# DYNAMICS OF RADIATING STARS IN THE STRONG GRAVITY REGIME

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# Dynamics of radiating stars in the strong gravity regime

by

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As the candidate's supervisors, we have approved this dissertation for submission.

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# Abstract

We model the dynamics of a spherically symmetric radiating dynamical star, emitting outgoing null radiation, with three spacetime regions. The local internal atmosphere is a two-component system consisting of standard pressure-free, null radiation and an additional string fluid with energy density and nonzero pressure obeying all physically realistic energy conditions. The middle region is purely radiative which matches to a third region which is the Schwarzschild exterior. A large family of solutions to the field equations are presented for various realistic equations of state. A comparison of our solutions with earlier well known results is undertaken and we show that all these solutions, including those of Husain, are contained in our family. We then generalise our class of solutions to higher dimensions and consider the effects of diffusive transport We also study the gravitational collapse in the context of the cosmic censorship conjecture. We outline the general mathematical framework to study the conditions on the mass function so that future directed nonspacelike geodesics can terminate at the singularity in the past. Mass functions for several equations of state are analysed using this framework and it is shown that the collapse in each case terminates at a locally naked central singularity. These singularities are strong curvature singularities which implies that no extension of spacetime through them is possible. These results are then extended to modified gravity. We establish the result that the standard Boulware-Deser spacetime can radiate. This allows us to model the dynamics of a spherically symmetric radiating dynamical star in five-dimensional Einstein-Gauss-Bonnet gravity with three spacetime regions. Finally, the junction conditions are derived entirely in five-dimensional Einstein-Gauss-Bonnet gravity via the matching of two spacetime region leading to a model for a radiating star in higher order gravity.

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# Declaration 2 - Publications

The content contained within the pages of this dissertation is partially based upon the following research papers (published and in preparation) in peer-reviewed journals:

**Publication 1**

Brassel B P, Maharaj S D and Govender G, Analytical models for gravitating radiating systems, *Advances in Mathematical Physics* **2015**, 274251 (2015).

**Publication 2**

Govender G, Brassel B P and Maharaj S D, The effect of a two-fluid atmosphere on relativistic stars, *Eur. Phys. J. C* **75**, 324 (2015).

**Publication 3**

Brassel B P, Maharaj S D and Goswami R, Diffusive and dynamical radiating stars with realistic equations of state, *Gen. Relativ. Gravit.* **37**, 49 (2017).

**Publication 4**

Brassel B P, Goswami R and Maharaj S D, Collapsing radiating stars with various equations of state, *Phys. Rev. D* **95**, 124051 (2017).

**Publication 5**

Brassel B P, Maharaj S D and Goswami R, The Boulware-Deser class of spacetimes radiates, *Gen. Relativ. Gravit.* **49**, 101 (2017).

**Publication 6**

Brassel B P, Maharaj S D and Goswami R, Gravitational collapse to regular black holes in five-dimensional Einstein-Gauss-Bonnet gravity, in preparation (2017).

# Dedication

*Dedicated to the*

***Carcharodon carcharias***

*May the dark blue worlds they reside in continue to exist sempiternally.*

# In Memoriam

*James Roy Horner*

*(1953 - 2015)*

*Flights of angels, dear sweet and beautiful soul.*

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*Adspicit lucem in caelestis... sic itur ad astra.*

Byron P. Brassel

August 2017

# Contents

# Chapter 1

# Introduction

The general theory of relativity has so far been the most quintessential in describing the notion of gravity in the context of strong gravitational fields. General relativity describes the interaction between bodies in the universe as a result of their gravitational fields and with this, the richness of gravity becomes more evident and it can no longer be regarded as an amorphous force. It is part of a more efficacious structure, the spacetime manifold. The reader is encouraged to seek out Eddington (1923), Poison (2004) and Straumann (2004) for further insights into the principles of general relativity.

The very first solutions of Einstein's field equations are the Schwarzschild exterior and interior solutions (Schwarzschild 1916a, 1916b), and the Reissner-Nordström solution (Nordström 1918, Reissner 1916). The exterior Schwarzschild solution is a vacuum solution which describes the gravitational field in the exterior spacetime of a spherical, uncharged and non-rotating body whereas the interior Schwarzschild solution describes the interior of an object in the limit when the mass density is a constant. The Reissner-Nordström solution is a static solution to the Einstein-Maxwell field equations and represents the exterior gravitational field of a charged, non-rotating body.

In general, a realistic astrophysical star would consist of a rapidly rotating ball of perfect fluid matter and radiation, all entangled together. However, within the context of general relativity, modeling such systems is extremely complicated and analytically difficult. Therefore we impose spherical symmetry and use a special type of energy momentum tensor, that gives rise to the generalised Vaidya spacetime. However, such models are more general than those which contain perfect fluid matter and no radiation. The matter distribution for a model in cosmology or astrophysics is usually described by a relativistic fluid. Radiation and matter are assigned to an energy momentum tensor (which can take on several forms depending on whether the fluid distribution is static or radiating). It is conserved throughout the spacetime and is the source of the gravitational field for the Einstein field equations in the same way that the mass density is a source in Newtonian gravity.

The geometry outside a spherically symmetric radiating star is described by the Vaidya spacetime (Vaidya 1951) and it defines outgoing null radiation. It is written in terms of the mass of the radiating body and the Petrov-Pirani-Penrose classification of the metric is of type D (Petrov 1954, 2002, Pirani 1957, Penrose 1960). The result (Vaidya 1951) provided an advance in the modeling process and the possibility then arose to study the interior of radiating stars by matching the interior solution to the radiating exterior (de Oliviera 1985, 1986, 1988, Kramer 1992, Govender *et al* 1998). The complete derivation of the junction conditions for a shear-free radiating star was provided by Santos (1985). The important result that followed was that the pressure on the boundary of the radiating star should be nonzero in general, and proportional to the heat flux. It should be noted that this framework describes only the emission of pressureless null radiation (photons) into the exterior region of the dissipating star, and no other outflow of any other type of observable radiation. The effects of radiation are important in the latter stages of gravitational collapse and allows for a surrounding zone of radiation.

In recent times, Maharaj *et al* (2012) have generalised the Santos junction condition by matching the interior geometry of a manifold, containing a shear-free heat conducting fluid, to the exterior geometry described by the generalised Vaidya metric which contains the additional type II null fluid. This result provided an even wider array of opportunities and possibilities with regards to the modeling of relativistic objects in astrophysics and cosmology. More importantly, the result provided a more general exterior region for a radiating star, which is made up of a two-fluid system: a combination of the standard null radiation as in the case of the original Santos framework, and an additional more general fluid distribution which can be taken to be another form of radiation or, perhaps more interestingly, a field of particles such as neutrinos or other exotic non-interacting matter. Another interesting feature of the generalised junction condition is the fact that the radiating fluid pressure at the boundary is not only proportional to the heat flux, but also coupled to the non-vanishing energy density of the type II null fluid.

A significant amount of study on relativistic radiating stars has been carried out in the standard Santos framework. The junction conditions were generalised, for example, to include the effects of an electromagnetic field as well as shearing anisotropic stresses during dissipative stellar collapse by de Oliveira *et al* (1987) and Maharaj and Govender (1999). Analytical solutions for shear-free non-adiabatic collapse in the presence of electric charge were obtained by Pinheiro and Chan (2013). Schäfer and Goenner (2000) studied a highly idealised model with constant luminosity radiating away its mass showing that an event horizon never forms. Nonlinear models for relativistic stars in the shear-free regime were found with heat flow by Misthry *et al* (2008) using a transformation that reduced the boundary condition to a simpler form in the conformally flat zone. Abebe *et al* (2014, 2014) utilised the Lie symmetry analysis for studies on geodesic models and radiating Euclidean stars with an equation of state. Govender *et al* (2015) modeled the physical behaviour at the surface of a radiating star. They investigated the effect of the exterior energy density on the temporal evolution of the radiating fluid pressure, luminosity, gravitational redshift and mass flow at the boundary of a relativistic star.

It is often required that for a radiating stellar model to have a more realistic physical form, a barotropic equation of state must be imposed on the fluid distribution. In most cases the equation of state becomes a Cauchy-Euler differential equation. Several attempts have been made to model such situations. Wagh *et al* (2001) made use of a linear equation of state in spacetimes which are shear-free and Goswami and Joshi (2004) studied the gravitational collapse of an isentropic perfect fluid distribution with a linear equation of state. An important note to make is that the notion of a type II fluid existing in the exterior region of the radiating star has been studied in isolation without any direct connection to the interior matter conglomeration. Nonstatic spherically symmetric solutions to Einstein's field equations with a null fluid source were obtained by Husain (1996) in general for such an exterior fluid with a polytropic equation of state $P = k\rho^a$. He demonstrated that for a linear equation of state ($a = 1$) and varying values of the constant $k$, the metrics were either asymptotically flat ($1/2 < k < 1$) or cosmological ($0 < k < 1/2$). The value $k = 1$ yielded the charged Vaidya solution. Finally, it was shown that in the long time limit, the asymptotically flat spacetimes were hairy black hole solutions. Dawood and Ghosh (2004) characterised a large family of solutions to Einstein's equations representing a spherically symmetric type II fluid, and showed that the well known dynamical black hole solutions are a particular case of this larger family. Ghosh and Dawood (2008) then generalised these results to higher dimensions. An appraisal was conducted by Wang and Wu (1999) where the ideas of Husain and others were extended and further classes of solution were obtained. Contained within these results are the well known monopole solution, the de Sitter and anti-de Sitter models, the charged Vaidya solution, the Husain solution and the radiating dyon solution. It should be noted that these solutions were obtained via a method of assuming a series form for the gravitational mass function in the field equations. In our study below, we will attempt to integrate the field equations directly subsequent to assuming an equation of state.

Another important notion to consider is that of gravitational collapse. Maharaj and Govender (2005) studied collapse models with an internal isotropic pressure and vanishing Weyl stresses and probed the dynamical stability of the dissipating stellar fluid.

They found that close to the centre, the configuration was more unstable. The thermal evolution of a radiating fluid is vital in any stellar model and Martinez (1996), Herrera and Santos (1997) and Govender *et al* (1999) studied the explicit role of relaxation and mean collision time in these frameworks. A further investigation of these ideas was carried out by Naidu *et al* (2006), Naidu and Govender (2008) and Maharaj *et al* (2012) where the latter authors investigated the gravitational collapse of a radiating sphere evolving into a final static configuration described by the interior Schwarzschild solution. More recently, Mkenyeleye *et al* (2014) studied the gravitational collapse of the Vaidya spacetime in the context of the cosmic censorship hypothesis. Developing a general mathematical template, they showed that there exist classes of generalised Vaidya mass functions in which the collapse reaches an end state with a locally naked central singularity.

When a massive star of mass greater than 8 solar masses reaches the end of the luminous phase of its life, it experiences an inwardly directed gravitational collapse. This is a very violent process which occurs on timescales of the order of seconds and is observed as a type II supernova. The entire collapse process is usually divided into an early, intermediate and late stage. The effects of radiation are important in the later stages of gravitational contraction when an immense amount of energy is ejected from the star in the form of neutrinos or photons.

The notion of collapse was first brought to light by Oppenheimer and Snyder (1939), and they described the free-fall contraction of a spherical body in which pressure forces were completely overwhelmed by the gravitational forces. The equations of collapse, were analysed analytically by Misner (1965), Shapiro and Teukolsky (1983), Goswami and Joshi (2004), Misthry *et al* (2008) and Maharaj *et al* (2012) and numerically by May and White (1966), Bodenheimer *et al* (2007), Kuroda and Umeda (2010) and Müller *et al* (2012). These works have produced significant new insights into gravitational collapse. A massive stellar object (or supermassive object of 40 solar masses or more), in its very long life, will exist in a state of suspended collapse, converting its hydrogen into helium, carbon, neon, oxygen, magnesium and silicon through nucleosynthesis creating an internal pressure gradient resulting in the release of outward energy (radiation, convection and conduction). Thermonuclear fusion ends at iron-56, the most bound nuclear species. Beyond iron, fusion is no longer exothermic. The reader is encouraged to seek out Glendenning (2000) for further insights into gravitational contraction.

The process of gravitational collapse is usually complicated. Once the hydrogen has burned out in the core, the next phase of thermonuclear burning - helium - commences (hydrogen in some surrounding shell will continue to burn). The helium which builds up in the core undergoes an increasingly intense compression, until these helium atoms commence fusion into heavier elements like carbon, neon, oxygen and so on. Concentric burning shells are created as one element after the other is synthesized. Enormous amounts of gamma rays in the core produce electron-positron pairs which annihilate, producing neutrino pairs. At the exhaustion of each elemental fuel, the core contracts further until the ignition temperature for the next step in the next chain is attained. Each successive burning stage is quicker than the preceding one. Iron-56 is the end point of nucleosynthesis (the result of silicon burning). Burning in the outer shells of the star add to the now iron core's mass and since iron will not easily fuse to form a heavier element, it remains as such and a hydrodynamical instability sets in, where the inward pressure of gravity in the star will begin to overwhelm the outward energy being released. Gravity crushes the core to such an extent that electrons become relativistic and the pressure they provide increases less rapidly with increasing density.

The inward gravity crushes the iron core very quickly, and it becomes extremely hot ($\sim 10^{11} K$). The infalling material in the core, overshoots the equilibrium configuration and rebounds from the stiffened core, acting in a similar way to a piston. The catastrophic result of this induces what is called a post-bounce-pre-supernova shockwave which will propagate outward from some point within the collapsing core reaching relativistic speeds. The heat energy released in the process is transported away from the core through the interior and across the stellar surface by neutrinos and it is this shockwave that drives the outflow of the neutrinos. A physically reasonable relativistic model for gravitational collapse should include these features. In view of this, Glass (1990) modeled the emission of neutrinos in dissipative collapse and Herrera and Núñez (1987) and Barreto (1993) investigated the associated shock structure and propagation in the interior of a radiating star. As this shockwave travels outward, its energy is dissipated by neutrino losses and by photodisintegration of all the nuclei in its path, and will eventually stall. The plasma material that surrounded the pre-collapsed core will also, very quickly, fill the space now available, and this notion produces a decompression shockwave which travels outward at the speed of sound in the diffuse stellar material, and this material now begins to freefall. The freefalling material is seized as it meets the stalled shock front, and this turns the latter into an accretion shock, which is heated by this infalling matter. A rarefied bubble-like region develops between the highly dense core and the accreting shock front. Neutrino pairs diffusing from the extremely hot interior, annihilate, heat up and expand the bubble. Through a complex sequence of events, i.e. some interplay of convection and neutrino heating, a fraction of the deeply intense gravitational binding energy of the dense core remnant is transported to the accretion front. This small fraction provides the kinetic energy for the ejection of all but the core remnant of the progenitor star in what is called a type II supernova explosion. The resulting supernova remnants and this core remnant are pushed away from each other during this phase. The reader is encouraged to seek out Smoller and Temple (1997) and Temple and Smoller (1999) for further information on general relativistic shock wave theory.

What is left behind is a very dense, compact and hot core remnant which becomes, if the mass of the initial star was large, a proto-neutron star. Otherwise it becomes a white dwarf. The resulting compact object is usually supported by either electron degeneracy pressure in the case of a white dwarf or neutron degeneracy pressure in the case of a neutron star. Over an interval of a few seconds, this proto-neutron star loses its trapped neutrinos and cools, and at this point, the collapsed core remnant has reached its equilibrium configuration composition of neutrons, protons, hyperons, leptons and perhaps quarks. Thus, the neutron star is born. The radius of such a star is around $10km$ and it has a density in the region of $10^{14}$ times greater than that of the Earth. It may itself manifest into a pulsar or magnetar at a later point. The limit of neutron degeneracy pressure is known as the Tolman-Oppenheimer-Volkoff limit and if the neutron star is more massive than this limit, it must undergo a further collapse to some denser and more compact form, which is the hypothetical quark/ultra-compact star. The final state for a collapsed star is an astrophysical black hole (also referred to as a collapsar) which inherits the mass, angular momentum and electric charge (if any) of the initial object. This is characterised in the famous no-hair conjecture proposed by John Wheeler (Misner *et al* 1973).

A renewed interest in higher order theories of gravity has arisen in recent times. The reason for studying these new theories is the fact that conventional Einstein gravity has shortcomings. An example is the fact that the late time expansion of the universe is noted in observations, but isn't a direct consequence of standard general relativity. Many results are reported in the literature on solutions in EGB gravity. The well known Boulware-Deser solution (Boulware and Deser 1985) was an early higher dimensional analogue of the vacuum Schwarszchild solution from general relativity. Bhawal (1990) studied the higher dimensional geodesic motion of a Boulware-Deser black hole spacetime and performed comparisons with the higher dimensional Schwarszchild geometry. More recently Davis (2003) derived the generalised Israel junction conditions on a membrane and Anabalon *et al* (2009) found a vacuum solution in EGB gravity with the Kerr-Schild ansatz in five-dimensional space. Recent investigations (Maharaj *et al* 2014, Chilambwe *et al* 2015, Hansraj *et al* 2015) have reported new solutions to the EGB field equations for a static spherically symmetric interior of a perfect fluid. The notion of gravitational collapse has also been looked upon. Maeda (2006) studied the gravitational contraction of dust in EGB gravity, and efforts have been made to find asymptotically AdS black hole solutions in EGB gravity (Wheeler 1986, Wiltshire 1986, Cai 2002). Ghosh *et al* (2014) studied the gravitational contraction of a spherical cloud of inhomogeneous dust in EGB theory, and Ghosh and Maharaj (2014) presented null dust solutions in third order Lovelock gravity for a spherically symmetric string cloud background in arbitrary dimensions. Upon finding black hole solutions, an important task is to study the conserved charges such as the angular momentum; Peng (2014) looked at quasi-local conserved charges of dyonic rotating black holes in both EGB gravity and four-dimensional conformal Weyl gravity. Dawood and Ghosh (2004) characterised a large family of solutions to Einstein's equations for a spherically symmetric type II fluid, and showed that the well known black hole solutions are a particular case of this larger family. Ghosh and Dawood (2008) further generalised these results to higher dimensions. Ghosh and Dadhich (2002) studied the gravitational collapse of a type II fluid in higher dimensions and noted that due to the presence of strange quark matter, as well as the higher dimensions, there was a shrinkage of the initial data space.

This dissertation is organised as follows:

- Chapter 1: Introduction.

- Chapter 2: In this chapter the relevant theoretical concepts inherent with relativity theory are presented for the reader's perusal. We define the differentiable manifold as well as the spacetime manifold. Pertinent aphorisms and definitions are then underscored and the Einstein field equations are formulated from these. Further definitions relating to higher order theories of gravity are shown, with particular emphasis on Einstein-Gauss-Bonnet gravity, where we then present the field equations for that theory. The complete theoretical concepts of the junction conditions are then realised in full, followed by a brief description of the energy conditions for a realistic stellar model.

- Chapter 3: This chapter is the first major feature of this work. Dynamical radiating stars with outgoing null radiation are discussed in detail. Considering three separate concentric zones, the generalised Vaidya spacetime is used as a specific class of spacetimes to model an entire realistic, astrophysical radiating star in general relativity. The matching conditions are provided for these three zones and a complete mass function is described in full. Solutions for this mass function for various realistic equations of state are then presented in full and we demonstrate that many of the other seminal solutions of the past are contained in ours. We then generalise these results to higher dimensions. Finally, we consider the notion of diffusion, generating the relevant partial differential equation and solving it for various realistic equations of state. It is important to note that this work was published in *General Relativity and Gravitation* (Brassel *et al*, 2017).

- Chapter 4: In this chapter we consider the gravitational collapse of generalised Vaidya spacetimes. The collapsing model is formulated in detail as well as the notion of the naked singularity. We then discuss the conditions for the formation of a naked singularity, as well as discussing its structure, nature and strength. We then present the end states of our generalised Vaidya mass functions from Chapter 3 for various equations of state, showing that the final outcome of collapse is a strong, central naked singularity. We give a brief discussion on mass functions where naked singularities can be eliminated due to the presence of higher dimensions. The work in this chapter has been published in *Physical Review D* (Brassel *et al* 2017).

- Chapter 5: We demonstrate in this chapter, that the Boulware-Deser class of spacetimes can radiate. Using techniques analagous to those in Chapter 3, we model a five-dimensional astrophysical radiating star with three concentric zones using one class of Boulware-Deser metric. Again, the matching conditions for the three zones are formulated and the complete, continuous mass function is presented. Solutions for various realistic equations of state are then systematically presented for our radiating metric. We show that several previous results are contained within ours before extending them to higher dimensions. The work in this chapter was published in *General Relativity and Gravitation* (Brassel *et al* 2017).

- Chapter 6: In this chapter we completely model a radiating star in five-dimensional Einstein-Gauss-Bonnet gravity. We consider the smooth matching of two spacetimes, namely the shear-free interior and the pure Vaidya exterior, in the context of Einstein-Gauss-Bonnet gravity and we present the resulting Santos (1985) junction conditions from first principles, including the additional modified condition for the continuity of the scalar curvature and its first derivative. Solutions for the boundary condition as well as the third junction condition are then obtained.

- Chapter 7: Conclusion.

# Chapter 2

# Theoretical concepts of gravitation

## 2.1 Introduction

Einstein's theory of general relativity has, so far, been the most successful theory describing matter fields in strong gravitational fields. The reader is referred to the books by Shapiro and Teukolsky (1983) and Glendenning (2000) for excellent reviews on the physics of compact objects, black holes and relativistic stellar processes. Within this chapter the relevant background theory which allows us to generate a spherical model for a radiating star or cosmological system is presented. In Sec. 2.2 the definitions for a differentiable manifold and spacetime manifold are given with the latter discussed in detail, pertaining to the relevant curvature tensors and connections leading to the formulation of the Einstein field equations. For extensive details on manifolds and tensor analysis, the reader is referred to Bishop and Goldberg (1968), Misner *et al* (1973) and Foster and Nightingale (1994). A brief discussion is given on modified gravity in Sec. 2.3, specifically Einstein-Gauss-Bonnet gravity. The Lovelock term and Lanczos tensor are introduced and used to adduce the Einstein-Gauss-Bonnet field equations. The reader is encouraged to seek out Papantonopoulus (2015) for further details on these notions and modified theories in general. The theory of the junction conditions are presented in Sec. 2.4 followed by a description of the energy conditions which make a stellar model realistic.

## 2.2 Differentiable manifold

Let $\mathcal{R}^N$ denote an $N$-dimensional Euclidean space, i.e., the set of all $N$-tuples, represented as $(x^1, x^2, ..., x^N)$ such that, with the usual topology of the open interval $\frac{1}{2}\mathcal{R}^N$ denoting the lower half of $\mathcal{R}^N$ with $x^1 \leq 0$, $-\infty < x^i < \infty$ for $i \in \{1, 2, ..., N\}$. We then have the following definitions:

**Definition 1**: A map $\phi$ from some open set $\mathcal{O} \in \mathcal{R}^N$ to another open set $\mathcal{O}' \in \mathcal{R}^M$ is said to be of differentiability class $C^w$ if the coordinates $(x^{1'}, x^{2'}, ..., x^{N'})$ of the image point $\phi(p)$ in $\mathcal{O}'$ are $w$-times continuous differentiable functions of the coordinates $(x^1, x^2, ..., x^N)$ of $p$ in the original set $\mathcal{O}$. An $N$-dimensional differentiable manifold is a set that is simply locally similar to an open set of Euclidean space $\mathcal{R}^N$.

**Definition 2**: If there exists a map $C^w$ $\forall w \geq 0$, then it is a $C^\infty$ (infinitely differentiable) map. A manifold with such a mapping is said to be a *smooth* manifold.

**Definition 3**: A function $f$ with the mapping $f : \mathcal{R}^N \longrightarrow \mathcal{R}$ is said to be locally Lipschitz if, for each open set $U \in \mathcal{O}$ with compact closure (the set closure of $U$ is compact), there exists some constant $L$, such that for each pair of points $p, q \in U$, $|f(p) - f(q)| \leq L|p - q|$.

## 2.3 Spacetime manifold

For the purposes of this dissertation, unless otherwise stated, we assume that spacetime, denoted by $\mathcal{M}$, in general is an $N$-dimensional, oriented, paracompact, Hausdorff, $C^\infty$ pseudo-Riemannian manifold endowed with a symmetric, nondegenerate, smooth metric tensor field $\boldsymbol{g}$. In local regions, the manifold $(\mathcal{M}, \boldsymbol{g})$ is homeomorphic and diffeomorphic to Euclidean space $\mathcal{R}^N$ which implies that it may be covered by overlapping coordinate patches so that special relativity holds locally. The tensor field $\boldsymbol{g}$ has a signature $(-++\cdots+_N)$ and encodes the properties and dynamics of the gravitational field. For detailed treatments on spacetime geometry, the reader is referred to the standard textbooks in differential geometry such as Hawking and Ellis (1973), de Felice and Clarke (1990), do Carmo (1992), Foster and Nightingale (1994) and Straumann (2004).

### 2.3.1 Metric tensor

The idea of distance between two infinitesimally separated points on the spacetime manifold is defined by the metric tensor. This tensor acts on pairs of vectors to give a numerical value, and is symmetric in its indices. In terms of the coordinate basis, the metric tensor is defined as

$$\boldsymbol{g} = g_{ab} dx^a \otimes dx^b, \tag{2.3.1}$$

where $g_{ab} = g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right)$. The above can be written as

$$g(\mathbf{V}, \mathbf{W}) = g_{ab} V^a W^b,$$

for any two vectors $\mathbf{V}$ and $\mathbf{W}$, and this can be expressed as the distance between two infinitesimally separated points on the manifold as

$$ds^2 = g_{ab} dx^a dx^b. \tag{2.3.2}$$

The matrix $[g_{ab}]$ is regular with inverse $g^{ab}$ such that

$$g_{ab} g^{bc} = \delta_a{}^c, \tag{2.3.3}$$

where $\delta_a{}^c$ is the Kronecker delta. Both tensors $g_{ab}$ and $g^{ab}$ may be used to define relationships between covariant and contravariant vectors in the following way

$$X_a = g_{ab} X^b, \qquad X^a = g^{ab} X_b. \tag{2.3.4}$$

For a tensor $T$ of rank two, the relationships are given as follows

$$T_{ab} = g_{ac}g_{bd}T^{cd}, \quad T^{ab} = g^{ac}g^{bd}T_{cd}, \quad T^a{}_b = g^{ac}T_{cb}. \tag{2.3.5}$$

The metric is indefinite since the magnitude of the nonzero vector can be positive, zero or negative.

### 2.3.2 The metric connection

The metric tensor $\boldsymbol{g}$ can have a torsion-free connection $\nabla$ which is unique such that

$$\nabla \boldsymbol{g} = 0 \qquad \text{or} \qquad g_{ab;c} = 0, \tag{2.3.6}$$

where ; denotes covariant differentiation. The metric connection coefficient $\Gamma$ is defined in terms of the metric tensor and its derivatives (Boothby 1986) by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\frac{\partial g_{cd}}{\partial x^b} + \frac{\partial g_{db}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d}\right) \equiv \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}), \tag{2.3.7}$$

where commas denote partial differentiation. The coefficients $\Gamma^a{}_{bc}$ are also known as the Christoffel symbols of the second kind and are components of the Levi-Civita connection (Christoffel 1869, Levi-Civita 1917). The connection coefficients entirely describe the curvature of the coordinate system and since they are not tensors, can be transformed to vanish under a suitable coordinate transformation.

Given a vector field $X^a$, the covariant derivative is defined in the following way

$$\nabla_b X^a = \partial_b X^a + \Gamma^a{}_{bc}X^c, \quad \nabla_b X_a = \partial_b X_a - \Gamma^c{}_{ab}X_c, \tag{2.3.8}$$

where $\partial_b = \frac{\partial}{\partial x^b}$. For a mixed tensor $T^a{}_b$, we thus have

$$\nabla_c T^a{}_b = \partial_c T^a{}_b + \Gamma^a{}_{cd}T^d{}_b - \Gamma^d{}_{ab}T^b{}_d. \tag{2.3.9}$$

### 2.3.3 Geodesics

Within Euclidean space, a *geodesic* is defined by two properties which are equivalent to each other. First, its tangent vector points in the same direction and secondly, it is the curve of shortest length between any two points. The geodesic is considered, on a torsion-free manifold, as a curve $x^a(u)$ described by a parameter $u$ by the fixed direction of its associated tangent vector. It satisfies the condition

$$\frac{d\mathbf{t}}{du} = \lambda(u)\mathbf{t},$$

where $\lambda(u)$ is a parametric function of $u$. In general, the equations satisfied by null and non-null geodesics, parametrised by some parameter $u$, are given by

$$\frac{d^2x^a}{du^2} + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = \lambda(u)\frac{dx^a}{du}. \tag{2.3.10}$$

This curve can always be parametrised such that $\lambda(u)$ can vanish. The geodesics are then defined as the following equations

$$\frac{d^2x^a}{du^2} + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0, \tag{2.3.11}$$

and in this case, $u$ is is called an affine parameter. Thus, on a manifold $(\mathcal{M}, \boldsymbol{g})$, a geodesic can be timelike, spacelike or null if its tangent vector is timelike, spacelike or null respectively.

### 2.3.4 Curvature tensors

The Riemann curvature (or Riemann-Christoffel) tensor is a rank four tensor which is a tensor field that measures the extent to which the metric tensor is not locally isometric to that of Euclidean space. It can be defined for any pseudo-Riemannian manifold or any manifold with an affine connection. It is defined (in terms of Christoffel symbols) as

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}, \tag{2.3.12}$$

where $\Gamma^d{}_{ac,b} = \partial_b\Gamma^d{}_{ac}$. The curvature tensor represents the tidal force experienced by any rigid body moving along a geodesic, and is a way to capture a measure of the intrinsic curvature. It also measures noncommutativity of the covariant derivative, and as such is the integrability condition for the existence of an isometry with Euclidean (or flat) space. The symmetry properties inherent with the curvature tensor are observed by changing from mixed components, $R^a{}_{bcd}$, say, to covariant components $R_{abcd} = g_{ae}R^e{}_{bcd}$. It can easily be shown that

$$R_{abcd} = -R_{bacd} = -R_{abdc}, \tag{2.3.13a}$$

$$R_{abcd} = R_{cdab}. \tag{2.3.13b}$$

The tensor is therefore antisymmetric in each of the index pairs $(a, b, c, d)$ and is symmetric under the interchange of any two pairs of indices with one another. The cyclic sum (obtained by permuting any three indices) of the components of $R_{abcd}$ is zero

$$R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{2.3.14}$$

The Bianchi identity

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0, \tag{2.3.15}$$

can also be proved by use of the curvature tensor.

The Ricci (or Ricci-curvature) tensor is a rank two tensor which provides a way of measuring the degree to which the geometry determined by a Riemannian metric differs from ordinary Euclidean space. It is defined on a pseudo-Riemannian manifold as a trace of the Riemann curvature tensor and can be written as

$$R_{ab} = g^{dc}R_{dacb} = R^c{}_{acb}. \tag{2.3.16}$$

Using (2.3.12) we have

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac}. \end{aligned} \tag{2.3.17}$$

The Ricci tensor is symmetric, i.e. $R_{ab} = R_{ba}$ and upon its contraction, we acquire

$$R = g^{ab}R_{ab} = g^{ac}g^{bd}R_{abcd}, \tag{2.3.18}$$

which is called the Ricci scalar of the space. The Kretschmann invariant (or scalar) is defined as the square of the Riemann tensor and can be used as an indicator of curvature singularities in the manifold. It is given by

$$K = R_{abcd}R^{abcd}, \tag{2.3.19}$$

and as such, is called a quadratic invariant. In some higher order theories of gravity, a possible invariant is given by

$$C_{abcd}C^{abcd}, \tag{2.3.20}$$

where $C_{abcd}$ is the Weyl (deformation) tensor, defined as

$$\begin{aligned} C_{abcd} &= R_{abcd} + \frac{1}{N-2}[g_{ad}R_{bc} - g_{ac}R_{bd} + g_{bc}R_{ad} - g_{bd}R_{ac}] \\ &\quad + \frac{1}{(N-1)(N-2)}[g_{ac}g_{bd} - g_{ad}g_{cb}]R, \end{aligned} \tag{2.3.21}$$

where $N$ corresponds to the spacetime dimension. It can be related to the Kretschmann invariant (Cherubini *et al* 2002) by

$$K = C_{abcd}C^{abcd} + \frac{4}{N-2}R_{ab}R^{ab} - \frac{2}{(N-2)(N-1)}R^2. \tag{2.3.22}$$

### 2.3.5 The Einstein tensor

The Einstein tensor is a rank two tensorial quantity used to express curvature on a pseudo-Riemannian manifold. It can be derived from the Bianchi identity (2.3.15) and is expressed in terms of the Ricci tensor (2.3.17) and Ricci scalar (2.3.18) as

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}. \tag{2.3.23}$$

In general relativity (and higher order theories) the Einstein tensor describes the curvature of spacetime in a way which is consistent with physical and energy considerations. The Einstein tensor is symmetric, so $G_{ab} = G_{ba}$ and is divergenceless, i.e.

$$\nabla_a G^{ab} = 0. \tag{2.3.24}$$

The trace of the Einstein tensor can be computed by contraction with the metric tensor $g^{ab}$. On an $N$-dimensional manifold

$$g^{ab}G_{ab} = g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R,$$

which then gives

$$G = R - \frac{1}{2}(NR) = \frac{2-N}{2}R. \tag{2.3.25}$$

It is interesting to note that when $N = 4$, the trace of the Einstein tensor is the negative of $R$, the trace of the Ricci tensor. Thus, it is often called the *trace-reversed Ricci tensor*. With regards to uniqueness, Lovelock (1969, 1971) showed that, in a four-dimensional differentiable manifold, the Einstein tensor is the only tensorial and divergence free function of the metric tensor (and at most their first and second partial derivatives).

### 2.3.6 Energy momentum tensor

The matter distribution for a model in astrophysics and cosmology is usually described by a relativistic fluid. For an erudition on relativistic fluid dynamics and magneto-fluid dynamics, the reader is referred to Anile (1989). The energy momentum tensor (or stress-energy tensor) is an attribute to radiation and matter in the spacetime manifold, and is the source of the gravitational field in the Einstein field equations. Denoted by $T^{ab}$ it can be described as the $a$-th component of four-momentum across some surface with constant $x^b$, so that we have

1. $T^{00}$ is the flux density of the 0-th component of four-momentum across the time surface ($x^0$), called the *energy density*.
2. $T^{0i} = T^{i0}$ is the energy flux density across a surface of constant $x^i$, called the *heat conduction*.
3. $T^{ij}$ is the flux of the $i$-th momentum across the $j$-surface, called the *stress*.
4. $T^{ii}$ is the *pressure* in the $i$-th direction (there is no summing over $i$).

The energy momentum tensor for uncharged matter, for example, is defined as

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab}. \tag{2.3.26}$$

In the above $\rho$ is the energy density, $p$ is the isotropic (kinetic) pressure, $q^a$ is the heat flux vector $(q^a u_a) = 0$ and $\pi^{ab}$ is the anisotropic pressure (stress) tensor ($\pi^{ab} u_a = 0 = \pi^a{}_a$). These quantities are measured relative to a comoving fluid four-velocity $\boldsymbol{u}$ which is unit and timelike ($u^a u_a = -1$). In perfect fluids there are no stress and heat conduction terms ($q^a = 0, \pi^{ab} = 0$) and so the stress-energy tensor is isentropic. Hence, for a perfect fluid, the energy momentum tensor (3.3.4$a$) becomes

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}. \tag{2.3.27}$$

Since the energy momentum tensor is conserved throughout the spacetime manifold, we have that

$$\nabla_a T^{ab} = 0. \tag{2.3.28}$$

### 2.3.7 Einstein field equations

Derived in four seminal papers by Albert Einstein (1915), the *Einstein field equations* (or gravitational field equations) describe the fundamental interaction of gravity as a consequence of spacetime being curved by matter and energy (described by the energy momentum tensor $T_{ab}$). Using equations (2.3.24) and (2.3.28) we have the following

$$\nabla_a G^{ab} = 0 = \nabla_a T^{ab}, \tag{2.3.29}$$

which can be written naturally in the following way

$$R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab}, \tag{2.3.30}$$

where $\kappa$ is the constant given by $\kappa = \frac{8\pi\tilde{G}}{c^4}$, $\tilde{G}$ is a gravitational constant and $c$ is the speed of light. Using (2.3.23) we can write (2.3.30) as

$$G_{ab} = \kappa T_{ab}. \tag{2.3.31}$$

These equations (2.3.31) are known as the *Einstein field equations* and are a set of ten highly nonlinear, hyperbolic-elliptic partial differential equations, with twenty independent components of the Riemann tensor $R_{abcd}$. (2.3.31) is also a tensor equation and along with the geodesic equation, forms the fundamental mathematical formulation of general relativity. Including the cosmological constant term $\Lambda$ in the field equations gives

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}, \tag{2.3.32}$$

and since $\Lambda$ is constant, the conservation law of energy is unaffected. If the energy momentum tensor $T_{ab}$ vanishes, the Einstein field equations become the vacuum field equations and are written as

$$R_{ab} = 0. \tag{2.3.33}$$

The field equations can be written in trace-reversed form as was the case with the Ricci tensor, previously. Computing the trace with respect to the metric on both sides of the equation (2.3.32) gives, in $N$-dimensional space

$$R_{ab} - \frac{\Lambda g_{ab}}{\frac{N}{2} - 1} = \kappa \left( T_{ab} - \frac{1}{N-2} T g_{ab} \right). \tag{2.3.34}$$

Letting $N = 4$ gives

$$R_{ab} - \Lambda g_{ab} = \kappa \left( T_{ab} - \frac{1}{2} T g_{ab} \right). \tag{2.3.35}$$

Setting $T_{ab} = 0$ and $\Lambda = 0$ in the above equation gives (2.3.33). The original field equations can be regained by reversing the trace once again. The form of the field equations above is useful in certain applications.

## 2.4 Modified theories of gravity: Einstein-Gauss-Bonnet gravity

Higher order theories of gravity have been the subject of much study. One approach to modify general relativity is the introduction of nonlinear forms of the Riemann and Ricci tensor, and the Ricci scalar. The second order equations of motion resulting from linear forms is advantageous in four dimensions; however as shown by Lovelock (1971, 1972) it is possible to introduce a polynomial form of the Lagrangian which is of quadratic order. This form generates the Einstein-Gauss-Bonnet (EGB) action. Curvature terms which are quadratic in the spacetime appear as corrections to Einstein gravity, and this theory can be considered a consequence of low energy string theory (Fradkin and Tseytlin 1985, Metsaev *et al* 1987). These higher order curvature terms will have no consequence in four-dimensional gravity unless some surface term is involved. An interesting point to note is that the equations of motion which result from the EGB action are still second order and quasilinear. If the higher order quantities are vanquished or absent, conventional Einstein gravity is regained (Deser and Yang 1989). The modified form of the Einstein-Hilbert action in five dimensions is

$$S = \frac{1}{16\pi}\int \sqrt{-g}\,[(R - 2\Lambda + \alpha L_{GB})]\, d^5x + S_{matter}, \tag{2.4.1}$$

which is called the Gauss-Bonnet action where $\alpha$ is the Einstein-Gauss-Bonnet (EGB) coupling constant, $R$ is the Ricci scalar, $L_{GB}$ is the Lovelock term and $\Lambda$ is the cosmological constant. The above action has no direct effect in dimensions of four or less (it is in fact a total derivative in four dimensions) since the Lovelock term does not contribute to the field equations, but is generally nonzero in dimensions higher than four.

### 2.4.1 Lovelock term

The Lovelock (or Gauss-Bonnet) term $L_{GB}$ is derived from the Lagrangian

$$\mathcal{L}_{GB} = \sqrt{-g}\,(R - 2\Lambda + \alpha L_{GB})\,, \tag{2.4.2}$$

of the theory and we have that $L_{GB} \propto \mathcal{L}_{GB}$. It is given by

$$L_{GB} = R^2 + R_{abcd}R^{abcd} - 4R_{cd}R^{cd}, \tag{2.4.3}$$

and is essentially a linear combination of quadratic terms in curvature. It is also the dimensionally extended version of the Euler density. The cogency of the Lovelock term lies in the fact that the equations of motion are second order and quasilinear despite the fact that the Langrangian is quadratic in the Riemann-curvature tensor, the Ricci tensor and the Ricci scalar. The EGB theory with a Lagrangian given by a linear combination of the Einstein-Hilbert Lagrangian and the Lagrangian $\mathcal{L}_{GB}$ is the most trivial truncation of the Lovelock model (Padmanabhan and Kothawala 2013).

### 2.4.2 Lanczos tensor

The Lanczos (or Gauss-Bonnet) tensor $H_{ab}$ is defined (Lanczos 1932, 1938) in terms of the metric tensor, Riemann tensor, Ricci tensor and Ricci scalar, and the Lovelock term, as

$$H_{ab} = g_{ab}L_{GB} - 4RR_{ab} + 8R_{ac}R^{c}{}_{b} + 8R_{acbd}R^{cd} - 4R_{acde}R_{b}{}^{cde}. \tag{2.4.4}$$

Moreover, the above can also be written in terms of the following rank four tensor

$$P_{abcd} \equiv R_{abcd} + g_{ad}R_{bc} - g_{ac}R_{bd} - g_{bd}R_{ac} + g_{bc}R_{ad} + \frac{1}{2}R(g_{ac}g_{bc} - g_{bc}g_{ad}), \tag{2.4.5}$$

since

$$H_{ab} = P_{acde}R_{b}{}^{cde} - \frac{1}{2}g_{ab}L_{GB}. \tag{2.4.6}$$

Interestingly, the tensor $P_{abcd}$ has several properties:

- Firstly, it is divergence free, i.e.

$$\nabla^{d}P_{abcd} = 0.$$

- It has the same index symmetries as the Riemann tensor.

- Tracing two of its indices yields

$$P^{b}{}_{acb} = G_{ac}, \tag{2.4.7}$$

  which yields the divergence free property of the Einstein tensor. It is interesting to note that the tensor $P_{abcd}$ could be argued as the curvature tensor associated with the Einstein tensor in the same way that the Riemann tensor is the curvature tensor associated with the Ricci tensor.

- In four dimensions $H_{ab} = 0$, hence we have the Lovelock identity from (2.4.6) as

$$P_{acde} R_b{}^{cde} = \frac{1}{2} g_{ab} L_{GB}. \tag{2.4.8}$$

For extensions of the above equation, the reader is encouraged to seek out Edgar and Hoglund (2002).

### 2.4.3 Einstein-Gauss-Bonnet field equations

In dimensions four or more, Einstein gravity can be considered a particular case of Lovelock gravity and EGB gravity in particular. The EGB field equations may finally be written as

$$\mathcal{G}_{ab} = \kappa T_{ab}, \tag{2.4.9}$$

where

$$\mathcal{G}_{ab} = G_{ab} - \frac{\alpha}{2} H_{ab}. \tag{2.4.10}$$

In the above, $G_{ab}$ is the Einstein tensor, $T_{ab}$ is the energy momentum tensor and $H_{ab}$ is the Lanczos tensor. In the limit where $\alpha \to 0$, conventional Einstein gravity will be regained.

## 2.5 Theory of junction conditions

The model for a relativistic, dissipating radiating star with outgoing null radiation was completed by Santos (1985) by analysing the junction conditions at the stellar surface. The important result that followed was that the pressure on the boundary of the radiating star should be nonzero in general, and proportional to the heat flux. Consider two $N$-dimensional spacetime manifolds $\mathcal{M}^{\pm}$ with oriented boundaries $\Sigma^{\pm}$ such that there exists a diffeomorphism between $\Sigma^{-}$ and $\Sigma^{+}$. Each is endowed with symmetric, nondegenerate, smooth metric tensor fields $\boldsymbol{g}^{\pm}$. Upon successful matching, the resulting spacetime $(\mathcal{M}, \boldsymbol{g})$ is the disjoint union of the spacetimes $\mathcal{M}^{\pm}$ with points on their oriented boundaries $\Sigma^{\pm}$ such that the junction conditions are satisfied (Israel

1966). Individual points on $\mathcal{M}^{\pm}$ are labelled by $\{x^a_{\pm}\}$ where $a \in \{0, ..., N-1\}$. Let $g_{ij}$ be the intrinsic metric to $\Sigma^{\pm}$ so that

$$ds^2_{\Sigma^{\pm}} = g_{ij} d\xi^i d\xi^j. \tag{2.5.1}$$

The intrinsic coordinates to $\Sigma^{\pm}$ are given by $\{\xi^i_{\pm}\}$ where $i \in \{1, ..., N-1\}$. The two embeddings are given by the maps

$$\Psi^{\pm} : \Sigma \longrightarrow \mathcal{M}^{\pm}, \tag{2.5.2a}$$

$$\xi^i \mapsto x^a_{\pm} = \Psi^i_{\pm}(\xi^i), \tag{2.5.2b}$$

such that $\Sigma^{\pm} \equiv \Psi^{\pm}(\Sigma) \subset \mathcal{M}^{\pm}$. The diffeomorphism from $\Sigma^+$ to $\Sigma^-$ is $\Psi^- \circ {\Psi^+}^{-1}$. The metrics in the regions $\mathcal{M}^{\pm}$ are of the form

$$ds^2_{\pm} = g_{ab} d\chi^a_{\pm} d\chi^b_{\pm}, \tag{2.5.3}$$

where $\chi^a_{\pm}$ are the coordinates in $\mathcal{M}^{\pm}$ with $a \in \{0, ..., N-1\}$. The requirement here is that the metrics (2.5.1) and (2.5.3) match smoothly across $\Sigma^{\pm}$. This generates the *first junction condition*

$$(ds^2_-)_{\Sigma^-} = (ds^2_+)_{\Sigma^+} = ds^2_{\Sigma}, \tag{2.5.4}$$

which implies that $\Sigma^- \equiv \Sigma^+$, hence from here on both boundaries $\Sigma^{\pm}$ can be represented by $\Sigma$. The above condition (2.5.4) is also referred to as the *first fundamental form*. The coordinates of $\Sigma$ on the spacetimes $\mathcal{M}^{\pm}$ are consequently given by $\chi^a_{\pm} = \chi^a_{\pm}(\xi^i_{\pm})$. The *second junction condition* is acquired by requiring the continuity of the extrinsic curvature of $\Sigma$ across the boundary. This results in

$$K^-_{ij} = K^+_{ij}, \tag{2.5.5}$$

where

$$K^{\pm}_{ij} \equiv -n^{\pm}_a \frac{\partial^2 \chi^a_{\pm}}{\partial \xi^i_{\pm} \partial \xi^j_{\pm}} - n^{\pm}_a \Gamma^a{}_{bc} \frac{\partial \chi^b_{\pm}}{\partial \xi^i_{\pm}} \frac{\partial \chi^c_{\pm}}{\partial \xi^j_{\pm}}. \tag{2.5.6}$$

The above condition is called the *second fundamental form* where $n^{\pm}_a(\chi^b_{\pm})$ are components of the vector normal to $\Sigma$. The junction conditions (2.5.4) and (2.5.5) are homologous with those generated by O'Brien and Synge (1952) and Lichnerowicz (1955).

Further insights into the junction conditions can be found in the texts by Raju (1982), Lake (1987) and Husain (1996).

In higher dimensional modified theories of gravity two further matching conditions need to be satisfied at the boundary. First, the continuity of the scalar curvature

$$R^{\pm}\Big|_{\Sigma} = 0, \tag{2.5.7}$$

as well as its first derivative, which is a directional derivative along the radial coordinate perpendicular to the boundary

$$\nabla_a R^{\pm}\Big|_{\Sigma} = 0. \tag{2.5.8}$$

These two extra conditions make the notion of matching a stellar interior with an exterior quite restrictive. Further insights into the matching conditions in modified gravity can be found in Clifton (2006), Deruelle *et al* (2008), Clifton *et al* (2013) and Ganguly *et al* (2014).

## 2.6 Energy conditions

In addition, for a stellar model to be deemed realistic, it must also adhere to the so-called energy conditions of general relativity, as well as not being in violation of the law of causality. A relativistic fluid must obey the three energy conditions:

(i) *The weak energy condition*: For any future pointing timelike vector $w^a$, the total energy density $T_{ab}w^a w^b \geq 0$, at each event in the spacetime.

(ii) *The strong energy condition*: For any future pointing timelike unit vector $w^a$, the stresses of the matter, at each event in the spacetime are restricted by the condition $2T_{ab}w^a w^b + T \geq 0$ where $T$ is the trace of the energy momentum tensor $T_{ab}$.

(iii) *The dominant energy condition*: For any future pointing timelike vector $w^a$, the four-momentum density vector $T_{ab}w^b$ must be future pointing and timelike, or null at each event in the spacetime.

A detailed discussion of the energy conditions is contained in Hawking and Ellis (1973) and Kolassis *et al* (1988). To further elucidate on the above three conditions, we express them in terms of the matter variables for an imperfect fluid, so that

(a) *The weak energy condition*: $\rho - p + \Delta \geq 0$.

(b) *The strong energy condition*: $2p + \Delta \geq 0$.

(c) *The dominant energy condition*: $\rho - 3p + \Delta \geq 0$,

where $\rho$ is the energy density, $p$ is the pressure and $q$ is the heat flux. We have defined

$$\Delta = \sqrt{(p+q)^2 - 4q^2},$$

in the above. In the absence of heat flux we obtain $\rho \geq 0$ and $p \geq 0$. The speed of sound for a relativistic fluid is given by

$$\frac{dp}{d\rho} = c_s^2. \tag{2.6.1}$$

A violation of causality would result if the above quantity is, at any point in time, negative or greater than the speed of light.

# Chapter 3

# Diffusive and dynamical radiating stars with realistic equations of state

## 3.1 Introduction

The Vaidya metric (Vaidya 1951) describes the geometry outside a spherically symmetric radiating star and it defines outgoing null radiation. Though this outside geometry and the matching conditions have been studied in detail, the main problem is as follows:

*How do we model a realistic collapsing astrophysical star with a core null fluid and a string fluid which matches to the intermediate Vaidya spacetime enclosed by the Schwarzschild exterior?*

This is a key question for a better understanding of the dynamics, thermodynamics and gravitational collapse in realistic astrophysical stars, in the context of general relativity. The class of spacetimes that are natural candidates for models of such stellar interiors are *generalised Vaidya spacetimes*. The matter field in these spacetimes have two components: A general type I matter field (whose energy momentum tensor has a timelike and three spacelike eigenvectors), that describes null fluid matter, and also a type II matter field (whose energy momentum tensor has double null eigenvectors) that describes null radiation and a string fluid. Such a stellar interior can then be naturally matched to an external radiating zone described by the Vaidya spacetime, and finally the radiation zone can be matched smoothly with the vacuum Schwarzschild exterior, as we explain in later sections of this dissertation.

The intent of this chapter is to generate solutions to the generalised Vaidya stellar interior with a string fluid and null matter for various thermodynamically realistic equations of state. It turns out that a direct integration of the resulting partial differential equations is possible in general for the linear, quadratic and polytropic equations. Our solutions for the linear cases generalise all of those obtained by Husain and others as well as the complete summary of solutions presented in Wang and Wu (1999), and are therefore the most general solutions known. For pedagogical completeness, we also further generalise all of our results in the higher dimensional generalised Vaidya spacetimes.

This chapter is organised as follows: In the next section we give a complete outline of how to model an isolated spherical and physically realistic radiating astrophysical star via the generalised Vaidya geometry. In the following section we describe the generalised Vaidya spacetime in detail by analysing the Einstein field equations. We present the relevant aphorisms indicative with the geometry of the generalised Vaidya metric and make mention of the energy conditions for a physically reasonable model. A note on equations of state is presented with various cases discussed. In Sec. 3.4 we systematically present solutions to the Einstein field equations for the gravitational mass function by assuming several different equations of state. The succeeding Sec. 3.5 then deals with the higher dimensional spacetime and field equations. The solutions are summarised in detail for higher dimensions and the masses are tabulated. Finally an analysis if the effects of diffusion on our model is undertaken. We will present several classes of solutions to the diffusion equation using the various equations of state.

### 3.1.1 Generalised Vaidya spacetimes

The generalisation of the Vaidya spacetime was given in detail by Wang and Wu (1999) and includes most of the known solutions of Einstein's field equations with the additional Type II fluid. The notion that the energy momentum tensor is linear in terms of the gravitational mass for these matter fields, engenders this generalisation of the spacetime. The generalised Vaidya metric in single (collapsing and exploding) null coordinates $(v, \mathsf{r}, \theta, \phi)$ is given as

$$ds^2 = -\left(1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}}\right) dv^2 + \epsilon 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1.1}$$

where $\epsilon = \pm 1$. Here the function $m(v, \mathsf{r})$ describes the Misner-Sharp mass of the stellar interior and can be obtained via integrating the Einstein field equations with combinations of perfect fluid and null matter sources.

## 3.2 The model of a dynamic and radiating relativistic fluid star

Any isolated spherically symmetric astrophysical star with outgoing null radiation is a combination of three distinct concentric zones: the innermost zone is the stellar interior where there are two component matter sources, namely null fluid matter along with radiation. The middle zone is purely a radiation zone while the outermost zone is the vacuum Schwarzschild exterior that extends roughly to a radius of 1 light year (for solar mass stars) beyond which galactic dynamics take over. In this section we briefly outline how to model all three of these zones under a combined framework using a generalised Vaidya class of metric.

### 3.2.1 Stellar interior

As described earlier, the best possible candidate for the spacetime of a stellar interior is the class of generalised Vaidya spacetimes (3.1.1), and the mass function $m(v, \mathsf{r})$ can be uniquely obtained via the Einstein field equations with the two component matter sources. Let $m(v, \mathsf{r})$ be one such solution for a given combination of fluid and radiation fields. This solution then completely describes the solution of the interior of the star, up to a boundary layer given by $\mathsf{r} = \mathsf{r}_b$. Beyond this boundary we enter a pure radiation zone.

### 3.2.2 Radiation zone

In this zone the matter field is a single component null matter field and the spacetime is well described by the Vaidya metric

$$ds^2 = -\left(1 - \frac{2m_1(v)}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{3.2.1}$$

We can naturally relate the Vaidya mass function $m_1(v)$ in the radiation zone to the generalised Vaidya mass function in the stellar interior in the following way

$$m_1(v) = m(v, \mathsf{r}_b).$$

This radiation zone continues until some retarded null coordinate value $v = V_0$, beyond which the spacetime is Schwarzschild (as dictated by Birkhoff's theorem).

### 3.2.3 Schwarzschild exterior

This region is well described by the exterior static subset of the completely extended Schwarzschild manifold, and the metric is given by

$$ds^2 = -\left(1 - \frac{2M}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{3.2.2}$$

Here the Schwarzschild mass $M$ is related to the Vaidya mass $m_1(v)$ by

$$M = m_1(V_0).$$

### 3.2.4 Matching conditions at the boundary layers: Complete mass function

We note here that the spacetime is divided into three distinct regions for our above mentioned stellar model: the interior region, the radiation zone and the Schwarzschild exterior region. The first boundary layer between the inner and the intermediate zone, given by $\mathsf{r} = \mathsf{r}_b$, is a timelike boundary, whereas the second boundary given by $v = V_0$ is a null boundary. *The important point that all the three zones are described by the same class of metric makes the matching conditions across these boundaries extremely transparent.* To match the first fundamental form all we need is the mass function to be continuous across these boundaries. Hence the complete $C^2$ mass function for an isolated stellar model can be given in the following form:

$$m(v, \mathsf{r}) = \begin{cases} m(v, \mathsf{r}) & \mathsf{r} \le \mathsf{r}_b \ , v \le V_0 \\ m_1(v) \equiv m(v, \mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b \ , v \le V_0 \\ M \equiv m_1(V_0) \equiv m(V_0, \mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b \ , v > V_0 \end{cases} \tag{3.2.3}$$

We can easily check that this mass function is a solution to the Einstein field equations in all the three zones mentioned above, and hence it completely describes the spacetime of an isolated collapsing star. To match the second fundamental form, we need the partial derivatives of the mass functions across the boundaries to be continuous. These conditions are given by

$$\frac{\partial}{\partial v} m(v, \mathsf{r}_b) = \frac{\partial}{\partial v} m_1(v), \tag{3.2.4a}$$

$$\left. \frac{\partial}{\partial \mathsf{r}} m(v, \mathsf{r}) \right|_{\mathsf{r}=\mathsf{r}_b} = 0, \tag{3.2.4b}$$

$$\left. \frac{\partial}{\partial v} m_1(v) \right|_{v=V_0} = 0. \tag{3.2.4c}$$

where $\mathsf{r} = \mathsf{r}_b$ is the timelike boundary (from equating (3.1.1) to (3.2.1)) and $v = V_0$ is the null boundary (from equating (3.2.1) to (3.2.2)). These boundaries serve as the matching surfaces for the three concentric regions which can be seen in Figure 3.1.

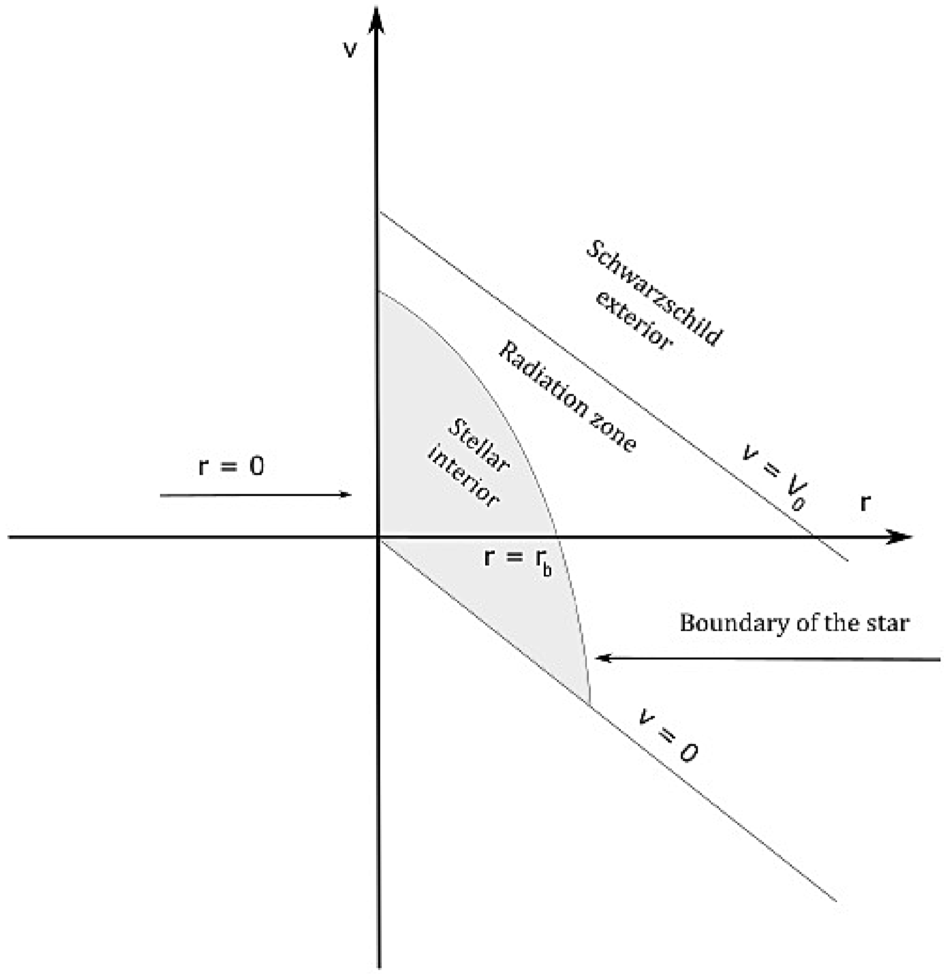


Figure 3.1: Depiction of spacetime divided into the three distinct regions (in retarded time coordinates with $\epsilon = 1$).

It is therefore necessary to find physically relevant mass functions, with the structure of (3.2.3), to model a dynamical radiating star which is isolated. We achieve this by imposing specific equations of state.

## 3.3 Generalised Vaidya spacetime: Field equations and energy conditions

The line element for all three regions belongs to the generalised Vaidya class given by (3.1.1). Note that $m(v, \mathsf{r})$ is the mass of the star and is related to the gravitational energy within a given radius $\mathsf{r}$ (Poisson and Israel 1990, Lake and Zannias 1991). The nonvanishing connection coefficients (2.3.7) are given by

$$\Gamma^0{}_{00} = \frac{1}{\mathsf{r}}(\mathsf{r}m_{\mathsf{r}} - m) \qquad \Gamma^1{}_{01} = -\frac{1}{\mathsf{r}}(\mathsf{r}m_{\mathsf{r}} - m)$$

$$\Gamma^1{}_{00} = \frac{1}{\mathsf{r}^3}(2mm_{\mathsf{r}} - \mathsf{r}^2 m_{\mathsf{r}} - \mathsf{r}^2 m_v - 2m^2 + \mathsf{r}m) \qquad \Gamma^0{}_{22} = \mathsf{r}^2$$

$$\Gamma^0{}_{33} = \mathsf{r}\sin^2\theta \qquad \Gamma^2{}_{12} = \Gamma^3{}_{13} = \frac{1}{\mathsf{r}}$$

$$\Gamma^1{}_{22} = 2m - \mathsf{r} \qquad \Gamma^3{}_{23} = \cot\theta$$

$$\Gamma^1{}_{33} = \sin^2\theta(2m - \mathsf{r}) \qquad \Gamma^2{}_{33} = -\sin\theta\cos\theta$$

where we have used the notation

$$m_v = \frac{\partial m}{\partial v}, \qquad m_{\mathsf{r}} = \frac{\partial m}{\partial \mathsf{r}}.$$

From the above we have the following quantities

$$R^0{}_0 = R^1{}_1 = \frac{m_{\mathsf{rr}}}{\mathsf{r}}, \tag{3.3.1a}$$

$$R^1{}_0 = \frac{2m_v}{\mathsf{r}^2}, \tag{3.3.1b}$$

$$R^2{}_2 = R^3{}_3 = \frac{2m_{\mathsf{r}}}{\mathsf{r}^2}, \tag{3.3.1c}$$

with the Ricci scalar

$$R = \frac{2}{\mathsf{r}^2}(\mathsf{r}m_{\mathsf{rr}} + 2m_{\mathsf{r}}).$$

The Einstein tensor components are

$$G^0{}_0 = G^1{}_1 = -\frac{2m_{\mathsf{r}}}{\mathsf{r}^2}, \tag{3.3.2a}$$

$$G^1{}_0 = \frac{2m_v}{\mathsf{r}^2}, \tag{3.3.2b}$$

$$G^2{}_2 = G^3{}_3 = -\frac{m_{\mathsf{rr}}}{\mathsf{r}}. \tag{3.3.2c}$$

The energy momentum tensor is defined by

$$T_{ab} = T^{(n)}_{ab} + T^{(m)}_{ab}, \tag{3.3.3}$$

where

$$\begin{aligned} T^{(n)}_{ab} &= \mu l_a l_b, \\ T^{(m)}_{ab} &= (\tilde{\rho} + P)(l_a n_b + l_b n_a) + P g_{ab}. \end{aligned}$$

In the above

$$l_a = \delta^0_a, \qquad n_a = \frac{1}{2}\left[1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}}\right]\delta^0_a + \delta^1_a,$$

with $l_c l^c = n_c n^c = 0$ and $l_c n^c = -1$. The null vector $l^a$ is a double null eigenvector of the energy momentum tensor (3.3.3). Hence the nonzero components are given by

$$T^0{}_0 = -\tilde{\rho}, \tag{3.3.4a}$$

$$T^1{}_0 = -\mu, \tag{3.3.4b}$$

$$T^2{}_2 = T^3{}_3 = P, \tag{3.3.4c}$$

The Einstein field equations ($G^a{}_b = \kappa T^a{}_b$) become

$$\mu = -2\frac{m_v}{\kappa \mathsf{r}^2}, \tag{3.3.5a}$$

$$\tilde{\rho} = 2\frac{m_{\mathsf{r}}}{\kappa \mathsf{r}^2}, \tag{3.3.5b}$$

$$P = -\frac{m_{\mathsf{rr}}}{\kappa \mathsf{r}}, \tag{3.3.5c}$$

which describe the gravitational behaviour of a string fluid (Glass and Krisch 1998, 1999).

The energy conditions for this kind of fluid are

1. The weak and strong energy conditions:

$$\mu \geq 0, \qquad \tilde{\rho} \geq 0, \qquad P \geq 0 \qquad (\mu \neq 0). \tag{3.3.6}$$

2. The dominant energy condition:

$$\mu \geq 0, \qquad \tilde{\rho} \geq P \geq 0 \qquad (\mu \neq 0). \tag{3.3.7}$$

In the case when $m = m(v)$ the above energy conditions all reduce to $\mu \geq 0$, and if $m = m(r)$, then $\mu = 0$ and the matter field becomes a Type I fluid. For the purposes of many applications, it is a requirement that the matter distribution satisfy an equation of state

$$P = P(\tilde{\rho}), \tag{3.3.8}$$

on physical grounds. Sometimes the linear $\gamma$-law equation of state

$$P = (\gamma - 1)\tilde{\rho}, \tag{3.3.9}$$

where $0 < \gamma < 1$, is assumed in cosmology when probing the dynamics of matter on galactic and extragalactic length scales. The case $\gamma = 1$ corresponds to dust (vanishing pressure); $\gamma = 2$ gives a stiff equation of state in which the speed of sound and light speed are equal; $\gamma = 4/3$ corresponds to radiation. In the limit when $\gamma = 0$, the fluid pressure is negative, $p = -\tilde{\rho}$ (since $\tilde{\rho} > 0$). This is the characteristic property of the so-called dark energy or the existence of a possible scalar field that is responsible for the accelerated expansion of the universe. Often the particular equation of state

$$P = k\tilde{\rho}^{\gamma},$$

is assumed in relativistic astrophysics; this is called the polytropic equation of state. It is commonly used to model electron degenerate and neutron degenerate gases in white dwarfs and neutron stars, respectively.

## 3.4 Solutions with equations of state

In this section we will impose various equations of state upon the system (3.3.5).

### 3.4.1 Case I(a): Linear

If we assume a linear equation of state $P = k\tilde{\rho}$ to the field equations (3.3.5), we have

$$m_{\mathsf{rr}} + \frac{2k}{\mathsf{r}} m_{\mathsf{r}} = 0, \tag{3.4.1}$$

which is a second order linear partial differential equation. Since we are differentiating with respect to one variable, we can treat it as an ordinary differential equation, in which case it's a weaker variant of the Cauchy-Euler equation. The above equation can be solved via reduction of order and has two solutions. For the case when $k = \frac{1}{2}$, the solution is given by

$$m(v, \mathsf{r}) = c_1(v) \ln(\mathsf{r}) + c_2(v),$$

where $c_1(v)$ and $c_2(v)$ are functions of integration. For $k \neq \frac{1}{2}$ we have the solution

$$m(v, \mathsf{r}) = c_1(v) \frac{\mathsf{r}^{1-2k}}{1-2k} + c_2(v). \tag{3.4.2}$$

Hence we have (for the latter case)

$$\mu = -\left[ \frac{2\dot{c}_1}{\kappa(1-2k)\mathsf{r}^{1+2k}} + \frac{2\dot{c}_2}{\kappa \mathsf{r}^2} \right], \tag{3.4.3a}$$

$$\tilde{\rho} = \frac{2c_1}{\kappa \mathsf{r}^{2+2k}}, \tag{3.4.3b}$$

$$P = \frac{2c_1 k}{\kappa \mathsf{r}^{2+2k}}. \tag{3.4.3c}$$

We have that all the energy conditions are satisfied if $c_1(v) \geq 0$ and $\dot{c}_2(v) < 0$.

### 3.4.2 Case I(b): Generalised linear

Imposing the condition $P = k\tilde{\rho} + k_2$ yields

$$m_{\mathsf{rr}} + \frac{2k}{\mathsf{r}} m_{\mathsf{r}} + \kappa k_2 \mathsf{r} = 0, \tag{3.4.4}$$

which can be solved via reduction of order. Letting $y(v, \mathsf{r}) = m_{\mathsf{r}}$ yields the first order equation

$$y' + \frac{2k}{\mathsf{r}} y + \kappa k_2 \mathsf{r} = 0, \tag{3.4.5}$$

which in turn has the solution

$$y(v, \mathsf{r}) = \frac{-\kappa k_2}{2k+2} \mathsf{r}^2 + c_1 \mathsf{r}^{-2k}, \tag{3.4.6}$$

where $c_1 = c_1(v)$ is a function of integration. Again, two cases arise. When $k = \frac{1}{2}$ the general solution for $m(v, \mathsf{r})$ is given by

$$m(v, \mathsf{r}) = c_1(v) \ln(\mathsf{r}) + c_2(v) - \frac{\kappa k_2 \mathsf{r}^3}{9},$$

where $c_2 = c_2(v)$ is a further integration function. The general solution for the mass when $k \neq \frac{1}{2}$ is

$$m(v, \mathsf{r}) = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}^3 + \frac{c_1 \mathsf{r}^{1-2k}}{1-2k} + c_2. \tag{3.4.7}$$

Thus we have (for the latter case)

$$\mu = -\left[\frac{2\dot{c}_1}{\kappa(1-2k)\mathsf{r}^{1+2k}} + \frac{2\dot{c}_2}{\kappa \mathsf{r}^2}\right], \tag{3.4.8a}$$

$$\tilde{\rho} = \frac{2c_1}{\kappa \mathsf{r}^{2+2k}} - \frac{k_2}{k+1}, \tag{3.4.8b}$$

$$P = \frac{k_2}{k+1} + \frac{2c_1 k}{\kappa \mathsf{r}^{2+2k}}, \tag{3.4.8c}$$

which contains the system (3.4.3) as well as several of the seminal other cases summarised in Wang and Wu (1999). This list of possible solutions is presented in Table 3.1. Again, the energy conditions are satisfied for $c_1(v) \geq 0$ and $\dot{c}_2(v) < 0$.

### 3.4.3 Case II(a): Quadratic

If we impose the quadratic equation of state $P = k\tilde{\rho}^2$ on the system (3.3.5), we have

$$m_{\mathsf{rr}} + \frac{\eta}{\mathsf{r}^3} m_{\mathsf{r}}^2 = 0, \tag{3.4.9}$$

where $\eta = 4k/\kappa$. Reducing the order of the above equation with $y(v, \mathsf{r}) = m_{\mathsf{r}}$ gives

$$y' + \frac{\eta}{\mathsf{r}^3} y^2 = 0, \tag{3.4.10}$$

which is a separable equation in $y$. The solution is

$$y(v, \mathsf{r}) = -\frac{2\mathsf{r}^2}{c_1 \mathsf{r}^2 + \eta}, \tag{3.4.11}$$

where $c_1 = c_1(v)$ is again, an integration function. Hence, the general solution for $m$ is

$$m(v, \mathsf{r}) = c_2 - 2\left(\frac{\mathsf{r}}{2c_1} - \frac{\sqrt{\eta}\arctan\left(\frac{\sqrt{2}\sqrt{c_1}\mathsf{r}}{\sqrt{\eta}}\right)}{2\sqrt{2}c_1^{3/2}}\right), \tag{3.4.12}$$

where $c_2 = c_2(v)$ is a second integration function. Hence the field equations give

$$
\begin{aligned}
\mu &= -\frac{2}{\kappa \mathsf{r}^2}\left[\dot{c}_2 + \frac{\dot{c}_1 \mathsf{r}}{c_1^2} + \frac{\dot{c}_1 \mathsf{r}}{2c_1\left(1+\frac{2c_1\mathsf{r}^2}{\eta}\right)}\right. \\
&\quad \left. - \frac{3\sqrt{2}\sqrt{\eta}\dot{c}_1}{4\sqrt{c_1}^5}\arctan\left(\frac{\sqrt{2}\sqrt{c_1}\mathsf{r}}{\sqrt{\eta}}\right)\right], && (3.4.13a) \\
\tilde{\rho} &= \frac{4}{\kappa(2c_1\mathsf{r}^2+\eta)}, && (3.4.13b) \\
P &= \frac{4\eta}{\kappa(2c_1\mathsf{r}^2+\eta)^2}. && (3.4.13c)
\end{aligned}
$$

For the above, the energy conditions are satisfied for the following restriction: $\eta \geq 0$ or $c_1(v) \geq 0$ and for any $c_2(v)$.

### 3.4.4 Case II(b): Generalised quadratic

Imposing the condition $P = k\tilde{\rho}^2 + k_2\tilde{\rho} + k_3$ yields

$$
m_{\mathsf{rr}} + \frac{\eta}{\mathsf{r}^3}m_{\mathsf{r}}^2 + \frac{2k_2}{\mathsf{r}}m_{\mathsf{r}} + k_3\kappa\mathsf{r} = 0, \tag{3.4.14}
$$

which can be solved again by reducing the order. Doing so with $y(v,\mathsf{r}) = m_{\mathsf{r}}$ yields

$$
y' + \frac{\eta}{\mathsf{r}^3}y^2 + \frac{2k_2}{\mathsf{r}}y + k_3\kappa\mathsf{r} = 0. \tag{3.4.15}
$$

Equation (3.4.15) is a nonlinear Riccati equation. Integration yields

$$
\begin{aligned}
y(v,\mathsf{r}) &= -\frac{1}{\eta}\left(\mathsf{r}^2\tan\left(\sqrt{k_3\kappa\eta - k_2^2 - 2k_2 - 1}(\ln\mathsf{r} - c_1)\right)\right. \\
&\quad \left. \times\sqrt{k_3\kappa\eta - k_2^2 - 2k_2 - 1}\right), && (3.4.16)
\end{aligned}
$$

where $c_1 = c_1(v)$ is a function of integration. Thus, the solution for $m$ can be expressed as a quadrature

$$
m(v,\mathsf{r}) = -\frac{1}{\eta}\int\left(\mathsf{r}^2\tan\left(\sqrt{\zeta}\,(\ln\mathsf{r} - c_1)\right)\sqrt{\zeta}\right)d\mathsf{r} + c_2, \tag{3.4.17}
$$

where $c_2 = c_2(v)$ is a second integration function. In the above we have set $\zeta = k_3\kappa\eta - k_2^2 - 2k_2 - 1$ for convenience. So we have the result

$$\begin{aligned}
\mu &= \frac{2}{\kappa\eta\mathsf{r}^2}\left[\frac{\partial}{\partial v}\int\left(\mathsf{r}^2\sqrt{\zeta}\tan\left(\sqrt{\zeta}(\ln\mathsf{r}-c_1)\right)\right)d\mathsf{r}\right] \\
&\quad +\frac{2}{\kappa\eta\mathsf{r}^2}\dot{c}_2, \qquad (3.4.18a)\\
\tilde{\rho} &= \frac{2}{\kappa\eta\mathsf{r}^2}\left(\sqrt{\zeta}\mathsf{r}^2\tan\left(\sqrt{\zeta}(\ln\mathsf{r}-c_1)\right)\right), \qquad (3.4.18b)\\
P &= \frac{1}{\kappa\eta}\left[2\sqrt{\zeta}\tan(\sqrt{\zeta})\times\zeta\sec^2\left(\sqrt{\zeta}(\ln\mathsf{r}-c_1)\right)\right]. \qquad (3.4.18c)
\end{aligned}$$

The energy conditions are satisfied as in the previous case for any $c_2(v)$ and the condition: $\eta \geq 0$ or $c_1(v) \geq 0$.

### 3.4.5 Case III: Polytropic

If we finally impose the equation of state $P = k\tilde{\rho}^\gamma$, we have

$$m_{\mathsf{rr}} + k\kappa\left(\frac{2}{\kappa}\right)^\gamma \mathsf{r}^{1-2\gamma}m_{\mathsf{r}}^\gamma = 0. \qquad (3.4.19)$$

Reducing the order of the above equation with $y(v,\mathsf{r}) = m_{\mathsf{r}}$ gives

$$y' + k\kappa\left(\frac{2}{\kappa}\right)^\gamma \mathsf{r}^{1-2\gamma}y^\gamma = 0, \qquad (3.4.20)$$

which is a separable equation in $y$. Therefore the solution is

$$y(v,\mathsf{r}) = \left[(\gamma+1)k\kappa\left(\frac{2}{\kappa}\right)^\gamma\frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1\right]^{\frac{1}{1-\gamma}}, \qquad (3.4.21)$$

where $c_1 = c_1(v)$ is a function resulting from the integration process. So the solution for the mass $m$ is

$$\begin{aligned}
m(v,\mathsf{r}) &= \int\left[(\gamma+1)k\kappa\left(\frac{2}{\kappa}\right)^\gamma\right.\\
&\quad \left.\times\frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1\right]^{\frac{1}{1-\gamma}} d\mathsf{r} + c_2, \qquad (3.4.22)
\end{aligned}$$

Table 3.1: Known solutions contained within the system (3.4.8).

| Solution | $m(v,\mathsf{r})$ | $c_1(v)$ and $c_2(v)$ | $k$-indices |
|---|---|---|---|
| Monopole | $\frac{a\mathsf{r}}{2}$ | $c_1(v)=\frac{a}{2}$, $c_2(v)=0$ | $k, k_2=0$ |
| Charged Vaidya | $g(v)-\frac{q(v)^2}{2\mathsf{r}}$ | $c_1=\frac{q(v)^2}{2}$, $c_2=g(v)$ | $k=1$, $k_2=0$ |
| dS/AdS | $\frac{\Lambda}{6}\mathsf{r}^3$ | $c_1(v)=c_2(v)=0$ | $k=$ const., $k_2=-\frac{\Lambda(k+1)}{\kappa}$ |
| Husain | $g(v)-\frac{q(v)}{(2k-1)\mathsf{r}^{2k-1}}$ | $c_1(v)=\frac{-q(v)}{2}$, $c_2(v)=g(v)$ | $k, k_2=$ const. |

where $c_2=c_2(v)$ is a second integration function. It should be noted that this solution was first presented by Husain (1996). The field equations yield

$$\mu = -\frac{2}{\kappa\mathsf{r}^2}\left(\frac{\partial}{\partial v}\int\left[(\gamma+1)k\kappa\left(\frac{2}{\kappa}\right)^{\gamma} \times\frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma}+(1-\gamma)c_1\right]^{\frac{1}{1-\gamma}}d\mathsf{r}\right)-\frac{2}{\kappa\mathsf{r}^2}\dot{c}_2, \tag{3.4.23a}$$

$$\tilde{\rho} = \frac{2}{\kappa\mathsf{r}^2}\left[(\gamma+1)k\kappa\left(\frac{2}{\kappa}\right)^{\gamma}\frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma}+(1-\gamma)c_1\right]^{\frac{1}{1-\gamma}}, \tag{3.4.23b}$$

$$P = \frac{1}{(1-\gamma)\mathsf{r}^{2k}}\left[(\gamma+1)k\kappa\left(\frac{2}{\kappa}\right)^{\gamma}\frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma}+(1-\gamma)c_1\right]^{\frac{\gamma}{1-\gamma}} \times(\gamma+1)k\left(\frac{2}{\kappa}\right)^{\gamma}. \tag{3.4.23c}$$

A summary of the above solutions can be found in Table 3.2. The weak and strong energy conditions are satisfied when $c_1(v)\geq 0$ and for any $c_2(v)$. The dominant energy condition is satisfied for $c_1(v)\geq 0$ and the further restriction: $\frac{2}{\kappa r^{2k}}\geq\frac{(\gamma+1)}{(1-\gamma)}\left(\frac{2}{\kappa}\right)^{\gamma}$.

Table 3.2: Equations of state and the gravitational mass.

| Equation of state | $P = P(\tilde{\rho})$ | $m(v, \mathsf{r})$ | |
|---|---|---|---|
| Linear | $P = k\tilde{\rho}$ | $m(v,\mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v),$ | $(k = \frac{1}{2})$ |
| | | $m(v,\mathsf{r}) = c_1(v)\frac{\mathsf{r}^{1-2k}}{1-2k} + c_2(v),$ | $(k \neq \frac{1}{2})$ |
| Generalised linear | $P = k\tilde{\rho} + k_2$ | $m(v,\mathsf{r}) = c_1(v)\ln(\mathsf{r})$ | |
| | | $+c_2(v) - \frac{k_2\kappa\mathsf{r}^3}{9},$ | $(k = \frac{1}{2})$ |
| | | $m(v,\mathsf{r}) = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}^3$ | |
| | | $+\frac{c_1(v)\mathsf{r}^{1-2k}}{1-2k} + c_2(v)$ | $(k \neq \frac{1}{2})$ |
| Quadratic | $P = k\tilde{\rho}^2$ | $m(v,\mathsf{r}) = c_2(v) - 2\left(\frac{\mathsf{r}}{2c_1(v)}\right.$ | |
| | | $\left. - \frac{\sqrt{\eta}\arctan\left(\frac{\sqrt{2}\sqrt{c_1(v)}\mathsf{r}}{\sqrt{\eta}}\right)}{2\sqrt{2}c_1(v)^{3/2}}\right)$ | |
| Generalised quadratic | $P = k\tilde{\rho}^2$ | $m(v,\mathsf{r}) = c_2(v)$ | |
| | $+k_2\tilde{\rho} + k_3$ | $-\frac{1}{\eta}\int\left(\mathsf{r}^2\tan\left(\sqrt{\zeta}\,(\ln\mathsf{r} - c_1(v))\right)\sqrt{\zeta}\right)d\mathsf{r}$ | |
| Polytropic | $P = k\tilde{\rho}^\gamma$ | $m(v,\mathsf{r}) = \int\left[(\gamma+1)k\kappa\left(\frac{2}{\kappa}\right)^\gamma\right.$ | |
| | | $\left.\times\frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1(v)\right]^{\frac{1}{1-\gamma}} d\mathsf{r} + c_2(v)$ | |

Table 3.3: Equations of state and the higher dimensional gravitational mass.

| Equation of state | $P = P(\tilde{\rho})$ | $m(v, \mathsf{r})$ |
|---|---|---|
| Linear | $P = k\tilde{\rho}$ | $m(v,\mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v), \qquad (k = \frac{1}{N-2})$ |
| | | $m(v,\mathsf{r}) = c_1(v)\frac{\mathsf{r}^{1-(N-2)k}}{1-(N-2)k} + c_2(v), \quad (k \neq \frac{1}{N-2})$ |
| Generalised linear | $P = k\tilde{\rho} + k_2$ | $m(v,\mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v)$ |
| | | $-\frac{\kappa k_2}{(N-2)k+N-2}\frac{\mathsf{r}^{N-1}}{N-1}, \qquad (k = \frac{1}{N-2})$ |
| | | $m(v,\mathsf{r}) = -\frac{\kappa k_2}{(N-2)k+N-2}\frac{\mathsf{r}^{N-1}}{N-1}$ |
| | | $+c_1(v)\frac{\mathsf{r}^{1-(N-2)k}}{1-(N-2)k} + c_2(v) \qquad (k \neq \frac{1}{N-2})$ |
| Quadratic | $P = k\tilde{\rho}^2$ | $m(v,\mathsf{r}) = (2-N)\int \frac{\mathsf{r}^{N-2}}{c_1(v)(N-2)\mathsf{r}^{N-2}+\eta}d\mathsf{r} + c_2(v)$ |
| Generalised quadratic | $P = k\tilde{\rho}^2$ | $m(v,\mathsf{r}) = -\frac{1}{2\eta}\int \left[\left(\mathsf{r}^{N-2}\tan(\sqrt{\varsigma}(\ln\mathsf{r} - c_1(v))\right)\right.$ |
| | $+k_2\tilde{\rho} + k_3$ | $\left.\times(\sqrt{\varsigma} + N - 2 + \xi)\right] d\mathsf{r} + c_2(v)$ |
| Polytropic | $P = k\tilde{\rho}^\gamma$ | $m(v,\mathsf{r}) = \int \left[\kappa k(\gamma+1)\left(\frac{N-2}{\kappa}\right)^\gamma\right.$ |
| | | $\left.\times\frac{\mathsf{r}^{N-2-\gamma(N-2)}}{N-2-\gamma(N-2)} + (1-\gamma)c_1(v)\right]^{\frac{1}{1-\gamma}} d\mathsf{r} + c_2(v)$ |

## 3.5 Higher dimensional Vaidya spacetime

Higher dimensional Vaidya spacetimes have a variety of physical applications. For example, the thermodynamics of spacetime, entropy and the existence of horizons have been studied in detail by Debnath (2014). Also Mkenyeleye et al. (2015) considered gravitational collapse in higher dimensional Vaidya spacetimes. It is also interesting to note that solutions have been found in alternate theories of gravity. For example, Dominguez and Gallo (2006) found families of radiating black hole solutions for various equations of state in higher dimensional Einstein-Gauss-Bonnet gravity. Collapse and other physical features are affected by the presence of higher dimensions. The $N$-dimensional generalised Vaidya metric is given by

$$ds^2 = -\left(1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}^{N-3}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2 d\Omega^2_{N-2}, \tag{3.5.1}$$

where

$$d\Omega^2_{N-2} = \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta^j)\right](d\theta^i)^2.$$

The nonvanishing Ricci tensor components are given by

$$\begin{aligned} R^0{}_0 &= R^1{}_1 = \frac{m_{\mathsf{rr}}}{\mathsf{r}^{(N-3)}} - \frac{(N-4)m_{\mathsf{r}}}{\mathsf{r}^{(N-2)}}, && (3.5.2a)\\ R^1{}_0 &= \frac{(N-2)m_v}{\mathsf{r}^{(N-2)}}, && (3.5.2b)\\ R^2{}_2 &= R^3{}_3 = \ldots = R^{\theta(N-2)}{}_{\theta(N-2)} = \frac{2m_{\mathsf{r}}}{\mathsf{r}^{(N-2)}}, && (3.5.2c) \end{aligned}$$

with the Ricci scalar

$$R = \frac{2m_{\mathsf{rr}}}{\mathsf{r}^{(N-3)}} + \frac{4m_{\mathsf{r}}}{\mathsf{r}^{(N-2)}}. \tag{3.5.3}$$

The nonvanishing components of the Einstein tensor are

$$\begin{aligned} G^0{}_0 &= G^1{}_1 = -\frac{(N-2)m_{\mathsf{r}}}{\mathsf{r}^{(N-2)}}, && (3.5.4a)\\ G^1{}_0 &= \frac{(N-2)m_v}{\mathsf{r}^{(N-2)}}, && (3.5.4b)\\ G^2{}_2 &= G^3{}_3 = \ldots = G^{\theta(N-2)}{}_{\theta(N-2)} = -\frac{m_{\mathsf{rr}}}{\mathsf{r}^{(N-3)}}. && (3.5.4c) \end{aligned}$$

The Einstein field equations are thus

$$\mu = -\frac{(N-2)m_v}{\kappa \mathsf{r}^{N-2}}, \tag{3.5.5a}$$

$$\tilde{\rho} = \frac{(N-2)m_{\mathsf{r}}}{\kappa \mathsf{r}^{N-2}}, \tag{3.5.5b}$$

$$P = -\frac{m_{\mathsf{rr}}}{\kappa \mathsf{r}^{N-3}}. \tag{3.5.5c}$$

As in section 3.4 we can find solutions to the field equations with various equations of state for the higher dimensional Vaidya spacetime (3.5.1). The results are presented in Table 3.3 for particular equations of state.

## 3.6 Diffusion

The notion of diffusion is an important one in regards to the understanding of many physical systems. The ideas of diffusion have been applied to fields as diverse as the stock exchange, kinetic theory and physiology. Vilenken (1981) characterised string evolutions as the formation of Brownian trajectories in an attempt to introduce diffusion into the description of cosmic strings. Calogero (2011) presented a new model to describe the dynamics of particles undergoing diffusion in general relativity. It was shown that in the flat Robertson-Walker spacetime, either unlimited expansion or the formation of a singularity may occur, depending on the initial value of the cosmological scalar field. If we assume that string diffusion is likened to point particle diffusion then we have

$$\frac{\partial}{\partial v} n = \mathcal{D}\nabla^2 n, \tag{3.6.1}$$

where $\nabla^2 = r^{-2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$ and $\mathcal{D}$ is the positive coefficient of self-diffusion, which we treat as a constant. In classical transport theory the diffusion equation is derived beginning with Fick's law

$$\vec{J}_{(n)} = -\mathcal{D}\vec{\nabla} n, \tag{3.6.2}$$

where $\vec{\nabla}$ is a purely spatial gradient. The 4-current conservation $J^a_{(n);a} = 0$, where

$$\begin{aligned} J^a_{(n);a} &= (n, \vec{J}_{(n)}) \\ &= n\frac{\partial}{\partial u} - \mathcal{D}\left(\frac{\partial n}{\partial \mathsf{r}}\right)\left(\frac{\partial}{\partial \mathsf{r}}\right), \end{aligned} \tag{3.6.3}$$

then yields the diffusion equation (3.6.1). Rewriting the field equations (3.3.5a) and (3.3.5b) as $m_v = -\kappa\mu\mathsf{r}^2$ and $m_{\mathsf{r}} = \kappa\tilde{\rho}\mathsf{r}^2$, we can express the integrability condition for $m$ as

$$\frac{\partial\tilde{\rho}}{\partial v} + \frac{1}{\mathsf{r}^2}\frac{\partial}{\partial \mathsf{r}}(\mathsf{r}^2\mu) = 0. \tag{3.6.4}$$

If we compare the diffusion equation (3.6.1) ($n$ replaced with $\rho$) with $\tilde{\rho}_v$ in equation (3.6.4) above, we get

$$\frac{\partial m}{\partial v} = \mathcal{D}\mathsf{r}^2\frac{\partial\tilde{\rho}}{\partial \mathsf{r}}. \tag{3.6.5}$$

Solving the above equation (3.6.5) for the mass function $m(v, \mathsf{r})$ will provide solutions for the Einstein equations. Recall that the equations of state presented earlier were of the form $F_1(m', m'') = 0$. Using the field equation (3.3.5b) and substituting into (3.6.5) we get the following

$$\frac{\partial m}{\partial v} = \frac{2\mathcal{D}}{\kappa}\left(\frac{\partial^2 m}{\partial \mathsf{r}^2} - \frac{4}{\mathsf{r}}\frac{\partial m}{\partial \mathsf{r}}\right), \tag{3.6.6}$$

which is of the functional form $F_2(\dot{m}, m', m'') = 0$. In order to solve (3.6.6) entirely we require a functional form $F_3(\dot{m}, m') = 0$. This entails isolating $\frac{\partial^2 m}{\partial \mathsf{r}^2}$ in each equation of state and substituting into (3.6.6). We will consider some cases below.

### 3.6.1 Linear

If we begin with the linear equation of state $P = k\rho$ we have from before

$$m_{\mathsf{rr}} + \frac{2k}{\mathsf{r}}m_{\mathsf{r}} = 0,$$

which we can substitute into (3.6.6) to finally get

$$\frac{\partial m}{\partial v} - \frac{\alpha}{\mathsf{r}}\frac{\partial m}{\partial \mathsf{r}} = 0, \tag{3.6.7}$$

where $\alpha = -\frac{2\mathcal{D}}{\kappa}(2k+4)$ for convenience. Equation (3.6.7) can be solved using the method of characteristics and the solution is given by

$$m(v, \mathsf{r}) = \mathcal{F}\left(\frac{1}{2}\mathsf{r}^2 + \alpha v\right), \tag{3.6.8}$$

which is an infinite family of solutions. To check for consistency, we simply have to substitute (3.6.8) into (3.4.1). In doing so we get

$$\mathsf{r}^2\mathcal{F}'' + (1+2k)\mathcal{F}' = 0, \tag{3.6.9}$$

which is a consistency condition on $\mathcal{F}$. It turns out that a solution is only possible for the case when $k = -\frac{1}{2}$. It is given by

$$\mathcal{F}\left(\frac{1}{2}\mathsf{r}^2 + \alpha v\right) = l_1\left(\frac{1}{2}\mathsf{r}^2 + \alpha v\right) + l_2, \tag{3.6.10}$$

where $l_1$ and $l_2$ are constants.

### 3.6.2 Generalised linear

For the generalised linear equation of state the resulting partial differential equation becomes

$$\frac{\partial m}{\partial v} - \frac{\alpha}{\mathsf{r}}\frac{\partial m}{\partial \mathsf{r}} + \kappa k_2 \mathsf{r} = 0, \tag{3.6.11}$$

and can be solved in the same way as above giving the solution

$$m(v, \mathsf{r}) = \mathcal{F}\left(\frac{1}{2}\mathsf{r}^2 + \alpha v\right) - \frac{\kappa k_2}{3\alpha}\mathsf{r}^3. \tag{3.6.12}$$

The above solution is consistent if and only if

$$\mathsf{r}^2\mathcal{F}'' + (1+2k)\mathcal{F}' - \left(\frac{2k_2\kappa}{\alpha} + \frac{2kk_2\kappa}{\alpha} - k_2\kappa\right)\mathsf{r} = 0. \tag{3.6.13}$$

Then we must have

$$\mathcal{F}'' = (1+2k)\mathcal{F}' = \left(\frac{2k_2\kappa}{\alpha} + \frac{2kk_2\kappa}{\alpha} - k_2\kappa\right) = 0. \tag{3.6.14}$$

This implies that $\mathcal{F}$ has the same form as (3.6.10) and $k = -\frac{1}{2}$. We also have

$$k_2\kappa\left(1 - \frac{1}{\alpha}\right) = 0. \tag{3.6.15}$$

Since $k_2 \neq 0$, we have that $\left(1 - \frac{1}{\alpha}\right) = 0$ which implies $\alpha = 1$ and so $\kappa = -6\mathcal{D}$. This is also a generalisation of the first result (3.6.10) where $k = -\frac{1}{2}$ with the added restriction that $\kappa = -6\mathcal{D}$.

### 3.6.3 Generalised quadratic

Isolating $m_{\mathsf{rr}}$ in the equation (3.4.14) and substituting into (3.6.6) the resulting partial differential equation is given by

$$\frac{\partial m}{\partial v} - \Theta\eta\frac{1}{\mathsf{r}^3}\left(\frac{\partial m}{\partial \mathsf{r}}\right)^2 - \Theta\left(\frac{2k_2+4}{\mathsf{r}}\right)\frac{\partial m}{\partial \mathsf{r}} - \Theta k_3\kappa\mathsf{r} = 0, \tag{3.6.16}$$

where $\Theta = -\frac{2\mathcal{D}}{\kappa}$ and $\eta = \frac{4k}{\kappa}$. This equation above cannot be solved via the method of characteristics and so another approach is needed. If we assume a separable solution for the mass function $m$ of the form

$$m(v, \mathsf{r}) = a(v) + b(\mathsf{r}),$$

then we can express (3.6.16) as two ordinary differential equations

$$\frac{da}{dv} = c, \tag{3.6.17a}$$

$$\Theta\eta\frac{1}{\mathsf{r}^3}\left(\frac{db}{d\mathsf{r}}\right)^2 - \Theta\left(\frac{2k_2+4}{\mathsf{r}}\right)\frac{db}{d\mathsf{r}} - \Theta k_3\kappa\mathsf{r} = c, \tag{3.6.17b}$$

where $c$ is a constant. Both of these equations can be analysed independently and used to yield a solution for the master equation (3.6.16). Solving equations (3.6.17a) and (3.6.17b) yields the final expression for the mass function as

$$\begin{aligned} m(v, \mathsf{r}) = {} & cv + \varepsilon - \frac{(2k_2+4)}{6\eta}\mathsf{r}^3 \pm \left[\frac{1}{6\alpha\beta\eta}(\alpha\mathsf{r}^2 + \beta\mathsf{r})^{\frac{3}{2}}\right. \\ & - \frac{\beta\sqrt{\alpha\mathsf{r}}}{8\alpha^{\frac{5}{2}}\eta\sqrt{\beta}}\left(1 + \frac{\alpha\mathsf{r}}{\beta}\right)^{\frac{3}{2}} + \frac{\beta}{16\alpha^{\frac{5}{2}}\eta}\sqrt{1 + \frac{\alpha\mathsf{r}}{\beta}} \\ & \left. - \frac{\beta}{2\alpha^{\frac{5}{2}}\eta}\ln\left(\sqrt{1 + \frac{\alpha\mathsf{r}}{\beta}} + \frac{\sqrt{\alpha\mathsf{r}}}{\sqrt{\beta}}\right) + \frac{\zeta}{2\beta\eta}\right], \end{aligned} \tag{3.6.18}$$

where $\alpha = \Theta^2(2k_2+4)^2 - 4\Theta^4 k_3\kappa$, $\beta = 4c$ and, as before $\eta = \frac{4k}{\kappa}$. In the above $\varepsilon$ and $\zeta$ are constants of integration. The mass functions (3.6.18) are other solutions to the diffusion equation (3.6.6) which we believe are new. The functional form (3.6.18) has to be consistent with the equation of state (3.4.14).

### 3.6.4 Polytropic

Special mention should be made about this particular case. The partial differential equation resulting from substitution of the expression (3.4.19) for the polytrope is

$$\frac{\partial m}{\partial v} - \beta\kappa k\left(\frac{2}{\kappa}\right)\mathsf{r}^{2-2\gamma}\left(\frac{\partial m}{\partial \mathsf{r}}\right)^{\gamma} - \beta\frac{4}{\mathsf{r}}\frac{\partial m}{\partial \mathsf{r}} = 0, \tag{3.6.19}$$

which is a first order, degree $\gamma$ nonlinear equation. The above equation can only be solved for specific values of the constant $\gamma$ but not in general. Specifically, we have shown that it only admits general closed form analytical solutions only for $0 < \gamma \leq 2$. The case $\gamma > 3$ is highly nonlinear and not easy to analyse.

## 3.7 Discussion

In this chapter we considered a spherically symmetric radiating star. We noted that any astrophysical star is a combination of three distinct concentric zones: the innermost two-component matter zone, the middle radiation zone and the outermost zone which is the vacuum Schwarzschild exterior. A large family of solutions to the field equations were presented for various thermodynamically realistic equations of state. We showed that it was possible to obtain solutions via a direct integration of simple second order differential equations. Note that many of our solutions cannot be found using the Wang and Wu (1999) approach; they assumed a series form of the mass function which is restrictive. Several other mass functions have been shown to exist in four and higher dimensions which are physically reasonable. It was also possible to obtain several diffusive solutions for the mass function via a substitution of the above mentioned second order equations into the diffusion equation. We can easily show that a dynamical radiating star is possible by matching the mass function (3.2.3) at the two boundaries. We illustrate this with the generalised linear equation of state

$$P = k\tilde{\rho} + k_2. \tag{3.7.1}$$

At the first interface $\mathsf{r} = \mathsf{r}_b$, between the two-component region and the null Vaidya zone, the mass function is

$$m_1(v) = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}_b^3 + \frac{c_1(v)\mathsf{r}_b^{1-2k}}{1-2k} + c_2(v). \tag{3.7.2}$$

At the second interface, between the Vaidya zone and the vacuum exterior the mass function is

$$M = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}_b^3 + \frac{c_1(V_0)\mathsf{r}_b^{1-2k}}{1-2k} + c_2(V_0). \tag{3.7.3}$$

Clearly the forms (3.7.2) and (3.7.3) are always possible since $c_1(v)$ and $c_2(v)$ are arbitrary functions. A comparison with earlier well known results was undertaken and we showed that our solutions generalise all of the earlier ones, including those of Husain (1996). We then generalised our results to higher dimensional spacetimes. Additionally, diffusive solutions are also possible at each interface for the same reasons. These (for the generalised linear case) are given by

$$m_1(v, \mathsf{r}) = \mathcal{F}\left(\frac{1}{2}\mathsf{r}_b^2 + \alpha v\right) - \frac{\kappa k_2}{3\alpha}\mathsf{r}_b^3, \tag{3.7.4a}$$

$$M = \mathcal{F}\left(\frac{1}{2}\mathsf{r}_b^2 + \alpha V_0\right) - \frac{\kappa k_2}{3\alpha}\mathsf{r}_b^3, \tag{3.7.4b}$$

at the first and second interfaces respectively.

Another important observation that transparently comes out of our analysis here is the non-linear nature of gravity. Even though the energy momentum tensor can be written as a combination of radiation and matter parts, these quantities intertwine in the metric in such a way as to give physically interesting solutions that can model a dynamic star. If we switch off the radiation part completely, then the field equations force the remaining matter to obey an equation of state $\rho + p_\mathsf{r} = 0$, ($p_\mathsf{r}$ is the radial pressure) which is that of an anisotropic de Sitter-like space, and hence not appropriate for stellar modeling.

This work can further be enhanced by considering the notion of gravitational collapse; whether or not there are special classes of Vaidya mass functions for which a collapse comes to an end as a naked singularity or not. This will be discussed in the next chapter.

# Chapter 4

# Radiating stars undergoing gravitational collapse

## 4.1 Introduction

The notion of radiating stars emitting null radiation, collapsing under their own gravity has been an ongoing area of interest in the relativistic setting. Chan (1997) studied the collapse of a shearing isotropic fluid undergoing radial heat flow with outgoing radiation. Govender (2013) studied the collapse of a spherically symmetric star dissipating in the form of a radial heat flux. An interior was matched to the generalised Vaidya spacetime consisting of a two-fluid atmosphere of null radiation and a string fluid. In higher order theories, Ghosh and Maharaj (2012) found null dust solutions in metric $f(R)$ gravity with constant scalar curvature which described the collapse of null dust matter in an AdS background. Goswami *et al* (2014) studied the physically realistic collapse scenario of massive stars in $f(R)$ gravity. They presented the extra matching conditions required in modified gravity and mentioned the restrictive nature of these impositions (which will be relevant in a later chapter).

Generalised Vaidya spacetimes have been widely used in the study of regular and dynamical black holes (Dawood and Ghosh 2004, Hayward 2006) as well as black holes with trapped regions (Frolov 2014). The Vaidya-Papapetrou model (Papapetrou 1985, Dwivedi and Joshi 1989) is one of the earliest to counter the cosmic censorship conjecture (CCC). Here, a physically reasonable matter field satisfying the energy conditions was found in a shell focusing central singularity ($v = 0$, $\mathsf{r} = 0$) which was formed by imploding shells of radiation. Radially injected radiation flows into a region which is initially flat, and is focused into a central singularity of increasing mass. A central singularity was shown to become a node with a definite tangent for families of nonspacelike geodesics, for some nonvanishing measure of parameters in the model. Thus, the singularity is naked as in families of future directed nonspacelike geodesic curves going to future null infinity, terminate at the central singularity in the past. A comprehensive analysis on censorship violation is given by Joshi (1993, 2007). Mkenyeleye *et al* (2014) studied the gravitational collapse of Vaidya spacetimes in the context of the CCC. A general mathematical framework was developed to study the conditions on the mass function where future directed nonspacelike geodesics can terminate at the central singularity in the past. We will use this framework extensively in this chapter. The results obtained were further generalised in higher dimensions in Mkenyeleye *et al* (2015). Maharaj *et al* (2012) showed that the generalised Vaidya model can be matched smoothly to a heat conducting interior in the Santos (1985) framework. The physical behaviour of this model was analysed in detail by Govender *et al* (2015) where they investigated the effect of the exterior energy density on the temporal evolution of the radiating fluid pressure, luminosity, gravitational redshift, mass flow and collapse rate at the boundary of a relativistic star.

The main intent of this chapter is to study the gravitational collapse of generalised Vaidya spacetimes in the context of the CCC. The mass functions obtained in Chapter 3 for various equations of state will be analysed using the general framework developed in Mkenyeleye *et al* (2014). It will be shown that for each mass function, the collapse terminates with a local central singularity, which is naked. We also calculate the strength of the naked singularities and show that they are strong curvature singularities and there does not exist an extension of spacetime through these singularities. This Chapter is organised as follows: In the next section we present a complete outline of how to model an isolated spherical and physically reasonable radiating astrophysical star via the generalised Vaidya geometry. In the following section we describe the generalised Vaidya spacetime by analysing the field equations; the relevant aphorisms indicative with the geometry of the generalised Vaidya metric are presented and we mention the energy conditions for a physically reasonable model. In Sec. 4.2 we systematically present the mathematical framework as done by Mkenyeleye *et al* (2014) for a collapsing model followed by the conditions for the formation of a locally naked singularity and its strength. In Sec. 4.3 and 4.4 the conditions for the formation of a locally naked singularity as well as the strength of such a singularity are presented in detail. The following section details the end state of generalised mass functions found in Chapter 3 for various equations of state. Finally, we discuss the higher dimensional spacetime and how the consequences of the model may change due to the presence of higher dimensions.

## 4.2 Collapsing model

We will examine the gravitational contraction of imploding matter and radiation described by the generalised Vaidya spacetime. Here, a thick shell of radiation and type I matter collapses at the centre of symmetry (Joshi 1993). If $K^a$ is the tangent to non-spacelike geodesics with $K^a = \frac{dx^a}{d\hat{k}}$, where $\hat{k}$ is the affine parameter, then $K^a{}_{;b}K^b = 0$ and

$$g_{ab}K^aK^b = \mathcal{B}, \tag{4.2.1}$$

where $\mathcal{B}$ is a constant which characterises different classes of geodesics. Null geodesics are characterised by $\mathcal{B} = 0$ while $\mathcal{B} < 0$ applies to timelike geodesics. The equations for the quantities $\frac{dK^v}{d\hat{k}}$ and $\frac{dK^{\mathsf{r}}}{d\hat{k}}$ are calculated from the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^{\varrho}} - \frac{d}{d\hat{k}}\left(\frac{\partial L}{\partial \dot{x}^{\varrho}}\right) = 0, \tag{4.2.2}$$

where the Lagrangian is

$$L = \frac{1}{2} g_{ab}\dot{x}^a \dot{x}^b. \tag{4.2.3}$$

In the above equations the dot denotes differentiation with respect to the affine parameter $\hat{k}$. These equations are given by

$$\frac{dK^v}{d\hat{k}} + \left(\frac{m(v,\mathsf{r})}{\mathsf{r}^2} - \frac{m'(v,\mathsf{r})}{\mathsf{r}}\right)(K^v)^2 - \frac{l^2}{\mathsf{r}^3} = 0, \tag{4.2.4a}$$

$$\begin{aligned} \frac{dK^{\mathsf{r}}}{d\hat{k}} + \frac{\dot{m}(v,\mathsf{r})}{\mathsf{r}}(K^v)^2 - \frac{l^2}{\mathsf{r}^3}\left(1 - \frac{m(v,\mathsf{r})}{\mathsf{r}}\right) & \\ -\mathcal{B}\left(\frac{m(v,\mathsf{r})}{\mathsf{r}^2} - \frac{m'(v,\mathsf{r})}{\mathsf{r}}\right) &= 0. \end{aligned} \tag{4.2.4b}$$

Joshi (1993) gave the components $K^\theta$ and $K^\phi$ of the tangent vector as

$$K^\theta = \frac{l\cos\varphi}{\mathsf{r}^2\sin^2\theta}, \tag{4.2.5a}$$

$$K^\phi = \frac{l\sin\varphi\cos\phi}{\mathsf{r}}, \tag{4.2.5b}$$

where $l$ is the impact parameter and $\varphi$ is the isotropy parameter defined by $\sin\phi\tan\varphi = \cot\theta$.

Following Dwivedi and Joshi (1989), we can write $K^v$ as

$$K^v = \frac{P}{\mathsf{r}}, \tag{4.2.6}$$

where $P = P(v,\mathsf{r})$ is some arbitrary function. Therefore, $g_{ab}K^aK^b = \mathcal{B}$ gives

$$K^{\mathsf{r}} = \frac{P}{2\mathsf{r}}\left[1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}}\right] - \frac{l^2}{2\mathsf{r}P} + \frac{\mathcal{B}\mathsf{r}}{2P}. \tag{4.2.7}$$

Using (4.2.6), we get

$$\frac{dK^v}{d\hat{k}} = \frac{d}{d\hat{k}}\left(\frac{P}{\mathsf{r}}\right) = \frac{1}{\mathsf{r}}\frac{dP}{d\hat{k}} - \frac{P}{\mathsf{r}^2}\frac{d\mathsf{r}}{d\hat{k}}. \tag{4.2.8}$$

Thus,

$$\frac{dP}{d\hat{k}} = \frac{1}{\mathsf{r}}\left(\mathsf{r}^2\frac{dK^v}{d\hat{k}} + P\frac{d\mathsf{r}}{d\hat{k}}\right). \tag{4.2.9}$$

If we substitute equations (4.2.4a), and (4.2.7) into the above equation (4.2.9), the differential equation satisfied by the function $P$ results in

$$\frac{dP}{d\hat{k}} = \frac{P^2}{2\mathsf{r}^2}\left(1 - \frac{4m(v,\mathsf{r})}{\mathsf{r}} + 2m'(v,\mathsf{r})\right) + \frac{l^2}{2\mathsf{r}^2} + \frac{\mathcal{B}}{2}. \tag{4.2.10}$$

If the mass function $m(v,\mathsf{r})$ and initial conditions are defined, the function $P(v,\mathsf{r})$ can then be found.

## 4.3 The conditions for a locally naked singularity

In this section we analyse how the final fate of collapse is determined in terms of a naked singularity or a black hole, given the generalised Vaidya mass function. If there exist families of future directed nonspacelike trajectories reaching observers faraway in spacetime, which terminate in the past at the singularity, then the singularity forming as the final state of collapse is naked. If no such families exist and an event horizon forms at a sufficiently early time to cover the singularity, we then have a black hole. The equation of null geodesics for the metric (3.1.1) is given by

$$\frac{dv}{d\mathsf{r}} = \frac{2\mathsf{r}}{\mathsf{r} - 2m(v,\mathsf{r})}. \tag{4.3.1}$$

This equation has a singularity at $v = 0$ and $\mathsf{r} = 0$ and its nature can be analysed using the standard techniques associated with the theory of differential equations (Tricomi 1961, Perko 1991, Roberts and Stewart 1991).

### 4.3.1 Structure of the central singularity

Equation (4.3.1) can generally be written in separable form

$$\frac{dv}{d\mathsf{r}} = \frac{\hat{M}(v,\mathsf{r})}{\hat{N}(v,\mathsf{r})}, \tag{4.3.2}$$

with its singularity at $v = \mathsf{r} = 0$, where the functions $\hat{M}$ and $\hat{N}$ are vanquished. Thus, the analysis of the existence and uniqueness of the solution to this differential equation should be carefully considered. If we introduce the new independent variable $t$ with differential $dt$ such that

$$\frac{dv}{\hat{M}(v,\mathsf{r})} = \frac{d\mathsf{r}}{\hat{N}(v,\mathsf{r})} = dt, \tag{4.3.3}$$

the differential equation (4.3.2) may be replaced by

$$\frac{dv(t)}{dt} = \hat{M}(v,\mathsf{r}), \tag{4.3.4a}$$

$$\frac{d\mathsf{r}(t)}{dt} = \hat{N}(v,\mathsf{r}). \tag{4.3.4b}$$

It is important to note that all solutions of the equation (4.3.2) are indeed solutions to (4.3.4) and hence, studying the behaviour of this system of equations near the singular point $v = \mathsf{r} = 0$ is necessary in the $(v, \mathsf{r})$ plane. It can be easily seen that the singular point of (4.3.2) is a fixed point of the system (4.3.4). In order to find the necessary and sufficient conditions for the existence of solutions to this system in the region of the fixed point $v = \mathsf{r} = 0$, (4.3.4) can be written as a differential equation of the vector $\mathbf{x}(t) = [v(t), \mathsf{r}(t)]^T$ on $\Re^2$ as

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)). \tag{4.3.5}$$

Several definitions and theorems were given by Mkenyeleye *et al* (2014) on the methodologies of showing the existence and uniqueness of solutions to the above system (4.3.4) and equation (4.3.5).

### 4.3.2 Nature of the fixed point $v = \mathsf{r} = 0$

Since the partial derivatives of the functions $\hat{M}$ and $\hat{N}$ exist and are continuous in the neighbourhood of the fixed point, the system can be linearised near the fixed point and whence, the general behaviour of this system near the singularity is homologous to the characteristic equations

$$\frac{dv}{dt} = Av + B\mathsf{r}, \tag{4.3.6a}$$

$$\frac{d\mathsf{r}}{dt} = Cv + D\mathsf{r}, \tag{4.3.6b}$$

where $A = \hat{M}_v(0,0)$, $B = \hat{M}_{\mathsf{r}}(0,0)$, $C = \hat{N}_v(0,0)$ and $D = \hat{N}_{\mathsf{r}}(0,0)$, with the following

$$\hat{M}_v = \frac{\partial \hat{M}}{\partial v}, \qquad \hat{M}_{\mathsf{r}} = \frac{\partial \hat{M}}{\partial \mathsf{r}},$$

and

$$\hat{N}_v = \frac{\partial \hat{N}}{\partial v}, \qquad \hat{N}_{\mathsf{r}} = \frac{\partial \hat{N}}{\partial \mathsf{r}}.$$

In the above, $AD - BC \neq 0$. Considering a linear transformation of the type

$$\phi = \alpha v + \nu \mathsf{r}, \tag{4.3.7a}$$

$$\psi = \tau v + \omega \mathsf{r}, \tag{4.3.7b}$$

where $\alpha\omega - \nu\tau \neq 0$, as well as the equation

$$\frac{d\psi}{d\phi} = \frac{\chi_2 \psi}{\chi_1 \phi}, \tag{4.3.8}$$

the system (4.3.6) can be expressed in the reduced form

$$\frac{d\phi}{dt} = \chi_1 \phi, \tag{4.3.9a}$$

$$\frac{d\psi}{dt} = \chi_2 \psi, \tag{4.3.9b}$$

where $\chi_1$ and $\chi_2$ are suitable real constants. Using the systems (4.3.6), (4.3.7) and (4.3.9) it can be shown that

$$\alpha(Av + B\mathsf{r}) + \nu(Cv + D\mathsf{r}) = \chi_1(\alpha v + \nu \mathsf{r}), \tag{4.3.10a}$$

$$\tau(Av + B\mathsf{r}) + \omega(Cv + D\mathsf{r}) = \chi_2(\tau v + \omega \mathsf{r}). \tag{4.3.10b}$$

Equating coefficients of $v$ and $\mathsf{r}$ in (4.3.10) above, we acquire

$$\begin{aligned} (A - \chi_1)\alpha + C\nu &= 0, \\ B\alpha + (D - \chi_1)\nu &= 0, \end{aligned} \tag{4.3.11}$$

as well as

$$\begin{aligned} (A - \chi_2)\tau + C\omega &= 0, \\ B\tau + (D - \chi_2)\omega &= 0. \end{aligned} \tag{4.3.12}$$

If the values of $\alpha$, $\nu$, $\tau$ and $\omega$ are not all zero, then the above equations may be satisfied if the determinant of the coefficients vanishes, i.e.

$$\begin{vmatrix} A-\chi & C \\ B & D-\chi \end{vmatrix} = 0,$$

or

$$\chi^2 - (A+D)\chi + Ad - BC = 0. \tag{4.3.13}$$

The above is the characteristic equation, with eigenvalues $\chi_1$ and $\chi_2$, given by

$$\chi = \frac{1}{2}\left((A+D) \pm \sqrt{(A-D)^2 + 4BC}\right). \tag{4.3.14}$$

The singularity of (4.3.6) can be classified as a node if $(A-D)^2+4BC \geq 0$ and $BC > 0$. It is otherwise a center of focus. In equation (4.3.1) we have that $M(v, \mathsf{r}) = 2\mathsf{r}$ and $N(v, \mathsf{r}) = \mathsf{r} - 2m(v, \mathsf{r})$. If $v = 0$ and $\mathsf{r} = 0$ at the central singularity we can define the following limits:

$$m_0 = \lim_{\substack{v\to 0 \\ \mathsf{r}\to 0}} m(v, \mathsf{r}), \tag{4.3.15a}$$

$$\dot{m}_0 = \lim_{\substack{v\to 0 \\ \mathsf{r}\to 0}} \frac{\partial}{\partial v} m(v, \mathsf{r}), \tag{4.3.15b}$$

$$m_0' = \lim_{\substack{v\to 0 \\ \mathsf{r}\to 0}} \frac{\partial}{\partial \mathsf{r}} m(v, \mathsf{r}). \tag{4.3.15c}$$

The null geodesic equation can then be linearised near the central singularity as

$$\frac{dv}{d\mathsf{r}} = \frac{2\mathsf{r}}{(1-2m_0')\mathsf{r} - 2\dot{m}_0 v}. \tag{4.3.16}$$

It can be clearly seen that this equation has a singularity at $v = 0$ and $\mathsf{r}$. It is possible to determine the nature of this singularity by observing the discriminant value of the characteristic equation. By making use of (4.3.14), the roots of the characteristic equation are

$$\chi = \frac{1}{2}\left((1-2m_0') \pm \sqrt{(1-2m_0')^2 - 16\dot{m}_0}\right). \tag{4.3.17}$$

For the singularity at $v = 0$ and $\mathsf{r} = 0$ to be a node, we must have that

$$(1 - 2m_0')^2 - 16\dot{m}_0 \geq 0, \qquad \dot{m}_0 > 0. \tag{4.3.18}$$

Hence, if the mass function $m(v, \mathsf{r})$ is chosen such that the above condition (4.3.18) is satisfied, the singularity at the origin will then be a node and outgoing nonspacelike geodesics can exit the singularity with a defined value of the tangent.

### 4.3.3 Existence of outgoing nonspacelike geodesics

We can now choose a generalised Vaidya mass function with the properties:

1. The mass function $m(v, \mathsf{r})$ obeys all the physically reasonable energy conditions throughout the spacetime.

2. The partial derivatives of the mass function must exist and are continuous on the entire spacetime.

3. The limits of the partial derivatives of the mass function $m(v, \mathsf{r})$ at the central singularity obey the conditions: $(1 - 2m_0')^2 - 16\dot{m}_0 \geq 0$ and $\dot{m}_0 > 0$.

A mass function with the above properties would ensure the existence and uniqueness of the solutions of the null geodesic equation in the immediate vicinity of the central singularity. Also, the central singularity will be a node of $C^1$ solutions with definite tangents.

If we consider the tangents of these curves at the singularity, we can find the condition for the existence of outgoing radial nonspacelike geodesics from the nodal singularity. Let $X$ be the tangent to the radial null geodesic. If the limiting value of $X$ is finite and positive at the singular point, we can then see that the outgoing future directed null geodesics terminate in the past at the central singularity. The existence of these radial null geodesics characterises the nature (a naked singularity or a black hole) of the collapsing solutions. To determine the nature of the limiting value of $X$

at $v = 0$ and $\mathsf{r} = 0$, we define the following:

$$X_0 = \lim_{\substack{v \to 0 \\ \mathsf{r} \to 0}} X = \lim_{\substack{v \to 0 \\ \mathsf{r} \to 0}} \frac{v}{\mathsf{r}}. \tag{4.3.19}$$

Using (4.3.16) and l'Hôpital's rule (for the $C^1$ null geodesics) we acquire

$$X_0 = \lim_{\substack{v \to 0 \\ \mathsf{r} \to 0}} \frac{v}{\mathsf{r}} = \frac{dv}{d\mathsf{r}} = \frac{2}{(1 - 2m_0') - 2\dot{m}_0 \left(\frac{v}{\mathsf{r}}\right)}, \tag{4.3.20}$$

which in turn simplifies to

$$X_0 = \frac{2}{(1 - 2m_0') - 2\dot{m}_0 X_0}. \tag{4.3.21}$$

Solving the above for $X_0$ gives the following

$$X_0 = b_{\pm} = \frac{(1 - 2m_0') \pm \sqrt{(1 - 2m_0')^2 - 16\dot{m}_0}}{4\dot{m}_0}. \tag{4.3.22}$$

If one or more positive real roots exist for (4.3.21), then the singularity may be locally naked if the null geodesic lies outside the trapped region. In the following subsection we will consider the dynamics of the trapped region to find conditions for the existence of such geodesics.

### 4.3.4 Apparent horizon

The causal behaviour of the trapped surfaces developing within the spacetime usually decides the occurrence of a naked singularity or black hole during the collapse evolution. The apparent horizon is the boundary of the trapped surface region within the spacetime. The equation of the apparent horizon for the generalised Vaidya spacetime is given as

$$\frac{2m(v, \mathsf{r})}{\mathsf{r}} = 1. \tag{4.3.23}$$

Hence, the slope of the apparent horizon can be calculated in the following manner:

$$2\frac{dm}{d\mathsf{r}} = 1, \tag{4.3.24a}$$

$$2\left(\frac{\partial m}{\partial v}\right)\left(\frac{dv}{d\mathsf{r}}\right)_{AH} + 2\frac{\partial m}{\partial \mathsf{r}} = 1, \tag{4.3.24b}$$

which gives the slope of the apparent horizon at the central singularity ($v \longrightarrow 0$, $\mathsf{r} \longrightarrow 0$) as

$$\left(\frac{dv}{d\mathsf{r}}\right)_{AH} = \frac{1-2m_0'}{2\dot{m}_0}. \tag{4.3.25}$$

All of the above can now be stated in the following proposition.

**Proposition 1.** Consider a collapsing generalised Vaidya spacetime from some regular epoch, with a mass function $m(v,\mathsf{r})$ that obeys all the physically reasonable energy conditions and is differentiable in the entire spacetime. The central singularity is locally naked with outgoing $C^1$ radial null geodesics escaping to the future if the following conditions:

1. The limits of the partial derivatives of the mass function $m(v,\mathsf{r})$ at the central singularity obey the conditions: $(1-2m_0')^2 - 16\dot{m}_0 \geq 0$ and $\dot{m}_0 > 0$,
2. There exists at least one root $X_0$ (real and positive) of equation (4.3.21),
3. At least one of the positive real roots is less than $\left(\frac{dv}{dr}\right)_{AH}$ at the central singularity,

are satisfied.

## 4.4 Strength of the singularity

If we consider the null geodesics parametrised by the affine parameter $\hat{k}$ and terminating at the shell focusing singularity $v = \mathsf{r} = \hat{k} = 0$, we can compute the strength of the singularity (according to Tipler (1977)). The strength of the singularity is the measure of its destructive capacity in the sense of whether the extension of spacetime is possible through them or not (Dadhich and Ghosh 2001). Following Clarke and Krolack (1985) and Mkenyeleye *et al* (2014), a singularity would be strong if the condition

$$\lim_{\hat{k}\to 0} \hat{k}^2\psi = \lim_{\hat{k}\to 0} \hat{k}^2 R_{ab}K^aK^b > 0, \tag{4.4.1}$$

where $R_{ab}$ is the Ricci tensor, is satisfied. It is interesting to note that this is the sufficient condition for the singularity to be Tipler strong (Tipler 1977). Using equations

(3.3.1), (4.2.6) and (4.2.7) we can find the scalar $\psi = R_{ab}\hat{k}^a K^b$ as

$$\psi = (2\dot{m}_0)\left(\frac{P}{\mathsf{r}^2}\right)^2, \tag{4.4.2}$$

and so therefore

$$\hat{k}^2\psi = (2\dot{m}_0)\left(\frac{\hat{k}P}{\mathsf{r}^2}\right)^2. \tag{4.4.3}$$

Using l'Hôpital's rule and equations (4.2.6) and (4.2.7), we can evaluate the limit along nonspacelike geodesics as $\hat{k} \to 0$ as

$$\lim_{\hat{k}\to 0}\hat{k}^2\psi = (2\dot{m}_0)\lim_{\hat{k}\to 0}\left(\frac{\hat{k}P}{\mathsf{r}^2}\right)^2. \tag{4.4.4}$$

If we make the assumption that $P \neq 0, \infty$, then we can use l'Hôpital's rule to obtain

$$\lim_{\hat{k}\to 0}\left(\frac{\hat{k}P}{\mathsf{r}^2}\right) = \lim_{\hat{k}\to 0}\left(\frac{Pd\hat{k}}{2\mathsf{r}d\mathsf{r}}\right). \tag{4.4.5}$$

From equation (4.2.6), $\frac{P}{\mathsf{r}} = \frac{dv}{d\hat{k}}$ and so therefore

$$\lim_{\hat{k}\to 0}\left(\frac{\hat{k}P}{\mathsf{r}^2}\right) = \frac{1}{2}\frac{dv}{d\hat{k}}\frac{d\hat{k}}{d\mathsf{r}} = \frac{1}{2}\frac{dv}{d\mathsf{r}} = \frac{1}{2}X_0. \tag{4.4.6}$$

Thus, we finally acquire

$$\lim_{\hat{k}\to 0}\hat{k}^2\psi = \frac{1}{4}X_0^2(2\dot{m}_0). \tag{4.4.7}$$

It can be observed that the strength of the central singularity depends on the limit of the derivative of the mass function only with respect to $v$ and the limiting value $X_0$.

Given a suitable mass function, it can be shown that

$$\lim_{\hat{k}\to 0}\hat{k}^2\psi = \frac{1}{4}X_0^2(2\dot{m}_0) > 0. \tag{4.4.8}$$

If this above condition is indeed satisfied for some positive and real root $X_0$, we can conclude that the naked singularity observed is strong. An interesting note is that when the energy conditions are satisfied, and if a naked singularity is developed as an end state of collapse, the naked singularity is always strong.

## 4.5 The end state of generalised Vaidya spacetimes for several equations of state

In this section we will present solutions to the Einstein field equations (3.3.5) for various equations of state, as found in chapter 3. A direct integration of the resulting partial differential equations was possible in general for the linear, quadratic and polytropic equations. Those solutions for the linear cases generalise all of those obtained by Husain (1996) and others as well as the complete summary of solutions presented in Wang and Wu (1999), and are therefore the most general solutions known. Also, using equation (4.3.21) the equations of the tangents to the null geodesics at the central singularity are calculated for these various Vaidya mass functions. We show that it is possible to obtain at least one real and positive value of $X_0$ for each mass function. Below, we include each solution with its equation of state.

1. For the linear equation of state $P = k\tilde{\rho}$ the mass function found is

$$m(v, \mathsf{r}) = c_1(v)\frac{\mathsf{r}^{1-2k}}{1-2k} + c_2(v). \tag{4.5.1}$$

   Using (4.3.21) we obtain the following

$$2X_0^{2k+1} + 2\dot{c}_2 X_0^2 - X_0 + 2 = 0. \tag{4.5.2}$$

   When $\dot{c}_2 = 0.01$ and $k = -1$, the above equation becomes a cubic and it is possible to find two positive and real roots. One such root is $X_0 = 2.862560272$ which means the singularity is naked. We also have that $c_2 = 0.01v$ and $c_1 = v^{2k}$ (from (4.3.15)) for $v > 0$. In Figure 4.1 a naked singularity forms at the origin and we have a static distribution of matter focused into this central singularity of growing mass. Null radiation shells fall through this static distribution terminating at the singularity. For $v > 0$ and those values of $c_1$ and $c_2$ we have infalling light-like matter described by the generalised Vaidya metric, specifically (4.5.1), reaching the singularity. We also have that $\lim_{\hat{k}\to 0} \hat{k}^2\psi = \frac{1}{4}X_0^2(2\dot{m}_0) = 0.04097 > 0$ so the condition for a strong singularity is satisfied. The real root is less than the slope at the apparent horizon

$$X_0 < \left(\frac{dv}{d\mathsf{r}}\right)_{AH} = 37.79632254,$$

thus the third condition of **Proposition 1** is satisfied. Therefore the central singularity is naked and strong with outgoing $C^1$ radial null geodesics escaping to the future.

2. For the generalised linear equation of state $P = k\tilde{\rho} + k_2$ we have

$$m(v, \mathsf{r}) = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}^3 + \frac{c_1 \mathsf{r}^{1-2k}}{1-2k} + c_2. \tag{4.5.3}$$

By using (4.3.21) the equation we obtain is identical to (4.5.2) and so all the conditions of **Proposition 1** are satisfied as well. An interesting observation is that this solution generalised all those contained in Wang and Wu (1999) and Husain (1996), so those should satisfy all these conditions too.

3. The quadratic equation of state $P = k\tilde{\rho}^2$ gives

$$m(v, \mathsf{r}) = c_2 - 2\left(\frac{\mathsf{r}}{2c_1} - \frac{\sqrt{\eta}\arctan\left(\frac{\sqrt{2}\sqrt{c_1}\mathsf{r}}{\sqrt{\eta}}\right)}{2\sqrt{2}c_1^{3/2}}\right), \tag{4.5.4}$$

where $\eta = 4k/\kappa$. Making use of (4.3.21) we have

$$2\left[\dot{c}_2 + \frac{\dot{c}_1}{2c_1^2}\right]X_0^2 - X_0 + 2 = 0. \tag{4.5.5}$$

If we let $\dot{c}_2 = 0.01$ and $\dot{c}_1 = 0$ (so $\dot{m}_0 > 0$) the above equation reduces to

$$2(0.01)X_0^2 - X_0 + 2 = 0, \tag{4.5.6}$$

which admits two positive real roots. One of these roots is $X_0 = 2.087121525$ so we have a naked singularity. We also have that $c_2(v) = 0.01v$ and $c_1 = \text{const.}$ and in Figure 4.2, we have a dynamical type I and light-like fluid distribution focused into the central singularity of growing mass forming at the origin. Again,

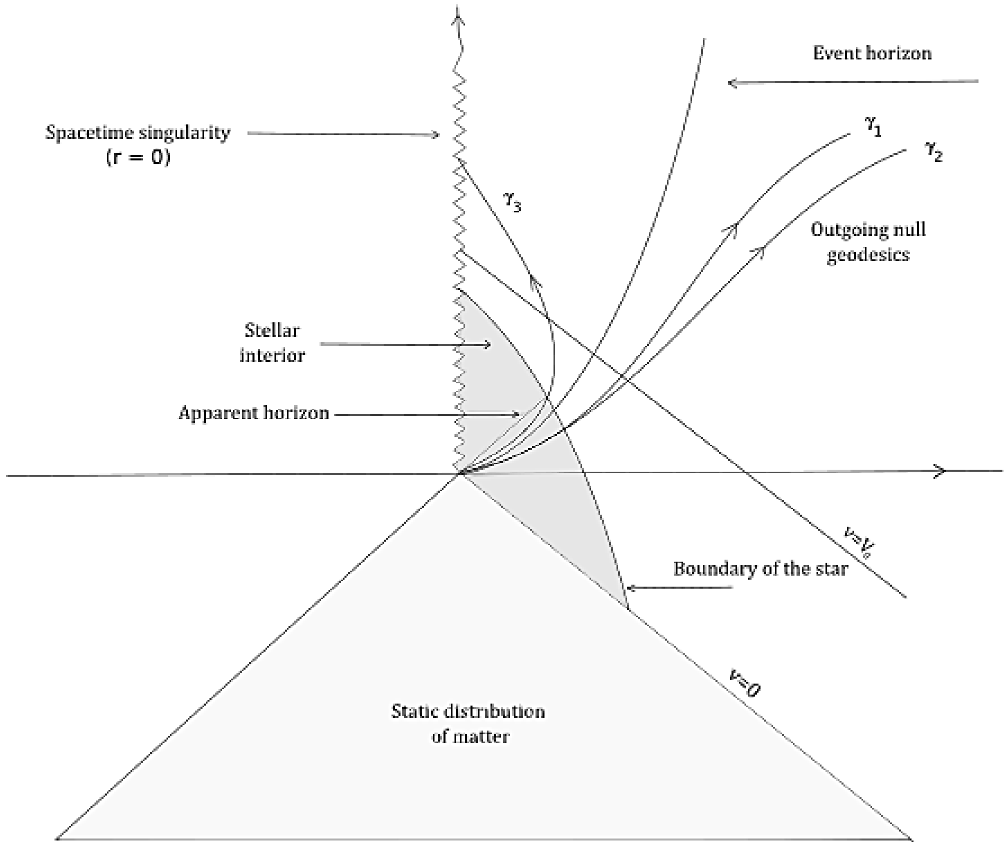


Figure 4.1: Linear: Here $v = 0$ depicts the first collapsing null ray falling into the central singularity through a static distribution of matter. A naked singularity forms at the origin with families of trajectories $\gamma_1$ and $\gamma_2$ escaping to infinity from the singularity. A shell of null radiation falls through a static distribution of matter, and into the singularity. Nonspacelike curves such as $\gamma_3$ which are emitted after the event horizon, cross the apparent horizon and fall back into the singularity.

at $v = 0$, a shell of radiation falls through this type I fluid and into the central singularity. In the region where $v > 0$ and for those values of $c_1$ and $c_2$ we have the generalised Vaidya solution (4.5.4) collapsing to the singularity. Also $\lim_{\hat{k}\to 0} \hat{k}^2\psi = \frac{1}{4}X_0^2(2\dot{m}_0) = 0.02178 > 0$: thus this singularity is indeed strong. Finally, this $X_0$ is also less than $\left(\frac{dv}{dr}\right)_{AH} = 50$ so the third condition of **Proposition 1** is satisfied.

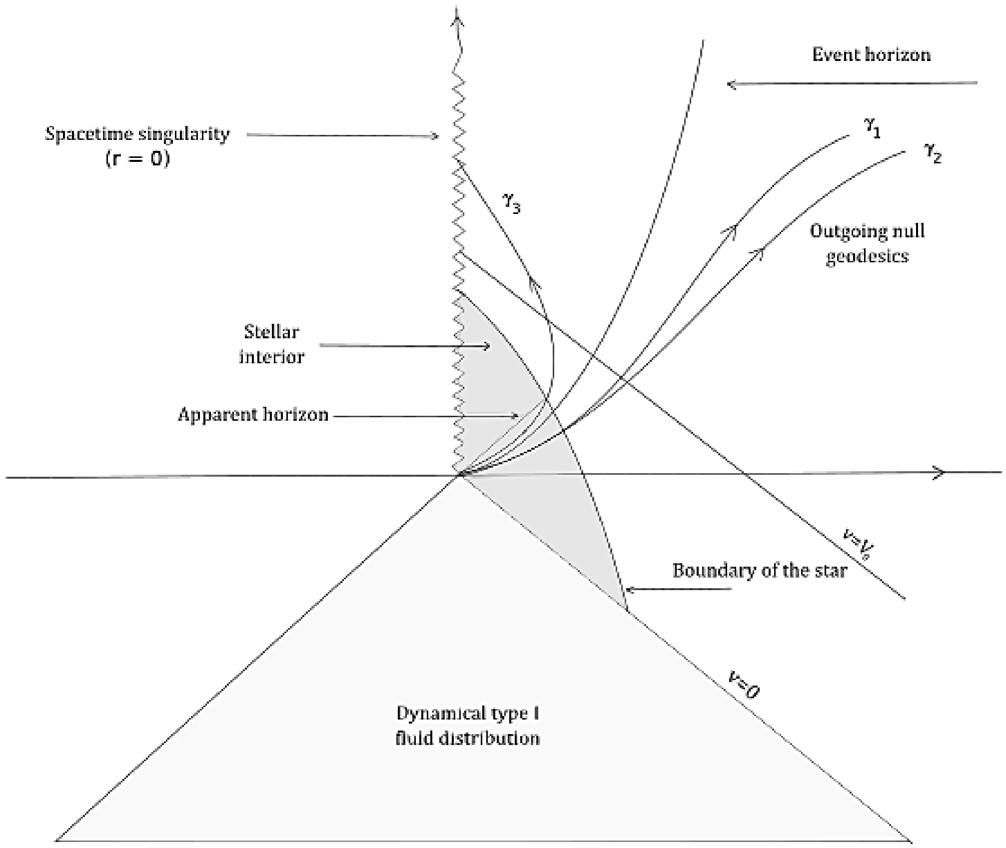


Figure 4.2: Quadratic: Here $v = 0$ depicts the first collapsing null ray falling into the central singularity, superposed onto a dynamic and collapsing distribution of type I matter. In this case the naked singularity forms at the origin as before with escaping null geodesic trajectories $\gamma_1$ and $\gamma_2$. However due to the form of the mass function (3.4.12) for the quadratic equation of state here, we instead have an injected radiation flow into an initially radiated region (consisting of a type I fluid) focused into the central singularity of growing mass, as opposed to a static region in the preceding case Figure 4.1.

4. The generalised quadratic case $P = k\tilde{\rho}^2 + k_2\tilde{\rho} + k_3$ yields

$$m(v, \mathsf{r}) = -\frac{1}{\eta}\int \left(\mathsf{r}^2 \tan\left(\sqrt{\zeta}\,(\ln \mathsf{r} - c_1)\right)\sqrt{\zeta}\right) d\mathsf{r} + c_2, \qquad (4.5.7)$$

where again, $\eta = 4k/\kappa$. We have set $\zeta = k_3\kappa\eta - k_2^2 - 2k_2 - 1$ for convenience.

The equation (4.3.21) becomes the following

$$2\dot{c}_2 X_0^2 - X_0 + 2 = 0,$$

which, if we set $\dot{c}_2 = 0.01$, becomes identical to (4.5.6). Thus, all the conditions will be satisfied for this case.

5. For the polytropic equation of state $P = k\tilde{\rho}^{\gamma}$, we have a mass function of the form

$$\begin{aligned} m(v, \mathsf{r}) \quad &= \quad \int \Bigg[ (\gamma + 1)k\kappa \left(\frac{2}{\kappa}\right)^{\gamma} \\ &\times \frac{\mathsf{r}^{2-2\gamma}}{2 - 2\gamma} + (1 - \gamma)c_1 \Bigg]^{\frac{1}{1-\gamma}} d\mathsf{r} + c_2. \end{aligned} \tag{4.5.8}$$

It should be noted that this solution was first presented by Husain (1996). Using (4.3.21) we have

$$2\dot{c}_2 X_0^2 - X_0 + 2\left[(1-\gamma)c_1\right]^{\frac{1}{1-\gamma}} X_0 + 2 = 0. \tag{4.5.9}$$

If we let $c_1 = 1$, $\dot{c}_2 = 0.01$ with $\gamma > 0$ we have the following

$$2(0.01)X_0^2 - X_0 + 2\left[(1-\gamma)c_1\right]^{\frac{1}{1-\gamma}} X_0 + 2 = 0,$$

which is not solvable for any integer $\gamma > 0$. If we let $\gamma = \frac{1}{2}$, (4.5.9) will admit two positive real roots. One of these is $X_0 = 5$, thus the singularity is naked. Also we have $\lim_{\hat{k}\to 0} \hat{k}^2 \psi = \frac{1}{4} X_0^2 (2\dot{m}_0) = \frac{1}{8} > 0$ so the condition for a strong singularity is satisfied. Finally, $\left(\frac{dv}{d\mathsf{r}}\right)_{AH} = 25 > X_0$ so the final condition is thus, satisfied.

In all the above cases $c_1 = c_1(v)$ and $c_2 = c_2(v)$ are integration functions. A summary of the algebraic equations for $X_0$ for each equation of state is presented in Table 3.1.

## 4.6 Higher dimensional Vaidya spacetime

In this section we briefly show that it is possible to extend the analysis of the collapse dynamics of our generalised Vaidya masses to higher dimensions, since the presence of

$N > 4$ dimensions can have an effect on the physical contracting features of radiating stars (Mkenyeleye *et al* 2015). The general theory of collapse extends naturally to higher dimensions as presented by Mkenyeleye *et al* (2015). For the linear and generalised linear equations of state, there exist at least two positive and real roots for equation (4.3.21) for all dimensions greater than four. Some examples are provided in Table 4.1. The quadratic, generalised quadratic and polytropic equations of state all yielded equations identical to their four dimensional analogues, so these are omitted from the table. It is also important to note that **Proposition 1** will of course be satisfied for all the cases. That is to say, a naked singularity forms in each case which is naked and indeed strong.

A summary of the algebraic equations of tangents $X_0$ to the singularity curve values for the above higher dimensional mass functions is presented in Table 4.2. We are also in the position to state this theorem:

**Theorem 1.** Consider an $N \geq 4$ generalised Vaidya spacetime which is undergoing gravitational collapse, with a mass function $m(v, \mathsf{r})$ obeying all physically reasonable energy conditions and is continuously differentiable throughout the entire spacetime. If the local central singularity is naked, and $\dot{c}_2 > 0$, which is to say that $\dot{m}_0 > 0$, this naked singularity will always be strong.

### 4.6.1 Mass functions in $N \geq 4$ dimensions where naked singularities can be eliminated

The existence of a naked sigularity is not guaranteed for any generalised Vaidya mass function. As noted in Mkenyeleye *et al* (2015), the possible transition to higher dimensions does not necessarily preserve the existence of a naked singularity. They showed that the value of the tangent to the null outgoing geodesic from a central singularity can be greater than the slope of the apparent horizon curve of the singularity. Hence, the outgoing null direction is contained within the trapped region which implies the singularity is causally cut off from an external observer. It turns out that one can prove that for such a mass function, there exists an open set of mass functions in the functional space. Let $p(z)$ be a complex polynomial of degree $n \geq 1$ with $m$ distinct roots $\{\omega_1, ..., \omega_m\}$ with $(1 \leq m \leq n)$. Defining the quantity $R_0(p)$ as follows:

$$R_0(p) = \begin{cases} \frac{1}{2}, & \text{if } m = 1 \\ \frac{1}{2}\min|\omega_1, ..., \omega_m|, & i \leq j \leq m \text{ , if } m > 1 \end{cases} \tag{4.6.1}$$

we can state the well known theorem from complex analysis (Alen 2015):

**Theorem 2.** Let $p(z)$ be a polynomial of degree $n \geq 1$, with real coefficients $\{\hat{\mu}_k\}$. Suppose $\omega$ is a real root of $p(z)$ of multiplicity one. Then for any $\varepsilon$ with $0 \leq \varepsilon \leq R_0(p)$, there exists a $\delta(\varepsilon) > 0$ such that any polynomial $q(z)$ with real coefficients $\{\hat{\nu}_k\}$ and $|\hat{\mu}_k - \hat{\nu}_k| \leq \delta$ has a real root $\hat{\beta}$ with $|\omega - \hat{\beta}| \leq \varepsilon$.

From the above theorem, we see that if a polynomial $p(z)$ with a real coefficient admits a real root $\omega$ of multiplicity one, then any polynomial $q(z)$ which is obtained by small, real perturbations to the coefficients of $p(z)$ will also have a real root in the neighbourhood of $\omega$. This means that not only is the root dependant continuously on coefficients, but also remains real under sufficiently small perturbations of coefficients. This result directly translates to this problem of an open set of mass functions in the mass functional space. A specific example of this is shown in Mkenyeleye *et al* (2015). In summary, for a given mass function, it is possible (but not guaranteed) for a naked singularity to form in four dimensions, however a transition to higher dimensions may (however not necessarily) yield a final outcome which is a black hole.

## 4.7 Discussion

In this chapter we detailed the general mathematical framework to describe the gravitational collapse of a generalised Vaidya spacetime in the context of the cosmic censorship conjecture (CCC). The structure of the central singularity was studied in order to show that it can be a node with outgoing null geodesics emerging from a singular point with a definite value of the tangent, depending on the parameters in the problem and the nature of the generalised Vaidya mass function in question.

We considered a spherically symmetric radiating star. We made note that any astrophysical star is a combination of three distinct concentric zones: the innermost two-component matter zone, the middle radiation zone and the outermost zone which is the vacuum Schwarzschild exterior. The mass functions obtained in Chapter 3 for various equations of state were analysed using this mathematical framework and it was shown that in each case, the collapse terminates with a local central singularity which is naked. The strength of these naked singularities were calculated and it was shown that they are strong curvature singularities and no extension of spacetime through them exists. This has consequences which are far reaching as their presence will no longer make the global spacetime asymptotically simple. This is to say that the theorems of black hole dynamics may require some reformulation. With this, for any realistic mass function, there exists an open set for which the central singularity is naked in the parameter space, and the CCC is violated. That is, the occurence of a naked singularity is a phenomenon which can be referred to as "stable", despite any changes in the matter field due to a combination of a radiation-like field with a collapsing perfect fluid. With regard to pure type I matter fields, this result is well known (Joshi 2007, Goswami and Joshi 2007). For completeness, we considered higher dimensional spacetimes and discussed briefly where and under what conditions naked singularities may be eliminated.

It is important to note that during the later stages of gravitational collapse, the generalised Vaidya spacetime is more realistic and physically reasonable than pure dustlike matter or perfect fluid fields. Any collapsing star must radiate and so there exists a combination of a perfect fluid and lightlike matter for this period in the evolution of the star.

Table 4.1: Solutions for algebraic equations of tangents $X_0$ for different dimensions with $k = -1$, $\dot{c}_2 = 0.01$

| Dimensions | Equation for tangent to the singularity curve $X_0$ | Roots |
|---|---|---|
| $N = 5$ | $2X_0^{-2} + 2(0.01)X_0^2 - X_0 + 2 = 0$ | $X_0 = 2.45276$ |
| | | $X_0 = 47.9119$ |
| $N = 6$ | $2X_0^{-3} + 2(0.01)X_0^2 - X_0 + 2 = 0$ | $X_0 = 2.27356$ |
| | | $X_0 = 47.9129$ |
| $N = 7$ | $2X_0^{-4} + 2(0.01)X_0^2 - X_0 + 2 = 0$ | $X_0 = 2.18335$ |
| | | $X_0 = 47.9129$ |
| $N = 8$ | $2X_0^{-5} + 2(0.01)X_0^2 - X_0 + 2 = 0$ | $X_0 = 2.1362$ |
| | | $X_0 = 47.9129$ |

Table 4.2: Equations of tangents $X_0$ to the singularity curve and values of $\lim_{\hat{k}\to 0} k^2\psi$ for several generalised higher dimensional Vaidya mass functions.

| Equation of state | Equation for the tangent to the singularity curve $X_0$ | $\lim_{\hat{k}\to 0} \hat{k}^2\psi$ |
|---|---|---|
| Linear | $2X_0^{(N-2)k+1} + 2\dot{c}_2 X_0^2 - X_0 + 2 = 0$ | $\frac{1}{4}X_0^2(2\dot{c}_2)$ |
| Generalised linear | $2X_0^{2k+1} + 2\dot{c}_2 X_0^2 - X_0 + 2 = 0$ | $\frac{1}{4}X_0^2(2\dot{c}_2)$ |
| Quadratic | $2\left[\dot{c}_2 + \frac{\dot{c}_1}{2c_1}\right] X_0^2 - X_0 + 2 = 0$ | $\frac{1}{4}X_0^2\left(2\left[\dot{c}_2 + \frac{\dot{c}_1}{2c_1}\right]\right)$ |
| Generalised quadratic | $2\dot{c}_2 X_0^2 - X_0 + 2 = 0$ | $\frac{1}{4}X_0^2(2\dot{c}_2)$ |
| Polytropic | $2\dot{c}_2 X_0^2 - \left(1 - 2\left[(1-\gamma)c_1\right]^{\frac{1}{1-\gamma}}\right) + 2 = 0$ | $\frac{1}{4}X_0^2(2\dot{c}_2)$ |

# Chapter 5

# A class of radiating Boulware-Deser spacetimes

## 5.1 Introduction

The need to study higher order theories of gravity has been more prominent in recent times. The well known Boulware-Deser solution (Boulware and Deser 1985) was an early higher dimensional analogue of the vacuum Schwarszchild solution from general relativity and although the outside geometry as well as the matching conditions have been studied extensively in general relativity with the Vaidya metric (Vaidya 1951), we are faced with the following issue in the Einstein-Gauss-Bonnet EGB theory: *How do we model a five-dimensional realistic, collapsing astrophysical star, emitting null radiation with a core containing a null fluid and a string fluid which matches to an intermediate radiating Boulware-Deser spacetime enclosed by the Boulware-Deser vacuum exterior?* This question is vital for a better understanding of the thermodynamics, dynamics and gravitational collapse of realistic stars, in the context of EGB gravity. The class of spacetimes which forms a natural candidate for models of such interiors are the *radiating Boulware-Deser spacetimes.* The idea of a radiating Vaidya-like Boulware-Deser spacetime was first brought to light by Kobayashi (2004). The matter fields in these spacetimes will be analagous to those in the four-dimensional generalised Vaidya metrics with type I and type II matter distributions. A general type I matter field (whose energy momentum tensor has three spacelike and one timelike eigenvector), describes null matter, and a type II matter field (whose energy momentum tensor has double null eigenvectors) describes a string fluid and null radiation. A stellar interior with a type II distribution can be matched naturally to an external radiating zone described by a pure radiating Boulware-Deser spacetime, and then finally, this radiation zone can be matched smoothly to the conventional Boulware-Deser vacuum exterior.

In the five-dimensional Boulware-Deser spacetime the constant $\tilde{M}$ can be related to the mass within a hypersurface. We show that it is possible for $\tilde{M}$ to depend on the spacetime coordinates; the variable $\tilde{M}$ is consistent with the field equations with a modified energy momentum tensor. Hence, with a type II fluid (consisting of a null fluid and a string source), the standard Boulware-Deser spacetime radiates. In this chapter we generate solutions to the radiating interior Boulware-Deser spacetime with null matter and a string fluid for various thermodynamically realistic equations of state. It turns out that it is possible to directly integrate the resulting partial differential equations for linear, quadratic and polytropic equations of state. In recent times Dominguez and Gallo (2006) found solutions to the EGB equations which represented dynamic black holes as well as EGB versions of the original Vaidya (dS/AdS) solution, the monopole and the Husain black hole Husain (1996). Our solutions for the linear cases as well as the total solution set are the EGB analogues of those found in Husain (1996), Wang and Wu (1999) and Chapter 3 of this dissertation respectively. We also further generalise our results in higher dimensions for pedagogical completeness.

This chapter is organised as follows: In the next section, an outline of the theory of EGB gravity is presented as well as the general modified field equations. The Boulware-Deser spacetime is discussed in the following section followed by a detailed description of a radiating Boulware-Deser metric in Sec. 5.3. Here, the relevant definitions relating to the modified geometry of the spacetime are presented along with the EGB field equations and energy conditions for a physically reasonable model. In Sec. 5.4 a complete description of how to model an isolated, spherical five-dimensional astrophysical radiating star via the Boulware-Deser geometry is given. In the section following this, solutions to the EGB field equations for the gravitational mass are systematically presented for various realistic equations of state. In Sec. 5.6 the higher dimensional analogue of the Boulware-deser metric is discussed and the generalised solutions for equations of state are tabulated.

## 5.2 The Boulware-Deser spacetime

A static, spherically symmetric, exterior vacuum solution of the modified action (2.4.1) was first given by Boulware and Deser (1985). The form is given by

$$ds^2 = -f(\mathsf{r})dt^2 + \frac{1}{f(\mathsf{r})}d\mathsf{r}^2 + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2), \tag{5.2.1}$$

where

$$f(\mathsf{r}) = 1 + \frac{\mathsf{r}^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16\tilde{M}\alpha}{\mathsf{r}^4}}\right).$$

In the line element (5.2.1), $\tilde{M}$ is the gravitational constant mass of the 5-dimensional hypersurface. For our purposes it will be prudent to express (5.2.1) in retarded coordinates. Utilising the transformation

$$v = t - \int \frac{d\mathsf{r}}{f(\mathsf{r})},$$

the Boulware-Deser metric (5.2.1) becomes

$$ds^2 = -f(\mathsf{r})dv^2 + \epsilon 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2), \tag{5.2.2}$$

where $\epsilon = \pm 1$ and $(x^a) = (v, \mathsf{r}, \theta, \phi, \psi)$. As $\tilde{M}$ is a constant mass, all the components of (2.4.10) vanish since the Boulware-Deser spacetime is vacuum.

## 5.3 A radiating Boulware-Deser interior

If we consider a radiating inhomogeneous spacetime in EGB gravity we then can obtain a Vaidya-like metric by allowing the mass function to depend on both the retarded null coordinate and the radius of the star

$$\tilde{M} \longrightarrow M(v, \mathsf{r}). \tag{5.3.1}$$

Thus we will have

$$ds^2 = -f(v,\mathsf{r})dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2), \tag{5.3.2}$$

where

$$f(v,\mathsf{r}) = 1 + \frac{\mathsf{r}^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16M(v,\mathsf{r})\alpha}{\mathsf{r}^4}}\right).$$

The nonzero components of the connection coefficients (2.3.7) are given by

$$\Gamma^0{}_{00} = \tfrac{1}{4\alpha^2\mathsf{r}^2}\left(1+\tfrac{16\alpha M}{\mathsf{r}^4}\right)^{-\frac{1}{2}}\left[-\sqrt{1+\tfrac{16\alpha M}{\mathsf{r}^4}}\mathsf{r}^3 + \mathsf{r}^3 + 4\alpha M_\mathsf{r}\right] \qquad \Gamma^1{}_{01} = -\Gamma^0{}_{00}$$

$$\begin{aligned}\Gamma^1{}_{00} = {} & \tfrac{1}{8\alpha^2\mathsf{r}^2}\left(1+\tfrac{16\alpha M}{\mathsf{r}^4}\right)^{-\frac{1}{2}}\Bigg[\sqrt{1+\tfrac{16\alpha M}{\mathsf{r}^4}}\mathsf{r}^5 + 2\alpha\mathsf{r}^2 M_\mathsf{r}\sqrt{1+\tfrac{16\alpha M}{\mathsf{r}^4}} \\ & -\mathsf{r}^5 - 2\alpha\mathsf{r}^2 M_\mathsf{r} - 2\alpha\mathsf{r}^3 - 8\alpha^2 M_\mathsf{r} - 8\alpha^2 M_v - 8\alpha\mathsf{r}M \\ & +2\alpha\mathsf{r}^3\sqrt{1+\tfrac{16\alpha M}{\mathsf{r}^4}}\Bigg]\end{aligned} \qquad \Gamma^0{}_{22} = \mathsf{r}$$

$$\Gamma^1{}_{22} = \tfrac{\mathsf{r}}{4\alpha}\left(\sqrt{1+\tfrac{16\alpha M}{\mathsf{r}^4}}\mathsf{r}^2 - \mathsf{r}^2 - 4\alpha\right) \qquad \Gamma^3{}_{23} = \Gamma^4{}_{24} = \cot\theta$$

$$\Gamma^0{}_{33} = \mathsf{r}\sin^2\theta \qquad \Gamma^1{}_{33} = \sin^2\theta\Gamma^1{}_{22}$$

$$\Gamma^4{}_{34} = \cot\phi \qquad \Gamma^0{}_{44} = \mathsf{r}\sin^2\theta\sin^2\phi$$

$$\Gamma^2{}_{12} = \Gamma^3{}_{13} = \Gamma^4{}_{14} = \tfrac{1}{\mathsf{r}} \qquad \Gamma^2{}_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3{}_{44} - \sin\phi\cos\phi \qquad \Gamma^1{}_{44} = \sin^2\phi\Gamma^1{}_{33}$$

$$\Gamma^3{}_{44} = -\sin\phi\cos\phi$$

for the metric (5.3.2). In the above, we have the following

$$M_v = \frac{\partial M}{\partial v}, \qquad M_\mathsf{r} = \frac{\partial M}{\partial \mathsf{r}}.$$

Using the above connection coefficients, the components of the Ricci tensor (2.3.17) which do not vanish are as follows

$$
\begin{aligned}
R^0{}_0 &= R^1{}_1 = \frac{(\mathsf{r}^4+16\alpha M)^{-1}}{\alpha \mathsf{r}^3\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}}\left[-\mathsf{r}^7\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}+\alpha \mathsf{r}^5 M_{\mathsf{rr}}+\mathsf{r}^7\right.\\
&\quad -\alpha \mathsf{r}^4 M_{\mathsf{r}}+16\alpha^2 \mathsf{r} M M_{\mathsf{rr}}-8\alpha \mathsf{r} M_{\mathsf{r}}^2+48\alpha^2 M M_{\mathsf{r}}+24\alpha \mathsf{r}^3 M\\
&\quad \left.-16\alpha \mathsf{r}^3 M\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}\right], \qquad (5.3.3a)\\
R^1{}_0 &= \frac{3M_v}{\mathsf{r}^3\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}}, \qquad (5.3.3b)\\
R^2{}_2 &= R^3{}_3 = R^4{}_4 = \frac{8\alpha M+\mathsf{r}^4-\mathsf{r}^4\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}+2\alpha \mathsf{r} M_{\mathsf{r}}}{\alpha \mathsf{r}^4\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}}. \qquad (5.3.3c)
\end{aligned}
$$

Using (5.3.3) and (2.3.18), the Ricci scalar is thus

$$
\begin{aligned}
R &= \frac{(\mathsf{r}^4+16\alpha M)^{-1}}{\alpha \mathsf{r}^4\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}}\left[-5\mathsf{r}^8\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}+2\alpha \mathsf{r}^6 M_{\mathsf{rr}}+5\mathsf{r}^8\right.\\
&\quad +4\alpha \mathsf{r}^5 M_{\mathsf{r}}+32\alpha^2 \mathsf{r}^2 M M_{\mathsf{rr}}-16\alpha^2 \mathsf{r}^2 M_{\mathsf{r}}^2+120\alpha \mathsf{r}^4 M+192\alpha^2 \mathsf{r} M M_{\mathsf{rr}}\\
&\quad \left.+384\alpha^2 M^2-80\alpha \mathsf{r}^4\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}\right]. \qquad (5.3.4)
\end{aligned}
$$

Now making use of (5.3.3) and (5.3.4), the nonvanishing components of the Einstein tensor $G^a{}_b$ are calculated as

$$
\begin{aligned}
G^0{}_0 &= G^1{}_1 = -\frac{3}{2\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}}\left[\frac{1}{\alpha \mathsf{r}^2}+\frac{8M}{\mathsf{r}^4}\right.\\
&\quad \left.+\frac{2M_{\mathsf{r}}}{\mathsf{r}^3}-\mathsf{r}^4\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}\right], \qquad (5.3.5a)\\
G^1{}_0 &= \frac{3M_v}{\mathsf{r}^3\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}}, \qquad (5.3.5b)\\
G^2{}_2 &= G^3{}_3 = G^4{}_4 = -\frac{-1}{2\mathsf{r}^8\alpha+32\alpha^2\mathsf{r}^4 M}\\
&\quad \times\left[-3\mathsf{r}^8\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}\right.\\
&\quad +2M_{\mathsf{rr}}\alpha \mathsf{r}^6+3\mathsf{r}^8+32\alpha^2\mathsf{r}^2 M_{\mathsf{rr}}-16\alpha^2\mathsf{r}^2 M_{\mathsf{r}}^2\\
&\quad +72\alpha \mathsf{r}^4 M+128\alpha^2 \mathsf{r} M_{\mathsf{r}}+128\alpha^2 M^2\\
&\quad \left.-48\alpha \mathsf{r}^4 M\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}\right], \qquad (5.3.5c)
\end{aligned}
$$

The nonzero components of the Lanczos tensor (2.4.4) become

$$\begin{aligned}
H^0{}_0 &= H^1{}_1 = -\frac{3}{\alpha^2\mathsf{r}^3(\mathsf{r}^4+16\alpha M)} \\
&\quad\times\left[\mathsf{r}^7\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}\right. \\
&\quad+2\alpha\mathsf{r}^4 M_{\mathsf{r}}\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}} \\
&\quad-\mathsf{r}^7-2\alpha\mathsf{r}^4M_{\mathsf{r}}-32\alpha^2 MM_{\mathsf{r}} \\
&\quad\left.+8\alpha\mathsf{r}^3M\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}-16\alpha\mathsf{r}^3M\right], \qquad (5.3.6a)\\
H^1{}_0 &= \frac{6M_v\left[\mathsf{r}^4\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}-\mathsf{r}^4-16\alpha M\right]}{\alpha\mathsf{r}^3(\mathsf{r}^4+16\alpha M)}, \qquad (5.3.6b)\\
H^2{}_2 &= H^3{}_3 = H^4{}_4 = \frac{\left(1+\frac{16\alpha M}{\mathsf{r}^4}\right)^{-\frac{1}{2}}}{\alpha^2\mathsf{r}^4(\mathsf{r}^4+16\alpha M)} \\
&\quad\times\left[2\alpha\mathsf{r}^6M_{\mathsf{rr}}\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}\right. \\
&\quad+3\mathsf{r}^8\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}-2\alpha\mathsf{r}^6M_{\mathsf{rr}}-3\mathsf{r}^8 \\
&\quad+32\alpha^2\mathsf{r}^2MM_{\mathsf{rr}}-32\alpha^2\mathsf{r}^2MM_{\mathsf{rr}} \\
&\quad+16\alpha^2\mathsf{r}^2M^2-128\alpha^2\mathsf{r}M_{\mathsf{r}}-128\alpha^2M^2 \\
&\quad\left.+48\alpha\mathsf{r}^4M\sqrt{1+\frac{16\alpha M}{\mathsf{r}^4}}-72\alpha\mathsf{r}^4M\right], \qquad (5.3.6c)
\end{aligned}$$

Using the above expressions (5.3.5) and (5.3.6), we can calculate the nonzero components of (2.4.10). It is remarkable to note that despite the complexity of the nonzero components of the Einstein and Lanczos tensors, their combinations yield rather simple expressions. These are given by

$$\begin{aligned}
\mathcal{G}^0{}_0 &= \mathcal{G}^1{}_1 = -\frac{3}{r^3}M_{\mathsf{r}}, \qquad (5.3.7a)\\
\mathcal{G}^1{}_0 &= \frac{3}{\mathsf{r}^3}M_v, \qquad (5.3.7b)\\
\mathcal{G}^2{}_2 &= \mathcal{G}^3{}_3 = \mathcal{G}^4{}_4 = -\frac{1}{\mathsf{r}^2}M_{\mathsf{rr}}. \qquad (5.3.7c)
\end{aligned}$$

The modified curvature components (5.3.7) are generated by an appropriate matter field. Comparing (5.3.7) with the field equations (2.4.9) gives rise to an energy momentum tensor of the form

$$T_{ab} = \mu l_a l_b + (\tilde{\rho} + P)(l_a n_b + l_b n_a) + P g_{ab}, \tag{5.3.8}$$

where

$$\begin{aligned} l_a &= \delta^0_a, \\ n_a &= \frac{1}{2}\left[1 + \frac{\mathsf{r}^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16M\alpha}{\mathsf{r}^4}}\right)\right]\delta^0_a + \delta^1_a, \end{aligned}$$

with $l_c l^c = n_c n^c = 0$ and $l_c n^c = -1$. The null vector $l^a$ is a double null eigenvector of the energy momentum tensor (5.3.8). Using the form of the energy momentum tensor (5.3.8) with (5.3.7), we acquire the EGB field equations $\mathcal{G}^a{}_b = \kappa T^a{}_b$ in the form

$$\mu = -\frac{3}{\kappa \mathsf{r}^3} M_v, \tag{5.3.9a}$$

$$\tilde{\rho} = \frac{3}{\kappa \mathsf{r}^3} M_{\mathsf{r}}, \tag{5.3.9b}$$

$$P = -\frac{1}{\kappa \mathsf{r}^2} M_{\mathsf{rr}}. \tag{5.3.9c}$$

As $M_v \neq 0$, in general it is clear that the Boulware-Deser spacetime radiates. When $\tilde{\rho} = P = 0$, the above expressions reduce to the single solution obtained for the radiating Boulware-Deser metric when $M = M(v)$. Furthermore when $\mu = \tilde{\rho} = P = 0$, we regain the vacuum case with constant mass.

The energy conditions for this kind of fluid are

1. The weak and strong energy conditions:

$$\mu \geq 0, \qquad \tilde{\rho} \geq 0, \qquad P \geq 0 \qquad (\mu \neq 0). \tag{5.3.10}$$

2. The dominant energy condition:

$$\mu \geq 0, \qquad \tilde{\rho} \geq P \geq 0 \qquad (\mu \neq 0). \tag{5.3.11}$$

In the case when $M = M(v)$ the above energy conditions all reduce to $\mu \geq 0$, and if $M = M(\mathsf{r})$, then $\mu = 0$ and the matter field becomes a type I fluid.

Finally, we are in the position to state the following theorem:

**Theorem 1.** Consider the five-dimensional spacetime

$$ds^2 = -\left[1 + \frac{\mathsf{r}^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16M\alpha}{\mathsf{r}^4}}\right)\right] dv^2 - 2dvd\mathsf{r}$$
$$+\mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2),$$

from a regular epoch, where $M = M(v, \mathsf{r})$ ($v$ is the null retarded time coordinate and $\mathsf{r}$ is the radius) is differentiable in the entire spacetime, and obeys all physically reasonable energy conditions. This spacetime is then consistent with an energy momentum tensor which is a unique combination of the type I and type II matter fields. This geometry represents a solution to the EGB field equations with a superposition of null radiation and a string fluid. In the relevant limit, we regain the radiating case ($M = M(v)$) and the Boulware-Deser spacetime ($M = \tilde{M} = \text{const.}$) when $\alpha \neq 0$, and Einstein gravity when $\alpha = 0$.

## 5.4 The model for an isolated, radiating and dynamic star in five dimensions

Any spherically symmetric five-dimensional astrophysical star is a combination of three distinct concentric zones: the innermost zone is the stellar interior where there are two component matter sources, namely null fluid matter together with radiation. The middle zone is a purely radiative zone while the outermost zone is the vacuum Boulware-Deser exterior (5.2.1) that extends approximately to a radius of one light year (for solar mass stars) beyond which galactic dynamics will take over. In this section we briefly outline how to model all three of these zones under a combined framework using a generalised Boulware-Deser class of metric.

### 5.4.1 Stellar interior: $M = M(v, \mathsf{r})$

As described earlier, the best possible candidate for the spacetime of a stellar interior with the mass parameter (5.3.1) is

$$ds^2 = -f(v, \mathsf{r})dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2), \tag{5.4.1}$$

where

$$f(v, \mathsf{r}) = 1 + \frac{\mathsf{r}^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16M(v, \mathsf{r})\alpha}{\mathsf{r}^4}}\right).$$

The mass function $M$ which depends on the coordinates $v$ and $\mathsf{r}$ can be uniquely obtained via the Einstein field equations with the two component matter sources. Let $M(v, \mathsf{r})$ be one such solution for a given combination of fluid and radiation fields. This solution then completely describes the solution of the interior of the star, up to a boundary surface given by $\mathsf{r} = \mathsf{r}_b$. Beyond this boundary we enter a pure radiation zone.

### 5.4.2 Radiation zone: $M = M(v)$

In this zone the matter field is a single component null matter field and the spacetime is well described by the radiating Boulware-Deser metric

$$ds^2 = -f_1(v, \mathsf{r})dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2), \tag{5.4.2}$$

with

$$f_1(v, \mathsf{r}) = 1 + \frac{\mathsf{r}^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16M_1(v)\alpha}{\mathsf{r}^4}}\right).$$

We can naturally relate the Boulware-Deser mass function $M_1(v)$ in the radiation zone to the generalised Boulware-Deser mass function in the stellar interior in the following way

$$M_1(v) = M(v, \mathsf{r}_b), \tag{5.4.3}$$

so that $M$ is a function of the retarded coordinate $v$. This radiation zone continues until some retarded null coordinate value $v = V_0$, beyond which the spacetime is the conventional Boulware-Deser vacuum (as dictated by Birkhoff's theorem).

### 5.4.3 Boulware-Deser exterior: $M = \tilde{M}$

This vacuum region is described by the exterior static subset of the completely extended Boulware-Deser manifold, and the metric is given by

$$ds^2 = -\tilde{f}(\mathsf{r})dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2), \tag{5.4.4}$$

with

$$\tilde{f}(\mathsf{r}) = 1 + \frac{\mathsf{r}^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16\tilde{M}\alpha}{\mathsf{r}^4}}\right).$$

Here the static mass $\tilde{M}$ is related to the radiating Boulware-Deser mass $M_1(v)$ by

$$\tilde{M} = M_1(V_0), \tag{5.4.5}$$

which is constant.

### 5.4.4 Matching conditions at the boundary surfaces: Complete mass function

We note here that the spacetime is divided into three distinct regions for our above mentioned stellar model: the interior region, the radiation zone and the vacuum Boulware-Deser exterior region. The first boundary surface between the inner and the intermediate zone, given by $\mathsf{r} = \mathsf{r}_b$, is a timelike boundary, whereas the second boundary surface given by $v = V_0$ is a null boundary. *The important point that all the three zones are described by the same class of metric makes the matching conditions between boundaries extremely transparent.* To match the first fundamental form all we need is the mass function to be continuous across these boundaries. Hence the complete $C^2$ mass function for an isolated stellar model can be given in the following form:

$$M(v,\mathsf{r}) = \begin{cases} M(v,\mathsf{r}) & \mathsf{r} \le \mathsf{r}_b\ , v \le V_0 \\ M_1(v) \equiv M(v,\mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b\ , v \le V_0 \\ \tilde{M} \equiv M_1(V_0) \equiv M(V_0,\mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b\ , v > V_0 \end{cases} \tag{5.4.6}$$

We can easily check that this mass function is a solution to the EGB field equations in all three zones mentioned above, and hence it completely describes the spacetime of an isolated collapsing star. To match the second fundamental form, we need the partial derivatives of the mass functions across the boundaries to be continuous. These conditions are given by

$$\frac{\partial}{\partial v}M(v,\mathsf{r}_b) = \frac{\partial}{\partial v}M_1(v), \tag{5.4.7a}$$

$$\left.\frac{\partial}{\partial \mathsf{r}}M(v,\mathsf{r})\right|_{\mathsf{r}=\mathsf{r}_b} = 0, \tag{5.4.7b}$$

$$\left.\frac{\partial}{\partial v}M_1(v)\right|_{v=V_0} = 0. \tag{5.4.7c}$$

where $\mathsf{r} = \mathsf{r}_b$ is the timelike boundary (from equating (5.2.2) to (5.4.2)) and $v = V_0$ is the null boundary (from equating (5.4.2) to (5.4.4)). These boundaries serve as the matching surfaces for the three concentric regions which can be seen in Figure 5.1.

It is therefore necessary to find physically relevant mass functions, with the structure of (5.4.6), to model a dynamical radiating star which is isolated. We achieve this by imposing specific equations of state.

## 5.5 Solutions with equations of state

In this section, we will consider various equations of state to solve the system (5.3.9).

### 5.5.1 Case I(a): $P = k\tilde{\rho}$

If we assume a linear equation of state $P = k\tilde{\rho}$ in the field equations where $k$ is a constant, we then arrive at

$$\mathsf{r}^2 M_{\mathsf{rr}} + 3k\mathsf{r}M_{\mathsf{r}} = 0, \tag{5.5.1}$$

which is a second order linear partial differential equation. Since the derivatives occur in one variable, we can integrate it as an ordinary differential equation via reduction

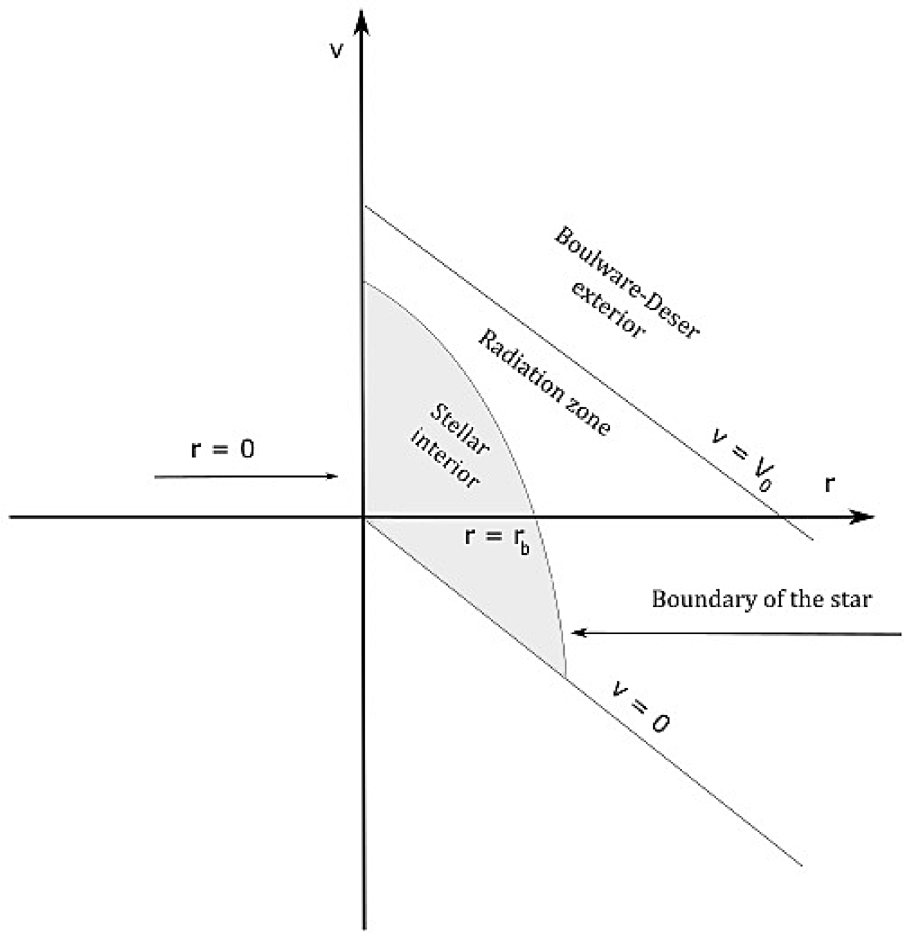


Figure 5.1: Depiction of a five-dimensional spacetime divided into the three distinct regions (in retarded coordinates with $\epsilon = 1$).

of order. There are two solutions. For the case when $k = \frac{1}{3}$, the solution is given by

$$M(v, \mathsf{r}) = c_1(v) \ln(\mathsf{r}) + c_2(v),$$

where $c_1(v)$ and $c_2(v)$ are functions of integration. For $k \neq \frac{1}{3}$, the second solution is

$$M(v, \mathsf{r}) = c_1(v) \frac{\mathsf{r}^{1-3k}}{1-3k} + c_2(v). \tag{5.5.2}$$

### 5.5.2 Case I(b): $P = k\tilde{\rho} + k_2$

Assuming $P = k\tilde{\rho} + k_2$ in the field equations (5.3.9) yields

$$M_{\mathsf{rr}} + \frac{3k}{\mathsf{r}}M_{\mathsf{r}} + \kappa k_2 \mathsf{r}^2 = 0, \tag{5.5.3}$$

which takes the form of a Cauchy-Euler equation. If we let $y(v, \mathsf{r}) = m_{\mathsf{r}}$ we get the first order linear equation

$$y' + \frac{3k}{\mathsf{r}}y = -\kappa k_2 \mathsf{r}^2, \tag{5.5.4}$$

which can be easily integrated to give

$$y = -\kappa k_2 \frac{\mathsf{r}^3}{3k+1} + \frac{c_1(v)}{\mathsf{r}^{3k}}. \tag{5.5.5}$$

Again, two cases arise. When $k = \frac{1}{3}$ the first solution for $M(v, \mathsf{r})$ is given by

$$M(v, \mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v) - \frac{\kappa k_2 \mathsf{r}^4}{16}.$$

The second solution for the mass with $k \neq \frac{1}{3}$ is given by

$$M(v, \mathsf{r}) = c_2(v) + c_1(v)\frac{\mathsf{r}^{1-3k}}{1-3k} - \frac{\kappa k_2}{4(3k+1)}\mathsf{r}^4, \tag{5.5.6}$$

which contains (5.5.2) as well as several of the seminal others contained in Dominguez and Gallo (2006). It should be noted that this solution also contains the EGB analogues of those found in Husain (1996) and Wang and Wu (1999). These are summarised in Table 5.1.

### 5.5.3 Case II(a): $P = k\tilde{\rho}^2$

If we assume a quadratic equation of state $P = k\tilde{\rho}^2$, in the field equations (5.3.9), we have

$$\mathsf{r}^5 M_{\mathsf{rr}} + \frac{9k}{\kappa}\mathsf{r}M_{\mathsf{r}}^2 = 0, \tag{5.5.7}$$

which is a second order nonlinear equation. A reduction of the order yields the following

$$y' + \frac{\eta}{\mathsf{r}^4}y^2 = 0, \tag{5.5.8}$$

which is a first order nonlinear equation with $\eta = 9k/\kappa$. Integration yields

$$y = \frac{-1}{\frac{\eta}{3\mathsf{r}^3} + c_1(v)}. \tag{5.5.9}$$

Therefore the mass can be expressed as

$$M(v, \mathsf{r}) = -\int \left( \frac{1}{\frac{\eta}{3\mathsf{r}^3} + c_1(v)} \right) d\mathsf{r} + c_2(v). \tag{5.5.10}$$

The integral on the right hand side of the above expression admits two solutions. When the constant $c_1 = 0$ the solution is

$$M(v, \mathsf{r}) = -\frac{3}{4\eta}\mathsf{r}^4 + c_2(v). \tag{5.5.11}$$

When $c_1 \neq 0$ the above integral can be evaluated via partial fraction decomposition. The final expression for the mass $M$ is given by

$$\begin{aligned} M(v, \mathsf{r}) \;\; &= \;\; c_2(v) - \frac{\mathsf{r}}{3c_1(v)} + \left(\frac{\eta}{3c_1(v)}\right)^{1/3} \left(\frac{1}{3\eta}\right) \\ &\times \left[ \frac{1}{2} \ln \left( \frac{\left(\mathsf{r} + \left(\frac{\eta}{3c_1(v)}\right)^{1/3}\right)^2}{\mathsf{r}^2 - \left(\frac{\eta}{3c_1(v)}\right)^{1/3} + \left(\frac{\eta}{3c_1(v)}\right)^{2/3}} \right) \right. \\ &\left. + \sqrt{3} \arctan \left( \frac{2\mathsf{r} - \left(\frac{\eta}{3c_1(v)}\right)^{1/3}}{\sqrt{3}\left(\frac{\eta}{3c_1(v)}\right)^{1/3}} \right) \right]. \end{aligned} \tag{5.5.12}$$

As far as we are aware, this solution is not found anywhere in the previous literature.

### 5.5.4 Case II(b): $P = k\tilde{\rho}^2 + k_2\tilde{\rho} + k_3$

Imposing the equation of state $P = k\tilde{\rho}^2 + k_2\tilde{\rho} + k_3$ in the field equations (5.3.9) yields

$$M_{\mathsf{rr}} + \frac{9k}{\kappa \mathsf{r}^4} M_{\mathsf{r}}^2 + \frac{3k_2}{\mathsf{r}} M_{\mathsf{r}} + k_3 \kappa \mathsf{r} = 0. \tag{5.5.13}$$

Reducing the order of the above equation yields

$$y' + \frac{3k_2}{\mathsf{r}} y + k_3 \kappa \mathsf{r} = -\frac{\eta}{\mathsf{r}^4} y^2, \tag{5.5.14}$$

with $\eta = 9k/\kappa$, which is a Riccati differential equation. Integration of the above equation gives

$$y = -\frac{\mathsf{r}^3}{2\eta} \tan\left(\frac{1}{2}\sqrt{\varepsilon}(\ln(\mathsf{r}) - c_1(v))\right) \times \left(-\frac{\mathsf{r}^3}{2\eta}(\sqrt{\varepsilon} + 3 + 3k_2)\right). \tag{5.5.15}$$

In the above expression, we have the following: $\varepsilon = 4\beta\eta - 9k_2^2 - 109 - 18k_2$ and $\beta = \kappa k_3$. Hence, the expression for the mass is given by

$$M(v, \mathsf{r}) = -\int \left[\frac{\mathsf{r}^3}{2\eta} \tan\left(\frac{1}{2}\sqrt{\varepsilon}(\ln(\mathsf{r}) - c_1(v))\right) \times \left(\frac{\mathsf{r}^3}{2\eta}(\sqrt{\varepsilon} + 3 + 3k_2)\right)\right] d\mathsf{r} + c_2(v). \tag{5.5.16}$$

It should be noted that when $k_2 = k_3 = 0$ in the above solution (5.5.16), we regain Case II(a), which is to be expected.

### 5.5.5 Case III: $P = k\tilde{\rho}^\gamma$

If we assume the equation of state $P = k\tilde{\rho}^\gamma$, where $\gamma \in \mathbb{R}$, we then have

$$M_{\mathsf{rr}} + k\kappa \left(\frac{3}{\kappa}\right)^\gamma \mathsf{r}^{2-3\gamma} M_{\mathsf{r}}^\gamma = 0. \tag{5.5.17}$$

Reducing the order of the above equation yields

$$y' + k\kappa \left(\frac{3}{\kappa}\right)^\gamma \mathsf{r}^{2-3\gamma} y^\gamma = 0,$$

which is a separable equation. Its general solution is given by

$$y = \left[(\gamma - 1)\left(k\kappa \left(\frac{3}{\kappa}\right)^\gamma \frac{\mathsf{r}^{3-3\gamma}}{3 - 3\gamma} + c_1(v)\right)\right]^{\frac{1}{1-\gamma}}. \tag{5.5.18}$$

Therefore we can express the mass $M$ as

$$M(v, \mathsf{r}) = c_2(v) + \int \left[(\gamma - 1)\left(k\kappa \left(\frac{3}{\kappa}\right)^\gamma \times \frac{\mathsf{r}^{3-3\gamma}}{3 - 3\gamma} + c_1(v)\right)\right]^{\frac{1}{1-\gamma}} d\mathsf{r}. \tag{5.5.19}$$

Table 5.1: Solutions contained within the system (5.5.6).

| Solution | $M(v,\mathsf{r})$ | $c_1(v)$ and $c_2(v)$ | $k$-indices |
|---|---|---|---|
| Monopole-EGB | $\frac{a\mathsf{r}}{2}$ | $c_1(v)=\frac{a}{2}$, $c_2(v)=0$ | $k, k_2=0$ |
| Charged Vaidya-EGB | $g(v)-\frac{q(v)^2}{2\mathsf{r}}$ | $c_1=\frac{q(v)^2}{2}$, $c_2=g(v)$ | $k=1$, $k_2=0$ |
| dS/AdS | $\frac{\Lambda}{6}\mathsf{r}^3$ | $c_1(v)=c_2(v)=0$ | $k=$ const. |
| | | | $k_2=-\frac{\Lambda(k+1)}{\kappa}$ |
| Husain-EGB | $g(v)-\frac{q(v)}{(3k-1)\mathsf{r}^{3k-1}}$ | $c_1(v)=\frac{-q(v)}{2}$, $c_2(v)=g(v)$ | $k, k_2=$ const. |
| Boulware-Deser-Wheeler | $M_0$ | $c_1=c_2=0$ | |

It should be noted that in the works of Wang and Wu (1999), Ghosh and Dadhich (2002) and Dominguez and Gallo (2006), it appears that solutions only for a linear and/or generalised linear equation of state are provided in conventional general relativity in the Vaidya spacetime. In Chapter 3 we further found solutions for both linear cases as well as quadratic and generalised quadratic equations of state. It should be noted that Husain (1996) found a general integral quadrature similar to ours above for the polytropic equation of state in the generalised Vaidya spacetime, however ours differs in the fact that we are dealing with the EGB theory of gravity. In the case of Wang and Wu (1999), a series solution approach was used to obtain solutions, whereas in our case, a direct integration of the EGB field equations was undertaken. Also we have not assumed separability of the mass functions. This, in a sense, makes our solutions (which are the EGB analogues of those found in Wang and Wu (1999)) more general.

## 5.6 Higher dimensional Boulware-Deser spacetime

Higher dimensional Boulware-Deser spacetimes have been studied in various physical scenarios. Bhawal (1990) studied the geodesic motion inside a Boulware-Deser black hole in arbitrary dimensions and Dominguez and Gallo (2006) found solutions of radiating black holes for certain equations of state. Ghosh and Dadhich (2002) also considered type II black hole solutions and gravitational collapse with a quark equation of state in higher dimensions. Dadhich and Pons (2015) found static black hole solutions in both Einstein and EGB gravity in higher dimensions by considering the topology of the product of two spheres $S^n \times S^n$. This topology comprised of black rings and branes and new solutions were obtained, also, for constant curvature. Gravitational collapse as well as other features may be affected by additional dimensions.

The $N$-dimensional Boulware-Deser metric is given by

$$ds^2 = -f(\mathsf{r})dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2 d\Omega^2_{n-2}, \tag{5.6.1}$$

with

$$d\Omega^2_{N-2} = \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta^j)\right](d\theta^i)^2,$$

and where

$$f(\mathsf{r}) = 1 + \frac{\mathsf{r}^2}{2\hat{\alpha}}\left(1 - \sqrt{1 + \frac{8\hat{\alpha}}{N-3}\left(\frac{2M}{\mathsf{r}^{N-1}}\right)}\right).$$

In the above $\hat{\alpha} = \alpha(N-3)(N-4)$. If we consider an inhomogeneous radiating metric with

$$M \longrightarrow M(v, \mathsf{r}),$$

the nonzero components of (2.4.10) are given by

$$\mathcal{G}^0{}_0 = \mathcal{G}^1{}_1 = -\frac{(N-2)M_\mathsf{r}}{\mathsf{r}^{(N-2)}}, \tag{5.6.2a}$$

$$\mathcal{G}^1{}_0 = \frac{(N-2)M_v}{\mathsf{r}^{(N-2)}}, \tag{5.6.2b}$$

$$\mathcal{G}^2{}_2 = \mathcal{G}^3{}_3 = \ldots = \mathcal{G}^{\theta(N-2)}{}_{\theta(N-2)} = -\frac{M_{\mathsf{rr}}}{\mathsf{r}^{(N-3)}}. \tag{5.6.2c}$$

The EGB field equations are thus

$$
\begin{aligned}
\mu &= \frac{(N-2)M_v}{\kappa \mathsf{r}^{N-2}}, && (5.6.3a)\\
\tilde{\rho} &= \frac{(N-2)M_{\mathsf{r}}}{\kappa \mathsf{r}^{N-2}}, && (5.6.3b)\\
P &= -\frac{M_{\mathsf{rr}}}{\kappa \mathsf{r}^{N-3}}. && (5.6.3c)
\end{aligned}
$$

As in the previous section, we find solutions to the EGB field equations and these are presented in Table 5.2. We do not give the details of the integrations as they are similar to the five-dimensional case. It is also important to note that **Theorem 1** can be extended to hold in higher dimensions. We state this as follows:

**Theorem 2.** Consider an $N$-dimensional spacetime given by

$$ds^2 = -f(\mathsf{r})dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2 d\Omega^2_{n-2},$$

with

$$d\Omega^2_{N-2} = \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta^j)\right](d\theta^i)^2,$$

and

$$f(\mathsf{r}) = 1 + \frac{\mathsf{r}^2}{2\hat{\alpha}}\left(1 - \sqrt{\frac{8\hat{\alpha}}{N-3}\left(\frac{2M}{\mathsf{r}^{N-1}}\right)}\right),$$

where $\hat{\alpha} = \alpha(N-3)(N-4)$ and $M = M(v,\mathsf{r})$, which obeys all physically reasonable energy conditions and is differentiable in the entire spacetime. This spacetime is then consistent with an energy momentum tensor which is a unique combination of the type I and type II matter fields, and represents a solution to the EGB field equations with a superposition of null radiation and a string fluid. Again, in the relevant limit, we regain the radiating case ($M = M(v)$) and the Boulware-Deser spacetime ($M = \tilde{M} =$ const.) when $\alpha \neq 0$, and the Einstein gravity when $\alpha = 0$.

## 5.7 Discussion

We have established the principal result that the standard Boulware-Deser spacetime can be made to radiate. In this chapter we considered a five-dimensional spherically symmetric radiating star in Einstein-Gauss-Bonnet (EGB) gravity. We noted

Table 5.2: Equations of state and the higher dimensional gravitational mass.

| Equation of state | $P = P(\tilde{\rho})$ | $M(v,\mathsf{r})$ |
|---|---|---|
| Linear | $P = k\tilde{\rho}$ | $M(v,\mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v), \qquad (k = \frac{1}{N-2})$ |
| | | $M(v,\mathsf{r}) = c_1(v)\frac{\mathsf{r}^{1-(N-2)k}}{1-(N-2)k} + c_2(v), \quad (k \neq \frac{1}{N-2})$ |
| Generalised linear | $P = k\tilde{\rho} + k_2$ | $M(v,\mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v)$ |
| | | $-\frac{\kappa k_2}{(N-2)k+N-2}\frac{\mathsf{r}^{N-1}}{N-1}, \qquad (k = \frac{1}{N-2})$ |
| | | $M(v,\mathsf{r}) = -\frac{\kappa k_2}{(N-2)k+N-2}\frac{\mathsf{r}^{N-1}}{N-1}$ |
| | | $+c_1(v)\frac{\mathsf{r}^{1-(N-2)k}}{1-(N-2)k} + c_2(v) \qquad (k \neq \frac{1}{N-2})$ |
| Quadratic | $P = k\tilde{\rho}^2$ | $M(v,\mathsf{r}) = (2-N)\int \frac{\mathsf{r}^{N-2}}{c_1(v)(N-2)\mathsf{r}^{N-2}+\eta}d\mathsf{r} + c_2(v)$ |
| | | $\eta = \frac{k(N-2)^2}{\kappa}$ |
| Generalised quadratic | $P = k\tilde{\rho}^2 +$ | $M(v,\mathsf{r}) = -\frac{1}{2\eta}\int \left[\left(\mathsf{r}^{N-2}\tan(\frac{1}{2}\sqrt{\varsigma}(\ln\mathsf{r} - c_1(v))\right)\right.$ |
| | $k_2\tilde{\rho} + k_3$ | $\left.\times(\sqrt{\varsigma} + N - 2 + \xi)\right] d\mathsf{r} + c_2(v)$ |
| | | $\varsigma = 4\beta\eta - N^2 - 2N\xi - \xi^2 + 4\xi - 4,$ |
| | | $\xi = k_2(N-2),\ \beta = \kappa k_3$ |
| Polytropic | $P = k\tilde{\rho}^{\gamma}$ | $M(v,\mathsf{r}) = \int \left[\kappa k(\gamma+1)\left(\frac{N-2}{\kappa}\right)^{\gamma}\right.$ |
| | | $\left.\times\frac{\mathsf{r}^{N-2-\gamma(N-2)}}{N-2-\gamma(N-2)} + (1-\gamma)c_1(v)\right]^{\frac{1}{1-\gamma}} d\mathsf{r} + c_2(v)$ |

that any astrophysical star is a combination of three concentric zones: the innermost two-component zone of matter which can be modeled by an inhomogeneous radiating Boulware-Deser metric, the radiation zone in the middle and the outermost zone which is the Boulware-Deser vacuum exterior. A large family of solutions to the EGB field equations were presented for various realistic equations of state. It was shown that solutions were possible via a direct integration of second order differential equations. Many of these solutions cannot be found by the approach used by Wang and Wu (1999) in conventional Einstein gravity; they assumed a restrictive series form of the mass function. Other mass functions have been shown to exist in five and higher dimensions which are physically reasonable. It is easy to show the existence of a dynamical star which is radiating, by matching the mass function (5.4.6) at the two boundaries. We illustrate this notion with the generalised linear equation of state

$$P = k\tilde{\rho} + k_2.$$

At the first interface $\mathsf{r} = \mathsf{r}_b$, between the two-component region and the null Boulware-Deser zone, the mass function is written as

$$M_1(v) = c_2(v) + c_1(v)\frac{\mathsf{r}_b^{1-3k}}{1-3k} - \frac{\kappa k_2}{4(3k+1)}\mathsf{r}_b^4. \tag{5.7.1}$$

At the second interface, between this null zone and the vacuum exterior region, the mass function is

$$\tilde{M} = c_2(V_0) + c_1(V_0)\frac{\mathsf{r}_b^{1-3k}}{1-3k} - \frac{\kappa k_2}{4(3k+1)}\mathsf{r}_b^4. \tag{5.7.2}$$

It is clear that the forms (5.7.1) and (5.7.2) are always possible due to the freedom permitted by the integration functions $c_1(v)$ and $c_2(v)$. A comparison with earlier results was undertaken and we showed that our solutions generalise earlier results in EGB gravity, including the EGB analogue of Husain's solution. We then generalised our results to higher dimensions.

An important point to note is the nonlinear nature of gravity, and even more specifically, modified gravity. Despite the fact that the energy momentum tensor can be written as a combination of radiation, matter and modified curvature parts, these quantities intertwine in the metric in such a way as to give physically interesting and reasonable solutions that can be used to model a dynamic star in dimensions five or higher. If the radiation part is absent, for example, then the EGB field equations force the matter that remains to obey an equation of state $\tilde{\rho} + P_{\mathsf{r}} = 0$ ($P_{\mathsf{r}}$ is the radial pressure), which is that of an AdS-like space, and is not appropriate for stellar modeling. It is also important to note that if the Gauss-Bonnet connection term tends to zero ($\alpha \to 0$), Einstein gravity is regained.

# Chapter 6

# Junction conditions in 5-$D$ Einstein-Gauss-Bonnet gravity

## 6.1 Introduction

The study of radiating stars has been an ongoing and salient endeavour in the context of general relativity due to the many applications in the area of relativistic stellar astrophysics. These include dynamical stability, surface luminosity, relaxation effects, particle production at the surface of the star as well as the important idea of gravitational collapse. Di Prisco *et al* (2007), Pinheiro and Chan (2008), Herrera *et al* (2009) and Govender *et al* (2015) have studied some of these features in detail. With regards to gravitational collapse, Goswami and Joshi (2007), Joshi (2007) and Mkenyeleye *et al* (2014, 2015) have studied the final end state of radiating stars with particular emphasis on naked singularity formation. We have also provided new details on these notions in a previous chapter.

Shear-free spacetimes are extensively used to model the interior of relativistic stars which, in the form of radial heat flow, dissipate null radiation. The heat flows outward from the much hotter centre towards the stellar surface. There exist many models on radiative gravitational collapse that have been studied earlier. These include the models by Santos (1985), Glass (1990), Deng and Mannheim (1990, 1991), Stephani *et al* (2003), Ivanov (2012) and Sharif and Yousaf (2012). An important requirement for all these models is that the interior spacetime must match at the stellar boundary to the exterior radiating spacetime.

The complete model for a radiating star with outgoing null radiation undergoing dissipation was given by Santos (1985) by use of the junction conditions at the stellar surface. By matching a purely radiating Vaidya exterior spacetime to a shear-free interior spacetime, it was shown that at the stellar surface, the pressure was nonvanishing and proportional to the magnitude of the heat flux. By investigating the boundary condition, Kramer (1992) and Maharaj and Govender (1997) generated, from a static model, radiating spheres by allowing certain parameters to become functions of time. Geodesic fluid trajectories were assumed by Kolassis *et al* (1988) and Thirukkanesh and Maharaj (2009) to produce new radiating models. There are solutions for shear-free interiors which are conformally flat which generate radiating stellar models. These are contained in Herrera *et al* (2004), Herrera *et al* (2005), Maharaj and Govender (2005) and Misthry *et al* (2008). de Oliviera *et al* (1985) and Nogueira and Chan (2004) assumed a model which was initially in a static configuration before a gradual radiating collapse of the sphere commenced. In recent times the generalised junction conditions were analysed by Maharaj *et al* (2012) where a shear-free interior metric was matched to a generalised Vaidya spacetime. The physical features of the resulting model were discussed in detail by Govender *et al* (2015).

Some work on the junction conditions has been done in higher order and modified theories of gravity. Doležel (2002) analysed the junction conditions in the context of the Randall-Sundrum model with the addition of the Gauss-Bonnet interaction. Davis (2003) derived the Israel junction conditions in spacetimes of dimensions greater than four and applied these to certain EGB brane world scenarios. Mena (2012) studied the junction of a collapsing interior fluid and exterior vacuum as well as the gravitational collapse of such a model in higher dimensions. Sharif and Abbas (2013) formulated the field equations for the shear-free spherical interior geometry of a radiating star and studied the effect of charge on dissipative collapse. An important note which needs to be made and is often not included in such models is the inclusion of the additional matching conditions (2.5.7) and (2.5.8). These were described briefly in Clifton (2006), Clifton *et al* (2013) and Ganguly *et al* (2014).

In Sec. 6.2 the relevant tensorial quantities and the EGB field equations are derived for both the five-dimensional shear-free interior and the exterior radiating Vaidya spacetime. In the following section the matching conditions are presented in detail and the junction conditions are derived completely for the matching of both spacetimes at the boundary interface. In Sec. 6.4 a summary is presented of the junction conditions as well as certain properties of the various junction conditions. Sec. 6.5 then deals with the finding of solutions to both the boundary condition as well as the condition for the continuity of scalar curvature. Finally, a brief remark is made on the structure of the scalar curvature condition in Sec. 6.6 and the possibility of finding further solutions.

## 6.2 EGB field equations

We take the interior spacetime $\mathcal{M}^-$ to be the five-dimensional shear-free line element in comoving coordinates $(x^a) = (t, r, \theta, \phi, \psi)$,

$$ds^2 = -A(r,t)^2 dt^2 + B(r,t)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2)\right]. \quad (6.2.1)$$

Interior solutions for this metric were obtained by Brassel *et al* (2015) in four-dimensional Einstein gravity. The fluid five-velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta^a_0.$$

The matter field is an imperfect fluid with energy-momentum tensor

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a, \quad (6.2.2)$$

and the heat flow vector takes the form

$$q^a = (0, q, 0, 0, 0),$$

since $q^a u_a = 0$. Due to the spherical symmetry of the star, the heat is assumed to flow in a radial direction outwards from the interior on physical grounds. The nonvanishing connection coefficients (2.3.7) are given by

$$
\begin{aligned}
\Gamma^0{}_{00} &= \frac{\dot{A}}{A} & \Gamma^0{}_{01} &= \frac{A'}{A} \\
\Gamma^0{}_{11} &= \frac{B\dot{B}}{A^2} & \Gamma^0{}_{22} &= r^2\frac{B\dot{B}}{A^2} \\
\Gamma^0{}_{33} &= r^2\sin^2\theta\frac{B\dot{B}}{A^2} & \Gamma^1{}_{00} &= \frac{AA'}{B^2} \\
\Gamma^0{}_{44} &= \frac{1}{A^2}(r^2\sin^2\theta\sin^2 p\phi) & \Gamma^1{}_{44} &= \frac{1}{B}(-r\sin^2\theta(B'r+B)) \\
\Gamma^1{}_{11} &= \frac{B'}{B} & \Gamma^1{}_{22} &= -r^2\left(\frac{B'}{B}+\frac{1}{r}\right) \\
\Gamma^1{}_{33} &= -r^2\sin^2\theta\left(\frac{B'}{B}+\frac{1}{r}\right) & \Gamma^1{}_{01} &= \frac{\dot{B}}{B} \\
\Gamma^2{}_{02} &= \frac{\dot{B}}{B} & \Gamma^3{}_{03} &= \frac{\dot{B}}{B} \\
\Gamma^2{}_{12} &= \frac{B'}{B}+\frac{1}{r} & \Gamma^3{}_{13} &= \frac{B'}{B}+\frac{1}{r} \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta & \Gamma^3{}_{23} &= \cot\theta \\
\Gamma^4{}_{04} &= \frac{\dot{B}}{B} & \Gamma^4{}_{14} &= \frac{B'}{B}+\frac{1}{r} \\
\Gamma^4{}_{24} &= \cot\theta & \Gamma^2{}_{44} &= -\sin\theta\sin^2\phi\cos\theta \\
\Gamma^3{}_{44} &= -\sin\phi\cos\phi & \Gamma^4{}_{34} &= \cot\phi
\end{aligned}
$$

for the metric (6.2.1). In the above, dots and primes denote differentiation with respect to $t$ and $r$, respectively. Using the above Christoffel symbols and the definition for the Ricci tensor (2.3.17), we can write the nonvanishing Ricci tensor components as follows

$$
\begin{aligned}
R^{-}_{00} &= \frac{AA''}{B^2} - 4\frac{\ddot{B}}{B} + 4\frac{\dot{A}\dot{B}}{B} + \frac{AA'B'}{rB^3} + 3\frac{AA'}{rB^2}, && (6.2.3a)\\
R^{-}_{01} &= 3\left(\frac{\dot{B}B'}{B^2} - \frac{\dot{B}'}{B} + \frac{A'\dot{B}}{AB}\right), && (6.2.3b)\\
R^{-}_{11} &= 3\frac{\dot{B}^2}{A^2} + \frac{A'}{A}\frac{B'}{B} - \frac{3}{r}\frac{B'}{B} - B\dot{B}\frac{\dot{A}}{A^3} - \frac{A''}{A}\\
&\quad + \frac{B\ddot{B}}{A^2} + 3\frac{B'^2}{B^2} - 3\frac{B''}{B}, && (6.2.3c)\\
R^{-}_{22} &= r^2\frac{B\ddot{B}}{A^2} - r^2 B\dot{B}\frac{\dot{A}}{A^3} + 3r^2\frac{\dot{B}^2}{A^2} - r^2\frac{A'}{A}\frac{B'}{B} - r\frac{A'}{A}\\
&\quad - 5r\frac{B'}{B} - r^2\frac{B''}{B}, && (6.2.3d)\\
R^{-}_{33} &= \sin^2\theta R^{-}_{22}, && (6.2.3e)\\
R^{-}_{44} &= \sin^2\phi R^{-}_{33}. && (6.2.3f)
\end{aligned}
$$

Making use of (6.2.3), and the definition (2.3.18), we obtain the Ricci scalar

$$
\begin{aligned}
R^{-} &= 8\frac{\ddot{B}}{AB} - 6\frac{B''}{B^3} + 12\frac{\dot{B}^2}{A^2B^2} - 2\frac{A''}{AB^2} - 4\frac{A'B'}{AB^3}\\
&\quad - 8\frac{\dot{A}\dot{B}}{A^3B} - \frac{18}{r}\frac{B'}{B^3} - \frac{6}{r}\frac{A'}{AB^2}. && (6.2.4)
\end{aligned}
$$

Now using (6.2.3), along with (6.2.4), we obtain the nonvanishing Einstein tensor components

$$
\begin{aligned}
G^{-}_{00} &= 3\left(2\frac{\dot{B}^2}{B^2} - \frac{A^2B''}{B^3} - \frac{3}{r}\frac{A^2B'}{B^3}\right), && (6.2.5a)\\
G^{-}_{01} &= 3\left(\frac{A'\dot{B}}{AB} - \frac{\dot{B}'}{B^2} + \frac{B'\dot{B}}{B^2}\right), && (6.2.5b)\\
G^{-}_{11} &= 3\left(\frac{B'^2}{B^2} + \frac{A'B'}{AB} - \frac{B\ddot{B}}{A^2} - \frac{\dot{B}^2}{A^2} + \frac{B\dot{A}\dot{B}}{A^3}\right.\\
&\quad \left. + \frac{2}{r}\frac{B'}{B} + \frac{1}{r}\frac{A'}{A}\right), && (6.2.5c)\\
G^{-}_{22} &= -3r^2\frac{B\ddot{B}}{A^2} + 3r^2 B\dot{B}\frac{\dot{A}}{A^3} - 3r^2\frac{\dot{B}^2}{A^2}\\
&\quad + 2r\frac{A'}{A} + 4r\frac{B'}{B} + r^2\frac{A''}{A} - r^2\frac{B'^2}{B^2} + 2r^2\frac{B''}{B} + r^2\frac{A'B'}{AB}, && (6.2.5d)\\
G^{-}_{33} &= \sin^2\theta G^{-}_{22}, && (6.2.5e)\\
G^{-}_{44} &= \sin^2\phi G^{-}_{33}. && (6.2.5f)
\end{aligned}
$$

When shearing stresses are absent in the fluid ($\pi^{ab} = 0$), the nonvanishing components of the energy momentum tensor (2.3.26) are written in the following way:

$$
\begin{aligned}
T_{\bar{00}} &= \rho A^2, && (6.2.6a)\\
T_{\bar{01}} &= -AB^2 q, && (6.2.6b)\\
T_{\bar{11}} &= pB^2, && (6.2.6c)\\
T_{\bar{22}} &= pB^2 r^2, && (6.2.6d)\\
T_{\bar{33}} &= \sin^2\theta T_{22}, && (6.2.6e)\\
T_{\bar{44}} &= \sin^2\phi T_{33}, && (6.2.6f)
\end{aligned}
$$

where $q = q^a q_a$. Furthermore, it is quite evident from (6.2.6b) that in the limit when the fluid is not conducting heat ($q = 0$), the energy momentum tensor has diagonal components. The higher order curvature contribution is contained in the Lanczos tensor. The nonvanishing components of the Lanczos tensor (2.4.4) are given by the following

$$
\begin{aligned}
H_{00}^{-} &= 4\left(3\frac{B''B'^2A^2}{B^7} - 3\frac{B''\dot{B}^2}{B^5}3\frac{B'^4A^2}{B^8} + 3\frac{\dot{B}^4}{A^2B^4} + \frac{6}{r}\frac{A^2B''B'}{B^6}\right.\\
&\left.-3\frac{A^2B'^3}{B^7} - \frac{9}{r}\frac{\dot{B}^2B'}{B^5} + \frac{6}{r^2}\frac{A^2B'^2}{B^2}\right), \qquad \text{(6.2.7a)}\\
H_{01}^{-} &= 4\left(3\frac{\dot{B}'B'^2}{B^5} - 3\frac{\dot{B}'\dot{B}^2}{A^2B^3} + \frac{6}{r}\frac{\dot{B}'B'}{B^4} - 3\frac{\dot{B}B'^2A'}{AB^5} + 3\frac{\dot{B}^3A'}{A^3B^3}\right.\\
&\left.-\frac{6}{r}\frac{\dot{B}B'A'}{AB^4} - 3\frac{\dot{B}B'^3}{B^5} + 3\frac{\dot{B}^3B'}{A^2B^4} - \frac{6}{r}\frac{\dot{B}B'^2}{B^5}\right), \qquad \text{(6.2.7b)}\\
H_{11}^{-} &= 4\left(3\frac{\ddot{B}B'^2}{A^2B^3} - 3\frac{A'B'^3}{AB^5} - 3\frac{\dot{A}\dot{B}B'^2}{A^3B^3} + 3\frac{\dot{B}^2A'B'}{A^3B^3} - 3\frac{\ddot{B}\dot{B}^2}{A^4B} + 3\frac{\dot{A}\dot{B}^3}{A^5B}\right.\\
&\left.-\frac{9}{r}\frac{A'B'^2}{AB^4} + \frac{6}{r}\frac{\ddot{B}B'}{A^2B^2} - \frac{6}{r}\frac{\dot{A}\dot{B}}{A^3B^2} + \frac{3}{r}\frac{\dot{B}^2A'}{A^3B^2} - \frac{6}{r^2}\frac{A'B'}{AB^3}\right), \qquad \text{(6.2.7c)}\\
H_{22}^{-} &= 4\left(\frac{r^2A''\dot{B}^2}{A^3B^2} - \frac{r^2A''B'^2}{AB^4} - 2\frac{r^2\dot{B}'^2}{A^2B^2} + 4\frac{r^2\dot{B}'\dot{B}B'}{A^2B^3} - 2\frac{r^2B''A'B'}{AB^4}\right.\\
&-2\frac{r^2B''A'B'}{A^3B^2} + \frac{r^2\dot{A}\dot{B}B'^2}{A^3B^3} - 4\frac{r\dot{A}\dot{B}B'}{A^3B^2} - 2\frac{r^2\dot{B}^2B'^2}{A^2B^4} - 2\frac{r^2\dot{B}^2A'^2}{A^4B^2}\\
&+2\frac{rA'B'^2}{AB^4} + 2\frac{rA'\dot{B}^2}{A^3B^2} + 2\frac{r^2B''\ddot{B}}{A^2B^2} - \frac{r^2\ddot{B}B'^2}{A^2B^3} - 3\frac{r^2\ddot{B}\dot{B}^2}{A^4B}\\
&\left.-2\frac{rA''B'}{AB^3} - 2\frac{rA'B''}{AB^3} + 4\frac{r\ddot{B}B'}{A^2B^2} + 3\frac{r^2A'B'^3}{AB^5} + 3\frac{r^2\dot{A}\dot{B}^3}{A^5B}\right), \qquad \text{(6.2.7d)}\\
H_{33}^{-} &= \sin^2\theta H_{22}^{-}, \qquad \text{(6.2.7e)}\\
H_{44}^{-} &= \sin^2\phi H_{33}^{-}. \qquad \text{(6.2.7f)}
\end{aligned}
$$

In the above, the components (6.2.7e) and (6.2.7f) play no explicit part in the field equations.

Using (6.2.5) and (6.2.6) we obtain the EGB field equations (2.4.9) as

$$\rho = \frac{6\dot{B}^2}{A^2B^2} - \frac{1}{B^2}\left(\frac{3B''}{B} + \frac{9B'}{rB}\right) + \frac{\alpha}{A^2}H^-_{00}, \tag{6.2.8a}$$

$$\begin{aligned} p &= \frac{3}{A^2}\left(\frac{-\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{\dot{A}\dot{B}}{AB}\right) \\ &+ \frac{3}{B^2}\left(\frac{B'^2}{B^2} + \frac{A'B'}{AB} + \frac{A'}{rA} + \frac{2B'}{rB}\right) + \frac{\alpha}{B^2}H^-_{11}, \end{aligned} \tag{6.2.8b}$$

$$\begin{aligned} p &= \frac{-3\ddot{B}}{BA^2} + \frac{3\dot{A}\dot{B}}{BA^3} - \frac{3\dot{B}^2}{A^2B^2} + \frac{2A'}{rAB^2} \\ &+ \frac{4B'}{rB^3} + \frac{A''}{AB^2} - \frac{B'^2}{B^4} + \frac{2B''}{B^3} + \frac{A'B'}{AB^3} + \frac{\alpha}{r^2B^2}H^-_{22}, \end{aligned} \tag{6.2.8c}$$

$$q = -\frac{3}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right) - \frac{\alpha}{AB^2}H^-_{01}. \tag{6.2.8d}$$

The system (6.2.8) are highly nonlinear, coupled partial differential equations that describe the dynamics of the matter field in the interior of the radiating star.

The line element for the exterior manifold $\mathcal{M}^+$ is taken to be

$$ds^2 = -\left(1 - \frac{m(v)}{\mathsf{r}^2}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2), \tag{6.2.9}$$

which is the five-dimensional purely radiating Vaidya metric. The solution was first found in a general relativistic setting by Vaidya (1951, 1953). In the above, $m = m(v)$ is the Newtonian mass function of the star as measured by an observer at infinity. The Vaidya solution describes the exterior gravitational field of a radiating star. The above solution has been the subject of much analysis with insights by de Oliveira *et al* (1985), Kolassis *et al* (1988), Kramer (1992) and Thirukkanesh and Maharaj (2009) in conventional general relativity. Kobayashi (2005) and Dominguez and Gallo (2006) considered this metric in EGB gravity. In the latter paper by Dominguez and Gallo, the Born-Infeld-Vaidya and the Bonnor-Vaidya solutions for black holes were discussed in a modified gravity setting. The nonvanishing connection coefficients for (6.2.9) are

$\Gamma^0{}_{00} = -\frac{m}{\mathsf{r}^3}$ $\qquad$ $\Gamma^1{}_{01} = \frac{m}{\mathsf{r}^3}$

$\Gamma^1{}_{00} = \frac{1}{\mathsf{r}^5}\left(\frac{1}{2}m_v\mathsf{r}^3 - \mathsf{r}^2m + 2m\right)$ $\qquad$ $\Gamma^0{}_{22} = \mathsf{r}$

$\Gamma^2{}_{12} = \Gamma^3{}_{13} = \Gamma^4{}_{14} = \frac{1}{\mathsf{r}}$ $\qquad$ $\Gamma^1{}_{22} = \frac{1}{\mathsf{r}}(m - \mathsf{r}^2)$

$$\Gamma^0{}_{33} = \mathsf{r}\sin^2\theta \qquad \Gamma^3{}_{23} = \Gamma^4{}_{24} = \cot\theta$$

$$\Gamma^1{}_{33} = \tfrac{1}{\mathsf{r}}(\sin^2\theta(m - \mathsf{r}^2)) \qquad \Gamma^2{}_{33} = \sin\theta\cos\theta$$

$$\Gamma^0{}_{44} = \mathsf{r}\sin^2\theta\sin^2\phi \qquad \Gamma^4{}_{34} = \cot\phi$$

$$\Gamma^1{}_{44} = \tfrac{1}{\mathsf{r}}(\sin^2\theta\sin^2\phi(m - \mathsf{r}^2)) \qquad \Gamma^2{}_{44} = -\sin\theta\cos\theta\sin^2\phi$$

$$\Gamma^3{}_{44} = -\sin\phi\cos\phi$$

The single nonvanishing Ricci tensor component is given by

$$R^+_{00} = -\frac{3}{2}\frac{m_v}{\mathsf{r}^3}, \tag{6.2.10}$$

and the Ricci scalar is

$$R^+ = 0. \tag{6.2.11}$$

The only nonvanishing component of the Einstein tensor $G^+_{ab}$ is hence

$$G^+_{00} = -\frac{3}{2\mathsf{r}^3}\frac{dm}{dv}, \tag{6.2.12}$$

and the nonzero components of the energy momentum tensor (3.3.3) are given by

$$T^+_{00} = \epsilon + \rho\left(1 - \frac{1}{\mathsf{r}}\right), \tag{6.2.13a}$$

$$T^+_{01} = \rho, \tag{6.2.13b}$$

$$T^+_{22} = \mathsf{r}^2 P, \tag{6.2.13c}$$

$$T^+_{33} = \sin^2\theta T^+_{22}, \tag{6.2.13d}$$

$$T^+_{44} = \sin^2\phi T^+_{33}. \tag{6.2.13e}$$

The nonzero Lanczos tensor components are given by

$$H^+_{00} = 2\left(-\frac{3m}{\mathsf{r}^7}\frac{dm}{dv} - \frac{6m^2}{\mathsf{r}^8} + \frac{6m^3}{\mathsf{r}^{10}}\right), \tag{6.2.14a}$$

$$H^+_{01} = -\frac{12m^2}{\mathsf{r}^8}, \tag{6.2.14b}$$

$$H^+_{22} = -\frac{20m^2}{\mathsf{r}^6}, \tag{6.2.14c}$$

$$H^+_{33} = \sin^2\theta H^+_{22}, \tag{6.2.14d}$$

$$H^+_{44} = \sin^2\phi H^+_{33}. \tag{6.2.14e}$$

Therefore the EGB field equations (2.4.9) are given by

$$\epsilon = \alpha\left[H^+_{00} - H^+_{01}\left(1 - \frac{m}{\mathsf{r}^2}\right)\right] - \frac{3}{2\mathsf{r}^3}\frac{dm}{dv}, \tag{6.2.15a}$$

$$\rho = \alpha H^+_{01}, \tag{6.2.15b}$$

$$P = \frac{\alpha}{\mathsf{r}^2}H^+_{22}, \tag{6.2.15c}$$

where $\epsilon$ is the energy density of the null dust radiation; $\rho$ and $P$ are the null string energy density and null string pressure respectively. It is interesting to note that both $\rho$ and $P$ are nonvanishing despite the one nonvanishing Einstein tensor component, unlike the situation in general relativity. This is a direct consequence of the existence of the Lanczos tensor (2.4.4) in this modified theory.

## 6.3 Matching of the spacetimes

In this section we now present the junction conditions (Darmois 1927, Israel 1966) to match the two spherically symmetric spacetimes (6.2.1) and (6.2.9) on a hypersurface $\Sigma$ as done by Santos (1985) and Bonnor *et al* (1989) in four-dimensions.

The intrinsic metric to $\Sigma$ is given by

$$ds^2_\Sigma = -d\tau^2 + \mathcal{R}^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2), \tag{6.3.1}$$

where $\mathcal{R} = \mathcal{R}(\tau)$ and coordinates $\xi^i = (\tau, \theta, \phi, \psi)$. It is important to note that the coordinate $\tau$ is defined only on $\Sigma$. The surface $\Sigma$ is the boundary of the interior distribution of matter (6.2.1) and in this case is given by

$$f(r,t) = r - r_\Sigma = 0,$$

where $r_\Sigma$ is a constant. The vector orthogonal to $\Sigma$ is $\frac{\partial f}{\partial \chi^a_-}$ and is given by

$$\frac{\partial f}{\partial \chi^a_-} = (0, 1, 0, 0, 0),$$

therefore the unit vector normal to the surface $\Sigma$ takes the form

$$n^-_a = [0, B(r_\Sigma, t), 0, 0, 0]. \tag{6.3.2}$$

For the interior manifold $\mathcal{M}^-$ the first junction condition (2.5.4), for the line elements (6.2.1) and (6.3.1) yields the following restrictions

$$A(r_\Sigma, t)\dot{t} = 1, \tag{6.3.3a}$$

$$r_\Sigma B(r_\Sigma, t) = \mathcal{R}(\tau), \tag{6.3.3b}$$

where, in the above, dots refer to differentiation with respect to $\tau$. The extrinsic curvature components $K_{ij}^-$ of $\Sigma$ can be obtained by using (2.5.6), (6.2.1) and (6.3.2). The lengthy calculation eventually yields

$$K_{\tau\tau}^- = \left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma, \tag{6.3.4a}$$

$$K_{\theta\theta}^- = (r(rB)')_\Sigma, \tag{6.3.4b}$$

$$K_{\phi\phi}^- = \sin^2\theta K_{\theta\theta}^-, \tag{6.3.4c}$$

$$K_{\psi\psi}^- = \sin^2\phi K_{\phi\phi}^-, \tag{6.3.4d}$$

which are valid on the surface $\Sigma$. In the above, primes denote differentiation with respect to the variable $r$.

For the exterior spacetime $\mathcal{M}^+$, the equation of the surface $\Sigma$ is given by

$$f(\mathsf{r}, v) = \mathsf{r} - \mathsf{r}_\Sigma = 0,$$

hence the vector orthogonal to $\Sigma$ is

$$\frac{\partial f}{\partial \chi_+^a} = \left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0, 0\right).$$

The unit normal vector is then

$$n_a^+ = \left(1 - \frac{m}{\mathsf{r}_\Sigma^2} + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{-\frac{1}{2}} \left(-\frac{d\mathsf{r}_\Sigma}{dv}, 1, 0, 0, 0\right), \tag{6.3.5}$$

on $\Sigma$. The first junction condition (2.5.4) for the spacetimes (3.1.1) and (6.3.1) gives the following

$$\mathsf{r}_\Sigma = \mathcal{R}(\tau), \tag{6.3.6a}$$

$$\left(1 - \frac{m}{\mathsf{r}^2} + 2\frac{d\mathsf{r}}{dv}\right)_\Sigma = \left(\frac{1}{\dot{v}^2}\right)_\Sigma. \tag{6.3.6b}$$

The unit normal vector (6.3.5) can be rewritten, using (6.3.6b) in the following simpler form

$$n_a^+ = (-\dot{\mathsf{r}}, \dot{v}, 0, 0, 0). \tag{6.3.7}$$

Using (2.5.6), (3.1.1) and the above equation (6.3.7), we can calculate the nonvanishing extrinsic curvature components to $\Sigma$. The lengthy and intricate calculation finally yields

$$\begin{aligned} K^+_{\tau\tau} &= \left(\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\mathsf{r}^3}\right)_\Sigma, & (6.3.8a)\\ K^+_{\theta\theta} &= \left(\frac{\dot{v}}{\mathsf{r}}(\mathsf{r}^2 - m) + \mathsf{r}\dot{\mathsf{r}}\right)_\Sigma, & (6.3.8b)\\ K^+_{\phi\phi} &= \sin^2\theta K^+_{\theta\theta}, & (6.3.8c)\\ K^+_{\psi\psi} &= \sin^2\phi K^+_{\phi\phi}, & (6.3.8d) \end{aligned}$$

which are valid on the surface $\Sigma$.

The first junction condition (2.5.4) yields the equations (6.3.3) and (6.3.6), which are summarised below

$$\begin{aligned} A(r_\Sigma, t)\dot{t} &= 1, & (6.3.9a)\\ r_\Sigma B(r_\Sigma, t) &= \mathcal{R}(\tau), & (6.3.9b)\\ \mathsf{r}_\Sigma(v) &= \mathcal{R}(\tau), & (6.3.9c)\\ \left(1 - \frac{m}{\mathsf{r}^2} + 2\frac{d\mathsf{r}}{dv}\right)_\Sigma &= \left(\frac{1}{\dot{v}^2}\right)_\Sigma. & (6.3.9d) \end{aligned}$$

Since the variable $\tau$ was only defined as an intercessor, it can be eliminated from the above equations. Thus, *the necessary and sufficient conditions on the spacetimes for the validity of the first junction condition (2.5.4) are*

$$\begin{aligned} (Adt)_\Sigma &= \left(1 - \frac{m}{\mathsf{r}^2} + 2\frac{d\mathsf{r}}{dv}\right)^{\frac{1}{2}}_\Sigma, & (6.3.10a)\\ (rB)_\Sigma &= \mathsf{r}_\Sigma(v). & (6.3.10b) \end{aligned}$$

Equating the appropriate extrinsic curvature components (6.3.4) and (6.3.8), yields the second junction condition (2.5.5) as

$$\left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma = \left(\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\mathsf{r}^3}\right)_\Sigma , \tag{6.3.11a}$$

$$(r(rB)')_\Sigma = \left(\frac{\dot{v}}{\mathsf{r}}(\mathsf{r}^2 - m) + \mathsf{r}\dot{\mathsf{r}}\right)_\Sigma . \tag{6.3.11b}$$

We can obtain an expression for $m(v)$ in terms of $A$ and $B$ only by eliminating $\mathsf{r}$, $\dot{\mathsf{r}}$ and $\dot{v}$ from equation (6.3.11b) above. The mass function can be written, with the aid of (6.3.9), after a long calculation as

$$m(v) = \left(\frac{r^4B^2}{A^2}\dot{B}^2 - 2r^3BB' - r^4B'^2\right)_\Sigma . \tag{6.3.12}$$

The expression above is interpreted as the total gravitational mass of the star within the surface $\Sigma$. The mass function (6.3.12) is the five-dimensional analogue of the mass function first obtained by Hernandez and Misner (1966) and Cahill and McVittie (1970) for spheres of radius $r$ within $\Sigma$. From equations (6.3.9a), (6.3.9b) and (6.3.9c) we have

$$\dot{\mathsf{r}}_\Sigma = \left(\frac{r\dot{B}}{A}\right)_\Sigma ,$$

and substituting the mass function (6.3.12) into (6.3.11b), using the expression for $\dot{\mathsf{r}}_\Sigma$ above, we get

$$\dot{v}_\Sigma = \left(1 + r\frac{B'}{B} + r\frac{\dot{B}}{A}\right)_\Sigma^{-1} . \tag{6.3.13}$$

Differentiating the above expression with respect to $\tau$ and using (6.3.9a), we acquire

$$\begin{aligned} \ddot{v}_\Sigma &= \left[\frac{1}{A}\left(1 + r\frac{B'}{B} + r\frac{\dot{B}}{A}\right)^{-2}\right. \\ &\quad \left.\times \left(r\frac{\dot{B}B'}{B^2} - r\frac{\dot{B}'}{B} + r\frac{\dot{A}\dot{B}}{A} - r\frac{\ddot{B}}{B}\right)\right]_\Sigma . \end{aligned} \tag{6.3.14}$$

Upon substituting (6.3.9b), (6.3.9c), (6.3.12), (6.3.13) and (6.3.14) into (6.3.11a), we obtain

$$\begin{aligned} \left(-\frac{1}{B}\frac{A'}{A}\right)_\Sigma &= \left[\left(1 + r\frac{B'}{B} + r\frac{\dot{B}}{B}\right)^{-1}\right. \\ &\quad \times \left(r\frac{\dot{B}B'}{AB^2} - r\frac{\dot{B}'}{AB} + r\frac{\dot{A}\dot{B}}{A^3} - r\frac{\ddot{B}}{A^2}\right. \\ &\quad \left.\left. - r\frac{\dot{B}^2}{A^2B} + 2\frac{B'}{B^2} + r\frac{B'^2}{B^3}\right)\right]_\Sigma . \end{aligned}$$

Multiplying this above expression by $1 + r\left(\frac{B'}{B}\right) + r\left(\frac{\dot{B}}{A}\right)$ and simplifying, we acquire the result

$$
\begin{aligned}
\Bigg[ -\frac{3}{r}\frac{A'}{AB^2} - 3\frac{A'B'}{AB^3} - 3\frac{A'\dot{B}}{A^2B^2} + 3\frac{\dot{B}'}{AB^2} - 3\frac{\dot{B}B'}{AB^3} & \\
-3\frac{\dot{A}\dot{B}}{A^3B} + 3\frac{\ddot{B}}{A^2B} + 3\frac{\dot{B}^2}{A^2B^2} - \frac{6}{r}\frac{B'}{B^3} - 3\frac{B'^2}{B^4}\Bigg]_{\Sigma} &= 0,
\end{aligned}
$$

which is equivalent to

$$
\left[\left(qB + \frac{\alpha}{AB}H^{-}_{01}\right) - \left(p - \frac{\alpha}{B^2}H^{-}_{11}\right)\right]_{\Sigma} = 0,
$$

where we have utilised the field equations (6.2.8b) and (6.2.8d). The above simplifies to

$$
p_{\Sigma} = \left(qB + \alpha\left[\frac{1}{AB}H^{-}_{01} + \frac{1}{B^2}H^{-}_{11}\right]\right)_{\Sigma}. \tag{6.3.15}
$$

It is interesting to note that the above condition (6.3.15) can be expressed as

$$
p_{\Sigma} = (p_q + p_{\alpha})_{\Sigma},
$$

which is to say that the pressure due to the heat flux coupled with the pressure due to the higher order curvature terms contribute to the pressure at the boundary. This is a fundamental distinction between EGB gravity and standard general relativity. *Therefore, the necessary and sufficient conditions on the spacetimes for the second junction condition (2.5.5) to hold are*

$$
\begin{aligned}
m(v) &= \left(\frac{r^4B^2}{A^2}\dot{B}^2 - 2r^3BB' - r^4B'^2\right)_{\Sigma}, & (6.3.16a)\\
p_{\Sigma} &= \left(qB + \alpha\left[\frac{1}{AB}H^{-}_{01} + \frac{1}{B^2}H^{-}_{11}\right]\right)_{\Sigma}. & (6.3.16b)
\end{aligned}
$$

The final two conditions deal with the continuity of the scalar curvature (and its derivative) across the boundary. Utilising (2.5.7) yields the following additional restriction

$$
\begin{aligned}
\Bigg[4\frac{\ddot{B}}{A} - 3\frac{B''}{B} + 6\frac{\dot{B}^2}{A^2B} - \frac{A''}{AB} - 2\frac{A'B'}{AB^2} & \\
-4\frac{\dot{A}\dot{B}}{A^3} - \frac{3}{r}\left(3\frac{B'}{B} + \frac{A'}{AB}\right)\Bigg]_{\Sigma} &= 0. & (6.3.17)
\end{aligned}
$$

Continuity of the derivative (2.5.8) gives

$$\begin{aligned}
&\left[4\frac{\ddot{B}'}{A} - 4\frac{A'\ddot{B}}{A^2} - 3\frac{B'''}{B^2} + 6\frac{B'B''}{B^3} + 12\frac{\dot{B}\dot{B}'}{A^2B}\right. \\
&-12\frac{A'\dot{B}^2}{A^3B} - 6\frac{B'\dot{B}^2}{A^2B^2} - \frac{A'''}{AB} + \frac{A'A''}{A^2B} - \frac{A''B'}{AB^2} \\
&-2\frac{A'B''}{AB^2} + 2\frac{A'^2B'}{A^2B^2} + 4\frac{A'B'^2}{AB^3} - 4\frac{\dot{A}'\dot{B}}{A^3} - 4\frac{\dot{A}\dot{B}'}{A^3} \\
&+12\frac{A'\dot{A}\dot{B}}{A^4} + \frac{3}{r}\left(3\frac{B'}{B} - 3\frac{B''}{B^2} + 6\frac{B'^2}{B^3} + \frac{1}{r}\frac{A'}{AB}\right. \\
&\left.\left. - \frac{A''}{AB} + \frac{A'^2}{A^2B} + \frac{A'B'}{AB^2}\right)\right]_{\Sigma} = 0. \qquad (6.3.18)
\end{aligned}$$

An important note is that (6.3.17) and (6.3.18) are only relevant in higher order gravity.

In the above, (6.3.17) and (6.3.18) arise in EGB gravity since $\alpha \neq 0$. When $\alpha = 0$, i.e. in 5-D Einstein gravity, these conditions do not arise. Also, (6.3.16b) becomes

$$p_{\Sigma} = (qB)_{\Sigma},$$

which itself is the five-dimensional analogue of the result by Santos (1985).

## 6.4 Summary of the complete junction conditions

The necessary and sufficient conditions on the spacetimes for all the three junction conditions to hold are

1. *The first fundamental form*:

$$(Adt)_{\Sigma} = \left(1 - \frac{m}{\mathsf{r}^2} + 2\frac{d\mathsf{r}}{dv}\right)_{\Sigma}^{\frac{1}{2}}, \qquad (6.4.1a)$$

$$(rB)_{\Sigma} = \mathsf{r}_{\Sigma}(v). \qquad (6.4.1b)$$

2. *The second fundamental form*:

$$m(v) = \left(\frac{r^4B^2}{A^2}\dot{B}^2 - 2r^3BB' - r^4B'^2\right)_{\Sigma}, \qquad (6.4.2a)$$

$$\begin{aligned}
p_{\Sigma} &= \left(qB + \alpha\left[\frac{1}{AB}H_{01}^{-} + \frac{1}{B^2}H_{11}^{-}\right]\right)_{\Sigma} \\
&= (p_q + p_{\alpha})_{\Sigma}, \qquad (6.4.2b)
\end{aligned}$$

3. *Continuity of the scalar curvature*:

$$\begin{aligned}
&\left[4\frac{\ddot{B}}{A} - 3\frac{B''}{B} + 6\frac{\dot{B}^2}{A^2B} - \frac{A''}{AB} - 2\frac{A'B'}{AB^2}\right. \\
&\qquad \left. -4\frac{\dot{A}\dot{B}}{A^3} - \frac{3}{r}\left(3\frac{B'}{B} + \frac{A'}{AB}\right)\right]_\Sigma \quad = \quad 0, \qquad (6.4.3a)
\end{aligned}$$

$$\begin{aligned}
&\left[4\frac{\ddot{B}'}{A} - 4\frac{A'\ddot{B}}{A^2} - 3\frac{B'''}{B^2} + 6\frac{B'B''}{B^3} + 12\frac{\dot{B}\dot{B}'}{A^2B}\right. \\
&-12\frac{A'\dot{B}^2}{A^3B} - 6\frac{B'\dot{B}^2}{A^2B^2} - \frac{A'''}{AB} + \frac{A'A''}{A^2B} - \frac{A''B'}{AB^2} \\
&-2\frac{A'B''}{AB^2} + 2\frac{A'^2B'}{A^2B^2} + 4\frac{A'B'^2}{AB^3} - 4\frac{\dot{A}'\dot{B}}{A^3} - 4\frac{\dot{A}\dot{B}'}{A^3} \\
&+12\frac{A'\dot{A}\dot{B}}{A^4} + \frac{3}{r}\left(3\frac{B'}{B} - 3\frac{B''}{B^2} + 6\frac{B'^2}{B^3} + \frac{1}{r}\frac{A'}{AB}\right. \\
&\qquad \left.\left. - \frac{A''}{AB} + \frac{A'^2}{A^2B} + \frac{A'B'}{AB^2}\right)\right]_\Sigma \quad = \quad 0. \qquad (6.4.3b)
\end{aligned}$$

With regards to the above conditions, some important points can be made:

- These junction conditions hold only at the boundary of the radiating star.

- Both the heat flux and spacetime curvature contribute to the pressure at the boundary. The fundamental distinction between general relativity and higher order gravity are the modified curvature components, relating to $\alpha$, contributing to the field equations. This has a profound influence on the dynamics of the radiating star and the physical features.

- In the Einstein limit, i.e. when $\alpha \to 0$, equation (6.4.2b) reduces to the well known result of Santos (1985)

$$p_\Sigma = (qB)_\Sigma,$$

  in five dimensions. An important note to make is that the third junction condition (6.4.3) does not arise in Einstein gravity.

- In the absence of a heat flux, i.e. when $q = 0$ and $p_q = 0$ the pressure at the boundary of the star is a function only of the modified curvature of the theory

$$p_\Sigma = \left(\alpha\left[\frac{1}{AB}H^-_{01} + \frac{1}{B^2}H^-_{11}\right]\right)_\Sigma = p_\alpha.$$

This quantity $p_\alpha$ can be regarded as a pressure due to the presence of modified curvature where this modified curvature behaves like a heat flux, thus the null radiation on the exterior is generated by higher order curvature terms only.

- Importantly, solving (6.4.2b) and either one of (6.4.3a) or (6.4.3b) for the gravitational potentials $A$ and $B$ will complete the model for a radiating relativistic star in EGB gravity. The restrictive nature of the third junction condition lies in the fact that two partial differential equations need to be solved in addition to the boundary condition.

## 6.5 A solution

In this section we will present a solution to the two conditions (6.4.2b) and (6.4.3a). Due to the highly nonlinear nature of the equations as well as the fact that they both contain two dependent variables, a choice needs to be made for one of these dependant variables such that the equations may be integrated to yield solutions for the remaining variable.

### 6.5.1 $B(r,t) = \beta r$

If we assume that $B(r,t)$ is a linear function in $r$ such that $B(r,t) = \beta r$ with $\beta$ being a constant, the two junction conditions (6.4.2b) and (6.4.3a) become

$$(2\beta^4 r^6 + 18\alpha)A' + (3\beta^2 r^3 + 2\beta^4 r^6)A = 0, \tag{6.5.1a}$$

$$r^2 A'' + 5rA' + 9A = 0, \tag{6.5.1b}$$

respectively. Both these equations need to be solved independently. Beginning with the second equation (6.5.1b), it is easy to notice that it is a second order Cauchy-Euler equation. Letting $A = r^m$, generates the corresponding characteristic equation

$$m^2 + 4m + 9 = 0,$$

with the complex roots

$$m = -2 \pm i\sqrt{5}.$$

Hence, the solution for $A(r,t)$ is given by

$$A(r,t) = \frac{c_1(t)}{r^2}\sin(\sqrt{5}\ln r) + \frac{c_2(t)}{r^2}\cos(\sqrt{5}\ln r), \tag{6.5.2}$$

where $c_1(t)$ and $c_2(t)$ are functions of the integration processes. So we have the following potentials

$$\begin{aligned} A(r,t) &= \frac{c_1(t)}{r^2}\sin(\sqrt{5}\ln r) + \frac{c_2(t)}{r^2}\cos(\sqrt{5}\ln r), && (6.5.3a) \\ B(r,t) &= \beta r. && (6.5.3b) \end{aligned}$$

If we now consider the boundary condition (6.5.1a), we can see that it is a separable first order equation. It can hence be written as

$$\int \frac{dA}{A} = -\beta\left[\underbrace{\int\left(\frac{3r^3 dr}{\tilde{\beta}r^6 + 18\alpha}\right)}_{I_1} + \underbrace{\int\left(\frac{\hat{\beta}r^6 dr}{\tilde{\beta}r^6 + 18\alpha}\right)}_{I_2}\right],$$

where we have set $\tilde{\beta} = 2\beta^4$ and $\hat{\beta} = 2\beta^2$ for convenience. We will consider both integrals separately due to their complexity. The first integral $I_1$ becomes

$$\begin{aligned} I_1 = \; & \frac{18^{\frac{2}{3}}}{72}\tilde{\beta}^{-1}\left(\frac{\alpha}{\tilde{\beta}}\right)^{-\frac{1}{3}}\left[-2\ln\left(r^2 + 18^{\frac{1}{3}}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{3}}\right)\right. \\ & + \ln\left(r^4 - 18^{\frac{1}{3}}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{3}} r^2 + 18^{\frac{2}{3}}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{2}{3}}\right) \\ & \left. + 2\sqrt{3}\arctan\left(\frac{1}{27}\left(\frac{\alpha}{\tilde{\beta}}\right)^{-\frac{1}{3}}\sqrt{3}\left(18^{\frac{2}{3}}r^2 - 9\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{3}}\right)\right)\right]. \end{aligned} \tag{6.5.4}$$

The second, more complicated integral, $I_2$ takes the form

$$\begin{aligned} I_2 &= \frac{-1}{72\tilde{\beta}}\left[3^{\frac{2}{3}}2^{\frac{5}{6}}\tilde{\beta}\left(4\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}2^{\frac{1}{3}}3^{\frac{2}{3}}\right.\right. \\ &\times\arctan\left(\frac{1}{6}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}3^{\frac{2}{3}}2^{\frac{5}{6}}\left(2^{\frac{1}{6}}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}3^{\frac{5}{6}}+2r\right)\right) \\ &-2\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}2^{\frac{1}{3}}3^{\frac{2}{3}}\arctan\left(\frac{3\sqrt{3}\alpha}{3^{\frac{1}{3}}2^{\frac{2}{3}}r^2\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{2}{3}}\tilde{\beta}-3\alpha}\right) \\ &+4\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}3^{\frac{2}{3}}2^{\frac{1}{3}}\arctan\left(\frac{1}{6}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}3^{\frac{2}{3}}2^{\frac{5}{6}}r\right) \\ &+3\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}2^{\frac{1}{3}}3^{\frac{1}{6}}\ln\left[r^2+3^{\frac{5}{6}}2^{\frac{1}{6}}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}r+3^{\frac{2}{3}}2^{\frac{1}{3}}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{3}}\right] \\ &-3\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}2^{\frac{1}{3}}3^{\frac{1}{6}}\ln\left[r^2-3^{\frac{5}{6}}2^{\frac{1}{6}}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{6}}r+3^{\frac{2}{3}}2^{\frac{1}{3}}\left(\frac{\alpha}{\tilde{\beta}}\right)^{\frac{1}{3}}\right] \\ &\left.\left.-12(3^{\frac{1}{3}}2^{\frac{1}{6}})\right)\right]. \end{aligned} \tag{6.5.5}$$

Therefore the complete solution for the differential equation for $A$ is given by

$$A(r,t) = \tilde{C}_1(t)\exp[-\beta(I_1+I_2)], \tag{6.5.6}$$

where $\tilde{C}_1(t)$ is a function of integration. Thus the full solution to the boundary condition (6.4.2b) is given by

$$A(r,t) = \tilde{C}_1(t)\exp[-\beta(I_1+I_2)], \tag{6.5.7a}$$

$$B(r,t) = \beta r, \tag{6.5.7b}$$

where $I_1$ and $I_2$ are given by equations (6.5.4) and (6.5.5) respectively.

An important point that needs to be made about the solutions (6.5.3) and (6.5.7) is the choice of the potential $B(r,t)$. In both cases, the linear choice for $B$ implies that the heat flux, $q$ will vanish entirely, since $H^-_{01} = 0$ in the field equations. The consequence of this is the fact that the pressure at the boundary will be a function of the modified curvature inherent with the theory, and will not vanish. So, in essence, this modified curvature acts like a heat flux. The solutions (6.5.3) and (6.5.7) complete the model for a five-dimensional radiating relativistic star in EGB gravity.

## 6.6 Solutions for the continuity of scalar curvature

If we focus only on the condition (6.4.3a), it is possible to find some generic solutions. For example, if we let $B(r,t) = \beta$ where $\beta$ is a constant, then equation (6.4.3a) reduces to

$$A'' + \frac{3}{r}A' = 0,$$

which can be integrated via a reduction of order. Letting $y(r,t) = A'(r,t)$, the above equation reduces to

$$y' + \frac{3}{r}y = 0,$$

which can be integrated to give

$$y(r,t) = \frac{c_1}{r^3},$$

where $c_1 = c_1(t)$ is an integration function. Finally, the solution for $A(r,t)$ is given by

$$A(r,t) = \frac{-c_1(t) + c_2(t)r^2}{r^2}, \tag{6.6.1}$$

where $c_2(t)$ is a second integration function. Thus, in summary we have

$$A(r,t) = \frac{-c_1(t) + c_2(t)r^2}{r^2}, \tag{6.6.2a}$$

$$B(r,t) = \beta. \tag{6.6.2b}$$

If $c_1(t) = 0$ and $c_2(t) = 1$ in the above solution, the particle motion in the model will become geodesic. Again, this choice of $B(r,t)$ will cause the heat flux to vanish, and so this solution above is more cosmological than astrophysical. It is interesting to note that the condition (6.4.3a) can be written in the following form

$$A'' + \left(2\frac{B'}{B} + \frac{3}{r}\right)A' + \left(3\frac{B''}{B} + \frac{9}{r}\frac{B'}{B}\right)A = 0, \tag{6.6.3}$$

therefore, for suitable choices of one of the variables $A$ and $B$, it may be possible to obtain further solutions to the above equation. We will demonstrate this below.

### 6.6.1 Case I

First, we let

$$3\frac{B''}{B} + \frac{9}{r}\frac{B'}{B} = 0,$$

in equation (6.6.3). The above equation is a second order linear partial differential equation with radial derivatives. Reducing the order with the transformation $y(r,t) = B'(r,t)$ yields

$$y' + \frac{3}{r}y = 0,$$

which is a separable equation with a solution given by

$$y = \frac{c_1(t)}{r^3}.$$

Therefore, the solution for $B$ is given by

$$B(r,t) = c_2(t) - \frac{c_1(t)}{2r^2}. \tag{6.6.4}$$

In the above, $c_1(t)$ and $c_2(t)$ are functions of integration. Using, (6.6.4) the equation (6.6.3) becomes

$$A'' + \left(\frac{4c_1}{c_2r^3 - c_1r} + \frac{3}{r}\right)A' = 0. \tag{6.6.5}$$

The above equation can again be solved via a reduction of order. Letting $z(r,t) = A'(r,t)$ yields

$$z' + \left(\frac{4c_1}{c_2r^3 - c_1r} + \frac{3}{r}\right)z = 0, \tag{6.6.6}$$

which is a separable equation. The solution for $A$ is hence given by

$$A(r,t) = \frac{1}{4}r^4 + \int\left(\frac{r^4}{(c_1(t)r^2 - c_2(t))^2}\right)^{c_1(t)} dr + c_3(t), \tag{6.6.7}$$

where $c_3(t)$ is a further integration function. So, we have

$$\begin{aligned} A(r,t) &= \frac{1}{4}r^4 + \int\left(\frac{r^4}{(c_1(t)r^2 - c_2(t))^2}\right)^{c_1(t)} dr + c_3(t), && (6.6.8a)\\ B(r,t) &= c_2(t) - \frac{c_1(t)}{2r^2}. && (6.6.8b)\end{aligned}$$

For different values of the integration function $c_1(t)$, it may be possible to integrate (6.6.8a) entirely. This is beyond the scope of this dissertation and is the subject of ongoing work.

### 6.6.2 Case II

We now let

$$2\frac{B'}{B} + \frac{3}{r} = 0, \tag{6.6.9}$$

in equation (6.6.3). The above equation can be integrated to give

$$B(r,t) = c_1(t)r^{-\frac{3}{2}}. \tag{6.6.10}$$

With this, equation (6.6.3) becomes

$$A'' - \left(\frac{45}{2}r + 18\right)A = 0, \tag{6.6.11}$$

which is an Airy differential equation. It can be solved via a series solution approach to give

$$A(r,t) = c_2(t)\mathrm{Ai}\left(\frac{3^{\frac{2}{3}}(5r-4)}{2^{\frac{1}{3}}5^{\frac{2}{3}}}\right) + c_3(t)\mathrm{Bi}\left(\frac{3^{\frac{2}{3}}(5r-4)}{2^{\frac{1}{3}}5^{\frac{2}{3}}}\right), \tag{6.6.12}$$

where $\mathrm{Ai}(r)$ are Airy functions of the first kind and $\mathrm{Bi}(r)$ are Airy functions of the second kind (or Bairy functions). The reader is encouraged to seek out Airy (1838), Vallée and Soares (2004) and Press *et al* (2007) for further insights into these types of equations. Therefore, the complete solutions are

$$A(r,t) = c_2(t)\mathrm{Ai}\left(\frac{3^{\frac{2}{3}}(5r-4)}{2^{\frac{1}{3}}5^{\frac{2}{3}}}\right) + c_3(t)\mathrm{Bi}\left(\frac{3^{\frac{2}{3}}(5r-4)}{2^{\frac{1}{3}}5^{\frac{2}{3}}}\right), \tag{6.6.13a}$$

$$B(r,t) = \frac{c_1(t)}{r^{\frac{3}{2}}}. \tag{6.6.13b}$$

In the above $c_1(t)$, $c_2(t)$ and $c_3(t)$ are functions of integration. An important note to consider here are the Airy functions. Since these are series solutions, it is difficult to model a physically reasonable stellar model. The reason for this is the fact that Airy functions cannot be truncated to become a polynomial.

## 6.7 Discussion

The main intent of this chapter was the calculation of the junction conditions for two five-dimensional spacetime geometries in EGB gravity. The interior spacetime was taken to be the shear-free line element and the outside geometry was that of a pure Vaidya class. The EGB field equations were derived for both metrics and the matching was performed using the standard Santos framework, with the addition of the junction condition for the continuity of the scalar curvature and its derivative. A summary of the three junction conditions was then provided before a solution was obtained for both the boundary condition and the scalar curvature. Some important points can now be elucidated upon:

- For these solutions to be entirely valid, they have to be consistent with each other. This is already the case for the gravitational potential $B(r,t)$, since the same choice was made for both conditions (6.4.2b) and (6.4.3a). For the potentials $A(r,t)$ to be consistent, special choices for the integration functions $c_1(t)$, $c_2(t)$ and $\tilde{C}_1(t)$ need to be made. This is beyond the scope of this dissertation and is the subject of ongoing work.

- The linear choice of the potential $B(r,t) = \beta r$ has some repercussions. First of all, this choice causes the heat flux $q$ to vanish in the model. Secondly, the pressure at the boundary of the star is then simply a function of the modified curvature of the theory as a result.

Finally, a trivial solution was presented for the continuity of scalar curvature. Since the choice for $B$ in this case was constant, the heat flux $q$ will again vanish in the stellar model. We have also shown that it is possible to write equation (6.4.3a) in a form which may yield solutions more easily. Two such solutions were presented, however for these to be complete, the boundary condition needs to be solved and made consistent with them.

The most important aspect of this chapter is the fact that a complete model for a relativistic radiating star has been obtained in five-dimensional EGB gravity, for the first time. This should now have effects on the observable quantities measured on the surface of the star as well as in its surrounding region. These include the surface redshift and the luminosity which play a critical role in understanding the formation of black holes as a result of gravitational collapse (Chan 1997, 2003). This research is ongoing work.

# Chapter 7

# Conclusion

In this dissertation we have studied spherically symmetric radiating stars emitting null radiation in the strong gravity regime. We have generated several new classes of solutions for the class of generalised Vaidya spacetimes describing the complete model of a radiating star in four and higher dimensions. We demonstrated that the differential equations resulting from the assumption of an equation of state could be integrated directly, and several of the well known seminal solutions from earlier are contained within our class. We then considered the idea of gravitational collapse where we analysed the end state of these generalised Vaidya spacetimes, demonstrating that each collapse terminated with the formulation of a naked singularity. This idea was extended to an higher order theory, namely Einstein-Gauss-Bonnet (EGB) gravity, where we modeled a five-dimensional radiating star using one class of the Boulware-Deser metric. Several solutions were obtained for various realistic equations of state and it was shown that they generalise many solutions found by others, prior. We then extended these results to higher dimensions for pedagogical completeness. Finally, we generated and analysed the junction conditions resulting from the matching of two five-dimensional spacetimes in EGB gravity. We demonstrated that a solution can be found for both the boundary condition as well as the extra condition resulting from the matching of the scalar curvature across the boundary.

Below is an overview of our study:

- In chapter 2, we introduced the relevant formalisms and aphorisms for differential geometry and general relativity. These notions were also extended to a particular case of higher order gravity, which is the Einstein-Gauss-Bonnet theory. The theory of the junction conditions (Darmois 1927, Israel 1966, Santos 1985) resulting from the matching of two general spherically symmetric spacetimes, were presented in detail from first principles. We also showed that in higher order theories of gravity, a third matching condition needs to be satisfied to complete the model of a radiating star. In summary, the junction conditions in the strong gravity regime are given by

$$\begin{aligned}
(ds^2_-)_{\Sigma^-} = (ds^2_+)_{\Sigma^+} &= ds^2_{\Sigma}, \\
K^{\pm}_{ij}\Big|_{\Sigma} &= 0, \\
R^{\pm}\Big|_{\Sigma} &= 0, \\
\nabla_a R^{\pm}\Big|_{\Sigma} &= 0.
\end{aligned}$$

  Finally, the chapter closes with a brief portraiture of the energy conditions for a stellar model.

- Chapter 3 commenced with one of the main results obtained in this dissertation. The complete model of a radiating and dynamic relativistic star was described under one general class of generalised Vaidya metric. The spacetime region was divided into three concentric regions. The stellar interior was a two-component system consisting of standard null, pressure-free radiation and an additional string fluid with energy density and nonzero pressure obeying all physical conditions. The middle region was a purely radiative one which matched smoothly to the vacuum Schwarzschild exterior. The complete $C^2$ mass function was given as:

$$m(v, \mathsf{r}) = \begin{cases} m(v, \mathsf{r}) & \mathsf{r} \leq \mathsf{r}_b \ , v \leq V_0 \\ m_1(v) \equiv m(v, \mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b \ , v \leq V_0 \\ M \equiv m_1(V_0) \equiv m(V_0, \mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b \ , v > V_0 \end{cases}$$

Several solutions were obtained for the mass function $m(v, \mathsf{r})$ for various realistic equations of state including the polytrope which was first analysed by Husain (1996). We demonstrated that it was possible for the second order equations to be integrated directly. It was then shown that one of our classes of solutions contained the monopole, charged Vaidya, dS/AdS, and Husain solutions. An important note that requires mentioning is the fact that many of our solutions cannot be found using the approach of Wang and Wu (1999). They assumed a series form of the mass function which was restrictive. We naturally extended these notions to higher dimensional spacetimes and showed that physically reasonable solutions exist. The possibility also existed for the generation of diffusive solutions for the mass function via a substitution of the above mentioned second order equations into the diffusion equation.

- A complete analysis of the gravitational collapse of our generalised Vaidya spacetimes, obtained for various equations of state, was performed in chapter 4. The general mathematical framework to describe the gravitational collapse of a generalised Vaidya spacetime in the context of the cosmic censorship conjecture (CCC) was presented in detail. The collapsing model was described (as shown in Joshi (1993) and Mkenyeleye (2014, 2015)) before considering the conditions for the formation of a locally naked central singularity. The singularity structure was investigated in order to show that it can be a node with outgoing null geodesics emerging from a singular point with a definite value of the tangent, depending on the parameters in the problem and the nature of the generalised Vaidya mass function. The nature and strength of the singularity and the apparent horizon were then described. The various generalised Vaidya solutions obtained in chapter 3 were then analysed and it was shown that the end state of collapse for each of these spacetimes is a locally, strong naked singularity. During the latter stages of gravitational collapse, the generalised Vaidya spacetime is more realistic and physically reasonable than pure pressureless matter or perfect fluid fields since any collapsing star must radiate. The higher dimensional metric was then discussed briefly and it was proved that the spacetime dimensions do have an effect on the existence of a naked singularity; in higher dimensions it is possible for naked singularities to be eliminated.

- In chapter 5 we considered EGB gravity. We introduced the Boulware-Deser spacetime (Boulware and Deser 1985) and showed that it can radiate by allowing the mass function $M$ to depend on both the radial coordinate and retarded time, creating the inhomogeneity. These notions were then formulated into a theorem. We then modeled a five-dimensional relativistic radiating astrophysical star in isolation using one class of generalised Boulware-Deser spacetime. The spacetime was divided into three concentric regions. In the stellar interior there was a two-component matter source consisting of null fluid matter together with radiation. The middle zone was once again a pure zone of radiation which then matched smoothly to the Boulware-Deser vacuum exterior spacetime. It was proved that the complete $C^2$ mass function is given by:

$$M(v,\mathsf{r}) = \begin{cases} M(v,\mathsf{r}) & \mathsf{r} \leq \mathsf{r}_b\ , v \leq V_0 \\ M_1(v) \equiv M(v,\mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b\ , v \leq V_0 \\ \tilde{M} \equiv M_1(V_0) \equiv M(V_0,\mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b\ , v > V_0 \end{cases}$$

  A large family of solutions to the EGB field equations were then presented for various realistic equations of state. As was the case in Chapter 3 for the generalised Vaidya spacetime, it was possible to obtain solutions via a direct integration of the second order differential equations. These results are essentially the EGB analogues of those found in Chapter 3, in Einstein gravity. With that, some important points can be made:

- Many of these solutions cannot be found using the approach of Wang and Wu (1999) in general relativity. They assumed a restrictive series form for the mass function which was also separable, therefore our solutions can be regarded as more general than those found in the past (Husain 1996, Wang and Wu 1999, Dominguez and Gallo 2006).
- In Einstein gravity, Husain (1996) found a solution for the polytropic equation of state in the generalised Vaidya spacetime which was in quadrature form. We found the EGB analogue of this solution in the Boulware-Deser spacetime.
- In the limit when $\alpha \longrightarrow 0$, conventional Einstein gravity is regained in five dimensions.

Finally, our model was extended to higher dimensional spacetimes and it was possible to state the following theorem:

**Theorem.** Consider an $N$-dimensional spacetime given by

$$ds^2 = -f(\mathsf{r})dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2 d\Omega^2_{n-2},$$

with

$$d\Omega^2_{N-2} = \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta^j)\right](d\theta^i)^2,$$

and

$$f(\mathsf{r}) = 1 + \frac{\mathsf{r}^2}{2\hat{\alpha}}\left(1 - \sqrt{\frac{8\hat{\alpha}}{N-3}\left(\frac{2M}{\mathsf{r}^{N-1}}\right)}\right),$$

where $\hat{\alpha} = \alpha(N-3)(N-4)$ and $M = M(v,\mathsf{r})$, which obeys all physically reasonable energy conditions and is differentiable in the entire spacetime. This spacetime is then consistent with an energy momentum tensor which is a unique combination of the type I and type II matter fields, and represents a solution to the EGB field equations with a superposition of null radiation and a string fluid. Again, in the relevant limit, we regain the radiating case ($M = M(v)$) and the Boulware-Deser spacetime ($M = \tilde{M} = \text{const.}$) when $\alpha \neq 0$, and the Einstein gravity when $\alpha = 0$.

A paramount point that needs to be made is the nonlinear nature of gravity, specifically modified and higher order theories of gravity. As mentioned in the chapter, despite the fact that the energy momentum tensor can be written as a combination of radiation, matter and modified curvature parts, these quantities are married in the metric in such a way as to provide physically interesting and reasonable solutions. These can be used to model a dynamic star in dimensions five or higher.

- The complete model of a five-dimensional radiating relativistic star in EGB gravity theory is shown from first principles in chapter 6. The smooth matching of two spherically symmetric spacetimes, namely the shear-free interior and the pure Vaidya exterior is considered in the context of EGB gravity. The Santos (1985) junction conditions are derived with the inclusion of the additional junction condition for the continuity of the Ricci scalar across the boundary, as well as its first derivative. It was mentioned that the additional junction condition adds a restriction to the model in the sense that an extra consistency equation (either one of the scalar curvature or its derivative) needs to be solved over and above the original boundary condition. The equation for the continuity of the scalar curvature was simpler than that of its derivative, so this was chosen in the subsequent analysis. These two highly nonlinear partial differential equations are:

$$p_\Sigma = \left( qB + \alpha \left[ \frac{1}{AB} H^-_{01} + \frac{1}{B^2} H^-_{11} \right] \right)_\Sigma ,$$

$$\left[ 4\frac{\ddot{B}}{A} - 3\frac{B''}{B} + 6\frac{\dot{B}^2}{A^2 B} - \frac{A''}{AB} - 2\frac{A'B'}{AB^2} \right.$$
$$\left. -4\frac{\dot{A}\dot{B}}{A^3} - \frac{3}{r}\left( 3\frac{B'}{B} + \frac{A'}{AB} \right) \right]_\Sigma = 0.$$

A linear function in $r$ was chosen for the potential $B$, such that $B(r,t) = \beta r$ which reduced the above equations to

$$\begin{aligned}(2\beta^4 r^6 + 18\alpha)A' + (3\beta^2 r^3 + 2\beta^4 r^6)A &= 0,\\ r^2 A'' + 5rA' + 9A &= 0.\end{aligned}$$

Subsequent solutions were then found for the gravitational potential $A$. In summary, these are

$$\begin{aligned}A(r,t)_{BC} &= \frac{c_1(t)}{r^2}\sin(\sqrt{5}\ln r) + \frac{c_2(t)}{r^2}\cos(\sqrt{5}\ln r),\\ A(r,t)_{SC} &= \tilde{C}_1(t)\exp[-\beta(I_1 + I_2)],\\ B(r,t) &= \beta r,\end{aligned}$$

where the subscripts "BC" and "SC" stand for the "boundary condition" and "scalar curvature" respectively. $I_1$ and $I_2$ are given by equations (6.5.4) and (6.5.5) respectively. For these solutions to be valid in their entirety, they both have to be consistent with each other. This involves tweaking the various integration functions such that this consistency may be found. An important note which requires mentioning is the fact that this assumption for the variable $B$ will result in the heat flux $q$ vanishing which implies that the radiating star is being held together by the higher order curvature terms.

A solution was presented for the third junction condition for a constant choice for $B$ and again, this choice causes the heat flux to vanish at the boundary. The vital result of this chapter lies in the fact that a complete model for a five-dimensional radiating star in EGB gravity has now been obtained for the first time. A subsequent analysis of the physical conditions will be the subject of future work.

The investigations presented in this dissertation and the results generated form a paramount part of a wide selection of models which can be used in astrophysics and cosmology, within the frameworks of general relativity and modified gravity. Within the context of astrophysics and the modeling of a radiating star in EGB gravity, these investigations may be enhanced by the inclusion of the following features:

- Studying the dynamical stability of the dissipating fluid within the star in the context of non-adiabatic gravitational collapse. To achieve this, we would require a detailed analysis of the behaviour of the effective adiabatic index $\Gamma_{eff}$, the Weyl tensor $C_{abcd}$ and the rate of collapse $\Theta = u^a{}_{;a}$.

- Including the effects of shearing anisotropic stresses in the interior radiating fluid. This is a more general and realistic way of depicting stellar fluids in relativistic astrophysics.

This research will be carried out in future work.

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