diameter on a connected, locally connected metric frame. The purpose herein is to present and study the properties for the analogue of Kelley's metric from [18]. As a result, the constructed compatible diameter will successfully answer the question posed by Whyburn [30], in the point-free context.

Throughout the rest of this chapter we will assume that (L, d) is a connected, locally connected metric frame.

Definition 4.1.1. Let $0 \neq x, y \in (L, d)$, where x, y connected and let $\varepsilon > 0$. We define an $R(\varepsilon, x, y)$ *chain* from x to y, to be a finite collection $\{a_i\}_{i=1}^{n}$ of elements of L such that,

(1) each a_i is connected,

(2) $d(a_i) < \frac{\varepsilon}{4}$ for $i = 1, ..., n$,

(3) $x = a_1$, $y = a_n$ and for $i = 1, ..., n-1$, $a_i \wedge a_{i+1} \neq 0$,

(4) whenever $1 \leq j < i \leq n$, we have $\varepsilon - 2d(a_j) - 2d(a_i) - \Sigma_{k=j+1}^{i-1} d(a_k) > 0$.

Definition 4.1.2. Let $a \in L$ be arbitrary and $\varepsilon > 0$. An $R(\varepsilon)$ cover of a is a subset $S \subseteq L$ consisting of connected elements such that

(1) $a \leq \bigvee S$,

(2) $d(s) < \frac{\varepsilon}{4}$ for all $s \in S$,

(3) for any $s, t \in S$, there exists an $R(\varepsilon, s, t)$ chain from s to t consisting only of elements from S.

Proposition 4.1.3. *If S is an $R(\varepsilon)$ cover of any $a \in L$, then $\bigvee S$ is connected.*

Proof.

Let $a \in L$ be arbitrary and suppose that S is a $R(\varepsilon)$ cover of a. Fix $s \in S$ and take any $t \in S$. Then there exists an $R(\varepsilon, s, t)$ chain, say, $\{a_1, ..., a_n\}$ of connected elements with $a_1 = s$ and $a_n = t$, such that $a_i \in S$ for all i. Let $c_t = a_1 \vee a_2 \vee ... \vee a_n$. Then it follows from Lemma 2.1.34 that c_t is connected, since each a_i is connected and for $i = 1, ..., n-1$, $a_i \wedge a_{i+1} \neq 0$. Also $s \leq c_t$ for all t, so $\bigvee\{c_t \mid t \in S\}$ is connected. But $\bigvee\{c_t \mid t \in S\} = \bigvee S$, so $\bigvee S$ is connected.

□

Proposition 4.1.4. *Let $a, b \in L$ such that $a \wedge b \neq 0$. If S_1 is an $R(\varepsilon_1)$ cover of a and S_2 an $R(\varepsilon_2)$ cover of b, then $S_1 \cup S_2$ is an $R(\varepsilon_1 + \varepsilon_2)$ cover of $a \vee b$.*

Proof.

(1) Certainly, $a \vee b \leq (\bigvee S_1) \vee (\bigvee S_2) = \bigvee(S_1 \cup S_2)$.

(2) Take any $s \in S_1 \cup S_2$, then $s \in S_1$ or $s \in S_2$. This implies that $d(s) < \frac{\varepsilon_1}{4}$ or $d(s) < \frac{\varepsilon_2}{4}$, hence $d(s) < \frac{\varepsilon_1 + \varepsilon_2}{4}$.

(3) Now take any $s, t \in (S_1 \cup S_2)$. Assume that both s, t are in S_1. Then since S_1 is an $R(\varepsilon_1)$ cover of a, there exists an $R(\varepsilon, s, t)$ chain from s to t consisting only of elements from S_1 and hence $S_1 \cup S_2$. Thus, this chain will be an $R(\varepsilon_1 + \varepsilon_2, s, t)$ chain. We similarly obtain a chain if both s and t are in S_2. We may therefore assume that $s \in S_1$ and $t \in S_2$. Now, $0 \neq a \wedge b \leq (\bigvee S_1) \wedge (\bigvee S_2) = \bigvee\{x \wedge y \mid x \in S_1, y \in S_2\}$, and this implies that $a \wedge b \wedge x \wedge y \neq 0$ for some $x \in S_1$ and $y \in S_2$. Now there exists an $R(\varepsilon_1, s, x)$ chain, $\{a_i\}_{i=1}^{n}$ from s to x, and there exists and $R(\varepsilon_2, y, t)$ chain, say $\{b_j\}_{j=1}^{m}$, from y to t. Thus, $\{a_1, a_2, ..., a_n, b_1, b_2, ..., b_m\}$ is an $R(\varepsilon_1 + \varepsilon_2, s, t)$ chain from s to t consisting of elements from $S_1 \cup S_2$. To see this, we note that conditions (1), (2) and (3) of Definition 4.1.1 are clearly true. For condition (4), re-name the chain $\{a_1, a_2, ..., a_n, b_1, b_2, ..., b_m\}$ as $\{a_1, a_2, ..., a_n, a_{n+1}, a_{n+2}, ..., a_{n+m}\}$. Hence $a_1 = s$, $a_n = x$, $a_{n+1} = y$ and $a_{n+m} = t$. We

need to show that, whenever $1 \leq j < i \leq n+m$, then we have

$$\varepsilon_1 + \varepsilon_2 - 2d(a_j) - 2d(a_i) - \sum_{k=j+1}^{i-1} d(a_k) > 0. \tag{4.1}$$

If $i \leq n$, then equation (4.1) is true since, $\varepsilon_1 - 2d(a_j) - 2d(a_i) - \sum_{k=j+1}^{i-1} d(a_k) > 0$. Similarly, if $n+1 \leq j$, then equation (4.1) is true. Hence we may assume that $i > n$ and $j \leq n$. Now,

$$\begin{aligned}
&2d(a_j) + 2d(a_i) + \sum_{k=j+1}^{i-1} d(a_k) \\
&= 2d(a_j) + d(a_{j+1}) + ... + d(a_n) + d(a_{n+1}) + ... + d(a_{i-1}) + 2d(a_i) \\
&\leq [2d(a_j) + d(a_{j+1}) + ... + 2d(a_n)] + [2d(a_{n+1}) + d(a_{n+2}) + ... + 2d(a_i)] \\
&< \varepsilon_1 + \varepsilon_2.
\end{aligned}$$

So equation (4.1) is satisfied, and thus $S_1 \cup S_2$ is an $R(\varepsilon_1 + \varepsilon_2)$ cover of $a \vee b$.

□

Definition 4.1.5. Let (L, d) be a connected, locally connected metric frame. Define $\rho : L \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ by

$$\rho(x) = \begin{cases} \infty & \text{, if there exists no } R(\varepsilon) \text{ cover of } x. \\ \inf\{\varepsilon | \text{ there exists an } R(\varepsilon) \text{ cover of } x\} & \text{, otherwise.} \end{cases}$$

for any $x \in L$.

Lemma 4.1.6. *For any $c \in L$, $d(c) \leq \rho(c)$.*

Proof.

If $\rho(c) = \infty$, then we are done. Suppose $\rho(c) < \infty$. Take any $c \in L$ and let S be any $R(\varepsilon)$ cover of c. Let $\delta > 0$ be arbitrary. Since d is a metric diameter, there exists $u, v \leq c$, $d(u), d(v) < \frac{\delta}{3}$ such that $d(c) - \frac{\delta}{3} < d(u \vee v)$. We may assume, by Lemma 2.3.7, that u, v

may be chosen to be connected in L. Now $u \wedge s \neq 0$, for some $s \in S$ and $v \wedge t \neq 0$, for some $t \in S$. It follows that there exists an $R(\varepsilon, s, t)$ chain from s to t consisting of elements from S. Let this chain be denoted by $\{s = a_1, a_2, ..., a_n = t\}$, then $u \vee a_1 \vee .... \vee a_n \vee v$ is connected, and

$$\begin{aligned} d(c) - \frac{\delta}{3} < d(u \vee v) &\leq d(u) + d(a_1) + d(a_2) + ... + d(a_n) + d(v) \\ &= d(u) + d(v) + [d(a_1) + d(a_2) + ... + d(a_n)] \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \varepsilon. \end{aligned}$$

So $d(c) < \delta + \varepsilon$. Since this inequality is true for arbitrary $\delta > 0$, we have that $d(c) \leq \varepsilon$. Thus $d(c) \leq \inf\{\varepsilon \mid \text{there exists an } R(\varepsilon) \text{ cover of } c\} = \rho(c)$.

□

Lemma 4.1.7. *If $c \in L$ is connected, then $d(c) \leq \rho(c) \leq 4d(c)$.*

Proof.

Take any $\varepsilon > 4d(c)$ (that is, $d(c) < \frac{\varepsilon}{4}$), then $S = \{c\}$ is an $R(\varepsilon)$ cover of c, so $\rho(c) \leq \varepsilon$. Since this is true for all $\varepsilon > 4d(c)$, it follows that $\rho(c) \leq 4d(c)$. By Lemma 4.1.6, we have that $d(c) \leq \rho(c)$. Hence $d(c) \leq \rho(c) \leq 4d(c)$.

□

Theorem 4.1.8. *ρ is a diameter on L.*

Proof.

(M1) Take any $\varepsilon > 0$. Since $U^d_{\frac{\varepsilon}{4}} = \{u \in L |\ d(u) < \frac{\varepsilon}{4}\}$ is a cover of L, we can find $0 \neq u \in L$ such that $d(u) < \frac{\varepsilon}{4}$. By local connectedness of L, u is a join of connected elements from L, so there exists a connected element $c \in L$ such that $c \neq 0$ and $c \leq u$. Hence $d(c) < \frac{\varepsilon}{4}$. Now $S = \{c\}$ is an $R(\varepsilon)$ cover of 0, thus it follows that $\rho(0) < \varepsilon$. Since this is true for all $\varepsilon > 0$, we must have $\rho(0) = 0$.

(M2) Let $a, b \in L$ and suppose $a \leq b$. We will show that $\rho(a) \leq \rho(b)$. If $\rho(b) = \infty$, then we are done. So assume that $\rho(b) < \infty$. If S is an $R(\varepsilon)$ cover of b, then it is also an $R(\varepsilon)$

cover of a, so $\rho(a) < \varepsilon$. Since this is true for all ε for which there exists an $R(\varepsilon)$ cover of b, we must have that $\rho(a) \leq \rho(b)$.

(M3) Suppose $a \wedge b \neq 0$ for $a, b \in L$. We need to show that $\rho(a \vee b) \leq \rho(a) + \rho(b)$. If either $\rho(a) = \infty$ or $\rho(b) = \infty$, then we are done. So we may assume that $\rho(a) < \infty$ and $\rho(b) < \infty$. Suppose that S_1 is an $R(\varepsilon_1)$ cover of a, and that S_2 is an $R(\varepsilon_2)$ cover of b. Then by Proposition 4.1.4, $S_1 \cup S_2$ is an $R(\varepsilon_1 + \varepsilon_2)$ cover of $a \vee b$. It follows that $\rho(a \vee b) \leq \varepsilon_1 + \varepsilon_2$, and this implies that $\rho(a \vee b) - \varepsilon_1 \leq \varepsilon_2$. By fixing ε_1 and varying ε_2 we obtain that $\rho(a \vee b) - \varepsilon_1 \leq \rho(b)$, and so $\rho(a \vee b) - \rho(b) \leq \varepsilon_1$. Now varying ε_1 we have, $\rho(a \vee b) - \rho(b) \leq \rho(a)$. Hence $\rho(a \vee b) \leq \rho(a) + \rho(b)$, as required.

(M4) Let $\varepsilon > 0$ be arbitrary, we shall show that $U^{\rho}_{\varepsilon} = \{u \in L |\ \rho(u) < \varepsilon\}$ is a cover of L. Take any $a \in U^d_{\frac{\varepsilon}{4}}$, then $d(a) < \frac{\varepsilon}{4}$. Since L is a locally connected frame, a can be expressed as the join of connected elements from L, that is, there exists $S \subseteq L$ where S consists of connected elements from L such that $a = \bigvee\{x\ |x \in S\} = \bigvee S$. Now for each $x \in S$, $x \leq a$. Let $S_a = \{x \mid x \text{ is connected and } x \leq a\}$. Notice that $S_a \neq \emptyset$, since $0 \in S_a$ for all $a \in U^d_{\frac{\varepsilon}{4}}$. Now $U^d_{\frac{\varepsilon}{4}}$ is a cover of L, therefore $\bigcup_{d(a)<\frac{\varepsilon}{4}} S_a$ is also a cover of L.
Claim: $\bigcup_{d(a)<\frac{\varepsilon}{4}} S_a \subseteq U^{\rho}_{\varepsilon}$.
Take any $x \in \bigcup_{d(a)<\frac{\varepsilon}{4}} S_a$, then $x \in S_a$ for some $a \in U^d_{\frac{\varepsilon}{4}}$. So $d(x) \leq d(a) < \frac{\varepsilon}{4}$ and x is connected in L. Hence by Lemma 4.1.7 and since $4d(c) < \varepsilon$, we have that $\rho(x) \leq 4d(x) < \varepsilon$. Thus $\rho(x) < \varepsilon$ and so $x \in U^{\rho}_{\varepsilon}$, as required.
Now, since $\bigcup_{d(a)<\frac{\varepsilon}{4}} S_a$ is a cover of L and $\bigcup_{d(a)<\frac{\varepsilon}{4}} S_a \subseteq U^{\rho}_{\varepsilon}$, this implies that U^{ρ}_{ε} is indeed a cover of L.

Hence we have shown that ρ is a diameter on L.

□

Corollary 4.1.9. *ρ is a compatible diameter on L.*

Proof.

To show that for each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd_{\rho} a\}$, where $x \lhd_{\rho} a$ means that there exists U^{ρ}_{ε} such that $U^{\rho}_{\varepsilon} x \leq a$. By Lemma 4.1.6, we have that $d \leq \rho$, therefore $U^{\rho}_{\varepsilon} \subseteq U^{d}_{\varepsilon}$

and this implies that $U_\varepsilon^\rho x \leq U_\varepsilon^d x$. Since d is compatible, $x \lhd_d a$ implies $x \lhd_\rho a$. Thus ρ is a compatible diameter.

□

We recall from Theorem 2.3.5, that for any diameter on a frame and in particular, ρ on L from Theorem 4.1.8, that $\tilde{\rho}$ defined by

$$\tilde{\rho}(a) = \inf_{\varepsilon>0} \sup\{\rho(u \vee v) \mid u, v \leq a \text{ and } \rho(u), \rho(v) < \varepsilon \}, \tag{4.2}$$

for $a \in L$, is a metric diameter on L with the property that $\tilde{\rho} \leq \rho$. Thus $(L, \tilde{\rho})$ a metric frame.

The following result investigates the relationship between the metric diameters d and $\tilde{\rho}$. Thereafter, we establish that $\tilde{\rho}$ is a compatible metric diameter.

Lemma 4.1.10. *$d \leq \tilde{\rho}$.*

Proof.

Take any $a \in L$, and let $\varepsilon > 0$ be arbitrary. L is locally connected and d is a metric diameter, therefore by Lemma 2.3.7, there exists u, v, connected such that $u, v \leq a$, $d(u), d(v) < \frac{\varepsilon}{4}$ and $d(a) - \frac{\varepsilon}{4} < d(u \vee v)$. Thus by Lemma 4.1.6,

$$d(a) < \frac{\varepsilon}{4} + d(u \vee v) \leq \frac{\varepsilon}{4} + \rho(u \vee v).$$

Now $d(u), d(v) < \frac{\varepsilon}{4}$ implies that $4d(u) < \varepsilon$ and $4d(v) < \varepsilon$, and since u and v are connected, it follows from Lemma 4.1.7 that $\rho(u) < \varepsilon$ and $\rho(v) < \varepsilon$. So we have,

$$\begin{aligned} d(a) < \frac{\varepsilon}{4} + d(u \vee v) \leq \frac{\varepsilon}{4} + \rho(u \vee v) &\leq \frac{\varepsilon}{4} + \sup\{\rho(r \vee s) \mid r, s \leq a \text{ and } \rho(r), \rho(s) < \varepsilon\} \\ &< \varepsilon + \sup\{\rho(r \vee s) \mid r, s \leq a \text{ and } \rho(r), \rho(s) < \varepsilon\}. \end{aligned}$$

Since this is true for all $\varepsilon > 0$, $d(a) \leq \inf_{\varepsilon>0} \sup\{\rho(r \vee s) \mid r, s \leq a \text{ and } \rho(r), \rho(s) < \varepsilon\}$. So $d \leq \tilde{\rho}$.

□

Corollary 4.1.11. *$\tilde{\rho}$ is a compatible metric diameter.*

Proof.

It is known that $\tilde{\rho}$, defined in equation (4.2), is a metric diameter. To show that $\tilde{\rho}$ is compatible, we will show that for each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd_{\tilde{\rho}} a\}$. By Lemma 4.1.10, $d \leq \tilde{\rho}$, therefore $U_\varepsilon^{\tilde{\rho}} \subseteq U_\varepsilon^d$ and this implies that $U_\varepsilon^{\tilde{\rho}} x \leq U_\varepsilon^d x$. Since d is compatible, $x \lhd_d a$ implies $x \lhd_{\tilde{\rho}} a$. Thus $\tilde{\rho}$ is a compatible diameter. Hence $\tilde{\rho}$ is a compatible metric diameter.

□

4.2 Relationship with Kelley's Construction

We now discuss Kelley's metric as found in [18].

Definition 4.2.1 ([18]). Let (X, d) be a connected, locally connected metric space and $x, y \in X$. An *$R(\varepsilon, x, y)$ chain* is a collection of connected sets $A_1, A_2, ..., A_n$, with $x \in A_1$, $y \in A_n$, $A_i \cap A_{i+1} \neq 0$ for $i = 1, ..., n-1$, and satisfying the following inequalities :

1. $d(A_i) < \frac{\varepsilon}{4}$, for $i = 1, ..., n$,

2. $\varepsilon - 2d(A_j) - 2d(A_i) - \Sigma_{k=j+1}^{i-1} d(A_k) > 0$, for all $i, j, 1 \leq j < i \leq n$,

where $d(A_i)$ denotes the diameter of A_i with respect to the metric d.

Definition 4.2.2 ([18]). Let (X, d) be a connected, locally connected metric space. Then for any $x, y \in X$, d^* is a metric on X given by

$$d^*(x, y) = \inf\{\varepsilon > 0 \mid \text{there exists a } R(\varepsilon, x, y) \text{ chain}\}.$$

Thus (X, d^*) is a metric space.

Theorem 4.2.3 ([18]). *(X, d) is homeomorphic to (X, d^*).*

Let (X, d) be a connected, locally connected metric space. Then (X, d^*), where d^* is Kelley's metric defined in Definition 4.2.2, is a connected, locally connected metric space, since it was shown that (X, d^*) is homeomorphic to (X, d). For any open subset $A \subseteq X$, let $d^*(A)$ denote the d^*-diameter of A, where

$$d^*(A) = \sup\{d^*(x, y) \mid x, y \in A\}.$$

Then $(\mathcal{O}X, d^*)$ is a metric frame and is connected and locally connected.

Now $\mathcal{O}X$ is a connected, locally connected metric frame with diameter d. So the diameter ρ (as defined earlier in Definition 4.1.5) is a compatible diameter on the frame $\mathcal{O}X$, where for any $A \in \mathcal{O}X$,

$$\rho(A) = \inf\{\varepsilon \mid \text{there exists an } R(\varepsilon) \text{ cover of } A\},$$

or $\rho(A) = \infty$, if there exists no $R(\varepsilon)$ cover of A. Hence $(\mathcal{O}X, \tilde{\rho})$ is a metric frame, where $\tilde{\rho}$ is a compatible metric diameter defined earlier in equation (4.2) given by

$$\tilde{\rho}(A) = \inf_{\varepsilon > 0} \sup\{\rho(U \vee V) \mid U, V \subseteq A \text{ and } \rho(U), \rho(U) < \varepsilon\},$$

for any $A \in \mathcal{O}X$.

Proposition 4.2.4. *Let (X, d) be a connected, locally connected metric space. For any $A \in \mathcal{O}X$, $d^*(A) = \tilde{\rho}(A)$.*

Proof.

We first show that $\tilde{\rho}(A) \leq d^*(A)$. Let $\varepsilon > 0$. We need to show that

$$\sup\{\rho(U \cup V) | \, U, V \subseteq A, U, V \text{ is open connected, } \rho(U), \rho(V) < \frac{\varepsilon}{8}\} < d^*(A) + \varepsilon.$$

Take any $U, V \subseteq A$, with U, V open connected and $\rho(U), \rho(V) < \frac{\varepsilon}{8}$. We show that $\rho(U \cup V) < d^*(A) + \varepsilon$. Pick $x \in U$ and $y \in V$. Now,
$d^*(x, y) = \inf\{\varepsilon > 0 \mid \text{there exists an } R(\varepsilon, x, y) \text{ chain}\}$. Thus there exists an $R(\alpha, x, y)$ chain $\{A_1, ..., A_n\}$ such that $\alpha < d^*(x, y) + \frac{\varepsilon}{2}$. Let $\beta = \alpha + \frac{\varepsilon}{2}$, so $\alpha < \beta$.
Claim: $\{U = A_0, A_1, ..., A_n, A_{n+1} = V\} \subseteq \mathcal{O}X$ is an $R(\beta)$ cover of $U \cup V$.

(1) $U \cup V \subseteq U \cup A_1 \cup ... \cup A_n \cup V$.

(2) $d(U) \leq \rho(U) < \frac{\varepsilon}{8} < \frac{\varepsilon}{8} + \frac{\alpha}{4} = \frac{\beta}{4}$. Similarly $d(V) < \frac{\beta}{4}$, and $d(A_i) < \frac{\alpha}{4} < \frac{\beta}{4}$.

(3) We must show that for any $S, T \in \{U, A_1, ..., A_n, V\}$, there exists an $R(\beta, S, T)$ chain from S to T consisting only of elements from $\{U = A_0, A_1, ..., A_n, A_{n+1} = V\}$. It suffices to show that $\beta - 2d(A_j) - 2d(A_i) - \sum_{k=j+1}^{i-1} d(A_k) > 0$, whenever $0 \leq j < i \leq n+1$. We will show this in 4 cases.

Case 1: Set $j = 0$ and $i = n+1$.
$\beta - 2d(U) - 2d(V) - \sum_{k=1}^{n} d(A_k) = (\frac{\varepsilon}{2} - 2d(U) - 2d(V)) + (\alpha - \sum_{k=1}^{n} d(A_k)) > 0$, since $\frac{\varepsilon}{2} - 2d(U) - 2d(V) > 0$ and $\alpha - \sum_{k=1}^{n} d(A_k) > 0$.

Case 2: Set $j = 0$ and let $0 < i < n+1$.
Now, $\beta - 2d(U) - 2d(A_i) - \sum_{k=1}^{i-1} d(A_k) = (\frac{\varepsilon}{2} - 2d(U)) + (\alpha - 2d(A_i) - \sum_{k=1}^{i-1} d(A_k))$. Since $d(U) \leq \rho(U) < \frac{\varepsilon}{8}$, then $2d(U) < \frac{\varepsilon}{4}$. Hence $\frac{\varepsilon}{2} - 2d(U) > 0$. Also,

$$\begin{aligned} \alpha - 2d(A_i) - \sum_{k=1}^{i-1} d(A_k) &= \alpha - 2d(A_i) - d(A_1) - \sum_{k=2}^{i-1} d(A_k) \\ &\geq \alpha - 2d(A_i) - 2d(A_1) - \sum_{k=2}^{i-1} d(A_k) > 0. \end{aligned}$$

Thus, $\beta - 2d(U) - 2d(A_i) - \sum_{k=1}^{i-1} d(A_k) > 0$.

Case 3: Set $0 < j < n+1$ and $i = n+1$.
Similarly, as in Case 2, it follows that $\beta - 2d(A_j) - 2d(V) - \sum_{k=j+1}^{n} d(A_k) > 0$.

Case 4: Let $1 \leq j < i \leq n$.
Since $\{A_1, ..., A_n\}$ is an $R(\alpha, x, y)$ chain and $\alpha < \beta$, then
$\beta - 2d(A_j) - 2d(A_i) - \Sigma_{k=j+1}^{i-1} d(A_k) > \alpha - 2d(A_j) - 2d(A_i) - \Sigma_{k=j+1}^{i-1} d(A_k) > 0$.

Hence it follows from the 4 cases that whenever $0 \leq j < i \leq n+1$, then
$\beta - 2d(A_j) - 2d(A_i) - \sum_{k=j+1}^{i-1} d(A_k) > 0$.

Thus $\{U = A_0, A_1, ..., A_n, A_{n+1} = V\} \subseteq \mathcal{O}X$ is an $R(\beta)$ cover of $U \cup V$, as claimed. Now,

$$\begin{aligned}\rho(U \cup V) \leq \beta = \alpha + \frac{\varepsilon}{2} \\ &< d^*(x,y) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\leq d^*(A) + \varepsilon.\end{aligned}$$

Hence $\sup\{\rho(U \cup V) |\ U, V \subseteq A, U, V$ open connected, $\rho(U), \rho(V) < \frac{\varepsilon}{8}\} \leq d^*(A) + \varepsilon$. Thus $\inf_{\varepsilon>0} \sup\{\rho(U \cup V) |\ U, V \subseteq A, U, V$ open connected, $\rho(U), \rho(V) < \varepsilon\} \leq d^*(A)$, that is, $\tilde{\rho}(A) \leq d^*(A)$.

We now show that $d^*(A) \leq \tilde{\rho}(A)$. Take any $x, y \in A$ and $\varepsilon > 0$. We will show that $d^*(x,y) \leq \sup\{\rho(U \cup V) |\ U, V \subseteq A, U, V$ open connected, $\rho(U), \rho(V) < \varepsilon\}$. Choose $\delta = \frac{\varepsilon}{8}$. By the local connectedness of (X, d), there exists connected C_x and C_y such that $x \in C_x \subseteq A \cap S(x, \delta)$, $y \in C_y \subseteq A \cap S(y, \delta)$. Hence $d(C_x) < \frac{\varepsilon}{4}$, $d(C_y) < \frac{\varepsilon}{4}$. Now,

$$\begin{aligned}d^*(x,y) &= \inf\{\alpha > 0 |\ \text{there exists an } R(\alpha, x, y) \text{ chain in } (X, d)\} \\ &\leq \inf\{\alpha > 0 |\ \text{there exists an } R(\alpha) \text{ cover of } C_x \cup C_y\} = \rho(C_x \cup C_y).\end{aligned}$$

This is true, since for any $R(\alpha)$ cover, $\mathcal{S}$, of $C_x \cup C_y$, we can choose $S, T \in \mathcal{S}$ such that $x \in S$ and $y \in T$. Hence there exists an $R(\alpha, x, y)$ chain from x to y, since $\mathcal{S}$ is an $R(\alpha)$ cover. Hence

$$\begin{aligned}d^*(x,y) &\leq \rho(C_x \cup C_y) \\ &\leq \sup\{\rho(U \cup V) |\ U, V \subseteq A, U, V \text{ open connected}, \rho(U), \rho(V) < \varepsilon\},\end{aligned}$$

since $\rho(C_x) \leq 4d(C_x) < \varepsilon$ and similarly $\rho(C_y) < \varepsilon$. Thus $d^*(x,y)$ is a lower bound of $\sup\{\rho(U \cup V) |\ U, V \subseteq A, U, V$ open connected, $\rho(U), \rho(V) < \varepsilon\}$. Hence $d^*(A) \leq \tilde{\rho}(A)$.

$\square$

4.3 The Main Results

Let (L, d) be a connected, locally connected metric frame.

In this section we present more properties of the newly constructed metric frame $(L, \tilde{\rho})$. But first we begin with a generalisation of Kelley's [18] result, showing that an analogously defined *spherical neighbourhood* in L has Property S, if (L, d) has Property S. We begin with the following definition.

Definition 4.3.1. Let $0 \neq a \in (L, d)$ and $\epsilon > 0$. An *ϵ-neighbourhood* of $a \in L$, is defined as follows

$$V_\varepsilon(a) = \{x \in L |\ x \text{ connected, there exists an } R(\varepsilon, x, y) \text{ chain from } x \text{ to } y \text{ with } y \wedge a \neq 0\}.$$

Lemma 4.3.2. *Let $z = \bigvee V_\varepsilon(a)$. Then $a \leq z$ and z is connected in L.*

Proof.

By local connectedness of L, $S = \{t \in L \mid t \text{ is connected and } d(t) < \frac{\varepsilon}{4}\}$ is a cover of L. Now $a = a \wedge \bigvee S = \bigvee\{a \wedge t \mid t \text{ is connected and } d(t) < \frac{\varepsilon}{4}\}$. For any t connected with $d(t) < \frac{\varepsilon}{4}$ and $a \wedge t \neq 0$, we have that $t \in V_\varepsilon(a)$ (since $\{t\}$ is an $R(\varepsilon, t, t)$ chain from t to t). Now $a \leq \bigvee\{t \in L \mid t \text{ is connected}, a \wedge t \neq 0,\ d(t) < \frac{\varepsilon}{4}\} \leq \bigvee V_\varepsilon(a) = z$. Thus $a \leq z$. We now show that z is connected. Take any $x \in V_\varepsilon(a)$. Then x is connected and there exists an $R(\varepsilon, x, y)$ chain, C, from x to some connected y such that $y \wedge a \neq 0$. Any $t \in C$ is also an $R(\varepsilon, t, y)$ chain to y, so $t \in V_\varepsilon(a)$. Thus $\bigvee C \leq \bigvee V_\varepsilon(a) = z$. Also, each $x \in V_\varepsilon(a)$ is part of a chain C, so $x \leq \bigvee C$. Thus,

$$\begin{aligned}
z &= \bigvee V_\varepsilon(a) \\
&= \bigvee\{\bigvee C |\ C \text{ is an } R(\varepsilon, x, y) \text{ chain from a connected } x \text{ to connected } y \text{ with } y \wedge a \neq 0\} \\
&= \bigvee\{a \vee \bigvee C |\ C \text{ is an } R(\varepsilon, x, y) \text{ chain from a connected } x \text{ to connected } y,\ y \wedge a \neq 0\}.
\end{aligned}$$

The last equality holds since $a \leq z$, and is connected since $a \vee \bigvee C$ is connected for any $R(\varepsilon, x, y)$ chain. Hence z is connected. $\square$

Lemma 4.3.3. *Let $l > 0$ be arbitrary and take any $a \in L$. Suppose $x_1, x_2, ..., x_p$ is an $R(\varepsilon, x_1, x_p)$ chain such that $x_1 \wedge a \neq 0$ and $x_p \wedge b \neq 0$ for some $b \in L$ such that $d(b) < l$. Suppose*

$$l < \frac{1}{2}[\varepsilon - 2d(x_j) - \sum_{k=j+1}^{p} d(x_k)],$$

whenever $1 \leq j \leq p$. Then $b \in V_\varepsilon(a)$.

Proof.

We will show that $\{x_1, x_2, ..., x_p, b\}$ is an $R(\varepsilon, x_1, b)$ chain. Let $x_{p+1} = b$. For $1 \leq j < i \leq p$, we have $2d(x_j) + 2d(x_i) + \sum_{k=j+1}^{i-1} d((x_k) < \varepsilon$, since $x_1, x_2, ..., x_p$ is an $R(\varepsilon, x_1, x_p)$ chain. For $1 \leq j < i = p + 1$,

$$\begin{aligned} 2d(x_j) + 2d(x_{p+1}) + \sum_{k=j+1}^{p} d((x_k) &< 2d(b) + \varepsilon - 2l \\ &< 2l + \varepsilon - 2l = \varepsilon. \end{aligned}$$

So we have an $R(\varepsilon, x_1, b)$ chain. Thus $b \in V_\varepsilon(a)$.

□

Theorem 4.3.4. *Let (L, d) be a connected, locally connected metric frame. Let $a \in L$, $\varepsilon > 0$ and $z = \bigvee V_\varepsilon(a)$. If (L, d) has Property S, then $\downarrow z$ has Property S (with respect to d).*

Proof.

Suppose l is arbitrary and assume $\frac{l}{9} < \frac{\varepsilon}{4}$. To show that $\downarrow z$ has Property S, we must show that its top element, z, can be written as a finite join of connected elements $y \in \downarrow z$ such that $d(y) < l$. Since (L, d) has Property S, we can find $b_1, b_2, ..., b_n$ connected in L such that $b_1 \vee ... \vee b_n = 1$ and $d(b_i) < \frac{l}{9} < \frac{\varepsilon}{4}$, for $i = 1, ..., n$.
Since $\{b_1, b_2, ..., b_n\}$ is a cover of L, there exists b_m such that $b_m \wedge a \neq 0$. Hence $b_m \in V_\varepsilon(a)$, since $\{b_m\}$ is an $R(\varepsilon, b_m, b_m)$ chain from b_m to b_m. Choose those $b_i's$ such that $b_i \wedge a \neq 0$ and call them $b_{i_1}, b_{i_2}, ..., b_{i_k}$. Let $T = \{b_{i_1}, b_{i_2}, ..., b_{i_k}\}$, then T is non-empty. Now $b_{i_j} \in T$,

implies $b_{i_j} \in V_\varepsilon(a)$, thus we have that $a \leq \bigvee T$. Let

$$c_j = b_{i_j} \vee \bigvee\{x \in L \mid x \text{ connected }, x \wedge b_{i_j} \neq 0,\ d(x) < \frac{4l}{9}\}.$$

By local connectedness of L, $\{x \in L \mid x \text{ connected}, d(x) < \frac{l}{9}\}$ is a cover of L. Hence $S = \{x \in L \mid x \text{ connected }, x \wedge b_{i_j} \neq 0,\ d(x) < \frac{4l}{9}\}$ is non-empty. For any $x \in S$, $x \vee b_{i_j}$ is connected. Hence $0 \neq b_{i_j} \leq \bigwedge x \vee b_{i_j}$ implies that $\bigvee_{x \in S}(x \vee b_{i_j})$ is connected, and so c_j is connected. We will now show that $d(c_j) < l$, and $z = c_1 \vee ... \vee c_k$. Firstly,

<u>Claim 1</u>: $d(c_j) < l$.

Take any $\delta > 0$. By Lemma 2.3.7, since d is a metric diameter, there exists $u, v \leq c_j$, u, v connected and $d(u), d(v) < \delta$ and $d(c_j) - \delta < d(u \vee v)$. Now $u \wedge (b_{i_j} \vee x_1) \neq 0$ for some x_1, therefore this implies that either $u \wedge b_{i_j} \neq 0$ or $u \wedge x_1 \neq 0$. Similarly, $v \wedge b_{i_j} \neq 0$ or $v \wedge x_2 \neq 0$ for some suitable x_2.

Case 1: Assume $u \wedge b_{i_j} \neq 0$ and $v \wedge b_{i_j} \neq 0$, then $d(c_j) < \delta + d(u \vee v)$. Now

$$\begin{aligned} d(u \vee v \vee b_{i_j}) &\leq d(u) + d(v \vee b_{i_j}) \qquad (\text{since } u \wedge (v \vee b_{i_j}) \neq 0) \\ &\leq d(u) + d(v) + d(b_{i_j}) \\ &< 2\delta + \frac{l}{9}. \end{aligned}$$

This implies,

$$\begin{aligned} d(c_j) &< \delta + d(u \vee v) \\ &\leq \delta + d(u \vee v \vee b_{i_j}) \\ &\leq \delta + 2\delta + \frac{l}{9} \\ &= 3\delta + \frac{l}{9}. \end{aligned}$$

Case 2: Assume $u \wedge b_{i_j} \neq 0$ and $v \wedge x_2 \neq 0$, then

$$\begin{aligned} d((u \vee b_{i_j}) \vee (v \vee x_2)) &\leq d(u \vee b_{i_j}) + d(v \vee x_2) \qquad (\text{since } (u \vee b_{i_j}) \wedge (v \vee x_2) \neq 0) \\ &\leq d(u) + d(b_{i_j}) + d(v) + d(x_2) \\ &< \delta + \frac{l}{9} + \delta + \frac{4l}{9} \\ &= 2\delta + \frac{5l}{9}. \end{aligned}$$

Case 3: Assume $u \wedge x_1 \neq 0$ and $v \wedge x_2 \neq 0$, then

$$\begin{aligned} d(u \vee x_1 \vee b_{i_j} \vee (v \vee x_2)) &\leq d(u \vee x_1 \vee b_{i_j}) + d(v) + d(x_2) \\ &\leq d(u) + d(x_1) + d(b_{i_j}) + d(v) + d(x_2) \\ &< \delta + \frac{4l}{9} + \frac{l}{9} + \delta + \frac{4l}{9} = 2\delta + l. \end{aligned}$$

Thus in Case 1 $d(c_j) < 3\delta + \frac{l}{9}$, and in Case 2 and Case 3 we have that $d(c_j) < 2\delta + l$. Hence $d(c_j) < l$, as claimed.

We note that for any connected $x \in L$, such that $x \wedge b_{i_j} \neq 0$ and $d(x) < \frac{l}{9}$, we have that $d(x) < \frac{\varepsilon}{4}$. Now $\{x, b_{i_j}\}$ is an $R(\varepsilon, x, b_{i_j})$ chain with $b_{i_j} \wedge a \neq 0$, since $d(x) < \frac{\varepsilon}{4}$, $d(b_{i_j}) < \frac{\varepsilon}{4}$, and $2d(x) + 2d(b_{i_j}) < \frac{2l}{9} + \frac{2l}{9} = \frac{4l}{9} < \varepsilon$. Thus $x \in V_\varepsilon(a)$. So $x \leq \bigvee V_\varepsilon(a) = z$. Also, $b_{i_j} \in V_\varepsilon(a)$ implies that $b_{i_j} \leq z$, and so $x \vee b_{i_j} \leq z$, for all such $x \in L$. Therefore $c_j \leq z$ and thus $c_1 \vee c_2 \vee ... \vee c_k \leq z$ with c_j connected and $d(c_j) < l$.

It remains for us to show that $z \leq c_1 \vee c_2 \vee ... \vee c_k$. If we show this then $z = c_1 \vee c_2 \vee ... \vee c_k$ and hence $\downarrow z$ will have Property S.

Now, take any $x \in V_\varepsilon(a)$. Then x is connected and there exists a $R(\varepsilon, x, y)$ chain, $\{x = a_1, a_2, ..., a_n = y\}$, from x to some connected $y \in L$ such that $y \wedge a \neq 0$. Let $y = x_1, x_2 = a_2, x_3 = a_3, ..., x_{n-1} = a_{n-1}$, and $x_n = x$; that is, we shall consider the collection $\{y = x_1, x_2, ..., x_n = x\}$. If $x \leq b_{i_1} \vee b_{i_2} \vee ... \vee b_{i_k}$, then $x \leq c_1 \vee c_2 \vee ... \vee c_k$. So assume $x \nleq b_{i_1} \vee b_{i_2} \vee ... \vee b_{i_k}$. Choose p such that

$$\frac{1}{2}[\varepsilon - 2d(x_j) - \sum_{k=j+1}^{i-1} d(x_k)] > \frac{l}{9}, \tag{4.3}$$

is satisfied for $1 \leq j < i \leq p+1$, but such that equation (4.3) is not satisfied for $i = p+2$ and $j = M \leq p+1$. We now show the existence of p.

<u>Existence of p</u> :

For $i = 1$, we have that the left hand side of equation (4.3) is $\frac{1}{2} > \frac{l}{9}$. So equation (4.3) is satisfied.

For $i = 2$, the left hand side is $\frac{1}{2}[\varepsilon - 2d(x_1)] = \frac{1}{2}\varepsilon - d(x_1) > \frac{1}{2}\varepsilon - \frac{\varepsilon}{4} = \frac{\varepsilon}{4}$. But $\frac{\varepsilon}{4} > \frac{l}{9}$, hence equation (4.3) is satisfied.

If equation (4.3) is satisfied for all i where $1 \leq i \leq n+1$, then for $1 \leq j < i = n+1$, we have

$$\frac{1}{2}[\varepsilon - 2d(x_j) - \sum_{k=j+1}^{n} d(x_k)] > \frac{l}{9}. \tag{4.4}$$

Now since $\{b_1, b_2, ..., b_n\}$ is a cover of L, $x \leq \bigvee\{b_q \mid x \wedge b_q \neq 0\}$. Hence there exists a b_q such that $x \wedge b_q \neq 0$ and $b_q \notin \{b_{i_1}, b_{i_2}, ..., b_{i_k}\}$. By Lemma 4.3.3, we see that equation (4.4) implies that $b_q \in V_\varepsilon(a)$, but this implies that b_q is some b_{i_j}, which is a contradiction. Hence there must be such a p as in equation (4.3).

Let $w = x_{p+1} \vee x_{p+2} \vee ... \vee x_n$. We note that w is connected, since $\{x_{p+1}, ..., x_n\}$ is an $R(\varepsilon, x_{p+1}, x_n)$ chain.

<u>Claim 2</u>: $d(w) < \frac{4l}{9}$.

(1) We will show that $M \neq p+1$:

If $M = p+1$, then equation (4.3) is not satisfied for $j = M = p+1$ and $i = p+2$, so $\frac{1}{2}[\varepsilon - 2d(x_{p+1})] \leq \frac{l}{9} < \frac{\varepsilon}{4}$. Now $d(x_{p+1}) < \frac{\varepsilon}{4}$. Thus $\varepsilon - 2d(x_{p+1}) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$ and $\frac{1}{2}[\varepsilon - 2d(x_{p+1})] > \frac{\varepsilon}{4}$, which is a contradiction.

(2) We now show that $d(x_{p+1}) < \frac{2l}{9}$:

We have from equation (4.3) that for $j = M \leq p+1$ and $i = p+2$, that

$$\frac{1}{2}[\varepsilon - 2d(x_M) - \sum_{k=M+1}^{p+1} d(x_k)] \leq \frac{l}{9}. \tag{4.5}$$

Now $M \neq p+1$ implies that $M < p+1$, so

$$\varepsilon - 2d(x_M) - 2d(x_{p+1}) - \sum_{k=M+1}^{p} d(x_k) > 0. \tag{4.6}$$

Multiplying equation (4.5) by 2, gives us

$$\varepsilon - 2d(x_M) - \sum_{k=M+1}^{p+1} d(x_k) \leq \frac{2l}{9}, \tag{4.7}$$

and equation (4.7) - equation (4.6) yields,

$$-d(x_{p+1}) + 2d(x_{p+1}) < \frac{2l}{9}.$$

Hence $d(x_{p+1}) < \frac{2l}{9}$, as required.

(3) Lastly, we show $\sum_{k=p+2}^{n} d(x_k) < \frac{2l}{9}$:

We know that,

$$\varepsilon - 2d(x_M) - 2d(x_n) - \sum_{k=M+1}^{n-1} d(x_k) > 0. \tag{4.8}$$

Therefore, equation (4.7) - equation (4.8) yields,

$$2d(x_n) - \sum_{k=p+1}^{n-1} d(x_k) < \frac{2l}{9}.$$

Thus $\sum_{k=p+2}^{n} d(x_k) < \frac{2l}{9}$.

From (1), (2), and (3), we obtain

$$\begin{aligned} d(w) &\leq d(x_{p+1}) + d(x_{p+2}) + ... + d(x_n) \\ &= d(x_{p+1}) + \sum_{k=p+2}^{n} d(x_k) \\ &< \frac{2l}{9} + \frac{2l}{9} = \frac{4l}{9}, \end{aligned}$$

as claimed.

We now show that $w \wedge b_{i_j} \neq 0$ for some b_{i_j}. From equation (4.3), we know that for $1 \leq j \leq p$,

$$\frac{l}{9} < \frac{1}{2}[\varepsilon - 2d(x_j) - \sum_{k=j+1}^{p} d(x_k)],$$

and we have $\{y = x_1, x_2, ..., x_p, w\}$, where $w = x_{p+1} \vee x_{p+2} \vee ... \vee x_n$. Now $x_p \leq \bigvee\{b_i \mid x_p \wedge b_i \neq 0\}$ and $w \wedge x_p \neq 0$, since $x_p \wedge x_{p+1} \neq 0$. Hence $w \wedge \bigvee\{b_i \mid x_p \wedge b_i \neq 0\} \neq 0$, and therefore there exists b_i such that $x_p \wedge b_i \neq 0$ and $w \wedge b_i \neq 0$. By Lemma 4.3.3, $x_p \wedge b_i \neq 0$ must imply that $b_i \in V_\varepsilon(a)$. So $b_i = b_{i_j}$ for some j. Thus $w \wedge b_{i_j} \neq 0$, w is connected and $d(w) < \frac{4l}{9}$. Hence $w \leq c_j$ and so $x \leq c_j$. Thus $z \leq c_1 \vee c_2 \vee ... \vee c_k$ and so $z = c_1 \vee c_2 \vee ... \vee c_k$. Hence $\downarrow z$ has Property S, with respect to d.

□

Proposition 4.3.5. *(L, d) has Property S iff $(L, \tilde{\rho})$ has Property S.*

Proof.
($\Longrightarrow$) Let $\varepsilon > 0$ be arbitrary and suppose that (L, d) has Property S. Then, there exists a cover $\{a_i\}_{i=1}^n$ of L, consisting of connected elements such that $d(a_i) < \frac{\varepsilon}{4}$ for $i = 1, ..., n$. Since every a_i is connected, it follows from Lemma 4.1.7 that $\rho(a_i) \leq 4d(a_i) < \varepsilon$, for each $i = 1, ..., n$. Thus $\tilde{\rho}(a_i) < \varepsilon$ for each i, since it is known that $\tilde{\rho} \leq \rho$.

($\Longleftarrow$) Let $\varepsilon > 0$ be arbitrary and suppose that $(L, \tilde{\rho})$ has Property S. Then there exists a cover $\{a_i\}_{i=1}^n$ of L, consisting of connected elements such that $\tilde{\rho}(a_i) < \varepsilon$ for $i = 1, ..., n$. Now $d \leq \tilde{\rho}$, by Lemma 4.1.10. Hence for $i = 1, ..., n$, $d(a_i) < \varepsilon$, and thus (L, d) has Property S.

□

Corollary 4.3.6. *If $(L, \tilde{\rho})$ has Property S then $\downarrow z$ has Property S with respect to $\tilde{\rho}$, where $z = \bigvee V_\varepsilon(a)$, for $0 \neq a \in L$ and $\varepsilon > 0$.*

Proof.

Let l be arbitrary and suppose that $(L, \tilde{\rho})$ has Property S, then Proposition 4.3.5 implies that (L, d) has Property S. By Theorem 4.3.4, it follows that $\downarrow z$ has Property S with respect to d. So there exists $\{a_1, ..., a_n\}$ where a_i is connected in L, $a_i \leq z$, $d(a_i) < \frac{l}{4}$ for $i = 1, ..., n$ and $z = \bigvee_{i=1}^n a_i$. Hence by Lemma (4.1.7) and the definition of $\tilde{\rho}$ we see that,

$$\tilde{\rho}(a_i) \leq \rho(a_i) \leq 4d(a_i) < l.$$

Thus $\downarrow z$ has Property S with respect to $\tilde{\rho}$.

□

Proposition 4.3.7. *Given any $a \in (L, d)$ and $\varepsilon > 0$ there exists z such that $a \leq z$, and $d(z) \leq d(a) + 3\varepsilon$, with $\downarrow z$ having Property S (with respect to d).*

Proof.

By Lemma 4.3.2 and Theorem 4.3.4, for $a \in L$, we have $a \leq \bigvee V_\varepsilon(a) = z$ and $\downarrow z$ has Property S with respect d. Take any $\delta > 0$, with $\delta < \frac{\varepsilon}{4}$. L is locally connected and d is a metric diameter, therefore by Lemma 2.3.7 there exists u, v connected in L, $u, v \leq z$, $d(u), d(v) < \delta$, and $d(z) - \delta < d(u \vee v)$. Now we have $\{u, x = x_1, x_2, ..., x_n\}$ and $x_n \wedge a \neq 0$, where $\{x_1, ..., x_n\}$ is an $R(\varepsilon, x_1, x_n)$ chain, and $\{v, y = y_1, y_2, ..., y_m\}$ and $y_m \wedge a \neq 0$, where

$\{y_1, ..., y_m\}$ is an $R(\varepsilon, y_1, y_m)$ chain.

$$\begin{aligned}
d(z) &< \delta + d(u \vee v) \\
&\leq \delta + d((u \vee x_1 \vee ... \vee x_n) \vee a \vee (v \vee y_1 \vee ... \vee y_m)) \\
&\leq \delta + d(u \vee x_1 \vee ... \vee x_n) + d(a) + d(v \vee y_1 \vee ... \vee y_m) \\
&\leq \delta + d(u) + d(x_1) + ... + d(x_n) + d(a) + d(v) + d(y_1) + ... + d(y_m) \\
&< \delta + d(u) + \varepsilon + d(a) + d(v) + \varepsilon \\
&< \delta + \delta + \varepsilon + d(a) + \delta + \varepsilon \\
&= 3\delta + d(a) + 2\varepsilon \\
&< \frac{3\varepsilon}{4} + d(a) + 2\varepsilon \\
&< d(a) + 3\varepsilon
\end{aligned}$$

□

Theorem 4.3.8. *$(L, \tilde{\rho})$ is uniformly locally connected.*

Proof.

Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{36}$. Take $a \in L$ with $\tilde{\rho}(a) < \delta$. By Lemma 2.3.6, since L is locally connected then

$$\tilde{\rho}(a) = \inf_{\varepsilon > 0} \sup\{\rho(u \vee v) \mid u, v \text{ connected}, \ u, v \leq a, \text{ and } \rho(u), \rho(v) < \varepsilon\}.$$

So there exists $\varepsilon' > 0$ such that for all connected $u, v \in L$, $u, v \leq a$, $\rho(u), \rho(v) < \varepsilon'$ and $\rho(u \vee v) < \delta$ (we may assume that $\varepsilon' < \frac{\varepsilon}{36}$). Fix $u \leq a$, u is connected and $\rho(u) < \varepsilon'$. Then for all $v \leq a$, v connected and $\rho(v) < \varepsilon'$, $\rho(u \vee v) < \delta$. For each such v, let T_v be a $R(\beta_v)$ cover of $u \vee v$, where $\beta_v < \delta$. Now $u, v \leq u \vee v \leq \bigvee T_v$, so there exists connected $c_v, s_v \in T_v$ such that $u \wedge c_v \neq 0$ and $v \wedge s_v \neq 0$. Since T_v is an $R(\beta_v)$ cover of $u \vee v$, and $c_v, s_v \in T_v$, there exists an $R(\beta_v, c_v, s_v)$ chain in T_v, say, $a_1^v, a_2^v, ..., a_n^v$, where $a_1^v = c_v$, $a_n^v = s_v$ and $a_i^v \wedge a_{i+1}^v \neq 0$ for $i = 1, .., n-1$. Let $z_v = u \vee a_1^v \vee a_2^v \vee ... \vee a_n^v \vee v$. Let $c = \bigvee\{z_v \mid v \leq a, v \text{ is connected and } \rho(v) < \varepsilon\}$. c is connected since $u \leq z_v$, for all such

v. We now show that $a \leq c$. Now $U^d_{\frac{\varepsilon'}{4}}$ is a cover of L, thus

$$\begin{aligned} a &= a \wedge \bigvee U^d_{\frac{\varepsilon'}{4}} \\ &= \bigvee \{a \wedge s \mid d(s) < \frac{\varepsilon'}{4}\}. \end{aligned}$$

By local connectedness of L, a is a join of connected elements $x \in L$ such that $d(x) < \frac{\varepsilon'}{4}$. For each such x, $x \leq z_x$, and $\rho(x) < \varepsilon'$. Hence $a \leq c$. We note that,

$$\begin{aligned} d(z_v) &\leq d(u) + d(a_1^v) + d(a_2^v) + ... + d(a_n^v) + d(v) \\ &< \beta_v + \varepsilon' + \varepsilon' \\ &< \delta + 2\varepsilon'. \end{aligned}$$

We now show that $\tilde{\rho}(c) < \varepsilon$. By Lemma 2.3.7, there exists $s, t \leq c$, $d(s), d(t) < \varepsilon'$ and $d(c) - \varepsilon' < d(s \vee t)$. Now $s, t \leq c$, hence there exists z_{v1}, z_{v2} such that $v_1, v_2 \leq a$ and connected, $s \wedge z_{v1} \neq 0$, $t \wedge z_{v2} \neq 0$, $\rho(v_1), \rho(v_2) < \varepsilon'$. So,

$$\begin{aligned} d(c) <& \ \varepsilon' + d(s \vee t) \\ \leq& \ \varepsilon' + d(s \vee z_{v1} \vee z_{v2} \vee t) \\ \leq& \ \varepsilon' + d(s) + d(z_{v1}) + d(z_{v2}) + d(t) \quad \text{(since } u \vee v \leq z_{v1}, z_{v2}\text{)} \\ <& \ \varepsilon' + \varepsilon' + (\delta + 2\varepsilon') + (\delta + 2\varepsilon') + \varepsilon' \\ <& \ \frac{7\varepsilon}{36} + \frac{\varepsilon}{36} + \frac{\varepsilon}{36} = \frac{9\varepsilon}{36} = \frac{\varepsilon}{4}. \end{aligned}$$

Thus by Lemma 4.1.7 and since c is connected, $\rho(c) \leq 4d(c) < \varepsilon$. Hence $\tilde{\rho}(c) < \varepsilon$, and so we have shown that $(L, \tilde{\rho})$ is uniformly locally connected.

□

The significance of the constructed compatible metric diameter $\tilde{\rho}$, from the given d, on a connected locally connected frame L, is that it has much stronger properties than the original d. This is exemplified in the three results which follow. We believe these results would be relevant in further developing the theory of *Peano* frames; that is, the compact, connected, locally connected and metrizable frames.

Proposition 4.3.9. *Let (L, d) be connected and have Property S. Then there exists a compatible metric diameter $\tilde{\rho}$ on L such that*

(1) $(L, \tilde{\rho})$ has Property S.

(2) If $\varepsilon > 0$, then for all $a \in L$ with $\tilde{\rho}(a) < \frac{\varepsilon}{4}$, there exists z connected, $a \leq z$ with $\tilde{\rho}(z) < \varepsilon$ and $\downarrow z$ having Property S with respect to $\tilde{\rho}$.

Proof.

(1) Follows from Proposition 4.3.5.

(2) Take $\varepsilon > 0$ and $a \in L$ such that $\tilde{\rho}(a) < \frac{\varepsilon}{4}$. Then by Lemma 4.1.10, $d(a) \leq \tilde{\rho}(a) < \frac{\varepsilon}{4}$. Choose $\varepsilon' > 0$ such that $d(a) + 3\varepsilon' < \frac{\varepsilon}{4}$. By Proposition 4.3.7 there exists z such that $a \leq z$, z connected and $d(z) \leq d(a) + 3\varepsilon' < \frac{\varepsilon}{4}$ and $\downarrow z$ has Property S with respect to d. Then since z is connected $\tilde{\rho}(z) \leq 4d(z) < \varepsilon$, and so $\downarrow z$ has Property S with respect to $\tilde{\rho}$.

□

Corollary 4.3.10. *Let (L, d) be connected and have Property S. Then there exists a compatible metric diameter $\tilde{\rho}$ such that $(L, \tilde{\rho})$ has Property S, and for every $\varepsilon > 0$ there exists a finite number of connected elements $z_1, ..., z_n$ such that $\bigvee_{i=1}^{n} z_i = 1$, $\tilde{\rho}(z_i) < \varepsilon$ for each i, and $\downarrow z_i$ has Property S with respect to $\tilde{\rho}$ for each i.*

Proof.

Take any $\varepsilon > 0$. Then $(L, \tilde{\rho})$ has Property S, by (1) Proposition 4.3.9. Hence there exists connected $a_1, ..., a_n$ in L such that $\bigvee_{i=1}^{n} a_i = 1$ and $\tilde{\rho}(a_i) < \frac{\varepsilon}{4}$ for each i. By (2) of Proposition 4.3.9, for each i there exists connected z_i such that $a_i \leq z_i$, $\tilde{\rho}(z_i) < \varepsilon$ with $\downarrow z_i$ having Property S with respect to $\tilde{\rho}$.

□

Definition 4.3.11. Let S be a sublocale of a frame L. S is called a *continuum* if S is compact, connected and locally connected.

Corollary 4.3.12. *Let (L, d) be compact, connected and locally connected. Then there exists a compatible metric diameter $\tilde{\rho}$ on L such that*

(1) $(L, \tilde{\rho})$ has Property S.

(2) For every $\varepsilon > 0$, L is a finite join of open connected sublocales $S_1, ..., S_n$, with diam$(S_i) < \varepsilon$ (where diam(S_i) is the diameter of S_i inherited from $\tilde{\rho}$) and such that S_i has Property S with respect to $\tilde{\rho}$, for each i.

(3) For each $\varepsilon > 0$, $L = C_1 \vee ... \vee C_n$, where each C_i is a continuum, each C_i has Property S and diam$(C_i) < \varepsilon$ for each i (where diam(C_i) is the diameter of C_i inherited from $\tilde{\rho}$).

Proof.

(1) (L, d) has Property S by Theorem 3.1.8. Hence it follows by Proposition 4.3.5 that $(L, \tilde{\rho})$ has Property S .

(2) Take $\varepsilon > 0$. Then by Corollary 4.3.10, since (L, d) has Property S, there exists connected elements $z_1, ..., z_n$ such that $\bigvee_{i=1}^{n} z_i = 1$, $\tilde{\rho}(z_i) < \varepsilon$ for each i, and $\downarrow z_i$ has Property S with respect to $\tilde{\rho}$ for each i. Then $L = \mathfrak{o}(z_1) \vee ... \vee \mathfrak{o}(z_n)$, where each $\mathfrak{o}(z_i)$ is a open connected sublocale, and by Proposition 3.2.7, diam$(\mathfrak{o}(z_i)) = \tilde{\rho}(z_i) < \varepsilon$. Since $\downarrow z_i \cong \mathfrak{o}(z_i)$, it follows that $\mathfrak{o}(z_i)$ has Property S for each i.

(3) From (2), it follows that $L = \overline{\mathfrak{o}(z_1)} \vee ... \vee \overline{\mathfrak{o}(z_n)}$, where by Proposition 2.2.29 each $\overline{\mathfrak{o}(z_i)}$ is connected. Since each $\overline{\mathfrak{o}(z_i)}$ is a closed sublocale and L is compact, Proposition 2.2.13 implies that each $\overline{\mathfrak{o}(z_i)}$ is compact. Furthermore, since each $\mathfrak{o}(z_i)$ has Property S and because closures preserve Property S, each $\overline{\mathfrak{o}(z_i)}$ has Property S . Hence each $\overline{\mathfrak{o}(z_i)}$ is locally connected by Proposition 3.1.2, and it follows from Corollary 3.2.11 that diam$(\overline{\mathfrak{o}(z_i)}) =$ diam$(\mathfrak{o}(z_i)) < \varepsilon$, for each i.

$\square$

Chapter 5

On Dense Metric Sublocales of Metric Frames and S-metrizability

In this chapter we discuss and provide equivalent characterisations of S-metrizability for metric frames. In [14], Garcia-Maynez showed that S-metrizability of a topological space X is equivalent to the space having a perfect locally connected metrizable compactification. In order to generalise this result, we begin with a discussion on metric frame completions and compactifications of frames. Following this, we present a study on dense metric sublocales, and later provide an intrinsic characterisation of S-metrizability in terms of the Wallman basis of a frame.

5.1 Compactifications

This section is intended to be an introduction to compactifications of frames, for the purpose of ensuring that the required theory for this chapter is presented. In particular, we will discuss the Wallman compactification. The following definitions appear in [6].

First recall that a frame homomorphism $h : L \longrightarrow M$ is called dense if, $h(x) = 0_M$ implies that $x = 0_L$.

Definition 5.1.1. A *compactification* of a frame M is a compact regular frame L together with a dense onto homomorphism $h : L \longrightarrow M$, denoted by (L, h).

Definition 5.1.2. Let $h : L \longrightarrow M$ be a compactification of M, and $h_* : M \longrightarrow L$ be the right adjoint of h. Then the compactification (L, h) is said to be *perfect* with respect to an element $u \in M$, if

$$h_*(u \vee u^*) = h_*(u) \vee h_*(u^*).$$

The compactification (L, h) is said to be a *perfect compactification* of M, if it is perfect with respect to every element $u \in M$.

Theorem 5.1.3 ([6]). *Let $h : L \longrightarrow M$ be a compactification of M, and h_* be the right adjoint of h. Then $h : L \longrightarrow M$ is a perfect compactification if and only if h_* preserves disjoint binary joins.*

Proposition 5.1.4. *Let $h : L \longrightarrow M$ be a* perfect compactification *of M. If $x \in L$ is connected, then $h(x)$ is connected in M.*

Proof.
Suppose that $x \in L$ is connected. Since $h : L \longrightarrow M$ is a *perfect compactification* of M, then by Theorem 5.1.3, h_* preserves disjoint binary joins. Hence by Proposition 2.1.33, $h(x)$ is connected in M.

$\square$

Banaschewski [7] introduced the concept of a *strong inclusion* on a frame and showed that every strong inclusion on a frame M gives rise to a compactification of M, and vice versa. We discuss this next:

Definition 5.1.5. A *strong inclusion* on a frame M is a binary relation $\blacktriangleleft$ on M such that:

1. if $x \leq a \blacktriangleleft b \leq y$ then $x \blacktriangleleft y$,

2. $\blacktriangleleft$ is a sublattice of $M \times M$,

3. $a \blacktriangleleft b \Longrightarrow a \prec b$,

4. $a \blacktriangleleft b \Longrightarrow a \prec c \prec b$, for some $c \in M$,

5. $a \blacktriangleleft b \Longrightarrow b^* \blacktriangleleft a^*$,

6. for each $a \in M$, $a = \bigvee\{x \in M \mid x \blacktriangleleft a\}$.

Let $S(M)$ be the set of all strong inclusions on M. Let $K(M)$ be the set of all compactifications of M, partially ordered by $(L, h) \leq (N, f)$ if and only if there exists a frame homomorphism $g : L \longrightarrow N$ making the following diagram commute:

$$\begin{tikzcd} L \arrow[r, "g"] \arrow[d, "h"'] & N \arrow[d, "f"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

Banaschewski [7], showed that $K(M)$ is isomorphic to $S(M)$ by defining maps $K(M) \longrightarrow S(M)$ and $S(M) \longrightarrow K(M)$, which are inverses of each other and are order preserving.

The map $K(M) \longrightarrow S(M)$ is defined as follows: For a compactification $h : L \longrightarrow M$ with right adjoint h_*, define for any $x, y \in M$, $x \blacktriangleleft y$ to mean that $h_*(x) \prec h_*(y)$. Then $\blacktriangleleft$ is a strong inclusion on M. Thus every compactification on M gives rise to a strong inclusion on M.

For the map $S(M) \longrightarrow K(M)$, let $\blacktriangleleft$ be any strong inclusion on M. Let γM be the set of all strongly regular ideals of M (That is, the ideals J of M such that $x \in J$ implies there exists $y \in J$ with $x \blacktriangleleft y$). Then the join map $\bigvee : \gamma M \longrightarrow M$ is dense and onto and γM is a regular subframe of the frame of ideals of M, $\mathcal{I}(M)$. Hence $\bigvee : \gamma M \longrightarrow M$ is a compactification of M associated with the given $\blacktriangleleft$.

Theorem 5.1.6 ([7]). *Let M be a frame. Let (L, h) be a compactification of M associated with strong inclusion $\blacktriangleleft_1$, and let (N, f) be a compactification of M associated with strong inclusion $\blacktriangleleft_2$. If $\blacktriangleleft_1 = \blacktriangleleft_2$, then $L \cong N$.*

We now study a particular case of compactifications, the Wallman compactification, as presented in [4]. The Wallman compactification for frames was first introduced by Johnstone [16].

Definition 5.1.7. For any frame M, $B \subseteq M$ is called a *Wallman basis* of M if:

1. The bottom and top elements of M are in B, and $a, b \in B$ implies that $a \vee b \in B$ and $a \wedge b \in B$.

2. For every $a \in M$, $a = \bigvee\{b \in B \mid b \prec_B a\}$, where $b \prec_B a$ means that there exists $c \in B$ such that $b \wedge c = 0$ and $c \vee a = 1$.

3. For $a, b \in B$ such that $a \vee b = 1$, there exists $c, d \in B$ such that $c \wedge d = 0$ and $a \vee c = b \vee d = 1$.

Proposition 5.1.8 ([4]). *Let M be a regular frame and B a Wallman basis for M. Define $a \blacktriangleleft_B b$ in M by*

$$a \blacktriangleleft_B b \text{ iff there exists } c \in B \text{ such that } a \prec_B c \prec_B b.$$

Then $\blacktriangleleft_B$ is a strong inclusion on M.

From Proposition 5.1.8, the corresponding compactification associated with this Wallman basis B, denoted $\gamma_B M$, is called the *Wallman compactification* of M. Here $\gamma_B M$ consists of all strongly regular ideals of M associated with $\blacktriangleleft_B$ and we have the join map $\bigvee : \gamma_B M \longrightarrow M$.

Proposition 5.1.9 ([4]). *Let B be a Wallman basis of M, then $k(B)$ is a basis for $\gamma_B M$ where $k : M \longrightarrow \gamma_B M$ is the right adjoint of $\bigvee : \gamma_B M \longrightarrow M$.*

5.2 The completion of metric frames

Using the approach of Banaschewski and Pultr [9], we shall define a metric frame completion from a uniform frame completion. To do this, we will present the definition of a complete uniform frame and discuss the completion of a uniform frame in the terminology of Kříž [19].

Recall the definition of a *uniformity* on a frame L, as presented earlier in Chapter 2 (Definition 2.3.9). We note that a frame L together with a specified uniformity, $\mathcal{U}L$, is a *uniform frame*. We denote a uniform frame by the pair $(L,\mathcal{U}L)$. Any element $U \in \mathcal{U}L$ is called a *uniform cover* of L.

Definition 5.2.1. For uniform frames $(L,\mathcal{U}L)$ and $(M,\mathcal{U}M)$, a *uniform frame homomorphism* $h : L \longrightarrow M$ is a frame homomorphism such that whenever $A \in \mathcal{U}L$, then $h[A] = \{h(a) \mid a \in A\} \in \mathcal{U}M$.

Definition 5.2.2. Let $(L,\mathcal{U}L)$ and $(M,\mathcal{U}M)$ be uniform frames. The map $h : L \longrightarrow M$ is called a *uniform frame surjection* if, h is an onto uniform frame homomorphism such that the covers $h[A]$ generate $\mathcal{U}M$, for $A \in \mathcal{U}L$.

Definition 5.2.3.

1. A uniform frame $(M,\mathcal{U}M)$ is *complete* whenever any dense uniform frame surjection $h : L \longrightarrow M$ is an isomorphism.

2. A *uniform frame completion* of M is a complete uniform frame L together with a dense surjection $h : L \longrightarrow M$.

Definition 5.2.4. A uniformity is said to be *totally bounded* if it has a basis consisting of some of the finite covers.

Proposition 5.2.5 ([10]). *Let $(M,\mathcal{U}M)$ be a uniform frame. $\mathcal{U}M$ is totally bounded if and only if the completion of $(M,\mathcal{U}M)$ is compact.*

From an earlier discussion in Chapter 2, we have seen that every metric frame is a uniform frame, since the covers of a metric frame define a uniformity. Hence every metric frame has a uniform frame completion. Using this approach, of obtaining a uniform frame completion for a metric frame, Banaschewski and Pultr [9] showed the following:

Theorem 5.2.6. *Any metric frame M has an essentially unique completion.*

We now present the definitions for a metric frame completion which naturally arises from the definition of a complete metric space.

Definition 5.2.7. A frame homomorphism $h : (L, \rho) \longrightarrow (M, d)$ between metric frames is called a *dense surjection* if, h is dense, onto, and $\rho(a) = d(h(a))$, for $a \in L$.

Remark 5.2.8. We note that for any $a \in L$, $\rho(a) = d(h(a))$, from Definition 5.2.7, defines a metric diameter on L. This fact is found in [9].

Definition 5.2.9.

1. A metric frame M is called *complete* if, any dense surjection $h : L \longrightarrow M$ is an isomorphism.
2. For any metric frame M, a complete metric frame L together with a dense surjection $h : L \longrightarrow M$ is called a *metric frame completion* of M

Remark 5.2.10. Since every metric frame has an essentially unique completion, throughout the rest of this thesis, given any metric frame (M, d), we shall denote its metric frame completion by $(\tilde{M}, \tilde{d})$, where $h : (\tilde{M}, \tilde{d}) \longrightarrow (M, d)$ is a dense surjection with $\tilde{d}(a) = d(h(a))$, for $a \in L$.

5.3 Dense metric sublocales

This section is dedicated to studying properties of dense metric sublocales, for the purpose of generalising a result by Garcia-Maynez [14]. Amongst other properties, we show that every metric frame is a dense metric sublocale of its metric frame completion and we show that metric frames are perfect extensions of their uniformly locally connected dense metric sublocales.

We first recall the following definition from Pultr and Picado [20].

Definition 5.3.1 ([20])**.** Let (L, ρ) be a metric frame and $h : L \longrightarrow M$ be an onto frame homomorphism. For $a \in M$, let

$$d(a) = \inf\{\rho(x) \mid a \leq h(x), x \in L\},$$

then d is a compatible metric diameter on M, and (M, d) is called a *metric sublocale* of (L, ρ).

Recall that a frame homormorphism $h : L \longrightarrow M$ is dense, if whenever $h(x) = 0_M$, then $x = 0_L$. This merits the following definition.

Definition 5.3.2. Let (M, d) be a *metric sublocale* of (L, ρ) with onto frame homomorphism $h : (L, \rho) \longrightarrow (M, d)$. If h is dense, then (M, d) is called a *dense metric sublocale* of (L, ρ).

The following fact, although simple to prove, is important to take note of as it is frequently required later.

Lemma 5.3.3. *Let $h : L \longrightarrow M$ be a dense frame homomorphism, $c \in M$ be connected and h_* be the right adjoint of h . Then $h_*(c)$ is connected in L.*

Proof.
Let $c \in M$ be connected. Will show that $h_*(c)$ is connected in L. Let $h_*(c) = s \vee t$, where $s \wedge t = 0_L$ for $s, t \in L$. Then by Fact 2.1.14, $c = h(s \vee t) = h(s) \vee h(t)$, and we have that $h(s) \wedge h(t) = 0_M$. Since $c \in M$ is connected, then $h(s) = 0_M$, say. Hence $s = 0_L$ because h is a dense map. Thus $h_*(c)$ is connected in L.

□

Proposition 5.3.4. *Let (M, d) be a dense* metric sublocale *of (L, ρ). If (M, d) is uniformly locally connected then (L, ρ) is uniformly locally connected.*

Proof.

Let $\varepsilon > 0$ be given and suppose that (M, d) is uniformly locally connected. Since (M, d) is a dense metric sublocale of (L, ρ), then we have a dense onto frame homomorphism $h : (L, \rho) \longrightarrow (M, d)$, with $d(a) = \inf\{\rho(x) \mid a \leq h(x)\}$, for $a \in M$. (M, d) is uniformly locally connected, hence there exists $\delta > 0$ such that if $d(a) < \delta$ then there exists a connected $c \in M$ such that $a \leq c$ and $d(c) < \varepsilon$. Take $a \in L$ such that $\rho(a) < \delta$. Now $h(a) \in M$ and $d(h(a)) \leq \rho(a) < \delta$. Thus by the uniform local connectedness of (M, d), there exists a connected $c \in M$ such that $h(a) \leq c$ and $d(c) < \varepsilon$. Now $h_*(h(a)) \leq h_*(c)$, therefore by Fact 2.1.14, $a \leq h_*(c)$. It follows that $h_*(c)$ is connected in L, by Lemma 5.3.3. We now show that $\rho(h_*(c)) < \varepsilon$. By the definition of d on M, $d(c) < \varepsilon$ implies that there exists $y \in L$ such that $c \leq h(y)$ and $d(c) \leq \rho(y) < \varepsilon$. Now $hh_*(c) = c$, therefore $(hh_*(c)) \wedge (h(y))^* \leq h(y) \wedge (h(y))^* = 0_M$. Hence $(hh_*(c)) \wedge (h(y))^* = 0_M$, and this implies

$$
\begin{aligned}
& (hh_*(c)) \wedge h(y^*) = 0_M \quad \text{(By Proposition 2.1.22)} \\
\Longrightarrow\ & h(h_*(c) \wedge y^*) = 0_M \\
\Longrightarrow\ & h_*(c) \wedge y^* = 0_L \quad \text{(Since } h \text{ is dense)} \\
\Longrightarrow\ & h_*(c) \leq y^{**}
\end{aligned}
$$

Thus $\rho(h_*(c)) \leq \rho(y^{**}) = \rho(y) < \varepsilon$, by Proposition 2.3.8. Hence (L, ρ) is uniformly locally connected.

□

Recall that given any metric frame (M, d), we denote its metric frame completion by $(\tilde{M}, \tilde{d})$, where $h : (\tilde{M}, \tilde{d}) \longrightarrow (M, d)$ is a dense surjection with $\tilde{d}(a) = d(h(a))$, for $a \in \tilde{M}$. We now show that every metric frame (M, d) is a dense metric sublocale of its metric frame completion.

Theorem 5.3.5. *Let (M, d) be a metric frame. (M, d) is a dense metric sublocale of its metric frame completion $(\tilde{M}, \tilde{d})$, where $h : \tilde{M} \longrightarrow M$ is a dense onto frame homomor-*

phism with $\tilde{d}(a) = d(h(a))$*, for* $a \in \tilde{M}$*.*

Proof.

Define $\sigma : M \longrightarrow \mathbb{R}_+$, by $\sigma(a) = \inf\{\tilde{d}(x) \mid a \leq h(x)\}$, for $a \in M$. By [20], we know that σ is a compatible metric diameter on M. Hence (M, σ) is a metric frame and (M, σ) is a metric sublocale of $(\tilde{M}, \tilde{d})$, since Definition 5.3.1 is satisfied. To see why $\sigma = d$, take any $a \in M$, then

$$\begin{aligned} \sigma(a) &\leq \tilde{d}(h_*(a)) \quad \text{(since } a \leq hh_*(a).\text{)} \\ &= d(hh_*(a)) \\ &= d(a). \quad \text{(by Fact 2.1.14, since } h \text{ is dense and onto.)} \end{aligned}$$

Now take any $x \in \tilde{M}$ such that $a \leq h(x)$, then $d(a) \leq d(h(x)) = \tilde{d}(x)$. This implies that $d(a) \leq \inf\{\tilde{d}(x) \mid a \leq h(x)\} = \sigma(a)$. Thus for any $a \in M$, $\sigma(a) = d(a)$. So (M, d) is a dense metric sublocale of its completion $(\tilde{M}, \tilde{d})$.

□

Corollary 5.3.6. *Let* (M, d) *be a metric frame. If* (M, d) *is uniformly locally connected then its metric frame completion is uniformly locally connected.*

Proof.

Suppose that (M, d) is uniformly locally connected. By Theorem 5.3.5, (M, d) is a dense metric sublocale of its metric frame completion, $(\tilde{M}, \tilde{d})$. It follows from Proposition 5.3.4, that $(\tilde{M}, \tilde{d})$ is uniformly locally connected.

□

The next result gives an equivalent and useful criterion for determining whether a metric frame is a dense metric sublocale of some other metric frame.

Theorem 5.3.7. *Let* (M, d) *and* (L, ρ) *be metric frames with dense onto frame homomorphism* $h : L \longrightarrow M$*.* (M, d) *is a dense metric sublocale of* (L, ρ) *if and only if for all* $a \in L$*,* $\rho(a) = d(h(a))$*.*

Proof.

($\Longrightarrow$)Let (M, d) be a dense metric sublocale of (L, ρ), then we have a dense onto frame homomorphism $h : (L, \rho) \longrightarrow (M, d)$ with $d(x) = \inf\{\rho(y) \mid x \leq h(y)\}$, for $x \in M$. We will show that $d(h(y)) = \rho(y)$, for all $y \in L$. Since $h(y) \leq h(y)$, for all $y \in L$, we must have that $d(h(y)) \leq \rho(y)$. It remains for us to show that

$$\rho(y) \leq d(h(y)) = \inf\{\rho(x) \mid h(y) \leq h(x)\}, \text{ for all } y \in L.$$

Let $y \in L$ be arbitrary, we will show that $\rho(y) \leq \rho(x)$ for all $x \in L$ such that $h(y) \leq h(x)$. Now $h(y) \leq h(x)$ implies that $h(y) \wedge (h(x))^* \leq h(x) \wedge (h(x))^* = 0_M$. Therefore

$$\begin{aligned} & h(y) \wedge (h(x))^* = 0_M \\ \Longrightarrow\ & h(y) \wedge h(x^*) = 0_M \quad \text{(By Proposition 2.1.22)} \\ \Longrightarrow\ & h(y \wedge x^*) = 0_M \\ \Longrightarrow\ & y \wedge x^* = 0_L \quad \text{(Since } h \text{ is dense)} \\ \Longrightarrow\ & y \leq x^{**}. \end{aligned}$$

By Proposition 2.3.8, $\rho(y) \leq \rho(x^{**}) = \rho(x)$. Hence $\rho(y) \leq d(h(y))$ and therefore $d(h(y)) = \rho(y)$, for all $y \in L$.

($\Longleftarrow$) Suppose that (M, d) and (L, ρ) are metric frames with dense onto frame homomorphism $h : L \longrightarrow M$ such that for all $a \in L$, $\rho(a) = d(h(a))$. We want to show that (M, d) is a dense metric sublocale of (L, ρ). Define $\sigma : M \longrightarrow \mathbb{R}_+$, by
$\sigma(a) = \inf\{\rho(x) \mid a \leq h(x), x \in L\}$, for $a \in M$. Then σ is a compatible metric diameter on M. Hence (M, σ) is a metric frame and (M, σ) is a metric sublocale of (L, ρ), since Definition 5.3.1 is satisfied. It remains for us to show that $\sigma = d$. This follows by a similar argument as in the proof of Theorem 5.3.5. Hence (M, d) is a dense metric sublocale of (L, ρ)

$\square$

We now aim to show that metric frames are *perfect extensions* of their uniformly locally connected dense metric sublocales. We first recall relevant theory from Baboolal [4].

Definition 5.3.8. Let L and M be frames.

1. $h : L \longrightarrow M$ is called an *extension* of M, if h is a dense onto homomorphism.

2. $h : L \longrightarrow M$ is called a *perfect extension* of M, if $h : L \longrightarrow M$ is an extension of M and if the right adjoint, h_*, of h satisfies the following: For all $a \in M$,

$$h_*(a \vee a^*) = h_*(a) \vee h_*(a^*).$$

Definition 5.3.9. Let $h : L \longrightarrow M$ be an extension of M. M is said to be locally connected in L, if L has a basis B of connected elements such that $h(b)$ is connected in M, for every $b \in B$.

Proposition 5.3.10 ([4]). *Let $h : L \longrightarrow M$ be an extension of M. Then M is locally connected in L if and only if $h : L \longrightarrow M$ is a perfect extension of M and L is locally connected.*

Proposition 5.3.11. *If (M, d) is a dense metric sublocale of (L, ρ) and (M, d) is uniformly locally connected, then (L, ρ) is a perfect extension of (M, d).*

Proof.
Suppose that (M, d) is a dense metric sublocale of (L, ρ), then by Theorem 5.3.7 we have that $\rho(a) = d(h(a))$, where $h : L \longrightarrow M$ is a dense onto frame homomorphism. By Proposition 5.3.10, it suffices to show that M is locally connected in L.
Let $B = \{h_*(c) \mid c \text{ is connected in } M\}$, where h_* is the right adjoint of h. We know, by Lemma 5.3.3, that $h_*(c)$ is connected in L when $c \in M$ is connected. Thus we need to show that B is a basis of L. Take any $y \in L$, and we will show that $y = \bigvee\{h_*(c) \mid c \in B' \subseteq B\}$. Now $y = \bigvee\{z \in L \mid z \triangleleft_\rho y\}$, since (L, ρ) is a metric frame. Take any z, such that $z \triangleleft_\rho y$ then there exists $\varepsilon > 0$ such that $U^\rho_\varepsilon z \leq y$. Since (M, d) is uniformly locally connected, there exists $\delta > 0$ such that if $d(a) < \delta$, then there exists a connected $c \in M$ with $a \leq c$

and $d(c) < \varepsilon$. Let $C = \{c \in M \mid c \text{ is connected in } M,\ d(c) < \varepsilon\}$. Then $U_\delta^d \leq C$ (that is, U_δ^d refines C), since for any $x \in U_\delta^d$, the uniform local connectedness of (M, d) implies that there exists $c \in C$ such that $x \leq c$. Consider $h_*(C) = \{h_*(c) \mid c \in C\}$. We now show that U_δ^ρ is a refinement of $h_*(C)$. Take $s \in U_\delta^\rho$, then $\rho(s) < \delta$ and thus $d(h(s)) < \delta$. Again, by the uniform local connectedness of (M, d), there exists $c \in C$ such that $h(s) < c$, and therefore $s \leq h_*(c)$. Hence $U_\delta^\rho \leq h_*(C)$, as required. Now $1_L = \bigvee U_\delta^\rho \leq \bigvee h_*(C)$. Therefore $h_*(C)$ is a cover of L. We know that $\rho(a) = d(h(a))$ for $a \in L$, since (M, d) is a dense metric sublocale of (L, ρ), hence for all $c \in C$, $d(c) < \varepsilon$ implies $\rho(h_*(c)) < \varepsilon$. Since $U_\varepsilon^\rho z \leq y$ (that is, $\bigvee\{u \in U_\varepsilon^\rho \mid u \wedge z \neq 0_L\} \leq y$. Also, $\bigvee h_*(C) = 1_L$ and so

$$z \leq \bigvee\{h_*(c) \mid c \in C,\ h_*(c) \wedge z \neq 0_L\} \leq y.$$

Now $y = \bigvee\{z \in L \mid z \lhd_\rho y\}$, and for each $z \lhd_\rho y$, $z \leq \bigvee\{h_*(c) \mid c \in C,\ h_*(c) \wedge z \neq 0_L\} \leq y$. Hence y is a join of elements of the type $h_*(c)$, where $c \in M$ is connected. Thus B is a basis, as required.

□

In [14], Garcia-Maynez showed that a topological space X is S-metrizable if and only if X has a perfect locally connected metrizable compactification. For the remainder of this section, our purpose is to obtain the analogue of Garcia-Maynez's result for frames. We now provide the frame analogue definition for S-metrizability of spaces from [14], followed by recalling essential results that will be required for our proof.

Definition 5.3.12. Let (M, d) be a metric frame. (M, d) is *S-metrizable*, if there exists a compatible metric diameter ρ on M such that (M, ρ) has Property S.

We have noted earlier that every metric frame is a uniform frame, hence the following Proposition follows from Baboolal [5].

Proposition 5.3.13 ([5])**.** *If $h : (L, \rho) \longrightarrow (M, d)$ is a dense surjection, and (M, d) has Property S, then so does (L, ρ).*

Theorem 5.3.14. *Let (M, d) be a connected metric frame. If (M, d) has Property S, then its metric frame completion is compact, locally connected and connected.*

Proof.
Let (M, d) be a metric frame with Property S. We will show that the completion $\tilde{M}$ of M is compact, locally connected and connected. Let $h : (\tilde{M}, \tilde{d}) \longrightarrow (M, d)$ be a dense surjection. By Proposition 5.3.13, since (M, d) has Property S, then so will $(\tilde{M}, \tilde{d})$. Thus $(\tilde{M}, \tilde{d})$ is locally connected by Proposition 3.1.2. We now show that $\tilde{M}$ is compact. Since (M, d) has Property S, then it is totally bounded. Now (M, d) is a uniform frame since $\{U_\varepsilon^d \mid \varepsilon > 0\}$ defines a uniformity on M. Thus by Proposition 5.2.5, the uniform frame completion $\tilde{M}$ of M is compact. By defining $\tilde{d}(x) = d(h(x))$, for $x \in M$, we know that $\tilde{d}$ is a metric diameter. Hence $(\tilde{M}, \tilde{d})$ is a metric frame completion which is now compact. It remains for us to show that $\tilde{M}$ is connected. Now (M, d) is a connected frame, therefore 1_M is connected in M. We show that $1_{\tilde{M}}$ is connected in $\tilde{M}$. Suppose that $1_{\tilde{M}} = a \vee b$, $a \wedge b = 0_{\tilde{M}}$, for $a, b \in \tilde{M}$. Now $h(1_{\tilde{M}}) = h(a \vee b) = h(a) \vee h(b)$, therefore $1_M = h(a) \vee h(b)$, with $h(a) \wedge h(b) = h(0_{\tilde{M}}) = 0_M$. $h(a) = 0_M$, say, since 1_M is connected in M. Hence $a = 0_{\tilde{M}}$, since h is dense. Thus $1_{\tilde{M}}$ is connected in $\tilde{M}$ and so $\tilde{M}$ is connected.

□

We shall recall a compatible diameter constructed by Baboolal [2] and its properties. The following diameter is essential in the proof of this sections main result.

Definition 5.3.15 ([2], Theorem 3.5)**.** Let (M, d) be locally connected (which we may assume is bounded by 1). For each $a \in M$, let $C_a = \{c \mid c \text{ connected and } a \leq c\}$.
Define $\sigma_d : M \longrightarrow \mathbb{R}^+$ by

$$\sigma_d(a) = \begin{cases} 1, & \text{if } C_a = \emptyset. \\ \inf\{d(c) \mid a \leq c, c \text{ connected}\}, & \text{if } C_a \neq \emptyset. \end{cases}$$

Then σ_d is a compatible diameter.

Remark 5.3.16.

(1) From [23], we recall that if d is a diameter on a frame M, then for $a \in M$, $\tilde{d}$ defined by

$$\tilde{d}(a) = \inf_{\varepsilon>0} \sup\{d(u \vee v)|\ u, v \leq a, d(u), d(v) < \varepsilon\}$$

is a metric diameter with the property that $\tilde{d} \leq d$. Thus, analogously, given σ_d, we obtain the metric diameter $\tilde{\sigma}_d$ with $\tilde{\sigma}_d \leq \sigma_d$. Hence $(M, \tilde{\sigma}_d)$ is a metric frame.

(2) In [2], it was shown that $d \leq \tilde{\sigma}_d \leq \sigma_d$.

(3) We observe that for any connected $c \in (M, d)$, $d(c) = \sigma_d(c)$. Hence by (2) of this Remark , it follows that $d(c) = \tilde{\sigma}_d(c)$.

Theorem 5.3.17 ([2]). *Let (M, d) be a locally connected metric frame. Then $(M, \tilde{\sigma}_d)$ is uniformly locally connected*

The equivalence of metric frames (M, d) and $(M, \tilde{\sigma}_d)$ having Property S, is now deduced.

Theorem 5.3.18. *If (M, d) is locally connected then, (M, d) has Property S iff $(M, \tilde{\sigma}_d)$ has Property S.*

Proof.
$(\Longrightarrow)$ Suppose that (M, d) has Property S and let $\varepsilon > 0$ be given. Then there exists a finite cover $\{a_i\}_{i=1}^n$ of M, such that a_i is connected and $d(a_i) < \varepsilon$ for $i = 1, ..., n$. Since a_i is connected for each i, it follows that $\tilde{\sigma}_d(a_i) = d(a_i) < \varepsilon$. Thus $(M, \tilde{\sigma}_d)$ has Property S.

$(\Longleftarrow)$ Suppose $(M, \tilde{\sigma}_d)$ has Property S and let $\varepsilon > 0$ be given. By an analogous argument, as given above in the forward direction, it follows immediately that (M, d) has Property S.

□

We now prove the main result of this section.

Theorem 5.3.19. *Let (M, d) be a connected, locally connected metric frame. (M, d) is S-metrizable if and only if (M, d) has a perfect locally connected metrizable compactification.*

Proof.

($\Longrightarrow$) Suppose that (M, d) is S-metrizable. Then there exists a compatible metric diameter, ρ, such that (M, ρ) has Property S. By Theorem 5.3.18, this implies that $(M, \tilde{\sigma}_\rho)$ has Property S, where $\tilde{\sigma}_\rho$ is the construction from Definition 5.3.15 . Hence by Theorem 5.3.14, the metric frame completion of $(M, \tilde{\sigma}_\rho)$ is compact, locally connected and connected. By Theorem 5.3.17, we know that $(M, \tilde{\sigma}_\rho)$ is uniformly locally connected, and since the metric frame completion of $(M, \tilde{\sigma}_\rho)$ is a dense metric sublocale of $(M, \tilde{\sigma}_\rho)$, then Proposition 5.3.11 implies that the metric frame completion of $(M, \tilde{\sigma}_\rho)$ is a perfect extension of $(M, \tilde{\sigma}_\rho)$. The completion of $(M, \tilde{\sigma}_\rho)$ is a metric frame and hence regular, since every metric frame is a regular frame. Therefore (M, d) has a perfect locally connected metrizable compactification.

($\Longleftarrow$) Suppose $h : (L, \rho) \longrightarrow (M, d)$ is a perfect locally connected metrizable compactification of (M, d). Then $h : L \longrightarrow M$ is a perfect extension of M, and (L, ρ) is compact, regular and locally connected metric frame. By Theorem 3.1.8, since (L, ρ) is compact and locally connected then (L, ρ) has Property S. For any $a \in M$, let $d_\rho(a) = \inf\{\rho(x) \mid a \leq h(x)\}$, then d_ρ is a compatible metric diameter on M. We now show that (M, d_ρ) has Property S. Let $\varepsilon > 0$. Since (L, ρ) has Property S, there exists connected $x_1, ..., x_n \in L$, such that $\bigvee_{i=1}^n x_i = 1_L$, and $\rho(x_i) < \varepsilon$ for $i = 1, ..., n$. Now $h(x_i) \in M$, for $i = 1, ..., n$, and $\bigvee_{i=1}^n h(x_i) = h(\bigvee_{i=1}^n x_i) = h(1_L) = 1_M$. By the definition of d_ρ, $d_\rho(h(x_i)) \leq \rho(x_i) < \varepsilon$ for $i = 1, ..., n$. (L, h) is a perfect compactification of M, therefore by Proposition 5.1.4, $h(x_i)$ is connected in M for $i = 1, ..., n$. Thus (M, d_ρ) has Property S and we have shown that (M, d) is S-metrizable.

$\square$

5.4 The Wallman compactification and dense sublocales of compact metric frames

In this section, our purpose is to provide an intrinsic characterisation of S-metrizability in terms of the Wallman basis of a frame. The main result of this section is the frame analogue of the characterisation in spaces by Garcia-Maynez [14].

We first aim is to show that every compact metric frame is a Wallman compactification of each of its dense sublocales. In order to do so, we will generalise a result of Steiner [27]. We begin by stating Steiner's original result from spaces, followed by some theory required for our generalisation.

Proposition 5.4.1 ([27]). *If (X, d) is a compact metric space, then it has a base $\mathcal{B}$ of open regular sets which satisfies the following: $B_1, B_2 \in \mathcal{B}$ implies that $B_1 \cap B_2 \in \mathcal{B}$ and $B_1 \cup B_2 \in \mathcal{B}$. We say that $\mathcal{B}$ is a ring consisting of regular open sets.*

Definition 5.4.2. An element a of a frame M is called *regular* if $a = a^{**}$.

Remark 5.4.3. We note the following:

1. If X is a topological space, then an open set U is said to be regular open if $U = int(\overline{U})$.
2. It can be shown that an open set $U \in \mathcal{O}X$ is regular open if and only if $U = U^{**}$, where U^* refers to the pseudocomplement of U in the frame $\mathcal{O}X$.
   Thus an open set U is *regular open* if and only if $U \in \mathcal{O}X$ is a regular element.

Definition 5.4.4. Let M be a frame and $B \subseteq M$. B is called a *ring* in M, if $b_1, b_2 \in B$ implies that $b_1 \wedge b_2 \in B$ and $b_1 \vee b_2 \in B$.

Theorem 5.4.5 ([8], **(Boolean ultrafilter theorem - BUT)**). *Every non trivial Boolean algebra contains an ultrafilter (That is, a maximal proper filter).*

Lemma 5.4.6 ([8]). *The following are equivalent:*

1. *Every non trivial Boolean algebra contains an ultrafilter.*

2. *Every compact regular frame M is spatial.*

3. $\Sigma M \neq \emptyset$*, for every non-trivial, compact regular* M*.*

We now generalise Steiner's result.

Proposition 5.4.7. *If* (M, d) *is a compact metric frame, then* M *has a base* B *of regular elements, and* B *is a ring.*

Proof.
If (M, d) is a compact metric frame then (M, d) is compact regular, since every metric frame is regular. If we assume the Boolean ultrafilter theorem, then by Lemma 5.4.6, M is spatial. Thus

$$\eta : M \longrightarrow \mathcal{O}\Sigma M, \text{ given by } \eta(a) = \Sigma_a = \{\psi : M \longrightarrow \underline{2} \mid \psi(a) = 1\}, \text{ for } a \in M,$$

is an isomorphism. From [9], $(\Sigma M, \rho)$ is a metric space with metric given by

$$\rho(\xi, \eta) = \inf\{d(a) \mid \xi(a) = 1 = \eta(a)\}, \text{ for } \xi, \eta \in \Sigma M,$$

and τ_ρ (the topology on ΣM generated by ρ) is exactly $\mathcal{O}\Sigma M$. Furthermore, since M is compact, $\mathcal{O}\Sigma M$ is compact and therefore ΣM is compact. So $(\Sigma M, \rho)$ is a compact metric space and by Proposition 5.4.1, has a ring base $\mathcal{B}$ consisting of regular open sets of ΣM. Each $\Sigma_a \in \mathcal{B}$ is regular open in ΣM, so $\Sigma_a \in \mathcal{O}\Sigma M$ is a regular element of the frame $\mathcal{O}\Sigma M$. Since η is an isomorphism, $\eta^{-1}(\mathcal{B}) = B$ is a ring base for M consisting of regular elements. We can assume that $0_M, 1_M$ is also in B, without loss of generality, since $B \cup \{0_M, 1_M\}$ is still a ring base for M.

□

The existence of a ring basis of regular elements for a compact frame, is now guaranteed by Proposition 5.4.7. This enables us to present the following result.

Proposition 5.4.8. *Let (M,d) be a dense metric sublocale of (L,ρ), with a dense onto frame homomorphism $h:(L,\rho)\longrightarrow(M,d)$ where $d(a)=\inf\{\rho(x)\mid a\leq h(x)\}$, for $a\in M$. Suppose that L is compact and let B be a ring basis of regular elements of L. Then $h(B)$ is a Wallman basis of M.*

Proof.

(1): Take any $h(b_1),h(b_2)\in h(B)$, for $b_1,b_2\in B$. Then $h(b_1)\wedge h(b_2)=h(b_1\wedge b_2)$, and since B is a ring, $h(b_1\wedge b_2)\in h(B)$. Now $h(b_1)\vee h(b_2)=h(b_1\vee b_2)\in h(B)$, since B is a ring. Also, $0_M=h(0_L)\in h(B)$ and $1_M=h(1_L)\in h(B)$.

(2): Take any $w\in L$. We will show that $w=\bigvee\{h(b)\mid b\in B, h(b)\prec_{h(B)} w\}$. Since h is onto, $w=h(a)$ for some $a\in L$ and $a=\bigvee\{b\mid b\in B, b\prec a\}$, since L is regular and B is a basis of L.

<u>Claim 1</u>: $b\prec a \Longleftrightarrow b\prec_B a.$ (5.1)

For $b\prec a$, we have $b^*\vee a=1_L$. Now $b^*=\bigvee\{c\mid c\in B, c\leq b^*\}$, so by the compactness of L, we have $c_1\vee c_2\vee ...\vee c_n\vee a=1_L$, for suitable $c_i\leq b^*$ and $c_i\in B$ for $i=1,...,n$. Since B is closed under finite joins, then $c=c_1\vee c_2\vee ...\vee c_n\in B$, and so $c\vee a=1_L$ with $c\in B$ and $c\leq b^*$. Hence $c\wedge b=0_L$. Thus for $b\prec a$, we have shown that there exists $c\in B$ such that $b\wedge c=0_L$ and $c\vee a=1_L$. Hence $b\prec_B a$.

Now $b\prec_B a$ implies $b\prec a$ is immediate, hence $b\prec a$ if and only if $b\prec_B a$.

We also note that $b\prec_B a$ implies $h(b)\prec_{h(B)} h(a)$, since for the $c\in B$ such that $b\wedge c=0_L$ and $c\vee a=1_L$, we have that $h(b)\wedge h(c)=0_M$, $h(c)\vee h(a)=1_M$ and $h(c)\in h(B)$. Thus

$$\begin{aligned} w=h(a)&=h(\bigvee\{b\in B\mid b\prec a\})\\ &=h(\bigvee\{b\in B\mid b\prec_B a\})\\ &=\bigvee\{h(b)\mid b\in B, b\prec_B a\}\\ &\leq\bigvee\{h(b)\mid b\in B, h(b)\prec_{h(B)} h(a)\}\\ &=\bigvee\{h(b)\mid b\in B, h(b)\prec_{h(B)} w\}\\ &\leq w. \end{aligned}$$

So $w = \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} w\}$, as required.

(3): Take any $h(a), h(b) \in h(B)$ with $a, b \in B$, such that $h(a) \vee h(b) = 1_M$. Then $h(a \vee b) = 1_M$. We have to show that there exists $h(c), h(d) \in h(B)$ such that $h(c) \wedge h(d) = 0_M$ and $h(c) \vee h(a) = 1_M = h(d) \vee h(b)$. Now $a \vee b \in B$, so $a \vee b$ is regular.

<u>Claim 2</u>: If $x \in L$ is regular and $h(x) = 1_M$, then $x = 1_L$. (5.2)

Assume that $h(x) = 1_M$ with x regular. Then,

$$\begin{aligned} & (h(x))^* = 0_M \\ \Longrightarrow\ & h(x^*) = 0_M \quad \text{(by Proposition 2.1.22)} \\ \Longrightarrow\ & x^* = 0_L \quad \text{(since } h \text{ is dense)} \\ \Longrightarrow\ & x^{**} = 1_L. \end{aligned}$$

Since x is regular, $x = 1_L$, as claimed.

Hence $h(a \vee b) = 1_M$ implies $a \vee b = 1_L$. Now $a = \bigvee\{x \mid x \in B, x \prec_B a\}$, and $b = \bigvee\{y \mid y \in B,\ y \prec_B b\}$, therefore

$$\bigvee\{x \mid x \in B,\ x \prec_B a\} \vee \bigvee\{y \mid y \in B,\ y \prec_B b\} = 1_L.$$

Since M is compact, there exists $x \in B$ with $x \prec_B a$, and there exists $y \in B$ with $y \prec_B b$ such that $x \vee y = 1_L$. $x \prec_B a$ implies that there exists $c \in B$, such that $x \wedge c = 0_L$ and $c \vee a = 1_L$, and $y \prec_B b$ implies that there exists $d \in B$ such that $y \wedge d = 0_L$ and $d \vee b = 1_L$. Now, $c \wedge d = (c \wedge d) \wedge (x \vee y) = (c \wedge d \wedge x) \vee (c \wedge d \wedge y) = 0_L$. Hence $h(c) \wedge h(d) = h(c \wedge d) = 0_M$. Furthermore, $h(c) \vee h(a) = 1_M$, since $c \vee a = 1_L$ and $h(d) \vee h(b) = 1_M$, since $d \vee b = 1_L$. Hence condition (3) is satisfied.

We have shown that $h(B)$ is a Wallman basis of M.

□

Proposition 5.4.9. *With the conditions as in Proposition 5.4.8, the Wallman compact-*

ification $\gamma_{h(B)}M$ of M is isomorphic to L (as frames).

Proof.

By Proposition 5.1.8, $h(B)$ determines a strong inclusion on M given by: $x \blacktriangleleft y$ for $x, y \in M$ if and only if there exists $h(b)$ for $b \in B$, such that $x \prec_{h(B)} h(b) \prec_{h(B)} y$. Thus, $\gamma_{h(B)}M = \{J \mid J \text{ is a strongly regular ideal}\}$, where J is said to be strong regular if $x \in J$ implies there exists $y \in J$ such that $x \blacktriangleleft y$. $\gamma_{h(B)}M$ is a compact regular frame and the join map

$$\bigvee : \gamma_{h(B)}M \longrightarrow M$$
$$J \mapsto \bigvee J$$

makes $\gamma_{h(B)}M$ a compactification of M. We will show that $\gamma_{h(B)}M \cong L$. Let h_* be the right adjoint of h. We note that $h : L \longrightarrow M$ is a compactification of M (since L is a compact regular frame), and this induces a strong inclusion $\blacktriangleleft_1$ on M given by:

$$x \blacktriangleleft_1 y \iff h_*(x) \prec h_*(y).$$

It suffices to show that $\blacktriangleleft = \blacktriangleleft_1$, for then by Theorem 5.1.6, $\gamma_{h(B)}M \cong L$. So suppose that $x \blacktriangleleft_1 y$, for $x, y \in M$. Then $h_*(x) \prec h_*(y)$ and therefore there exists $z \in L$ such that $h_*(x) \prec z \prec h_*(y)$, since $\prec$ interpolates in compact regular frames by Proposition 2.1.31. Now $h_*(x) \prec z$ implies $h_*(x)^* \vee z = 1_L$, and so $h_*(x)^* \vee \bigvee\{b \in B \mid b \leq z\} = 1_L$. Since L is compact and B is closed under finite joins, it follows that $h_*(x)^* \vee b = 1_L$, for some $b \in B$ with $b \leq z$. Now,

$$
\begin{aligned}
& h_*(x) \prec b \leq z \prec h_*(y) \\
\implies & h_*(x) \prec b \prec h_*(y) \quad (b \in B) \\
\implies & h_*(x) \prec_B b \prec_B h_*(y) \quad \text{(by equation (5.1))} \\
\implies & hh_*(x) \prec_{h(B)} h(b) \prec_{h(B)} hh_*(y) \\
\implies & x \prec_{h(B)} h(b) \prec_{h(B)} y \quad \text{(by Propostion 2.1.14)} \\
\implies & x \blacktriangleleft y.
\end{aligned}
$$

Now suppose $x \blacktriangleleft y$, for $x, y \in M$. Then there exists $b_1 \in B$ such that

$$x \prec_{h(B)} h(b_1) \prec_{h(B)} y.$$

$x \prec_{h(B)} h(b_1)$ implies there exists $c_1 \in B$ such that $x \wedge h(c_1) = 0_M$ and $h(c_1) \vee h(b_1) = 1_M$. Now $h(h_*(x) \wedge c_1) = hh_*(x) \wedge h(c_1) = x \wedge h(c_1) = 0_M$. So, $h_*(x) \wedge c_1 = 0_L$, since h is a dense map. Furthermore, $c_1 \vee b_1 \in B$ and is therefore regular, so by equation (5.2), since $h(c_1 \vee b_1) = h(c_1) \vee h(b_1) = 1_M$, we must have that $c_1 \vee b_1 = 1_L$. Hence we have shown that $h_*(x) \prec b_1$. Now, we observe that

$$
\begin{aligned}
& h(b_1) \leq y \\
\implies & b_1 \leq h_*(y) \\
\implies & h_*(x) \prec b_1 \prec h_*(y) \\
\implies & h_*(x) \prec h_*(y) \\
\implies & x \blacktriangleleft_1 y.
\end{aligned}
$$

Hence, we have shown that $\gamma_{h(B)}M \cong L$.

$\square$

From the above Proposition, we have established that every compact metric frame is a

Wallman compactification of each of its dense sublocales. We now focus on providing a characterisation of S-metrizability in terms of the Wallman basis of a frame, which is the main result in this section.

For the remainder of this section, let M be a locally connected frame. We briefly state required theory from [4].

Definition 5.4.10. Let $B \subseteq M$ be a Wallman basis. B is *locally connected* if each component of each element of B is also in B.

Definition 5.4.11. A basis B of M is *uniformly connected* if whenever A is finite, $\bigvee A = 1$ and $A \subseteq B$, then there exists finite $C \subseteq B$, such that every $c \in C$ is connected and $C \leq A$.

Definition 5.4.12. An element $0 \neq c \in M$ is a *component* of an element $u \in M$ if:

1. c is connected and $c \leq u$,
2. c is maximally connected in u (that is, whenever $c \leq x \leq u$ and x is connected in M, then $c = x$).

Remark 5.4.13. We note that if c_α and c_β are components of $u \in M$, and $c_\alpha \neq c_\beta$, then $c_\alpha \wedge c_\beta = 0$

Definition 5.4.14. Let $\gamma_B M$ be the Wallman compactification associated with a Wallman basis B. An ideal $J \in \gamma_B M$ is said to be *insular* if whenever $x \in J$, there exists $y \in J$ having finitely many componects, such that $y \in B$ and $x \blacktriangleleft y$.

Theorem 5.4.15 ([4]). *Let B be a locally connected Wallman basis for the locally connected frame M. Then the following are equivalent:*

1. $\bigvee : \gamma_B M \longrightarrow M$ *is a perfect locally connected compactification of M.*
2. *B is uniformly connected.*
3. *Every ideal J in $\gamma_B M$ is insular.*

Although the following Lemma is known, it is difficult to find in the literature. We will therefore, provide a proof for completeness.

Lemma 5.4.16. *Let M be a locally connected frame and c be a component of $v \in M$. Then $v \leq c \vee c^*$.*

Proof.

By the local connectedness of M, $v = \bigvee_{\alpha \in I} c_\alpha$, where c_α are the components of v. Now $c = c_\alpha$, for some $\alpha \in I$. For $\beta \neq \alpha$, $c_\beta \wedge c_\alpha = 0_M$, so $c_\beta \leq c^*$. This implies that $\bigvee_{\beta \neq \alpha} c_\beta \leq c^*$, therefore $v = c \vee (\bigvee_{\beta \neq \alpha} c_\beta) \leq c \vee c^*$. □

Next we shall show that S-metrizability of a locally connected frame ensures the existence of a countable locally connected and uniformly connected Wallman basis. Before doing this, we need the following two propositions on *countability.*

Proposition 5.4.17. *Every compact metric frame has a countable base.*

Proof.

Let (M, d) be a compact metric frame. For each $n \in \mathbb{N}$, $U^d_{\frac{1}{n}} = \{x \in M \mid d(x) < \frac{1}{n}\}$ is a cover of M. So by compactness of M, there exists a finite cover $F_n \subseteq U^d_{\frac{1}{n}}$, of M.
Let $B = \bigcup_{n=1}^{\infty} F_n$. Then B is countable. We shall show that B is a base for M. Take any $a \in M$, then $a = \bigvee\{x \in M \mid x \lhd_d a\}$. Now for any $x \lhd_d a$, there exists $\varepsilon > 0$, such that $U^d_\varepsilon x \leq a$. Take $n \in \mathbb{N}$, such that $\frac{1}{n} < \varepsilon$, then $U^d_{\frac{1}{n}} x \leq a$. Since F_n is a cover of M,

$$x = x \wedge \bigvee\{y \mid y \in F_n\} = \bigvee\{x \wedge y \mid y \in F_n, y \neq 0\}.$$

Now, $y \in F_n$ and $x \wedge y \neq 0$ implies that $y \leq a$ and therefore

$$x \leq \bigvee\{y \in F_n \mid x \wedge y \neq 0\} \leq a.$$

Since a is a join of the $x's$, it follows that a is a join of elements that come from B, since each $y \in F_n$ is in B. So B is a countable base.

□

Proposition 5.4.18. *If (M, d) is a compact locally connected metric frame, then each $u \in M$ has only countably many components.*

Proof.

Since M is locally connected, $u = \bigvee_{\alpha \in I} c_\alpha$, where c_α are the components of u. Let B be a countable base of M. The existence of a countable base follows from Proposition 5.4.17. Each c_α is a join of elements from B, so we can choose any $b_\alpha \in B$ such that $b_\alpha \leq c_\alpha$. Whenever $\alpha, \beta \in I$ and $\alpha \neq \beta$, then $c_\alpha \wedge c_\beta = 0$, therefore $b_\alpha \neq b_\beta$. Thus if I were uncountable, then $\{b_\alpha\}_{\alpha \in I}$ would be uncountable. But $\{b_\alpha\}_{\alpha \in I} \subseteq B$, and B is countable. Hence $\{b_\alpha\}_{\alpha \in I}$ is countable, which is a contradiction. Thus I is countable.

□

Proposition 5.4.19. *If M is S-metrizable then M has a countable, locally connected and uniformly connected Wallman basis.*

Proof.

Assume that (M, d) is S-metrizable. Then by Theorem 5.3.19, (M, d) has a perfect locally connected metrizable compactification (just take the completion of (M, d)). Call it (L, ρ) and let $h : (L, \rho) \longrightarrow (M, d)$ be a dense surjection where $\rho(a) = d(h(a))$, for all $a \in L$. We know by Proposition 5.4.7 and Proposition 5.4.17, that whenever L is a compact metric frame, then L has a countable ring basis, call it B_0, consisting of regular elements. Let

$$C_0 = \{c \in L \mid c \text{ is a component of some } b \in B_0\},$$

and let $B_1 = \langle B_0 \cup C_0 \rangle$, where $\langle B_0 \cup C_0 \rangle$ denotes the ring generated by B_0 and C_0. We will now show that B_1 is the smallest ring containing B_0 and C_0. Since $B_1 = \langle B_0 \cup C_0 \rangle$, we have that

$$B_1 = \{x \in L \mid x \text{ is a finite join of elements } y, \text{ where } y = \bigwedge_{i=1}^{n} t_i, \ t_i \in B_0 \cup C_0\}.$$

Take any $x, y \in B_1$, then $x = \bigvee_{i=1}^{n} x_i$, where $x_i = s_1^i \wedge ... \wedge s_{k_i}^i$, for $s_j^i \in B_0 \cup C_0$, and $y = \bigvee_{i=1}^{m} y_i$, where $y_i = t_1^i \wedge ... \wedge t_{q_i}^i$, for $t_{q_i}^i \in B_0 \cup C_0$. Thus $x \vee y = \bigvee_{i=1}^{n} x_i \vee \bigvee_{i=1}^{m} y_i$, with x_i and y_i as described above, so $x \vee y \in B_1$. Now, $x \wedge y = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (x_i \wedge y_i)$, where $x_i \wedge y_i = s_1^i \wedge ... \wedge s_{k_i}^i \wedge t_1^i \wedge ... \wedge t_{q_i}^i$. So $x \wedge y \in B_1$. Hence B_1 is a ring containing B_0 and

C_0, and B_1 is the smallest ring containing B_0 and C_0.

We now show that B_1 consists of regular elements. We first note that if x and y are regular then $x \wedge y$ is regular. For if $x = x^{**}$ and $y = y^{**}$, then by Lemma 2.1.19, $(x \wedge y)^{**} = x^{**} \wedge y^{**} = x \wedge y$ and so $x \wedge y$ is regular. If $c \in C_0$, then c is a component of some $b \in B_0$. Now $c \leq b$ implies that $c^{**} \leq b^{**} = b$, so $c \leq c^{**} \leq b$. Now, c is connected therefore c^{**} is connected by Lemma 2.1.34. Since c is a component we must have that $c = c^{**}$. Hence c is regular. Thus $B_0 \cup C_0$ consists of regular elements and finite meets of elements from $B_0 \cup C_0$ are regular. Let

$$H_1 = \{x \in L \mid x \text{ is a finite meet of elements from } B_0 \cup C_0\}.$$

Then H_1 consists of regular elements. For each $m > 1$, let

$$H_m = \{x \in L \mid x \text{ is a join of at most } m \text{ elements from } H_1\}.$$

We prove by induction that each H_m consists of regular elements. Let $m > 1$ and assume H_{m-1} consists of regular elements. Let $x \in H_m$, then there exists $h_1, h_2, ..., h_m \in H_1$ such that $x = h_1 \vee h_2 \vee ... \vee h_m$. Take any h_k for $1 \leq k \leq m$. Now,

$$\begin{aligned} h_k &= b_1 \wedge ... \wedge b_t \wedge c_1 \wedge ... \wedge c_s \quad (\text{where } b_i \in B_0, c_j \in C_0) \\ &= b \wedge c_1 \wedge ... \wedge c_s, \end{aligned}$$

where $b = b_1 \wedge ... \wedge b_t \in B_0$, since B_0 is a ring. Each c_i is a component of some $v_i \in B_0$, so

$$\begin{aligned} h_k &= b \wedge c_1 \wedge ... \wedge c_s \\ &\leq b \wedge v_1 \wedge ... \wedge v_s = d_k \in B_0. \end{aligned}$$

Claim: $d_k \leq h_k \vee h_k^*$.

$h_k \vee h_k^* = (b \wedge c_1 \wedge ... \wedge c_s) \vee (b \wedge c_1 \wedge ... \wedge c_s)^*$. Now $h_k = b \wedge c_1 \wedge ... \wedge c_s \leq c_i$, for $i = 1, ..., s$. So $c_i^* \leq h_k^*$, for each i, and thus $c_1^* \vee ... \vee c_s^* \leq h_k^*$. Hence,

$$\begin{aligned} h_k \vee h_k^* &\geq (b \wedge c_1 \wedge ... \wedge c_s) \vee (c_1^* \vee ... \vee c_s^*) \\ &= (b \vee (c_1^* \vee ... \vee c_s^*)) \wedge (c_1 \vee (c_1^* \vee ... \vee c_s^*)) \wedge \ ... \ \wedge (c_s \vee (c_1^* \vee ... \vee c_s^*)) \\ &\geq b \wedge (c_1 \vee c_1^* \vee ... \vee c_s^*) \wedge (c_2 \vee c_1^* \vee ... \vee c_s^*) \wedge \ ... \ \wedge (c_s \vee c_1^* \vee ... \vee c_s^*) \\ &\geq b \wedge (c_1 \vee c_1^*) \wedge (c_2 \vee c_2^*) \wedge \ ... \ \wedge (c_s \vee c_s^*) \quad \text{(By Lemma 5.4.16)} \\ &\geq b \wedge v_1 \wedge v_2 \wedge \ ... \ \wedge v_s = d_k. \end{aligned}$$

Thus proving the claim that $d_k \leq h_k \vee h_k^*$.

We now show that x is regular. Firstly, $x = h_1 \vee h_2 \vee ... \vee h_m \leq d_1 \vee d_2 \vee ... \vee d_m$. Hence $x^{**} \leq (d_1 \vee d_2 \vee ... \vee d_m)^{**} = d_1 \vee d_2 \vee ... \vee d_m$, since $d_i \in B_0$ and B_0 is a ring of regular elements. Fix any i, $1 \leq i \leq m$. Now $x = h_i \vee \bigvee_{j \neq i} h_j$, hence

$$\begin{aligned} x \wedge h_i^* &\leq \bigvee_{j \neq i} h_j \\ \implies (x \wedge h_i^*)^{**} &\leq (\bigvee_{j \neq i} h_j)^{**} = \bigvee_{j \neq i} h_j \quad \text{(by the induction hypothesis)} \\ \implies x^{**} \wedge h_i^{***} &\leq \bigvee_{j \neq i} h_j \quad \text{(by Lemma 2.1.19)} \\ \implies x^{**} \wedge h_i^* &\leq \bigvee_{j \neq i} h_j \quad \text{(by Lemma 2.1.19)} \end{aligned}$$

Hence for all i, we have $x^{**} \wedge h_i^* \leq \bigvee_{j \neq i} h_j$. Now,

$$\begin{aligned} x^{**} &\leq d_1 \vee d_2 \vee ... \vee d_m \\ &\leq (h_1 \vee h_1^*) \vee (h_2 \vee h_2^*) \vee ... \vee (h_m \vee h_m^*) \\ &= (h_1 \vee ... \vee h_m) \vee (h_1^* \vee ... \vee h_m^*) \\ &= x \vee h_1^* \vee h_2^* ... \vee h_m^*. \end{aligned}$$

Therefore,

$$\begin{aligned}
x^{**} &= x^{**} \wedge (x \vee h_1^* \vee h_2^* ... \vee h_m^*) \\
&= (x^{**} \wedge x) \vee (x^{**} \wedge h_1^*) \vee (x^{**} \wedge h_2^*) \vee ... \vee (x^{**} \wedge h_m^*) \\
&\leq x \vee \bigvee_{j \neq 1} h_j \vee \bigvee_{j \neq 2} h_j \vee ... \vee \bigvee_{j \neq m} h_j \\
&\leq x.
\end{aligned}$$

Since $x \leq x^{**}$, we conclude that $x = x^{**}$, and so x is regular.

Thus by induction on m, H_m consists of regular elements for every $m > 1$. Thus $B_1 = \langle B_0 \cup C_0 \rangle$ consists of regular elements. Let $B_2 = \langle B_1 \cup C_1 \rangle$, where C_1 consists of components of elements from B_1. By a similar argument in which we showed that B_1 consists of regular elements, we can show that B_2 consists of regular elements. Thus $B = \bigcup_{n=0}^{\infty} B_n$, consists of regular elements. Also, B is a ring basis since $B_n \subseteq B_{n+1}$ and since each B_n is a ring basis. Hence by Proposition 5.4.8, $h(B)$ is a Wallman basis for (M, d).

Claim: $h(B)$ is countable.

B_0 is countable and by Proposition 5.4.18, since (L, ρ) is compact and locally connected, it follows that C_0 is countable. Thus the ring generated by B_0 and C_0 is countable. So B_1 is countable. It follows that all $B_n's$ are countable. Hence $B = \bigcup_{n=0}^{\infty} B_n$ is countable. In addition, $h(B)$ would then be a countable base, as claimed.

We now show that $h(B)$ is a locally connected base. Take any $h(b) \in h(B)$, where $b \in B$. Let w be a component of $h(b)$. We will show that $w \in h(B)$. Now, $b \in B_n$ for some n. We know that $b = \bigvee_\alpha \{c_\alpha \mid c_\alpha \text{ is a component of } b\}$, therefore $h(b) = \bigvee_\alpha \{h(c_\alpha) \mid c_\alpha \text{ is a component of } b\}$. Since (L, ρ) is a perfect compactification, then by Proposition 5.1.4, each $h(c_\alpha)$ is connected in M. Now $w \leq h(b)$ implies $w \wedge h(c_\alpha) \neq 0_M$, for some component c_α of b. Therefore $w \leq w \vee h(c_\alpha) \leq h(b)$, with $w \vee h(c_\alpha)$ connected in M. Since w is a component of $h(b)$, $h(c_\alpha) \leq w$. Also,

$$w = w \wedge h(b) = (w \wedge h(c_\alpha)) \vee \bigvee_{\beta \neq \alpha} (w \wedge h(c_\beta)).$$

Furthermore,

$$(w \wedge h(c_\alpha)) \wedge \bigvee_{\beta \neq \alpha} (w \wedge h(c_\beta)) = w \wedge (h(c_\alpha) \wedge \bigvee_{\beta \neq \alpha} h(c_\beta)) = 0_M.$$

Whenever $\beta \neq \alpha$, then $h(c_\alpha) \wedge h(c_\beta) = h(c_\alpha \wedge c_\beta) = h(0_L) = 0_M$. So since w is connected and $w \wedge h(c_\alpha) \neq 0_M$, we must have that $\bigvee_{\beta \neq \alpha}(w \wedge h(c_\beta)) = 0_M$. Hence $w = w \wedge h(c_\alpha) \leq h(c_\alpha)$, and therefore $w = h(c_\alpha)$. But c_α is a component of $b \in B_n$ for some n, so $c_\alpha \in B_{n+1} \subseteq B$. Thus $w = h(c_\alpha)$ with $c_\alpha \in B$, showing that $h(B)$ is a locally connected basis.

Lastly, we show that $h(B)$ is a uniformly connected base. We have that $h : (L, \rho) \longrightarrow (M, d)$ is a perfect locally connected metrizable compactification of M, therefore by Proposition 5.4.9, the Wallman compactification $\gamma_{h(B)}M \cong L$, as frames. Thus $\gamma_{h(B)}M$ is a perfect locally connected compactification of M. By Theorem 5.4.15, $h(B)$ is uniformly connected. Thus $h(B)$ is a countable, locally connected and uniformly connected Wallman base for M.

□

The following metrization theory from [20], is required for our main result:

Definition 5.4.20. A subset $X \subseteq M$ is said to be *locally finite* if there exists a cover W of M such that each $w \in W$ meets only finitely many elements from X.

Definition 5.4.21. A basis B of M is said to be $\sigma-$*locally finite* if $B = \bigcup_{n=1}^{\infty} B_n$ and each subset B_n is locally finite.

Theorem 5.4.22 ([20]). *Let M be a regular frame. M is metrizable if and only if M has a $\sigma-$locally finite basis.*

We now establish our main result in this section, which is the analogue of a result from [14].

Theorem 5.4.23. *Let M be a locally connected frame. The following are equivlent:*

(1) M is S-metrizable.

(2) M has a countable locally connected and uniformly connected Wallman basis.

(3) M has a countable locally connected Wallman basis B such that every ideal J of $\gamma_B M$ is insular.

Proof.

(1) $\Longrightarrow$ (2): Follows from Proposition 5.4.19.

(2) $\Longleftrightarrow$ (3): Follows from Theorem 5.4.15.

(2) $\Longrightarrow$ (1): Suppose then that M has a countable locally connected and uniformly connected Wallman basis B. By Theorem 5.4.15, $\bigvee : \gamma_B M \longrightarrow M$ is a perfect locally connected compactification of M. From Proposition 5.1.9, $k(B)$ is a basis for $\gamma_B M$, where $k : M \longrightarrow \gamma_B M$ is the right adjoint of $\bigvee : \gamma_B M \longrightarrow M$. Since B is countable, then $k(B)$ is countable. Thus $\gamma_B M$ has a countable basis and hence by Theorem 5.4.22 $\gamma_B M$ must be metrizable, since it is regular . So M has a perfect locally connected metrizable compactification and hence by Theorem 5.3.19 is S-metrizable.

□

Chapter 6

Locally Non-Separating Sublocales

In [12], Curtis introduced the concept of a *locally non-separating* remainder in order to study the hyperspace of a non-compact space X. Using the property of a locally non-separating remainder, Curtis established the conditions under which a *Peano* compactification of a connected space X would exist. In this chapter we discuss the analog of the concept of locally non-separating sets, in frames. We shall begin with a discussion of properties of sublocales, followed by a section in which we define a locally non-separating sublocale and thereafter provide a generalisation for a special case of Curtis's result.

6.1 Some notes on sublocales

The purpose of this section is to provide the prerequisite theory on sublocales that is required to generalise a result of Curtis found in [12]. In the following discussion, we will be concerned with the *supplement* of a sublocale and the *difference* of a sublocale amongst other properties and results.

We state the following definitions from Plewe [21].

Definition 6.1.1. Let R and S be sublocales of a frame L.

1. The *difference* of S from R, denoted $R \setminus S$, is given by

$$R \setminus S = \bigvee \{T \mid T \text{ is a sublocale of } R,\ T \wedge S = \{1\}\}.$$

2. The *supplement* of S, denoted $\sup(S)$, is given by

$$\sup(S) = \bigwedge\{T \mid T \text{ is a sublocale of } L,\ T \vee S = L\}.$$

3. A sublocale S is called *complemented* in L if, $S \wedge \sup(S) = \{1\}$.

Remark 6.1.2.

1. We recall from Chapter 2 that *meets* of sublocales are the same as *intersections*. For the remainder of this chapter, we shall denote meets of sublocales by intersections for notational consistency.

2. $R \setminus S$ and $\sup(S)$, as defined above, are indeed sublocales of L.

3. Since $\mathcal{S}(L)$ is a co-frame, we must have that $S \vee \sup(S) = L$.

4. By Proposition 2.2.14, we know that for any $a \in L$, $\mathfrak{o}(a)$ and $\uparrow a$ are complements of one another in $\mathcal{S}(L)$. Hence it follows that each of $\mathfrak{o}(a)$ and $\uparrow a$ are complemented in L.

Lemma 6.1.3 ([21]). *Let S be a sublocale of a frame L. Then* $\sup(S) = L \setminus S$.

Lemma 6.1.4. *If T is a complemented sublocale of a frame L, and S is any sublocale of L, then $S \setminus (L \setminus T) = S \cap T$.*

Proof.
Since T is a complemented sublocale, then $T \cap \sup(T) = \{1\}$ and $T \vee \sup(T) = L$, which by Lemma 6.1.3 means that $T \cap (L \setminus T) = \{1\}$ and $T \vee (L \setminus T) = L$. Now $S \cap T \subseteq S$ is a sublocale of S, and $(S \cap T) \cap (L \setminus T) = \{1\}$. So $S \cap T \subseteq S \setminus (L \setminus T)$. For the other inclusion,

$$\begin{aligned} S \setminus (L \setminus T) &= \bigvee\{A \mid A \text{ is a sublocale of } S,\ A \cap (L \setminus T) = \{1\}\} \\ &= \bigvee\{A \mid A \text{ is a sublocale of } S,\ A \subseteq T\} \\ &\subseteq S \cap T. \end{aligned}$$

$\square$

We note the following useful property on differences of sublocales.

Lemma 6.1.5. *Let A be a sublocale of L, and for any subset B of L, let $\{\mathfrak{o}(b) \mid b \in B\}$ be a collection of open sublocales in L. Then*

$$(\bigvee_{b\in B} \mathfrak{o}(b)) \setminus A = \bigvee_{b\in B} (\mathfrak{o}(b) \setminus A).$$

Proof.

For all $b' \in B$, we note that $\mathfrak{o}(b') \subseteq \bigvee_{b\in B} \mathfrak{o}(b)$. Take any sublocale T of $\mathfrak{o}(b')$, such that $T \cap A = \{1\}$, then $T \subseteq \bigvee_{b\in B} \mathfrak{o}(b)$ and therefore $T \subseteq (\bigvee_{b\in B} \mathfrak{o}(b)) \setminus A$. So we have shown that $\mathfrak{o}(b') \setminus A \subseteq (\bigvee_{b\in B} \mathfrak{o}(b)) \setminus A$. It follows that $\bigvee_{b\in B}(\mathfrak{o}(b') \setminus A) \subseteq (\bigvee_{b\in B} \mathfrak{o}(b)) \setminus A$.
We now show the reverse inclusion that $(\bigvee_{b\in B} \mathfrak{o}(b)) \setminus A \subseteq \bigvee_{b\in B}(\mathfrak{o}(b) \setminus A)$. Take any sublocale T of $\bigvee_{b\in B} \mathfrak{o}(b)$, such that $T \cap A = \{1\}$. Then,

$$\begin{aligned} T =&\ T \cap \bigvee_{b\in B} \mathfrak{o}(b) \\ =& \bigvee_{b\in B} (T \cap \mathfrak{o}(b)) \qquad \text{(by Proposition 2.2.17)} \end{aligned}$$

Since for any $b' \in B$, $T \cap \mathfrak{o}(b') \subseteq \mathfrak{o}(b')$, and $T \cap \mathfrak{o}(b') \cap A = \{1\}$, then $T \cap\ \mathfrak{o}(b') \subseteq \mathfrak{o}(b') \setminus A$. Hence for any $b' \in B$, $T \cap \mathfrak{o}(b') \subseteq \bigvee_{b\in B}(\mathfrak{o}(b) \setminus A)$, and thus $T \subseteq \bigvee_{b\in B}(\mathfrak{o}(b) \setminus A)$, as required. So $(\bigvee_{b\in B} \mathfrak{o}(b)) \setminus A \subseteq \bigvee_{b\in B}(\mathfrak{o}(b) \setminus A)$, and hence equality holds.

$\square$

Recall, from chapter 2, a sublocale S of a frame L is called dense, if $\overline{S} = L$.

Lemma 6.1.6. *S is a dense sublocale of L if and only if S meets every non-trivial open sublocale of L.*

Proof.

($\Longrightarrow$) Let S be a dense sublocale of L and let $U \neq \{1\}$ be an open sublocale of L. Then $U = \mathfrak{o}(a)$, for some $a \in L$ and $a \neq 0_L$. Suppose $S \cap \mathfrak{o}(a) = \{1\}$, then $S \subseteq \uparrow a$. So

$L = \overline{S} \subseteq \overline{\uparrow a} = \uparrow a$. Hence $a = 0_L$, which is a contradiction. Thus $S \cap \mathfrak{o}(a) \neq \{1\}$, as required.

($\Longleftarrow$) Let S be an arbitrary sublocale of L and suppose that every non-trivial open sublocale of L meets S. Since $S \subseteq \overline{S} = \uparrow (\bigwedge S)$, we have $S \cap \mathfrak{o}(\bigwedge S) = \{1\}$. Thus $\bigwedge S = 0$, by the hypothesis. Hence $\overline{S} = \uparrow 0 = L$, and so S is dense in L.

□

Some theory on *simple chains* is presented next, so that we can present an equivalent characterisation of a connected frame in terms of a *simple chain of open sublocales.*

Definition 6.1.7. Let $a, b \in L$ with $a \neq 0 \neq b$. A *simple chain* in L joining a to b, is a finite set of elements $\{x_1, x_2, ..., x_n\}$ from L such that $a \wedge x_1 \neq 0$, $x_n \wedge b \neq 0$, and $x_i \wedge x_{i+1} \neq 0$ for $i = 1, 2, ..., n-1$.

Proposition 6.1.8. *L is a connected frame if and only if whenever $X \subseteq L$ with $\bigvee X = 1$, $a, b \in L$ and $a \neq 0 \neq b$, there is a simple chain with elements in X joining a to b.*

Proof.
($\Longrightarrow$) Take $X \subseteq L$ with $\bigvee X = 1$, $a, b \in L$ and $a \neq 0 \neq b$. Let

$$u = \bigvee \{x \in X \mid \text{there exists a simple chain in } X \text{ joining } a \text{ to } x\}.$$

Since X is a cover and $a \neq 0$, there exists $x \in X$ such that $a \wedge x \neq 0$. The singleton set $\{x\}$ is a simple chain joining a to x, therefore $u \neq 0$. Let

$$v = \bigvee \{x \in X \mid \text{there is no simple chain in } X \text{ joining } a \text{ to } x\}.$$

Then $u \vee v = 1$. If $u \wedge v \neq 0$, then there exists $x, y \in X$ such that $x \wedge y \neq 0$ and such that there is a simple chain in X joining a to x, and there is no simple chain in X joining a to y. Let $x_1, x_2, ..., x_n$ be a simple chain joining a to x. But since $x \wedge y \neq 0$, $\{x_1, x_2, ..., x_n, x\}$ would be a simple chain in X joining a to y, and this is a contradiction. Hence $u \wedge v = 0$. Since L is connected and $u \neq 0$, we must have $v = 0$. Hence for every $x \in X$, there is a simple chain in X joining a to x. Since $b \neq 0$, we have that $b \wedge y \neq 0$ for some $y \in X$. Now there is a simple chain in X joining a to y, and since $y \wedge b \neq 0$, there is a simple chain in X joining a to b.

($\Longleftarrow$) Assume that whenever $X \subseteq L$ is a cover of L, $a, b \in L$ and $a \neq 0 \neq b$, there is a simple chain with elements in X joining a to b. Suppose that $a \vee b = 1$, $a \wedge b = 0$. If $a \neq 0$ and $b \neq 0$, then $X = \{a, b\}$ is a cover of L. Hence there is a simple chain in X joining a to b. This means that $a \wedge b \neq 0$, which is a contradiction. Hence either $a = 0$ or $b = 0$. Thus L is connected.

□

The above proposition can now be phrased in the language of sublocales. For the reader's convenience, we recall the following facts about open sublocales from Proposition 2.2.15:

1. $\mathfrak{o}(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} \mathfrak{o}(a_i)$, where $a_i \in L$ for all $i \in I$.

2. $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \wedge \mathfrak{o}(b)$, for $a, b \in L$.

3. $\mathfrak{o}(a) = \mathfrak{o}(b) \iff a = b$, for $a, b \in L$.

Proposition 6.1.9. *L is connected if and only if whenever $X \subseteq L$ with $L = \bigvee\{\mathfrak{o}(x) \mid x \in X\}$ and $\mathfrak{o}(a), \mathfrak{o}(b) \neq \{1\}$, then there exist $\mathfrak{o}(x_1), \mathfrak{o}(x_2), ..., \mathfrak{o}(x_n)$, $x_i \in X$ such that $\mathfrak{o}(a) \cap \mathfrak{o}(x_1) \neq \{1\}$, $\mathfrak{o}(x_n) \cap \mathfrak{o}(b) \neq \{1\}$ and $\mathfrak{o}(x_i) \cap \mathfrak{o}(x_{i+1}) \neq \{1\}$, for $i = 1, ..., n-1$.*

Proof.
($\Longrightarrow$) Assume that L is connected. Let $X \subseteq L$ with $L = \bigvee\{\mathfrak{o}(x) \mid x \in X\}$ and $\mathfrak{o}(a), \mathfrak{o}(b) \neq \{1\}$, for $a, b \in L$. Then $L = \mathfrak{o}(\bigvee\{x \mid x \in X \subseteq L\})$, but $L = \mathfrak{o}(1)$. Thus $1 = \bigvee\{x \mid x \in X \subseteq L\}$; that is, $\bigvee X = 1$. Now $\mathfrak{o}(a), \mathfrak{o}(b) \neq \{1\}$, and this implies $a, b \neq 0$. Thus by Proposition 6.1.8, there is a simple chain in X joining a to b. Hence, there exists $x_1, x_2, ..., x_n \in X$ such that $a \wedge x_1 \neq 0$, $x_n \wedge b \neq 0$, and $x_i \wedge x_{i+1} \neq 0$ for $i = 1, 2, ..., n-1$. Thus we have $\mathfrak{o}(x_1), \mathfrak{o}(x_2), ..., \mathfrak{o}(x_n)$, $x_i \in X$, such that $\mathfrak{o}(a \wedge x_1) \neq \{1\}$, $\mathfrak{o}(x_n \wedge b) \neq \{1\}$ and $\mathfrak{o}(x_i \wedge x_{i+1}) \neq \{1\}$ for $i = 1, 2, ..., n-1$. Hence $\mathfrak{o}(a) \cap \mathfrak{o}(x_1) \neq \{1\}, \mathfrak{o}(x_n) \cap \mathfrak{o}(b) \neq \{1\}$

and $\mathfrak{o}(x_i) \cap \mathfrak{o}(x_{i+1}) \neq \{1\}$, for $i = 1, ..., n-1$, as required.

($\Longleftarrow$) Assume the condition and suppose that $a \vee b = 1$, $a \wedge b = 0$, for $a, b \in L$. If $a \neq 0$ and $b \neq 0$, then $X = \{a, b\}$ is a cover of L, hence $\mathfrak{o}(a \vee b) = \mathfrak{o}(1)$ and therefore $L = \mathfrak{o}(a) \vee \mathfrak{o}(b)$, where $\mathfrak{o}(a), \mathfrak{o}(b) \neq \{1\}$. Thus there exists a simple chain of open sublocales in $X \subseteq L$ joining $\mathfrak{o}(a)$ to $\mathfrak{o}(b)$. This means that $\mathfrak{o}(a) \cap \mathfrak{o}(b) \neq \{1\}$, thus $a \wedge b \neq \{1\}$, which is a contradiction. Hence either $a = 0$ or $b = 0$, and so L is connected.

□

The results which follow, are concerned with useful properties of the images of sublocales under the right adjoint of a given frame homomorphism.

For the remainder of this section, $h_* : M \to L$ shall denote the right adjoint of $h : L \to M$, where h is a frame homomorphism.

Proposition 6.1.10. *If $h : L \to M$ is any frame homomorphism, $a \in L$, and T is a sublocale of M, then :*

(1) $h_*(T) \subseteq \uparrow a \Longleftrightarrow T \subseteq \uparrow h(a)$,

(2) $h_*(T) \cap \uparrow a = \{1_L\} \Longleftrightarrow T \cap \uparrow h(a) = \{1_M\}$,

(3) $h_*(T) \subseteq \mathfrak{o}(a) \Longleftrightarrow T \subseteq \mathfrak{o}(h(a))$.

Proof.

(1) :

$$\begin{aligned} h_*(T) \subseteq \uparrow a &\Longleftrightarrow h_*(t) \in \uparrow a \quad \text{for all } t \in T \\ &\Longleftrightarrow a \leq h_*(t) \quad \text{for all } t \in T \\ &\Longleftrightarrow h(a) \leq t \quad \text{for all } t \in T \\ &\Longleftrightarrow t \in \uparrow h(a) \quad \text{for all } t \in T \\ &\Longleftrightarrow T \subseteq \uparrow h(a). \end{aligned}$$

(2) : ($\Longrightarrow$) Suppose $h_*(T) \cap \uparrow a = \{1_L\}$. Take any $t \in T \cap \uparrow h(a)$. Then $h(a) \leq t$, so $a \leq h_*(t)$. This implies that $h_*(t) \in h_*(T) \cap \uparrow a$, thus $h_*(t) = 1_L$. So $1_M = hh_*(t) \leq t$, therefore $t = 1_M$ and $T \cap \uparrow h(a) = \{1_M\}$.
($\Longleftarrow$) Suppose $T \cap \uparrow h(a) = \{1_M\}$ and take $h_*(t) \in h_*(T) \cap \uparrow a$. Then $a \leq h_*(t)$, so $h(a) \leq t$. Thus $t \in T \cap \uparrow h(a)$, and so $t = 1_M$. Then $h_*(t) = 1_L$, hence $h_*(T) \cap \uparrow a = \{1_L\}$.

(3) : Since $\mathfrak{o}(a)$ and $\uparrow a$ are complements of one another in $\mathcal{S}(L)$ by Proposition 2.2.14, $h_*(T) \subseteq \mathfrak{o}(a)$ if and only if $h_*(T) \cap \uparrow a = \{1_L\}$. From part (2), $h_*(T) \cap \uparrow a = \{1_L\}$ if and only if $T \cap \uparrow h(a) = \{1_M\}$. Now $T \cap \uparrow h(a) = \{1_M\}$ if and only if $T \subseteq \mathfrak{o}(h(a))$, since $\mathfrak{o}(h(a))$ and $\uparrow h(a)$ are complements. Hence we have shown that $h_*(T) \subseteq \mathfrak{o}(a)$ if and only if $T \subseteq \mathfrak{o}(h(a))$.

□

Proposition 6.1.11 ([20]). *Let $h : L \to M$ be a frame homomorphism and T be a sublocale of M, then $h_*(T)$ is a sublocale of L.*

Proposition 6.1.12. *If $h : L \to M$ is any frame homomorphism and T is a connected sublocale of M, then $h_*(T)$ is a connected sublocale of L.*

Proof.
By Proposition 6.1.11, $h_*(T)$ is a sublocale of L. We will show that $h_*(T)$ is connected in L. Suppose $h_*(T) \subseteq \mathfrak{o}(a) \vee \mathfrak{o}(b)$ and $h_*(T) \cap \mathfrak{o}(a) \cap \mathfrak{o}(b) = \{1_L\}$ for $a, b \in L$. Then $h_*(T) \subseteq \mathfrak{o}(a \vee b)$ and $h_*(T) \cap \mathfrak{o}(a \wedge b) = \{1_L\}$. By (3) of Proposition 6.1.10, $T \subseteq \mathfrak{o}(h(a \vee b))$, therefore $T \subseteq \mathfrak{o}(h(a)) \vee \mathfrak{o}(h(b))$. Also, since $\mathfrak{o}(a \wedge b)$ and $\uparrow (a \wedge b)$ are complements in $\mathcal{S}(L)$, then

$$
\begin{aligned}
h_*(T) \cap \mathfrak{o}(a \wedge b) = \{1_L\} &\implies h_*(T) \subseteq \uparrow (a \wedge b) \\
&\implies T \subseteq \uparrow h(a \wedge b) \quad \text{(by (1) of Proposition 6.1.10)} \\
&\implies T \cap \mathfrak{o}(h(a \wedge b)) = \{1_M\} \\
&\implies T \cap \mathfrak{o}(h(a)) \cap \mathfrak{o}(h(b)) = \{1_M\}.
\end{aligned}
$$

By the connectedness of T, we have $T \cap \mathfrak{o}(h(a)) = \{1_M\}$, say. Hence $T \subseteq \uparrow h(a)$, and by (1) of Proposition 6.1.10, $h_*(T) \subseteq \uparrow a$. Thus $h_*(T) \cap \mathfrak{o}(a) = \{1_L\}$ and hence we have shown that $h_*(T)$ is connected as a sublocale in L.

□

6.2 Locally non-separating sublocales

In this section we will present the definition of a locally non-separating sublocale and prove natural consequences of it. Throughout this section, we shall assume that L is a locally connected frame.

Definition 6.2.1. A non-trivial sublocale A of L is called *locally non-separating sublocale* in L, if whenever $\{1\} \neq U \subseteq L$ is an open connected sublocale then $U \setminus A \neq \{1\}$ and $U \setminus A$ is connected as a sublocale.

The following result is required to show that every non-trivial sublocale of a locally non-separating sublocale is locally non-separating.

Proposition 6.2.2. *Let S and T be sublocales of L. $S \subseteq \overline{T}$ if and only if for every non-trivial open sublocale $\mathfrak{o}(a)$ of L such that $\mathfrak{o}(a) \cap S \neq \{1\}$ then $\mathfrak{o}(a) \cap T \neq \{1\}$.*

Proof.
($\implies$) Suppose that $S \subseteq \overline{T}$, and let $\mathfrak{o}(a)$ be a non-trivial open sublocale of L such that $\mathfrak{o}(a) \cap S \neq \{1\}$. Take $x \in \mathfrak{o}(a) \cap S$ such that $x \neq 1$. Then $x = a \to x$. Now $x \in \overline{T} = \uparrow 0_T$, where $0_T = \bigwedge T \in T$, therefore $0_T \leq x$. Now $a \to 0_T \in \mathfrak{o}(a) \cap T$. If $a \to 0_T = 1$, then

$a \leq 0_T \leq x$. This implies that $a \to x = 1$ and hence $x = 1$, which is a contradiction. So $a \to 0_T \neq 1$, and therefore $\mathfrak{o}(a) \cap\ T \neq \{1\}$.

($\Longleftarrow$) Suppose that for every non -trivial open sublocale $\mathfrak{o}(a)$ of L, whenever $\mathfrak{o}(a) \cap S \neq \{1\}$ then $\mathfrak{o}(a) \cap T \neq \{1\}$. Take any $x \in S$. We will show that $0_T \leq x$, where $0_T = \bigwedge T$. We may assume that $x \neq 1$. Thus we will show that $0_T \to x = 1$. Assume that $0_T \to x \neq 1$. Now $0_T \to x \in \mathfrak{o}(0_T) \cap S$, so by the hypothesis $\mathfrak{o}(0_T) \cap T \neq \{1\}$. Hence there exists $t \in T$, $t \in \mathfrak{o}(0_T)$ with $t \neq 1$. Since $t \in \mathfrak{o}(0_T)$, $t = 0_T \to t$, and since $0_T \leq t$, it follows that $t = 1$, which is a contradiction. Thus $0_T \to x = 1$ and $0_T \leq x$, as required.

□

Proposition 6.2.3. *Let A and B be sublocales of L such that $\{1\} \neq B \subseteq A$. If A is locally non-separating in L then B is locally non-separating in L.*

Proof.

Take any non-trivial open connected sublocale U of L. Now $U \setminus A \neq \{1\}$, and

$$\begin{aligned} U \setminus A &= \bigvee\{T \mid T \text{ is a sublocale of } \ U,\ T \cap A = \{1\}\} \\ &\leq \bigvee\{T \mid T \text{ is a sublocale of } \ U,\ T \cap B = \{1\}\} \\ &= U \setminus B. \end{aligned}$$

So $U \setminus B \neq \{1\}$. We will now show that $U \setminus B$ is connected. In order to show this, we show that $U \setminus A \subseteq U \setminus B \subseteq \overline{U \setminus A}$. By Proposition 2.2.30 it will follow that $U \setminus B$ is connected, since $U \setminus A$ is connected.

Claim: $U \subseteq \overline{U \setminus A}$.

Let $U = \mathfrak{o}(x)$, for some $x \in L$. Since $U \neq \{1\}$, then $x \neq 1$. Let $\mathfrak{o}(z) \neq \{1\}$ and $\mathfrak{o}(z) \cap\ \ U \neq \{1\}$. We shall show that $\mathfrak{o}(z) \cap\ \ (U \setminus A) \neq \{1\}$, for then it follows by Proposition 6.2.2 that $U \subseteq \overline{U \setminus A}$. Now $\mathfrak{o}(z) \cap U = \mathfrak{o}(z) \cap \mathfrak{o}(x) = \mathfrak{o}(z \wedge x) \neq \{1\}$. By local connectedness of L, $z \wedge x = \bigvee\{w \in C \mid C \subseteq L,\ C$ consists of connected elements$\}$. So $\mathfrak{o}(z \wedge x) = \bigvee\{\mathfrak{o}(w) \mid w \in C \subseteq L,\ C$ consists of connected elements$\}$. Since $\mathfrak{o}(z \wedge x) \neq \{1\}$, there exists $w \in C$, $w \leq z \wedge x$ such that $\mathfrak{o}(w) \neq \{1\}$ and $\mathfrak{o}(w)$ connected. Since A is locally non-separating, we have that $\mathfrak{o}(w) \setminus A \neq \{1\}$, that is,

$$\bigvee\{T \mid T \text{ is a sublocale of } \mathfrak{o}(w),\ T \cap A = \{1\}\} \neq \{1\}.$$

Hence, there exists a sublocale T of $\mathfrak{o}(w)$, $T \cap A = \{1\}$ such that $T \neq \{1\}$. Now $\mathfrak{o}(w) \subseteq \mathfrak{o}(z)$ is a sublocale of $\mathfrak{o}(z)$, so T is a sublocale of $\mathfrak{o}(z)$. Also, $\mathfrak{o}(w)$ is a sublocale of $\mathfrak{o}(x) = U$, and this implies that T is a sublocale of U. So $\{1\} \neq T \subseteq \mathfrak{o}(z) \cap (U \setminus A)$ and hence $\mathfrak{o}(z) \cap (U \setminus A) \neq \{1\}$. Thus by Proposition 6.2.2, $U \subseteq \overline{U \setminus A}$, as claimed. Hence $\{1\} \neq U \setminus A \subseteq U \setminus B \subseteq U \subseteq \overline{U \setminus A}$, and therefore by Proposition 2.2.30, $U \setminus B$ is connected.

$\square$

Theorem 6.2.4. *Let $B \subseteq L$ be a base of L consisting of connected elements. Suppose $A \neq \{1\}$ is a sublocale of L and that $\mathfrak{o}(b) \setminus A \neq \{1\}$ is connected for each $b \in B$. Then A is locally non-separating in L.*

Proof.

Let U be a non-trivial open connected sublocale of L. Then $U = \mathfrak{o}(a)$ for some $a \in L$. We will show that $U \setminus A \neq \{1\}$ and $U \setminus A$ is connected as a sublocale. Since B is a base, $a = \bigvee\{b \mid b \in B'\}$ for some $B' \subseteq B$, therefore $\mathfrak{o}(a) = \bigvee\{\mathfrak{o}(b) \mid b \in B'\}$. By Lemma 6.1.5, $U \setminus A = (\bigvee\{\mathfrak{o}(b) \mid b \in B'\}) \setminus A = \bigvee_{b \in B' \subseteq B}(\mathfrak{o}(b) \setminus A)$. Hence $U \setminus A \neq \{1\}$, since $\mathfrak{o}(b) \setminus A \neq \{1\}$, for all $b \in B'$.

We now show that $U \setminus A$ is connected. Now $a = \bigvee\{b \mid b \in B'\}$, for some $B' \subseteq B$. Thus we have $U \setminus A = \bigvee\{\mathfrak{o}(b) \setminus A \mid b \in B'\}$. Now the collection $\{\mathfrak{o}(b) \setminus A \mid b \in B'\}$ is a collection of connected sublocales of L. We shall use Proposition 2.2.31 to show that $\bigvee\{\mathfrak{o}(b) \setminus A \mid b \in B'\}$ is connected. Take any $\mathfrak{o}(b_i) \setminus A$, $\mathfrak{o}(b_j) \setminus A$ from this collection. Then $b_i, b_j \in B'$. Now the frame $\downarrow a$ is connected, since a is connected. Also $B' \subseteq \downarrow a$, and $a = \bigvee\{b \mid b \in B'\}$ makes B' a cover of $\downarrow a$ consisting of connected elements. From Proposition 6.1.8 there exists a simple chain $b_1, b_2, ..., b_n$ of elements from B' such that $b_i \wedge b_1 \neq 0$, $b_k \wedge b_{k+1} \neq 0$ for $k = 1, 2, ..., n-1$, and $b_n \wedge b_j \neq 0$.

<u>Claim</u>: $b_k \wedge b_{k+1} \neq 0 \Longrightarrow (\mathfrak{o}(b_k) \setminus A) \cap (\mathfrak{o}(b_{k+1}) \setminus A) \neq \{1\}$.

$b_k \wedge b_{k+1} = \bigvee\{c \mid c \text{ connected}, c \in C\}$ for some $C \subseteq B$. Pick any $0 \neq c \in C$, then $c \leq b_k \wedge b_{k+1}$. Now $\mathfrak{o}(c) \setminus A \neq \{1\}$, so there exists a sublocale T such that $T \subseteq \mathfrak{o}(c), T \cap A = \{1\}$ and $T \neq \{1\}$. Then $T \subseteq \mathfrak{o}(b_k), T \cap A = \{1\}$ and $T \neq \{1\}$, so T is a sublocale of $\mathfrak{o}(b_k) \setminus A$. Similarly T is a sublocale of $\mathfrak{o}(b_{k+1}) \setminus A$. Hence $(\mathfrak{o}(b_k) \setminus A) \cap (\mathfrak{o}(b_{k+1}) \setminus A) \neq \{1\}$, as claimed.

Thus $\mathfrak{o}(b_i) \setminus A$, $\mathfrak{o}(b_1) \setminus A$, $\mathfrak{o}(b_2) \setminus A$, . . . , $\mathfrak{o}(b_n) \setminus A$, $\mathfrak{o}(b_j) \setminus A$ is a simple chain of connected sublocales joining $\mathfrak{o}(b_i) \setminus A$ to $\mathfrak{o}(b_j) \setminus A$. By Proposition 2.2.31, $\bigvee\{\mathfrak{o}(b) \setminus A \mid b \in B'\}$ is connected, so $U \setminus A$ is connected. Hence A is locally non-separating in L. $\square$

6.3 A Peano compactification with a locally non-separating remainder

Curtis established, in [12], that a connected space X having a Peano compactification with a specified *locally non-separating remainder* is equivalent to the space X being S-metrizable. In this section we provide a generalisation of the above result under the assumption of L being a *regular continuous* frame. In order to do so, we first define a *locally non-separating remainder* of a frame and recall important known theory.

Definition 6.3.1. Let S be a sublocale of L. Then $L \setminus S$ is called a *locally non-separating remainder* if $L \setminus S$ is locally non-separating in L.

We recall the following definition from Banaschewski [7]:

Definition 6.3.2. A regular frame L is said to be *continuous* if for every $a \in L$, we can write a as

$$a = \bigvee\{x \in L \mid x << a\},$$

where $x << a$ means that whenever $a \leq \bigvee S$, for some $S \subseteq L$, then there exists $s_1, s_2, ..., s_n \in S$ such that $x \leq s_1 \vee ... \vee s_n$.

Proposition 6.3.3. *Let $h : L \to M$ be an onto frame homomorphism, then for all $a \in L$*

we have that

$$h_*(\mathfrak{o}(h(a))) = \mathfrak{o}(a) \cap h_*(M).$$

Proof.

We observe that $h_*(\mathfrak{o}(h(a))) \subseteq \mathfrak{o}(a) \cap\ h_*(M)$, since $h_*(\mathfrak{o}(h(a))) \subseteq \mathfrak{o}(a)$ by (3) of Proposition 6.1.10 and $h_*(\mathfrak{o}(h(a))) \subseteq h_*(M)$.

We now show that $\mathfrak{o}(a) \cap h_*(M) \subseteq h_*(\mathfrak{o}(h(a)))$. Take $h_*(x) \in \mathfrak{o}(a) \cap h_*(M)$, for some $x \in M$. Then $h_*(x) = a \to h_*(x)$. We shall show that $x = h(a) \to x$. Now for $y \in M$,

$$\begin{aligned}
y \leq h(a) \to x &\iff y \wedge h(a) \leq x \\
&\iff hh_*(y) \wedge h(a) \leq x \quad \text{(since } h \text{ is onto and by Proposition 2.1.14)} \\
&\iff h(h_*(y) \wedge a) \leq x \\
&\iff h_*(y) \wedge a \leq h_*(x) \\
&\iff h_*(y) \leq a \to h_*(x) = h_*(x) \\
&\iff y \leq x \quad \text{(since } h \text{ is onto).}
\end{aligned}$$

Hence we have shown that $x = h(a) \to x$, so $x \in \mathfrak{o}(h(a))$. Hence $h_*(x) \in h_*(\mathfrak{o}(h(a)))$ and therefore $\mathfrak{o}(a) \cap h_*(M) \subseteq h_*(\mathfrak{o}(h(a)))$.

□

Recall that a frame homomorphism $h : L \to M$ is said to be *open* precisely when $h_*(U)$ is an open sublocale of L, for every open sublocale U of M.

Corollary 6.3.4. *If $h : L \to M$ is an onto frame homomorphism and $h_*(M)$ is an open sublocale of L, then h is an open map.*

Proof.

Let U be an open sublocale of M. Since h is onto, pick $a \in L$ such that $U = \mathfrak{o}_M(h(a))$. Then, by Proposition 6.3.3, $h_*(U) = \mathfrak{o}_L(a) \cap h_*(M)$; an open sublocale of L. Therefore h is an open map. □

Given a compactification of a non-compact regular continuous frame, Baboolal [3], obtained the following characterisation.

Theorem 6.3.5 ([3]). *Let $h : L \to M$ be any compactification of M, where M is non-compact. Then M is regular continuous if and only if $M \cong \downarrow a$ for some $a \in L$.*

Remark 6.3.6.

We note from the proof of Theorem 6.3.5 that $a = \bigvee\{h_*(x) \mid x << 1_M\}$, and that the frame isomorphism is given by

$$g : \downarrow a \longrightarrow M$$
$$x \mapsto h(x).$$

Proposition 6.3.7. *Let $h : L \to M$ be a compactification of M, where M is non-compact and regular continuous. If $a = \bigvee\{h_*(x) \mid x << 1_M\}$, then $h_*(M) = \mathfrak{o}(a)$. Hence h is an open map.*

Proof.

From Theorem 6.3.5, we note that the map

$$g : \downarrow a \longrightarrow M$$
$$x \mapsto h(x)$$

is frame isomorphism. Since g is an isomorphism, $g(a) = 1_M$, but $g(a) = h(a)$ therefore $h(a) = 1_M$.Thus $h : L \to M$ factors as

$$\begin{tikzcd} L \arrow[rr, "h"] \arrow[dr, "f"'] & & M \\ & \downarrow a \arrow[ur, "g"'] & \end{tikzcd}$$

where f is the frame map $f(x) = a \wedge x$, and g the frame isomorphism. We note that $gf = h$, since $gf(x) = g(f(x)) = g(a \wedge x) = h(a \wedge x) = h(a) \wedge h(x) = 1_M \wedge h(x) = h(x)$, for $x \in L$. So since $h = gf$, it follows from Lemma 2.1.15 that $h_* = f_* g_*$. Thus $h_*(M) = f_*(g_*(M)) = f_*(\downarrow a)$, since g is an isomorphism making $g_*(M) = \downarrow a$. Now for $b \in L$, $c \in \downarrow a$,

$$b \leq f_*(c) \Longleftrightarrow f(b) \leq c \Longleftrightarrow a \wedge b \leq c \Longleftrightarrow b \leq a \to c.$$

Thus $f_*(c) = a \to c$. Hence $f_*(\downarrow a) = \{f_*(c) \mid c \leq a\} = \{a \to c \mid c \leq a\} \subseteq \mathfrak{o}(a)$. But also, $\mathfrak{o}(a) \subseteq \{a \to c \mid c \leq a\}$, since by Proposition 2.1.24, $a \to (x \wedge a) = (a \to x) \wedge (a \to a) = (a \to x) \wedge 1_L = a \to x$. So $a \to x \in f_*(\downarrow a)$. Thus $f_*(\downarrow a) = \mathfrak{o}(a)$, and hence $h_*(M) = \mathfrak{o}(a)$.

□

Definition 6.3.8. Let $h : L \longrightarrow M$ be a compactification of M. The compactification (L, h) is called a *Peano compactification* of M, if L is compact, connected, locally connected and metrizable.

The following theorem is the main result of this chapter.

Theorem 6.3.9. *Suppose that M is a non-compact, connected and regular continuous frame. Then M has a Peano compactification $h : L \to M$ with a locally non-separating remainder $L \setminus h_*(M)$ if and only if M is S-metrizable.*

Proof.
($\Longrightarrow$) Suppose that M has a Peano compactification $h : L \to M$ with a locally non-separating remainder $L \setminus h_*(M)$. Then for any non-trivial open connected sublocale U of L, we have that $U \setminus (L \setminus h_*(M)) \neq \{1_L\}$ and is connected. By Proposition 6.3.7, $h_*(M) = \mathfrak{o}(a)$, where $a = \bigvee\{h_*(x) \mid x << 1_M\}$, and so $h_*(M)$ is complemented. Thus Lemma 6.1.4 implies that $U \setminus (L \setminus h_*(M)) = U \cap h_*(M)$, and so $U \cap h_*(M) \neq \{1_L\}$ and is connected as a sublocale in L. Take any compatible metric diameter ρ on L, and let $\varepsilon > 0$. Since L is compact and locally connected, then L must have Property S by Theorem 3.1.8. So there exists connected $a_1, a_2, ..., a_n \in L$, such that $a_1 \vee ... \vee a_n = 1_L$, and $\rho(a_i) < \varepsilon$ for $i = 1, ..., n$. Let d be the metric diameter on M induced by ρ, given by $d(h(a)) = \rho(a)$

for $a \in L$. Then $d(h(a_i)) = \rho(a_i) < \varepsilon$, for $i = 1, ..., n$. Now $\mathfrak{o}(a_i) \cap h_*(M) \neq \{1_L\}$ and is connected for each i. Hence $\mathfrak{o}(a_i) \cap \mathfrak{o}(a) = \mathfrak{o}(a_i \wedge a) \neq \{1_L\}$ and is connected. Thus $a_i \wedge a \neq 0_L$ is connected in L for each i. Now $\downarrow a \cong M$ via the map g, by Theorem 6.3.5. So $h(a_i \wedge a) = g(a_i \wedge a)$ is connected in M, and since $h(a) = 1_M$, we conclude that $h(a_i)$ is connected in M for each i. Then $1_M = h(a_1) \vee ... \vee h(a_n)$, and $d(h(a_i)) < \varepsilon$ for each i. So (M, d) has Property S, and hence M is S-metrizable.

($\Longleftarrow$) Suppose that (M, d) is S-metrizable. Then by Theorem 5.3.19, (M, d) has a perfect connected, locally connected metrizable compactification $h : (L, \rho) \longrightarrow (M, d)$. Hence (L, ρ) is a Peano compactification of M. We need to show that $L \setminus h_*(M)$ is locally non-separating. Now, by Proposition 6.3.7, $h_*(M) = \mathfrak{o}(a)$, where $a = \bigvee\{h_*(x) \mid x << 1_M\}$. Since $h_*(M)$ is complemented, for any non-trivial open connected sublocale U of L, Lemma 6.1.4 implies that $U \setminus (L \setminus h_*(M)) = U \cap h_*(M)$. Now $h_*(M)$ is dense in L, since

$$\begin{aligned} \overline{h_*(M)} &= \overline{\mathfrak{o}(a)} = \uparrow a^* \quad \text{(by Proposition 2.2.24)} \\ &= \uparrow (\bigvee\{h_*(x) \mid x << 1_M\})^* \\ &= \uparrow \bigwedge\{h_*(x)^* \mid x << 1_M\} \quad \text{(by Proposition 2.1.21)} \\ &= \uparrow \{0_L\} = L \quad (\text{since } 1_M << 1_M \text{ and } h_*(1_M)^* = 0_L). \end{aligned}$$

So by Lemma 6.1.6, $U \setminus (L \setminus h_*(M)) \neq \{1_L\}$. We now show that $U \setminus (L \setminus h_*(M))$ is connected in L. Since $U \neq \{1_L\}$ is an open connected sublocale, then $U = \mathfrak{o}(x)$, for some connected $x \in L$, $x \neq 0_L$. Now since $h : L \longrightarrow M$ is a perfect compactification and $x \in L$ is connected, then by Proposition 5.1.4, $h(x)$ is connected in M. Thus $\mathfrak{o}(h(x))$ is connected as a sublocale in M, and by Proposition 6.1.12, $h_*(\mathfrak{o}(h(x)))$ is connected in L as a sublocale. But $h_*(\mathfrak{o}(h(x))) = \mathfrak{o}(x) \cap h_*(M) = U \cap h_*(M)$, by Proposition 6.3.3. So $U \cap h_*(M) = U \setminus (L \setminus h_*(M)$ is connected. Thus $L \setminus h_*(M)$ is a locally non-separating remainder. $\square$

Remark 6.3.10. We note that Curtis's result in spaces required fewer assumptions and hence is more general. The analog of his result in frames, without the additional assumption of M being a regular continuous frame, remains an open problem.

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