An investigation into learner understanding of the properties of selected quadrilaterals using manipulatives in a grade eight mathematics class

by

Rakesh Issardeo Singh

Submitted in partial fulfilment of the academic
requirements for the degree of
Master of Education in the
School of Science, Mathematics, and Technology Education
Faculty of Education
University of KwaZulu-Natal

November 2006

ABSTRACT

Benchara Blandford as quoted in Griffiths & Howson (1974) has provided the researcher with the inspiration to seek new methods of trying to improve the teaching and learning of geometry:

> "To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions is both interesting and highly educational".

Freudenthal (1973), who states that "the child should not be deprived of this privilege", further echoes this thinking. Recent literature on mathematics education, more especially on the teaching and learning of geometry, indicates a dire need for further investigations into the possibility of devising new strategies, or even improving present methods, in order to curb the problems that most learners have in geometry. It would seem that most educators and textbooks eschew the use of concrete manipulatives to teach important geometrical concepts, as they feel it is time-consuming and unnecessary since it creates noisier classrooms. In some cases, the educators have not been trained in the use of these manipulatives.

This study intends highlighting the many uses that tangrams (a Chinese puzzle) have in enhancing learners' understanding of the properties of the square and rectangle, including the properties of their diagonals. The researcher also intends showing that the tangram pieces are an important cog in the wheel that keeps the geometry thinking and reasoning

process ticking. The days of "kill and drill" are over because the tangram will soon become an interesting and stimulating manipulative that can effectively be used to teach important geometrical concepts and definitions. Not only will learners find it fun to work with, but it will also provide an alternate means of learning since it is not monotonous. It will create an environment which learners will find relaxing and enjoyable to work in, and consequently, promote collaborative learning. The tangram can be used as an important assessment tool; however, this investigation goes beyond the scope and intention of this study.

Several useful implications have evolved from this study which may influence both the teaching and learning of geometry in school. Perhaps the suggestions made may be useful not only to educators, but to important stakeholders in policy-making as well. If these ideas can be incorporated in drafting the geometry curriculum, I am sure geometry will not be regarded as the stumbling block for many aspiring mathematics learners who are striving for an "A" symbol in the mathematics examination. The researcher has used action research and a task-based interview process with ten grade 8 learners to show that the use of manipulatives, namely tangrams, has been effective in enhancing learners understanding of the properties of the square and rectangle. In addition, tangrams can go a long way in helping learners achieve van Hiele level 3. The learners interviewed were able to develop a good understanding of the properties of the square and rectangle resulting in remarkable improved pre-test scores. Furthermore, the investigation reaffirmed the practice that learners can be effectively taught from the general to the more specific, enabling them to develop a better understanding of concepts being taught.

PREFACE

The work described in this thesis was carried out in the School of Science, Mathematics and Technology Education, University of KwaZulu-Natal, from **March 2005** to **November 2006** under the supervision of Dr **V. Mudaly** (appointed supervisor).

This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

Rakesh Issardeo Singh

November 2006

Dr Vimolan Mudaly

(Supervisor)

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGEMENT

- I am eternally grateful to my supervisor and mentor Dr Vimolan Mudaly, for his constant encouragement, time, dedication, and guidance throughout my studies.
- To my wife Jenny for her moral support, and her invaluable contribution towards my dissertation.
- To my two children, Sunveer and Urishka Singh, for their patience at being 'fatherless' for prolonged periods.
- To Sunil Kheru, for consenting to the use of his facilities and acceding to my numerous requests for assistance.
- To Paul Singh for his assistance in proof reading and editing my dissertation.
- To all other members of my family for their support and assistance rendered.
- To those whose names I have omitted.
- To all the learners who have consented to being interviewed.

Dedication

To:

My late mother, Dayawathey Singh, who was the original source of my inspiration.

CHAPTER 1

OVERVIEW

The following Chinese proverb provided the researcher with the ideal inscription to introduce the chosen topic and intentions in using manipulatives, namely the tangram: *'I hear, and I forget. I see and I remember. I do, and I understand."* We often hear things and forget because we do not commit them to memory easily. If we see things, we also tend to forget them because we see so many different things everyday, but if we actually do things ourselves then we tend to internalise these things and retain it for longer periods because we understand them. Experiential education is based on the idea that active involvement enhances students' learning. One way of bringing experience to bear on students' mathematical understanding is the use of manipulatives and engaging learners in activities that will allow them to make conclusions from their practical observations.

According to the Van Hieles'report, the main reason for the failure of the traditional school geometry curriculum was attributed to the fact that the curriculum was presented at a higher level than those of the learners. In other words, the learners could not understand the teacher and the teacher could not understand why they could not understand! Therefore, I intend showing that by using basic manipulatives like the tangram and paper-folding one can start developing basic concepts by using resources that learners are able to feel, touch and visualise.

It is no secret that high school geometry with its formal proof is considered difficult and is very detached from practical life. Many teachers and experts have tried different teaching methods and programmes to make learners understand formal geometry, sometimes with success and sometimes not. Many geometry problems in textbooks follow the typical questioning format of, "calculate the area, circumference, perimeter and radius" of this figure. More emphasis is placed on calculations and the use of formulae rather than the analysis of concepts, making conjectures about the properties, testing them, analysing various types of figures and shapes experimentally.

Representations have always been a very important part of teaching mathematics. These manipulatives are not only visual but also provide hands-on experiences, which help teachers in relaying important topics and concepts to the students. By using manipulatives, learners are more likely to remember what they have learned, and can recall content material when completing homework and during tests. As children build and experiment with manipulative materials, they discover and develop richer ways of thinking about mathematical concepts such as number, size and shape.

The **manipulatives**[1] discussed throughout this thesis refers to the tangible materials that the learners physically handled to assist them to see actual examples of mathematical principles at work. Research indicates that manipulatives are particularly useful in helping children's transition from the concrete to the abstract level. Manipulatives are especially useful in introducing or reinforcing a mathematical concept according to Hartshorn and Boren (1990).

---

[1] Annexure A

From the researcher's experience and interaction with many mathematics teachers during meetings and workshops it is commonly agreed that many learners find mathematics, especially geometry, dull, boring and irrelevant to their lives. There is also consensus that many learners find geometry difficult to understand. Manipulatives help relieve boredom in learners as they offer a change from the textbook (abstract) method of learning, thereby allowing learners to explore and use their imagination. Manipulatives provide a picture of a mathematics concept that appeals to visual/spatial learners and they provide stimulation for those who are not. Visualisation is the natural way one begins to think, since before actually verbalising (using words), images first need to emerge before one can write or speak. Manipulatives can also be placed within cooperative groups, which is appealing to the interpersonal learners.

Many factors can be given for feelings of negativity and the rejection towards geometry. However, this research attempts to show that the use of manipulatives, namely tangrams, will contribute to an increased understanding of geometric definitions and properties. This will create an environment that will lead to active participation and discovery, which will make learning more enjoyable and meaningful. The intention of the researcher is to show that manipulatives are tools, which would increase the understanding of mathematical concepts instead of being used as a mere teaching tool. The seven pieces that make up the tangram set have value well beyond their small size. One of their most important values, other than providing educational entertainment to students, is the introduction of geometric properties and theorems.

Tangrams can be used to provide numerous worthwhile mathematical experiences for children. They not only offer reluctant teachers a simple but exciting means of introducing geometric concepts, but they are also excellent vehicles for learners and teachers alike to engage in tasks that foster spatial visualisation. According to Rigdon (2000) the concept of tangrams originated from the following Chinese folktale:

> "A young boy named Tan wished to give the emperor a beautiful tile. As he carries the tile to the palace, he accidentally drops it and it breaks into seven pieces. Tan tries and tries to restore the tile to its original shape. In the process, he finds out that he can create all sorts of fascinating pictures with the seven pieces of this tile. The seven tile pieces are what are now called tangrams".

This story illustrates that tangrams can be used as a method of guided discovery as well as for enjoyment, since many pictures can be formed. It also helps learners to see why they are able to form a square, which forms the basis for this particular study. By having learners use investigation, a teacher is able to establish a trusting classroom atmosphere as well as have the learners see that abstract concepts are very meaningful. (Thatcher, 2001)

According to Ernest (1986), "*The success of all mathematics teaching depends to a large extent on the active involvement of the learner. Children learn mathematics by doing and by making the concepts and skills of mathematics their own.*" Further, he believes that games and manipulatives encourage the active involvement of children thereby making them more receptive to learning and increasing their motivation.

Students' understanding will increase if they are actively engaged in tasks and experiences designed to deepen and connect their knowledge of mathematical concepts. The researcher has actively engaged learners by asking probing questions, which has encouraged discussion and collaboration between learners. Learners were asked to justify their thinking and to suggest alternatives where necessary.

According to Stevens (2004), "visual-spatial ability is how we understand and manipulate what we see which is fundamental to almost all school learning. In science, you cannot understand how the solar system works, unless you can visualize the interrelationships among the planets. In mathematics, you need to picture what a fraction means or what it means to carry a remainder". Further, Stevens says, "even though visual-spatial skills figure prominently in aptitude tests and are closely linked to success in many professions, schools neglect teaching them".

Moses (1990) highlights the fact that the spatial development in the primary school was compromised for other disciplines of mathematics. According to her, the NCTM Commission on Standards for School Mathematics (1989) clearly states that teachers should devote less attention to complex paper-and-pencil computations and rote learning of rules. The time currently spent on these topics should instead be devoted to areas such as geometry and problem solving. Learners should be afforded the opportunity to visualise and represent geometric figures with special attention to developing spatial sense. Moses (1990) further criticises the fact that school mathematics textbooks contain few activities that deal with spatial sense. Wheatley (1990) concurs with the above assertion that a review of the United States school mathematics shows that rules, procedures and analytical reasoning dominate the curriculum and that little attention is given to spatial visualization.

According to van Niekerk (1997), South Africa inherited its geometry from England at a time when the teaching of geometry in England was more conservative than in any other country in the world. In the 1930s, Euclid was followed more closely in England than elsewhere. Any informal approach to the teaching of geometry in high school was regarded as a waste of time. Theorems were introduced as early as possible. She states that van Zyl (1942) felt that the earlier introduction of informal geometry was desirable but the fact that primary school teachers were not trained to facilitate this at university put the brakes on this idea.

According to research done by Nakin (2003) in his thesis *Creativity and Divergent Thinking in Geometry Education,* he found that the mathematics syllabi of several universities in South Africa, as evidenced in their calendars - UDW(1998), RAU(1999), Vista (1996), Transkei (1997) and North (2000) reflected no geometry. He therefore concluded that university graduates were not prepared for the teaching of geometry in the schools. Even in the primary schools, the teaching of geometry especially, informal geometry, is underemphasized.

Nakin (2003), goes on to pose the question: "*Could the high failure rate in matric paper 2, of which geometry forms the bulk, be a result of the status quo as described above or do teachers emphasise paper 1 at the expense of paper 2 (they handle paper 1 first and then run out of time for paper 2)?*" However, as this did not form part of his study, he did not present an answer to this relevant question. The researcher is of the view that many teachers are afraid to engage in too much geometry since they lack the confidence in teaching it. Furthermore, since they find many learners having difficulty understanding geometry, they opt to skim the surface of geometry and focus on algebra.

According to Freudenthal (1971), learners failed geometry because it was taught in such a manner that its deductivity could not be reinvented by the learner, but rather imposed. For Freudenthal, starting with axioms and theorems is a wrong approach to the teaching of geometry (Streefland, 1993). Indeed, starting with the fine polished product such as axioms and theorems (the generalizations), denies learners the opportunity of finding out how such theorems or axioms were arrived at. The starting point should be from the child's everyday life experiences of spatial objects (his reality).

Dina van Hiele-Geldof, a student of Freudenthal's experiments, emphasised the importance of the re-invention of geometry and not its imposition during instruction. She did not start with definitions of triangles, squares and rectangles but allowed children to discover the properties of these shapes and hence be in a position to define them by themselves (re-invention/self-discovery). This is a more meaningful approach to the teaching of geometry. The importance of this approach is supported by Presmeg (1989) who stated that, seeing patterns in spatial objects in various geometric shapes and lines of symmetry can empower learners with skills that are essential for solving problems in Euclidean geometry.

Hershkowitz et al (1996) stated that visual education is important for learners to be able to interact effectively with shapes. Some learners who are not gifted at symbolic thinking need to visualize and interact with a given phenomenon (actually see, feel or have a mental picture) in order to understand it. Visual ways of thinking and reasoning are thus important for such learners. Herschkowitz et al (1996) stated that visual thinking and reasoning may be acquired through a well-planned visual education but criticizes the fact that visual education is often a neglected area in curricula.

The researcher therefore intends showing that the simple direct instruction and "kill and drill" and rote memorisation methods of teaching have become the mathematics classes of the past. By using manipulatives, today's classes will be all about guided discovery and co-operative learning. My conjecture is that introducing the use of tangrams and other concrete tools as mathematical tools will provide students with a stimulus and the enthusiasm to learn. This would create an environment that would encourage learning and consequently make geometry a fun subject.

These manipulatives will not only provide a stimulating and interactive classroom environment but also create a level of trust in the classroom. If tangrams are among the manipulatives, then the learners will commit concepts taught to memory thereby retaining the information for longer periods.

Different manipulative materials engage children in different types of thinking illustrated by the old saying: "Give a person a hammer, and the whole world looks like a nail". Similarly, if the learner is given tangram pieces, their discoveries could be endless and enjoyable, making geometrical relationships relevant to them. This research hopes to corroborate the assertion that the use of manipulatives such as the tangram will not merely accomplish the learning of geometrical properties more rapidly and effectively, but also engage learners in creative thinking.

This thesis consists of six chapters including the introduction. Chapter Two focuses on the literature review undertaken. Chapter Three describes the theoretical and conceptual framework within which this study was undertaken. Chapter Four describes the research design, and Chapter Five is an analysis of the data collected. Chapter Six concludes this thesis, providing some recommendations.

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

In her article *"Math Curse or Math Anxiety*?" hosted by the National Council of Teachers of Mathematics, Stuart (1998) provides data about the usefulness of manipulatives as a tool for instruction:

> "As manipulatives and cooperative groups become more widely used in mathematics classes, I wanted to know whether students perceived these aides and situations as being useful learning tools. Three-fourths of the students thought that using manipulatives when learning a new mathematical concept was helpful. Most of the comments indicated that using manipulatives first helped students see the origin of the numbers in the formulas. Fewer than one-fourth of the students said that manipulatives were not helpful learning tools, stating that they were confusing".

Towards the end of the article she quotes Williams (1988) as cited in Stuart (1998) paraphrasing the Chinese proverb: "Tell me mathematics and I forget; show me mathematics and I may remember; involve me...and I will be less likely to have math anxiety" to reinforce her beliefs of the value of manipulatives.

The idea that physical objects might play an important role in the learning process is a relatively new idea, according to Resnick (1998). Until the 19th century, formal education focused almost exclusively on lectures and recitations. One of the first advocates for "hands-on learning" was the Swiss educator Johann Heinrich Pestalozzi (1746-1827).

Pestalozzi asserted that students needed to learn through their senses and through physical activity, arguing for "*things before words, concrete before abstract*".

Maria Montessori developed materials for older children and inspired a network of schools in which manipulative materials played a central role. In her effort to create an education of the "*senses*", Montessori developed new materials and activities to help children develop their sensory capabilities. She hoped that her materials would put children in control of the learning process, enabling them to learn through personal investigation and exploration. This research intends to show that in using tangrams, learners will be able to identify the properties of the square and rectangle through manipulatives and exploration.

Piaget provided an epistemological foundation for the afore-mentioned educational ideas. Piaget theorised that children must first construct knowledge through "concrete operations" before moving on to "formal operations". During the past decade, a new wave of research has suggested that Piaget, if anything, understated the importance of concrete operations. Papert (1980), for example, have argued for a "revaluation of the concrete", suggesting that "abstract reasoning" should not be viewed as more advanced than, or superior to, concrete manipulations.

The Van Hiele theory supplies an important explanation of why learners' performance in grade 12 geometry is far worse than in algebra. Research by De Villiers & Njisane (1987) has shown that about 45% of black pupils in grade 12 in KwaZulu Natal had only mastered Van Hiele Level 2 or lower, whereas the examination assumed mastery at Level 3 and above. Similar low Van Hiele levels among secondary pupils were found by Malan

(1986), Smith & De Villiers (1990) and Govender (1995), as cited in De Villiers (1996a). In a paper presented by De Villiers (1996), he says,

> "...the transition from Level 1 to Level 2 poses specific problems to second language learners, since it involves the acquisition of the technical terminology by which the properties of figures need to be described and explored. This requires sufficient time which is not available in the presently overloaded secondary curriculum. It seems clear that no amount of effort and fancy teaching methods at the secondary school will be successful unless we embark on a major revision of the primary school geometry".

While I agree with De Villiers that misconceptions are inherited from the primary school, I also assert that using manipulatives, despite the shortcomings of primary school geometry, will give learners a more practical understanding of geometry concepts.

Research in South Africa and elsewhere by De Villiers and Njisane (1987), Smith (1987), Senk (1989) and Usiskin (1982) have indicated that many secondary learners are on Van Hiele level 1 and 2. They believe that in order for learners to cope with secondary school geometry learners have to receive adequate experience on the visual and analysis levels to prepare for van Hiele ordering level.

Both Pestalozzi in the 19th century and Montessori in the early 20th century advocated the active involvement of children in the learning process. In every decade since 1940, the National Council of Teachers of Mathematics (NCTM) has encouraged the use of manipulatives at all grade levels. Every recent issue of the "Arithmetic Teacher" has described uses of manipulatives. In fact the entire February 1986 issue considered answers to the practical questions of why, when, how and with whom manipulatives should be used.

Tapson (1985) believes that not enough spatial work is done with learners. He believes the reason for many learners failing to get started in geometry is because they are unable to visualise them and he therefore recommends that educators try to give learners more practical spatial experiences.

Baez (1997) explored the causes and effects of the lack of manipulative materials in the mathematics classrooms. After much research, she concluded that manipulative aids provide for both the improvement of teaching and consequently, better student understanding of the concepts covered. She believes that the use of devices such as games and pictures will represent a shift from transmission of knowledge practices to student-centred practices.

According to Vithal R (1992) in Moodley et al, she believes that the main advantage of games in the formal teaching situation is that they make learning fun. Games offer pupils the opportunity to learn in a controlled and structured activity with minimum teacher direction. It provides a vehicle for presenting abstract concepts in a practical and concrete form.

Chattin-McNichols (1992) claims that one of the pioneers of the use of manipulatives, was Montessori. The Montessori programme relies on student participation in different activities. Teacher presentation is minimal, whilst students "create" their own learning. Learners work individually or in small groups for three to four hours each day. Learners cooperate with, rather than compete against each other. The researcher's perception of manipulative use is similar to that of the Montessori assumptions that manipulatives, classroom interactions and student-centred learning will combine to create a more beneficial learning environment, which will make geometry more interesting and meaningful.

In the National Council of Teachers of Mathematics Principles and Standards for School Mathematics (2000), representation is identified as one of the important processes in the teaching and learning of mathematics. Thus, the researcher envisages manipulatives to be one of the many representations the teachers have at their disposal for teaching mathematics. I believe that being able to teach effectively requires educators to choose the kinds of representations that will support meaningful mathematics learning in the classrooms, and tangrams can provide that opportunity. Since it is advantageous for students to internalise their own representations of mathematics concepts, interacting with a tool like tangrams during mathematics experiences may be a powerful way for internalising the different cognitive abstractions that mathematics demand. According to Pape (2001), the kinds of external representations the teacher uses during instruction for demonstration or for student exploration have a direct impact on students' learning during mathematics.

According to the NCTM (1989, 1991 & 1995), American students were "at risk" in mathematics when compared to the Japanese students in the Third International Mathematics and Science Study (TIMMS) report. The report has generated much concern about the teaching and learning of mathematics in America. It was noted that the traditional mathematics teaching and learning in the elementary grades was a factor affecting students' future poor performances in middle and high school mathematics. However, this was not the case in the Japanese schools since they placed emphasis in elementary mathematics in developing mathematical thinking by exploring, developing and understanding concepts and discovering multiple solutions to the same problem. Obviously, it was engaging learners in manipulative activities to achieve the desired results that accomplished this. According to Nohda (1992) pupils in grade one start with extended tangram activities as well as other planar and spatial activities. Therefore, it is not surprising that Japanese schoolchildren consistently out-perform schoolchildren from other countries.

According to Burton (1985) by using such tangible items as beans, pebbles, sticks, cubes or shells to solve addition or subtraction algorithms, one can make a record of real-world events. This can be very difficult especially in young children since they are not capable at this stage to think or reason in abstract terms. Through the manipulation of objects, children are able to see and truly understand various relationships and patterns in mathematics (Baratta-Lorton, 1995). As Charlesworth (1984) states, children must first be equipped with developmental pre-requisites before they are able to carry out certain pen and pencil tasks. Stone (1987) reinforces this claim by saying that the use of manipulatives in the classroom can inspire learners by taking into account their needs, interests and abilities.

Using tangrams gives students an opportunity to use a manipulative set to construct an understanding of geometric ideas. It can also help students develop their spatial skills. They are practical, versatile and can create vivid images for students who are visually oriented. Besides moving the pieces around, they can be combined to create other shapes, thus exploring area, perimeter, congruency, similarity and symmetry.

Many sceptics still object to the use of manipulatives because they feel uncomfortable with the fact that manipulatives can make previously difficult work, appear easy, and can therefore mask a lack of understanding. Ball (1992) gives the example of students carrying out a subtraction correctly with manipulatives by following rules they had memorized. However, once removed from the manipulatives, they reverted to their previous mistakes. This implies therefore, that the manipulatives obviously do not cause the mistake, since this mistake happened even before manipulatives existed. In addition, when mistakes are made the mere mention of the manipulatives by the teacher is often enough for the student to think about it, perhaps draw a sketch, and make the correction without any further guidance. In other words, manipulatives can provide a student-controlled home base to which they can retreat as necessary, until they no longer need it. One needs to remember that mathematical understanding can only be acquired through arduous struggle and the use of manipulatives makes it a little more interesting and enjoyable to overcome this struggle.

Manipulatives, because they are, in Papert's (1980) terminology, "*objects to think with and objects to talk about*", have the potential to improve understanding in most classes. They are often a teacher's first step away from traditional instruction, and they can lead to decisive changes in classroom culture: the introduction of collaborative work, the option for students to create exercises instead of merely solving them, an opportunity for the teachers to enjoy a new window on student understanding or lack of it, and so on.

Ball (1992) expresses some caution when she warns against the "*magical hopes*" that many teachers have about manipulatives. She writes, "*manipulatives -- and the underlying notion that understanding comes through the finger tips -- have become part of educational dogma*". She gives several examples of the mere use of manipulatives failing to deliver understanding and concludes that manipulatives cannot be used effectively without adequate teacher preparation, and without a better understanding of how children learn. However, she does acknowledge that "*manipulatives undoubtedly have a role to play... by enhancing the modes of learning and communication available to students*". The researcher does agree to a certain extent with Ball's concern and insights and do not claim magical powers for manipulatives but this is a tool, which could provide another dimension to the traditional "chalk and talk" method.

Resnick (1998) presents the following advantages of manipulative use:

- Manipulatives are extraordinary tools to help reach weaker students, but that is not their only purpose: they are a useful way to improve education in any math class.
- Manipulatives provide an environment to teach math as well as pedagogy to teachers. Often, teachers are ineffective because of their own limited understanding of the material.
- Manipulatives do not make math "easy", and teachers may need to learn something in order to use them. The increased understanding will serve them whether or not they use manipulatives in their class in the immediate future.
- There is no sense in using manipulatives in a "do as I say" algorithmic model which only perpetuates antiquated pedagogy. It is far more effective to use them as a setting for problem solving, discussion, communication, and reflection.
- Manipulatives should be a complement to, not a substitute for other representations. In particular, Cartesian graphing and other pictorial representations are extremely important.

- Deliberate attention must be paid to help students transfer what they know in the context of the manipulatives to other representations, including symbolic, numerical, and graphical. Transfer does not happen spontaneously.

One can deduce from this that manipulatives have an important role to play in the conceptual understanding of mathematics. The NCTM (1989) reform documents strongly recommend using manipulatives and other tools in order to involve students in actively learning mathematics.

In a letter addressed to parents, Burns (n.d.) gives five reasons why manipulative materials are useful in school:

- Firstly, manipulatives help make abstract ideas concrete as they give students ways to construct physical models of abstract mathematical ideas.
- Secondly, manipulatives lift mathematics off textbook pages as words and symbols only represent ideas and ideas exist in a child's mind and manipulatives help construct an understanding of ideas that they can connect to mathematical vocabulary and symbols.
- Thirdly, manipulatives build students' confidence by giving them a way to test and confirm their reasoning because if students have physical evidence of how their thinking works, their understanding is more robust.
- Fourthly, manipulatives are useful tools for solving problems because manipulative materials serve as concrete models to use to solve problems.
- Lastly, manipulatives make learning mathematics interesting as they intrigue and motivate while helping students learn.

Semadeni (1984) wrote a concise condensation and evaluation of the principles behind Dienes work, with some improvements and additional applications. The most important aspect of Diene's approach is that he emphasises using manipulatives to provide a concrete referent for a concept, usually at more than one level, instead of a referent for a given abstract idea or procedure. In keeping with Piaget's thoughts, Dienes and Semadeni view new knowledge as the extension of old knowledge into new areas. For example, learning the nature of positive and negative integers is an extension of the nature of natural numbers. Dienes established several principles on which to base teaching mathematics. The ones that apply to the use of manipulatives are:

- **Permanence Principle**: When extending knowledge into a new area, pick an extension of properties that is similar to the rules the students are already comfortable with. Explore familiar, manipulative examples and extend the concept to generic numbers, and then explore examples of the new domain. For example, the subtraction of fractions could be introduced through subtraction of whole numbers, and then extended through questions.
- **Mathematical Variability Principle:** To enhance the full understanding of a concept, you need to vary all the variables possible so students can understand which properties are constant. In other words, you proceed from the specific concrete examples demonstrated with the manipulatives, to the abstract, generic properties of the concept.
- **Multiple Embodiment Principle:** In order to abstract a mathematical concept, you should demonstrate it concretely with as many different situations as possible, again changing variables, so that the student can abstract what its "purely structural" properties are. Semadeni emphasises that the Permanence Principle be carried out with extensive manipulatives, because the extension of known rules

could foster the habit of applying old rules to new situations without checking to see whether it is reasonable or not.

The principle that has engendered the most controversy is the "Multiple Embodiment Principle". Many educators do not think it applies to the use of manipulatives. Vest (1985) maintains that students do not need to experience different types of concrete developments, for each complex algorithm, as one should suffice. Jackson (1979) concurred with this, because using more than one manifestation of a concept required advanced formal skills including reflective thinking, generalizing and transferring of knowledge. These tasks would render the "concrete" experience more distracting and confusing than useful. Some may disagree with this thinking, and would therefore try to incorporate the reflection of problems as early as possible. Indeed, reflection is not that sophisticated a skill, so long as the concept is not too sophisticated (Freudenthal, 1981 and Skemp, 1972).

Juraschek (1983) explored the application of Piaget's "learning cycles" to mathematics in the middle schools. A "learning cycle" has three stages, namely: exploration, concept introduction and application. Concrete exploration should be designed so learners confront information slightly beyond their understanding. Ideally, the student should feel this lack of understanding (a manifestation of "reflection") so that the teacher can initiate the second "concept introduction" phase. The teacher then introduces the concept that will allay that lack of understanding and explains it; the student, finally, applies and practises the use of the skill, proving and re-proving to himself the validity of the concept.

By keeping the 'application" stage concrete, the connection between the learner's "simple intuitive" understanding and the mathematics he is expected to learn has a much better chance of being established. Here, the child's intuitive models can change and advance. However, if the "application" is abstract, then the student is required not only to toggle between being introduced to the concept and applying it, but must simultaneously leap from concrete to symbolic applications.

2.2 CRITIQUE OF VAN HIELE THEORY

According to Pegg and Davey (1991) the ideas of van Hiele, have been the catalyst for much of the renewed interest in the teaching of geometry during the 1980's, evolving largely as a reaction to the deficiencies perceived in the views of Piaget. It can further be noted that Land (1990) has studied the van Hiele levels of theory outside of geometry in algebra (exponential and logarithmic functions) and Nixon (2002) in higher arithmetic (sequences and series) wherein the existence of levels has been validated.

While van Hiele (1986) specifically identified discontinuity between levels as the most distinctive property of the levels of thinking, the autonomy of the levels does not seem to be as distinct. Burger and Shaughnessy (1986) state that they failed to detect the discontinuity and instead found that the levels appear dynamic rather than static and of a more continuous nature than their discrete descriptions would lead one to believe. Students may move back and forth between levels quite a few times while they are in transition from one level to the next.

Fuys, Geddes and Tischler (1988) also found a significantly sized group of students who made some progress toward level two with familiar shapes such as squares and rectangles; however, they encountered difficulties with unfamiliar figures. They concluded that progress was marked by frequent instability and oscillation between levels.

Gutierrez, Jaime and Fortuny (1991) also found that the levels were not as autonomous in that people do not behave in a single, linear manner, which the assignment of one single level would lead us to believe. They identified students who could be coded 100%, 85%, <40% and <15% for levels 1, 2, 3 and 4 respectively implying that students develop more than one level at the same time. In other words, van Hiele's broad statements are not as black and white as they are often portrayed. Other findings from their research revealed the following:

- There is a possibility that a student can develop two consecutive levels of thinking simultaneously. However, the acquisition of a lower level is more complete than a higher level when this happens.
- Depending on the level of the problem, some students use several levels at the same time.
- The above does not reject the hierarchical nature of the level but highlights the complexity of the human reasoning process ie., people do not behave in a simple linear manner as mentioned earlier.
- A student may operate on a certain level on one task but may be on a different level at a different task.

The van Hiele theory offers a theoretical framework for the teaching and learning of geometry. Indeed, it is essential that the teacher and the learner first find common ground as a basis for learning. Once this has been established, instruction can then be taken to higher grounds.

Treffers (1987) points out that the van Hiele theory was proposed at a time when geometry was not part of the primary school curriculum in the Netherlands. He further concedes that the theory lacked clarity about how to concretely shape the phenomenological exploration at the first level, and which didactical acts should be performed to raise pupils as efficiently as possible from one level to the next. Even van Hiele (1986) has doubted the existence or testability of levels higher than the fourth and considered them as of no practical value.

A question often asked, 'Is it the level of the student or the level of response that should matter?' The SOLO (Structure of Observed Learning Outcome) taxonomy has been proposed as a more realistic model for assessing and classifying students' responses in geometry and mathematics in general (Biggs, 1996, Pegg & Davey, 1998, Pegg, 2003).

However, Usiskin (1982) commends the van Hiele theory *"for its elegance, comprehensiveness, and wide applicability"*. The implication for teachers is that whereas the van Hiele theory explains geometric thought development from a macroscopic perspective, there could be variations to be considered when one takes a closer look at the microscopic level.

Although it may seem that this theory is not perfect, but based on other research, it seems to model the progress of geometrical thinking. The important point is that the majority of geometry taught before high school does not foster students into a higher level of geometrical thinking.

A personal investigation of my learners on the Van Hiele levels revealed that their difficulties were based on some of the barriers that Van Hiele proposes as opposed to learners' lack of intelligence or my "unclear" explanation of the subject matter. These barriers are real and until we decide to give merit to understanding how and why students think the way they do, the researcher believes one will continue to be discouraged in one's efforts to educate them.

Van Hiele-based tasks and units have proven successful in helping students grow in their geometric reasoning. These kinds of activities should be open-ended and build on learners' prior knowledge. In general, geometry should begin less formally with decreased emphasis on proof initially, and encourage learners' to create informal arguments. Engaging learners with meaningful hands-on-approach using manipulatives could achieve this. Previous research (eg., Senk, 1989; Usisken, 1982 as cited in Fuys, 1985; Fuys et al., 1988) support these ideas that lower Van Hiele level experiences are lacking. If this could be improved, it would improve learners' success in the more formal high school geometry course.

Recent educational theories promote developing conceptual understanding rather than teaching procedures and memorizing facts and formula. Hiebert (1986) states that, conceptual knowledge can be regarded as a connected web of knowledge, linking relationships between the individual facts and discrete pieces of information. Conceptual knowledge grows by the construction of new knowledge and the relationships between constructed concepts are strengthened when one practises with tasks involving those concepts. Therefore, it is very important to devise appropriate tasks to relay certain concepts and accomplish effective teaching. Meaningful educational activities and cognitive tools might improve students' active involvements in the teaching-learning process and encourage their reflections on the concepts and relations to be investigated. When students perform tasks that they perceive as purposeful and authentic, they show greater interest and accept more responsibility for their own learning and hence set their own personal meaningful goals (Jones et al., 1997; Savery & Duffy, 1995).

According to Heddens (2005), manipulative materials are concrete models that involve mathematical concepts appealing to several senses, including the socio-cultural needs that learners can touch and move. These are physical objects, such as tangrams, base-ten blocks, and geometric solids that can make abstract ideas and symbols more meaningful and understandable to students. Whilst it is difficult to demonstrate a mathematical concept directly with the help of manipulatives, it is likely for a learner to construct a concept or discover a mathematical relationship through appropriate use of manipulatives with an adequate task. Lesh (1979) suggested that manipulative materials could be used as an intermediary between the real world and the mathematical world. Using manipulatives according to Driscoll (1983), Sowell (1989), and Suydam (1986), benefits students across grade level and ability level. Every student should be given an opportunity to play with manipulatives.

An extended literature review revealed the following findings:

- Del Grande (1990) states: "*Geometry has been difficult for pupils due to an emphasis on the deductive aspects of the subject and a neglect of the underlying* *spatial abilities, acquired by hands-on activities that are necessary prerequisites for understanding and mastery of geometry concepts".* Rowan (1990) concurs with this viewpoint in that students should first meet geometric ideas through hands-on experiences with the geometric nature of their surroundings. The ability to name geometric figures should emanate from experiences that lead to the development of the underlying concepts.

- Chester et al. (1991) used manipulatives with an experimental group to teach a geometry unit and found that after teaching the same unit to a control group using the traditional lecture-style instruction the results in a post-test was significantly higher in the experimental group than in the control group.
- Garrity (1998) carried out an action research to determine whether "hands-on-learning" with manipulatives, improve test scores of secondary education geometry students. This study documented the difficulty of high school students to understand geometry problems and sought to improve this ability by implementing a constructivist approach, which included manipulatives, co-operative learning, and real-life problem solving. The study was conducted with 47 high school students in geometry classes. This group was the experimental group while the other was the control group. The conclusion was that the scores of the experimental group, which interacted with manipulatives, were higher than those of the control group. This lead to the conclusion that the traditional teaching methods are less effective when compared to using manipulatives.
- Steele (1993) reinforced the findings of the previously mentioned researchers discovered that students were more engaged and motivated when actively involved in the learning process when using manipulatives and when working in co-operative groups. The researcher concludes that mathematics instruction needs

to move away from the traditional lecturing, rote memorisation and move towards student-centred activities.

- Driscoll (1988) - Usiskin and Skenk as mentioned in Driscoll (1988) conducted a study of several thousand high school geometry students to determine what changes in van Hiele levels occur during the year with geometry. They also wanted to determine how well the van Hiele levels of students entering high school geometry could predict the level of their proof skills at the end of their year in geometry. In order to make these conclusions, the researchers began by determining how many of the students fitted into the van Hiele model and of those who did, what their levels were. Their findings were that over half of those students were at level 1 or below. The study determined that during the year of geometry, more than 50% of the students at the lowest level moved to levels 2 or 3, but a third of them remained at level one. They found the van Hiele model to be a successful tool to predict geometry performance.
- Teeguarden (1999) states that in most cases learners as well as educators view mathematics as only computation and skill-based exercises. There are many activities such as Tangram Art which can enrich the curriculum and provide an opportunity for learners to see that mathematics and creativity really do go hand in hand i.e., there is a correlation between mathematics, especially geometry, and art – a manifestation of creativity.
- According to Dunkels (1990), a Chinese Emperor, Tan, accidentally broke his mirror into seven pieces. He then instructed (as a game) people to create designs using all the seven shapes. This is how he believes tangrams originated. However, there are many other views as well.

- Suydam and Higgins (1977), in a review of activity-based mathematics learning in grade eight, determined that mathematics achievement increased when manipulatives were used.
- Sowell (1989) performed a meta-analysis of 60 studies to examine the effectiveness of various types of manipulatives with kindergarten through to post-secondary students. Although these studies indicate that manipulatives can be effective, they suggest that many teachers have not used manipulatives. Sowell found that long-term use of manipulatives was more effective than short-term use.
- Gilbert and Bush (1988) examined the recognition, availability, and use of 11 manipulatives among primary school teachers in 11 states. Results indicated that inexperienced teachers tended to use manipulatives more often than experienced teachers did. A possible explanation for this could be that experienced teachers lack the training that the more recent graduates have had.

2.3 A CRITIQUE OF MANIPULATIVE USE

Although the use of manipulatives in combination with other methods can enrich and deepen learners' understanding, relying only on manipulatives as a means of instruction can also be ineffective. Students may lose the opportunity for deeper conceptual learning if manipulatives are used without further formal discussion, abstraction, and mathematical connection.

Indeed manipulatives are not the 'be-all and end-all' in teaching mathematics. They can be a waste of time and effort. Concept development cannot replace learning algorithms for computation-children must have a strong command of computation to apply the concepts. However, indications are there that they can be very useful in middle and secondary education if they are wisely planned and executed to build a firm, concrete model for abstract mathematical concepts. They do not always succeed but when they fail, it is generally because of the following reasons according to Wiebe (1983):

- The child is not developmentally ready for the concept
- The child has not mastered prerequisite concepts
- The model used is too abstract for the student
- The instruction shifts to symbolic before the child has developed the cognitive concrete model to embrace the new concept
- The gap between the model and its symbolic representation is too large.

It can therefore be concluded, that the manipulative must be a model from which the child can gather meaning from his actions. Without a firm grasp of what a child can understand, it is easy to slip abstraction into teaching, especially if making the lesson manipulative-based is expected to compensate for any abstractions. It is the educator's responsibility to detect the developmental level of the children they are teaching, and work to build the child's insight to accommodate new horizons.

According to Judith Bellonio (2001), she gives four possible reasons why more teachers do not use manipulatives in their lessons, namely:

- Lack of training_- many teachers feel that they do not know how to teach using manipulatives and are therefore hesitant to use them in the classroom. I tend to disagree with this thinking since workshops could be arranged to teach the use of manipulatives. In addition, the companies that distribute the manipulatives provide books, pamphlets and videos on the most effective use of the materials.

There are also countless articles on using manipulatives in mathematics teaching journals such as *Arithmetic Teacher.*

- Availability of manipulatives - Even without resources, any teacher can easily obtain a can of buttons or straws. This should not be used as an excuse for not using manipulatives. The teacher should seek innovative ways to come up with suitable manipulatives that are readily available and inexpensive. .
- Noisier lessons_- lessons using manipulatives may be noisier and not as neat. However, using manipulatives works nicely in a cooperative learning setting. I always believed that a noisy class in mathematics, is a class that is actively engaged, since learners are engaged in constructive discussions surrounding the given problem or investigation.
- Fear of breakdown of classroom management- manipulatives requires a great deal of prior planning and organisation. If the lesson is well prepared then there will be no reason to fear a breakdown in classroom management.

Clements (1999) recommends that manipulatives should not be used as an end, without careful thought, rather than as a means to an end. He states further that a manipulatives physical nature does not carry the meaning of a mathematical idea but it should be used in the context of educational tasks to actively engage children's thinking with teacher guidance.

However, the researcher believes that there are no obstacles that cannot be overcome to make the use of manipulatives enjoyable, fun, and beneficial in the learning of mathematics. Furthermore, the researcher strongly believes that if students are having fun, then they are more likely to learn. Therefore, any manipulative that could provide fun and learning opportunities simultaneously is likely to be more effective than the traditional "chalk and talk" or "kill and drill" method.

Suydam and Higgins (1976) believe that lessons involving manipulative materials, if utilised effectively, will produce greater mathematical achievement than lessons in which manipulatives are not used. In fact, their meta-analysis of the studies using manipulatives verified them. They gave the following suggestions, in the same report, on good use of manipulatives:

- Manipulative materials should be used frequently in a total mathematics programme in a way consistent with the goals of the programme.
- Manipulative materials should be used in conjunction with other aids, including pictures, diagrams, textbooks, films, and similar materials.
- Manipulative materials should be used appropriate to mathematics content, and mathematics content should be adjusted to capitalize on manipulative approaches.
- Manipulative materials should be used in conjunction with exploratory and inductive approaches.
- The simplest possible materials should be employed.
- Finally, manipulative materials should be employed with programmes that encourage results to be recorded symbolically.

Heddens (2005) further confirms these views by arguing that using manipulatives will help students learn:

- To relate real world situations to mathematics symbolism.
- To work together cooperatively in solving problems.
- To discuss mathematical ideas and concepts.

- To verbalise their mathematics thinking.
- To make presentations in front of a large group.
- That there are many different ways to solve problems.
- That mathematics problems can be symbolised in many different ways.
- That they can solve mathematics problems without just following teachers' directions.

Clements and McMillen (1996) proposed that using manipulatives does not always guarantee conceptual understanding since students often fail to link their action with manipulatives to describe their actions. Furthermore, students sometimes used manipulatives in a rote manner according to Hiebert and Wearne (1992). However, I am not in agreement with the views of these authors since I believe that the effective utilisation of manipulatives will enhance learning if the role of the teacher and the aims and the potential of the tasks involved are properly defined. One should not use manipulatives simply to keep learners occupied. Careful planning and preparation are necessary to ensure that the tasks involving the use of manipulatives would be helpful and meaningful in understanding mathematical concepts.

Research in England, Japan, China and the United States supports the contention that mathematics understanding will be more effective if manipulative materials are used (Canny, 1984; Clements & Battista, 1990; Dienes, 1960; Driscoll, 1981; Fennema, 1972; 1973; Mouly, 1978; Skemp, 1987; Sugiyama, 1987; Suydam, 1984).

2.4 PIAGETIAN THEORY

According to Piaget, a child's ability to perform a given cognitive task depends on his level of intellectual development. Depending on the nature of the task, a child cannot perform that task unless he is biologically mature enough to perform such a task. This has important implications for instruction, which includes geometry learning. If a child cannot perform a task on an abstract level, the child should first work on a concrete level and as he matures, only then, must he be given tasks on an abstract level.

According to Piaget, the development of the thinking process occurs through consecutive stages, which depend on biological maturity. According to Clements & Battista (1992), at stage one, the child's thinking is non-reflective, unsystematic and illogical. Various pieces of data collected or examples examined are treated as separate, unrelated events. Exploration proceeds randomly without a plan. Conclusions may be contradictory. For example, when three angles of a triangle are put together so that they are adjacent, students were shown what happened for one triangle and asked to predict what would happen for others. Many stage1 learners failed to generalise the pattern and were not sure if it would form a straight angle if the order of the angles were changed. They ignored the size of angles and did not attempt to determine why the pattern occurred.

At stage two, students not only use empirical results to make predictions but also try to justify their predictions. They anticipate results in their search for information and think logically only about premises that they believe in. In the angles task, students attempt to analyse the angles for each new example. However, since they do not see the sizes of the three angles as being interdependent, they are often misled by the appearance of the angles. However, they gradually establish a relationship among the three angles. The induction that leads students to believe that the angles of any triangle yield a straight angle guides their thinking about the angles of new triangles.

Only at stage three do students progress beyond a belief that something is simply always true to making a logical conclusion that it must necessarily be true. The student is capable of formal deductive reasoning based on any assumptions and so is capable of operating explicitly with a mathematical system. For example, students progress beyond an empirical generalisation that the sum of the angles of a triangle is a straight angle to a belief, based on logical reasoning, that it must necessarily be so. They see this relationship as necessary because they understand that the angles of a triangle form complementary parts whose union is a straight angle. Furthermore, they can deduce that three angles that sum to more than 180 degrees cannot possibly belong to the same triangle.

Piaget's theory has contributed to the field of education by giving description to children's thinking. In terms of geometry, there has been evidence overall to suggest that Piaget's theory has really been effective (Clements & Battista, 1992). Piaget claims that students' development in their thinking is of a physiological nature that just happens, as they grow older. Thus, it is not something that teaching can affect or improve. Piaget would say that a child's growth is already dictated and not reversible through planned instructional techniques or activities. Pegg and Davey (1998) concur with Clements and Battista (1992), questioning whether Piaget's work has really changed classroom instruction, and they share the sceptic's doubts about Piaget's ideas of topological primacy.

Van der Sandt (2000), as stated in Piaget and Inhelder (1971), laments that the inability to draw a shape accurately reflects the inadequacy of mental tools for spatial development because drawing is an act of representation and not perception. Sfard (1991) argues that abstract mathematical notions can be conceived in two different ways: namely, operationally as processes and structurally as objects. Learners,' firstly become familiar with mathematical concepts by using the processes or operations, highlighted in this study by the use of manipulatives, Their conception is later , detached from the process and seen as a new object belonging to a particular category of concepts through reflection on these actions. Hence, it is important to encourage learners to reflect on actions they make in order to be able to perceive mathematical processes as objects.

2.5 COMPARISON OF VAN HIELE THEORY TO PIAGETIAN THEORY

This comparison has really intrigued many a researcher. One of the more obvious differences between van Hiele and Piaget's theory is that the former describes *levels* of thinking while the latter describes the *stages* of development. According to Glaserfeld and Kelley (1982), a stage *"designates a stretch of time that is characterized by a qualitative change that differentiates it from adjacent periods and constitutes one step in a progression"* while a level is not defined in terms of time but rather *"implies a specific degree or height of some measurable feature or performance"*. For example, van Hiele's levels are described according to the ways a student is reasoning about a figure. A particular level is possible at any age and can change at any time, indicated by a student's reasoning from a different source (according to how a figure looks, to what properties it has). In Piaget's case, he asserts that infancy is the period when children operate in the *sensorimotor* stage, characterized by children's first interaction with the physical environment in developing motor skills.

Clements & Battista (1992) claim that both theorists promote students' ownership in building understanding; further "both theorists believe that a critical instructional dilemma is teaching about objects that are not yet objects of reflection for students" (Clements & Battista, 1992). This analogy takes one back to the idea of 'level mismatch', where students will struggle to make sense of ideas being taught at higher levels/stages than they have achieved. Furthermore, they claim that in both cases, the theorists support the conflicts that arise for students in trying to think at opposing levels, and such conflicts are an essential and healthy part of the learning process. In addition, they suggest that van Hiele and Piaget do not support the notion that clear explanations are what define good teaching. Clements & Battista finally conclude that both theorists avoid two perspectives on teaching: the first is the perception that a lower level is inferior once a higher level is achieved, and the other is the attempt to force students quickly through succeeding levels once their current level has been established. In other words, neither Piaget nor van Hiele views their theories as an avenue to speed up development.

Pandiscio and Orton (1998) assert that the one difference between the two theorists is the theorists' stance on students' movement among levels or stages. They claim that Piaget would suggest that this is dependent on activity, whereas van Hiele would suggest it is dependent on language. Another important distinction that Pandiscio & Orton make is that the purposes of the two theories are quite different. They suggest that van Hiele was trying to help teachers to improve instruction by describing levels of thinking for students, whereas Piaget was interested more simply in describing the progression of thinking and when it could be expected to occur. Simply stated, the van Hiele model is a theory that can inform instruction and Piaget's model is a theory of development.

Clements and Battista (1992) also noted that there were marked differences in the way that Piaget and van Hiele viewed students' development and thinking about reasoning and proof. They indicate that van Hiele's view is that growth is dependent on increasing understanding of geometric knowledge and relationships, while Piaget believes that logical operations develop in students, which is independent of the content to which they are applied. This suggests that van Hiele would say that a student was ready to prove something if his understanding of the content is at an appropriate level (formal deduction), implying that instruction is a controllable factor that can prepare them for this kind of reasoning. However, Piaget would argue that understanding content is unrelated to a child's readiness for formal argumentation.

According to Lehrer, Jenkins, and Osana (1998), "*children's thinking was much more sensitive to context than to grade; age-related trends suggested that, if anything, experiences in school and in the world did little to change children's conceptions of shape*". This would seem to contradict Piaget's theory, because Piaget suggested that development is related to a child's age. Lehrer et al, also asserted that the detail in which children think about space is greater than what van Hiele explains in his theory. We therefore note that van Hiele (1986), in similar vein to Piaget, was interested in general features, and his model reflects this.

2.6 THE SOLO TAXONOMY

The SOLO (Structure of the Observed Learning Outcome) Taxonomy is a theoretical framework that attempts to provide a language for categorising the levels of students' responses at the various stages in the development of understanding (Biggs and Collis, 1982). Biggs and Collis (1991) consolidated the evolving structure of the SOLO Taxonomy, consisting of developmental modes and levels within each mode. Biggs and Collis' (1982) SOLO Taxonomy concentrates on student responses as opposed to levels of thinking or stages of development. In describing the taxonomy, they say it is "*the only instrument available for assessing quality retrospectively in an objective and systematic way that is also easily understandable by both teacher and student*". They were interested in describing pre-existing criteria that could be used for qualitative evaluation of students' responses in any subject area, not just mathematics.

Similar to Piaget's work, Biggs and Collis (1991) proposed five modes of functioning (also linked to certain age ranges) where learning takes place; *sensorimotor, ikonic, concrete symbolic, formal* and *postformal.* They then suggest five levels of response within a particular mode: prestructural, unistructural, multistructural, relational and extended abstract. A prestructural response indicates a student is actually operating at the preceding mode. Unistructural, multistructural and relational responses indicate the expected mode. Extended abstract responses indicate a student operating in the succeeding mode. So essentially, the goal of the taxonomy is to take observable behaviours and classify them by modes of thinking and level of response.

Biggs and Collis (1982) outline the response levels in the following way. A student giving a prestructural response would either not respond or give information irrelevant or unrelated to the question. A unistructural response would indicate a student gave one piece of information relevant to the question. A multistructural response would elicit several pieces of information relevant to the question. A relational response would indicate relationships among these pieces of information and fit them together as a coherent whole.

Finally, an extended abstract response is evident when a student is "able to derive a general principle from the integrated data that can be applied to new situations" (Olive, 1991).

The following table extracted from Biggs and Collis (1991) best illustrates the different levels:

MODE		STRUCTURAL LEVEL (SOLO)
Next	5	*Extended abstract:* The learner now generalises the structure to take in new and more abstract features, representing a new and higher mode of operation
Target	4	*Relational:* The learner now integrates the parts with each other, so that the whole has a coherent structure and meaning.
	3	*Multistructural:* The learner picks up more and more relevant or correct features but does not integrate them.
	2	*Unistructural:* The learner focuses on the relevant domain, and picks up one aspect and works with it.
Previous	1	*Prestructural:* The task is engaged, but the learner is distracted or misled by an irrelevant aspect belonging to a previous stage or mode.

Table1 Models and levels in the SOLO Taxonomy

Biggs and Collis (1982) intended for the SOLO Taxonomy to be of particular interest to the educator. They hoped for teachers to use it in evaluating their instructional decisions. Such decisions might include teacher intentions, remediation, analysis of curriculum and instructional tasks. The other obvious way this structure can benefit education is as a tool to evaluate students' answers in other ways besides the traditional quantitative method (number correct).

2.7 COMPARISON OF VAN HIELE TO SOLO TAXONOMY THEORY

Both models give teachers a way to assess students' reasoning. In addition, the goal of both models was to empower teachers with a resource to guide assessment as well as instruction. However, as mentioned previously, this was not an objective that Piaget was interested in with his theory of development. In both the models, the extremes of the levels indicate similar responses. Van Hiele's formal deduction is evident at the SOLO higher end as opposed to very limited reasoning at the lower end.

Jurdak (1989) suggests a comparison of the theories of Biggs and Collis and van Hiele. He claims a distinction is "classifying learning outcomes by looking at their structure rather than classifying individuals by looking at indicators of some cognitive abilities". He used the "Identifying and Defining Task" from Burger and Shaughnessy's (1986) study to try to make a correspondence between the two models. He claims that if a correspondence exists, it would alleviate the need associated with the van Hiele model to devise indicators for different tasks. Jurdak felt that he was reasonably able to match up the two sets with the exception of two levels. These exceptions were the SOLO Taxonomy's prestructural level (indicates no response or an irrelevant answer) and the van Hiele model's level of rigor (comparing axiomatic systems). The existence of this highest van Hiele level has also been questioned or noted as difficult to assess, in the research of the van Hiele model (Usiskin, 1982).

One of the distinct differences lies in the fact that the SOLO Taxonomy is not subject specific, whereas van Hiele's motivation to devise a theory grew out of his own frustration in attempting to teach geometry. The SOLO Taxonomy makes provisions for classifying students in transition. Biggs and Collis even assigned numbers to these transitional levels:

1A (prestructural to unistructural), 2A (unistructural to multistructural), 3A (multistructural to relational), 4A (relational to extended abstract). On the other hand, van Hiele did not assume that these "in-between" levels even existed. In fact, this has been a great source of controversy emanating from van Hiele's theory.

Van Hiele claims that the levels are of a discrete nature and that there is a "jump" from one level to the next. However, Burger and Shaughnessy (1986) contend that the levels seem more continuous because they found it difficult in assessment to choose between levels and often identified students oscillating between levels, even on the same question. They suggested that this kind of behaviour made sense coming from students transitioning between levels, particularly between levels one and two; however, a larger sample size of college-level students seemed to be transitioning between levels three and four. This claim has been supported by other researchers such as Usiskin (1982), Fuys, Geddes and Tischler (1988), Gutierrez, Jamie and Fortuny (1991). Since van Hiele's assumption of discrete levels has generated such a controversy and has not been supported by the research, it would seem reasonable to suggest an alternative to this part of the theory.

Clements, Battista, and Sarama (2001) extracted the strengths of both Piaget's theory and the van Hiele model and used a synthesis of the two as a basis for their research on the LOGO Geometry project. They proposed a different approach that does not assume that students think only one way at a time, but instead that they can reason at multiple levels at the same time.

The SOLO Taxonomy has strength in its generality, as it is not specific to geometry or even mathematics, and therefore, does not require revamping for different content objectives, as Van Hiele levels would indicate. Similarly, it does not appear to be anything more than a qualitative rubric for evaluating student responses. Classifying a student by response level and mode of functioning, gives more precision in assessing reasoning. The SOLO Taxonomy does allow for the possibility of evaluating students who might exhibit transitional thinking unlike van Hiele's original model.

In evaluating the van Hiele model as a theory, we can conclude that it has strengths that set it apart from the SOLO Taxonomy and Piaget's theory. Firstly, it is much easier to understand from the perspective of a classroom teacher. This can be attributed to the fact that the van Hiele's were both teachers and their theory grew out of this context. Secondly, it was written specifically for geometry as opposed to the SOLO Taxonomy, which applies across a variety of subject areas and Piaget's theory that applies to several areas of mathematics. Thirdly, from the perspective of the classroom teacher, the van Hiele's theory offers the greater possibility of meeting the challenge of students' varying levels of reasoning within a geometry class. Van Hiele's greatest contribution with his theory is that differences in reasoning level are under the teacher's control and can be facilitated with appropriate instruction.

CHAPTER 3

THEORETICAL AND CONCEPTUAL FRAMEWORK

3.1 CONSTRUCTIVISM

This research has been framed within the constructivist approach, which highlights the view that learning is an active process and, that learners construct their own meaning. This implies that learners themselves are responsible for their own learning. Constructivism argues that learners do not remember content exactly as presented, but that they interpret instructional situations in many different ways. New knowledge is interpreted, based on pre-existing knowledge. (Oliver, 1992).

In the constructivist theory it is accepted that "*the learners have their own ideas, that these persist despite teaching and that they develop in a way characteristic of the person and the way they experience things leads inevitably to the idea that, in learning, people construct their own meaning*" (Brooks,1994). Learners interpret and integrate new ideas into existing networks of knowledge. For example, when a learner, according to Nakin (2003), hears the word "triangle", and depending on his experiences, he may think of a tricycle, a tripod, or a triangular road sign. In any event, the learner comes to understand that a triangle has a "three" of something. In support of his afore-mentioned assertion, he goes on to quote from Yackel et al (1990) who states that in tackling new mathematical tasks, children use strategies that have already been internalised in order to complete such tasks.

The importance of constructing own meaning was further emphasised by Benchara Blanford in 1908 as quoted in Griffiths & Howson (1974) as follows:

> "To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions is both interesting and highly educational".

Hans Freudenthal (1973), also strongly criticises the traditional practice of the direct provision of geometry definitions when he says:

> ".....most definitions are not preconceived but the finishing touch of the organising activity. The child should not be deprived of this privilege.... Traditional instruction is different. Rather than giving the child the opportunity to organise spatial experiences, the subject matter is offered as a preorganised structure. All concepts, definitions, and deductions are preconceived by the teacher, who knows what is its use in every detail – or rather by the textbook author who has carefully built all his secrets into the structure".

Scott (1987) considers the following key to the construction of meaning by learners:

- "*That, which is already in the learner's mind, matters*". This further reinforces the idea that the learner's pre-existing knowledge is important. Therefore, when recognising the different quadrilaterals, the learners will try to recall their real life experiences with shapes that they interact with daily, for example, a ruler (rectangle), a lunch box (square), and a slice of bread.

- Individuals construct their own meaning. Each learner may be at a different learning stage due to environmental, societal difference, cultural difference or his or her mental abilities.
- The construction of meaning is a continuous and active one. Learners generate ideas and test or evaluate them and then review and reaffirm these ideas or hypotheses. Learners will continue exploring with the tangram and paper folding and discover most of the properties by themselves and they can reaffirm their conclusions by testing them with their manipulatives.

Since with the constructivist perspective, the learner is responsible for his own learning, the role of the educator in the learning process needs to be modified. The educator does not merely transfer knowledge but acts as a facilitator for the learner to construct knowledge. In an editorial by Clements and Battista (1991), it stated that, *"effective teachers are those who can stimulate students to learn mathematics. Education research offers compelling evidence that students learn mathematics well only when they construct their own mathematical understanding"*. Therefore, by allowing learners to explore with manipulatives, it could provide the stimulus for learners to construct their own knowledge.

The role of the teacher in a constructivist setting is to assist the students in developing new insights and connecting them with previous learning. The activities are student centred and students are encouraged to ask their own questions, carry out their own experiments, make their own analogies, and arrive at their own deductions and conclusions. According to Brooks and Brooks (1993), the following summarises some of the recommended characteristics of a constructivist teacher:

- Become one of many resources that the student may learn from, and not the primary source of information.
- Engage students in experiences that challenge previous conceptions of their existing knowledge.
- Allow student responses to drive lessons and seek elaboration of the students' initial responses. Allow students some thinking time after posing questions;
- Encourage the spirit of questioning by asking thoughtful, open-ended questions;
- Encourage thoughtful discussion among students.
- Use cognitive terminology such as "classify", "analyse" and "create" when framing tasks;
- Encourage and accept student autonomy and initiative. Be willing to let go of classroom control;
- Use raw data and primary sources, along with manipulative, interactive physical materials;
- Do not separate knowing from the process of finding out.
- Finally, insist on clear expression from students. When students can communicate their understanding, then they have truly learned.

Clearly, a lesson based on a constructivist approach differs greatly from the traditional "teacher-as-lecturer" class type. Further, the concept of using manipulatives has also reinforced it as an important tool in achieving the constructivist ideal of constructing own meaning.

Jonassen (1994) also provides the following guidelines on constructivism for instructional design as follows:

- One should provide multiple representations of reality and also represent the natural complexity of the real world;
- One should focus on knowledge construction, not reproduction;
- One should present authentic tasks (contextualising rather than abstracting instruction);
- One should provide real-world, case-based learning environment, rather than pre-determined instructional sequences;
- It is also important to foster reflective practice and enable context and content dependent knowledge construction and
- One should always support collaborative construction of knowledge through social negotiation.

Echoing similar sentiments, Wilson & Cole (1991) isolated the following concepts which are similar to the afore-mentioned and which are central to constructivist design:

- One should embed learning in a rich authentic problem-solving environment;
- Secondly one should provide for authentic versus academic contexts for learning;
- Thirdly one should provide for learner control; and
- Finally, one should use errors as a mechanism to provide feedback on learners' understanding.

On similar lines, Honebein (1996) describes seven goals for the design of constructivist learning environments:

- Provide experience with the knowledge construction process.
- Provide experience in and appreciation for multiple perspectives.
- Embed learning in realistic and relevant contexts.

- Encourage ownership and voice in the learning process.
- Embed learning in social experience.
- Encourage the use of multiple modes of representations.
- Encourage self-awareness in the knowledge construction process.

From all of the above, it is obvious that constructivism is a process of guiding the learner from what is presently known, to what is to be known. It is also increasingly clear that the use of manipulatives is essential in providing a healthy constructivist-learning environment.

Even though many traditional teachers use concrete materials (manipulatives) to introduce ideas whenever possible, they use them only for an introduction. This kind of teacher devalues children's intuitive thinking about what is meaningful to them. Learners are made to feel that their intuitive ideas and methods are worthless and have very little to do with *real* mathematics. In contrast, in constructivist teaching, the teacher encourages learners to use their own methods for solving problems. The teacher does not ask them to take over someone else's thinking, but instead he encourages them to refine their own thinking. Tasks are given to get learners to invent, discover, or adopt techniques that are more sophisticated. By giving learners appropriate tasks to do and the opportunities to discuss what they are doing, the constructivist teacher guides their attention towards what is important. It therefore requires the constructivist teacher to set tasks that will help learners develop their understanding in a way that will make their mathematical thinking more powerful.

Conceptual understanding underpins constructivism. According to Schifter (1993), in the construction of knowledge, learners construct understandings as they attempt to make sense of their experiences. In this regard, learners bring to bear, a web of prior understandings that are used in interpreting new experiences. This shows that knowledge is not received from one person to another in intact form.

3.2 RADICAL AND SOCIAL CONSTRUCTIVISM

Two forms of constructivism have relevance in mathematics education, namely radical constructivism and social constructivism. Radical constructivism has emerged from Jean Piaget's theory of cognitive development (see 2.5) and had been developed by Ernst von Glaserfeld, as cited in Ernest (1995). According to Piaget, *"in order to know objects, the subject must act upon them, and therefore transform them; he must displace, connect, combine, take apart, and reassemble them"* (Piaget, 1983). Knowledge, therefore, arises from the interactions between the individual and the object. Therefore, this form of constructivism emphasises the individuals' interaction with the world.

Social constructivism, on the other hand, emphasises the social nature of learning, that is, that *"human subjects are formed through their interactions with each other, as well as by their individual processes"* (Ernest, 1995). Vygotsky, born in the U.S.S.R. in 1896, is responsible for the social development theory of learning. Central to Vygotsky's theory is his belief that biological and cultural development do not occur in isolation (Driscoll, 1994).

Vygotsky's social development theory challenges traditional teaching methods. Historically, schools have been organised around recitation teaching, that is, the teacher disseminates knowledge to be memorised by the learners, who in turn recite the information back to the teacher (Hausfather, 1996). Social constructivist scholars' view learning as an active process where learners should learn to discover principles, concepts, and facts for themselves. Hence, the importance of encouraging guesswork and intuitive thinking in learners is emphasised (Brown et al., 1989; Ackerman, 1996; Gredler, 1997). Vygotsky identified the level of potential development as the *"zone of proximal development"*. The Zone of Proximal Development (ZPD) is the difference between an individual's current level of development and his or her potential level of development (Vygotsky, 1978). Vygotsky argues that at any given time in development there are certain problems that children are not able to solve, and therefore all they need is structure, clues, reminders, help with remembering details, arrangement, and so on (Vygotsky, 1978). According to Rogoff (1998), children make the most significant development when they participate in activities slightly beyond their competence with the aid of adults and other skilled children.

3.3 COMPARISON BETWEEN VYGOTSIAN THEORY AND PIAGETIAN THEORY

Vygotsky approached development differently from Piaget. Piaget believed that cognitive development consisted of four main periods of cognitive growth: sensorimotor, preoperational, concrete operations and formal operations (Saettler, 1990). Piaget's theory suggests that development has as its goal, an endpoint. Vygotsky, on the other hand, believed that development is a process that should be analysed, instead of obtaining an end product. According to Vygotsky, the development process that begins at birth and continues until death is too complex to be defined by stages (Driscoll, 1994; Hausfather, 1996).

Vygotsky believed that development was dependent on social interaction and that social learning leads to cognitive development. He called this phenomenon 'the Zone of Proximal Development'. He believed that this zone bridges the gap between what is known and what can be known, and that learning occurred in this zone (Vygotsky, 1978).

According to Vygotsky, humans use tools that develop from a culture, such as speech and writing to mediate their social environments. Initially children develop these tools to serve solely as social functions, ways to communicate needs. Vygotsky believed that the internalisation of these tools led to higher thinking skills. When Piaget observed young children participating in egocentric speech in their preoperational stage, he believed it was a phase that disappeared once the child reached the stage of concrete operations. On the other hand, Vygotsky viewed this egocentric speech as a transition from social speech to internalised thoughts (Driscoll, 1994). Therefore, Vygotsky believed that thought and language could not exist without each other.

3.4 CONCEPTUAL FRAMEWORK

The van Hiele theory of learning geometry provided this study with a useful conceptual framework for interpreting and analysing the learners' levels of understanding. According to van Hiele (cited in de Lange, 1996) the process of learning proceeds through three levels:

- A learner reaches the first level of thinking as soon as he/she can manipulate the known characteristics of a pattern that is familiar to him/her;
- As soon as he/she learns to manipulate the interrelatedness of the characteristics he/she will have reached the second level;
- He/she will reach the third level of thinking when he/she starts manipulating the intrinsic characteristics of relations.

Traditional instruction is inclined to start at the second or third level, while the realistic approach starts from the first level.

In order to understand whether learners have achieved the desired level in geometry, it is necessary to discuss the various van Hiele levels of understanding in Van Hiele (1986) as cited in Mudaly (1998):

- Level 1: **Visualisation or Recognition:**

  The learner can recognise for example, a square, but will not be able to list any properties of the square. It therefore means that a learner recognises specified shapes holistically, but not by its properties. In this stage of the learners' interaction with manipulatives, namely, tangrams the learner engages in *explorative* activities, attempting to find common relationships that may lead to a hypothesis. He may even be able to compare or sort shapes based on appearance as a whole. By using the tangram the learner may be able to fit two squares into a rectangle or two isosceles triangles into a square. Therefore, by trial and error the learners may be able to determine areas of shapes by placing their tangrams into the rectangles and comparing it to a square.

- Level 2: **<u>Analysis level</u>:**

  At this level, learners begin to understand that the shapes that they are working with through observation and playing around with have certain properties. Learners' can identify properties of figures and recognise them by their properties. They can now see that the square is made up of equal sides, or rectangles have opposite sides equal. However, the learner is not able to find the relevant links between the different figures, for example, the relationship between the rectangle and the square. By interacting with tangrams learners will be able to use appropriate vocabulary for parts, relationships - for example, opposite sides are equal, and diagonals bisect each other.

- Level 3: **<u>Ordering or Informal deduction</u>:**

  At this level, learners discern the relationships between and within geometric figures: for example, if the opposite sides of a quadrilateral are parallel then that quadrilateral is a parallelogram, or that square is a rectangle. Precise definitions are understood and accepted. These definitions are referred to when learners talk about the shapes. However, learners cannot employ deductive strategies to solve geometric problems at this level. They may be able to follow a proof but may not be able to prove. Therefore, by exploring with manipulatives, the learner begins to assimilate the various informal facts that become available through exploration. Most of the learners interviewed were able to explain why all squares are rectangles and that not all rectangles are squares by manipulating the tangram pieces.

- Level 4: **Deduction:**

  At this stage, the learner understands the significance of deduction as a means of solving geometric problems. However, it would be presumptuous to think that the learners in this study would be able to reach this stage, as they are not expected to construct proofs at this level.

According to van Hiele theory, a learner cannot advance from one stage to another if he has not mastered the previous stage. It is therefore important in any analysis of learners' achievement to ensure learners do not omit any stage during their exploration with manipulatives. Four important characteristics of the theory are summarised as follows by Usiskin (1982):

- **Fixed order** – the order in which learners' progress through the thought levels is invariant. In other words, a pupil cannot be at level *n* without having passed through level *n-1*.
- **Adjacency** – at each level of thought that which was intrinsic in the preceding level becomes extrinsic in the current level.
- **Distinction** – each level has its own linguistic symbols and own network of relationships connecting those symbols.
- **Separation** – two persons who reason at different levels cannot understand each other.

The characteristics of van Hiele's theory are further reinforced by Mayberry (1983) who believes that instruction should be geared towards finding out the level at which a child operates and then building up from there. Otherwise, the child and the teacher may be at different wavelengths and instruction is bound to fail. A child at level n will answer most questions at that level but may not be able to answer questions at level $n + 1$.

Van Hiele (1986) states that if a child concludes that "every square is a rhombus", this is not because of maturity but is a result of the learning process. These instructional phases can be summarised as follows:

- Phase one: i*information* - the learners get acquainted with material for instruction. In the study, the researcher got the learners to acquaint themselves with the different pieces of the tangram sets by making different shapes and patterns.
- Phase two: *bound orientation* - the learners explore the field of inquiry through carefully guided activities. The learners in the research had to work through a series of activities in the interview schedule.
- Phase three: *explication* - the learners and the teacher discuss the object of the study. Language appropriate to that particular level is stressed. At the interview session the researcher got the learners to explain certain conclusions and assertions in their own words.
- Phase four: *free orientation* - the learners learn by general tasks to find different types of solutions. In addition to giving learners tangram pieces, the researcher also requested them to confirm their results by using paper-folding tasks.
- Phase five: *integration* - students build an overview of all they have learned of the subject. At this stage, rules may be composed and memorised (after using the

tangrams learners were asked to draw outlines of their tangram pieces on paper and test their assertions to check whether their results were always true).

The van Hiele theory therefore, offers a theoretical framework for the teaching and learning of geometry. Furthermore, it is important that the teacher and the learner first find common ground as a basis for learning. Once this has been achieved, instruction can then be taken to higher grounds. The theory points out the levels of geometric thinking a child goes through and that it is through instruction that a learner will proceed from a lower level to a higher one. Thus for effective teaching and learning to occur in the mathematics classroom the educator must first be mindful that learners differ in capabilities – this implies that some learners rely on symbolic thinking whilst others need to visualise a problem as suggested by Krutetskii (1976); Mayer (1989); Mouly (1978); Hershkowitz, et al (1996); Wilson (1990). Therefore visualisation and the hands-on approach does not harm the gifted learner but if ignored from the curricula it will limit the chance of success for learners that rely on visual/concrete thinking in geometry.

CHAPTER 4

RESEARCH DESIGN AND METHODOLOGY

4.1 INTRODUCTION

This study entailed seeking answers to the following critical questions:

1. Can grade 8 learners determine the properties of the square and rectangle using tangrams?
2. Will the manipulation of the tangram pieces enable grade 8 learners to improve their understanding of the properties of the diagonals of the square and rectangle?
3. Do learners attain van Hiele level 3 after their interactions with manipulatives?

The purpose of this research study was to determine whether manipulatives, namely tangrams, could be useful as a mathematical tool in enhancing the learners understanding of the properties of squares and rectangles. Furthermore, this study will determine whether van Hiele level 3 was achieved with manipulatives.

Because of the proposed questions, it was advisable to use an action research method and the method of qualitative analysis by means of one-to-one task based interviews. A qualitative research methodology was implemented, because qualitative research is not concerned with verifying existing theories and hypotheses, but rather with discovery, as proposed by Hitchoch and Hughes (1995). Strauss & Corbin (1990) further defines qualitative research, as "*any kind of research that produces findings not arrived at by means of statistical procedures or other means of quantification*". Whereas quantitative researchers seek casual determination, predication, and generalisation of findings, qualitative researchers seek instead illumination, understanding, and extrapolation to similar situations. Therefore, it is clear that qualitative analysis results in a different type of knowledge than does quantitative inquiry.

Furthermore, qualitative research relies on data in which meanings are expressed through words and other symbols, thus the qualitative researcher looks for patterns, themes and exceptions to the rule (Hitchoch & Hughes, 1995; Neuman, 2000; Welman & Kruger, 2001). Bell (1993), elaborates further that researchers adopting the qualitative perspective are more concerned with the individuals' perceptions of the world. Hence, the key to understanding qualitative research lies with the idea that meaning is socially constructed by individuals in their interaction with their world, moreover, the language of qualitative research is one of interpretation (Neuman, 2000; Merriam & Associates, 2002).

4.2 ACTION RESEARCH

As Cohen and Manion (1994) stated, *"action research is a small-scale intervention in the functioning of the real world and a close examination of the effects of such intervention"*. This method makes it easier to document the rich data that individual children display when playing around with the tangrams. According to Myers (1985), action research refers to research done by educators using the classroom as their focus of attention. The primary purpose of action research is to investigate issues of immediate concern and to incorporate the results into future teaching, rather than being primarily concerned with the development of results for the profession. Myers (1985) states further that the findings of action research can be much more readily applied, since they come from the same environment where the research was performed, in contrast to research that was done in a different context.

Furthermore, the researcher chose action research because there is a need for improvement in learning strategies in a mathematics classroom. Action research relies foremost on observation and behavioural data and is therefore more flexible and adaptable to changes if the need arises. The researcher feels that there is a need for some kind of change, or improvement in teaching and learning of geometry, especially the properties of the square and rectangle. The researcher is therefore in a position to translate his ideas into action in his own class (Cohen & Manion, 1994).

Action research is appropriate whenever specific knowledge is required for a specific problem in a specific situation; or when a new approach is to be grafted onto an existing system (Cohen & Manion, 1994). Cohen & Manion also mentions that action research could also be used to improve teaching and learning methods where the researcher in this study intends showing that the traditional method of teaching geometry properties could be effectively replaced by a guided discovery method using manipulatives.

Action research, according to McNiff (2002), is open-ended and does not begin with a fixed hypothesis but rather an idea that you develop. McNiff goes on to explain that this "research process is the developmental process of following through the idea, seeing how it goes, and continually checking whether it is in line with what you wish to happen". It is therefore a form of self-evaluation. The basic principles underpinning action research involves, identifying a problematic issue (in this study the method of teaching geometry); imagining a possible solution (in this study using manipulatives); trying it out (tangrams were used to implement the strategy); evaluating it – did it work? (interviews were conducted to ascertain whether the strategy really worked); and finally changing practice in the light of the evaluation (adapted from McNiff, 2002).

4.3 METHODOLOGICAL APPROACH

According to Cohen, Manion, (2000) the purpose of the research should assist in deciding the methodology and design of the research study. Thus, the purpose of this study is to determine whether the use of manipulatives was effective in improving learner understanding of the properties of the square and rectangle in a grade 8 class in a secondary school in KwaZulu-Natal, South Africa. Hence, the interpretative research paradigm was employed. Moreover, an interpretative researcher studies a text such as a conversation or a drawing to elicit subtle verbal communications and to attain meanings embedded in the subjective, personal explanations of actions or techniques employed (Pillay, 2004).

This research focused on the learner and to understand and interpret the learners' understanding, reasoning, and techniques employed with respect to the properties of the square and the rectangle. The learners in this research study were probed with respect to their responses to the interview schedule questions to assist the researcher in gaining a more adept understanding about their prior knowledge, existing knowledge, and the new knowledge gained after interacting with the tangram pieces.

4.4 RESEARCH DESIGN

According to Pillay (2004), research design refers to the logical structure of the inquiry so that unequivocal conclusions are afforded. In addition, research design proceeds in an orderly and specific manner as represented in the table on the following page:

One can clearly see from the table that each item in the flow chart depends upon the successful completion of all the previous items. Therefore, it is important not to skip a single step and to ensure that each step is thoroughly completed. Thus, methods are selected because they will furnish the necessary information required to produce a complete piece of research as suggested by Bell (2005).

Define Goals and Objectives
↓
Design Methodology
↓
Design Feasibility
↓
Develop Instruments
↓
Select Sample
↓
Conduct Pilot Test
↓
Revise Instruments
↓
Conduct Research
↓
Analyse Data
↓
Prepare report

Table 4. Research design flowchart

For the purpose of this study, the research instruments consisted of a pre-test, an interview schedule, and an hour-long interview using an audio tape to assist in transcription. Before going into any detail about the instruments, the researcher will firstly describe the sampling of learners.

4.5 SAMPLING

Apart from the research methodology and instrumentation, the research study also relies on the quality of the sample chosen. Questions of sampling arise directly from defining the population on which the research will be based or focused on. Hence, sampling needs to be considered early on in the research plan. Cohen & Manion, et al., (2000) propose that judgements need to be made based on four key factors in sampling, namely, the sample size, the representativeness and parameters of the sample, access to the sample and the sampling strategy. Furthermore, the researcher must decide whether to use a probability or non-probability sample.

Within a probability sample, the chances of members of a wider population being selected are known, whereas in a non-probability sample the chances of members of the wider population being selected for the sample are unknown (Cohen, Manion, 2000). Since this study is a relatively small-scale research study, a non-probability sampling strategy was utilised. Despite the disadvantages that may arise from their non-representativeness, non-probability sampling is far less complicated to set up. Furthermore, there are several types of non-probability sampling: convenience sampling; quota sampling; dimensional sampling; purposive sampling and snowball sampling (Cohen, Manion, 2000). Each sample only represents itself in a similar population rather than representing the undifferentiated population. Thus, the sampling strategy used in this research study was purposive sampling.

4.6 CONTEXTUAL FACTORS – THE SCHOOL AND THE LEARNERS

The school chosen was an urban secondary school in KwaZulu-Natal, South Africa. This school is an ex- model C school situated east of Durban, a city in KwaZulu-Natal. A model C school was a school in the pre-apartheid era, before 1994, where only Whites were permitted to attend and the learners' parents and sponsorships provided funding for most of the resources and facilities. It is now referred to as ex-model C because it is partially subsidised by the state. Most of the learners in this school come from middle to upper middle-income families. This school offers mathematics as one of the compulsory subjects in the curriculum from grade eight to grade twelve. This school is unique because it is co-educational (most ex-model C schools were boys only or girls only schools). The school's pupil mix consists of learners ranging across the racial spectrum, namely Whites, Blacks, Indians, and Coloureds. The school also had a 100% pass rate in the last 10 years in the matriculation examinations. The schools teacher-pupil ratio compared to other public schools was much lower, namely 1:32 as compared to 1:45.

Permission was sought from the principal and consent was obtained from the ten learners' parents. Further, the Department of Education granted permission to proceed with this research study. This school was selected because of convenience to the researcher and that it was easily accessible and learners generally displayed no serious disciplinary problems.

Since the grade eight learners had come from different primary schools, a pre-test was considered essential and given to a grade 8 class to establish their existing knowledge of the properties of the square and rectangle. It also served to establish at what level they were performing at when using the van Hiele theory levels. Learners were chosen after the pre-test was administered and the results ranked using the following criteria:

- Properties of square;
- Properties of rectangle;
- Van Hiele level 3 understanding (refer to table below showing results of pre-test).
- Learners showing little or no understanding to the above criteria were placed into a broad group.
- Ten participating learners were selected from the above group.

Thus, the learners selected were chosen using the purposive sampling strategy based on the needs of the research. The participants had little or no knowledge of the properties of the square and the rectangle.

CRITERIA	**No Understanding**	**Little Understanding**	**Good Understanding**	**Comprehensive Understanding**
Properties of square	31% (9)	48% (14)	17% (5)	4% (1)
Properties of rectangle	52% (15)	31% (9)	14% (4)	3% (1)
Van Hiele Level 3	31% (9)	59% (17)	10% (3)	0% (0)

Table 3. Analysis of pre-test results in a grade 8 class of 29 learners

The researcher first piloted this study by giving a worksheet containing relevant questions pertaining to the eventual interview. The reason for doing this was to make sure the questions probed would eventually provide sufficient data to extrapolate into useful conclusions. After completing the pilot with a grade 8 class, the researcher then prepared an interview schedule that was used during the hour-long interview with each of the ten participants. A dictaphone was used to record the one-to-one interviews. The interview process was conducted over a three-month period, as it was a challenge to arrange appointments with the learners to be interviewed as most of them were involved in extra-curricular activities and these interviews could only be conducted after school hours. The findings will be analysed in the next chapter to evaluate the success or failure of the use of manipulatives and to evaluate the attainment of van Hiele 3.

CHAPTER 5

DATA ANALYSIS

5.1 OVERVIEW

The primary aim of this research was to determine whether manipulatives could enhance the learners' ability in understanding the properties of squares and rectangles. Furthermore, the research attempted to determine whether learners could attain van Hiele level 3 of understanding. The method of qualitative analysis, by means of a one-to-one task based interviews was used. This made it easier to document the high level of information that individual learners displayed when working and manipulating the tangram pieces. Furthermore, this method allowed the researcher sufficient time to observe, probe and to take note of how each learner answered questions based on their manipulation of the tangram pieces and their discoveries. The researcher acknowledges that the classroom situation is dynamic, due to the interaction of learners with each other, the educator, the subject content, and the environment. By reducing the number of external variables, one narrows the focus, giving generalisations based on findings during task-based interviews greater credibility. Therefore, the researcher chose a one-to-one interview.

However, when the researcher first conceptualised the idea of using one-to-one task based interviews, the researcher did not think that a video recorder was necessary, but in hindsight the researcher has realised that a video recorder should have been used since it could be used to capture important facial expressions or attempts that could be overlooked when transcribing. All ten learners chosen, were given a brief history of tangrams and thereafter they were allowed to experiment with their tangrams making different shapes and patterns before the start of the interviews. They were given opportunities to work in groups without the limitation of working within a framework of prescribed examples before the actual interviews started. In their interactions in groups, valuable vocabulary was exchanged between learners and this was documented and taken into account when the researcher designed his interview schedule.

The pre-test results indicated that while some of the learners could recall some of the required mathematics vocabulary, the majority could not. It was accepted that when some learners gave answers like "the diagonals cut each other exactly in the middle", they really meant that the diagonals bisected each other. The ten selected learners (see 4.6 for criteria) were given time to play around with the tangram pieces before the actual interviews begun. While playing around with the pieces the learners were encouraged to freely explore and discover some properties and patterns. By involving learners in this practice, the researcher was able to observe and, through questioning, informally assess the learners' methods and strategies.

5.2 PROPERTIES OF THE SQUARE USING TANGRAMS

5.2.1 Description of the triangles

All ten participants were able to identify correctly, that the triangles concerned were isosceles. However, it was interesting to note that 40% of the learners were able to indicate further that this was a right-angled triangle as well. One learner also described it as a right and acute-angled triangle. It was clear that these learners possessed the required mathematics vocabulary to answer this question. It also indicated that they possessed the required pre-knowledge to continue with the task. When they were asked to justify their answers as to why they chose an isosceles triangle, all ten of them placed the two triangles from their tangram set over the other. Six of them actually rotated the triangles, convinced themselves of their finding, and then stated that the sides were the same. The learners were familiar with the concept and properties of an isosceles triangle. The following is an excerpt from an interview with one of the participants:

RESEARCHER: *What type of triangles are these? (Researcher holds up 2 of the smallest triangles)*

LEARNER: *Acute………………Isosceles………..right-angled triangles.*

RESEARCHER: *Can you justify your answer?*

LEARNER: *I rotated the one triangle while the other triangle stayed in its original position. Two sides on the one triangle were equal in length because they still fitted on each other exactly.*

5.2.2 Magnitude of the angles of the isosceles triangles

All ten learners were able to correctly state that the two triangles had 45° angles and one a 90° angle. This once again confirmed the researcher's pre-supposition that these learners were quite familiar with the properties of an isosceles triangle. When the learners were asked to justify their answers, 80% of the learners placed the two triangles one on top of the other and thereafter they placed it side-by-side to form a square. They then concluded that the resultant angle was 90° **(refer figure 1.1)**. When probed further, as to how convinced they were that it was exactly 90°, 20% of the learners stated that they rotated the two triangles clockwise and it still matched each other showing that the acute angles were equal. They then drew a straight line, took one of the acute angles of the isosceles triangle, placed it on the straight line, and marked it. The next acute angle was placed next to the first marked angle and this angle was marked as well and continued this process until all acute angles were marked. **(Refer to figure 1.2)**. They then concluded that the sum of the four equal angles was 180°. A natural extrapolation of this result showed that each had to be 45°. As for the largest angle in the triangle, they then placed the 90° angles of the isosceles triangle on their sketch in figure 1.2 and completed the straight line and hence they concluded that the angles had to be 90°.

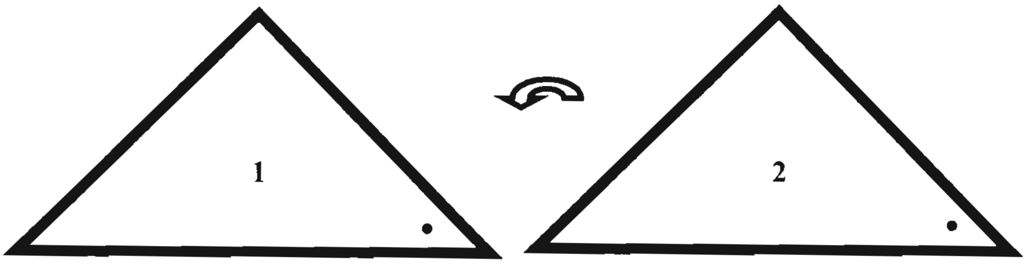


Figure 1.1 (a). Two isosceles triangles placed side by side

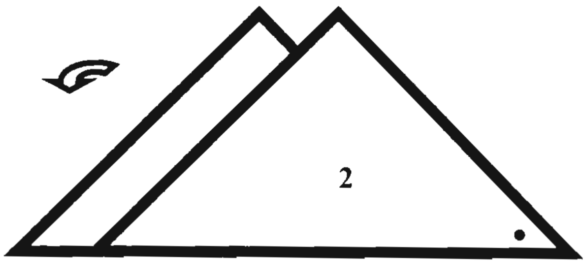


Figure 1.1 (b). Two isosceles triangles super-imposed

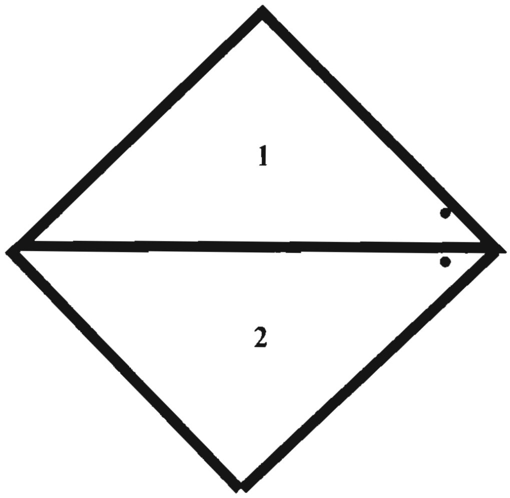


Figure 1.1(c). Two isosceles triangles flipped

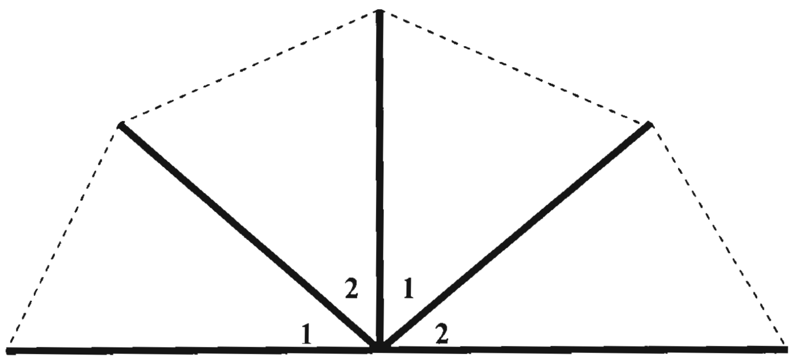


Figure 1.2 Two isosceles triangles placed on a straight line and marked

The following is an excerpt from one of the interviews conducted:

RESEARCHER: *What is the size of each angle of the triangle?*

LEARNER: *90° ………; 45° ………….; 45° ……………………*

RESEARCHER: *Can you proof this?*

LEARNER: *If I place these triangles side by side they form a square, so each angle will be 90° in a square; but this is an isosceles triangle, therefore the two equal angles must be 45° each so that they* *will add to* 90°.

Quite evidently, the learners' ability to rationalise mathematically could not be underestimated. This was a complex sequence of rational statements, which led the student to the answer. The researcher believes that this can be attributed to the high visual nature of the activity and the hypotheses. Therefore, this form of teaching may be far more effective in a classroom.

5.2.3 Effects of rotating the tangram pieces within the square

The learners were asked to place the smaller triangles within the square (learners were asked to draw an outline of the square tangram piece) and state their observation. All ten learners were able to predict correctly that the triangles fitted exactly into the square and further stated that this occurred when they rotated the triangles through 180°. This has good possibilities for the teaching of transformational geometry, which is now part of the Outcome Based Education (OBE) curriculum. They also concluded that the angles and sides were still equal irrespective of the rotation.

5.2.4 Magnitude of each angle of the square without measuring

All ten learners were able to state that each angle was 90°. They proved this by using their previous deduction that the isosceles had two 45° angles and one 90° angle. When they placed the 90° angle on one angle of the square, they fitted exactly. They confirmed this by placing the same 90° angle on each of the other three angles and they found that it was equal, (**refer to figure 1.3**). However, what was interesting to note was that one of the ten learners used the end of a page and said that it was 90° and that the square's angle fitted exactly on the edge of the page, therefore it was 90°.

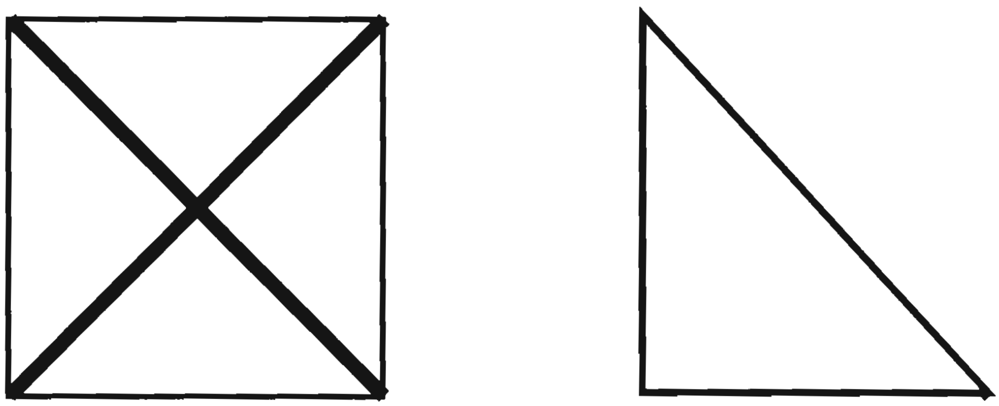


Figure 1.3 One isosceles triangle placed on diagonals & rotated

The following is an excerpt from one of the interviews conducted:

RESEARCHER: *Can you justify your answer? (90° angle in a square)*

LEARNER: *In an isosceles triangle two angles are equal and because this triangle had a 90° angle, I know this because I placed the angle on the corner of my page (learner demonstrates) and it was square. Thereafter, I placed this angle on the square and it fitted.*

This actually showed that the learners are capable of innovating in a classroom if opportunities were provided to them.

5.2.5 Observation of lines joining opposite vertices

Once again, 100% of the learners were able to correctly state that these lines were equal in length. However, one learner further stated that this line is longer than the sides of the square and that it divided the square into two triangles. When asked to justify their answers, 90% of the learners used their triangles from their tangram set and placed the hypotenuse of the triangle onto the diagonal (**refer to figure 1.3***)* and found that it fitted exactly on each of the two lines. The one learner who stated the additional properties of the diagonal approached the justification differently. He stated that if a circle was drawn around the square then this line (the diagonal) would be the 'diameter' because it passes through the centre. Therefore, the line drawn from the other two vertices will also pass through the centre and it will be a 'diameter'. Since 'diameters' are equal then these two lines must be equal. This was not an expected response and it shows that learners are able to draw conjectures if they are given adequate activities to discover and explore for themselves. This further provides evidence that learners are not empty vessels ready to be filled up with knowledge.

The following are excerpts from interviews conducted:

RESEARCHER: *What can you say about the length of the lines joining the opposite vertices of the square?*

LEARNER: *It is longer than the other sides of the square. It divides the square into two equal triangles. These lines are also equal in length.*

RESEARCHER: *Can you prove why these two lines (referring to the diagonals) are equal.*

LEARNER: *If a circle is drawn around the square then this line will be the diameter because it will pass through the centre of the circle, therefore the line drawn from the other two vertices will also pass through the centre and it will also be a diameter. Because diameters are equal then these lines will also be equal.*

Although it is true that the learner had no way of showing that a circle passes through the vertices of the square, it was the correct assumption. The researcher did not probe this any further because it was not part of the study.

When asked to name this line that joined the opposite vertices of the square, 50% of the learners were able to correctly state that it was called diagonals, while the others gave answers like, 'slant line', 'oblique line', 'transversal line' and 'cross line'. Their enunciation did not matter in their understanding of the concepts the researcher was attempting to establish. Of the 50% who stated the name, only one person was able to get the spelling of the word 'diagonals' correct. This shows that not much emphasis was placed on writing in mathematics, and this might only affect their communicating mathematically but not their understanding. However, this aspect of investigation was not within the scope of this study.

The following summarises the results of the participants' knowledge of the properties of the square after their interaction with tangrams:

- 100% of the learners were able to state that all the sides were equal.
- 100% of the learners were able to state that all the angles were 90° each although some stated that all angles were equal, implying that each angle was 90°.
- 100% of the learners were able to state that the diagonals bisected the area of the square.
- However, strangely enough only 40% of the learners were able to state that the opposite sides were parallel. In hindsight, the researcher realises that this property was not investigated well enough.

Reflecting on the answers obtained from the participants it was noted that the use of the tangrams assisted the learners in determining the properties of the square.

5.3 TANGRAMS TO DEDUCE PROPERTIES OF DIAGONALS – SQUARE

The following observations enabled the researcher to answer the second critical question regarding the diagonals of the square. When learners were asked about their observations about the areas of the two isosceles triangles that made up the square, all ten learners were able to correctly state that the areas of the two tangram triangles were exactly the same since they occupied the same space when they were placed on top of each other. Eighty percent of the learners were able to state that the diagonals of a square bisected the area of the square, while one learner stated that the diagonal made the area bigger. On further enquiry, this learner actually meant that one area became two, while another learner merely stated that the diagonal divided the area into two equal parts. It was quite apparent that mathematics vocabulary was not used in most cases, since only one learner used the word, 'bisect'. When asked what the relationship between the two diagonals was, all of them were able to state that they cut each other in half or bisected each other. Here again, it was evident that the learners were not au fait with the mathematics vocabulary.

The following represents excerpts from interviews conducted:

RESEARCHER: *So what does this tell us about the diagonals of the square and its area.*

LEARNER 1: *The diagonal ....cuts the area of the square in half.*

LEARNER 2: *....it halves the area of the square.*

LEARNER 3: *....it divides the area of the square into two equal pieces.*

LEARNER 4: *....the diagonals divide the square into two equal triangles.....*

LEARNER 5: *. ...it makes it bigger.*

RESEARCHER: *What do you mean by 'makes it bigger?'*

LEARNER 5: *I mean the squares area becomes divided into two areas.*

LEARNER 6: *It forms two equal parts.*

When asked about the size of the angles around the point of intersection of the diagonals, all ten learners were able to state correctly that they were 90° each. In justifying their answers, some learners used the square tangram piece and proved that the angles were 90 degrees (**refer to figure 1.4**) – they placed the square piece on each of the angles namely 1, 2, 3 and 4. Some of the learners placed all the angles on each other (**refer to figure 1.4**) and found that they were all equal. They then placed it on a straight line and found that two angles fitted exactly, therefore they were 90° each. One learner folded the outlined square and found that it fitted and then folded it a second time and still found that it also fitted. The learner then placed the midpoint angles on one angle of the tangram square and found that it fitted exactly, and therefore he deduced it was all 90° each. This was again a complex deductive process.

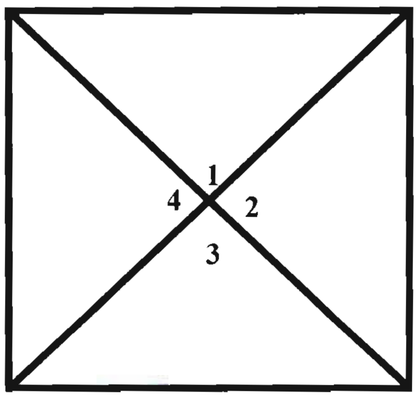


Figure 1.4 Square tangram piece

The following summarises the results of their conclusions about the diagonals of a square:

- 100% of the learners were able to state that the diagonals bisected the area of the square, although not using the same vocabulary, some stated "cut in half", whilst others stated "cut equally".
- 60% of the learners were able to state that "the diagonals bisected each other", not necessarily using the same vocabulary.
- 100% of the learners were able to state that "the diagonals meet", "bisect or intersect at 90 degrees".
- 50% of the learners were able to state, "the diagonals were equal in length".

Once again, it must be noted that learners lacked correct mathematics vocabulary. Nonetheless, it must be highlighted that learners were able to work with the tangram manipulatives to discover the properties of the square as dictated to by the diagonals.

5.4 TANGRAMS USED – PROPERTIES OF RECTANGLE

When learners were questioned about their observations regarding the sides and angles of the rectangle, 100% of the learners were correctly able to state that all angles were equal to 90°, and that the opposite sides of the rectangle were equal. When asked to justify their observations all the participants were able to use their tangram pieces to confirm their results. Some of the learners placed their tangram pieces onto their outlined rectangle and flipped the pieces and found that the long opposite sides were equal. They then flipped the pieces onto the other side, to prove that the two shorter sides were equal. Other learners rotated their tangram pieces 180° about its centre and used this to prove that the opposite sides were the same length.

All ten learners used the square tangram piece to prove that each angle of the rectangle was 90°. They placed one angle of the square onto each angle of the rectangle and used this to prove that each angle of the rectangle was 90°.

This is convincing proof that the tangram pieces were used effectively to determine the properties of the rectangle, thereby answering the stated critical question that tangrams do enhance learner understanding of the properties of the rectangle.

5.5 TANGRAMS USED – PROPERTIES OF RECTANGLE DIAGONALS

When the learners were asked about the relationship between the diagonals of the square and rectangle all ten learners were able to answer correctly that there was a difference, namely, that the square's diagonal intersected at 90° whereas the rectangle did not. However, not all stated the other difference, namely, that the diagonals of a square bisected the interior angles whereas in a rectangle it did not. The researcher acknowledges this omission as a shortfall as it was not analysed further. The following are excerpts from interviews conducted:

RESEARCHER: *Do you think that the diagonals of a rectangle will have the same properties of the square?*

LEARNER 1: *The diagonal of the rectangle and square cut the figures into two triangles; the diagonals in both figures are equal; the angles around the centre in a rectangle will not be equal; they both will bisect each other.*

LEARNER 2: *Lengths are the same; angles around the midpoint are different; they cut each other equally.*

LEARNER 3: *The diagonal lines will be the same length; the four triangles in the Rectangle will not be the same as in the square; the angles around the point of intersection of the diagonals in a rectangle won't be.*

LEARNER 4: *Diagonals in both cases are equal in length; diagonals in both cases cut each other equally; diagonals in both cases bisect the area; in a rectangle the diagonals don't intersect at 90°.*

LEARNER 5: *They will still bisect each other and have the same length but they don't meet at right angles.*

LEARNER 6: *Diagonals bisect the area in both shapes; the diagonals are equal in both shapes; diagonals don't meet at 90° in a rectangle but does meet at 90° in a square; the rectangle has longer diagonals than the square.*

LEARNER 7: *They both bisect each other; they both bisect the area; they both don't meet at 90°, only a square does.*

LEARNER 8: *The diagonals will both be equal in length; they both will bisect each other; the angles around the midpoint of the diagonals in a rectangle will not be equal.*

5.6 TANGRAMS USED TO ATTAIN VAN HIELE LEVEL 3

This question was probed using the following question: "Can we call a square a rectangle?" Ninety percent of the learners answered this question in the affirmative. The one learner who answered differently did have a logical explanation. He stated that a square could not be a rectangle because a rectangle must have two long sides and two short sides, whereas in a square all the sides are of the same length. This brings to fore the debate about definitions between inclusive or exclusive and partitional and hierarchical definitions. According to De Villiers (1994), hierarchical is when the given definitions provide for the inclusion of special cases, eg. a parallelogram is defined so as to include squares, rhombi and rectangles, whereas partitional definitions, is when learners prefer to define a parallelogram as a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal.

However, in a constructivist approach, learners are allowed to formulate their own definitions irrespective of whether they are partitional or hierarchical. Moreover, one can see that hierarchical definitions are more economical than partitional, and furthermore partitional, besides being longer will have to include additional properties to ensure the exclusion of special cases. An advantage of hierarchical definitions is that all theorems proved for that concept automatically apply to its special cases, for example: if we prove that the diagonals of a parallelogram bisect each other, we can immediately conclude that it is true for rectangles, rhombi and squares. If we classified them partitionally, we would have to prove separately in each case, for parallelograms, rhombi and squares (De Villiers, 1996b).

A second question was posed whether a rectangle was a square, and 90% of the learners answered "no" while one learner stated ",yes". On further questioning and enquiry the one learner, stated that two squares made one rectangle; therefore a rectangle has squares, and hence we could call a rectangle a square. It was clear from the answer that learners do not look at concepts holistically, but rather in parts. The following excerpt from an interview conducted represents justification of above:

RESEARCHER : *Can we call a rectangle a square?*

LEARNER: *Yes. If we divide the rectangle exactly in the centre you get two squares, therefore the rectangle is made up of squares and it can be called a square.*

Although not necessarily correct, the learner actually believed that the longer side of the rectangle was twice the length of the shorter side.

Although only two questions probing the attainment of van Hiele level 3 was asked, the researcher regards this as being adequate proof that learners have attained van Hiele level 3, as this study concentrated only on squares and rectangles. The final interview question was to elicit from learners whether they have truly attained van Hiele level 3. The learners were required to complete following question: "a square is a rectangle......" (complete). This ultimately summarised the learners' ability to attain van Hiele level 3. This statement was correctly answered by 90% of the learners, further providing proof that they have attained the required level. This indicated that many learners have the ability of attaining level 3 if they are provided ample opportunities to explore and discover properties by using manipulatives. Yet again, there was evidence that these grade eight learners were able to grapple with their manipulatives to arrive at this complex deduction. The following are excerpts from interviews conducted:

LEARNER 1: A square is a rectangle *with all sides equal.*

LEARNER 2: A square is a rectangle *with all sides equal in length.*

LEARNER 3: A square is a rectangle *with all sides equal in length and diagonals equal in length.*

LEARNER 4: A square is a rectangle *having no long and short sides but all equal sides.*

LEARNER 5: A square is a rectangle *with smaller sides but equal sides as well.*

LEARNER 6: A square is a rectangle *having all sides equal.*

LEARNER 7: A square is a rectangle *having all sides equal in length.*

LEARNER 8: A square is a rectangle *because it has the same properties with one addition that the square has four equal sides.*

LEARNER 9: A square is a rectangle *but with all sides equal.*

The questions relating to the definition of the rectangle and the square were not precisely answered as most learners gave all the properties of the two figures instead of giving a clear and concise description of both figures. The following are excerpts from interviews conducted:

RESEARCHER: *Can you define a square for me?*

LEARNER 1: *A square has four equal sides and four 90° angles. When the diagonals cut each other 90° angles are formed around the centre. When the diagonal cuts the square two equal triangles are formed.*

LEARNER 2: *A square is a shape that has four 90° angles and all sides are equal.*

LEARNER 3: *A square is a shape that has 4 equal sides and 4 corner angles are 90°.*

LEARNER 4: *A square has all sides equal and each angle equal to 90°.*

LEARNER 5: *A square has four equal sides and four equal angles.*

LEARNER 6: *A square is a diagram/object which has 4,90° angles and 4 equal sides.*

LEARNER 7: *A square must have all the sides equal and each angle must be equal to 90°.*

LEARNER 8: *A square has four equal sides and four 90° angles.*

LEARNER 9: *A square has four equal sides and four angles of 90° each.*

LEARNER 10: *A square is a four-sided figure with equal sides and equal angles.*

The following are excerpts from the researcher's interview relating to the definition of a rectangle:

RESEARCHER: *Can you define a rectangle?*

LEARNER 1: *A rectangle is a shape with opposite sides that are equal. All angles are equal to $90°$. A rectangle is not a square.*

LEARNER 2: *A rectangle is a shape that has four $90°$ angles and opposite sides equal.*

LEARNER 3: *A rectangle is a stretched square with opposite sides the same length but all 4 sides don't have the same length, but each angle must be $90°$.*

LEARNER 4: *A rectangle has opposite sides equal and each angle equal to $90°$.*

LEARNER 5: *A rectangle is a figure with 4 equal $90°$ angles and parallel sides are equal.*

LEARNER 6: *A rectangle is an object, which has two equal long sides and two equal short sides and all $90°$ angles.*

LEARNER 7: *A rectangle has all the angles equal to $90°$ and 2 short and 2 long equal sides.*

LEARNER 8: *A rectangle has 2 long equal sides and 2 short equal sides with each angle equal to $90°$.*

LEARNER 9: *A rectangle is a quadrilateral shape with opposite sides equal and each angle $90°$.*

LEARNER 10: *A rectangle is a four-sided figure with the parallel sides equal and each angle measuring $90°$.*

These learners' definitions were irrelevant to this research, and therefore it did not necessitate the intervention of the researcher. However, it must be acknowledged that the activity could have provided a useful platform for the establishment of concise definitions.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 DEFINITIONS

What emerged from this study was the fact that many learners are still unable to state clear definitions, since most of them listed properties of the figure instead of defining it appropriately. According to De Villiers (1994), many grade eight learners memorize given formal definitions and then regurgitate them in the examination. This surely is a recipe for disaster since the learners are unlikely to accept, understand, and appreciate them. De Villiers (1994) goes further to suggest that formal definitions of quadrilaterals in the secondary school are not necessarily required. In fact, he quotes a teaching experiment (as reported in Human & Nel et al., 1989) which indicated that the question of formal definitions should not be approached before grade ten, and then only if learners are actively involved in the process of constructing definitions for the various quadrilaterals. This view concurs with the results of this study since learners were encouraged to give their own definitions once they had completed the tasks given to them. Learners were able to construct their own definitions for the rectangle and the square once they had actively involved themselves through guided discovery in the task-based interviews.

By allowing learners to be actively involved in discovering the different properties of the square and rectangle, it was evident that they were able to understand and give meaning to the concepts being observed. Providing learners with ready-made definitions could have led to misconceptions that there is a single correct definition for each concept. In addition, if we want learners to understand the need for definitions, we should ensure that they are actively involved in its construction and use. Learners are often unable to grasp concepts because they are presented in an abstract form. Therefore, the researcher tends to agree with De Villiers (1994) who states that defining concepts accurately in mathematics is certainly not an easy task, and is only developed after vast experience and practice. It is beneficial to provide opportunities for the learners to discover their own definitions through self-exploration so that the definitions become more meaningful.

The researcher observed that redundancy, or a lack of economy was evident in definitions and properties, suggesting that teachers have to be aware of the adequacy, sufficiency and equivalence of some properties and definitions as pointed out by De Villiers (2003). What was also quite evident was the lack of usage of proper mathematical terminology. Teachers should be cognisant of the fact that language plays a vital role in mathematics. They need to ensure that they describe geometrical shapes and properties by their correct terminology. In cases where teachers themselves have a problem expressing concepts and definitions accurately and correctly, will ultimately lead to learners also learning and adopting those inaccuracies and misconceptions.

It was perceived that definitions were problematic for some learners at this stage since they are still immature in their thinking as far as geometry is concerned. Some learners had difficulty recalling definitions whilst others gave all the properties of the square and rectangle when asked for the definitions. However, it was observed that learners were quite comfortable working with the concrete tangram pieces rather than recalling definitions. Therefore, I envisage that emphasis should not be placed on formal definitions at this stage but rather that educators should provide sufficient opportunities for learners to interact with concrete manipulatives so that later on they will be able to understand definitions much better.

6.2 RECOMMENDATIONS

The following recommendations can be made from this study:

- Many learners have underdeveloped visualisation skills because not enough practical experience is provided for learners to explore, experiment and discover properties for themselves. It is therefore necessary to provide sufficient opportunities to hone on these skills.
- Many teachers avoid concrete manipulatives because they believe it is time-consuming or unnecessary. This study proves that learners, who knew very little about squares, rectangles and certainly nothing of diagonals from their pre-test, were able to grasp and list these properties with ease. The researcher therefore suggests that workshops and training be made available for teachers on how and when to use concrete manipulatives. This will enable them to become confident in the use of manipulatives and they will begin to appreciate its value in teaching many abstract concepts in mathematics.
- The tangram pieces are inexpensive manipulatives that could easily be made in classrooms. Learners can use it for different concepts, especially in geometry - for example, transformation geometry, tessellations, similarity, congruence, properties of quadrilaterals and so on. Teachers could make difficult concepts look easy by using the tangram pieces in the presentation of their lessons.
- The researcher was able to obtain valuable information regarding diagonals of square and rectangle which otherwise would have been a long and arduous

process to achieve if the traditional "chalk and talk" method was used. The researcher expresses this with conviction because this section was taught using both methods, and the values of using tangram pieces cannot be underestimated.

- Learners should be encouraged to have a vocabulary list of the words they will encounter in the learning of daily geometry.
- When learners invent or discover relationships and properties themselves, they are likely to retain that knowledge for a longer period than if they merely listened to a lesson.
- Manipulatives could train learners to look for patterns/invariants in a given shape, be it in two or three dimensions, which could prove an invaluable step towards the solution of geometrical problems (riders) in high school.
- That teacher training colleges and workshops be set up to assist the teachers in adjusting from the tradition of following the text books slavishly, to developing new resources like manipulatives which can be used to teach new concepts in mathematics.
- Using manipulatives, as a mathematical tool will help produce learners who will be able to reason and thereby become creative problem solvers.
- Informal geometry, interacting with manipulatives, forms the basis for the learning of formal geometry.
- Educators must be aware that learners are capable of complex reasoning processes and are able to rationalise in their own words in order to attain van Hiele 3. Recommend a research were a class is observed using manipulatives instead of a one-to-one interview, as this could have revealed interesting results since group dynamics always presents interesting findings.

- The results of this study support initiation of other studies to replicate these findings not only in geometry but also in other areas of mathematics as well.
- Future studies may also want to further test the validity of this research by having learners switch after a period of time and change from the use of manipulatives to normal '*kill and drill*' method and compare the results to establish whether this method is more effective or not.

6.3 LIMITATIONS

- This study was conducted using considerable time constraints as the researcher found it very difficult to conduct the interviews, since the school has a very intense extra-curricular programme and this meant that interviews had to be conducted after school hours depending on availability of participants.
- This study focussed on the value of manipulatives enhancing learning, therefore a video-recorder would have been ideal since the participants' facial expressions as well as their failed attempts could have been recorded. The researcher did acknowledge this in hindsight.
- This study sample comprised of only ten learners; therefore, the results could not be generalised but it is useful in determining that manipulatives can assist in making teaching simpler and learning much easier.
- As already stated, this is a study of a specific focus, and that the data used in this analysis was based solely on learner accounts in the interviews.
- Even a qualitative analysis is jeopardised because the researcher, the author of this study, was the participant in this study who analysed the classroom experiences.

- Further research, using the computer software programme involving tangrams, could be undertaken to determine whether similar or better results could be achieved.
- Further research needs to be conducted in order to determine whether examination and test results improve if learners are exposed to these types of manipulatives.

6.4 CONCLUSIONS

In conclusion, the researcher affirms that in order to achieve satisfactory results in mathematics geometry and promote active participation in mathematics the authorities need to make some crucial changes to the way mathematics, especially geometry, is taught at high school level. Emphasis should not be placed on trying at all costs to complete a syllabus, but rather to spend more time especially in grade 8 and 9 to allow learners to discover and experiment with different manipulatives so that later on in high school, concepts become easier to grasp. When learners discover concepts for themselves, they internalise and retain them for longer periods, as compared to concepts merely taught to them. As stated previously, when concepts and definitions are provided to the learners beforehand, they tend to see very little meaning in them since they do not understand its origins nor can they comprehend its relevance. It is therefore essential that adequate time be devoted to learner discoveries to ensure that the learners enjoy the learning process, rather than seeing it as a dull and boring subject. The results of this study confirms my initial speculation that geometry can be presented in a form that is much easier to understand when concrete manipulatives are used, since this provides a visual vehicle that drives the concepts in a subtle way across to the learners. It was ascertained that learners were able to accept this method much more easily compared to the abstract form of dishing-out theorems and axioms. It is therefore strongly recommended that concrete manipulatives be used as often as possible to ensure geometry lessons become more interesting, enjoyable and beneficial to the learner. Manipulatives certainly afford learners an opportunity to demonstrate their thinking.

The results of this study should help educators determine an effective way of approaching mathematics instruction by using manipulatives. Clearly more evidence would provide stronger support for my claims that active engagement and self-discovery with manipulatives are the key components for making the teaching and learning of geometry more enjoyable and interesting.

References

Ackerman, P. L. (1996). A theory of adult intellectual development: process, personality, interests, and knowledge. *Intelligence*, 22, 229-259.

Baez, A. (1997). *Critical thinking through manipulatives: Staff development intervention for middle grades.* Unpublished master's thesis, University of Massachusetts, Boston.

Ball, D. (1992). Magical hopes: Manipulatives and the reform of math education. *American Educator*, Summer, 14-18; 46-47.

Baratta-Lorton, M. (1995). *Mathematics their way.* Menlo Park, CA: Addison-Wesley.

Bell, J. (1993). *Doing your research project: A guide to first time researchers in education and social science*. Buckingham: Open University Press.

Bell, J. (2005). *Doing your research project: A guide for first time researchers in education, health and social science.* Berkshire, England: Open University Press.

Bellonio, J. L. (2001). *Multi-sensory manipulatives in mathematics: linking the abstract to the concrete.* Yale: New Haven Teachers Institute

Biggs, J., and Collis,K (1982). *Evaluating the quality of learning: The SOLO taxonomy.* New York: Academic Press.

Biggs, J., and Collis,K (1991). *Multimodal learning and the quality of intelligent behavior.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Biggs, J. (1996). Enhancing teaching through constructive alignment. *Higher Education*, 32, 347-464.

Brooks, D. W. (1994). *The nature of conceptual learning and thinking.* Unpublished master's thesis, University of Durban Westville, South Africa.

Brooks, J. G., & Brooks,M.G. (1993). *In search of understanding: The case for constructivist classrooms.* Alexandria, VA: Association for Supervision and Curriculum Development.

Brown, J. S., Collins,A., & Duguid,P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32-42.

Burger, W. F., & Shaughnessy, J.M. (1986). Characterising the Van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17, 31-48.

Burns, M. (n.d.). *Manipulatives in today's classroom*. Retrieved April 14, 2006, from

http://teacher.scholastic.com/com/products/instructor/letter.htm

Burton, G. M. (1985). *Towards a good beginning: teaching early childhood mathematics.* Menlo Park, CA: Addison-Wesley.

Canny, M. E. (1984). The relationship of manipulative materials to achievement in three areas of fourth-grade mathematics: computation, concept development and problem solving, Dissertation Abstracts International.

Charlesworth, R. (1984). Kindergarten mathematics: Step-by-step from concrete materials to paper and pencil. National Association for the Education of Young Children, Los Angeles, ERIC Document Reproduction Service No. ED 252 433.

Chattin-McNichols, J. (1992). *Montessori Programs in public school.* Unpublished master's thesis, University of Illinois at Urbana-Champaign Children's Research Centre.

Chester, J., Davis,J & Reglin,G. (1991). *Math manipulatives use and math achievement of third grade students.* Unpublished master's thesis, University of North Carolina, USA.

Clements, D. H. (1999). "Concrete" manipulatives, concrete ideas. *Contemporary Issues in Early Childhood,* 1(1), 45-60.

Clements, D. H., & Battista, M.T. (1990). Constructive learning and teaching. *Arithmetic Teacher,* 38, 34-35.

Clements, D. H., & Battista, M.T. (1991). Using spatial imagery in geometric reasoning. *Arithmetic Teacher,* 39(3), 18-21.

Clements, D. H., & Battista, M.T. (1992). *Geometry and spatial reasoning.* New York: Macmillan.

Clements, D. H., Battista, M.T., & Sarama,, J. (Eds). (2001). *Journal for research in Mathematics Education Monograph 10: Logo and Geometry.* Reston, VA: NTCM.

Clements, D. H., & McMillen, S. (1996). Rethinking "concrete" manipulatives. *Teaching Children Mathematics,* 2(5), 270-279.

Cohen, L. M., L. (1994). *Research methods in education.* London: Routledge.

Cohen, L. M., L. (2000). *Research methods in education.* London: Routledge Falmer.

De Lange, J. (1996). *Using and applying mathematics in education*, Kluwer Academic Press.

De Villiers, M. (1994). The role and function of proof in mathematics. *Pythagoras,* 24, 17-23.

De Villiers, M. (1996a). *The future of secondary school geometry.* SOSI Geometry Imperfect Conference Pretoria, UNISA.

De Villiers, M. (1996b). Why proof in dynamic geometry ? Proceedings of topic group 8: Proofs and proving: Why, When and How ? AMESA, Centrahil.

De Villiers, M. (2003). *Rethinking proof with the geometer's sketchpad.* USA: Key Curriculum Press.

De Villiers, M. D., & Njisane, R.M. (1987). *The development of geometric thinking among black high school pupils in Kwazulu-Natal (Republic of South Africa).* Conference of the international group for the psychology of mathematics education, Montreal.

Del Grande, J. (1990). Spatial sense. *Arithmetic Teacher,* 37(6), 14-19.

Dienes, Z. P. (1960). *Building up mathematics.* London: Hutchinson Educational.

Driscoll, M. (1988). *Research within reach: Secondary school mathematics.* Virginia: NTCM.

Driscoll, M. J. (1981). *Research within reach: elementary school mathematics.* Reston, VA: National Council of Teachers of Mathematics & CEMREL, Inc.

Driscoll, M. J. (1983). *Research within reach: elementary school mathematics and reading.* St Louis: CEMREL.

Driscoll, M. P. (1994). *Psychology of learning for instruction.* Needham, Ma: Allyn & Bacon.

Dunkels, A. (1990). Making and exploring tangrams. *Arithmetic Teacher,* 37(6), 38-42.

Ernest, P. (1986). Games: a rationale for their use in the teaching of mathematics in school. *Mathematics in School,* 15(1), 2-5.

Ernest, P. (1995). *The one and the many.* Hillsdale,NJ: Lawrence Erlbaum Associates.

Fennema, E. H. (1972). Models and mathematics. *Arithmetic Teacher,* 19, 635-640.

Fennema, E. H. (1973). Manipulative in the classroom. *Arithmetic Teacher*, 20, 350-352.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics,* 3, 413-435.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht: Reidel.

Freudenthal, H. (1981). Major problems of mathematics education. *Education in Science and Mathematics,* 12(2), 5.

Fuys, D. (1985). Van Hiele levels of thinking in geometry. *Education and Urban Society,* 17(4), 447-462.

Fuys, D., Geddes,D., & Tischler,R (1988). *Journal of Research in Mathematics Education Monograph 3: The van Hiele model of thinking in geometry among adolescents.* Reston: NCTM.

Garrity, C. (1998). *Does the use of hands-on learning with manipulatives improve the test scores of secondary education geometry students?* Unpublished master's thesis, University of Chicago, Illinois.

Gilbert, R., & Bush, W. (1988). Familiarity, availibility and use of manipulative devices in mathematics at the primary level. *School Science and Mathematics,* 88, 459-469.

Glaserfeld, E. V., & Kelly,M.F (1982). On the concepts of period, phase, stage and level. *Human Development,* 25, 152-160.

Govender, M. (1995). *Pupils' proof-writing achievement in circle geometry.* Unpublished B.Ed mini dissertation, University of Durban-Westville, Durban.

Gredler, M.E. (1997). *Learning and instruction: Theory into practice (3rd ed).* Upper Saddle River, NJ: Prentice-Hall.

Griffiths, H. B., & Howson, A.G. (1974). *Mathematics: society and curricula.* Cambridge University Press.

Gutierrez, A., Jaime,A., & Fortuny, J.M. (1991). An alternative paradigm to evaluate the acquisition of the van Hiele levels. *Journal for Research in Mathematics Education,* 22(3), 237-251.

Hartshorn, R., & Boren, S. (1990). *Experiential learning of mathematics: Using manipulatives.* Retrieved September 1, 2006, from http://www.ericdigests.org/pre-9217/math.htm

Hausfather, S. J. (1996). Vygotsky and schooling: creating a social contest for learning. *Action in Teacher Education,* 18, 1-10.

Heddens, J. W. (2005). *Improving mathematics teaching by using manipulatives.* Retrieved September 1, 2006, from http://www.fed.cuhk.edu.hk/~fllee/mathfor/edumath/9706/13hedden.html

Hershkowitz, R. & Parzysz, B. (Eds.). (1996). *Space and shape: International handbook of mathematics education. Part 1.* Dordrecht: Kluwer Academic Publishers.

Hiebert, J. (1986). *Conceptual and procedural knowledge: the case of mathematics.* London: Lawrence Earlbaum Associates.

Hiebert, J., & Wearne,D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for Research in Mathematics Education,* 23, 98-122.

Hitchock, G., & Hughes, D. (1995). *Research and the teacher.* London: Falmer Press.

Honebein, P. (1996). *Seven goals for the design of constructivist learning environments.* New York: Educational Technology Publications.

Human, P.G. & Nel, J.H. With M.D. de Villiers, T.P. Dreyer & S.F.G. Wessels. (1989). *Alternative instructional strategies for geometry education: A theoretical & empirical study.* RUMEUS: University of Stellenbosch.

Jackson, R. (1979). Hands-on Math: Misconceptions and abuses. *Learning*, 7, 76-78.

Jonassen, D. H. (1994). Thinking technology: toward a constructivist design model. *Educational Technology*, 34(3), 34-37.

Jones, K. (1997). Some lessons in mathematics: a comparison of mathematics teaching in Japan and America. *Mathematics Teacher*, 159, 6-9.

Juraschek, W. (1983). Piaget and middle school mathematics. *School Science and Mathematics*, 83(1), 1.

Jurdak, M. (1989). *Van Hiele levels and the Solo taxonomy.* In proceedings of the 13th International Conference fo the Psychology of Mathematics Education, Paris.

Kruteskii, V. A. (1976). *The psychology of mathematics ability in schoolchildren.* Chicago: University of Chicago Press.

Land, J. E. (1990). *Appropriateness of the Van Hiele model for describing students' cognitive processes on algebra tasks as typified by college students' learning.* Unpublished master's thesis, University of Massachusetts, Boston.

Lehrer, R., Jenkins, M., & Osana, H. (1998). *Longitudinal study of children's reasoning and geometry.* Mahwah,NJ: Lawrence Erlbaum Associates.

Lesh, R. A.. (1979). *Applied problem solving in early mathematics learning.* Northwestern University.

Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, 14(1), 58-69.

Mayer, R. E. (1989). Models for understanding. *Review of Educational Research*, 59(1), 43-64.

McNiff, J. (2002). *Action research for professional development – Concise advise for new action researchers.* Retrieved November 19, 2006, from http://www.jeanmcniff.com

Merriam, S. B., & Associates. (2002). *Qualitative research in practice: examples for discussion and analysis.* San Francisco: Jossey-Bay Publishers.

Moses, B. (1990). Developing thinking in the middle grades: designing a spatial station. *Arithmetic Teacher*, 37(6), 59-60.

Mouly, G. J. (1978). *Educational research: The art and science of investigation.* Boston: Allyn and Bacon.

Mudaly, V. (1998). *Pupils' needs for conviction and explanation within the context of dynamic geometry.* Unpublished master's thesis, University of Durban-Westville, Durban.

Myers, M. (1985). *The teacher-researcher: How to study writing in the classroom.* Urbana, IL: National Council of Teachers of English.

Nakin, J. B. N. (2003). *Creativity and divergent thinking in geometry education.* Unpublished doctoral thesis, UNISA, Pretoria.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston,VA: NCTM.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston,VA: NCTM

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA.: NCTM.

Neuman, W. (2000). *Social research methods.* Boston: Allyn and Bacon.

Nixon, E. G. (2002). *An investigation of the influence of visualization, exploring patterns and generalization of thinking levels in the formation of the concepts of sequences and series.* Unpublished master's thesis, UNISA, Pretoria.

Nohda, N. (1992). Geometry teaching in Japanese school mathematics. *Pythagoras*, 28(4), 18-25.

Olive, J. (1991). Logo programming and geometric understanding: an in-depth study. *Journal for Research in Mathematics Education*, 22(2), 90-111.

Oliver, A. I. (1992). *Handling pupil's misconceptions.* Pietermaritzburg: Shuter & Shooter.

Pandiscio, E., & Orton,R.E. (1998). Geometry and metacognition: an analysis of Piaget's and van Hiele's perspectives. *Focus on Learning Problems in Mathematics*, 20(2-3), 78-87.

Pape, S. J. (2001). *Using technology to help form representations of mathematical concepts.* Retrieved September 14, 2006, from http://www.enc.org/topics/edtech/learning/documents/0,1946,FOC-000703-index,00.shtmx

Papert, S. (1980). *Mindstorms: children, computers and powerful ideas.* New York: Basic Books.

Pegg, J. (2003). *Assessment in Mathematics: a developmental approach.* Melbourne: Information Age Publishing.

Pegg, J., & Davey, G. (1991). Understanding geometric levels of students. *The Australian Mathematics Teacher*, 47(2), 10-13.

Pegg, J., & Davey, G. (1998). *A synthesis of two models: Interpreting student understanding in geometry.* Mahwah, NJ, US: Lawrence Erlbaum Associates.

Piaget, J., & Inhelder,B. (1971). *The psychology of the child.* New York: Basic Books.

Piaget, J. (1983). *Dialogue 111. Jean Piaget's views on the psychology of language and thought.* New York: Plenum Press.

Pillay, T. (2004). *Below the surface: African learner's experiences of schooling in a predominantly Indian school in KwaZulu-Natal.* Unpublished master's thesis, University of KwaZulu-Natal: Durban.

Presmeg, N. E. (1989). Visualization in multicultural mathematics classrooms. *Focus on Learning Problems in Mathematics*, 2(1), 17-24.

Resnick, M. (1998). *Digital manipulatives: new toys to think with.* In proceedings of CHI'98. Retrieved September 1, 2006, from http://citeseer.ist.psu.edu/resnick98digital.html

Rigdon, D. (2000). Tackling tangrams. *Teaching Children Mathematics*, 6(5), 304-305.

Rogoff, B. (1998). Cognition as a collaborative process. In W. Damon, D. Kuhn, & R.S. Siegler (Eds). *Handbook of Child Psychology.* Toronto: John Wiley & Sons.

Rowan, T. E. (1990). The geometry standards in K-8 mathematics. *Arithmetic Teacher*, 37(6), 24-27.

Saettler, P. (1990). *The evolution of American educational technology*. Egnlewood,Co: Libraries Unlimited.

Savery, J. R., & Duffy,T.M. (1995). Problem-based learning: an instructional model and its constructivist framework. *Educational Technology*, Sept-Oct, 31-38.

Schifter, D., & Fosnot,C.T. (1993). *Reconstructing mathematics education: stories of teachers meeting the challenge of reform.* New York: Teachers College Press.

Scott, P. (1987). *A constructivist view of learning and teaching in science*. Children's Learning in Science Project, Centre for studies in science and mathematics education,University of Leeds, England.

Semadeni, Z. (1984). A principle of concretization permanence for the formation arithmetical concepts. *Education in Science and Mathematics*, 15(4), 11.

Senk, S. L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education*, 20(3), 309-321.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies In Mathematics*, 22, 1-36.

Skemp, R. (1972). *Schematic learning: the process of learning mathematics.* New York: Pergamon Press.

Skemp, R. (1987). *The Psychology of teaching mathematics* (revised American edition). Hillsdale,NJ: Lawrence Erlbaum.

Smith, E. C. (1987). *'n Kwaliatiewe vergelyking van enkele van Hiele-toetsinstrumente.* Unpublished master's thesis, University of Stellenbosch, South Africa.

Sowell, E. J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education*, 20, 498-505.

Steele, D. (1993). *What mathematics students can teach us about educational engagement : Lessons from the middle school.* Paper presented at a meeting of the American Educational Research Association, Atlanta, Georgia.

Stevens, D. (2004). *Students interacting with video games at Lexia Learning Systems.* Retrieved August 26, 2005, from http://gseweb.harvard.edu/news/features/stevens01012004.html

Stone, J. I. (1987). Early childhood math: make it manipulative! *Young Children*, 42(6), 16-23.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* Newbury Park, CA: Sage Publications, Inc.

Streefland, L. (1993). The design of a mathematics course: a theoretical reflection. *Educational Studies in Mathematics*, 25, 109-135.

Stuart, V. B. (1998). Math curse or math anxiety? Retrieved September 21, 2006, from http://mathforum.org/t2t/faq/faq.manipulatives.html

Sugiyama, Y. (1987). *Comparison of word problems in textbooks between Japan and the U.S.* In Proceedings of U.S. - Japan Seminar on Problem Solving, Carbondale, Southern University.

Suydam, M. N., & Higgins,J.L. (1976). Review and synthesis of studies of activity-based approaches to mathematics teaching. Recommendations from research, Columbus, OHIO: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Suydam, M. N., & Higgins,J.L. (1977). *Activity-based learning in elementary school mathematics: recommendations from research.* Columbus, OHIO: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Suydam, M. N. (1984). Research report: manipulatve materials. *Arithmetic Teacher*, 31, 27.

Suydam, M. N. (1986). Research report: manipulative materials and achievement. *Arithmetic Teacher*, 33(10), 32.

Tapson, F. (1985). Cutting and sticking. *Mathematics in School*, 14(1), 18.

Teeguarden, J. E. (1999). *Math-art in the elementary classroom. Creativity and mathematics education.* Proceedings of the international conference, Meunster,Germany.

Thatcher, D. (2001). The tangram Conundrum. *Mathematics Teaching in the Middle School*, 6(7), 397.

Treffers, A. (1987). *Three dimensions: a model of goal and theory description in mathematics instruction - the Wiskobas Project.* Dordrecht: Reidel.D.

Usiskin, Z. (1982). *Van Hiele levels of achievement in secondary school geometry.* CDASSG Project, University of Chicago, Department of Education. (ERIC Document Reproduction Service No. ED 220 288).

Van der Sandt, S. (2000). *An analysis of geometry learning in a problem solving context from a social cognitive perspective*. Unpublished master's thesis, University of Potchefstroom, South Africa.

Van Hiele, P. M. (1986). *Structure and insight: a theory of mathematics education.* Orlando: Academic Press Inc.

Van Niekerk, H. M. (1997). *A subject didactical analysis of the development of the spatial knowledge of young children through a problem-centred approach to mathematics education and learning.* Unpublished master's thesis, University of Potchefstroom, South Africa.

Van Zyl, A. J. (1942). *Mathematics at the cross-roads.* Cape Town: Maskew Miller.

Vest, F. (1985). Using physical models to explain a division algorithm. *School Science and Mathematics*, 85(3), 3.

Vygotsky, L. S. (1978). *Mind in society: the development of higher psychological processes.* Cambridge, Massachusetts: Harvard University Press.

Welman, J., & Kruger, S.J. (2001). *Research methodology.* Cape Town: Oxford University Press.

Wheatley, G. H. (1990). Spatial sense and mathematics. *Arithmetic Teacher*, 37(6), 10-11.

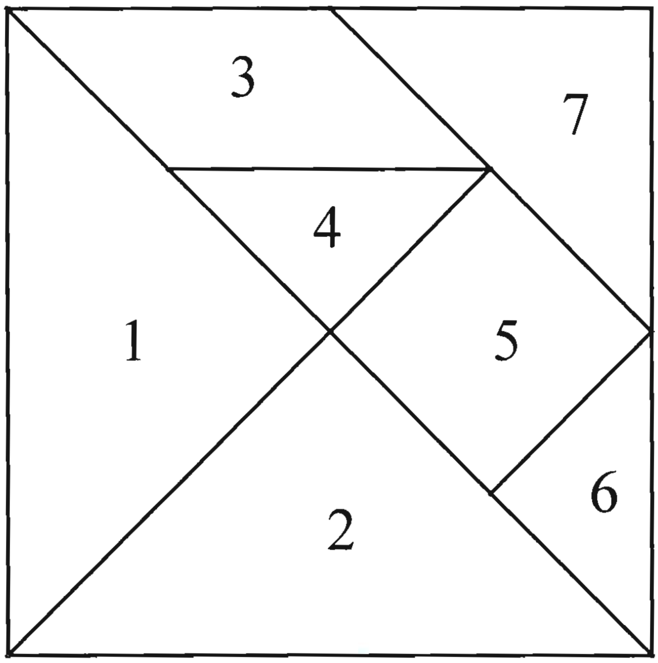
Wiebe, J. H. (1983). Physical models for symbolic representations in arithmetic. *School Science and Mathematics*, 83(6), 10.

Wilson, B., & Cole,P. (1991). A review of cognitive teaching models. *Educational Technology Research and Development*, 39(4), 59-61.

Wilson, M. (1990). Measuring a van Hiele geometry sequence: A reanalysis. *Journal for Research in Mathematics Education*, 21(3), 230-237.

Yackel, E., Cobb,P., Wood,T., Wheatley,G., & Merkel,G (1990). The importance of social interaction in children's construction of mathematical knowledge. In T.J. Cooney, & C.R. Hirsch (Eds). *Teaching and learning mathematics in the 1990s.* Virginia: The Natiional Council of Teachers of Mathematics.

APPENDIX A – TANGRAM – THE MANIPULATIVE



APPENDIX B - LETTER OF CONSENT

Dear ____

I am an M.Ed student at UKZN interested in the value of using manipulatives to understand mathematics geometry. As part of my research, I would like to interview your son/daughter____ as part of my study. He/she will be interviewed after writing a test.

The interview will take approximately 60 minutes and will be tape recorded. The data from the interview will only be used for my research purposes and will not be used for any other purpose without your consent. The children are not obliged to answer all the questions that I ask them and they are free to withdraw from the interview at any time. Please note that no real names will be used in any material that I write up and every attempt will be made to keep the material confidential.

Thank you for your assistance. If you require any further information, please feel free to contact my course supervisor, Dr Vimolan Mudaly at 082-9770577 or contact me. My contact details are - Rakesh Singh, cell phone number 082-5804787.

Yours sincerely
R.I.Singh

I ______________________________agree to allow my
child_____________________to participate in this study. I understand that my child's real name will not be used in any write-up and that their responses will be treated confidentially. I also understand that they are free to withdraw from the study at any time.

Name: ____________________________ Date:____________________________

I thank you in anticipation.

APPENDIX C- PILOT WORKSHEET

Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler.

Instruction:

A. Make an outline of the square tangram piece on a clean sheet of paper and then follow the steps and answer the questions that follow:

1. Fold this square along the diagonal:
   - 1.1 What do you observe about the sides of the square?
   - 1.2 What can you deduce about the angles? Can you prove this?
   - 1.3 What can you say about the area of these two folded triangles?

2. Now, unfold and fold along the other diagonal:
   - 2.1 What do you observe about the sides?
   - 2.2 What do you observe about the angles?
   - 2.3 What can you say about the two triangles?
   - 2.4 What is the relationship between 1.1 and 2.1?
   - 2.5 What is the relationship between 1.2 and 2.2?
   - 2.6 What is the relationship between 1.3 and 2.3?
   - 2.7 Write out your conclusions in three full sentences.

B. Make a second outline of the square tangram piece:

3. Cut the first outline along the first diagonal and the second outline along a different diagonal.

   Now, place all four cut-outs with all diagonals on each other. What do you notice about the all the diagonals? Write out your conclusion in a full sentence.

4. Take any half of the square and crease it lightly to find the midpoint of the longest side. Mark this point by using a dot. Measure the distance from this point to the three vertices.

4.1 What do you notice? What does this tell us about the diagonals of a square?

4.2 What is the size of the angles formed around the midpoint? Can you prove this?

4.3 Write out your conclusions about the diagonals in two full sentences.

APPENDIX D - INTERVIEW SCHEDULE

Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler.

1. THE SQUARE

Make an outline of the square tangram piece on a clean sheet of paper. Select the two smallest triangles from your tangram set and then answer the following questions:

1.1 What type of triangles are these? Can you justify your answer?

1.2 What is the size of each angle of the triangles? Can you prove this?

1.3 Take the two smallest triangles and place it onto the outlined square such that it fits exactly. Now rotate the two triangles 900 about its centre. What can you deduce about the sides of the square?

1.4 Rotate the triangles another 900 clockwise about its centre. What do you observe about the sides of the square?

1.5 Without measuring, what can you say about the angles of the square? Can you prove this?

1.6 What can you say about the length of the line joining the opposite vertices of the square? Can you prove this? What do we call this line?

1.7 What can you say about the area of these two triangles forming the square? What does this tell you about the diagonals of a square and its area?

1.8 Can you list all the properties you have observed about the square?

1.9 Take one of the square outlines and cut along one of the diagonals.
Take any half of the square and crease lightly along the diagonals to find the midpoint of the longest side. Mark this point by using a dot. Measure the distance from this midpoint, to the three vertices of the half square.

1.9.1 What do you notice? What does this tell us about the diagonals of a square? Does this work for the other half as well? Check.

1.9.2 What is the size of the angles formed around the midpoint? Can you prove this?

1.9.3 List your conclusions about the diagonals of a square?

1.10 List only those properties that are necessary to define a square?

2. THE RECTANGLE

Make a rectangle using these three pieces of your tangram set, viz., square, two triangles. Make an outline of this rectangle on a clean sheet of paper and then answer the following questions:

2.1 What do you notice about the opposite sides of the rectangle? Can you prove this using your outline?

2.2 What can you say about the angles of the rectangle? Can you prove this using your outline?

2.3 Do you think that the diagonals of the rectangle will have the same properties as the square? If yes/no state the similarities or differences.

2.4 Can you list any other similarities and or differences between the square and rectangle?

2.5 Can we call a square a rectangle? Justify your answer.

2.6 Can we call a rectangle a square? Justify your answer.

2.7 Can you state the necessary properties that will enable you to define a rectangle?

2.8 Can you define a rectangle?

APPENDIX E – PRE-TEST

The following test must be answered on the question paper.

1. Write down a definition for the term quadrilaterals.

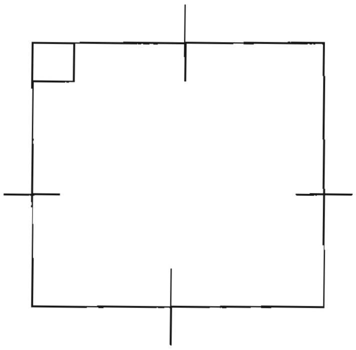
____

____(2)

2. Choose one of the terms from the list below which best describes the geometrical shape illustrated. Provide a reason for your choice.

Kite, rectangle, trapezium, square, hexagon, octagon, parallelogram, rhombus, pentagon.

2.1

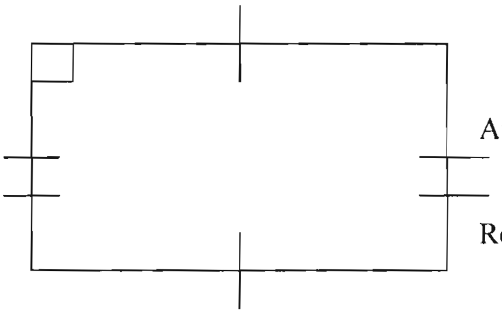


Answer: ____________________

Reason: ____________________

(2)

2.2

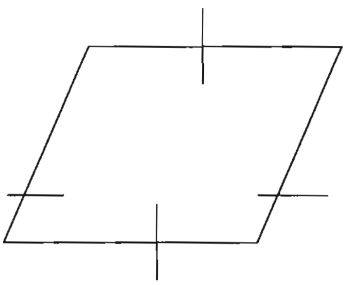


Answer: ____________________

Reason: ____________________

(2)

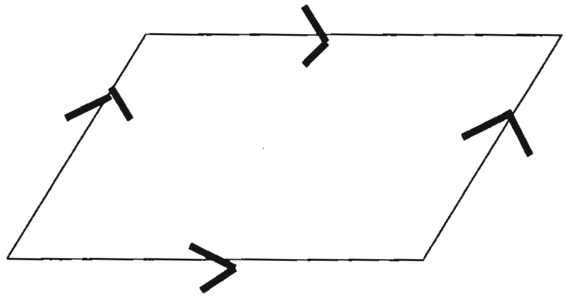
2.3



Answer: ____________________

Reason: ____________________(2)

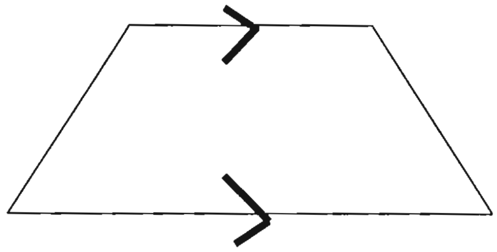
2.4



Answer _______________

Reason _______________(2)

2.5



Answer _______________

Reason _______________

(2)

3. Answer the following questions with a YES or NO , giving a reason for your choice.

3.1 Is a rectangle always a parallelogram? ________; Why?

____(2)

3.2 Is a square always a rectangle? _____________;

Why____(2)

3.3 Is a rhombus always a square? _____________; Why?

____(2)

3.4 Is a parallelogram always a square? _________; Why?

____(2)

3.5 Is a parallelogram always a rectangle? ________; Why?

____(2)

3.6 Is a rhombus always a parallelogram? ________; Why?

____(2)

3.7 Is a square always a parallelogram? _________; Why?

____(2)

3.8 Is a rectangle always a square? ____________; Why?

____(2)

3.9 Is a square always a rhombus? ____________; Why?

____(2)

3.10 Is a parallelogram always a rhombus? ________; Why?

____(2)

3.11 Is a rectangle always a rhombus? __________; Why?

____(2)

4. Underline the odd word in the following list and give a reason for your choice:

Rectangle; Parallelogram; Rhombus. - Reason: ______________________

____(2)

5. Now underline a different odd word not chosen in 4 avove and give a reason:

Rectangle; Parallelogram; Rhombus. - Reason: ______________________

____(2)

6. Underline the odd word from the list below and give a reason for your choice:

Square; Rectangle; Rhombus - Reason: ______________________

____(2)

7. Now underline a different odd word not chosen in 6 above and give a reason:

Square; Rectangle; Rhombus - Reason: ______________________

____(2)

8. List 4 properties of a square.

____

____

____

____(4)

9. List 4 properties of a rectangle.

____

____

____

____(4)

10. Can you think of any other properties of squares and rectangles not listed in 8 & 9.

____

____

____

____

Bonus (4)

INTERVIEW 1

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 1: Acute, isosceles, right-angled triangle.

RESEARCHER: Can you justify your answer?

LEARNER 1: I rotated the one triangle while the other triangle stayed in its original position. Two sides on the one triangle were equal in length because they still fitted on

each other exactly.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 1: Two 45 degree angles and one 90 degree angle.

RESEARCHER: Can you prove this?

LEARNER 1: When the triangles are placed side by side, the smaller angles touch, they form a square. In a square all angles are equal to 90 degrees. The smaller angles are equal and when they are put together they make a 90 degree angle, so they must 45 degrees each.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 1: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 1: They are equal.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 1: They are still equal.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 1: They are 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 1: I already deduced that one of the angles of the isosceles triangle of my tangram was 90 degrees. When I placed the two triangles onto my outlined square I noticed the two 90 degree angles fitted exactly onto two angles of the square. When I rotated the two triangles I found that the same two right angles of the triangles fitted onto the other two angles of the square. Therefore each angle of the square must be 90 degrees.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 1: They are equal.

RESEARCHER: Can you prove this?

LEARNER 1: When the plastic triangles was placed on the outlined square the longest side of the plastic triangles fitted on the opposite vertices. When I placed the same plastic triangle on the other opposite vertices the longest side of the plastic triangle still fitted exactly. Therefore I say that both lines are equal.

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 1: Diagonal line.

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 1: They are the same.

RESEARCHER: How do know this?

LEARNER 1: I placed the two triangles on each other and they fitted exactly.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 1: The diagonal cuts the area of the square in half.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 1: All sides are equal; all angles are equal; a diagonal cuts a square into equal parts; a square has four sides; the opposite sides are parallel.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 1: Each part of the diagonals fit exactly on each other. They are equal.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 1: They cut each other equally.

RESEARCHER: What is the size of the angles around the centre of the point of intersection of your diagonals?

LEARNER 1: They are 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 1: All the four triangles fitted on each other exactly and the centre point met as well. When I placed the centre on the tangram square angle it fitted. Therefore each angle around the centre must be 90 degrees.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 1: The diagonals cut each other equally; the diagonals cut the square into two equal triangles; there are two diagonals; the diagonals cut the angles around the midpoint into four 90 degree angles.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 1: They are equal.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 1: I placed the three tangram pieces on the outlined rectangle and they fitted. When I flipped these three pieces I found that it still fitted exactly.

That told me that the long sides were equal. I did the same for the two shorter sides and it worked. Therefore I say the opposite sides are equal.

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 1: They are each equal to 90 degrees.

RESEARCHER: Can you prove this using your tangram pieces?

LEARNER 1: When I placed the plastic square on the drawn rectangle the square angles fitted on two angles of the rectangle. I then took the plastic square and placed it on the other two angles and they also fitted exactly. Therefore I say that each angle must be 90 degrees.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 1: No.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 1: The diagonal of the rectangle and square cut into two equal triangles; the diagonals will both be equal in length; they both will bisect each other; the angles around the centre will not be equal.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 1: All the angles will not be equal around the point of intersection; all the sides are not the same; all the angles are the same.

RESEARCHER: Can we call a square a rectangle?

LEARNER 1: Yes.

RESEARCHER: Justify your answer.

LEARNER 1: The square and the rectangle have 90 degree angles; the diagonals bisect each other; opposite sides are equal.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 1: No.

RESEARCHER: Why do you say this?

LEARNER 1: In a square all sides are equal; in a square angles around the centre is 90 degrees.

RESEARCHER: Now can you define a rectangle?

LEARNER 1: A rectangle is a shape with opposite sides that are equal. All angles are equal to 90 degrees. A rectangle is not a square.

RESEARCHER: Can you define a square for me.

LEARNER 1: A square has four equal sides and four 90 degree angles. When the diagonals cut each other 90 degree angles are formed around the centre. When the diagonal cuts the square two equal triangles are formed.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 1: A square is a rectangle with all sides equal.

INTERVIEW 2

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 2: Isosceles, right-angled triangle.

RESEARCHER: Can you justify your answer?

LEARNER 2: Two of the sides are equal because if you place then on top of each other they are equal and if you turn them around they are still equal.

(the learner meant rotating the triangles while they were placed on top of each other)

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 2: 90 degrees; 45 degrees; 45 degrees.

RESEARCHER: Can you prove this?

LEARNER 2: If I place these triangles side by side they form a square, so each angle will be 90 degrees in a square; but this is an isosceles triangle therefore the two equal angles must be 45 degrees each so that they will add up to 90 degrees.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 2: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 2: All the sides are equal.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 2: They are still equal because they are merely flipped around.

(learner has actually identified that transformation has occurred-reflection)

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 2: They are equal to 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 2: When I turned the two triangles 90 degrees it fit into the picture of the square, therefore each angle is 90 degrees because the 90 degree angle of the right angled triangles fitted on each angle of the square.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 2: It is longer than the other sides of the square. It divides the square into two triangles. They are equal to each other.

RESEARCHER: Can you prove that these two lines are equal?

LEARNER 2: If a circle is drawn around the square then this line will be the diameter because it passes through the centre, therefore the line drawn from the other two vertices will also pass through the centre and it will also be a diameter. Therefore diameters are equal meaning that these lines will be equal in length.

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 2: Diagonally. (researcher had indicated the correct name-diagonal)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 2: They are the same.

RESEARCHER: How do know this?

LEARNER 2: If you place the triangles one on top of the other they fit exactly, therefore they are equal.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 2: It divides the area of the square into two equal pieces.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 2: It has four equal sides; it has four equal angles which are 90 degrees; a squares diagonal line divides it into two equal pieces.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 2: They are all equal.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 2: It goes through the midpoint. All parts are equal.

RESEARCHER: What does the diagonals do to each other?

LEARNER 2: They cut each other in half.

RESEARCHER: What is the size of the angles around the point of intersection of your diagonals?

LEARNER 2: They are 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 2: At the midpoint there are two straight lines, therefore 180 degrees each and because these are all equal angles, I placed them on each other and found they were all equal, each angle around the midpoint must be 90 degrees each.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 2: They go through the midpoint; at the midpoint of the diagonal it takes the same distance to go to one end as the other; they are equal to each other; they divide the area of the square equally.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 2: They are equal.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 2: I placed the three pieces of my tangram into the outlined rectangle and it fitted. When I turned it 180 degrees to the right it still fitted exactly, therefore the opposite sides are equal.

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 2: They are equal to 90 degrees each.

RESEARCHER: Can you prove this using your tangram pieces or your outline?

LEARNER 2: When I fold my outlined rectangle once the opposite angles are the same (learner actually means the co-interior angles) and when I fold it the other way all four angles are the same. I have formed this shape using a square and I have proved that each angle in a square is 90 degrees, therefore each angle in the rectangle must be 90 degrees.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 2: No.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 2: Lengths are the same; angles around the midpoint are different; they cut each other equally.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 2: The square and rectangles have all angles equal to 90 degrees; square has all sides equal; rectangle has opposite sides equal; square and rectangle has equal diagonals; diagonals divide the areas equally; the angles around the midpoint of the diagonals in a square are 90 degrees each while in a rectangle the opposite angles (meaning vertically opposite angles) are equal.

RESEARCHER: Can we call a square a rectangle?

LEARNER 2: Yes.

RESEARCHER: Justify your answer.

LEARNER 2: The opposite sides of a square are equal which is similar to a rectangle and each angle in a square is 90 degrees similar to rectangle; the diagonals in a square are equal similar to a rectangle.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 2: No.

RESEARCHER: Why do you say this?

LEARNER 2: A square has all sides equal and all angles around the diagonal equal.

RESEARCHER: Now can you define a rectangle?

LEARNER 2: A shape that has four 90 degree angles and opposite sides equal.

RESEARCHER: Can you define a square for me.

LEARNER 2: A shape that has four 90 degree angles and all sides are equal.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 2: A square is a rectangle with all sides equal in length.

INTERVIEW 3

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 3: Isosceles, right-angled triangle, acute angled triangle.

RESEARCHER: Can you justify your answer?

LEARNER 3: When you put both triangles together you get a perfect square, therefore right angle; and when you put them on top of each other all the sides are the same, therefore isosceles triangle.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 3: 45 degrees each and one 90 degree angle.

RESEARCHER: Can you prove this?

LEARNER 3: When I placed the two triangles on each other all the angles were equal. When I placed the the two triangles side by side they formed a perfect square, therefore I say there is one 90 degree angle and two 45 degree angles because two equal angles formed a 90 degree angle.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 3: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 3: All the sides are the same length. All are still the same.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 3: They are all still the same, they are equal.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 3: They are all 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 3: The two 90 degree angles of the triangles fitted on the two angles of the square and since the other two were made up of two 45 degree angles and they fitted the other two corner angles of the square. Therefore each angle is 90 degrees each.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 3: The length is exactly the same.

RESEARCHER: Can you prove that these two lines are equal?

LEARNER 3: I placed one of the tangram triangles on one line and thereafter took the same triangle and placed it on the other line and I found that it was the same length.

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 3: Diagonale. (researcher had indicated the correct name-diagonal)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 3: They each take up the same area.

RESEARCHER: How do know this?

LEARNER 3: If I put both triangles together they are exactly the same.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 3: It halves the area of the square.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 3: It has 90 degree corner angles; two corners are made up of two 45 degree angles; two triangles in equal area form the square; all the sides are equal.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you

end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 3: They are all the same.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 3: They split each other exactly in half.

RESEARCHER: What is the size of the angles around the point of intersection of your diagonals?

LEARNER 3: They are 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 3: If you fold the outlined square in half it folds exactly with each other and if you fold it half again it folds exactly. Therefore if I place the triangle from the tangram on the midpoint angles I find that they are all 90 degrees.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 3: They divide the square exactly in half; they make 90 degree angles at the midpoint; they are equal to each other in length.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 3: They are the same length.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 3: If you put the pieces of the tangram they align exactly. If you put them on top of each other they are exactly the same, so if you put them next to each other they are still the same length. Therefore I say that the opposite sides are equal.

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 3: They are all 90 degrees each.

RESEARCHER: Can you prove this using your tangram pieces or your outline?

LEARNER 3: Since I already proved that the angles of a square was 90 degrees, so I placed the square on each angle of the rectangle and I found that it fitted exactly.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 3: No.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 3: The diagonal lines will be the same length; the four triangles in the rectangle will not be the same as in the square; the angles around the point of intersection of the diagonals in a rectangle won't be 90 degrees each; the diagonals do split each other in half in both cases.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 3: In a rectangle the two long sides and two short sides are equal whereas in the square all the sides are equal; all the angles in a square and rectangle are the same, that is they are all 90 degrees each.

RESEARCHER: Can we call a square a rectangle?

LEARNER 3: Yes.

RESEARCHER: Justify your answer.

LEARNER 3: The opposite sides in a square are equal just like a rectangle; each angle in a square is 90 degrees just like a rectangle; the diagonals cut each other in half in a square just like a rectangle.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 3: No.

RESEARCHER: Why do you say this?

LEARNER 3: All four sides aren't the same length in a rectangle.

RESEARCHER: Now can you define a rectangle?

LEARNER 3: A rectangle is a stretched square with opposite sides the same length but all four sides don't have to be the same length, but each angle must be 90 degrees.

RESEARCHER: Can you define a square for me.

LEARNER 3: A square is a shape that has four equal sides and all four corner angles are 90 degrees.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 3: A square is a rectangle all sides equal and diagonals meeting at 90 degrees.

INTERVIEW 4

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 4: Isosceles triangle.

RESEARCHER: Can you justify your answer?

LEARNER 4: By placing the two triangles on top of each other, the triangles fitted exactly. When I took the top triangle and turned it around and placed it on the other triangle it still fitted exactly, therefore the two triangles are equal on both triangles.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 4: One 90 degree angle and two 45 degree angles.

RESEARCHER: Can you prove this?

LEARNER 4: Putting the two triangles side by side I noticed they formed a right angle and since these two similar angles form a right angle each one has to be 45 degrees each. When I placed the biggest angles of each triangles next to each other I noticed it formed a straight line, therefore each angle had to be 90 degrees each.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 4: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 4: The sides are all the same length.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 4: They are all still the same, they are equal.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 4: They are all 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 4: Since I already proved that one of the angles of the tangram triangle was 90 degrees, I rotated this triangle and found that it fitted exactly on each angle of the square. Therefore each angle must be 90 degrees.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 4: They are equal in length.

RESEARCHER: Can you prove that these two lines are equal?

LEARNER 4: I used one of the triangles from my tangram set and placed it on one line and then took the same triangle and placed it on the other line it was the same.

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 4: Transversal. (researcher had indicated the correct name-diagonal)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 4: They are equal.

RESEARCHER: How do know this?

LEARNER 4: If I put both triangles together they are exactly the same.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 4: The diagonals divides the square into two equal triangles.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 4: All the sides are equal; each angle of the square is equal to 90 degrees; opposite sides are equal; the diagonals bisects the area of the square.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 4: All parts of the diagonals are equal.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 4: They cut each other exactly in half.

RESEARCHER: What is the size of the angles around the point of intersection of your diagonals?

LEARNER 4: Each angle is 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 4: I used the triangle from the tangram and placed the 90 degree angle on each of the four angles and they fitted exactly.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 4: They bisect the area of the square; they cut each other exactly in half; they meet at 90 degree angles.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 4: They are the equal.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 4: When I placed the tangram pieces first they fitted exactly. Thereafter I turned the pieces around and it still fitted exactly, meaning the opposite sides are equal. (the learner had flipped the tangram pieces in both directions)

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 4: They are 90 degrees each.

RESEARCHER: Can you prove this using your tangram pieces or your outline?

LEARNER 4: Each angle of the square fitted exactly on each angle of the rectangle and I have already proven that each angle of the square is 90 degrees..

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 4: No.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 4: Diagonals in both cases are equal in length; diagonals in both cases cut each other equally; diagonals in both cases bisect the area; in a rectangle the diagonals don't intersect at 90 degrees.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER4: In a rectangle the opposite angles sides are equal but in a square all the sides are equal; each angle in a square and rectangle is 90 degrees; in a rectangle and square the opposite sides are parallel.

RESEARCHER: Can we call a square a rectangle?

LEARNER 4: Yes.

RESEARCHER: Justify your answer.

LEARNER 4: because the square has all the properties of the rectangle like: opposite sides equal; opposite sides parallel; each angle is 90 degrees; diagonals bisect each other; diagonals are equal in length.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 4: No.

RESEARCHER: Why do you say this?

LEARNER 4: Because a rectangle has two long sides and two short sides.

RESEARCHER: Now can you define a rectangle?

LEARNER 4: A rectangle has opposite sides equal and each angle equal to 90 degrees.

RESEARCHER: Can you define a square for me.

LEARNER 4: A square has all the sides equal and each angle equal to 90 degrees.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 4: A square is a rectangle having no long and short sides but all equal sides.

INTERVIEW 5

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 5: Right angled Isosceles triangle.

RESEARCHER: Can you justify your answer?

LEARNER 5: In a square the corner angles are 90 degrees. When I placed these two triangles together they formed a square, therefore right-angled triangle. Because these are two equal triangles, I placed them on each other that is how I know that they are equal, this tells me that they are isosceles because two sides are equal.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 5: Two 45 degree angles and one 90 degree angle.

RESEARCHER: Can you prove this?

LEARNER 5: As I said earlier on that when I placed them on top of each other they fitted exactly and when I placed them together they formed a square. This

therefore tells me that there are two equal angles forming a 90 degree angle and this therefore indicates to me that each angle has to be 45 degrees to give you 90 degree.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 5: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 5: The sides are all the same length.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 5: They are all still the same, they are equal.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 5: They are all 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 5: I took the triangle from the tangram and placed the 90 degree angle of this triangle on each angle of the square and they fitted exactly.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 5: They will have the same length.

RESEARCHER: Can you prove that these two lines are equal?

LEARNER 5: If you put the two triangles into your outlined square they fit exactly on this middle line. If you rotate these same two triangles they will fit on the other middle line exactly. This tells me that they are equal in length.

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 5: Slant line. (researcher had indicated the correct name-diagonal)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 5: They are equal in area.

RESEARCHER: How do know this?

LEARNER 5: If I put both triangles on top of each other they are exactly the same.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 5: It makes it bigger.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 5: all the corners are 90 degrees; its diagonals have the same length; all the sides are equal; a square can form two isosceles.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 5: They are all the same.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 5: They are equal and cut each other.

RESEARCHER: What is the size of the angles around the point of intersection of your diagonals?

LEARNER 5: Each angle is right angle.

RESEARCHER: Can you prove this?

LEARNER 5: There were four angles when I cut them up. I placed two angles next to each other and they formed a straight line. When I placed them on top of each other they were the same. This means they are two equal angles adding up to 180 degrees. This therefore means each angle has to be 90 degrees. I did the same for the other two angles.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 5: They form two equal triangles; the lines bisect each other; they are equal on a straight line; the bisect point is 90 degrees..

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 5: They are the equal in length.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 5: If I place the tangram pieces the other sides they still remain the same. (the learner had indicated that she had flipped and then rotated the tangram pieces 180 degrees)

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 5: They are 90 degrees each.

RESEARCHER: Can you prove this using your tangram pieces or your outline?

LEARNER 5: In fact I can use two ways to prove this. Firstly I took the triangle and placed the right angle of the triangle on each angle and it fitted; secondly I took the square and placed each angle of the square on each angle of the rectangle and it fitted exactly.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 5: No not all.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 5: They will still bisect each other and they are the same length but they don't meet at right angles.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 5: Only two sides are equal in in rectangle; in a square all the sides are equal.

RESEARCHER: Can we call a square a rectangle?

LEARNER 5: Yes.

RESEARCHER: Justify your answer.

LEARNER 5: If you put two squares together it forms a rectangle. The square has all the things a rectangle has.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 5: No.

RESEARCHER: Why do you say this?

LEARNER 5: A square is smaller than a rectangle; a rectangle has two long sides and two short sides while a square has all sides the same.

RESEARCHER: Now can you define a rectangle?

LEARNER 5: A figure with four equal 90 degree angles and parallel sides are equal in length.

RESEARCHER: Can you define a square for me.

LEARNER 5: A square has four equal sides and four equal angles.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 5: A square is a rectangle with smaller sides.

INTERVIEW 6

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 6: Isosceles triangle.

RESEARCHER: Can you justify your answer?

LEARNER 6: I took the two triangles from the tangram set and placed on top of each other they fitted perfectly. I then picked up the top triangle and turned it around and placed back on top of the other triangle and it still fitted perfectly. This tells me therefore it is isosceles because two sides are equal.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 6: 45 degrees; 45 degrees and 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 6: When I joined the two triangles it formed a 90 degree angle. This led me to the conclusion that each angle must be 45 degrees because I discovered that these were two equal angles. The other angle was 90 degrees because when I placed it on the corner of my page it fitted exactly.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 6: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 6: They now become equal.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 6: Nothing has changed, they are still equal.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 6: They are all 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 6: I already proved that one of the angles of the tangram triangle was 90 degrees. I took this angle and placed it on each angle of the square and they fitted exactly.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 6: They are equal in length.

RESEARCHER: Can you prove that these two lines are equal?

LEARNER 6: I used one triangle from the tangram and it fitted properly on this middle line. I then took this same triangle and placed it on the other middle line it stll fitted properly. Therefore I say they are equal in length.

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 6: Oblique line. (researcher had indicated the correct name-diagonal)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 6: They have the same area.

RESEARCHER: How do know this?

LEARNER 6: By placing the two triangles on top of each other I found that they occupied the same area.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 6: It forms two equal parts.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 6: It has four 90 degree angles; two right angles isosceles triangles formed a square; when a square is divided by a diagonal line its parts still remains the same; all the sides of a square are equal; the sides are parallel to each other.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 6: They are all of the same length.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 6: They bisect or cut each other equally.

RESEARCHER: What is the size of the angles around the point of intersection of your diagonals?

LEARNER 6: Each angle is 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 6: I placed all the midpoint angles on an angle of a square on my tangram set and it fitted.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 6: It cut the area of a square in half; it cut each other; it met at right angles.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 6: The two long sides and two short sides are equal.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 6: When I rotated the three pieces of my tangram onto the outlined rectangle it still fitted exactly, this tells me that the opposite sides are equal.

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 6: Each angle is 90 degrees.

RESEARCHER: Can you prove this using your tangram pieces or your outline?

LEARNER 6: When I placed the square angle on each angle of the rectangle it fitted exactly.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 6: No, some are the same.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 6: Diagonals bisect the area in both shapes; diagonals are equal in both shapes; diagonals don't meet at 90 degrees in a rectangle but meets at 90 degrees in a square; the rectangle has longer diagonals than a square.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 6: Two squares form a rectangle; the sides of a square are all equal; the angles of a square and rectangle are 90 degrees each.

RESEARCHER: Can we call a square a rectangle?

LEARNER 6: Yes.

RESEARCHER: Justify your answer.

LEARNER 6: Two squares form a rectangle; all that a rectangle has a square has.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 6: Yes.

RESEARCHER: Why do you say this?

LEARNER 6: If we divide the rectangle exactly in the centre you would get two squares.

RESEARCHER: Now can you define a rectangle?

LEARNER 6: It is an object which has two equal long sides and two equal short sides; it has all 90 degree angles.

RESEARCHER: Can you define a square for me.

LEARNER 6: It is a diagram/object which has four 90 degree angles and four equal sides.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 6: A square is a rectangle having all the sides equal.

INTERVIEW 7

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 7: Isosceles triangle.

RESEARCHER: Can you justify your answer?

LEARNER 7: If you place one triangle on top of the other, both triangles fit exactly on top of each other, whereas if you turn the top triangle on the opposite side and place it again above the bottom triangle they still fit exactly, therefore it shows that one side is equal to two.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 7: One 90 degree angle and two 45 degree angles.

RESEARCHER: Can you prove this?

LEARNER 7: If you place one triangle above the other, the angles are directly above each other, whereas if you turn the top triangle on its opposite side, it still remains directly above the bottom one. However, when you place the triangles together, it forms a square and a square has all 90 degree angles. Since two equal angles form a 90 degree angle, each one must be 45 degrees.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 7: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 7: The sides are all equal in length.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 7: They still remain equal.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 7: They are all 90 degrees each.

RESEARCHER: Can you prove this?

LEARNER 7: Yes, because each time I rotated the square, the triangle corners fitted equally at the corners of the drawn square, forming the 90 degree angle.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 7: They are equal.

RESEARCHER: Can you prove that these two lines are equal?

LEARNER 7: When I placed the triangle piece against the middle line it was equal in length when I placed the same triangle piece on the other middle line.

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 7: Diagonally. (researcher had indicated the correct name-diagonal)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 7: They are equal.

RESEARCHER: How do know this?

LEARNER 7: When I placed the two triangles on top of each other they occupy the same amount of space.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 7: It bisects the area of the square.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 7: Each angle is 90 degrees; each side is the same length; the diagonal bisects the area of the square; the length of the diagonals are equal in length.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 7: All parts of the diagonals are equal.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 7: They cut each other equally.

RESEARCHER: What is the size of the angles around the point of intersection of your diagonals?

LEARNER 7: Each angle is 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 7: When I placed all the angles together I found that they were equal. I then took two of these equal angles and placed them on a straight line and I found that they fitted exactly, therefore they had to be 90 degrees each. I then did the same for the other two angles and found it was also 90 degrees each.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 7: They bisect each other; they meet at 90 degrees; they bisect the area of the square.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 7: The two short sides are equal and the two long sides are equal.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 7: When you flip over the shorter sides and longer sides the tangram pieces still fit, therefore I say the opposite sides are equal.

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 7: Each angle is 90 degrees.

RESEARCHER: Can you prove this using your tangram pieces or your outline?

LEARNER 7: The square was used to form the rectangle and it had all 90 degree angles, therefore the rectangle must also have all 90 degree angles.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 7: No, not all.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 7: They both bisect each other; they both bisect the area; they both don't meet at 90 degrees, only the square does.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 7: A square has all sides equal whereas a rectangle has two long sides and two short sides that are equal; both the square and the rectangle has all 90 degree angles.

RESEARCHER: Can we call a square a rectangle?

LEARNER 7: Yes.

RESEARCHER: Justify your answer.

LEARNER 7: The square has all the properties of the rectangle.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 7: No.

RESEARCHER: Why do you say this?

LEARNER 7: All sides aren't equal in a rectangle and the diagonals don't meet at 90 degrees in a rectangle.

RESEARCHER: Now can you define a rectangle?

LEARNER 7: A rectangle has all the angles equal to 90 degrees and two short sides which are equal and two long sides which are equal.

RESEARCHER: Can you define a square for me.

LEARNER 7: A square must have all the sides equal and each angle must be equal to 90 degrees.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 7: A square is a rectangle having all the sides equal in length.

INTERVIEW 8

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 8: Isosceles triangle.

RESEARCHER: Can you justify your answer?

LEARNER 8: I rotated the one triangle while the other triangle stayed in its original position. Two sides on the one triangle were equal in length because they still fitted each other exactly.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 8: Two 45 degree angles and one 90 degree angle.

RESEARCHER: Can you prove this?

LEARNER 8: In an isosceles triangle two angles are equal and because there is one 90 degree angle, I know this because I placed on the corner of my page and it was square, therefore the other two must be 45 degrees each.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 8: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 8: They are equal.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 8: They are still equal.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 8: They are 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 8: They have two right angled triangles that fit exactly into the square. When I rotated these two triangles the right angles fitted on the other two angles of the square.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 8: They are equal.

RESEARCHER: Can you prove this?

LEARNER 8: When I placed one triangle on the middle line it fit exactly. When I placed the same triangle on the other middle line it still fit exactly, therefore they are equal

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 8: Vertices. (The researcher had corrected the participant by indicating the correct term, namely, diagonals)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 8: They are the same.

RESEARCHER: How do know this?

LEARNER 8: I placed the two triangles on each other and they fitted exactly.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 8: The diagonal cuts the area of the square in half.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 8: All sides are equal; all angles are equal to 90 degrees; a diagonal cuts a square into equal parts; the two triangles put into the square have the same area.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you

end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 8: They are equal.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 8: They bisect each other.

RESEARCHER: What is the size of the angles around the centre of the point of intersection of your diagonals?

LEARNER 8: They are 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 8: When I placed the angle of the square tangram piece on all four center angles they fitted exactly, therefore they must be all 90 degrees.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 8: They have four equal angles at the centre; they bisect the area of the square; they are equal to each other in length.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 8: They are equal.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 8: By putting the pieces into the rectangle they fit perfectly and if you turn it around then they still fit perfectly. Therefore they must be equal.

(The learner meant rotating the three pieces 180 degrees anti-clockwise)

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 8: They are each equal to 90 degrees.

RESEARCHER: Can you prove this using your tangram pieces?

LEARNER 8: The square piece of the tangram fits exactly on each angle of the rectangle.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 8: No.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 8: The diagonals will both be equal in length; they both will bisect each other; the angles around the midpoint will not be equal.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 8: All the sides of the square are equal but rectangle is not; the angles of the square and rectangle are 90 degrees each.

RESEARCHER: Can we call a square a rectangle?

LEARNER 8: No.

RESEARCHER: Justify your answer.

LEARNER 8: The rectangle has two long sides and two short sides.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 8: No.

RESEARCHER: Why do you say this?

LEARNER 8: In a square all sides are equal whereas in a rectangle they are not.

RESEARCHER: Now can you define a rectangle?

LEARNER 8: A rectangle has two equal long sides and two equal short sides; each angle of a rectangle is equal to 90 degrees.

RESEARCHER: Can you define a square for me.

LEARNER 8: A square has four equal sides and four 90 degree angles.

RESEARCHER: Finally, can you define for me a square using a rectangle, that starts as follows: A square is a rectangle ....

LEARNER 8: A square is a rectangle because it has the same properties with one addition that the square has four equal sides.

INTERVIEW 9

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 9: Isosceles triangle.

RESEARCHER: Can you justify your answer?

LEARNER 9: When I rotated the triangles they fitted exactly over each other, even after I turned them.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 9: Two 45 degree angles and one 90 degree angle.

RESEARCHER: Can you prove this?

LEARNER 9: When I placed the two triangles together the angles coincided, which means they are equal in size. So when I placed the two smaller angles next to each other they formed a 90 degree angle, therefore each small angle must be 45 degrees each while the third angle must be 90 degrees.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 9: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 9: They are equal.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 9: They are still equal.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 9: They are all equal in length which is 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 9: If you put the two isosceles triangles into the outlined square, they fit exactly into each corner even if you rotate them to the opposite corners.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 9: It is the same length.

RESEARCHER: Can you prove this?

LEARNER 9: By placing the isosceles triangle on the one line joining the opposite vertices I found that it was the same as the other line joining the other opposite vertices. Therefore I say they are equal.

RESEARCHER: What do we call this line that joins the opposite vertices?

LEARNER 9: Vertically opposite line. (the learner was given the correct term-diagonals)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 9: They are the same.

RESEARCHER: How do know this?

LEARNER 9: I placed the two triangles on each other and they fitted exactly.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 9: The diagonal bisects the area of the square.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 9: Opposite vertices lines are equal to each other; all four sides are equal; each angle is 90 degrees.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you

end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 9: Each part of the diagonals fit exactly on each other. They are equal.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 9: They cut each other equally.

RESEARCHER: What is the size of the angles around the centre of the point of intersection of your diagonals?

LEARNER 9: They are 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 9: If you take one isosceles triangle from your tangram and place the 90 degree angle on the centre angle it will fit exactly. I then tried the same on the other three angles and it still fitted exactly, therefore they are all 90 degrees.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 9: The diagonals bisect the area; each angle around the point of intersection is 90 degrees; the diagonals.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 9: They are equal.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 9: If you rotate the three tangram pieces it will fit exactly. If you rotate them again they will still fit exactly.

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 9: They are each equal to 90 degrees.

RESEARCHER: Can you prove this using your tangram pieces?

LEARNER 9: When I placed the square tangram angles on each angle of the rectangle they fitted exactly. I confirmed this by placing the two 45 degree angles on one angle of the rectangle and they still fitted exactly.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 9: No.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 9: The diagonals of the square and rectangle are equal; diagonals of square meet at 90 degrees but in rectangle the diagonals don't meet at 90 degrees.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 9: All the angles will not be equal around the point of intersection; all the sides are not the same; all the angles are the same.

RESEARCHER: Can we call a square a rectangle?

LEARNER 9: Yes.

RESEARCHER: Justify your answer.

LEARNER: the square has opposite sides equal; it has opposite angles equal; it has diagonals equal; it has each angle equal to 90 degrees.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 9: No.

RESEARCHER: Why do you say this?

LEARNER 9: In a square all sides are equal, in a rectangle it is not.

RESEARCHER: Now can you define a rectangle?

LEARNER 9: A rectangle is a quadrilateral shape with opposite sides equal and each angle 90 degrees.

RESEARCHER: Can you define a square for me.

LEARNER 9: A square has four equal sides and four 90 degree angles.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 9: A square is a rectangle but with all sides equal.

INTERVIEW 10

(Requirements: tangram pieces; exam pad paper; pencil; scissors and ruler)

RESEARCHER: Hold up the two smallest triangles from your tangram set. What type of triangles are these?

LEARNER 10: Isosceles triangle.

RESEARCHER: Can you justify your answer?

LEARNER 10: When I turned the triangles aound they fitted exactly over each other enen after I flipped them around.

RESEARCHER: What is the size of each angle of the triangles?

LEARNER 10: Two 45 degree angles and one 90 degree angle.

RESEARCHER: Can you prove this?

LEARNER 10: When I placed the two triangles together the angles fitted on top of each other exactly, which means they are equal in size. So when I placed the two smaller angles next to each other they formed a 90 degree angle, therefore each small angle must be 45 degrees each while the third angle must be 90 degrees.

RESEARCHER: Now make an outline of the square tangram piece on a clean sheet of paper. Take the two smallest triangles that you are holding up and place it onto your outlined square. Do the triangles fit exactly onto your outlined square?

LEARNER 10: Yes.

RESEARCHER: Now rotate the two triangles 90 degrees clockwise about its centre.

What can you deduce about the sides of the square?

LEARNER 10: They are equal.

RESEARCHER: Now rotate the two triangles another 90 degrees clockwise about its centre. What do you observe about the sides of the square?

LEARNER 10: They are the same length.

RESEARCHER: Without measuring, what can you say about the angles of the square?

LEARNER 10: They are all equal in length which is 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 10: If you put the two isosceles triangles into the outlined square, they fit exactly into each corner even if you rotate them to the opposite corners.

RESEARCHER: Now join the opposite vertices of your outlined square.

What can you say about the length of the lines joining the opposite vertices of the square?

LEARNER 10: It is the same length.

RESEARCHER: Can you prove this?

LEARNER 10: By placing the isosceles triangle on the one line joining the opposite vertices I found that it was the same as the other line joining the other opposite vertices. Therefore, I say they are equal.

RESEARCHER: What do we call the line that joins the opposite vertices?

LEARNER 10: Slanted line. (the learner was given the correct term-diagonals)

RESEARCHER: What do you observe about the areas of the two triangles you are working with?

LEARNER 10: They are the same.

RESEARCHER: How do know this?

LEARNER 10: I placed the two triangles on each other and they fitted exactly.

RESEARCHER: So what does this tell us about the diagonals of the square and its area?

LEARNER 10: The diagonal cuts the area of the square.

RESEARCHER: Now can you state in point form all the properties of the square you have discovered?

LEARNER 10: Opposite vertices lines are equal to each other; all four sides are equal; each angle is 90 degrees.

RESEARCHER: Now mark clearly on your outlined square each part of the diagonal and the centre where they intersect. You may use different colours, letters or numbers. Thereafter cut along each of the diagonals so that you end up with four triangles. Now place the triangles on each other. What do you notice?

LEARNER 10: Each part of the diagonals fit exactly on each other. They are equal.

RESEARCHER: So what does this tell us about the diagonals of a square?

LEARNER 10: They cut each other equally.

RESEARCHER: What is the size of the angles around the centre of the point of intersection of your diagonals?

LEARNER 10: They are 90 degrees.

RESEARCHER: Can you prove this?

LEARNER 10: If you take one isosceles triangle from your tangram and place the 90 degree angle on the centre angle it will fit exactly. I then tried the same on the other three angles and it still fitted exactly, therefore they are all 90 degrees.

RESEARCHER: Now can you state in point form your conclusions about the diagonals of a square?

LEARNER 10: The diagonals bisect the area; each angle around the point of intersection is 90 degrees; the diagonals.

RESEARCHER: Now make an outline of a rectangle using the same two triangles and square piece from your tangram set. What do you observe about the opposite sides of the rectangle?

LEARNER 10: They are equal.

RESEARCHER: Can you prove this using your outline and tangram pieces?

LEARNER 10: If you rotate the three tangram pieces it will fit exactly. If you rotate them again they will still fit exactly.

RESEARCHER: What can you say about the size of the angles of the rectangle?

LEARNER 10: They are each equal to 90 degrees.

RESEARCHER: Can you prove this using your tangram pieces?

LEARNER 10: When I placed the square tangram angles on each angle of the rectangle they fitted exactly. I confirmed this by placing the two 45 degree angles on one angle of the rectangle and they still fitted exactly.

RESEARCHER: Do you think that the diagonals of a rectangle will have the same properties of a square?

LEARNER 10: No.

RESEARCHER: If yes/no, can you state any similarities or differences between the diagonals of the square and rectangle?

LEARNER 10: The diagonals of the square and rectangle are equal; diagonals of square meet at 90 degrees but in rectangle, the diagonals do not meet at 90 degrees.

RESEARCHER: Can you list any other similarities or differences between the square and rectangle?

LEARNER 10: All the angles will not be equal around the point of intersection; all the sides are not the same; all the angles are the same.

RESEARCHER: Can we call a square a rectangle?

LEARNER 10: Yes.

RESEARCHER: Justify your answer.

LEARNER 10: the square has opposite sides equal; it has opposite angles equal; it has diagonals equal; it has each angle equal to 90 degrees.

RESEARCHER: Then can we call a rectangle a square?

LEARNER 10: No.

RESEARCHER: Why do you say this?

LEARNER 10: In a square all sides are equal, in a rectangle it is not.

RESEARCHER: Now can you define a rectangle?

LEARNER 10: A rectangle is a quadrilateral shape with opposite sides equal and each angle 90 degrees.

RESEARCHER: Can you define a square for me.

LEARNER 10: A square has four equal sides and four 90 degree angles.

RESEARCHER: Finally, can you define for me a square using a rectangle, that is start as follows:

A square is a rectangle ....

LEARNER 10: A square is a rectangle but with all sides equal.