

**ITERATIVE METHODS FOR APPROXIMATING
SOLUTIONS OF CERTAIN OPTIMIZATION PROBLEMS
AND FIXED POINT PROBLEMS**

by

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As the candidate's supervisor I have approved this dissertation for submission.

Dr. O.T. Mewomo

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Dedication

To my late father, Mr. N.I. Jolaoso
who passed away on 13/07/2015.

Abstract

In this dissertation, we present an iterative method for approximating a common element of the set of solutions of split equalities for generalized equilibrium problem, monotone variational inclusion problem and fixed point problem for k demi-contractive mapping in Hilbert space without prior knowledge of the operator norms. We also give numerical example of our main theorem and use Matlab version 2014a to show how the sequence values, that is, $\|x_{n+1} - x_n\|$ are affected by the number of iterations. This is done in order to see how the initial values affect the number of iterations.

Also, we introduce another iterative scheme which do not require a prior knowledge of the operator norms for approximating a common element of the set of solutions of split equalities for finite family of generalized mixed equilibrium problems and fixed point problem for k -strictly pseudo-nonspreading multi-valued mapping of type-one in Hilbert space. We also give numerical example of our main theorem and use Matlab version 2014a to show how the sequence values are affected by the number of iterations.

Furthermore, we extend our work to reflexive Banach space by introducing an iterative method for approximating a common fixed point of quasi-Bregman nonexpansive mapping which also solve finite system of variational inequality problems and convex minimization problems. We give application of our result to approximating a common zeroes of an infinite family of Bregman inversely strongly monotone operators which are also solutions to the set of finite system of convex minimization problems and variational inequality problems and for approximating a common solution of finite system of equilibrium problems in real reflexive Banach space. Our results in this dissertation extend and improve some recent results in literature.

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All praises and adorations belong to Almighty God, the Lord of mankind, jinns and all that exist. Peace and Blessing be upon His noble prophet and messenger, Muhammad, his households, companions and those who follow their footsteps till the Day of Resurrection.

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Declaration

I declare that this dissertation, in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents my own work and I have made proper reference of the works of others wherever they are been used in the text.

Jolaoso Lateef Olakunle

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1.1 General Introduction

An optimization problem is one where the values of a given function $f : \mathbb{R} \rightarrow \mathbb{R}$ are to be maximized or minimized over a given nonempty set $D \subset \mathbb{R}$. The function f is called the objective function and the set D is called the constraint set. Optimization problems can be formulated as minimization problems, variational inequality problems, equilibrium problems, min-max problems, etc. Examples of optimization problems include utility maximization, expenditure minimization, profit maximization, cost minimization, portfolio choice, electricity value maximization, among others.

Let C be a nonempty, closed and convex subset of a Hilbert space H . The Variational Inequality Problem (VIP) is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C, \quad (1.1.1)$$

where $A : C \rightarrow H$ is a nonlinear mapping. The set of solutions of (1.1.1) is denoted by $\text{VI}(C,A)$. The variational inequality theory was introduced by Stampacchia [112] in the early 1960's to study some problems in partial differential equations with applications drawn from elasticity and potential theory. The first general theorem for the existence and uniqueness of solution of VIP was proved by Lions and Stampacchia [72] in 1967. Since then, the VIP have played fundamental and important roles in the study of wide range of problems arising in optimization, control theory, economics, operation research, management science, physics, mechanics, elasticity, transportation and other branches of mathematical and engineering sciences, (see [2], [5], [13], [19], [39], [42], [45], [46], [65], [81], [96]).

Also, the Equilibrium Problem (EP) is to find $x \in C$ such that

$$g(x, y) \geq 0, \quad \forall y \in C, \quad (1.1.2)$$

where $g : C \times C \rightarrow \mathbb{R}$ is a bifunction and C is a nonempty closed convex subset of a Hilbert space H . The set of solutions of (1.1.2) is denoted by $\text{EP}(g)$.

The EP was first introduced by Blum and Oettli [15] in 1994 as generalization of optimization and variational inequality problems. They discussed existence theorems and variational principle for the equilibrium problem. Since then, various generalizations of equilibrium problem have been introduced and studied by many authors. The EP has a great influence in the study of several problems arising in engineering and sciences such as nonlinear analysis, optimization, economics, finance, game theory, physics, image reconstruction, etc, (see [15], [40], [46], [59], [62], [76], [83]).

Let X be a nonempty set and T a mapping of X into X . A point $x \in X$ is called a fixed point of T if $Tx = x$. However, if T is a multi-valued mapping, that is, T is mapping X into the collection of nonempty subsets of X , then a point $x \in X$ is called a fixed point of T if $x \in Tx$. The set of fixed points of T is denoted by $F(T)$. Also, a topological space (X, τ) is said to have a fixed point property if every continuous mapping $T : X \rightarrow X$ on it has a fixed point. The problem of investigating sufficient conditions for the existence of a fixed point for a mapping is one of the most vigorous among the fundamental branches of topology and functional analysis.

In particular, fixed point theorems have extensive applications in proving existence and uniqueness of solutions of various functional equations. These theorems have found applications in the theory of differential and integral equations, dynamical systems, game theory and mathematical economics among others (see [7], [90], [118], [123], [64], [44]).

One of the cornerstones in the history of fixed point theorems is the work of Banach [7] in 1922 which states that every contraction mapping T (i.e, T satisfying $d(Tx, Ty) \leq kd(x, y)$, $0 \leq k < 1$, $\forall x, y \in X$) of a complete metric space (X, d) into itself has a unique fixed point (we shall give a further explanation on this theorem in Chapter 2).

This theorem is known as the Banach contraction mapping principle. The theorem is constructive in its nature and provides a procedure to arrive at the required fixed point by using the convergence of the Picard's iteration. This theorem has been broadly used in the study of solutions of various operator equations, including numerical approximations (cf. Agarwal et al. [3, 4], Kirk and Sims [66] and Zeidler [123]). Some important generalizations of the above theorem include contractive mappings by Edelstein [43] and nonexpansive mappings by Browder [24]. These generalizations lead to the introduction of other fixed point iteration procedures such as the Krasnoselskij iteration, the Mann iteration and the Ishikawa iteration.

In 1967, Bregman [18] introduced a nice and effective method for designing and analyzing the feasibility and optimization algorithms using the so called Bregman distance function D_f (see, Definition 2.3.9). This method opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms for solving various optimization problems and for computing fixed points of nonlinear mappings (see eg. [101, 102, 103, 104, 114]).

In 2003, Butnariu, Iusem and Zalinescu [27] studied several notions of convex analysis, uniformly convexity at a point, total convexity on bounded sets and sequential consistency which are useful in establishing convergence properties for fixed point and optimization algorithms in infinite-dimensional Banach spaces. They established the connections between these concepts and used these relations in order to obtain improved convergence results concerning the outer-Bregman projection algorithm for solving convex feasibility problems and the generalized proximal point algorithm for optimization problems in Banach spaces. Also, in 2005, Butnariu and Resmerita [29] presented a Bregman-type iterative algorithms and studied the convergence of the Bregman-type iterative method for solving operator equations. Resmerita [105] further investigated the existence of totally convex functions in Banach spaces and established continuity and stability properties of the Bregman projections.

1.2 Research Motivation

In [82], Moudafi introduced the notion of Split Equality Problem (SEP) which is a generalization of the split feasibility problem introduced by Censor and Elfving [33]. Let C and Q be two nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear operators and let $U : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ be two nonlinear mappings with $F(U) \neq \emptyset$, $F(T) \neq \emptyset$ respectively. The SEP is to find $x \in C := F(U)$, $y \in Q := F(T)$ such that:

$$Ax = By, \quad (1.2.1)$$

where $F(U)$ and $F(T)$ denote the fixed point sets of U and T respectively. The SEP allows asymmetric and partial relations between the variables x and y . The interest is to cover many situations, for instance, in decomposition method for PDE's, applications in game theory and in intensity-modulated radiation therapy.

For solving the SEP, Moudafi [82] introduced the following alternating iterative method:

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases} \quad (1.2.2)$$

for firmly quasi-nonexpansive operators U and T , where

$$\{\gamma_k\} \subset \left(\epsilon, \min\left(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\right) - \epsilon \right)$$

is a nondecreasing sequence and λ_A, λ_B stand for the spectral radii of A^*A and B^*B respectively.

Furthermore, Moudafi and Al-Shemas [86] introduced the following simultaneous iterative method which also solve the SEP for firmly quasi-nonexpansive operators U and T :

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \quad (1.2.3)$$

and $\{\gamma_k\}$ is a nondecreasing sequence such that

$$\gamma_k \in \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon \right),$$

where λ_A and λ_B are spectral radii of A^*A and B^*B respectively.

Note that in the iterative methods (1.2.2) and (1.2.3), the determination of the step-size $\{\gamma_k\}$ depends on the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B). In order to implement the iterative schemes for solving the SEP, one has to first compute or estimate the operator norms of A and B which is not an easy work in practice.

To overcome this difficulty, López et al. [74] and Zhao and Yang [125] presented helpful methods for estimating the step-sizes which does not require a prior knowledge of the operator norm for solving the SEP. In this direction, Zhao [124] studied the SEP and presented the following step-size which guarantee convergence of the iterative scheme without a prior information about the operator norms of A and B ,

$$\gamma_k \in \left(0, \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} \right).$$

In 2011, Shehu [108] introduced an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of a generalized equilibrium problem and the set of solutions of a variational inclusion problem in a real Hilbert space. In particular, he obtained the following algorithm and proved a strong convergence theorem under suitable conditions:

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 & y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T[\alpha_n f(x_n) + (1 - \alpha_n) J_\lambda^T(u_n - \lambda f u_n)] & \forall n \geq 1. \end{cases} \quad (1.2.4)$$

Motivated by the results mentioned above, we present a new iterative scheme for finding a common element of the set of solutions of split equality generalized equilibrium problem, split equality monotone variational inclusion problem and split equality fixed point problem for demi-contractive mappings in Hilbert space and we prove a strong convergence theorem for the sequence generated by our iterative scheme without a prior knowledge of the operator norms.

Recently, Ma et al. [75] introduced the following scheme for obtaining a weak and a strong convergence results for a set of solutions of split equality mixed equilibrium problem under some certain conditions:

$$\begin{cases} F(u_n, u) + \phi(u) - \phi(u_n) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0 & \forall u \in C, \\ G(v_n, v) + \psi(v) - \psi(v_n) + \frac{1}{r_n} \langle v - v_n, v_n - y_n \rangle \geq 0 & \forall v \in Q, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) T(u_n - \rho_n A^*(Au_n - Bv_n)), \\ y_{n+1} = \alpha_n v_n + (1 - \alpha_n) S(v_n - \rho_n B^*(Au_n - Bv_n)) & \forall n \geq 1, \end{cases} \quad (1.2.5)$$

where $T : H_1 \rightarrow H_1, S : H_2 \rightarrow H_2$ are single-valued nonexpansive mappings. Also, Li and Liu [73] obtained a weak convergence result for approximating a common element of

the set of solutions of equilibrium problem and the set of common fixed point of k -strictly pseudo-nonspreading multi-valued mappings in Hilbert space.

Motivated by the works of [75] and [73], we introduce a new iterative scheme for finding a common element of the set of solutions of split equality for finite family of generalized mixed equilibrium problems and the set of common fixed points of k -strictly pseudo-nonspreading multi-valued mappings of type-one without prior knowledge of the operator norms in real Hilbert space.

Furthermore, Kassay, Reich and Sabach [58] proposed the following algorithms for solving systems of variational inequality problems corresponding to finitely many Bregman inverse strongly monotone mappings, pseudo-monotone mappings and hemi-continuous mappings:

$$\left\{ \begin{array}{l} x_0 \in C = \bigcap_{i=1}^N C_i, \\ y_n^i = T_n^i(x_n + e_n^i), \\ C_n^i = \{z \in C_i : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in C : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = Proj_{C_n \cap Q_n}^f(x_0), \quad n \geq 1, \end{array} \right. \quad (1.2.6)$$

and

$$\left\{ \begin{array}{l} x_0 \in C = \bigcap_{i=1}^N C_i, \\ y_n^i = Res_{\lambda_n^i B_i}(x_n + e_n^i), \\ C_n^i = \{z \in C_i : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in C : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = Proj_{C_n \cap Q_n}^f(x_0), \quad n \geq 1, \end{array} \right. \quad (1.2.7)$$

where $T_n^i : E \rightarrow E$ and $B_i : E \rightarrow 2^{E^*}$ are given operators for each $i = 1, 2, \dots, N$ and the sequence of errors $\{e_n\} \subset E$ satisfies $\|e_n^i\| < \epsilon$ and $\lim_{n \rightarrow \infty} e_n^i = 0$ for each $x_0 \in C$. In particular, they proved the following theorem.

Theorem 1.2.1. *Let $C_i, i = 1, 2, \dots, N$ be N nonempty, closed and convex subsets of E such that $C := \bigcap_{i=1}^N C_i$. Let $B_i : C_i \rightarrow E^*, i = 1, 2, \dots, N$ be N monotone and hemicontinuous mappings with $V := VI(C_i, B_i) \neq \emptyset$. Let $\{\lambda_n^i\}_{n \in \mathbb{N}}, i = 1, 2, \dots, N$ be N sequence of positive real numbers and satisfy $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$. Let $f : E \rightarrow \mathbb{R}$ be a Legendre function which is bounded uniformly Fréchet differentiable and totally convex on bounded subsets of E . Suppose ∇f^* is bounded on bounded subsets of E^* . If for each $i = 1, 2, \dots, N$ the sequence of errors $\{e_n^i\} \subset E$ satisfies $\lim_{n \rightarrow \infty} e_n^i = 0$, then for each $x_0 \in C$ there are sequence $\{x_n\}_{n \in \mathbb{N}}$ which satisfy (1.2.7). Each of such sequence $\{x_n\}_{n \in \mathbb{N}}$ converges strongly as $n \rightarrow \infty$ to $Proj_V^f(x_0)$.*

Motivated by the result mentioned above, we propose a new iterative algorithm for finding a common element of the set of fixed points of infinite family of quasi-Bregman nonexpansive mappings which is a common solution to a finite system of variational inequalities problems and also a solution to a finite system of convex minimization problems for convex and lower semicontinuous functions $\phi_i : C \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\text{dom} \phi_i \cap \text{dom} f \neq \emptyset, i = 1, 2, \dots, N$ in reflexive Banach space.

1.3 Objectives

The main objectives of this study are:

- i. to introduce an iterative scheme that solves the split equality generalized equilibrium problem, split equality variational inclusion problem and split equality fixed point problem for demi-contractive mappings without prior knowledge of the operator norm and also give numerical example that justify our result in a real Hilbert space.
- ii. to introduce an iterative scheme for solving the split equality generalized mixed equilibrium problem and split equality fixed point problem for k -strictly pseudo-nonspreading multi-valued mappings of type-one without prior knowledge of the operator norm and also give numerical example that justify our result in a real Hilbert space.
- iii. to generate an iterative scheme that solve the variational inequality problem, convex minimization problem and quasi-Bregman nonexpansive mapping in a real reflexive Banach space and also give applications of our result to solutions of other problems in a real reflexive Banach space.

1.4 Organization of study

This dissertation is organize as follow.

In Chapter 2, we give a background overview of some definitions and results required in achieving our results. We also give a brief survey of some classes of equilibrium problem and variational inequality problem. Further, we discuss some iterative methods for solving fixed point problems, equilibrium problems and variational inequality problems.

In Chapter 3, we introduce an iterative scheme for finding a common element of the set of solutions of split equality generalized equilibrium problem, split equality monotone variational inclusion problem and split equality fixed point problem for demi-contractive mappings in Hilbert space. We state and prove a strong convergence theorem for the sequence generated by our scheme. We also give numerical example of our main theorem in the realm of two-dimensional real Hilbert space and we use matlab version R2014a to show how the sequence values are affected by the number of iterations.

In Chapter 4, we introduce a new iterative scheme for finding a common element of the set of solutions of split equality for finite family of generalized mixed equilibrium problem and the set of common fixed point of k -strictly pseudo-nonspreading multi-valued mappings of type-one without prior knowledge of the operator norm in Hilbert space. We state and prove a strong convergence theorem for the sequence generated by our scheme and we give numerical example of our main theorem. We also use matlab version R2014a to show how

the sequence values are affected by the number of iterations.

In Chapter 5, we consider the real reflexive Banach space. We propose a new iterative scheme for finding a common element of the set of fixed points of infinite family of quasi-Bregman nonexpansive mappings which is also a common solution to a finite system of variational inequality problems and finite system of convex minimization problems for convex and lower semicontinuous functions. Using the $prox_{\lambda}^{\phi}$ operator introduced by Bauschke et al. [10], we show that this iterative scheme converges strongly to a common element of the three sets. Furthermore, we apply our results to the approximation of the common zeroes of a finite family of Bregman inverse strongly monotone operators and also to a finite system of equilibrium problems in real reflexive Banach space.

In Chapter 6, we give the conclusion, contributions to knowledge and some area of further research of our work.

In this chapter, we present some basic concepts that are relevant to this study. Throughout this dissertation, unless stated otherwise, H denotes a real Hilbert space and H^* the topological dual space of H .

2.1 Basic Definitions

We recall some basic definitions and results in functional analysis that are required for our work.

Definition 2.1.1. Let C be a nonempty, closed and convex subset of H . A function $f : C \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be

- i. convex, if for any $x, y \in C$ and $t \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),$$

- ii. lower semicontinuous on C , if for $\alpha \in \mathbb{R}$, the set $\{x \in C : f(x) \leq \alpha\}$ is closed in C ,
- iii. concave if $-f$ is convex,
- iv. upper semicontinuous on C , if $-f$ is lower semicontinuous on C .

Lemma 2.1.1. *Let C be a nonempty closed convex subset of H , then the following result holds $\forall x, y \in H$ and $t \in (0, 1]$:*

- i. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,$

$$ii. \quad \|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2.$$

Definition 2.1.2. Let X be a normed linear space. A mapping $T : X \rightarrow X$ is said to be:

- i. continuous at an arbitrary point $x_0 \in X$, if for each $\epsilon > 0$, there exist a real number $\delta > 0$ such that for $x \in X$

$$\|x - x_0\| < \delta \implies \|T(x) - T(x_0)\| \leq \epsilon, \quad (2.1.1)$$

- ii. L -Lipschitz if there exists a real constant $L > 0$ such that

$$\|T(x) - T(y)\| \leq L\|x - y\|, \quad \forall x, y \in X, \quad (2.1.2)$$

- iii. contraction if it is Lipschitz with $L \in [0, 1)$,

- iv. strict contractive if it is Lipschitz with $L \in (0, 1)$.

Definition 2.1.3. Let $T : H \rightarrow H$ be a nonlinear mapping. Then T is called

- i. monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in H, \quad (2.1.3)$$

- ii. α -strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|x - y\|^2, \quad x, y \in H, \quad (2.1.4)$$

- iii. β -inverse strongly monotone, if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \beta\|Tx - Ty\|^2, \quad \forall x, y \in H, \quad (2.1.5)$$

- iv. firmly nonexpansive, if it is β -inverse strongly monotone with $\beta = 1$.

Remark 2.1.2. It is easy to observe that every β -inverse strongly monotone mapping T is monotone and $\frac{1}{\beta}$ Lipschitz.

Definition 2.1.4. A multi-valued mapping $M : H \rightarrow 2^H$ is called monotone if for all $x, y \in H$ such that $u \in Mx$ and $v \in My$, then

$$\langle x - y, u - v \rangle \geq 0. \quad (2.1.6)$$

Definition 2.1.5. A multi-valued monotone mapping $M : H \rightarrow 2^H$ is said to be maximal if the graph of M (denoted by $G(M)$) is not properly contained in the graph of any other monotone mapping. It is known that a multi-valued mapping M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$ for every $(y, v) \in G(M)$ implies that $u \in Mx$.

We recall that a point $x \in H$ is said to be a fixed point of a mapping $T : H \rightarrow H$ if $Tx = x$ and the set of fixed points of T is denoted by $F(T)$.

Example 2.1.3.

- i. If $H = \mathbb{R}$ and $T(x) = x^2 + 9x + 16$, then $F(T) = \{-4\}$;
- ii. If $H = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$;
- iii. Given an initial value problem

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t)), \\ x(t_0) = x_0. \end{cases} \quad (2.1.7)$$

This system can be transformed into the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds.$$

To establish the existence of solution to system (2.1.7), we consider the operator $T : X \rightarrow X$ ($X := C[a, b]$) defined by

$$Tx = x_0 + \int_{t_0}^t f(s, x(s))ds.$$

Then finding a solution to the initial value problem (2.1.7) amounts to finding a fixed point of T .

Definition 2.1.6. Let H be a real Hilbert space. The mapping $T : H \rightarrow H$ is said to be:

- i. nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in H,$$

- ii. quasi-nonexpansive if, $F(T) \neq \emptyset$ and

$$\|Tx - Tp\| \leq \|x - p\|, \quad \forall x \in H, \quad p \in F(T),$$

- iii. firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \quad \forall x, y \in H, \quad (2.1.8)$$

- iv. k -strictly pseudo-contractive mapping if for $k \in [0, 1)$, we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2, \quad \forall x, y \in H,$$

- v. k demi-contractive if $F(T) \neq \emptyset$ and for $k \in [0, 1)$, we have

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \quad \forall x \in H, \quad p \in F(T). \quad (2.1.9)$$

Remark 2.1.4.

- i. It is clear that in a real Hilbert space H , (2.1.8) is equivalent to the definition of firmly nonexpansive mapping in Definition 2.1.3(iv).
- ii. Also (2.1.9) is equivalent to

$$\langle Tx - p, x - p \rangle \|x - p\|^2 \geq \frac{1 - k}{2} \|x - Tx\|^2, \quad \forall x \in H, p \in F(T).$$

We note that the following inclusions hold for the classes of the mappings:

$$\begin{aligned} \text{firmly nonexpansive} &\subset \text{nonexpansive} \subset \text{quasi nonexpansive} \subset k \text{ strictly} \\ &\text{pseudo-contractive} \subset k \text{ demi-contractive.} \end{aligned} \quad (2.1.10)$$

We illustrate these by the following examples.

Example 2.1.5. Let $X = l_\infty$ and $C := \{x \in l_\infty : \|x\|_\infty \leq 1\}$. Define $T : C \rightarrow C$ by $Tx = (0, x_1^2, x_2^2, x_3^2, \dots)$ for $x = (x_1, x_2, x_3, \dots)$ in C . Then, it is clear that T is continuous and maps C into C . Moreover, $Tp = p$ if and only if $p = 0$. Futhermore,

$$\begin{aligned} \|Tx - p\|_\infty &= \|Tx\|_\infty = \|(0, x_1^2, x_2^2, x_3^2, \dots)\|_\infty \\ &\leq \|(x_1, x_2, x_3, \dots)\|_\infty = \|x\|_\infty \\ &= \|x - p\|_\infty, \quad x \in C. \end{aligned} \quad (2.1.11)$$

Therefore, T is quasi-nonexpansive. However, T is not nonexpansive.

For if $x = (\frac{3}{4}, 0, 0, 0, \dots)$ and $y = (\frac{1}{2}, 0, 0, 0, \dots)$, it is clear that x and y belong to C . Furthermore, $\|x - y\|_\infty = \|(\frac{1}{4}, 0, 0, 0, \dots)\|_\infty = \frac{1}{4}$ and $\|Tx - Ty\|_\infty = \|(0, \frac{5}{16}, 0, 0, \dots)\|_\infty = \frac{5}{16} > \frac{1}{4} = \|x - y\|_\infty$. Thus T is quasi-nonexpansive but not nonexpansive.

The following example is a k strictly pseudo-contractive mapping which is not quasi-nonexpansive for $k \in [0, 1)$.

Example 2.1.6. Let H be the real line together with the usual norm and $C = \mathbb{R}$. Define $T : C \rightarrow C$ by

$$T(x) = -3x. \quad (2.1.12)$$

Indeed, $F(T) = \{0\}$. Thus for $x \in \mathbb{R}$, we have

$$|Tx - 0|^2 = |-3x - 0|^2 = 9|x - 0|^2 > |x - 0|^2,$$

which implies that T is not quasi-nonexpansive. Also

$$|Tx - Ty|^2 = |-3x + 3y|^2 = 9|x - y|^2,$$

and

$$|x - Tx - (y - Ty)|^2 = |x + 3x - (y + 3y)|^2 = 16|x - y|^2.$$

Thus

$$\begin{aligned} |Tx - Ty|^2 &= |x - y|^2 + 8|x - y|^2 \\ &= |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2. \end{aligned} \quad (2.1.13)$$

Hence, T is $\frac{1}{2}$ strictly pseudo-contractive mapping.

The following example is k demi-contractive mapping which is not k strictly pseudo-contractive mapping.

Example 2.1.7. [109] Let $H = \mathbb{R}$ and $C = [-1, 1]$. Define $T : C \rightarrow C$ by

$$Tx = \begin{cases} \frac{2}{3}x \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases} \quad (2.1.14)$$

Clearly $F(T) = \{0\}$. For $x \in C$, we have

$$\begin{aligned} |Tx - 0|^2 &= \left|\frac{2}{3}x \sin\left(\frac{1}{x}\right)\right|^2 \\ &\leq \left|\frac{2}{3}x\right|^2 \\ &\leq |x|^2 \\ &\leq |x - 0|^2 + k|Tx - x|^2 \quad \forall k \in [0, 1). \end{aligned}$$

Thus T is k demi-contractive for $k \in [0, 1)$. To see that T is not k strictly pseudo-contractive, choose $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, then

$$\begin{aligned} |Tx - Ty|^2 &= \left|\frac{4}{3\pi} \sin\left(\frac{\pi}{2}\right) - \frac{4}{9\pi} \sin\left(\frac{3\pi}{2}\right)\right|^2 \\ &\leq \left|\frac{4}{3\pi} + \frac{4}{9\pi}\right|^2 \\ &= \frac{256}{81\pi^2}. \end{aligned}$$

However,

$$\begin{aligned} |x - y|^2 + |x - Tx - (y - Ty)|^2 &= \left|\frac{2}{\pi} - \frac{2}{3\pi}\right|^2 + \left|\frac{2}{\pi} - \frac{4}{3\pi} \sin\left(\frac{\pi}{2}\right) - \left(\frac{2}{3\pi} - \frac{4}{9\pi} \sin\left(\frac{3\pi}{2}\right)\right)\right|^2 \\ &\leq \left|\frac{4}{3\pi}\right|^2 + \left|\frac{4}{9\pi}\right|^2 \\ &= \frac{160}{81\pi^2}. \end{aligned}$$

Thus, $|Tx - Ty|^2 > |x - y|^2 + |x - Tx - (y - Ty)|^2$. Hence T is not k strictly pseudo-contractive mapping for $k \in [0, 1)$.

2.2 Metric Projection

The theory of metric projection is very important in fixed point theory. In this section, we briefly look at the characterization of the metric projection.

Definition 2.2.1. Let C be a nonempty, closed and convex subset of H . For every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C, \quad (2.2.1)$$

where P_C is called the metric projection of H onto C .

A very important inequality that characterizes the metric projection is stated below.

Proposition 2.2.1. [9] *Let C be a nonempty closed convex subset of a Hilbert H . For arbitrary $x \in H$ and $z \in C$. Then, $z = P_C x$ if and only if*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \quad (2.2.2)$$

From Proposition 2.2.1, we deduce that:

- i. The metric projection is firmly nonexpansive, that is, for all $x, y \in H$

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle.$$

- ii. For all $x \in H$ and $y \in C$,

$$\|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2. \quad (2.2.3)$$

- iii. If C is a closed subspace, then P_C coincides with the orthogonal projection from H onto C , that is, $x - P_C x$ is orthogonal to C . Thus, for any $y \in C$,

$$\langle x - P_C x, y \rangle = 0.$$

If C is a closed convex subset with a particular simple structure, then the projection P_C has a closed form expression as describe below (see [80]):

1. If $C = \{x \in H : \|x - u\| \leq r\}$ is a closed ball centred at $u \in H$ with radius $r > 0$, then

$$P_C x = \begin{cases} u + \frac{r(x-u)}{\|x-u\|}, & \text{if } x \notin C, \\ x, & \text{if } x \in C. \end{cases}$$

2. If $C = [a, b]$ is a closed rectangle in \mathbb{R}^n , where $a = (a_1, a_2, \dots, a_n)^T$ and $b = (b_1, b_2, \dots, b_n)^T$, then for $1 \leq i \leq n$, $P_C x$ has the i^{th} coordinate given by

$$(P_C x)_i = \begin{cases} a_i, & \text{if } x_i \leq a_i, \\ x_i, & \text{if } x_i \in [a_i, b_i], \\ b_i, & \text{if } x_i > b_i. \end{cases}$$

3. If $C = \{y \in H : \langle a, y \rangle = \alpha\}$ is a hyperplane with $a \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_C x = x - \frac{\langle a, x \rangle - \alpha}{\|a\|^2} a.$$

4. If $C = \{y \in H : \langle a, y \rangle \leq \alpha\}$ is a closed halfspace, with $a \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_C x = \begin{cases} x - \frac{\langle a, x \rangle - \alpha}{\|a\|^2} a, & \text{if } \langle a, x \rangle > \alpha, \\ x, & \text{if } \langle a, x \rangle \leq \alpha. \end{cases}$$

5. If C is the range of a $m \times n$ matrix A with full column rank, then

$$P_C x = A(A^*A)^{-1}A^*x,$$

where A^* is the adjoint of A .

2.3 Some Notions on Geometric Properties of Banach Spaces

We recall that a Banach space E is a complete normed vector space.

Example 2.3.1.

i. The space $l_p(\mathbb{R})$ defined by

$$l_p(\mathbb{R}) = \{\bar{x} = (x_1, x_2, x_3, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}, \quad (2.3.1)$$

together with norm $\|\cdot\|_p : l_p(\mathbb{R}) \rightarrow [0, \infty)$, defined by

$$\|\bar{x}\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p},$$

is a Banach space for $1 < p < \infty$.

ii. The space $l_\infty(\mathbb{R})$ defined by

$$l_\infty(\mathbb{R}) := \{\bar{x} = (x_1, x_2, x_3, \dots), x_i \in \mathbb{R} : \bar{x} \text{ is bounded}\}, \quad (2.3.2)$$

together with the function $\|\cdot\|_\infty : l_\infty(\mathbb{R}) \rightarrow [0, \infty)$, defined by

$$\|\bar{x}\|_\infty = \sup_{i \geq 1} |x_i|,$$

is a Banach space.

- ii. The space $C[a, b]$ of all real-valued continuous function on $[a, b]$ together with the function $\|\cdot\| : C[a, b] \rightarrow \mathbb{R}$ defined by

$$\|f\|_2 = \left(\int_a^b |f(t)|^2 dt \right)^{1/2},$$

is not complete and thus, not a Banach space.

Definition 2.3.1. A Banach space E is called uniformly convex if given any $\epsilon > 0$, there exist $\delta > 0$ such that for all $x, y \in E$ satisfying $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, we have

$$\frac{\|x + y\|}{2} < 1 - \delta.$$

Proposition 2.3.2. [41] *The l_p spaces are uniformly convex for $1 < p < \infty$.*

Definition 2.3.2. A normed linear space X is called strictly convex if for all $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\alpha x + (1 - \alpha)y\| < 1$, for all $\alpha \in (0, 1)$.

Proposition 2.3.3. [35] *Every uniformly convex space is strictly convex.*

Remark 2.3.4. The space l_∞ is not strictly convex. To see this, if we consider $\bar{u} = (1, 1, 0, 0, 0, \dots)$ and $\bar{v} = (-1, 1, 0, 0, 0, \dots)$. Both $\bar{u}, \bar{v} \in l_\infty$. Taking $\epsilon = 1$, then $\|\bar{u}\|_\infty = 1 = \|\bar{v}\|_\infty$ and $\|\bar{u} - \bar{v}\|_\infty = 2 > \epsilon$. However, $\|\frac{\bar{u} + \bar{v}}{2}\|_\infty = 1$. Thus l_∞ is not strictly convex.

Definition 2.3.3. Let E be a real Banach space. The space E^* of all linear continuous functionals on E is called the dual space of E . For $f \in E^*$ and $x \in E$, the value of f at x is denoted by $\langle f, x \rangle$.

Remark 2.3.5.

1. The dual E^* is a Banach space with respect to the norm

$$\|f\|_{E^*} = \sup\{\langle f, x \rangle : \|x\| \leq 1\}.$$

2. The dual space of E^* is E^{**} , the bidual space of E . Since, in general, $E \subseteq E^{**}$, we say that E is reflexive if $E = E^{**}$.
3. A uniformly convex Banach space is strictly convex and reflexive. The concept of uniformly convex and strictly convex Banach spaces are equivalent in finite dimensional spaces.

Definition 2.3.4. Let E^* be the dual space of a real Banach space. The multi-valued mapping $J : E \rightarrow 2^{E^*}$ defined by

$$Jx = \{f \in E^* : \langle f, x \rangle = \|x\| \cdot \|f\|, \|x\| = \|f\|\} \quad (2.3.3)$$

is called the normalized duality mapping of E .

Remark 2.3.6.

1. It is well known that if E^* is strictly convex, then J is single-valued. We shall denote this by j in the sequel.
2. For reflexive Banach spaces, the assumption on strict convexity is not an essential restriction, since E and E^* can be equivalently re-normed as strictly convex spaces such that the duality mapping is preserved.

In a Banach space E , beside the strong convergence defined by the norm, that is, $\{x_n\} \subset E$ converges strongly to a if and only if $\|x_n - a\| \rightarrow 0$, as $n \rightarrow \infty$, we shall often consider the weak convergence, corresponding to the weak topology in E . We say that $\{x_n\} \subset E$ converges weakly to a if for any $f \in E^*$,

$$\langle f, x_n \rangle \rightarrow \langle f, a \rangle, \quad n \rightarrow \infty. \quad (2.3.4)$$

We shall denote by $x_n \rightharpoonup x$ and $x_n \rightarrow x$, the weak and the strong convergence of $\{x_n\}$ to x respectively.

Remark 2.3.7. Every weak convergence sequence $\{x_n\}$ in a Banach space is bounded. Further, if $x_n \rightharpoonup a$, then $\|a\| \leq \liminf \|x_n\|$.

Definition 2.3.5. A Banach space E is called smooth if for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = \langle f, x \rangle = 1$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(t) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in E, \|x\| = 1, \|y\| = t \right\}. \quad (2.3.5)$$

The Banach space E is called uniformly smooth if

$$\lim_{t \rightarrow 0} \frac{\rho_E(t)}{t} = 0, \quad (2.3.6)$$

see [14]. Henceforth, E denotes a real reflexive Banach space with the dual space E^* and C a nonempty closed convex subset of E . We shall also denote the value of the functional $y^* \in E^*$ at $x \in E$ by $\langle y^*, x \rangle$ and assume that the mapping $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. We also denote the domain of f by $\text{dom} f$, where $\text{dom} f = \{x \in E : f(x) < \infty\}$.

Definition 2.3.6. Let $x \in \text{int}(\text{dom} f)$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \forall y \in E\}$$

and the Fréchet conjugate of f is the function $f^* : E^* \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(y^*) = \sup \{\langle y^*, x \rangle - f(x) : x \in E\}.$$

Definition 2.3.7. Let $x \in \text{int}(\text{dom}f)$, for any $y \in E$, the directional derivative of f at x is defined by

$$f^\circ(x, y) := \lim_{t \rightarrow 0^+} \frac{f(x + ty) - f(x)}{t}. \quad (2.3.7)$$

If the limit in (2.3.7) exists as $t \rightarrow 0$ for each y , then the function f is said to be Gâteaux differentiable at x . In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y \rangle := f^\circ(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \text{int}(\text{dom}f)$. When the limit as $t \rightarrow 0$ in (2.3.7) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is Fréchet differentiable at x . It is well known that f is Gâteaux (resp. Fréchet) differentiable at $x \in \text{int}(\text{dom}f)$ if and only if the gradient ∇f is norm-to-weak* (resp. norm-to-norm) continuous at x (see [11]).

Definition 2.3.8. The function f is called Legendre if it satisfies the following two conditions:

- (C1) the function f is Gâteaux differentiable, $\text{int}(\text{dom} f) \neq \emptyset$ and $\text{dom} \nabla f = \text{int}(\text{dom} f)$,
- (C2) the function f^* is Gâteaux differentiable, $\text{int}(\text{dom} f^*) \neq \emptyset$ and $\text{dom} \nabla f^* = \text{int}(\text{dom} f^*)$.

The notion of Legendre function in infinite dimensional spaces was first introduced by Bauschke, Borwein and Combettes in [11]. Their definition is equivalent to conditions (C1) and (C2) because the space E is assumed to be reflexive (see [11], Theorem 5.4 and 5.6, p. 634). It is also well known that in reflexive Banach space $\nabla f = (\nabla f^*)^{-1}$ (see [16], p. 83). When this fact is combined with conditions (C1) and (C2), we obtain

$$\begin{aligned} \text{ran} \nabla f &= \text{dom} \nabla f^* = \text{int}(\text{dom} f)^*, \\ \text{ran} \nabla f^* &= \text{dom} \nabla f = \text{int}(\text{dom} f). \end{aligned}$$

It also follows that f is Legendre if and only if f^* is Legendre (see [11], Corollary 5.5, p. 634) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains. When the Banach space E is smooth and strictly convex, in particular, a Hilbert space, the function $\frac{1}{p} \|\cdot\|^p$ with $p \in (1, \infty)$ is Legendre (cf. [8], Lemma 6.2, p. 639). For further details on Legendre functions, see, [8, 11].

Definition 2.3.9. Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom}f \times \text{int}(\text{dom}f) \rightarrow [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \quad (2.3.8)$$

is called the Bregman distance with respect to f , (see [18, 34]).

The Bregman distance does not satisfy the well-known properties of a metric, but it has the following important property which is called the three point identity: for any $x \in \text{dom}f$ and $y, z \in \text{int}(\text{dom}f)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle. \quad (2.3.9)$$

Definition 2.3.10. Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function f is called:

- i. totally convex at x if its modulus of total convexity at $x \in \text{int}(\text{dom} f)$, that is, the bifunction $v_f : \text{int}(\text{dom} f) \times [0, +\infty) \rightarrow [0, +\infty)$, defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom} f, \|y - x\| = t\}, \quad (2.3.10)$$

is positive for any $t > 0$,

- ii. totally convex if it is totally convex at every point $x \in \text{int}(\text{dom} f)$,
- iii. totally convex on bounded subset B of E , if $v_f(B, t)$ is positive for any nonempty bounded subset B , where the function $v_f : \text{int}(\text{dom} f) \times [0, +\infty) \rightarrow [0, +\infty)$ is defined by

$$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap \text{int}(\text{dom} f)\}, \quad t > 0. \quad (2.3.11)$$

For further details and examples on totally convex functions, see [17, 26, 29].

Definition 2.3.11. [26, 103] Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function f is called:

- i. cofinite if $\text{dom} f^* = E^*$,
- ii. coercive if $\lim_{\|x\| \rightarrow +\infty} \left| \frac{f(x)}{\|x\|} \right| = +\infty$,
- iii. sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n \rightarrow \infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (2.3.12)$$

Definition 2.3.12. Let $T : C \rightarrow C$ be a mapping, a point $x^* \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to x^* and $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Definition 2.3.13. Let C be a nonempty, closed and convex subset of E . A mapping $T : C \rightarrow \text{int}(\text{dom} f)$ is called

- i. Bregman Firmly Nonexpansive (BFNE for short) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \quad \forall x, y \in C. \quad (2.3.13)$$

- ii. Bregman Strongly Nonexpansive (BSNE) with respect to a nonempty $\hat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x), \quad (2.3.14)$$

for all $p \in \hat{F}(T)$ and $x \in C$ and if whenever $\{x_n\}_{n=1}^{\infty} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n \rightarrow \infty} \left(D_f(p, x_n) - D_f(p, Tx_n) \right) = 0,$$

it follows that

$$\lim_{n \rightarrow \infty} D_f(Tx_n, x_n) = 0.$$

iii. Quasi-Bregman Nonexpansive (QBNE) if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x) \quad \forall x \in C, p \in F(T). \quad (2.3.15)$$

From the Definition 2.3.9, it is clear that (2.3.13) is equivalent to

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x). \quad (2.3.16)$$

It was remarked in [58] that in the case where $\hat{F}(T) = F(T)$, the following inclusion holds

$$BFNE \subset BSNE \subset QBNE. \quad (2.3.17)$$

2.4 Equilibrium and Variational Inequality Problems

In this section, we give brief survey of some classes of equilibrium and variational inequality problems. Throughout this section, C is a nonempty closed and convex subset of a Hilbert space H .

2.4.1 Equilibrium problem

In 1994, Blum and Oettli [15] introduced the following abstract Equilibrium Problem (in short EP). Given a bifunction $F : C \times C \rightarrow \mathbb{R}$, the EP is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \quad (2.4.1)$$

In solving the EP, it is assumed that the bifunction F satisfied the following:

- L1. $F(x, x) = 0 \quad \forall x \in C$,
- L2. F is monotone, i.e $F(x, y) + F(y, x) \leq 0 \quad \forall x, y \in C$,
- L3. $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ for each $x, y, z \in C$,
- L4. for each $x \in C$, the function $y \mapsto F(x, y)$ is convex, lower semicontinuous.

The set of solution to (2.4.1) is denoted by $EP(F)$.

Generalized Equilibrium Problem (GEP)

In 1999, Moudafi and Théra [87] introduced the GEP which is to find $x \in C$ such that :

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad y \in C, \quad (2.4.2)$$

where $A : C \rightarrow H$ is a nonlinear mapping. The set of solutions to (2.4.2) is denoted by $GEP(F)$.

The EP and GEP have potential and useful applications in nonlinear analysis and mathematical economics as seen below (see Blum and Oettli [15]):

- (1) *Optimization*: Let $\phi : C \rightarrow \mathbb{R}$ be a convex and lower semi-continuous function, the minimization problem is to find $\bar{x} \in C$ such that

$$\phi(\bar{x}) \leq \phi(y) \quad \forall y \in C. \quad (2.4.3)$$

Setting $F(x, y) := \phi(y) - \phi(x)$, problem (2.4.3) coincides with (2.4.1). The function F is monotone in this case.

- (2) *Saddle point problem*: Let $\varphi : C_1 \times C_2 \rightarrow \mathbb{R}$. Then $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is called a saddle point of the function φ if and only if for $(\bar{x}_1, \bar{x}_2) \in C_1 \times C_2$,

$$\varphi(\bar{x}_1, y_2) \leq \varphi(y_1, \bar{x}_2) \quad \forall (y_1, y_2) \in C_1 \times C_2. \quad (2.4.4)$$

Setting $C = C_1 \times C_2$ and define $F : C \times C \rightarrow \mathbb{R}$ by

$$F\left((x_1, x_2), (y_1, y_2)\right) := \varphi(y_1, x_2) - \varphi(x_1, y_2).$$

Then $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of (2.4.1) if and only if (\bar{x}_1, \bar{x}_2) satisfies (2.4.4). F is monotone in this case.

- (3) *Nash equilibria in non-cooperative game*: Let I be a finite index set (the set of players). For every $i \in I$, let there be given a set C_i (the strategy set of the i -th player). Let $C := \prod_{i \in I} C_i$. For every $i \in I$, let there be given a function $f_i : C \rightarrow \mathbb{R}$ (the loss function of the i -th player depending on the strategies of all players). For $x = (x_i)_{i \in I} \in C$, we define $x^i = (x_j)_{j \in I, j \neq i}$. The point $\bar{x} = (\bar{x}_i)_{i \in I} \in C$ is called Nash equilibrium if and only if for all $i \in I$, there holds

$$f_i(\bar{x}_i) \leq f_i(\bar{x}^i, y_i) \quad \forall y_i \in C_i, \quad (2.4.5)$$

(that is, no player can reduce his loss by varying his strategy alone).

Define $F : C \times C \rightarrow \mathbb{R}$ by

$$F(x, y) := \sum_{i \in I} \left(f_i(x^i, y_i) - f_i(x) \right).$$

Then $x \in C$ is a Nash equilibrium if and only if x fulfills (2.4.1). Indeed: If (2.4.5) holds for all $i \in I$, then it is obvious that (2.4.1) is fulfilled. If for some $i \in I$, we choose $y \in C$ in such a way that $\bar{x}^i = y^i$. Then

$$F(\bar{x}, y) = f_i(\bar{x}^i, y_i) - f_i(\bar{x}).$$

Hence, (2.4.1) implies (2.4.5) for all $i \in I$. F in this case is not automatically monotone.

- (4) *Fixed Point Problem (FPP)*: Let $T : C \rightarrow C$ be a given mapping. The fixed point problem is to find $x \in C$ such that

$$x = Tx. \quad (2.4.6)$$

Setting $F(x, y) = \langle x - Tx, y - x \rangle$. Then x solves (2.4.6) if and only if x is a solution of (2.4.1). Indeed: (2.4.6) \Rightarrow (2.4.1) is obvious. And if (2.4.1) is satisfied, then choose $\bar{y} = T\bar{x}$ to obtain

$$0 \leq F(\bar{x}, \bar{y}) = -\|\bar{x} - T\bar{x}\|^2, \quad (2.4.7)$$

hence, $\bar{x} = T\bar{x}$. So (2.4.1) \Rightarrow (2.4.6). In this case F is monotone if and only if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 \quad \forall x, y \in C,$$

hence in particular if T is nonexpansive.

- (5) *Convex differentiable optimization*: Besides the straightforward connection between optima and equilibria given in (1), there is a more subtle connection in the convex differentiable case. Let $\Phi : C \rightarrow \mathbb{R}$ be convex and Gâteaux differentiable, with Gâteaux differential $D_\Phi(x) \in H^*$ at x . Consider the problem

$$\min\{\Phi(x) : x \in C\}. \quad (2.4.8)$$

It is well known from convex analysis that \bar{x} is a solution of (2.4.8) if and only if \bar{x} satisfy the variational inequality

$$\bar{x} \in C, \quad \langle D_\Phi(\bar{x}), y \rangle \geq 0 \quad \forall y \in C.$$

Upon setting $F(x, y) := \langle D_\Phi(x), y - x \rangle$ this becomes an example of our equilibrium problem (2.4.1). The function F is monotone in this case, since the mapping $x \mapsto D_\Phi(x)$ is monotone, i.e

$$\langle D_\Phi(y) - D_\Phi(x), y - x \rangle \geq 0 \quad \forall x, y \in C.$$

- (6) *Variational operator inequalities*: Let $E : C \rightarrow H^*$ be a given mapping. It is required to find $\bar{x} \in H$ such that

$$\bar{x} \in C, \quad \langle E\bar{x}, y - \bar{x} \rangle \geq 0 \quad \forall y \in C. \quad (2.4.9)$$

We set $F(x, y) := \langle Ex, y - x \rangle$. Then clearly (2.4.9) \iff (2.4.1).

- (7) *Complementarity problems*: This is a special case of the previous example (6). Let C be a closed convex cone with $C^* := \{x^* \in H^* : \langle x^*, y \rangle \geq 0 \quad \forall y \in C\}$ denoting its polar cone. Let $A : C \rightarrow H^*$ be a given mapping. It is required to find $\bar{x} \in H$ such that

$$\bar{x} \in C, \quad A\bar{x} \in C^*, \quad \langle A\bar{x}, \bar{x} \rangle \geq 0. \quad (2.4.10)$$

It is easily seen that (2.4.10) is equivalent with (2.4.9). Obviously, (2.4.10) \Rightarrow (2.4.9). If (2.4.9) holds, then setting in turn $y := 2\bar{x}$ and $y := 0$, we obtain from (2.4.9) that $\langle A\bar{x}, \bar{x} \rangle = 0$ and thereby, $\langle A\bar{x}, y \rangle \geq 0, \quad \forall y \in C$. Hence (2.4.9) \implies (2.4.10).

Mixed Equilibrium Problem(MEP)

In 2008, Ceng and Yao [32] studied the MEP which is to find $x \in C$, such that

$$F(x, y) + \phi(y) - \phi(x) \geq 0 \quad \forall y \in C, \quad (2.4.11)$$

where $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ is a nonlinear functional. The set of solutions of the MEP is denoted by $\text{MEP}(F, \phi)$.

Generalized Mixed Equilibrium Problem(GEMEP)

Also, Peng and Yao [97] studied the GEMEP which is to find $x \in C$, such that

$$F(x, y) + \langle Ax, y - x \rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C, \quad (2.4.12)$$

where $A : C \rightarrow H$ is a nonlinear mapping and $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ is a nonlinear functional. The set of solutions of GEMEP is denoted by $\text{GEMEP}(F, A, \phi)$.

2.4.2 Variational inequality problem

In 1967, Lions and Stampacchia [72] studied the following problem: For a given $f \in H^*$, find $x \in C$ such that

$$\phi(x, y - x) \geq \langle f, y - x \rangle \quad \forall y \in C, \quad (2.4.13)$$

where $\phi(\cdot, \cdot) : H \times H \rightarrow \mathbb{R}$ is a bilinear form. The inequality (2.4.13) is termed as variational inequality which characterizes the classical Signorini problem of electro-statics, that is, the analysis of a linear elastic body in contact with a rigid frictionless foundation. If the bilinear form is continuous, then by Riesz - Fréchet theorem, we have

$$\phi(x, y) = \langle Ax, y \rangle \quad \forall x, y \in H, \quad (2.4.14)$$

where $A : H \rightarrow H^*$ is a continuous linear operator. Then Problem 2.4.13 is equivalent to the following problem: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq \langle f, y - x \rangle \quad \forall y \in C. \quad (2.4.15)$$

If $f \equiv 0 \in H^*$, then (2.4.15) reduces to the following classical Variational Inequality Problem (VIP) studied by Hartmann and Stampachia [50]: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad y \in C, \quad (2.4.16)$$

where $A : C \rightarrow H$ is a nonlinear mapping. The set of solutions of the VIP is denoted by $\text{VIP}(C, A)$. In many important applications, the convex set C also depends implicitly on the solution of the VIP.

Mixed Variational Inequality Problem

In 2001, Noor [91] consider the following mixed variational inequality problem: Find $x \in C$ such that

$$\langle Ax, y - x \rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C, \quad (2.4.17)$$

where $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ is a nonlinear functional.

Monotone Variational Inclusion Problem(MVIP)

The monotone variational inclusion problem is to find $x \in H$ such that

$$0 \in f(x) + T(x), \quad (2.4.18)$$

where 0 is the zero vector in H , $f : H \rightarrow H$ is a single-valued nonlinear mapping and $T : H \rightarrow 2^H$ is a set-valued mapping. The set of solutions to the MVIP (2.4.18) is denoted by $I(f, T)$ and note that for $f = 0$ in (2.4.18), we obtain the Variation Inclusion Problem. The MVIP generalizes the classical variational inequality problem and the zero problem for nonlinear mapping. For more information on MVIP and VIP (see, [25, 48, 61, 71, 83, 91, 92, 93, 94, 95, 110, 111] and the references therein).

2.5 Iterative Methods

In this section, we give a brief survey of some iterative methods for solving fixed point problems, variational inequalities and equilibrium problems.

2.5.1 Picard iteration

The Banach contraction mapping principle whose short form was given in Section 1.1 will be reformulated here in its complete form.

Lemma 2.5.1. [14] *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a contraction mapping satisfying*

$$d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in X, \quad (2.5.1)$$

with $k \in [0, 1)$ fixed. Then:

- a. T has a unique fixed point;
- b. The Picard iteration associated to T , i.e, the sequence $\{x_n\}$ defined by

$$x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, \dots \quad (2.5.2)$$

converges to x^* , for any initial guess $x_0 \in X$;

- c. The following a priori and a posteriori error estimates holds:

$$d(x_n, x^*) \leq \frac{k^n}{1-k} \cdot d(x_0, x_1), \quad n = 0, 1, 2, \dots \quad (2.5.3)$$

$$d(x_n, x^*) \leq \frac{k}{1-k} \cdot d(x_{n-1}, x_n), \quad n = 0, 1, 2, \dots \quad (2.5.4)$$

d. The rate of convergence is given by

$$d(x_n, x^*) \leq k \cdot d(x_{n-1}, x^*) \leq k^n d(x_0, x^*), \quad n = 1, 2, \dots \quad (2.5.5)$$

Remark 2.5.2. [14]

- i. The a priori estimate (2.5.3) shows that, when starting from an initial guess $x_0 \in X$, the approximation error of the n^{th} iterate is completely determined by the contraction coefficient k and the initial displacement $d(x_0, x_1)$.
- ii. Similarly, the a posteriori estimate (2.5.4) shows that, in order to obtain the desired error approximation of the fixed point by means of Picard iteration, that is, to have $d(x_n, x^*) < \epsilon$, we need to stop the iterative process at the first n for which the displacement between the two consecutive iterates is at most $\frac{(1-k)\epsilon}{k}$.
So, the a posteriori estimation offers a direct stopping criterion for the iterative approximation of fixed point by Picard iteration.

By slightly weakening the contraction condition in Lemma 2.5.1, the Picard iterations need not converge to a fixed point of the operator T as seen in the following example.

Example 2.5.3. [14] If $X = [1, \infty)$ and let $T : X \rightarrow X$ be defined by

$$T(x) = x + \frac{1}{x}$$

then:

- i. T is not a contraction mapping,
- ii. T is strict contractive,
- iii. $F(T) = \emptyset$,
- iv. The Picard iteration associated to T does not converge for any $x_0 \in [1, \infty)$. Indeed, if the Picard iteration $\{x_n\}$, $x_{n+1} = x_n + \frac{1}{x_n}$, $n \geq 0$ would be convergent, then its limit l would satisfy $\frac{1}{l} = 0$ which is not possible.

2.5.2 Krasnoselskij iteration

If the Picard iteration formula (2.5.2) is replaced by the following formula: For $x_0 \in C$,

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n) \quad n \geq 0, \quad (2.5.6)$$

then, the iterative sequence converges to the unique fixed point. In general, if X is a normed linear space and T is a nonexpansive mapping, the following generalization of

(2.5.6) which has proved successful in the approximation of a fixed point of T (when it exists) was given by Schaefer [107]: For $x_0 \in C$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n \geq 0, \quad (2.5.7)$$

and $\lambda \in (0, 1)$. This formula is known as the Krasnoselskij iteration.

Remark 2.5.4.

- i. It is easy to see that the Krasnoselskij iteration $\{x_n\}$ given by (2.5.7) is exactly the Picard iteration corresponding to the averaged operator

$$T_\lambda = (1 - \lambda)I + \lambda \cdot T, \quad (2.5.8)$$

where I is the identity operator.

- ii. For $\lambda = 1$, the Krasnoselskij iteration reduces to Picard iteration.

2.5.3 Mann iteration

The most general iterative formula for approximation of fixed points of nonexpansive mapping called the Mann iteration formula due to Mann [79] is the following: For $x_0 \in C$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \quad (2.5.9)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

- i. $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- ii. $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Remark 2.5.5. If the sequence $\{\alpha_n\} = \{\lambda\}$, then the Mann iteration process obviously reduces to the Krasnoselskij iteration.

2.5.4 Ishikawa iteration

In 1974, Ishikawa [54] enlarged and improved the Mann iterative algorithm to a new iterative algorithm which generates the sequence $\{x_n\}$ defined by: For $x_0 \in C$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n], \quad (2.5.10)$$

where

- i. $0 \leq \alpha_n \leq \beta_n \leq 1$,

ii. $\lim_{n \rightarrow \infty} \beta_n = 0,$

iii. $\sum_{n \geq 1} \alpha_n \beta_n = \infty.$

If we write (2.5.10) in a system form as:

$$\begin{cases} y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \end{cases} \quad (2.5.11)$$

then, we can regard the Ishikawa iteration as a two-step Mann iteration with two different parameter sequences.

In the last three decades, both the Mann and the Ishikawa schemes have been successfully used by various authors to approximate fixed points of various classes of operators in Banach spaces. As a matter of fact, the Mann iteration may fail to converge while the Ishikawa iteration can still converge for a Lipschitz pseudo-contractive mapping in Hilbert space. To obtain strong convergence of Mann iteration to a fixed point of k -strictly pseudo-contractive maps, additional conditions (such as compactness) are required on the operator T or the subset C .

2.5.5 Implicit iteration

An iterative method for solving the problem of approximating a fixed point of a mapping T which may have multiple solutions is to replace it by a family of perturbed problems admitting a unique solution, and then to get a particular solution as the limit of these perturbed solutions as the perturbation vanishes. For example, given a nonempty closed and convex set $C \subseteq H$, $T : C \rightarrow C$, $u \in C$ and $t \in (0, 1)$, Browder [21, 22, 23] studied the approximating curve $\{z_t\}$ defined by

$$z_t = tu + (1 - t)Tz_t, \quad (2.5.12)$$

that is, z_t is the unique fixed point of the contraction $tu + (1 - t)T$. He proved that if the underlying space H is Hilbert, $\{z_t\}$ converges strongly to the fixed point of T closest to u as $t \rightarrow 0$.

2.5.6 Halpern explicit iteration

Halpern [49] introduced the explicit iterative algorithm which generates a sequence via the recursive formula

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T x_n, \quad n \geq 0, \quad (2.5.13)$$

where the initial guess $x_0 \in C$ and $u \in C$ are arbitrarily fixed and the sequence $\{\alpha_n\}$ is contained in $(0, 1)$, for finding a fixed point of a nonexpansive mapping $T : C \rightarrow C$ with $F(T) \neq \emptyset$. This iterative method is commonly known as the Halpern iteration.

2.5.7 Viscosity approximation method

In [85], Moudafi presented the following implicit and explicit recursions as generalization of the results of Browder [21] and Halpern [49]:

$$z_t = tf(x_t) + (1 - t)Tx_t \quad (2.5.14)$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad (2.5.15)$$

where $\{\alpha_n\} \subset [0, 1]$, $t \in (0, 1]$, $f : C \rightarrow C$ is a contraction mapping and $T : C \rightarrow C$ is a nonexpansive mapping. He proved that if the set of fixed point of $T \cap F(T)$ is not empty, the recursions (2.5.14) and (2.5.15) converge strongly to the fixed point of T which solve the variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0 \quad \forall x \in F(T), \quad (2.5.16)$$

where I is the identity operator. This method has been developed and generalized by Takahashi and Takahashi [115] and Xu [120].

2.5.8 Hybrid iteration

The hybrid iterative method is also known as the outer-approximation method. This type of algorithm was introduced by Haugazeau [51] in 1968 and was successfully generalized and extended by Bauschke and Combettes [12], Combettes [37], Nakajo and Takahashi [88], Kikkawa and Takahashi [63]. In 2004, Nakajo and Takahashi [88] introduced and studied the following iterative method for a nonexpansive mapping T over a Hilbert space:

$$\begin{cases} x_0 & = x \in C \subseteq H, \\ w_n & = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n & = \{z \in C : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n & = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} & = P_{C_n \cap Q_n} x. \end{cases} \quad (2.5.17)$$

They proved that the sequence $\{x_n\}$ generated by (2.5.17) converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ denotes the metric projection from H onto $F(T)$.

The classical VIP that was introduced by Lions and Stampacchia [72] can be re-written as a fixed point problem of the form: find $x \in C$ such that

$$x = P_C(I - \lambda T)x, \quad (2.5.18)$$

where $\lambda > 0$ and I is the identity mapping. Using this fixed point formulation, we can have an iterative algorithm which generates the sequence $\{x_n\}$ given by

$$x_{n+1} = P_C(I - \lambda T)x_n, \quad (2.5.19)$$

where $x_0 \in C$ is given and $\lambda > 0$, (see, [6] and [47]).

There are several iterative methods that have been developed for approximating solutions of the fixed point problem for nonlinear mappings, variational inequality problem and equilibrium problem. It is of further interest to develop and study iterative methods for approximating common element of the set of solutions of these problems. In this direction, Takahashi and Toyoda [116] in 2003, considered the problem of finding a common element of the set solutions of a fixed point problem for nonexpansive mapping T on C and variational inequality with α -inverse strongly monotone mappings and developed the following algorithm:

$$\begin{cases} x_0 \in C; \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(x_n - \lambda_n Ax_n), \end{cases} \quad (2.5.20)$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequence of real numbers. They proved that under certain appropriate conditions on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (2.5.20) converges strongly to $z \in F(T) \cap VIP(C, A)$.

On the other hand, Takahashi and Takahashi [115] in 2007, proposed the following iterative scheme for approximating the common element of the set of solutions of EP and fixed point problem for a nonexpansive mapping T in Hilbert space:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \end{cases} \quad (2.5.21)$$

where $f : C \rightarrow C$ is a contraction mapping and $\{\alpha_n\} \subset [0, 1)$, $r_n > 0$. They proved that under some suitable conditions on $\{\alpha_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ and $\{u_n\}$ generated by (2.5.21) converges strongly to $z \in F(T) \cap EP(F)$, where $z = P_{F(T) \cap EP(F)}f(z)$.

For more related works on iterative methods for approximating common solutions of fixed point problem, variational inequality problem and equilibrium problem, see [1, 36, 38, 57, 60, 78, 100, 121] and reference therein.

We now state a very important result which will be used to established the strong convergence of our iterative schemes introduce in this dissertation.

Lemma 2.5.6. [119] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n\delta_n \quad \forall n \geq 0,$$

where $\{t_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathbb{R} such that:

- i. $\sum_{n=0}^{\infty} t_n = \infty$,
- ii. $\limsup_{n \rightarrow \infty} \delta_n \leq 0$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Approximation of Common Solution of Split Equalities for Generalized Equilibrium Problem, Monotone Variational Inclusion Problem and Fixed Point Problem in Hilbert Spaces

In this chapter, we introduce an iterative algorithm for finding a common element of the set of solutions of split equality generalized equilibrium problem, split equality monotone variational inclusion problem and split equality fixed point problem for k demi-contractive mapping without a prior knowledge of the operator norm in a real Hilbert space. We obtain a strong convergence result and give numerical example of our result in two-dimensional real Hilbert space.

3.1 Introduction

In 2011, Moudafi [84] introduced the following Split Monotone Variational Inclusion Problem (SMVIP): Find $x \in H_1$ such that

$$0 \in f_1(x) + B_1(x),$$

and

$$y = Ax \in H_2 \quad \text{solves} \quad 0 \in f_2(y) + B_2(y),$$

where $f_1 : H_1 \rightarrow H_1$ and $f_2 : H_2 \rightarrow H_2$ are given single-valued operators, $A : H_1 \rightarrow H_2$ is a bounded linear operator, $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ are multi-valued maximal monotone mappings. He introduced an iterative method for solving SMVIP which can be seen as an important generalization of an iterative method by Censor et al. [33] for solving split variational inequality problem.

Let H_1 , H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be nonlinear mappings. Assume $F : C \times C \rightarrow \mathbb{R}$ and $G : Q \times Q \rightarrow \mathbb{R}$ are nonlinear bifunctions, $\phi : C \rightarrow H_1$, and $\psi : Q \rightarrow H_2$ are nonlinear mappings, $f_1 : C \rightarrow H_1$ and $f_2 : Q \rightarrow H_2$ are inverse strongly monotone mappings and $T_1 : H_1 \rightarrow 2^{H_1}$, $T_2 : H_2 \rightarrow 2^{H_2}$ are maximal monotone mappings. Let $S_1 : C \rightarrow H_1$ and $S_2 : Q \rightarrow H_2$ be demi-contractive mappings. We define the following:

- I. *The Split Equality Monotone Variational Inclusion Problem* (SEMVIP) is to find $x^* \in C$ and $y^* \in Q$ such that

$$0 \in f_1(x^*) + T_1(x^*), \quad (3.1.1)$$

$$0 \in f_2(y^*) + T_2(y^*), \quad (3.1.2)$$

and

$$Ax^* = By^*.$$

If we consider (3.1.1) and (3.1.2) separately, we have that (3.1.1) is a MVIP with its solution set $I(f_1, T_1)$ and (3.1.2) is a MVIP with its solution set $I(f_2, T_2)$.

- II. *The Split Equality Generalized Equilibrium Problem* (SEGEP) is to find $x^* \in C$ and $y^* \in Q$ such that

$$F(x^*, x) + \langle \phi x^*, x - x^* \rangle \geq 0, \quad (3.1.3)$$

$$G(y^*, y) + \langle \psi y^*, y - y^* \rangle \geq 0, \quad (3.1.4)$$

and

$$Ax^* = By^*.$$

If we also consider (3.1.3) and (3.1.4) separately, we have that (3.1.3) is a GEP with its solution set $EP(F, \phi)$ and (3.1.4) is a GEP with its solution set $EP(G, \psi)$.

- III. *The Split Equality Fixed Point Problem* (SEFPP) is to find $x^* \in C$ and y^* such that

$$S_1 x^* = x^*, \quad (3.1.5)$$

$$S_2 y^* = y^*, \quad (3.1.6)$$

and

$$Ax^* = By^*.$$

If we consider (3.1.5) and (3.1.6) separately, we have that (3.1.5) is a FPP with its solution set $F(S_1)$ and (3.1.6) is a FPP with its solution set $F(S_2)$.

3.2 Preliminaries

In this section, we present some lemmas which are needed for our result in this chapter. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively.

Lemma 3.2.1. [52] (*Demiclosedness principle*) *Let C be a nonempty, closed and convex subset of H and $S : C \rightarrow C$ be a k demi-contractive mapping with $F(S) \neq \emptyset$, then $I - S$ is demiclosed at 0, i.e if $x_n \rightharpoonup x^* \in C$ and $x_n - Sx_n \rightarrow 0$, then $x^* = Sx^*$.*

For solving the equilibrium problem, we assume that the bifunction $F : C \times C \rightarrow \mathbb{R}$ satisfies the assumptions L1 - L4 in Section 2.4.1.

Lemma 3.2.2. [15] *Let C be a nonempty closed and convex subset of H and let F be a bifunction which satisfies conditions (L1) - (L4). Let $r > 0$ and $x \in H$, then there exist $u \in C$ such that*

$$F(u, y) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0 \quad \forall y \in C.$$

Lemma 3.2.3. [38] *Assume that $F : C \times C \rightarrow \mathbb{R}$ satisfies (L1) - (L4). For $r > 0$ and $x \in H$, define a resolvent function $T_r : H \rightarrow C$ as follows*

$$T_r(x) = \{u \in C : F(u, y) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \quad \forall y \in C\}$$

Then the following holds:

- i. T_r is single-valued;
- ii. T_r is firmly nonexpansive, i.e for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- iii. $F(T_r) = EP(F)$;
- iv. $EP(F)$ is closed and convex.

We recall that a set-valued mapping $T : H \rightarrow 2^H$ is monotone if for all $x, y \in H$, with $u \in T(x)$ and $v \in T(y)$ then

$$\langle x - y, u - v \rangle \geq 0,$$

and T is maximal monotone if the graph of T ($G(T) = \{(x, y) : y \in T(x)\}$) is not properly contain in the graph of any other monotone mapping. It is also known that T is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$ for all $(y, v) \in G(T)$ implies $u \in T(x)$. The resolvent operator J_λ^T associated with T and λ is the mapping $J_\lambda^T : H \rightarrow H$ defined by

$$J_\lambda^T(u) = (I + \lambda T)^{-1}(u), \quad u \in H, \quad \lambda > 0. \quad (3.2.1)$$

It is well-known that the resolvent operator J_λ^T is single-valued, nonexpansive and 1-inverse strongly monotone (see, [20]) and that a solution of MVIP is a fixed point of $J_\lambda^T(I - \lambda f)$, $\forall \lambda > 0$, that is,

$$0 \in f(x) + T(x) \Leftrightarrow x = J_\lambda^T(I - \lambda f)(x).$$

If f is α -inverse strongly monotone mapping with $0 < \lambda < 2\alpha$, then $J_\lambda^T(I - \lambda f)$ is nonexpansive and $I(f, T)$ is closed and convex.

Lemma 3.2.4. [69] *Let $T : H \rightarrow 2^H$ be a maximal monotone mapping and $f : H \rightarrow H$ be a Lipschitz continuous mapping. Then the mapping $G := T + f : H \rightarrow 2^H$ is a maximal monotone mapping.*

3.3 Main Results

In this section, we state and prove a strong convergence theorem for approximating a common solution of SEMVIP, SEGEP and SEFPP without prior knowledge of the operator norms.

Theorem 3.3.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed and convex subsets of H_1 and H_2 respectively, $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear operators. Let $F : C \times C \rightarrow \mathbb{R}$ and $G : Q \times Q \rightarrow \mathbb{R}$ be bifunctions satisfying (L1)-(L4). Let $\phi : C \rightarrow H_1$ be α_1 -inverse strongly monotone mapping, $\psi : Q \rightarrow H_2$ be α_2 -inverse strongly monotone mapping, $f_1 : C \rightarrow H_1$ be μ -inverse strongly monotone mapping, $f_2 : C \rightarrow H_2$ be ν -inverse strongly monotone mapping, $T_1 : H_1 \rightarrow 2^{H_1}$ and $T_2 : H_2 \rightarrow 2^{H_2}$ be two multi-valued maximal monotone mappings. Let $S_1 : H_1 \rightarrow H_1$ and $S_2 : H_2 \rightarrow H_2$ be demi-contractive mappings with constants k_1 and k_2 respectively such that $I - S_1$ and $I - S_2$ are demiclosed at 0 and $F(S_1) \neq \emptyset, F(S_2) \neq \emptyset$. Let $(\{x_n\}, \{y_n\})$ be the sequence generated for $(x_0, y_0) \in C \times Q$ defined by*

$$\left\{ \begin{array}{l} w_n = J_\lambda^{T_1}(I - \lambda f_1)((1 - t_n)x_n - \gamma_n A^*(A(1 - t_n)x_n - B(1 - t_n)y_n)), \\ z_n = J_\lambda^{T_2}(I - \lambda f_2)((1 - t_n)y_n + \gamma_n B^*(A(1 - t_n)x_n - B(1 - t_n)y_n)), \\ F(u_n, u) + \langle \phi w_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - w_n \rangle \geq 0 \quad \forall u \in C, \\ G(v_n, v) + \langle \psi z_n, v - v_n \rangle + \frac{1}{r_n} \langle v - v_n, v_n - z_n \rangle \geq 0 \quad \forall v \in Q, \\ x_{n+1} = (1 - \beta_n)S_1 u_n + \beta_n u_n; \quad \forall n \geq 0, \\ y_{n+1} = (1 - \delta_n)S_2 v_n + \delta_n v_n, \quad \forall n \geq 0, \end{array} \right. \quad (3.3.1)$$

$\{\gamma_n\}$ is a positive real sequence such that

$$\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon \right), \quad n \in \Omega,$$

otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$, A^* and B^* are adjoints of A and B respectively. Suppose $\{r_n\} \subset (0, \infty)$, $\{t_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

- i. $\lim_{n \rightarrow \infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$;
- ii. $0 < k_1 \leq \liminf_{n \rightarrow \infty} \beta_n$ and $0 < k_2 \leq \liminf_{n \rightarrow \infty} \delta_n$;
- iii. $0 < \lambda < 2\mu, 2\nu$;
- iv. $0 < r_n < 2\alpha_i, i = 1, 2$.

If $\Gamma := \left(F(S_1) \cap EP(F, \phi) \cap I(f_1, T_1) \right) \times \left(F(S_2) \cap EP(G, \psi) \cap I(f_2, T_2) \right) \neq \emptyset$, then the sequence $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. Let $(x^*, y^*) \in \Gamma$, $a_n = (1 - t_n)x_n$ and $b_n = (1 - t_n)y_n$, then

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|J_\lambda^{T_1}(I - \lambda f_1)(a_n - \gamma_n A^*(Aa_n - Bb_n)) - x^*\|^2 \\
&\leq \|a_n - \gamma_n A^*(Aa_n - Bb_n) - x^*\|^2 \\
&= \|a_n - x^*\|^2 + \gamma_n^2 \|A^*(Aa_n - Bb_n)\|^2 - 2\gamma_n \langle a_n - x^*, A^*(Aa_n - Bb_n) \rangle \\
&= \|a_n - x^*\|^2 + \gamma_n^2 \|A^*(Aa_n - Bb_n)\|^2 - 2\gamma_n \langle Aa_n - Ax^*, Aa_n - Bb_n \rangle.
\end{aligned}$$

But

$$2\langle Aa_n - Ax^*, Aa_n - Bb_n \rangle = \|Aa_n - Ax^*\|^2 + \|Aa_n - Bb_n\|^2 - \|Bb_n - Ax^*\|^2. \quad (3.3.2)$$

Thus, we have

$$\begin{aligned}
\|w_n - x^*\|^2 &\leq \|a_n - x^*\|^2 + \gamma_n^2 \|A^*(Aa_n - Bb_n)\|^2 - \gamma_n \|Aa_n - Ax^*\|^2 \\
&\quad - \gamma_n \|Aa_n - Bb_n\|^2 + \gamma_n \|Bb_n - Ax^*\|^2.
\end{aligned} \quad (3.3.3)$$

Similarly, following the same steps as above, we obtain

$$\begin{aligned}
\|z_n - y^*\|^2 &\leq \|b_n - y^*\|^2 + \gamma_n^2 \|B^*(Aa_n - Bb_n)\|^2 + \gamma_n \|Aa_n - By^*\|^2 \\
&\quad - \gamma_n \|Bb_n - By^*\|^2 - \gamma_n \|Aa_n - Bb_n\|^2.
\end{aligned} \quad (3.3.4)$$

Adding (3.3.3) and (3.3.4) and noting that $Ax^* = By^*$, we have

$$\begin{aligned}
\|w_n - x^*\|^2 + \|z_n - y^*\|^2 &\leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - \gamma_n \left(2\|Aa_n - Bb_n\|^2 \right. \\
&\quad \left. - \gamma_n (\|A^*(Aa_n - Bb_n)\|^2 \right. \\
&\quad \left. + \|B^*(Aa_n - Bb_n)\|^2) \right).
\end{aligned} \quad (3.3.5)$$

Therefore,

$$\|w_n - x^*\|^2 + \|z_n - y^*\|^2 \leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2. \quad (3.3.6)$$

Also from (3.3.1), we can write $u_n = T_{r_n}^F(w_n - r_n\phi w_n)$ and $v_n = T_{r_n}^G(z_n - r_n\psi z_n)$. Thus, for all $n \geq 0$, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}^F(w_n - r_n\phi w_n) - x^*\|^2 \\
&= \|T_{r_n}^F(w_n - r_n\phi w_n) - T_{r_n}^F(x^* - r_n\phi x^*)\|^2 \\
&\leq \|(I - r_n\phi)w_n - (I - r_n\phi)x^*\|^2 \\
&\leq \|(w_n - x^*) - r_n(\phi w_n - \phi x^*)\|^2 \\
&= \|w_n - x^*\|^2 - 2r_n\langle w_n - x^*, \phi w_n - \phi x^* \rangle + r_n^2\|\phi w_n - \phi x^*\|^2 \\
&\leq \|w_n - x^*\|^2 - 2r_n\alpha_1\|\phi w_n - \phi x^*\|^2 + r_n^2\|\phi w_n - \phi x^*\|^2 \\
&= \|w_n - x^*\|^2 - r_n(2\alpha_1 - r_n)\|\phi w_n - \phi x^*\|^2 \\
&\leq \|w_n - x^*\|^2 \quad (\text{since } 2\alpha_1 > r_n).
\end{aligned} \tag{3.3.7}$$

Following similar process as above, we have that

$$\begin{aligned}
\|v_n - y^*\|^2 &\leq \|z_n - y^*\|^2 - r_n(2\alpha_2 - r_n)\|\psi z_n - \psi y^*\|^2 \\
&\leq \|z_n - y^*\|^2.
\end{aligned} \tag{3.3.8}$$

Thus, from (3.3.6), (3.3.7) and (3.3.8), we have

$$\begin{aligned}
\|u_n - x^*\|^2 + \|v_n - y^*\|^2 &\leq \|w_n - x^*\|^2 + \|z_n - y^*\|^2 \\
&\leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2.
\end{aligned} \tag{3.3.9}$$

Further, from (3.3.1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n)S_1u_n + \beta_nu_n - x^*\|^2 \\
&= \|(1 - \beta_n)(S_1u_n - x^*) + \beta_n(u_n - x^*)\|^2 \\
&= (1 - \beta_n)\|S_1u_n - x^*\|^2 + \beta_n\|u_n - x^*\|^2 - \beta_n(1 - \beta_n)\|S_1u_n - u_n\|^2 \\
&\leq (1 - \beta_n)\left(\|u_n - x^*\|^2 + k_1\|u_n - S_1u_n\|^2\right) + \beta_n\|u_n - x^*\|^2 \\
&\quad - \beta_n(1 - \beta_n)\|S_1u_n - u_n\|^2 \\
&= \|u_n - x^*\|^2 + (1 - \beta_n)(k_1 - \beta_n)\|u_n - S_1u_n\|^2 \\
&\leq \|u_n - x^*\|^2 \quad (\text{since } 0 < k_1 \leq \liminf_{n \rightarrow \infty} \beta_n).
\end{aligned} \tag{3.3.10}$$

Similarly as above, we obtain

$$\begin{aligned}
\|y_{n+1} - y^*\|^2 &\leq \|v_n - y^*\|^2 + (1 - \delta_n)(k_2 - \delta_n)\|S_2v_n - v_n\|^2 \\
&\leq \|v_n - y^*\|^2 \quad (\text{since } 0 < k_2 \leq \liminf_{n \rightarrow \infty} \delta_n).
\end{aligned} \tag{3.3.11}$$

Therefore, from (3.3.9), (3.3.10) and (3.3.11), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq \|u_n - x^*\|^2 + \|v_n - y^*\|^2 \\
&\leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2.
\end{aligned} \tag{3.3.12}$$

But

$$\begin{aligned}
\|a_n - x^*\|^2 &= \|(1 - t_n)x_n - x^*\|^2 \\
&= \|(1 - t_n)(x_n - x^*) - t_nx^*\|^2 \\
&\leq (1 - t_n)\|x_n - x^*\|^2 + t_n\|x^*\|^2,
\end{aligned} \tag{3.3.13}$$

and

$$\begin{aligned}
\|b_n - y^*\|^2 &= \|(1 - t_n)y_n - y^*\|^2 \\
&= \|(1 - t_n)(y_n - y^*) - t_n y^*\|^2 \\
&\leq (1 - t_n)\|y_n - y^*\|^2 + t_n\|y^*\|^2.
\end{aligned} \tag{3.3.14}$$

Thus, adding (3.3.13) and (3.3.14), we have

$$\begin{aligned}
\|a_n - x^*\|^2 + \|b_n - y^*\|^2 &\leq (1 - t_n)\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right) \\
&\quad + t_n(\|x^*\|^2 + \|y^*\|^2).
\end{aligned} \tag{3.3.15}$$

Hence, it follows from (3.3.12) and (3.3.15) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq (1 - t_n)\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right) \\
&\quad + t_n(\|x^*\|^2 + \|y^*\|^2) \\
&\leq \max\left\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2, \|x^*\|^2 + \|y^*\|^2\right\} \\
&\quad \vdots \\
&\leq \max\left\{\|x_0 - x^*\|^2 + \|y_0 - y^*\|^2, \|x^*\|^2 + \|y^*\|^2\right\}.
\end{aligned}$$

This implies that $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is bounded and consequently $\{x_n\}$, $\{y_n\}$, $\{Ax_n\}$, $\{By_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{z_n\}$ are bounded. Also since A , A^* , B and B^* are linear mappings, we obtained from (3.3.5), (3.3.7) and (3.3.8) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq (1 - t_n)\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right) + t_n(\|x^*\|^2 + \|y^*\|^2) \\
&\quad - (1 - t_n)\gamma_n\left(2\|Ax_n - By_n\|^2 - \gamma_n(\|A^*(Ax_n - By_n)\|^2\right. \\
&\quad \left.+ \|B^*(Ax_n - By_n)\|^2)\right) - r_n\left((2\alpha_1 - r_n)\|\phi w_n - \phi x^*\|^2\right. \\
&\quad \left.+ (2\alpha_2 - r_n)\|\psi z_n - \psi y^*\|^2\right) \\
&\quad - (1 - \beta_n)\left((k_1 - \beta_n)\|S_1 u_n - u_n\|^2\right. \\
&\quad \left.+ (k_2 - \beta_n)\|S_2 v_n - v_n\|^2\right).
\end{aligned} \tag{3.3.16}$$

Now, we divide the rest of the prove into two cases.

Case A: Assuming $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is monotonically decreasing, then

$$\left(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\right) - \left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right) \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{3.3.17}$$

Putting $\|x_n - x^*\|^2 + \|y_n - y^*\|^2$ to be $\rho_n(x^*, y^*)$, then it follows from (3.3.16) that

$$\begin{aligned}
\rho_{n+1}(x^*, y^*) &\leq (1 - t_n)\rho_n(x^*, y^*) + t_n(\|x^*\|^2 + \|y^*\|^2) \\
&\quad - (1 - t_n)\gamma_n\left(2\|Ax_n - By_n\|^2 - \gamma_n(\|A^*(Ax_n - By_n)\|^2\right. \\
&\quad \left.+ \|B^*(Ax_n - By_n)\|^2)\right).
\end{aligned}$$

Putting $K_n = \|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2$, we have

$$\gamma_n \left(2\|Ax_n - By_n\|^2 - \gamma_n K_n \right) \leq \rho_n(x^*, y^*) - \frac{1}{1-t_n} \rho_{n+1}(x^*, y^*) + \frac{t_n}{1-t_n} (\|x^*\|^2 + \|y^*\|^2).$$

By the condition

$$\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon \right), \quad n \in \Omega,$$

we have

$$\lim_{n \rightarrow \infty} K_n = \lim_{n \rightarrow \infty} (\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2) = 0, \quad (3.3.18)$$

observe that $Ax_n - By_n = 0$, if $n \notin \Omega$, hence

$$\lim_{n \rightarrow \infty} \|A^*(Ax_n - By_n)\| = 0, \quad (3.3.19)$$

and

$$\lim_{n \rightarrow \infty} \|B^*(Ax_n - By_n)\| = 0. \quad (3.3.20)$$

Also from (3.3.16), we obtain

$$\begin{aligned} \rho_{n+1}(x^*, y^*) &\leq (1-t_n)\rho_n(x^*, y^*) + t_n(\|x^*\|^2 + \|y^*\|^2) - (1-t_n)\gamma_n^2 \left(\|A^*(Ax_n - By_n)\|^2 \right. \\ &\quad \left. + \|B^*(Ax_n - By_n)\|^2 \right) - r_n \left((2\alpha_1 - r_n)\|\phi w_n - \phi x^*\|^2 \right. \\ &\quad \left. + (2\alpha_2 - r_n)\|\psi z_n - \psi y^*\|^2 \right). \end{aligned}$$

Set $\Upsilon_n = (2\alpha_1 - r_n)\|\phi w_n - \phi x^*\|^2 + (2\alpha_2 - r_n)\|\psi z_n - \psi y^*\|^2$, then we have from (3.3.18),

$$\begin{aligned} \Upsilon_n &\leq (1-t_n)\rho_n(x^*, y^*) - \rho_{n+1}(x^*, y^*) + t_n(\|x^*\|^2 + \|y^*\|^2) \\ &\quad - (1-t_n)\gamma_n^2 \left(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2 \right) \longrightarrow 0, \\ &\quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.3.21)$$

Hence, from condition (iv), we obtain

$$\lim_{n \rightarrow \infty} \|\phi w_n - \phi x^*\|^2 = 0, \quad (3.3.22)$$

and

$$\lim_{n \rightarrow \infty} \|\psi z_n - \psi y^*\|^2 = 0. \quad (3.3.23)$$

Let $\Lambda_n = (1-\beta_n) \left((k_1 - \beta_n)\|S_1 u_n - u_n\|^2 + (k_2 - \beta_n)\|S_2 v_n - v_n\|^2 \right)$.

Then from (3.3.16) and (3.3.18), we have

$$\begin{aligned} \Lambda_n &\leq (1-t_n)\rho_n(x^*, y^*) - \rho_{n+1}(x^*, y^*) + t_n(\|x^*\|^2 + \|y^*\|^2) \\ &\quad - (1-t_n)\gamma_n^2 \left(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2 \right) \\ &\quad - r_n \left((2\alpha_1 - r_n)\|\phi w_n - \phi x^*\|^2 \right. \\ &\quad \left. + (2\alpha_2 - r_n)\|\psi z_n - \psi y^*\|^2 \right) \longrightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.3.24)$$

Thus by condition (ii), we obtain

$$\lim_{n \rightarrow \infty} \|S_1 u_n - u_n\| = 0, \quad (3.3.25)$$

and

$$\lim_{n \rightarrow \infty} \|S_2 v_n - v_n\| = 0. \quad (3.3.26)$$

Also from (3.3.1), recall that $a_n = (1 - t_n)x_n$ and $b_n = (1 - t_n)y_n$, then we have

$$\begin{aligned} \|w_n - x^*\|^2 &= \|J_\lambda^{T_1}(I - \lambda f_1)(a_n - \gamma_n A^*(Aa_n - Bb_n) - x^*)\|^2 \\ &\leq \langle w_n - x^*, a_n - \gamma_n A^*(Aa_n - Bb_n) - x^* \rangle \\ &= \frac{1}{2} \left(\|w_n - x^*\|^2 + \|a_n - \gamma_n A^*(Aa_n - Bb_n) - x^*\|^2 - \|w_n - a_n \right. \\ &\quad \left. - \gamma_n A^*(Aa_n - Bb_n)\|^2 \right) \\ &\leq \frac{1}{2} \left(\|w_n - x^*\|^2 + \|a_n - x^*\|^2 + \gamma_n^2 \|A^*(Aa_n - Bb_n)\|^2 \right. \\ &\quad \left. + 2\gamma_n \|a_n - x^*\| \|A^*(Aa_n - Bb_n)\| - [\|w_n - a_n\|^2 \right. \\ &\quad \left. + \gamma_n^2 \|A^*(Aa_n - Bb_n)\|^2 - 2\gamma_n \|w_n - a_n\| \|A^*(Aa_n - Bb_n)\|] \right) \\ &= \frac{1}{2} \left(\|w_n - x^*\|^2 + \|a_n - x^*\|^2 + 2\gamma_n \|a_n - x^*\| \|A^*(Aa_n - Bb_n)\| \right. \\ &\quad \left. - \|w_n - a_n\|^2 + 2\gamma_n \|w_n - a_n\| \|A^*(Aa_n - Bb_n)\| \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \|w_n - a_n\|^2 &\leq \|a_n - x^*\|^2 - \|w_n - x^*\|^2 \\ &\quad + 2\gamma_n \|A^*(Aa_n - Bb_n)\| [\|w_n - a_n\| + \|a_n - x^*\|]. \end{aligned} \quad (3.3.27)$$

In a similar way as (3.3.27), we obtain

$$\begin{aligned} \|z_n - b_n\|^2 &\leq \|b_n - y^*\|^2 - \|z_n - y^*\|^2 \\ &\quad + 2\gamma_n \|B^*(Aa_n - Bb_n)\| [\|z_n - b_n\| + \|b_n - y^*\|]. \end{aligned} \quad (3.3.28)$$

Adding (3.3.27) and (3.3.28), we have

$$\begin{aligned} \|w_n - a_n\|^2 + \|z_n - b_n\|^2 &\leq (\|a_n - x^*\|^2 + \|b_n - y^*\|^2) - (\|w_n - x^*\|^2 + \|z_n - y^*\|^2) \\ &\quad + 2\gamma_n \|A^*(Aa_n - Bb_n)\| [\|w_n - a_n\| + \|a_n - x^*\|] \\ &\quad + 2\gamma_n \|B^*(Aa_n - Bb_n)\| [\|z_n - b_n\| + \|b_n - y^*\|]. \end{aligned}$$

From (3.3.6), (3.3.19) and (3.3.20), we have

$$\begin{aligned} \|w_n - a_n\|^2 + \|z_n - b_n\|^2 &\leq (\|a_n - x^*\|^2 + \|b_n - y^*\|^2) - (\|a_n - x^*\|^2 + \|b_n - y^*\|^2) \\ &\quad + 2\gamma_n \|A^*(Aa_n - Bb_n)\| [\|w_n - a_n\| + \|a_n - x^*\|] \\ &\quad + 2\gamma_n \|B^*(Aa_n - Bb_n)\| [\|z_n - b_n\| + \|b_n - y^*\|] \\ &\quad \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.3.29)$$

Hence,

$$\lim_{n \rightarrow \infty} \|w_n - a_n\| = 0, \quad (3.3.30)$$

and

$$\lim_{n \rightarrow \infty} \|z_n - b_n\| = 0. \quad (3.3.31)$$

But

$$\|a_n - x_n\| = t_n \|x_n\| \rightarrow 0, n \rightarrow \infty, \quad (3.3.32)$$

and

$$\|b_n - y_n\| = t_n \|y_n\| \rightarrow 0, n \rightarrow \infty. \quad (3.3.33)$$

Therefore from (3.3.30), (3.3.31), (3.3.32) and (3.3.33), we have

$$\|w_n - x_n\| \leq \|w_n - a_n\| + \|a_n - x_n\| \rightarrow 0, n \rightarrow \infty, \quad (3.3.34)$$

and

$$\|z_n - y_n\| \leq \|z_n - b_n\| + \|b_n - y_n\| \rightarrow 0, n \rightarrow \infty. \quad (3.3.35)$$

Also, from (3.3.1)

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}^F(w_n - r_n \phi w_n) - x^*\|^2 \\ &\leq \|T_{r_n}^F(w_n - r_n \phi w_n) - T_{r_n}^F(x^* - r_n \phi x^*)\|^2 \\ &\leq \langle u_n - x^*, (w_n - r_n \phi w_n) - (x^* - r_n \phi x^*) \rangle \\ &= \frac{1}{2} \left(\|u_n - x^*\|^2 + \|(w_n - r_n \phi w_n) - (x^* - r_n \phi x^*)\|^2 - \|w_n - r_n \phi w_n \right. \\ &\quad \left. - (x^* - r_n \phi x^*) - (u_n - x^*)\|^2 \right) \\ &= \frac{1}{2} \left(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|(w_n - r_n \phi w_n) - (x^* - r_n \phi x^*) \right. \\ &\quad \left. - (u_n - x^*)\|^2 \right) \\ &= \frac{1}{2} \left(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|w_n - u_n\|^2 + 2r_n \langle w_n - u_n, \phi w_n - \phi x^* \rangle \right. \\ &\quad \left. - r_n^2 \|\phi w_n - \phi x^*\|^2 \right) \\ &\leq \frac{1}{2} \left(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|w_n - u_n\|^2 \right. \\ &\quad \left. + 2r_n \langle w_n - u_n, \phi w_n - \phi x^* \rangle \right). \end{aligned} \quad (3.3.36)$$

Therefore,

$$\|w_n - u_n\|^2 \leq \|w_n - x^*\|^2 - \|u_n - x^*\|^2 + 2r_n \|w_n - u_n\| \|\phi w_n - \phi x^*\|. \quad (3.3.37)$$

Following similar argument as above, we obtain

$$\|z_n - v_n\|^2 \leq \|z_n - y^*\|^2 - \|v_n - y^*\|^2 + 2r_n \|z_n - v_n\| \|\psi z_n - \psi y^*\|. \quad (3.3.38)$$

Adding (3.3.37) and (3.3.38), we have

$$\begin{aligned} \|w_n - u_n\|^2 + \|z_n - v_n\|^2 &\leq (\|w_n - x^*\|^2 + \|z_n - y^*\|^2) - (\|u_n - x^*\|^2 + \|v_n - y^*\|^2) \\ &\quad + 2r_n \left(\|w_n - u_n\| \|\phi w_n - \phi x^*\| \right. \\ &\quad \left. + \|z_n - v_n\| \|\psi z_n - \psi y^*\| \right). \end{aligned} \quad (3.3.39)$$

It follows from (3.3.7), (3.3.8), (3.3.16), (3.3.22) and (3.3.23) that

$$\begin{aligned} \|w_n - u_n\|^2 + \|z_n - v_n\|^2 &\leq (\|w_n - x^*\|^2 + \|z_n - y^*\|^2) - (\|w_n - x^*\|^2 + \|z_n - y^*\|^2) \\ &\quad + 2r_n \left(\|w_n - u_n\| \|\phi w_n - \phi x^*\| + \|z_n - v_n\| \|\psi z_n - \psi y^*\| \right) \\ &\longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \|w_n - u_n\| = 0, \quad (3.3.40)$$

and

$$\lim_{n \rightarrow \infty} \|z_n - v_n\| = 0. \quad (3.3.41)$$

Again from (3.3.1), (3.3.25) and (3.3.26), we have

$$\begin{aligned} \|x_{n+1} - u_n\| &= \|(1 - \beta_n)S_1 u_n + \beta_n u_n - u_n\| \leq (1 - \beta_n) \|S_1 u_n - u_n\| \\ &\longrightarrow 0, \quad \text{as } n \rightarrow \infty \end{aligned} \quad (3.3.42)$$

and

$$\begin{aligned} \|y_{n+1} - v_n\| &= \|(1 - \delta_n)S_2 v_n + \delta_n v_n - v_n\| \leq (1 - \delta_n) \|S_2 v_n - v_n\| \\ &\longrightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (3.3.43)$$

therefore from (3.3.34), (3.3.35), (3.3.40), (3.3.41), (3.3.42) and (3.3.43) we obtain

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|u_n - w_n\| + \|w_n - x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty \quad (3.3.44)$$

and

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - v_n\| + \|v_n - z_n\| + \|z_n - y_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.3.45)$$

Since $(\{x_n\}, \{y_n\})$ is bounded, there exists a subsequence $(\{x_{n_j}\}, \{y_{n_j}\})$ of $(\{x_n\}, \{y_n\})$ such that $(\{x_{n_j}\}, \{y_{n_j}\})$ converges weakly to $(\bar{x}, \bar{y}) \in C \times Q$. From (3.3.34) and (3.3.35), we have $(\{w_{n_j}\}, \{z_{n_j}\})$ converges weakly to (\bar{x}, \bar{y}) . Also by (3.3.40) and (3.3.41), we

have $(\{u_{n_j}\}, \{v_{n_j}\})$ converges weakly to (\bar{x}, \bar{y}) . Furthermore, by (3.3.25), (3.3.26), the demi-closedness of $I - S_1$ at 0 and the demi-closedness of $I - S_2$ at 0, we have $(\bar{x}, \bar{y}) \in F(S_1) \times F(S_2)$.

Also since $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are bounded linear mappings and $\{Ax_{n_j}\} \rightharpoonup A\bar{x}$ and $\{By_{n_j}\} \rightharpoonup B\bar{y}$ and by the weakly lower semicontinuity of the squared norm, we have

$$\|A\bar{x} - B\bar{y}\|^2 \leq \liminf_{n \rightarrow \infty} \|Ax_{n_j} - By_{n_j}\|^2 = 0,$$

thus

$$A\bar{x} = B\bar{y}. \quad (3.3.46)$$

We now show that $\bar{x} \in I(f_1, T_1)$ and $\bar{y} \in I(f_2, T_2)$. Since T_1 is $\frac{1}{\mu}$ Lipschitz monotone and the domain of T_1 is H_1 , we obtain from Lemma 3.2.4 that $f_1 + T_1$ is maximal monotone. Let $(u, w) \in G(T_1, f_1)$, i.e $w - f_1 u \in T_1(u)$.

Putting $a_{n_j} = (1 - t_{n_j})x_{n_j}$ and $c_{n_j} = a_{n_j} - \gamma_{n_j}A^*(Aa_{n_j} - Bb_{n_j})$, then

$$w_{n_j} = J_{\lambda}^{T_1}(I - \lambda f_1)c_{n_j},$$

which implies that

$$(I - \lambda f_1)c_{n_j} \in (I + \lambda T_1)w_{n_j}.$$

Applying the maximal monotonicity of $(T_1 + f_1)$, we obtain

$$\langle u - w_{n_j}, w - f_1 u - \frac{1}{\lambda}(c_{n_j} - \lambda f_1 c_{n_j} - w_{n_j}) \rangle \geq 0,$$

and so

$$\begin{aligned} \langle u - w_{n_j}, w \rangle &\geq \langle u - w_{n_j}, f_1 u + \frac{1}{\lambda}(c_{n_j} + \lambda f_1 c_{n_j} - w_{n_j}) \rangle \\ &= \langle u - w_{n_j}, f_1 u - f_1 w_{n_j} + f_1 w_{n_j} - f_1 c_{n_j} + \frac{1}{\lambda}(c_{n_j} - w_{n_j}) \rangle \\ &\geq 0 + \langle u - w_{n_j}, f_1 w_{n_j} - f_1 c_{n_j} \rangle + \langle u - w_{n_j}, \frac{1}{\lambda}(c_{n_j} - w_{n_j}) \rangle. \end{aligned} \quad (3.3.47)$$

Since

$$\|c_{n_j} - a_{n_j}\| = \gamma_{n_j} \|A^*(Aa_{n_j} - Bb_{n_j})\| \longrightarrow 0, \quad \text{as } j \rightarrow \infty$$

and by (3.3.30)

$$\|w_{n_j} - c_{n_j}\| \leq \|w_{n_j} - a_{n_j}\| + \|a_{n_j} - c_{n_j}\| \longrightarrow 0, \quad \text{as } j \rightarrow \infty, \quad (3.3.48)$$

it follows that

$$\lim_{j \rightarrow \infty} \|f_1 w_{n_j} - f_1 c_{n_j}\| = 0, \quad (3.3.49)$$

and since $w_{n_j} \rightharpoonup \bar{x}$, therefore

$$\lim_{j \rightarrow \infty} \langle u - w_{n_j}, w \rangle = \langle u - \bar{x}, w \rangle \geq 0. \quad (3.3.50)$$

Using the maximal monotonicity of $f_1 + T_1$, we obtain

$$0 \in (T_1 + f_1)\bar{x} \quad (3.3.51)$$

which implies that $\bar{x} \in I(f_1, T_1)$.

Following similar argument as above, we obtain that $\bar{y} \in I(f_2, T_2)$.

Next, we show that $\bar{x} \in EP(F, \phi)$ and $\bar{y} \in EP(G, \psi)$. Since $u_n = T_{r_n}^F(w_n - r_n \phi w_n)$, $n \geq 0$, we have that for any $p \in C$,

$$F(u_n, p) + \langle \phi w_n, p - u_n \rangle + \frac{1}{r_n} \langle p - u_n, u_n - w_n \rangle \geq 0.$$

By replacing n by n_j in the last inequality and using assumption L2, we obtain

$$\langle \phi w_{n_j}, p - u_{n_j} \rangle + \frac{1}{r_{n_j}} \langle p - u_{n_j}, u_{n_j} - w_{n_j} \rangle \geq F(p, u_{n_j}). \quad (3.3.52)$$

Let $z_q = qp + (1 - q)\bar{x}$ for all $q \in (0, 1]$ and $p \in C$. Since C is a convex set, thus $z_q \in C$, hence, by (3.3.52), we have

$$\begin{aligned} \langle z_q - u_{n_j}, \phi z_q \rangle &\geq \langle z_q - u_{n_j}, \phi z_q \rangle - \langle z_q - u_{n_j}, \phi w_{n_j} \rangle - \langle z_q - u_{n_j}, \frac{u_{n_j} - w_{n_j}}{r_{n_j}} \rangle + F(z_q, u_{n_j}) \\ &= \langle z_q - u_{n_j}, \phi z_q - \phi u_{n_j} \rangle + \langle z_q - u_{n_j}, \phi u_{n_j} - \phi w_{n_j} \rangle - \langle z_q - u_{n_j}, \frac{u_{n_j} - w_{n_j}}{r_{n_j}} \rangle \\ &\quad + F(z_q, u_{n_j}). \end{aligned}$$

From (3.3.40), we obtain that $\|\phi u_{n_j} - \phi w_{n_j}\| \rightarrow 0$, as $j \rightarrow \infty$ and by the monotonicity of ϕ we obtain

$$\langle z_q - u_{n_j}, \phi z_q - \phi u_{n_j} \rangle \geq 0,$$

then using assumption L4 in (3.3.52), we obtain (noting that $u_{n_j} \rightharpoonup \bar{x}$)

$$\langle z_q - \bar{x}, \phi z_q \rangle \geq F(z_q, \bar{x}) \quad j \rightarrow \infty,$$

hence, using assumption L1, we have

$$\begin{aligned} 0 &= F(z_q, z_q) \\ &= F(z_q, qp + (1 - q)\bar{x}) \\ &= qF(z_q, p) + (1 - q)F(z_q, \bar{x}) \\ &\leq qF(z_q, p) + (1 - q)\langle z_q - \bar{x}, \phi z_q \rangle \\ &= qF(z_q, p) + (1 - q)q\langle p - \bar{x}, \phi z_q \rangle \\ &= q\left(F(z_q, p) + (1 - q)\langle p - \bar{x}, \phi z_q \rangle\right). \end{aligned} \quad (3.3.53)$$

Letting $q \rightarrow 0$ and using assumption L3, we have that for each $p \in C$,

$$0 \leq F(\bar{x}, p) + \langle p - \bar{x}, \phi \bar{x} \rangle, \quad (3.3.54)$$

which implies that $\bar{x} \in EP(F, \phi)$.

Following similar approach as above, we obtain that $\bar{y} \in EP(G, \psi)$. Therefore, $(\bar{x}, \bar{y}) \in \Gamma$. We now show that $(\{x_n\}, \{y_n\})$ converges strongly to (\bar{x}, \bar{y}) . From (3.3.12), we obtain

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 + \|y_{n+1} - \bar{y}\|^2 &\leq \|(1 - t_n)x_n - \bar{x}\|^2 + \|(1 - t_n)y_n - \bar{y}\|^2 \\ &= (1 - t_n)^2 (\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2) + t_n^2 (\|\bar{x}\|^2 + \|\bar{y}\|^2) \\ &\quad - 2t_n(1 - t_n) (\langle x_n - \bar{x}, \bar{x} \rangle + \langle y_n - \bar{y}, \bar{y} \rangle) \\ &\leq (1 - t_n) (\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2) + t_n (t_n (\|\bar{x}\|^2 + \|\bar{y}\|^2) \\ &\quad - 2(1 - t_n) (\langle x_n - \bar{x}, \bar{x} \rangle + \langle y_n - \bar{y}, \bar{y} \rangle)). \end{aligned} \quad (3.3.55)$$

Therefore by Lemma 2.5.6, we have

$$\lim_{n \rightarrow \infty} (\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2) = 0. \quad (3.3.56)$$

Hence

$$\lim_{n \rightarrow \infty} \|x_n - \bar{x}\|^2 = \lim_{n \rightarrow \infty} \|y_n - \bar{y}\|^2 = 0, \quad (3.3.57)$$

which implies that $(\{x_n\}, \{y_n\}) \rightarrow (\bar{x}, \bar{y})$, $n \rightarrow \infty$.

Case B: Assume that $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is not monotonically decreasing. Set $\rho_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$ and let $\tau : \mathbb{N} \rightarrow \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \rho_k \leq \rho_{k+1}\}.$$

Obviously, τ is a non-decreasing sequence such that $\tau(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $\rho_{\tau(n)} \leq \rho_{\tau(n)+1}$, $\forall n \geq n_0$.

Let $K_{\tau(n)} = \|A^*(Ax_{\tau(n)} - By_{\tau(n)})\|^2 + \|B^*(Ax_{\tau(n)} - By_{\tau(n)})\|^2$ and $\tau(n) \in \Omega$, then it follows from (3.3.16) that

$$\begin{aligned} \gamma_{\tau(n)} \left(2\|Ax_{\tau(n)} - By_{\tau(n)}\|^2 - \gamma_{\tau(n)} K_{\tau(n)} \right) &\leq \rho_{\tau(n)} - \frac{1}{(1 - t_{\tau(n)})} \rho_{\tau(n)+1} \\ &\quad + \frac{t_{\tau(n)}}{(1 - t_{\tau(n)})} (\|x^*\|^2 + \|y^*\|^2), \end{aligned}$$

which implies that

$$\epsilon K_{\tau(n)} \leq \rho_{\tau(n)} - \frac{1}{(1 - t_{\tau(n)})} \rho_{\tau(n)+1} + \frac{t_{\tau(n)}}{(1 - t_{\tau(n)})} (\|x^*\|^2 + \|y^*\|^2).$$

From the condition that

$$\gamma_{\tau(n)} \in \left(\epsilon, \frac{2\|Ax_{\tau(n)} - By_{\tau(n)}\|^2}{\|A^*(Ax_{\tau(n)} - By_{\tau(n)})\|^2 + \|B^*(Ax_{\tau(n)} - By_{\tau(n)})\|^2} - \epsilon \right), \quad \tau(n) \in \Omega,$$

we have

$$\lim_{n \rightarrow \infty} K_{\tau(n)} = \lim_{n \rightarrow \infty} \|A^*(Ax_{\tau(n)} - By_{\tau(n)})\|^2 + \|B^*(Ax_{\tau(n)} - By_{\tau(n)})\|^2 = 0 \quad (3.3.58)$$

Observe that $Ax_{\tau(n)} - By_{\tau(n)} = 0$, if $\tau(n) \notin \Omega$.

Hence

$$\lim_{n \rightarrow \infty} \|A^*(Ax_{\tau(n)} - By_{\tau(n)})\|^2 = 0, \quad (3.3.59)$$

$$\lim_{n \rightarrow \infty} \|B^*(Ax_{\tau(n)} - By_{\tau(n)})\|^2 = 0. \quad (3.3.60)$$

Following the same argument as in **Case A**, we conclude that there exists a subsequence of $(\{x_{\tau(n)}\}, \{y_{\tau(n)}\})$ denoted as $(\{x_{\tau(n)}\}, \{y_{\tau(n)}\})$ for ease of notation which converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$.

Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq [\|x_{\tau(n)+1} - \bar{x}\|^2 + \|y_{\tau(n)+1} - \bar{y}\|^2] - [\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2] \\ &\leq (1 - t_{\tau(n)}) [\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2] + t_{\tau(n)}^2 (\|\bar{x}\|^2 + \|\bar{y}\|^2) \\ &\quad - 2t_{\tau(n)}(1 - t_{\tau(n)}) [\langle x_{\tau(n)} - \bar{x}, \bar{x} \rangle + \langle y_{\tau(n)} - \bar{y}, \bar{y} \rangle] - [\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2], \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2 &\leq t_{\tau(n)} (\|\bar{x}\|^2 + \|\bar{y}\|^2) - 2(1 - t_{\tau(n)}) [\langle x_{\tau(n)} - \bar{x}, \bar{x} \rangle \\ &\quad + \langle y_{\tau(n)} - \bar{y}, \bar{y} \rangle] \longrightarrow 0, \end{aligned}$$

hence,

$$\lim_{n \rightarrow \infty} \left(\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2 \right) = 0, \quad (3.3.61)$$

therefore,

$$\lim_{n \rightarrow \infty} \rho_{\tau(n)} = \lim_{n \rightarrow \infty} \rho_{\tau(n)+1} = 0. \quad (3.3.62)$$

Moreover, for $n \geq n_0$, it is easily observed that $\rho_{\tau(n)} \leq \rho_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\rho_j > \rho_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.

Consequently, for all $n \geq n_0$,

$$0 \leq \rho_n \leq \max\{\rho_{\tau(n)}, \rho_{\tau(n)+1}\} = \rho_{\tau(n)+1}. \quad (3.3.63)$$

Thus, $\lim \rho_n = 0$. That is $(\{x_n\}, \{y_n\})$ converges strongly to (\bar{x}, \bar{y}) . □

The following consequences are obtained from Theorem 3.3.1.

Corollary 3.3.2. Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed and convex subsets of H_1 and H_2 respectively, $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear operators. Assume that $F : C \times C \rightarrow \mathbb{R}$ and $G : Q \times Q \rightarrow \mathbb{R}$ are bifunctions satisfying (L1)-(L4). Let $\phi : C \rightarrow H_1$ be α_1 -inverse strongly monotone mapping, $\psi : Q \rightarrow H_2$ be α_2 -inverse strongly monotone mapping, $f_1 : C \rightarrow H_1$ be μ -inverse strongly monotone mapping, $f_2 : C \rightarrow H_2$ be ν -inverse strongly monotone mapping, $T_1 : H_1 \rightarrow 2^{H_1}$ and $T_2 : H_2 \rightarrow 2^{H_2}$ be two multi-valued maximal monotone mappings. Let $S_1 : H_1 \rightarrow H_1$ and $S_2 : H_2 \rightarrow H_2$ be nonexpansive mappings such that $F(S_1) \neq \emptyset$, $F(S_2) \neq \emptyset$. Let $(\{x_n\}, \{y_n\})$ be the sequence generated for $(x_0, y_0) \in C \times Q$ defined by

$$\left\{ \begin{array}{l} w_n = J_{\lambda}^{T_1}(I - \lambda f_1)((1 - t_n)x_n - \gamma_n A^*(A(1 - t_n)x_n - B(1 - t_n)y_n)), \\ z_n = J_{\lambda}^{T_2}(I - \lambda f_2)((1 - t_n)y_n + \gamma_n B^*(A(1 - t_n)x_n - B(1 - t_n)y_n)), \\ F(u_n, u) + \langle \phi w_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - w_n \rangle \geq 0 \quad \forall u \in C, \\ G(v_n, v) + \langle \psi z_n, v - v_n \rangle + \frac{1}{r_n} \langle v - v_n, v_n - z_n \rangle \geq 0 \quad \forall v \in Q, \\ x_{n+1} = (1 - \beta_n)S_1 u_n + \beta_n u_n, \\ y_{n+1} = (1 - \delta_n)S_2 v_n + \delta_n v_n, \quad \forall n \geq 0, \end{array} \right. \quad (3.3.64)$$

$\{\gamma_n\}$ is a positive real sequence such that

$$\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon \right), \quad n \in \Omega,$$

otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$, A^* and B^* are adjoints of A and B respectively. Suppose $\{r_n\} \subset (0, \infty)$, $\{t_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions

- (i) $\lim_{n \rightarrow \infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$;
- (ii) $0 < \lambda < 2\mu, 2\nu$;
- (iii) $0 < r_n < 2\alpha_i, i = 1, 2$.

If $\Gamma := (F(S_1) \cap EP(F, \phi) \cap I(f_1, T_1)) \times (F(S_2) \cap EP(G, \psi) \cap I(f_2, T_2)) \neq \emptyset$, then

the sequence $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.

Corollary 3.3.3. Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed and convex subsets of H_1 and H_2 respectively, $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear operators. Assume that $F : C \times C \rightarrow \mathbb{R}$ and $G : Q \times Q \rightarrow \mathbb{R}$ are bifunctions satisfying (L1)-(L4). Let $\phi : C \rightarrow H_1$ be α_1 -inverse strongly monotone mapping, $\psi : Q \rightarrow H_2$ be α_2 -inverse strongly monotone mapping, $T_1 : H_1 \rightarrow 2^{H_1}$ and $T_2 : H_2 \rightarrow 2^{H_2}$ be two multi-valued maximal monotone mappings. Let $S_1 : H_1 \rightarrow H_1$ and $S_2 : H_2 \rightarrow H_2$ be demi-contractive mappings with constants k_1 and k_2 respectively where $k = \max\{k_1, k_2\}$ such that $I - S_1$ and $I - S_2$ are demiclosed at 0 and $F(S_1) \neq \emptyset$, $F(S_2) \neq \emptyset$.

Let $(\{x_n\}, \{y_n\})$ be the sequence generated for $(x_0, y_0) \in C \times Q$ defined by

$$\left\{ \begin{array}{l} w_n = J_\lambda^{T_1}((1-t_n)x_n - \gamma_n A^*(A(1-t_n)x_n - B(1-t_n)y_n)), \\ z_n = J_\lambda^{T_2}((1-t_n)y_n + \gamma_n B^*(A(1-t_n)x_n - B(1-t_n)y_n)), \\ F(u_n, u) + \langle \phi w_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - w_n \rangle \geq 0 \quad \forall u \in C, \\ G(v_n, v) + \langle \psi z_n, v - v_n \rangle + \frac{1}{r_n} \langle v - v_n, v_n - z_n \rangle \geq 0 \quad \forall v \in Q, \\ x_{n+1} = (1 - \beta_n)S_1 u_n + \beta_n u_n, \\ y_{n+1} = (1 - \delta_n)S_2 v_n + \delta_n v_n, \quad \forall n \geq 0, \end{array} \right. \quad (3.3.65)$$

$\{\gamma_n\}$ is a positive real sequence such that

$$\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \epsilon \right), \quad n \in \Omega,$$

otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$, A^* and B^* are adjoints of A and B respectively. Suppose $\{r_n\} \subset (0, \infty)$, $\{t_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions

- (i) $\lim_{n \rightarrow \infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$;
- (ii) $0 < k_1 \leq \liminf_{n \rightarrow \infty} \beta_n$ and $0 < k_2 \leq \liminf_{n \rightarrow \infty} \delta_n$;
- (iii) $0 < r_n < 2\alpha_i, i = 1, 2$.

If $\Gamma := (F(S_1) \cap EP(F, \phi) \cap I(T_1)) \times (F(S_2) \cap EP(G, \psi) \cap I(T_2)) \neq \emptyset$, then

the sequence $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.

3.4 Numerical Example

In this section, we give a numerical example of our Theorem 3.3.1. Using Matlab version 2014a, we show how the sequence values are affected by the number of iterations. This is done in order to see how the initial values and tolerance levels affect the number of iterations.

Let $H_1 = H_2 = H_3 = \mathbb{R}^2$, together with the usual norm on \mathbb{R}^2 . Let the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $\langle \hat{x}, \hat{y} \rangle = \hat{x} \cdot \hat{y} = x_1y_1 + x_2y_2$, for all $\hat{x} = (x_1, x_2)$ and $\hat{y} = (y_1, y_2)$. Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $G : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $F(\hat{x}, \hat{y}) = -3\hat{x}^2 + \hat{x}\hat{y} + 2\hat{y}^2$, $G(\hat{x}, \hat{y}) = -4\hat{x}^2 + \hat{x}\hat{y} + 3\hat{y}^2$ respectively. Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\phi(\hat{x}) = \hat{x}$ and $\psi(\hat{x}) = 2\hat{x}$ respectively. It is easy to check that F and G satisfy condition L1 - L4. For each $\hat{w}_n \in \mathbb{R}^2$, $r > 0$, Lemma 3.2.3 ensures that there exists $\hat{u}_n \in \mathbb{R}^2$ such that for any $\hat{x} \in \mathbb{R}^2$

$$F(\hat{u}_n, \hat{x}) + \langle \phi\hat{w}_n, \hat{x} - \hat{u}_n \rangle + \frac{1}{r_n} \langle \hat{x} - \hat{u}_n, \hat{u}_n - \hat{w}_n \rangle \geq 0. \quad (3.4.1)$$

$$\begin{aligned} &\iff (-3u_{n,1}^2 + u_{n,1}x_1 + 2x_1^2, -3u_{n,2}^2 + u_{n,2}x_2 + 2x_2^2) + \left((w_{n,1}, w_{n,2})(x_1 - u_{n,1}, x_2 - u_{n,2}) \right) \\ &\quad + \frac{1}{r_n} \left((x_1 - u_{n,1}, x_2 - u_{n,2})(u_{n,1} - w_{n,1}, u_{n,2} - w_{n,2}) \right) \geq 0, \\ &\iff \left(-3r_n u_{n,1}^2 + r_n x_1 u_{n,1} + 2r_n x_1^2 + r_n x_1 w_{n,1} - r_n w_{n,1} u_{n,1}, -3r_n u_{n,2}^2 + r_n x_2 u_{n,2} \right. \\ &\quad \left. + 2r_n x_2^2 + r_n x_2 w_{n,2} - r_n w_{n,2} u_{n,2} \right) + \left(x_1 u_{n,1} - x_1 w_{n,1} - u_{n,1}^2 + u_{n,1} w_{n,1}, \right. \\ &\quad \left. x_2 u_{n,2} - x_2 w_{n,2} - u_{n,2}^2 + u_{n,2} w_{n,2} \right) \geq 0, \\ &\iff \left(-3r_n u_{n,1}^2 + r_n x_1 u_{n,1} + 2r_n x_1^2 + r_n x_1 w_{n,1} - r_n w_{n,1} u_{n,1} + x_1 u_{n,1} - x_1 w_{n,1} - u_{n,1}^2 \right. \\ &\quad \left. + u_{n,1} w_{n,1}, -3r_n u_{n,2}^2 + r_n x_2 u_{n,2} + 2r_n x_2^2 + r_n x_2 w_{n,2} - r_n w_{n,2} u_{n,2} + x_2 u_{n,2} \right. \\ &\quad \left. - x_2 w_{n,2} - u_{n,2}^2 + u_{n,2} w_{n,2} \right) \geq 0, \\ &\iff \left(2r_n x_1^2 + r_n x_1 u_{n,1} + x_1 u_{n,1} + r_n x_1 w_{n,1} - x_1 w_{n,1} - 3r_n u_{n,1}^2 - r_n w_{n,1} u_{n,1} - u_{n,1}^2 \right. \\ &\quad \left. + u_{n,1} w_{n,1}, 2r_n x_2^2 + r_n x_2 u_{n,2} + x_2 u_{n,2} + r_n x_2 w_{n,2} - x_2 w_{n,2} - 3r_n u_{n,2}^2 - r_n w_{n,2} u_{n,2} \right. \\ &\quad \left. - u_{n,2}^2 + u_{n,2} w_{n,2} \right) \geq 0, \\ &\iff \left(2r_n x_1^2 + ((r_n + 1)u_{n,1} + (r_n - 1)w_{n,1})x_1 - 3r_n u_{n,1}^2 - r_n w_{n,1} u_{n,1} - u_{n,1}^2 + u_{n,1} w_{n,1}, \right. \\ &\quad \left. 2r_n x_2^2 + ((r_n + 1)u_{n,2} + (r_n - 1)w_{n,2})x_2 - 3r_n u_{n,2}^2 - r_n w_{n,2} u_{n,2} - u_{n,2}^2 + u_{n,2} w_{n,2} \right) \\ &\quad \geq 0. \end{aligned}$$

Let $G(x_1) = 2r_n x_1^2 + ((r_n + 1)u_{n,1} + (r_n - 1)w_{n,1})x_1 - 3r_n u_{n,1}^2 - r_n w_{n,1} u_{n,1} - u_{n,1}^2 + u_{n,1} w_{n,1}$, and $G(x_2) = 2r_n x_2^2 + ((r_n + 1)u_{n,2} + (r_n - 1)w_{n,2})x_2 - 3r_n u_{n,2}^2 - r_n w_{n,2} u_{n,2} - u_{n,2}^2 + u_{n,2} w_{n,2}$.

Then $G(x_1)$ and $G(x_2)$ are quadratic functions of x_1, x_2 respectively with coefficients:

$$a_1 = 2r_n, \quad b_1 = (r_n + 1)u_{n,1} + (r_n - 1)w_{n,1}, \quad c_1 = -3r_n u_{n,1}^2 - r_n w_{n,1} u_{n,1} - u_{n,1}^2 + u_{n,1} w_{n,1},$$

$$a_2 = 2r_n, \quad b_2 = (r_n + 1)u_{n,2} + (r_n - 1)w_{n,1}, \quad c_2 = -3r_n u_{n,2}^2 - r_n w_{n,2} u_{n,2} - u_{n,2}^2 + u_{n,2} w_{n,2}.$$

So the determinant Δ_1 of $G(x_1)$ is obtained as follow

$$\begin{aligned} \Delta_1 &= b_1^2 - 4a_1c_1 \\ &= \left((r_n + 1)u_{n,1} + (r_n - 1)w_{n,1} \right)^2 + 8r_n(3r_n u_{n,1}^2 + r_n w_{n,1} u_{n,1} + u_{n,1}^2 - u_{n,1} w_{n,1}) \\ &= w_{n,1}^2 (r_n - 1)^2 + 2u_{n,1} w_{n,1} (r_n^2 - 1) + (r_n + 1)^2 u_{n,1}^2 + 24r_n^2 u_{n,1}^2 + 8r_n^2 w_{n,1} u_{n,1} \\ &\quad + 8r_n u_{n,1}^2 - 8r_n u_{n,1} w_{n,1} \\ &= w_{n,1}^2 (r_n - 1)^2 + r_n^2 u_{n,1}^2 + 2r_n u_{n,1}^2 + u_{n,1}^2 + 2r_n^2 u_{n,1} w_{n,1} - 2u_{n,1} w_{n,1} + 24r_n^2 u_{n,1}^2 \\ &\quad + 8r_n^2 w_{n,1} u_{n,1} - 8r_n u_{n,1} w_{n,1} \\ &= w_{n,1}^2 (r_n - 1)^2 + 10r_n^2 u_{n,1} w_{n,1} - 8r_n u_{n,1} w_{n,1} - 2u_{n,1} w_{n,1} + 25r_n^2 u_{n,1}^2 + 10r_n u_{n,1}^2 + u_{n,1}^2 \\ &= w_{n,1}^2 (r_n - 1)^2 + 2u_{n,1} w_{n,1} (5r_n^2 - 4r_n - 1) + u_{n,1}^2 (25r_n^2 + 10r_n + 1) \\ &= w_{n,1}^2 (r_n - 1)^2 + 2u_{n,1} w_{n,1} (r_n - 1) (5r_n + 1) + u_{n,1}^2 (5r_n + 1)^2 \\ &= \left(w_{n,1} (r_n - 1) + u_{n,1} (5r_n + 1) \right)^2 \geq 0. \end{aligned}$$

Thus $\Delta_1 \geq 0 \quad \forall x_1 \in \mathbb{R}$ and if (3.4.1) has at most one solution in \mathbb{R} , then $\Delta_1 \leq 0$, so we obtain $\Delta_1 = 0$. Thus

$$u_{n,1} = \frac{(1 - r_n)}{5r_n + 1} w_{n,1}. \quad (3.4.2)$$

Following similar approach as above, we obtain

$$u_{n,2} = \frac{(1 - r_n)}{5r_n + 1} w_{n,2}. \quad (3.4.3)$$

Thus from (3.4.2) and (3.4.3), we have

$$\widehat{u}_n = (u_{n,1}, u_{n,2}) = \left(\frac{(1 - r_n)}{5r_n + 1} w_{n,1}, \frac{(1 - r_n)}{5r_n + 1} w_{n,2} \right) = \frac{(1 - r_n) \widehat{w}_n}{5r_n + 1}. \quad (3.4.4)$$

Similarly, we obtain that

$$\widehat{v}_n = (v_{n,1}, v_{n,2}) = \left(\frac{(1 - 2r_n)}{7r_n + 1} z_{n,1}, \frac{(1 - 2r_n)}{7r_n + 1} z_{n,2} \right) = \frac{(1 - 2r_n) \widehat{z}_n}{7r_n + 1}.$$

Now, let $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T_1(\widehat{x}) = (-2x_1, -3x_2)$, $T_2(\widehat{x}) = (x_2 - x_1, x_2)$ respectively. From (3.2.1), we obtain the resolvent mappings associated with T_1 and T_2 as thus

$$\begin{aligned} J_\lambda^{T_1}(\widehat{x}) &= (I + \lambda T_1)^{-1}(\widehat{x}) \\ &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2\lambda & 0 \\ 0 & -3\lambda \end{pmatrix} \right]^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 2\lambda & 0 \\ 0 & 1 - 3\lambda \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(1-2\lambda)(1-3\lambda)} \begin{pmatrix} 1-3\lambda & 0 \\ 0 & 1-2\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{1-2\lambda} & 0 \\ 0 & \frac{1}{1-3\lambda} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \left(\frac{x_1}{1-2\lambda}, \frac{x_2}{1-3\lambda} \right). \tag{3.4.5}
\end{aligned}$$

Similarly as (3.4.5), we also obtain

$$J_\lambda^{T_2}(\hat{x}) = \left(\frac{x_1(1+\lambda) - \lambda x_2}{1-\lambda^2}, \frac{x_2}{1+\lambda} \right). \tag{3.4.6}$$

Let $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $f_1(x_1, x_2) = (2x_1, 2x_2)$ and $f_2(x_1, x_2) = (-x_1, -x_2)$. Using (3.4.5) and (3.4.6), we obtain

$$J_\lambda^{T_1}(I - \lambda f_1)\hat{x} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1-2\lambda}{1-3\lambda} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(x_1, \frac{(1-2\lambda)x_2}{1-3\lambda} \right)$$

and

$$J_\lambda^{T_2}(I - \lambda f_2)\hat{x} = \begin{pmatrix} \frac{1+\lambda}{1-\lambda} & \frac{-\lambda}{1-\lambda} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\frac{(1+\lambda)x_1 - \lambda x_2}{1-\lambda}, x_2 \right).$$

Also let $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$A(\hat{x}) = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$B(\hat{x}) = \begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Furthermore, let $S_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $S_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $S_1(\hat{x}) = -2\hat{x}$ and $S_2(\hat{x}) = \frac{-3}{2}\hat{x}$. Indeed, $S_1\hat{p} = \hat{p}$ if and only $\hat{p} = 0$. It follows that

$$\|S_1\hat{x} - S_1\hat{p}\|^2 = \|-2\hat{x} - 0\|^2 = 4\|\hat{x} - 0\|^2$$

and

$$\|\hat{x} - S_1\hat{x}\|^2 = \|\hat{x} - (-2\hat{x})\|^2 = \|3\hat{x}\|^2 = 9\|\hat{x} - 0\|^2.$$

Hence,

$$\begin{aligned}
\|S_1\hat{x} - \hat{p}\|^2 &= 4\|\hat{x} - 0\|^2 = \|\hat{x} - 0\|^2 + 3\|\hat{x} - 0\|^2 \\
&= \|\hat{x} - \hat{p}\|^2 + \frac{1}{3}\|\hat{x} - S_1\hat{x}\|^2.
\end{aligned}$$

Thus, S_1 is $\frac{1}{3}$ demi-contractive mapping. Also, $S_2\hat{q} = \hat{q}$ if and only if $\hat{q} = 0$. Thus

$$\|S_2\hat{y} - S_2\hat{q}\|^2 = \left\| -\frac{3}{2}\hat{y} - 0 \right\|^2 = \frac{9}{4}\|\hat{y} - 0\|^2$$

and

$$\|\hat{y} - S_2\hat{y}\|^2 = \left\| \hat{y} - \left(-\frac{3}{2}\hat{y}\right) \right\|^2 = \left\| \frac{5}{2}\hat{y} \right\|^2 = \frac{25}{4}\|\hat{y} - 0\|^2 \quad (3.4.7)$$

Hence

$$\begin{aligned} \|S_2\hat{y} - \hat{q}\|^2 &= \frac{9}{4}\|\hat{y} - 0\|^2 = \|\hat{y} - 0\|^2 + \frac{5}{4}\|\hat{y} - 0\|^2 \\ &= \|\hat{y} - \hat{q}\|^2 + \frac{1}{5}\|\hat{y} - S_2\hat{y}\|^2. \end{aligned}$$

Thus, S_2 is $\frac{1}{5}$ demi-contractive mapping.

By choosing $t_n = \frac{1}{n+1}$, $\beta_n = \frac{5n-1}{6(n+1)}$ and $\delta_n = \frac{2}{3(1+\frac{1}{n})}$, $r_n = \frac{n+1}{5n+3}$,

our iterative scheme (3.3.1) becomes: for $\hat{x}_0 \in \mathbb{R}^2$ and $\hat{y}_0 \in \mathbb{R}^2$

$$\left\{ \begin{array}{l} \hat{w}_n = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1-2\lambda}{1-3\lambda} \end{pmatrix} \left(\frac{n}{n+1}\right) (\hat{x}_n - \gamma_n A^T (A\hat{x}_n - B\hat{y}_n)), \\ \hat{z}_n = \begin{pmatrix} \frac{1+\lambda}{1-\lambda} & \frac{-\lambda}{1-\lambda} \\ 0 & 1 \end{pmatrix} \left(\frac{n}{n+1}\right) (\hat{y}_n + \gamma_n B^T (A\hat{x}_n - B\hat{y}_n)), \\ \hat{u}_n = \left(\frac{2n+1}{5n+4}\right) \hat{w}_n, \\ \hat{v}_n = \left(\frac{3n+1}{12n+10}\right) \hat{z}_n, \\ \hat{x}_{n+1} = \left(\frac{n+7}{6(n+1)}\right) S_1 \hat{u}_n + \left(\frac{5n-1}{6(n+1)}\right) \hat{u}_n, \quad \forall n \geq 0, \\ \hat{y}_{n+1} = \left(\frac{n+3}{3(n+1)}\right) S_2 \hat{v}_n + \left(\frac{2n}{3(n+1)}\right) \hat{v}_n, \quad \forall n \geq 0, \end{array} \right. \quad (3.4.8)$$

where A^T and B^T are transposes of A and B respectively and

$$\gamma_n \in \left(\epsilon, \frac{2\|A\hat{x}_n - B\hat{y}_n\|^2}{\|A^*(A\hat{x}_n - B\hat{y}_n)\|^2 + \|B^*(A\hat{x}_n - B\hat{y}_n)\|^2} - \epsilon \right), \quad n \in \Omega,$$

otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : A\hat{x}_n - B\hat{y}_n \neq 0\}$,

Case A

(i) Take $\hat{x}_0 = (1, 0.5)^T$, $\hat{y}_0 = (-0.5, 2.2)^T$ and $\lambda = 0.002$.

(ii) Take $\hat{x}_0 = (10, -5.78)^T$, $\hat{y}_0 = (-0.278, 1)^T$ and $\lambda = 0.1$.

Case B

- (i) Take $\hat{x}_0 = (0.5, 0.003)^T$, $\hat{y}_0 = (0.3, 0.005)^T$ and $\lambda = 0.001$.
- (ii) Take $\hat{x}_0 = (1.1, 0.2)^T$, $\hat{y}_0 = (1, -0.2)^T$ and $\lambda = 0.001$.

The Matlab version used is R2014a and the execution times with different tolerance levels are as follows:

1. (case A(i), $\varepsilon = 10^{-4}$) and execution time is 0.044 sec.
2. (case A(i), $\varepsilon = 10^{-6}$) and execution time is 0.045 sec.
3. (case A(i), $\varepsilon = 10^{-12}$) and execution time is 0.047 sec.
4. (case A(ii), $\varepsilon = 10^{-4}$) and execution time is 0.044 sec.
5. (case A(ii), $\varepsilon = 10^{-6}$) and execution time is 0.046 sec.
6. (case A(ii), $\varepsilon = 10^{-12}$) and execution time is 0.050 sec.
7. (case B(i), $\varepsilon = 10^{-4}$) and execution time is 0.044 sec.
8. (case B(i), $\varepsilon = 10^{-6}$) and execution time is 0.046 sec.
9. (case B(i), $\varepsilon = 10^{-12}$) and execution time is 0.049 sec.
10. (case B(ii), $\varepsilon = 10^{-4}$) and execution time is 0.004 sec.
11. (case B(ii), $\varepsilon = 10^{-6}$) and execution time is 0.044 sec.
12. (case B(ii), $\varepsilon = 10^{-12}$) and execution time is 0.047 sec.

See Figure 6.1, Figure 6.2, Figure 6.3 and Figure 6.4 to see how the sequence values are affected by the number of iteration.

**Approximation of Common Solution of Split Equalities for
Generalized Mixed Equilibrium Problem and Fixed Point Problem
for Multi-Valued Mappings in Hilbert Spaces**

In this chapter, we introduce an iterative algorithm for approximating a common element of the set of solutions of split equalities for finite family of generalized mixed equilibrium problem and the set of common fixed points of k -strictly pseudo-nonspreading multi-valued mappings of type-one, without prior knowledge of the operator norm in real Hilbert space. We state and prove a strong convergence theorem for the sequence generated by our iterative algorithm and give numerical example of our main theorem.

4.1 Introduction

Let X be a normed space, C a nonempty closed subset of X and let $CB(X)$ denote the family of nonempty closed and bounded subsets of X . The Hausdorff metric on $CB(X)$ is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

for $A, B \in CB(X)$, where $d(x, B) = \inf\{\|x - y\| : y \in B\}$.

Also, a subset C of X is called *proximal* if for each $x \in X$, there exist $y \in C$ such that

$$\|x - y\| = \inf\{\|x - u\| : u \in C\} = d(x, C).$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximal. A multi-valued mapping $T : X \rightarrow 2^X$ is said to be of type-one if for any given $x, y \in X$,

$$\|u - v\| \leq H(Tx, Ty), \tag{4.1.1}$$

for all $u \in Tx$ and $v \in Ty$, [56].

Let $T : C \rightarrow CB(X)$ be a multi-valued mapping, T is called L -Lipschitzian if there exists $L \geq 0$ such that

$$H(Tx, Ty) \leq L\|x - y\| \quad \forall x, y \in C. \quad (4.1.2)$$

T is said to be nonexpansive if $L = 1$, while T is said to be a contraction if $L \in [0, 1)$. If $F(T) \neq \emptyset$ and for all $p \in F(T)$, then T is said to be quasi-nonexpansive if

$$H(Tx, Tp) \leq \|x - p\| \quad \forall x \in C. \quad (4.1.3)$$

Let E be a real smooth, strictly convex and reflexive Banach space, and let j denote the duality mapping of E and C a nonempty, closed and convex subset of E . A single-valued mapping $T : C \rightarrow C$ is said to be nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \quad \forall x, y \in C, \quad (4.1.4)$$

where

$$\phi(x, y) = \|x\|^2 - 2\langle x, jy \rangle + \|y\|^2 \quad \forall x, y \in C.$$

This class of mappings is deduced from the class of single-valued firmly nonexpansive mappings, see for example [53, 67]. Observe that if E is a real Hilbert space, then j is the identity and

$$\phi(x, y) = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 = \|x - y\|^2.$$

Thus (4.1.4) becomes

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \quad \forall x, y \in C. \quad (4.1.5)$$

It is shown in [70] that (4.1.5) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle \quad \forall x, y \in C. \quad (4.1.6)$$

We observe that if T is a single-valued nonspreading mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.

A multi-valued mapping $T : C \rightarrow CB(X)$ is said to be nonspreading if $2\|u - v\|^2 \leq \|u - y\|^2 + \|v - x\|^2$ for $u \in Tx$ and $v \in Ty$. Following the terminology used by Browder-Petryshyn [25], we say a multi-valued mapping $T : C \rightarrow CB(X)$ is k -strictly pseudo-nonspreading mapping of type-one if there exist $k \in [0, 1)$ such that

$$\|u - v\|^2 \leq \|x - y\|^2 + k\|x - u - (y - v)\|^2 + 2\langle x - u, y - v \rangle, \quad \forall x, y \in C, \quad (4.1.7)$$

where $u \in Tx$ with $\|x - u\| = d(x, Tx)$ and $v \in Ty$ with $\|y - v\| = d(y, Ty)$.

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed convex subsets of H_1 and H_2 respectively. Let $F : C \times C \rightarrow \mathbb{R}$ and $G : Q \times Q \rightarrow \mathbb{R}$ be two nonlinear bifunctions, $T : C \rightarrow C$ and $P : Q \rightarrow Q$ be two nonlinear mappings and $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$

and $\varphi : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex mappings such that $C \cap \text{dom}(\phi) \neq \emptyset$ and $Q \cap \text{dom}(\varphi) \neq \emptyset$. Let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be bounded linear mappings. The so-called Split Equality Generalized Mixed Equilibrium Problem (SEGMEP) is to find $x^* \in C$ and $y^* \in Q$ such that:

$$\begin{aligned} F(x^*, x) + \langle T(x^*), x - x^* \rangle + \phi(x) - \phi(x^*) &\geq 0 & \forall x \in C, \\ G(y^*, y) + \langle P(y^*), y - y^* \rangle + \varphi(y) - \varphi(y^*) &\geq 0, & \forall y \in Q \end{aligned} \quad (4.1.8)$$

and

$$Ax^* = By^*.$$

4.2 Preliminaries

In this section we shall state some well known results which will be used in the sequel to obtain our result in this chapter.

Lemma 4.2.1. [73] *Let C be a nonempty closed convex subset of a real Hilbert space and $S : C \rightarrow CB(X)$ be a k strictly pseudo-nonspreading multi-valued mapping and $F(S) \neq \emptyset$ with $Sp = \{p\}$ for $p \in F(S)$, then $F(S)$ is closed and convex.*

Lemma 4.2.2. [73] *Let C be a nonempty closed and convex subset of a real Hilbert space H and $S : C \rightarrow CB(X)$ be a k strictly pseudo-nonspreading multi-valued mapping and $F(S) \neq \emptyset$, then $(I - S)$ is demiclosed at 0.*

To solve the equilibrium problem, we assume that the bifunction $F : C \times C \rightarrow \mathbb{R}$ satisfies the conditions L1 - L4 in Section 2.4.1.

Lemma 4.2.3. [99] *Let C be a nonempty closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let $B : C \rightarrow E^*$ be a continuous and monotone mapping, $\phi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function and $F : C \times C \rightarrow \mathbb{R}$ be a bifunction which satisfies (L1)-(L4). Let $r > 0$ be any given number and $x \in E$ be any given point. Then the following holds:*

1. *there exist $z \in C$ such that*

$$F(z, y) + \langle Bz, y - z \rangle + \phi(y) - \phi(z) + \frac{1}{r} \langle y - z, jz - jx \rangle \geq 0 \quad \forall y \in C, \quad (4.2.1)$$

where $j : E \rightarrow 2^{E^}$ is the normalized duality mapping which is defined by*

$$j(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|\} \quad \forall x \in E.$$

2. *If we define a resolvent mapping $T_r^F : H \rightarrow C$ by*

$$T_r^F(x) = \{z \in C : F(z, y) + \langle Tz, y - z \rangle + \phi(y) - \phi(z) + \frac{1}{r} \langle y - z, jz - jx \rangle \geq 0, \quad \forall y \in C\},$$

then the following results hold;

- i. for each $x \in C, T_r^F(x) \neq \emptyset$,
- ii. T_r^F is singled-valued,
- iii. T_r^F is firmly nonexpansive, i.e for any $x, y \in H$

$$\langle T_r^F x - T_r^F y, jT_r^F x - jT_r^F y \rangle \leq \langle T_r^F x - T_r^F y, jx - jy \rangle,$$

- iv. $F(T_r^F) = GMEP(F, B, \phi)$ and
- v. $GMEP(F, B, \phi)$ is closed and convex.

We note that if E is a real Hilbert space then j becomes the identity mapping and T_r^F is defined as:

$$T_r^F(x) = \{z \in C : F(z, y) + \langle Tz, y - z \rangle + \phi(y) - \phi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C\}.$$

Lemma 4.2.4. [117] *Let C be a nonempty, closed and convex subset of H . Assume that the $F : C \times C \rightarrow \mathbb{R}$ satisfies L1 – L4 and $r > 0$. Then $\forall x \in H$ and $q \in F(T_r^F)$,*

$$\|T_r^F x - x\|^2 \leq \|x - q\|^2 - \|T_r^F x - q\|^2.$$

4.3 Main Result

In this section, we state and prove our main result in this chapter.

Theorem 4.3.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed convex subsets of H_1 and H_2 respectively. Assume that for $i = 1, 2, \dots, m, l = 1, 2, \dots, N$, $F_i : C \times C \rightarrow \mathbb{R}$ and $G_l : Q \times Q \rightarrow \mathbb{R}$ are bifunctions which satisfy (L1) – (L4) and the mappings $T_i : C \rightarrow C$ and $P_l : Q \rightarrow Q$ are continuous monotone mappings. Let $\phi_i : C \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_l : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex mappings such that $C \cap \text{dom}\phi_i \neq \emptyset$ and $Q \cap \text{dom}\varphi_l \neq \emptyset$ for $i = 1, 2, \dots, m, l = 1, 2, \dots, N$. Let $CB(C)$ and $CB(Q)$ be closed and bounded subsets of C and Q respectively and let $S_1 : C \rightarrow CB(C)$ and $S_2 : Q \rightarrow CB(Q)$ be two k strictly pseudo-nonspreading multi-valued mappings of type-one with constants k_1 and k_2 respectively, where $k_1, k_2 \in [0, 1)$. Assume $F(S_1) \neq \emptyset$ with $S_1 p = \{p\}$ for all $p \in F(S_1)$ and $F(S_2) \neq \emptyset$ with $S_2 q = \{q\}$ for all $q \in F(S_2)$. Let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear operators and $\Gamma := (F(S_1) \cap_{i=1}^m GMEP(F_i, T_i, \phi_i)) \times (F(S_2) \cap_{l=1}^N GMEP(G_l, P_l, \varphi_l)) \neq \emptyset$. Let $(x_0, y_0) \in H_1 \times H_2$ and the iterative scheme $(\{x_n\}, \{y_n\})$ be defined as follows:*

$$\left\{ \begin{array}{l} w_n = \alpha_n u + (1 - \alpha_n) x_n; \\ z_n = \alpha_n v + (1 - \alpha_n) y_n, \\ u_n = T_{r_{m,n}}^{F_m} \circ T_{r_{m-1,n}}^{F_{m-1}} \circ \dots \circ T_{r_{1,n}}^{F_1} (w_n - \gamma_n A^*(Aw_n - Bz_n)) \quad \text{for } r_{i,n} > 0, \\ v_n = T_{r_{N,n}}^{G_N} \circ T_{r_{N-1,n}}^{G_{N-1}} \circ \dots \circ T_{r_{1,n}}^{G_1} (z_n + \gamma_n B^*(Aw_n - Bz_n)) \quad \text{for } r_{l,n} > 0, \\ x_{n+1} = (1 - \beta_n) u_n + \beta_n d_n, \\ y_{n+1} = (1 - \delta_n) v_n + \delta_n c_n, \end{array} \right. \quad (4.3.1)$$

for every $u \in C, v \in Q, d_n \in S_1 u_n$, with $\|u_n - d_n\| = d(u_n, S_1 u_n)$, $c_n \in S_2 v_n$, with $\|v_n - c_n\| = d(v_n, S_2 v_n)$, $n \geq 0$ and $\{\gamma_n\}$ is a positive real sequence such that

$$\gamma_n \in \left(\epsilon, \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \epsilon \right), \quad n \in \Omega.$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequence in $(0, 1)$ such that:

- i. $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- ii. $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- iii. $\beta_n \in (a, 1 - k_1) \subseteq (0, 1)$ for some $a > 0$,
- iv. $\delta_n \in (b, 1 - k_2) \subseteq (0, 1)$ for some $b > 0$.

Then the sequence $(\{x_n\}, \{y_n\})$ defined by the iterative scheme (4.3.1) converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. : Let $(x^*, y^*) \in \Gamma$, $\Theta_n^m = T_{r_{n,m}}^{F_m} T_{r_{n,m-1}}^{F_{m-1}} \dots T_{r_{n,2}}^{F_2} T_{r_{n,1}}^{F_1}$, where $\Theta_n^0 = I$ and $\psi_n^N = T_{r_{n,N}}^{G_N} T_{r_{n,N-1}}^{G_{N-1}} \dots T_{r_{n,2}}^{G_2} T_{r_{n,1}}^{G_1}$, where $\psi_n^0 = I$. Putting $a_n = w_n - \gamma_n A^*(Aw_n - Bz_n)$ and $b_n = z_n + \gamma_n B^*(Aw_n - Bz_n)$. Then,

$$\begin{aligned} \|u_n - x^*\|^2 &= \|\Theta_n^m a_n - x^*\|^2 \\ &= \|T_{r_{n,m}}^{F_m} \Theta_n^{m-1} a_n - x^*\|^2 \\ &\leq \|\Theta_n^{m-1} a_n - x^*\|^2 \\ &\vdots \\ &\leq \|a_n - x^*\|^2 \\ &= \|w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*\|^2 \\ &= \|w_n - x^* - \gamma_n A^*(Aw_n - Bz_n)\|^2 \\ &\leq \|w_n - x^*\|^2 + \gamma_n^2 \|A^*(Aw_n - Bz_n)\|^2 - 2\gamma_n \langle w_n - x^*, A^*(Aw_n - Bz_n) \rangle \\ &= \|w_n - x^*\|^2 + \gamma_n^2 \|A^*(Aw_n - Bz_n)\|^2 - 2\gamma_n \langle Aw_n - Ax^*, Aw_n - Bz_n \rangle \\ &= \|w_n - x^*\|^2 + \gamma_n^2 \|A^*(Aw_n - Bz_n)\|^2 - \gamma_n \left(\|Aw_n - Ax^*\|^2 + \|Aw_n - Bz_n\|^2 \right. \\ &\quad \left. - \|Bz_n - Ax^*\|^2 \right) \\ &= \|w_n - x^*\|^2 + \gamma_n^2 \|A^*(Aw_n - Bz_n)\|^2 - \gamma_n \|Aw_n - Ax^*\|^2 - \gamma_n \|Aw_n - Bz_n\|^2 \\ &\quad + \gamma_n \|Bz_n - Ax^*\|^2. \end{aligned} \tag{4.3.2}$$

Similarly, we have $\|v_n - y^*\|^2 \leq \|b_n - y^*\|^2$ and

$$\begin{aligned} \|v_n - y^*\|^2 &\leq \|z_n - y^*\|^2 + \gamma_n^2 \|B^*(Aw_n - Bz_n)\|^2 + \gamma_n \|Aw_n - By^*\|^2 - \gamma_n \|Bz_n - By^*\|^2 \\ &\quad - \gamma_n \|Aw_n - Bz_n\|^2. \end{aligned} \tag{4.3.3}$$

Adding (4.3.2) and (4.3.3) and noting that $Ax^* = By^*$, we have

$$\begin{aligned} \|u_n - x^*\|^2 + \|v_n - y^*\|^2 &\leq \|w_n - x^*\|^2 + \|z_n - y^*\|^2 - \gamma_n \left(2\|Aw_n - Bz_n\|^2 \right. \\ &\quad \left. + \gamma_n (\|A^*(Aw_n - Bz_n)\|^2 \right. \\ &\quad \left. + \|B^*(Aw_n - Bz_n)\|^2) \right). \end{aligned} \tag{4.3.4}$$

Therefore,

$$\|u_n - x^*\|^2 + \|v_n - y^*\|^2 \leq \|w_n - x^*\|^2 + \|z_n - y^*\|^2.$$

Also,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n)u_n + \beta_n d_n - x^*\|^2 \\ &= \|(1 - \beta_n)(u_n - x^*) + \beta_n(d_n - x^*)\|^2 \\ &= (1 - \beta_n)\|u_n - x^*\|^2 + \beta_n\|d_n - x^*\|^2 - \beta_n(1 - \beta_n)\|d_n - u_n\|^2 \\ &\leq (1 - \beta_n)\|u_n - x^*\|^2 + \beta_n H^2(S_1 u_n, S_1 x^*) - \beta_n(1 - \beta_n)\|d_n - u_n\|^2 \\ &\leq (1 - \beta_n)\|u_n - x^*\|^2 + \beta_n \left(\|u_n - x^*\|^2 + k_1 \|u_n - d_n - (x^* - x^*)\|^2 \right. \\ &\quad \left. + 2\langle u_n - d_n, x^* - x^* \rangle \right) - \beta_n(1 - \beta_n)\|d_n - u_n\|^2 \\ &= (1 - \beta_n)\|u_n - x^*\|^2 + \beta_n\|u_n - x^*\|^2 + \beta_n k_1 \|u_n - d_n\|^2 \\ &\quad - \beta_n(1 - \beta_n)\|d_n - u_n\|^2 \\ &= \|u_n - x^*\|^2 - \beta_n(1 - \beta_n - k_1)\|u_n - d_n\|^2. \end{aligned} \quad (4.3.5)$$

Hence,

$$\|x_{n+1} - x^*\|^2 \leq \|u_n - x^*\|^2. \quad (4.3.6)$$

Similarly as (4.3.5), we obtain

$$\|y_{n+1} - y^*\|^2 \leq \|v_n - y^*\|^2 - \delta_n(1 - \delta_n - k_2)\|v_n - c_n\|^2. \quad (4.3.7)$$

Hence,

$$\|y_{n+1} - y^*\|^2 \leq \|v_n - y^*\|^2. \quad (4.3.8)$$

Thus,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq \|u_n - x^*\|^2 + \|v_n - y^*\|^2 \\ &\leq \|w_n - x^*\|^2 + \|z_n - y^*\|^2. \end{aligned} \quad (4.3.9)$$

But

$$\begin{aligned} \|w_n - x^*\|^2 &= \|\alpha_n u + (1 - \alpha_n)x_n - x^*\|^2 \\ &= \|\alpha_n(u - x^*) + (1 - \alpha_n)(x_n - x^*)\|^2 \\ &= \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|(x_n - x^*)\|^2 - \alpha_n(1 - \alpha_n)\|x_n - u\|^2 \\ &\leq \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2, \end{aligned} \quad (4.3.10)$$

and

$$\begin{aligned} \|z_n - y^*\|^2 &= \|\alpha_n v + (1 - \alpha_n)y_n - y^*\|^2 \\ &= \|\alpha_n(v - y^*) + (1 - \alpha_n)(y_n - y^*)\|^2 \\ &= \alpha_n\|v - y^*\|^2 + (1 - \alpha_n)\|y_n - y^*\|^2 - \alpha_n(1 - \alpha_n)\|y_n - v\|^2. \\ &\leq \alpha_n\|v - y^*\|^2 + (1 - \alpha_n)\|y_n - y^*\|^2. \end{aligned} \quad (4.3.11)$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 + \alpha_n \|v - y^*\|^2 \\
&\quad + (1 - \alpha_n) \|y_n - y^*\|^2 \\
&= \alpha_n (\|u - x^*\|^2 + \|v - y^*\|^2) + (1 - \alpha_n) (\|x_n - x^*\|^2 \\
&\quad + \|y_n - y^*\|^2) \\
&\leq \max\{\|u - x^*\|^2 + \|v - y^*\|^2, \|x_n - x^*\|^2 + \|y_n - y^*\|^2\} \\
&\leq \max\{\|u - x^*\|^2 + \|v - y^*\|^2, \|x_0 - x^*\|^2 + \|y_0 - y^*\|^2\}.
\end{aligned}$$

Therefore, $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ and consequently $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$, $\{Aw_n\}$ and $\{Bz_n\}$ are bounded.

Also, from (4.3.4), (4.3.5), (4.3.7), (4.3.10) and (4.3.11), we obtain,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq \alpha_n (\|u - x^*\|^2 + \|v - y^*\|^2) + (1 - \alpha_n) (\|x_n - x^*\|^2 \\
&\quad + \|y_n - y^*\|^2) - \alpha_n (1 - \alpha_n) (\|x_n - u\|^2 + \|y_n - v\|^2) \\
&\quad - \gamma_n \left(2\|Aw_n - Bz_n\|^2 + \gamma_n (\|A^*(Aw_n - Bz_n)\|^2 \right. \\
&\quad \left. + \|B^*(Aw_n - Bz_n)\|^2) \right) - \beta_n (1 - \beta_n - k_1) \|u_n - d_n\|^2 \\
&\quad - \delta_n (1 - \delta_n - k_2) \|v_n - c_n\|^2. \tag{4.3.12}
\end{aligned}$$

We now divide the rest of the proof into two cases:

Case1: Assume that $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is monotonically decreasing.

Putting $\rho_n(x^*, y^*) := \|x_n - x^*\|^2 + \|y_n - y^*\|^2$, we have:

$$\begin{aligned}
\rho_{n+1}(x^*, y^*) &\leq \rho_n(x^*, y^*) + \alpha_n (\|u - x^*\|^2 + \|v - y^*\|^2) - \alpha_n (\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&\quad - \alpha_n (1 - \alpha_n) (\|x_n - u\|^2 + \|y_n - v\|^2) - \gamma_n \left(2\|Aw_n - Bz_n\|^2 \right. \\
&\quad \left. + \gamma_n (\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2) \right) \\
&\quad - \beta_n (1 - \beta_n - k_1) \|u_n - d_n\|^2 - \delta_n (1 - k_2 - \delta_n) \|v_n - c_n\|^2. \tag{4.3.13}
\end{aligned}$$

Clearly, $(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2) - (\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \rightarrow 0$ as $n \rightarrow \infty$.

Putting $K_n = \|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2$ and $n \in \Omega$, then it follows from 4.3.13 that

$$\begin{aligned}
\gamma_n \left(2\|Aw_n - Bz_n\|^2 - \gamma_n K_n \right) &\leq \alpha_n (\|u - x^*\|^2 + \|v - y^*\|^2) - \alpha_n (1 - \alpha_n) (\|x_n - u\|^2 \\
&\quad + \|y_n - v\|^2) - \alpha_n (\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&\quad + \rho_n(x^*, y^*) - \rho_{n+1}(x^*, y^*),
\end{aligned}$$

which implies that

$$\begin{aligned}
\gamma_n K_n &\leq \alpha_n (\|u - x^*\|^2 + \|v - y^*\|^2) - \alpha_n (1 - \alpha_n) (\|x_n - u\|^2 + \|y_n - v\|^2) \\
&\quad - \alpha_n (\|x_n - x^*\|^2 + \|y_n - y^*\|^2) + \rho_n(x^*, y^*) - \rho_{n+1}(x^*, y^*) \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

By the condion

$$\gamma_n \in \left(\epsilon, \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \epsilon \right), \quad n \in \Omega,$$

we have that,

$$\lim_{n \rightarrow \infty} K_n = \lim_{n \rightarrow \infty} (\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2) = 0. \quad (4.3.14)$$

Note that $Aw_n - Bz_n = 0$, if $n \notin \Omega$. Thus,

$$\lim_{n \rightarrow \infty} \|A^*(Aw_n - Bz_n)\| = \lim_{n \rightarrow \infty} \|B^*(Aw_n - Bz_n)\| = 0. \quad (4.3.15)$$

Also from (4.3.12), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq \alpha_n(\|u - x^*\|^2 + \|v - y^*\|^2) + (1 - \alpha_n)(\|x_n - x^*\|^2 \\ &\quad + \|y_n - y^*\|^2) - \alpha_n(1 - \alpha_n)(\|x_n - u\|^2 + \|y_n - v\|^2) \\ &\quad - \gamma_n^2 \left(\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2 \right) \\ &\quad - \beta_n(1 - \beta_n - k_1)\|u_n - d_n\|^2 \\ &\quad - \delta_n(1 - \delta_n - k_2)\|v_n - c_n\|^2. \end{aligned} \quad (4.3.16)$$

Therefore from (4.3.15), (4.3.16) and since $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\beta_n(1 - \beta_n - k_1)\|u_n - d_n\|^2 \longrightarrow 0, \quad \text{as } n \rightarrow \infty \quad (4.3.17)$$

and

$$\delta_n(1 - \delta_n - k_2)\|v_n - c_n\|^2 \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.3.18)$$

By conditions iii. and iv., we obtain

$$\lim_{n \rightarrow \infty} \|u_n - d_n\| = \lim_{n \rightarrow \infty} \|v_n - c_n\| = 0. \quad (4.3.19)$$

Also from (4.3.1), we have that

$$\|w_n - x_n\| = \|\alpha_n u + (1 - \alpha_n)x_n - x_n\| = \alpha_n \|u - x_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (4.3.20)$$

and

$$\|z_n - y_n\| = \|\alpha_n v + (1 - \alpha_n)y_n - y_n\| = \alpha_n \|v - y_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.3.21)$$

Since,

$$\begin{aligned} \|a_n - x_n\|^2 &= \|w_n - \gamma_n A^*(Aw_n - Bz_n) - x_n\|^2 \\ &= \|w_n - x_n\|^2 + \gamma_n^2 \|A^*(Aw_n - Bz_n)\|^2 - 2\gamma_n \langle w_n - x_n, A^*(Aw_n - Bz_n) \rangle \\ &\leq \|w_n - x_n\|^2 + \gamma_n^2 \|A^*(Aw_n - Bz_n)\|^2 + 2\gamma_n \|w_n - x_n\| \|A^*(Aw_n - Bz_n)\|. \end{aligned}$$

It follows from (4.3.15) and (4.3.20) that

$$\|a_n - x_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.3.22)$$

Similarly,

$$\|b_n - y_n\|^2 \leq \|z_n - y_n\|^2 + \gamma_n^2 \|A^*(Aw_n - Bz_n)\|^2 + 2\gamma_n \|z_n - y_n\| \|A^*(Aw_n - Bz_n)\|.$$

From (4.3.21), we obtain

$$\|b_n - y_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (4.3.23)$$

By the nonexpansivity of $T_{r_n, m}^{F_i}$, for $i = 1, 2, \dots, m$, we know that

$$\|\Theta_n^m a_n - x^*\|^2 = \|T_{r_n, m}^{F_m} \Theta_n^{m-1} a_n - x^*\|^2 \leq \|\Theta_n^{m-1} a_n - x^*\|^2 \leq \dots \leq \|u_n - x^*\|^2. \quad (4.3.24)$$

Also, by the nonexpansivity of $T_{r_n, N}^{G_l}$ for $l = 1, 2, \dots, N$, we know that

$$\|\psi_n^N b_n - y^*\|^2 = \|T_{r_n, N}^{G_N} \psi_n^{N-1} b_n - y^*\|^2 \leq \|\psi_n^{N-1} b_n - y^*\|^2 \leq \dots \leq \|v_n - y^*\|^2. \quad (4.3.25)$$

From Lemma 4.2.4, (4.3.6), (4.3.8), (4.3.22), (4.3.23), (4.3.24) and (4.3.25) we have that

$$\begin{aligned} \|u_n - \Theta_n^{m-1} a_n\|^2 + \|v_n - \psi_n^{N-1} b_n\|^2 &= \|T_{r_n, m}^{F_m} \Theta_n^{m-1} a_n - \Theta_n^{m-1} a_n\|^2 \\ &\quad + \|T_{r_n, N}^{G_N} \psi_n^{N-1} b_n - \psi_n^{N-1} b_n\|^2 \\ &\leq \|\Theta_n^{m-1} a_n - x^*\|^2 - \|u_n - x^*\|^2 + \|\psi_n^{N-1} b_n - y^*\|^2 \\ &\quad - \|v_n - y^*\|^2 \\ &\leq \|a_n - x^*\|^2 - \|u_n - x^*\|^2 + \|b_n - y^*\|^2 \\ &\quad - \|v_n - y^*\|^2 \\ &= \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - (\|u_n - x^*\|^2 \\ &\quad + \|v_n - y^*\|^2) \\ &\leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - (\|x_{n+1} - x^*\|^2 \\ &\quad + \|y_{n+1} - y^*\|^2) \\ &= \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - (\|x_n - x^*\|^2 \\ &\quad + \|y_n - y^*\|^2) + (\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\ &\quad - (\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2) \longrightarrow 0, \\ &\text{as } n \rightarrow \infty. \end{aligned} \quad (4.3.26)$$

Thus,

$$\lim_{n \rightarrow \infty} \|u_n - \Theta_n^{m-1} a_n\| = \lim_{n \rightarrow \infty} \|v_n - \psi_n^{N-1} b_n\| = 0. \quad (4.3.27)$$

Similarly, we have that

$$\begin{aligned} \|\Theta_n^{m-1} a_n - \Theta_n^{m-2} a_n\|^2 + \|\psi_n^{N-1} b_n - \psi_n^{N-2} b_n\|^2 &= \|T_{r_n, m-1}^{F_{m-1}} \Theta_n^{m-2} a_n - \Theta_n^{m-2} a_n\|^2 \\ &\quad + \|T_{r_n, N-1}^{G_{N-1}} \psi_n^{N-2} b_n - \psi_n^{N-2} b_n\|^2 \\ &\leq \|\Theta_n^{m-2} a_n - x^*\|^2 - \|\Theta_n^{m-1} a_n - x^*\|^2 \\ &\quad + \|\psi_n^{N-2} b_n - y^*\|^2 - \|\psi_n^{N-1} b_n - y^*\|^2 \\ &\leq \|a_n - x^*\|^2 - \|u_n - x^*\|^2 \\ &\quad + \|b_n - y^*\|^2 - \|v_n - y^*\|^2 \\ &\leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2 \\ &\quad - (\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2) \longrightarrow 0, \\ &\text{as } n \rightarrow \infty. \end{aligned} \quad (4.3.28)$$

Thus,

$$\lim_{n \rightarrow \infty} \|\Theta_n^{m-1} a_n - \Theta_n^{m-2} a_n\| = \lim_{n \rightarrow \infty} \|\psi_n^{N-1} b_n - \psi_n^{N-2} b_n\|^2 = 0. \quad (4.3.29)$$

In a similar way, we can verify that

$$\lim_{n \rightarrow \infty} \|\Theta_n^{m-2} a_n - \Theta_n^{m-3} a_n\| = \dots = \lim_{n \rightarrow \infty} \|\Theta_n^2 a_n - \Theta_n^1 a_n\| = 0$$

and

$$\lim_{n \rightarrow \infty} \|\psi_n^{N-2} b_n - \psi_n^{N-3} b_n\|^2 = \dots = \lim_{n \rightarrow \infty} \|\psi_n^2 b_n - \psi_n^1 b_n\|^2 = 0. \quad (4.3.30)$$

From (4.3.27), (4.3.29), (4.3.30), we can conclude that

$$\lim_{n \rightarrow \infty} \|\Theta_n^i a_n - \Theta_n^{i-1} a_n\| = 0, \quad i = 1, 2, 3, \dots, m, \quad (4.3.31)$$

and

$$\lim_{n \rightarrow \infty} \|\psi_n^l a_n - \psi_n^{l-1} a_n\| = 0, \quad l = 1, 2, \dots, N. \quad (4.3.32)$$

Thus,

$$\begin{aligned} \|u_n - a_n\| &\leq \|u_n - \Theta_n^{m-1} a_n\| + \|\Theta_n^{m-1} a_n - \Theta_n^{m-2} a_n\| + \|\Theta_n^{m-2} a_n - \Theta_n^{m-3} a_n\| \\ &\quad + \dots + \|\Theta_n^1 a_n - a_n\| \longrightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned} \quad (4.3.33)$$

and

$$\begin{aligned} \|v_n - b_n\| &\leq \|v_n - \psi_n^{N-1} b_n\| + \|\psi_n^{N-1} b_n - \psi_n^{N-2} b_n\| + \|\psi_n^{N-2} b_n - \psi_n^{N-3} b_n\| \\ &\quad + \dots + \|\psi^1 b_n - b_n\| \longrightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.3.34)$$

Therefore, from (4.3.22) and (4.3.33) we obtain

$$\|u_n - x_n\| \leq \|u_n - a_n\| + \|a_n - x_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.3.35)$$

Similarly, from (4.3.23) and (4.3.34), we have

$$\|v_n - y_n\| \leq \|v_n - b_n\| + \|b_n - y_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.3.36)$$

Also,

$$\|x_{n+1} - u_n\| = \|(1 - \beta_n)u_n + \beta_n d_n - u_n\| = \beta_n \|(u_n - d_n)\|, \quad (4.3.37)$$

and from (4.3.19), we have that

$$\|x_{n+1} - u_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.3.38)$$

Therefore, from (4.3.35) and (4.3.38)

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|u_n - x_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.3.39)$$

Similarly, from (4.3.19) we have that

$$\|y_{n+1} - v_n\| = \|(1 - \delta_n)v_n + \delta_n c_n - v_n\| = \delta_n \|v_n - c_n\| \longrightarrow 0, \text{ as } n \rightarrow \infty. \quad (4.3.40)$$

Therefore, from (4.3.36) and (4.3.40), we have

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - v_n\| + \|v_n - y_n\| \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.3.41)$$

Since $(\{x_n\}, \{y_n\})$ are bounded, there exist subsequence $(\{x_{n_j}\}, \{y_{n_j}\})$ such that $\{x_{n_j}\}$ converges weakly to $\bar{x} \in C$ and $\{y_{n_j}\}$ converges weakly to $\bar{y} \in Q$. From (4.3.35), we have $\{u_{n_j}\}$ converges weakly to \bar{x} and by the demi-closedness of $I - S_1$ at 0 and (4.3.19), we have that $\bar{x} \in F(S_1)$. Similarly, $\{v_{n_j}\}$ converges weakly to \bar{y} and by the demi-closedness of $I - S_2$ at 0, we have that $\bar{y} \in F(S_2)$.

Also since $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are bounded linear mapping and $\{Aw_{n_j}\} \rightharpoonup A\bar{x}$ and $\{Bz_{n_j}\} \rightharpoonup B\bar{y}$ and by the weak lower semicontinuity of the squared norm, we have

$$\|A\bar{x} - B\bar{y}\|^2 \leq \liminf_{n \rightarrow \infty} \|Aw_{n_j} - Bz_{n_j}\|^2 = 0. \quad (4.3.42)$$

Hence,

$$A\bar{x} = B\bar{y}. \quad (4.3.43)$$

We now prove that , $\bar{x} \in \cap_{i=1}^m GMEP(F_i, T_i, \phi_i)$ and $\bar{y} \in \cap_{l=1}^N GMEP(G_l, P_l, \varphi_l)$.

For each $i = 1, 2, \dots, m$, from Lemma 4.2.3, we have that $\forall u \in C$

$$F_i(\Theta_n^i a_n, u) + \langle T_i \Theta_n^i a_n, u - \Theta_n^i a_n \rangle + \phi_i(u) - \phi_i(\Theta_n^i a_n) + \frac{1}{r_{n,i}} \langle u - \Theta_n^i a_n, \Theta_n^i a_n - \Theta_n^{i-1} a_n \rangle \geq 0.$$

Replacing n by n_j and using (L2) and by the monotonicity of T_i , we obtain

$$\begin{aligned} \phi(u) - \phi(\Theta_{n_j}^i a_{n_j}) + \frac{1}{r_{n_j,i}} \langle u - \Theta_{n_j}^i a_{n_j}, \Theta_{n_j}^i a_{n_j} - \Theta_{n_j}^{i-1} a_{n_j} \rangle &\geq -F_i(\Theta_{n_j}^i a_{n_j}, u) - \langle T_i \Theta_{n_j}^i a_{n_j}, \\ &u - \Theta_{n_j}^i a_{n_j} \rangle \\ &= F_i(u, \Theta_{n_j}^i a_{n_j}) + \langle T_i u, \Theta_{n_j}^i a_{n_j} - u \rangle. \end{aligned}$$

Thus, for each $i = 1, 2, \dots, m$,

$$F_i(u, \Theta_{n_j}^i a_{n_j}) + \langle T_i u, \Theta_{n_j}^i a_{n_j} - u \rangle - \phi_i(u) + \phi_i(\Theta_{n_j}^i a_{n_j}) - \frac{1}{r_{n_j,i}} \langle u - \Theta_{n_j}^i a_{n_j}, \Theta_{n_j}^i a_{n_j} - \Theta_{n_j}^{i-1} a_{n_j} \rangle \leq 0.$$

By L4,

$$\frac{\Theta_{n_j}^i a_{n_j} - \Theta_{n_j}^{i-1} a_{n_j}}{r_{n_j,i}} \longrightarrow 0, \quad \text{for each } i = 1, 2, \dots, m,$$

and since

$$u_{n_j} = \Theta_{n_j}^i a_{n_j} \longrightarrow \bar{x},$$

by the proper lower semicontinuity of ϕ_i , it follows that for each $i = 1, 2, \dots, m$,

$$F_i(u, \bar{x}) + \langle T_i u, \bar{x} - u \rangle + \phi_i(\bar{x}) - \phi_i(u) \leq 0, \quad \forall u \in C.$$

Putting $z_t = tu + (1-t)\bar{x}$ for all $t \in (0, 1]$ and $u \in C$, since C is a convex set, thus $z_t \in C$, for each $i = 1, 2, \dots, m$, hence

$$F_i(z_t, \bar{x}) + \langle T_i z_t, \bar{x} - z_t \rangle + \phi_i(\bar{x}) - \phi_i(z_t) \leq 0.$$

So, from (L1), we have for each $i = 1, 2, \dots, m$,

$$\begin{aligned} 0 &= F_i(z_t, z_t) + \langle T_i z_t, z_t - z_t \rangle + \phi_i(z_t) - \phi_i(z_t) \\ &= F_i(z_t, tu + (1-t)\bar{x}) + \langle T_i z_t, tu + (1-t)\bar{x} - z_t \rangle + \phi_i(tu + (1-t)\bar{x}) - \phi_i(z_t) \\ &= tF_i(z_t, u) + (1-t)F_i(z_t, \bar{x}) + t\langle T_i z_t, u - z_t \rangle + (1-t)\langle T_i z_t, \bar{x} - z_t \rangle + t\phi(u) \\ &\quad + (1-t)\phi(\bar{x}) - (t\phi(z_t) + (1-t)\phi_i(z_t)) \\ &= t\left(F_i(z_t, u) + \langle T_i z_t, u - z_t \rangle + \phi_i(u) - \phi_i(z_t)\right) + (1-t)\left(F_i(z_t, \bar{x}) + \langle T_i z_t, \bar{x} - z_t \rangle\right. \\ &\quad \left.+ \phi_i(\bar{x}) - \phi_i(z_t)\right) \\ &\leq F_i(z_t, u) + \langle T_i z_t, u - z_t \rangle + \phi_i(u) - \phi_i(z_t). \end{aligned} \tag{4.3.44}$$

Hence, we have that for each $i = 1, 2, \dots, m$,

$$F_i(z_t, u) + \langle T_i z_t, u - z_t \rangle + \phi_i(u) - \phi_i(z_t) \geq 0, \quad \forall u \in C. \tag{4.3.45}$$

Letting $t \rightarrow 0$ in (4.3.44), therefore $z_t \rightarrow \bar{x}$. For each $i=1, 2, \dots, m$, using the condition (L4) and the proper lower semi-continuity of ϕ , we have

$$F_i(\bar{x}, u) + \langle T_i \bar{x}, u - \bar{x} \rangle + \phi_i(u) - \phi_i(\bar{x}) \geq 0, \quad \forall u \in C,$$

which shows that $\bar{x} \in GMEP(F_i, T_i, \phi_i)$, for each $i = 1, 2, \dots, m$. Hence,

$$\bar{x} \in \bigcap_{i=1}^m GMEP(F_i, T_i, \phi_i).$$

Following similar argument as the proof above, we have that

$$\bar{y} \in \bigcap_{l=1}^N GMEP(G_l, P_l, \varphi_l).$$

We now show that $(\{x_n\}, \{y_n\})$ strongly converges to (\bar{x}, \bar{y}) .

From (4.3.10) and (4.3.11), we have that

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 + \|y_{n+1} - \bar{y}\|^2 &\leq \|w_n - \bar{x}\|^2 + \|z_n - \bar{y}\|^2 \\ &= (1 - \alpha_n)^2(\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2) + \alpha_n^2(\|u - \bar{x}\|^2 \\ &\quad + \|v - \bar{y}\|^2) + 2\alpha_n(1 - \alpha_n)(\langle u - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &\quad + \langle v - \bar{y}, y_{n+1} - \bar{y} \rangle) \\ &\leq (1 - \alpha_n)\left(\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2\right) + \alpha_n\left(2(1 - \alpha_n)\right. \\ &\quad \left.(\langle u - \bar{x}, x_{n+1} - \bar{x} \rangle + \langle v - \bar{y}, y_{n+1} - \bar{y} \rangle) + \alpha_n(\|u - \bar{x}\|^2\right. \\ &\quad \left.+ \|v - \bar{y}\|^2)\right). \end{aligned} \tag{4.3.46}$$

It is clear that $(\langle u - \bar{x}, x_{n+1} - \bar{x} \rangle + \langle v - \bar{y}, y_{n+1} - \bar{y} \rangle) \rightarrow 0$ and since $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, it follows from Lemma 2.5.6 that

$$\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.3.47)$$

Therefore

$$\lim_{n \rightarrow \infty} \|x_n - \bar{x}\| = \lim_{n \rightarrow \infty} \|y_n - \bar{y}\| = 0.$$

Hence, $(\{x_n\}, \{y_n\})$ converges strongly to (\bar{x}, \bar{y}) .

Case 2: Assume that $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is not a monotonically decreasing sequence. Set $\rho_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$ and let $\tau : \mathbb{N} \mapsto \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large enough n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \rho_k \leq \rho_{k+1}\}.$$

We see that $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $\rho_{\tau(n)} \leq \rho_{\tau(n)+1}$, for $n \geq n_0$.

It follows from (4.3.13) that

$$\begin{aligned} \gamma_{\tau(n)}^2 (\|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 + \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2) &\leq \alpha_{\tau(n)} (\|u - x^*\|^2 + \|v - y^*\|^2) \\ &\quad - \alpha_{\tau(n)} (1 - \alpha_{\tau(n)}) (\|x_{\tau(n)} - u\|^2 \\ &\quad + \|y_{\tau(n)} - v\|^2) + (1 - \alpha_{\tau(n)}) \\ &\quad \rho_{\tau(n)}(x^*, y^*) - \rho_{\tau(n)+1}(x^*, y^*). \end{aligned}$$

Using the fact that $\lim_{n \rightarrow \infty} \alpha_{\tau(n)} = 0$, therefore,

$$\gamma_{\tau(n)}^2 (\|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 + \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2) \rightarrow 0.$$

By the condition

$$\gamma_{\tau(n)} \in \left(\epsilon, \frac{2\|Aw_{\tau(n)} - Bz_{\tau(n)}\|^2}{\|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 + \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2} - \epsilon \right), \quad \tau(n) \in \Omega,$$

we can conclude that

$$\lim_{n \rightarrow \infty} [\|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 + \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2] = 0.$$

Note that $Aw_{\tau(n)} - Bz_{\tau(n)} = 0$ if $\tau(n) \notin \Omega$. Thus,

$$\lim_{n \rightarrow \infty} \|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\| = \lim_{n \rightarrow \infty} \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\| = 0.$$

By following the same argument as in Case 1, we see that

$$\lim_{n \rightarrow \infty} \|u_{\tau(n)} - d_{\tau(n)}u_{\tau(n)}\| = \lim_{n \rightarrow \infty} \|v_{\tau(n)} - c_{\tau(n)}v_{\tau(n)}\| = 0,$$

and $(\{x_{\tau(n)}\}, \{y_{\tau(n)}\})$ converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$. Now for all $n \geq n_0$, we have from (4.3.46),

$$\begin{aligned}
0 &\leq \|x_{\tau(n)+1} - \bar{x}\|^2 + \|y_{\tau(n)+1} - \bar{y}\|^2 - (\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2) \\
&\leq (1 - \alpha_{\tau(n)})[\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2] + 2\alpha_{\tau(n)} \left(\langle u - \bar{x}, x_{\tau(n)+1} - \bar{x} \rangle + \langle v - \bar{y}, \right. \\
&\quad \left. y_{\tau(n)+1} - \bar{y} \rangle \right) - (\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2) \\
&= 2\alpha_{\tau(n)} \left(\langle u - \bar{x}, x_{\tau(n)+1} - \bar{x} \rangle + \langle v - \bar{y}, y_{\tau(n)+1} - \bar{y} \rangle \right) - \alpha_{\tau(n)} (\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2) \\
&= \alpha_{\tau(n)} \left(2[\langle u - \bar{x}, x_{\tau(n)+1} - \bar{x} \rangle + \langle v - \bar{y}, y_{\tau(n)+1} - \bar{y} \rangle] - \|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2 \right).
\end{aligned}$$

Therefore,

$$\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2 \leq 2(\langle u - \bar{x}, x_{\tau(n)+1} - \bar{x} \rangle + \langle v - \bar{y}, y_{\tau(n)+1} - \bar{y} \rangle).$$

Thus,

$$\lim_{n \rightarrow \infty} (\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2) = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - \bar{x}\|^2 = \lim_{n \rightarrow \infty} \|y_{\tau(n)} - \bar{y}\|^2 = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \rho_{\tau(n)} = \lim_{n \rightarrow \infty} \rho_{\tau(n)+1}.$$

Futhermore, for $n \geq n_0$, it is easy to observed that $\rho_{\tau(n)} \leq \rho_{\tau(n)+1}$ if $n \neq \tau(n)$, (that is $\tau(n) < n$) because $\rho_j > \rho_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \rho_n \leq \max\{\rho_{\tau(n)}, \rho_{\tau(n)+1}\} = \rho_{\tau(n)+1}.$$

So, $\lim \rho_n = 0$. That is $(\{x_n\}, \{y_n\})$ converges strongly to (\bar{x}, \bar{y}) . This complete the proof. \square

We now give the following consequences obtained from Theorem 4.3.1.

Corollary 4.3.2. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed convex subsets of H_1 and H_2 respectively. Assume that for $i = 1, 2, \dots, m, l = 1, 2, \dots, N$, $F_i : C \times C \rightarrow \mathbb{R}$ and $G_l : Q \times Q \rightarrow \mathbb{R}$ are bifunctions which satisfy (L1) – (L4) and the mappings $T_i : C \rightarrow C$ and $P_l : Q \rightarrow Q$ be continuous monotone mappings. Let $\phi_i : C \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_l : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex mappings such that $C \cap \text{dom}\phi_i \neq \emptyset$ and $Q \cap \text{dom}\varphi_l \neq \emptyset$ for $i = 1, 2, \dots, m, l = 1, 2, \dots, N$. Let $S_1 : C \rightarrow C$ and $S_2 : Q \rightarrow Q$ be two k strictly pseudo-nonspreading, (single-valued) mappings with constants k_1 and k_2 respectively, where $k_1, k_2 \in [0, 1)$ and $F(S_1) \neq \emptyset, F(S_2) \neq \emptyset$. Suppose $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are two bounded linear operators and*

$\Gamma := (F(S_1) \cap_{i=1}^m \text{GMEP}(F_i, T_i \phi_i)) \times (F(S_2) \cap_{i=1}^N \text{GMEP}(G_i, P_i, \varphi_i)) \neq \emptyset$. Let $(x_0, y_0) \in C \times Q$ and the iterative scheme $(\{x_n\}, \{y_n\})$ be defined as follows:

$$\left\{ \begin{array}{l} w_n = \alpha_n u + (1 - \alpha_n)x_n, \\ z_n = \alpha_n v + (1 - \alpha_n)y_n, \\ u_n = T_{r_{m,n}}^{F_m} \circ T_{r_{m-1,n}}^{F_{m-1}} \circ \dots \circ T_{r_{1,n}}^{F_1}(w_n - \gamma_n A^*(Aw_n - Bz_n)), \quad \text{for } r_{i,n} > 0, \\ v_n = T_{r_{N,n}}^{G_N} \circ T_{r_{N-1,n}}^{G_{N-1}} \circ \dots \circ T_{r_{1,n}}^{G_1}(z_n + \gamma_n B^*(Aw_n - Bz_n)) \quad \text{for } r_{l,n} > 0, \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n S_1 u_n, \\ y_{n+1} = (1 - \delta_n)v_n + \delta_n S_2 v_n, \end{array} \right. \quad (4.3.48)$$

for every $u \in C$, $v \in Q$, $n \geq 0$, and $\{\gamma_n\}$ is a positive real sequence such that

$$\gamma_n \in \left(\epsilon, \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \epsilon \right), \quad n \in \Omega.$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequence in $(0,1)$, such that:

- i. $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- ii. $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- iii. $\beta_n \in (a, 1 - k_1) \subseteq (0, 1)$ for some $a > 0$,
- iv. $\delta_n \in (b, 1 - k_2) \subseteq (0, 1)$ for some $b > 0$.

Then the sequence $(\{x_n\}, \{y_n\})$ defined by the iterative scheme (4.3.48) converges strongly to $\{\bar{x}, \bar{y}\} \in \Gamma$.

Corollary 4.3.3. Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed convex subsets of H_1 and H_2 respectively. Assume that $F : C \times C \rightarrow \mathbb{R}$ and $G : Q \times Q \rightarrow \mathbb{R}$ are two bifunctions which satisfy (L1)–(L4) and the mappings $T : C \rightarrow C$ and $P : Q \rightarrow Q$ be continuous monotone mappings. Let $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex mappings such that $C \cap \text{dom}\phi \neq \emptyset$ and $Q \cap \text{dom}\varphi \neq \emptyset$. Let $CB(C)$ and $CB(Q)$ be closed and bounded subsets of C and Q respectively and let $S_1 : C \rightarrow CB(C)$ and $S_2 : Q \rightarrow CB(Q)$ be two k strictly pseudo-nonspreading multi-valued mappings of type-one with constants k_1 and k_2 respectively, where $k_1, k_2 \in [0, 1)$. Assume $F(S_1) \neq \emptyset$ with $S_1 p = \{p\}$ for $p \in F(S_1)$ and $F(S_2) \neq \emptyset$ with $S_2 q = \{q\}$ for $q \in F(S_2)$. Let $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear operators and $\Gamma := (F(S_1) \cap \text{GMEP}(F, T, \phi)) \times (F(S_2) \cap \text{GMEP}(G, P, \varphi)) \neq \emptyset$. Let $(x_0, y_0) \in C \times Q$ and the iterative scheme $(\{x_n\}, \{y_n\})$ be defined as follows:

$$\left\{ \begin{array}{l} w_n = \alpha_n u + (1 - \alpha_n)x_n, \\ z_n = \alpha_n v + (1 - \alpha_n)y_n, \\ u_n = T_{r_n}^F(w_n - \gamma_n A^*(Aw_n - Bz_n)), \quad \text{for } r_n > 0, \\ v_n = T_{r_n}^G(z_n + \gamma_n B^*(Aw_n - Bz_n)), \quad \text{for } r_n > 0, \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n d_n, \\ y_{n+1} = (1 - \delta_n)v_n + \delta_n c_n, \end{array} \right. \quad (4.3.49)$$

for every $u \in C$, $v \in Q$, $d_n \in S_1 u_n$, with $\|u_n - d_n\| = d(u_n, S_1 u_n)$, $c_n \in S_2 v_n$, with $\|v_n - c_n\| = d(v_n, S_2 v_n)$, $n \geq 0$, and $\{\gamma_n\}$ is a positive real sequence such that

$$\gamma_n \in \left(\epsilon, \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \epsilon \right), \quad n \in \Omega.$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequence in $(0,1)$ such that:

- i. $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- ii. $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- iii. $\beta_n \in (a, 1 - k_1) \subseteq (0, 1)$ for some $a > 0$,
- iv. $\delta_n \in (b, 1 - k_2) \subseteq (0, 1)$ for some $b > 0$.

Then the sequence $(\{x_n\}, \{y_n\})$ defined by the iterative scheme (4.3.49) converges strongly to $\{\bar{x}, \bar{y}\} \in \Gamma$.

Remark 4.3.4.

1. If $F_i = G_l = T_i = P_l = 0$ in Theorem 4.3.1, we obtain a result for solving Split Equality for system of Convex Minimization Problem $SECM P(\phi_i, \varphi_l)$ which is to find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} \phi_i(x) &\geq \phi_i(x^*), \quad \forall x \in C, \\ \varphi_l(y) &\geq \varphi_l(y^*), \quad \forall y \in Q, \end{aligned}$$

and

$$Ax^* = By^*.$$

2. If $F_i = G_l = \phi_i = \varphi_l = 0$ in Theorem 4.3.1, then we obtain a result for solving Split Equality for system of Variational Inequality Problems $SEVIP(T_i, P_l)$ which is to find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} \langle T_i x^*, x - x^* \rangle &\geq 0, \quad \forall x \in C \\ \langle P_l y^*, y - y^* \rangle &\geq 0, \quad \forall y \in Q, \end{aligned}$$

and

$$Ax^* = By^*.$$

3. If $H_2 = H_3$ and $B = I$ in Theorem 4.3.1, we obtain a result for solving system of Split Generalized Mixed Equilibrium Problem $SGMEP(F_i, G_l, T_i, P_l, \phi_i, \varphi_l)$ which is to find $x^* \in C$ such that,

$$\begin{aligned} F_i(x^*, x) + \langle T_i x^*, x - x^* \rangle + \phi_i(x^*) - \phi_i(x) &\geq 0 \quad \forall x \in C \quad \text{and} \\ Ax^* = y^* \in Q \quad \text{solves} \quad G_l(y^*, y) + \langle P_l y^*, y - y^* \rangle + \varphi_l(y^*) - \varphi_l(y) &\geq 0 \quad \forall y \in Q. \end{aligned}$$

4.4 Numerical Example

In this section, we give numerical example of Theorem 4.3.1. Using Matlab version 2014a, we show how the sequence values are affected by the number of iterations.

Let $H_1 = H_2 = H_3 = \mathbb{R}$. Define $F_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by $F_i(z, y) = -\frac{1}{2}iz^2 + \frac{1}{2}iy^2$, $T : \mathbb{R} \rightarrow \mathbb{R}$ by $T_i(z) = iz$ and $\phi_i : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ by $\phi_i(z) = \frac{1}{2}iz^2$, $i = 1, 2, \dots, m$. For each $r_i > 0$ and $x \in \mathbb{R}$, Lemma 4.2.3 ensures that there exist $z \in \mathbb{R}$ such that for any $y \in \mathbb{R}$,

$$\begin{aligned}
 & F_i(z, y) + \langle T_i, y - z \rangle + \phi_i(y) - \phi_i(z) + \frac{1}{r} \langle z - x, y - z \rangle \geq 0, \\
 \iff & -\frac{1}{2}iz^2 + \frac{1}{2}iy^2 + iz(y - z) + \frac{1}{2}iy^2 - \frac{1}{2}iz^2 + \frac{1}{r}(z - x)(y - z) \geq 0, \\
 \iff & -\frac{1}{2}iz^2 + \frac{1}{2}iy^2 + izy - iz^2 + \frac{1}{2}iy^2 - \frac{1}{2}iz^2 + \frac{1}{r}(zy - z^2 - xy + xz) \geq 0, \\
 \iff & -2iz^2 + iy^2 + izy + \frac{1}{r}(zy - z^2 - xy + xz) \geq 0, \\
 \iff & -2irz^2 + iry^2 + iryz + yz - z^2 - xy + xz \geq 0, \\
 \iff & iry^2 + iryz + yz - xy - 2irz^2 - z^2 + xz \geq 0, \\
 \iff & iry^2 + (irz + z - x)y - (2irz^2 + z^2 - xz) \geq 0.
 \end{aligned}$$

Putting

$$F(y) = iry^2 + (irz + z - x)y - (2irz^2 + z^2 - xz),$$

then $F(y)$ is a quadratic function of y with coefficients:

$$a = ir, \quad b = irz + z - x, \quad \text{and} \quad c = (2irz^2 + z^2 - xz).$$

We then compute the discriminant Δ of F as follows:

$$\begin{aligned}
 \Delta &= b^2 - 4ac \\
 &= (irz + z - x)^2 + 4 \cdot ir(2irz^2 + z^2 - xz) \\
 &= i^2r^2z^2 + irz^2 - irzx + irz^2 + z^2 - xz - irxz - xz + x^2 + 8i^2r^2z^2 + 4irz^2 \\
 &\quad - 4irxz \\
 &= 9i^2r^2z^2 + 6irz^2 - 6irxz - 2xz + z^2 + x^2 \\
 &= x^2 - 6irxz - 2xz + 9i^2r^2z^2 + 6irz^2 + z^2 \\
 &= x^2 - 2(3irz + z)x + (3irz + z)^2 \\
 &= \left(x - (3irz + z)\right)^2.
 \end{aligned} \tag{4.4.1}$$

Thus, $\Delta \geq 0$ for all $y \in \mathbb{R}$. If it has atmost one solution in \mathbb{R} , $\Delta \leq 0$, therefore

$$\Delta = \left(x - (3irz + z)\right)^2 = 0.$$

Hence

$$z = \frac{x}{3ir + 1}.$$

We thus obtained the resolvent function for F_i as

$$T_{r_i}^{F_i} = \frac{x}{3ir + 1}, \quad \text{for } i = 1, 2, \dots, m. \quad (4.4.2)$$

Also, let $G_l : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by $G_l(q, s) = -3lq^2 + 2lqs + ls^2$, $P_l : \mathbb{R} \rightarrow \mathbb{R}$ by $P_l(q) = 2lq$ and $\varphi_l : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ by $\varphi_l(q) = lq^2$, for $l = 1, 2, \dots, N$. For each $r_l > 0$ and $x \in \mathbb{R}$, Lemma 4.2.3 ensures that there exist $q \in \mathbb{R}$ such that for any $s \in \mathbb{R}$,

$$\begin{aligned} & G_l(q, s) + \langle P_l q, s - q \rangle + \varphi_l(s) - \varphi_l(q) + \frac{1}{r_l} \langle s - q, q - x \rangle \geq 0, \\ \Leftrightarrow & -3lq^2 + 2lqs + ls^2 + 2lq(s - q) + ls^2 - lq^2 + \frac{1}{r}(s - q)(q - x) \geq 0, \\ \Leftrightarrow & -3lq^2 + 2lqs + ls^2 + 2lqs - 2lq^2 + ls^2 - lq^2 + \frac{1}{r}(sq - sx - q^2 + qx) \geq 0, \\ \Leftrightarrow & -6lq^2 + 2ls^2 + 4lqs + \frac{1}{r}(sq - sx - q^2 + qx) \geq 0, \\ \Leftrightarrow & -6lrs^2 + 2lrs^2 + 4lrs + sq - sx - q^2 + qx \geq 0, \\ \Leftrightarrow & 2lrs^2 + (4lrq + q - x)s - 6lrs^2 - q^2 + qx \geq 0. \end{aligned}$$

Putting

$$G(s) = 2lrs^2 + (4lrq + q - x)s - 6lrs^2 - q^2 + qx,$$

then $G(s)$ is a quadratic function with coefficients:

$$a = 2lr, \quad b = 4lrq + q - x, \quad c = -(6lrs^2 + q^2 - qx).$$

We then compute its discriminant Δ as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (4lrq + q - x)^2 + 4 \cdot 2lr(6lrs^2 + q^2 - qx) \\ &= 16l^2r^2q^2 + 4lrs^2 - 4lrs^2 + 4lrs^2 + q^2 - qx - 4lrs^2 - qx + x^2 + 48l^2r^2q^2 + 8lrs^2 \\ &\quad - 8lrs^2 \\ &= 64l^2r^2q^2 + 16lrs^2 - 16lrs^2 - 2qx + q^2 + x^2 \\ &= x^2 - 16lrs^2 - 2qx + 64l^2r^2q^2 + 16lrs^2 + q^2 \\ &= x^2 - 2(8lrs^2 + q)x + (8lrs^2 + q)^2 \\ &= \left(x - (8lrs^2 + q) \right)^2. \end{aligned} \quad (4.4.3)$$

Thus $\Delta \geq 0$ for all $s \in \mathbb{R}$ and if it has atmost one solution in \mathbb{R} , then $\Delta \leq 0$.

So we obtain

$$\left(x - (8lrs^2 + q) \right)^2 = 0.$$

Hence

$$q = \frac{x}{8lr + 1}$$

we thus obtained the resolvent function for G_l as

$$T_{r_l}^{G_l} x = \frac{x}{8lr + 1}, \quad \text{for } l = 1, 2, \dots, N. \quad (4.4.4)$$

Let $S_1 : \mathbb{R} \rightarrow CB(\mathbb{R})$ with the usual metric on \mathbb{R} be defined by

$$S_1 x = \begin{cases} x, & (-\infty, 0), \\ [-2x, 0] & [0, \infty). \end{cases} \quad (4.4.5)$$

To see that S_1 is k strictly pseudo-nonspreading multi-valued mapping of type-one, let $u \in S_1 x, v \in S_1 y$. For $x, y \in (-\infty, 0)$, then $u = x$ and $v = y$.

Thus

$$|u - v|^2 = |x - y|^2 + k|x - u - (y - v)|^2 + 2\langle x - u, y - v \rangle \quad \forall k \in [0, 1),$$

because $|u - v|^2 = |x - y|^2$ and $k|x - u - (y - v)|^2 = 2\langle x - u, y - v \rangle = 0$.

For all $x, y \in [0, \infty)$, choose $u = 0$ and $v = 0$ since

$$d(x, S_1 x) = \inf\{|x + 2x|, |x - 0|\} = |x - 0|,$$

and

$$d(y, S_1 y) = \inf\{|y + 2y|, |y - 0|\} = |y - 0|.$$

Clearly, S_1 is k strictly pseudo-nonspreading multi-valued mapping of type-one.

For $x \in (-\infty, 0)$ and $y \in [0, \infty)$, then $u = x$ and $v = 0$. Thus

$$\begin{aligned} |u - v|^2 &= |x - 0|^2 = |x|^2 = |x - y + y|^2 \\ &= |x - y|^2 + |y|^2 + 2(x - y)(y) \\ &= |x - y|^2 + \frac{1}{2}|y|^2 + \frac{1}{2}|y|^2 + 2(x - y)(y) \\ &= |x - y|^2 + \frac{1}{2}|y|^2 + (y)\left(\frac{1}{2}(y) + 2(x - y)\right) \\ &= |x - y|^2 + \frac{1}{2}|y|^2 + (y)\left(2x - \frac{3y}{2}\right) \\ &\leq |x - y|^2 + \frac{1}{2}|y|^2 \quad (\text{since } x \in (-\infty, 0)) \\ &= |x - y|^2 + \frac{1}{2}|x - x - (y - 0)|^2 + 0 \\ &= |x - y|^2 + \frac{1}{2}|x - u - (y - v)|^2 + 2(x - u)(y - v). \end{aligned}$$

Thus, we obtain that S_1 is $\frac{1}{2}$ strictly pseudo-nonspreading multi-valued mapping of type-one $\forall x, y \in \mathbb{R}$.

Also, let $S_2 : \mathbb{R} \rightarrow CB(\mathbb{R})$ with the usual metric on \mathbb{R} be defined by

$$S_2x = \begin{cases} [x, -\frac{3x}{2}] & (0, \infty), \\ [-\frac{3x}{2}, 0] & (-\infty, 0]. \end{cases} \quad (4.4.6)$$

For all $x, y \in (0, \infty)$, choose $u = x$ and $v = y$ since

$$d(x, S_2x) = \inf\{|x - x|, |x + \frac{3x}{2}|\} = |x - x|,$$

and

$$d(y, S_2y) = \inf\{|y - y|, |y + \frac{3y}{2}|\} = |y - y|.$$

Thus

$$|u - v|^2 = |x - y|^2 + k|x - u - (y - v)|^2 + 2\langle x - u, y - v \rangle \quad \forall k \in [0, 1),$$

since $|u - v|^2 = |x - y|^2$ and $k|x - u - (y - v)|^2 = 2\langle x - u, y - v \rangle = 0$.

Thus S_2 is k -strictly pseudo-nonspreading multi-valued mapping of type-one.

For all $x, y \in (-\infty, 0]$, choose $u = 0$ and $y = 0$ since

$$d(x, S_2x) = \inf\{|x + \frac{3x}{2}|, |x - 0|\} = |x - 0|,$$

and

$$d(y, S_2y) = \inf\{|y + \frac{3y}{2}|, |y - 0|\} = |y - 0|.$$

Clearly S_2 is k -strictly pseudo-nonspreading multi-valued mapping of type-one.

For $x \in (0, \infty)$ and $y \in (-\infty, 0]$, then $u = x$ and $v = 0$. Thus

$$\begin{aligned}
|u - v|^2 &= |x - 0|^2 = |x|^2 = |x - y + y|^2 \\
&= |x - y|^2 + |y|^2 + 2(x - y)(y) \\
&= |x - y|^2 + \frac{1}{2}|y|^2 + \frac{1}{2}|y|^2 + 2(x - y)(y) \\
&= |x - y|^2 + \frac{1}{2}|y|^2 + (y)\left(\frac{1}{2}(y) + 2(x - y)\right) \\
&= |x - y|^2 + \frac{1}{2}|y|^2 + (y)\left(2x - \frac{3y}{2}\right) \\
&\leq |x - y|^2 + \frac{1}{2}|y|^2 \quad (\text{since } y \in (-\infty, 0]) \\
&= |x - y|^2 + \frac{1}{2}|x - x - (y - 0)|^2 + 0 \\
&= |x - y|^2 + \frac{1}{2}|x - u - (y - v)|^2 + 2(x - u)(y - v).
\end{aligned}$$

Thus, S_2 is $\frac{1}{2}$ strictly pseudo-nonspreading multi-valued mapping of type-one $\forall x, y \in \mathbb{R}$.

Let $A : \mathbb{R} \rightarrow \mathbb{R}$ and $B : \mathbb{R} \rightarrow \mathbb{R}$ by $A(x) = \frac{x}{2}$ and $B(x) = 3x$, then A and B are bounded linear operators with adjoints $A^*(x) = \frac{x}{2}$ and $B^*(x) = 3x$ respectively.

Chooosen $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1 - \frac{1}{n}}{5(1 + \frac{1}{n})}$, $\delta_n = \frac{2}{7(1 + \frac{1}{n})}$, $r_{i,n} = \frac{n}{in+3}$, $i = 1, 2, \dots, 5$, and $r_{l,n} = \frac{n}{ln+3}$, $l = 1, 2, \dots, 5$, then our iterative scheme (4.3.1) becomes: for x_0, y_0, u and $v \in \mathbb{R}$,

$$\left\{ \begin{array}{l}
w_n = \frac{1}{n+1}u + \left(\frac{n}{n+1}\right)x_n, \\
z_n = \frac{1}{n+1}v + \left(\frac{n}{n+1}\right)y_n, \\
u_n = \left(\frac{1}{\prod_{i=1}^5 3ir_{i,n} + 1}\right)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\
v_n = \left(\frac{1}{\prod_{l=1}^5 8lr_{l,n} + 1}\right)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\
x_{n+1} = \frac{4n+6}{5(n+1)}u_n + \frac{n-1}{5(n+1)}d_n, \quad \forall n \geq 0, \\
y_{n+1} = \frac{5n+7}{7(n+1)}v_n + \frac{2n}{7(n+1)}c_n, \quad \forall n \geq 0,
\end{array} \right. \quad (4.4.7)$$

where $d_n \in S_1 u_n$ such that $d_n = u_n$ if $u_n < 0$ and $d_n = 0$ if $u_n \geq 0$, also $c_n \in S_2 v_n$ such that $c_n = v_n$ if $v_n < 0$ and $c_n = 0$ if $v_n \geq 0$, $A^* = A$, $B^* = B$ and

$$\gamma_n \in \left(\epsilon, \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \epsilon \right), \quad n \in \Omega.$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.

Case I.

- (a) Take $x_0 = 1, y_0 = -1, u = 0.5$ and $v = 1$.
- (b) Take $x_0 = 1, y_0 = 1, u = 0.5$ and $v = 1$.

Case II.

- (a) Take $x_0 = 0.02, y_0 = 0.3, u = 0.03$ and $v = 0.005$.
- (b) Take $x_0 = 0.1, y_0 = 1, u = 0.2$ and $v = 0.2$.

The Matlab version used is R2014a and the execution times are as follows:

1. (case Ia, $\varepsilon = 10^{-4}$) and execution time is 0.024 sec.
2. (case Ia, $\varepsilon = 10^{-6}$) and execution time is 0.025 sec.
3. (case Ib, $\varepsilon = 10^{-4}$) and execution time is 0.019 sec.
4. (case Ib, $\varepsilon = 10^{-6}$) and execution time is 0.025 sec.
5. (case IIa, $\varepsilon = 10^{-4}$) and execution time is 0.019 sec.
6. (case IIa, $\varepsilon = 10^{-6}$) and execution time is 0.025 sec.
7. (case IIb, $\varepsilon = 10^{-4}$) and execution time is 0.021 sec.
8. (case IIb, $\varepsilon = 10^{-6}$) and execution time is 0.025 sec.

See Figure 6.5, Figure 6.6, Figure 6.7 and Figure 6.8 to see how the sequence values are affected by the number of iterations.

**Approximation of Common Fixed Points for Bregman Mappings
and Common solutions of Convex Minimization and Variational
Inequality Problems in Reflexive Banach Spaces**

In this chapter, we propose an iterative algorithm for approximating a common fixed point of an infinite family of quasi-Bregman nonexpansive mappings which is also a solution to finite systems of convex minimization problems and variational inequality problems in real reflexive Banach spaces. We obtain a strong convergence result and give applications of our result to finding zeroes of infinite family of Bregman inverse strongly monotone operators and finite system of equilibrium problems in real reflexive Banach spaces.

5.1 Introduction

Let A be a monotone mapping defined from C into E^* and $N_C\{q\}$ be a normal cone to C at $q \in E$, i.e $N_C\{q\} = \{\xi \in E^* : \langle \xi, q - u \rangle \geq 0, \quad \forall u \in C\}$. Define a mapping M by

$$Mq = \begin{cases} Aq + N_C\{q\}, & q \in C \\ \emptyset, & q \notin C. \end{cases} \quad (5.1.1)$$

Then M is maximal monotone and $x^* \in M^{-1}(0^*) \iff x^* \in \text{VI}(C, A)$, (see, e.g [106]). The resolvent of M is the operator $\text{Res}_M^f : E \rightarrow 2^E$ defined as follow (see [10])

$$\text{Res}_M^f(x) = (\nabla f + M)^{-1} \circ \nabla f(x). \quad (5.1.2)$$

It is well-known that Res_M^f is BFNE and single-valued on its domain if $f|_{\text{int}(\text{dom}f)}$ is strictly convex (see [10], Proposition 3.8(iv)).

For any $r > 0$, the Yosida approximation of A is defined by

$$M_r(x) = \frac{1}{r} \left(\nabla f(x) - \nabla f(\text{Res}_{rM}^f(x)) \right), \quad (5.1.3)$$

(see [103]). From Proposition 2.7 in [103], we know that $(\text{Res}_{rM}^f(x), M_r(x)) \in G(M)$ and $0^* \in Mx$ if and only if $0^* \in M_r x$ for all $r > 0$. Also, it is known that if $F(\text{Res}_{rM}^f) \neq \emptyset$, for all $x \in E$ and $q \in F(\text{Res}_{rM}^f)$, we have

$$D_f(q, \text{Res}_{rM}^f(x)) + D_f(\text{Res}_{rM}^f(x), x) \leq D_f(q, x), \quad (5.1.4)$$

(see [103]).

Consider the following constrained convex minimization problem:

$$\text{minimize} \{ \phi(x) : x \in C \}, \quad (5.1.5)$$

where $\phi : C \rightarrow \mathbb{R} \cup +\infty$ is a real-valued convex function. We say that the minimization problem is consistent if the minimization problem has a solution. The gradient projection method for finding the approximate solutions of the constrained convex minimization problem in Hilbert spaces is based on the behavior of the gradient of the objective function $\nabla \phi$ such as strongly monotone and L-Lipschitzian (see, [30, 31, 113] and reference therein). However, there are several difficulties on extending this method to Banach spaces, (see for example [29]).

One way to overcome these difficulties is to use the proximal operator introduced by Bauschke, Borwein and Combettes (see [10]). If $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is a Legendre and convex function, then the operator $\text{prox}_\lambda^\phi : E \rightarrow 2^E$ is given by

$$\text{prox}_\lambda^\phi = \left(\partial(f + \lambda\phi) \right)^{-1} \circ \nabla f, \quad \lambda > 0 \quad (5.1.6)$$

and is well defined, where $\phi : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and convex function such that $\text{dom}(f) \cap \text{dom}(\phi) \neq \emptyset$. The fixed point of prox_λ^ϕ , $F(\text{prox}_\lambda^\phi)$ is a solution of (5.1.5) (see [10], Proposition 3.22).

In order to prove the convergence of the iterates of prox_λ^ϕ , we need nonexpansivity properties of this resolvent operator. Bauschke, Borwein and Combettes [10] introduced the class of Bregman firmly nonexpansive operators and proved that the resolvent prox_λ^ϕ belongs to this class under the range condition: $\text{ran}(\text{prox}_\lambda^\phi) \subset \text{int}(\text{dom}f)$ and proved many other properties of this resolvent (see [10]).

5.2 Preliminaries

In this section, we shall state some known results that will be used in the sequel.

Definition 5.2.1. Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in \text{int}(\text{dom}f)$ onto the nonempty, closed and convex subset $C \subset \text{int}(\text{dom}f)$ is defined as the necessarily unique vector $\text{Proj}_C^f(x) \in C$ satisfying

$$D_f(\text{Proj}_C^f(x), x) = \inf \{ D_f(y, x) : y \in C \}. \quad (5.2.1)$$

It is known from [29] that $z = \text{Proj}_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0 \quad \text{for all } y \in C. \quad (5.2.2)$$

We also have

$$D_f(y, \text{Proj}_C^f(x)) + D_f(\text{Proj}_C^f(x), x) \leq D_f(y, x) \quad \text{for all } x \in E, y \in C. \quad (5.2.3)$$

Lemma 5.2.1. [101] (Characterization of Bregman Projection): *Let f be totally convex on $\text{int}(\text{dom}f)$. Let C be a nonempty, closed and convex subset of $\text{int}(\text{dom}f)$ and $x \in \text{int}(\text{dom}f)$, if $\omega \in C$, then the following conditions are equivalent:*

- i. *the vector ω is the Bregman projection of x onto C , with respect to f ,*
- ii. *the vector ω is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0 \quad \forall y \in C,$$

- iii. *the vector ω is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x) \quad \forall y \in C.$$

Lemma 5.2.2. [98] *Let E be a real reflexive Banach space, let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper semicontinuous function, then $f^* : E^* \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper weak* lower semicontinuous and convex function. Thus for all $z \in E$ one has*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_f(z, x_i). \quad (5.2.4)$$

Lemma 5.2.3. [68] *Let E be a reflexive Banach space, let $f : E \rightarrow \mathbb{R}$ be a strong coercive Bregman function and let $V_f : E \times E^* \rightarrow [0, +\infty)$ be defined by*

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad x \in E, \quad x^* \in E^*, \quad (5.2.5)$$

then the following assertions hold:

- i. *$D_f(x, \nabla f(x^*)) = V_f(x, x^*)$ for all $x \in E$ and $x^* \in E^*$,*
- ii. *$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.*

Lemma 5.2.4. [29] *Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function whose domain contains at-least two points. Then the following statements holds:*

- i. *f is sequentially consistent if and only if it is totally convex on bounded subsets.*
- ii. *If f is lower semicontinuous, then f is sequential consistent if and only if it is uniformly convex on bounded subsets.*

iii. If f is uniformly strictly convex on bounded subsets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain, and the Fréchet derivative ∇f is uniformly continuous on bounded subsets.

Lemma 5.2.5. [89] Let $r > 0$ be a constant and let $f : E \rightarrow \mathbb{R}$ be a continuous uniformly convex function on bounded subsets of E . Then

$$f\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \leq \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - x_j\|), \quad (5.2.6)$$

for all $i, j \in \mathbb{N} \cup 0$, $x_k \in B_r$, $\alpha_k \in (0, 1)$ and $k \in \mathbb{N} \cup 0$ with $\sum_{k=0}^{\infty} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f .

Lemma 5.2.6. [101] If $f : E \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E , then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* .

Lemma 5.2.7. [26] The function $f : E \rightarrow \mathbb{R}$ is totally convex on bounded subsets if and only if it is sequentially consistent.

Lemma 5.2.8. [103] Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 5.2.9. [77] Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Consider the integer $\{m_k\}$ defined by

$$m_k = \max\{j \leq k : a_j < a_{j+1}\}.$$

Then $\{m_k\}$ is a nondecreasing sequence verifying $\lim_{n \rightarrow \infty} m_n = \infty$, and for all $k \in \mathbb{N}$ the following estimate hold:

$$a_{m_k} \leq a_{m_k+1}, \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

We prove the following lemma.

Lemma 5.2.10. Let $f : E \rightarrow \mathbb{R}$ be a convex and Gâteaux differentiable function. Let $\phi : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous and convex function such that $\text{dom}(f) \cap \text{dom}(\phi) \neq \emptyset$ and $\text{ran}(\text{prox}_\lambda^\phi) \subset \text{int}(\text{dom} f)$. For all $x \in E$, $u \in F(\text{prox}_\lambda^f)$ and $\lambda > 0$, then we have the following

$$D_f(u, \text{prox}_\lambda^\phi(x)) + D_f(\text{prox}_\lambda^\phi(x), x) \leq D_f(u, x). \quad (5.2.7)$$

Proof. Since prox_λ^f is a BFNE operator, it follows from (2.3.16) that for all $x, y \in E$,

$$\begin{aligned} D_f(\text{prox}_\lambda^\phi(x), \text{prox}_\lambda^\phi(y)) + D_f(\text{prox}_\lambda^\phi(y), \text{prox}_\lambda^\phi(x)) &\leq D_f(\text{prox}_\lambda^\phi(x), y) - D_f(\text{prox}_\lambda^\phi(x), x) \\ &\quad + D_f(\text{prox}_\lambda^\phi(y), x) \\ &\quad - D_f(\text{prox}_\lambda^\phi(y), y). \end{aligned}$$

Letting $y = u \in F(\text{prox}_\lambda^\phi)$, we have

$$D_f(\text{prox}_\lambda^\phi(x), u) + D_f(u, \text{prox}_\lambda^\phi(x)) \leq D_f(\text{prox}_\lambda^\phi(x), u) - D_f(\text{prox}_\lambda^\phi(x), x) + D_f(u, x) - D_f(u, u).$$

Thus

$$D_f(u, \text{prox}_\lambda^\phi(x)) + D_f(\text{prox}_\lambda^\phi(x), x) \leq D_f(u, x).$$

□

5.3 Main result

We are now in the position to state and prove the main result of this chapter.

Theorem 5.3.1. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E . Let $f : E \rightarrow \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . For each $l = 1, 2, \dots, N$, let $\phi_l : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions such that $\text{dom}f \cap \text{dom}\phi_l \neq \emptyset$ and $\text{ran}(\text{prox}_\lambda^{\phi_l}) \subset \text{int}(\text{dom}f)$. For $k = 1, 2, \dots, m$ let $A_k : C \rightarrow E^*$ be continuous and monotone operators and let $\{T_i\}_{i=1}^\infty$ be sequence of uniformly continuous quasi-Bregman nonexpansive mappings from C into itself, such that $F(T_i) = \hat{F}(T_i)$ for all $i \geq 1$. Suppose $\Gamma := \left(\bigcap_{i=1}^\infty F(T_i) \right) \cap \left(\bigcap_{i=1}^N F(\text{prox}_\lambda^{\phi_i}) \right) \cap \left(\bigcap_{k=1}^m VI(C, A_k) \right) \neq \emptyset$. For $u, x_0 \in C$, let $\{x_n\}$ be iteratively generated by*

$$\begin{cases} u_n &= \text{prox}_{\lambda_n}^{\phi_N} \circ \text{prox}_{\lambda_n}^{\phi_{N-1}} \circ \dots \circ \text{prox}_{\lambda_n}^{\phi_1} \left(\nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n)) \right), \\ y_n &= \text{Res}_{r_n B_m}^f \circ \text{Res}_{r_n B_{m-1}}^f \circ \dots \circ \text{Res}_{r_n B_1}^f u_n, \\ x_{n+1} &= \nabla f^* \left(\beta_{0,n} \nabla f(y_n) + \sum_{i=1}^\infty \beta_{i,n} \nabla f(T_i y_n) \right), \quad n \geq 0, \end{cases} \quad (5.3.1)$$

where

$$B_k x = \begin{cases} A_k x + N_C \{x\}, & x \in C \\ \emptyset, & x \notin C, \end{cases} \quad (5.3.2)$$

for $k = 1, 2, \dots, m$ and $N_C \{x\}$ is a normal cone to C at $x \in E$ and $\lambda_n, r_n > 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ such that

$$i. \lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=0}^\infty \alpha_n = \infty,$$

$$ii. \liminf_{n \rightarrow \infty} \beta_{i,n} > 0 \quad \text{and} \quad \sum_{i=0}^\infty \beta_{i,n} = 1,$$

$$iii. \liminf_{n \rightarrow \infty} r_n \geq 0.$$

Then $\{x_n\}$ converges strongly to $Proj_{\Gamma}^f u$, where $Proj_{\Gamma}^f$ is the Bregman projection of C onto Γ .

Proof. Let $x^* = Proj_{\Gamma}^f u$. Putting $\Theta_n^N = prox_{\lambda_n^{\phi_N}} prox_{\lambda_n^{\phi_{N-1}}} \dots prox_{\lambda_n^{\phi_1}}$, where $\Theta_n^0 = I$ and $\Psi_n^m = Res_{r_n B_m}^f Res_{r_n B_{m-1}}^f \dots Res_{r_n B_1}^f$, where $\Psi_n^0 = I$. Let $w_n = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n))$, then

$$\begin{aligned}
D_f(x^*, u_n) &= D_f(x^*, \Theta_n^N w_n) \\
&\leq D_f(x^*, \Theta_n^{N-1} w_n) \\
&\vdots \\
&\leq D_f(x^*, w_n) \\
&= D_f(x^*, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n))) \\
&\leq \alpha_n D_f(x^*, u) + (1 - \alpha_n) D_f(x^*, x_n).
\end{aligned} \tag{5.3.3}$$

Also,

$$\begin{aligned}
D_f(x^*, x_{n+1}) &= D_f\left(x^*, \nabla f^*(\beta_{0,n} \nabla f(y_n) + \sum_{i=1}^{\infty} \beta_{i,n} \nabla f(T_i y_n))\right) \\
&\leq \beta_{0,n} D_f(x^*, y_n) + \sum_{i=1}^{\infty} \beta_{i,n} D_f(x^*, T_i y_n) \\
&\leq \beta_{0,n} D_f(x^*, y_n) + \sum_{i=1}^{\infty} \beta_{i,n} D_f(x^*, y_n) \\
&= D_f(x^*, y_n) \\
&= D_f(x^*, \Psi_n^m u_n) \\
&\leq D_f(x^*, \Psi_n^{m-1} u_n) \\
&\vdots \\
&\leq D_f(x^*, u_n).
\end{aligned} \tag{5.3.4}$$

Therefore, it follows from (5.3.3) and (5.3.4) that

$$\begin{aligned}
D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, u) + (1 - \alpha_n) D_f(x^*, x_n) \\
&\leq \max\{D_f(x^*, u), D_f(x^*, x_n)\} \\
&\vdots \\
&\leq \max\{D_f(x^*, u), D_f(x^*, x_0)\}.
\end{aligned} \tag{5.3.5}$$

Therefore $\{D_f(x^*, x_n)\}$ is bounded and by Lemma 5.2.8, we obtain that $\{x_n\}$ is bounded. Furthermore, let $s = \sup\{\|\nabla f(y_n)\|, \|\nabla f(T_i y_n)\|\}$ and $\rho_s^* : E^* \rightarrow \mathbb{R}$ be the gauge of uniform convexity of the conjugate function f^* . From Lemma 5.2.3, Lemma 5.2.5, (5.3.1)

and (5.3.4), we have

$$\begin{aligned}
D_f(x^*, x_{n+1}) &= D_f\left(x^*, \nabla f^*(\beta_{0,n}\nabla f(y_n) + \sum_{i=1}^{\infty} \beta_{i,n}\nabla f(T_i y_n))\right) \\
&\leq V_f\left(x^*, \beta_{0,n}\nabla f(y_n) + \sum_{i=1}^{\infty} \beta_{i,n}\nabla f(T_i y_n)\right) \\
&= f(x^*) - \langle x^*, \beta_{0,n}\nabla f(y_n) + \sum_{i=1}^{\infty} \beta_{i,n}\nabla f(T_i y_n) \rangle + f^*\left(\beta_{0,n}\nabla f(y_n) \right. \\
&\quad \left. + \sum_{i=1}^{\infty} \beta_{i,n}\nabla f(T_i y_n)\right) \\
&\leq \beta_{0,n}f(x^*) + \sum_{i=1}^{\infty} \beta_{i,n}f(x^*) - \beta_{0,n}\langle x^*, \nabla f(y_n) \rangle - \sum_{i=1}^{\infty} \beta_{i,n}\langle x^*, \nabla f(T_i y_n) \rangle \\
&\quad + \beta_{0,n}f^*(\nabla f(y_n)) + \sum_{i=1}^{\infty} \beta_{i,n}f^*(\nabla f(T_i y_n)) \\
&\quad - \beta_{0,n} \sum_{i=1}^{\infty} \beta_{i,n}\rho_s^* \left(\|\nabla f(y_n) - \nabla f(T_i y_n)\| \right) \\
&= \beta_{0,n} \left[f(x^*) - \langle x^*, \nabla f(y_n) \rangle + f^*(\nabla f(y_n)) \right] + \sum_{i=1}^{\infty} \beta_{i,n} \left[f(x^*) - \langle x^*, \nabla f(T_i y_n) \rangle \right. \\
&\quad \left. + f^*(\nabla f(T_i y_n)) \right] - \beta_{0,n} \sum_{i=1}^{\infty} \beta_{i,n}\rho_s^* \left(\|\nabla f(y_n) - \nabla f(T_i y_n)\| \right) \\
&= \beta_{0,n}D_f(x^*, y_n) + \sum_{i=1}^{\infty} \beta_{i,n}D_f(x^*, T_i y_n) - \beta_{0,n} \sum_{i=1}^{\infty} \beta_{i,n}\rho_s^* \left(\|\nabla f(y_n) - \nabla f(T_i y_n)\| \right) \\
&\leq \beta_{0,n}D_f(x^*, y_n) + \sum_{i=1}^{\infty} \beta_{i,n}D_f(x^*, y_n) - \beta_{0,n} \sum_{i=1}^{\infty} \beta_{i,n}\rho_s^* \left(\|\nabla f(y_n) - \nabla f(T_i y_n)\| \right). \\
&= D_f(x^*, y_n) - \beta_{0,n} \sum_{i=1}^{\infty} \beta_{i,n}\rho_s^* \left(\|\nabla f(y_n) - \nabla f(T_i y_n)\| \right).
\end{aligned}$$

Thus

$$\begin{aligned}
D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, u) + (1 - \alpha_n) D_f(x^*, x_n) \\
&\quad - \beta_{0,n} \sum_{i=1}^{\infty} \beta_{i,n}\rho_s^* \left(\|\nabla f(y_n) - \nabla f(T_i y_n)\| \right). \tag{5.3.6}
\end{aligned}$$

We now divide the rest of the proof into two cases.

Case I: Assume that $\{D_f(x^*, x_n)\}$ is monotonically decreasing. Then $\{D_f(x^*, x_n)\}$ converges and $D_f(x^*, x_{n+1}) - D_f(x^*, x_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus from (5.3.6), using the fact that $\lim_{n \rightarrow \infty} \alpha_n = 0$, we obtain

$$\lim_{n \rightarrow \infty} \beta_{0,n} \sum_{i=1}^{\infty} \beta_{i,n}\rho_s^* \left(\|\nabla f(y_n) - \nabla f(T_i y_n)\| \right) = 0. \tag{5.3.7}$$

Since $\liminf_{n \rightarrow \infty} \beta_{i,n} > 0$ and by the property of ρ_s^* , we have

$$\lim_{n \rightarrow \infty} \|\nabla f(y_n) - \nabla f(T_i y_n)\| = 0, \quad \forall i = 1, 2, \dots \quad (5.3.8)$$

Since f is strongly coercive and uniformly convex on bounded subsets of E , f^* is uniformly Fréchet differentiable on bounded sets, then from (5.3.8), we obtain

$$\lim_{n \rightarrow \infty} \|y_n - T_i y_n\| = 0, \quad \forall i = 1, 2, \dots \quad (5.3.9)$$

We note that

$$\begin{aligned} \lim_{n \rightarrow \infty} D_f(w_n, x_n) &= \lim_{n \rightarrow \infty} [D_f(\nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n)), x_n)] \\ &\leq \lim_{n \rightarrow \infty} [\alpha_n D_f(u, x_n) + (1 - \alpha_n) D_f(x_n, x_n)] \\ &= 0. \end{aligned} \quad (5.3.10)$$

By Lemma 5.2.7, we have

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \quad (5.3.11)$$

Since f is uniformly Fréchet differentiable on bounded subset of E , then by Lemma 5.2.4, we have

$$\lim_{n \rightarrow \infty} \|\nabla f(w_n) - \nabla f(x_n)\| = 0. \quad (5.3.12)$$

Also, since f is uniformly Fréchet differentiable on bounded subset of E , then f is uniformly continuous on bounded subset of E , thus we have

$$\lim_{n \rightarrow \infty} \|f(w_n) - f(x_n)\| = 0. \quad (5.3.13)$$

From the three-point identity property of the Bregman distance function (2.3.9), we know that

$$\begin{aligned} D_f(x^*, w_n) - D_f(x^*, x_n) &= \langle \nabla f(x_n) - \nabla f(w_n), x^* - w_n \rangle - D_f(w_n, x_n) \\ &= \langle \nabla f(x_n) - \nabla f(w_n), x^* - w_n \rangle + f(x_n) - f(w_n) \\ &\quad + \langle \nabla f(x_n), w_n - x_n \rangle. \end{aligned} \quad (5.3.14)$$

Thus, it follows from (5.3.11), (5.3.12), (5.3.13) and (5.3.14) that

$$\lim_{n \rightarrow \infty} [D_f(x^*, w_n) - D_f(x^*, x_n)] = 0. \quad (5.3.15)$$

Since $x^* \in F(\text{prox}_{\lambda_n}^{\phi_N})$ for all $n \geq 0$, we observe that

$$D_f(x^*, \Theta_n^N w_n) = D_f(x^*, \text{prox}_{\lambda_n}^{\phi_N} \Theta_n^{N-1} w_n) \leq D_f(x^*, \Theta_n^{N-1} w_n) \leq \dots \leq D_f(x^*, w_n) \quad (5.3.16)$$

By Lemma 5.2.10, (5.3.4), (5.3.15) and (5.3.16), we obtain

$$\begin{aligned} D_f(u_n, \Theta_n^{N-1} w_n) &\leq D_f(x^*, \Theta_n^{N-1} w_n) - D_f(x^*, u_n) \\ &\leq D_f(x^*, w_n) - D_f(x^*, x_{n+1}) \\ &= D_f(x^*, w_n) - D_f(x^*, x_n) + D_f(x^*, x_n) \\ &\quad - D_f(x^*, x_{n+1}) \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (5.3.17)$$

Thus

$$\lim_{n \rightarrow \infty} \|u_n - \Theta_n^{N-1} w_n\| = 0. \quad (5.3.18)$$

Similarly, we have

$$\begin{aligned} D_f(\Theta_n^{N-1} w_n, \Theta_n^{N-2} w_n) &\leq D_f(x^*, \Theta_n^{N-2} w_n) - D_f(x^*, \Theta_n^{N-1} w_n) \\ &\leq D_f(x^*, w_n) - D_f(x^*, x_{n+1}) \\ &= D_f(x^*, w_n) - D_f(x^*, x_n) + D_f(x^*, x_n) \\ &\quad - D_f(x^*, x_{n+1}) \longrightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \|\Theta_n^{N-1} w_n - \Theta_n^{N-2} w_n\| = 0. \quad (5.3.19)$$

In a similar way, we can verify that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\Theta_n^{N-2} w_n - \Theta_n^{N-3} w_n\| &= \lim_{n \rightarrow \infty} \|\Theta_n^{N-3} w_n - \Theta_n^{N-4} w_n\| = \\ &= \cdots = \lim_{n \rightarrow \infty} \|\Theta_n^1 w_n - w_n\| = 0. \end{aligned} \quad (5.3.20)$$

Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_n - w_n\| &\leq \lim_{n \rightarrow \infty} \|u_n - \Theta_n^{N-1} w_n\| + \lim_{n \rightarrow \infty} \|\Theta_n^{N-1} w_n - \Theta_n^{N-2} w_n\| \\ &\quad + \cdots + \lim_{n \rightarrow \infty} \|\Theta_n^1 w_n - w_n\| = 0. \end{aligned} \quad (5.3.21)$$

Therefore, from (5.3.11) and (5.3.21), we have

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| \leq \lim_{n \rightarrow \infty} \|u_n - w_n\| + \lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \quad (5.3.22)$$

Furthermore, since f is uniformly Fréchet differentiable on bounded subset of E , then by Lemma 5.2.4, we have

$$\lim_{n \rightarrow \infty} \|\nabla f(u_n) - \nabla f(x_n)\| = 0. \quad (5.3.23)$$

Also, since f is uniformly Fréchet differentiable on bounded subset of E , then f is uniformly continuous on bounded subset of E , thus we have

$$\lim_{n \rightarrow \infty} \|f(u_n) - f(x_n)\| = 0. \quad (5.3.24)$$

Also, from the three-point identity of Bregman distance function, we obtain

$$\begin{aligned} D_f(x^*, x_n) - D_f(x^*, u_n) &= \langle \nabla f(u_n) - \nabla f(x_n), x^* - x_n \rangle - D_f(x_n, u_n) \\ &= \langle \nabla f(u_n) - \nabla f(x_n), x^* - x_n \rangle + f(u_n) - f(x_n) \\ &\quad + \langle \nabla f(u_n), x_n - u_n \rangle. \end{aligned} \quad (5.3.25)$$

It follows from (5.3.22), (5.3.23), (5.3.24) and (5.3.25) that

$$\lim_{n \rightarrow \infty} \left[D_f(x^*, x_n) - D_f(x^*, u_n) \right] = 0. \quad (5.3.26)$$

Also, since $x^* \in F(\text{Res}_{r_n B_m}^f)$, we observe that

$$D_f(x^*, \Psi_n^m u_n) = D_f(x^*, \text{Res}_{r_n B_m}^f \Psi_n^{m-1} u_n) \leq D_f(x^*, \Psi_n^{m-1} u_n) \leq \dots \leq D_f(x^*, u_n) \quad (5.3.27)$$

Hence, from (5.3.4), (5.3.26) and (5.3.27), we have

$$\begin{aligned} D_f(y_n, \Psi_n^{m-1} u_n) &= D_f(\text{Res}_{r_n, B_m}^f \Psi_n^{m-1} u_n, \Psi_n^{m-1} u_n) \\ &\leq D_f(x^*, \Psi_n^{m-1} u_n) - D_f(x^*, y_n) \\ &\leq D_f(x^*, u_n) - D_f(x^*, x_{n+1}) \\ &= D_f(x^*, u_n) - D_f(x^*, x_n) + D_f(x^*, x_n) \\ &\quad - D_f(x^*, x_{n+1}) \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (5.3.28)$$

By Lemma 5.2.7, we have

$$\lim_{n \rightarrow \infty} \|y_n - \Psi_n^{m-1} u_n\| = 0. \quad (5.3.29)$$

Hence

$$\lim_{n \rightarrow \infty} \|\nabla f(y_n) - \nabla f(\Psi_n^{m-1} u_n)\| = 0. \quad (5.3.30)$$

Similarly

$$\begin{aligned} D_f(\Psi_n^{m-1} u_n, \Psi_n^{m-2} u_n) &\leq D_f(x^*, \Psi_n^{m-2} u_n) - D_f(x^*, \Psi_n^{m-1} u_n) \\ &\leq D_f(x^*, u_n) - D_f(x^*, x_{n+1}) \\ &= D_f(x^*, u_n) - D_f(x^*, x_n) + D_f(x^*, x_n) \\ &\quad - D_f(x^*, x_{n+1}) \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (5.3.31)$$

By Lemma 5.2.7, we have

$$\lim_{n \rightarrow \infty} \|\Psi_n^{m-1} u_n - \Psi_n^{m-2} u_n\| = 0, \quad (5.3.32)$$

and hence,

$$\lim_{n \rightarrow \infty} \|\nabla f(\Psi_n^{m-1} u_n) - \nabla f(\Psi_n^{m-2} u_n)\| = 0. \quad (5.3.33)$$

In a similar way, we can verify that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\Psi_n^{m-2} u_n - \Psi_n^{m-3} u_n\| &= \lim_{n \rightarrow \infty} \|\Psi_n^{m-3} u_n - \Psi_n^{m-4} u_n\| = \\ &= \dots = \lim_{n \rightarrow \infty} \|\Psi_n^1 u_n - u_n\| = 0. \end{aligned} \quad (5.3.34)$$

Therefore, we conclude that

$$\lim_{n \rightarrow \infty} \|\Psi_n^k u_n - \Psi_n^{k-1} u_n\| = 0, \quad \forall k = 1, 2, \dots, m. \quad (5.3.35)$$

Hence

$$\lim_{n \rightarrow \infty} \|\nabla f(\Psi_n^k u_n) - \nabla f(\Psi_n^{k-1} u_n)\| = 0, \quad \forall k = 1, 2, \dots, m. \quad (5.3.36)$$

From (5.3.29) - (5.3.36), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|y_n - u_n\| &\leq \lim_{n \rightarrow \infty} \|y_n - \Psi_n^{m-1} u_n\| + \lim_{n \rightarrow \infty} \|\Psi_n^{m-1} u_n - \Psi_n^{m-2} u_n\| + \\ &\dots + \lim_{n \rightarrow \infty} \|\Psi_n^1 u_n - u_n\| = 0. \end{aligned} \quad (5.3.37)$$

Thus

$$\lim_{n \rightarrow \infty} \|\nabla f(y_n) - \nabla f(u_n)\| = 0. \quad (5.3.38)$$

Therefore, from (5.3.22) and (5.3.37), we have

$$\|y_n - x_n\| \leq \|y_n - u_n\| + \|u_n - x_n\| \longrightarrow 0, \quad n \rightarrow \infty. \quad (5.3.39)$$

Since T_i is uniformly continuous for $i = 1, 2, \dots$, it follows from (5.3.9) and (5.3.39) that

$$\|x_n - T_i x_n\| \leq \|x_n - y_n\| + \|y_n - T_i y_n\| + \|T_i y_n - T_i x_n\| \longrightarrow 0, \quad n \rightarrow \infty, \forall i \geq 1.$$

Since $\{x_n\}$ is bounded and E is reflexive, there exist subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $q \in C$ as $n \rightarrow \infty$. Since $\hat{F}(T_i) = F(T_i)$, then $q \in F(T_i)$ for all $i \geq 1$. Hence, $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

We now show that $q \in \bigcap_{i=1}^N F(\text{prox}_{\lambda_n}^{\phi_i})$. From (5.3.18), (5.3.19) and (5.3.20), we observe that

$$\lim_{n \rightarrow \infty} \|\Theta_n^l w_n - \Theta_n^{l-1} w_n\| = 0, \quad \text{for } l = 1, 2, \dots, N. \quad (5.3.40)$$

Thus, from (5.3.40), we have

$$\begin{aligned} \|\Theta_n^1 w_n - w_n\| &= \|\text{prox}_{\lambda_n}^{\phi_1} w_n - w_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \\ \|\Theta_n^2 w_n - w_n\| &\leq \|\Theta_n^2 w_n - \Theta_n^1 w_n\| + \|\Theta_n^1 w_n - w_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \\ &\vdots \\ \|\Theta_n^{N-1} w_n - w_n\| &\leq \|\Theta_n^{N-1} w_n - \Theta_n^{N-2} w_n\| + \|\Theta_n^{N-2} w_n - \Theta_n^{N-3} w_n\| + \dots \\ &\quad + \|\Theta_n^2 w_n - \Theta_n^1 w_n\| + \|\Theta_n^1 w_n - w_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \\ \|\Theta_n^N w_n - w_n\| &\leq \|\Theta_n^N w_n - \Theta_n^{N-1} w_n\| + \|\Theta_n^{N-1} w_n - \Theta_n^{N-2} w_n\| + \\ &\quad \dots + \|\Theta_n^1 w_n - w_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence, it follows that

$$\lim_{n \rightarrow \infty} \|\Theta_n^l w_n - w_n\| = \lim_{n \rightarrow \infty} \|\text{prox}_{\lambda_n}^{\phi_l} \Theta_n^{l-1} w_n - w_n\| = 0, \quad \text{for } l = 1, 2, \dots, N. \quad (5.3.41)$$

Without loss of generality, let $\{w_{n_j}\}$ be subsequence of $\{w_n\}$ such that $w_{n_j} \rightharpoonup q$, for each $l = 1, 2, \dots, N$. From (5.3.41) and the nonexpansiveness of $\text{prox}_{\lambda_n}^{\phi_l}$, for $l = 1, 2, \dots, N$, we have

$$\begin{aligned} \|\text{prox}_{\lambda_{n_j}}^{\phi_l} w_{n_j} - w_{n_j}\| &= \|\text{prox}_{\lambda_{n_j}}^{\phi_l} w_{n_j} - \text{prox}_{\lambda_{n_j}}^{\phi_l} \Theta_{n_j}^{l-1} w_{n_j}\| + \|\text{prox}_{\lambda_{n_j}}^{\phi_l} \Theta_{n_j}^{l-1} w_{n_j} - w_{n_j}\| \\ &\leq \|\Theta_{n_j}^{l-1} w_{n_j} - w_{n_j}\| + \|\text{prox}_{\lambda_{n_j}}^{\phi_l} \Theta_{n_j}^{l-1} w_{n_j} - w_{n_j}\| \longrightarrow 0, \end{aligned} \quad (5.3.42)$$

as $j \rightarrow \infty$, for each $l = 1, 2, \dots, N$. It follows that $q \in F(\text{prox}_{\lambda_n}^l)$, for each $l = 1, 2, \dots, N$. Therefore, $q \in \bigcap_{l=1}^N F(\text{prox}_{\lambda_n}^l)$.

Next we show that $q \in \bigcap_{k=1}^m VI(C, A_k)$. By (5.1.1), we have that B_k is maximal monotone for each $k = 1, 2, \dots, m$. We note that $\Psi_n^k u_n = \text{Res}_{r_n B_k}^f \Psi_n^{k-1} u_n$ and so, by the Yosida approximation (5.1.3), we have

$$\|B_{r_n} \Psi_n^{k-1} u_n\| = \frac{1}{r_n} \|\nabla f(\Psi_n^{k-1} u_n) - \nabla f(\Psi_n^k u_n)\|. \quad (5.3.43)$$

From (5.3.36), (5.3.43) and $\liminf_{n \rightarrow \infty} r_n > 0$, $k = 1, 2, \dots, m$, we have

$$\lim_{n \rightarrow \infty} \|B_{r_n} \Psi_n^{k-1} u_n\| = 0. \quad (5.3.44)$$

Let $(w_1, w_2) \in G(B_k)$ for each $k = 1, 2, \dots, m$, then it follows from the monotonicity of B_k , $k = 1, 2, \dots, m$ that

$$\langle w_2 - B_{r_n} \Psi_n^{k-1} u_n, w_1 - \Psi_n^k u_n \rangle \geq 0. \quad (5.3.45)$$

Since $\|y_{n_j} - x_{n_j}\| \rightarrow 0$, $j \rightarrow \infty$, then $y_{n_j} = \Psi_{n_j}^k u_{n_j} \rightarrow q$, thus from (5.3.44) and (5.3.45), we obtain

$$\langle w_2, w_1 - q \rangle \geq 0. \quad (5.3.46)$$

Hence $q \in B_k^{-1}(0^*)$ for each $k = 1, 2, \dots, m$. It follows that $q \in VI(C, A_k)$ for each $k = 1, 2, \dots, m$. Therefore, $q \in \bigcap_{k=1}^m VI(C, A_k)$.

On the other hand, from (5.3.4), we obtain

$$\begin{aligned} D_f(x^*, x_{n+1}) &\leq D_f(x^*, u_n) \\ &= D_f\left(x^*, \Theta_n^N(\nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n)))\right) \\ &\leq D_f\left(x^*, \Theta_n^{N-1}(\nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n)))\right) \\ &\quad \vdots \\ &= D_f\left(x^*, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n))\right) \\ &\leq V_f\left(x^*, \alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n) - \alpha_n(\nabla f(u) - \nabla f(x^*))\right) \\ &\quad + \langle \alpha_n(\nabla f(u) - \nabla f(x^*)), x_{n+1} - x^* \rangle \\ &= V_f\left(x^*, \alpha_n \nabla f(x^*) + (1 - \alpha_n) \nabla f(x_n)\right) + \alpha_n \langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^* \rangle \\ &\leq \alpha_n V_f(x^*, \nabla f(x^*)) + (1 - \alpha_n) V_f(x^*, \nabla f(x_n)) + \alpha_n \langle \nabla f(u) - \nabla f(x^*), \\ &\quad x_{n+1} - x^* \rangle \\ &= (1 - \alpha_n) D_f(x^*, x_n) + \alpha_n \langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^* \rangle. \end{aligned}$$

It follows from the definition of Bregman projection (5.2.2) that

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^* \rangle = \langle \nabla f(u) - \nabla f(x^*), q - x^* \rangle \leq 0. \quad (5.3.47)$$

Therefore, using Lemma 2.5.6, (5.3.47) and (5.3.47), we obtain $D_f(x^*, x_n) \rightarrow 0$, $n \rightarrow \infty$. Thus $x_n \rightarrow x^*$, $n \rightarrow \infty$.

Case II: Suppose there exist a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(x^*, x_{n_i}) < D_f(x^*, x_{n_i+1}) \quad (5.3.48)$$

for all $k \in \mathbb{N}$. Then by Lemma 5.2.9, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$,

$$D_f(x^*, x_{m_k}) \leq D_f(x^*, x_{m_k+1}) \quad \text{and} \quad D_f(x^*, x_{m_k+1}),$$

for all $k \in \mathbb{N}$. Furthermore, we obtain

$$\begin{aligned} D_f(x^*, x_{m_k}) - D_f(x^*, T_i x_{m_k}) &= D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) + D_f(x^*, x_{m_k+1}) \\ &\quad - D_f(x^*, T_i x_{m_k}) \\ &\leq D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) + \alpha_n(D_f(x^*, u) \\ &\quad - D_f(x^*, x_{m_k})) \rightarrow 0, \quad k \rightarrow \infty. \end{aligned}$$

It then follows that

$$\lim_{k \rightarrow \infty} D_f(T_i x_{m_k}, x_{m_k}) = 0. \quad (5.3.49)$$

Following the same argument as in Case I, we obtain

$$\limsup_{k \rightarrow \infty} \langle \nabla f(u) - \nabla f(x^*), x_{m_k+1} - x^* \rangle \leq 0, \quad (5.3.50)$$

and

$$D_f(x^*, x_{m_k+1}) \leq (1 - \alpha_{m_k})D_f(x^*, x_{m_k}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(x^*), x_{m_k} - x^* \rangle. \quad (5.3.51)$$

Thus

$$\alpha_{m_k} D_f(x^*, x_{m_k}) \leq D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(x^*), x_{m_k} - x^* \rangle.$$

Since $D_f(x^*, x_{m_k}) \leq D_f(x^*, x_{m_k+1})$ and $\alpha_{m_k} > 0$, we get

$$D_f(x^*, x_{m_k}) \leq \langle \nabla f(u) - \nabla f(x^*), x_{m_k} - x^* \rangle. \quad (5.3.52)$$

It then follows from (5.3.50) that $D_f(x^*, x_{m_k}) \rightarrow 0$, $k \rightarrow \infty$. From (5.3.51) and (5.3.52), we have

$$D_f(x^*, x_{m_k+1}) \rightarrow 0, \quad k \rightarrow \infty. \quad (5.3.53)$$

Since $D_f(x^*, x_k) \leq D_f(x^*, x_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $x_k \rightarrow x^*$ as $k \rightarrow \infty$. Hence both cases imply that $\{x_n\}$ converges strongly to $x^* = \text{proj}_\Gamma^f u$. \square

We now state the following consequences of Theorem 5.3.1.

Corollary 5.3.2. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E . Let $f : E \rightarrow \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $\phi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous function such that $\text{dom}f \cap \text{dom}\phi \neq \emptyset$ and $\text{ran}(\text{prox}_{\lambda}^{\phi}) \subset \text{int}(\text{dom}f)$. Let $A : C \rightarrow E^*$ be continuous and monotone operator and let $\{T_i\}_{i=1}^{\infty}$ be sequence of uniformly continuous quasi-Bregman nonexpansive mappings from C into itself, such that $F(T_i) = \hat{F}(T_i)$ for all $i \geq 1$. Suppose $\Gamma := (\cap_{i=1}^{\infty} F(T_i)) \cap F(\text{prox}_{\lambda_n}^{\phi}) \cap VI(C, A) \neq \emptyset$. For $u, x_0 \in C$, let $\{x_n\}$ be iteratively generated by*

$$\begin{cases} u_n &= \text{prox}_{\lambda_n}^{\phi} \left(\nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n)) \right), \\ y_n &= \text{Res}_{r_n B}^f u_n, \\ x_{n+1} &= \nabla f^* \left(\beta_{0,n} \nabla f(y_n) + \sum_{i=1}^{\infty} \beta_{i,n} \nabla f(T_i y_n) \right), \quad n \geq 0, \end{cases} \quad (5.3.54)$$

where

$$Bx = \begin{cases} Ax + N_C\{x\}, & x \in C \\ \emptyset, & x \notin C, \end{cases} \quad (5.3.55)$$

and $N_C\{x\}$ is a normal cone to C at $x \in E$ and $\lambda_n, r_n > 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ such that:

- i. $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- ii. $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $\sum_{i=0}^{\infty} \beta_{i,n} = 1$,
- iii. $\liminf_{n \rightarrow \infty} r_n \geq 0$.

Then $\{x_n\}$ converges strongly to $\text{Proj}_{\Gamma}^f u$, where Proj_{Γ}^f is the Bregman projection of C onto Γ .

Corollary 5.3.3. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E . Let $f : E \rightarrow \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . For each $l = 1, 2, \dots, N$, let $\phi_l : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions such that $\text{dom}f \cap \text{dom}\phi_l \neq \emptyset$ and $\text{ran}(\text{prox}_{\lambda}^{\phi_l}) \subset \text{int}(\text{dom}f)$. For $k = 1, 2, \dots, m$ let $A_k : C \rightarrow E^*$ be continuous and monotone operators and let $\{T_i\}_{i=1}^{\infty}$ be sequence of uniformly continuous Bregman strongly nonexpansive mappings from C into itself, such that $F(T_i) = \hat{F}(T_i)$ for all $i \geq 1$. Suppose $\Gamma := \left(\cap_{i=1}^{\infty} F(T_i) \right) \cap \left(\cap_{l=1}^N F(\text{prox}_{\lambda_n}^{\phi_l}) \right) \cap \left(\cap_{k=1}^m VI(C, A_k) \right) \neq \emptyset$. For $u, x_0 \in C$, let $\{x_n\}$ be iteratively generated by*

$$\begin{cases} u_n &= \text{prox}_{\lambda_n}^{\phi_N} \circ \text{prox}_{\lambda_n}^{\phi_{N-1}} \circ \dots \circ \text{prox}_{\lambda_n}^{\phi_1} \left(\nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n)) \right), \\ y_n &= \text{Res}_{r_n B_m}^f \circ \text{Res}_{r_n B_{m-1}}^f \circ \dots \circ \text{Res}_{r_n B_1}^f u_n, \\ x_{n+1} &= \nabla f^* \left(\beta_{0,n} \nabla f(y_n) + \sum_{i=1}^{\infty} \beta_{i,n} \nabla f(T_i y_n) \right), \quad n \geq 0, \end{cases} \quad (5.3.56)$$

where

$$B_k x = \begin{cases} A_k x + N_C\{x\}, & x \in C \\ \emptyset, & x \notin C, \end{cases} \quad (5.3.57)$$

for $k = 1, 2, \dots, m$ and $N_C\{x\}$ is a normal cone to C at $x \in E$ and $\lambda_n, r_n > 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ such that:

- i. $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- ii. $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $\sum_{i=0}^{\infty} \beta_{i,n} = 1$,
- iii. $\liminf_{n \rightarrow \infty} r_n \geq 0$.

Then $\{x_n\}$ converges strongly to $\text{Proj}_{\Gamma}^f u$, where Proj_{Γ}^f is the Bregman projection of C onto Γ .

5.4 Applications

5.4.1 Zeroes of Bregman inversely strongly monotone operators

Let the Legendre function f be such that

$$\text{ran}(\nabla f - S) \subseteq \text{ran}(\nabla f), \quad (5.4.1)$$

the operator $S : E \rightarrow 2^{E^*}$ is called Bregman Inversely Strongly Monotone (BISM) if $(\text{dom} S) \cap (\text{int}(\text{dom} f)) \neq \emptyset$ and for any $x, y \in \text{int}(\text{dom} f)$ and each $\xi \in Sx, \eta \in Sy$, we have

$$\langle \xi - \eta, (\nabla f(x) - \xi) - \nabla f^*(\nabla f(y) - \eta) \rangle \geq 0. \quad (5.4.2)$$

This class of operators was introduced by Butnariu and Kassay (see [28]). For any operator $S : E \rightarrow 2^{E^*}$, the anti-resolvent $S^f : E \rightarrow 2^E$ of S is defined by

$$S^f := \nabla f^* \circ (\nabla f - S). \quad (5.4.3)$$

Observe that $\text{dom} S^f \subseteq (\text{dom} S) \cap (\text{int}(\text{dom} f))$ and $\text{ran} S^f \subseteq \text{int}(\text{dom} f)$. The operator S is BISM if and only if the anti-resolvent S^f is a single-valued BFNE operator (see [28]). Some examples of BISM operators can be seen in [28]. From the definition of anti-resolvent and ([28], Lemma 3.5), we obtain the following proposition.

Proposition 5.4.1. *Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a Legendre function and let $S : E \rightarrow 2^{E^*}$ be a BISM operator such that $S^{-1}(0)^* \neq \emptyset$. Then the following statements hold:*

- i. $S^{-1}(0)^* = F(S^f)$,
- ii. for any $u \in S^{-1}(0)^*$ and $x \in \text{dom} S^f$, we have

$$D_f(u, S^f x) + D_f(S^f x, x) \leq D_f(u, x).$$

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E , then the anti-resolvent S^f of S is single-valued BSNE operator which satisfies

$$F(S^f) = \hat{F}(S^f)$$

(see [102], Lemma 1.3.2).

In Theorem 5.3.1, if we let $T_i = S_i^f$ and let f be the Legendre function such that (5.4.1) is satisfied, then we obtain the following result for approximating a common zeroes of an infinite family of Bregman inversely strongly monotone operators which is also a solution to the set of finite families of convex minimization problem and variational inequalities problems.

Theorem 5.4.2. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E . Let $f : E \rightarrow \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E satisfying the range condition. For each $l = 1, 2, \dots, N$, let $\phi_l : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions such that $\text{dom}f \cap \text{dom}\phi_l \neq \emptyset$ and $\text{ran}(\text{prox}_{\lambda}^{\phi_l}) \subset \text{int}(\text{dom}f)$. For $k = 1, 2, \dots, m$ let $A_k : C \rightarrow E^*$ be continuous and monotone operators and let $\{T_i\}_{i=1}^{\infty} = \{S_i^f\}_{i=1}^{\infty}$ such that $F(T_i) = \hat{F}(T_i)$ for all $i \geq 1$. Suppose $\Gamma := \left(\bigcap_{i=1}^{\infty} F(T_i) \right) \cap \left(\bigcap_{l=1}^N F(\text{prox}_{\lambda_n}^{\phi_l}) \right) \cap \left(\bigcap_{k=1}^m VI(C, A_k) \right) \neq \emptyset$. For $u, x_0 \in C$, let $\{x_n\}$ be iteratively generated by*

$$\begin{cases} u_n &= \text{prox}_{\lambda_n}^{\phi_N} \circ \text{prox}_{\lambda_n}^{\phi_{N-1}} \circ \dots \circ \text{prox}_{\lambda_n}^{\phi_1} \left(\nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n)) \right), \\ y_n &= \text{Res}_{r_n B_m}^f \circ \text{Res}_{r_n B_{m-1}}^f \circ \dots \circ \text{Res}_{r_n B_1}^f u_n, \\ x_{n+1} &= \nabla f^* \left(\beta_{0,n} \nabla f(y_n) + \sum_{i=1}^{\infty} \beta_{i,n} \nabla f(T_i y_n) \right), \quad n \geq 0, \end{cases} \quad (5.4.4)$$

where

$$B_k x = \begin{cases} A_k x + N_C \{x\}, & x \in C \\ \emptyset, & x \notin C, \end{cases} \quad (5.4.5)$$

for $k = 1, 2, \dots, m$ and $N_C \{x\}$ is a normal cone to C at $x \in E$ and $\lambda_n, r_n > 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ such that:

- i. $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- ii. $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $\sum_{i=0}^{\infty} \beta_{i,n} = 1$,
- iii. $\liminf_{n \rightarrow \infty} r_n \geq 0$.

Then $\{x_n\}$ converges strongly to $\text{Proj}_{\Gamma}^f u$, where Proj_{Γ}^f is the Bregman projection of C onto Γ .

5.4.2 Equilibrium problem

Let C be a nonempty closed and convex subset of the Banach space E and $g : C \times C \rightarrow \mathbb{R}$ be a bifunction. We recall that the equilibrium problem (EP) is to find $x \in C$ such that

$$g(x, y) \geq 0 \quad \forall y \in C. \quad (5.4.6)$$

Let $\bar{x} \in C$, setting

$$g(\bar{x}, \bar{y}) := \phi(\bar{y}) - \phi(\bar{x}) \quad \forall \bar{y} \in C,$$

the equilibrium problem (5.4.6) coincides with the convex minimization problem (5.1.5) and the function g satisfy conditions L1 - L4 of Section 2.4.1 (see, [15]). The resolvent of the bifunction g is the function $Res_g^f : E \rightarrow 2^C$ defined by (see, [101])

$$Res_g^f(x) = \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \quad \forall y \in C\}. \quad (5.4.7)$$

Proposition 5.4.3. (see [101]) *Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a coercive and Legendre function. If the bifunction $g : C \times C \rightarrow \mathbb{R}$ satisfies conditions L1 - L4, then*

1. Res_g^f is single-valued,
2. Res_g^f is BFNE,
3. $F(Res_g^f) = EP(g)$,
4. $EP(g)$ is a closed and convex subset of C ,
5. for all $x \in E$ and $q \in F(Res_g^f)$,

$$D_f(q, Res_g^f(x)) + D_f(Res_g^f(x), x) \leq D_f(q, x).$$

Setting $\phi_l = g_l$, $l = 1, 2, \dots, N$ in Theorem 5.3.1, then, we have an iterative algorithm for approximating a common solution of infinitely family of quasi-Bregman nonexpansive mappings which is also a common solution of finite systems of equilibrium problems and variational inequality problems in reflexive Banach spaces.

Conclusion, Contribution to Knowledge and Future Research

6.1 Conclusion

In this dissertation, we introduced some iterative schemes for approximating common element of set of solutions of minimization problems, variational inequality problems, monotone variational inclusion problems, generalized equilibrium problems, generalized mixed equilibrium problems and fixed point problems. We obtained strong convergence results of the sequences generated by our iterative schemes without compactness assumption. We also gave numerical examples of our main results in Chapter 3 and Chapter 4 in real Hilbert spaces and show how the sequence values are affected by the number of iterations. We gave application of our result in Chapter 5 to finding zeroes of Bregman inversely strongly monotone operators in real reflexive Banach space.

6.2 Contribution to Knowledge

The following are contribution to knowledge:

1. Our Theorem 3.3.1 improve the result of Shehu [108] from approximation of common solution of generalized equilibrium problem, monotone variational inclusion problem and fixed point problem to common solution of split equalities for generalized equilibrium problem, monotone variational inclusion problem and fixed point problem in Hilbert space. Also the class of k demi-contractive mapping considered in our work is larger than the class of nonexpansive mapping consider in [108].
2. In [73], the authors obtained weak convergence result on approximation of common solution of equilibrium problem and fixed point problem of k -strictly pseudo-

nonspreading multi-valued mapping in Hilbert space, while, our Theorem 4.3.1 present a strong convergence result. We also extend the work of [73] to split equalities for generalized mixed equilibrium problem and fixed point problem of k -strictly pseudo-nonspreading multi-valued mapping of type-one in Hilbert space.

3. Also, our iterative scheme (4.3.1) improved (1.2.5) presented by Ma et al. [75] in the sense that (4.3.1) does not require a prior knowledge of the operator norm and the condition of demi-compactness on the mappings.
4. Our Theorem 5.3.1 complements and extends many recent results in literature. For example, the result of Kassay, Reich and Sabach [58] is extended to approximation of common solution of finite systems of variational inequalities problems, convex minimization problems and common fixed point of infinite families of quasi-Bregman nonexpansive mapping. Also, our Theorem 5.3.1 extend the results on convex minimization problem in Hilbert space (for instance, [113, 31, 30]) to reflexive Banach space.

The following are submitted research articles based on our work for publication:

1. L.O. Jolaoso, F.U. Ogbuisi and O.T. Mewomo, Operator norm independent solution of split equality of systems of generalized mixed equilibrium problem and split equality fixed point problem for certain multi-valued maps, submitted to *Boletín de la Sociedad Matemática Mexicana*.
2. L.O. Jolaoso, F.U. Ogbuisi and O.T. Mewomo, On split equality for certain nonlinear optimization and fixed point problems, submitted to *Arkiv for Matematik*.
3. L.O. Jolaoso, F.U. Ogbuisi and O.T. Mewomo, An iterative method for solving minimization, variational inequality and fixed point problems in reflexive Banach spaces, submitted to *Computation and Applied Mathematics*.

6.3 Future research

In the future, we will like to extend our study of fixed point theorems for set-valued mappings with weak contractions, generalized weak contractions, asymptotic contractions, generalized asymptotic contractions and give numerical examples and applications of our results in real Hilbert and reflexive Banach spaces.

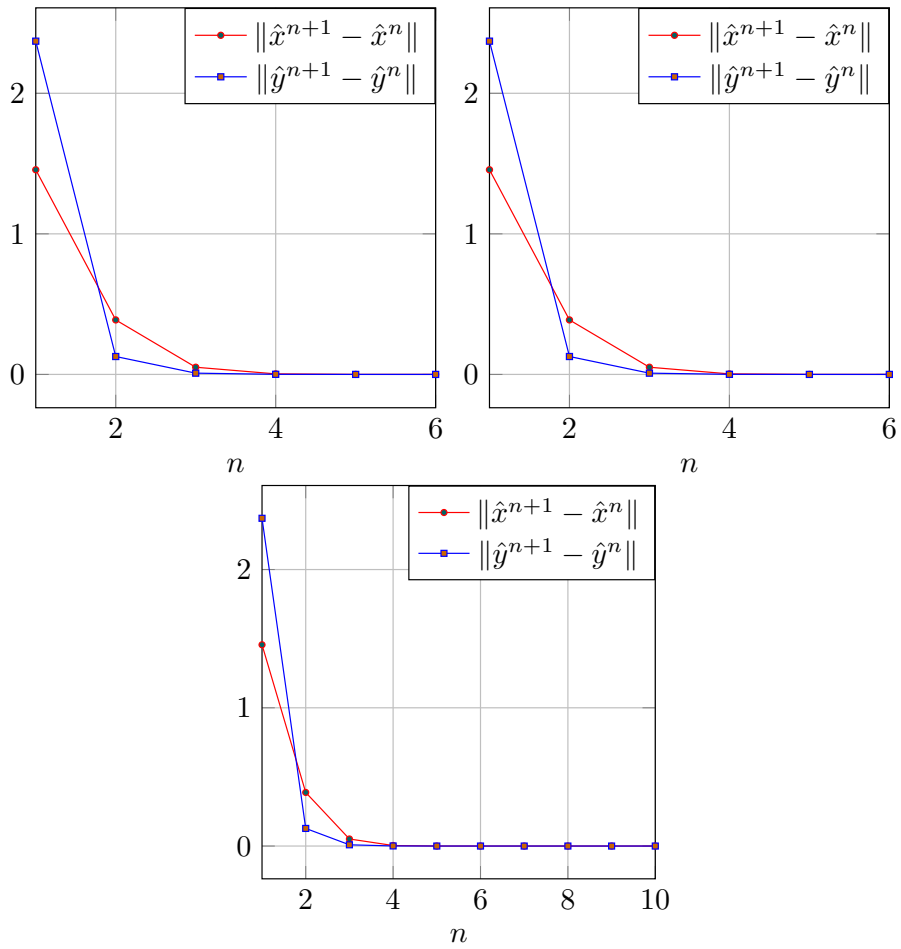


Figure 6.1: Errors: Case A(i), $\varepsilon = 10^{-4}$ (left, 0.044sec), $\varepsilon = 10^{-6}$ (right, 0.045sec), $\varepsilon = 10^{-12}$ (bottom, 0.047sec).

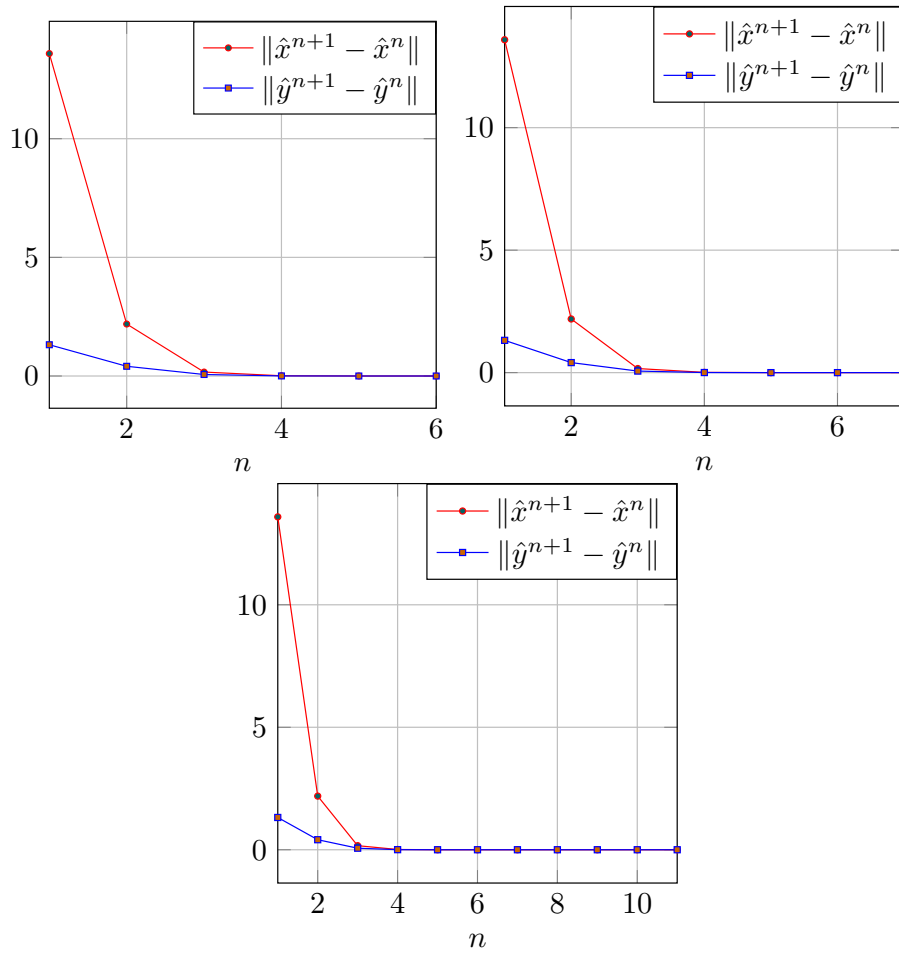


Figure 6.2: Errors: Case A(ii), $\epsilon = 10^{-4}$ (left, 0.044sec), $\epsilon = 10^{-6}$ (right, 0.046sec), $\epsilon = 10^{-12}$ (bottom, 0.051sec).

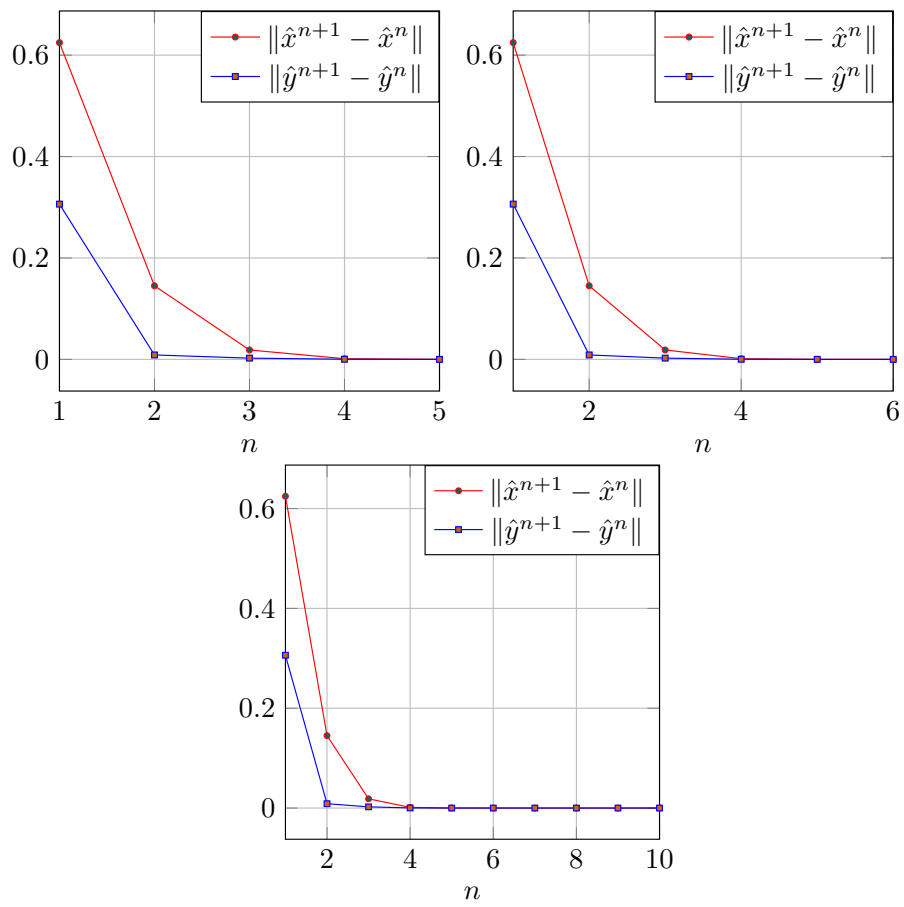


Figure 6.3: Errors: Case B(i), $\varepsilon = 10^{-4}$ (left, 0.044sec), $\varepsilon = 10^{-6}$ (right, 0.046sec), $\varepsilon = 10^{-12}$ (bottom, 0.049sec).

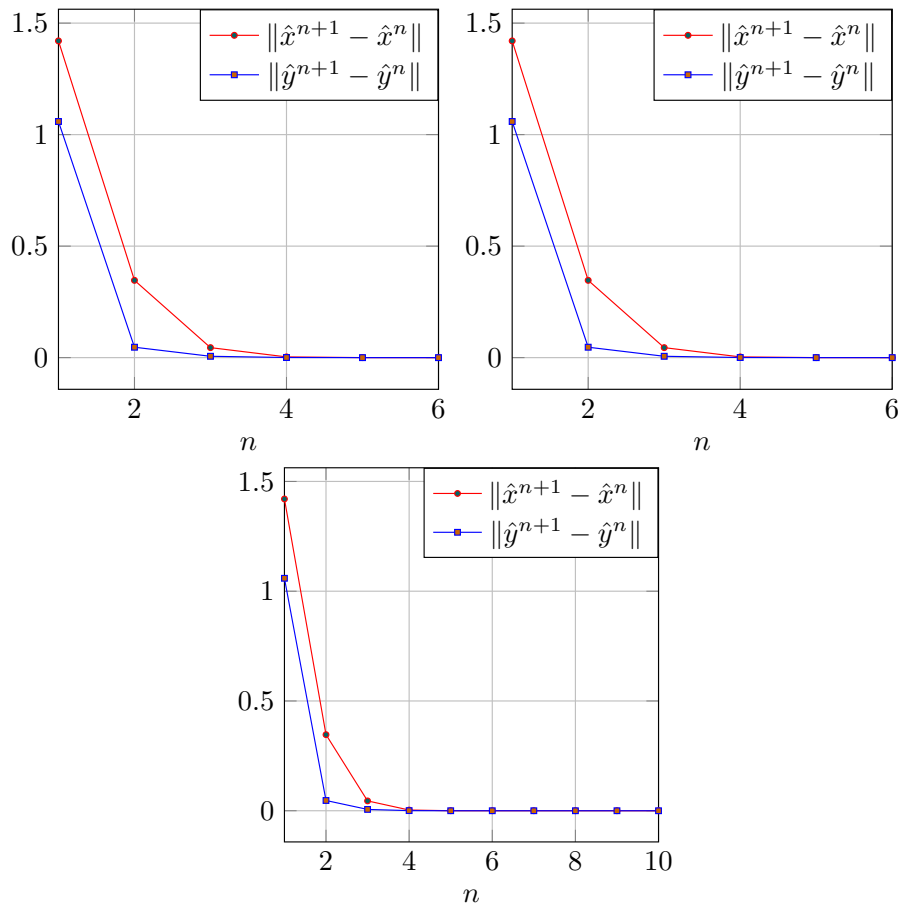


Figure 6.4: Errors: Case B(ii), $\varepsilon = 10^{-4}$ (left, 0.044sec), $\varepsilon = 10^{-6}$ (right, 0.044sec), $\varepsilon = 10^{-12}$ (bottom, 0.047sec).

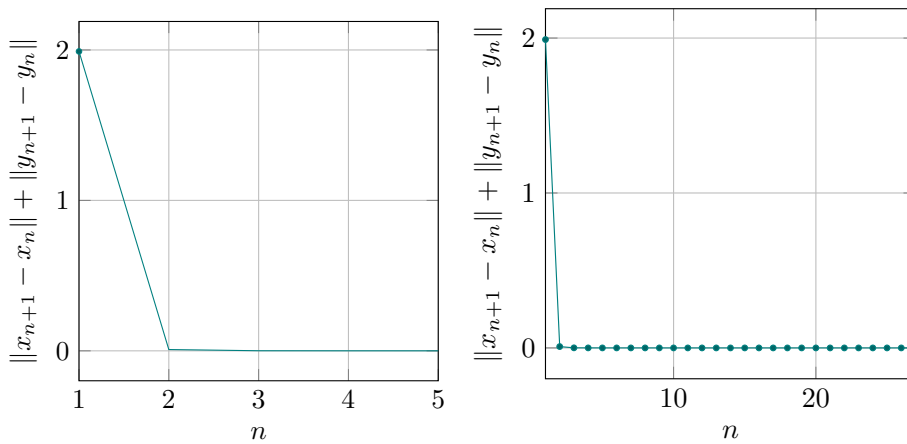


Figure 6.5: Errors: Case I(a), $\varepsilon = 10^{-4}$ (left, 0.024sec) and $\varepsilon = 10^{-6}$ (right, 0.025sec).

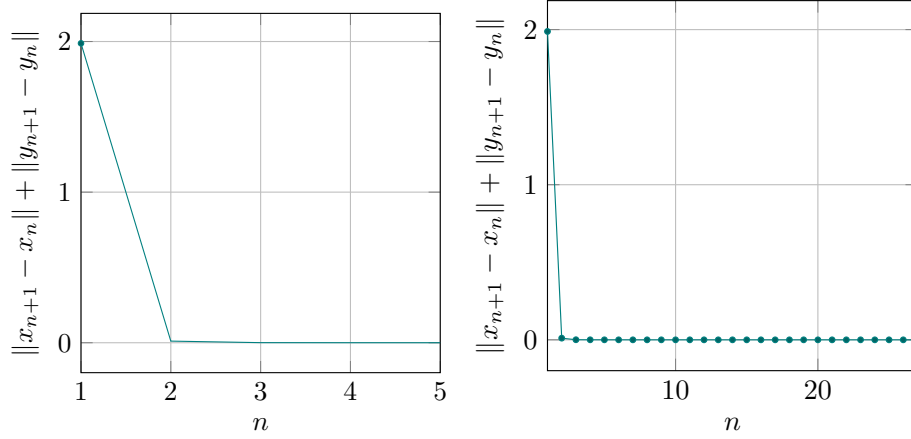


Figure 6.6: Errors: Case I(b), $\varepsilon = 10^{-4}$ (left, 0.019sec) and $\varepsilon = 10^{-6}$ (right, 0.025sec).

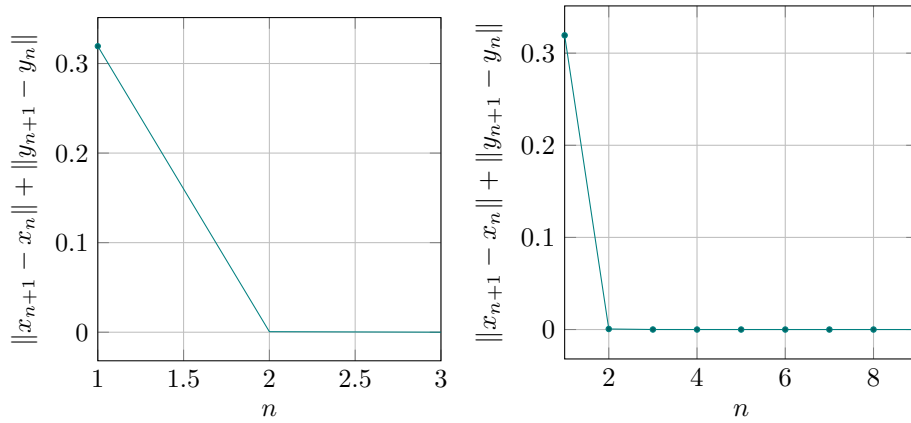


Figure 6.7: Errors: Case II(a), $\varepsilon = 10^{-4}$ (left, 0.019sec) and $\varepsilon = 10^{-6}$ (right, 0.025sec).

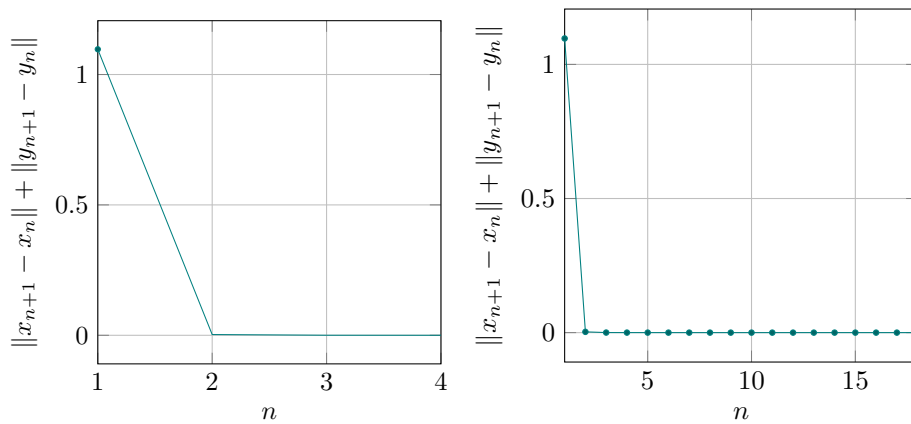


Figure 6.8: Errors: Case II(b), $\varepsilon = 10^{-4}$ (left, 0.021sec) and $\varepsilon = 10^{-6}$ (right, 0.025sec).

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