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An analysis of Symmetries and Conservation laws of some classes of PDEs that arise in Mathematical Physics and Biology

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As the candidate's supervisors, we have approved this dissertation for submission.

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Declaration of Publications

Details of contribution to publications presented in this thesis:

Chapter 2

J. E. Okeke, R. Narain, K. S. Govinder. An analysis of nonlinear PDE arising in elasto-plastic flow. *Phys. D: Nonlinear Phenomena.*, (2016), submitted.

Chapter 3

J. E. Okeke, R. Narain, K. S. Govinder. Exact solutions of a generalised Boussinesq equation with damping term and a system of variant Boussinesq equations via double reduction theory. *Appl. Math. J. Chinese Univ. Ser. A*, (2016), submitted.

Chapter 4

J. E. Okeke, R. Narain, K. S. Govinder. A group theoretic analysis of the generalized Gardner equation with arbitrary order nonlinear terms. *Int J Non Linear Mech.* (2016), submitted.

Chapter 5

J. E. Okeke, R. Narain, K. S. Govinder. A group theoretic analysis and complex dynamics of a diffusive Caughley prey - predator model. *Nonlinear Differential Equations Appl.* (2016), submitted.

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Dedication

To Lord Jesus Christ, my Saviour and my redeemer. To my lovely husband Obiora Collins, my sweet son Chukwubuikem Collins, my mum Elizabeth Okeke, my parents in law, my sister Ifeoma, my brother Adinuba and all my siblings.

Abstract

In this thesis, the symmetry properties and the conservation laws for a number of well-known PDEs which occur in certain areas of mathematical physics are studied. We focus on wave equations that arise in plasma physics, solid physics and fluid mechanics. Firstly, we carry out analyses for a class of non-linear partial differential equations, which describes the longitudinal motion of an elasto-plastic bar and anti-plane shearing deformation. In order to systematically explore the mathematical structure and underlying physics of the elasto-plastic flow in a medium, we generate all the geometric vector fields of the model equations. Using the classical Lie group method, it is shown that this equation does not admit space dilation type symmetries for a specific parameter value. On the basis of the optimal system, the symmetry reductions and exact solutions to this equation are derived. The conservation laws of the equation are constructed with the help of Noether's theorem

We also consider a generalized Boussinesq (GB) equation with damping term which occurs in the study of shallow water waves and a system of variant Boussinesq equations. The conservation laws of these systems are derived via the partial Noether method and thus demonstrate that these conservation laws satisfy the divergence property. We illustrate the use of these conservation laws by obtaining several solutions for the equations through the application of the double reduction method, which encompasses the association of symmetries and conservation laws.

A similar analysis is performed for the generalised Gardner equation with dual power law nonlinearities of any order. In this case, we derive the conservation laws of the system via the Noether approach after increasing the order and by the use of the multiplier method. It is observed that only the Noether's approach gives a unified treatment to the derivation of

conserved vectors for the Gardner equation and can lead to local or an infinite number of nonlocal conservation laws. By investigating the solutions using symmetry analysis and double reduction methods, we show that the double reduction method yields more exact solutions; some of these solutions cannot be recovered by symmetry analysis alone.

We also illustrate the importance of group theory in the analysis of equations which arise during investigations of reaction-diffusion prey-predator mechanisms. We show that the Lie analysis can help obtain different types of invariant solutions. We show that the solutions generate an interesting illustration of the possible behavioural patterns.

Contents

Introduction

Towards the end of the 19th century a Norwegian mathematician, Sophus Lie introduced the theory of continuous groups, now known as Lie group theory. In the literature, symmetry analysis is one of the most powerful techniques widely used for finding closed form solutions of differential equations [51, 62, 61, 31, 33]. Investigations of these differential equations plays an important role in understanding physical phenomena. A symmetry has the very important property of invariance such that mapping a differential equation from one form to another leaves its fundamental properties unchanged. In the case of scalar ODEs, symmetries are used for reduction of order [9]. With PDEs it is desirable to reduce the PDE to an ODE or at least reduce the number of independent variables. This can be achieved by constructing invariants of the symmetries which will in turn be used to reduce the underling system of PDEs to ODEs which if solved, could yield an exact solution to the PDE.

In the mid 1980s, Peter G. L. Leach and his student, Fazal M. Mahomed [49] introduced Lie group theory to South Africa. Thereafter, researchers from different disciplines in which differential equations play an important role started developing an interest in this area of study. Lie group theory has vast applications in fluid mechanics, solid-state mechanics, modern physics, biological and physical systems, to name a few. Recently, many applications of Lie group theory to DEs have been performed in various fields of natural sciences and engineering. These includes generating new solutions from known ones, linearization of ODEs and PDEs, construction of equivalence groups, solving group classification problems, reductions of PDEs (by invariant or similarity solutions), construction of generalised local symmetries and nonlocal symmetries, solving initial and boundary value problems, approximate symmetries, symmetries of stochastic differential equations, symmetries of integro-differential equations, symmetries of difference equations, symmetries of functional differential equations, symmetries of geodesic equations, construction of conservation laws, construction of invariants of algebraic and differential equations and so on.

Quite often, one may obtain a set of symmetries admitted by a PDE and face some indecisiveness as to which symmetries or linear combinations thereof to use for reduction of the original equation. We may proceed first by finding the commutator relation on the span of symmetries of the Lie algebra and based on non-abelian subalgebras we proceed to an "optimal subalgebra". An optimal system assists in this regard as it provides a (non-unique) list of all possible combinations of Lie subalgebras partitioned into disjoint classes [61]. This list contains conjugacy inequivalent subalgebras of the Lie algebra $\mathfrak{g}$ corresponding to the Lie symmetries [63] such that every Lie subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list and no two Lie subalgebras in the list are equivalent to each other.

The popular methods for constructing optimal system of one dimensional subalgebras include the use of a global matrix of adjoint transformations as suggested by Ovsiannikov [62] and the use of adjoint representation table presented by Olver [61]. In [63], a detailed analysis of the classification of the optimal system for all two-, three- and four-dimensional real algebras are provided.

In the analysis of differential equations, conservation laws play a significant role, particularly with regard to integrability and linearisation, constants of motion, analysis of solutions and numerical solution methods. The existence of a large number of conservation laws for a PDE indicates its integrability [9]. We also note that without these conserved vectors (integrals of motion), an understanding of the problem would be incomplete [25]. Different methods for obtaining conservation laws of PDEs have been developed. One such method is Noether's theorem which provides an elegant and constructive way of finding conservation laws for a system of PDEs which has a Lagrangian formulation [5]. The central problem in the calculus of variations is the determination of a Lagrangian, so that the differential equation is regarded as an

Euler-Lagrange equation. This problem is regarded as the inverse problem in the calculus of variations [59, 18].

There are also methods to obtain conservation laws without making use of a Lagrangian. The direct method introduced by Laplace [42, 4] is used to construct conserved quantities. The multipliers approach involves writing a conservation law in characteristic form, where the characteristics are the multipliers of the differential equations [67]. A recent method for constructing conservation laws without the use of a Lagrangian was provided by Ibragimov [67, 32]. Another way of constructing conservation laws for a system of PDEs without the existence of Lagrangians is via the partial Noether approach, introduced by Kara and Mahomed [35]. It works like the Noether approach for differential equations with " a partial Lagrangian".

It is well known that for variational systems a conservation law is associated with a Noether symmetry. Recently, this idea of associating conservation laws with Noether symmetries was extended to Lie-Bäcklund symmetries [36] and non-local symmetries [69]. The association of symmetry with a conserved vector leads to the idea of double reduction theory for PDEs [68]. The fundamental basis of this method is that when a symmetry is associated with a conserved vector, a double reduction transformation exists. The association of symmetry with the conserved vector firstly reduces the number of independent variables and secondly reduces the order of the differential equation. It is worth noting that the double reduction theory yields a new way of finding invariants and exact solutions of PDEs which may not be obtained using classical symmetry analysis [55].

Motivation

Numerous real-world models are formulated in terms of partial differential equations (PDEs). The complexity of these models is always increasing, either due to addition of new parameters, arbitrary elements or functions which may not be straight forwardly determined. Quite often these complexities contribute to added difficulty in solving the partial differential equations of the models. In such situations, the ability to obtain physical information into the real features of the models becomes the sole test for a comprehensive technique suitable for solving the complicated systems.

Methods often employed to analyse these differential equations include symmetry analysis, dynamical system analysis and conservation laws. Although these approaches have different emphases and directions of investigation, but in some cases they should not be regarded as completely independent as they complement each other.

It is well known that differential equations have a number fundamental structures, that is, symmetries and conservation laws. The existence of infinitely many generalized symmetries is of great important, especially in situations of physical and mathematical interest, where symmetries are used to reduce the number of unknown functions. Therefore, understanding the symmetry structure of physical systems is important. Conservation laws play a key role in the analysis of differential equations, particularly in studies of existence, uniqueness and stability of solutions.

Aims and objectives of thesis

The objectives of the thesis are firstly, to derive the Lie point symmetries and conservation laws of some classes of nonlinear PDEs that arise in physics. The importance of studying these categories of equations, are due to their appearance in different branches of science and engineering including plasma physics, fluid dynamics, quantum theory and solid state physics.

Secondly, to investigate the integrability of these nonlinear PDEs via symmetry analysis, conservation laws and double reduction theory.

Thirdly, to illustrate the significance of group theory in the analysis of mathematical models in ecology.

The outline of the thesis

The thesis is structured as follows:

In Chapter 1, we introduce the basic concepts, definitions and theorems that are needed to tackle our investigation.

In Chapter 2, we present and classify all the geometric vector fields of a class of nonlinear partial differential equations, which describes the longitudinal motion of an elasto-plastic bar and anti-plane shearing deformation. Symmetry reductions of the underlying equation are performed to obtain invariant solutions. The conservation laws associated with Noether symmetries of the equations are constructed.

In Chapter 3, we consider a generalised Boussinesq (GB) equation with damping term and a system of variant Boussinesq (VB) equations. As the GB equation is not derived from a variational principle we construct its conservation laws using the partial Noether method. The derived conserved vectors are adjusted to satisfy the divergence condition. These conservation laws are utilised to obtain a double reduction of the equation. As a result, some invariants and exact solutions are found. A similar analysis is performed for the system of VB equations to obtain exact solutions of the system.

In Chapter 4, we investigate and analyse the symmetry and conservation laws of a Gardner equation with dual power law nonlinearity of any order which has applications in quantum field theory, solid state, plasma and fluid physics. We derive the symmetry generators of the equation in terms of its arbitrary parameters and used them to obtain symmetry reductions and exact solutions. Furthermore, the conservation laws of the equation are constructed via the Noether approach after increasing the order and by the use of the multiplier method. Noether approach gives rise to some local and an infinite number of nonlocal conservation laws. The importance of these conservation laws in finding exact solutions is proved via double reduction theory, which involves the conserved vector and its associated symmetry.

In Chapter 5, we investigate the dynamical complexity of a diffusive Caughley prey - predator model, which describes the interaction between elephants and trees within a space domain. A one dimensional optimal system of Lie subalgebras is constructed via symmetry analysis. We use the optimal system to reduce this system of nonlinear PDEs to different systems of PDEs. Using the fact that the commutator relation contains information about further reduction, we further reduce these equations to systems of ODEs. Stability analysis of great biological significance is given for the travelling wave of the reduced model. Effects of diffusion on the structure and form of tree-elephant ecosystems are studied.

The conclusions are summarized in Chapter 6.

Chapter 1

Preliminaries

In this chapter, we present the basic notations, definitions and theorems of the fundamental concepts that will be used throughout this thesis. Let $x^i, i = 1, 2, \dots n$, be n independent variables and $u^\alpha, \alpha = 1, 2, \dots m$, be m dependent variables. The collection of kth-order derivatives, $k \geq 1$, is denoted by $u_{(k)}$. Subscripts denote partial derivatives. The summation convention is adopted in which there is summation over repeated upper and lower indices. We denote $\mathcal{A}$ to be the universal vector space of differential functions. The basic operators defined in $\mathcal{A}$ are stated below.

Definition 1.1. The *Euler-Lagrange operator* is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \dots i_s}}, \quad \alpha = 1, \dots, m. \tag{1.1}$$

The operator (1.1) is sometimes referred to as the Euler operator, named after Euler (1744) who first introduced it in a geometrical manner of the one-dimensional case. It is also called the *Lagrange operator*, bearing the name of Lagrange (1762) who considered the multi-dimensional case and established its use in a variational sense (see for example, [24] for a history of the calculus of variations). Following Lagrange, equation (1.1) is frequently referred to as a variational derivative. In the modern literature, the terminology Euler-Lagrange and variational derivative are used interchangeably as (1.1) usually arises in considering a variational problem.

Definition 1.2. The Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in \mathcal{A}. \tag{1.2}$$

The operator is an abbreviated form of the following infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_s}}, \tag{1.3}$$

where $\zeta^\alpha_{i_1 \ldots i_s}$ are defined by

$$\zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j) \tag{1.4}$$

$$\zeta^\alpha_{i_1 \ldots i_s} = D_{is}(\zeta^\alpha_{i_1 \ldots i_{s-1}}) - u_{ji_1 \ldots i_{s-1}} D_{i_s}(\xi^j), \quad s > 1. \tag{1.5}$$

The Lie-Bäcklund operator (1.3) in its characteristic form is

$$X = \xi^i \frac{\partial}{\partial x^i} + W^\alpha \frac{\partial}{\partial u^\alpha} + D_i(W^\alpha) \frac{\partial}{\partial u_i{}^\alpha} + D_i D_j(W^\alpha) \frac{\partial}{\partial u^\alpha_{ij}} + \ldots. \tag{1.6}$$

In (1.6), W^α are the Lie characteristic functions given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j, \quad \alpha = 1, 2, \ldots, m. \tag{1.7}$$

Definition 1.3. The Noether operator associated with a Lie-Bäcklund operator X is defined by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i{}^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{ii_1 \ldots i_s}}, \quad i = 1, \ldots, n \tag{1.8}$$

where the Euler-Lagrange operator $\frac{\delta}{\delta u^\alpha_i}$ is given by

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s \geq 1} (-1)^s D_{j_1} \ldots D_{j_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{ij_1 \ldots j_s}}, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m. \tag{1.9}$$

and similarly for the other Euler-Lagrange operators with respect to higher order derivatives. The operator (1.8) is named the Noether operator and was given in recognition of Noether's contribution.

1.1 Noether Identity

Theorem 1.1. *The Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity*

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.10}$$

Here, $D_i(\xi^i)$ is a differential function which is a sum of functions obtained by total derivations D_i of differential functions ξ^i. That is, $D_i(\xi^i)$ is a divergence of the vector $\xi = (\xi^1, \ldots, \xi^n)$, in other words, $div\xi$ whereas, $D_i N^i$ is an operator obtained as a sum of products of operators D_i on N^i, that is, it is scalar product of vector operators $D = (D_1, \ldots, D_n)$ and $N = (N^1, \ldots, N^n)$. The identity (1.10) is called the Noether identity because of its close relation to Noether's theorem.

1.2 Lie point symmetries

Consider a k^{th} order system of partial differential equations of independent variables $x = (x^1, x^2, \ldots, x^n)$ and r dependent variables $u = (u^1, u^2, \ldots, u^r)$

$$E^\alpha(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m. \tag{1.11}$$

The variables $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., $kth-$ order partial derivatives, respectively, that is

$$u_i{}^\alpha = D_i(u^\alpha), \quad u^\alpha_{ij} = D_j D_i(u^\alpha), \ldots, \tag{1.12}$$

with the total differentiation operator with respect to x^i given by,

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \ldots, \quad i = 1, \ldots, n. \tag{1.13}$$

The Lie point symmetry of equation (1.11) is a generator X of the form (1.3) that satisfies

$$X^{[k]} E_{|_{E=0}} = 0, \tag{1.14}$$

where $X^{[k]}$ is the kth prolongation of X defined by

$$X^{[k]} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta^\alpha \frac{\partial}{\partial u^\alpha_i} + \ldots + \zeta^\alpha_{i_1 \ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_k}}. \tag{1.15}$$

This means that equation (1.11) is invariant under the action of the generator X.

1.3 Conservation laws

Definition 1.4. A conserved vector of (1.11) is n-tuple $T = (T^1, \dots, T^n)$, $T^j = T^j(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) \in \mathcal{A}, \quad j = 1, ..., n$ such that

$$D_i(T^i) = 0 \tag{1.16}$$

is satisfied for all solutions of (1.11) .

Remark: When Definition 1.4 is satisfied, (1.16) is called a conservation law for (1.11).

1.3.1 Approaches to construct conservation laws

In this section various approaches used to construct conservation laws in this work are discussed.

Noether's approach

An elegant and constructive way of finding conservation laws is by means of Noether's theorem [5].

Definition 1.5. If there exists a function $L = L(x, u, u_{(1)}, u_{(2)}, \dots, u_{(l)}) \in \mathcal{A}$, $l \leq k$, such that (1.11) are equivalent to

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \dots, m, \tag{1.17}$$

then L is called a Lagrangian of (1.11) and (1.17) are the corresponding Euler-Lagrange differential equations.

Definition 1.6. A Lie-Bäcklund operator X of the form (1.3) is called a Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$ if there exists a vector $B = (B^1, \dots, B^n)$, $B^i \in \mathcal{A}$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \tag{1.18}$$

If in equation ((1.18)), $B^i = 0, \quad i = 1, \dots, n$, then X is referred to as a strict Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$.

Theorem 1.2. *For each Noether symmetry generator X associated with a given Lagrangian $L \in \mathcal{A}$, corresponding to the Euler-Lagrange differential equations, there corresponds a vector $T = (T^1, \ldots, T^n), \quad T^i \in \mathcal{A}$, with defined by T^i*

$$T^i = N^i(L) - B^i, \quad i = 1, \ldots, n, \tag{1.19}$$

which is a conserved vector for the Euler-Lagrange differential equations (1.17)*, that is, $D_i(T^i) = 0$ on the solutions of* (1.17)*.*

In the Noether approach we find $L = L(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(l-1)})$ and then (1.18) is used for the determination of the Noether symmetries. Finally (1.19) yields the corresponding Noether conserved vectors. The characteristics W^α of the Noether symmetry generator are the characteristics of the conservation law.

1.4 Partial Noether approach

If the standard Lagrangian does not exist or is difficult to find, then we may be able to write its partial Lagrangian and derive the conservation laws by the partial Noether approach introduced by Kara and Mahomed [35]

Definition 1.7. Suppose that the *kth*-order differential system (1.11) can be written as

$$E_\alpha \equiv E_\alpha^0 + E_\alpha^1 = 0, \quad \alpha = 1, \ldots, m. \tag{1.20}$$

A function $L = L(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) \in \mathcal{A}, s \leq k$ is called a partial Lagrangian of system (1.11) if it can be written as $\delta L/\delta u^\alpha = f_\alpha^\beta E_\beta^1$ provided that $E_\beta^1 \neq 0$, L for some β.

Here f_α^β is an invertible matrix.

Definition 1.8. The operator X defined in (1.6) satisfying

$$X(L) + LD_i(\xi^i) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i(B^i),\ i = 1, 2, \ldots n,\ \alpha = 1, 2, \ldots m \tag{1.21}$$

is a partial Noether operator corresponding to the partial Lagrangian L.

If the B^i's are identically zero, then the Lie-Bäcklund operator X is called a *strict partial Noether operator*.

Theorem 1.3. *The conserved vector of the system* (1.11) *associated with a partial Noether operator X corresponding to the partial Lagrangian L is determined from* (1.19).

Here also $W^\alpha \in \mathcal{A}$ are the characteristics of the conservation law. We can use the partial Noether approach for equations that have Lagrangian formulations.

1.5 Characteristic method

A conservation law can be written in characteristic form [61, 67] as

$$D_i T^i = Q^\alpha E_\alpha \tag{1.22}$$

where Q^α are the characteristics. The characteristics are the multipliers which make the equation "exact".

1.6 Symmetry and conservation law relation

The fundamental relation between the Lie-Bäcklund symmetry generator X and the conserved vector T for a differential equation is governed by [36]

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, ..., n. \tag{1.23}$$

The joint conditions (1.23) together with (1.16) are used to find conserved vectors T^i.

1.6.1 Double reduction of PDEs

Here we give some basic theorems and definitions on double reductions.

Theorem 1.4. *Suppose that X is any Lie Bäcklund symmetry of equation* (1.11) *and $T^i, i = 1, ..., n$ are the components of its conserved vectors. Then [10]*

$$T^{*i} = (T^i, X) = X(T^i) + T^i D_j \xi^j - T^j D_j \xi^i, \ i = 1, ..., n, \tag{1.24}$$

constitute the components of a conserved vector of (1.11)*, i.e,* $D_i T^{*i}|_{(1.11)} = 0$.

In general, we can transform a conservation law to canonical form via a mapping of its symmetry into canonical term.

Definition 1.9. Consider a scalar pde $F = 0$ with $n = 2, (x^1, x^2) = (t, x)$ which admits a symmetry X associated with a conserved vector (T^t, T^x). In terms of the canonical variables r, s obtained by mapping X to $Y = \frac{\partial}{\partial s}$, the conservation laws can be expressed as [68]

$$D_r T^r + D_s T^s = 0, \tag{1.25}$$

with T^r and T^s given as

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r) D_x(s) - D_x(r) D_t(s)}, \tag{1.26}$$

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r) D_x(s) - D_x(r) D_t(s)}. \tag{1.27}$$

This now allows for a double reduction of the PDE.

Theorem 1.5. *A PDE of order with two independent variables, which admits a symmetry* X *that is associated with a conserved vector* T*, can be reduced to an ODE of order, namely* $T^r = k$*, where* T^r *is defined in* (1.26) *[10].*

Chapter 2

An analysis of nonlinear PDE arising in elasto-plastic flow

In the present chapter, we use the group analysis and the Noether's approach to study the generalised fourth order nonlinear partial differential equation arising in elasto-plastic flow [83, 3]

$$u_{tt} + \alpha u_{xxxx} - \gamma (u_x^n)_x = 0, \tag{2.1}$$

where α, γ are constants, u_{xxxx} is the dispersive term and $n > 0$. The equation (2.1) describes the propagation of the wave in the medium with the dispersive effect. It is used to study longitudinal motion of elasto-plastic microstructure models [3]

The instability of its special solution and ordinary stain solution were studied in [3]. The sufficient conditions for the nonexistence of the solution of equation (2.1) for $n = 2$ were given in [84, 14]. Yan [81] studied the equation (2.1) with the viscous damping term, by using the direct reduction method. They show that this equation is not integrable under the sense of Ablowitzs conjecture and found four families of exact solutions. The work of Yan [81] was extended by Wu and Fan [80] via the same method and presented the solutions for the equation for $n \geq 3$. However, none of these studies categorizes analytic, exact or invariant solutions or studies the underlying symmetries and conservation laws of the equation (2.1). The nonlinear equation (2.1) has extensive physical applications, any additional exact solutions could be of interest.

Here, we will investigate symmetry and conservation law classification of the generalised nonlinear equation (2.1). We will show that the particular case $n = 1$ is special as it is the only case that the dilation symmetry in time and space are lost. Furthermore, we shall study the equation (2.1) for integrability and for exact solutions using a combination of Lie classical method and several other methods including Extended tanh method.

The chapter is organized as follows. In Section 2.1, we present all of the vector fields of (2.1), which split into two different cases namely: $n \neq 1$ and $n = 1$. In Section 2.2, the optimal system of one dimensional subalgebras of (2.1) is constructed. Using the optimal system of subalgebras, symmetry reductions and exact group invariant solutions are obtained. A similar analysis is presented in Section 2.3 for the special case of n, that is $n = 1$. In Section 2.4, variational conservation laws are obtained using Noether Theorem. A brief discussion and conclusion is given in the last Section.

2.1 Lie point symmetries

In this section, we present the Lie point symmetry generators admitted by equation (2.1). The Lie point symmetries admitted by equation (2.1) are generated by a vector field of the form

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.2}$$

and we need to solve for the coefficient functions $\xi^1(t,x,u),\ \xi^2(t,x,u),\ \eta(x,t,u)$.

X must satisfy Lie's symmetry condition (1.14), i.e,

$$X^{[4]}\left[u_{tt} + \alpha u_{xxxx} - \gamma(u_x^n)_x = 0,\right]\Big|_{(2.1)} = 0, \tag{2.3}$$

where $X^{[4]}$ is the fourth prolongation of the operator X defined by

$$\begin{aligned} X^{[4]} = X &+ \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} \\ &+ \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \zeta_{xxxx}\frac{\partial}{\partial u_{xxxx}} \end{aligned} \tag{2.4}$$

and the coefficients $\zeta_t,\ \zeta_x,\ \zeta_{tt},\ \zeta_{xx},\ \zeta_{xxx}$ and ζ_{xxxx} are given by

$$\zeta_t = D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2),$$

$$
\begin{aligned}
\zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2),\\
\zeta_{xx} &= D_x(\zeta_x - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2),\\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2),\\
\zeta_{xxxx} &= D_x(\zeta_{xxx}) - u_{xxxt} D_x(\xi^1) - u_{xxxx} D_x(\xi^2).
\end{aligned}
$$

Here $D_x,\ D_t$ denote the total derivative operators defined by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + \cdots, \quad D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{2.5}$$

Expansion and separation of (2.3) with respect to the powers of different derivatives of u yields an over determined system in the unknown coefficients ξ^1, ξ^2 and η

$$
\begin{aligned}
&\eta_u = \frac{\xi^1_t(n-3)}{2(n-1)}, \quad \eta_x = 0, \quad \eta_{tt} = 0, \quad \xi^1_u = 0,\\
&\xi^1_x = 0, \quad \xi^1_{tt} = 0, \quad \xi^2_t = 0, \quad \xi^2_u = 0, \quad \xi^2_x = \frac{1}{2}\xi^1_t.
\end{aligned} \tag{2.6}
$$

Solving the overdetermined system (2.6) for $\xi^1(t,x,u),\ \xi^2(t,x,u)$ and $\eta(t,x,u)$, we obtain

$$\xi^1(t,x,u) = C_1 + tC_5, \tag{2.7}$$

$$\xi^2(t,x,u) = C_2 + \frac{1}{2}xC_5, \tag{2.8}$$

$$\eta(t,x,u) = \frac{1}{2}\left(\frac{n-3}{n-1}\right)uC_5 + C_3 + tC_4, \tag{2.9}$$

$C_1, \cdots, C_5$ are constants. From the governing equations (2.9), it can be observed that there are two cases appearing: Case (i) $n \neq 1$ and Case (ii) $n = 1$.

2.1.1 Case (i) $n \neq 1$

The Lie point symmetry generators of (2.1) for this case is a five-dimensional Lie algebra spanned with the following basis

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = \frac{\partial}{\partial u}, X_4 = t\frac{\partial}{\partial u},\\
&X_5 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{1}{2}\mu u\frac{\partial}{\partial u},
\end{aligned} \tag{2.10}
$$

where $\mu = \frac{n-3}{n-1}$. The commutation relation between these five operators (2.10) is presented in Table 2.1, where each entry, A_{ij}, constitutes the Lie Bracket $[X_i, X_j]$ of two infinitesimal generators from (2.10) for $1 \leq i, j \leq 5$.

2.1.2 Case (ii) $n = 1$

This case gives rise to the linear wave equation of (2.1)

$$u_{tt} + \alpha u_{xxxx} - \gamma u_{xx} = 0, \tag{2.11}$$

which admits four Lie point symmetries. These are X_1 and X_2 of Case (i) and $X_3 = u\frac{\partial}{\partial u}$ as well as an infinite symmetry $X_4 = F_1(t, x)\frac{\partial}{\partial u}$, where $F_1(t, x)$ is a solution of equation (2.11) and hence called the solution symmetry. This symmetry always arises in the event that the equation in question is linear and is not used to reduce the original PDE. The commutation relations satisfied by these four operators are presented in Table 2.4.
Obviously, from equation (2.10) when $n = 1$, the dilations in space and time are lost. This seems outstanding and distinguishes the symmetry structure of (2.1) for $n = 1$ from any other values of n.

2.2 Symmetry reductions and group-invariant solutions for equation (2.1)

In the previous section, we present the symmetry classification of the fourth order nonlinear equation (2.1). In this section, by using the optimal system, we give some group-invariant solutions.

2.2.1 One-dimensional optimal system of subalgebras

It is well known that the Lie group theory method plays an important role in finding exact solutions of differential equations. Any linear combination of symmetry generators is also a symmetry generator, there are always infinitely many different symmetry subgroups for a differential equation. Therefore, a method determining which subgroups would give basically different types of solutions is necessary and significant for a complete understanding of the list of invariant solutions.

In this section we discuss in brief an optimal system of one-dimensional subalgebras of the symmetry group admitted by equation (2.1) which was proposed by Olver [61]. This optimal system shall be used in the next section for reduction and construction of invariant solutions. Firstly, we find the optimal system of the Lie algebra possessed by equation (2.1) for $n \neq 1$. In constructing one dimensional optimal system of symmetry group $\langle X_1,\ X_2,\ X_3,\ X_4,\ X_5\rangle$, admitted by equation (2.1), we consider the general operator

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5, \tag{2.12}$$

where a_i are arbitrary real constants and try to simplify as many of the coefficients a_i as possible through application of adjoint actions to (2.12) to obtain a new simpler operator. The adjoint representation of the Lie algebra is defined as

$$Ad(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2[X_i, [X_i, X_j]] - \cdots . \tag{2.13}$$

Using equation (2.13) together with the commutator Table 2.1 we obtain all the adjoint representation of Lie algebra of equation (2.1) presented in Table 2.2.

Table 2.1: The commutation relations satisfied by symmetries of (2.1).

$[,]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	X_3	X_1
X_2	0	0	0	0	$\frac{1}{2}X_2$
X_3	0	0	0	0	$\frac{1}{2}\mu X_3$
X_4	$-X_3$	0	0	0	$\frac{1}{2}(\mu-2)X_4$
X_5	$-X_1$	$-\frac{1}{2}X_2$	$-\frac{1}{2}\mu X_3$	$-\frac{1}{2}(\mu-2)X_4$	0

Table 2.2: Adjoint representation of generators for (2.1).

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	X_3	$X_4-\epsilon X_3$	$X_5-\epsilon X_1$
X_2	X_1	X_2	X_3	X_4	$X_5-\frac{1}{2}\epsilon X_2$
X_3	X_1	X_2	X_3	X_4	$X_5-\frac{1}{2}\mu\epsilon X_3$
X_4	$X_1+\epsilon X_3$	X_2	X_3	X_4	$X_5-\frac{1}{2}(\mu-2)\epsilon X_4$
X_5	$e^{\epsilon}X_1$	$e^{\frac{1}{2}\epsilon}X_2$	$e^{\frac{1}{2}\mu\epsilon}X_3$	$e^{\frac{1}{2}(\mu-2)\epsilon}X_4$	X_5

In what follows, after some calculations, it turns out that the optimal system of one dimensional subalgebras of (2.1) is

$$X_2,\ X_3+\epsilon X_2,\ X_1+\epsilon X_2,\ bX_1+aX_2+X_4,\ X_5, \tag{2.14}$$

where $\epsilon=\pm1$, and a,b are arbitrary real constants.

2.2.2 Symmetry Reduction and Invariant Solution

In the present subsection, we employ the optimal system of one-dimensional subalgebras to reduce equation (2.1) to ordinary differential equations and find their exact solutions. The canonical variables and the group invariants corresponding to the subalgebras (2.14) are determined and presented in Table 2.3.

Invariance under X_2

The substitution of the group invariant solution of X_2 into (2.1) gives rise to an ordinary differential equation whose general solution is the linear function in t,

$$u(t,x) = c_1 t + c_2, \tag{2.15}$$

where c_1, c_2 are constants.

Invariance under $X_3 + \epsilon X_2$

Similarly, in this case the group invariant solution corresponding to the symmetry generator $X_3 + \epsilon X_2$ leads to the linear solution in t and x variable,

$$u(t,x) = c_1 t + \frac{x}{\epsilon} + c_2, \tag{2.16}$$

where c_1, c_2 are constants.

Invariance under $X_1 + \epsilon X_2$

The group invariant solution of the subalgebra $X_1 + \epsilon X_2$ reduces (2.1) to

$$\epsilon^2 w''(z) + \alpha w''''(z) - n\gamma w'(z)^{n-1} w''(z) = 0. \tag{2.17}$$

The ode (2.17) possesses the following solutions:

Case 1 Exact solution by integration: Integrating the above ode (2.17) once with respect to z and setting the constant of integration to zero yields the nonlinear ode

$$\epsilon^2 w'(z) + \alpha w'''(z) - \gamma w'^n(z) = 0. \tag{2.18}$$

Substituting $w'(z) = J(z)$ in (2.18), multiplying by $J'(z)$ and then once more integrating with respect to z, taking the constant of integration to zero, we obtain

$$J(z) = \left(\frac{(n+1)\epsilon^2}{2\gamma} \sec^2\left(\frac{(n-1)\epsilon}{2\sqrt{\alpha}} z\right) \right)^{\frac{1}{n-1}}, \quad \alpha > 0 \tag{2.19}$$

$$J(z)=\left(\frac{(n+1)\epsilon^2}{2\gamma}\operatorname{sech}^2(\frac{(n-1)\epsilon}{2\sqrt{\alpha}}z)\right)^{\frac{1}{n-1}},\quad \alpha<0. \tag{2.20}$$

Integrating (2.19) and (2.20), we obtain the solutions of (2.1) in terms of Hypergeometric function given respectively as

$$u(t,x)=\frac{\left(\frac{(n+1)\epsilon^2}{2\gamma}\right)^{\frac{1}{n-1}}2\sqrt{\alpha}\sin\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\ {}_2F_1\left(\frac{1}{2},\frac{n+1}{2(n-1)};\frac{3}{2};\sin^2\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\right)}{(n-1)\epsilon},\quad \alpha>0 \tag{2.21}$$

and

$$u(t,x)=\frac{\left(\frac{(n+1)\epsilon^2}{2\gamma}\right)^{\frac{1}{n-1}}2\sqrt{\alpha}\sinh\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\ {}_2F_1\left(\frac{1}{2},\frac{n+1}{2(n-1)};\frac{3}{2};-\sinh^2\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\right)}{(n-1)\epsilon},\quad \alpha<0, \tag{2.22}$$

where c is a constant.

Case 2 Exact solution by extended-tanh method: In this case, we seek solutions of equation (2.17) by extended-tanh method proposed by Fan [20]. The method mainly consists of the following steps:

- Assuming the solution of (2.17) can be expressed as

$$w(z)=A_0+\sum_{i=1}^{m}G^{i-1}(z)\left(A_iG(z)+B_i\sqrt{R+G^2(z)}\right), \tag{2.23}$$

and the new variable $G=G(z)$ satisfies the following first order nonlinear ode:

$$G'(z)-(R+G^2(z))=0, \tag{2.24}$$

which admits several types of solutions

$$\begin{aligned}
G(z) &= -\sqrt{-R}\tanh(\sqrt{-R})z,\ G(z)=-\sqrt{-R}\coth(\sqrt{-R})z,\ R<0,\\
G(z) &= -\frac{1}{z},\ R=0,\\
G(z) &= \sqrt{R}\tanh(\sqrt{-R})z,\ G(z)=-\sqrt{-R}\cot(\sqrt{-R})z,\ R>0.
\end{aligned} \tag{2.25}$$

The parameters $A_0, A_i, B_i, i = 1, 2, ..., m$ and R are constants to be determined later, and m is a positive integer. Letting $w'(z) = J(z)$, in (2.17) and integrating it once with respect to z while setting the constant of integration to zero, results in

$$\epsilon^2 J(z) + \alpha J''(z) - \gamma J(z)^n = 0. \tag{2.26}$$

On balancing the term with the highest-order derivative, $J''(z)$ with the nonlinear term $J(z)^n$ appearing in (2.26), we get $m = \frac{2}{n-1}$. Since m is not an integer for any value of $n > 0$, we make the following transformation

$$J(z) = v(z)^{\frac{2}{n-1}}, \tag{2.27}$$

then substitute (2.27) into (2.26) and obtain

$$\begin{aligned}(n-1)^2 v(z)^2(\epsilon^2 - \gamma v(z)^2) + \alpha(2(3-n)v'(z)^2 \\ +2(n-1)v(z)v''(z)) = 0.\end{aligned} \tag{2.28}$$

Now balancing $v(z)^4$ and $v(z)v''(z)$ we find that $m = 1$. So we assume that

$$J(z) = \left(A_0 + A_1 G(z) + B_1\sqrt{R + G(z)^2}\right)^{\frac{2}{n-1}}. \tag{2.29}$$

- Substituting (2.23) into (2.28) and using (2.24), collecting all terms with the same powers of G^k and $G^k\sqrt{R+G^2}$ together, and equating each coefficient of them to zero, yield a set of the following algebraic equations for A_0, A_1, B_1 and R:

$$
\begin{aligned}
&(i)\ (n-1)^2\left(-6A_0^2B_1^2\gamma R + A_0^2c^2 + A_0^4(-\gamma) + B_1^2c^2R - B_1^4\gamma R^2\right)\\
&\quad + 2\alpha(3-n)\left(A_1^2R^2\right) + 2\alpha(n-1)\left(B_1^2R^2\right) = 0,\\
&(ii)\ (n-1)^2\left(-12A_1A_0B_1^2\gamma R + 2A_1A_0c^2 - 4A_1A_0^3\gamma\right) + 2\alpha(n-1)\left(2A_0A_1R\right) = 0,\\
&(iii)\ (n-1)^2\left(-6A_0^2B_1^2\gamma - 6A_1^2B_1^2\gamma R + A_1^2c^2 - 6A_0^2A_1^2\gamma + B_1^2c^2 - 2B_1^4\gamma R\right)\\
&\quad + 2\alpha(n-1)\left(2A_1^2R + 3B_1^2R\right) + 2\alpha(3-n)\left(2A_1^2R + B_1^2R\right) = 0,\\
&(iv)\ (n-1)^2\left(-12A_0A_1B_1^2\gamma - 4A_0A_1^3\gamma\right) + 2\alpha\left(2A_0A_1\right)(n-1) = 0,\\
&(v)\ 2\alpha(n-1)\left(2A_1^2 + 2B_1^2\right) + 2\alpha(3-n)\left(A_1^2 + B_1^2\right) + (n-1)^2\left(-6A_1^2B_1^2\gamma + A_1^4(-\gamma) - B_1^4\gamma\right) = 0,\\
&(vi)\ (n-1)^2\left(2A_0B_1c^2 - 4A_0^3B_1\gamma - 4A_0B_1^3\gamma R\right) + 2\alpha(n-1)\left(A_0B_1R\right) = 0,\\
&(vii)\ (n-1)^2\left(2A_1B_1c^2 - 12A_0^2A_1B_1\gamma - 4A_1B_1^3\gamma R\right)\\
&\quad + 2\alpha(n-1)\left(3A_1B_1R\right) + 2\alpha(3-n)\left(3A_1B_1R + 2A_1B_1R\right) = 0,\\
&(viii)\ 2\alpha(n-1)\left(2A_0B_1\right) + (n-1)^2\left(-4A_0B_1^3\gamma - 12A_0A_1^2B_1\gamma\right) = 0,\\
&(ix)\ 2\alpha(n-1)\left(4A_1B_1\right) + 2\alpha(3-n)\left(2A_1B_1\right) + (n-1)^2\left(-4A_1^3B_1\gamma - 4A_1B_1^3\gamma\right) = 0.
\end{aligned}
\tag{2.30}
$$

Solving the resultant algebraic equations (2.30), we obtain the following results:

Subcase 1:

$$
A_0 = \pm\frac{\epsilon}{\sqrt{\gamma}}, A_1 = 0, B_1 = 0. \tag{2.31}
$$

This case leads to the following linear solution

$$
u(t,x) = \pm\frac{\epsilon(x-\epsilon t)}{\sqrt{\gamma}}. \tag{2.32}
$$

Subcase 2:

$$
A_0 = 0,\ A_1 = 0,\ B_1 = \pm\sqrt{\frac{2\alpha(n+1)}{\gamma(n-1)^2}},\ R = \frac{\epsilon^2(n-1)^2}{4\alpha}. \tag{2.33}
$$

From (2.24) and (2.33), we deduce the travelling wave solutions of (2.1) as follows:

$$
\begin{aligned}
u_1(t,x) = &\pm \frac{\left(\frac{(n+1)\epsilon^2}{2\gamma}\right)^{\frac{1}{n-1}} 2\sqrt{\alpha}\sin\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\ {}_2F_1\left(\frac{1}{2},\frac{n+1}{2(n-1)};\frac{3}{2};\sin^2\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\right)}{(n-1)\epsilon},\\
&\alpha > 0,\\
u_2(t,x) = &\pm \frac{\left(\frac{(n+1)\epsilon^2}{2\gamma}\right)^{\frac{1}{n-1}} 2\sqrt{\alpha}\cos\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\ {}_2F_1\left(\frac{1}{2},\frac{n+1}{2(n-1)};\frac{3}{2};\cos^2\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\right)}{(n-1)\epsilon},\\
&\alpha > 0,\\
u_3(t,x) = &\pm \frac{\left(\frac{(n+1)\epsilon^2}{2\gamma}\right)^{\frac{1}{n-1}} 2\sqrt{\alpha}\sinh\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\ {}_2F_1\left(\frac{1}{2},\frac{n+1}{2(n-1)};\frac{3}{2};-\sinh^2\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\right)}{(n-1)\epsilon},\\
&\alpha < 0,\\
u_4(t,x) = &\pm \frac{\left(\frac{(n+1)\epsilon^2}{2\gamma}\right)^{\frac{1}{n-1}} 2\sqrt{\alpha}\cosh\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\ {}_2F_1\left(\frac{1}{2},\frac{n+1}{2(n-1)};\frac{3}{2};\cosh^2\left(\frac{(n-1)\epsilon(x-\epsilon t+c)}{2\sqrt{\alpha}}\right)\right)}{(n-1)\epsilon},\\
&\alpha < 0.
\end{aligned} \tag{2.34}
$$

Invariance under $aX_2 + bX_1 + X_4$

The group invariant solution arising from the subalgebra $aX_2+bX_1+X_4$ results to the following nonlinear ordinary differential equation

$$\frac{1}{b} + \left(\frac{a}{b}\right)^2 w'' + \alpha w'''' - n\gamma w'^{n-1} w'' = 0. \tag{2.35}$$

Integrating (2.35) yields a third order equation

$$\frac{z}{b} + \left(\frac{a}{b}\right)^2 w' + \alpha w''' - w'^n = c, \tag{2.36}$$

where c is a constant of integration. Further reduction of (2.36) using its symmetry $\dfrac{\partial}{\partial w}$ gives

$$\frac{r}{b} + \left(\frac{a}{b}\right)^2 \psi + \alpha\psi'' - \psi^n = c, \tag{2.37}$$

where ψ is a function of r.

Invariance under X_5

The substitution of the group invariant solution of the symmetry X_5 into (2.1) yields the following nonlinear ordinary differential equation

$$\frac{1}{4}\mu\left(\mu-2\right)w-\frac{1}{2}\left(\mu+\frac{3}{2}\right)zw'+\frac{1}{4}z^{2}w''+\alpha w''''-n\gamma w'^{n-1}w''=0. \qquad (2.38)$$

The reduced nonlinear Ode (2.37) and (2.38) are quite challenging to solve analytically. The numerical investigation of the reduced equations might be of great importance in understanding the complex dynamics in elasticity and plasticity.

Table 2.3: Subalgebras, Canonical variables and Group invariants for the subalgebras (2.14).

Subalgebra	Canonical variable	Group invariants.
X_2	$z=t$	$u=w(z)$
$X_3+\epsilon X_2$	$z=t$	$u=w(z)+\frac{x}{\epsilon}$
$X_1+\epsilon X_2$	$z=x-\epsilon t$	$u=w(z)$
$aX_2+bX_1+X_4$	$z=x-\frac{a}{b}t$	$u=w(z)+\frac{1}{2b}t^2$
X_5	$z=\frac{x}{\sqrt{t}}$	$u=w(z)t^{\frac{\mu}{2}}$

2.3 Symmetry reductions and group-invariant solutions for special case, $n=1$

The infinite symmetry, X_4 obtained in this case is not used to reduce the original PDE. Therefore, we consider the symmetries $X_1,\ X_2,\ X_3$, for the reduction. Without the infinite symmetry, X_4 , we observed from the commutator Table 2.4, that the Lie algebra for this special case, $n=1$, forms an Abelian subalgebra, hence, we consider a linear combination $dX_1+kX_2+lX_3$, where l,k and d are constants.

Table 2.4: The commutation relations satisfied by symmetries of (2.1) for $n = 1$.

$[,]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_4
X_2	0	0	0	X_4
X_3	0	0	0	$-X_4$
X_4	$-X_4$	$-X_4$	X_4	0

Invariance under $dX_1 + kX_2 + lX_3$

This subalgebra gives rise to the group invariant solution

$$u(t,x) = w(z)e^{\frac{lt}{d}}, \tag{2.39}$$

where $z = dx - kt$. Equation (2.39) reduces (2.11) to the following fourth order linear ordinary differential equation

$$l^2 w - 2kldw' + d^2(k^2 - \gamma d^2)w'' + \alpha d^6 w'''' = 0. \tag{2.40}$$

Solving the above equation (2.40) and applying condition (2.39) results in

$$u(t,x) = e^{\frac{lt}{d}} \sum_{i=1}^{4} C_i e^{\lambda_i (dx-kt)}, \tag{2.41}$$

where C_i are constants and λ_i are the roots of the polynomial in y given by $\alpha d^6 y^4 + d^2(k^2 - \gamma d^2)y^2 - 2kldy + l^2 = 0$, $i = 1, ...4$.

2.4 The Conservation Laws

A Lagrangian of equation (2.1) is given by

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{n+1}u_x^{n+1}. \tag{2.42}$$

The corresponding Euler equation satisfies

$$\frac{\delta L}{\delta u} = 0, \tag{2.43}$$

where $\dfrac{\delta L}{\delta u}$ is defined by

$$\frac{\delta L}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}+D_t^2\frac{\partial}{\partial u_{tt}}+D_x^2\frac{\partial}{\partial u_{xx}}+D_xD_t\frac{\partial}{\partial u_{tx}}. \tag{2.44}$$

The Noether symmetry operator $X=\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\zeta_t\frac{\partial}{\partial u_t}+\zeta_x\frac{\partial}{\partial u_x}+\zeta_{xx}\frac{\partial}{\partial u_{xx}}$ corresponding to the Lagrangian of (2.1) according to (1.18) satisfies the equation

$$X(L)+L\left[D_t(\xi^1)+D_x(\xi^2)\right]=D_t(B^1)+D_x(B^2), \tag{2.45}$$

where $B^1(t,x,u)$ and $B^2(t,x,u)$ are gauge functions. Expansion of equation (2.45) with the Lagrangian (2.42) yields

$$\begin{aligned} u_t\zeta_t \quad - \quad & \alpha u_{xx}\zeta_{xx}-\gamma u_x^n\zeta_x+\left(\frac{1}{2}u_t^2-\frac{1}{2}\alpha u_{xx}^2-\frac{1}{n+1}u_x^{n+1}\right)\left(\xi_t^1+u_t\xi_u^1+\xi_x^2+u_x\xi_u^2\right)\\ &=B_t^1+u_tB_u^1+B_x^2+u_xB_u^2. \end{aligned} \tag{2.46}$$

The expansion and separation of (2.46) with respect to different derivatives of u results in an overdetermined system of equations for ξ^1,ξ^2,η,B^1 and B^2. The expansion of (2.46) and separation by derivatives of u give rise to the set of over-determined linear equations

$$\begin{aligned} &-\frac{\xi_u^1}{2}=0,\frac{\gamma n\xi_u^1}{1+n}=0,\frac{\alpha\xi_u^1}{2}=0,2\alpha\xi_u^1=0,2\alpha\xi_x^1=0,\\ &\alpha\xi_{xx}^1=0,\alpha\xi_{uu}^1=0,2\alpha\xi_{xu}^1=0,\frac{-\xi_u^2}{2}=0,\frac{\gamma n\xi_u^2}{1+n}=0,\\ &\frac{5\alpha\xi_u^2}{2}=0,\alpha\xi_{uu}^2=0,-\xi_t^2=0,\gamma\xi_x^1=0,-\alpha\eta_{xx}=0,\\ &2\alpha\eta_{xu}=\alpha\xi_{xx}^2,\alpha\eta_{uu}=2\alpha\xi_{xu}^2,\eta_u+\frac{\xi_x^2}{2}-\frac{\xi_t^1}{2},\gamma\eta_u=\frac{\gamma n\xi_x^2-\gamma\xi_t^1}{1+n},\\ &\alpha\eta_u=\frac{3\alpha\xi_x^2}{2}-\frac{\alpha\xi_t^1}{2},B_u^1=\eta_t,-\gamma\eta_x=0,-B_u^2=0,B_t^1+B_x^2=0. \end{aligned} \tag{2.47}$$

The solution of the above overdetermined system (2.47) is

$$\xi^1=c_1,\ \xi^2=c_2,\ \eta=c_3+tc_4,\ B^1=uc_4,\ B^2=0, \tag{2.48}$$

where $c_1,\ c_2,\ \cdots c_4$ are constants. The conserved vectors of (2.1) for the second order Lagrangian (2.42) is determined by (1.19)

$$T^t \;=\; B^1-L\xi^1-W\frac{\partial L}{\partial u_t},$$

$$T^x = B^2 - L\xi^2 - W\left(\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}}\right) - D_x(W)\frac{\partial L}{\partial u_{xx}}, \tag{2.49}$$

where $W = \eta - \xi^1 u_t - \xi^2 u_x$. Equations (2.49) together with (2.48), yield the conserved vectors and gauge functions for equation (2.1) presented in Table 2.5.

Conservation Laws for the special case, $n = 1$

In this case, the Lagrangian is given by

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\gamma u_x^2. \tag{2.50}$$

Similarly, by applying the same procedure, we obtain the strict Noether operators corresponding to the Lagrangian (2.50) as

$$\xi^1 = c_1, \xi^2 = c_2, \eta = c_3, B^1 = uc_4, B^2 = 0, \tag{2.51}$$

Hence, by invoking (2.49) we obtain the conserved vectors corresponding to the Noether symmetry operators (2.51). These conserved quantities are presented in the Table 2.6.

Table 2.5: The conservation laws of equation (2.1).

Symmetry generator	T_x	T_t
X_1	$\alpha(u_t u_{xxx} - u_{xx}u_{xt} - \gamma u_x^n u_t)$	$\frac{1}{2}(\alpha u_{xx}^2 + u_t^2) + \frac{1}{n+1}\gamma u_x^{n+1}$
X_2	$\alpha\left(u_x u_{xxx} - \frac{1}{2}{u_{xx}}^2\right) - \frac{1}{2}u_t^2 - \frac{1}{(n+1)}n\gamma u_x^{n+1}$	$u_x u_t$
X_3	$-\gamma {u_x}^n + \alpha u_{xxx}$	u_t
X_4	$t(-\gamma {u_x}^n + \alpha u_{xxx})$	$tu_t - u$

Table 2.6: The conservation laws of of equation (2.1) for special case, $n = 1$.

Symmetry generator	T_x	T_t
X_1	$\alpha(u_t u_{xxx} - u_{xx}u_{xt} - \gamma u_x u_t)$	$\frac{1}{2}(\alpha u_{xx}^2 + u_t^2 + \gamma u_x^2)$
X_2	$\alpha\left(u_x u_{xxx} - \frac{1}{2}{u_{xx}}^2\right) - \frac{1}{2}(u_t^2 - \gamma u_x^2)$	$u_x u_t$
X_3	$-\gamma u_x + \alpha u_{xxx}$	u_t

2.5 Discussion and Conclusion

In this chapter, using the group methods, and the Noether approach, a nonlinear PDE found in elasto-plastic flow is studied. We derived all the geometric vector fields of the model equations. We found that the analysed model does not admit space dilation type symmetries as a result of the linearity of the equation when $n = 1$. In addition, on the basis of symmetries, the optimal system is constructed. Based on the optimal system, some exact solutions are presented. Meanwhile, more exact solutions, which include soliton solutions are constructed using the Extended tanh method. Finally, conservations laws are derived. These results can be used to study deformity in elastic and plastic medium.

Chapter 3

Exact solutions of a generalised Boussinesq equation with damping term and a system of variant Boussinesq equations via double reduction theory

In this chapter, we consider two systems of NLEEs found in mathematical physics. The first one is the generalised Boussinesq (GB) equation with damping term [82, 45]

$$u_{tt} + 2\rho u_{xxt} + \beta u_{xxxx} + \gamma (u^n)_{xx} = 0, \tag{3.1}$$

where ρ, β, γ are constants and n is a nonzero real number. Equation (3.1) is widely used as a model to describe natural phenomena in many scientific fields such as plasma waves, solid physics and fluid mechanics [82]. It is to be noted that when $\rho = 0,\ \beta = -1,\ \gamma = 1$, and $n = 3$, in (3.1) we obtain the modified Boussinesq equation [23]. The modified Boussinesq equation is used as a model to describe the temporal evolution of nonlinear finite amplitude waves on a density front in a rotating fluid. Exact travelling wave solutions for (3.1) were studied in [45] using the extended tanh method [20]. Yan *et al.* [82] investigated the solitary wave solutions of the equation (3.1) for $n = 3$ using the direct method [16, 15, 46].

The second system is the variant Boussinesq (VB) equations [64, 72]

$$\begin{aligned} u_t + (uv)_x + v_{xxx} &= 0, \\ v_t + u_x + vv_x &= 0, \end{aligned} \tag{3.2}$$

described as a model for water waves, where $v(t, x)$ represents the velocity and $u(t, x)$ represents the total depth. Solitary wave solutions and multi-solitary wave solutions of the system (3.2) were obtained in [84] using the homogeneous balance method [74]. Fu *et al.* [22] examined the system (3.2) for periodic wave solutions using the ansatz method [29]. Conservation laws for the system were derived by Naz *et al.* [56] by increasing the order of the equation and using Noether's approach.

In this study, the conservation laws of the GB equation (3.1) which are not derived from a variational principle are constructed for the first time using the partial Lagrangian method. The GB equation contains an odd order term which consists of mixed derivative i.e. the derivative of both t and x, getting its standard Lagrangian is not possible and thus Noether approach is not applicable for finding its conservation laws. The partial Noethers approach is then used to derive the conservation laws. These conserved vectors constructed by partial Noether's theorem failed to satisfy the divergence property. A number of extra terms arise because of the odd order term which consists of mixed derivative. These extra terms contribute to the trivial part of the conserved vector and need to be adjusted to satisfy the divergence property. After construction of conservation laws the solutions of the GB equation are derived by double reduction theory. A similar analysis is performed for the system of VB equations. A similar analysis is performed for the system of VB equations (3.2) to obtain exact solutions of the system. The chapter is organised as follows: In the next Section, the Lie point symmetries of the GB equation are obtained. In Section 3.2, the conservation laws of the GB equation are derived. Section 3.3 discusses the double reduction and exact solutions of the GB equation while Section 3.4 deals with the exact solutions of a system of VB equations using double reduction theory. Concluding remarks are presented in Section 3.5.

3.1 Lie Symmetries of the GB equation

The Lie point symmetries admitted by (3.1) are generated by a vector field of the form

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{3.3}$$

The operator X satisfies the Lie symmetry condition (1.14)

$$X^{[4]}\left[u_{tt} + 2\rho u_{xxt} + \beta u_{xxxx} + \gamma(u^n)_{xx} = 0,\right]\Big|_{(3.1)} = 0, \tag{3.4}$$

where $X^{[4]}$ is the fourth prolongation of the operator X defined by

$$\begin{aligned} X^{[4]} = X &+ \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xt}\frac{\partial}{\partial u_{xt}} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} \\ &+ \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \zeta_{xxt}\frac{\partial}{\partial u_{xxt}} + \zeta_{xxxx}\frac{\partial}{\partial u_{xxxx}} \end{aligned} \tag{3.5}$$

and the coefficients $\zeta_t,\ \zeta_x,\ \zeta_{tt},\ \zeta_{xx},\ \zeta_{xt},\ \zeta_{xxt},\ \zeta_{xxx}$ and ζ_{xxxx} are given by

$$\begin{aligned} \zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\ \zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \\ \zeta_{xx} &= D_x(\zeta_x - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2), \\ \zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2), \\ \zeta_{xt} &= D_x(\zeta_t) - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2), \\ \zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2), \\ \zeta_{xxt} &= D_t(\zeta_{xx}) - u_{xxt} D_t(\xi^1) - u_{xxx} D_x(\xi^2), \\ \zeta_{xxxx} &= D_x(\zeta_{xxx}) - u_{xxxt} D_x(\xi^1) - u_{xxxx} D_x(\xi^2), \end{aligned}$$

Here $D_x,\ D_t$ denote the total derivative operators defined by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + \cdots, D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{3.6}$$

Expansion and separation of (3.4) with respect to the powers of different derivatives of u yields an over determined system in the unknown coefficients ξ^1, ξ^2 and η. This system can not be presented here due to its lengthy calculations, we present only the results and refer the reader to [61] for details.

Solving the overdetermined system for arbitrary parameters gives two different cases as follows:
Case (1): Provided $\rho\beta\gamma(n-1) \neq 0$, we have the following three Lie point symmetries:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = t\frac{\partial}{\partial t} - \frac{u}{(n-1)}\frac{\partial}{\partial u} + \frac{x}{2}\frac{\partial}{\partial x}. \tag{3.7}$$

Case (2): When $\rho\beta\gamma \neq 0,\ n = 1$, we obtain, in addition to X_1 and X_2, another symmetry $X_4 = u\dfrac{\partial}{\partial u}$ and an infinite-dimensional symmetry, $X_5 = F_1(t,x)\dfrac{\partial}{\partial u}$ which is expected as (3.1) is now linear.

3.2 Conservation laws of the GB equation

A conserved vector corresponding to a conservation law of the GB equation (3.1) is a 2−tuple (T^t, T^x), such that

$$D_t T^t + D_x T^x = 0 \tag{3.8}$$

along the solutions of the equation.

Conservation laws of the GB equation via partial Noether's method

As equation (3.1) does not have a standard Lagrangian due to the presence of an odd order term u_{xxt} hence, is not derivable from a variational principle. We investigate the conserved quantities via the partial Noether approach using the partial Lagrangian [35]. This study of the conserved vectors of the equation (3.1) has not been previously conducted. The equation (3.1) possesses a partial Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}\beta u_{xx}^2 + \frac{1}{2}\gamma n u^{n-1} u_x^2. \tag{3.9}$$

The associated partial Euler-Lagrange equation is

$$\frac{\delta L}{\delta u} = 2\rho u_{xxt} + \gamma\frac{1}{2}n(n-1)u^{n-2}u_x^2, \tag{3.10}$$

where $\dfrac{\delta L}{\delta u}$ is defined by

$$\frac{\delta L}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_x D_t\frac{\partial}{\partial u_{tx}}, \tag{3.11}$$

and

$$
\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u}, \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x}.
\end{aligned}
$$

The partial Noether operator is given by

$$
X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} \tag{3.12}
$$

and satisfies the partial Noether's operator (1.21)

$$
X^{[2]}(L) + L(D_t \xi^1 + D_x \xi^2) = (\eta - u_t \xi^1 - u_x \xi_2) \frac{\delta L}{\delta u} + B_t^1 + u_t B_u^1 + B_x^2 + u_x B^2 u, \tag{3.13}
$$

where B^1 and B^2 are gauge functions. Separating (3.13), after expansion by the of derivatives of u, with the partLagrangian (3.9) yields the following overdetermined system:

$$
\begin{aligned}
&\rho \xi^1 = 0, \rho \xi^2 = 0, \rho \xi_u^1 = 0, \rho \xi_x^1 = 0, \xi_u^2 = 0, \beta \xi_{xx}^1 = 0, \beta \xi_{uu}^1 = 0, \beta \xi_{uu}^2 = 0, \beta \xi_{xu}^1 = 0, \\
&\beta \eta_{xx} = 0, \frac{1}{2} n(n-1) \gamma u^{n-2} \xi^1 - n \gamma u^{n-1} \xi_u^1 = 0, \eta_u + \frac{1}{2} \xi_x^2 - \frac{1}{2} \xi_t^1 = 0, -B_u^1 + \eta_t = 0, \\
&B_t^1 + B_x^2 = 0, \beta(2\eta_{uu} - 4\xi_{xu}^2) = 0, -B_u^2 + n\gamma u^{n-1} \eta_x = 0, -\beta(\eta_u + \frac{3}{2} \xi_x^2 - \frac{1}{2} \xi_t^1) = 0, \\
&- n\gamma u^{n-1} \xi_x^1 - \xi_t^2 = 0, n\gamma u^{n-1} \eta_u - \frac{1}{2} n\gamma u^{n-1} \xi_x^2 - n\gamma u^{n-1} \xi_x^2 + \frac{1}{2} n\gamma u^{n-1} \xi_t = 0.
\end{aligned} \tag{3.14}
$$

The solution of the system (3.14) yields the following partial Noether operators and gauge functions

$$
\begin{aligned}
&\xi^1 = \xi^2 = 0,\ \eta = c_1 + tc_3 + x(c_3 + tc_4),\ B^1 = u(c_3 + xc_4) + F(t, x), \\
&B^2 = u^n \gamma (c_2 + tc_4) + G(t, x),\ F_t(t, x) + G_x(t, x) = 0,
\end{aligned} \tag{3.15}
$$

where $c_1,\ c_2,\ c_3,\ c_4$ are constants. Without loss of generality, we set
$F(t, x) = G(t, x) = 0$ as $F_t(t, x) + G_x(t, x) = 0$ and obtain the partial Noether operators X_i of (3.1) presented in Table 3.1. The conserved vectors of (3.1) for the second order partial Lagrangian (3.9) is determined by

$$
\begin{aligned}
T^t &= B^1 - L\xi^1 - W \frac{\partial L}{\partial u_t}, \\
T^x &= B^2 - L\xi^2 - W \left(\frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} \right) - D_x(W) \frac{\partial L}{\partial u_{xx}},
\end{aligned} \tag{3.16}
$$

where $W = \eta - \xi^1 u_t - \xi^2 u_x$. Thus, by adoption of (3.16), the conserved quantities corresponding to each of the four partial Noether operators in Table 3.1 are given by: (i) $X_1 = \frac{\partial}{\partial u}, B^1 = 0, B^2 = 0, \xi^1 = 0, \xi^2 = 0, \eta = 1, W = 1$

$$\begin{aligned} T_1^t &= -u_t, \\ T_1^x &= -\gamma n u^{n-1} u_x - \beta u_{xxx}. \end{aligned} \tag{3.17}$$

The total divergence is

$$\begin{aligned} D_t T_1^t + D_x T_1^x &= -u_{tt} - \gamma n u^{n-1} u_{xx} - \gamma n(n-1) u^{n-2} u_x^2 - \beta u_{xxxx}, \\ &= -u_{tt} - \gamma (u^n)_{xx} - \beta u_{xxxx}, \\ &= 2\rho u_{xxt}. \end{aligned} \tag{3.18}$$

Extra terms emerge that require further analysis. These terms can be absorbed into the conservation law by making an adjustment

$$D_t T_1^t + D_x T_1^x \quad = 2\rho D_t u_{xx}. \tag{3.19}$$

This can be written as

$$D_t(T_1^t - 2\rho u_{xx}) + D_x T_1^x = 0. \tag{3.20}$$

Hence, the modified conserved vector viz., $(\tilde{T_1^t}, \tilde{T_1^t})$ that satisfies the divergence property $D_t \tilde{T_1^t} + D_x \tilde{T_1^t} = 0$ along the equation is given by

$$\begin{aligned} \tilde{T_1^t} &= -u_t - 2\rho u_{xx}, \\ \tilde{T_1^x} &= -\gamma n u^{n-1} u_x - \beta u_{xxx}, \\ &= -\gamma (u^n) x - \beta u_{xxx}. \end{aligned} \tag{3.21}$$

The same applies to the following results below.

(ii) $X_2 = x\frac{\partial}{\partial u}, B^1 = 0, B^2 = u^n \gamma, \xi^1 = 0, \xi^2 = 0, \eta = x, W = x$,

with

$$\begin{aligned} T_2^t &= -x u_t, \\ T_2^x &= \gamma u^n - x(\gamma n u^{n-1} u_x + \beta u_{xxx}) + \beta u_{xxx}, \end{aligned} \tag{3.22}$$

we obtain

$$\begin{aligned} D_t T_2^t + D_x T_2^x &= -x(u_{tt} + \alpha u_{xxxx} + \gamma (u^n)_{xx}), \\ &= x(2\rho u_{xxt}). \end{aligned} \tag{3.23}$$

This can be adjusted as

$$D_t T_2^t + D_x T_2^x = D_t 2\rho x u_{xx}, \tag{3.24}$$

which simplifies further to

$$\begin{aligned} &D_t(T_2^t - 2\rho x u_{xx}) + D_x T_2^x = 0, \\ &D_t(-x(u_t + 2\rho u_{xx})) + D_x(\gamma u^n - x(\gamma(u^n)_x + \beta u_{xxx}) + \beta u_{xxx}) = 0. \end{aligned} \tag{3.25}$$

The new form of conserved vector becomes

$$\begin{aligned} \tilde{T_2^t} &= -x(u_t + 2\rho u_{xx}), \\ \tilde{T_2^x} &= \gamma u^n - x(\gamma(u^n)_x + \beta u_{xxx}) + \beta u_{xxx}. \end{aligned} \tag{3.26}$$

(iii) $X_3 = t\frac{\partial}{\partial u}, B^1 = u, B^2 = 0, \xi^1 = 0, \xi^2 = 0, \eta = t, W = t,$

the conserved vector is

$$\begin{aligned} T_3^t &= u - tu_t, \\ T_3^x &= -t(\gamma n u^{n-1} u_x + \beta u_{xxx}), \end{aligned} \tag{3.27}$$

and satisfies

$$\begin{aligned} D_t T_3^t + D_x T_3^x &= -t(u_{tt} + \beta u_{xxxx} + (u^n)_{xx}), \\ &= t(2\rho u_{xxt}). \end{aligned} \tag{3.28}$$

The adjustment leads to

$$D_t T_3^t + D_x T_3^x = D_x 2\rho t u_{xt}, \tag{3.29}$$

so that

$$\begin{aligned} &D_t T_3^t + D_x(T_3^x - 2\rho t u_{xt} = 0, \\ &D_t(u - tu_t) + D_x(-t(\gamma(u^n)_x + \beta u_{xxx} + 2\beta u_{xt})) = 0. \end{aligned} \tag{3.30}$$

This gives rise to the new form of the conserved vector

$$\tilde{T_3^t} = u - tu_t, \tilde{T_3^x} = -t(\gamma(u^n)_x + \beta u_{xxx} + 2\beta u_{xt}). \tag{3.31}$$

(iv) $X_4 = xt\frac{\partial}{\partial u}, B^1 = ux, B^2 = tu^n\gamma, \xi^1 = 0, \xi^2 = 0, \eta = xt, W = xt,$

results to the conserved vector

$$\begin{aligned} T_4^t &= ux - xtu_t, \\ T_4^x &= \gamma t u^n - xt(\gamma n u^{n-1} u_x + \beta u_{xxx}) + t\beta u_{xxx}. \end{aligned} \tag{3.32}$$

Table 3.1: The partial Noether operators and gauge terms of (3.1)

X_i	operator	gauge function
X_1	$\frac{\partial}{\partial u}$	$B^1 = B^2 = 0$
X_2	$x\frac{\partial}{\partial u}$	$B^1 = B^2 = 0$
X_3	$t\frac{\partial}{\partial u}$	$B^1 = u, B^2 = 0$
X_4	$xt\frac{\partial}{\partial u}$	$B^1 = xu, B^2 = \gamma t u^n$

which gives

$$\begin{aligned} D_t T_4^t + D_x T_4^x &= -xt(u_{tt} + \beta u_{xxxx} + (u^n)_{xx}), \\ &= xt(2\rho u_{xxt}). \end{aligned} \tag{3.33}$$

The redefinition leads to

$$D_t T_4^t + D_x T_4^x = D_x(2\rho(xtu_{xt} - tu_t)), \tag{3.34}$$

and further simplifies to

$$D_t T_4^t + D_x(T_4^x - 2\rho(xtu_{xt} - tu_t)) = 0, \tag{3.35}$$

so that

$$\begin{aligned} \tilde{T}_4^t &= ux - xtu_t, \\ \tilde{T}_4^x &= \gamma t u^n - xt(\gamma(u^n)_x + \beta u_{xxx} + 2\rho u_{xt}) + 2\rho t u_t + t\beta u_{xxx}. \end{aligned} \tag{3.36}$$

3.3 Double reduction and exact solutions of the GB equation

When a PDE of order n with two independent variables, admits a symmetry X that is associated with a conserved vector T, then it can be reduced to an ODE of order $n-1$. [10]. Now, we utilize the relationship between the conservation laws and the Lie point symmetries of equation (3.1) to obtain its doubly reduced equation which is easily solved to find exact solutions. A Lie point symmetry X of the GB equation (3.1) is associated with its conserved vector (T^t, T^x) if (1.23)

$$X_i^{[2]}\begin{pmatrix}T_i^t\\ T_i^x\end{pmatrix}+(D_t\xi_t^1+D_x\xi_x^2)\begin{pmatrix}T_i^t\\ T_i^x\end{pmatrix}-\begin{pmatrix}D_t\xi_t^1 & D_x\xi_t^1\\ D_t\xi_x^2 & D_x\xi_x^2\end{pmatrix}\begin{pmatrix}T_i^t\\ T_i^x\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \tag{3.37}$$

For each $X_i,\ i=1,2$ of equation (3.7) and the conserved vectors $(\tilde{T^t}, \tilde{T^x})$ of equations (3.21), (3.26), (3.31) and (3.36), equation (3.37) becomes

$$\begin{aligned}&X_i^{[3]}\begin{pmatrix}\tilde{T^t}\\ \tilde{T^x}\end{pmatrix}+(D_t\xi_t^1+D_x\xi_x^2)\begin{pmatrix}\tilde{T^t}\\ \tilde{T^x}\end{pmatrix}-\begin{pmatrix}D_t\xi_t^1 & D_x\xi_t^1\\ D_t\xi_x^2 & D_x\xi_x^2\end{pmatrix}\begin{pmatrix}\tilde{T^t}\\ \tilde{T^x}\end{pmatrix}\\ &=X_i^{[3]}\begin{pmatrix}\tilde{T^t}\\ \tilde{T^x}\end{pmatrix}+(0+0)\begin{pmatrix}\tilde{T^t}\\ \tilde{T^x}\end{pmatrix}-\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{T^t}\\ \tilde{T^x}\end{pmatrix}\\ &=0.\end{aligned}$$

Therefore, the symmetry generators X_1 and X_2 are associated with the four conserved vectors. Thus we can perform the double reduction by a combination of the two generators,

$$X = X_1 + cX_2 \tag{3.38}$$

using any one of the four conservation laws. Mapping (4.55) to

$$Y = \frac{\partial}{\partial s} \tag{3.39}$$

yields the canonical coordinates

$$s = t, r = x - ct, w(r) = u. \tag{3.40}$$

The conservation law $\tilde{T} = (\tilde{T^t},\ \tilde{T^x})$ is rewritten as $D_rT^r + D_sT^s = 0$. By using the formulas (1.26) and (1.27), a double reduction by $\tilde{T_1} = (\tilde{T_1^t},\ \tilde{T_1^x})$ results in the reduced conserved form

$$T_1^r \quad = \quad c^2w_r - 2\rho cw_{rr} + \gamma(w^n)_r + \beta w_{rrr}, \tag{3.41}$$

$$T_1^s \quad = \quad -cw_r + 2\rho w_{rr}. \tag{3.42}$$

Since (4.57) does not depend on s, the reduced conserved vector becomes

$$D_rT^r = 0, \tag{3.43}$$

which implies that

$$c^2 w_r - 2\rho c w_{rr} + \gamma(w^n)_r + \beta w_{rrr} = k, \tag{3.44}$$

where k is a constant. Equation (4.59) is a third order ODE which is a double reduction of the fourth order PDE (3.1). Integrating (4.59) once with respect to r while setting the constant of integration to zero, results in

$$c^2 w - 2\rho c w_r + \gamma w^n + \beta w_{rr} = 0. \tag{3.45}$$

We seek solutions of equation (4.60) by the extended $\left(\frac{G'}{G}\right)$ – expansion method [26]. The method mainly consists of the following steps: Suppose that the solution of (4.60) can be expressed as

$$w(r) = a_0 + \sum_{i=1}^{m} a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{i-1} \sqrt{\nu\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}, \tag{3.46}$$

with the new variable $G = G(r)$ satisfying

$$G''(r) + \mu G(r) = 0, \tag{3.47}$$

where $'$ means $\frac{d}{dr}$.

The parameters $a_i,\ b_i\ (i = 1, 2, ..., m)$ and a_0 are constants to be determined, such that $\mu \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (4.60).

Balancing w_{rr} with w^n in (4.60), we obtain an irreducible fraction, $m = \frac{2}{n-1}$, for some n. Therefore we make the following transformation

$$w(r) = h(r)^{\frac{2}{n-1}}, \tag{3.48}$$

and then substitute (3.48) into (4.60) to obtain

$$\begin{aligned}&(n-1)^2(c^2 h(r)^2 + \gamma h(r)^4) - 4\rho c(n-1)h(r)h'(r)\\&+\beta(2(3-n)h'(r)^2 + 2(n-1)h(r)h''(r)) = 0.\end{aligned} \tag{3.49}$$

Now balancing $h(r)^4$ and $h(r)h''(r)$ we find $m=1$. Thus, we assume that

$$h(r) = a_0 + a_1\left(\frac{G'}{G}\right) + b_1\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}. \tag{3.50}$$

Substituting (3.50) into (3.49) and using (3.47), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)^k$ and $\left(\frac{G'}{G}\right)^k\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$ together, and equating each coefficient of them to zero, yield a set of algebraic equations for a_0, a_1, b_1 and μ:

$(i)\left(\frac{G'}{G}\right)^0$: $a_0^2(n-1)^2\left(6b_1^2\gamma v + c^2\right) - 2\beta a_1^2\mu^2(n-3) + 4a_1a_0\rho c\mu(n-1) + a_0^4\gamma(n-1)^2$
$+\ b_1^1(n-1)v\left(2\beta\mu + b_1^2\gamma(n-1)v + c^2(n-1)\right) = 0,$

$(ii)\left(\frac{G'}{G}\right)^1$: $2(n-1)\left(a_1a_0\left(2\beta\mu + 6b_1^2\gamma(n-1)v + c^2(n-1)\right) + 2\rho c\left(a_1^2\mu + b_1^2 v\right)\right.$
$\left. +\ 2a_1a_0^3\gamma(n-1)\right) = 0,$

$(iii)\left(\frac{G'}{G}\right)^2$: $\frac{1}{\mu}a_1^2\mu\left(8\beta\mu + 6b_1^2\gamma(n-1)^2v + c^2(n-1)^2\right) + 6a_0^2\gamma(n-1)^2\left(a_1^2\mu + b_1^2v\right)$
$+\ 4a_1a_0\rho c\mu(n-1) + b_1^2v\left(2b_1^2\gamma(n-1)^2v + 4\beta\mu n + c^2(n-1)^2\right) = 0,$

$(iv)\left(\frac{G'}{G}\right)^3$: $(n-1)\left(a_0a_1\left(\beta\mu + a_1^2\gamma\mu(n-1) + 3b_1^2\gamma(n-1)v\right) + \rho c\left(a_1^2\mu + b_1^2v\right)\right)$
$= 0,$

$(v)\left(\frac{G'}{G}\right)^4$: $\frac{1}{\mu^2}2a_1^2\mu\left(3b_1^2\gamma(n-1)^2v + \beta\mu(n+1)\right) + a_1^4\gamma\mu^2(n-1)^2 + b_1^2v\left(b_1^2\gamma(n-1)^2v\right.$
$\left. +\ 2\beta\mu(n+1)\right) = 0,$

$(vi)\left(\frac{G'}{G}\right)^0\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$: $2b_1(n-1)\left(2a_1\rho c\mu + a_0\left(\beta\mu + 2b_1^2\gamma(n-1)v + c^2(n-1)\right)\right.$
$\left. +2a_0^3\gamma(n-1)\right) = 0,$

$(vii)\left(\frac{G'}{G}\right)^1\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$: $2b_1\left(a_1\left(2b_1^2\gamma(n-1)^2v + \beta\mu(n+3) + c^2(n-1)^2\right)\right.$
$\left. +2a_0\rho c(n-1) + 6a_1a_0^2\gamma(n-1)^2\right) = 0,$

$(viii)\left(\frac{G'}{G}\right)^2\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$: $b_1(n-1)\left(2a_1\rho c\mu + a_0\left(\beta\mu + 3a_1^2\gamma\mu(n-1)\right.\right.$
$\left.\left. +b_1^2\gamma(n-1)v\right)\right) = 0,$

$(ix)\left(\frac{G'}{G}\right)^3\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$: $a_1b_1\left(a_1^2\gamma\mu(n-1)^2 + b_1^2\gamma(n-1)^2v + \beta\mu(n+1)\right)$
$= 0.$

Solving the resultant algebraic equations, we obtain the following results:

Case 1:

$$\begin{aligned} b_1 &= 0,\ a_0 = \pm\sqrt{\frac{-c^2}{4\gamma}},\ a_1 = \pm\frac{4(n+1)\rho}{(n-1)(n+3)}\sqrt{\frac{-1}{\gamma}},\ \mu = -\frac{(n+3)^2(n-1)^2c^2}{64(n+1)^2\rho^2}, \\ \beta &= \frac{8\rho^2(n+1)}{(n+3)^2}. \end{aligned} \tag{3.51}$$

Since $\mu < 0$, from equations (3.48), (3.50) and (3.51), when $\beta = \dfrac{8\rho^2(n+1)}{(n+3)^2}$ the GB equation (3.1) has the following solution:

$$u_1 = \left(\pm\sqrt{\frac{-c^2}{4\gamma}}\left(1 \pm \frac{A\cosh(\sqrt{-\mu}(x-ct)) + B\sinh(\sqrt{-\mu}(x-ct))}{B\cosh(\sqrt{-\mu}(x-ct)) + A\sinh(\sqrt{-\mu}(x-ct))}\right)\right)^{\frac{2}{n-1}}, \tag{3.52}$$

where $A,\ B$ are arbitrary constants.

Case 2:

$$\begin{aligned} &a_0 = \pm\sqrt{\frac{-c^2}{4\gamma}},\ a_1 = \pm\frac{2(n+1)\rho}{(n-1)(n+3)}\sqrt{\frac{-1}{\gamma}},\ b_1 = \pm\sqrt{\frac{c^2}{4\nu\gamma}},\ \mu = -\frac{(n+3)^2(n-1)^2c^2}{16(n+1)^2\rho^2}, \\ &\beta = \frac{8\rho^2(n+1)}{(n+3)^2}. \end{aligned} \tag{3.53}$$

This case leads to the following solution

$$\begin{aligned} u_2 = \Bigg(\pm\sqrt{\frac{c^2}{4\gamma}}\Bigg(i \pm i\frac{A\cosh(\sqrt{-\mu}(x-ct)) + B\sinh(\sqrt{-\mu}(x-ct))}{B\cosh(\sqrt{-\mu}(x-ct)) + A\sinh(\sqrt{-\mu}(x-ct))} \\ + \sqrt{1 - \left(\frac{A\cosh(\sqrt{-\mu}(x-ct)) + B\sinh(\sqrt{-\mu}(x-ct))}{B\cosh(\sqrt{-\mu}(x-ct)) + A\sinh(\sqrt{-\mu}(x-ct))}\right)^2}\Bigg)\Bigg)^{\frac{2}{n-1}} \end{aligned} \tag{3.54}$$

Case 3:

$$a_0 = a_1 = \rho = 0,\ b_1 = \pm\sqrt{\frac{-(n+1)c^2}{2\nu\gamma}},\ \mu = \frac{(n-1)^2c^2}{4\beta}. \tag{3.55}$$

From (3.55), the solutions of (3.1) are as follows:

$$\begin{aligned} u_3 &= \left(\frac{-(n+1)c^2}{2\gamma}\left(1 - \left(\frac{A\cosh(\sqrt{-\mu}(x-ct)) + B\sinh(\sqrt{-\mu}(x-ct))}{B\cosh(\sqrt{-\mu}(x-ct)) + A\sinh(\sqrt{-\mu}(x-ct))}\right)^2\right)\right)^{\frac{1}{n-1}}, \\ &\beta < 0, \end{aligned} \tag{3.56}$$

$$\begin{aligned} u_4 &= \left(\frac{-(n+1)c^2}{2\gamma}\left(1 + \left(\frac{A\cos(\sqrt{\mu}(x-ct)) - B\sin(\sqrt{\mu}(x-ct))}{B\cos(\sqrt{\mu}(x-ct)) + A\sin(\sqrt{\mu}(x-ct))}\right)^2\right)\right)^{\frac{1}{n-1}}, \\ &\beta > 0. \end{aligned} \tag{3.57}$$

Remark: These results are a generalisation of those covered in [82, 45]. In particular, the cases $A = 0,\ B \neq 0$ and $A \neq 0,\ B = 0$, with $2\rho = \alpha,\ \mu = -\dfrac{(n+3)^2(n-1)^2c^2}{4(n+1)^2\alpha^2}$ in (3.52)–(3.57), contain the results of Chen *et al.* [45], who applied the extended-tanh method developed by Fan [20] to explore some exact solutions of the GB (3.1) equation.

Further, if $n = 3,\ A = 0,\ B \neq 0,\ 2\rho = \alpha,\ \mu = -\dfrac{(n+3)^2(n-1)^2c^2}{4(n+1)^2\alpha^2}$, then (3.52) becomes

$$u_1 = \pm\sqrt{\frac{-c^2}{4\gamma}}\left(1 \pm \tanh(\sqrt{-\mu}(x-ct))\right). \tag{3.58}$$

This is a form of solitary wave solution of the GB equation (3.1) obtained by Yan *et al.* [82], who used both the direct method by Clarkson and Kruskal [16, 16] and the improved direct method by Lou [46].

3.4 Double reduction and exact solutions of a system of VB equations

The conservation laws of the system (3.2) are given by [22]

$$(T_1^t, T_1^x) = (v, u + \frac{1}{2}v^2) \tag{3.59}$$

$$(T_2^t, T_2^x) = (v, uv + v_{xx}) \tag{3.60}$$

$$(T_3^t, T_3^x) = (uv, \frac{1}{2}u^2 + uv^2 - \frac{1}{2}v_x^2 + vv_{xx}), \tag{3.61}$$

with the corresponding multipliers

$$\begin{aligned} Q_1 &= [0,1] \\ Q_2 &= [1,0] \\ Q_3 &= [v,u]. \end{aligned}$$

We apply the double reduction to the conserved vector T^3 in equation (3.61) to investigate exact solutions of the system. Equation (3.2) admits the four Lie point symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t} \\ X_2 &= \frac{\partial}{\partial x} \\ X_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial v} \\ X_4 &= \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v}. \end{aligned} \tag{3.62}$$

It can be easily shown that X_1 and X_2 are associated with the conserved vector $T_3 = (T_3^t, T_3^x)$ in equation (3.61). We consider a linear combination $X = X_1 + cX_2$ and transform this generator to its canonical form via

$$r = x - ct, s = t, q(r) = v, w(r) = u. \tag{3.63}$$

The components of the reduced conserved form are given by (1.26), (1.27)

$$T_r^3 = cwq - \frac{1}{2}w^2 - wq^2 + \frac{1}{2}q_r^2 - qq_{rr} \tag{3.64}$$

$$T_s^3 = -wq, \tag{3.65}$$

where the reduced conserved form satisfies

$$D_r T_3^r = 0. \tag{3.66}$$

Thus, the double reduced equation is given as

$$cwq - \frac{1}{2}w^2 - wq^2 + \frac{1}{2}q_r^2 - qq_{rr} = k_1, \tag{3.67}$$

where k_1 is a constant. Differentiating (3.67) implicitly with respect to r results in

$$cw_r q + cwq_r - ww_r - w_r q^2 - 2wqq_r - qq_{rrr} = 0. \tag{3.68}$$

Since the multipliers of the conserved vector (3.61) are $q_1 = v$ and $q_2 = u$, we can also obtain a reduced conserved form for the equation

$$v(u_t + vu_x + v_{xxx}) - u(v_t + uu_x - vv_x) = 0. \tag{3.69}$$

The above equation, (3.69), in the canonical variables (3.63) is given as

$$cqw_r - q^2 w_r - qq_{rrr} - cwqr + ww_r = 0. \tag{3.70}$$

Substituting for q_{rrr} from (3.68) into (3.70) yields the first order ODE

$$cq_r - w_r - qq_r = 0. \tag{3.71}$$

Integrating (3.71) with respect to r results in

$$w = cq - \frac{1}{2}q^2 + k_2, \tag{3.72}$$

where k_2 is a constant of integration. The substitution of (3.72) into (3.67) gives

$$qq_{rr} - \frac{1}{2}q_r^2 - \frac{3}{8}q^4 + cq^3 - \frac{1}{2}q^2\left(c^2 - k_2\right) + \frac{1}{2}(k_2^2 + 2k_1) = 0. \tag{3.73}$$

Using further symmetry analysis, this ODE (3.73) can be reduced to the quadrature

$$\int \frac{dq}{\sqrt{\frac{1}{4}q^4 - cq^3 + q^2(c^2 - k) + k_3 q + k_2^2 + 2k_1}} = r + k_4, \tag{3.74}$$

where k_3, k_4 are constants of integration.

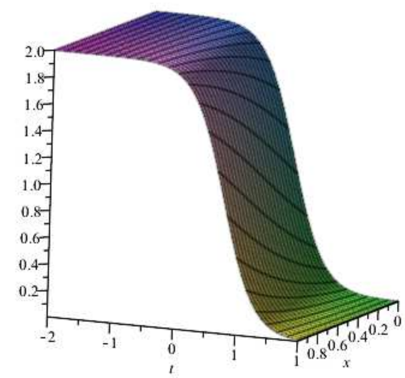
3.5 Discussion

The double reduction theory based on the association of Lie point symmetries and conservation laws was utilised to construct new exact solutions of the GB equation and a system of VB equations. Firstly, the GB equation was considered and the conservation laws were computed via the partial Noether's approach. The derived conserved vectors failed to satisfy the divergence relation due to the presence of the mixed derivative term. The conserved vectors were then adjusted to absorb the extra term. As a result new forms of the conserved vectors satisfying the divergence condition were found. To the best of our knowledge, these conserved vectors have not been reported in the literature.

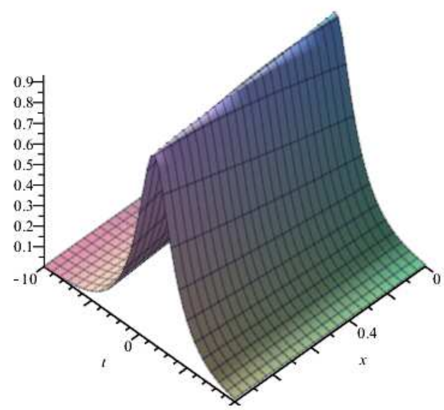
The importance of these conservation laws was illustrated by finding several exact travelling wave solutions for the GB equation through the application of the double reduction method.

The solutions obtained behave as solitary and periodic waves for different values of special parameters involved. The important kink solitary waves, bell shaped solitary waves and periodic travelling waves can be obtained from the solutions (3.52), (3.56) and (3.57) respectively as shown in Figure 4.1. We have shown that our results were not only a generalisation of the work previously done by some authors but also contain some new exact solutions.

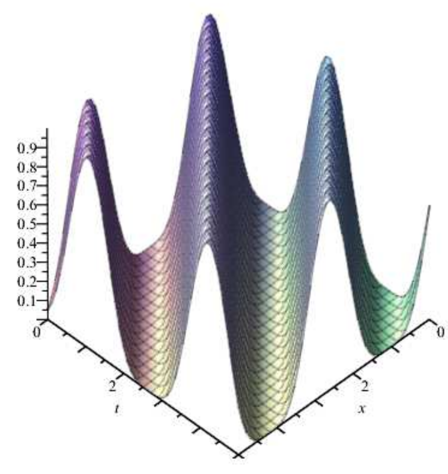
A similar analysis is carried out to obtain new exact solutions for a system of VB equations. It is worthy to note that the solutions of the system of VB equations presented in [74] using the $(\frac{G'}{G})$ – expansion method can also be obtained from the resultant reduced equations of the system found here via double reduction theory. Note, however, that our approach is simpler than that used in [73]. Hence, the double reduction method is an effective and convenient method which allows us to solve certain complicated nonlinear differential equations in mathematical physics. The new solutions presented in this paper may be used to study disturbance or wave propagation problems in fluid mechanics and space plasma physics.



(a)



(b)



(c)

Figure 3.1: (a) Kink shaped solitary wave solution for u_1, with $n = 3, c = 2, \gamma = -1, \mu = -1, \rho = \frac{3}{4}, \beta = \frac{1}{2}, A = 1, B = 2$; (b) Bell shaped solitary wave solution of u_3, with $n = 5, c = 1, \gamma = -3, \mu = -1, \rho = 0, \beta = -4, A = 1, B = 2$; (c) Periodic wave solution of u_4, with $n = \frac{1}{2}, c = 1, \gamma = -\frac{3}{4}, \mu = 1, \rho = 0, \beta = \frac{1}{16}, A = B = 1$.

Chapter 4

A group theoretic analysis of the generalised Gardner equation with arbitrary order nonlinear terms

In this chapter, we consider a nonlinear evolution equation (NLEE) that is widely used in various branches of physics, such as solid-state physics, plasma physics, fluid physics and quantum field theory and which describes a variety of wave phenomena in plasma and solid state. This is the generalised form of the Gardner equation with nonlinear terms of any order given by [22] [45]

$$u_t + (a + bu^n + cu^{2n})u_x + ku_{xxx} = 0. \tag{4.1}$$

The profile $u(t, x)$ is the amplitude of the relevant wave mode and variables x and t represent spatial and temporal variables respectively. The term u_t is the evolution term while b and c represent the coefficients of the dual-power law nonlinearities. Then a and k are the coefficients of dispersion terms. The parameter $n \geq 1$ represents the power law nonlinearity parameter. This nonlinearity is introduced in a generalised setting that could represent the regular Korteweg-de Vries (KdV) equation or the modified Korteweg-de Vries (mKdV) depending on the values of n and the constants of the equation. For example, when $n = 1, b \neq 0, c \neq 0$, we get the famous combined KdV-mKdV equation [22] given by,

$$u_t + (a + bu + cu^2)u_x + ku_{xxx} = 0. \tag{4.2}$$

When $n = 1, b \neq 0, c = 0$, we obtain the KdV equation [77],

$$u_t + (a + bu)u_x + ku_{xxx} = 0, \tag{4.3}$$

and when $n = 1, b = 0, c \neq 0$ yields the mKdV equation

$$u_t + (a + cu^2)u_x + ku_{xxx} = 0. \tag{4.4}$$

The Gardner equation (4.1) has been investigated as a generalised KdV-mKdV equation with high-order nonlinear terms [45] and the Benjamin-Bona-Mahoney (BBM) equation, with dual-power law nonlinearity [6]. Different versions of equation (4.1) have been studied for exact solutions [7], [52], [76], [75].

Most of these studies are based on the "travelling wave" type solutions via some well known substitutions. Here the solutions will be obtained via the Lie symmetry approach and a double reduction which involves the association of symmetries with conservation laws. As such association is not limited to the travelling wave symmetries, there exist some possibilities of additional solutions different from travelling wave solutions. This will be investigated with the scaling symmetries which might lead to new exact solutions. Finally it is worth noting that the improved generalised Riccati equation mapping method leads to important singular soliton solutions.

The organization of the chapter is as follows: In the next Section, we present a Lie symmetry analysis of the Gardner equation (4.1). In Section 4.2, the conservation laws of the equation are derived. In Section 4.3 the exact solutions of the equation are discussed. Concluding remarks are presented in Section 4.4.

4.1 Lie Symmetries of the Gardner equation

The Lie point symmetries admitted by (4.1) are generated by a vector field of the form

$$X = \xi^1(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{4.5}$$

The operator X satisfies the Lie symmetry condition [61], [9]

$$X^{[3]}\left[u_t + (a + bu^n + cu^{2n})u_x + ku_{xxx}\right]\big|_{(4.1)} = 0, \tag{4.6}$$

where $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{xxx} \frac{\partial}{\partial u_{xxx}}. \tag{4.7}$$

Expansion and separation of (4.6) with respect to the powers of different derivatives of u yields an over determined system in the unknown coefficients ξ^1, ξ^2 and η. Solving the overdetermined system for arbitrary parameters gives the 2-dimensional trivial Lie algebra spanned by the vector fields of translation with respect to the independent variables

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}. \tag{4.8}$$

The Lie algebra is extended in the following three cases:

Case (1) $n \neq 1, \;\; c = 0, \;\; b \neq 0, \;\; k \neq 0$. This condition reduces (4.1) to the generalised version of the KdV equation

$$u_t + (a + bu^n)u_x + ku_{xxx} = 0. \tag{4.9}$$

The additional operator obtained in this case is

$$X_3 = -3nt\frac{\partial}{\partial t} - (2ant + nx)\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{4.10}$$

Case (2) $n \neq 1, \;\; c \neq 0, \;\; b = 0, \;\; k \neq 0$. In this case (4.1) becomes the generalised MKdV equation

$$u_t + (a + cu^{2n})u_x + ku_{xxx} = 0. \tag{4.11}$$

The additional operator is

$$X_3 = -3nt\frac{\partial}{\partial t} - (2ant + nx)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \tag{4.12}$$

Case (3) $n = 1, \;\; c \neq 0, \;\; b \neq 0, \;\; k \neq 0$. This case corresponds to the combined KdV-mKdV equation given in (4.2). The additional operator is

$$X_3 = -6ct\frac{\partial}{\partial t} + (b^2t - 4act - 2cx)\frac{\partial}{\partial x} + (b + 2cu)\frac{\partial}{\partial u}. \tag{4.13}$$

The Lie symmetry algebra for Cases (1) and (2) exhibits the same commutative property. The commutator table for both these cases are presented in Table 4.1 while the commutator table for Case (3) is given in Table 4.2.

Table 4.1: Commutator table of symmetry operators of (4.1)for Case 1 & Case (2)

[,]	X_1	X_2	X_3
X_1	0	0	$-3nX_1 - 2anX_2$
X_2	0	0	$-nX_2$
X_3	$3nX_1 + 2anX_2$	nX_2	0

Table 4.2: Commutator table of symmetry operators of (4.1)for Case (3)

[,]	X_1	X_2	X_3
X_1	0	0	$(b^2 - 4ac)\, X_2 - 6cX_1$
X_2	0	0	$-2cX_2$
X_3	$6cX_1 - (b^2 - 4ac)\, X_2$	$2cX_2$	0

4.2 Conservation laws of the Gardner equation

A conserved vector corresponding to a conservation law of the Gardner equation (4.1) is a $2-$tuple (T^t, T^x) , such that

$$D_t T^t + D_x T^x = 0 \tag{4.14}$$

along the solutions of the equation. The Gardner equation (4.1) is a third order partial differential equation and its conservation laws cannot be computed directly by the Noether or partial Noether approach. Hence, the conservation laws for (4.1) will be derived by two methods namely (i) the multiplier approach and (ii) Noether approach (after increasing its order by one).

(i) The multiplier approach: Consider the multiplier Λ of order up to two, viz., $\Lambda = (t, x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$ for (4.1). The conserved vector (T^t, T^x) of (4.1) satisfies the divergence relation

$$D_tT^t + D_xT^x = \Lambda(u_t + (a + bu^n + cu^{2n})u_x + ku_{xxx}) = 0. \tag{4.15}$$

Moreover, we have

$$\frac{\delta}{\delta u}\Lambda(u_t + (a + bu^n + cu^{2n})u_x + ku_{xxx}) = 0. \tag{4.16}$$

After a lengthy calculation, we observe that Λ is of order zero or two in derivatives with respect to u. The obtained forms of $\Lambda = \Lambda_i$ are

$$\Lambda_1 = 1, \tag{4.17}$$

$$\Lambda_2 = u, \tag{4.18}$$

$$\Lambda_3 = \frac{c}{k(2n+1)}u^{2n+1} + \frac{b}{k(n+1)}u^{n+1} + u_{xx}. \tag{4.19}$$

For $n = 1$, we obtain in addition to the three multipliers (4.17)−(4.19) above the following multipliers

$$\begin{aligned}
\Lambda_4 &= (2cu + b)(ta - x + tu(b + cu)) + 6cktu_{xx}, \\
\Lambda_5 &= -\frac{1}{36k}(6c^2u^5 + 15bcu^4 + 12acu^3 + 10b^2u^3 + 24cku^2u_{xx} \\
&\quad - 12ckuu_x^2 + 18abu^2 + 24bkuu_{xx} - 6bku_x^2 - 36ku_{tx}), \\
\Lambda_6 &= -\frac{1}{144c^2k}\left(40c^4u^7 + 140bc^3u^6 + 144a^5c^3u + 174b^2c^2u^5 + 264c^3ku^4uxx + 85b^3cu^4\right. \\
&\quad -48c^3ku^3u_x^2b + 360abc^2 + 528bc^2ku^3u_{xx} - 72bc^2ku^2u_x^2 + 144a^2c^2u^3 + 252ab^2cu^3 \\
&\quad +10b^4u^3 + 432ac^2ku^2u_{xx} + 288b^2cku^2u_{xx} - 36b^2ckuu_x^2 + 432c^2k^2uu_{xx}^2 - 144c^2k^2u_x^2 \\
&\quad -144c^2ku^2u_{tx} + 216a^2bcu^2 + 18a^2b^3u^2 + 288c^2kuu_tu_x + 432abckuu_{xx} + 24b^3kuu_{xx} \\
&\quad \left.-6b^3ku_x^2 + 216bck^2u_{xx}^2 - 144bckuu_{tx} + 144bcku_tu_x - 36b^2ku_{tx} + 432cku_{tt}\right).
\end{aligned} \tag{4.20}$$

Now, we calculate the corresponding conserved density T^t and flux T^x by substituting (4.17)−(4.20) into (4.15). For Λ_1 we obtain the obvious conserved vector

$$\begin{aligned}
T_1^t &= u, \\
T_1^x &= \frac{c}{2n+1}u^{2n+1} + \frac{b}{n+1}u^{n+1} + au + ku_{xx}.
\end{aligned} \tag{4.21}$$

Λ_2 gives rise to

$$\begin{aligned} T_2^t &= \frac{1}{2}u^2, \\ T_2^x &= \frac{c}{2n+2}u^{2n+2} + \frac{b}{n+2}u^{n+2} + kuu_{xx} - \frac{1}{2}ku_x^2 + \frac{1}{2}au^2, \end{aligned} \tag{4.22}$$

whereas Λ_3 results in

$$\begin{aligned} T_3^t &= \frac{1}{k(2n+1)(2n+2)}cu^{2n+2} + \frac{1}{k(n+1)(n+2)}bu^{n+2} + \frac{1}{2}uu_{xx}, \\ T_3^x &= \frac{1}{2k(2n+1)^2}c^2u^{4n+2} + \frac{1}{k(2n+1)(n+1)}bcu^{3n+2} + \frac{1}{k(2n+1)(2n+2)}acu^{2n+2} \\ &+ \frac{1}{2k(n+1)^2}b^2u^{2n+2} + \frac{1}{2n+1}cu^{2n+1}u_{xx} + \frac{1}{k(n+1)(n+2)}abu^{n+2} \\ &+ \frac{1}{n+1}bu^{n+1}u_{xx} + \frac{1}{2}au_x^2 + \frac{1}{2}ku_{xx}^2 + \frac{1}{2}u_tu_x - \frac{1}{2}uu_{tx}. \end{aligned} \tag{4.23}$$

We construct $(T_4^t, T_4^x) - (T_6^t, T_6^x)$ using the same approach.

(ii) The Noether approach: To apply Noether's method, we increase the order of equation (4.1) by one. Let $u = U_x$, then equation (4.1) becomes

$$U_{tx} + (a + bU_x^n + cU_x^{2n})U_x + kU_{xxxx} = 0. \tag{4.24}$$

A standard Lagrangian for equation (4.24) is

$$L = \frac{1}{2}U_tU_x + \left(\frac{1}{2}a + \frac{1}{(n+1)(n+2)}bU_x^n + \frac{U_x^2n}{(2n+1)(2n+2)}\right)U_x^2 - \frac{1}{2}kU_{xx}^2 \tag{4.25}$$

and the associated Euler-Lagrange equation is

$$\frac{\delta L}{\delta U} = 0,$$

where $\dfrac{\delta L}{\delta U}$ is defined by

$$\frac{\delta L}{\delta U} = \frac{\partial}{\partial U} - D_t\frac{\partial}{\partial U_t} - D_x\frac{\partial}{\partial U_x} + D_t^2\frac{\partial}{\partial U_{tt}} + D_x^2\frac{\partial}{\partial U_{xx}} + D_xD_t\frac{\partial}{\partial U_{tx}}, \tag{4.26}$$

and

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U}, \\ D_x &= \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + U_{xx}\frac{\partial}{\partial U_x}. \end{aligned}$$

The Lie-Bäcklund operator X defined in (4.5) is a Noether operator corresponding to the Lagrangian L of equation (4.25) if it satisfies [34]

$$X(L) + L(D_t\xi^1 + D_x\xi^2) = D_tB^1 + D_xB^2, \tag{4.27}$$

where $B^1(t,x,U), B^2(t,x,U)$ are the gauge terms. The conserved vectors for the second−order Lagrangian L (4.25) is given by (1.19)

$$\begin{aligned} T^t &= B^1 - L\xi^1 - w\frac{\delta L}{\delta U_t} - D_t(w)\frac{\delta L}{\delta Utt} - D_x(w)\frac{\delta L}{\delta U_{tx}}, & (4.28)\\ T^x &= B^2 - L\xi^2 - w\frac{\delta L}{\delta U_x} - D_t(w)\frac{\delta L}{\delta Utx} - D_x(w)\frac{\delta L}{\delta U_{xx}}, & (4.29)\end{aligned}$$

where $w = \eta - U_t\xi^1 - U_x\xi^2$ is the characteristic function.

Substituting (4.25) into (4.27) and solving yields the following Noether coefficient functions and gauge terms

$$\begin{gathered} \xi^1 = c_1,\ \xi^2 = c_2,\ \eta = m(t),\ B^1 = f(t,x), \\ B^2 = \frac{1}{2}m_t(t) + g(t,x),\ f_t(t,x) + g_x(t,x) = 0, \end{gathered} \tag{4.30}$$

where $c_1,\ c_2,\ c_3$ are arbitrary constants and $m(t),\ f(t,x),\ g(t,x)$ are arbitrary gauge functions. We set $f(t,x)$ and $g(t,x)$ to zero as they contribute to the trivial part of the conserved vector. This yields the following conserved vectors for the Gardner equation (4.1):

$$\begin{aligned} T_1^t &= \frac{1}{2}au^2 + \frac{1}{(n+1)(n+2)}bu^{n+2} + \frac{1}{(2n+1)(2n+2)}cu^{2n+2} - \frac{1}{2}ku_x^2, \\ T_1^x &= -\frac{1}{2}\left(\int u_t dx\right)^2 - \left(au + \frac{1}{(n+1)}bu^{n+1} + \frac{1}{2n+1}cu^{2n+1} + ku_{xx}\right)\int u_t dx \\ &\quad + ku_tu_x, & (4.31)\\ T_2^t &= \frac{1}{2}u^2, \\ T_2^x &= \frac{1}{2}au^2 + \frac{1}{n+2}bu_x^{n+2} + \frac{1}{2n+2}cu_x^{2n+2} + kuu_{xx} - \frac{1}{2}ku_x^2, & (4.32)\end{aligned}$$

and for the arbitrary function $m(t)$

$$\begin{aligned} T_m^t &= m(t)u - D_x\frac{1}{2}\int um(t)dx, \\ T_m^x &= m(t)\left(au + \frac{1}{n+1}u^{n+1} + \frac{1}{2n+1}cu^{2n+1} + ku_{xx}\right) \end{aligned}$$

$$- m_t(t)\int u dx + D_t\frac{1}{2}\int um(t)dx. \tag{4.33}$$

Remark: The local conserved vector (4.32) is the same as the conserved vector (4.22) obtained using the multiplier approach while the nonlocal conserved vector (4.31) and (4.33) cannot be obtained using the multiplier approach. For $m(t) = 1$, the nonlocal conserved vector (4.33) gives a local conserved vector

$$\begin{aligned} T_3^t &= u, \\ T_3^x &= au + \frac{1}{n+1}bu^{n+1} + \frac{1}{2n+1}cu^{2n+1} + ku_{xx}. \end{aligned} \tag{4.34}$$

This coincides with the local conserved vector obtained in (4.21). Hence, for arbitrary value of $m(t)$ infinitely many nonlocal conservation laws exist for the Gardner equation (4.1).

4.3 Exact solutions of the Gardner equation

In this section we will discuss exact solutions of the Gardner equation (4.1) via the Lie point symmetry approach, double reduction method and improved generalised Riccati mapping method.

4.3.1 Symmetry reduction and exact solutions of Gardner equation (4.1)

A linear combination of the symmetries (4.8): $X = X_1 + \lambda X_2$, where λ is a constant can be used to find travelling wave solutions. The similarity variables for X obtained by solving the characteristic equation

$$\frac{dt}{1} = \frac{dx}{\lambda} = \frac{du}{u} \tag{4.35}$$

are given by

$$s = t, r = x - \lambda t, w(r) = u. \tag{4.36}$$

These similarity variables reduce the Gardner equation (4.1) to the third order ODE

$$(a-\lambda)w_r + cw_r w^{2n} + bw_r w^n + kw_{rrr} = 0. \tag{4.37}$$

Integrating (4.37) twice with respect to r yields

$$\begin{aligned} &\frac{(a-\lambda)}{2}w^2 + \frac{c}{(2n+1)(2n+2)}w^{2n+2} + \frac{b}{(n+1)(n+2)}w^{n+2} + \frac{k}{2}w_r^2 \\ &= wk_1 + k_2, \end{aligned} \tag{4.38}$$

where k_1, k_2 are constants of integration. The solution for (4.38) in terms of the original variables is

$$\int \frac{du}{\sqrt{\frac{\lambda-a}{k}u^2 - \frac{2c}{k(2n+1)(2n+2)}u^{2n+2} - \frac{2b}{k(n+1)(n+2)}u^{n+2} - \frac{2k_1}{k}u + k_2}} = x - \lambda t + k_3, \tag{4.39}$$

where k_3 is a constant of integration. Setting the constants of integration to zero in (4.39) and simplifying we obtained the following solutions to the Gardner equation (4.1)

$$\begin{aligned} u_1(t,x) &= \left\{ \frac{(\lambda-a)(n+1)(n+2)sech\left(n\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)}{\pm\sqrt{\frac{b^2(2n+1)+c(\lambda-a)(n+1)(n+2)^2}{2n+1}} + bsech\left(n\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)} \right\}^{\frac{1}{n}}, \lambda > a, \\ u_2(t,x) &= \left\{ \frac{(\lambda-a)csch^2\left(\frac{n}{2}\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)}{2\left(\frac{-b}{(n+1)(n+2)} \pm \sqrt{\frac{(a-\lambda)c}{(2n+1)(n+1)}}\coth\left(\frac{n}{2}\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)\right)} \right\}^{\frac{1}{n}}, \lambda > a, \\ u_3(t,x) &= \left\{ \frac{(a-\lambda)csc^2\left(\frac{n}{2}\sqrt{\frac{a-\lambda}{k}}(x-\lambda t)\right)}{2\left(\frac{-b}{(n+1)(n+2)} \pm \sqrt{\frac{(a-\lambda)c}{(2n+1)(n+1)}}\cot\left(\frac{n}{2}\sqrt{\frac{a-\lambda}{k}}(x-\lambda t)\right)\right)} \right\}^{\frac{1}{n}}, \lambda < a. \end{aligned} \tag{4.40}$$

Symmetry reduction and exact solutions of the special cases of the Gardner equation (4.1)

In this subsection, we perform the symmetry reductions and construct exact group-invariant solutions of the special cases of the Gardner equation (4.1) discussed in Section 4.2.

Case (1) $n \neq 1, \quad c = 0, \quad b \neq 0, \quad k \neq 0.$

1.(a) : $X_1 + \lambda X_2$ (Travelling wave solutions): Here the travelling wave solution is obtained by setting $c = 0$ in (4.39) and is given as

$$\int \frac{du}{\sqrt{\frac{\lambda-a}{k}u^2 - \frac{2b}{k(n+1)(n+2)}u^{n+2} - \frac{2k_1}{k}u + k_2}} = x - \lambda t + k_3. \tag{4.41}$$

Setting the constants of integration k_1, k_2, k_3 in (4.41) to zero gives

$$\begin{aligned}
u_{1^1}(t,x) &= \left\{\frac{(\lambda-a)}{2b}(n+1)(n+2)sech^2\left(\frac{n}{2}\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)\right\}^{\frac{1}{n}}, \lambda > a,\\
u_{2^1}(t,x) &= \left\{\frac{(a-\lambda)}{2b}(n+1)(n+2)csch^2\left(\frac{n}{2}\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)\right\}^{\frac{1}{n}}, \lambda > a,\\
u_{3^1}(t,x) &= \left\{\frac{(\lambda-a)}{2b}(n+1)(n+2)csc^2\left(\frac{n}{2}\sqrt{\frac{a-\lambda}{k}}(x-\lambda t)\right)\right\}^{\frac{1}{n}}, \lambda < a.
\end{aligned} \tag{4.42}$$

1.(b) : X_3: The symmetry generator X_3 in (4.10) gives rise to the invariants

$$u(x,t) = w(r)t^{-\frac{2}{3n}}, \quad r = xt^{-\frac{1}{3}} - at^{\frac{2}{3}}. \tag{4.43}$$

Substitution of (4.43) into equation (4.9) results in the ODE

$$3nkw_{rrr} + 3nw^n w_r - nrw_r - 2w = 0. \tag{4.44}$$

Case 2 $n \neq 1, \quad c \neq 0, \quad b = 0, \quad k \neq 0.$

2.(a) : $X_1 + \lambda X_2$ (Travelling wave solutions): Here the travelling wave solution is obtained by setting $b = 0$ in (4.39) and is given by

$$\int \frac{du}{\sqrt{\frac{\lambda-a}{k}u^2 - \frac{2c}{k(2n+1)(2n+2)}u^{2n+2} - \frac{2k_1}{k}u + k_2}} = x - \lambda t + k_3. \tag{4.45}$$

Setting the constants of integration k_1, k_2, k_3 in (4.41) to zero gives

$$\begin{aligned}
u_{1^2}(t,x) &= \left\{\pm\sqrt{\frac{(\lambda-a)(2n+1)(n+1)}{c}}sech\left(n\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)\right\}^{\frac{1}{n}}, \lambda > a,\\
u_{2^2}(t,x) &= \left\{\pm\sqrt{\frac{(a-\lambda)(2n+1)(n+1)}{c}}csch\left(n\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)\right\}^{\frac{1}{n}}, \lambda > a,\\
u_{3^2}(t,x) &= \left\{\pm\sqrt{\frac{(\lambda-a)(2n+1)(n+1)}{c}}csc\left(n\sqrt{\frac{a-\lambda}{k}}(x-\lambda t)\right)\right\}^{\frac{1}{n}}, \lambda < a.
\end{aligned} \tag{4.46}$$

2.(b) : X_3: The symmetry generator X_3 in (4.12) gives rise to the invariants

$$u(x,t) = w(r)t^{-\frac{1}{3n}}, \quad r = xt^{-\frac{1}{3}} - at^{\frac{2}{3}}. \tag{4.47}$$

Substitution of this solution (4.47) into equation (4.11) results in the ODE

$$3nkw_{rrr} + 3nw^n w_r - nrw_r - w = 0. \tag{4.48}$$

Case (3): $n = 1, \quad c \neq 0, \quad b \neq 0, \quad k \neq 0.$

3.(a) : $X_1 + \lambda X_2$ (Travelling wave solutions): Here the travelling wave solution is obtained by putting $n = 1$ in (4.39) and is given as

$$\int \frac{du}{\sqrt{\frac{\lambda-a}{k}u^2 - \frac{2c}{12k}u^4 - \frac{2b}{6k}u^3 - \frac{2k_1}{k}u + k_2}} = x - \lambda t + k_3. \tag{4.49}$$

Setting the constants of integration k_1, k_2, k_3 in (4.49) to zero gives

$$\begin{aligned}
u_{1^3}(t,x) &= \frac{6(\lambda-a)sech\left(\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)}{\pm\sqrt{b^2+6c(\lambda-a)} + bsech\left(\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)}, \lambda > a,\\
u_{2^3}(t,x) &= \frac{(\lambda-a)csch^2\left(\frac{1}{2}\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)}{2\left(\frac{-b}{6} \pm \sqrt{\frac{(a-\lambda)c}{6}}\coth\left(\frac{1}{2}\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)\right)}, \lambda > a,\\
u_{3^3}(t,x) &= \frac{(a-\lambda)csc^2\left(\frac{1}{2}\sqrt{\frac{a-\lambda}{k}}(x-\lambda t)\right)}{2\left(\frac{-b}{6} \pm \sqrt{\frac{(a-\lambda)c}{6}}\cot\left(\frac{1}{2}\sqrt{\frac{a-\lambda}{k}}(x-\lambda t)\right)\right)}, \lambda < a.
\end{aligned} \tag{4.50}$$

3.(b) : X_3: The symmetry generator X_3 in (4.13) gives rise the invariants

$$u(x,t) = w(r)t^{-\frac{1}{3}} - \frac{b}{2c}, r = xt^{-\frac{1}{3}} - at^{\frac{2}{3}} + \frac{b^2 t^{\frac{2}{3}}}{4c}. \tag{4.51}$$

Substitution of this (4.51) into equation (4.2) results in the ODE

$$3kw_{rrr} + 3cw^2 w_r - rw_r - w = 0. \tag{4.52}$$

Integrating the above equation (4.52) gives

$$3kw_{rr} + cw^3 - rw = k_1, \tag{4.53}$$

where k_1 is a constant of integration. Equation (4.53) is known as the first *Painlevé transcendent*. Its solutions $w = \sigma(r)$ are meromorphic in the entire complex plane, but are essentially new functions that cannot be expressed in any standard form [61].

Remark: It should be noted that the reduced equations (5.13), (4.48), and (4.53) are to some extent highly nonlinear and hence quite challenging to solve analytically. The next logical step would be to look at numerical solutions of the reduced equations.

4.3.2 Exact solutions of the Gardner equation by double reduction theory

For the conserved vectors (4.21), (4.22) and (4.23) we will apply double reduction theory to obtain the doubly reduced Gardner equation (4.1) which is easily solved to find exact solutions. A Lie point symmetry X of the Gardner (4.1) is associated with its conserved vector (T^t, T^x) if (1.23)

$$X_i^{[2]} \begin{pmatrix} T_i^t \\ T_i^x \end{pmatrix} + (D_t\xi_t^1 + D_x\xi_x^2) \begin{pmatrix} T_i^t \\ T_i^x \end{pmatrix} - \begin{pmatrix} D_t\xi_t^1 & D_x\xi_t^1 \\ D_t\xi_x^2 & D_x\xi_x^2 \end{pmatrix} \begin{pmatrix} T_i^t \\ T_i^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.54}$$

Double reduction via X_1, X_2

It is observed from (4.54) that only the Lie point symmetries of the Gardner equation given in (4.8) are associated with the three conserved vectors (4.21), (4.22) and (4.23). Thus we can perform the double reduction by a combination of the two generators,

$$X = X_1 + \lambda X_2. \tag{4.55}$$

Hence, using the canonical coordinates (4.36), formulas (1.26) and (1.27), a double reduction by $T_1 = (T_1^t,\ T_1^x)$ results in the conserved form

$$T_1^r = (\lambda - a)w - \frac{1}{2n+1}cw^{2n+1} - \frac{1}{n+1}bw^{n+1} - kw_{rr}, \tag{4.56}$$

$$T_1^s = -\frac{1}{2n+1}cw^{2n+1} - \frac{1}{n+1}bw^{n+1} - aw - kw_{rr}. \tag{4.57}$$

Since (4.57) does not depend on s, the reduced conserved vector becomes

$$D_r T_1^r = 0, \tag{4.58}$$

which implies that

$$(\lambda-a)w-\frac{1}{2n+1}cw^{2n+1}-\frac{1}{n+1}bw^{n+1}-kw_{rr}=k_1, \tag{4.59}$$

where k_1 is a constant. Equation (4.59) is a second order ODE which is a double reduction of the third order PDE (4.1). Integrating (4.59) twice with respect to r, leads to following solution of the Gardner equation (4.1)

$$\begin{aligned}\int\left(\frac{\lambda-a}{k}u^2-\frac{2c}{k(2n+1)(2n+2)}u^{2n+2}-\frac{2b}{k(n+1)(n+2)}u^{n+2}-\frac{2k_1}{k}u+c_1\right)^{-\frac{1}{2}}du \\ =x-\lambda t+d_1,\end{aligned} \tag{4.60}$$

where $c_1,\;\; d_1$ are constants. Similarly, from (1.26) the reduced form of the conserved vector (3.26) is

$$T_2^r \;\;=\;\; \frac{1}{2}(\lambda-a)w^2-\frac{c}{2n+2}w^{2n+2}-\frac{b}{n+2}w^{n+2}-kww_{rr}+\frac{1}{2}kw_r^2 \tag{4.61}$$

and thus the reduced conserved form $T_2^r=k_2$ gives

$$\frac{1}{2}(\lambda-a)w^2-\frac{c}{2n+2}w^{2n+2}-\frac{b}{n+2}w^{n+2}-kww_{rr}+\frac{1}{2}kw_r^2=k_2 \tag{4.62}$$

where k_2 is a constant. The solution of (4.62) represented by the original variables forms the solution of the Gardner equation (4.1) and is given by

$$\begin{aligned}&\int\left[k(n^2+3n+2)\right]^{\frac{1}{2}}\left[(\lambda-a)n^2u^2+3(\lambda-a)nu^2-2cu^{2n+2}-2bu^{n+2}\right.\\ &\left.+2(\lambda-a)u^2+2k_2n^2+6k_2n+k_2kn^2u+3k_2knu+4k_2+2k_2ku+c_2\right]^{-\frac{1}{2}}du=x-\lambda t+d_2.\end{aligned}$$

For the conserved vector (4.23), we obtain the reduced vector

$$\begin{aligned}T_3^r \;\;=\;\; &\lambda\left[\frac{1}{k(2n+1)(2n+2)}cw^{2n+2}+\frac{1}{k(n+1)(n+2)}bw^{n+2}+\frac{1}{2}ww_{rr}\right]\\ &-\frac{1}{2k(2n+1)^2}c^2w^{4n+2}-\frac{1}{k(2n+1)(n+1)}bcw^{3n+2}-\frac{1}{k(2n+1)(2n+2)}acw^{2n+2}\\ &-\frac{1}{2k(n+1)^2}b^2w^{2n+2}-\frac{1}{2n+1}cw^{2n+1}w_{rr}-\frac{1}{k(n+1)(n+2)}abw^{n+2}\\ &-\frac{1}{n+1}bw^{n+1}w_{rr}-\frac{1}{2}aw_r^2+\frac{1}{2}kw_{rr}^2+\lambda\frac{1}{2}w_r^2+\lambda\frac{1}{2}ww_{rr}.\end{aligned}$$

Hence, a double reduction of the third order PDE (4.1) is

$$\lambda\left[\frac{1}{k(2n+1)(2n+2)}cw^{2n+2}+\frac{1}{k(n+1)(n+2)}bw^{n+2}+\frac{1}{2}ww_{rr}\right]$$

$$
\begin{aligned}
&-\frac{1}{2k(2n+1)^2}c^2w^{4n+2}-\frac{1}{k(2n+1)(n+1)}bcw^{3n+2}-\frac{1}{k(2n+1)(2n+2)}acw^{2n+2}\\
&-\frac{1}{2k(n+1)^2}b^2w^{2n+2}-\frac{1}{2n+1}cw^{2n+1}w_{rr}-\frac{1}{k(n+1)(n+2)}abw^{n+2}\\
&-\frac{1}{n+1}bw^{n+1}w_{rr}-\frac{1}{2}aw_r^2+\frac{1}{2}kw_{rr}^2+\lambda\frac{1}{2}w_r^2+\lambda\frac{1}{2}ww_{rr}=k_3,
\end{aligned}
\tag{4.63}
$$

where k_3 is a constant.

Double reduction via the scaling symmetry operator X_3

Case (1) $n \neq 1, \quad c = 0, \quad b \neq 0, \quad k \neq 0.$

We obtain from (1.23)

$$
X_3^{[2]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - 4n\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} -3n & 0 \\ -2an & -n \end{pmatrix}\begin{pmatrix} T_i^t \\ T_i^x \end{pmatrix} = \begin{pmatrix} (2-n)u \\ (2-n)(bu^{n+1}+au+ku_{xx}) \end{pmatrix},
$$

where $X_3^{[2]}$ is given by

$$
X_3^{[2]} = -3nt\frac{\partial}{\partial t} - (2ant+nx)\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} + (nu_x+2u_x)\frac{\partial}{\partial u_x} + (nu_{xx}+2u_{xx})\frac{\partial}{\partial u_{xx}}.
$$

Case (2) $n \neq 1, \quad c \neq 0, \quad b = 0, \quad k \neq 0.$

In the same manner, we obtain

$$
X_3^{[2]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - 4n\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} -3n & 0 \\ -2an & -n \end{pmatrix}\begin{pmatrix} T_i^t \\ T_i^x \end{pmatrix} = \begin{pmatrix} (1-n)u \\ (1-n)(cu^{2n+1}+au+ku_{xx}) \end{pmatrix},
$$

where $X_3^{[2]}$ is given by

$$
X_3^{[2]} = -3nt\frac{\partial}{\partial t} - (2ant+nx)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + (nu_x+u_x)\frac{\partial}{\partial u_x} + (nu_{xx}+u_{xx})\frac{\partial}{\partial u_{xx}}.
$$

Hence, doubly reducing the Gardner equation (4.1) via the scaling symmetry X_3 with the conserved vector (T_1^t, T_1^x) (4.21) is only achievable when either $c = 0, \quad n = 2$ or $b = 0, \quad n = 1$. This means that the power law nonlinearity term must be quadratic.

Let us consider $b = 0, \quad n = 1$. We transform the generator X_3 to its canonical form $Y = \frac{\partial}{\partial s}$ by solving

$$
\frac{dt}{-3t} = -\frac{dx}{2at+x} = \frac{du}{u} = \frac{dr}{1} = \frac{ds}{1} = \frac{dw}{1}.
\tag{4.64}
$$

The solution of (4.64) yields the invariants of X_3 given by

$$r = \frac{x}{t^{\frac{1}{3}}} - at^{\frac{2}{3}}, \quad w(r) = ut^{\frac{1}{3}}, \quad s = -\frac{1}{3}\ln(t). \tag{4.65}$$

Using the above canonical variables (4.65) with the formulas (1.26) and (1.27), we obtained the reduced conserved vector for (4.21) written as $D_r T_1^r + D_s T_1^s = 0$ with

$$T_1^r = cw^3 + kw_{rr}, \tag{4.66}$$

$$T_1^s = w. \tag{4.67}$$

The reduced equation becomes

$$cw^3 + kw_{rr} = k_1, \tag{4.68}$$

where k_1 is a constant. The integration of the above equation (4.68) results in

$$\int \frac{dw}{\sqrt{\frac{2}{3k}wk_1 - \frac{c}{6k}w^4}} = r + k_2, \tag{4.69}$$

where k_2 is a constant of integration.

Remark: The solution (4.60) of the Gardner equation (4.1) obtained by double reduction is same as the solution (4.39) derived by the Lie symmetry method. The second solution of Gardner equation (4.1) given in (4.63) cannot be constructed by the Lie symmetry method.

4.3.3 Exact solutions of the Gardner equation using the improved generalised Riccati equation mapping method

We note that the Lie symmetry and double reduction approach do not lead to the formation of singular soliton solutions to the Gardner equation (4.1). The singular solitons provide a possible analytical explanation to the formation of freak or abnormal waves that are a threat even to large ships and ocean liners. In order to find the singular soliton solutions, we apply the improved generalised Riccati equation mapping method to the reduced Gardner equation (4.37).

We seek solutions of reduced equation (4.37) by the improved generalised Riccati equation mapping method [87]. Our main goal is to derive exact or at least approximate solutions, if possible, for the ODE (4.37). We express the solution $w(r)$ of equation (4.37) in the finite series form

$$w(r) = \sum_{i=-m}^{m} a_i \psi^i, \tag{4.70}$$

where the new function $\psi = \psi(r)$ satisfies the Riccati equation

$$\psi' = \mu + \beta\psi + (\nu - 1)\psi^2. \tag{4.71}$$

The parameters μ, β, ν are constants and both a_{-m}, a_m are not zero. The positive integer m can be determined by balancing the highest order derivatives with nonlinear terms appearing in ODE (4.37). Integrating (4.37) once and setting the constant of integration to zero results in

$$(a - \lambda)w + \frac{1}{2n+1} c w^{2n+1} + \frac{1}{n+1} b w^{n+1} + k w_{rr} = 0. \tag{4.72}$$

Balancing w_{rr} with w^{2n+1} in (4.72), we obtain an irreducible fraction, $m = \frac{1}{n}$, for some n. Therefore we make the following transformation

$$w(r) = h(r)^{\frac{1}{n}} \tag{4.73}$$

and then substitute (4.73) into (4.72) to obtain

$$n^2(a - \lambda)h^2 + \frac{1}{2n+1} n^2 c h^4 + \frac{1}{n+1} n^2 b h^3 + k(1-n)h_r^2 + knhh_{rr} = 0. \tag{4.74}$$

Now balancing $h(r)^4$ and hh_{rr} we find $m = 1$. Thus, we assume that

$$h(r) = a_{-1}\psi^{-1} + a_0 + a_1\psi. \tag{4.75}$$

Substituting equations (4.75) together with (4.71) into (4.74) yields an algebraic equation involving powers of ψ. Equating the coefficients of each power of ψ to zero gives a system of algebraic equations for a_{-1}, a_0, a_1 with solution

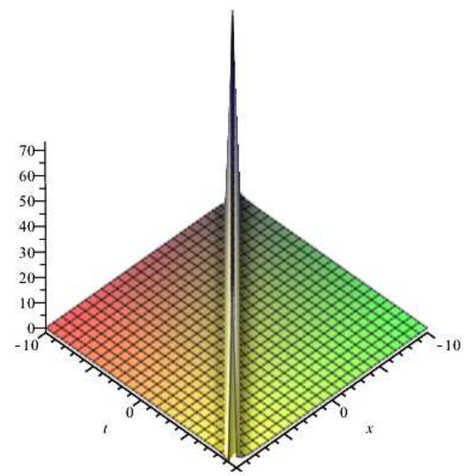
$$\begin{aligned} &a_0 = a_0, \quad a_1 = \frac{(\nu - 1)a_0}{\beta}, \quad a_{-1} = \frac{\mu a_0}{\beta}, \quad a = \frac{n^2\lambda - k(\beta^2 - 4\mu(\nu - 1))}{2\mu}, \\ &b = \frac{k(n+1)(n+2)\beta^2}{n^2 a_0}, \quad c = -\frac{k(n+1)(2n+1)\beta^2}{n^2 a_0}. \end{aligned} \tag{4.76}$$

From (4.76), (4.75), (4.73) and (4.36), we obtain multiple explicit solutions of the Gardner equation (4.1). We follow [87], [50], [86] and consider four cases. These can be found in Appendix 4.5.1.

Note that we do not find a Type 4 solution.

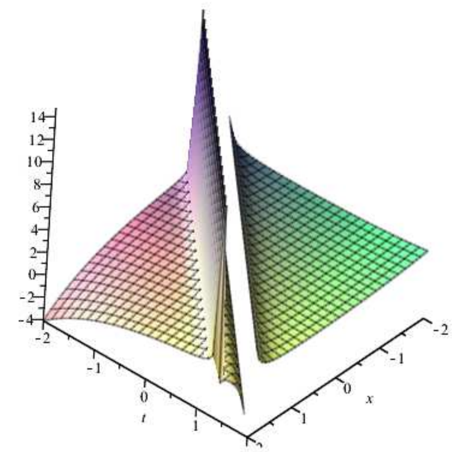
It is well known that the presence of dispersion without nonlinearity kills the solitary wave as different Fourier harmonics start propagating at different group velocities. On the other hand, introducing nonlinearity without dispersion also destroys the propagation of solitary waves, since the pulse energy is frequently pumped into higher frequency modes. Thus, in order to maintain solitary waves both dispersion and nonlinearity must be introduced. We maintained a delicate balance between the nonlinearity effect of dual power law terms, u^{2n} and u^n, and the dispersive effect of u_{xxx} which resulted in solitons or solitary waves of the Gardner equation (4.1). The solitons are waves that are characterised by retaining their identity upon interacting with other solitons without change of shape and velocity properties. The type of travelling wave solutions depends on the values of the physical parameters. Solutions u_1, u_3, in (4.77), u_6, u_7, in (4.78), u_8, u_{10} in (4.79) describe the soliton solution [17]. Figure 4.1 (a) shows the soliton obtained from solution u_8 in (4.79). Solutions $u_9, u_{11} text and u_{12}$ in (4.79) represent exact singular soliton solutions (as shown in Figure 4.1 (b)). Solutions u_{25} and u_{26} in (4.82) have the exact solitary wave of kink type [41], graphically represented in Figure 4.1(c). Solutions u_2, u_4 in (4.77) and u_5 in (4.78) represent the solitary wave solutions of singular kink type (see Figure 4.1 (d)). The solutions of u_{13}, u_{15}, u_{18}, u_{19} in (4.80), u_{20}, u_{22} in (4.81) are periodic wave solutions. Figure 4.1 (e) shows the periodic travelling wave solution obtained from u_{13} in (4.80). Solutions u_{14}, u_{16}, u_{17} in (4.80) and u_{21}, u_{23} and u_{24} in (4.81) describe singular periodic wave solutions as shown in Figure 4.1 (f).

Note that if $\mu = 0$ in $u_1, u_2, \quad u_4$ in (4.77), u_5, u_9, u_{12} in (4.78) and when $\mu = 0$, with $F = 0$ in u_6 in (4.78), our obtained solutions are identical to those the solutions obtained by Li *et al* [45] using the substitution method. Thus their solutions are a subset of our results.



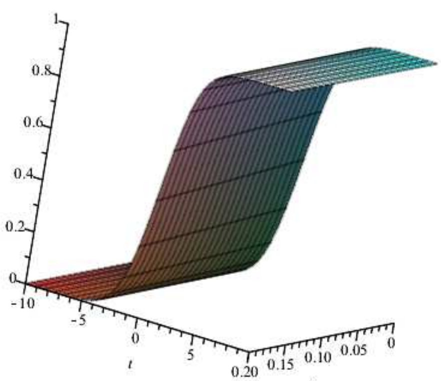
70
60
50
40
30
20
10
0
-10
t
0
x
0
-10

(a)



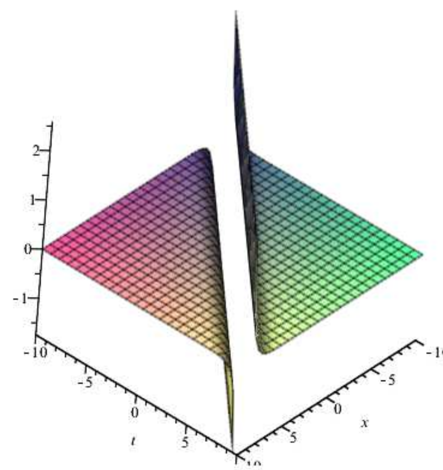
14
12
10
8
6
4
2
0
-2
-4
-2
-1
0
t
1
x
0
-1
-2
1

(b)



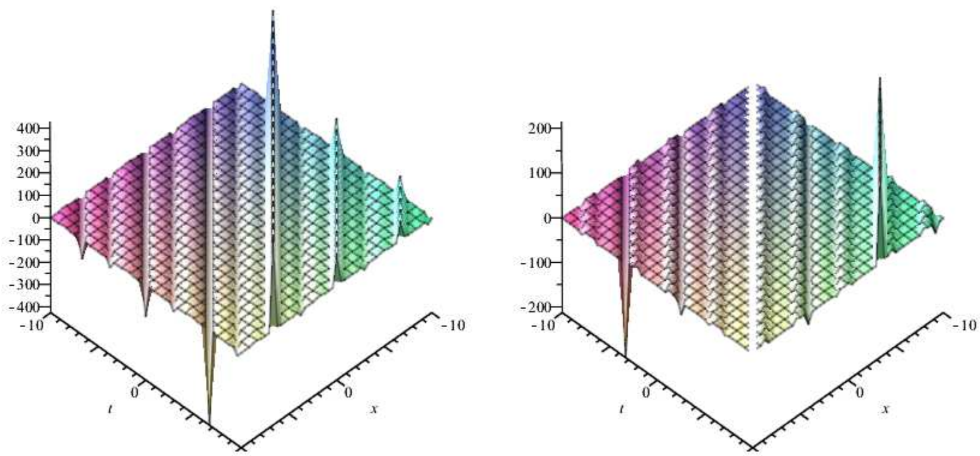
1
0.8
0.6
0.4
0.2
0
-10
-5
0
t
5
0.20
0.15
0.10
0.05
0
x

(c)



2
1
0
-1
-10
-5
0
t
5
x
0
5
-5
-10

(d)



(e) (f)

Figure 4.1: (a) Soliton shaped solitary wave solution for (4.79), with $n = 1, a_0 = 1, \beta = 1, \lambda = 1, \mu = -1, \nu = 2$. (b) Singular soliton shaped solitary wave solution of u_9 in (4.78) with same fixed values in (a). (c) Kink type solitary wave solution for $u_2$6in (4.82) with $n = 1, a_0 = 1, \beta = 1, d = 1, \lambda = 1$. (d) Singular kink type solitary wave solution for u_2 in (4.77) with same fixed values in (c). (e) Periodic wave solution of u_{13} in (4.80), with $n = 1, a_0 = 1, \beta = 1, \lambda = 1, \mu = 1, \nu = 2$. (f) Singular periodic wave solution of u_{16} in (4.80), with same fixed values in (e).

4.4 Conclusion

In this chapter, using the group method, the third order Gardner equation with dual power law nonlinearity as a generalised setting was studied. In the sense of geometric symmetry, all the vector fields of the equation are presented. The Gardner equation possesses only translational symmetries. These symmetries reduce the nonlinear third order PDE to an integrable third order ODE and we obtained one independent solution. As the Gardner equation is a third order PDE hence, not derivable from a variational principle, its conservation laws were discussed via the multiplier method and Noether approach (after increasing the order by one). We have shown that there are six local conservation laws when the power law nonlinearity is quadratic and only the Noether approach leads to a infinite number of nonlocal conservation laws. The importance of these conservation laws was demonstrated using double reduction theory which is based on the association of symmetries with conserved vectors. As a result, three independent solutions were obtained. Furthermore, we derived additional nontrivial conservation laws and scaling symmetries for some special cases of the Gardner equation. Similar studies are carried out for these special cases. Moreover, the singular soliton and singular periodic solutions were extracted for the Gardner equation with the aid of the improved generalised Riccati method. Certain constraints or compatibility conditions of the parameters of the Gardner equation had to be satisfied to ensure the existence of these solutions. To understand the behaviour of the solutions, we plotted the graphs of the solution surfaces for some special parameter values. Most of the solutions presented have not been reported in literature and many known solutions are only special cases of these. These solutions include important soliton solutions and nontrivial solutions in terms of special functions which are meromorphic in the entire complex plane. Remarkably, such solutions, having special functions together with arbitrary parameters can be used as a benchmark for solving other related model problems and assessing numerical and approximate analytical methods for nonlinear equations describing solitons in wave mechanics. Apart from complementing results in the literature, the solutions are also useful in the analysis of wave propagation on physical phenomena. This study presents new ways of finding more exact solutions of PDEs and these solutions may not be obtained from symmetry analysis.

4.5 Appendix

4.5.1 Explicit solutions of the Gardner equation using improved generalised Riccati equation mapping method

Type 1 When $\Omega = \beta^2 - 4\mu(\nu-1) > 0, \quad \beta(\nu-1) \neq 0, \quad (\text{or} \;\; \mu(\nu-1) \neq 0)$

$$
\begin{aligned}
u_1(t,x) &= a_0^{\frac{1}{n}} \left[\frac{-2\mu(\nu-1)}{\beta\left(\beta + \sqrt{\Omega}\tanh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})\right)} + 1 - \frac{1}{2\beta}\left(\beta + \sqrt{\Omega}\tanh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})\right) \right]^{\frac{1}{n}}, \\
u_2(t,x) &= a_0^{\frac{1}{n}} \left[\frac{-2\mu(\nu-1)}{\beta\left(\beta + \sqrt{\Omega}\coth(\frac{\sqrt{\Omega}(x-\lambda t)}{2})\right)} + 1 - \frac{1}{2\beta}\left(\beta + \sqrt{\Omega}\coth(\frac{\sqrt{\Omega}(x-\lambda t)}{2})\right) \right]^{\frac{1}{n}}, \\
u_3(t,x) &= a_0^{\frac{1}{n}} \left[\frac{-2\mu(\nu-1)}{\beta\left(\beta + \sqrt{\Omega}\left(\tanh(\sqrt{\Omega}(x-\lambda t)) \pm i\operatorname{sech}(\sqrt{\Omega}(x-\lambda t))\right)\right)} \right. \\
&\quad \left. +1 - \frac{1}{2\beta}\left(\beta + \sqrt{\Omega}\left(\tanh(\sqrt{\Omega}(x-\lambda t)) \pm i\operatorname{sech}(\sqrt{\Omega}(x-\lambda t))\right)\right) \right]^{\frac{1}{n}}, \\
u_4(t,x) &= a_0^{\frac{1}{n}} \left[\frac{-2\mu(\nu-1)}{\beta\left(\beta + \sqrt{\Omega}\left(\coth(\sqrt{\Omega}(x-\lambda t)) \pm \operatorname{csch}(\sqrt{\Omega}(x-\lambda t))\right)\right)} \right. \\
&\quad \left. +1 - \frac{1}{2\beta}\left(\beta + \sqrt{\Omega}\left(\coth(\sqrt{\Omega}(x-\lambda t)) \pm \operatorname{csch}(\sqrt{\Omega}(x-\lambda t))\right)\right) \right]^{\frac{1}{n}},
\end{aligned}
\tag{4.77}
$$

$$
\begin{aligned}
u_5(t,x) &= a_0^{\frac{1}{n}} \left[\frac{-4\mu(\nu-1)}{\beta\left(2\beta+\sqrt{\Omega}\left(\tanh(\frac{\sqrt{\Omega}(x-\lambda t)}{4}) \pm \coth(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\right)\right)} \right. \\
&\qquad \left. +1 - \frac{1}{4\beta}\left(2\beta+\sqrt{\Omega}\left(\tanh(\frac{\sqrt{\Omega}(x-\lambda t)}{4}) \pm \coth(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\right)\right)\right]^{\frac{1}{n}}, \\
u_6(t,x) &= a_0^{\frac{1}{n}} \left[\frac{2\mu(\nu-1)}{\beta}\left(-\beta+\frac{\sqrt{\Omega(E^2+F^2)}-E\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))}{E\sinh(\sqrt{\Omega}(x-\lambda t))+F}\right)^{-1} \right. \\
&\qquad \left. +1+\frac{1}{2\beta}\left(-\beta+\frac{\sqrt{\Omega(E^2+F^2)}-E\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))}{E\sinh(\sqrt{\Omega}(x-\lambda t))+F}\right)\right]^{\frac{1}{n}}, \\
u_7(t,x) &= a_0^{\frac{1}{n}} \left[\frac{2\mu(\nu-1)}{\beta}\left(-\beta-\frac{\sqrt{\Omega(F^2-E^2)}+E\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))}{E\sinh(\sqrt{\Omega}(x-\lambda t))+F}\right)^{-1} \right. \\
&\qquad \left. +1+\frac{1}{2\beta}\left(-\beta-\frac{\sqrt{\Omega(F^2-E^2)}+E\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))}{E\sinh(\sqrt{\Omega}(x-\lambda t))+F}\right)\right]^{\frac{1}{n}},
\end{aligned}
\tag{4.78}
$$

where E and F are two non-zero real constants satisfying $F^2 - E^2 > 0$,

$$
\begin{aligned}
u_8(t,x) &= a_0^{\frac{1}{n}}\left[\frac{\sqrt{\Omega}\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})-\beta\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{2\beta\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}+1\right.\\
&\left.+\frac{2\mu(\nu-1)}{\beta}\left(\frac{\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{\sqrt{\Omega}\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})-\beta\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}\right)\right]^{\frac{1}{n}},\\
u_9(t,x) &= a_0^{\frac{1}{n}}\left[-\frac{\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})-\sqrt{\Omega}\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{2\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}+1\right.\\
&\left.-\frac{2\mu(\nu-1)}{\beta}\left(\frac{\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})-\sqrt{\Omega}\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}\right)\right]^{\frac{1}{n}},\\
u_{10}(t,x) &= a_0^{\frac{1}{n}}\left[\frac{\sqrt{\Omega}\sinh(\sqrt{\Omega}(x-\lambda t))-\beta\cosh(\sqrt{\Omega}(x-\lambda t))\pm i\sqrt{\Omega}}{2\beta\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}+1\right.\\
&\left.+\frac{2\mu(\nu-1)}{\beta}\left(\frac{\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{\sqrt{\Omega}\sinh(\sqrt{\Omega}(x-\lambda t))-\beta\cosh(\sqrt{\Omega}(x-\lambda t))+\pm i\sqrt{\Omega}}\right)\right]^{\frac{1}{n}},\\
u_{11}(t,x) &= a_0^{\frac{1}{n}}\left[\frac{-\beta\sinh(\sqrt{\Omega}(x-\lambda t))+\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))\pm\sqrt{\Omega}}{2\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}+1\right.\\
&\left.+\frac{2\mu(\nu-1)}{\beta}\left(\frac{\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{-\beta\sinh(\sqrt{\Omega}(x-\lambda t))+\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))\pm\sqrt{\Omega}}\right)\right]^{\frac{1}{n}},\\
u_{12}(t,x) &= a_0^{\frac{1}{n}}\left[\frac{-2\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})+2\sqrt{\Omega}\cosh^2(\frac{\sqrt{\Omega}(x-\lambda t)}{4})-\sqrt{\Omega}}{4\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})}+1\right.\\
&\left.+\frac{4\mu(\nu-1)}{\beta}\left(\frac{\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})}{-2\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})+2\sqrt{\Omega}\cosh^2(\frac{\sqrt{\Omega}(x-\lambda t)}{4})-\sqrt{\Omega}}\right)\right]^{\frac{1}{n}}.
\end{aligned}
\tag{4.79}
$$

Type 2 When $\Omega < 0, \quad \beta(\nu-1) \neq 0, \quad (\text{or} \quad \mu(\nu-1) \neq 0)$

$$u_{13}(t,x) = a_0^{\frac{1}{n}} \left[\frac{2\mu(\nu-1)}{\beta\left(-\beta+\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}\right)\right)} + 1 + \frac{1}{2\beta}\left(-\beta+\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}\right)\right)\right]^{\frac{1}{n}},$$

$$u_{14}(t,x) = a_0^{\frac{1}{n}} \left[-\frac{2\mu(\nu-1)}{\beta\left(\beta+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}\right)\right)} + 1 - \frac{1}{2\beta}\left(\beta+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}\right)\right)\right]^{\frac{1}{n}},$$

$$\begin{aligned} u_{15}(t,x) = a_0^{\frac{1}{n}} \Bigg[& \frac{2\mu(\nu-1)}{\beta\left(-\beta+\sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}(x-\lambda t)) \pm \sec(-\sqrt{\Omega}(x-\lambda t))\right)\right)} \\ & +1+\frac{1}{2\beta}\left(-\beta+\sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}(x-\lambda t)) \pm \sec(-\sqrt{\Omega}(x-\lambda t))\right)\right)\Bigg]^{\frac{1}{n}}, \end{aligned}$$

$$\begin{aligned} u_{16}(t,x) = a_0^{\frac{1}{n}} \Bigg[& -\frac{2\mu(\nu-1)}{\beta\left(\beta+\sqrt{-\Omega}\left(\cot(\sqrt{-\Omega}(x-\lambda t)) \pm \csc(-\sqrt{\Omega}(x-\lambda t))\right)\right)} \\ & +1-\frac{1}{2\beta}\left(\beta+\sqrt{-\Omega}\left(\cot(\sqrt{-\Omega}(x-\lambda t)) \pm \csc(-\sqrt{\Omega}(x-\lambda t))\right)\right)\Bigg]^{\frac{1}{n}}, \end{aligned}$$

$$\begin{aligned} u_{17}(t,x) = a_0^{\frac{1}{n}} \Bigg[& \frac{4\mu(\nu-1)}{\beta\left(-2\beta+\sqrt{-\Omega}\left(\tan\left(\frac{\sqrt{-\Omega}}{4}(x-\lambda t)\right) - \cot\left(\frac{\sqrt{-\Omega}}{4}(x-\lambda t)\right)\right)\right)} \\ & +1+\frac{1}{4\beta}\left(-2\beta+\sqrt{-\Omega}\left(\tan\left(\frac{\sqrt{-\Omega}}{4}(x-\lambda t)\right) - \cot\left(\frac{\sqrt{-\Omega}}{4}(x-\lambda t)\right)\right)\right)\Bigg]^{\frac{1}{n}}, \end{aligned}$$

$$\begin{aligned} u_{18}(t,x) = a_0^{\frac{1}{n}} \Bigg[& \frac{2\mu(\nu-1)}{\beta}\left(-\beta+\frac{\pm\sqrt{-\Omega(E^2-F^2)}-E\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))}{E\sin(\sqrt{-\Omega}(x-\lambda t))+F}\right)^{-1} \\ & +1+\frac{1}{2\beta}\left(-\beta+\frac{\pm\sqrt{-\Omega(E^2-F^2)}-E\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))}{E\sin(\sqrt{-\Omega}(x-\lambda t))+F}\right)\Bigg]^{\frac{1}{n}}, \end{aligned}$$

$$\begin{aligned} u_{19}(t,x) = a_0^{\frac{1}{n}} \Bigg[& \frac{2\mu(\nu-1)}{\beta}\left(-\beta-\frac{\pm\sqrt{-\Omega(E^2-F^2)}+E\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))}{E\sin(\sqrt{-\Omega}(x-\lambda t))+F}\right)^{-1} \\ & +1+\frac{1}{2\beta}\left(-\beta-\frac{\pm\sqrt{-\Omega(E^2-F^2)}+E\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))}{E\sin(\sqrt{-\Omega}(x-\lambda t))+F}\right)\Bigg]^{\frac{1}{n}}, \end{aligned} \tag{4.80}$$

where E and F are two non-zero real constants satisfying $E^2 - F^2 > 0$,

$$
\begin{aligned}
u_{20}(t,x) &= a_0^{\frac{1}{n}} \left[-\frac{\sqrt{-\Omega}\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}) + \beta\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{2\beta\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})} \right. \\
&\left. \quad +1 - \frac{2\mu(\nu-1)}{\beta}\left(\frac{\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{\sqrt{-\Omega}\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}) + \beta\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})} \right) \right]^{\frac{1}{n}}, \\
u_{21}(t,x) &= a_0^{\frac{1}{n}} \left[\frac{-\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}) + \sqrt{-\Omega}\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{2\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})} \right. \\
&\left. \quad +1 + \frac{2\mu(\nu-1)}{\beta}\left(\frac{\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{-\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}) + \sqrt{-\Omega}\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})} \right) \right]^{\frac{1}{n}}, \\
u_{22}(t,x) &= a_0^{\frac{1}{n}} \left[-\frac{\sqrt{-\Omega}\sin(\sqrt{-\Omega}(x-\lambda t)) + \beta\cos(\sqrt{-\Omega}(x-\lambda t)) \pm \sqrt{-\Omega}}{2\beta\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})} \right. \\
&\left. \quad +1 - \frac{2\mu(\nu-1)}{\beta}\left(\frac{\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}(x-\lambda t)) + \beta\cos(\sqrt{-\Omega}(x-\lambda t)) \pm \sqrt{-\Omega}} \right) \right]^{\frac{1}{n}}, \\
u_{23}(t,x) &= a_0^{\frac{1}{n}} \left[\frac{-\beta\sin(\sqrt{-\Omega}(x-\lambda t)) + \sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t)) \pm \sqrt{-\Omega}}{2\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})} \right. \\
&\left. \quad +1 + \frac{2\mu(\nu-1)}{\beta}\left(\frac{\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{-\beta\sin(\sqrt{-\Omega}(x-\lambda t)) + \sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t)) \pm \sqrt{-\Omega}} \right) \right]^{\frac{1}{n}}, \\
u_{24}(t,x) &= a_0^{\frac{1}{n}} \left[\frac{-2\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{4}) + 2\sqrt{-\Omega}\cos^2(\frac{\sqrt{-\Omega}(x-\lambda t)}{4}) - \sqrt{-\Omega}}{4\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})} \right. \\
&\left. \quad +1 + \frac{4\mu(\nu-1)}{\beta}\left(\frac{\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})}{-2\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{4}) + 2\sqrt{-\Omega}\cos^2(\frac{\sqrt{-\Omega}(x-\lambda t)}{4}) - \sqrt{-\Omega}} \right) \right]^{\frac{1}{n}}.
\end{aligned}
\tag{4.81}
$$

Type 3 When $\mu = 0 \ \ \beta(\nu - 1) \neq 0$,

$$
\begin{aligned}
u_{25}(t,x) &= a_0^{\frac{1}{n}} \left[1 - \left(\frac{d}{d + \cosh(\beta(x-\lambda t) - \sinh(\beta(x-\lambda t)}\right)\right]^{\frac{1}{n}}, \\
u_{26}(t,x) &= a_0^{\frac{1}{n}} \left[1 - \left(\frac{\cosh(\beta(x-\lambda t) + \sinh(\beta(x-\lambda t)}{d + \cosh(\beta(x-\lambda t) + \sinh(\beta(x-\lambda t)}\right)\right]^{\frac{1}{n}},
\end{aligned}
\tag{4.82}
$$

where d is an arbitrary constant.

Type 4 When $\mu = \beta = 0, \ \ \nu \neq 1$,

There is no solution to the Gardner equation (4.1) for Type 4

Chapter 5

A group theoretic analysis and complex dynamics of a diffusive Caughley prey - predator model

5.1 Introduction

Elephant is the largest living land mammal, found in tropical regions of Africa and Asia. Elephants are browsing animals, they consume a large amount of vegetation, and also pull down very large trees. In Southern Africa, elephants are a major component of game reserves which are mostly open to the public and tourists. These reserves have only a limited amount of vegetation and the number of elephants is growing exponentially [12]. This scenario poses unique problems to reserve managers and conservationists fear that elephant will irreversibly change the habitat by destroying all the trees. Habitat loss is one of the key threats that is placing the elephants future at risk. By the 1970s, the decline in elephant numbers across the continent had provoked serious concern about the long-term survival of the species [43]. In 1976 Caughley proposed a simple predator-prey model for elephant-tree dynamics. The significance of this model was to show that elephants and trees coexist in a stable limit cycle. The model is based on the well known predator-prey models developed by Lotka in 1925 and independently by Volterra in 1927. Caughley's version is an example of a more realistic predator-prey model, but its exact form is unique in the literature. The model is given by the system of differential equations [13]:

$$\begin{aligned} u' &= u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right), \\ v' &= v\left(-A + \frac{ku}{v+B}\right), \end{aligned} \tag{5.1}$$

where $u = u(t)$ is the density of the trees (in trees/km^2), $v = v(t)$ is the density of elephants(in elephants/km^2), t is time (in year) and $' = \frac{d}{dt}$. For the trees, a is the natural rate of increase (in year$^-1$), K is the tree carrying capacity of the environment (in trees/km^2), c is the instantaneous rate of elimination of trees by elephant (in trees/(elephant year)) and g determines the threshold above which tree destruction depends on elephant density alone (in trees/km^2). For the elephants, A is their rate of decrease (in year$^-1$) in the absence of trees, k is the rate at which this decrease is ameliorated at a given ratio of trees to elephants (in elephant/(tree year)), and B determines the threshold above which this amelioration depends only on the density of trees (in elephant/km^2). Duffy et al [19] investigated the model (5.1) with real data and found that it exhibits limit cycle solutions for certain parameter conditions.

The aforementioned study help biologists and ecologists to understand the dynamics that evolve between elephants and trees temporally without any spatial dimensions. From a biological perspective, elephants and trees are spread out over a two-dimensional landscape and typically interact with the physical environment and other organisms in their spatial neighbourhood. System of partial differential equations (PDEs) will provide more information to explain the population distribution, the wave speed and the effects of diffusivity of each species over a space domain.

Solutions of PDEs have contributed tremendously to the understanding of diverse and complex systems. The travelling wave is one kind of special solutions to the evolutionary systems which provides information regarding how population densities disperse over space as time evolves [39]. Travelling waves and their spread is beneficial in allocating appropriate number of elephants to a reasonably space of forest at a particular time to ensure a stable coexistence of both species [53]. It is a standard approach to consider travelling wave solutions when dealing with diffusive prey - predator systems [30]. Lie group analysis plays a central role in the analysis of nonlinear

PDEs, especially in determining their exact solutions. This powerful and prolific method has been effectively used to construct invariant solutions frequently used as basis for simplification of numerical and dynamical system analysis [70].

Spatial pattern formation arose from the observation in chemistry by Turing [71] that diffusion can also destabilize equilibrium solution, a scenario well known as Turing instability. This phenomenon, known as Turing or diffusion-driven instability, is commonly found across many fields of study and has been playing a significant role in the mechanism for pattern formation in numerous embryological and ecological contexts. Presently, the knowledge of patterns and mechanisms of spatial dispersal of interacting species is an issue of concern in conservation biology, ecology, and biochemical reactions [60].

Based on these discussions above, we consider the extended Caughley model that includes diffusive effects given by [66]

$$\begin{aligned} u_t &= a_1(u_{xx}+u_{yy}) + u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right), \\ v_t &= a_2(v_{xx}+v_{yy}) + v\left(-A + \frac{ku}{v+B}\right), \end{aligned} \tag{5.2}$$

where the tree density and the elephant density are function of space and time:$u(t,x,y)$ and $v(t,x,y)$ and; parameters a_1 and a_2 are the diffusion coefficients for the tree and elephant, respectively. Other parameters are the same definition as those above.

Robert Willie [79] studied the model (5.2) using functional analysis approach and investigated the stability of the global asymptotic dynamics of the model with varied diffusion. In [66] the numerical analysis for this model (5.2) are given by methods of an algebraic transformations together with a spectral method.

In this study we perform symmetry analysis and obtain many reductions of the Caughley system (5.2). We analyse each reduction and investigate the feasibility of its resulting invariant solution. This includes the generalised travelling wave solution. Our goal is to extract crucial information regarding how tree and elephant populations disperse over space. This study is very vital in taking some control measures to ensure stable coexistence of both species. To achieve this, we perform the dynamical system analysis of the travelling wave model (5.23) to investigate its global behaviour along the equilibrium points of the system. Moreover, we investigate the necessary conditions for Turing instability and perform extensive numerical simulations from both the mathematical and the biological points of view in order to examine the role of diffusion coefficients in pattern formations of the Caughley model. This chapter is organised as follows. In Section 5.2, the Lie symmetry analysis of the model (5.2) is performed. The resultant reduced equations are analysed. In Section 5.3, the Turing instability and the effects of diffusion on emergence of spatial patterns of the diffusive model and the reduced travelling wave model is analysed. We conclude our work in Section 5.4

5.2 Lie point Symmetries of the Caughley model (5.2)

In this section, we provide classical Lie symmetry analysis of the system (5.2). The Lie point symmetries admitted by (5.2) are generated by the vector field of the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v},$$

where ξ^1, ξ^2, ξ^3, η^1, η^2 functions of (t, x, y, u, v) are the infinitesimals of the Lie group of transformation of (5.2). The operator X satisfies the Lie symmetry condition (1.14)

$$\begin{aligned} X^{[2]}\left(\Delta_1\right)|_{\Delta_1=0} &= 0, \\ X^{[2]}\left(\Delta_2\right)|_{\Delta_2=0} &= 0, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \Delta_1 &= u_t - a_1(u_{xx} + u_{yy}) - u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right) = 0, \\ \Delta_2 &= v_t - a_2(v_{xx} + v_{yy}) - v\left(-A + \frac{ku}{v+B}\right) = 0, \end{aligned}$$

and $X^{[2]}$ is the second prolongation of equation (5.3). Expansion and separation of (5.3) with respect to the powers of different derivatives of u and v yields an over determined system in unknown coefficients ξ^1, ξ^2, η^1, and η^2.

$$
\begin{aligned}
&a_1\eta^1_v = 0, a_2\eta^2_u = 0, a_1\xi^1_u = 0, a_1\xi^1_v = 0, a_1\xi^2_u = 0,\\
&a_1\xi^2_v = 0, a_2\xi^1_u = 0, a_2\xi^1_v = 0, a_2\xi^2_u = 0, a_2\xi^1_x + a_2\xi^2_t = 0,\\
&2a_1\xi^1_t - a_1\eta^1_u = 0, 2a_2\xi^2_x - a_2\eta^2_v = 0, \eta^1_u + a_1\xi^1_{xx} - \xi^1_t - 2a_1\eta^1_{tu} + a_1\xi^1_{tt} = 0,\\
&a_2\xi^2_v = 0, \eta^2_v + a_2\xi^1_{xx} - \xi^1_t - 2a_2\eta^2_{tv} + a_2\xi^1_{tt} = 0, a_2\xi^2_{xv} = 0,\\
&a_2\eta^2_{xx} + a_2\eta^2_{tt} - \eta^2_t + \left(\frac{kB}{(B+u)^2}\right)v\eta^1 - \left(A - \frac{ku}{B+u}\right)\eta^2 = 0,\\
&a_1\eta^1_{xx} + a_1\eta^1_{tt} - \eta^1_t - \left(\frac{cv}{g+v} + \frac{2au}{K} - a\right)\eta^1 - \left(\frac{cg}{(g+v)^2}\right)u\eta^2 = 0
\end{aligned}
\tag{5.4}
$$

Solving the overdetermined system for arbitrary parameters gives the 4-dimensional algebra spanned by the following vector fields

$$
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}. \tag{5.5}
$$

The symmetries $X_1,\ X_2$ and X_3 can be found by inspection as the diffusive Caughley model (5.2) is independent of $t,\ x$ and y. The last symmetry, X_4 represents Laplacian in x, y space variables and always arises in the event that the equation in question is diffusive.

The commutation relations between these vector fields are given in Table 5.1, the entry in row i and the column j representing $[X_i, X_j]$. From the above commutator table, one can see that the

Table 5.1: Commutator relation for (5.5)

[,]	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	$-X_3$
X_3	0	0	0	X_2
X_4	0	X_3	$-X_2$	0

operators X_i $(i = 1, 2, ..., 4)$ form a Lie algebra, which is a four dimensional symmetry algebra.

5.2.1 Optimal system of one dimensional subalgebras

The detailed discussion on optimal system of one dimensional subalgebras can be found in[61, 63] as well as Chapter 2. Here we adopt Olver's approach and also make use of the well known study by Patera and Winternitz [63] to obtain optimal system of the Lie algebra admitted by (5.2).

In finding the optimal system, we consider the commutator relation presented in Table 5.1 together with the adjoint representation given in Table 5.2. We observed from the commutator Table 5.1 that the generator X_1 is an Abelian element that satisfies $[X_1, X_i] = 0,\ i = 2, 3, 4$. Hence X_1 is identified as the center of the Lie algebra (5.5). Thus the four dimensional Lie algebra (5.5) is decomposable into a sum of three dimensional subalgebras spanned by the generators X_2, X_3, X_4 and one dimensional subalgebra spanned by X_1. From Patera and Winternitz classification, Lie algebra (5.5) corresponds to $A_{3,6} \oplus A_1$. Based on [61] and [63] techniques, we obtain an extensive optimal system of one dimensional subalgebras given as

$$bX_1 + X_4,\ X_1,\ eX_1 + X_2, \tag{5.6}$$

where $b,\ e$ are constants.

Table 5.2: Adjoint representation for (5.5)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_1	X_2	X_3	$\epsilon X_3 + X_4$
X_3	X_1	X_2	X_3	$X_4 - \epsilon X_2$
X_4	X_1	$\cos(\epsilon)X_2 - \sin(\epsilon)X_3$	$\sin(\epsilon)X_2 + \cos(\epsilon)X_3$	X_4

5.2.2 Symmetry reductions of the Caughley model (5.2)

In this subsection, we study symmetry reductions of the Caughley model (5.2) using every element in the optimal system (5.6). Solving associated characteristic equations, we obtain the representation of an invariant solution. Substituting this representation into the system (5.2), we obtain the reduced equations which could be solved to obtain the exact or numerical solutions.

Reduction via the subalgebra $bX_1 + X_4$.

The characteristic equation is given as

$$\frac{dt}{b} = \frac{dx}{y} = -\frac{dy}{x} = \frac{du}{0} = \frac{dv}{0}. \tag{5.7}$$

Solving the above equation (5.7), we obtain the following four invariants:

$$r = x^2 + y^2, \ s = -b\tan^{-1}(\frac{x}{y}) + t, \quad u = u(r, s), \quad v = v(r, s). \tag{5.8}$$

In terms of the new variables r, s, $u_t = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = u_s$, $u_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = 2xu_r - \frac{1}{r}byu_s$, $u_y = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = 2yu_r + \frac{1}{r}bxu_s$, $v_t = \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} = v_s$, $v_x = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} = 2xv_r - \frac{1}{r}byv_s$, $v_y = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} = 2yv_r + \frac{1}{r}bxv_s$. Following this procedure one finds that $u_{xx} = \frac{1}{r^2}\left[b^2y^2u_{ss} + 2bxyu_s\right.$ $\left.+2r(ru_r + 2x(xru_{rr} - byu_{rs}))\right]$,

$u_{yy} = \frac{1}{r^2}\left[b^2x^2u_{ss} - 2bxyu_s + 2r(ru_r + 2y(yru_{rr} + bxu_{rs}))\right]$,

$v_{xx} = \frac{1}{r^2}\left[b^2y^2v_{ss} + 2bxyv_s + 2r(rv_r + 2x(xrv_{rr} - byv_{rs}))\right]$,

$v_{yy} = \frac{1}{r^2}\left[b^2x^2v_{ss} - 2bxyv_s + 2r(rv_r + 2y(yrv_{rr} + bxv_{rs}))\right]$.

In the light of these, the Caughley model (5.2) reduces to second order PDE in two new independent variables r, s given as,

$$\begin{aligned} ru_s &= a_1(b^2u_{ss} + 4r(u_r + ru_{rr})) + ru\left(a - \frac{au}{K} - \frac{cv}{u+g}\right), \\ rv_s &= a_2(b^2v_{ss} + 4r(v_r + rv_{rr})) + rv\left(-A + \frac{ku}{v+B}\right). \end{aligned} \tag{5.9}$$

The above system (5.9) admitted only principal Lie symmetry, $\frac{\partial}{\partial s}$ which gives rise to the invariants

$$z = r, u = u(z), \quad v = v(z). \tag{5.10}$$

The transformation (5.10) reduces the Caughley system (5.2) to second order ode

$$\begin{aligned}
4a_1(u' + zu'') + u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right) &= 0,\\
4a_2(v' + zv'') + v\left(-A + \frac{ku}{v+B}\right) &= 0,
\end{aligned} \tag{5.11}$$

where $'$ means total derivative with respect to z.

Reduction via the subalgebra X_1.

The invariants of this generator are

$$r = x,\ \ s = y,\ \ u = u(r, s),\ \ v = v(r, s). \tag{5.12}$$

They reduce the system (5.2) to the following equations:

$$\begin{aligned}
a_1(u_{rr} + u_{ss}) + u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right) &= 0,\\
a_2(v_{rr} + v_{ss}) + v\left(-A + \frac{ku}{v+B}\right) &= 0.
\end{aligned} \tag{5.13}$$

Following similar procedure as done previously, we shall investigate system (5.13) under the scope of Lie symmetry. It admitted three - dimensional Lie algebra spanned by the generators

$$v_1 = \frac{\partial}{\partial r}, v_2 = \frac{\partial}{\partial s}, v_3 = -r\frac{\partial}{\partial s} + s\frac{\partial}{\partial r}. \tag{5.14}$$

Without the central element, this Lie algebra (5.14) is identical to the Lie algebra admitted by the Caughley model (5.2) which is given in (5.5). Hence, the optimal system of one dimensional subalgebras for the Lie algebra(5.14) is given as

$$v_1, v_3. \tag{5.15}$$

Reduction with v_1, through its invariants

$$z = s,\ \ u = u(z), v = v(z), \tag{5.16}$$

gives rise to the equation

$$\begin{aligned}
a_1u'' + u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right) &= 0,\\
a_2v'' + v\left(-A + \frac{ku}{v+B}\right) &= 0.
\end{aligned} \tag{5.17}$$

The generator v_3, having the invariants

$$z = r^2 + s^2,\ u = u(z),\ v = v(z), \tag{5.18}$$

reduced system (5.13) to the same system of equations (5.11).

Reduction via the subalgebra $eX_1 + X_2$.

The characteristic equation is

$$\frac{dt}{e} = \frac{dx}{1} = \frac{dy}{0} = \frac{du}{0} = \frac{dv}{0}. \tag{5.19}$$

Solving the above system yields the following four invariants

$$r = ex - t, \quad s = y, \quad u = u(r,s), \quad v = v(r,s). \tag{5.20}$$

Applying the above transformation to (5.2) we obtain

$$\begin{aligned} -u_r &= a_1(e^2 u_{rr} + u_{ss}) + u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right), \\ -v_r &= a_2(e^2 v_{rr} + v_{ss}) + v\left(-A + \frac{ku}{v+B}\right). \end{aligned} \tag{5.21}$$

The reduced equation (5.21) posses the symmetries $\Gamma_1 = \frac{\partial}{\partial r},\ \Gamma_2 = \frac{\partial}{\partial s}$, which form an Abelian group. So we consider a linear combination $\beta\Gamma_1 + \Gamma_2$ and obtain a travelling wave transformation

$$z = \beta r - s, \quad \beta \in \Re \ \ u, = u(z), \quad v = v(z). \tag{5.22}$$

The functions $u = u(z)$ and $v = v(z)$ represent the travelling wave solution where the variable $z = \alpha x - y - \beta t$ is a moving frame with speed $\beta > 0$, $\alpha = \beta e$. Substitution of the transformation (5.22) in (5.21) gives the travelling wave system

$$\begin{aligned} -\beta u' &= a_1(\alpha^2 + 1)u'' + u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right), \\ -\beta s v' &= a_2(\alpha^2 + 1)v'' + v\left(-A + \frac{ku}{v+B}\right), \end{aligned} \tag{5.23}$$

where $'$ means total derivative with respect to z.

5.2.3 Analysis of the travelling wave solution of the Caughley system

We have presented a complete list of invariant solutions for the spatial diffusive nonlinear PDE system (5.2) and the corresponding reduced systems of ODEs. At times, it is useful to write down the exact solutions of these reduced systems of ODEs, but often times, some basic general features of these solutions can be obtained from qualitative analyses. As the different invariant solutions of subalgebras from the optimal system (5.6) are associated with fundamentally and qualitatively different physical aspects, their qualitative behaviours can be similar since they represent the solution of the same physical system of equations (5.2) which describes the dynamics between elephants and trees. Therefore, it is sufficient to investigate one of the representatives of the optimal system.

Without loss of generality, we investigate the system (5.23) which represents the travelling wave of the diffusive system (5.2) using dynamical system analysis.

An important aspect of this approach is to investigate under what conditions will the equilibrium solutions exist. Ecologically, under what conditions do both elephants and their food source (trees) maintain mutual symbiosis. If an equilibrium solution exits, an interesting view of the possible behavioural patterns can emerge from the various types of solutions.

Existence of positive equilibrium solutions (biomass)

We rewrite the system (5.23) as a $4-$ dimensional system of first order equations, by letting $u_1 = u'$ and $v_1 = v'$ which becomes:

$$\begin{aligned} u' &= u_1, \\ u_1' &= -\frac{1}{a_1(\alpha^1+1)}\left(\beta u_1 + u\left(a - \frac{au}{K} - \frac{cv}{u+g}\right)\right), \\ v' &= v_1, \\ v_1' &= -\frac{1}{a_2(\alpha^1+1)}\left(\beta v_1 + v\left(-A + \frac{ku}{v+B}\right)\right). \end{aligned} \tag{5.24}$$

Model (5.24) has the following equilibrium solutions:

- (i) The trivial equilibrium $E_0 = (u^0, u_1^0, v^0, v_1^0) = (0,0,0,0)$,

- (ii) The predator free equilibrium $E_1 = (u^o, u_1^o, v^o, v_1^o) = (K, 0, 0, 0)$,

- (iii) The steady state of coexistence (interior equilibrium)

$$E_2 = (u^*, u_1, v^*, v_1) = (u^*, 0, v^*, 0).$$

We do not focus on the solutions (i) and (ii), due to their trivial nature. We discuss only existence of the interior positive equilibrium solution.

Theorem 5.1. *There exits an interior equilibrium solution $E_2 = (u^*, u_1, v^*, v_1) = (u^*, 0, v^*, 0)$ if and only if the elephant death rate A is less than a threshold value $\frac{kK}{B}$, i.e $(A < \frac{kK}{B})$, where $u^* == \frac{-\theta \pm \sqrt{4KaA^2(Bc+ag)+\theta^2}}{2aA}$ and $v^* = \frac{ku^*-AB}{A}$, where $\theta = aA(g-K) + ckK$.*

Proof. The interior equilibrium solution $E_2 = (u^*, 0, v^*, 0)$ is obtained by setting the left hand side of (5.24) to zero, i.e.,

$$\begin{aligned} &a\left(1 - \frac{u}{K}\right) - \frac{cv}{u+g} = 0, \\ &A(v+B) - ku = 0. \end{aligned} \tag{5.25}$$

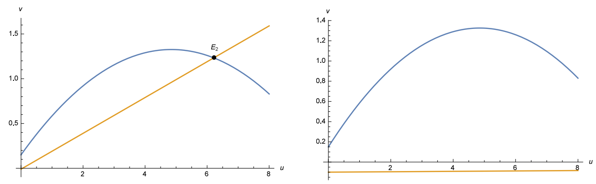
The solution of the system (5.25) is

$$u^* = \frac{-\theta \pm \sqrt{4KaA^2(Bc+ag)+\theta^2}}{2aA}, \tag{5.26}$$

$$v^* = \frac{ku^* - AB}{A}. \tag{5.27}$$

Since we are interested in biological meaningful solution, we consider the positive solution $u^* = \frac{-\theta + \sqrt{4KaA^2(Bc+ag)+\theta^2}}{2aA}$. It is clear from (5.27) that $v^* > 0$ iff $AB < ku^*$, which implies that $AB < ku^* < kK$. Therefore, the interior positive solution, E_2 exists iff $A < \frac{kK}{B}$. □

Figure (5.1) shows the region where the positive interior equilibrium of the model (5.24) is lost.



(a) (b)

Figure 5.1: Nullclines and possible coexistence equilibrium solution E_2 of the system (5.24) for (a) $A = 0.05, B = 0.01, k = 0.01, K = 10, a = 0.2, g = 0.3, c = 0.4$. (b) $A = 0.05, B = 0.01, k = 0.0001, K = 10, a = 0.2, g = 0.3, c = 0.4$.

Stability analysis

Here we discuss the dynamics of the system (5.24) in the context of stability and instability of equilibrium solutions to determine the behaviour of solutions near equilibrium points. The local asymptotic stability analysis of an equilibrium point is determined by the Jacobian matrix J obtained by the linearisation of the system (5.24) around the equilibrium point. If all the eigenvalues of the Jacobian matrix, say JE of system (5.24) evaluated at equilibrium point E have negative real parts then the equilibrium E is stable, otherwise it is unstable. Analytically, equilibrium solutions in which the solutions that start " near" them move toward the equilibrium solution are said to be asymptotically stable equilibrium points or asymptotically stable equilibrium solutions. While equilibrium solutions in which solutions that start " near" them move away from the equilibrium solution are said to be unstable equilibrium points or unstable equilibrium solutions. In other words local stability of equilibrium point E implies that all the roots of the characteristic polynomial of the Jacobian matrix JE are negative real parts.

The Jacobian matrix of the system (5.24) takes the form

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ M & -\delta_1\beta & \frac{c\delta_1 u}{u+g} & 0 \\ \hline 0 & 0 & 0 & 1 \\ -\frac{\delta_2 kv}{v+B} & 0 & N & -\delta_2\beta \end{array}\right), \tag{5.28}$$

where the constants $\delta_1 = \frac{1}{a_1(\alpha^2+1)}, \quad \delta_2 = \frac{1}{a_2(\alpha^2+1)}$
and the functions $M = -\delta_1 \left[a(1 - \frac{2u}{K}) - \frac{cv}{u+g} + \frac{cuv}{(u+g)^2} \right]$,

$N = -\delta_2 \left[\frac{ku}{v+B} - \frac{kuv}{(v+B)^2} - A \right].$

At equilibrium point E_0, the matrix (5.28) becomes

$$J_{E_0} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -a\delta_1 & -\beta\delta_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & A\delta_2 & -\beta\delta_2 \end{array} \right). \tag{5.29}$$

The characteristic equation of (5.29) is

$$p(\lambda) = \lambda^4 + \beta(\delta_1 + \delta_2)\lambda^3 + (a\delta_1 - \delta_2(A - \beta^2\delta_1))\lambda^2 + (a - A)\beta\delta_1\delta_2\lambda - aA\delta_1\delta_2,$$

which gives the eigenvalues

$$\lambda_{1,2} = \frac{\delta_1 \left(-\beta \mp \sqrt{\beta^2 - \frac{4a}{\delta\ 1}} \right)}{2}, \tag{5.30}$$

$$\lambda_{3,4} = \frac{\delta_2 \left(-\beta \mp \sqrt{\beta^2 + \frac{4A}{\delta_2}} \right))}{2}. \tag{5.31}$$

Observing from equation (5.30), if $\beta \geq 2\sqrt{\frac{a}{\delta_1}}$, then λ_1 and λ_2 are a pair of real negative eigenvalues. When $0 < \beta < 2\sqrt{\frac{a}{\delta_1}}$ then λ_1 and λ_2 are a pair of complex conjugate eigenvalues with real negative part. From the equation (5.31), the eigenvalue λ_3 is always negative while λ_4 is always positive. Therefore, system (5.24) is always unstable around E_0 which is, in fact, a saddle point and has the stable manifold which is the plane $(u, u_1) \longrightarrow (0, 0)$, while unstable manifold is the plane (v, v_1). When $0 < \beta < 2\sqrt{\frac{a}{\delta_1}}$, the trivial equilibrium E_0 is a spiral point on the stable manifold and such spiral behaviour should not occur with $v > 0$ [27]. Thus, we suggest that the biological relevant travelling wave solution with non-negative (u, v) exists and has the minimum speed of the system (5.24) given by

$$\beta \geq 2\sqrt{\frac{a}{\delta_1}}. \tag{5.32}$$

The condition (5.32) justifies the instability of E_0 and indicates the presence of elephants (since the manifold (v, v_1) is unstable.) even though trees are absent. Ecologically, this can happen when there are other sources of food available for the elephants. It follows that the extinction of both species is difficult in the model (5.24). One can observe from (5.32) that the speed at which trees are exhausted by elephants depends heavily on the growth rate and the diffusion coefficient of the trees. Thus, we establish the following results.

Theorem 5.2. *The trivial equilibrium point E_0 is a saddle point. The nonnegative solutions of* (5.24) *which correspond to travelling wave solutions of the model* (5.2) *exist with a minimum speed, β satisfying*

$$\beta \geq 2\sqrt{\frac{a}{\delta_1}}. \tag{5.33}$$

Theorem 5.3. *The predator free equilibrium point E_1 is unstable. It is a saddle point if $A < \frac{kK}{B}$ and $\beta^2 \geq \frac{4}{\delta_2}(\frac{kK}{B} - A)$. In addition, if the minimum speed, β of the system* (5.24) *satisfies*

$$\beta \geq \sqrt{\frac{4}{\delta_2}(\frac{kK}{B} - A)} \tag{5.34}$$

then the nonnegative solutions of (5.24) *which correspond to travelling wave solutions of system* (5.2) *must exist.*

Proof. The Jacobian matrix (5.28) in the small neighbourhood of equilibrium point E_1 is

$$J_{E_1} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ a\delta_1 & -\beta\delta_1 & \frac{cK\delta_1}{g+K} & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -\delta_2(\frac{kK}{B} - A) & -\beta\delta_2 \end{array}\right). \tag{5.35}$$

It follows that the characteristic equation of (5.35) is

$$\begin{aligned} p(\lambda) = \lambda^4 + \beta(\delta_1 + \delta_2)\lambda^3 - (a\delta_1 + \delta_2(A - \frac{kK}{B} - \beta^2\delta_1))\lambda^2 \\ - \beta\delta_1\delta_2(a + A - \frac{kK}{B})\lambda + a\delta_1\delta_2(A - \frac{kK}{B}), \end{aligned} \tag{5.36}$$

and the corresponding eigenvalues are given by

$$\begin{aligned} \lambda_{1,2} &= \frac{\delta_1\left(-\beta \mp \sqrt{\beta^2 + \frac{4a}{\delta_1}}\right)}{2}, \\ \lambda_{3,4} &= \frac{\delta_2\left(-\beta \mp \sqrt{\beta^2 - \frac{4}{\delta_2}(\frac{kK}{B} - A)}\right)}{2}. \end{aligned}$$

We observe that λ_2 is always positive and this implies that E_1 is unstable. In addition, if $\frac{kK}{B} < A$, the manifold (v, v_1) is unstable since λ_3 is always negative and λ_4 is positive. On the other hand, if $A < \frac{kK}{B}$ then λ_3 and λ_4 are a pair of (i) negative eigenvalues when $\beta^2 \geq \frac{4}{\delta_2}(\frac{kK}{B} - A)$ (ii) complex conjugate eigenvalues with real negative part when $0 < \beta^2 < \frac{4}{\delta_2}(\frac{kK}{B} - A)$. Therefore, following the similar argument for E_0, a necessary condition for the existence of non-negative solutions is $\beta^2 \geq \frac{4}{\delta_2}(\frac{kK}{B} - A)$. In this situation the manifold (v, v_1) is stable. Consequently, if $A < \frac{kK}{B}$ and $\beta^2 \geq \frac{4}{\delta_2}(\frac{kK}{B} - A)$ then E_1 is a saddle point with unstable manifold (u, u_1) and stable manifold (v, v_1). □

Theorem 5.4. *For $A < \frac{kK}{B}$, the steady state E_2 is asymptotically stable if $\alpha_1 + \alpha_2 > 0$, $\alpha_3\alpha_4(\delta_1 + \delta_2) + c_3^2 s^2 \delta_1\delta_2(\alpha_1 + \alpha_2) > 0$ and $\delta_1 > \delta_2$, where the functions $\alpha_1 = \frac{-au^*}{K} + \frac{cv^*u^*g}{(u^*+g)^2}$, $\alpha_2 = -\frac{ku^*v^*}{(v^*+B)^2}$, $\alpha_3 = \frac{kv^*}{v^*+B}$, $\alpha_4 = -\frac{cu^*}{u^*+g}$. Otherwise, it is unstable.*

Proof. Similarly, the Jacobian matrix at the equilibrium point E_2 takes the form

$$J_{E_2} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ M_2 & -\delta_1\beta & \frac{\delta_1 cu^*}{u^*+g} & 0 \\ \hline 0 & 0 & 0 & 1 \\ -\frac{\delta_2 kv^*}{v^*+B} & 0 & N_2 & -\delta_2\beta \end{array}\right), \tag{5.37}$$

where $M_2 = -\delta_1\left[\frac{cu^*v^*}{(u^*+g)^2} - \frac{au^*}{K}\right]$, $N_2 = \delta_2\left[\frac{ku^*v^*}{(v^*+B)^2}\right]$.

The characteristic polynomial associated with (5.37) is given by

$$p(\lambda) = d_4\lambda^4 + d_3\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0. \tag{5.38}$$

The Coefficients d_0, d_1, d_2, d_3, and d_4 are defined as

$d_0 = \delta_1\delta_2(\alpha_1\alpha_2 - \alpha_3\alpha_4)$, $d_1 = \beta\delta_1\delta_2(\alpha_1 + \alpha_2)$, $d_2 = \alpha_1\delta_1 + \alpha_2\delta_2 + \beta^2\delta_1\delta_2$, $d_3 = \beta(\delta_1 + \delta_2)$ $d_4 = 1$.

Based on the Routh-Hurwitz criteria [40],[58], all roots of the characteristic polynomial (5.38), which are the eigenvalues of the system (5.37) have negative real parts if and only if $d_n > 0$, $d_3d_2 > d_4d_1$ and $d_3d_2d_1 > d_4d_1^2 + d_3^2d_0$. Clearly, $d_1, ..., d_4 > 0$, for $\alpha_1 + \alpha_2 > 0, \delta_1 > \delta_2$.

$$\begin{aligned} d_0 &= \delta_1\delta_2(\alpha_1\alpha_2 - \alpha_3\alpha_4), \\ &= \delta_1\delta_2\left(\frac{u^*a}{K}\left(\frac{ku^*v^*}{(v^*+B)^2}\right) + \frac{cku^*v^*}{(u^*+g)(v^*+B)}\left(1 - \frac{v^*}{(v^*+B)}\frac{u^*}{(u^*+g)}\right)\right) > 0, \end{aligned} \tag{5.39}$$

Thus we show that (i) $d_3d_2 > d_4d_1$ and (ii) $d_3d_2d_1 > d_4d_1^2 + d_3^2d_0$

(i) $d_3d_2 > d_4d_1$, now

$$\begin{aligned} d_3d_2 - d_4d_1 &= \beta(\alpha_1\delta_1^2 + \alpha_2\delta_2^2 + \delta_1\delta_2\beta^2(\delta_1 + \delta_2)), \\ &> 0, \end{aligned} \tag{5.40}$$

since

$$\delta_1 > \delta_2, \quad \alpha_2 < 0 \ \text{ and } \ \alpha_1 + \alpha_2 > 0, \tag{5.41}$$

$(ii) d_3d_2d_1 > d_4d_1^2 + d_3^2d_0$ implies that

$$\begin{aligned} d_3d_2d_1 - d_4d_1^2 - d_3^2d_0 &= \beta^2\delta_1\delta_2\left((\alpha_1\delta_1 - \alpha_2\delta_2)^2\right. \\ &\left.(\delta_1 + \delta_2)\left(\alpha_3\alpha_4(\delta_1 + \delta_2) + \beta^2\delta_1\delta_2(\alpha_1 + \alpha_2)\right)\right) \\ &> 0 \end{aligned} \tag{5.42}$$

as long as (5.41) holds and $\alpha_3\alpha_4(\delta_1 + \delta_2) + c_3^2s^2\delta_1\delta_2(\alpha_1 + \alpha_2) > 0$, since $\alpha_3\alpha_4 < 0$. $\square$

5.3 Turing instability and Pattern formation

5.3.1 Turing instability

A steady state is Turing unstable if it is stable as a solution of the temporal system, i.e. the system without diffusion terms, but unstable as a solution of the reaction-diffusion system [85, 28, 65]. In other words, the Turing instability implies that the steady state is stable in the local system and will become unstable due to diffusion of populations. As a result of the instability of the reaction-diffusion system, spatial pattern formations usually occur.

In this subsection we establish the condition necessary and sufficient for Turing instability to occur as a result of the introduction of diffusion. Observe that the three biologically meaningful equilibria of the system (5.24) correspond to the equilibria of the diffusion model (5.2) as well as the non-diffusive model (5.1) in the first quadrant $\Re^2_+$, namely

- Total extinction of the two species: $E_0(0, 0)$.
- Predator free steady state: $E_1(K, 0)$.

- Coexistence of species with interior equilibrium population: $E_2(u^*, v^*)$.

Hence, the conditions stated in theorems of subsection 5.2.3 also apply here. From this point onwards we focus on the study of the conditions for Turing instability around the positive interior equilibrium point E_2, because of its biological relevance. Next we derive the necessary and sufficient conditions for Turing instability to occur in (5.2) where the positive interior equilibrium point $E_2(u^*, v^*)$ is stable in the absence of diffusion and unstable due to the addition of diffusion, under a small perturbation to $E_2(u^*, v^*)$. This is achieved by first linearizing the model (5.2) around $E_2(u^*, v^*)$ for both space and time-dependent fluctuations. This is given as

$$\begin{aligned} u(t,x,y) &= u^* + \bar{u}(t,x,y), \quad \bar{u}(t,x,y) \ll u^*, \\ v(t,x,y) &= v^* + \bar{v}(t,x,y), \quad \bar{u}(t,x,y) \ll u^*, \end{aligned} \tag{5.43}$$

$$\text{with} \begin{pmatrix} \bar{u}(t,x,y) \\ \bar{v}(t,x,y) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} e^{\mu t + (\omega_x x + \omega_y y)i}, \tag{5.44}$$

where μ is the perturbation growth rate in time $t; \beta_1, \beta_2$ are the corresponding amplitudes; $\omega = (\omega_x, \omega_y)$ is the wave number. Substituting equations (5.43)-(5.44) into (5.2) and neglecting all nonlinear terms in u and v, one obtains the characteristic equation,

$$|J - \mu I - \omega^2 D| \tag{5.45}$$

where

$$J = \begin{pmatrix} \frac{cv}{(g+u)^2} - \frac{au}{K} & -\frac{cu}{g+u} \\ \frac{kv}{B+v} & -\frac{kuv}{(B+v)^2} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_4 \\ \alpha_3 & \alpha_2 \end{pmatrix}, D = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \tag{5.46}$$

and I is a 2×2 identity matrix. The characteristic polynomial of (5.45) is

$$\mu^2 + \sigma_1(\omega^2)\mu + \sigma_0(\omega^2), \tag{5.47}$$

where

$$\begin{aligned} \sigma_1(\omega^2) &= \omega^2(a_1 + a_2) - (\alpha_1 + \alpha_2), \\ \sigma_0(\omega^2) &= \omega^4 a_1 a_2 - \omega^2(\alpha_1 a_2 + \alpha_2 a_1) + \alpha_1\alpha_2 - \alpha_3\alpha_4. \end{aligned} \tag{5.48}$$

The eigenvalues of (5.47) are given by

$$\mu_{1,2}(\omega^2) = \frac{-\sigma_1(\omega^2) \pm \sqrt{(\sigma_1(\omega^2)^2 - 4\sigma_0(\omega^2)}}{2}. \tag{5.49}$$

In the absence of diffusion ($a_1 = 0, a_2 = 0$), the steady state E_2 is stable if and only if the real part of the eigenvalues is negative,

$$\mu_{1,2}(\omega^2) < 0. \tag{5.50}$$

This is possible when $\sigma_1(\omega^2) > 0$ and $\sigma_0(\omega^2) > 0$. From equation (5.48), this implies that $\alpha_1 + \alpha_2 < 0$ and $\alpha_1\alpha_2 - \alpha_3\alpha_4 > 0$. Hence, the conditions for the steady state E_2 to be stable for the non-diffusive model of (5.2) are given by

$$\alpha_1 + \alpha_2 < 0 \tag{5.51a}$$

$$\alpha_1\alpha_2 - \alpha_3\alpha_4 > 0 \tag{5.51b}$$

With diffusion ($a_1 \neq 0, a_2 \neq 0$) we look for conditions where the steady state E_2 will be unstable for certain ω values. Meaning, that around the steady state E_2, we require atleast one of the real values of the eigenvalues

$$\mu_{1,2}(\omega^2) > 0, \text{ for } \ \omega \neq 0. \tag{5.52}$$

This can only be achieved when $\sigma_1(\omega^2) < 0$ or $\sigma_0(\omega^2) < 0$ for $\omega \neq 0$, this implies that diffusion-driven instability occurs when either $\sigma_1(\omega^2) < 0$ or $\sigma_0(\omega^2) < 0$. Since $\alpha_1 + \alpha_2 < 0$ from equation (5.51) and $\omega^2(a_1 + a_2) > 0$ for all $\omega \neq 0$, $\sigma_1(\omega^2) > 0$. So $\sigma_0(\omega^2) < 0$ for $\omega \neq 0$ is the only condition that will give rise to diffusion-driven instability and this happens when

$$a_2\alpha_1 + a_1\alpha_2 > 0. \tag{5.53}$$

We obtain from the conditions (5.51a) and (5.53):

$$\frac{a_2}{a_1} > -\frac{\alpha_2}{\alpha_1} > 1. \tag{5.54}$$

Thus, we further require

$$a_2\alpha_1 + a_1\alpha_2 > 0, \ a_2 > a_1. \tag{5.55}$$

According to (5.55), we observe that one of the necessary conditions for Turing instability is $a_2 > a_1$ which indicates that diffusivity of the elephant is greater than that of the trees. It must be noted that diffusive instability cannot occur for $a_1 = a_2$ and also for large diffusion coefficient of the trees provided the diffusion coefficient of the elephant be large enough. The inequalities (5.55) are necessary but, however, not sufficient conditions for $\sigma_0(\omega^2) < 0$, and we must require that the minimum of $\sigma_0(\omega^2)$ be negative for non zero ω. To achieve this, we need to find the critical wave number (ω_T^2) such that $\sigma_0(\omega^2 = \omega_T^2) < 0$. This critical value occurs when

$$\frac{\partial \sigma_0}{\partial \omega^2} = 0, \tag{5.56}$$

this when solved for ω^2 yields

$$\omega^2 = \omega_T^2 = \frac{\alpha_2 \alpha_1 + a_1 \alpha_2}{2 a_1 a_2} > 0. \tag{5.57}$$

Therefore, the condition $\sigma_0(\omega^2 = \omega_T^2) < 0$ turns into

$$(a_2 \alpha_1 + a_1 \alpha_2)^2 - 4 a_1 a_2 (\alpha_1 \alpha_2 - \alpha_3 \alpha_4) > 0. \tag{5.58}$$

Finally, we can summarize all the Turing instability conditions obtained from the entire analysis as follows:

(i) $\alpha_1 + \alpha_2 < 0$,
i.e, $\frac{cu^* v^*}{(u^*+g)^2} - \frac{au^*}{K} - \frac{ku^* v^*}{(v^*+B)^2} < 0$.

(ii) $\alpha_1 \alpha_2 - \alpha_3 \alpha_4 > 0$,.
i.e, $\frac{u^* a}{K}\left(\frac{ku^* v^*}{(v^*+B)^2}\right) + \frac{cku^* v^*}{(u^*+g)(v^*+B)}\left(1 - \frac{v^*}{(v^*+B)}\frac{u^*}{(u^*+g)}\right) > 0$.

(iii) $a_2 \alpha_1 + a_1 \alpha_2 > 0,\ a_2 > a_1$
i.e, $a_2\left(\frac{cu^* v^*}{(u^*+g)^2} - \frac{au^*}{K}\right) - a_1\left(\frac{ku^* v^*}{(v^*+B)^2}\right) > 0,\ a_2 > a_1$.

(iv) $a_2 \alpha_1 + a_1 \alpha_2 - 2\sqrt{a_1 a_2 (\alpha_1 \alpha_2 - \alpha_3 \alpha_4)} > 0,\ a_2 > a_1$
i.e, $a_2\left(\frac{cu^* v^*}{(u^*+g)^2} - \frac{au^*}{K}\right) - a_1\left(\frac{ku^* v^*}{(v^*+B)^2}\right)$
$- 2\sqrt{a_1 a_2\left(\frac{u^* a}{K}\left(\frac{ku^* v^*}{(v^*+B)^2}\right) + \frac{cku^* v^*}{(u^*+g)(v^*+B)}\left(1 - \frac{v^*}{(v^*+B)}\frac{u^*}{(u^*+g)}\right)\right)} > 0,\ a_2 > a_1$.

If the above four conditions (i) $-$ (iv) are satisfied then the spatially homogeneous stable state E_2 becomes unstable to perturbation within the range of wave numbers $\omega_1^2 < \omega^2 < \omega_2^2$ where

$\omega_1^2,\ \omega_2^2$ are the roots of the equation $\sigma_0(\omega^2) = 0$ and are given as

$$\omega_{1,2}^2 = \frac{a_1\alpha_2 + a_2\alpha_1 \pm \sqrt{(a_1\alpha_2 + a_2\alpha_1)^2 - 4a_1a_2(\alpha_1\alpha_2 - \alpha_3\alpha_4)}}{2a_1a_2}. \tag{5.59}$$

The Turing-bifurcation curve is obtained by setting (5.58) to zero and is given by the following equation

$$P(\Gamma) = \frac{(\Gamma\alpha_1 + \alpha_2)^2}{4\Gamma} - (\alpha_1\alpha_2 - \alpha_3\alpha_4) = 0 \text{ with } \Gamma = \frac{a_2}{a_1}. \tag{5.60}$$

To find an estimate value of the ratio of diffusivity Γ_c above which the Turing instability occurs, in Fig 5.2 we plot a graph of (5.60). The parameter values are chosen to satisfy the Turing instability conditions (i) $-$ (iv) and are given by

$$a = 0.1, \quad A = 0.5, \quad B = 0.09, \quad k = 0.8, \quad K = 50, \quad g = 0.5, \quad c = 0.1. \tag{5.61}$$

With this set of parameter values, the coexistence state $E_2(u^*, v^*) = E_2(0.9383, 1.4114)$. The above set of parameter values (5.61) should be used in subsequent simulations in this subsection. The results of figure 5.2 indicate that when $\Gamma > 34.6$, the Turing pattern appears to emerge. In figures 5.3 (a) and (b), we observed that where the ratio of diffusivity Γ is above its critical value $\Gamma_c = 34.6$, there is a range of values for the wave number ω for which $\sigma_0 < 0$ and $Re(\mu) > 0$, i.e, the Turing patterns emerge. Figure 5.3 (c) is plotted for comparison, we see that $\sigma_0 = Re(\mu) = 0$ at certain range of values for the wave number ω.

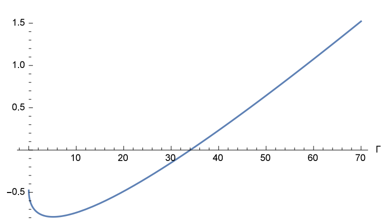
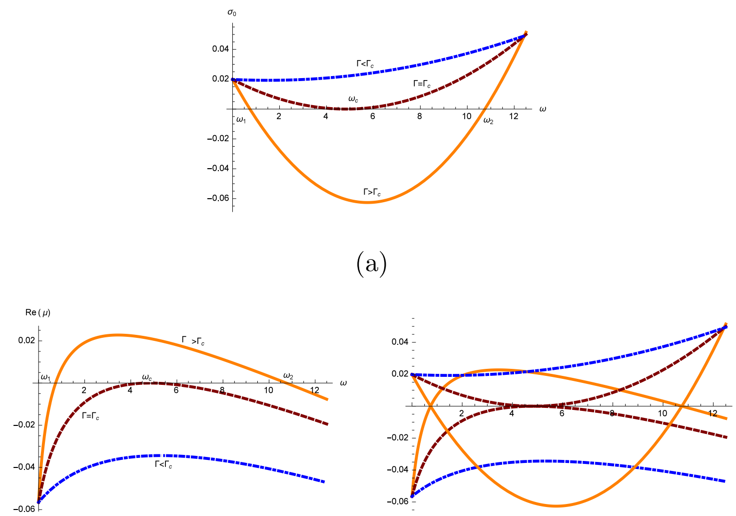


Figure 5.2: Emergence of Turing pattern corresponding to equation (5.60)

5.3.2 Pattern structure

Turing spacial patterns for the Caughley model (5.2)

We have demonstrated earlier in the subsection 5.3.1 that the existence and non existence of



(a)

(b) (c)

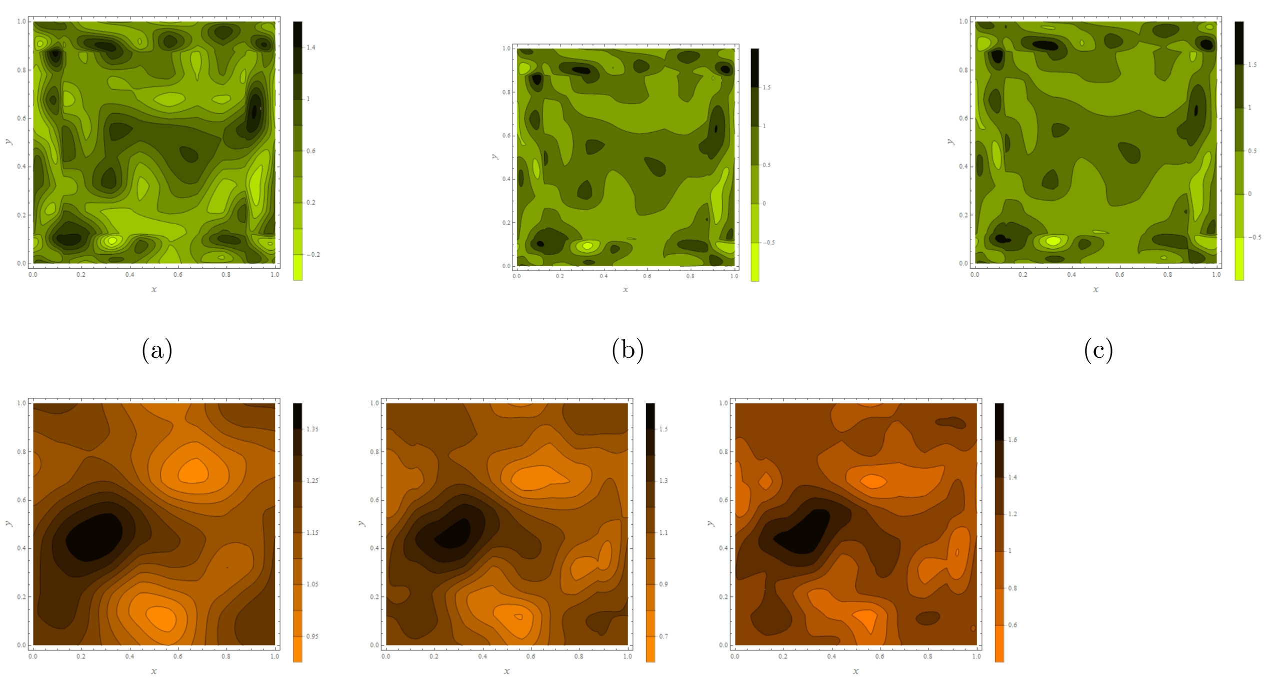
Figure 5.3: (a) σ_0 (b) $Re(\mu)$ (c) $\sigma_0 = Re(\mu)$

Turing patterns are dependent upon the magnitude of diffusion coefficient ratio Γ. In this subsection 5.3.2, we carry out numerical simulations of the diffusive Caughley model (5.2) in two dimensional space within the Turing region for different values of the diffusion coefficient ratio Γ and for different times. The spacial structures developed by trees shown in green color and that of elephants shown in blown color at different instants of time and for different diffusion coefficients are exhibited in Figs 5.4a-f and Figs 5.5 a-f. It is interesting to note that for $\Gamma = 125$, the random initial perturbations lead to the formation of labyrinth patterns of long stripes with spots see figures 5.4 (a), (d) and figures 5.5 (a), (d). However, as the value of Γ increases from 125 to 250, the number of the spotted area in the space region also increases and the number of labyrinth decreases see figures 5.4 (b), (e) and figures 5.5 (b), (e). Finally, if we increase the diffusion coefficient to 500, system (5.2) exhibits a pattern transition from labyrinth to spotted pattern figures 5.4 (c), (f) and figures 5.5 (c), (f).

Ecologically, spotted patterns are isolated zones with high population densities [21].

Turing wave patterns for the Caughley model (5.2)

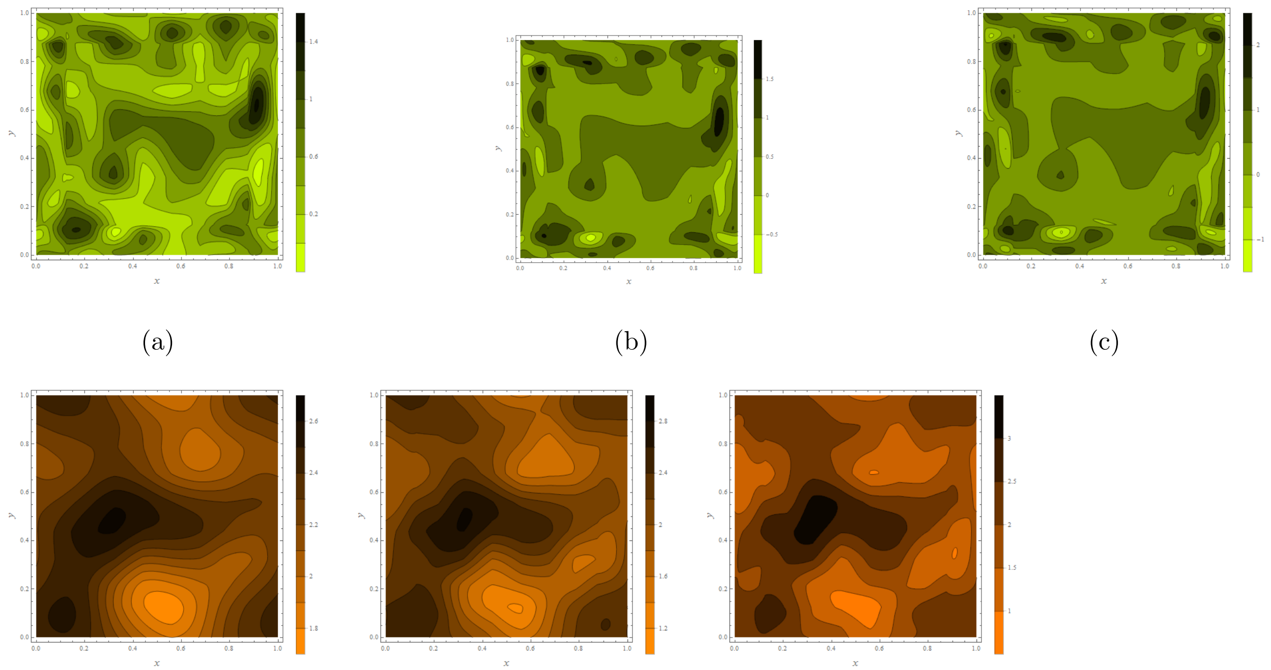
To learn more about these dynamical processes in terms of Turing wave patterns, in figures 5.7 and 5.6, we plot the travelling wave profile of system (5.23) for elephant and trees within the Turing space with different ratio of diffusion. From a realistic biological point of view, we take the non-trivial state for the coexistence of tree and elephant, $E_2(u^*, v^*) = (0.9383, 1.4114)$ as the initial condition. In order to avoid negative population density which is not biologically sensible, we consider the minimum speed condition (5.32) and choose the minimum speed $\beta = 2\sqrt{aa_1}$. Other parameters are fixed as in (5.61). In Figs 5.7 and 5.6, we notice that the patterns formed by the travelling wave system (5.23) are characterised by chaotic waves with wavelength decreasing as the ratio of diffusion coefficient Γ increases. The waves travel a lesser distance in space over a given time as Γ increases. One can conclude that the interaction and distribution of elephants and trees within a two dimensional land scape is being regulated by ratio of diffusivity. This scenario observed in figures 5.7 and 5.6 due to the change in ratio of diffusion coefficient Γ can also lead to the existence of isolated zones over a region with high population population density. Hence, one can interpret that diffusion can lead to clumps of high tree and elephant populations in areas where respectively, elephant and tree densities are low as observed in figures 5.4 (c), (f) and figures 5.5 (c), (f).



(a) (b) (c)

(d) (e) (f)

Figure 5.4: Different patterns observed in the diffusion model for elephant and tree when $t = 400$ years. (a);(d): $\Gamma = 125$, (b);(e): $\Gamma = 250$,(c);(f): $\Gamma = 500$.



(a) (b) (c)

(d) (e) (f)

Figure 5.5: Different patterns observed in the diffusion model for elephant and tree when $t = 800$ years. (a);(d): $\Gamma = 125$, (b);(e): $\Gamma = 250$,(c);(f): $\Gamma = 500$.

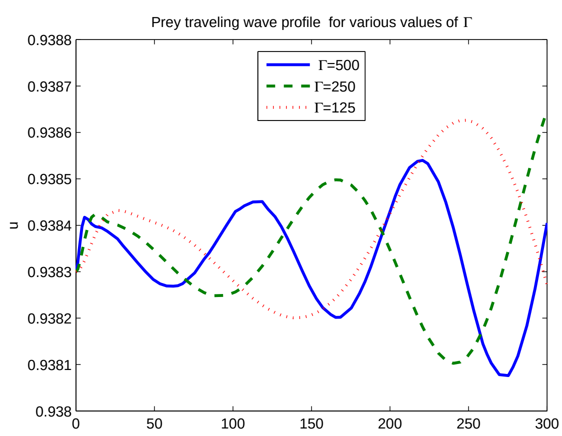


Figure 5.6: Tree travelling wave profile in (5.23) within the Turing region for various values of diffusion parameter, Γ.

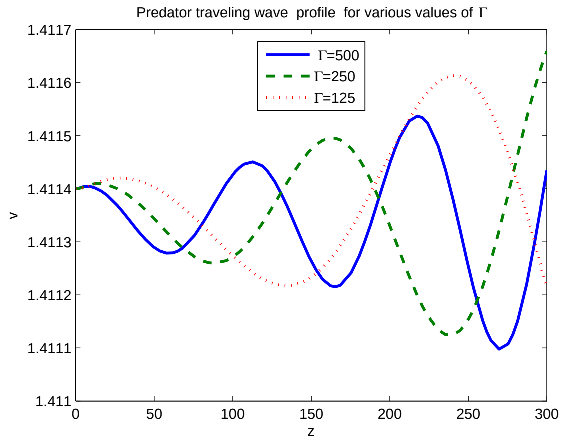


Figure 5.7: Elephant travelling wave profile in (5.23) within the Turing region for various values of diffusion parameter, Γ.

5.4 Conclusion

In this paper, the complex dynamics of a reaction diffusion predator- prey Caughley type model was investigated through exhaustive analysis and carefully designed numerical simulations. Using symmetry analysis, an optimal system comprising of a complete set of representations of invariant solutions for the spatially diffusive Caughley type model (5.2) was proposed. These representations gave rise to reduction of this system of nonlinear second order PDEs (5.2) into a number of reduced systems of second order ODEs. The reduced systems provide basis for further analysis of the model. Through the analysing of one of the reduced systems of the ODEs we proved the existence of a travelling wave solution thus showing that the coexistence of both elephants and trees in the system is only possible if the elephant death rate, A is less than a survival threshold $\frac{kK}{B}$. Moreover, a minimum wave speed which regulates the stable growth of the two species populations was derived.

Furthermore, a detailed study of the effects of diffusion on the stability of the system was carried out. In subsection 5.3.1, mathematical analysis showed that the diffusion rates can destabilise the growth around spatially homogeneous solutions of the model which initiates the

Turing instability leading to different pattern structures. An important observation is noted when the elephant diffusion rate is large provided the tree diffusion rate is small enough; the Turing instability is more likely to occur, otherwise when both rates are equal.

Numerical simulations were carried out to support the analysis which confirmed that the system (5.2) exhibits a pattern transition from labyrinth pattern to spotted pattern (see Figures 5.4,5.5). The images represent an increase of values of the ratio of the elephant's diffusion rate versus the tree's diffusion rate. The same scenario was observed for the wave patterns obtained from the numerical solutions of the reduced system (5.23) (Figure 5.7), which reaffirms the consistency of the obtained invariant solutions with the solutions of the original system (5.2).

However, the results show that the system dynamics exhibits complex pattern transformation controlled by the diffusion. Therefore, one can predict that the effect of diffusion coefficient can be considered as an important mechanism for the appearance of different patterns of solutions of Caughley type predator-prey model, more so ecological models of similar type.

This varied approach presented in this study shows the interplays between the different methodologies which provide a comprehensive technique to solving complicated ecological systems.

Chapter 6

Conclusion

In this study we investigated the fundamental structures and underlining features of some important systems of PDEs in mathematical physics via symmetry analysis and conservation laws.

Moreover we demonstrated how the use of symmetry analysis in the study of mathematical models in ecology complements the mathematical techniques (qualitative and numerical analysis) traditionally used.

We began our investigation by analysing a nonlinear PDE, known as the Kumaroto-Shivinsky equation which describes elasto-plastic flow in the medium with dispersive effect. We investigated the symmetry classification of the equation and observed that the equation does not admit space dilation type symmetries for a specific parameter value. The symmetry reductions and exact solutions of the equation were obtained using an optimal system. The conservation laws were derived via Noether approach.

We investigated two important systems of nonlinear evolution equations found in mathematical physics, namely, the generalised Boussinesq (GB) equation with damping term and the Variant Boussinesq (VB) equation. The generalised Boussinesq (GB) equation with damping term is widely used to describe natural phenomena in many scientific fields such as plasma waves, solid physics and fluid mechanics. This equation is not derivable from a variational principle, and hence does not possess a standard Lagrangian. We examined the conserved vectors using the partial Noether approach.

We discovered that the derived conserved vectors failed to satisfy the divergence relation due to the presence of the mixed derivative term in the equation. The conserved vectors were then adjusted to absorb the extra term. As a result new forms of the conserved vectors satisfying the divergence condition were found. The importance of these conservation laws in finding the exact solutions were clearly shown via the double reduction method, which involves the relationship between conservation laws and symmetries. The solutions obtained portray physical features of the system. A similar analysis is carried out to obtain new exact solutions for the system of Variant Boussinesq (VB) equations.

We further studied a generalised third order Gardner equation with dual power law nonlinearity widely used in quantum field theory, solid state, plasma and fluid physics. Its conservation laws were discussed via the multiplier method and Noether approach (after increasing the order by one). We found that there are six local conservation laws when the nonlinear power law is quadratic. The Noether approach also leads to a number of nonlocal conservation laws. We investigated the exact solutions using both symmetry generators and the double reduction technique. We found that the double reduction method leads to more solutions than would be obtained by symmetry analysis alone.

Finally, the importance of group theory in the analysis of equations which arise during investigations of reaction-diffusion prey-predator mechanisms is illustrated. In this study symmetry analysis was employed to reduce the system. We showed the existence of a travelling wave solution thus indicating that the coexistence of both vegetation consuming mammals and the vegetation in the system is only possible if the mammals death rate is less than a survival threshold.

Furthermore, a detailed study of the effects of diffusion on the stability of the system was carried out. The diffusion rates can destabilise the growth around spatially homogeneous solutions of the model which initiates the Turing instability leading to different pattern structures. An important observation is that when the mammals diffusion rate is large, provided the vegetation diffusion rate is small enough; the Turing instability is more likely to occur, than when both rates are equal. Based on this analysis, important biological observations were made on the solutions of the reduced system for varied diffusion parameters.

Future research

The finite difference scheme and the numerical analysis of the models studied in thesis are not discussed. These fall outside the scope of this work.

The numerical solutions and other solutions of the models via finite difference scheme should be considered as future investigation, to confirm the consistency or similarity of the obtained exact solutions.

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