

**FIRST-YEAR ENGINEERING STUDENTS' CONCEPT
DEVELOPMENT OF INTEGRAL CALCULUS AT A
SOUTH AFRICAN UNIVERSITY OF TECHNOLOGY**

By

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A full thesis for the Degree of

DOCTOR OF EDUCATION
(Mathematics Education)

In the School of Education

University of Kwa-Zulu Natal

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2015

DECLARATION

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ABSTRACT

This thesis reports on a study to explore the development of the concept of integration among the first year engineering students at a South African university of technology. The study focused on concept definitions that were evoked through symbolic as well as visualisation of integrals. It further explored various concept images evoked the techniques of integration. A framework combining the Action-Process-Object-Schema (APOS) and the Three-Worlds of Mathematics (TWM) theories was adopted as a tool to analyse students' concept formation of an integral.

This was a qualitative case study that consisted of two phases. Firstly, a pilot phase was introduced as Phase 1 of the study to uncover issues that could be probed more deeply when the study was rolled out to a larger group of students. The activity sheet was administered and interviews were conducted with seven students who were willing to participate in the study. Secondly, as Phase 2 of the study, the modified activity sheet was then administered to 22 first year students who also volunteered to be in the study. The intention was to provide comprehensive investigation of concept development of integral calculus. Students were also organised into focus groups in order to explore emerging mental constructions during the discussions among the students.

The findings of the research indicated that students operated mainly at an action level of cognition for integral calculus. Their definition of an integral was restricted to the notion finding an integral with no association to the area below the graph of a function. Students mainly conceptualised an integral as an anti-derivative. With regard to techniques of integration, students relied on rules and algorithms without reflecting on objects and processes embedded within the rules. Cases of inadequate prerequisite schemas for integral calculus such as basic algebra, inverse trigonometric functions and some aspects of differentiation were also noted. Although there were notable strengths in skills such as completing a square and resolving fraction into partial fractions, there was little understanding of the underlying concepts. This study contributed by presenting a genetic decomposition for integration that is premised on APOS and TWM theories. While the action level of APOS was dominant, the proceptual-symbolic was the main prevalent world of mathematics learning.

ACKNOWLEDGEMENTS

I wish to express my gratitude towards the following who contributed to make this work a possibility:

I am grateful to my supervisor, Professor Deonarain Brijlall, for the help, support and encouragement he has provided me with throughout my research. His unceasing patience and care has greatly inspired me throughout our interaction. He made me acknowledge my potential and his guidance enabled me to realise my dreams.

I further thank the School of Education at the University of KwaZulu Natal for the academic support they provided and for admitting me into the doctoral programme. The orientation to a PhD through the 'Stimulating Knowledge Innovation through Life-long Learning' (SKILL) programme and the diversity of experience from the 2012 Cohort enriched my perspective on research. In addition, I am grateful for the doctoral grant the School provided during my studies.

I also thank Mangosuthu University of Technology for supporting my studies financially, granting me study leave and allowing me to collect data within the University. I take this opportunity also to thank the 2013 S2 electrical engineering students at this University for their willingness to be part of this study.

Further gratitude goes to Dr Carolyn Turnbull-Jackson for editing part of my thesis.

Lastly, I thank my husband Siphon James and my children Nkululeko, Malusi, Njabulo, Khula and Duduziwe for the love and support they cushioned me with. I could not have done it without their encouragement.

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CHAPTER ONE

INTRODUCTION

1.7. *Background information*

Engineering mathematics is fundamental to all engineering programmes at universities of technology. It facilitates the understanding of content for various subjects within the discipline and, as a result, it is a pre-requisite for advancement to senior levels of study. Poor performance in mathematics may present it as a gateway subject for first year students. As a mathematics lecturer at a university of technology, therefore, I felt that there was a need to explore students' learning of mathematics since "research into students' learning of mathematics helps to predict what they may learn about a specific mathematical concept and the conditions by which that learning takes place" (Arnon *et al.*, 2014, p. 27).

A dominant mathematical concept that students encounter within engineering mathematics at a university of technology is calculus. In their investigation on what calculus experts regard as essential learning points of the first year calculus, Sofronas *et al.* (2011) observed general consensus over three broad areas. These are: (i) mastery of fundamental concepts and-or skills, (ii) connections and relations between and among concepts and-or skills and, (iii) the ability to use the ideas of the calculus. There was also noteworthy agreement among the experts that derivatives, integrals and limits constitute the fundamental concepts and-or skills for first year calculus. In the understanding of an integral, in particular, experts cited the notion of an integral as an area, integral as net change or accumulated total change and the techniques of integration as necessary facets.

The school curriculum for mathematics in South Africa includes only two of these fundamental concepts or learning points, namely, derivatives and limits (Department of Education, 2003). The exclusion of the integration topic in school calculus results in students entering universities being highly underprepared for it. Also, the quality

of mathematics teaching at a school level which seems inadequate contributes to many difficulties exhibited by students when handling this aspect of the subject (Jojo, Maharaj, & Brijlall, 2013).

Several studies have investigated the teaching and learning of integral calculus (Berger, 2006; Brijlall & Bansilal, 2010; Habineza, 2010; Koepf & Ben-Israel, 1994); while Orton (1983b) established that students had difficulties in understanding an integral as the limit of a sum. In addition, some studies have advocated that mathematics teaching for engineering students should include both mathematical knowledge and mathematical thinking (Bennett, Moore, & Nguyen, 2011; Cardella & Atman, 2004; Huang, 2011).

It is the mathematical thinking that enables students to apply mathematics in different contexts beyond the mathematics classroom. Researchers refer to this ability to apply knowledge learned in one context to a new context as transfer of learning and is regarded as the central goal of education (Bennett, Moore, & Nguyen, 2011; Byrnes, 2001; Cui, Rebello, Fletcher, & Bennett, 2006; McKeough, Lupart, & Marini, 2013). The role of mathematics teaching to engineering students is, therefore, to assist students develop the ability to use the language of mathematics in solving engineering problems. Students should know the sets of tools that can be used to solve problems and, in addition, be able to apply such tools in other contexts (Bennett et al., 2011).

According to Berger (2006), acceptable application of mathematical tools depends on how a student has mentally constructed such tools or concepts. Bennett et al. (2011) mention the level of abstraction and problem solving as other factors impacting on the transfer of mathematical knowledge. Students tend to be more successful in transferring mathematical knowledge when dealing with algebraic representations, which are more abstract and, in most instances, independent of context. Transferring also gets hindered when students do not possess the problem solving abilities required in the other contexts of application.

Students studying for a national diploma in electrical engineering, for example, apply integration when analysing electric circuits; and in civil engineering, the application of integration is prominent in the calculation of deflection in beams. Students are

supposed to model engineering problems mathematically and utilise their knowledge and skills of mathematics to solve such problems.

Dewi and Kusumah (2014) distinguish between lower order and higher order mathematical thinking. They view lower order as encompassing the memorising and simple application of a given mathematical formula. At this level of thinking, the focus is on simple operations, the application of direct procedures and the use of algorithms. Higher order mathematical thinking includes a deeper understanding of mathematical ideas, extraction of implicit ideas from given data and formulating conjectures and analogies. It is also the thinking that displays logical reasoning, problem solving, mathematical communication and linking of mathematical ideas to other intellectual activities. The meaning that students make of integration and the higher order mathematical thinking enable students to successfully apply integration in other fields of study.

1.8. ***Problem statement***

This study was conducted in my fourteenth year as a mathematics lecturer at a university of technology. Prior to that, I had taught mathematics at high school level for two years and, at teacher-training colleges for eight years. The engineering programme consists of a two-year (four semesters) theoretical instruction and a one-year experiential or work-based learning. Mathematics teaching for engineering students spans over three of the four semesters of the theoretical component. Mathematics applications are expected throughout the students' theoretical instructions but the demand is noticeable in the third and fourth semesters.

My observation was that students struggled to apply integration post their mathematics learning. They often approached me with engineering problems which required the use of integration in order to solve. This indicated that the mathematics training that students received did not prepare them adequately to be able to apply integration in the field of engineering. Analysis of results over the previous four semesters showed that students' performance in electric circuit and heat transfer problems, which require application of integration, averaged 42%. This failure to

apply integration reflected poor mental constructions of the concept (Berger, 2006). Nguyen (2011) made a similar observation with regard to the application of integration in physics where students did not understand the meaning of integrands and could not view integration as a summation. Students were struggling to use integration as a tool in the engineering field, in spite of having successfully completed the calculus modules.

Furthermore, students' performance in assessments always revealed that students had difficulty in understanding and applying integration. This challenge of poor performance in integration was mainly noticed in the second semester of the first year studies. The mathematics module that students take in this semester is called 'Mathematics II', the 'II' designating the semester of study. Integration constitutes about 70% of this module, the other topics being hyperbolic functions, partial differentiation and first order differential equations (Msomi, 2011). Students often performed well in the other sections but struggled in integration. The poor performance contributed to a high failure rate in the subject. As a result, many students could not progress to advanced levels of study within engineering and, in some cases; they eventually dropped out of the university. I, therefore, became interested to know how students developed their knowledge and understanding of an integral and how teaching can be structured in order to enhance students' learning.

Within the mathematics education research community, discussions have ensued about students' conceptions of different mathematical concepts and about the development of such conceptions. As a result, an emerging trend in addressing difficulties in students' understanding of mathematical concepts is the exploration of how particular knowledge or concepts are constructed in the minds of students. This trend marks a shift from the previous approaches where the focus was on the revision of a curriculum, integration of technology in teaching or the identification of better teaching methods (Dubinsky & Lewin, 1986).

In exploring how concepts are constructed in the mind, some researchers in mathematics education have used Piaget's ideas on cognitive development to develop various theories that explain the learning processes in mathematics (Asiala,

Brown, et al., 1997). Among the theories developed to attempt to explain the learning processes in mathematics is the Action-Process-Object-Schema (APOS) theory (Asiala, Brown, et al., 1997; Dubinsky & McDonald, 2001). As confirmed by DeVries and Arnon (2004) and Berger (2006), APOS theory elaborates on the Piaget's cognitive theory, expanding it to advanced mathematics. APOS theory embraces social interactions among students as a principle of effective mathematics teaching. Such an approach is in line with the education theories of Piaget which purport that for deep conceptual understanding and positive relationships to develop in learning, interaction is essential.

APOS theory provides the levels of understanding or the nature of mental constructions that are necessary for students to learn mathematical concepts (Clark, Cordero, Cottrill, Czarnocha, DeVries, John, et al., 1997). In addition to the APOS theory is the work by Tall (2002), in which he defines three worlds of mathematics cognition (TWM), namely, the conceptual-embodied, the proceptual-symbolic and the axiomatic-formal worlds. Both APOS and the TWM theories require a teacher or researcher to provide a possible genetic decomposition (GD) for a particular mathematical concept. A GD of a concept consists of a description of possible actions, behaviours and reactions expected of a student who has developed the concept in question (DeVries & Arnon, 2004). It is a "structured set of constructs which might describe how the concept can develop in the mind of an individual" (Maharaj, 2010, p. 42). A detailed explanation of APOS and TWM theories will be provided when reviewing the related literature of this study.

Several researchers in mathematics education have used APOS and TWM theories to analyse how students construct knowledge in advanced mathematics. Brijlall and Maharaj (2009) used APOS theory to analyse how second-year university students specialising in the teaching of mathematics for an FET high school curriculum in South Africa construct the concept of continuity of a single-valued function. Stewart and Thomas (2007) applied APOS theory in the context of the TWM in analysing students' learning of linear algebra in their first year of study at the University of Auckland. In particular, Brijlall and Bansilal (2010) employed APOS theory to analyse how the mathematics teacher trainees developed their understanding of a

Riemann Sum. There was no evidence though, of corresponding APOS studies on engineering students' understanding of integral calculus, and particularly in the South African context.

The aim of this study, therefore, was to explore how engineering students at a university of technology construct knowledge as they learn integral calculus. A framework combining the APOS theory and TWM was used to analyse students' concept formation.

1.9. ***Research questions***

This study was aimed at answering the primary question: How do students construct mathematical meaning when learning integral calculus? The study had potential of contributing to the theory of understanding of how students learn mathematics in general, and integral calculus in particular. It could also to inform the development of appropriate pedagogical instruction, based on the theory developed. In the study, APOS approach was adopted to explore mental constructions displayed by students when learning integral calculus. To pursue the objectives of the main question of the study, the following sub-questions were addressed:

1. What concept definitions do students attach to an integral?
2. What concept images do students exhibit when employing techniques of integration?
3. In what worlds of mathematical thinking do students operate when they internalise integration? How do these worlds influence the learning of the integral calculus?
4. What genetic decomposition can be proposed for the construction of meaning in integration?

The first sub-question was about the concept definitions of an integral. Rasslan and Tall (200) maintain that all mathematical concepts, except the ones that are primitive,

have definitions. Students, however, seldom draw from a definition in order to conclude whether a particular idea is or is not an example of a concept. The participants in this study had been exposed to an integral as an area under the graph of a function. They had also been taught the Fundamental Theorem of Calculus (FTC) which then led to viewing an integral as an anti-derivative. Various techniques of integration had also been handled during lectures, including relevant applications of an integral such as calculating a mean or the mean of the squares of a function.

The second sub-question was about concept images that students evoked when handling problems in integration. Concept image, according to Tall and Vinner (1981), refers to the summative cognitive structure associated with a concept that an individual possesses. Included in the concept image are mental formulations, associated processes and properties and in some cases, personal concept definition by an individual. It was important to establish concept definitions and concept images of integration since these aspects form basis for concept formation in mathematics (Rösken & Rolka, 2007). APOS theory was used as a lens to analyse these aspects of concept definition and concept image.

The third sub-question was about the application of the TWM theory on students' internalisation of integral calculus. I was interested in investigating the perceptions of and reflections on the properties of an integral possessed by the participants in this study. The TWM theory provides for the conceptual-embodied, the proceptual-symbolic and the axiomatic-formal world (Tall, 2008). Understandably, for the group of students under study who were not taking calculus as a major, there was minimal emphasis on formal definitions and proofs. Nonetheless, there was still expectation that students should be able to display ability to think about mathematics symbolically.

The fourth sub-question informed the revision of the GD initially proposed. An activity sheet was designed to compare the mental constructions students seemed to be making to those that had been predicted in the hypothesised genetic decomposition for integration. The revised GD for integration, which combined the two theories used

in the study, was an important contribution to the analysis of concept development of integration.

1.10. ***Outline of the research process***

Here I outline the overall design of the research process to address the research questions mentioned above. When I expressed an interest in exploring my students' learning, my supervisor advised that I start with preliminary readings on constructivist theories in the learning of mathematics (Cooley, Trigueros, & Baker, 2007; Dubinsky, 1991b; Dubinsky & Lewin, 1986; Dubinsky & McDonald, 2001; Tall, 2002, 2007, 2008). In addition, I did reading on the construction of different mathematical concepts from the identified theories' perspective (Brijlall & Maharaj, 2010; Clark, Cordero, Cottrill, Czarnocha, DeVries, John, *et al.*, 1997; Dubinsky, Weller, McDonald, & Brown, 2005; Parraguez & Oktaç, 2010).

To work on my research proposal, I was supported by being admitted into the Stimulating Knowledge Innovation through Life-long Learning (SKILL) programme, a two week course on writing a good PhD proposal. I also joined a cohort doctoral support programme that is offered in my university of study (Samuel & Vithal, 2011). Having read on research methodologies (Cohen, Manion, & Morrison, 2011; Creswell, 2002; Leedy & Ormrod, 2005), I ultimately designed my research as a case study project.

Data for the study was collected through three distinct activities. Firstly, I started with hypothesising about mental constructs that students should exhibit when learning an integral. This hypothesis helped to formulate items or tasks that constituted the activity sheet that was the main research instrument. To validate the activity sheet, I conducted a "pilot" exercise which I termed Phase 1 of this study. Students had to respond to a carefully structured activity sheet. In addition, I interviewed them based on their responses to the tasks in the sheet. The aim was to validate the main research instrument and analysing data collected during this phase could result in the revision of both the activity sheet and the initial hypothesised genetic decomposition for integration.

The second research activity entailed Phase 2 of the study where the revised activity sheet was administered, followed by structured interviews as well. The last activity, still in Phase 2, was the structuring of focus groups that were video recorded. Students responded to structured mathematical items in groups and they were encouraged to discuss their solutions among each other and, in some instances, to the whole class. Analysis of data included interpretations and coding of written responses, transcription of both interviews and video recordings of students' discussions.

The final stage of my research process was the writing of the thesis. In this document I present findings and conclusions based on critical analysis and structuring of generated data, while evaluating, comparing and judging it against the existing theories. Finally, the thesis was subjected to a language and technical expert for editing purposes as well as through the Turn-it-in to guard against plagiarism. The editor's and Turn-it-in certificates are included on pages 199 and 200 of this thesis as Appendices A4 and A5 respectively. Due to unpredictable circumstances, this editing process did not include chapters one and eight. The two chapters were then rectified during the consultation process with my supervisor.

1.5 *Delineations*

This study reports only on data obtained from a single university of technology in South Africa. In particular, the investigation was conducted within an Electrical Engineering group of students that I was teaching in the years of the study. Furthermore, the study focuses only on the aspects of integration as they relate to the identified programme of study. As a result, the investigation was restricted to the area definition of an integral, graphical representations and techniques of integration for a single-variable function. Lastly, the focus of the study was on concept development and did not include performance in the subject. Consequently, the study does not report on interventions and possible impacts thereof.

1.6 Overview of chapters of the thesis

This thesis consists of eight chapters, the list of references and appendices. Briefly, the first four chapters outline the generalities of a research project. The next three chapters present the data of the study and their analysis. The last chapter provides some conclusions and recommendations on students' construction of meaning when learning integral calculus. I then suggest a genetic decomposition for integration, which is based on the APOS and TWM theoretical frameworks.

More elaborately, this first chapter introduces the thesis by providing the background, the research problem and the research questions for the study. In addition, it presents the outline of the research process, delineations of the study and the road map of the thesis. The second chapter contains a review of literature about students' construction of meaning when learning mathematics. It presents discussions on the teaching and learning of calculus in general. This section includes what experts in mathematics consider as essential for the learning of first year calculus. Next, the chapter provides discussions on the construction of meaning in various mathematics topics with a specific focus on the integral calculus. Furthermore, it provides reviews of the procedural and conceptual learning types, as well as the concept image and concept definition notions.

The third chapter discusses the theoretical frameworks used both in the generation and analysis of data for the study. The concept of an integral, a framework for research in mathematics education, APOS theory and the TWM theory are discussed. Ultimately, a proposed genetic decomposition for integration will be suggested. The fourth chapter outlines the research methods and research methodology adopted for this study. The chapter starts by contrasting some research paradigms, subsequently locating this study within an interpretivist paradigm. It then motivates for the qualitative case study as an appropriate strategy of inquiry for this study. In addition, the chapter includes a detailed discussion on the research methods used to collect data, the instrument and the analysis of data.

As stated, the next three chapters focus on the presentation and analysis of data for the study. The fifth chapter reports on the first phase of the study which was aimed at validating the research instrument for the study. Findings on what meaning students attached to an integral, through symbolic representation, graphical representations and techniques of integration are presented and analysed. This chapter also concludes with some recommendations on the research instrument.

The second phase is reported on in chapters six and seven. Firstly, the sixth chapter reports on data obtained through the research instrument and follow-up interviews. It contains discussions on the evoked concept definition and evoked concept images of an integral. Secondly, the seventh chapter presents data and findings from classroom collaborations which were structured into focus groups. This chapter presents types of embodied or symbolic conceptualisation which students displayed when interacting about the set of given tasks in integration.

The eighth chapter presents a summary of the findings and conclusions derived from the overall results of the study. It presents discussions which are structured according to the sub-questions of the study as indicated in Section 1.3 above. Findings and conclusions about the evoked concept definitions, evoked concept images and the worlds of mathematical thinking that students exhibited are discussed. A modified genetic decomposition for integration that combines APOS and the TWM theoretical frameworks is then suggested. This chapter concludes by stating limitations of the study and suggestions for further research in the teaching and learning of integral calculus.

CHAPTER TWO

LITERATURE REVIEW

2.1. Introduction

This study was aimed at exploring how engineering students at a university of technology developed the concept of integral calculus. In reviewing the literature I therefore, focused on research which had been conducted concerning mathematical meaning and how students construct such meaning when learning various topics in mathematics. To contextualise the study, I started by reviewing the literature on the teaching and learning of calculus, in general. I then summarised readings that report on construction of meaning in mathematics. After that I looked at various studies that have been conducted on trying to understand how students construct meaning of various topics in mathematics, both internationally and locally.

Finally, I discussed at studies that focused on the learning of integral calculus and the approaches these studies have taken. Emphasis is made on both the understanding of the construction of meaning and the learning of integral calculus because these form the basis for my study. I embedded my discussions within the historical development of integral calculus in order to support the expected constructs from students.

The section after this introduction discusses research on the teaching and learning of calculus. I then report on the construction of meaning in mathematics and the fourth section discusses research on students' construction of knowledge in some topics of mathematics, internationally and locally. In the fifth section I discuss investigations that have been conducted on students' learning of integral calculus. The sixth section provides a review of the procedural and conceptual learning concepts while the seventh section discusses concept image and concept definition.

2.2. Teaching and learning of calculus

Studies pertaining to the teaching and learning of specific key concepts in calculus might be classified into at least four categories (Habineza, 2015). Firstly, it is the studies that focus on the concept of a limit (Hardy, 2009; Scheja & Pettersson, 2010; Szydlik, 2000). Secondly, it is the studies on the concept of a derivative of a function (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Habre & Abboud, 2006; Orton, 1983a; Siyepu, 2013a). Thirdly, it is the category of studies on the concept of an integral (Haripersad et al., 2008; Orton, 1983b; Rasslan & Tall, 2002; Thompson & Silverman, 2008). Fourthly, we have a category dealing with a broad range of key aspects of calculus (Pettersson & Scheja, 2008; Tall, 1985, 1992, 1997; Uhlenbeck & Stroup, 2002; Zollman, 2014).

2.2.1 The concept of a limit

The limit concept remains difficult for most students and they struggle to understand this important mathematical aspect (Cottrill, et.al., 1996). The difficulties that students encounter when dealing with concepts such as differentiation, integration and continuity in calculus may be attributed to their difficulties with the understanding of limits (Ortron, 1983a; Tall, 1992).

Hardy (2009) reported on a study carried to investigate what instructors and students regarded as the knowledge to be learned about limits of functions in a college level course of calculus. This study was carried out through interviewing 28 recruits from a college Calculus II course. The context of calculus teaching in this instance was such that there was a disjoint between topics related to limits and limit concepts or its definition. Hardy (2009) observed that sections where intuitive ideas about limits were discussed were not linked to the teaching of the formal definition of a limit, in particular, the $\varepsilon - \delta$ definition. In brief, “the teaching of the formal definition and its uses is dissociated from the teaching of “finding” limits” (Hardy, 2009, p. 2). To postulate what instructors regarded as knowledge to be taught, (Hardy, 2009) used the analysis of past final examination papers and textbooks as a basis. For students, task-based interviews were conducted and their responses together with their expectations about tasks, was considered.

The result showed that, on the one hand, the instructors' models of knowledge to be learned seemed to be emphasising traditional learning rather than scientific, mathematical learning. The tendency was to adopt techniques because of their traditional applications instead of the analysis of the problem at hand. On the other hand, students displayed a strong reliance on their high school algebra when finding limits of rational expressions. To justify their choices of techniques to tackle given problems, students stated their beliefs and convictions that the chosen techniques were the ones applicable. The approach was mainly algorithmic, based on recalling a set of steps as provided by the textbook or the instructor. Hardy (2009) referred to this algorithmic approach as 'normal behaviour' instead of a 'mathematical behaviour' which requires mathematical reasoning for any approach chosen to solve a problem. The conclusion was that students exhibited normative behaviours which are built on routine tasks given by instructors.

The study by Scheja and Pettersson (2010) also expands on what it means to come to understand a particular mathematical concept. This study was undertaken with 20 undergraduate students of engineering at a Swedish university. Students were initially asked to explain the meanings of limit and integral concepts. They were then interviewed in order to explore, in greater detail, their understanding of the said concepts. Scheja and Pettersson (2010) upheld the notion that students' learning involved processes of approximation and feedback simultaneously. After trying out interpretations of learning materials, it is the responses received from instructors that will shape students' ideas about what it means to learn a particular subject or concept. This belief formation process is described as a process of contextualisation "through which students develop individual understandings of learning materials by putting it in a particular context or framework" (p. 225).

Scheja and Pettersson (2010) found that students' initial contextualisation of limit and integral concepts was mainly algorithmic. Students described the limit and integral concepts as tool or stepwise procedures used in solving some calculus problems. Such a procedural approach was perceived highly functional by students for the reason that they had been successful in their studies. Students interpreted their success as a measure of their understanding of limit and integral. It was through the probing by the interviewer that students were persuaded to explain their understanding of these concepts and how they connect to other mathematical

aspects. Scheja and Pettersson (2010) maintain that perceived demands of a situation may persuade a learner to begin to change the way of contextualising a concept. Such a change may then allow for the development of alternative conceptions which can influence the development of understanding.

The study by Szydlik (2000) also revealed that students “may view calculus as a collection of facts and procedures to be memorized and applied and claim that students neither understand nor value the theory underlying those facts and procedures” (p. 273). Szydlik (2000) designed a study aimed at investigating and eliciting both content beliefs and the nature of sources of convictions for students’ conceptions of real numbers, infinity and functions. Participants in that study were 577 second-semester calculus students who were enrolled in a traditional, standard calculus course at a university. These students were not calculus majors; therefore, their courses did not require them to do the $\varepsilon - \delta$ proofs in their courses. Nonetheless, they were exposed to the formal definition of limit and had used limit processes in the context of functions and sequences.

Three categories of responses emerged when students were asked to define a limit. Firstly, it was those students that viewed a limit as either being intuitively static. The provided definition in this case was: “The limit of a function is L if whenever x is close to the limiting value s , the function is close to L ” (p. 268). Secondly, students tended to view a limit as motion, stating that: “The limit of a function is L if the function is getting closer and closer to L as x approaches s (p. 268). Lastly, it was a category that consisted of incoherent or inappropriate responses where the limit was viewed as unreachable or perceived as a bound that cannot be crossed.

On the contrary, all students were fairly successful in applying techniques to solve the limit problems. Differences surfaced when students were asked to justify why their chosen methods worked. There were those who could justify their responses through a logical argument or deduction. In addition to this group, were students who were able to justify their answers using empirical evidence where the function at points close to the limiting value was evaluated. These students seemed to view calculus as both logical and consistent. As a result of their conviction, they had “access to formal definitions, power to solve limit problems, and concept images free

of major internal inconsistencies” (Szydlik, 2000, p. 273) the majority of students, though, based their reasoning on some form of external authority which was either the instructor or the textbook. Szydlik (2000) noted that students with such external source of conviction will have impediments when trying to make sense of mathematics. They will perpetually view mathematics as a collection of formulas to be committed to memory and applied without reflection.

In summary, there is general consensus among the mathematics community that the understanding of the limit is “a central concept and-or central skill that is critical to student comprehension of the first year calculus” (Sofronas et al., 2011, p. 139). The studies reviewed above indicate that the tendencies to adopt an algorithmic approach may deprive students of broader understanding of concepts. The expectations by instructors and the texts used in the teaching of the concepts contribute to how students ultimately perceive as what they are expected to know. Szydlik (2000) then suggest the adoption of pedagogical approaches that promote the discovery of ideas by students, in order for mathematics to be made a sense-making activity.

2.2.2 The concept of a derivative

In their investigation with expert mathematicians, Sofronas et al. (2011) established the understanding of a derivative as one of the essential fundamental concept and-or skill for the first-year calculus. In addition, their study indicated that understanding of a derivative involves the understanding of a derivative as a rate of change, graphical understanding of a derivative and mastery of derivative computations (Sofronas et al., 2011).

2.2.2.1 A derivative as a rate of change

In his study of students’ understanding of differentiation, Orton (1983a) urged that the foundations of ideas of rate of change should be laid throughout students’ schooling career. He maintained that important and fundamental concepts such as limits and rate of change should not be left until they are required to make sense of differentiation. Orton (1983a) conducted interviews with 110 students (55 males and 55 females) to test their understanding of rate of change, differentiation and its

applications. Included in the exploration was also the understanding of certain algebraic skills or processes.

Orton (1983a) identified the classification of the types of student errors as fundamental in analysing the results of the study. Subsequently, the types of errors emerging from students' thinking patterns were classified into structural, executive and arbitrary. Structural errors were those "which arose from some failure to appreciate the relationship involved in the problem or to grasp some principle essential to solution" (Orton, 1983b, p. 4). Arbitrary errors resulted when students "behaved arbitrarily" subsequently overlooking the constrained stipulated within what was given. Errors classified as executive were those where candidates displayed signs of understanding the involved principle but failed in the carrying out of manipulations.

Findings from the research by Orton (1983a) indicated that a large number of students struggled to apply the elementary rule, $\frac{\Delta y}{\Delta x}$, when dealing with the rate of change. In particular, there is a need for students to grasp the embedded differences between rate of change in straight lines and rate of change in curves. While an average rate of change in a curve can be calculated in the same way as for a straight line, there is also a notion of rate of change at a point on the curve. While the rate of change in a straight line is constant everywhere, every point on a curve may yield a different value for the rate of change. Orton (1983a) then recommended that real-life situations could be used to generate data for both linear and non-linear graphs to assist students in building an understanding of rate of change. He also emphasized the importance of paying attention to special points such as points of increase or points of decrease, stationary points, turning points and points of inflection when examining the rate of change at a point on a curve.

2.2.2.2 Graphical understanding of a derivative

Asiala et al. (1997) conducted a study to investigate graphical understanding of a function and its derivative possessed by calculus students. They interviewed 41 engineering, science and mathematics students who had been taught, at least, two semesters of single variable calculus at a large midwestern university. For the

general understanding of a function, their line of inquiry included aspects such as the understanding of the $y = f(x)$ notation, the ability to deal with an only graphically-represented graph, general understanding of functional notation and the ability to draw a graph of a function from specific information given about values of the function and its derivative. With regard to the understanding of a derivative of a function, they explored whether students appeared to understand that the value of $f'(x)$ is the slope of the tangent to the graph of the function at the point $(x, f(x))$. They also investigated students' ability to deal with a derivative of the function using only the graphical information and without making use of a defining expression. Included were also questions to assess students' ability to work with derivatives approaching infinity as well as to use the derivative to determine intervals of monotonicity for the function.

The observation by Asiala et al. (1997) was that the relationship between a derivative of a function at a point and the slope of the line tangent to the graph of the function at that point was key in graphical understanding of a derivative. This fact is fundamental for understanding the derivative as a function. It enables an understanding that for each point in the domain of the derivative, there is a corresponding value of the slope. According to Asiala et al. (1997), many students struggled to work with graphical representations of functions. When given a graph of a function, they could not determine derivatives at specified points but instead tried to formulate algebraic expressions for functions in order to differentiate them. Asiala et al. (1997) concluded that the noted difficulty was a result of a lack of a prerequisite process conception of a function by students.

Another study to examine students' conceptual understanding of a function and its derivative was conducted by Habre and Abboud (2006). They conducted an experiment with 89 students enrolled for the Calculus I course at the Lebanese American University in Beirut, Lebanon. Students were taken through a course that required them to reflect on their own thinking when responding to questions. The approach to the concept of a derivative did not follow the traditional trajectory of teaching this concept. Traditionally, the instructor starts with the analytical definition of a derivative of a function $f(x)$ at a point a which is $f'(x) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(x)}{h}$. Next, they proceed to do examples and then discuss the geometric meaning of a

derivative, the slope of a line tangent to $f(x)$ at $x = a$. On the main, in the traditional setting students are then expected to memorise formula and rules for differentiation. Thus typical assessment questions in such a setting range from finding the derivatives of various functions to working out an equation of tangent lines to the graph of a function at a given point (Habre & Abboud, 2006).

In their experiment, Habre and Abboud (2006) adopted an approach of first discussing the rate of change of a function at a given point as the limit of an average rate of change. They then proceeded to relate the result to the slope of a tangent line, which finally led to the analytical definition of the derivative. They integrated this sequencing of aspects with new methods of teaching and assessment where technologies such as graphic calculators and dynamic calculus computer software were used. The focus of experiment classes was mainly on the geometric aspects of derivative concepts.

Concluding their study, Habre and Abboud (2006) found that students generally had a poor response to the non-traditional approach that emphasised the graphical representation of the derivative. However, for better students, the approach proved to be valuable in supporting a strong understanding of the derivative as a rate of change. Habre and Abboud (2006) ascribed the observed lack of visual thinking to the traditional instructional background that most students were still offered in their schools. As a consequence, despite an instructional treatment that focused mainly on the geometric components of the calculus, students still adhered to the algebraic and analytical ways of thinking.

2.2.2.3 Computations in derivatives

Studies discussed above indicate that there is general agreement that the understanding of a derivative as a rate of change and its graphical definition may improve students' conceptualisation of the derivative generally (Asiala et al., 1997; Habre & Abboud, 2006; Orton, 1983a). Nonetheless, the ability to compute derivatives of elementary functions is still regarded as of the essence in the

understanding of a derivative (Orton, 1983a; Sofronas et al., 2011). As a result, Tall (1992) emphasises the use of three representations in calculus, namely, graphic, numeric and symbolic. He maintains that “graphics give *qualitative global insight* where numerics give *quantitative results* and symbolics give powerful *manipulative ability* (Tall, 1992, p. 9). Flexible movement between the three representations is important than either focusing on all three, which may be less natural, or focusing on the most useful, which restricts the conceptualisation (Tall, 1992).

In his study, Orton (1983a) observed that students struggled with the understanding of symbols of differentiation. Symbols such as δx and δy were not well-understood by students. Similarly, Tall (1992) asserted that the Leibniz notation, $\frac{dy}{dx}$, which proves to be almost indispensable in calculus, continued to cause misconceptions in calculus. Students could not ascertain whether it is a fraction or a single indivisible symbol. There was also confusion regarding the relation between the dx in $\frac{dy}{dx}$ and the dx in $\int f(x)dx$. Tall (1992) further refers to the confusion that usually arise as to whether the du in the equation $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ can be cancelled or not. Challenges in the understanding and use of symbols may have a negative impact on the manipulative facility of student.

Siyepu (2013b), for example, noted that students were failing “to link mathematical symbols and formulae with appropriate procedures to be applied” (p.191). Siyepu (2013b) reported on a study carried to investigate errors displayed in the derivatives of trigonometric functions. A qualitative case study approach was used to collect data from 30 students enrolled for mechanical engineering in a university of technology, South Africa. The data collected revealed that poor conceptualisation led to students’ poor understanding of differentiation. Siyepu (2013b) advocated that classroom interactions be structured such that there is focus on making sense of mathematical symbols, mathematical rules and formulae. Such an approach might support students in developing meaningful understanding of mathematics.

In brief, students’ understanding of a derivative remains “fundamental to deep comprehension of the first-year calculus” (Sofronas et al., 2011, p. 135). The concept of a derivative integrates the three representations, namely, the graphic, the

numeric and the symbolic. While focusing on symbols and manipulation may result in a lack of deep understanding of the concept, graphic representation which enhances understanding but may be impeded by under preparedness of students and insufficient time (Habre & Abboud, 2006).

2.2.3 The concept of an integral

Understanding the concept of an integral entails understanding an integral as net change or accumulated total change, the integral as an area and the competence with integration techniques (Sofronas et al., 2011). Such an approach is in contrast with the format in which this aspect is normally handled in calculus lectures, which is definition-theorem-proof-application (Habineza, 2015). The sequencing adopted in schools mainly promotes instrumental understanding instead of conceptual one. These dichotomous approaches to learning are sometimes referred to as surface learning and deep learning (Cano & Berbén, 2009; Haripersad, 2010).

2.2.3.1 An integral as accumulated total change

According to Thompson and Silverman (2008), the concept of accumulation is pivotal to the idea of integration and as such, it is core to the understanding and applications of an integral in calculus. Thompson and Silverman (2008) agree that “the mathematical idea of an accumulation function, represented as $F(x) = \int_a^x f(t)dt$, involves so many moving parts that it is understandable that students have difficulty understanding and employing it” (p. 1). They further maintain that students’ difficulties with the notion of accumulation functions are exacerbated by the way this aspect is taught. The teaching of a definite integral is not sufficient for students to understand the broad aspect of an accumulation function. In addition, the two aspects fundamental to understanding accumulation function, namely, limits and the use of notation, remain poorly understood by students.

Haripersad et al. (2008) conducted an experiment with 33 students to assess the impact of blended learning, in particular, the Web based learning (WBL) on students’ conceptual errors in calculus. WBL course allowed the researchers to use text or multimedia such as graphics, audio and videos to present the course content. It was possible, therefore, for students to visualise an area under the graph and

employ a grid to calculate its approximation. The Riemann method of slicing an area of an irregular region bounded by the graph of a function f and two vertical lines, $x = a$ and $x = b$, could also be visualised.

Results showed that students taught through the WBL learning committed fewer structural errors compared to those students who had been through traditional calculus lectures. According to Haripersad et al. (2008), structural errors indicate gaps in knowledge as a result of “students’ rote/mechanistic learning of elementary calculus – lack of understanding of concepts since pre-knowledge frames were not developed” (p. 315).

2.2.3.2 An integral as an area

Sealey (2006), for example, reported on how students used the area under the graph of a function as a tool for computing definite integrals. A teaching experiment methodology was implemented to students that were enrolled for a traditionally calculus course but concurrently registered for a calculus workshop. The calculus workshop was an additional instruction for those students who either regarded themselves as weak in mathematics and needed extra help with calculus or those who loved mathematics and wanted to enhance their knowledge of the subject.

Two groups of students were given a problem about the pressure exerted by water on the walls of a dam and a problem requiring the use of Hooke’s Law to calculate energy when a spring exerts a force, F , to move an object some distance, x , each. In calculating these physical quantities, students were encouraged to use the approximation framework instead of the definite integrals. The observation was that the group working on the water problem displayed a good understanding of the concepts involved. They broke the dam into horizontal slices, calculated the area of each and the corresponding approximate pressure. The picture of the dam seemed helpful in providing a conceivable context, thus enabling students to determine the pressure on each strip.

Students working on the spring problem did not consider the context of their problem but proceeded to draw a force versus displacement graph. They then attempted to set up an integral but were unsuccessful. They seemed not to know whether the

function to integrate was given by the formula for the force, $F = kx$, or the formula for energy, $E = Fd$. Failing to set up an integral, they then attempted to use the area under the graph of the force and displacement. Students maintained that the area under the curve was equal to energy but could not provide reasons for that assertion. When asked to justify their approach, they could only refer to the confirmation they had received from one of the research assistants.

Sealey (2006) hypothesised that the students' difficulties emanated from not understanding the structure of the Riemann sum. Students knew that for a definite integral there is summation, that is, $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^{i=n} f(x_i)\Delta x$ but misses the product component, ($f(x_i)\Delta x$), thereof. The study by Sealey (2006) emphasises the importance of the underlying structure for understanding an integral as an area under a curve. As a result, Sealey (2006) concluded that the area under a curve is necessary but not sufficient for students to understand the definite integral.

The lack of underlying structures necessary for the understanding of integrals was also observed by Orton (1983b). Concepts of limits, practical exercises of finding areas of irregular shapes and pictorial approaches to results of $\sum r$, $\sum r^2$ and $\sum r^3$ are neglected at schools. Such concepts constitute the "pre-knowledge frames" required to conceptually understand integrals.

2.2.3.3. Techniques of integration

Symbolic manipulation in integration remains of great interest because of the Fundamental Theorem of Calculus (FTC). The FTC reveals that symbolic manipulation in integration can be performed by anti-differentiation (Tall, 1993). Tall (1993) then suggest that appropriate conception of an integral should be applied for an appropriate purpose. For example, for conceptual insight, pictures and graphs should be used, while on the other hand, numerical calculations or symbolic manipulations will be applicable for productive calculations.

Tall (1993) argues that symbolic manipulation enables mathematicians to compress their thinking. Symbols are used flexibly, since they can represent a process or they can also be viewed as a single mental object. Tall (1993) maintains that students

who are unsuccessful in mathematics are those who limit their representations only to the approach that is procedurally driven. In such cases, students do not link the said procedures with the underlying concepts as single entities represented by manipulative symbols. Tall (199) motivates that the use of computers to carry out symbolic manipulations may be used to complement students' skills. Nonetheless, students still require some insight into how mathematical symbols are used. In this way, students' cognition will not be strained and their chances of developing more flexible thinking processes will be increased.

While the integration of computers is embraced, students in this study were still taught in a pen-and-paper mode due to the economical circumstances of the university where the study was conducted. Consequently, aptness with techniques of integration was still central in the calculus instruction. Techniques included in this study included integration of polynomials, trigonometric, exponential and logarithmic functions. Also included are techniques of using partial fractions in integration as well as integration by parts.

In summary, conceptual development of an integral requires a link with appropriate illustrations. The interdependence of these aspects of calculus is vital for students' learning. Sofronas et al. (2011) mentioned "connections and relationships between and among concepts as an overarching end goal" for students' learning of calculus (p. 144).

2.3 Construction of meaning in mathematics

According to Cooley *et al.* (2007), the foundation of mathematical learning is based upon the development and integration of mentally structured mathematical concepts. For successful construction of meaning, these structures or schema must both be stable and accessible when needed for reasoning within a mathematical context (Cooley et al., 2007). Dubinsky (1991b) asserts that a person's knowledge of a particular mathematical concept is his or her "tendency to invoke a schema in order

to understand, deal with, organize, or make sense out of a perceived problem situation” (p.102).

An almost similar conception of understanding is purported by Duffin and Simpson (2000) who cited “building”, “having” and “enacting” as the three components for understanding. “Building” is the construction of the connections between mental structures in order to respond to arising problems. “Having” is defined as the state of connections or their depth and breadth at any particular time. The last component, “enacting”, is the ability to use the connections at any moment in order to provide a solution to a problem or to answer a question. Thus, knowing involves two aspects, namely, acquisition or learning of a concept and the ability to access and use it when needed. Duffin and Simpson (2000) purport that “it is only through interpreting the physical manifestations of a learner's use of their understanding that the teacher can make any kind of judgement about the learner's existing understanding” (p. 419).

In addition, Dubinsky (1991b) asserts that mathematical knowledge is difficult to describe separately from the way it is constructed. He provides an insight into reflective abstraction as a framework for describing any mathematical concept and how such concept may be acquired by a student. Reflective abstraction is defined as “the construction of mental objects and of mental actions on these objects” (Dubinsky, 1991b, p. 102). It is the construction of logico-mathematical structures by an individual during the course of cognitive development. An elaborate conceptualisation of reflective abstraction will be included when discussing the theoretical framework for this study.

According to Tall (1990), there are three areas that might impact the teaching of mathematics and subsequently result in inconsistencies in how students learn concepts. Firstly, it is the area of the mind. Lecturers and students have experiences and beliefs that are not always in accord and might result in differences of understanding of mathematical concepts. Secondly, it is the mathematics itself consists of mathematical concepts that may be interpreted in different ways due to their complexity. Thirdly, it is the message or the packaging and delivery of the mathematical content (language and sequencing of aspects) may result in different understandings invoked in the minds of students (Tall, 1990).

The first two areas listed by Tall (1990) are the drivers for the development of various theories on how students construct meaning in mathematics. Some examples of these theories are the concept image and concept definition (Tall, 1991; Tall & Vinner, 1981; Vinner, 1983, 1991), the three worlds of mathematical learning (Gray & Tall, 2001; Tall, 2002) and the APOS theory by (Asiala, Brown, *et al.*, 1997). The third area is the driver of research on the impact of language in mathematics teaching and curriculum design (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Setati & Adler, 2000; Setati, Molefe, & Langa, 2008; Sfard, 2002).

For this study I only reviewed works which focused on how students construct meaning in mathematics. Although the impact of language in the teaching and learning of mathematics has a bearing on my investigation, it was not the focus of my study, therefore, did not form part of this literature review. In the next two sections I discuss the concepts “concept image” and “concept definition” and the three worlds of mathematical learning, and also indicate how they correlate to the theoretical framework to be used for this study, namely, the APOS theory.

2. 3.1 Concept image and concept definition

Tall and Vinner (1981) defined an individual’s concept image for a given concept as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). This concept image, according to Tall and Vinner, “is built up over years...” and changes “as the individual meets new stimuli and matures” (p. 152). As such, a concept image is embedded in a networking of different experiences and concepts with diverse relations between them (Rösken & Rolka, 2007). Tall and Vinner (1981) introduced the term “evoked concept image” to indicate that “portion of the concept image which is activated at a particular time” (p. 152). Thus, the evoked concept image is subject to the possessed concept image together with time and manner in which an individual is prompted to demonstrate the concept image (Habineza, 2010; Vinner, 1991).

Concept definition, on the other hand, is regarded to be the words used to specify a particular concept (Tall & Vinner, 1981). Tall and Vinner (1981) acknowledged that a concept definition can be personal or formal, the latter being a definition that is

purported by the mathematical community. Moreover, they mentioned that a concept definition by an individual is a part of the individual's concept image. This assertion was further confirmed by Vinner (1983) when he indicated that individuals produce definitions that are a description of their concept images.

The concepts of concept image and concept definition have been explored in relation to the integral concept (Habineza, 2010; Rasslan & Tall, 2002). Rasslan and Tall (2002), for example, conducted a study to investigate the cognitive schemes for the definite integral that are evoked by the high school students in the United Kingdom (UK). In the UK the concept of definite integral is taught in the last two years of schooling. Textbook used in schools at that time of investigation introduces integration through activities requiring students to estimate the area between a graph and the x-axis, using pictures and numerical methods. The definition of an integral is then provided as follows:

The symbol $\int_a^b f(x)dx$ denotes the precise value of the area under the graph of f between $x = a$ and $x = b$. It is known as the integral of y with respect to x over the interval from a to b . The integral can be found approximately by various numerical methods (Rasslan & Tall, 2002, p. 1).

Rasslan and Tall (2002) found that although students knew how to integrate, the majority of them were not “able (or willing) to explain the definition of a definite integral” (p. 7).

The notions of concept image and concept definition can be linked to the APOS theory (Asiala, Cottrill, et al., 1997). The total cognitive structure associated with a concept, for example, is developed through various experiences with the concept. These experiences may include usage of the concept in appropriate contexts and pictorial or symbolic representations which then lead to mental pictures developed (Tall & Vinner, 1981). The notion of various experiences with a concept correlates to transitions through the Action-Process and Process-Object stages of the APOS theory. Sfard (1991) referred to this stage of the route as “interiorisation”, where students are being familiarised with a process or concept and have developed mental representations thereof. The total cognitive structure of a concept will therefore correlate to the Schema of a concept. I maintain that it is the evoked

concept image that will indicate whether a candidate has an Action, Process, Object or Schema understating of a concept under study.

A related framework mentioned is what I call “Tall’s Three Worlds” (TTW) theory, wherein Tall (2004b) defines the three world of mathematical learning. The next section will focus on this theory.

2.3.2 TTW theory

Tall (2004b) differentiates between the three stages or worlds through which mathematical learning develops. These stages are the conceptual-embodied or embodied, the proceptual-symbolic or symbolic and the axiomatic-formal or formal world.

The embodied world refers to that stage of learning where operations are based on human perceptions and actions in a real-world context, but it also includes imagining the properties in the mind (Tall, 2004b). In this level of cognitive development the learner’s conceptions are dependent on the properties of objects and reflections on those properties (Tall, 2007). At this level a learner will still be expected to provide solutions through imagining a situation occurring and thinking through the consequences. Hence, this level includes enactive and iconic examples with an increasing inclusion of visual and spatial imagery (Tall, 2002). The knowledge of a physical drawing of a straight line, for example, will provide ability to conceptualise a complex fact that a line has length but no breadth (Tall, 2002).

The second world, the proceptual-symbolic or symbolic world, grows out of the embodied world and it involves the role of symbols and symbol-processing in different aspects of mathematics (Tall, 2004b). In this world actions “are encapsulated as concepts by using a symbol that allows us to switch effortlessly from processes to do mathematics to concepts to think about” (Tall, 2004b, p. 5). It is the world “where actions, processes and their corresponding objects are realised and symbolised” (Stewart & Thomas, 2007, p. 202). This level develops through several distinct stages. Examples are: arithmetic calculations which lead to algebraic manipulations then to limit concepts. Another example will be in operations, where learners start with normal addition and subtraction, then multiplication and division

and other related operations. This reaches its peak when differentiation and integration are included.

The axiomatic-formal or formal world is where thinking is predicated on definitions and proofs (Tall, 2007). It begins with formal set-theoretic definitions which are constructed through deductions made from the embodied experience. These definitions are then formulated to a complete systematic axiom theory. Formal proofs are subsequently used to construct meaning from set-theoretic definitions, and other properties deduced using formal proofs (Tall, 2002). In this case, the (non)existence of a derivative, for example, is established through proof. At this level mathematical conception is based on logical reasoning (Tall, 2008).

Tall (2008) outlines how the three world of mathematical learning relate to the APOS theory when studying the development of mathematical thinking. The following diagram depicts such interrelationship:

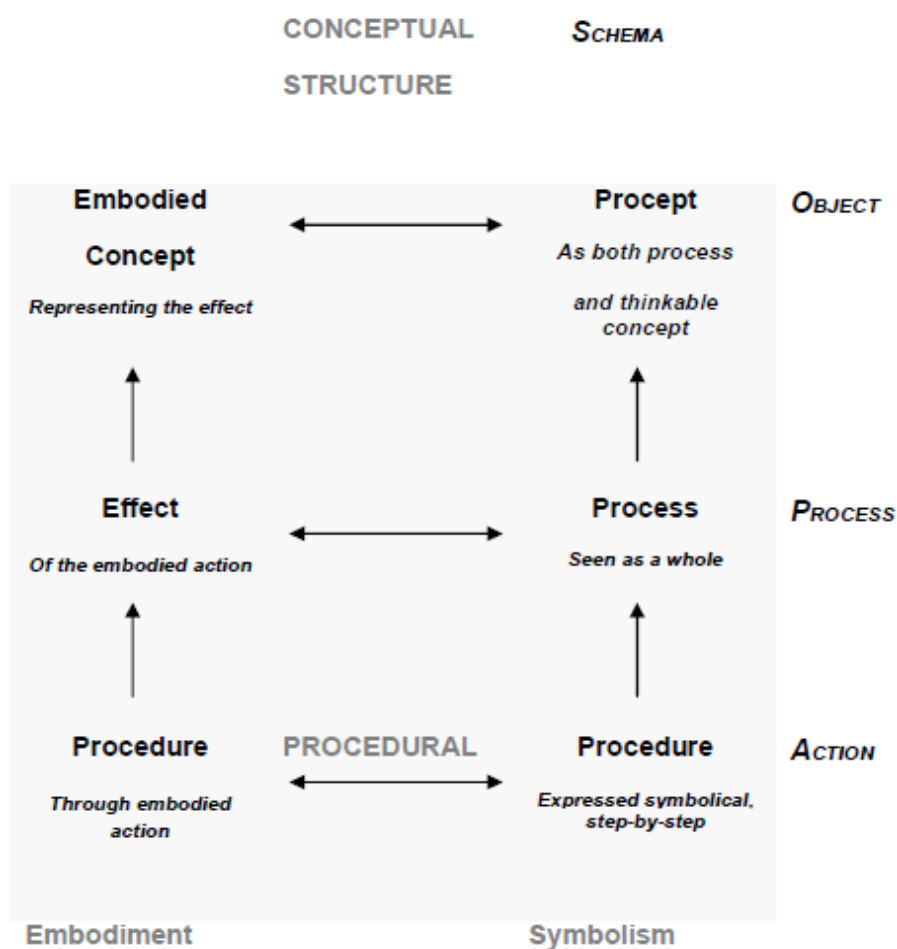


Figure 2.1. Procedural knowledge as part of conceptual knowledge (from Tall, 2008)

Tall (2008) purports that internalising an action into a process and encapsulating it into an object, with connections to other knowledge within a schema, is a form of compression. Compression is when the brain synthesises pieces of information “by connecting ideas together into thinkable concepts” (Tall, 2008, p. 10). He, moreover, argues that there is a correspondence between the symbolic and the embodiment compression. Both types of knowledge development start with procedures and for each subsequent stage in the symbolic compression, there is an embodied counterpart. While a procept, for example, refers to a symbolic process with thinkable concept, the embodied concept indicates the thinkable concept together with the effects of embodied action. The procept and the embodied concept can then be viewed as a process that has been encapsulated into an object, according to the APOS theory (Tall, 2008). Finally, a schema, as defined in the APOS theory, will indicate a fully developed conceptual structure which may be attainable through the embodied and symbolic worlds only (Figure 2.1). Tall’s TWM theory was considered as a secondary theoretical framework in this study and its link to APOS theory will be elaborated in the next chapter.

The APOS theory, as a main theoretical framework for this study, will be discussed thoroughly in Chapter 3. In the following section though, I give the definitions of the main aspects of this theory since the rest of my literature review reports on the use thereof.

2.3.3 APOS Theory

The APOS theory suggests that individuals use certain mechanisms to construct cognitive mental structures when learning mathematical concepts (DeVries & Arnon, 2004; Dubinsky & Lewin, 1986; Dubinsky *et al.*, 2005). These mechanisms are called interiorisation and encapsulation and the structures are actions, processes, objects and schemas (DeVries & Arnon, 2004; Dubinsky & Lewin, 1986; Dubinsky & McDonald, 2001). The structures are invoked accordingly in order to deal with perceived mathematical problem situations (Dubinsky & McDonald, 2001). The following are the definitions of these major stages of conception:

Action conception refers to that level of understanding where a person depends on detailed external cues in order to carry out transformation (Asiala, Brown, *et al.*,

1997; Dubinsky & McDonald, 2001; Maharaj, 2010). At this level a transformation can only be carried out one step at a time and without any mental image of the overall solution (DeVries & Arnon, 2004).

Process conception refers to a level of understanding where an individual would have repeated and reflected on an action, resulting in the internalisation thereof (Asiala, Brown, *et al.*, 1997; Maharaj, 2010). At this level of conception a person can perform transformations, predict outcomes and even reverse processes mentally, without external cues (Asiala, Brown, *et al.*, 1997; DeVries & Arnon, 2004).

Object conception is when a person views a process as a totality and is able to apply transformations on that totality (Brijlall & Bansilal, 2010; Meel, 2003). At this stage the person is said to have encapsulated a process into an object (Asiala, Brown, *et al.*, 1997; Brijlall, Maharaj, Bansilal, Mkhwanazi, & Dubinsky, 2011). When necessary, the person is able to de-encapsulate objects in order to access the underlying processes and actions (Parraguez & Oktaç, 2010).

A schema is a coherent framework of actions, processes and objects for a particular mathematical topic (Brijlall *et al.*, 2011; Meel, 2003). “Schemas themselves can be treated as objects and included in the organisation of “higher level” schemas” (Asiala, Cottrill, *et al.*, 1997, p. 8). This is called thematisation of schema (Asiala, Cottrill, *et al.*, 1997).

A concept that becomes relevant when using the APOS theory to analyse students’ understanding of a mathematical concept is a genetic decomposition (GD). According to Asiala, Brown, *et al.* (1997), a GD for a mathematical concept is a theoretical analysis that models the epistemology of the concept under review. This theoretical analysis outlines “the mental constructs that the student might make when learning a concept and accessing it when needed” (Jojo, 2011, p. 37). A researcher’s knowledge and experience informs the suggested action, process, object and schema conception of the concept.

Some of the studies where APOS theory has been used to analyse students’ understanding of mathematical concepts are Asiala, Cottrill, *et al.* (1997), DeVries and Arnon (2004), Parraguez and Oktaç (2010), Kabael (2011), Dubinsky and Wilson (2013) at an international level and Brijlall and Maharaj (2010), Maharaj

(2010), Jojo (2011), Brijlall and Ndlovu (2013) and Siyepu (2013a & b), within South Africa. In the next two sections I provide a review for some of these studies.

2.4 Application of the APOS in mathematics

2.4.1 Research on APOS theory in mathematics: Internationally

APOS Theory has diverse applications in mathematics education research. It has been used in many studies as a strictly developmental tool, a strictly analytical evaluative tool or as both (Arnon *et al.*, 2014). The study by Asiala, Cottrill, *et al.* (1997), for example, reported on the use of APOS framework to provide a deeper analysis of the epistemology of students' graphical conception of a function and its derivative concept.

Asiala, Cottrill, *et al.* (1997) conducted interviews on derivatives with 41 engineering, science and mathematics students who had completed at least two semesters of calculus. These students were taken through an instructional treatment that used the pedagogical strategy called the ACE Teaching Cycle. According to Asiala, Brown, *et al.* (1997), the ACE cycle is an instructional strategy consisting of Activities, Class Discussion and Exercises. The whole instruction design took into cognisance the GD of the derivative that the researchers had proposed.

The observation from this study was that some students relied on formulae to evaluate a function. Even when given a point $(5 ; 4)$ on the graph, these students expressed a need for an $f(x)$ in which to “plug-in” the x – value and calculate the corresponding y – value when asked the value of $f(5)$. Reliance on a formula was also displayed when responding to a question that required students to relate the slope of a tangent to the derivative. Although the given tangent line had two points on it, some students found firstly, the equation in the form $y = mx + c$, then differentiated it in order to determine its gradient. Asiala, Cottrill, *et al.* (1997) maintain, although with a lesser degree of certainty, that such students were “not able to use any process conception to solve the problem” (p. 12). The suggestion was to include the graphical representation of $(x ; y)$ when y is given by $f(x)$ in the genetic decomposition for this concept. Students should also be able to move

between several interpretations of $f'(a)$, bringing together ideas, for example, “of limit of different quotient, average velocity, marginal cost” (Asiala, Cottrill, *et al.*, 1997, p. 22).

The conclusion by Asiala, Cottrill, *et al.* (1997) also provided recommendations with regard to the pedagogical strategies used. The use of the ACE teaching cycle with carefully designed computer activities was reasonably effective in assisting students to “develop a relatively strong process conception of function and a graphical understanding of derivative” (p. 24). Students who had been taken through this treatment displayed strong process conception in the understanding of the $f(x)$ notation and in interpreting the relationship between the derivative, its graph and the graph of the function. The above report is an example of using APOS theory both as an analytical evaluative and a developmental tool.

Reporting on their study of students’ conceptualisation of a solution of system of equations, DeVries and Arnon (2004) also exhibited this dual usage of the APOS theory. They interviewed 12 students at a Teachers’ College shortly after finishing a one-semester linear algebra course. The focus of the interviews was to explore students’ conceptions or ideas about what a solution to a system of equations means. Analysis of students’ responses would also serve the purpose of developing a GD for this concept.

Although DeVries and Arnon (2004) concede that their instrument was not adequate in probing for deeper insight into their research questions, certain observations regarding students’ conceptualisation could still be made. For example, some students relied on memorised rules (without understanding) rather than reason on answering questions about a solution. When one student was asked whether $u + v$ a solution is if u and v are solutions, he could not justify his affirmative answer beyond the rule. Even when questioned further, he could only repeat the rule. A number of students also responded to the question, “What does a solution look like?” by directly solving the system of equations. According to DeVries and Arnon (2004) such students’ conception of solution developed out of using algorithms like the Gaussian method to solve the equation or system of equations. As a result, they are

at an Action level of development, being able to perform action only one step at a time.

The findings of this study also resulted in a formulation of an initial GD for a solution to systems of linear equations. At an Action level, DeVries and Arnon (2004) suggested that students should be enabled to identify the two functions, their common domain and co-domain, and a solution as that element of the domain which produces true equality when substituted. The Process level of development should involve students being assisted in identifying functions, domains and co-domains without actually substituting values into equations. Working on finite field like \mathbb{Z}_p^n , substituting and checking all elements for equality will constitute the Object level of development. Algorithms of solving systems of equations could only be applied when progressing into infinite fields, where substitution is inapplicable (DeVries & Arnon, 2004).

Parraguez and Oktaç (2010) applied APOS theory in a study they conducted with 10 undergraduate mathematics students in an American university. Their focus was on the possible concept construction of the vector space concept. Parraguez and Oktaç (2010) suggested that a set, a function and a binary operation schemas were fundamental to the learning of vector spaces. Hence, a student at an Action level would be able to apply binary operations schema to specific elements of a given set. The Process level would entail application of binary operations schema to specific elements, together with the development of an axiom schema. The Object level results from both the encapsulation, as purported by Dubinsky (1991a), and assimilation with the axiom schema. Still at the Object stage, Parraguez and Oktaç (2010) proposed the need to develop the concepts of a field, addition of vectors and multiplication of a vector by a scalar. These concepts, they suggested, should be augmented by coordination through distributive laws.

From the findings of Parraguez and Oktaç (2010), pedagogical suggestions on the teaching of the vector space concept emanated. The first suggestion was that, for students to develop the desired schema for vector spaces, flexibility in thinking about algebraic structures should be promoted during instruction. Secondly, a need to emphasise the relationship between the two vector space operations was highlighted (Parraguez & Oktaç, 2010). This possible improvement in instruction is attainable if

the APOS theory is integrated in teaching and pedagogical approaches. Besides, the analysis of students' responses would probably feed into the original genetic decomposition of the concept, which might result in further understanding of the construction of knowledge by students.

Kabael (2011) reported on the use of APOS theory to analyse how students, in Analysis 11 course in the mathematics education programme at a university in Turkey, generalised the function notion from single variable to two-variable function concepts. Interviews were conducted with six students whose conceptual levels were perceived as Process for both single and two-variable functions. These students were identified after being taught and tested on various representations of functions. Such included the use of a function machine, different representations (algebraic, geometric, set of triplets, table) and the drawing of special surfaces. A student at the Process level of conception was expected to be able to convert between the various representations of a function, namely, graphical representation, algebraic and table representations (Dubinsky, 1991b).

The findings indicated that students who had a schema conception of single-variable functions demonstrated good understanding of the notion of a two-variable function. On the other hand, those students whose understanding of a function concept was either at an action or process level displayed weak process conceptual level of the two-variable function. The conclusion reached by Kabael (2011) was that "there is a direct relationship between students' construction of the concept of a two-variable function and their conceptual levels of a general function concept" (Kabael, 2011, p. 494). In addition to the function concept, students require a schema of three-dimensional space in order to construct the concept of a two-variable function. A GD of a two-variable function concept could also be derived from the analysis of students' responses hence a recommendation to consider this GD when structuring the instruction.

Several other studies have heightened the importance of the APOS theoretical framework and how a corresponding GD informs teaching and improves learning (Clark, Cordero, Cottrill, Czarnocha, DeVries, St John, *et al.*, 1997; Martin, Loch, Cooley, Dexter, & Vidakovic, 2010). In their paper, Dubinsky, Dautermann, Leron, and Zazkis (1994) caution that the emerging GD of students' learning should not be

viewed as prescriptive. The GD should be considered as a guide to the cognitive development of a concept at that time.

2.4.2. Research on APOS theory in mathematics: South Africa

Research on students' concept development, using APOS theory, is also emerging within the South African context. In their study, Brijlall and Maharaj (2010) used a two-tiered approach of collaborative learning and structured worksheets, followed by interviews, to collect data from 12 second-year students studying for a qualification to teach mathematics in high school. The aim of their study was to investigate students' understanding of the concept of continuity. They structured their worksheets around inductive learning activities that promoted visualisation and verbalisation. In addition, these activities were aligned to the developmental stages contained in the APOS theory.

The findings of their study were that some students were able to use "symbols, verbal and written language, visual models and mental images to construct internal processes as a way of making sense of the concept of continuity of single-valued functions" (Brijlall & Maharaj, 2010, p. 47). This study is an example of integrating the learning theory into teaching and learning, thus using it as both an analytical and developmental tool. Their conclusion was that, based on the specific teaching methodology used, students were able to construct the concept of continuity successfully. Brijlall and Maharaj (2010) indicated a scope for additional research and analysis of the mental constructs of students, bearing in mind the teaching methodology used.

The question of a teaching approach was also investigated by Maharaj (2010), who focused on the concept of a limit of a function. He reported findings from a study where the APOS theory was used to investigate understanding of limits of functions by 891 undergraduate science students at a university in KwaZulu-Natal in South Africa. In this study the ACE teaching cycle was used, followed by a multiple choice question test and responses were analysed through the APOS theory framework. Students' responses indicated that less than three per cent of the students were not even at an action level of conceptualisation of limits of split-functions represented in symbolic form. Twenty-one per cent of students showed a potential of being at a

process level and 54, 2% a potential of a schema level. The conclusion was that students find it difficult to understand the limit of a function concept, “possibly a result of many students not having appropriate mental structures at the process, object, and schema levels” Maharaj (2010, p. 50).

A similar approach is found in a pilot study by Brijlall et al. (2011) to investigate pre-service students’ understanding of the relationship between $0, \bar{9}$ and 1. In this study the ACE and collaborative instructional approaches were used to collect data through questionnaires that were structured around the APOS theory. Interviews were also conducted in order for students to elucidate their responses. Findings were that, after the implementation of the worksheets, over 50% of students gave correct answers. Although a need to further validate responses is indicated, the researchers are of the view that APOS-designed worksheets might have impacted positively on students’ understanding of the equality between $0, \bar{9}$ and 1.

As purported by Arnon *et al.* (2014), Siyepu (2013a) used APOS theory as a tool to analyse students’ errors in their learning of derivatives of algebraic, exponential, logarithmic and trigonometric functions. Siyepu (2013a) designed his study according to the investigations cycle whose steps are:

- 1) Theoretical analysis of the content to be taught and learned;
- 2) Design and implementation of instruction; and
- 3) Collection and analysis of data.

He employed a case study method to investigate 20 students who were enrolled for chemical engineering in the extended curriculum programme at a university of technology in Western Cape, South Africa. The group consisted of students who were classified as ‘at-risk’. At-risk students are students who exhibit signs of not being successful in their schooling career, in spite of them having the necessary potential. According to Siyepu (2013a), such students usually achieve low in their academic work and are characterised by low confidence. Some of the factors that contribute to students being at-risk academically relate to family background as well as school experience (Choy, Horn, Nuñez, & Chen, 2000). Family background includes aspects such as low socio-economic status, single-parent families and first-

generation students. Factors relating to school experience are the changing of schools two or more times besides the normal progression of moving from one level to another, average performance of grades C or below from the sixth to the eighth grade and repeating one or more grades between the first and eighth grade (Choy et al., 2000). Mathematics curriculum and in particular, the taking of algebra in the eighth grade followed by the taking of advanced mathematics in high school greatly reduces future academic risks for students (Choy et al., 2000). The extended curriculum programme is, therefore, established to support such struggling students in their university studies.

Participants in the study by Siyepu (2013a) were all English second-language speakers. Three of them were from outside South Africa and those from within had obtained a Standard Grade level pass in their school mathematics. Standard Grade pass was designed for students with low abilities and such level allowed them access to diploma and certificate studies. Siyepu (2013a) employed activities, class discussions and exercises (ACE) teaching style to collect data.

Responses indicated that students exhibited the following types of errors: 1) Conceptual errors where students could neither grasp the concept nor identify the relationships involved in a problem; 2) Interpretation errors where students over generalise mathematical rules resulting in them failing to interpret a given problem correctly; 3) Linear extrapolation errors which are the generalisation of the distributive property, for example, $\sin(x + y) = \sin x + \sin y$; 4) Procedural errors where students err in computing or applying the algorithms even though they would have identified the concept correctly and 5) Arbitrary errors where students either transcribe sums incorrectly, do not present a complete solution or leave out certain questions unanswered.

Siyepu's conclusion was that most of the students were at action level or straddling between the action and process levels of APOS theory. He recommended that an ACE teaching cycle should be implemented in order to assist students to develop the required schema. Students should be encouraged to "self-reflect by trying to identify their errors on their own during class discussions" (Siyepu, 2013a, p. 590). He also suggested that the differentiation rules should be derived in order for students to develop full conceptions thereof.

The use of the APOS theory in integral calculus is reflected in a study by Brijlall and Bansilal (2010) which reports on development of understanding of the Riemann Sum. They worked with teacher trainees for high school mathematics at one university in South Africa. Having proposed a genetic decomposition of a Riemann Sum, Brijlall and Bansilal (2010) could only observe “partial understanding in the early stages of developing the concept” from their analysis (p. 137). The students could use the upper and lower sums to estimate the area of a region under a graph but only at an action level. There was no evidence of conceptual thinking at higher levels of cognition.

More work on the use of APOS in calculus is reported in other studies (Jojo, 2011; Jojo, Maharaj, & Brijlall, 2012). Nonetheless, there is no record of the analysis of students’ understanding of engineering mathematics. In particular, there is no analysis of how the essential integration concept is conceptualised by engineering students.

2.4.3 Summary

The studies cited above indicate that the APOS theory is a useful tool to use in analysing students’ construction of mathematical concepts. Dubinsky and McDonald (2001) support this thought when they state that, by using this theory, “the researcher can compare the success or failure of students on a mathematical task with the specific mental constructions they may or may not have” (p. 4). The analysis is also useful since it informs how instruction is to be structured and this, in turn, may result in improved performance (Asiala, Brown, *et al.*, 1997).

In their work, Dubinsky and McDonald (2001) provide an annotated bibliography of research that uses this theory in one way or the other. The list includes works by Carlson (1998) on the development of the function concept, Carmona (1996) on the concept of tangent and its relationship with the concept of derivative and a number of studies in high school mathematics and studies in many other mathematical concepts (Dubinsky & McDonald, 2001). What is notable is that there has been limited application of the APOS theory in calculus, and even fewer record of analysis in integral calculus. According to Dubinsky and McDonald (2001), for a doctoral thesis, Tostado (1995) used APOS to analyse students’ conception of a derivative

and a tangent in a graphical context, while Cottrill (1999) applied the APOS theory to the conceptualisation of a chain rule. It is the study by Brijlall and Bansilal (2010) on the genetic decomposition of the Riemann Sum, that relates to integral calculus but it does not extend to techniques of integration.

Since integral calculus is one of the fundamental mathematical concepts in engineering, this study aims at using the APOS theory to develop a genetic decomposition of techniques used in integral calculus and to analyse how students construct knowledge when they learn integral calculus.

2.5 Research in students' learning of integral calculus

Studies that have investigated the teaching and learning of integral calculus include works by Orton (1983b), Rösken and Rolka (2007), Pettersson and Scheja (2008), Mahir (2009), Huang (2010) and Habineza (2010). The following is the review of these studies.

Orton (1983b) reports on an investigation of 110 students' understanding of elementary calculus. Students worked through 38 items, 18 of which related to integration. Orton used Donaldson (in Orton, 1983b) to classify errors displayed by students into structural, arbitrary or executive types. As mentioned before, structural errors referred to failure by participants to establish relationships within the concept or to grasp the critical principles involved. Arbitrary errors were defined as errors resulting from sheer oversight of constraints given while executive errors were errors resulting from failure to carry out manipulations (Orton, 1983b).

The analysis of students' responses showed that students had serious difficulties with understanding integration as a limit of a sum. Students also struggled to find a relationship between a definite integral and areas under the curve. Orton (1983b) asserted that teachers of mathematics have realised these difficulties faced by students and have reacted in varying ways. These ways include a curriculum that avoids calculus to non-specialists, introducing integration strictly as an anti-differentiation (a rule) and building of a limit concept and related algebraic concepts over a period of years. Orton (1983b) emphasised that "rules without reason cannot be justified" (p. 11).

Orton (1983b) concluded his report by making recommendations towards a curriculum that promotes conceptual development of integration. He advocated for the inclusion of aspects such as limits and infinity, sets of polygons with an increasing number of edges and solids with an increasing number of faces and infinite series of fractions. Studying areas of irregular shapes, by counting the squares, could assist in discussing a limit from above and below, and finding the area of a circle by reassembling sectors into approximation of a rectangle also supports the notion of a limit. Orton (1983b) indicated the possibility of using a calculator in performing numerical integration. The need to derive results that students are using was also cited.

The study by Mahir (2009), that was conducted to investigate the conceptual and procedural performances of students on integration, found that students did not have a satisfactory conceptual understanding of integration. In this instance the research group consisted of students who had successfully completed calculus through instruction in one university. Students' conceptual understanding of integral-area relation, integral as an algebraic sum and the fundamental theorem of calculus was investigated. Mahir (2009) discouraged assessments of students that promote memorisation and advocated for the use diverse contexts when teaching the concept of integration.

On the other hand, the study by Huang (2010) study differs from the approach by Mahir (2009) in the sense that Huang conducted a quasi-experiment study. A group of students was split and procedure-based instruction was offered to one group and concept-based instruction to the other. The findings indicated that the conceptual group performed well in both classifications of knowledge while the procedure group displayed unsatisfactory conceptual understanding of the concept of integration (Huang, 2010).

According to Huang (2010), procedural knowledge includes

two main components; the first is the mathematical symbol representation system, which is the comprehension of mathematical symbols and awareness of symbol syntaxes. The second type consists of the algorithms or rules for solving mathematical tasks....In application, true mathematical understanding

has to be constructed on the connection of these two types of knowledge (p. 1).

The other study by Habineza (2010), which he conducted at the Kigali Institute of Education in Rwanda, used a teaching approach that was based on the theories of didactical situations in mathematics and zone of proximal development. Eleven student teachers were taught through the said teaching approach in order to develop the students' understanding of the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus.

The findings by Habineza (2010) were that student teachers' understanding of the definite and the indefinite integrals, through the teaching approach adopted, changed significantly from pseudo-objects to concept images that included "all the underlying concept layers" (p. iii) of the definite and indefinite integrals. However, there was little improvement in the students' understanding of the fundamental theorem of calculus.

In summary, the studies by Mahir (2009) and Huang (2010) looked at the level of understanding that students exhibited when learning integral calculus and how that level affected their performance. Habineza (2010) looked at a teaching model that will enhance students' understanding of some aspects of integral calculus. There is still no record of any study that worked towards the analysis of the actual understanding of the concept of integral calculus, techniques of integration in particular. This is regarded as a gap in the literature which this study attempts to fill.

2.6 Conclusion

The APOS theory is emerging as a critical tool to analyse students' learning in mathematics. It must be borne in mind that the analysis is not regarded as being conclusive but it suggests a possible trajectory that the development of a concept might follow (Dubinsky, 1991a).

This analysis becomes even more valuable since it embeds itself in the pedagogy of the concept under investigation (Parraguez & Oktac, 2009; De Vries & Arnon, 2004; Brijlall & Maharaj, 2009, 2010; Brijlall & Bansilal, 2010). It can be argued, therefore, that one of the results of exploring students' conceptualisation of a mathematical topic is ultimately the improved performance in class.

The fact that the existing literature on the learning of integral calculus says very little about the use of the APOS analysis in integral calculus indicates a gap in the literature and, therefore, the importance of this study. The next chapter deals with the theoretical framework informing this study.

CHAPTER THREE

THEORETICAL FRAMEWORKS

3.1 Introduction

In the last chapter I presented a review of the literature I deemed relevant to this study. In this chapter I will discuss theoretical frameworks that I used throughout the study. These frameworks grounded the interaction with students during the study, steered the generation of data and guided the analysis thereof.

In the section following this introduction, I discuss the concept of an integral within the context of this study. I then present the framework for research in mathematics education that I used to generate data. Within that presentation I elaborate on APOS theory as a tool to both generate and analyse data. Furthermore, I expand on reflective abstraction as constructions of mental objects. I then discuss Tall's three worlds of mathematical learning which pertain to conceptual construction in the learning of mathematics. In concluding the chapter I will indicate how these frameworks assisted me in answering the three main research questions mentioned for this study.

This chapter, therefore, is made up of five sections. Section two which comes after this introductory section will focus on the concept of an integral as it is defined in a mathematical context. I will indicate the various approaches to the presentation of integral calculus and compare them to the context of my study. In section three I will discuss a specific framework for research in mathematics education, define its components and indicate how it informed the work done in this study. Within that section I will also provide the description of an integral schema, together with the proposed genetic composition thereof. In addition, I will expound on the construction of meaning in mathematical learning through the framework of the APOS. Section four will indicate how Tall's three worlds of mathematical learning can be linked to the APOS theory and the proposed genetic decomposition. Lastly, in section five I will summarise the frameworks and indicate how they have been used to answer the research questions for this study.

3.2 Integrals

In South Africa, the concept of an integral is not included in the school mathematics curriculum (Department of Education, 2003). Students only encounter this concept for the first time in post school studies. For the institution where this study was conducted, the mathematics work programme is structured such that students are first taught the concept of a limit in brief (Msomi, 2011). The limit concept is then followed by a detailed teaching of differential calculus and applications thereof. It is after differential calculus that the concept of integration is introduced to students. Notions of an integral as an area and integral as summation are then introduced next. Throughout the teaching of integration, emphasis is placed on the techniques of integration, followed by applications in an engineering context. This background, and the following text on integrals, informed the genetic decomposition that I initially proposed and the analysis of evoked students' conception of the concept of integral.

3.2.1 Indefinite integrals

Engineering students at a university of technology in South Africa do not take mathematics as a major, hence aspects of proofs and in-depth analyses of concepts are not normally included in their mathematics curriculum. The focus of instruction is mainly on procedures and techniques of using integral calculus in solving problems (Msomi, 2011). Definitions of integration espoused during the teaching of this group of students are mainly operational definitions. The following are examples of such definitions.

Stroud and Booth (2007, p. 335), a textbook that was used for the course, defines integration as follows:

Integration is the inverse of differentiation. When we differentiate we start with an expression and proceed to find its derivative. When we integrate we start with the derivative and then find the expression from which it has been derived. (p. 335).

Stroud and Booth(2001) then continue to motivate for the inclusion of a constant of integration by showing that since $\frac{d}{dx}(x^4) = \frac{d}{dx}(x^4 + 2) = \frac{d}{dx}(x^4 - 5) = 4x^3$ then $\int 4x^3 dx$ is either x^4 or $x^4 + 2$ or $x^4 - 5$. They then argue that, since the

constant added to x^4 cannot be deduced when given $4x^3$ to integrate, it will be acknowledged by adding a “C” to the result of the integration, that is, $\int 4x^3 dx = x^4 + C$. They then refer to this integral as an indefinite integral “since normally we do not know the value of C” (p. 335).

Other authors like Smith and Minton (2002), also, define indefinite integral as follows:

Let F be any antiderivative of f . The indefinite integral of $f(x)$ (with respect to x), is defined by $\int f(x)dx = F(x) + c$ where c is any arbitrary constant (the constant of integration). (p. 324).

These two definitions restrict the indefinite integral to an antiderivative of a function. According to Orton (1983b) this approach leads to students who cannot justify the rules they are using and he advises that “if we wish to introduce calculus to non-specialists we need to think very hard about laying a satisfactory groundwork” (p.10).

A broader conception of an indefinite integral is provided by Koepf and Ben-Israel (1994) who indicated two definitions for an indefinite integral of a function f in an interval $[a, b]$. The first definition they provided was that of an indefinite integral as an antiderivative or a primitive of a function. As such, an indefinite integral of f is a function F satisfying the equation $F'(x) = f(x)$ at all points x in the interval $[a, b]$. This function F is defined up to a constant called the constant of integration.

The second definition, on the other hand, considers an indefinite integral as a definite integral over a variable interval $F(x) = \int_a^x f(t)dt$ and the lower endpoint a will determine the constant of integration (Habineza, 2010). A similar approach to the indefinite integral is stated in Stroud and Booth (2007) where they state that, “The total area under the curve and the x-axis up to a point P is given by the indefinite integral” (p. 348).

Habineza (2010) further purports that considering an indefinite integral as a definite integral over a variable offers a better way of understanding the function version of the Fundamental Theorem of Calculus (FTC). The function version of the FTC

states that “if $A(x) = \int_a^x f(t)dt$ represents the area under the curve of $f(x)$ then the derivative of the area function gives the function that delimitates that area:

$$\frac{dA(x)}{dx} = \frac{d}{dx} \int_a^x f(t)dt = f(x)”, (Habineza, 2010, p. 66).$$

Although all the above definitions were discussed during the teaching of the students, the first definition by Koepf and Ben-Israel (1994) and the definition by Stroud and Booth (2001) were used interchangeably.

3.2.2 Definite integral

The common approach to the introduction of a definite integral is that of computing the area under the graph of a function by dividing the area into strips (J. Stewart, 2009; Stroud & Booth, 2007). Stroud and Booth (2007) denote the width of these strips as ∂x and, invoking the definition of an indefinite integral as a total area under the curve, they deduce that “for an interval $[a, b]$, $a < b$, the required area is given by $A = \int_{x=b} ydx - \int_{x=a} ydx$ which is written as $A = \int_a^b ydx$ ” (p. 348). J. Stewart (2009), on the other hand, defines a definite integral as follows:

If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 (= a), x_1, x_2, \dots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i^{th} sub-interval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x. (p. 300).$$

Stroud and Booth (2001) provide a similar approach to that of J. Stewart (2009), where they present integration as a summation. They do not provide a mathematical definition but argue that if an interval $[a, b]$ is partitioned into subintervals of equal length, δx , the total area under the function $y = f(x)$ is then written as “ $A \approx \sum_{x=a}^{x=b} y \cdot \delta x$ where the symbol \sum represents ‘the sum of all terms of the form...’ ” (Stroud & Booth, 2007, p. 353). Making the strips narrow will then result in

$\lim_{\partial x \rightarrow 0} \sum_{x=a}^{x=b} y \cdot \partial x = A = \int_a^b y dx$. Stroud and Booth (2007) further omit the justification for the existence and the uniqueness of the limit. They immediately focus on procedures to determine the definite integral.

When teaching these students, I included the underlying aspects of a definite integral, like the properties of an integral and also the Fundamental Theorem of Calculus (FTC). The properties of a definite integral, which are stated as theorems in other texts, were also discussed. The proofs for these properties were not discussed with these students. The following is an example of properties which are stated as theorems, as it appears in Smith and Minton (2002):

Theorem 1 (Smith & Minton, 2002, pp. 356-357)

If f and g are integrable functions on the interval $[a, b]$ and c is any constant, then the following properties are true:

1. $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$,
2. $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$,
3. $\int_a^b cf(x) dx = c \int_a^b f(x) dx$ and
4. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, for any c in the interval $[a, b]$.

The formulas below follow from the definition of the integral:

1. For any integrable function f , if $a < b$, we have

$$\int_a^b f(x) dx = - \int_b^a f(x) dx \text{ and}$$

2. If a is defined then we have $\int_a^a f(x) dx = 0$, (Smith & Minton, 2002, p. 357).

Regarding the FTC, I adopted what Habineza (2010) refers to as the “fundamental theorem of calculus – version of the integral of the derivative (FTC-VID or FTC-VEA)” (p. 52). Habineza (2010) adopts the formulation provided by Smith and Minton (2002) stating that “If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of f , then $\int_a^b f(x) dx = F(b) - F(a)$ ” (Smith & Minton, 2002, p. 364).

These aspects discussed above are included in order to indicate the trajectory that was used to develop students' understanding of the definite integral. Properties of a definite integral enable students to simplify what might be difficult problems in some cases. The FTC is the main basis for evaluating a definite integral for engineering students. It also becomes important for these students to link the concept of an integral to an area since that is how it is mainly applied within the context of their fields of study.

3.3 Framework for research in mathematics education

This study was carried out in accordance with a specific framework for research and curriculum development in undergraduate mathematics education as proposed and used in various studies (Asiala *et al.*, 1996; Brijlall *et al.*, 2011; Clark, Cordero, Cottrill, Czarnocha, DeVries, St John, *et al.*, 1997; Dubinsky & McDonald, 2001; Maharaj, 2010). The framework consists of three components, namely, theoretical analysis, design and implementation of instruction, and observation and assessment of student learning. Figure 3.1 illustrates each of these components and the relationship among them.

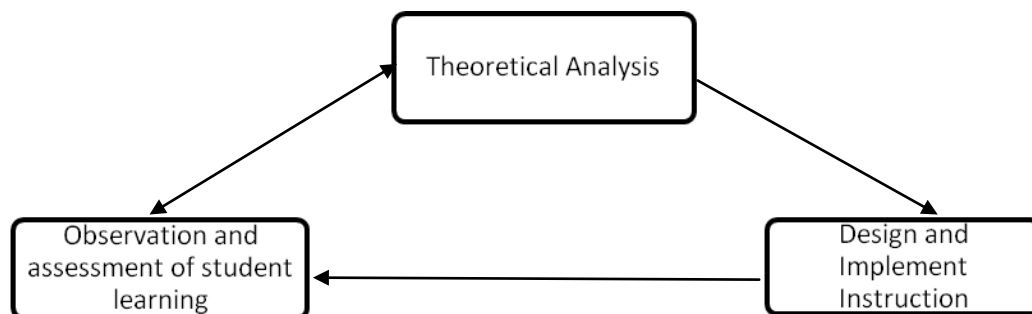


Figure 3.1: The framework for research and curriculum development (Asiala *et al.*, 1996).

According to Asiala *et al.* (1996), a researcher commences with a theoretical analysis, called a genetic decomposition, modelling the epistemology of the mathematics concept in question. In this study the question was what it meant to understand integration and how that understanding could be constructed by students. The analysis for the concept of integration was based primarily on a

particular theoretical perspective or learning theory, APOS theory, combined with the researcher's understanding of the concept in question through her experiences both as a student and a teacher of the concept (Asiala *et al.*, 1996; Dubinsky & McDonald, 2001). The theoretical perspective informed the design and implementation of the learning experiences during the instructional treatment. The theoretical perspective also guided the analysis of data collected (Bergsten, 2008; Maharaj, 2010).

The intention of instructional treatment was to “get students to make the proposed mental constructions and use them to construct an understanding of the concept as well as apply it in both mathematical and non-mathematical situations” (Dubinsky, 2001, p. 12). In the observation and assessment stage, researchers gather and analyse data generated during the instruction stage (Clark, Cordero, Cottrill, Czarnocha, DeVries, St John, *et al.*, 1997). This analysis of data tells something about the theoretical analysis in terms of mental constructions and also indicates any mathematics that the students might have learnt (Dubinsky, 2001). In the following three subsections I provide an in-depth elaboration on these components in relation to the study.

3.3.1 Theoretical analysis: APOS

This component of the framework is aimed at addressing the question on the nature of mental constructions constructed by students and the ways in which those constructions are made (Asiala *et al.*, 1996; Dubinsky & McDonald, 2001). As stated, answering this question requires a general theory influenced by the researcher's own understanding and previous experience with the particular mathematical concept.

The theoretical perspective adopted for this study, APOS theory, hypothesises that understanding a concept begins with constructing *actions*. Actions are external transformations dependant on explicit stimuli and guidance to perform operations. When actions are repeated and reflected upon, they are interiorised into *processes* where actions can be performed and even reversed, mentally. When individuals can view processes as a totality, applying transformations on them, processes are deemed to have been encapsulated into *objects*. Finally, actions, processes and objects, their interconnections and any other linking schema, constitute a *schema* for

that certain mathematical concept (Asiala, Brown, *et al.*, 1997; Dubinsky & McDonald, 2001; Maharaj, 2010). Following is the proposed hypothesised genetic decomposition (HGD) for integration.

At an Action level: The invoked concept image for integration is that of an antiderivative. Students at this level of conceptual understanding would have a simplistic notion of an integral as an area under the graph of a function. At this stage students know how to evaluate integrals only by following explicit algorithms that they have been taught. There is no vision of what succeeds any step they take and they do not have a conceptual understanding thereof.

At this stage integration is solely about identifying, from a catalogue of procedures, the one that will work in a given problem. When evaluating $\int(x^3 + 3x)dx$ for example, a student would invoke the rule on the integral of the sum, that is, $\int[f(x) + g(x)] dx = \int f(x)dx + \int g(x)dx$. The presented solution would be elaborate and display all steps taken, that is,

$$\begin{aligned} & \int(x^3 + 3x)dx \\ &= \int x^3 dx + \int 3x dx \dots \dots \dots (\text{step 1}) \\ &= \frac{1}{3+1} x^{3+1} + C_1 + \frac{3}{1+1} x^{1+1} + C_2 \dots (\text{step 2}) \\ &= \frac{1}{4} x^4 + \frac{3}{2} x^2 + C \dots \dots \dots (\text{step 3}) \end{aligned}$$

In this instance there is reliance on the algorithm for an integral. The student cannot process the integral of a sum as a single unit, neither can the integrals of x^3 and $3x$ be written without explicit definitions. The explicit definitions serve as external cues for the whole solution to be produced.

At a Process level: Through reflection and internal operation for an integral, integration is interiorised into process conception (Cooley *et al.*, 2007). Students still follow steps but display levels of understanding and adaptability in their approach to solutions. At this stage they are able to recognise errors in their presentations, although they may not succeed in explaining them. A student, for example, would be uncomfortable to give “zero” as an answer to the integral $\int_0^\pi \cos x dx$ but would

not be able to trace the cause for such a paradox. As a result of interiorised actions, they would be able to determine the integrals mentally and to reverse the process as well.

At an Object level: As stated, the object level of conceptualisation is a stage where a process is viewed as a totality and is encapsulated into mental objects. At this stage students would be able to find area for curves crossing an X-axis. A student, for example, would know that the integral $\int_0^{\pi} \cos x \, dx$ should be split into $\int_0^{\frac{\pi}{2}} \cos x \, dx$ and $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx$ for correct evaluation. As asserted by Mahir (2009), good understanding of differentiation rules is essential in solving integrals. Students would then be able identify cases that are the reversal of the chain rule, hence the “u-substitution” and cases requiring the application of formulas like $\int u \, dv = uv - \int v \, du$ (integration by parts).

Individuals would be said to possess a complete **schema** for integration when they display a coherent set of knowledge for the concept. Investigation by Sofronas *et al.* (2011) found that integral as an area, as an accumulated total change and “facility with integral techniques” (p. 139), were necessary components of understanding integration. Hiebert and Lefevre (1986), define the main concepts of integral calculus as the limit of Riemann sums, the integral as the area and the fundamental theorem of calculus. For purposes of this study, therefore, schema for integration would include conceptualisation of integral as an oriented area, the fundamental theorem of calculus and capability to use integral techniques.

Although the four stages, actions, processes, objects, and schema, are presented in a hierarchical, ordered list; it may be possible that individuals do not form constructions in such a linear manner (Dubinsky & McDonald, 2001). This statement agrees with Tall (1999) who purported that APOS theory would fall short in describing conceptual development in Geometry. Tall (1999) argued that geometry begins as object based, with processes like drawing, measuring and construction involved. According to Dubinsky and McDonald (2001), constructions of other various mathematical concepts become more dialectic than linear. This awareness was born in mind when interacting with data from the study.

The proposed initial HGD suggests that for students to succeed in integration, they must have developed the ability to explain, to recognise in other contexts and to derive consequences, for the function and the derivative schemas (Duffin & Simpson, 2000). Functions and derivatives are the building blocks in integral calculus as indicated by Jojo (2011) who states that “definitions of derivatives, integral functions, the relationships between average and instantaneous rates of change....and many other topics in calculus all require students to have a clear understanding of the concept of a function” (p. 45). This concept of building blocks in calculus is also endorsed by Haripersad, et.al. (2008). At an action level, therefore, I hypothesised that students should be able to respond to external stimuli such as graphs, pictures and formulae when dealing with integrals. As a result, such students’ conception of an integral may be limited to that of an integral as an anti-derivative.

A student at a process level of conception was expected to have interiorised basic actions of integration and thus able to perform and reverse actions mentally. As a result, students at this stage of conception were predicted to possess the ability to handle integrals of the form $\int f[g(x)].g'(x)dx$ with less difficulty. The concept image of an integral was also expected to have expanded to include perceptions of an integral as an area, without difficulty.

The object level of conceptualisation was deemed to include the ability to view an integral as an object. This level results from encapsulating processes and viewing them as objects. It was, therefore, expected that at this level, students should be able to manage problems requiring advanced techniques and comprehension for the level. Aspects such as integration by parts, where an integral is embedded within an integral, integration of the inverse trigonometric functions and using partial fractions with accuracy were expected from a student at this level of conceptualisation.

Students would be deemed to possess a schema for integration when they displayed a coherent set of knowledge for the integral concept. Figure 3.2 below, displays the initially hypothesised genetic decomposition as a diagram.

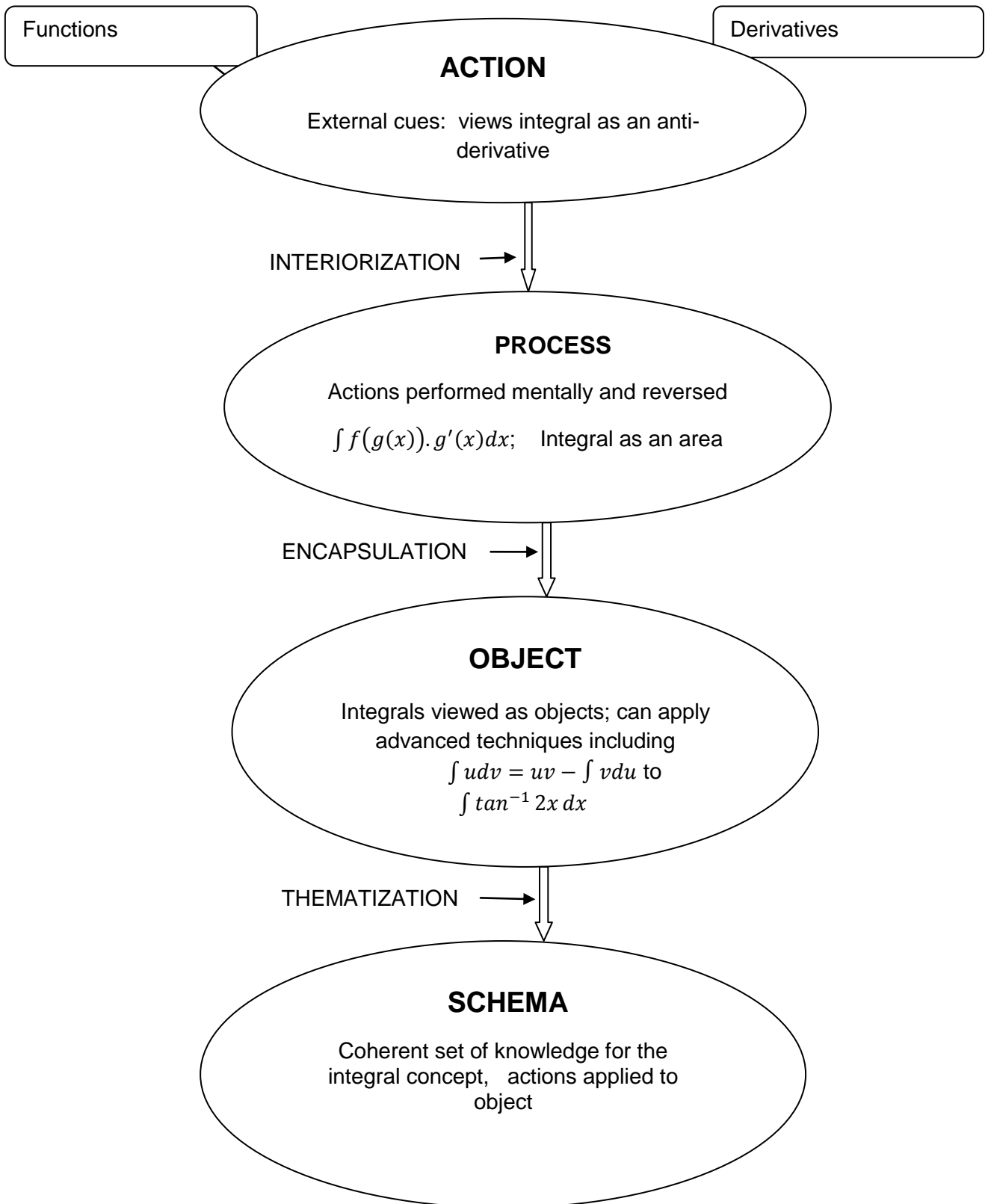


Figure 3.2. Initial hypothesised genetic decomposition for integration

3.3.2 Instructional treatment : ACE

As indicated in Figure 3.1, the second component for the adopted research framework involves the design and implementation of instruction, based on theoretical analyses. The theoretical analysis guides classroom interactions where learning and assessment materials are designed according to the proposed model.

For my study, students were taught in a group of 87, in regular lectures which occurred three times a week for the duration of three weeks. During the lessons, students were provided with tasks that were designed to induce mental constructions proposed in the initial genetic decomposition. Tasks were designed such that they provided students with experience in constructing actions corresponding to integration. Subsequently, this experience was augmented when students were asked to extend familiar actions to general processes. Students were then presented with higher order activities which required them to organise a variety of previously constructed schemas, like the derivatives of compositions of functions, the various rules for differentiation, derivatives of specific functions, into a schema applicable to integration problems. The focus of all interactions was not on the correctness of solutions but on the approach and procedure used to answer the question.

Interactions were also aimed at getting students to reflect on their work throughout the course. These interactions were designed according to a particular pedagogical approach called the ACE teaching style which many researchers in mathematics education have used (Asiala, Brown, *et al.*, 1997; Brijlall & Maharaj, 2010; Jojo, 2011). ACE is an acronym for activities, class discussion and exercises which are major components in this teaching style.

Students were given activities designed to help them make mental constructions according to the proposed genetic decomposition. The primary goal for the given activities was to provide students with experience in working with integrals rather than finding correct answers. The emphasis was therefore on collaborative learning, where students were explaining and justifying their approaches to other students within groups.

Class discussions again involved students working in teams to perform tasks that had been designed according to the proposed genetic decomposition of integration. Inter-group discussions were structured such that students could reflect on their work. As a lecturer I would randomly elaborate, probe, provide definitions and overviews of what students were discussing. Such interjections and guidance that can support students in understanding complex topics is referred to as scaffolding (Azevedo, Cromley, & Seibert, 2004; Brush & Saye, 2002). These discussions were video-taped and later analysed for emerging thought processes regarding concept development of integration.

Students also answered some exercises that were given as traditional homework. These exercises were completed outside classroom and without the lecturer's supervision. The main purpose of these tasks was to reinforce conceptions of integrals that students had developed, to expand cases of application of integration and to prepare for sections that would be studied later.

3.3.3 Collection and analysis of data

The third component of the adopted research framework is the collection and analysis of data. In this study, data were gathered using specially designed questions and student responses, in-depth interviews with students about their responses to the questions and focus group discussions, where written instruments were combined with interviews. Information about students, their pre-tertiary education and their performance in mathematics at school level was also considered. Such information could shed light on students' attitudes towards mathematics and the level of their preparedness as well as previous exposure to the concept of integration.

According to Asiala, Brown, *et al.* (1997), widening sources of information about student knowledge is likely to yield trustworthy conclusions about the phenomenon investigated. They purport that methods used in such a qualitative study do not provide clear-cut information leading to inexorable conclusions (Asiala, Brown, *et al.*, 1997). Different kinds of data were therefore useful in answering the two questions in this component of the framework: (1) how did mental constructions that students appeared to be making compare with those proposed in the theoretical analysis?

(2) how much of what mathematics were students appearing to be learning and using?

Analysis of data was aimed at establishing whether making, or failing to make the constructions proposed in the theoretical analysis of the integration concept, could reasonably explain why some students seemed to succeed in learning integration and others did not. Asiala, Brown, *et al.* (1997) concede that student learning is difficult to characterise in yes or no terms but that learning ranges in a spectrum from those who seem not to master a concept completely to those who exhibit mature understanding, consistent with the understanding of mathematicians. The goal of data analysis was therefore, to establish a similar spectrum in respect of the proposed theoretical analysis for integration. As such, what emanated from data could support or result in a revision of the theoretical analysis that had been proposed. Data analysis could also result in a revision of the general theoretical perspective, that is, a revision of the perspectives in APOS theory.

3.4 Transition to formal thinking in mathematics: TWM

Tall (2008) proposes an analysis of cognitive development that is complementary to APOS theory. He maintains that the cognitive development of an individual is premised on the fundamental mental structures, set-befores, that people are born with. These set-befores are: (1) recognition of patterns, similarities and differences; (2) repetition of sequence of actions until they become automatic and (3) language to describe and refine the way we think about things (Tall, 2008). He then describes these modes of thinking as the “Three Worlds of Mathematics”(TWM), which are the conceptual-embodied world, the perceptual-symbolic world and the axiomatic-formal world. This framework was integrated in the analysis of data as explained below.

3.4.1 The conceptual-embodied world

Conceptual-embodiment refers to the embodiment that is conceptualised through perceptions and reflections on the properties and representations of concepts (Tall, 2008, 2007). For integral calculus, cognitive development for the concept of integral as an area was considered. In this embodied world, the numerical value of an area under a continuous curve can be found by using small enough squares to cover it (Figure 3.3).

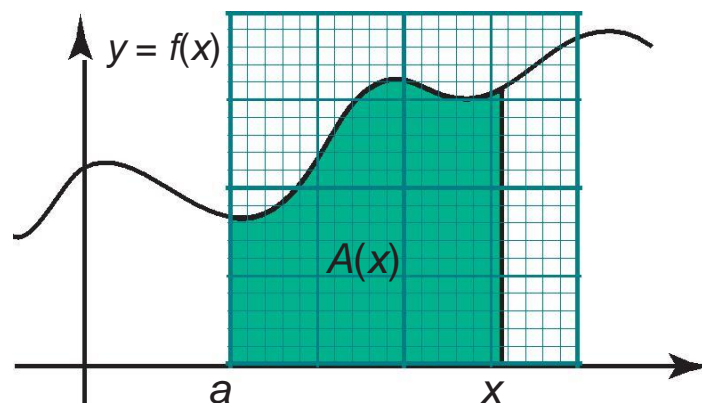


Figure 3.3 Measuring the area under the graph with a grid (Tall, 2002).

The area from a to x under the graph is a function $A(x) = \int_a^x f(t)dt$, an embodied notion of integral as an area which can lead to conceptualisation of more sophisticated approaches like the Riemann integration (Tall, 2002).

3.4.2 The proceptual-symbolic world

Proceptual symbolism or symbolism is when symbols are used as thinkable concepts (Tall, 2007, 2008). Brijlall and Maharaj (2013) point out that symbols may be viewed from analogue or symbolic perspectives. They state that “analogue codes represent the physical stimuli people observe in their environment” (p. 800). Alternatively, as symbolic codes, symbols may be some form of knowledge representation selected to characterise an aspect (Brijlall & Maharaj, 2013). In integral calculus, for example, an individual may perceive a symbol such as $\int f(x)dx$ as representing both a process to be carried out or the thinkable concept resulting from that process.

Such perceptions were noted when students were asked to state the difference in meaning between $\int f(x)dx$ and $\int_a^b f(x)dx$ during interviews. While some responded to Item 1 of the research questionnaire by stating the difference verbally, others had an urge to evaluate the integrals. Tall (2008) refers to such a “combination of symbol, process, and concept constructed from the process” as an elementary procept (p. 8). A procept is then defined as a collection of elementary procepts with the same output concept (Gray & Tall, 1994; Tall, 2007).

According to Tall (2004b) procepts begin with actions that are encapsulated as concepts and represented symbolically. This encapsulation occurs when the focus on symbols gets transferred from the physical meaning to a symbolic activity in mathematics (Tall, 2004a). Therefore, such symbols allow for students to switch seamlessly from procedures to do mathematics to concepts to think about.

3.4.3 The axiomatic-formal world

The third category of cognitive growth is the axiomatic-formal world or formal world. In formal mathematics, presentations start with formal definitions to concepts and proving theorems by mathematical proofs (Tall, 2004b 2007). Concept development does not start with practical objects of experience but with carefully formulated axioms which define mathematical structures in terms of specific properties. In this world, a statement is considered true either when it is assumed as an axiom or definition, or it can be deduced from existing axioms and definitions (Tall, 2004a). Since formal proofs in calculus were not included in the learning programme for students in this study, the analysis of data was not extended to this category.

3.4.4 Compression, connection and thinkable concepts

According to Tall (2007), the interiorisation of actions into processes and the encapsulation of processes into objects as described in APOS theory, is an example of compression of aspects into thinkable concepts. Such thinkable concepts are connected to other knowledge within a schema, that may also be encapsulated as an object. A procedure to find an integral, for example, which is a thinkable sequence of steps to do(action), progressively develops to give efficiency of choosing the most suitable procedure to employ for a given task. Subsequently, it gets condensed into a process and compressed into a procept to think about and to manipulate mentally (Tall, 2007).

Tall (2007) contends that the symbolic compression from procedure to process to object can be paralleled to embodied compression. He maintains that embodied compression shifts the focus from the steps in an action to the effect thereof and imagining the effect as an embodied process. Linking symbolism and embodiment can enable individuals to acquire conceptual embodiment as they mentally refer to the encapsulated process. Such conceptual knowledge will facilitate application of

thinkable concepts into real world. It will also assist individuals in establishing links within and between perceptual symbolism and conceptual embodiment (Figure 3.4).

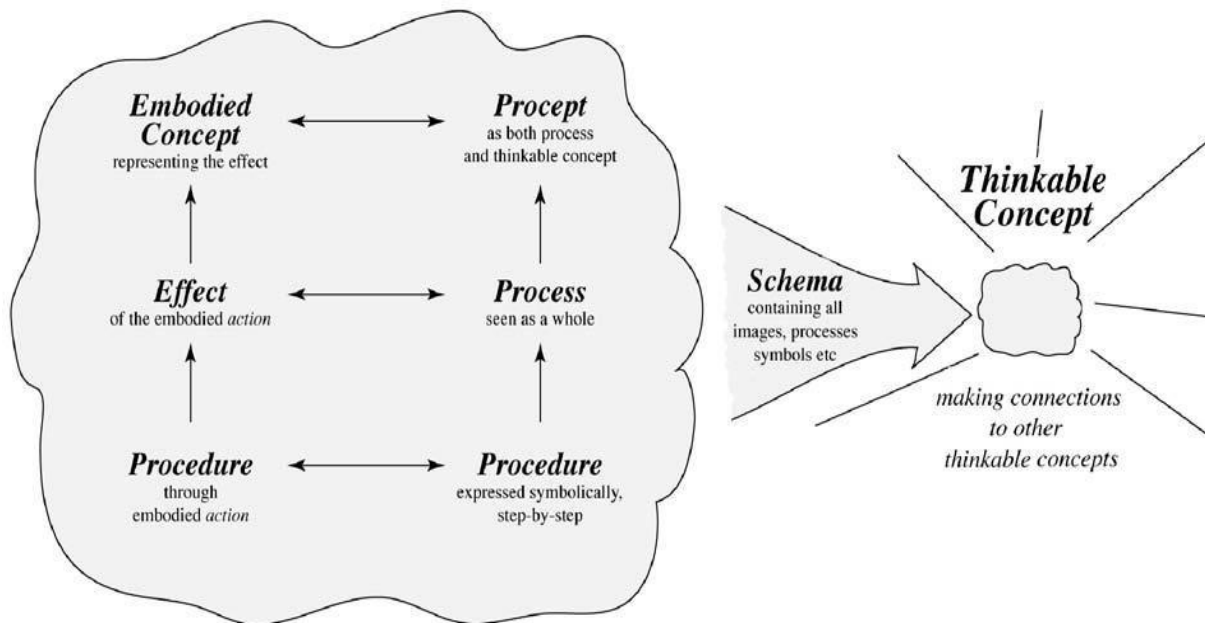


Figure 3.4 Compressing a schema into a thinkable concept (D. O. Tall, 2007).

Based on the approach by Tall (2008) on conceptual development and having initially proposed a genetic decomposition of integration using APOS theory, I then formulated a possible analysis for concept development that will integrate these theories. A broader perspective to conceptual development is necessary since individuals develop in different ways. According to Gray and Tall (1994), some stick to step-by-step procedures while others develop the ability to compress their knowledge into flexible use of symbols as procepts. Figure 3.5 represents the proposed integrative analysis for the concept of integration.

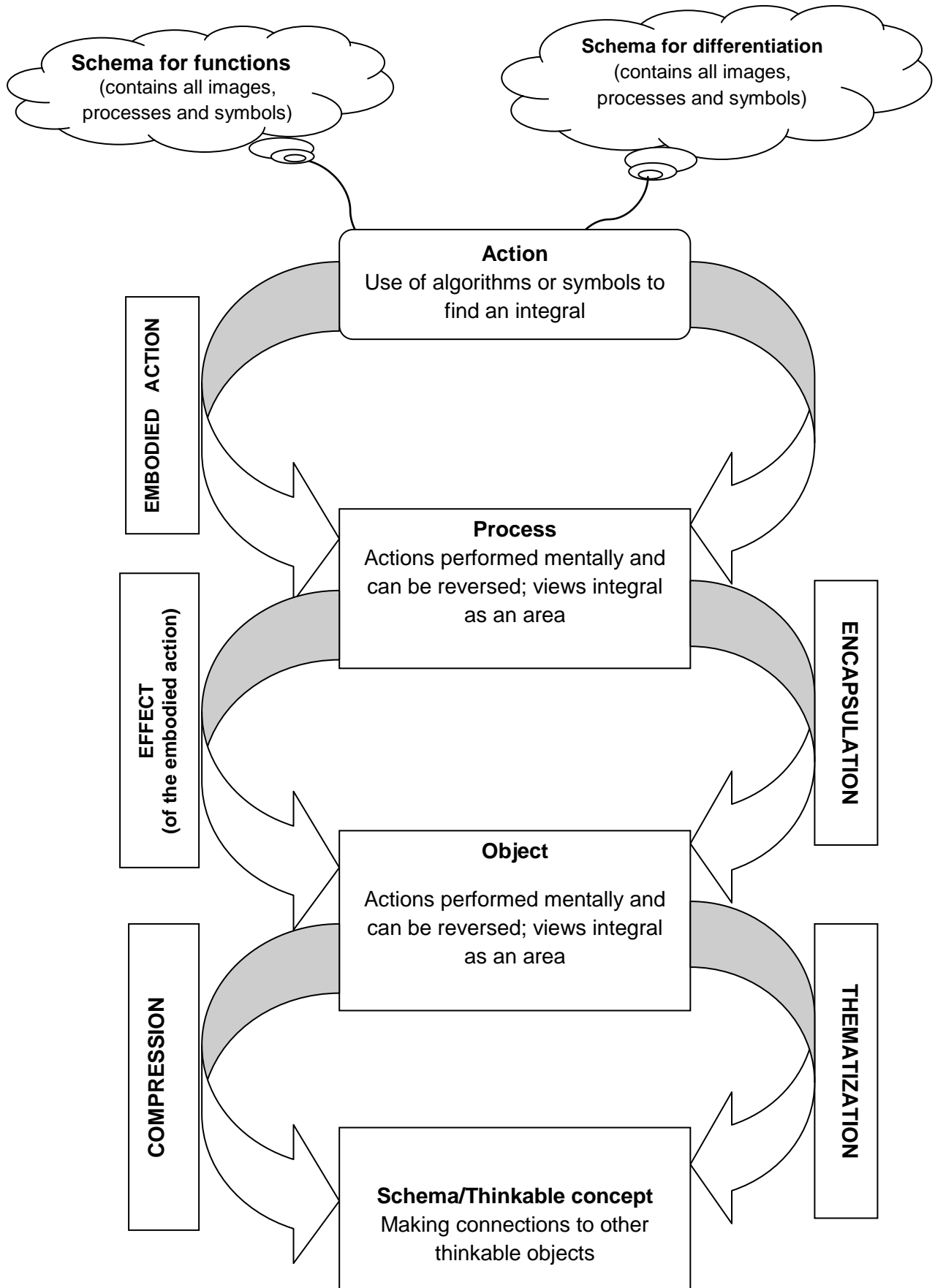


Figure 3.5 Integrated hypothesised genetic decomposition

The model in Figure 3.5 incorporates the developmental stages previously proposed in Figure 3.2. I, therefore, used this model to design all the activities and questionnaires, and to interpret and theorise conceptual development of integration in this study.

All activities in this study were aimed at determining whether students exhibited such constructs as suggested in the HGD.

3.5 Conclusion

In this chapter, I presented theoretical frameworks that I used to generate and analyse data during this study. Since the focus of investigation was on integral calculus, the first theoretical framework was about mathematical objects related to integrals and the fundamental theorem of calculus. Secondly, I discussed a theoretical framework for research in mathematics education as purported by Asiala, Brown, *et al.* (1997). I indicated how APOS theory was used as an analysis tool within this framework. This led me to refer to conceptual development according to APOS theory, as indicated by various authors and propose a genetic decomposition for integration (Asiala, Brown, *et al.*, 1997; Cooley *et al.*, 2007; Dubinsky & McDonald, 2001; Maharaj, 2010; Mahir, 2009). Further I expanded on the ACE teaching style as a style in which classroom interactions were structured (Asiala, Brown, *et al.*, 1997; Brijlall & Maharaj, 2010; Jojo, 2011). I then indicated how data was collected and analysed as maintained in Asiala, Brown, *et al.* (1997). APOS theory as a theoretical framework, therefore, was embedded the framework for reasearch as a third framework.

Lastly, I presented TTW as a model to analyse the construction of mathematical knowledge. I reflected on how TTW links with APOS theory and concluded by proposing an integrated genetic decomposition for integration. In the next chapter I present the methodology and the research methods that I used during this investigation.

CHAPTER FOUR

RESEARCH METHODOLOGY

4.1 *Introduction*

If you are going to pose yourself a problem and then come to a conclusion about it, you have to do *something* to come to that conclusion. That 'something' is your research method (Hofstee, 2006, p. 107).

This study was aimed at exploring how engineering students at a university of technology construct knowledge as they learn integral calculus. The desire was to see the meaning of integration from students' perspectives, within their world, and probably make discoveries that will contribute to the development of empirical knowledge about conceptual development of integration, for such a group of students.

In the previous chapter I presented theoretical frameworks or conceptual frameworks which guided my inquiry. According to Marshall and Rossman (2010), conceptual frameworks constitute the substantive focus of an inquiry with respect to the *what* question. Frameworks provide a detailed description of the issue that is explored. Critical for any research inquiry is the *how* question, that is, the methods for conducting the investigation. In this chapter, therefore, I will present the overall design of this study and specific research methods I used.

The next section will focus on research paradigms, including the design that I adopted for this study. I will expand on the strategy and the paradigm, indicating why I think it is an appropriate disposition. I will then discuss research methods utilised during the study. Here, I will give details of the participants in the study, methods for collecting data, the research instrument and how data were analysed. I will then highlight the delineations and limitations of this study. I will also mention ethical observations made. Lastly, I will provide a synopsis of the whole chapter on methodology, indicating how I see the proposed methodology suitable for this study.

4.2 Research design

Many authors distinguish between research methodology and research methods. Research methods are viewed as techniques or procedures used to collect and analyse data in a research project, while research methodology relates to a process of justifying the design of the research and the choice for particular methods to be adopted (Cohen *et al.*, 2011; King & Horrocks, 2010). Methodology, therefore, outlines the philosophical assumptions embedded in the approach of undertaking a particular research (King & Horrocks, 2010; Strasheim & Eiselen, 2011).

For purposes of this study, research design will encompass research methodology and research methods. Under research methodology, I will first articulate basic beliefs which guided the research process. Secondly, I will describe and justify the type of research that was conducted for this study. Research methods will provide details of data collection strategies.

4.2.1 Research Paradigms

According to Creswell (2013), a researcher's assumptions about knowledge claims, strategies of inquiry and methods of data collection, influence the choice of a research design. Such assumptions might be called paradigms and they address the following four questions that guide the approach to research: (1) what is the fundamental nature of reality (ontology)? ; (2) what is the nature of knowledge, how can it be acquired and communicated to other human beings? (epistemology)? ; (3) what values and value judgements go into the knowledge (axiology)? and (4), what process is followed in studying it or what are the most appropriate ways for investigating what can be known (methodology) (Cohen *et al.*, 2011; Guba & Lincoln, 1994; Neuman, 2006).

Neuman (2006) identifies three major paradigms or positions that are prevalent in response to the four questions mentioned above. These paradigms are the positivism, interpretivism or constructivism and critical theory. Guba and Lincoln (1994) include the fourth paradigm to this major group, namely, the post-positivism. Neuman (2006) states that, although the feminist and postmodern approaches are

also found in social research, most on-going studies are based on the first two which are the positivism paradigm and the interpretivism paradigm. Rubin and Rubin (2011) maintain that these two approaches reflect “major intellectual disagreements about the kind of information that researchers should be looking for and how they should go about obtaining it” (p. 19). For the purposes of this study I will restrict my discussion to these two paradigms and the post-positivism, since it indicates an intermediary phase between the two.

4.2.1.1 The positivism paradigm

There is a general agreement that positivism is an approach predominantly adopted in natural sciences (Guba & Lincoln, 1994; Hunt, 1991; Neuman, 2006; Noor, 2008; Shepard, Jensen, Schmoll, Hack, & Gwyer, 1993). In the positivism approach the model of natural sciences is emphasised whereby a researcher objectively collects data about a social phenomenon and then provides an explanation of that phenomenon, by arranging the data in cause and effect linkages (Noor, 2008). The expectation is that the researcher, the components of the phenomenon under investigation and the activity of investigating are independent and separate (Shepard *et al.*, 1993). An explanation of human behaviour is described through observations and scientific reasoning (Cohen *et al.*, 2011).

At the ontological level, positivism postulates naive realism where a single reality that is apprehendable, identifiable and measurable is assumed to exist. Knowledge of the “way things are” is not time or context-bound but can be generalised to cause-effect laws by immutable natural laws and mechanisms (Guba & Lincoln, 1994; Habineza, 2010). At the epistemological level, the positivism paradigm assumes the investigator and the investigated phenomenon to be independent entities (dualism) and the investigator being capable of objectively studying the phenomenon without influencing or being influenced by it. Replicable findings are considered true and provide evidence for theory non-falsification (Guba & Lincoln, 1994). At the axiological level, knowledge should be value free, “based on empirical evidence alone and without interference from moral-political values” (Neuman, 2006, p. 86).

Finally, at the methodological level, experimental and manipulative methods are designed to verify stated questions and/or hypotheses (Guba & Lincoln, 1994).

In summary, a positivist approach requires a researcher to begin with a cause-effect relationship within a social phenomenon. This relationship could be logically derived from a possible causal law in general theory. A researcher then measures aspects of the social phenomenon, examines evidence and replicates other researches, while remaining detached, neutral and objective throughout the process. The outcome could be the empirical test of and confirmation for the theoretical laws for that phenomenon.

4.2.1.2 The post-positivism paradigm

King and Horrocks (2010) refer to post-positivism paradigm as a “modified version of positivism” (p. 19). While maintaining some positivist elements such as being concerned with quantification and causal factors, post-positivists embrace approaches that contextualise theories and disciplines in larger social and historical contexts (Allmendinger, 2002; King & Horrocks, 2010; Ryan, 2006). Proponents of this paradigm emphasise the adoption of good principles which ensure that procedures, techniques and methods, while important, are always subject to ethical scrutiny (Ryan, 2006).

At the ontological level postpositivism, similar to positivism, postulates the existence of one true reality. The view by proponents for this paradigm is that such reality is imperfectly apprehendable or measurable because of basically flawed human intellectual mechanisms and the fundamentally intractable nature of phenomena (Guba & Lincoln, 1994). At the epistemological level, the perspective is that of modified dualism and objectivity. This means that dualism gets abandoned as considered not possible to maintain but objectivity remains a regulatory ideal. Special emphasis is placed on external guardians for objectivity which include critical traditions and critical community (such as editors, referees, and professional peers) (Guba & Lincoln, 1994; Habineza, 2010). At the axiological level, the values of a researcher are kept in check in order not to bias the study (Strasheim & Eiselen,

2011). Lastly, modified experimental and manipulative methods are designed and conducted to falsify, rather than to verify hypotheses (Guba & Lincoln, 1994). In this paradigm, the methodology redresses some of the concerns raised against the positivist paradigm by doing inquiry in more natural settings, collecting more situational information and reintroducing discovery as an element in inquiry. In the social sciences, emic instead of etic viewpoints, are solicited to assist in determining the meanings and purposes that people ascribe to their actions (Guba & Lincoln, 1994). There is a notable increase of the utilisation of qualitative techniques in this paradigm.

In summing up, research based on the post-positivism paradigm shares the same aim of explaining through prediction and control, as positivism. While acknowledging the researcher's connection to the phenomenon this time, there is still emphasis on objectivity to ensure validity and reliability during the research process. As all measurement is fallible, this paradigm emphasises the need to use **triangulation** across both quantitative and qualitative techniques in order to incorporate viewpoints of participants when investigating their actions. The post-positivism approach stresses the falsification of theory as opposed to theory verification in positivism.

4.2.1.3 The interpretivism paradigm

According to Strasheim and Eiselen (2011), in an interpretivist research there is interaction between a researcher and participants with the aim of understanding the phenomenon from the participants' viewpoint. Such research is generally idiographic where aspects of a social phenomenon are described by offering a detailed account of specific social settings, processes or relationships (King & Horrocks, 2010). It is also inductive where theory emerges from analysing the interpretations of the world by the participants.

The interpretivist paradigm upholds a view of multiple, equally valid and socially constructed realities. Realities are therefore relative, dependant for their form and content on the individual persons or groups constructing them. Realities are thus socially and experientially based, and local and specific in nature (Guba & Lincoln, 1994). The epistemology for this paradigm is that of transitional and subjectivist nature. The researcher and the participants are assumed to be interactively linked so

that the findings are literally created as the investigation proceeds. The interaction unearths deeper meaning and insight into the lived experience of participants (Guba & Lincoln, 1994). The axiological position is that the researcher's values and biases are inevitable and should be acknowledged and discussed at length (Strasheim & Eiselen, 2011). Finally, methodologies utilised are hermeneutical and dialectical in nature. The variable and personal nature of social constructions suggests that individual constructions can be elicited and refined only through interaction between and among, investigator and respondents. The varying constructions are interpreted using conventional hermeneutical techniques and are compared and contrasted through a dialectical interchange. The final objective is to distil a consensus construction that is "more informed and sophisticated than any of the predecessor constructions, including the etic construction of the investigator" (Guba & Lincoln, 1994, p. 111).

Briefly, interpretive research studies involve understanding a phenomenon subjectively, within cultural and contextual situations. Researchers do not impose their priori understanding of the phenomenon but derive categories and themes from the research field, through in-depth examination of and exposure to the phenomenon of interest. Researchers' prior assumptions, beliefs, values and interests always intervene to shape their investigations. According to (Guba & Lincoln, 1994), a researcher's intent should be revealed, since hiding it may be counterproductive towards the aim of uncovering and improving constructs.

In the next section, I present ontological, epistemological, axiological and methodological assumptions that were adopted for this study.

4.2.1.4. The paradigm of this study

As purported by (Asiala, Cottrill, *et al.*, 1997), it may not be possible to definitely explain the process of learning as students develop conceptual understanding of a mathematical concept. Findings of research can only represent the understanding and interpretations of the researcher combined with understanding of those being researched (Rubin & Rubin, 2011). Assertion by various researchers (Brijlall & Bansilal, 2010; Dubinsky *et al.*, 1994) that mathematical understanding is complex and that APOS theory is but one approach towards analysing the cognitive

development indicates that for this study there could be many truths. The findings that this study produced might not be the only possible explanation of knowledge construction process when students learn integral calculus. There might be other explanations, depending on the guiding theory other researchers might use.

At the ontological level, therefore, reality of conceptual development was relative, subject to the context of the participants at the time of investigation. Such reality was socially shaped and reshaped over time by the participants in this study, namely, the lecturer and the students. Students' prior knowledge, teaching approaches adopted for the module and the actual curriculum for the programme, constituted the context in which the perceived concept development was occurring.

Epistemologically, the phenomenon to be researched contained, to some extent, the researcher's influence as a person. How students' conceptual development of integration evolved was, to a greater extent, influenced by both teaching design and research instruments administered. At the axiological level, as the lecturer for this group of students and based on the theoretical frameworks adopted, there were preferred or expected responses from the participants. This predisposition to certain types of knowledge confirms the inevitability of the researcher's values and biases, and hence the need to discuss them (Strasheim & Eiselen, 2011). Triangulation of methods of data collection and reference to existing literature were incorporated to address this bias.

Finally, at the methodological level, hermeneutics was the adopted way of knowing about the phenomenon. Students' written responses to items in the research instrument were read and analysed with the aim of developing a deep understanding of imbedded meanings. Discourses among participants were analysed in-depth. Semi-structured interviews were also conducted in order to enrich the context of meanings further and to triangulate the emerging trends.

In ending, in this section, I have stated claims about knowledge, strategies of inquiry and methods of data collection. The stated claims locate this study within an interpretivist paradigm. In the next section I present the general strategy used to answer research problems in this study.

4.2.2 Strategy of inquiry: Qualitative case research

Given that interpretivism was adopted as the research paradigm for the study, an appropriate research strategy was the qualitative inquiry. Findings were arrived at without the use of statistical procedures or calculations. Data was in the form of text from students' written work and words and phrases from the interviews. Next, I indicate how qualitative design was applied in this study.

Many authors agree that in qualitative inquiry the researcher seeks to observe and interpret meanings in context (Corbin & Strauss, 2008; Hoepfl, 1997; King & Horrocks, 2010; Merriam, 1998; Ponterotto, 2005). Such an inquiry is characterised by rich, complete and detailed descriptions with notable interaction between a researcher and the participants (Ponterotto, 2005; Strasheim & Eiselen, 2011). In particular, a qualitative study is an empirical study because it involves collection, analysis and interpretations of primary data (Ponterotto, 2005).

There is a wide range of approaches to qualitative research. Strasheim and Eiselen (2011) cite case studies, ethnographic studies, phenomenological studies, action research and grounded theory as some of such approaches. For the purposes of this study, I will restrict my discussion to case studies as an approach followed. According to Creswell (2002), case studies may be intrinsic, instrumental or collective. An intrinsic case study refers to a case selected due to it being unusual and different from the norm. The goal of an intrinsic case study is to understand a case as a totality including its inner workings. When a case is used to illustrate and illuminate a particular issue, it is called instrumental. Unlike in intrinsic case studies, not all contexts of a chosen are significant to the study but those impacting on the issue that is investigated. Collective studies involve the description and comparison of multiple cases with the aim of providing insight into an issue (Creswell, 2002; Stake, 2013).

This was a single qualitative case study research to investigate concept development of integral calculus for first-year engineering students at a South African University of Technology. As Merriam (1998) defined it, a qualitative case study is "an intensive, holistic description and analysis of a single entity,

phenomenon or social unit” (p. 18). Creswell (2002) and Punch (2009) used four characteristics to define the nature of case studies. Firstly, the case under investigation, although not easily distinguishable from its context, should have clearly defined scope and boundaries pertaining to time, place, or some other physical boundaries. Secondly, in order to focus research and determine the unit of analysis, a case should be a case of something. Thirdly, a holistic approach is adopted by preserving “the wholeness, unity and integrity of the case” (Punch, 2009, p. 120). The last characteristic talks to triangulation which may be achieved by using multiple sources of data as well as multiple methods of collecting data. These characteristics applied in this study are as presented in the next paragraph.

The focus of this study was to answer “how” and “why” questions, therefore, a case study design was considered an appropriate strategy (Heck, 2006; Punch, 2009). Such questions would be answered through exploring how students constructed mathematical meaning in integral calculus and what influenced such constructions. The case, therefore, was that of engineering students’ learning of integral calculus. It is noted that contextual conditions for students, such as their schooling background, structuring of the instruction and the design of the curriculum, were relevant to students’ conceptual development. Nonetheless, the analysis of data focused on inferences that could be made from written responses and semi-structured interviews only.

Students’ responses to a structured worksheet provided initial data for understanding concept development in this case. Methodological triangulation was pursued by collecting additional data through semi-structured interviews and focus group observations. In Chapter 3 a model of cognition, called hypothesised genetic decomposition (HGD), was presented, indicating mental constructs that a student might make when developing understanding in integration. This HGD guided the analysis of data from written responses and interviews to address the how and the why questions of this study. In addition, interviews and focus group observations served to provide insight to the worlds of mathematical thinking of students as inquired in the third research question for this study.

In summary, this was a qualitative case research focused on answering how and why questions regarding students’ construction of mathematical meaning when

learning integral calculus. Triangulation was ensured by checking whether interviews confirmed analyses of written responses to the activity worksheets. Focus group observations were also conducted in order to decipher emerging nuances during group conversations. In the next section I describe the actual research methods used in this study.

4.2.3 Research methods

In sections 4.2.1 and 4.2.2 I presented the research paradigm and the strategy for inquiry for this study. This study was a qualitative case study located within an interpretivist paradigm. As indicated earlier, research methods refer to procedures and techniques utilised to generate and analyse data. In particular, I will discuss participants which include students, I as a lecturer-researcher and the calculus curriculum for this group under investigation. Next will be the presentation on different strategies employed to collect data for the study. The structure of the structured worksheet which served as a research instrument will then be discussed, ending with the process followed when analysing data.

4.2.3.1 Sampling strategy

Creswell (2013) maintains that in qualitative research, a researcher intentionally or purposefully selects participants and sites that would help explore the researched phenomenon in more depth. The phenomenon for this study was mathematical constructions displayed when students learn integral calculus; therefore, participants were first year students undertaking this module at a university of technology in South Africa.

The study composed of two phases. Phase 1 focused on validating the activity sheet, therefore the only criteria for participation was willingness by students. Seven students took part in this phase by responding to the questionnaire and being interviewed by the researcher. According to Stake (1978), Phase 1 was an instrumental case study with the primary interest being the validation of the main research instrument. The context of participants was, therefore, not included in the analysis of data.

The second phase was aimed at providing in-depth exploration of concept development of integral calculus. This phase consisted of group discussions, answering a research instrument and interviews and was undertaken with a different group of students yet given the same instruction as students in the first phase. A sample of 22 students participated in this phase. In both phases, students were selected from an electrical engineering class to respond to the activity sheet. These students had passed one semester module of calculus. They were still in their first year of study but studying a second semester module which consists mainly of integral calculus. Based on their responses, some participants were selected for interviews in order to expand on their answers. Sampling for participation in the study was, therefore, both voluntary and opportunistic. Opportunistic sampling is when new leads are followed as per emerging unexpected scenarios (Marshall & Rossman, 2010). In this instance, candidates to be interviewed were identified as per their responses to the structured activity sheet.

The biographical data of the 22 students who had volunteered to participate was taken into consideration as well. Eight of them were male and fourteen were female. Thirteen had matriculated in rural schools, seven in township and two in the former model C school. Eighteen had obtained a 50% or more pass mark in their matriculation mathematics, two had obtained marks ranging from 40% to 49% and two had other forms of entry requirements into tertiary mathematics. Six students had completed their matriculation at least six years before enrolling in this university. The selection of students for interviews was based on their written responses to the structured worksheet.

The role of a researcher in this study was that of a participant-observer since I served as both the lecturer for the module and the researcher for the phenomenon. As stated in Chapter Three, the whole lecturing-research activity was structured along a framework that encompassed theoretical analysis, instructional treatment and data analysis. My eleven years of experience in teaching this module, combined with the perspectives induced from the preliminary genetic decomposition, informed contents of both classroom activities and research instruments.

4.2.3.2 Data collection methods

Data was collected over two phases for this study. The first phase focused on: (1) validating the activity sheet and (2) evaluating the level of accuracy of the proposed genetic decomposition for integration. This phase was introduced as a “pilot interview” aimed at uncovering issues that could be probed more deeply when the study was rolled out to a larger group of students (Arnon *et al.*, 2014).

In this first phase, the main research instrument, aimed at assessing different cognitive levels in integration, was administered to seven students who had just completed a course in calculus. Students’ responses to the activity sheet were analysed, coded and scored according to the following five-point rubric adapted from Jojo, Brijlall, and Maharaj (2011):

Score	Assessment Criteria	Description of mental action
5	A complete response to all aspects of the item and indicating complete mathematical understanding of the concept assessed.	Made all mental constructions as suggested in the genetic decomposition.
4	A partially complete response with minor computational errors, demonstrating understanding of the main idea of the problem.	Understanding of the concept mostly conceptual.
3	Incomplete response to all aspects of the concept and incomplete reasoning.	Displaying few mental constructions, conceptual understanding at minimal level.
2	No reasoning to justify written response	Displaying few mental constructions, but at a procedural level.
1	No written response or completely principle error	No mental construction of a concept

Table 4.1: Scoring codes

Follow-up interviews were then conducted with individual students with the aim of describing how such students constructed the concept of integration. Interviews were held in the Library’s Learning Commons and were all videotaped. The main aim of

the interviews was to elicit students' understanding of integration based on their performance to tasks in the research instrument. Each student was, therefore, reminded of the task from the activity sheet, the response given and was requested to explain why such response was given. The collation of both analyses, namely, analysis of responses to the activity sheet and analysis of interviews, informed revisions to the activity sheet and the preliminary genetic decomposition. In the following semester, the study was then rolled out to a larger group of students as a second phase of this case study.

Typical of a case study, a variety of data collection techniques were combined in order to allow for in-depth analysis of students' concept development, and also to accommodate limitations relating to individual techniques. Multiple data collection methods are recommended in a case study to strengthen substantiation on and understanding of the phenomenon under study (Heck, 2006; Huberman & Miles, 2002; Mabry, 2008; Punch, 2009).

Firstly, students were individually made to complete an activity sheet comprising of tasks on integration. This activity was scheduled in a test format immediately after completing the section on integration in class. It was a two hour test scheduled on a Friday afternoon, a time found suitable for all the participants. Tasks were designed to examine specific mental constructions by providing an insight into students' knowledge and skills in relation to the preliminary genetic decomposition for integration. The researcher maintained the normal principles for individual interviewing by encouraging participants to respond to all tasks in the questionnaire, without giving any hints to the solutions. Attempts were made to ensure that all tasks were understood by providing oral explanations when needed.

Secondly, based on their responses to the research instrument, seven participants were invited for semi-structured interviews. Rubin and Rubin (2011) describe interviews as conversations in which an inquirer gently guides an interviewee to provide more depth and detail about the phenomenon under investigation. They further caution that in qualitative interviews questions need to match what each interviewee knows and is willing to share. Conversational, qualitative interviews may suffer from power imbalances, since the interviewer may be seen to be in a powerful

position (King & Horrocks, 2010). The lecturer-student relationship added a complication to the balance of power for this study.

To address the issue of power dynamics, students were made to select venues where they preferred to be interviewed. I had to ensure that the selected venues complied with the acceptable norms of interview environment, namely, they are comfortable, private and relatively quiet (King & Horrocks, 2010). Four students chose to be interviewed at the university residences where they were lodging. These residences have study rooms that were deemed suitable in which to hold interviews. The other two agreed to be interviewed in my boardroom at work, and one student was interviewed in my study room. In all cases, time suitable to the interviewees was agreed upon and all interviews were audio-recorded with consent of the participants.

To start interviews, brief descriptive information about the participants in relation to their mathematical background performance was asked. Included in this information were their performances in school mathematics and their scores in the first semester of university mathematics. The introductory phase was followed by semi-structured interview questions that focused on the definition of an integral, integral as an area and techniques of integration, in line with the worksheet. Participants were interviewed for more clarity and further explanations on their written responses. The level of abstraction, critical thinking and insightful conceptualisation were measured during the conversations.

The third technique used to collect data was the focus group discussions where students worked collaboratively to solve given problems. Focus groups are defined as “in-depth interviews employing relatively homogeneous groups to provide information around topics specified by the researchers” (Smithson, 2008, p. 358). This technique involves interviewing a group of people at the same time, with focus being on the interaction among the participants during such interview (Gibbs, 1997; Kitzinger, 1995). As a result of their collaborative nature, participation in such a research can be empowering to the participants (Gibbs, 1997).

For this study, focus groups meant participants responding to sets of questions on integral calculus as a group. They were encouraged to explain and defend their

approaches to each other. The lecturer served as a “soft scaffold”, as defined by McCosker and Diezmann (2009), through asking probing questions and providing explanation whenever necessary. The Learning Commons of the university’s library was used as a venue because of the available recording facilities. Two hour discussions were held on two Friday afternoons and were video-recorded. Photographs of students and their work were also taken. In order for all participants to be active, the size was kept to four members per group. According to Smithson (2008), smaller groups yield relevant data and allow space for all participants to express themselves. The analysis of discourses from these groups will be discussed in chapter seven.

4.2.3.3 The main research instrument: The activity sheet

As indicated in Chapter 3, the framework for research in mathematics education guided the whole process of data generation and data analysis. That framework involves three stages which are: (1) Theoretical analysis of the concept, based on the researcher’s knowledge and experience and the adopted theoretical framework; (2) Instructional treatment, where students are assisted and observed whether they are developing the mental constructions as predicted in the analysis of the investigated concept and (3), Collection and analysis of data with the aim of refining the initially proposed genetic decomposition. The main research instrument, included in this thesis as Appendix B1, pages 207-211, was used to collect data in the third stage of the framework.

The main research instrument consisted of five items, some of which had sub-items. Although items 1 and 2 contained simple functions, they dealt with the understanding of the meaning of an integral. Items 3 and 4 required students to choose appropriate techniques of integration based on the analysis of the integrand. Subsections of item 5 required an overall understanding of the concept. This research instrument was designed to elicit data with regards to the meaning students attached to integration. It was administered after students had been taught integral calculus in their normal mathematics course. Item 1, for example, required students to state, in their own understanding, the difference in meaning between $\int f(x)dx$ and $\int_a^b f(x)dx$. I was

of the view that the evoked concept image of an integral, through this item, would contribute to indicate the level of conceptualisation of concept definition.

The instrument also aimed at requiring students to apply various techniques for integration but with valid reasons. Item 3 of the questionnaire was:

A student asked to solve the integral $\int x e^{\frac{x}{2}} dx$ decided to use integration by parts and chose $e^{\frac{x}{2}}$ for a “ u ”.

3.1. Was this choice of a “ u ” appropriate?

3.2. Please support your answer.

3.3. Provide a solution for the same integral

Most respondents skipped 3.2 which required them to justify their choice and moved directly to question 3.3. This could be interpreted as the inclination towards procedural versus conceptual knowledge. The rest of the questionnaire targeted proficiency with the techniques of integration.

4.2.3.4 Data analysis

According to Creswell (2002) the analysis and interpretation of data in a qualitative research spans six stages. Researchers start by accumulating, organising and transcribing data for analysis. A decision on whether data would be manually or computer analysed is also made at this stage. The second stage involves exploring and coding data according to text segments identified. Thirdly, coded segments are then used to formulate themes that provide a broader description of the phenomenon under investigation, as well as contribute to key findings of a study. Such findings are then represented in narrative discussions such as a chronology, or in visual displays which include figures and diagrams in the fourth stage. It is from the findings, the researchers’ personal views and comparisons with literature that the interpretation gets made as a fifth stage. The sixth stage involves validating the accuracy of the findings mainly, through triangulation and auditing.

Analysis of data commenced with students’ written responses. For each item, students’ answers were analysed for emerging trends or themes. The themes were then categorised by using descriptors included in Rasslan and Tall (2002) as a

guide. Interviews were then held with selected students in order to provide clarity on the written responses.

The next level of analysis involved cross-referencing students' oral inputs from semi-structured interviews with the written text. The genetic decomposition was invoked as a tool to categorise students' conceptual development within the APOS theory. Inferring from the constructions made students' knowledge could be placed at the action, process or object level of conceptual development. The researcher was drawn to those cases that deviated from the pre-stated classification since they constituted areas for further probing.

Focus groups provided data that was mainly from the students' voice. Discussions among students on items contained in a research instrument that is appended here as Appendix B2, pages 214-217, were recorded and later transcribed into text. A coding framework was devised based on recurrent issues in the text and expectations from the theoretical framework. For example, when discussing the meaning of $\int_a^b f(x)dx$, some students preferred to show the meaning by invoking the Fundamental Theorem of Calculus (FTC), that is $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$. Common trends were to verbalise the FTC, to write it generally, or to define own functions and evaluate the definite integral. These types of responses formed distinct categories which were then contrasted against conceptual levels proposed in the hypothesised GD presented in Chapter 3.

4.3 Validity, credibility and trustworthiness of methods

The significance of a study depends on the validity claims that can be placed on the study and the standing these claims obtain when compared to other validity claims in the discourse to which the study is a contribution (Flyvbjerg, 2006). Validity in qualitative designs translates to credibility, transferability, dependability and confirmability (Creswell, 2013; Guba & Lincoln, 1994; Kretting, 1991). In a case study research, validity is increased by combining methods or sources of data that provide fuller picture of the phenomenon under investigation (Cohen et al., 2011; Jojo, 2011; Leedy & Ormrod, 2005). This approach of using two or more methods in

collecting data for a study is called triangulation (Cohen *et al.*, 2011; Rubin & Rubin, 2011).

Triangulation can be pursued by varying data sources, where data is collected from different persons or entities. The researcher checks the degree to which each source confirms, expands or disproves information from the other source (Mabry, 2008). Methodological triangulation is when data collected through one method is checked for consistency with data collected through another method. Triangulation is also achieved by collecting data at different times, using more than one data collector, or by referring to different theoretical frameworks (Creswell, 2013; Leedy & Ormrod, 2005; Mabry, 2008).

In this study, although participants had been subjected to the same instruction, they had varying capacities and different schooling backgrounds. In addition, different techniques were used to collect data. Students' written work was the first source of data. In-depth interviews were then conducted where participants elaborated on their written responses. Focus groups served to encourage students' voices as they engaged in discourses among themselves. These forms of triangulation, together with the exploration of meaning through the APOS framework and Tall's Three Worlds (2008), were included in order to address validity and credibility in the study.

Strategies and criteria to establish overall trustworthiness were crafted according to the guidelines provided by Krefting (1991) and Jojo (2011). Triangulation through employment of different methods to collect data addressed the aspect of credibility. To allow for any possible transferability, dense description of data was carried out. Extracts of students' written responses, verbatim quotes from interviews and sample dialogues from focus were included as form of data. This study adopted research strategies such as semi-structured interviews, written questions and observations in the form of focus (Arnon *et al.*, 2014). According to Krefting (1991) dense description of appropriate research methods indicate dependability of a study. Finally, as suggested by Krefting (1991), confirmability was ascertained by providing transcripts and sources for all claims made in the analysis.

4.4 Ethical issues

Ethical clearance was first sought from the university in which the degree is pursued. The university of registration granted ethical clearance: the protocol reference number is HSS/0135/012D. I then applied for ethical clearance from the university where the study was conducted, which was granted and communicated to me by the Research Directorate office of the university. These documents are included in this thesis as Appendix A1 on page 195 and Appendix A2 on page 196 respectively.

Although I taught the group, students were informed that participating in the research project was not linked to their study. Participation was completely voluntary and students could withdraw any time when they so wished. This explanation was captured in a letter of consent, Appendix A3 on pages 197-198, that was read to them and which they all signed. The letter also contained a brief explanation and context of the project. It notified the participants of the methods through which data would be collected and assured them that their identities would be protected.

4.5 Conclusion

In this chapter, I presented the methodologies adopted for this study. I indicated that the study was a qualitative case study located within an interpretivist paradigm. Under the heading of research methods I gave descriptions of the participants in this study and explained the data collection methods used and how data was analysed. I then explained how triangulation was employed in order to ensure validity and credibility. Lastly, I indicated how ethical considerations were observed for this study. In the next chapter I present the results of Phase 1 of the study, which are aimed at validating the main research instrument, the activity sheet.

CHAPTER 5

VALIDITY OF THE ACTIVITY SHEET

5.1 Introduction

In Chapter Four I presented the research design, methodologies and research techniques that I used for this study. I indicated that an interpretivist paradigm was adopted to conduct a qualitative study that is premised on an APOS theoretical framework. In addition, detailed descriptions of data collection tools, data sources and data analysis process were provided. In this chapter I present discussions and results from the first phase of the study. As stated in the previous chapter, this phase was introduced as a “pilot interview” aimed at uncovering issues that could be probed more deeply when the study was rolled out to a larger group of students (Arnon *et al.*, 2014).

The next section presents findings in relation to the different items of the activity sheet. Firstly, I present analysis of evoked meaning that seven students attached to an integral through symbolic (meaning attached to $\int f(x)dx$ and $\int_a^b f(x)dx$) and graphical representations (area bound by the graph of a function). Secondly, I present an exploration of students’ construction of meaning as they work with various techniques of integration. The last section reports on the overall conceptualisation of schematization of the concept of integration by students.

5.2 Meaning attached to an integral

5.2.1 Symbolic representation

All students responded to the first item. The analysis of the responses revealed two main trends. On the one hand there was a tendency to explain the symbol of the integral and on the other, reference to the area was made. The third category consisted of either a no response or a complete principle error. Table 5.1 shows the

distribution of students' responses on their understanding of $\int f(x)dx$ and $\int_a^b f(x)dx$:

Students' answers	The integral of the function $f(x)$	The area bounded by the graph and the x-axis	Other
Number of students: $\int f(x)dx$	4	2	1
Number of students: $\int_a^b f(x)dx$	2	4	1

Table 5.1: Students' answers to Item 1 on the "meaning of integration"

The four students who indicated the meaning of $\int f(x)dx$ as an integral of $f(x)$ focused on the symbol of an integral. Three of these stated that $\int f(x)dx$ meant "finding the integral of $f(x)$ " or "integrate the function $f(x)$ with respect to x ". The fourth one, Xolile, made reference to limits of integration by stating that:

- $\int f(x)dx$ is an integration of $f(x)$ without the limits which is a lower and an upper limit.

In this case it would infer that these four students had an action conception of APOS for an indefinite integral. They responded to the notational stimulus, explicating on the symbol for integration. Xolile's response indicated that she focused mainly on the symbol and the presence or absence of limits of integration. These students responded in a manner that was external to their cognition.

Simo and Menzi are the two students who defined $\int f(x)dx$ as an area below the graph. The following are the written responses they provided:

- $\int f(x)dx$ is the indefinite integral of $f(x)$. This integral represents the entire area below the graph. (Simo)
- $\int f(x)dx$ it is when you find the area without knowing the limits and we put C after we integrated for a constant value which we don't know. (Menzi)

According to the preliminary genetic decomposition, Simo and Menzi could be classified as displaying a process conception as their responses went beyond the notational symbolism. They had a notion of an integral as a procedure to determine an area, although not mentioning the orientation of such area. Nothing could be implied for the last student, Bonga, since he did not respond to this part of item 1.

With regard to the meaning of $\int_a^b f(x)dx$, four students referred to the area concept in their definition. In addition to Simo and Menzi were Ayanda who had referred to $\int f(x)dx$ as “*finding the integral of $f(x)$* ” and Bonga who had not responded in the first instance. In all responses there was no mention of oriented area. Simo seemed to be extending on the meaning he had provided for $\int f(x)dx$ above by stating that:

- $\int_a^b f(x)dx$ is a definite integral of $f(x)$. This means that the area calculated is between points $x=a$ and $x=b$.(Simo)

Similarly Menzi’s response was also linked to the meaning he attached to $\int f(x)dx$:

- It is when you are calculating the area given the limits.(Menzi)

Xolile and Gugu retained their point of view by stating that $\int_a^b f(x)dx$ is an integral with limits. In line with the meaning attached to $\int f(x)dx$, Xolile stated that:

- $\int_a^b f(x)dx$ is an integration of $f(x)$ with an upper and a lower limit.

The one student categorised as “other” defined $\int_a^b f(x)dx$ as *the mean or root of the mean of squares (RMS) of $f(x)$* (Faith).

The responses indicated that most students tended to link a definite integral with the notion of an area, without the mention of the orientation. Using the preliminary genetic decomposition provided, it also appeared that most students viewed an integral as an anti-derivative with limited extension to the area concept. With regard to $\int f(x)dx$, the main interpretation attached was that of “finding an integral of

$f(x)$ ". This could be interpreted as finding the anti-derivative of the function $f(x)$. It could be inferred from these responses that students relied on expressed limits of integration to construct meaning of an integral. In the absence of limits, the integral symbol assisted them to formulate whatever mental constructions, and mainly, finding the anti-derivative. Such conceptions placed them at the action – process level of concept development.

In follow-up interviews with Xolile and Simo, for example, when asked for the meaning of an integral in their understanding, they responded as follows:

Researcher: *"In your understanding, what is an integral?"*

Xolile: *"If you are given a derivative, then going back to the original function."*

Simo: *"(A) is the derivative of a function, then (B) is the derivative of a function but we must also find the area" {where (A) = $\int f(x)dx$ and (B) = $\int_a^b f(x)dx$ }.*

Notably, when asked the same question, student Faith responded by stating that: *"Integral actually gives an area"*, contrary to the answer given in the questionnaire of a mean or RMS.

It was decided that Item 1 would be kept for the main study, since it evoked students' associations with the symbolic, area, and anti-derivative notions of the integral. This item addressed the reliability of the research instrument since it was deemed appropriate to extract students' depictions on the concept definition of integration (Rösken & Rolka, 2007).

5.2.2 Graphical Representation

5.2.2.1 Item 2(2.1.1): The semicircle

Item 2 was designed to explore how students related a definite integral as the value of the area, when a graph is provided. Students were not required to evaluate the given definite integral $\int_0^5 \sqrt{25 - x^2} dx$ but to sketch the graph of a positive semicircle with radius equal to 5 with the function rule $y = \sqrt{25 - x^2}$ and state how the definite integral related to the graph without carrying out symbolic integration.

All students struggled to draw the graph of a semicircle until the researcher provided some hint. Errors in sketching the graph included the drawing of straight line graphs, circle graphs, quadratic functions and quarter of circles. Although graph and graph sketching is one of the schemas in the understanding of integration, this study did not include it. The researcher, therefore, decided to show students what the expected graph is. With regard to relating the integral $\int_0^5 \sqrt{25 - x^2} dx$ to the graph of the semicircle, only one student made reference to the area. Three students restricted their explanation to the sketch for the graph, two attempted to evaluate the integral and one student did not respond to this item.

The one student who made reference to the area omitted the implication of limits of integration. Her written response was:

Ayanda: *It determines the area covered by the function.*

When interviewed, Ayanda displayed difficulties in interpreting the limits of integration. When asked the implications of 0 and 5 in the integral $\int_0^5 \sqrt{25 - x^2} dx$ here response was:

Ayanda: *"0 and 5 mean ehh...to find the area of the shaded region from 0 to 5."*

Researcher: "Can you please point that on your sketch."

Ayanda: *"From 0 to 5 on the x-axis (pointing along the x-axis) and...(pause)... from 0 to 5 on the y-axis (pointing along the y-axis)."*

Researcher: "I see. By the way, which axis do limits of integration refer to?"

Ayanda: *"Hm.....the limits of integration...are referring to y-values."*

Ayanda's oral responses indicate that she did not possess an effective function schema necessary for the interpretation of the definite integral. This response also indicated difficulties with the conceptualisation of the symbol $\int_a^b f(x)dx$. There was no clarity in her understanding of the actual meaning of a and b as they appear in the symbol of a definite integral. She had an inclination that the definite integral pertained to the area under the graph but her function schema was not strong enough to enable for correct identification of the domain for the function. The whole presentation placed her in an action-process stage since the notion of an integral as

an area had been developed. The lack of a function schema impacted negatively on the encapsulation of the definite integral to an object conception of the area notion.

The other tendencies were to restrict the explanation to the sketch for the graph or to attempt to evaluate the given definite integrals. I will discuss two examples of explanations that focused on the sketch. The first such response was from Xolile.

Xolile: *This one will look like this*  *because it has limits of x , where $x=0$ and $x=5$*

According to Xolile, limits of integration meant the graph commences at $x = 0$. Based on her sketch of $y = \sqrt{25 - x^2}$, shown in Figure 5.1, it can be inferred that $x = 5$ was the other x -intercept.

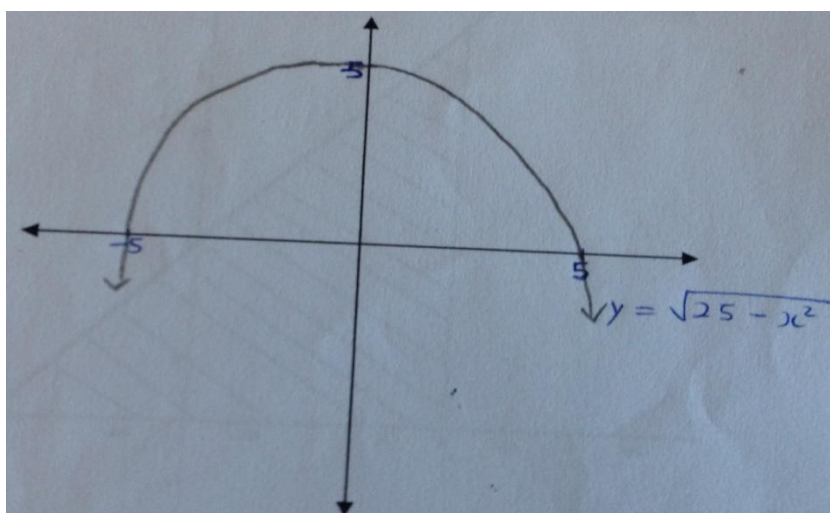


Figure 5.1: Xolile's response to the sketching of the graph

This view of $\int_0^5 \sqrt{25 - x^2} dx$ as a graph was supported in the follow-up interview with Xolile. When asked what $\int_0^5 \sqrt{25 - x^2} dx$ actually represented, Xolile responded as follows:

Xolile: *"I think it means the graph from 0 to 5....what is it?...half of... I mean, a quarter of a circle."*

Xolile still maintained that the definite integral means a graph. The interview seemed to have made her delve deeper since her domain was correct in the interview but not in her written graph as observed in Figure 5.1.

The second similar response was given by Simo, who drew a correct graph for the semicircle but then referred to $\int_0^5 \sqrt{25 - x^2} dx$ as a graph by writing that:

Simo: *The graph represents a half of the graph*

The first word “graph” is interpreted to refer to the given integral while the second one refers to the sketch of the semicircle. Simo’s explanation is limited to the sketch without linking it to an integral. During the interview, Simo restricted his explanation on the values of x ranging from ‘0’ to ‘5’ but made no mention of the corresponding area. To him, the ‘0’ and ‘5’ meant the start and end of the graph and he completely overlooked the integral symbol present in $\int_0^5 \sqrt{25 - x^2} dx$.

These two students’ conception was confined to the numerals, which were the limits of integration. They could not relate an integral to the oriented area since the area was not drawn, thus interpreted limits as only demarcating the domain for the graph. The numerals, ‘0’ and ‘5’ and the graph of the quarter circle, served as external cues. Accordingly, the expressed conceptualisation of a definite integral was limited to defining the graph within the given domain. There seemed to be no notion of an area in their understanding. Such responses placed them at an action stage of APOS regarding conceptualisation of an integral as an area since they relied on external stimuli to construct meaning.

Three students attempted to evaluate the integral, an indication that they were responding to the given formula by trying to apply some algorithms. Having drawn graphs of the semicircle, no attempt was made to answer the second part of Item 2 which asked for the relationship between the graph and the definite integral. Instead, students attempted to evaluate the integrals. I provide two of these attempted solutions. The first example is Figure 5.2, a presentation by Menzi:

$$\int_0^5 \sqrt{25-x^2} dx$$

$$\int_0^5 (25-x^2)^{1/2} dx$$

$$\left[\frac{2}{3} (25-x^2)^{3/2} \right]_0^5 = \frac{2}{3} (25-5^2)^{3/2} - \frac{2}{3} (25-0^2)^{3/2}$$

$$= 0 - \frac{250}{3} = -\frac{250}{3}$$

Figure 5.2: Menzi's response to Item 2.1.2

I observed that Menzi applied the power rule, that is, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ to line 4 of Figure 5.2. He treated $(25 - x^2)$ as a single variable and not a function. This shows that he did not possess a process conception of integration when using the power rule.

Faith possessed similar mental constructions as Menzi, as could be concluded from comparing their written responses. However, Faith committed a "sign" mistake at the final step resulting in her providing a different answer. Faith's response is shown below:

2.1.2 What is the relationship between $\int_0^5 \sqrt{25-x^2} dx$ and the graph you have just drawn?

$$\int_0^5 \sqrt{25-x^2} dx = \left[\frac{(25-x^2)^{3/2}}{3/2} - \frac{(25-0^2)^{3/2}}{3/2} \right]$$

$$= \int_0^5 (25-x^2)^{1/2} dx = 0 - 125 \times \frac{2}{3}$$

$$= -\frac{250}{3}$$

Figure 5.3: Faith's response to Item 2.1.2

In both of these solutions students were attempting to use the rule $\int [f(x)]^n f'(x) dx = \frac{1}{n+1} [f(x)]^{n+1} + C$, not considering the fact that the essential $f'(x)$ was missing in the given integral $\int_0^5 (25 - x^2)^{1/2} dx$. Both solutions

were therefore, firstly not necessary to answer the question and secondly, procedurally incorrect. These students perceived and reflected on a repeated action of integration and wanted to display mastery of the techniques. According to Dubinsky (1991b), such transformation of physical or mental objects which is a reaction to stimuli perceived by a subject as external, is considered to be an action. The students operated in the action stage of APOS for application of the power rule and were focused on solving an integral in a step-by step approach as prescribed by the “algorithm” they had adopted. They had not interiorised this action though and hence did not possess a process conception of the technique of integration.

In all the cases students showed inadequate conception of both an integral as an area and the actual use of algorithms. As stated, all seven students could not draw the graph of a semicircle without assistance. Although this was a cause for concern since students were expected to have done this type of graphs, this item was nevertheless, retained in the main research instrument with the analysis focusing on integration and not on the sketching of graphs.

5.2.2.2 Item 2 (2.2): The straight line

The next item consisted of a graph of $y = 2x + 3$ which was given with area shaded as shown in Figure 5.4 below. Students were asked to determine the area of the shaded region using integration (Figure 5.4).

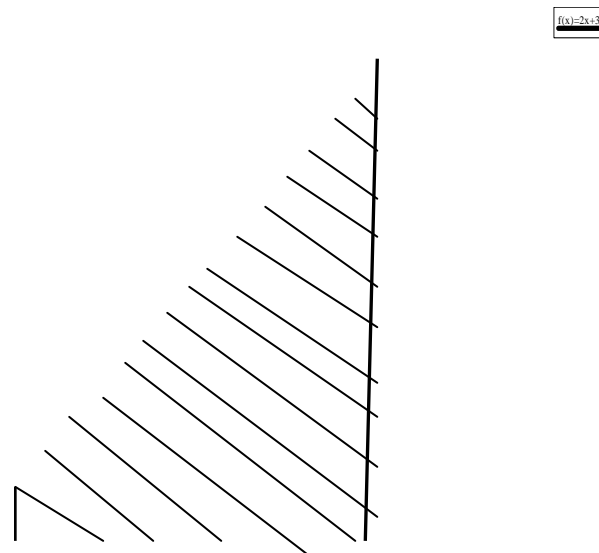


Figure 5.4: Sketch graph for Item 2.2.

One student did not respond to this question. The written responses displayed were coded into four categories. Description of categories and students' distribution is presented in Table 5.2.

Category	I	II	III	IV
Indicator	Employed correct integration techniques and answered correctly	Correct integration, but wrong limits	Inappropriate technique or principle error	Left Blank
Number of students	1	2	3	1

Table 5.2: Students' answers to Item 2.2

I paid my attention to Menzi, who answered this item correctly thus indicating mathematical understanding of the definite integral as an area. Menzi reflected on the presentation and properties of the given area and realised that the definite integral would have $x = -1$ and $x = 2$ as limits for integration. According to the hypothesised genetic decomposition, we could claim that this student had conceptually embodied integration. The interiorised action enabled the application of

an integral in finding an area, resulting in the correct answer. His response is presented in Figure 5.5 below.

The image shows a student's handwritten work on a piece of paper. The work is as follows:

$$\begin{aligned}
 y &= 2x + 3 \\
 &= \int 2x + 3 \\
 &= \int_{-1}^2 2x dx + \int_{-1}^2 3 dx \\
 &= 2 \int_{-1}^2 x + \int_{-1}^2 3 \\
 &= 2 \left[\frac{x^2}{2} \right]_{-1}^2 + [3x]_{-1}^2 \\
 &= [x^2]_{-1}^2 + [3x]_{-1}^2 \\
 &= 2^2 - (-1)^2 + [3(2) - 3(-1)] \\
 &= 3 + 9 \\
 &= 12
 \end{aligned}$$

Figure 5.5: Menzi's response to Item 2.2.

I observed that Menzi displayed an action conception of integration where he step-by-step solved the problem. I could not argue that he had a process conception as the question provided a hint for the use of integration to find the area. He missed out the "dx" in lines 2 and 4. On the same line, Menzi also missed the inclusion of brackets. He wrote $\int_{-1}^2 2x + 3$ instead of $\int_{-1}^2 (2x + 3)dx$. Nonetheless, I assumed that that was a slip as he included the "dx" elsewhere. He also applied the property of an integral, $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$ in line 3..

In an interview with Menzi, it emerged that the explicit sketch had guided his choice of approach. When questioned on his approach he said:

Researcher: Can you explain what is given in this sketch?

Menzi: *We have a straight line cutting from -1.5 on the x-axis and y-axis, and its area is shaded from -1.5 to 2 ... (pause)... actually it is shaded from -1 to 2.*

Researcher: So if I want to calculate this area, what should I do?

Menzi: *Find the integral from $x = -1$ to $x = 2$.*

Menzi's verbal answers and his indication on the sketch made me conclude that he had some knowledge of using an integral to find the area under a graph. The tendency to confuse x-intercepts with limits of integration displayed by Menzi was the characteristic of responses classified as Category II. Such students performed integration correctly, but used wrong limits of integration. They relied on visible cues without linking graphical representation with embodied concept of the area. They were placed at the action level of APOS, with the development towards a process stage.

It must be noted that the x-intercept, $-1,5$, could also be used in determine the area of the shaded region. In that instance, the student would need to first consider the area of the bigger triangle $[(-1,5,0), (2,0)$ and $(2,7)]$ and from it subtract the area of the smaller triangle, $[(-1,5,0), (-1,0)$ and $(-1,1)]$. Mathematically, a student would evaluate $\int_{-1,5}^2 (2x + 3)dx - \int_{-1,5}^{-1} (2x + 3)dx$. None of the respondents in Category II displayed this approach.

A different situation was experienced with Xolile who displayed a principle error in her written response. There was a tendency in her presentation to focus on the y-value when describing the shaded area. Such approach influenced her written solution as shown in Figure 5.6 below.

Use integration to find the shaded area.

$$\begin{aligned}
 \text{Area} &= \frac{1}{2} b \times h \\
 &= \frac{1}{2} \left(\frac{3}{2} \right) \times 7 \\
 &= \frac{21}{4} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{Area} &= \pi \int 2x + 3 \, dx \\
 &= \pi \left[\int 2x \, dx + \int 3 \, dx \right] \\
 &= \pi \left[x^2 + 3x \right]
 \end{aligned}$$

Figure 5.6: Xolile's response to Item 2.2.

Xolile presented two methods for calculating the area, namely, using a geometric figure and applying integration. Geometrically, Xolile deciphered the shaded area as a triangle instead of a trapezium thus applying the formula $\text{Area} = \frac{1}{2}(\text{base} \times \text{height})$ instead of $\text{Area} = \frac{1}{2}(\text{sum of } // \text{ sides}) \times \text{height}$.

In her alternate solution she erroneously applied integration, showing weak conceptualisation of the use thereof. When asked how she would calculate the given area, Xolile gave the following verbal response:

Xolile: *I want to work out the area between -1 and positive 2 on the x-axis and then on the y-axis up to positive 7. Actually, I think...hmm...if I read the statement, we are supposed to find the shaded area, and the shaded area is between -1 and 2 on x. So basically I think I must not worry about y.*

There seemed to be doubts in Xolile's thinking whether the y-value of '7' should be included in the solution or left out. This doubt confirms her use of '7' when finding the area of a triangle as shown in Figure 5.6. She seemed to have conceptualised neither the properties nor the representations of the integral-area concept. She had

not even displayed an action conception of integration. I, therefore, resorted to explaining her response using the TWM theory.

According to Tall (2007) and (2008), the embodiment of a concept, conceived through perceptions and reflections on the properties and presentations of the concept, is referred to as conceptual-embodiment. Xolile's presentation indicated reliance on external stimuli with a weak conception of both the graphical representation and the use of integration. She was placed at the lower action level of APOS. Ultimately this item was regarded as suitable in extracting perceptions on simple graphical application of integration and it was therefore not changed.

5.3 Techniques of integration

5.3.1 Integration by parts

The technique of integration by part plays a major role in engineering with applications in problems ranging from electric circuits, electromagnetic, digital signal processing, heat transfer and many more. This technique requires intuition and practice for a student to make a correct choice of a 'u' when applying the formula $\int u dv = uv - \int v du$ to solve an integral. Item 3 was designed to explore both students' insight on the use of the formula and ability in the application of the technique. It was important to know how students had conceptualised the two functions' behaviour when integrated.

Item 3

A student asked to solve the integral $\int x e^{\frac{x}{2}} dx$ decided to use integration by parts and chose $e^{\frac{x}{2}}$ for "u".

- 3.1. Was this choice of a "u" appropriate? Please support your answer.
- 3.2. Provide a solution for the same integral

Figure 5.7: Item 3 on integration by parts.

This item was scored according to the following rubric:

Score	1	2	3	4	5
Indicator	Yes to 3.1 and stuck to this choice	No to 3.1 but failed to solve the integral due to principle error OR no to 3.1, wrong reason	First yes, then no to 3.1 and solved the integral correctly	No to 3.1, correct justification, but failed to solve the integral because of constants.	No to 3.1, correct justification and solved the integral correctly
Number of students	3	1	2	1	0

Table 5.3: Scoring rubric for Item 3

A student scoring a four or a five would be placed at an object level of APOS for the understanding of exponential and x^n functions. I argue this on the fact that students require the process level of integration and differentiation as they would need to mentally decide on what will ‘cancel’ when the $\int u dv$ is determined. Also, they would need to carry out integration and differentiation procedures on the concerned functions, at an action level.

From written responses, only one student fitted this level of understanding, although the same student committed a computational error and hence could not provide a correct solution. Two students, Simo and Ayanda, only changed their choice of ‘u’ after encountering difficulties with what they had initially decided. They were both scored a three according to the devised rubric.

Simo then went on to provide a correct solution while Ayanda omitted a constant in her final answer. His reason for a choice of a ‘u’ was based on procedural workings when he wrote that:

Simo: *No, choosing x (initially he had put $e^{\frac{x}{2}}$) leads to simpler working.*

According to Simo, it was the ease to work with the functions that determined the choice of a ‘u’. He showed this by proceeding with his initial choice of $e^{\frac{x}{2}}$ for a ‘u’ and only changing his position when the working became complex. Simo seemed

not to possess the intuition of how the given functions would behave when integrated. His decision could only be derived after he had actually attempted to solve the problem given. Simo, therefore, could have interiorised the integration action but might not have developed his conceptualisation beyond the process level of APOS.

Ayanda, on the other hand, displayed an inclination of basing her argument on the nature of functions appearing in the integrand. She wrote that:

Ayanda: No (having written yes), so that function $x = dv$ increase the power of x which is not necessary.

Her written response is indicated in Figure 5.8 below:

3.2 $\int x e^{x/2} dx$

$x = u$ and $e^{x/2} = dv$
 $\frac{du}{dx} = 1$ $v = 2e^{x/2}$
 $du = dx$

$\int x e^{x/2} dx = u \cdot v - \int v \cdot du$
 $= x \cdot e^{x/2} - \int 2e^{x/2} \cdot 1 dx$
 $= x e^{x/2} - 2 \int e^{x/2} dx$
 $= x e^{x/2} - 4 e^{x/2} + C$
 $= e^{x/2} (x - 4) + C$

Figure 5.8: Ayanda's response to Item 3.2

Ayanda presented a correct approach to solving the given integral. She correctly determined that $v = 2e^{\frac{x}{2}}$, but erred when substituting such in the formula $\int u dv = uv - \int v du$ by omitting the coefficient two. Besides this omission, the rest of her presentation was correct. Ayanda seemed to have developed conceptual

understanding of the behaviour of functions x^n when integrated. Ayanda could, therefore, be placed at the process to object level of APOS.

Responses to Item 3 indicated that students did not consider behaviours of functions when making choices for 'u' in integration by parts. During follow-up interviews, when asked how they would decide on a choice of 'u', the following were some of the answers given:

Faith (Scored 1): *The one that will be a 'u' is because you cannot integrate further, and the one that will be a 'dv' is the one you can't differentiate further.*

Bonga (Scored 2): *I think $e^{\frac{x}{2}}$ is not 'x' to the power something. According to the priorities for 'u', 'x' to power something is chosen first as a 'u'.*

Faith's response indicated gaps in her reasoning when making a choice for a 'u'. The assertion that there are functions that cannot be integrated for the second time may be referring to functions whose integrals are not standard, as stated in Stroud and Booth (2007) that " $\int \ln x dx$ is not in our basic list of standard integrals" (p. 835). Such functions can be integrated though, albeit by using techniques of integration. Similarly, differentiability of functions had not been used as a criterion for assigning a function as a 'dv' when integrating by parts. Faith had grasped the procedure for integration by parts but displayed a lack of underlying reasoning behind the technique.

The lack of underlying reason resulted in Faith endorsing $e^{\frac{x}{2}}$ as a 'u' and sticking to such choice, misled by the fact that functions used in this item satisfied her stated criterion. The lack of mental constructions became apparent in her solution when she did not realise that her choices were leading to an infinitely increasing power of 'x'. Her solution is included as Figure 5.9 below:

3.2

$$\int x \cdot e^{\frac{x}{2}} dx$$

let $u = e^{\frac{x}{2}}$ $\int dv = \int x dx$
 $du = \frac{1}{2} e^{\frac{x}{2}} dx$ $v = \frac{1}{2} x^2$

$$\int u dv dx = uv - \int v du$$

$$\int x \cdot e^{\frac{x}{2}} dx = e^{\frac{x}{2}} \cdot \frac{1}{2} x^2 - \int \frac{1}{2} x^2 \cdot \frac{1}{2} e^{\frac{x}{2}} dx$$

$$\int x \cdot e^{\frac{x}{2}} dx = \frac{1}{2} x^2 e^{\frac{x}{2}} - \frac{1}{4} \int x^2 \cdot e^{\frac{x}{2}} dx$$

For $\int x^2 \cdot e^{\frac{x}{2}} dx$

let $u = x^2$ $dv = e^{\frac{x}{2}}$
 $du = 2x dx$ $v = \frac{1}{2} e^{\frac{x}{2}} dx$

$$\int x^2 \cdot e^{\frac{x}{2}} dx = \frac{1}{2} x^2 e^{\frac{x}{2}} - \int \frac{1}{2} e^{\frac{x}{2}} \cdot 2x dx$$

let $\int x \cdot e^{\frac{x}{2}} dx = I$

$$\int x \cdot e^{\frac{x}{2}} dx = \frac{1}{2} x^2 e^{\frac{x}{2}} - \frac{1}{4} \int x^2 \cdot e^{\frac{x}{2}} dx$$

$$I = \frac{1}{2} x^2 \cdot e^{\frac{x}{2}} - \frac{1}{4} \left[\frac{1}{2} x^2 e^{\frac{x}{2}} - \int I \right]$$

$$I = \frac{1}{2} x^2 e^{\frac{x}{2}} - \frac{1}{8} x^2 e^{\frac{x}{2}} - \frac{1}{4} I$$

$$\frac{5}{4} I = \frac{3}{8} x^2 e^{\frac{x}{2}}$$

$$\therefore I = \frac{3}{10} x^2 e^{\frac{x}{2}}$$

Figure 5.9: Faith's response to Item 3.2

Faith ultimately introduced an 'I' to substitute the original integral, which was a relevant approach since this integral had re-occurred on the right-hand side of her equations. However, an error in determining $\int e^{\frac{x}{2}} dx$ misled her. As can be seen from her solution, Faith wrote $\frac{1}{2} e^{\frac{x}{2}}$ instead of $2e^{\frac{x}{2}}$. Faith seemed to be confusing integration with differentiation here, an error that is located within calculus context. She appeared not to be sure of what to do with the derivative of $\frac{x}{2}$ which is $\frac{1}{2}$. She ended up multiplying by it instead of dividing as required when performing integration. Had she integrated her 'dv' correctly, she would have ended with $0=0$ and realised that she had used a dead-end reasoning. Faith could perform certain procedures in integration but had not adequately internalised processes to enable her to predict the behaviour of the given functions. Hers was mainly a procedural response thus placing her at the action level of APOS.

The other student, Bonga, seemed to be basing his reasoning on an algorithm where a priority order for 'u' is stated as "(a) $\ln x$ (b) x^n (c) e^{kx} " (Stroud & Booth, 2007, p. 837). Bonga seemed therefore, to be relying solely on external cues when the choice of 'u' is concerned. His approach would assist him in solving the problem but it was mainly procedural. Bonga was placed at the action-process level of APOS since his argument was based on properties, nonetheless external, of functions present in an integral.

Ultimately, Item 3 was retained for the main study as it provided indicators to the mental constructions the students formulated which were in line with the preliminary decomposition provided, and will provide evidence to verify our research questions. The only revision to the question was the splitting of 3.1, where justification for the choice of a 'u' was made a stand-alone sub-question.3.2.

5.3.2 Inverse of polynomials

This item focused on techniques on integration when the integrand is a multiplicative inverse of a polynomial. Students were given two integrals that required different approaches to work out.

Item 4

A student is given two integrals to evaluate:

$$(A) \int \frac{dx}{x^2-8x+25} \quad \text{and} \quad (B) \int \frac{10}{(x-1)(x^2+9)} dx$$

4.1 Work out the solution for integral (A) and (B)

4.2 Justify the choice of methods you picked to solve (A) and (B)

. **Figure 5.10: Item 4 on inverse of polynomials**

It was important for students to work out correct solutions based on the nature of an integrand. Emphasis was, therefore, placed on the justification of choices of approaches which students provided. The following rubric was used in scoring responses:

Score	1	2	3	4	5
Indicator	No justification for the technique chosen and a wrong technique used.	No justification for the technique, correct technique used but with principle errors	No justification for the technique, correct technique used solved correctly	Justification for the correct technique chosen, with computational errors in integration	Justified the choice for a technique and solved integrals correctly.
Number of students		1	4	1	1

Table 5.4: Scoring rubric for Item 4

The majority of students chose the correct techniques to solve these integrals but did not provide justification for their choices of an approach. Instead, the tendency was to state the approach adopted for each of the given integrals. Menzi, for example, provided correct solutions to both integrals (A) and (B) but his response to 4.2 was:

Menzi: *“(A) I integrated by completing the square; (B) I integrated by partial fractions”*

Menzi did not state reasons for choosing a particular approach in each case. He knew the approaches to be used though and correctly applied them. He was allocated a score of three and placed in an action-process level of APOS based on the solutions he provided.

The only student who provided some correct justification for the choice of a technique used and a correct solution was Simo. He based his justification on the nature of the integrand. The following is his written response:

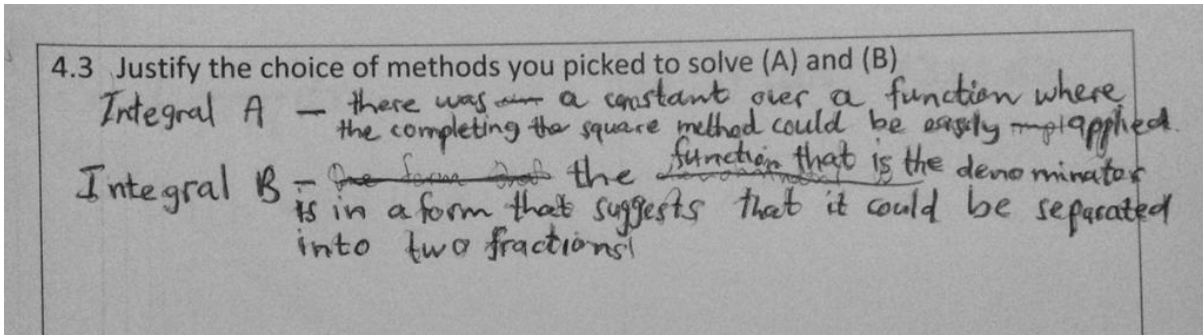


Figure 5.11: Simo's response to Item 4.2

Simo mentioned that the denominator in (B) was “in a form that suggests that it could be separated into two fractions”. He, therefore, correctly used integration by partial fractions to solve (B) and provided a correct solution. With regard to integral (A), Simo mentioned the use of completing the square but did not provide compelling reason for identifying it as an appropriate approach. He, nonetheless, provided a correct solution to the integral. Simo was awarded a score of four for his attempt to base his justification for the choice on the nature of the integrand. He seemed to have encapsulated the importance of analysing the form of an integrand when deciding on an approach to use.

Item 4 was deemed useful in extracting traits towards answering the research questions and was, therefore, not changed for the main study.

5.3.3 The schema for integration

Items that required overall schema for integration were numbered 5.1 to 5.6 and consisted of various techniques of integration. Students needed to be able to identify compositions of function and hence the reversal of the chain rule for differentiation.

Items 5.1 to 5.6

Determine the following integrals:

$$5.1 \quad \int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$$

$$5.4 \quad \int \tan^{-1}(3x) dx$$

$$5.2 \quad \int \frac{x-1}{\sqrt[3]{x^2}} dx$$

$$5.5 \quad \int \ln(x^2 - x + 2) dx$$

$$5.3 \quad \int \frac{e^x}{e^x+1} dx$$

$$5.6 \quad \int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}} dx$$

Figure 5.12: Item 5 on schema for integration

In scoring the sub-items, a distinction was made based on the nature of integrands. Sub-items 5.1, 5.3 and 5.6, for example, were an inverse of the chain rule and, therefore, a separate rubric was developed for them. Items 5.4 and 5.5 required the use of integration by parts while item 5.2 involved some algebraic simplification before the actual integration. Table 5.5 below indicates the scoring rubric used for the first cluster of items.

Score	1	2	3	4	5
Indicator	No attempt or a complete principle error.	Identified the product but could not interpret the composition.	Interpreted the composition well, integrated the inside function in the composition	Interpreted the composition well, errors in integrating the outside function	Provided a completely correct solution.
Number of students(5.1)	1	1	2	2	1
Number of students(5.3)	2	1		1	3
Number of students (5.6)	2	1	2	1	1

Table 5.5: Scoring rubric for Items 5.1, 5.3 and 5.6

In order to shorten the chapter, and since the purpose of this phase was to validate the research instrument, I give detailed analysis of two responses to sub-item 5.1

only. Results on the rest of the items are discussed at length in the second phase of the study.

Only one student solved item 5.1 correctly. The majority identified the product and the composition of functions within the integrand correctly but could not interpret it. Menzi, for example, realised that the integrand consisted of the product of $\frac{1}{\sqrt{x}}$ and $\sin\sqrt{x}$. He also realised that $\frac{d}{dx}(\sqrt{x})$ results in $\frac{1}{\sqrt{x}}$ (multiplied by some constant). Proceeding, he failed to interpret the composition $\sin\sqrt{x}$, instead he integrated the first power of $\sin\sqrt{x}$ to obtain $\frac{(\sin\sqrt{x})^2}{2}$. Menzi displayed some mental constructions as he correctly identified functions multiplied within an integrand. He seemed to be lacking in the conceptual understanding of interpreting and differentiating composite functions. He knew procedures for differentiating and integrating power functions and was subsequently scored a two for this item.

Simo, on the other hand, provided an example of an item that was scored four for Item 5.1. He identified the product and the composition correctly. He started by differentiating $x^{\frac{-1}{2}}$, but then changed and worked out the derivative of $x^{\frac{1}{2}}$ as can be seen in Figure 5.11 below. During the follow-up interview, Simo indicated that it was an error for him to work out the derivative of pointing at the derivative of $x^{\frac{-1}{2}}$ when he said:

Researcher: I see here you wrote out two derivatives, why?

Simo: *Oh..., here Mam...this is out Mam..(pointing at the derivative of $x^{\frac{-1}{2}}$). I made a mistake, kodwa ngabona ukuthi irong le function, ama exponents awafani (and I realised that this function is wrong, the exponents are not the same). Bekufanele ngi differentiate le (I was supposed to differentiate this one). I should have cancelled this (still pointing at the derivative of $x^{\frac{-1}{2}}$).*

After realising his mistake, Simo had then correctly differentiated the inside function of the composition. He further integrated the outside function correctly, by noting that $\int \sin x \, dx = -\cos x + C$. Simo's error emanated when he failed to "balance

constants” whereby he wrote a $\frac{1}{2}$ as a coefficient for the integral instead of a two. He seemed to have missed out the fact that he had introduced the $\frac{1}{2}$ and, therefore, needed to nullify it by multiplying by a 2.

Simo seemed to have engaged with this item at a process-object level. He could correctly identify which function to integrate and which to differentiate in the type of an integral in sub-item 5.1. Simo seemed to have encapsulated the chain rule, a differentiation technique underlying this type of an integral. He realised that if an integrand is a product of two functions, the inside function is differentiated while the outside integrated. He worked out the respective derivative and integral correctly. Simo seemed to be correctly reflecting on the procedure that gave rise to the integrand and was able to deconstruct such transformations. He did not display all mental constructions expected though, as he failed to address the constant that arose when he differentiated the inside function. His mental constructions placed him at the process-level of APOS.

Although most of the students in the pilot group could not provide correct solutions to item in this section of the activity sheet, the items were kept for the main study without change. These items required students to have developed a schema for the techniques of integral.

5.4. Conclusion

In this chapter I presented results on the first round of survey conducted to validate the research instrument for the study. I presented findings on the evoked conceptual meaning through symbols and through graphic representations of integrals as induced by Items 1 and 2 of the activity sheet. I further presented analysis of how students conceptualised integration when applying techniques like integration by parts, integration by partial fractions and integration using standard integrals. I ended with an analysis of students’ schema of integration and the ability to apply techniques of integration.

In most cases, written responses and interviews revealed that students operated at procedural level and had difficulty in justifying the approaches they had used. When persuaded to provide explanation, students used terms that were inaccurate, for

example, partial integration instead of integration by parts. This first phase of the study also revealed a need to delineate items that could be a distraction for the concept under investigation, such as the sketching of a semi-circle graph. Most items had responses which acknowledge the mental constructions desired in the genetic decomposition for this exploration, therefore, there were no major changes implied for the research instrument.

In the next chapter, I provide in-depth analysis of students' responses to the activity sheet, together with interviews of students who were selected on the basis of their written responses.

CHAPTER 6

FINDINGS AND ANALYSIS

6.1 *Introduction*

This chapter presents findings from data collected through activity sheets and follow-up interviews in the main study. Arnon *et al.* (2014) list interviews, written questions, classroom observations and textbook analysis as some of the data collection strategies to employ when conducting APOS-based research. They maintain that candidates to be interviewed may be selected based on “their responses to a written questionnaire or a previously administered exam, instructor feedback, or a combination of these criteria” (p. 95). In this study, candidates were selected based on their responses to the activity sheet resulting in those responses forming the basis for interview questions.

Activity sheets contained items that were structured to evoke concept images and concept definitions for an integral that students possessed. During interviews, I asked students to expound on their written responses in order to indicate even better their level of understanding. At all times the objective was to test the validity of the proposed genetic decomposition for integration and to identify mental structures exhibited by students. The report, therefore, will be presented thematically, providing supporting data across all sources. Excerpts from both written responses and interview transcripts will be provided to support interpretations made.

The next section after this introduction presents the evoked concept definition of an integral that students displayed. I discuss the evoked concept definition, firstly, through the symbols of integration and secondly, through graphical representations or visualisation. I then discuss the evoked concept images through symbolic representation. I firstly present concept images in relation to the nature of an integrand and secondly, in relation to the techniques of integration. Lastly, I provide the conclusion to this chapter.

6.2 Evoked concept definition of an integral

According to Rasslan and Tall (2002) all mathematical concepts, besides the primitive ones, have definitions. Nonetheless, students seldom refer to definitions when working with such concepts. Instead, responses are based on formulated mental pictures, properties and processes associated with the concepts. Students, who cannot define integrals meaningfully, have exhibited difficulties in interpreting problems that require the application of integration in wider contexts (Habineza, 2010; Orton, 1983b; Rasslan & Tall, 2002). Also for procedures to be learnt with meaning, they should be linked to concept image and definition of the concept under study (Jojo, 2011).

6.2.1 Evoked concept definition through the symbol of integral

The first two items of the questionnaire, Item 1 and Item 2, were designed to evoke students' concept definition of an integral. According to Pettersson and Scheja (2008), "the notion of integral can be seen as harbouring two concepts: the concept of definite integral (Haripersad et.al., 2008) and the concept of indefinite integral (anti-derivative)" (p. 772). While this is correct, conception of an integral as a continuous summation is fundamental for any mathematics student (Orton, 1983b). The following section reports on the possible concept definitions for definite and indefinite integrals that were displayed by the participants.

The task for Item 1 was as follows:

Item 1.

In your understanding, what is the difference between $\int f(x)dx$ and $\int_a^b f(x)dx$?

1.1 $\int f(x)dx$ means.....

1.2 $\int_a^b f(x)dx$ means.....

Students' responses to Item 1, which was aimed at evoking students' conception of an integral through symbolic notion, have been classified according to an approach used by Rasslan and Tall (2002) as follows:

6.2.1.1 Item 1.1: The meaning of $\int f(x)dx$

Students' answers	Frequency
Category I: $\int f(x)dx$ means an integral of $f(x)$ with respect to x .	4
Category II: II_a: $\int f(x)dx$ means integrate the function $f(x)$ with respect to x .	16
II_b: A specific function in place of $f(x)$ is provided. Example: Let $f(x) = y$; $\int y \cdot dx$; $= \frac{y^2}{2} + C$	1
Category III: Completely incorrect answer, (1/22) or No response Example: "I think here you just differentiate the $f(x)$. I mean you put $f(x)$ and differentiate $f(x)$."	1

Table 6.1: Categorisation of students' answers to Item 1(A)

All participants responded to Item 1.1. Category I contained all responses that presented $\int f(x)dx$ as an outcome or an entity. In this category $\int f(x)dx$ was viewed as a resulting solution from the process of integrating $f(x)$. A typical response in this category was, for example, " $\int f(x)dx$ is the indefinite integral of $f(x)$ " or "the integral of $f(x)$ with respect to x ". Responses classified into Category II presented $\int f(x)dx$ as a command or an instruction to carry out an operation. The majority of students responded by stating that " $\int f(x)dx$ means integrate the function $f(x)$ with respect to x ". One participant, who was also classified into to Category II, chose to provide a symbolical example as a means of defining $\int f(x)dx$ by putting $f(x) = y$ and then finding the anti-derivative.

In both these categories, the concept definition of an integral that was evoked was that of an anti-derivative. One participant, for example, explicitly demonstrated the notion of an anti-derivative when he wrote: " $\int f(x)dx$ means the integral of function $f(x)$ with respect to x . In other words the function $f(x)$ was differentiated now you have to bring back before it was differentiated". This is in line with what other researchers have observed (Habineza, 2010; Orton, 1983b; Rasslan & Tall, 2002). In analysing the understanding and misconceptions exhibited when students learn

integration, Orton (1983b) concluded that students struggle with conceptualising integration as a limit of a sum or area and as a consequence, some teachers resort to teaching integration as just an anti-derivative without any underlying reasons. Similarly, when interviewing student teachers on their conceptualisation of a definite and indefinite integrals, Habineza (2010) observed that the understanding displayed by his participants did not include underlying concepts of integrals but demonstrated orientation towards the anti-derivative.

This assertion was further confirmed during the follow-up interviews that were held with some of the participants. When student Sabelo, whose response fell into category II (see Figure 6.1), was asked to elaborate on his written response, the following dialogue ensued:

- L1: NJN:** What do you understand by integration?
- L2: Sabelo:** *So uma usebenzisa igama elithi integration, I don't think kwi basic English like uma nikhuluma. Kuqhamuka imaths nje kahle kahle.*
[Translation: When you use the word integration I don't think of basic English, like when you speak. I only think of Mathematics]
- L3: NJN:** What do you think of in mathematics?
- L4: Sabelo:** *Like ngiya understander ukuthi ama integration kahle kahle ahlukile, so...*
[Translation: Like, I understand that techniques of integration basically are different...]
- L5:** *So kahle kahle kuqhamuka isign ye integration, then i- function then i-*
- L6:** *instruction ukuthi integrater ngayiphi like inhlobo ye integration, like integration by parts.*
[Translation: So basically, I think of the sign of integration, then the function, then the instruction regarding the technique to apply, like integration by parts].
- L7:** *What I know is that when you integrate you are doing the inverse of differentiation.*

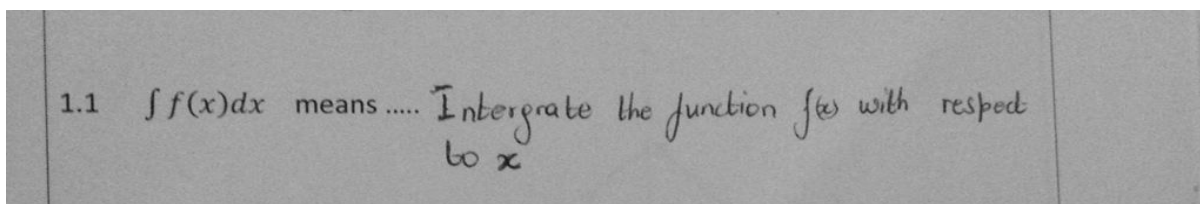


Figure 6.1: Sabelo's written response to Item 1(A)

From the interview, I observed that Sabelo referred to “the sign of integration” (L5). Here he was referring to the symbol ‘ \int ’. He further referred to an “instruction” regarding the technique of integration to be used (L6). Sabelo seemed to make sense of his mathematics via external cues. His conception of integration seemed to be confined to performing precise procedures or techniques of integration. This means that he had an action conception of the mathematical entity $\int f(x)dx$, as the symbol served as an external stimulus that triggered a reaction in Sabelo’s conception. As to the meaning of integration, in L 7 Sabelo said “*What I know is that when you integrate you are doing the inverse of differentiation*”. This statement tells that Sabelo regarded integration as a command to work out an anti-derivative for the given integrand.

The following extract from an interview with student Sbonelo, whose response fell into category I (see Figure 6.2), also emphasised the conception of an integral as an anti-derivative:

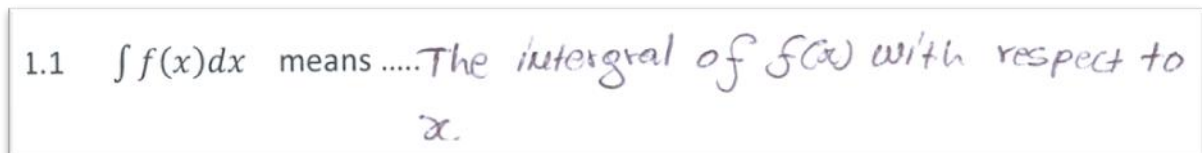
L8: NJN: What do you understand by integration?

L9: Sbonelo: *If you differentiate it’s like you are going forward and if you integrate it’s like.*

L10: *you are reversing what you have differentiated. The integral is a vice versa of differentiation.*

Extract 6.1: Interview with Sbonelo

Sbonelo’s notion of integration entails doing something and that is reversing differentiation (L10). His written response to the meaning of $\int f(x)dx$ had been as follows:



1.1 $\int f(x)dx$ meansThe integral of $f(x)$ with respect to x .

Figure 6.2: Sbonelo’s response to item 1(B)

What was noticed in Figure 6.2 was that I had classified the written response as an outcome but the interview showed that the student referred to the symbol as denoting a procedure. This, therefore, implies that the majority of students did infer

the entity $\int f(x)dx$ as an instruction to carry out an operation. The symbol ' \int ' served as an external cue for the operation to be carried out, which is to determine the anti-derivative for the given integrand.

6.2.1.2 Item 1.2: the meaning of $\int_a^b f(x)dx$

One student did not respond to this item and one gave a completely incorrect answer. The remaining 20 students referred to the procedure of integration of $f(x)$ and their responses can be classified as follows:

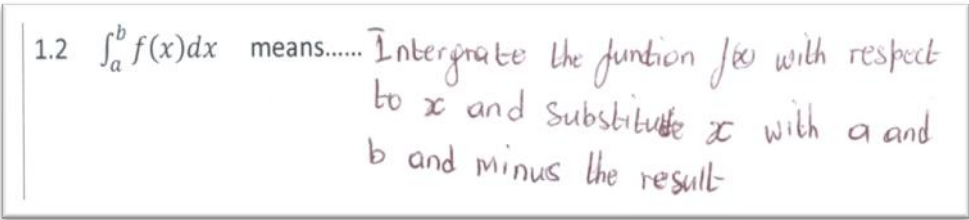
Students' answers	Frequency
<p>Category I: Integral of $f(x)$ from a to b</p> <p>I_a: Some reference to the Fundamental Theorem of calculus (FTC).</p> <p>Example 1:</p>  <p>Figure 6.3: Example of response to Item 1.2</p> <p>Example 2: Solution by a student who provided a specific function to show the FTC read as follows.</p> $\{ \text{Let } f(x) = y; \int_a^b y dx = \left[\frac{y^2}{2} \right]_a^b + C = \left[\frac{(b)^2}{2} \right] - \left[\frac{(a)^2}{2} \right] + C \}$	7
<p>I_b: No reference to the Fundamental Theorem of calculus</p> <p>Example: "the integral of $f(x)$ and is bounded from a to b"</p>	12
<p>Category II: Area bounded by $f(x)$ and the x-axis from a to b</p>	1
<p>Category III: Incorrect answer/no response</p>	2

Table 6.2: Categorisation of students' answers to Item 1(B)

From the distribution above, only one student referred the definite integral to the area under the graph. The majority cited “the integral” or “to integrate” in their definition (Category I). Responses classified as belonging category I_a were those that made reference to the FTC in some way as shown in Fig 6.3, while responses in categories I_b cited integration with no reference to the FTC. The distinction between categories I_a and I_b was actually blurred since when interviewed, even students who had not referred to the FTC showed thinking along that line.

In Example 2 of Category I_b of Table 6.2, the student decided to use an example to explain the meaning of $\int_a^b f(x)dx$. She chose $f(x) = y$. The solution she presented though depicted the use of y in a general form, $f(x) = y$, as a specific function of y . She presented an integral of a linear function of y oblivious of the meaning of a dx in the symbol. This implied that she had a poor concept image of function in terms of the association between the variables. She also inserted a constant of integration while working with a definite integral. She, therefore, had not conceptualised $\int_a^b f(x)dx$ as an object yet.

When asked to expatiate on his response, Muzi, who had not explicitly mentioned the FTC in his written response but had written: “*Integrate that is being bounded by a and b* ” stated the following:

L11: Researcher: What do limits mean in $\int_a^b f(x)dx$?

L12: Muzi: *The first place, just ignore the limits and putting them outside the brackets, then do your calculations. Then in the last step use the limits by opening the brackets*

L13: *and substitute the limit b to the first bracket then minus then substitute the limit a .*

L14: Researcher: What is that value that you get giving you?

L15: Muzi: *It gives you... if I am not mistaken, it's a gradient.*

Extract 6.2: Interview with Muzi

This extract showed that the notion of the FTC was evoked in Muzi. Embedded in his outline in L13: “*Substitute the limit b to the first bracket then minus then substitute the limit a*” is the procedure students follow when they apply the FTC. Muzi made no link between the integral and the area bound by the function. This was further confirmed by his assertion in L15 that the final value obtained would be giving a gradient. The hesitation and the use of the words, “*if I am not mistaken*”, might be interpreted as an indication of the lack of underlying conceptual understanding of evaluating a definite integral. Muzi relied on an algorithm to evaluate the integral and can therefore be classified as still operating at an action level.

Briefly, students’ conception of an integral was mainly procedural and the invoked concept definition was that of an integral as an anti-derivative. The notion of an area in a case of a definite integral, was mentioned by only one student and even then, it was presented as an alternate conception to that of a “bounded integral”.

6.2.2 Evoked concept definition through visualisation

6.2.2.1 Item 2.1

Item 2.1 provided situations that required students to link visual representation of graphs with the meaning of integration. In Item 2.1.1 they were asked to draw a graph of $y = \sqrt{25 - x^2}$ a semicircle with radius of five units and centred at the origin, and thereafter, in Item 2.1.2 they were asked to relate $\int_0^5 \sqrt{25 - x^2} dx$ to the graph they would have drawn. Students were not asked to evaluate the definite integral.

One student did not attempt this item. Fourteen students drew an incorrect graph instead of a semicircle and only seven drew the required graph. The following table summarises their responses to Item 2.1:

Students' answers	Frequency
Category I: Reference to the area	
I_a : Drew a correct graph	2
I_b : Drew an incorrect graph	3
Category II: No reference to the area	
II_a : Drew a correct graph	5
II_b : Drew an incorrect graph	11
Category III: No response	1

Table 6.3: Categorisation of students' answers to Item 2.1

It was concerning to realise that the majority of students at this level could not readily draw the graph of a semicircle. From the distribution above, five students made reference of the definite integral $\int_0^5 \sqrt{25-x^2} dx$ to the notion of an area. I will then focus my analysis on their interpretation of $\int_0^5 \sqrt{25-x^2} dx$ in relation to the graph drawn. The first respondent that I analysed is Max who gave the following as a response to Item 2.1:

A relationship is that $\int_0^5 \sqrt{25-x^2} dx$ wants integral of an area from a to b but $y = \sqrt{25-x^2}$ is a semicircle. So the second one wants half of the semicircle.

Max's response belonged to category I_a and he displayed complete conception of the definite integral as an area, as well as the significance and meaning of the limits of integration.

Max, firstly, referred to the definite integral denoted by the symbol $\int_0^5 dx$. He further referred to the procedure of integration which means that he had interiorised the

action, from the external stimulus $\int_0^5 dx$, to a process of being able to carry out a procedure mentally. He had further encapsulated this process as an object when he referred to the definite integral as an area. Max displayed a completely mathematically correct response and we observed that his knowledge of functions played a pivotal role in enhancing this correctness. I observed that having at least an object conception of functions is a prerequisite for success when working with techniques of integration.

The following, on the other hand, is an excerpt from an interview with Sbonelo, whose response fell into category I_b :

L16: NJN: Then what is the relationship between the drawn graph and the given integral?

L17: Sbonelo: *I did not know this thing I wrote here. (Sbonelo had written this “in this formula we take the area of a positive value of x”).*

L18: NJN: What does this thing gives you? (*meaning the definite integral $\int_0^5 \sqrt{25 - x^2} dx$).*

L19: Sbonelo: *We can draw the graph of the semicircle and then in the equation we substitute with the values of x they gave us.*

L20: NJN: In the graph, what does this integral define?

L21: Sbonelo: *I don't understand.*

Sbonelo had drawn an incorrect graph but the explanation he had written, as reflected in L17, meant that his response was classified as belonging to Category I_b . His verbal answers though, displayed a different conception. Firstly, he stated that he did not know the thing he had written (L17). Secondly, in L19 he made reference to substitution without indicating the process of integration first. It therefore could not be assumed that he was referring to substituting the given values of x into an integrated expression, a notion which students commonly use in relation to the FTC. Lastly, he stated explicitly that he did not understand what a definite integral defined in relation a graph drawn.

Sbonelo had not interiorised both the concept of a function, in this case a semicircle and the process of integration. He was aware of the action of substituting limits but he appeared to be overlooking the process of first finding the integral. Sbonelo was operating at an action conception level since his reasoning was based mainly on an algorithm (L19) with an intention to evaluate the integral.

This inclination to evaluate the definite integral was also displayed by Mvelase who responded to this item as follows:

2.1.2 What is the relationship between $\int_0^5 \sqrt{25-x^2} dx$ and the graph you have just drawn?

$$\int_0^5 \sqrt{25-x^2} dx = \int_0^5 (5-x) dx$$

$$= \int_0^5 5 dx - \int_0^5 x dx$$

$$= [5x - \frac{x^2}{2}]_0^5 = [5(5) - \frac{(5)^2}{2}] - [5(0) - \frac{(0)^2}{2}]$$

$$= 12,5$$

the values of my graph are positive and also my integral gives me a positive answer and I think that is the relationship between those two things.

6.4: Student Mvelase's response to item 2.1.2

Mvelase's response showed that his application of the FTC was correct. He displayed a process conceptualisation of the FTC. However, he did not have a process conception of the algebraic entities. In simplifying $\sqrt{25-x^2}$ he distributed the square root over subtraction. Also noticeable is the fact that Mvelase did not realise that $25-x^2$ is a different of two squares whose factors are $(5-x)$ and $(5+x)$, hence $(5-x)$ does not repeat. This awareness could have enabled him to realise that distributing a square root over subtraction was not a correct mathematical procedure. This error was disastrous given that this was a post school engineering mathematics student. The explanation he gave based the required relationship on signs of numbers. Although Mvelase had interiorised the FTC, he did not possess a similar prowess with regard to the practical meaning of integration.

6.2.2.2 Item 2.2

Item 2.2 sought to further elicit students' ability in the use of integration to find the area. This item included both the concept definition of an integral as an oriented area and the basic techniques for integration. The table below shows the categorisation of students' responses to this question:

Students' answers	Frequency
Category I: Evaluated a definite integral for the interval $[-1; 2]$	
<i>Category I_a</i> : Correct solution with notion of area reflected	8
<i>Category I_b</i> : Correct integration but no notion of area	3
<i>Category I_c</i> : Errors in integration but notion of area	2
<i>Category I_d</i> : Errors in integration and no notion of area	1
Category II: Evaluated a definite integral for the interval $[-1.5; 2]$	
<i>Category II_a</i> : Correct integration with notion of area	2
<i>Category II_b</i> : Errors in integration and no notion of area	3
Category III: Incorrect approach/Left blank	3

Table 6.4: Categorisation of students' answer to Item 2.2

Of the 22 participants, 19 responded by evaluating the definite integral. The variations within these 19 students' responses were in the limits of integration used and explicit reference to the notion of area. I will start by analysing those responses that fell into Category II. As can be seen from the distribution in the table, five students put -1.5 as a lower limit when evaluating the integral. This value was the x-intercept for the graph and not the lower bound of the shaded region. This means that this group of students lacked a coherent conception of the area of the shaded region. The external stimulus that they responded to was the intercepts, with no link to limits of integration. David was one of the students from *Category II_a*. He had

displayed the notion of an area and integrated correctly, but had used wrong limits of integration. The following is an extract from an interview with him:

L22: NJN: In item 2.2, what are the boundaries of the shaded area?

L23: David: *The boundaries are -1.5 and 2.*

L24: NJN: Where does the shading start?

L25: David: *Sorry, it's -1 here.*

Extract 6.3: Interview with David

The question asked by the researcher in L22 was meant to prompt David to realise the mistake he had made in the choice of limits of integration. David had not developed a complete schema of an integral as an area. His response in L23 indicated that, although he knew how to identify the limits of integration from a graph, this knowledge had not been interiorised. It was ultimately the probing in L24 that evoked this appropriate conception in David.

Next, I present students who were regarded as possessing a notion of an integral as an area. These were students who explicitly indicated that they were calculating the area and/or presented the final answer in square units. For example, a student in *Category I_a* integrated the given linear function correctly and used the correct limits for the integral. Furthermore, their responses correctly equated the area to the definite integral and the final answer was given with appropriate units. Figure 6.5 below shows two examples of such responses.

$ \begin{aligned} \text{Area} &= \int_a^b f(x) dx \\ &= \int_{-1}^2 [2x+3] dx \\ &= \int_{-1}^2 [2x+3] dx \\ &= \left[\frac{2x^2}{2} + 3x \right]_{-1}^2 \\ &= [(2)^2 + 3(2)] - [(-1)^2 + 3(-1)] \\ &= 10 + 2 = 12 \end{aligned} $ <hr/> <p>$\therefore \text{Area} = 12 \text{ units}^2$</p>	$ \begin{aligned} \text{Area} &= \int_{-1}^2 f(x) dx \\ &= \int_{-1}^2 [2x+3] dx \\ &= \int_{-1}^2 2x dx + \int_{-1}^2 3 dx \\ &= \frac{2x^2}{2} \Big _{-1}^2 + 3x \Big _{-1}^2 \\ &= 2^2 + 3[2] - [(-1)^2 - 3[-1]] \\ &= 12 \text{ units}^2 \end{aligned} $ <hr/>
--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

Figure 6.5: Sample correct response to item 2.2

The two solutions are similar but in the second case the student used the distributive property of integration over addition whereas, in the first case, integration was done over the sum. The two examples in Figure 6.5 indicate students who, according to APOS, possessed a complete schema of an integral as an area. Firstly, they knew that they needed to integrate in order to find the area. Secondly, they correctly identified the limits of integration and applied the FTC correctly. Finally, the significance of units was encapsulated in their final answer.

The following set of data is from students who were classified as having no notion of an integral as an area. They just calculated a definite integral, with no reference to area either as a concept or in units. The following is an example of a solution, by Sabelo, which fell within this category:

$$\begin{aligned}
 & \int_2^{-1} (2x+3) dx \\
 &= \left. \frac{2x^2+3x}{2} + C \right|_2^{-1} \\
 &= x^2+3x+C \Big|_2^{-1} \\
 &= [(2)^2+3(2)] - [(-1)^2+3(-1)] \\
 &= 12 \rightarrow
 \end{aligned}$$

Figure 6.6: Sample insufficient Sabelo's response to item 2.2

This written response depicted only the evaluation of a definite integral without any reference to the area, either as a concept or in units of the answer. The two other points notable from this presentation are, firstly, the insertion of a constant of integration, C , while dealing with a definite integral. The insertion of a constant of integration indicated a lack of encapsulation of the FTC in integration. Secondly, when finally evaluating the integral (lines 4 and 5 of Figure 6.6), Sabelo did not conventionally follow the notation as implied in the first three lines of the solution. According to the FTC, the upper limit is substituted first into the anti-derivative, minus substitution by the lower limit.

In short, Sabelo responded to a question of finding the area by evaluating a definite integral but omitted essential notational elements to reflect that he was calculating

the area. He seemed to know that to find an area between a curve and the x-axis one needed to evaluate the definite integral between the given limits. Sabelo's response indicated gaps in his schema of an area as a physical quantity with units. In addition, his text displayed that he did not possess full conceptualisation of the procedure of integration as it relates to the FTC. He seemed not to attach meaning to the order of symbols used in Line 3 of his solution (Figure 6.6). As such his written response seemed not to fully represent his possessed knowledge.

This inconsistency between the written text and possessed knowledge became evident in the follow-up interview with Sabelo. He confirmed his knowledge of the fact that evaluation of the definite integral in this item was actually calculating the area as reflected in Line 27 and Line 29 below.

L26: NJN: Ok. Then I gave you this one as Item 2.2. The question was: Use integration to find the shaded area. This is what you did. Is it true that when you do this you are finding the area?

L27: Sabelo: *Ya, it is true.*

L28: NJN: So if I go back to the previous question, what is then the relationship between the given integral and the graph? (referring to Item 2.1 which was the graph of a semicircle with radius 5 and students and the integral being

$$\int_0^5 \sqrt{25 - x^2} dx$$

L29: Sabelo: *We are finding the area that is being covered by the graph.*

L30: NJN: Where?

L31: Sabelo: *Above the x-axis*

L32: NJN: What guides us on the location of the area? What tells us where the area is?

L33: Sabelo: *The minimum value ka x, which is 5 and 0, the minimum and maximum.*

L34: NJN: Where is 5 and 0 on the graph itself?

L35: Sabelo: *Here (Pointing at 0 and 5 on the y-axis)*

L36: NJN: The 5 and 0, are they the X or Y values?

L37: Sabelo: *They are the X-values*

L38: NJN: So, which area will we be looking for then?

L39: Sabelo: *This part, (pointing at the portion in the first quadrant).*

Sabelo seemed to know that a definite integral gives the area under the graph (Line 29) but still showed gaps in his conceptualisation of the limits of integration as shown in Line 33 and Line 35. According to APOS theory, Sabelo had interiorised the action of integrating into a process and could relate the definite integral to an area. He displayed difficulties with interpreting graphical representations in both items thus putting him below the action level of some essential prerequisites for evaluating a definite integral as an area.

I further distinguished between incorrect answers and incorrect approaches. An incorrect answer was when students displayed accurate conceptualisation but had errors in evaluating an integral. This could be due to errors in integration, omitting brackets when applying the FTC or use of incorrect limits for integration. Such a student was said to possess a process conception of a definite integral but still failing to apply action on the process. Sabelo, therefore, was placed at a process conceptualisation for integration.

On the other hand, an incorrect approach was where students evaluated a completely different aspect for the given function. Such students were constrained by the absence of the formula for the area. Consequently, they retrieved whatever formula they could recall and evaluated it as an area. Such a practice indicated extreme reliance on external cues but with very limited conceptualisation demonstrated. According to Dubinsky (1991b), physical or mental transformations of physical or mental objects are considered to be at action stage of APOS theory when they are reactions to stimuli which the subject perceives as external. The following is an example of such a response which was classified into Category III of Table 6.4 above:

Integration to find the shaded area.

$$\begin{aligned}
 A &= \frac{1}{b-a} \int_{-1}^2 (2x+3)^2 dx \\
 &= \frac{1}{2-(-1)} \int_{-1}^2 (2x+3)^2 dx \\
 &= \frac{1}{3} \int_{-1}^2 (2x+3)(2x+3) dx \\
 &= \frac{1}{3} \int_{-1}^2 (4x^2 + 6x + 6x + 9) dx \\
 &= \frac{1}{3} \int_{-1}^2 (4x^2 + 12x + 9) dx \\
 &= \frac{1}{3} \left[\frac{4x^3}{3} + \frac{12x^2}{2} + 9x \right] \\
 &= \frac{1}{3} \left[\frac{4(2)^3}{3} + 6(2)^2 + 9(2) \quad \frac{4(1)^3}{3} + 6(1) + 9(1) \right] \\
 &= \frac{1}{3} [10,667 + 24 + 18] - [1,333 + 6 + 9] \\
 &= \frac{1}{3} [52,667 - 16,333] \\
 &= \underline{12,11 \text{ units}^2}
 \end{aligned}$$

Figure 6.7: Mpho's work: sample of an incorrect response to item 2.2

In the example reflected in Figure 6.7, Mpho evaluated either the mean value of the squares of $f(x)$. Notably integration procedure was carried out correctly, including the correct use of the FTC. A similar completely inappropriate conception was displayed by Zuzi who attempted to calculate the mean value of $f(x)$ by evaluating $\frac{1}{2-(-1)} \int_{-1}^2 (2x+3)dx$. These aspects, the mean value of the squares and the mean value of a function, are essential for engineering students and hence they are dealt with at this level of study. The use of the two aspects in this instance shows confusion, both with aspects themselves and with applications of integration.

In summary, students' responses showed a conception that is strongly biased towards algorithmic approach. In item 2.1, when asked the relationship between $\int_0^5 \sqrt{25-x^2} dx$ and the graph $y = \sqrt{25-x^2}$, 16 students gave answers that

reflected the inclination towards calculating an integral. Responses to Item 2.2 showed discrepancy between what students wrote and what they were actually thinking. This realisation makes follow-up interviews even more significant. What also emerged during interrogating this item is that some students were able to revert to Item 2.1 and correctly define $\int_0^5 \sqrt{25 - x^2} dx$ as an area bounded by the graph of $\int \sqrt{25 - x^2} dx$ and the x-axis.

6.3 Evoked concept images through symbolic representation

Items 3 and 4 were designed to explore students' concept images evoked when actually doing integration. Consequently, there was a strong emphasis on techniques of integration for these items. These items were not examining how students define an integral but what informs the choices students make regarding integration techniques to apply at any given time.

6.3.1 The nature of the integrand

In Item 3, students were asked to endorse the use of the technique of integration by parts to solve a given integral, following which they were asked to solve the integral. This is one of the most useful techniques of integration in integral calculus. Its important applications include: integrating differentials which include products, logarithms and inverse trigonometric functions (Fromhold, 2005). In applying the formula for integration by parts, which is $\int u dv = uv - \int v du$, the choice for a 'u' is informed by understanding functions appearing in the integrand. In particular, 'dv' must be integrable. The task for Item 3 was, therefore, as follows:

Item 3

A student asked to solve the integral $\int x e^{\frac{x}{2}} dx$ decided to use integration by parts and chose $e^{\frac{x}{2}}$ for a 'u'.

- 3.1. Was this choice of a 'u' appropriate?
- 3.2. Please support your answer to 3.1.
- 3.3. Now provide a solution for the same integral:

Figure 6.8: Item 3 of the Activity sheet

This item required the encapsulation of integrals for both elements in the integrand, namely, x and $e^{\frac{x}{2}}$ in order to split the integrand correctly. Although both x and $e^{\frac{x}{2}}$ are integrable, putting $dv = x$ would mean subsequently determining $v = \int x dx$ resulting in a higher power of x . Students' responses to Item 3 were classified as follows:

Students' answers	Frequency
Category I: Correct solution provided	
Category I_a : Reference to the nature of functions when choosing a "u"	1
Category I_b : Explanation for the choice of a "u" is purely algorithmic	3
Category II: Incorrect solution provided/Blank	
Category II_a : Correct choice for "u" but errors in integration	12
Category II_b : Incorrect choice of "u"	3
Category II_c : Blank	3

Table 6.5: Categorisation of students' answers to Item 2.2

Sixteen of the respondents made correct choice for a 'u' but fifteen gave algorithmic reasoning for such a choice. Their reasoning was mainly procedural and depended on the types of functions as a stimulus. The following are examples of the reasons provided:

- *Because if you do integration by parts you have 3 priorities to choose from, the first is $\ln x$, 2nd x^n and 3rd e^x . Therefore you see that this student does not follow this steps that why his/her answer will be wrong.*
- *Because our priority is starting by \ln , x , e^x ...therefore x comes first then e^x . This means that x must be equated to 'u'.*

The following is an extract from an interview with Bongani, whose written response fell into Category II_a :

L40: NJN: Why is the choice for 'u' not correct in this item?

L41: Bongani: *Because when you want to first prioritise by starting with $\ln x$, x and exponential function. So here the choice is wrong, so we must choose x as our 'u'.*

L42: NJN: Why are those priorities for u the way they are?

L43: Bongani: *We do not know why \ln is the first choice, etc. We just know the priorities.*

Extract 6.4: Interview with Bongani

The admission by Bongani in Line 43 that “*we do not know why \ln is the first choice*”, indicated a purely algorithmic approach without underlying reasons for methods applied. In such a case learning is highly mechanical and it focuses on the procedure only. At this stage, Bongani displayed action conception of integrating by parts since his decision was solely based on functions appearing in the integrand, with no mathematical basis to support the choice. *Among those who solved the integral correctly, only one gave a comprehensive reflection on the behaviour of the concerned functions when they are either integrated or differentiated (Figure 6.9).*

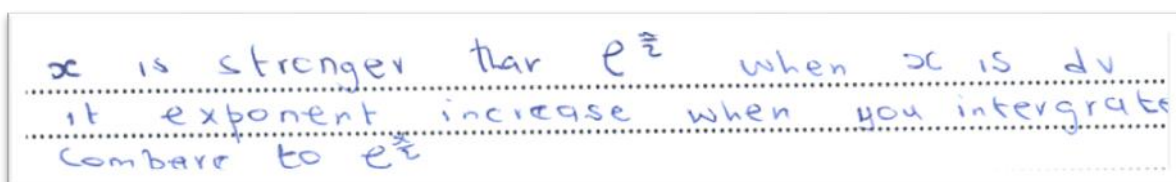


Figure 6.9: Tozi's response: Sample response to item 3.1

In brief, the majority of students relied on algorithms when doing integration by parts. They made no reference to the behaviour of functions in the integrand when either differentiated or integrated. They knew the order of priorities for a 'u' but were oblivious of a reason for such ordering. This level of comprehension placed them at the action stage of APOS theory.

This concept of how students reflected on an integrand when doing integration was further explored in Item 4. Students were given two integrals labelled (A) and (B)

and were required to state the technique to be used in solving the integrals, with justifications. Integral (A) was aimed at exploring integration by first completing the square while integral (B) required the use of partial fractions. They were then asked to provide solutions to the integrals. The following table presents categories that emerged from students' responses:

Category	Definition	Number Integral A	Number Integral B
1	Appropriate justification	8	1
1a	Correct technique and solution to the integral	5	0
1b	Correct technique but computational errors	3	1
2	Inappropriate or no justification	12	18
2a	Correct technique and solution to the integral	8	4
2b	Correct technique but computational errors	4	14
3	Incorrect technique/Blank	2	3

Table 6.6: Categorisation of students' answer to Item 2.2

Responses allocated to category 1 referred to the nature of the function as a guide for choosing a technique to use, while in category 2 there was either no reference to the nature of the integrand or no justification given. An example in the latter category was a student, in relation to integral (A), who stated that "*use completing the square in integral A because we will be able to find our A and Z to use the derived formulae in our final answer*". This student referred to the derived formulae or standard integrals but did not indicate reasons for making that choice. Ultimately, her solution to the item was classified into 2b because she omitted the constant of integration in the final answer.

The following figure (Figure 6.10) is an example of a response where Sbonelo was justifying his choices for the strategies to use in solving the integrals:

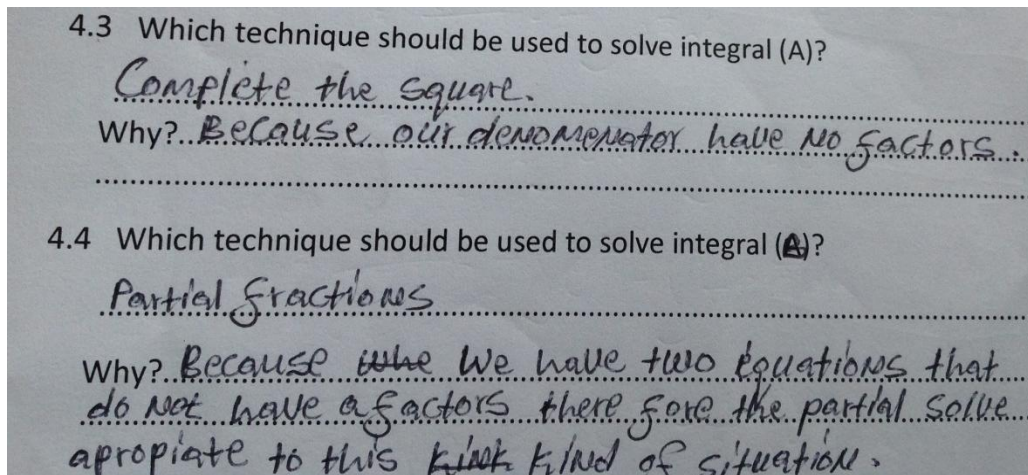


Figure 6.10 : Sbonelo's justification of strategies

Sbonelo stated that the denominator in integral (A) had no factors. In the follow-up interview with him he incorporated the aspect of the denominator being an irreducible quadratic expression. Sbonelo's solution to integral (A) was eventually classified into 1b since he omitted the constant of integration in his final answer (Figure 6.11 below).

4.5 Now, give a solution for integral (A): Please provide a reason for each step you write:

$$\int \frac{1}{x^2 - 8x + 25} dx$$

There is no factor for $x^2 - 8x + 25$ we use the Complete Square.

$$x^2 - 8x + 25$$

$$[x^2 - 8x + [4]^2 - [4]^2 + 25]$$

$$[x - 4]^2 + 9$$

$$[x - 4]^2 + 3^2$$

$$\int \frac{1}{[x - 4]^2 + 3^2} dx$$

$$\frac{1}{3} \int \frac{1}{[x - 4]^2 + 3^2} dx$$

$$\frac{1}{3} \tan^{-1} \left[\frac{x - 4}{3} \right]$$

Figure 6.11 : Sbonelo's response to Item 4.5

This response showed that the choice of the technique that Sbonelo made was informed by the nature of the fraction in the integrand. Sbonelo displayed a coherent collection of processes required to solve this type of a problem. His schema for the completion of a square enabled him to understand and ultimately integrate the derivative of an inverse trigonometric function. The omission of the constant of integration in the answer indicated shortcomings in his conception of an indefinite integral. During the follow-up interview, Sbonelo reiterated his reasoning for deciding on completing the square. He stated that the given denominator was not factorisable. When probed further, he confirmed that the technique of completing the square applies in irreducible quadratic expressions. He also realised the error made by omitting the constant of integration. According to Pettersson and Scheja (2008) and Mahir (2009), students' understanding of integrals is sometimes fragmented and

focuses mainly on procedures to solve tasks. Sabelo was placed at an entry level of the process stage in the proposed genetic decomposition.

On the contrary, the 12 students in Category 2 could not provide appropriate justification for the technique employed in solving integral (A). They all seemed to know that they needed to first complete a square but reasons for that approach included statements like “*because the numerator is 1*”, “*because our denominator is quadratic and our numerator is constant (number without any variable)*”, “*because we want to use the standard integrals*”. Nonetheless, this group succeeded to identify and apply the correct procedure for this item. Thembi, for example, wrote “*it be give us \tan^{-1} and its numerator is a constant*” as a justification for completing the square. Her solution to the item was similar to Sbonelo’s though, also omitting the constant of integration at the final answer.

A similar trend was displayed towards integral (B) that required the use partial fractions in order to solve. Nineteen out of the 22 students identified the need to use partial fraction, but could not state why they could use such a technique. Muzi, whose response was classified into 1b, stated that he would solve integral (B) by using partial fractions because “*it have two factors ...the denominator is a product*”. His explanation indicated some reflection on the nature of the integrand. The realisation that the denominator was a product was fundamental in adopting an appropriate technique. Other students gave responses such as: “*because we want to have a derivative of a function*”, “*find the value of A and B to this integral*”, “*I would say completing a square but it is a cubic function so I think using partial fractions, this is confusing*”. These responses did not reflect mental engagement with the structure of the functions involved. They were deemed inappropriate and thus classified into Category 2. Only two out of the 15 computational errors were on finding the partial fractions. The ultimate solution that Muzi presented had computational errors and is shown in Figure 6.12 below.

$$B \int \frac{10}{(x-1)(x^2+9)}$$

$$\int \frac{10}{(x-1)(x^2+9)}$$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \quad \dots L1$$

$$\text{let } x=1 \quad \dots L2$$

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

$$10 = A(x^2+9)$$

$$A = 1$$

$$x^2_0 = B+A$$

$$x_0 = -B+C$$

$$C_0, 10 = 9A - C$$

$$B = -1$$

$$C = -1$$

$$f'x = 2x$$

$$x+1 = \frac{2x}{2} + 1$$

$$\int \frac{10}{(x-1)(x^2+9)} = \int \frac{1}{x-1} + \frac{-x-1}{x^2+9} \quad \dots L3$$

$$= \ln|x-1| - \int \frac{x+1}{x^2+9} \quad \dots L4$$

$$= \ln|x-1| - \int \frac{\frac{1}{2}(2x)+1}{x^2+9} \quad \dots L5$$

$$= \ln|x-1| - \frac{1}{2} \int \frac{2x}{x^2+9} + \int \frac{1}{x^2+9} \quad \dots L6$$

$$B = \ln|x-1| - \frac{1}{2} \ln|x^2+9| + \frac{1}{3} \arctan\left(\frac{x}{3}\right)$$

Figure 6.12: Muzi's attempt of Integral(B)

Muzi displayed a complete schema for partial fractions as he correctly resolved $\frac{10}{(x-1)(x^2+9)}$ into its partial fractions which are $\frac{1}{x-1}$ and $\frac{-x-1}{x^2+9}$. He seemed not to be observing the conventional rules of mathematical writing in his response. For example, he constantly left out the 'dx' in his symbol of integration. Such an oversight might indicate a lack of fundamental perception of the symbol $\int dx$ as it is used in integral calculus. He also did not include square brackets in the right-hand side of line 3, that is, $\int \left[\frac{1}{x-1} + \frac{-x-1}{x^2+9} \right] dx$. Nonetheless, he did apply the linearity property of the integral (line 4). His computational error was algebraic and is in line 6 of his solution. When splitting the integral $-\int \frac{\frac{1}{2}(2x)+1}{x^2+9} dx$, Muzi did not distribute the negative sign as expected. The omissions he made reflected gaps in his conceptualisation of the underlying concepts applicable in this item. In addition,

Muzi's response also displayed inadequate conceptualisation of the restrictions on the domain of a logarithmic function. He repeatedly wrote $\ln(x - 1)$ instead of $\ln|x - 1|$, as a result providing a less accurate response. He also omitted the constant of integration in the final answer for this indefinite integral.

The errors displayed by Muzi, namely, the algebraic error with the signs, the non-use of the absolute value to restrict the domain of $\ln x$ and the omission of the constant of integration, were the ones common among students. In particular, students struggled to split the emerging integral $\int \frac{-1x-1}{(x^2+9)} dx$, ultimately writing it as $-\int \frac{x}{(x^2+9)} dx + \int \frac{1}{(x^2+9)} dx$ instead of $-\int \frac{x}{(x^2+9)} dx - \int \frac{1}{(x^2+9)} dx$. This mistake emanated from the failure to distribute signs when splitting the integrals.

In summary, students tended to adopt an algorithmic approach when using techniques of integration. The use of algorithms was not accompanied by the analytical knowledge of functions to integrate. In most cases, this absence of analytical approach did not hinder the presentation of a correct solution. According to APOS theory, such tendencies display integration as an action. Students knew the rules and how to apply them but there was no construction of meaning attached. For example, students could identify integrals that required either the completion of the square or integration by partial fractions with ease. They struggled to state why they were choosing a particular technique though. There were also omissions in the notation of integration, restrictions on the domain of a logarithmic function and constants of integration in an indefinite integral. Such omissions indicated gaps in concepts underlying the encapsulation of integration. Students, therefore, were still at the action stage but exhibited signs of process conceptualisation since they could readily detect the procedure to employ.

6.3.2 Techniques of integration

6.3.2.1 Reversal of the chain rule

The following section reports on facets of students' concept images revealed when using techniques of integration. Items 5.1, 5.3 and 5.6 were the reversal of the chain rule for differentiation. In order to succeed in these types of questions students needed to have a full conceptual development of a composition of functions. Item 5.2 examined how students manipulate algebraic powers when simplifying the integrand. Items 5.4 and 5.5 were included to examine how students understand the advanced application of integration by parts. The analysis will therefore be presented according to the following grouping of sub-items: 1) I will first give an analysis of responses to items 5.1, 5.3 and 5.6 then, 2) to items 5.4 and 5.5 and, 3) to item 5.2.

The following table is a presentation of categories that emerged from the responses provided by participants and distributions for items 5.1, 5.3 and 5.6:

Categories	Frequency for item 5.1	Frequency for item 5.3	Frequency for item 5.6
1. Interpreted the composition well			
1a. Provided a correct solution.	3	11	2
1b. Integrated the outside function in the composition with algebraic errors.	5		
2. Identified the product but could not interpret the composition.	2	1	4
3. No attempt or pseudo-conceptual answer.	12	10	16

Table 6.7: Categorisation of students' answers to Items 5.1, 5.3 and 5.6

From this distribution, eight students displayed the understanding of the composite function that appeared as an integrand in item 5.1. The following is an example of a response in category 1a:

Evaluate the given integrals:

$$\begin{aligned}
 5.1 \quad \int \frac{\sin\sqrt{x}}{\sqrt{x}} dx &= \int \sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \\
 \text{Let } f(x) &= \sqrt{x} \\
 f'(x) &= \frac{1}{2} \cdot x^{-1/2} \\
 &= \frac{1}{2\sqrt{x}} \\
 \int \sin\sqrt{x} \cdot \frac{1}{\sqrt{x}} &= \frac{1}{\frac{1}{2}} \int \sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \\
 &= 2 \int \sin\sqrt{x} \\
 &= \underline{-2 \cos\sqrt{x} + C}
 \end{aligned}$$

Figure 6.13: Daisy's solution to Item 5.1

Daisy seemed to comprehend the product appearing as an integrand. She also knew that she had to work out the derivative of \sqrt{x} . In line 5, Daisy seemed to be realising that she needed to divide by a $\frac{1}{2}$ in order to balance the constant that has emerged during differentiation. Daisy proceeded to write an answer that was correct ultimately.

Even though Daisy's final answer was correct, her work contained a number of errors. She omitted a dx in lines 1, 5 and 6. This omission indicates a gap in the conceptualisation of the notation and symbols of integrals. Such an error may be explained by the observation made by Tall (1992) that students have challenges in understanding and using symbols correctly in calculus.

A typical response falling into category 1b showed evidence of recognising a product, composition and the actual functions in the composition. An error occurring in the solution only related to the manipulation of constants. The following is Sabelo's response as an example of this category:

5.1 $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

~~$\int \cos \sqrt{x}$~~
 $\int (\sin(\sqrt{x})) (x^{-\frac{1}{2}}) dx$
 ~~$-\frac{\cos \sqrt{x}}{\sqrt{x}} + C$~~
 $-\cos \sqrt{x} + C$

Figure 6.14: Sabelo's response to Item 5.1

During the follow-up interview, Sabelo responded like this:

L44: NJN: What were you doing in this step? (meaning the first line that Sabelo wrote).

L45: Sabelo: *Oh ngihlukanisile, ngisuse u x ngawuletha ngaphezulu, then kwaba i product. Uma usebenzisa...mh..uya.. then... ama standard integral, gase ngasebensiza mastandard integral. Eka sine* (Oh I separated. I took x and brought it above then it became a product. If you use...then...standard integrals, I used standard integrals. The one for sine).

L46: NJN : What did you do with those?

L47: Sabelo: *Angithi Mam, i integral ithi... ifunction x iderivative* (Isn't it Mam, the integral says ..the function multiplied by the derivative).

L48: NJN: The derivative of what?

L49: Sabelo: *Of the angle.*

L50: NJN: Ok, good. What is your angle here?

L51: SabeloS: *I angle iwu.... $x^{\frac{1}{2}}$* (The angle is $x^{\frac{1}{2}}$).

L52: NJN: Good. And so what is the derivative of that angle?

L53: Sabelo: *Iwu $x^{\frac{-1}{2}}$* (It is $x^{\frac{-1}{2}}$)

L54: NJN: By the way, how do we find the derivative?

L55: Sabelo: *Si minasa ngo 1 i-exponent.* (We subtract 1 from the exponent).

During further discussions with him, Sabelo realised the mistake he was making while determining a derivative. That mistake had resulted in him omitting the constant 2 in his written solution. As a result of it, his response had been classified with students who interpreted the composition of functions well, found a correct derivative of the outside function in the composition but presented an incorrect solution due to an omission of some constant.

For Item 5.3, students who gave the correct answer without showing any workings were given the benefit of the doubt as having a conception of a composition. This assumption was made after interviews with some of the participants who had provided such a response. Music, for example, had given the following response to Item 5.3:

Figure 6.15: Muzi's response to Item 5.3

The following is an extract from an interview with Muzi, where he was requested to expand on his response:

L56: NJN: In 5.3, can you explain what you did?

L57: Muzi: *If you differentiate the denominator, it gives the numerator, so you just retain the function.*

Extract 6.5: Muzi's explanation on Item 5.3

The observation made from this statement was the apparent confidence with which Muzi tackled this item. 'So you just..' gave an impression of a statement which is common knowledge yet his proposition of '*just retain the function*' was incorrect and not what he had written as a response. Nonetheless, Muzi's written response proved that he could identify and apply correct procedure for this item. He differentiated the function $e^x + 1$ mentally and ascertained that the answer was the

numerator in the integrand. Muzi, therefore, had an object conception of integration for Item 5.3.

Item 5.6 contained an inverse of a trigonometric function and only two students were able to provide a correct solution to this item. Thirteen students presented solutions that displayed pseudo-conceptual understanding of the functions in the integrand. These were solutions with no indication of understanding of the relationship between $\sin^{-1} x$ and $\frac{1}{\sqrt{1-x^2}}$. Students displayed no conception of the fact that the latter function is a derivative of the former, even after expressing the integrand as a product. A typical example of such pseudo-conception is:

5.6 $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

$\int_0^{\frac{1}{2}} \sin^{-1} x \cdot (1-x^2)^{-\frac{1}{2}} dx$

$u = \sin^{-1} x \quad du = (1-x^2)^{-\frac{1}{2}} dx$

$du = \frac{-1}{1-x^2} dx \quad du = \left(\frac{1}{x^2}\right) dx$

$du = \frac{-1}{1-x^2} dx \quad du = \frac{1}{x^2} dx$

$\int u du = uv - \int v du$

$= \sin^{-1} x \cdot \cos^{-1} x - \int \cos^{-1} x \cdot \frac{-1}{1-x^2} dx$

$= \sin^{-1} x \cdot \cos^{-1} x + \int \frac{\cos^{-1} x}{1-x^2} dx$

$= \sin^{-1} x \cdot \cos^{-1} x + \int \frac{\cos^{-1} x}{1-x^2} dx$

Figure 6.16: Tholi's solution to Item 5.6 showing pseudo-conception

In this solution (Figure 6.16), Tholi explicitly wrote the integrand as a product of $\sin^{-1} x$ and $\frac{1}{\sqrt{1-x^2}}$, written as $(1-x^2)^{-\frac{1}{2}}$, in line 1 of Figure 6.16. The subsequent steps indicated mental processes which were not characteristic of conceptual understanding of the relationship between the two functions (Thomas, 2008). As a way out, she decided to apply integration by parts, which was also not

handled correctly. According to APOS theory, such a solution places a student below the action level of conception. The external cues present in the problem could not stimulate the conceptualisation required to handle this item.

The other type of a common error was when students had a conception of $\frac{1}{\sqrt{1-x^2}}$ being the derivative of $\sin^{-1} x$ but could not explicate the composite function in $\sin^{-1} x$ itself. The following is an example of a response by Nomsa, which fell under this category:

5.6 $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

$= \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \cdot \sin^{-1} x dx$

$= \frac{(\sin x)^{-1+1}}{-1+1}$

$= 0$

Figure 6.17: Nomsa's response to Item 5.6

When asked to explain her solution, Nomsa indicated that she did realise that of $\frac{1}{\sqrt{1-x^2}}$ is a derivative of $\sin^{-1} x$ when she said: *Since this (meaning $\frac{1}{\sqrt{1-x^2}}$) was the derivative, I ignored it and continued to integrate this second function.* Nomsa then could not view $\sin^{-1} x$ as a composite function, $(\sin^{-1} x)^1$, thus requiring the integration of the exponent $n = 1$. Instead she attempted to integrate the '-1' that designated the inverse function or the "arc sine". It was only during the interview that she recalled the actual meaning of the notation $\sin^{-1} x$, thus realising her error. Nomsa, therefore, seemed to lack the prerequisite constructs of notations for inverse trigonometric functions. As a result, she struggled to handle this integral. In addition, she displayed a weak schema for general algebra as seen in the third line of her

solution (Figure 6.17). Nomsa responded to the symbols in the item but displayed sub-minimal conceptions of both the foundational and underlying concepts involved. She had not attained the action level of conceptual development as required by APOS theory.

In brief, students' understanding of the composition of functions and the chain rule for differentiation is critical for success in working with techniques for integration. Students depicted some understanding of the chain rule but difficulties in identifying functions that have been composited (Item 5.6). Some students displayed gaps in their schema for inverse trigonometric functions, including both the notation and differentiation of these functions. Although their approach in Item 5.3 was procedural, it seemed to provide them with necessary skill and routines necessary in using techniques for integration.

6.3.2.2 Integration by parts

As stated above, Items 5.4 and 5.5 were aimed at further investigating the adeptness of students with the technique of integrating by parts. Twelve students did not respond to both of these items. Among those that responded, errors ranged from the writing of derivatives of the integrands (Figure 6.18) to discrepancies in working with coefficients (Figure 6.19). I start by analysing Figure 6.18 below which shows Themba's solutions to Items 5.4 and 5.4 respectively.

5.4 $\int \tan^{-1}(3x) dx$

$$\frac{1}{1 + (3x)^2} + C$$

$$\frac{1}{1 + 9x^2} + C$$

5.5 $\int \ln(x^2 - x + 2) dx$

$$\int \frac{2x - 1}{x^2 - x + 2} dx$$

Figure 6.18: Themba's solution to Items 5.4 and 5.5: Derivative of the integrand

In both items, Themba interpreted an integral as a derivative and hence attempted to differentiate the integrands. This type of an error is structural (Orton, 1983b) since

Themba seemed not to be grasping the principles involved in these items. In addition, the derivative that he wrote in Item 5.4 is also erroneous. He missed to differentiate the function $3x$ as required in the chain rule of differentiation. The HGD identified in Chapter 3 mentioned functions and differentiation as building blocks for the concept of integration. Themba's solution to Item 5.4 indicated gaps even with this essential underlying concept, which is differentiation.

Themba's solution to Item 5.5 indicates confusion with the symbols $\int dx$. Assuming that he viewed $\frac{2x-1}{x^2-x+2}$ as an integral of $\ln(x^2 - x + 2)$ with respect to x , he still proceeded to place such under the integral sign. The last line implied that further integration was still required. Again, Themba seemed not to have grasped the concept of integration involved in this item. In addition, he seemed to be struggling with mathematical symbols and their meaning. Themba's case was a typical example of confusion that students experienced with symbols.

Sabelo, whose solution to Item 5.4 is shown in Figure 6.19 below, displayed what Orton, (1983b) referred to as executive errors.

$$\begin{aligned}
 u &= \tan^{-1}(3x). \quad du = dx. \\
 du &= \frac{1}{1+9x^2} dx \quad v = x. \\
 \int u \cdot dv &= u \cdot v - \int v \cdot du. \\
 \int \tan^{-1}(3x) dx &= \tan^{-1}(3x) \cdot x - \int x \cdot \frac{1}{1+9x^2} dx \\
 &= \tan^{-1}(3x)x - \int \frac{x}{1+9x^2} dx. \\
 &= \tan^{-1}(3x)x - \frac{1}{18} \int \frac{18x}{1+9x^2} dx. \\
 &= \tan^{-1}(3x)x - \frac{1}{18} \ln|1+9x^2| + C
 \end{aligned}$$

Figure 6.19: Sabelo's solution to Items 5.4: Errors in constants

In line 2 of his solution, Sabelo wrote the derivative for $\tan^{-1}(3x)$ as $\frac{1}{1+9x^2}$ instead of $\frac{3}{1+9x^2}$ as is the case when applying the chain rule for differentiation. In

spite of this omission, Sabelo proceeded to work accurately until the end. He displayed good mastery of the use of symbols. In line 6, he remembered to balance the constants by inserting “18” both in the numerator and outside an integral as $\frac{1}{18}$. Sabelo seemed to have acquired the object level of conceptualisation in this instance.

In general, students struggled to retrieve knowledge skills required for Items 5.4 and 5.5 as individuals. The level of uncertainty prevailed even during the follow-up interviews where students clearly stated that they had forgotten how such integrals are handled. A different behaviour was exhibited when students were working as focus groups, resulting in an improved degree of success. In summary, most students had not interiorised the action on the technique of integration by parts.

6.4 Conclusion

In this chapter I reported on students’ responses to structured questions aimed at soliciting possessed conceptual knowledge and understanding of integration, and how such knowledge was retrieved when responding to tasks (Asiala, Brown, *et al.*, 1997). Students were presented with mathematical problem situations which included symbols of integration, graphical representation of functions and techniques of integration. Interviews conducted with selected students on some items assisted students to construct and reconstruct their mental objects regarding tasks presented.

Students’ construct of the meaning of integration was mainly procedural and based on integration as an inverse of differentiation. They could only link a definite integral to area when presented with external stimuli. Such conceptualisation of integration placed most of the participants at an action stage of APOS theory. The tendency to rely on procedure or algorithms was also displayed when applying integration techniques. As confirmed by Jojo (2011), applying the cycle of APOS theory assisted students to reflect and reconstruct manoeuvres important in integration of various functions.

The next chapter will report on findings and analysis of data collected using focus groups.

CHAPTER 7

FOCUS GROUPS ANALYSIS

7.1 *Introduction*

In this chapter I report on interactions among students during classroom activities. Classroom collaborations were designed into focus groups where students were requested to respond to a set of questions given as class exercises after they had been taught integral calculus. The teaching of calculus in this university was mainly based on paper and whiteboard. Lessons were structured in the form of discussions where the introduction was led by the lecturer and students were given tasks to investigate. Students were then given classroom activities to do as groups. These were again discussed in class and were followed by exercises to take home and do individually. The collection of this data was undertaken after the section on integration had been taught to completion.

For this stage of data collection, students were issued with the activity sheet attached here as Appendix 2B, page 216, to solve in focus groups. Students were encouraged to talk among themselves, ask questions and defend their approaches to members of their groups. Eventually, group representatives presented their group's work to the whole class. As purported by Brijlall *et al.* (2011), collaboration is when students work with others to achieve shared learning goals. Thus, the focus of this part of investigation was on conceptual understanding of integration that emerged as students were discussing among themselves.

Kitzinger (1995) maintains that data in focus group methodology is generated from communication among participants. Various mental constructions were displayed as students solved classroom exercises. In almost all the instances, students tended to use isiZulu to conduct their discussions. Occasionally, English words and expressions would be used when referring to mathematical concepts. The use of informal language when discussing mathematical concepts, such as hyperbolic functions by engineering students, was detected in the study by Brijlall (2014) when

working in groups. Adopting a Gurteen Knowledge Café model to encourage collaborative learning, Brijlall (2014) observed that students preferred a “language they found easy to understand” when explaining ideas to each other (p. 31). Such an approach facilitates both the explanations by the leader and comprehension by the rest of group members. For the ease of reading, I have translated most transcripts into English, while in few cases the translation is indicated in brackets within the extract.

It was noted that students did not always agree with each other. They sometimes misunderstood one another and at other times, provided justification for their own points of view. Due to time constraints, only sections A and B of the activity sheet were first tackled individually, while section C was solved directly as groups. The focus of my analysis is on the mathematics that prevailed in the arguments forwarded. Activities reported on were done in order to answer the third question of this study, which is: *“In what worlds of mathematical thinking do students operate when they internalise integration? How do these worlds influence the learning of the integral calculus?”*. In order to contribute meaningfully in the discussions, students were first asked to work on the items individually. They would then converge into groups to discuss their approaches to solving the items.

Calculus for engineers at a university of technology is mainly for application purposes and, therefore, students rarely use formal mathematical analysis. They are “more likely to use a combination of embodiment to imagine a situation and symbolism to model it to seek a solution” (Tall, 2007, p. 12). Nonetheless, Tall (2007) still maintains that “the categorisation of thinking into embodied, symbolic and formal is particularly appropriate in the calculus” (p. 9). The activities were, therefore, designed to elicit mental structures students possessed and the types of embodied or symbolic conceptualisation evoked when doing integration. Such mental structures would assist in analysing construction of mathematical understanding in integral calculus.

This chapter consists of five sections. After this introductory section I discuss how students seemed to link their integration to the chain rule in section two. In section

three I present conceptualisation of a negative area by students. I present perceptions of integration by parts in section four before concluding in section five.

7.2 Conception of inverse of the chain rule

In integration, students are expected to disaggregate an integrand, explain the components thereof and conceptualise any existing relationships between such components. The solution of encountered problems may require capability in the reversal of the chain rule or the use of techniques like integration by parts. The extent to which a student has interiorised an integration technique and is able to reverse processes mentally, determines the level of success when encountering tasks that require such expertise. With regard to reversing the chain rule, students were requested to work out (a) $\int \frac{\ln x}{x} dx$ and (b) $\int \frac{1}{x \ln x} dx$, in each case justifying the approach chosen.

Students were not given any hint on how to solve items but they had tables of standard integrals in their possession. Even though they were expected to work on this item as groups, they still took some time as individuals before working collaboratively. I discuss two discrete approaches which emerged with regard to solving item (b).

7.2.1 The power rule

Both items (a) and (b) required students to firstly, identify functions multiplied in the integrands and secondly, to comprehend the composite function within that product. In (a), for example, the integrand consisted of the product of $\frac{1}{x}$ and $\ln x$. Although this task could be solved using the u-substitution technique, all groups opted to use the table of standard integrals. Next, is an extract from a discussion of the solution to (a) within one of the groups, call it Group 1.

L1: Maggie: *This thing* (pointing at the integrand) *is, you see, $\frac{1}{x}$ times $\ln x$.*

L2: Roy: *So we agree that we are doing this rule?*

(At this stage Roy pointed at the first standard integral in the data sheet of their Study Guide. This standard integral is

$$\int [f(x)]^n f'(x) dx = \frac{1}{n+1} [f(x)]^{n+1} + C, n \neq -1.)$$

- L3: Roy:** *We are going to say, the answer is equal to 1 over...ehh what is "n"?...it is 1, so it is 1 plus 1,*
- L4:** *times,...what is f(x)?...it is ln x, 1 plus 1, plus C.*
- L5:** *So the final answer is 1 over 2 ln x squared plus C.*

Extract 7.1: Group 1's conversation about Item 4(a)



Figure 7.1: Group 1 discussion group

Maggie was the first to comment on the way forward in solving this problem. She correctly identified the two functions multiplied within the integrand as $\frac{1}{x}$ and $\ln x$. (line 3 of Extract 7.1). Roy then took the lead in discussing the solution further, identifying the standard integral applicable. We note that neither Maggie nor Roy explicitly categorised the functions $\frac{1}{x}$ and $\ln x$ to $f(x)$ and $f'(x)$ in line with the standard integral chosen. Roy solicited the group's endorsement by inserting leading questions such as: "what is 'n'?" and "what is 'f(x)'?" within his presentation (lines 3 and 4 in Extract 7.1). The whole group joined him in answering these "sub-questions". As such, although his voice was dominant, answers were provided in chorus form. This group ultimately presented a consensus solution as follows:

(a) $\int \frac{\ln x}{x} dx$

$\Rightarrow \int \frac{\ln x}{x} dx$

$\Rightarrow \int \frac{1}{x} \ln x dx$

$\Rightarrow \frac{1}{1+1} (\ln x)^{1+1}$

$= \frac{1}{2} (\ln x)^2 + C$

$f(x) = \ln x$
 $f'(x) = \frac{1}{x}$

Extract 7.2: Group 1's response to item 4(a)

The presentation by this group indicated that they had conceptually embodied the action of reversing the chain rule into a process. According to Tall (2007), conceptual-embodiment is when an individual's mental constructions are guided by in-depth perceptions and reflections on the nature or structure of a concept and various representations of such a concept. After tackling many tasks on integration using a variety of techniques the students in this group immediately identified the technique of integration required for this particular problem. The identification of the technique emanated from the identification of $f(x)$ and $f'(x)$ in the integrand, which was done mentally, as can be inferred from the verbal interactions in Extract 7.1 above.

In addition, they also demonstrated that they had encapsulated the chain rule into an object and could apply an action, in the form of reversal, to that encapsulated process. Writing the integrand as a product in line 2 of Extract 7.2 indicated that students perceived the nature of the integrand to be a product of two functions, thus expressing it in the exact form of a standard integral. The application of the identified standard integral further required conceptualisation of a composition within the product. In this case, it appeared that Roy figured out that $\ln x$ was the composite function with an exponent equal to 1, as stated in line 4 of both oral and written extracts (Extracts 7.1 and 7.2 above). It was also important to work out the

derivative of $f(x) = \ln x$, the inside function in the composition, checking whether the format in the standard integral was satisfied. The omission of a constant of integration in line 3 of Extract 7.2 was considered insignificant since they included it at the end.

Of further significance was the representation of the final answer, where $\ln x$ was put within brackets. Such representation indicated understanding of how a concept is to be represented. A student without that level of understanding may fail to present a correct solution to a problem of this nature. Suzan, for example, was one student in the group, who successfully conceptualised the composition within the integrand but provided $\frac{\ln x^2}{2} + C$ as her final answer. Although she understood that “the $f(x)$ was $\ln x$, and ‘ n ’ was 1”, and also knew the procedure for integrating $[f(x)]^n$, she had not conceptualised the fundamental difference between $\ln x^2$ and $\ln^2 x$. According to the standard integral the group was using, they were supposed to square $\ln x$, that is, $(\ln x)^2$. Written without brackets then $(\ln x)^2 = \ln x \times \ln x = \ln^2 x$. On the other hand, $\ln x^2$ is actually equal to $\ln(x \times x)$, which is not what the standard integral dictates. Maggie indicated the error to Suzan who then changed her answer and made it look like Roy’s.

Another group, call it Group 2, presented an incorrect answer to item 4(a). They did not show steps but simply wrote $\ln x + c$ as an answer. I noted this error when the group had already progressed to item 4(b). After they had deliberated on their solution to 4(b), I probed them on their solution to item 4(a). Pete articulated how they had separated the functions that are multiplied within the integrand and realised that $\frac{d}{dx}(\ln x) = \frac{1}{x}$. In addition though, this item required students to discern the composition in $\ln x$, which was $(\ln x)^1$. They would then be able to view the integral as the reverse of the chain rule. Group 2 seemed to have missed this critical aspect, concluding instead that $\ln x$ should remain unchanged during the integration. It was only when they were asked to provide the rule they were applying that Thabo started rejecting their solution. The whole group realised their error and

re-did the sum using the ‘u-substitution method’. Conceptualisation of the ‘u-substitution method’ is discussed in the next sub-section.

7.2.2 The ‘u-substitution method’

The same functions, $\ln x$ and x , were used in item 4(b) but were combined differently. Item 4(b) was $\int \frac{1}{x \ln x} dx$. Group 2 took time reflecting on this item, exploring various approaches to use until Sello identified the ‘u substitution method’ as appropriate. The rest of the group seemed not familiar with this approach, although it was one of the techniques that had been discussed during lessons in class. This is evident from the following discussion:

Line 1: **Thabo:** *What are we going to do? Maybe use integration by parts.*

Line 2: **Sello:** *Wait, Let us use substitution.*

Line 3: **Pete:** *Which one?*

Line 4: **Sello:** *Where you use (the rest of the group says: use ‘u’ and ‘v’) ...no, that is integration by parts. Substitution is where you convert*

Line 5: **Pete:** *You use that in differential equations*

Line 6: **Sello:** *No no no, this is integration by substitution, you don’t know it?*

Line 7: **Pete:** *Write it down*

Line 8: **Sello:** *It is not differential equations. Wait, wait, wait...*

Line 9: **Thabo:** *Show us the formula that you use.*

Line 10: **Sello:** *You substitute.....wait...it is almost like integration by parts, but it is not it exactly. You put ‘u’ equal to something, but I cannot remember well. Let us see,...*

Extract 7.3: Group 2’s conversation about Item 4(b)

Sello displayed some degree of confidence in his chosen approach. He was clear in his mind that the ‘u substitution method’ differed from integration by parts. He vehemently rejected the group’s suggestion to use ‘u’ and ‘v’, as indicated in line 4 of Extract 7.3. Sello eventually recalled how to proceed with the ‘u substitution method’ in this item. He started by splitting the integrand into a product, that is, $\int \frac{1}{x} \cdot \frac{1}{\ln x} dx$.

He then let $u = \ln x$. Differentiating, he obtained $\frac{du}{dx} = \frac{1}{x}$. He then proceeded to make dx the subject of the formula, obtaining $dx = xdu$. The next step was

substituting for $\ln x$ and dx into the integral. Sello wrote $\int \frac{1}{xu} \cdot x du$ and simplifying, the integral reduced to $\int \frac{1}{u} du$. This was a simpler integral to work out, giving $\ln u + C$ as the answer. The last step was to substitute the u in this last integral, yielding $\ln(\ln x) + C$ as the final answer.

The 'u substitution method' is used to transform an integral to another integral that is easier to work out. It is theoretically based on the chain rule for differentiation which states that $\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$. Integrating this equation yields $\int f'(g(x)) \cdot g'(x) dx = \int \frac{d}{dx} [f(g(x))] dx = f(g(x))$. Letting $u = g(x)$, thus $\frac{du}{dx} = g'(x)$, transforms the integral to $\int f'(u) \frac{du}{dx} dx = f(u)$. If we re-write $\frac{du}{dx} = g'(x)$ as $du = g'(x) dx$, the integral becomes $\int f'(u) du = f(u)$, which is a simpler integral in the variable 'u'.

Members of Sello's group displayed an action level of conceptualisation when approaching this item. Thabo, for example, asked for a formula that Sello was using (line 9 of Extract 7.3). Pete requested Sello to write down the substitution to which he was referring. These two students could only carry out the required integration by reacting to explicit external cues outlining steps to follow. Sello, on the other hand, seemed to have interiorized the action of integration into a process and was now attempting to retrieve it. This I deduce since Sello (and of course the rest of the class) confronted many tasks on integration and after working at an action level on those many tasks, Sello was able to sift the correct technique one should adopt for this particular task.

The 'u substitution method' requires a student to envision the $g(x)$ and the corresponding $g'(x)$ and identify them within an integrand. Sello ultimately recalled the procedure and executed it correctly. He was definitely at the process level of comprehension for this item. As such, he was confident enough to explain his approach to the whole class.

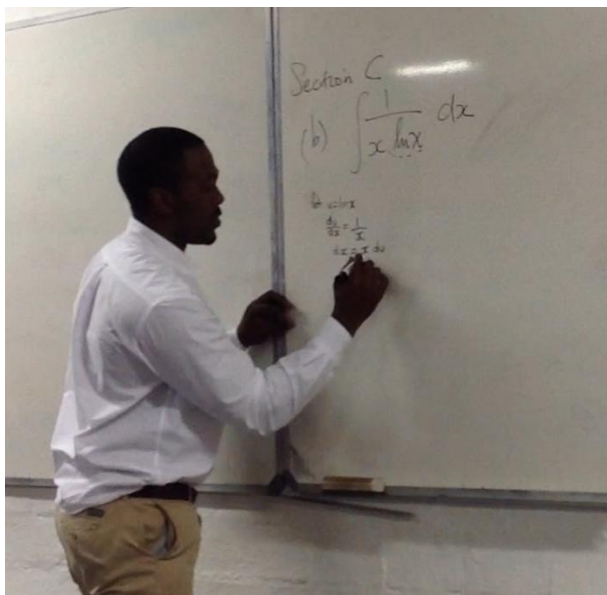


Figure 7.2: Sello's presentation to class

Sello gave an explanation for all the steps he was writing. Unlike on paper where he had first split the product in the integrand, on the board he started by doing the substitution directly. This move, however, did not create any confusion since most of the students had already attempted this item. Sello's presentations, both on paper and on the board, indicated the presence of reflections and perceptions on the properties of the integral concerned. He demonstrated greater power and precision when manipulating symbols. Sello was in the proceptual-symbolic or symbolic world of mathematical conception.

Firstly, Sello mentally identified the relationship between $\ln x$ and $\frac{1}{x}$ and was thus able to choose 'u' correctly. Secondly, he knew that in transforming the integral from the variable 'x' to the variable 'u', dx had to be expressed in terms of 'u' as well, hence he correctly determined $dx = x du$. The ultimate solution that he produced confirmed his level in computations and symbolic representations. Sello, therefore, was comfortable at both conceptual-embodied and symbolic stages of cognitive development (Tall, 2007, 2008). He could successfully reflect on the properties of the integrand and was able to perform the required manipulations of an integral.

What I observed for this task was that Sello took lead and drove the discussion to present the solution. This is one of the drawbacks of group work (Brijlall, 2014). However, the flip of the coin is that the others in the group could be peer taught into

the correct path. Otherwise, it could have been a long time before they might have arrived at any correct outcome.

Also, it must be made explicit, that Group 2, including Sello, did not attain an object stance of the technique of u-substitution. This I say due to the fact that the group could not apply actions on this process. For instance, they should have detected the domain for the expression $\ln(\ln x)$. They should have recognised that $x > 1$ for the entity $\ln(\ln x)$ to be defined or that they needed to take the absolute value of $\ln x$ in the brackets.

7.2.3 The multiplicative inverse of a function

It was interesting to note another group, Group 3, using a different approach to solving item 4(b). Zola, who was leading discussions for this item, was strongly challenged by group members when he presented the solution. Zola started by claiming that the standard integral applicable in this item was $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$. He proceeded to separate $\frac{1}{x}$ and $\ln x$ within the integrand, when he said:

Zola: Here is the rule, it says $f'(x)$ over $f(x)$ is the answer. Isn't when we split here it's going to be 1 over x times 1 over $\ln x$When we differentiate what will the derivative of $\ln x$ be?

The group then asked him to identify the $f(x)$ in the problem. When he pointed at the $\ln x$ in $\frac{1}{\ln x}$, the other students disputed that claim. An interesting dialogue ensued, with Zola attempting to defend his position:

Line 1: **Tebogo:** *It should be the whole thing as a function (referring to $\frac{1}{\ln x}$).*

Line 2: **Mike:** *It will be 1 over 1, and then 'x' will go above the line.*

Line 3: **Zola:** *We are using this rule which says $f'(x)$ over $f(x)$. Then here, $f(x)$...they say 1 over. So our $f(x)$ will be taken as....our $f(x)$ is $\ln x$.*

Line 4: **Daniel:** *1 over ln x will not yield 1 over x, it will be x because it will be 1 over 1 over x. Then x will go above the line* (Tebogo and Mike agreed with him).

Line 5: **Mike:** *There are two things here. Our f(x) should be 1 over ln x.*

Line 6: **Tebogo:** *Here is the rule, bafowethu (brothers), this first one.* (Here, Tebogo pointed at $\int [f(x)]^n f'(x) dx = \frac{1}{n+1} [f(x)]^{n+1} + C, n \neq -1$ in the tables of standard integrals).

Line 7: **Mike:** *It is not the first rule. I know the answer. It is not on the first rule.*

Extract 7.4: Group 3's conversation about Item 4(b)

Two misconceptions were displayed in this Extract 7.4. Firstly, the three students realised and agreed that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ but were struggling to conceptualise $\frac{1}{\ln x}$ as a composite function $f[g(x)]$ where $f(x) = \frac{1}{x}$ and $g(x) = \ln x$. They viewed $\frac{1}{\ln x}$ as a single entity, Lines 1 and 5 of Extract 7.4, and as such they could not detect the reversal of a chain rule in this item. With that fixation, they proceeded to differentiate $\frac{1}{\ln x}$ where the second misconception was displayed. Although the derivative of the reciprocal $\frac{1}{\ln x}$ was not required for this item, it was noted that students showed gaps in their knowledge of differentiation. In Line 4 of Extract 7.4, three students agreed that $\frac{d}{dx}\left(\frac{1}{\ln x}\right) = \frac{1}{\frac{1}{x}} = x$, evidence of an error in differentiating a multiplicative inverse of a function.

This showed that the prerequisite knowledge necessary for integration was lacking. Firstly, they could have exploited the quotient rule to arrive at the legitimate outcome or secondly, they could have used the chain rule. This indicated that within the schema for differentiation most students in this group did not display even an action level of understanding of the quotient and the chain rule. However, using the u-substitution would have led to a great deal of serious mathematics and would have highlighted an object conception of the technique of integration. As indicated in

Sello's presentation above, taking the u-substitution path, the following would have been the solution:

$$\begin{aligned}
 & \text{Let } u = \ln x \\
 & \frac{du}{dx} = \frac{1}{x} \\
 & xdu = dx \\
 & \therefore \int \frac{1}{x \ln x} dx = \int \frac{1}{xu} \times xdu \\
 & \qquad \qquad \qquad = \int \frac{1}{u} du \\
 & = \ln|u| + C \\
 & = \ln|\ln x| + C; x > 1
 \end{aligned}$$

Although correct, Zola's position seemed to be overpowered by the rest of the group. Some students seemed to have met this problem before and so they knew the answer but their arguments showed that they had not comprehended how it was arrived at. Zola continued to argue his point though, as captured in the following extract 7.5:

Line 8: **Zola:** *This thing is discrete (referring to '1' and 'lnx' in $\frac{1}{\ln x}$). You are taking it as a single entity. We are talking about the function. We know that $f(x)$ is...it is $\ln x$ (he is interrupted by members)*

Line 9: **Mike:** *How is it equal to $\ln x$?*

Line 10: **Tebogo:** *It is 1 over \ln , mfowethu (my brother)*

Line 11: **Daniel:** *If this is $f(x)$, take the whole thing as it is and deal with it. Don't separate it. I agree if we take the whole of 1 over $\ln x$.*

Line 12: **Zola:** *Here is the rule. Let us write it like this, so that it is the same as the rule (here Zola wrote:*

$$\int \left(\frac{1}{x} \div \ln x \right) dx = \int \left(\frac{1}{x} \times \frac{1}{\ln x} \right) dx = \int \left(\frac{1}{x} \times [\ln x]^{-1} \right) dx. \quad \text{These people are refuting something very simple. Now that we have separated it, the rule}$$

says 1 over $f(x)$, and the derivative of $f(x)$ will be 1 over x . Here is it. (at this stage Zola was pointing at the appropriate $f(x)$ and its derivative).

Line 13: **Daniel:** *It does not work if it is -1. This rule is out if 'n' is equal to -1*

Line 14: **Tebogo:** *Ohh...they don't consider 1 in the numerator. They just differentiate the function below. (Tebogo lingered on the page, showing hesitation. He was beginning to figure Zola's point out).*

Line 15: **Mike:** *So how do you deduce the answer?*

Line 16: **Tebogo:** *You take the function, this one already has \ln so it is $\ln(\ln x) + C$*

Extract 7.5: Group 3's further conversation about Item 4(b)



Figure 7.3: Discussion group 3

Initially Mike, Tebogo and Daniel were insisting that Zola should take $\frac{1}{\ln x}$ and not $\ln x$ as $f(x)$. In Line 9 of Extract 7.5, Mike questioned Zola's assertion in Line 8

that $f(x)$ is $\ln x$. Tebogo casually referred to ‘1 over \ln ’ in Line 8, but he was actually saying that $f(x)$ should be chosen to be ‘1 over $\ln x$ ’. Their view was supported by Daniel in Line 11 who insisted that the ‘1’ and ‘ $\ln x$ ’ in $\frac{1}{\ln x}$ were not to be separated. As indicated above, these students could not consider $\frac{1}{\ln x}$ as a composite function $(\ln x)^{-1}$. The appearance of $\frac{1}{x}$, which they knew is the derivative of $\ln x$, did not trigger them to isolate $\ln x$ in the expression $\frac{1}{\ln x}$. In addition, they were insisting on wrong differentiation of $\frac{1}{\ln x}$ and using the wrong answer to support their argument. This was seen in Line 14 where Daniel said that ‘then the ‘x’ will go over the line’, meaning $f(x)$ will not be $\frac{1}{x}$ which featured in the integrand, but ‘x’. Daniel displayed further weak conception in Line 23 but soon realised that the rule Zola was referring to actually applied when the power of $f(x)$ was -1.

Zola, on the other hand, was able to decode the integrand right from the beginning. In Line 13 he stated that $\ln x$ should be taken as $f(x)$. During discussions, Zola struggled to justify his reasoning, but eventually decided to explain symbolically as shown in Line 22. This symbolical representation of inserting a division sign assisted members of the groups in assimilating the given integrand as $\frac{f'(x)}{f(x)}$. The group ultimately rewrote the integrand in a fraction format with $\frac{1}{x}$ as the numerator and $f(x)$ as the denominator. Tebogo then volunteered to present this group’s work to the whole class.

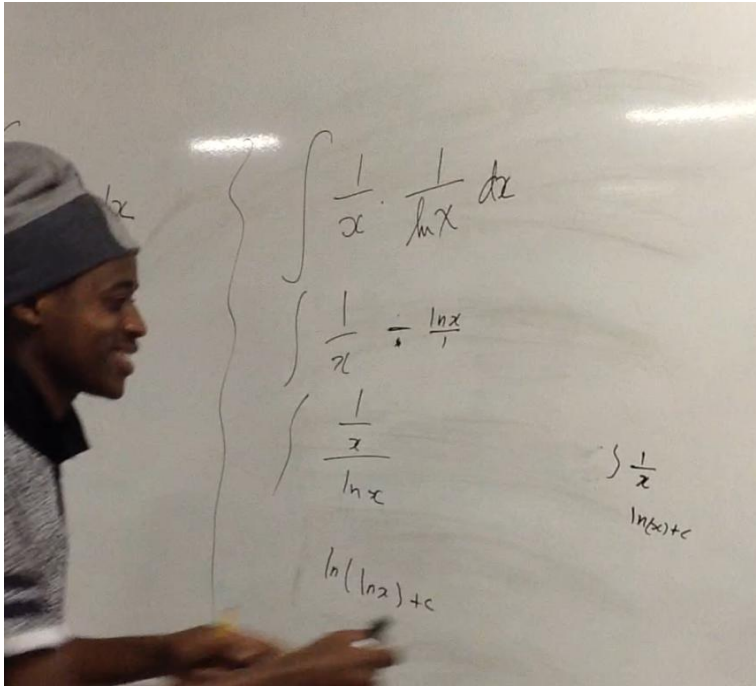


Figure 7.4: Tebogo's presentation to the class

Tebogo used the basic division symbol to assist him in rewriting the integral in the form of the standard integral. This presentation assisted him to transit to expressing the integrand in the form $\frac{f'(x)}{f(x)}$ as shown in his work (line 3 of Figure 7.4). He consistently omitted the dx throughout his working, though. This omission could be viewed as the lack of conceptual understanding of the symbol of integration. His focus was mainly on the procedure he had just learnt from his group, ensuring that he is able to arrive at the answer. To verify whether Tebogo understood what he was writing, I asked him how he moved from third line to the answer. His response was:

Tebogo: *You see, when we derive this (meaning differentiate $f(x)$), we get this* (here he was pointing at $\frac{1}{x}$). *So we just take the \ln .*

NJN: Why do you that, why do you just take $\ln f(x)$?

Tebogo: *This is according to the rule, rule 2 in the tables.*

At this stage the class implored Tebogo to write the said rule on the whiteboard, which he did. Tebogo displayed that prerequisite knowledge was vital in solving the tasks presently on hand. He displayed a process conception when reversing the

procedure of multiplication into division. His appropriate basic operations (multiplication and division) schema came to the rescue and he was able to then apply a rule from the table of standard integrals.

The tendency of students to memorise and just write answers deprives them of understanding mathematical procedures and constructions, underlying a particular approach. Mike, for example, claimed that he knew the answer but was unable to provide mathematical argument on how that answer could be obtained. As Huang (2010) observed, mathematical procedural understanding requires students to comprehend mathematical symbols and symbol syntaxes, master algorithms for solving mathematical tasks, and be able to connect the two. The omission of a dx , when Tebogo was writing the solution on the whiteboard, might also indicate a lack of understanding of the syntax in $\int dx$. Students seemed not to realise that omitting a dx was similar to uttering an incomplete sentence since the variable for integration was not indicated.

This tendency of not comprehending constructions underlying the use of $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$ was also displayed by Simo in another group when they were discussing the solution to the integral $\int \frac{e^x}{e^x+1} dx$. One of the group members, Thembi, presented the following as a solution:

The image shows a whiteboard with handwritten text. On the left, the integral $\int \frac{e^x}{e^x+1} dx$ is written. Below it, the solution $\ln(e^x+1) + C$ is written. To the right of the integral, the text "Group B" is written and underlined. The handwriting is somewhat blurry.

Figure 7.5: Thembi's solution to the group

Thembi wrote this answer very swiftly, resulting in the other members of the team showing discontent towards the provided answer. The following conversation then ensued:

Line 17: **Ruth:** *What do you do when you have the integral of derivative over a function?*

Line 18: **Simo:** *You simply say \ln of the derivative,... \ln of the function*

Line 19: **Ruth:** *So, what do we have here?*

Silence, then Thembi provides her explanation:

Line 20: **Thembi:** *Here the denominator is a function of the derivative. The $e^x + 1$ is the... function of the derivative. The derivative is e^x . When you differentiate $e^x + 1$ it's gonna give you e^x , it is the same as our numerator so in that, case it was supposed to be $\ln(e^x + 1) + C$*

Line 21: **Simo:** *We should continue integrate it because we simplify ..if you can put C la(here) , it seems like the function ends there. We should take this as a function (referring to $\ln(e^x + 1)$) and see.. differentiate it, and you can still integrate that function as well.*

Line 22: **Thembi:** *My friend this our function, is $e^x + 1$, our prime is gonna be e^x , right. When you differentiate this, it's gonna give you e^x , right? which is the same as the derivative. Meaning when you have something like this (at this point Thembi writes $\int \frac{f'(x)}{f(x)}$).....what are doing? This is the same as the \ln of $f(x)$, right. Which is, in this case it's gonna be a \ln of $(e^x + 1) + C$. OK?*

Line 23: **Simo:** *Ohh... only,... ya....Ok.*

Extract 7.6: Conversation on $\int \frac{e^x}{e^x+1} dx$

Ruth seemed to comprehend what Thembi had done and decided to explain to Simo through asking questions. In Line 18, Simo answered that if one is integrating a

quotient of a derivative and the function, “*you simply say....ln of the function*”. Simo appeared to agree with what had been done. Thembi had ‘simply written’ $\ln(e^x + 1) + C$ as an answer. However, Simo’s assertion that $\ln(e^x + 1)$ should be further differentiated and then integrated (Line 21), indicated deeper flaws in his comprehension of the procedure followed. Thembi then provided a breakdown of the solution (Line 22), outlining it step-by-step. Simo ultimately understood that the written answer was final, not requiring any further working out. It could be inferred that Simo was not even at the action stage of conception in so far as the application of this rule is concerned. This I gather since Thembi presented the solution step by step for Simo to understand the mathematical processes involved here. The reluctance contained in Line 23 was interpreted as indicating that Simo was still struggling to fathom Thembi’s explanation.

On the other hand, Thembi seemed to have interiorised the rule that she was talking about and wrote down the answer by omitting the intermediate steps of identifying the function and the derivative. She had done all of those steps mentally. It was only when questions were raised by Ruth and Simo that she indicated the other details which she had not written. She, therefore, demonstrated a process conception of the rule and its application. Thembi displayed entrenched symbolic conceptual development for this item. Nonetheless, I noted that in Line 20, Thembi struggled to verbally state whether $e^x + 1$ was a function or a derivative in the given integrand. She stated that ‘*the denominator is a function of the derivative*’, instead of the numerator being a derivative of the denominator. This gap could be assigned to weak conceptual embodiment of the concerned functions. It could also be assigned language incompetency, since her subsequent procedural explanation was correct.

7.3. Conception of integration by parts

To explore students’ conceptual understanding of the use of integration by parts, they were given carefully selected tasks. Next I discuss students’ attempts of some of them. One of the tasks required students to evaluate $\int \ln(x^2 - x + 2) dx$. They were not told which technique to use.

This problem provides a case where the integrand does not feature in the table of standard integrals. At this level of study students had dealt with the derivative of $\ln x$ and knew that it was $\frac{1}{x}$. Other functions in the same category as this one are inverse trigonometric functions like $\sin^{-1} x$, where $\frac{d}{dx}(\sin^{-1} x)$ is known to be $\frac{1}{\sqrt{1-x^2}}$, while the integral of $\sin^{-1} x$ is not readily known. Such a problem requires the use of integration by parts to solve. Xola, who was working as a pair with Lwazi, readily identified the technique applicable to this problem. He could not explain much about his choice and instead chose to lead his partner through the solution. The following is the conversation they had:

Line 24: **Xola:** *This is gonna be integration by parts. We say “u” will be $\ln(x^2 - x + 2)$ and “dv” will be “dx”. Do it. Use $\ln x^2$,*

Line 25: **Xola:** *Ya, write this as “u” and “dv” will be “dx”. We will get at the end but let’s give it a try.*

Line 26: **Lwazi:** *Hey, I am not sure about this!*

(At this stage Lwazi proceeded to differentiate $\ln(x^2 - x + 2)$)

Line 27: **Xola:** *No no no, It will be $2x$, du will be $2x-1$...No no my friend, if you are integrating this you write.....,*

Line 28: **Lwazi:** *Differentiation, we are not integrating. If you differentiate this, what is the answer?*

Line 29: **Xola:** *Yes differentiating I agree. Let me write it. It will be $2x-1$ over $x^2 - x + 2$*

Line 30: **Lwazi:** *If I am saying this, 1 over $x^2 - x + 2$, times $2x-1$, am I wrong if I say so.....?*

Line 31: **Xola:** *Well, it is the same, now continue. Write, $dv = dx$ and therefore $v=x$ because there is a 1 here and the integral of 1 is “x”. Then go to the formula:*

Line 32: **Lwazi:** *I am not sure about this bra..*

Extract 7.7: Conversation between Lwazi and Xola

Lwazi seemed to know that when using integration by parts, the “u” should be differentiated in order to determine the “du”. As by his confession in line 26, Lwazi’s answer was a mere response to the procedure of integrating by parts that he knew. He, nonetheless, displayed conceptual-embodiment of the chain rule for differentiation. He defended his approach when Xola stopped him as he was writing out the derivative of $\ln(x^2 - x + 2)$.

Although Xola displayed efficiency in choosing the suitable procedure to use for this task, an example of compression of aspects into thinkable concepts according to Tall (2007), the above extract reveals some gaps in his foundational conceptions. Firstly, he was using the terms integration and differentiation interchangeably, which is mathematically inaccurate. Lwazi corrected that error in line 28 when he emphasised that they were differentiating (the ‘u’) and not integrating. Secondly, he wanted to insist on a single representation of the derivative of $\ln(x^2 - x + 2)$. He did not wait for Lwazi to finish writing but assumed that it would be incorrect and so offered his “correct version” of the derivative. Lwazi then asked whether the derivative could not be equally written as a product of $\frac{1}{x^2 - x + 2}$ and $2x - 1$ (Line 30)? Xola continued to guide Lwazi in the use of integration by parts but struggled to manipulate the subsequent integral that arose (see Item 5.5 in Extract 7.7 below).

5.5 $\int \ln(x^2 - x + 2) dx$

$u = \ln(x^2 - x + 2) \quad \therefore \quad du = \frac{2x-1}{x^2-x+2} dx$

$\therefore \int u \cdot \frac{2x-1}{x^2-x+2} dx \quad v = x$

$= x \ln(x^2 - x + 2) - \int x \frac{2x-1}{x^2-x+2} dx$

$= x \ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2} dx$

$= x \ln(x^2 - x + 2) - \int \frac{x(2x-1)}{x^2 - x + 2} dx$

$= x \ln(x^2 - x + 2) - \int \frac{\frac{1}{2}(2x-1) + \frac{1}{2}}{x^2 - x + 2} dx - \int \frac{2x-1}{x^2-x+2} dx$

$= x \ln(x^2 - x + 2) - \frac{1}{2} \int \frac{2x-1}{x^2-x+2} dx - \frac{1}{2} \int \frac{1}{(x-1)^2+1} dx - \int \frac{2x-1}{x^2-x+2} dx$

$= x \ln(x^2 - x + 2) - \frac{1}{2} \ln(x^2 - x + 2) - \frac{1}{2} \tan^{-1}\left(\frac{x-1}{1}\right) - \ln(x^2 - x + 2) + C$

$= x \ln(x^2 - x + 2) - \frac{3}{2} \ln(x^2 - x + 2) - \frac{1}{2} \tan^{-1}(x-1) + C$

5.6 $\int_0^2 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-x^2}} \cdot \sin^{-1} x dx$

$= \frac{(\sin x)^{-1+1}}{-1+1}$

$= 0$

Figure 7.6: Lwazi and Xola’s solution on integration by parts

Xola and Lwazi applied the rule of integration by parts correctly. The first four lines of their solution indicated a proceptual-symbolism, which according to Tall (2007), is the use of symbols as thinkable concepts. Tall (2007) refers to an elementary procept as being the “combination of symbol, process, and concept constructed from the process” (p. 2). In this instance, the students moved flexibly between differentiating the ‘u’ and integrating ‘dv’ and structured their results correctly, in line with the rule for integrating by parts. They, therefore, possessed this elementary procept which enabled accuracy in working out the components of the integral.

In terms of APOS theory, this group displayed an action conception of integration by parts. They went about solving the task in a step-by-step manner. However, the error they displayed was in line 6 of Figure 7.6. They replaced the product of x and $2x - 1$ by the sum of x and $2x - 1$. This, of course, led to an incorrect solution to the task. In fact, the group showed an ineffective schema for basic algebra. They factorised $x^2 - x + 2$ to $(x - 1)^2 + 1$ in the denominators of the third and fourth terms. They, obviously, could not recall completing the square technique or the use of inspection to conclude the correct factors.

According to Gray and Tall (1994), individuals possess a precept if they have mastered the collection of elementary procepts with the same output concept. In this case, that would refer to mastery of all embedded integration techniques to solve a sum. Regarding Xola and Lizwi, they struggled to evaluate $\int \frac{2x^2 - x}{x^2 - x + 2} dx$ that arose when integrating by parts. They could not recognise equal degrees for the numerator and denominator, thus a need to first simplify by dividing the two expressions. As a result, their final solution was incorrect.



Figure 7.7: Discussions between Xola and Lwazi

Xola and Lwazi had skipped item 5.4 which was $\int \tan^{-1}(3x)dx$ but after working on item 5.5, they realised that the two problems required the same technique. What was noted was that Lwazi was more forthcoming and he voluntarily did all the writing. I only present their solution in Figure 7.8, since all their discussions were the steps that they eventually wrote down.

5.4 $\int \tan^{-1}(3x) dx$

$u = \tan^{-1}(3x) \quad dv = dx$

$du = \frac{1}{1+9x^2} dx \quad v = x$

$\int u dv = u \cdot v - \int v du$

$\int \tan^{-1}(3x) dx = \tan^{-1}(3x) \cdot x - \int x \cdot \frac{1}{1+9x^2}$

$= \tan^{-1}(3x) \cdot x - \int \frac{x}{1+9x^2} dx$

$= \tan^{-1}(3x) \cdot x - \frac{1}{18} \int 18x$

Figure 7.8: Whiteboard work emanating from discussions between Xola and Lwazi

I note that the rule of integration by parts was applied correctly, that is, a correct separation of the integrand into $u = \tan^{-1}3x$ and $dv = dx$. Nevertheless, both students could not realise the mistake when determining $\frac{d}{dx} \tan^{-1}(3x)$. They wrote $\frac{1}{1+9x^2}$ instead of $\frac{3}{1+9x^2}$, line 2 of Figure 7.8. This oversight persisted even when I tried to draw it to their attention, as can be derived in the conversation below:

NJN: There is a 3 here, what did you do with it?

Lwazi : We know that $\frac{d}{dx} (\tan^{-1}(x)) = \frac{1}{1+x^2}$, so here it is $\tan^{-1}(3x)$ so it is $\frac{1}{1+(3x)^2}$.

NJN: What if it was $\tan^{-1}(x^2)$?

Xola: It will be 1 over, in the place of 'x' we put x^2 , so it will be 'x' to the power 4.

NJN: Is that all?

Xola : *Ya...oh, there is an error here it is supposed to be times 2x . Oh, so we are supposed to say times 3.*

Xola and Lwazi responded interchangeably, an indication that both of them were equally confident of the approach they were using. Their presentation indicated that they had embodied the procedure of integration by parts. According to Jojo (2011), students operating in the action stage view a mathematical procedure as a series of individual steps. They mainly focus on producing a correct solution with less justification on how they produce such a solution. In addition to focusing on the steps, Xola and Lwazi displayed gaps in some underlying procedures required for this technique. In the second line of Figure 7.8 above, for example, having correctly set $\tan^{-1}(3x)$ as a “u”, Lwazi could not recognise the need to apply the chain rule for differentiation. Xola only realised the error when probed, and given $\tan^{-1}(x^2)$ as scaffolding.

In summary, Xola and Lwazi demonstrated an action conception of the technique of integration by parts. This we note in both Figure 7.6 and Figure 7.8. They worked, step by step, to arrive at their answer. In Figure 7.6 we note that they lacked an effective completion of a square schema in basic algebra. This impacted negatively on their solution. In Figure 7.8 they omitted to apply the chain rule for differentiation when evaluating $\frac{d}{dx} [\tan^{-1}(3x)]$. This could be as a resulting of focusing on the actual procedure of integrating by parts, thus paying less attention on the underlying procedures required. This was stressed by Brijlall and Maharaj (2015) in their study where they found similar omission by pre-service teachers when these teachers were confronted by problems involving infinite sets.

7.4 Conclusion

In this chapter I presented data from the focus groups conducted and analysis of mental structures that emerged during the discussions, using the TWM. I presented analyses of techniques employed by students when encountering integrals on the reversal of the chain rule. These techniques included the power rule, the ‘u-substitution method’ and the multiplicative inverse of a function. I ended with the

analysis of students' mental constructions when using the technique of integrating by parts.

Most of the discussions and written work indicated that students seemed to be operating in the conceptual-embodied world of cognitive development. Given integrals, students could reflect and perceive the properties, thus could decide on the correct technique to employ. The majority of presentations also revealed that most students struggled to interpret compositions, particularly in a case of an inverse function. Knowledge gaps in differentiation, symbolic notation and integration were also identified as having an effect on students' success to solve integrals.

In the next chapter, I provide an overall conclusion to my study. I will also include my pedagogical recommendations as well as areas for further research.

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 *Introduction*

In this chapter, I summarise the findings from Chapters 6 and 7. I then present conclusions from my study. The aim of the study was to analyse concept development of integration by first year engineering students at a university of technology, using APOS theory and Tall's Three Worlds of Mathematical (TWM). The primary question for the study was "How do students construct mathematical meaning when learning integral calculus?" Discussion of findings and conclusions are, therefore, arranged according to the sub-questions of the study which were:

1. What conceptual definitions do students attach to an integral?
2. What conceptual images do students exhibit when employing techniques of integration?
3. In what worlds of mathematical thinking do students operate when they internalise integration? How do these worlds influence the learning of the integral calculus?
4. What genetic decomposition can be proposed for the construction of meaning in integration?

As a result, I start by presenting findings and conclusions about the evoked conceptual definition of an integral. Next, I discuss findings and conclusions about conceptual images exhibited when students employed techniques of integration. The section after that includes findings and discussions on the worlds of mathematical thinking for integral conceptualisation. This section is followed by a modified genetic decomposition for integration. I then discuss limitations of this study and, finally, I include suggestions for future research in the teaching and learning of integral calculus.

8.2 Summary of findings and conclusions

8.2.1 What conceptual definitions do students attach to an integral?

The definition of a concept refers to words that an individual uses to explicate a particular concept (Habineza, 2010; Tall & Vinner, 1981). It denotes the mathematical meaning an individual attaches to a concept. There is agreement among the mathematics community that the definition of an integral includes notions of: integral as an area, integral as a continuous summation and integral as an anti-derivative (Habineza, 2010; Orton, 1983b; Pettersson & Scheja, 2008). In attempting to answer this question on students' conceptual definition of an integral I, therefore, used APOS theory to analyse student's conceptual meaning attached to both symbolical and graphical representations of an integral.

The common interpretation attached to symbols $\int f(x)dx$ and $\int_a^b f(x)dx$ was that of “finding an integral of $f(x)$ ” to the former and the application of the Fundamental Theorem of Calculus (FTC) to the latter. The meaning of the symbol $\int dx$ was that of an instruction to do something, hence most responses were restricted to “ $\int dx$ means I must find the integral” rather than the integral as an entity. There was limited extension to the notion of an area and, when the area was mentioned, it was mainly in the case of $\int_a^b f(x)dx$. In addition, students tended to omit the orientation of an area. In some instances, the area was presented as an alternate conception to that of a “bounded integral”. The mathematical meaning students attached to an integral was that of ‘doing something’, that something being the reversing of differentiation.

With regard to visual representation, students were presented with an equation of a semi-circle and an area under a straight line graph, with the expectation to link the areas to a practical meaning of an integral. Instead of stating the relationship between the graph of $y = \sqrt{25 - x^2}$ and $\int_0^5 \sqrt{25 - x^2} dx$ as was asked, students tended to evaluate the integral $\int_0^5 \sqrt{25 - x^2} dx$. Although students had interiorised

the Fundamental Theorem of Calculus (FTC), the presented solutions displayed that students did not possess a process conception of the power rule for integration, which they were applying to solve the integral. In some cases, students' responses showed gaps in their schema for basic algebraic entities. An example was when a student simplified $\sqrt{25-x^2}$ to $5-x$, thus proceeding with the FTC on $\int_0^5 (5-x)dx$.

As for the straight line graph, 19 out of the 22 students responded by evaluating the definite integral to determine the area under the graph. Noted misconceptions among the students' responses included the use of intercepts with the axis as limits of integration, mistakes in actual integration and no notion of the area when evaluating the definite integral. Examples of errors in the integration were flaws in the application of the FTC, such as the omission of brackets and the reversal of the order when substituting limits of integration. It was only during follow-up interviews that students realised such errors and corrected them. Additional errors were also detected in the assertions about the meaning of the quantity being evaluated. Students used expressions for the mean value or the mean of the squares of $f(x)$ to determine the area under the graph.

On the contrary, eight students' responses indicated a complete schema of an integral as an area. In this case, students evaluated the definite integral, with correct limits of integration, to determine the required area. They applied the FTC correctly and encapsulated the significance of units in their final answer.

In conclusion, the conceptual definition of an integral was mainly that of an anti-derivative. The findings indicated that students defined an integral based on procedural conception, that is, computations to perform as "directed" by symbols. They had a notion of an integral as a procedure to determine an area without the mentioning of the orientation thereof. Limits of integration in the symbol $\int_a^b f(x)dx$ did evoke the notion of an area but without orientation. The observation made was that most students possessed conception of an integral that was mainly algorithmic. As a result, they could apply the FTC with proficiency, albeit within erroneous conceptualisation of the context of application in some instances. Their construction

of meaning depended on external stimuli such as symbols for integration and graphical representation of functions. Through their ease in manipulating symbols, students displayed sufficient exposure to actions of integration. They also displayed a degree of interiorisation of these actions to a process conception thereof. They had not progressed to the object stage of development in APOS since they were still failing to apply actions on the possessed processes.

8.2.2 What conceptual images do students exhibit when employing techniques of integration?

Student's conceptual image represents the total cognitive structure associated with the concept. It includes all mental pictures and associated properties and processes and, it is entrenched in a networking of different experiences and concepts with diverse relations between them (Rösken & Rolka, 2007; Tall & Vinner, 1981). For engineering calculus at a university of technology, the cognitive structure includes such aspects as the definition of an integral, the applications of integration and the efficacy with techniques of integration. According to the hypothesised genetic decomposition for integration, the underlying network consists of both schema for functions and schema for differentiation. APOS theory was then used to look at students' responses and answers to relevant techniques of integration.

8.2.2.1 Procedure regarding integration by parts

Successful application of the formula $\int u \, dv = uv - \int v \, du$, the technique of integration by parts, depends on correct assigning of functions within integrands to a 'u' and a 'dv'. Students seemed not to connect their choices for a 'u' to the analytical knowledge of functions. There was a sole reliance on algorithms contained in textbooks. For example, out of the 16 students who chose a correct 'u' in the integral $\int x e^{\frac{x}{2}} \, dx$, only one referred to the nature of functions when justifying the choice. The rest of the students limited their arguments to the priority order as stated by Stroud and Booth (2007), that it is: (1) $\ln x$, (2) x^n and (3) e^{kx} (p. 837). Students knew the priorities for a 'u', however, the underlying conception for such

priorities was lacking. Presented interpretations were mainly procedural and were induced by types of functions in relation to the known priorities for a 'u'. There was no reference to the behaviour of functions when integrated or differentiated.

The shortcoming of not analysing the nature of functions in the integrand subsequently impacted on responses to Items 5.4 and 5.5, which also required the technique of integration by parts. Since the integrals $\int \tan^{-1}x \, dx$ and $\int \ln x \, dx$ were not included in the table of standard integrals, students could not readily discern integration by parts as an appropriate technique for these items. The tendency was not to attempt them, while other students presented errors that highlighted serious gaps in the conception of the overall integration procedure. Such gaps included the writing of derivatives of the integrands as a solution, as well as the omission of the chain rule when differentiating $\tan^{-1}3x$. Students' difficulties with handling the integrals $\int \tan^{-1}3x \, dx$ and $\int \ln(x^2 - x + 2) \, dx$ emanated from the lack of coordination of the nature of functions that are being integrated into the technique of integration by parts.

A different observation was made in the focus groups when students worked on the same problems. While they still could not provide reasons for assigning a given function as a 'u', they readily identified the need to use integration by parts and proceeded in an acceptable manner for both the tasks of Items 5.4 and 5.5. Exhibited errors, where they occurred, pertained to incorrect use of the chain rule for differentiation as well as flaws in basic algebraic procedures such as completing a square. This observation indicated the significance of diversifying interactions in the development of mathematical conceptions, as well as triangulating strategies in the collection of data. Nonetheless, students' conceptualisation seemed not to be extending beyond the step-by-step procedure of integration by parts.

The conclusion made was that students displayed integration as solely an action. There was no construction of meaning attached to the manipulation and application of the rules. Students could not, for example, justify the order of priorities for a 'u' when applying the technique of integrating by parts. Choices for a 'u' were based on the textbook stipulations and not embedded in mathematical conceptualisation of

functions involved. Consequently, students responded mechanically to mathematical challenges presented. Students had difficulty in handling tasks that could not be readily assimilated to their rules and algorithms. According to APOS theory, they were still at the action stage of conceptualisation for this aspect. Their conceptualisation depended on symbols and procedures, without internal reflections on embedded objects and processes.

8.2.2.2 Procedure regarding the reverse of a chain rule

Items 5.1, 5.3 and 5.6 of the activity sheet were focussed on the technique of reversing the chain rule when integrating. While most students were able to interpret the composition of functions in Items 5.1 and 5.3, which were $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$ and $\int \frac{e^x}{e^x+1} dx$ respectively, they depicted difficulties in handling Item 5.6 which was $\int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}} dx$. The result was that only two students succeeded to solve Item 5.6.

Regarding Item 5.1, three students presented correct responses while 12 either did not respond or presented completely incorrect responses. The common mistake in this item was the failure to balance the constant '2' which became necessary from differentiating the angle, \sqrt{x} . The derivative of \sqrt{x} is $\frac{1}{2\sqrt{x}}$ which necessitates multiplying by a '2' in order to retain the original integral. Another identified error was in the actual determination of the derivative of \sqrt{x} . Students wrote $(x)^{-\frac{1}{2}}$ instead of $\frac{1}{2}(x)^{-\frac{1}{2}}$ but were able to realise their mistakes when probed during follow-up interviews.

On the other hand, Item 5.3 seemed to invoke an object conception. Eleven students provided a correct response where all requisite manipulations were performed mentally. Follow-up interviews confirmed that students had performed the differentiation mentally and ascertained that the integrand satisfied the form $\int \frac{f'(x)}{f(x)} dx$.

Students displayed the object conception of Item 5.3 since they: (1) had interiorised the action of differentiating $e^x + 1$ into a process thus performing in

mentally; (2) had encapsulated the whole process of integrating $\frac{e^x}{e^x+1}$ and could action on it as an object. Although their approach was procedural, it seemed to be providing them with skill and routines necessary in using techniques for integration in this type of a task. This finding agrees with Rittle-Johnson and Alibali (1999) who maintained that procedural learning may improve conceptual adeptness when students start to reflect on the application of a procedure.

Some students displayed gaps in their schema for inverse trigonometric functions, including both the notation and differentiation of these functions. Responses to Item 5.6, where students failed to interpret the symbol '-1' in the notation $\sin^{-1}x$, displayed a lack of ability to interpret mathematical symbols. This observation agreed with the assertion by Huang (2010) that for the successful application of a procedure, a student should possess the comprehension of mathematical symbols as well as understand the symbol syntaxes. Students who misinterpreted the symbol '-1' in $\sin^{-1}x$ produced solutions such as $\int \sin^{-1}x, dx = \frac{(\sin x)^{-1+1}}{-1+1}$, indicating that they were interpreting '-1' as denoting a power of the sine function instead of an inverse. Consequently, the failure to understand the meaning of the index '-1' resulted in wrong attempts of Item 5.6. The lack of underlying conceptions, therefore, resulted in students failing to access the action level of cognition for this item.

8.2.2.3 Procedure regarding integration by first completing a square

Out of the 20 students who identified the completion of a square as the technique to use when handling the integral $\int \frac{dx}{x^2-8x+25}$, 13 presented the correct solution to the item. Among these 13 students, five could justify their choice for the technique while the other eight simply provided a correct solution. On the overall, 12 students could not provide the reason why they had to first complete the square in this item.

Responses indicated that students knew that they needed to complete the square. They also knew that the resulting integral would be an inverse of a trigonometric function, where the use of standard integrals applies. The response by one student

that “use completing the square....find our A and Z to use the derived formulae” confirmed the existence of such knowledge. The tendency by students was to state the approach instead of providing reasons for deciding on it. Interviews with those who had provided correct solutions confirmed assertions that students tend to focus on procedures to solve tasks, thus presenting fragmented understanding (Mahir, 2009; Pettersson & Scheja, 2008). A similar observation had been made from the participants in the first phase of the study.

Correct identification and use of a procedure indicated a level of procedural adeptness with the technique under investigation. Students could readily identify the denominator as being unfactorisable thus requiring the method of completing a square. In most cases, students could not provide reasoning behind the approach chosen. Nonetheless, they could perform the actions required for this technique. Students had also interiorised the standard integrals for the inverse trigonometric functions. Except for the omission of the constant of integration in some instances, they readily gave $\frac{1}{3} \tan^{-1} \left(\frac{x-4}{3} \right) + C$ as an answer to $\int \frac{1}{(x-4)^2+3} dx$. For this item, students seemed to be at the action-process stage of APOS theory.

8.2.2.4 Procedure regarding integration by partial fractions

Similar to integration by completing the square, a high number of students, nineteen, realised that they needed to solve integral 4(B) by using partial fractions. However, only one student referred to the form of the denominator when justifying the choice of a technique. Among the 19 students who had identified partial fractions as a suitable approach, two displayed challenges with resolving a fraction. The majority possessed the requisite skill, especially when handling a case of an irreducible quadratic expression.

Errors included the inclination to neglect imposing restrictions necessary for the domain of $y = \ln x$. Students also tended not to adhere to accurate integral notations such as not ensuring that the symbol ‘ \int ’ is always written with a ‘ dx ’, as well as the omission of a constant of integration while dealing with an indefinite

integral. There were also indications of weaknesses in the manipulation of algebraic signs. Students tended to write $-\int \frac{A+B}{c} dx$ as $-\int \frac{A}{c} dx + \int \frac{B}{c} dx$ instead of $-\int \frac{A}{c} dx - \int \frac{B}{c} dx$, resulting in an incorrect response. Students displayed an acceptable level of integrating by partial fraction even though with errors as indicated in this paragraph.

I, therefore, concluded that students were relying on procedures, without exhibiting signs of understanding. In addition, students had interiorised the procedures sufficiently and could promptly identify and apply the technique of integration by parts. Students possessed a complete schema for resolving fractions into partial fractions. With regards to using partial fractions in integration, I placed them at the action level of conceptual development with signs of advancing to process conceptualisation.

8.2.3 In what worlds of mathematical thinking do students operate when they internalise integration?

According to TWM theory, cognitive development of mathematical concepts can be classified into three worlds of knowing, namely, the conceptual-embodied, the proceptual-symbolic or the axiomatic-formal worlds (Tall, 2008). As stated before, in the conceptual-embodied world a student reflects on properties and presentations of a concept when formulating interpretations. Proceptual-symbolic world involves the shift of focus from physical meaning of symbols to mathematical concepts to think about (Tall, 2004a). The axiomatic-formal world emphasises the use of formal definitions to concepts which applies in advanced mathematics and was, therefore, not considered for this study.

When presented with integrals that required the reversal of the chain rule, students displayed mental constructions that were based on detailed discernments and considerations of the functions involved. Students could mentally identify the $f(x)$

and $f'(x)$ in the integrals $\int \frac{\ln x}{x} dx$ and $\int \frac{1}{x \ln x} dx$, hence decided on an

appropriate technique to use. Results for $\int \frac{\ln x}{x} dx$ indicated the recognition of the exponent '1' in $f(x) = \ln x$, thus correctly providing $\frac{1}{2}(\ln x)^2 + C$ as an answer.

Misconceptions with symbol syntaxes resulted in some students presenting $\ln x^2$ instead of $\ln^2 x$, an indication of weak precepts of algebraic symbols. When working in focus groups, basic errors such as the omission of constants of integration were not displayed.

The results indicated that students employed two approaches when dealing with the integral $\int \frac{1}{x \ln x} dx$. The first approach was the use of the 'u' substitution method, while other students opted viewed the given integral as an integral of a multiplicative inverse for $f(x) = \ln x$. The ability to transform integrals from the 'x' to the 'u' variable indicated proficiency with symbol manipulation. The 'u' substitution requires accurate analysis of a composition in the integrand and correct performance of differentiation. While signals of gaps were noted in handling restrictions of the domain of the function $y = \ln x$, most observed responses indicated that students were reflecting on the properties of the integrands and could also handle symbolical representations. Students were, therefore, deemed to be operating in both conceptual-embodied and proceptual-symbolic worlds of mathematical meaning.

Alternatively, the results indicated challenges in some students who opted for the approach of viewing $\int \frac{1}{x \ln x} dx$ as $\int \frac{f'(x)}{f(x)} dx$. The failure to conceive the composition in $\frac{1}{\ln x}$, which is $(\ln x)^{-1}$, as well as errors in determining $\frac{d}{dx} \left(\frac{1}{\ln x} \right)$ indicated weak conceptual-embodiment. Firstly, students could not perceive the embedded representation of a power in $\frac{1}{\ln x}$. As a result, they were persistent on

differentiating $\frac{1}{\ln x}$, instead of $\ln x$. Such an error signified poor perceptions of properties of integration. Secondly, gaps were also displayed in the underlying concepts of differentiation as students were insisting that $\frac{d}{dx}\left(\frac{1}{\ln x}\right) = \frac{1}{\frac{1}{x}} = x$, indicating a lack in the prerequisite knowledge necessary to carry out integration.

Nonetheless, results showed that the levels of operation for students were varied. The presentation and argument by Zola indicated advanced entrenching in both the embodied and symbolic worlds of thinking. When Zola could not justify his approach verbally, he opted for symbolic representation. His expression of the integral $\int \frac{1}{x \ln x} dx$ as $\int \left(\frac{1}{x} \div \ln x\right) dx = \int \left(\frac{1}{x} \times \frac{1}{\ln x}\right) dx = \int \left(\frac{1}{x} \times [\ln x]^{-1}\right) dx$ indicated an in-depth understanding of the integral. In addition, he succeeded to use the language of mathematical symbols to convey his thoughts. He was using symbols as thinkable concepts. A similar observation was made with respect to Thembi when working with $\int \frac{e^x}{e^x+1} dx$. Thembi could not state the relationship between the numerator and denominator functions verbally, but relied on symbols to explain her line of argument.

With regard to integration by parts, students displayed the ability to manipulate symbols and embedded procedures. For example, the technique of integration by parts gives rise to a 'u' and a 'dv' which require differing operations. Students managed that section of the task successfully. The tendency was to focus on step-by-step procedure to get a solution, subsequently omitting critical underlying aspects such as proper notation and correct differentiation. Students could work with symbols, the actual procedure and emerging concepts within the technique of integration by parts. They were using symbols as thinkable concepts, thus operating at a proceptual-symbolic world of mathematics learning (Tall, 2007). In short, students possessed the elementary procept for the technique of integration by parts.

Challenges observed included: (1) misconceptions with the syntax of symbols where some students expressed $\ln^2 x$ as $\ln x^2$, (2) failure to recognise embedded compositions such as $(\ln x)^1$ and $(\ln x)^{-1}$ when re-writing the integrals as $\int (\ln x \cdot \frac{1}{x}) dx$ and $\int (\frac{1}{\ln x} \cdot \frac{1}{x}) dx$ respectively, (3) sloppiness in using the symbols of integration and errors in the use of basic differentiation rules and (4), gaps in the underlying knowledge and skills such as the use of the chain rule in differentiation.

8.2.4 What genetic decomposition can be proposed for the construction of meaning in integration?

In Chapter 3, I provided a hypothesised genetic decomposition (HGD) for integration. In it I proposed that, for integration, students need to have complete schemas for functions and differentiation. For the action level of conceptualisation, I suggested that students should be proficient with the use of algorithms, including the understanding of symbols of integration. The interiorisation of steps in the algorithms is realised when a student can readily identify the most suitable technique applicable to a given task and apply it with precision. Such interiorisation results broadly from the ability to reflect on the properties of integrands involved and connect such reflections to symbols or techniques applicable.

Results confirmed that schemas for functions and differentiation were pre-requisites for learning integration in engineering mathematics. In particular, students require a complete schema for functional notation in order to be able to make a distinction between entities such as $f^{-1}(x)$ and $\frac{1}{f(x)}$, $[f(x)]^n$ and $f(x^n)$. Students should also have developed an object or process conception of a function which will enable them to decipher and reflect on fundamental features such as domain and graphical representations. Furthermore, students should possess a process conception of the composition of functions in order to be able to identify and interpret composite functions within integrands.

For the derivative schema, students should have an object level of conception for differentiation. Such conception should assist students to consider the behaviour of functions when applying techniques for integration such as integration by parts. In this case, the differentiated 'dv' denotes a process encapsulated into an object. Integrating the 'dv' will, therefore, symbolise further action on the said object. These observations led me to subsequently revise the HGD for integration as shown in Figure 8.1.

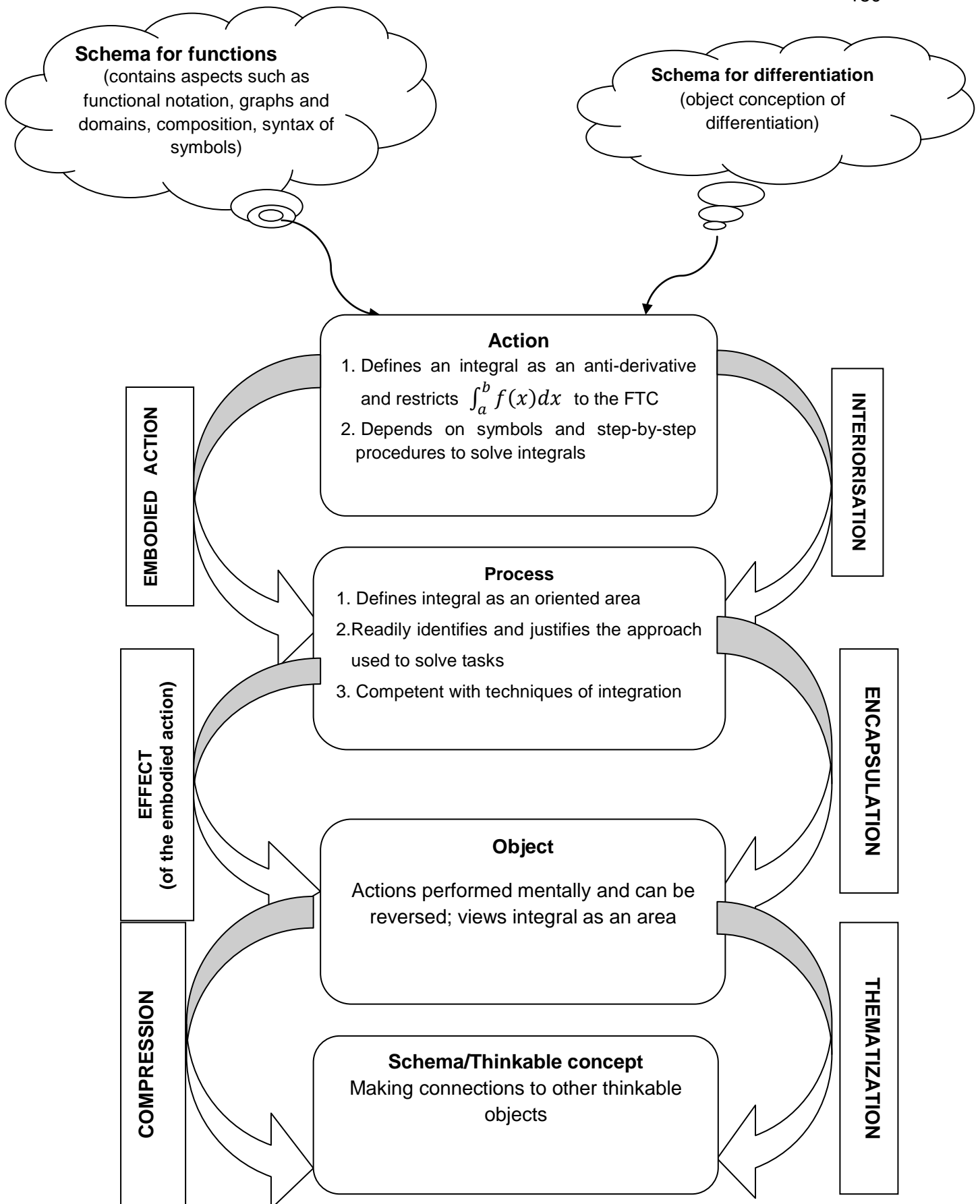


Figure 8.1 Model for the Genetic Decomposition for integration

8.3 Summary of contributions

8.3.1 Significance of findings

This study has presented mainly an APOS analysis of conceptual development when students learn integral calculus. Existing studies had provided analysis for aspects such as system of equations, vector spaces, function notation, concept of continuity and the chain rule in differentiation (Brijlall & Maharaj, 2010; Brijlall & Ndlovu, 2013; DeVries & Arnon, 2004; Jojo, 2011; Kabaël, 2011; Parraguez & Oktaç, 2010). In addition, the study has identified the proceptual-symbolic of the TWM theory as a dominant world of mathematical thinking for engineering students. Although there was evidence of conceptual-embodiment, students were mainly using symbols to formulate their thinking.

Also, the study indicated that students conceived an integral mainly as an anti-derivative. Their conception was based on algorithms and was mainly procedural. Students depended on external stimuli to construct meaning and invoke conceptual images. Students were at an action stage in the APOS levels of cognitive development. These contributions support the previous findings that students exhibit procedural tendencies in integration (Huang, 2010; Mahir, 2009; Orton, 1983b). Furthermore, the results showed that students could not define both definite and indefinite integrals thus extending the findings by Rasslan and Tall (2002).

Lastly, the study highlighted basic algebra, functions and differentiation as some of the mathematical concepts or structures fundamental for the learning of integration. As a result, students presenting weak or inaccessible such structures exhibited errors when conceptualising and tackling tasks in integration. This observation is consistent with the assertion by Cooley *et al.* (2007) on the successful construction of meaning in mathematics.

8.3.2. Recommendations

8.3.2.1 Foundational aspects in first year engineering mathematics

First year engineering mathematics should start by the revision of algebraic concepts such as exponential and logarithmic manipulations, completing a square and resolving fractions into partial fractions. In addition, properties of functions such as the notation, graphical representations and domain and range should be revised in the preliminary lectures. Appropriate instructional design should be employed to assist students attain an object level of understanding these concepts. Students should be able to apply actions on encapsulated processes of factorising algebraic expressions.

On the other hand, the object conception of all properties and graphical representations of functions will result in students operating in the object stage of integration. Students will be able to link an integral to the area concept, as well as incorporate the underlying restrictions when dealing with functions such as $f(x) = \ln x$. As a result, the recommendation is that these aspects be included as examinable content of the first year engineering mathematics.

8.3.2.2 Concept development in integration

To assist students in developing conceptual understanding beyond the action level, it is recommended that, firstly, graphical representations of areas be embedded within the introduction of the concept of integration. Students should be assisted to develop an object conception of an integral as an oriented area. Graphical representation will expose students to concepts of a negative area. Secondly, reasons for techniques adopted should be incorporated when solutions are presented. This will enhance proceptual-symbolism and build conceptual-embodiment into the formulation of thinking by students. Lastly, it is recommended that teaching strategies, such as collaborative learning, should be used when

teaching integral calculus. Students performed better when working collaboratively in focus groups than when working as individuals.

8.4 Limitations

This was a case study focusing on a sample of first year mathematics engineering students at a university of technology. As maintained by many authors, results may not be readily generalised to other groups of students or other universities (Baxter & Jack, 2008; Cohen *et al.*, 2011; Flyvbjerg, 2006). Results do, however, provide insight to students' conceptual development of integration, which may be considered when designing instructional offerings for this aspect.

Also, the sample consisted of volunteering students and data analysed was obtained from their written responses to the activity sheets, interviews held with some of them and video recordings of focus groups. The sampling method could have omitted informative cases that would have provided different perspectives to the investigation. Similarly, activity sheets were designed according to the proposed genetic decomposition for integration. Other sources of data such as responses in official test and examinations could provide other trends not realised through this set of activities.

Another limitation was the time factor. Although 23 students responded to the questionnaire, only seven could be interviewed. Some cases that had been identified to be interviewed did not honour the appointments and, since the term was approaching an end, they left for winter vacations.

Lastly, as stated by Asiala, Brown, *et al.* (1997), the genetic decomposition of a concept is not unique. It depends on the context of the study which includes students and their previous knowledge base and the researcher's expertise and experience with the concept. The genetic decomposition proposed in this study is based on a South African context of the education system. A different finding may ensue where students' prior knowledge significantly differs from the knowledge by participants in this study.

8.5 Suggestions for future research

As the case with many studies, this study identified further questions and areas for follow-up research as presented below. These questions are meant to promote reflective practices and inquiry among university mathematics lectures. The intention is that, when lecturers begin to consider how the process of knowledge construction evolves among students, they might provide instructional designs that are responsive to the needs of their classes. Certain recommendations have been made in this chapter. These recommendations are substantiated by findings from this study. The suggestion is that further justification would further clarify the stand that these recommendations advance. Some specific questions that can be explored are:

Question 1: *How do prerequisite algebraic skills affect the performance of engineering students when solving integration problems?*

This question connects the research with the first recommendation of the study. The algebraic skills we refer to are algebraic concepts such as exponential and logarithmic manipulations, completing a square and resolving fractions into partial fractions. I suggest that a quasi-empirical research method be adopted using control and experimental groups.

Question 2: *How will the prior introduction of graphical functional representation affect the conceptual understanding by engineering students of the integral concept?*

In this case the research is connected with the second recommendation of the study. The contention is that graphical representations of functions will result in students operating in the object stage of integration. Students will be able to link an integral to the area concept, as well as incorporate the underlying restrictions when dealing with functions such as $f(x) = \ln x$. Again, the suggestion is that a quasi-empirical research method be adopted using control and experimental groups.

8.6 Conclusion

I began this study with an acknowledgement of the significance of integral calculus for engineering students at a university of technology. I made the case that exploring construction of knowledge was a new trend towards addressing difficulties in students' understanding of mathematical concepts. In reviewing literature, I indicated an existing gap in research of studies, both nationally and internationally, that have explored conceptual developments in mathematics. There is no evidence of the application of APOS theory and TWM in integral calculus, and in particular, within a South African context. This study was designed to contribute some work towards filling this gap.

In the hypothesised genetic decomposition for integration, I had identified schemas for functions and differentiation as pre-requisite for a schema of integration. Results indicated syntax of symbols and algebraic algorithms as an additional schema. The proposed genetic decomposition for integration has, therefore, been modified: Students will be said to have schema for differentiation if they display process or object conception of a derivative. Such conception includes derivatives of functions such as $\ln x$ and inverse trigonometric functions and the use of the chain rule when differentiating. Regarding schema for functions, students need to display object conception of a composition of functions.

REFERENCES

- Allmendinger, P. (2002). Towards a post-positivist typology of planning theory. *Planning Theory*, 1(1), 77-99.
- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*: Springer Science & Business Media.
- Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. *Research in collegiate mathematics education*, 2(3), 1-32.
- Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1997). A framework for research and curriculum development in undergraduate mathematics education. *MAA NOTES*, 37-54.
- Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior*, 16(4), 399-431.
- Azevedo, R., Cromley, J. G., & Seibert, D. (2004). Does adaptive scaffolding facilitate students' ability to regulate their learning with hypermedia? *Contemporary Educational Psychology*, 29(3), 344-370.
- Baxter, P., & Jack, S. (2008). Qualitative case study methodology: Study design and implementation for novice researchers. *The qualitative report*, 13(4), 544-559.
- Bennett, A., Moore, T., & Nguyen, X. (2011, June). *A longitudinal study on students' development and transfer of the concept of integration*. Paper presented at the Conference & Exposition, American Society for Engineering Education, Vancouver, British Columbia, Canada.
- Berger, M. (2006). Making mathematical meaning: from preconcepts to pseudoconcepts to concepts. *Pythagoras*, 63, 14-21.
- Bergsten, C. (2008). On the influence of theory on research in mathematics education: the case of teaching and learning limits of functions. *ZDM*, 40(2), 189-199.
- Brijlall, D. (2014). Exploring the Gurteen Knowledge Café approach as an innovative teaching for learning strategy with first-year engineering students. *Groupwork*, 24(3), 26-45.

- Brijlall, D., & Bansilal, S. (2010, January). A genetic decomposition of the Riemann Sum by student teachers. In *Proceedings of the eighteenth annual meeting of the South African Association for research in mathematics, science and technology education* (pp. 131-138). University of KwaZulu-Natal, Durban, SA
- Brijlall, D., & Maharaj, A. (2009). Using an inductive approach for definition making: Monotonicity and boundedness of sequences. *Pythagoras*, 70, 68-79.
- Brijlall, D., & Maharaj, A. (2010). *An APOS analysis of students' construction of the concept of continuity of a singled-valued function*. Paper presented at the Proceeding of the 7th Southern Hemisphere Conference on Mathematics and Statistics Teaching and Learning. Stellenbosch, SA.
- Brijlall, D., & Maharaj, A. (2013). Pre-service teachers' understanding of continuity of functions in the context of assessment items. *South African Journal of Higher Education*, 27(4), 797-814.
- Brijlall, D., & Maharaj, A. (2015). Exploring pre-service teachers' mental constructions when solving problems involving infinite sets. *International Journal of Education Sciences*, 9(3), 273-281.
- Brijlall, D., Maharaj, A., Bansilal, S., & Mkhwanazi, T. Dubinsky, E. (2011, July). A pilot study exploring pre-service teachers understanding of the relationship between $0.\bar{9}$ and 1. In *Proceedings of the 17th Annual AMESA National Congress*. Witwatersrand, SA
- Brijlall, D., & Ndlovu, Z. (2013). High school learners' mental construction during solving optimisation problems in Calculus: a South African case study. *South African Journal of Education*, 33(2), 1-18..
- Brush, T. A., & Saye, J. W. (2002). A summary of research exploring hard and soft scaffolding for teachers and students using a multimedia supported learning environment. *The Journal of Interactive Online Learning*, 1(2), 1-12.
- Byrnes, J. P. (2001). *Cognitive development and learning in instructional contexts*. Boston, MA: Allyn & Bacon.
- Cano, F., & Berbén, A. B. G. (2009). University students' achievement goals and approaches to learning in mathematics. *British Journal of Educational Psychology*, 79(1), 131-153.
- Cardella, M. E., & Atman, C. J. (2004, December). *A qualitative study of the role of mathematics in engineering capstone design projects*. Paper presented at the

Proceedings of the 2004 International Conference on Engineering Education.
Kuala Lumpur, Malaysia

- Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. *CBMS Issues in Mathematics Education*, 7, 114-162.
- Carmona, G. (1996). *The concept of tangent and its relationship with the concept of derivative* (Unpublished doctoral thesis). Instituto Tecnológico de Mexico.
- Choy, S. P., Horn, L. J., Nuñez, A. M., & Chen, X. (2000). Transition to college: What helps at-risk students and students whose parents did not attend college. *New directions for institutional research*, 2000(107), 45-63.
- Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., St John, D., Tolia, G., & Vidakovic, D. (1997). Constructing a schema: The case of the chain rule? *The Journal of Mathematical Behavior*, 16(4), 345-364.
- Cohen, L., Manion, L., & Morrison, K. (2011). *Research methods in education*. London; New York: Routledge.
- Cooley, L., Trigueros, M., & Baker, B. (2007). Schema thematization: a framework and an example. *Journal for research in Mathematics Education*, 38(4), 370-392.
- Corbin, J. M., & Strauss, A. L. (2008). *Basics of qualitative research : techniques and procedures for developing grounded theory*. Los Angeles, Calif.: Sage Publications.
- Cottrill, J. (1999). *Students' understanding of the concept of chain rule in first year calculus and the relation to their understanding of composition of functions*. (Published Doctoral dissertation), Purdue University.
- Cottrill, J., Nichols, D., Dubinsky, E., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *The Journal of Mathematics Behaviour*, 15, 167-192.
- Creswell, J. W. (2002). *Educational research: Planning, conducting, and evaluating quantitative*. New Jersey: Upper Saddle River.
- Creswell, J. W. (2013). *Research design: Qualitative, quantitative, and mixed methods approaches*. London, UK : Sage publications.
- Cui, L., Rebello, N. S., Fletcher, P. R., & Bennett, A. G. (2006, April). *Transfer of learning from college calculus to physics courses*. Paper presented at the

- Proceedings of the Annual Meeting of the National Association for Research in Science Teaching. Baltimore, MD, USA.
- Department of Education. (2003). *Revised National Curriculum Statement Mathematics for Grades 10-12(general). Mathematics*. Pretoria, South Africa:
- Devries, D., & Arnon, I. (2004, July). *Solution-what does it mean? Helping linear algebra students develop the concept while improving research tools*. Paper presented at the Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education. Bergen, Norway.
- Dewi, N. R., & Kusumah, Y. (2014). Developing test of high order mathematical thinking ability in integral calculus subject. *International Journal of Education and Research*, 2(12), 101-108.
- Dubinsky, E. (1991a). Constructive aspects of reflective abstraction in advanced mathematics. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 160-202). New York: Springer.
- Dubinsky, E. (1991b). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-126). Netherlands: Springer.
- Dubinsky, E. (2001). Using a theory of learning in college mathematics courses. *MSOR Connections*, 1(2), 10-15.
- Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, 27(3), 267-305.
- Dubinsky, E., & Lewin, P. (1986). Reflective abstraction and mathematics education: The genetic decomposition of induction and compactness. *The Journal of Mathematical Behavior*. 5(1), 55-92.
- Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. *NEW ICMI STUDIES SERIES*, 7, 275-282.
- Dubinsky, E., Weller, K., Mcdonald, M. A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An Apos-Based analysis: Part 1. *Educational Studies in Mathematics*, 58(3), 335-359.

- Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior*, 32(1), 83-101. doi:http://dx.doi.org/10.1016/j.jmathb.2012.12.001
- Duffin, J. M., & Simpson, A. P. (2000). A search for understanding. *The Journal of Mathematical Behavior*, 18(4), 415-427.
- Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative inquiry*, 12(2), 219-245.
- Fromhold, A. T., Jr. (2005). Calculus Technique of Integration by Parts, Correlated with a Geometric Picture. *The Mathematics Teacher*, 99(1), 62-64. doi:10.2307/27971862
- Gibbs, A. (1997). Focus groups. *Social research update*, 19(8), 1-8.
- Gray, E., & Tall, D. (2001, May). *Relationships between embodied objects and symbolic procepts: An explanatory theory of success and failure in mathematics*. Paper presented at the PME CONFERENCE. Utrecht, The Netherlands
- Gray, E. M., & Tall, D. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for research in Mathematics Education*, 26(2) 115-141.
- Guba, E. G., & Lincoln, Y. S. (1994). Competing paradigms in qualitative research. *Handbook of qualitative research*, 2, 163-194.
- Habineza, F. (2010). *Developing first-year mathematics student teachers' understanding of the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus: An action Research Project in Rwanda*. (Unpublished doctoral thesis), University of KwaZulu-Natal, Pietermaritzburg.
- Habineza, F. (2015). A case study of analyzing student teachers' concept images of the definite integral. *Rwandan Journal of Education*, 1(2), 38-54.
- Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematical Behavior*, 25(1), 57-72.
- Hardy, N. (2009). Students' perceptions of institutional practices: the case of limits of functions in college level calculus courses. *Educational Studies in Mathematics*, 72(3), 341-358.

- Haripersad, R. (2010, July). *Deep and surface learning of elementary calculus concepts in a blended learning environment*. Paper presented at the Proceedings of the 7th WSEAS international conference on Engineering education. Corfu Island, Greece.
- Haripersad, R., Naidoo, R., Kartalopoulos, S., Buikis, A., Mastorakis, N., & Vladareanu, L. (2008, June). *Errors made by first year students in an integral calculus course using web-based learning*. Paper presented at the WSEAS International Conference Proceedings. Mathematics and Computers in Science and Engineering. Iasi, Romania.
- Heck, R. H. (2006). Conceptualizing and conducting meaningful research studies in education. In C.F. Conrad & R.C. Serlin (Eds.), *The Sage handbook for research in education: Engaging ideas and enriching inquiry* (pp. 373-392). Thousand Oaks, London, New Delhi: Sage Publications, Inc.
- Hiebert, J., & Lefevre, P. (1986). Concept and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp. 1-27). Hillsdale(NJ): Lawrence Erlbaum.
- Hoepfl, M. C. (1997). Choosing qualitative research: A primer for technology education researchers. *Journal of Technology Education*, 9(1), 47-63.
- Hofstee, E. (2006). *Constructing a good dissertation : a practical guide to finishing a Master's, MBA or PhD on schedule*. Sandton, South Africa: EPE.
- Huang, C. H. (2010, September). *Conceptual and procedural abilities of engineering students in integration*. Paper presented at the Proceedings of the Conference of the Joint International IGIP-SEFI Annual Conference. Trnava, Slovakia.
- Huang, C. H. (2011). *An exploratory study of engineering students' development of mathematical thinking*. Paper presented at the 1st World Engineering Education Flash Week, Lisbon.
- Huberman, M., & Miles, M. B. (2002). *The Qualitative Researcher's Companion*. Thousand Oaks, CA: SAGE Publications.
- Hunt, S. D. (1991). Positivism and paradigm dominance in consumer research: toward critical pluralism and rapprochement. *Journal of Consumer Research*, 18, 32-44.

- Jojo, Z. M. M. (2011). *An APOS exploration of conceptual understanding of the chain rule in calculus by first year engineering students*. (Published Doctoral dissertation), University of KwaZulu-Natal, Edgewood.
- Jojo, Z. M. M., Brijlall, D., & Maharaj, A. (2011, July). *The reliability of a research instrument used to measure mental constructs in the learning of the chain rule in calculus*. Paper presented at the Proceedings of the 16th Annual Congress of the Association of Mathematics Education of South Africa.
- Jojo, Z. M. M., Maharaj, A., & Brijlall, D. (2012). Reflective Abstraction and Mathematics Education: The Genetic Decomposition of the Chain Rule--Work in Progress. *Online Submission*.
- Jojo, Z. M. M., Maharaj, A., & Brijlall, D. (2013). From human activity to conceptual understanding of a mathematical concept. *Journal of Research in Mathematics Education*, 2(1), 77-99.
- Kabael, T. U. (2011). Generalizing Single Variable Functions to Two-Variable Functions, Function Machine and APOS. *Educational Sciences: Theory and Practice*, 11(1), 484-499.
- King, N., & Horrocks, C. (2010). *Interviews in Qualitative Research*. London, UK: SAGE Publications.
- Kitzinger, J. (1995). Qualitative research. Introducing focus groups. *BMJ: British medical journal*, 311(7000), 299-302.
- Koepf, W., & Ben-Israel, A. (1994). The Definite Nature of Indefinite Integrals. *The International DERIVE Journal*, 1(1), 115-131.
- Krefting, L. (1991). Rigor in qualitative research: The assessment of trustworthiness. *American journal of occupational therapy*, 45(3), 214-222.
- Leedy, P. D., & Ormrod, J. E. (2005). *Practical research. Planning and design* (8th ed.) Upper Saddle River, NJ: Prentice Hall.
- Mabry, L. (2008). Case study in social research. *The Sage handbook of social research methods*, 214-227.
- Maharaj, A. (2010). An APOS Analysis of Students' Understanding of the Concept of a Limit of a Function. *Pythagoras*, 71, 41-52.
- Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, 40(2), 201-211.

- Marshall, C., & Rossman, G. B. (2010). *Designing Qualitative Research*. Newbury Park, CA: SAGE Publications.
- Martin, W., Loch, S., Cooley, L., Dexter, S., & Vidakovic, D. (2010). Integrating learning theories and application-based modules in teaching linear algebra. *Linear Algebra and its Applications*, 432(8), 2089-2099.
- McCosker, N., & Diezmann, C. M. (2009). Scaffolding students' thinking in mathematical investigations. *Australian Primary Mathematics Classroom*, 14(3), 27-32.
- McKeough, A., Lupart, J. L., & Marini, A. (2013). *Teaching for transfer: Fostering generalization in learning*: Routledge.
- Meel, D. E. (2003). Models and theories of mathematical understanding: Comparing Pirie and Kieren's model of the growth of mathematical understanding and APOS theory. *CBMS Issues in Mathematics Education*, 12, 132-181.
- Merriam, S. B. (1998). *Qualitative Research and Case Study Applications in Education. Revised and Expanded from "Case Study Research in Education."*: ERIC.
- Msoni, A. M. (2011). *Mathematics 11 [Study Guide]*. Mangosuthu University of Technology, SA.
- Neuman, W. L. (2006). *Social research methods : qualitative and quantitative approaches* (6 ed.). Boston: Pearson Education, Inc.
- Nguyen, D. H. (2011). *Facilitating students' application of the integral and the area under the curve concepts in physics problems* (Published doctoral thesis), Kansas State University.
- Noor, K. B. (2008). Case study: A strategic research methodology. *American journal of applied sciences*, 5(11), 1602-1604.
- Orton, A. (1983a). Students' understanding of differentiation. *Educational Studies in Mathematics*, 14(3), 235-250. Retrieved from <http://www.jstor.org/stable/3482279>
- Orton, A. (1983b). Students' understanding of integration. *Educational Studies in Mathematics*, 14(1), 1-18.
- Parraguez, M., & Oktaç, A. (2010). Construction of the vector space concept from the viewpoint of APOS theory. *Linear Algebra and its Applications*, 432(8), 2112-2124. doi:<http://dx.doi.org/10.1016/j.laa.2009.06.034>

- Pettersson, K., & Scheja, M. (2008). Algorithmic contexts and learning potentiality: a case study of students' understanding of calculus. *International Journal of Mathematical Education in Science and Technology*, 39(6), 767-784.
- Ponterotto, J. G. (2005). Qualitative research in counseling psychology: A primer on research paradigms and philosophy of science. *Journal of Counseling Psychology*, 52(2), 126-136.
- Punch, K. F. (2009). *Introduction to research methods in education*: Sage.
- Rasslan, S., & Tall, D. (2002). *Definitions and images for the definite integral concept*. In A.D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the PME*, Norwich, UK. 4, 89-96.
- Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, 91(1), 175.
- Rösken, B., & Rolka, K. (2007). Integrating intuition: the role of concept image and concept definition for students' learning of integral calculus. *Montana Council of Teachers of Maths. Montana Maths. Enthusiast, Monograph*, 3, 181-204.
- Rubin, H. J., & Rubin, I. S. (2011). *Qualitative interviewing: The art of hearing data* (2nd ed.). London, UK: SAGE.
- Ryan, A. B. (2006). *Post-positivist approaches to research*. In: *Researching and Writing your Thesis: a guide for postgraduate students*. MACE: Mynooth Adult and Community Education, pp. 12-26.
- Samuel, M., & Vithal, R. (2011). Emergent frameworks of research teaching and learning in a cohort-based doctoral programme. *Perspectives in Education*, 29(3), 76-87.
- Scheja, M., & Pettersson, K. (2010). Transformation and Contextualisation: Conceptualising Students' Conceptual Understandings of Threshold Concepts in Calculus. *Higher Education*, 59(2), 221-241. Retrieved from <http://www.jstor.org/stable/25622177>.
- Sealey, V., (2006, November). Definite integral, Rieman sums and area under a curve: What is necessary and sufficient? In S. Alatorre, J. L. Cortina, M. Saiz, & A. Mendez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of*

- Mathematics Education*, (Vol 2, pp. 46-53). Universidad Perdagogica National, Merida, Yucatan, Mexico.
- Setati, M., & Adler, J. (2000). Between languages and discourses: Language practices in primary multilingual mathematics classrooms in South Africa. *Educational Studies in Mathematics*, 43(3), 243-269.
- Setati, M., Molefe, T., & Langa, M. (2008). Using language as a transparent resource in the teaching and learning of mathematics in a Grade 11 multilingual classroom. *Pythagoras: Teaching and learning mathematics in multilingual classrooms: Special Issue 67*, 14-25.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1-36.
- Sfard, A. (2002). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning *Learning Discourse* (pp. 13-57): Springer.
- Shepard, K. F., Jensen, G. M., Schmoll, B. J., Hack, L. M., & Gwyer, J. (1993). Alternative approaches to research in physical therapy: positivism and phenomenology. *Physical Therapy*, 73(2), 88-97.
- Siyepu, S. W. (2013a). An exploration of students' errors in derivatives in a university of technology. *The Journal of Mathematical Behavior*, 32(3), 577-592.
doi:<http://dx.doi.org/10.1016/j.jmathb.2013.05.001>
- Siyepu, S. W. (2013b). *Students' interpretations in learning derivatives in a university mathematics classroom*. Paper presented at the Proceedings of the 19th Annual Congress of the Association for Mathematics Education of South Africa, Cape Town: South Africa.
- Smith, R. T., & Minton, R. B. (2002). *Calculus: Single Variable* (2nd ed.): New York: McGraw-Hill.
- Smithson, J. (2008). Focus groups. *The Sage handbook of social research methods*, 357-370.
- Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, 30(2), 131-148.

- Stake, R. E. (1978). The case study method in social inquiry. *Educational researcher*, 7(2), 5-8.
- Stake, R. E. (2013). *Multiple case study analysis*: Guilford Press.
- Stewart, J. (2009). *Metric International Version Calculus* (6th ed.). Canada: Brooks/Cole.
- Stewart, S., & Thomas, M. O. (2007, July). *Embodied, symbolic and formal aspects of basic linear algebra concepts*. Paper presented at the PME CONFERENCE, Seoul, Korea.
- Strasheim, A., & Eiselen, R. (2011). *Overview of research methods 1*. [Lecture notes].
- Stroud, K. A., & Booth, D. J. (2007). *Engineering mathematics* (6th ed.): New York: Palgrave Macmillan.
- Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for research in Mathematics Education*, 258-276.
- Tall, D. (1985). Understanding the calculus. *Mathematics Teaching*, 110, 49-53.
- Tall, D. (1990). Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in mathematics*, 12(3), 49-63.
- Tall, D. (1991). The psychology of advanced mathematical thinking *Advanced mathematical thinking* (pp. 3-21): Springer.
- Tall, D. (1992). *Students' difficulties in calculus*. Paper presented at the ICME Working Group 3, Québec.
- Tall, D. (1993). Mathematical processes and symbols in the mind. In Z. A. Karian, *Symbolic computation in undergraduate mathematics education*. (MAA Notes 24)
- Tall, D. (1997). Functions and calculus *International handbook of mathematics education* (pp. 327-370): Springer.
- Tall, D. (1999, July). *Reflections on APOS theory in elementary and advanced mathematical thinking*. Paper presented at the Proceedings of the 23 rd Conference of the International Group for the Psychology of Mathematics Education, Haifa, Israel.
- Tall, D. (2002). *Using Technology to support an Embedded Approach to Learning Concepts in Mathematics*. Paper presented at the First Coloquio de Historia e Tecnologia no Ensino de Matematica, State University, Rio de Janeiro

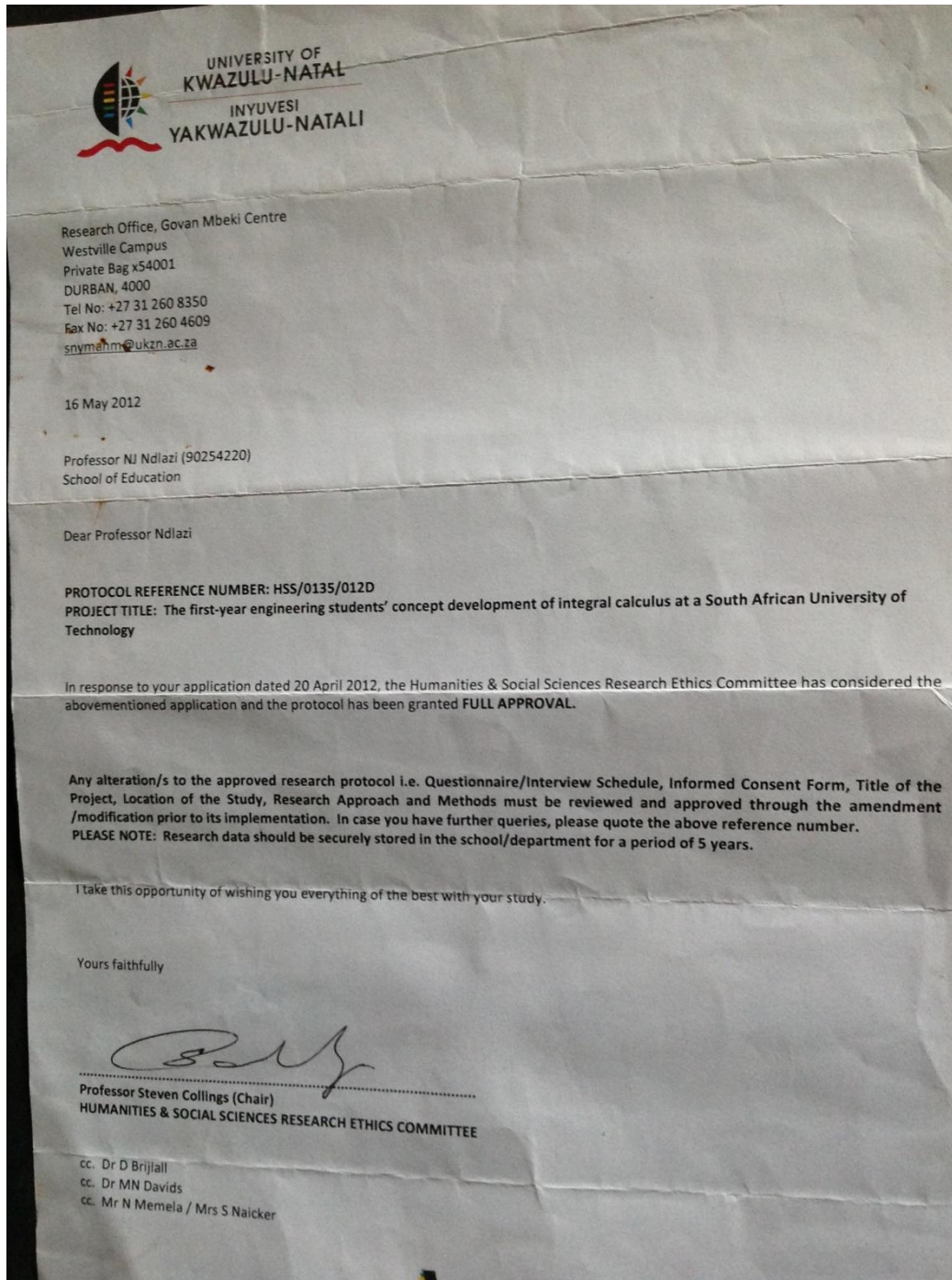
- Tall, D. (2004a). Building theories: The three worlds of mathematics. *For the Learning of Mathematics*, 24(1), 29-32.
- Tall, D. (2004b, July). *Thinking through three worlds of mathematics*. Paper presented at the Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway.
- Tall, D. (2007, February). *Embodiment, symbolism and formalism in undergraduate mathematics education*. Paper presented at the Plenary paper at 10 th Conference of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education, San Diego, Carlifonia.
- Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, 20(2), 5-24.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169.
- Thomas, M. O. (2008, 30 June-4 July). *Conceptual representations and versatile mathematical thinking*. Paper presented at the Proceedings of ICMI, Monterrey, Mexico.
- Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. *Making the connection: Research and teaching in undergraduate mathematics*, 73, 43-52.
- Uhlenbeck, K., & Stroup, W. (2002). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts: An instrumentalist investigation into calculus students' spontaneous reasoning. *Journal for research in Mathematics Education*, 40(4), 427-459.
- Vinner, S. (1983). Concept definition, concept image and the notion of function. *International journal of mathematical education in science and technology*, 14(3), 293-305.
- Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking*, (pp. 65-81). Kluwer, Dordrecht.
- Williams, S. R.-. (1991). Models of limit held by college calculus students. *Journal for research in Mathematics Education*, 219-236.

Zollman, A. (2014). *Re-conceptualizing procedural and conceptual knowledge in calculus*. Paper presented at the 41th Annual Meeting of the Research Council on Mathematics Learning, San Antonio, Texas

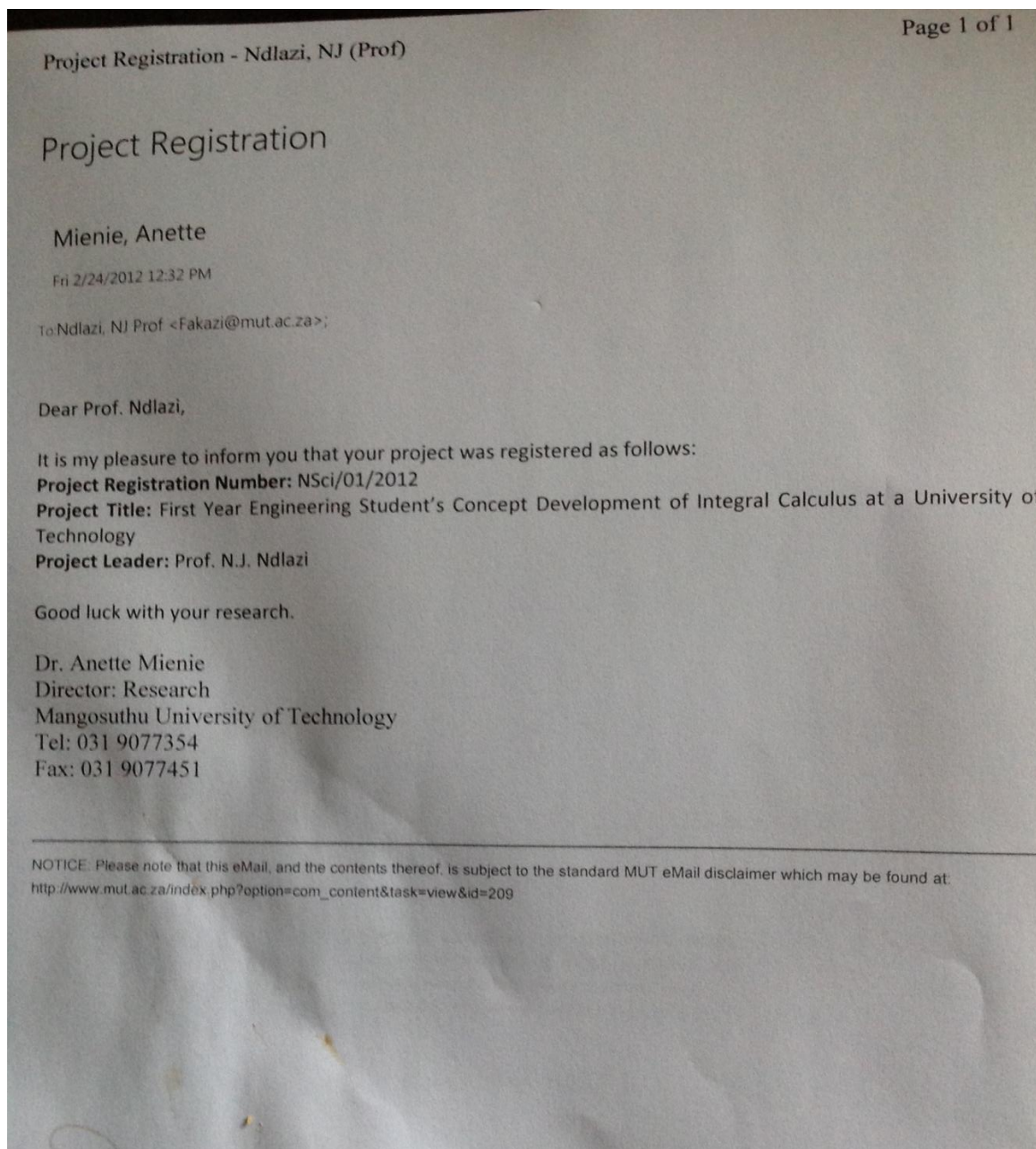
APPENDICES

APPENDIX A: Certificates and Letters of clearance

A1: Clearance certificate: UKZN



A2: Clearance certificate: MUT



A3: Consent letter



NJ Ndlazi

Faculty of Natural Sciences

Tel: 031-9077601

Fax: 031-9077611

Dear student

Consent Letter for the study:

First year engineering students' understanding of integral calculus.

You are being approached to take part in the above study. In this letter, I, the researcher, will describe to you the aims of the study, explain what is required from you if you agree to participate in the study, and explain how I will protect you in the study.

The study aims at investigating what happens when a student develops understanding of integration in a university of technology. I am interested in the conceptual development during the process of learning various techniques of integration. The study observes engineering classes at Mangosuthu University of Technology.

I intend to investigate this through analysing responses to specific relevant tasks, interviewing the chosen students and observing group discussions among students. As you will be attending the relevant course, I need your consent to collect the data.

This study is carried out by me, Mrs NJ Ndlazi, Mangosuthu University of Technology for my PhD studies under the supervision of Dr D Brijlall, University of KwaZulu Natal. If you have further questions about the project, you can direct them to me and my contact details are stated in the letterhead.

You will be requested to complete some questionnaires which have questions based on integration. These tasks will absolutely have no bearing to your course of study. I will also request to interview you. These interviews can be expected to last at least 30 minutes per time, at a time best suited to you.

I will keep all tape recordings of interviews and copies of materials and your work in safe keeping until the project has been completed, upon which they will be transferred to locked storage at University of KwaZulu-Natal for a five year period.

Before any results from the project are published, I will anonymise all references to you as a student. Thus, I assure you full confidentiality and anonymity.

Participation in the study is voluntary. If you choose not to participate in the study, it will not result in any form of disadvantage. If you choose to participate in the study, or parts thereof, you are free to withdraw at any stage and for any reason.

I (full name of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to allow the researcher to include me in the data collection.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

- I agree
- I do not agree to having the researchers obtain copies of my work upon request.
- I agree:
- I do not agree: to sit for a focused assessment on integration, which has no bearing towards my studies.
- I agree
- I do not agree: to being interviewed 3-4 times throughout the semester at times convenient to me.

.....

.....

(Signature of participant)

(Date)

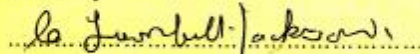
A4: Letter from Editor

TO WHOM IT MAY CONCERN

18 November 2015

This is to confirm that Chapters 2, 3 4, 6 and 7 of the thesis entitled "The first year engineering students' concept development of integral calculus at a South African University of Technology" have been edited to ensure technically accurate and contextually appropriate use of language.

Sincerely

A handwritten signature in black ink, appearing to read "Carolyn Turnbull-Jackson", written over a dotted horizontal line.

Dr Carolyn Turnbull-Jackson (PhD)

Language Editor

A5: Turn-It-In Certificate

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Originality Report

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THE FIRST-YEAR ENGINEERING STUDENT'S CONCEPT ...
 ENGINEERING STUDENT'S CONCEPT ...
 By Jubilee Ndazi

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THE FIRST-YEAR ENGINEERING STUDENT'S CONCEPT DEVELOPMENT OF INTEGRAL CALCULUS AT A SOUTH AFRICAN UNIVERSITY OF TECHNOLOGY By Nokwethemba Jubilee Ndazi (90254220) A full dissertation for the Degree of PhD (Mathematics Education) Supervisor: Professor Deonarian Briljall 2015 CHAPTER ONE INTRODUCTION 1.1. Background information Engineering mathematics is fundamental to all engineering programmes at universities of technology. It facilitates the understanding of content for various subjects within the discipline and, as a result, it is a pre-requisite for advancement to senior levels of study. Poor performance in mathematics may present it as a gateway subject for first year students. As a mathematics lecturer at a university of technology, therefore, I felt that there was a need to explore students' learning of mathematics since

"research into students' learning of mathematics helps to predict what they may learn about a specific mathematical concept and the conditions by which that learning takes place" 2

(Arnon et al., 2014, p. 27). A dominant mathematical concept that students encounter within engineering mathematics at a university of technology is integral calculus. Although calculus is introduced to learners at high school, two main reasons result in students that enter universities being highly underprepared for this specific topic. Firstly, the quality of mathematics teaching at a school level seems inadequate (Zingiswa Mybert Monica Jojo, Maharaj, & Briljall, 2013) and, secondly, integral calculus is not included in the South African school syllabus (Department of Education, 2003). Students, therefore, exhibit difficulties in handling this aspect of the subject. Several studies have investigated the teaching and learning of integral calculus (Berger, 2006; Deonarian Briljall & Bansal, 2010; Habineza, 2010; Koepf & Ben- Israel, 1994); while Orton (1983) established that students had difficulties in understanding an

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APPENDIX B: Research instruments

B1: The activity worksheet

Understanding of Integral Calculus

Student Number									
-----------------------	--	--	--	--	--	--	--	--	--

Date	DD		MM		YY				Duration	2 hours

Gender (Put an X in the correct box)

Male		Female	
------	--	--------	--

Type of school where matriculated (Put an X in the correct box)					
Rural		Ex-Model C		Township	Other (please specify)

Year in which you matriculated			

Symbol in matric mathematics

Notes for the participants

1. Please answer all questions as honestly as possible.
2. The duration of this questionnaire is **2 hours**.
3. This questionnaire does not form part of your assessment, but it is for research purposes only.
4. You may be invited for an interview based on your responses to this questionnaire.

Please Turn Over

Item 1.

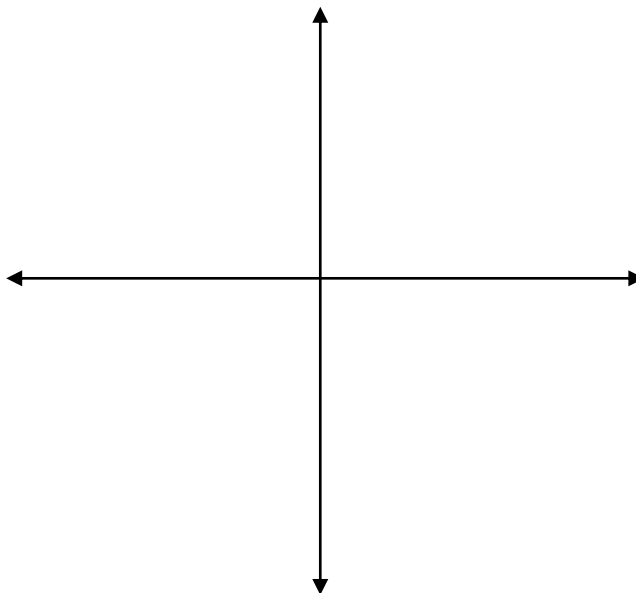
In your understanding, what is the difference between $\int f(x)dx$ and $\int_a^b f(x)dx$?

1.3 $\int f(x)dx$ means.....

1.4 $\int_a^b f(x)dx$ means.....

Item 2

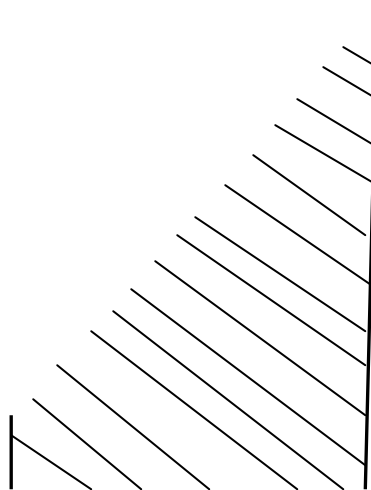
2.1.1 Sketch the graph for $y = \sqrt{25 - x^2}$:



2.1.2 How does $\int_0^5 \sqrt{25 - x^2} dx$ relate to the graph you have just drawn?

2.2. A sketch graph of $y = 2x + 3$ is shown below:

$$f(x) = 2x + 3$$



Use integration to find the shaded area.

Item 3

A student asked to solve the integral $\int x e^{\frac{x}{2}} dx$ decided to use integration by parts and chose $e^{\frac{x}{2}}$ for a “ u ”.

- 3.1. Was this choice of a “ u ” appropriate?
- 3.2. Please support your answer to 3.1.
- 3.3. Now provide a solution for the same integral:

Item 4

A student is given two integrals to evaluate:

(A) $\int \frac{dx}{x^2 - 8x + 25}$ and (B) $\int \frac{10}{(x-1)(x^2+9)} dx$

4.1 Work out the solution for integral (A):

4.2 Work out the solution for integral (B):

4.3 Justify the choice of methods you picked to solve (A) and (B)

Items 5.1 to 5.6

Determine the following integrals:

5.1 $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

5.2 $\int \frac{x-1}{\sqrt[3]{x^2}} dx$

5.3 $\int \frac{e^x}{e^x+1} dx$

5.4 $\int \tan^{-1}(3x) dx$

$$5.5 \quad \int \ln(x^2 - x + 2) dx$$

$$5.6 \quad \int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$$

Thank you for your time!!!

B2: Focus Groups discussion exercises

Understanding of Integral Calculus

The following questions are designed to explore your understanding of the concept of integration.

Please answer all questions to the best of your ability.

- For anonymity purposes, do not write your name on any of the pages of this worksheet.
- For each question show in detail how you obtained your answer.
- One student from each group may be chosen to present the negotiated answers on the board.

SECTION A

Item 1 (a & b)

Please explain to your classmates what, in your understanding, do the following mean: (a) $\int f(x)dx$ and (b) $\int_a^b f(x)dx$


Item 2

Find the area bounded by the graph $y = \cos x$ and the x-axis between $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ and then draw the graph of $y = \cos x$ to explain your answer.

Area =

=

Graph:



SECTION B

Item 3 (a & b)

Evaluate the following integrals:

In each case please explain the meaning of the sign, if you can

(a) $\int_0^4 \frac{1}{(x-2)^{2/3}} dx$

What can you say about the sign of the answer?

(b) $\int_{-1}^2 \frac{1}{x^2} dx$

What can you say about the sign of the answer?

SECTION C

Item 4 (a – d)

Evaluate the following integrals to the best of your ability, **in each case justifying why you chose a particular approach:**

(a) $\int \frac{\ln x}{x} dx$

(b) $\int \frac{1}{x \ln x} dx$

(c) $\int \ln x dx$

(d) A student asked to solve the integral $\int x e^{\frac{x}{2}} dx$ decided to use integration by parts and chose $e^{\frac{x}{2}}$ for a “ u ”.

(i) Was this choice of a “ u ” appropriate? Please support your answer.

(ii) Provide a solution for the same integral

Thank you for your time!!!

APPENDIX C: Sample of interview transcripts

C1: Interview with Student 1

Student Sabelo completed his matric in a rural school in 2008 with an F pass in mathematics (3- thirty something). Stayed at home for a year and then enrolled at an FET college up to an N4 level doing electrical engineering. At the university the candidate did the bridging course and got sixty percent. In maths 1 he got 50 percent.

(I: Interviewer; S: Sabelo)

I: You told me that you completed your matric in..

S: 2008

I: And what symbol did you get for mathematics there?

S: F

I: In 2008 which syllabus was it? Was it graded in F's or in 1's or 2's?

S: Ya, in 1's or 2's

I: So You got..

S: 3.

I: So you got a three, a thirty something?

S: Yes

I: Then you went to..

S; 2009 I was at home. 2010 I was studying at Berea Tech

I: Ok. What were you doing at Berea Tech

S: N4

I: N4. And which subjects were you doing for your N4?

S: Electrical, Electrotechnics, Electrical Eng Science, Maths and Electronics.

I: And then at MUT did you do Pretech?

S: Yes

I: Ok, And what did you get for your Maths in Pretech?

S: S sss.. I think,..... *angisakhumbuli*, it's sixty or fifty something.

I: Ok, alright, alright. And then in your Maths I, 'cause this must have been Maths 2. In Maths 1 can you remember how you fared?

S: Ya, I think it was around 50.

I: As well.

S: Ya

Referring to item1:

Item 1. *In your understanding, what is the difference in meaning between $\int f(x)dx$ and $\int_a^b f(x)dx$?*

I: OK fine. Ok. Now la we were talking about integration. I don't know whether you still remember this. The first question was: What is the difference between these two? And I see your responses here. My main interest to people here is to find out what is integration, when one sees integration what comes to their minds?

S: Mhm...(silence)....so....uma..

I: Ok you can speak in IsiZulu it's fine

S: So Uma usebenzisa igama elithi integration, I don't think kwi basic English like uma nikhuluma. Kuqhamuka imaths nje kahla kahle.

I: Kuqhamuka iMaths, in your case. And what in Mathematics?

S: Like.....

I: Kuqhamuka iMaths, kuqhamuka ini in Mathematics?

S: Like, you mean, ubuza....

I: Ukuthu uma kuthiwa integral, integration, yini, what is it, in your understanding.

S: Like ngiya understander ukuthi ama integration kahle kahle ahlukile, so...

I: Ok

S: So kahle kahle kuqhamuka isign ye integration, then i function then i instruction ukuthi integrater ngayiphi like inhlobo ye integration, like integration by parts,

I: Ok Fine, Mhlambe what I want to know is that when you are asked to integrate, what is it that you are actually asked to do? When you are asked to integrate, kusuke kufunwa ini,.... ye function?

S: What I know is that when you integrate kahle kahle you are doing i inverse ye differentiation.

I: Ok. Now here we were given this integral in (B). And then, of course there is a difference between these two, and I see your response. What does a and b represent, in your understanding?

S: Like when u integrater, then you subsitutor u "x", then you minus then you substitute by b, by a and b.

I: What do we call u a and b?

S: What I know is that kukhona imaximum value ka x then a minimum value ka x.

I: And, that is interesting that we say there is a maximum and a minimum, because that implies comparison somewhere. It's a maximum where?

S: Because it is written above.

I: Ok. Let us move to the next one (*Referring to item 2, 2.1.1*). First I asked you to sketch this graph, but I think I did explained how this graph should have looked like afterwards. And I asked this question: What is the relationship between this integral and the graph that you have drawn? Firstly, what would have been the correct graph? What type of a graph is this one?

S: A full circle

I: Not very correct, because there is a sqrt sign. It is a semicircle. Is it lying above or below the x-axis?

S: Oh...above.

I: Oh above, that's correct. So we were supposed to have something like this only. The second question was: What does this mean as far as our graph is concerned?

S: This whole..?

I: Ya, the integral

S: Integrate i function(*silence*), then you find the minimum and maximum i value ka x

I: And then you subtract.

S: Ya

I: Ok. Then I gave you this one...(*Item 2.2*). The question was: Use integration to find the shaded area. This is what you did. Is it true that when you do this you are finding the area?

S: Ya, it is true.

I: It's true. So if I go back to the previous question, what is then the relationship between the given integral and the graph?

S: We are finding the area that is being covered by the graph.

I: Where

S: Above the x-axis

I: What guides us on the location of the area? What tells us where the area is?

S: The minimum value ka x, which is 5 and 0, the minimum and maximum.

I: Where is 5 and 0 on the graph itself?

S: Here(Pointing on the y-axis)

I: The 5 and 0, are they the X or Y values?

S: They are the X-values

I: So, which area will we be looking for then?

S: This part, (*pointing at the portion in the first quadrant*).

I: Ok, in the first quadrant. That's correct. I am not going to ask you on the correctness of integration as yet. The next question was interesting. (*Reading Item 3*). Your answer was no. Why, what was your reason?

S: I can choose any of the function to be used because both of them can be integrated. But according to the rules of integration by parts, we can choose x to be u.

I: Ok. What are those rules? Do you recall them?

S: I don't think it is the rules, but they guide us to choose the simplest function to differentiate.

I: So how do you choose the simplest function? You look at the function that is easy to differentiate, easy to ..what do you look for?

S: Easy to differentiate

I: So, between these two functions, x and exponential function...?

S: x is easy to differentiate

I: Ok. So that should be the criteria. What about the exponential function, I thought it would be easier.

S: Ya. Both of them are easy....it's just that...like..abant'abaningi bayadideka when it comes to i function with e.

S: To differentiate it?

I: Ya

I: How do they show ukudideka?

S: Angazi kahle kahle. Kukhona esinye isibalo esakhishwa uDr Maal, sasinalo e. Wayengakasikhumbuzi nje ama rules, wathi asisenze abantu bahluleka.

I: To differentiate it?

S: Ya

I: Ok. By the way, what is the derivative of?

S: e^x ? It's.....it's e.

I: e^x . It's e^x .

S: Differentia...um' u differentia.....eyi ukuthi...oh ya

I: It remains the same. Actually u e is that special base

S: e is constant

I: So here your criteria is x is easier to differentiate

Moves to item 4

I: Are there any similarities? You said yes. And your answer was... which constant are you referring to?

S: The numerators

I: Differences? (*The student point at the highest powers*). You looked at the powers. Ok fine. For techniques, why did you choose “completing the square” for A?

S: Ngoba awekho ama factors la.

I: Because you can't factorise. In B: What technique do we use?

S: We use partial fractions.

I: And why?

S: Mh...mina engikwaziyo ukuthi once kwa khona ma products.

I: Actually ama factors.

S: Ya

I: Then you solved it, you completed the square. And here you used used partial fractions as you.

Moving to 5.1 – 5.6

I: What is important for me is how you find the answer. What were you doing in this step?

S: Oh ngihlukanisile, ngisuse u x ngawuletha ngaphezulu, then kwaba i product. Uma usebenzisa...mh..uya.. then... ama standard integral, gase ngasebensiza mastandard integral. Eka sine

I: Athini

S: Eka sine

I: What did you do with those?

S: Angithi Mam, i integral ithi... ifunction x iderivative

I: I derivative of??

S: Of the angle

I: Ok, good. What is your angle here?

S: I angle iwu.... $x^{\frac{1}{2}}$

I: Good. And so what is the derivative of that angle?

S: Iwu $x^{-\frac{1}{2}}$

I: Ok. Konje how do we find the derivative?

S: Si minasa ngo 1 i exponent.

I: Kuphela. Uma ku wu x^3 , what is the derivative?

S: x^2 kufanele kube wu x^2

I: Ok, What is the derivative of x^3 ?

S: U x^2

I: Ok. Of $\frac{1}{2}x^2$?

S: (*Laughing*), I am sorry Mam, of x^3 is $3x^2$

I: So $x^{\frac{1}{2}}$, what is the derivative? You said you angle is?

S: Kufanele kube wu... $\frac{1}{2} x^{-\frac{1}{2}}$.

I: Ok. So there is that $\frac{1}{2}$ missing here.

S: Ya

I: We just missed it kwi differentiation.

S: Ya.

I: So here (*item 5.2*), what did you do here, tell me?

Silence

I: You used long division.

S: Ya.

I: Changet that to

S: Ngithe $x^{\frac{3}{2}}$ ngase ngenza ilong division ngathola u $x^{-\frac{1}{2}}$

C2: Interview with Student 2

Student 2 completed his matric in a rural school in 2010, obtaining a D for mathematics. He joined the university the second semester of the following year and did the bridging course obtaining 60%. He got a 50% for mathematics 1 and has since obtained a 56% for mathematics 2.

I: What do you understand by integration?

S: If you differentiate it's like you are going forward and if you integrate it's like you are reversing what you have differentiated. The integral is a vice versa of differentiation.

I: What do limits mean?

S: The first place just ignore the limits and putting them outside the brackets, then do your calculations. Then to the last step use the limits by opening the brackets and substitute the limit b to the first bracket then minus then substitute the limit a to the bracket of the original equation

I: What is that value that you get give you?

S: It gives you.. if I am not mistaken, it's a gradient.

In item 2, the student could not draw the graph of the semicircle, so the focus was on the meaning of the definite integral.

S: The definite integral has limits, so it might shift the graph.

I: What do the limits give us?

S: They give us the values of x from point a to point b on the x -axis.

I: In item 2.2, what are the boundaries of the shaded area?

S: The boundaries it's 1 sorry it's 2, or I can say it's -1.5 and 2.

I: Where does the shading start?

S: Sorry, it's -1 here.

I: Is the used integration the way of calculating the area?

S: Yes.

I: So, how does it say about 2.1.2?

S: Compared to this one, this one is like an expression, you have got two terms and the root. Some formula must be applied when there is a root like this one to solve it.

I: But what would you be finding when you calculate all that?

S: You will be findingit's the area of the shaded part.

I: So what was supposed to be a response to 2.1.2?

S: The relationship is that from here it's ranging from 0 to 5. The graph is matching this equation. The x-axis is 5 and the first point of the x-axis is 0.

I: Item 3: Was it correct to use integration by parts?

S: When you have the two terms and maybe the other term has a function itself. And given the function maybe the two terms and some other function has a function, maybe the powers of a function.

I: Was it correct then here?

S: Yes

I: Is the choice of u correct? Please support your response

S: If you use integration by parts some part should be u and dv so that you can substitute u in the equation because u will make the x and dx will make the.. so in this equation if you come to du du is 1 so if you come to u u is the ln of x. Officially this cannot be solved by integration by parts.

I: It can't be solved?

S: yes it can't be solved.

I: But in your response you wrote the first property... what did you mean?

S: Right...if u look at u, (reads the response again). Oh.. remember in class there is something which guides you on which to apply u. The first is ln x, second a power of x..In this case there is no ln and no power of x, but there is a power of e which is the first property for u that was given in class.

I: Which functions are multiplied in this item?

S: It is x and $e^{x/2}$. Yes

I: The first one is x. In the priorities for u, which one will the x correspond to?

S: The second one.

I: What do you do to your u to get du?

S: From u I integrate it. Ya I differentiate it

I: And how do you get back to v from dv?

S: I integrate it.

I: You were integrating $e^{\frac{x}{2}}$. How did you go about?

S: Yes, if you integrate the number, the number you take out of the integral sign, integrate and times by that number. So I first integrated this number, I first integrated $\frac{x}{2}$ then the $\frac{1}{2}$ comes outside and integrated and then multiplied by

I: What did you do to the e itself?

S: To the e itself...

I: Maybe, what function are you integrating here?

S: It is $e^{\frac{x}{2}}$.

I: So what did you do with the e?

S: The e...

I: From what you did you just integrated $\frac{x}{2}$ and not $e^{\frac{x}{2}}$. Ok let us move to Item 4. What are the differences between A and B?

S: If you are given this form, you do it until you reach the step where you have to complete the square then find these other equations, the standard equations?

I: Which form are you referring to?

S: By looking at the denominator, if it is like this you have to solve it to reduce it so that it has at least two terms. B is right because there are two terms. Then we apply it in this form. These two require different formulas.

I: When do we complete the square and when do we write it as factors?

S: If you complete the square, you are trying to reduce the equation. There is a formula that requests you to put A and B.

I: What makes you decide to complete a square?

S: This equation tells me that the things I must use here is either I find the values of x or complete the square.

I: So why did you choose to complete a square here and not find the values of x?

S: Actually I just use it as a standard that if you come across an equation like this, you just complete the square, because if you find the values of x, it's like you are changing the given equation.

I: In doing B, this integral: $\int \frac{x+1}{x^2+1} dx$ arose. What are you doing here, then?

S: Here, you try to make the denominator and numerator common. For the second one, there is a formula that is used to solve it if it is in this form. We first distribute it over the denominator and then apply formulas to make it simpler. There is a rule that if you have something like this you can equate it to something and substitute other values.

I: Which other values did you substitute here?

S: Like if your equation comes to this form, then there is a formula to be applied. Like this one, this is the.. then if you come here, you just integrated then here there is some formula, arc tan.. Then here

I: So this comes from the formula?

S: Yes this comes from the formula.

I: You said here you just integrated. How did you integrate?

S: If something is A/B , then you integrate by.. you are trying.. oK if you look at the denominator and differentiate it, you get $2x$ which you have at the bottom part. If you differentiate the denominator you get the numerator, so we ..if we integrate this part, officially it will be like this one. So we're taking the original formula, if when differentiated gives us the numerator.

I: And the \ln , where does it come from?

S: It comes from that if something is A/B then it should be the \ln of a function.

I: Item 5: Briefly explain what you did here?

S: Here if you have this, the most easy part is to ..changed the root to the exponent half

I: Thereafter?

S: Thereafter, the integral of $\sin x$ is $-\cos x$. I integrated $\sin x$ and got the $-\cos x$. But if you apply implicitly, you have to integrate again. If you integrate $x^{\frac{1}{2}}$, it will give you something like, so officially tells us that this alone is a function and this is its derivative, so you retain the function.

I: The $x^{\frac{1}{2}}$ is a derivative of which function?

S: Of this (pointing at \sqrt{x} after the \cos)

I: In 5.2, can you explain what you did?

S: (*First simplified the exponents and then integrated*)

I: In 5.3?

S: If you differentiate the denominator, it gives the numerator, so you just retain the function.

I: In 5.4?

S: The integral of $\tan \text{ arc}$ is $\sec \text{ arc}$, then you differentiate $3x$.

I: Item 5.5?

S: Then if it was 1 over x , then if you trying to .. because if you integrate you are putting it in its original place. Then it should be x (Referring to 1 over x). So for this one if you are bringing it to its original it will be 1 over the denominator. Then you complete the square.

I: Do we still have the integral sign though?

S: Yes we still have an integral sign.

I: So how did it change from \ln to 1 over?

S: By In, what it means is that there are two things involved, like, ..officially In tells us that there is a 1 over that function.

I: Then 5.6?

S: If you look at some example that we looked at, we use the same method we used there to split this into its simpler form. ...If the power is positive half, if you raise it up it will be minus.

I: Then what did you do here?

S: I just opened the brackets for minus half.

I: Now in your Maths 3, are finding your Maths 2 helpful?

S: Yes it is helping me a lot.

I: What sections are you looking at presently?

S: The first order and second order differential equations.

I: Thank you very much

C3: Interview with Students 3 and 4

Sbonelo matriculated in a rural school in 2010 obtaining a D in Mathematics. He then did a bridging course at the university. For his maths 1 he got 53%.

Bongani matriculated in 2006 in a township school, and stayed home for two years. After that he did Electrical Engineering (N3-N5) at a technical college. At the university he was admitted directly to S1 where he obtained 60% for his mathematics. This student had done integration at N4 and N5 levels at a technical college.

I: What is it that we are looking for when we are integrating?

SS: I can say that we are reducing our equation.

I: From what to what?

SS: Eish

SB: When we talk about integration, we are looking for a smallest value possible.

I: Smallest value of what?

SB: Let's say maybe like you are given a certain application, and then if you try to solve the problem so then the integration is helping us to find that value.

I: In the two integrals in Item 1, what is the different in meaning?

SS: I think the difference is in A we are integrating with respect to x, while in B we integrate w.r.t x wher a and b are our x values.

I: And so what do we do with those a and bs

SB: It's where we substitute with those given values of x after integration.

I: And what is that giving us as far as the function is concerned?

SS: It gives the constant value.

I: Moving to item2, you were asked to draw this function.

SB: It is a semicircle.

I: What tells you that it is a semicircle?

SS: I think this is a Pythagoras equation.

I: Then what is the relationship between the draw graph and the given integral.

SS: I did not know this thing I wrote here.

I: What does this thing gives you?

SS: We can draw the graph of the semicircle and then in the equation we substitute with the values of x they gave us.

I: In the graph what does this integral define?

SS: I don't understand.

I: In 2.2, can you use the integration to find the area?

SB: Yes.

I: How do you use it?

SB: You first start by integrating the given function after that then you substitute by the given value between that particular part of the graph.

I: So back to item 2.1, what does the given the integral mean for the graph?

SB: We are requested to find the area under the graph.

I: Now item no 3, how do you know when to use integration by parts?

SB: When you have two functions

SS: We use IP when you two functions where you have u and yourif you have two constants, two values or two functions. But it depends on u and your v, just because we use the formula to calculate hthis thing. You have to integrate another and differentiate another. So you have to see in your functions that you can differentiate it or integrate it. If it possible you can use the IP.

I: If it is not possible??

SS: You can use another method. You can use the product rule. Eish we are talking about integration here.

I: In this item, why is the choice for u not correct in this item?

SB: Because when you want you first prioritise by starting with $\ln x$, x and exponential function. So here the choice is wrong, so we must choose x as our u .

I: Why are those priorities for u the way they are?

SS: We do not know why \ln is the first choice, etc. We just know the priorities.

I: Explain your solution Sbonelo.

SS: Here I put u as my x and then du , then I integrate this x , to get my du just because the equation grouping is holds this thing. I take my x as u then my dv is equal to this one. Then I integrate this one then I differentiate my u . Then I substitute in my equation to get this eventually.

SB: I did not attempt it because I had forgotten how to do after this step because by the time we were doing this thing it was long time ago so I had forgotten.

I: What if you need it now for your maths 3?

Sb: I will go back and remind myself

I: Item 4: What are your observed differences?

SB: A is integration by completing the square and B is integration by adjusting the numerator.

SS: B is partial fraction.

I: Why is A suitable for completing the square?

SB: Since we do not have the common factors of this equation, we can complete the square,

I: In B?

SS: We can use partial fractions because we have factors.

I: How do we complete the square?

SS: If you complete the square you have to use b of this equation, then minus it times $\frac{1}{2}$ minus this b times $\frac{1}{2}$ squared plus 25.

I: In 4.6:

SS: we used partial fractions.

I: Bongani you did not attempt it at all?

SB I was lost. I forgot the techniques we use.

I: Do you find these techniques appearing in your Maths 3?

SS: No they do not appear.

SB: But for partial fractions, the equation of coefficients does appear.

I: What do partial fractions help us to do?

SS: To find the values of A, B and C.

I: What are partial fractions?

SS: It's an equation that we use to find our values and then go back to substitute. It is the easy way to find our values.

I: *Explains the concept of partial fractions*

SS: *Let me ask something, if I add these two fractions will I get the original one?*

I: *Assist the students to add the fraction.*

I: Let us move to 5.1

SS: In 5.1, since there is this exponent I split it using the quotient rule.

I: How did you use the quotient rule?

SS: I said this one time the integration of this one, minus this one times the integration of this one, divided by this one squared.

I: When do we use the quotient rule?

SS: The quotient rule says.... *writes down the derivative of a quotient of f(x) and g(x).*

I: What does 'prime' mean in this rule?

SB: Prime stands for the derivative.

I: What would you be looking for when we use this formula?

SS: I think it's when you differentiate.

I: Good and what are we doing here?

SS: We are integrating, but it is difficult to integrate something that is in this form (meaning a quotient). We have to split this form first, using integration.

I: So, how did you split it?

SS: Here I used this formula of integration, $\int \frac{f(x)}{f'(x)}$, I then divided by n+1

I: What would you do here Bongani?

SB: I would use integration by parts. And then I will express u as x exp -1/2.

I: In 5.3, you just wrote the answer. How did you get it?

SS: I just guessed.

I: In 5.4, what did you do?

SB: This one we use trig functions as given from the data table.

I: What is happening here?

SB: The person is differentiating. Are you allowed to use differentiation when you are expected to integrate?

I: You are expected to integrate.

APPENDIX D: Samples of students' responses

Item 2

2.1.1 Sketch the graph for $y = \sqrt{25 - x^2}$:

$y = \sqrt{25 - x^2}$
 $0 = \sqrt{25 - x^2}$
 $= 5 + x$
 $-5 = x$

2.1.2 What is the relationship between $\int_0^5 \sqrt{25 - x^2} dx$ and the graph you have just drawn?

A relationship is that $\int_0^5 \sqrt{25 - x^2} dx$ is the area of the region bounded by the curve, the x-axis, and the y-axis. Since $y = \sqrt{25 - x^2}$ is a semi-circle, the integral represents the area of a quarter-circle.

Item 3

A student asked to solve the integral $\int x e^{\frac{x}{2}} dx$ decided to use integration by parts and chose $e^{\frac{x}{2}}$ for a "u".

3.1. Was this choice of a "u" appropriate?

Yes	No
<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>

3.2 Please support your answer for 3.1.

because its easy to differentiate ~~x~~ than $e^{\frac{x}{2}}$ since you are going to get x when differentiate x but it is easier to find integral of $e^{\frac{x}{2}}$ so $e^{\frac{x}{2}}$ would be appropriate for dv so that you will integrate and find ~~x~~ $e^{\frac{x}{2}}$

3.2. Now provide a solution for this same integral:

$$\int x e^{\frac{x}{2}} dx$$

~~$$u = e^{\frac{x}{2}} \quad dv = x$$~~

$$\int x \cdot e^{\frac{x}{2}} - \int \frac{x^2}{x^2} e^{\frac{x}{2}} dx$$

$$u = x \quad du = dx$$

$$dv = e^{\frac{x}{2}} \\ v = e^{\frac{x}{2}} \cdot \frac{x}{4} \\ = \frac{e^{\frac{x}{2}} x}{4}$$

$$\frac{x}{2} \cdot \frac{x}{2} \\ = \frac{x^2}{4}$$

$$x e^{\frac{x}{2}} - \frac{1}{4} \int x^2 \cdot e^{\frac{x}{2}} dx$$

$$x \cdot \frac{x}{4} e^{\frac{x}{2}} - \frac{1}{4} \int$$

$$u = x^2 \quad du = 2x dx$$

$$dv = e^{\frac{x}{2}} \\ = e^{\frac{x}{2}}$$

Confused | |
' |
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AM easily confused
Problem