On Common Fixed Points Approximation of Countable Families of Certain Multi-valued Maps in Hilbert Spaces

by

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As the candidate's supervisor, I have approved this dissertation for submission.

Dr. O. T. Mewomo.

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Abstract

Fixed point theory and its applications have been widely studied by many researchers. Different iterative algorithms have been used extensively to approximate solutions of fixed point problems and other related problems such as equilibrium problems, variational inequality problems, optimization problems and so on. In this dissertation, we first introduce an iterative algorithm for finding a common solution of multiple-set split equality mixed equilibrium problem and fixed point problem for infinite families of generalized k_i-strictly pseudo-contractive multi-valued mappings in real Hilbert spaces. Using our iterative algorithm, we obtain weak and strong convergence results for approximating a common solution of multiple-set split equality mixed equilibrium problem and fixed point problem. As application, we utilize our result to study the split equality mixed variational inequality and split equality convex minimization problems .

Also, we present another iterative algorithm that does not require the knowledge of the operator norm for approximating a common solution of split equilibrium problem and fixed point problem for infinite family of multi-valued quasi-nonexpansive mappings in real Hilbert spaces. Using our iterative algorithm, we state and prove a strong convergence result for approximating a common solution of split equilibrium problem and fixed point problem for infinite family of multi-valued quasi-nonexpansive mappings in real Hilbert spaces. We apply our result to convex minimization problem and also present a numerical example.

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Dedication

This work is dedicated to Almighty Allah and my beloved family.

Contents

Declaration

This dissertation, in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's work and where the work of others has been used in the text, proper reference has been made.

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CHAPTER 1

Introduction

1.1 General Introduction

Let X be a metric space and $T : X \to X$ be any nonlinear operator. We say that $p \in X$ is a fixed point of T if

$$T(p) = p, \tag{1.1}$$

and we denote by $F(T)$ the set of all fixed point of T. However, if T is a multivalued map, that is, from X to the collection of nonempty subsets of X, then a point p in X is called a fixed point of T if $p \in Tp$. Fixed point theory gives the conditions under which fixed point problems of both single-valued and multi-valued mappings have solutions. The theory of fixed point is a beautiful combination of analysis (real and functional), topology and geometry and its importance in functional analysis is due to its usefulness in the theory of ordinary and partial differential equations. It is an area of intensive research as it has wide application in establishing existence and uniqueness of solutions of diverse mathematical models such as solutions to variational analysis, optimization problems and equilibrium problems. These models represent various phenomena arising in fields such as steady temperature distribution, economic theories, optimal control of systems, radiation transfer, epidemics and flow of fluids. For instance, in connection with a problem of radiation transfer, we are led to the equation

$$y(x) = 1 + \int_0^1 \frac{sy(s)y(x)}{s+x}\varphi(s)ds,$$

where $\varphi : C[0,1] \to \mathbb{R}$. This system is an example of an integral equation of the form

$$y(x) = f(x) + \int_0^1 K(x, s, y(s))ds, \quad x \in [0,1]. \tag{1.2}$$

A special but important case of (1.2) is the nonlinear Fredholm equation

$$y(x) = f(x) + \lambda \int_0^1 K(x, s, y(s))ds,$$

where $\lambda \in \mathbb{R}$ is a given number. This system can be formulated into a fixed point problem

$$\varphi = T\varphi.$$

(see Section 1.3, pg 22-23 of [6] for details).

1.2 Fixed Point Iteration Procedure

Iterative algorithms plays a very important role in approximating solution of fixed point problem and other related problems such as equilibrium problems, variational inequality problems, optimization problems and so on. In this section, we give a brief introduction to some of the iterative algorithms used in fixed point theory (for details, see [6]).

1.2.1 Picard Iteration

Let (X, d) be a complete metric space, U a closed subset of X and $T : U \to U$ a selfmap possessing at least one fixed point $p \in F(T)$. For a given sequence $x_0 \in X$, we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive iteration method

$$x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, ... \tag{1.3}$$

The sequence defined by (1.3) is known as the sequence of successive approximations, or simply Picard iteration. When the contractive conditions on T are slightly weaker, then the Picard iteration need not converge to a fixed point of the operator T. As a result of this, other iteration procedures must be considered.
Throughout this dissertation, we denote by E a real normed space.

1.2.2 Krasnoselskii

Let $T : E \to E$ be a selfmap, $x_0 \in E$ and $\lambda \in [0,1]$. The sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, ... \tag{1.4}$$

is called the Krasnoselskii iteration. The Krasnoselskii iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.4) is exactly the Picard iteration corresponding to the averaged operation $T_\lambda = (1-\lambda)I + \lambda \dot{T}$ where I is the identity operator and for $\lambda = 1$, the Krasnolselskij iteration reduces to Picard iteration. Moreover, we have $F(T) = F(T_\lambda)$ for all $\lambda \in (0,1]$.

1.2.3 Mann Iteration Scheme

The Mann iteration process starting from $x_0 \in E$, is the sequence $\{x_n\}_{n=0}^{\infty} \in E$ defined by

$$x_{n+1} = (1-a_n)x_n + a_n T x_n, \quad n = 0, 1, 2, ..., \tag{1.5}$$

where $\{a_n\}_{n=0}^{\infty} \subset [0,1]$ satisfies certain appropriate conditions.
Considering $T_n = (1-a_n)I + a_n T$, then we have that $F(T) = F(T_n)$, for all $a_n \in (0,1]$. If the sequence $a_n = \lambda$ (constant), then the Mann iterative process obviously reduces to the Krasnoselskij iteration.

1.2.4 Ishikawa Iteration Procedure

The Ishikawa iteration was first used to establish the strong convergence of a fixed point for a Lipschitzian and pseudo-contractive selfmap of a convex and compact subset of a Hilbert space. It is defined as $x_0 \in X$ such that:

$$x_{n+1} = (1-a_n)x_n + a_n T[(1-b_n)x_n + b_n T x_n], \quad n = 0, 1, 2, ..., \tag{1.6}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty} \subset [0,1]$ satisfies certain appropriate conditions. Many authors have used both Mann and Ishikawa iterative schemes to approximate fixed point problem of various classes of operators in Hilbert spaces and Banach spaces (see [32, 54, 74] and the references therein).
If we rewrite (1.6) in a system form, we have

$$\begin{cases} y_n = (1-b_n)x_n + b_n T x_n, \\ x_{n+1} = (1-a_n)x_n + a_n T y_n, \quad n = 0, 1, 2, ... \end{cases} \tag{1.7}$$

then we regard the Ishikawa iteration as a sort of two-step Mann iteration with two different parameter sequences. Despite the similarities between the Mann and Ishikawa iteration, if $b_n = 0$, the Ishikawa iteration reduces to Mann iteration.
Replacing T by T^n in (1.7), we have the modified Ishikawa iterative scheme defined as follows:

$$\begin{cases} y_n = (1-b_n)x_n + b_n T^n x_n, \\ x_{n+1} = (1-a_n)x_n + a_n T^n y_n, \quad n = 0, 1, 2, ... \end{cases} \tag{1.8}$$

1.2.5 Viscosity Iteration Procedure

The viscosity method has been successfully applied to various problems coming from calculus of variations, minimal surface problems, plasticity theory and phase transition. It plays a central role in the study of degenerated elliptic and parabolic second order differential equations.

The viscosity iterative method was proposed by Moudafi [49]. Choose an arbitrary initial $x_0 \in H$, the sequence $\{x_n\}_{n=0}^{\infty}$ is constructed by:

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} T x_n, \forall\, n \geq 0, \tag{1.9}$$

where T is a nonexpansive self-mapping and f is a contraction with a coefficient $\mu \in [0,1)$ on H, the sequence $\{\varepsilon_n\}$ in $(0,1)$, such that

(i) $\lim_{n\to\infty} \varepsilon_n = 0$,

(ii) $\sum_{n=0}^{\infty} \varepsilon_n = \infty$,

(iii) $\lim_{n\to\infty} \left(\frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n+1}} \right) = 0$.

Then $\lim_{n\to\infty} x_n = x^*$, where $x^* \in C$ $(C = F(T))$ is the unique solution of the variation inequality

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \forall\, x \in F(T).$$

Many authors have considered the viscosity iterative algorithm to approximate solutions of fixed point problem and other related problems (see [2, 44, 72, 79, 82] and the references therein).

1.2.6 Other Important Fixed Point Iteration Procedures

The Kirk's iteration procedure which is defined as $x_0 \in E$ and

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + ... + \alpha_k T^k x_n,$$

where k is a fixed integer, $k \geq 1, \alpha_i \geq 0$, for $i = 0, 1, ...k, \alpha_i > 0$ and

$$\alpha_0 + \alpha_1 + ... + \alpha_k = 1.$$

This iteration procedure reduces to Picard iteration if $k = 0$ and to Krasnoselskij iteration if $k = 1$.

The Figueiredo iteration procedure is defined as $x_0 \in C$, where C is a closed, bounded and convex subset of Hilbert space H containing 0 and

$$x_n = T_n^{n^2} x_{n-1}, \quad n = 1, 2, ..., \text{ where } T_n x = \frac{n}{n+1} T x, \quad n \geq 1.$$

It is known that Figueiredo iteration converges strongly to a fixed point of nonexpansive operators $T : C \to C$.
The Halphern iteration for approximation of fixed point T is given by $x_0 \in C$:

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n,$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$.

1.3 Research Motivation

In 2015, Ma *et al.* [42] introduced a new algorithm for solving split equality mixed equilibrium problem and fixed point problem in the framework of infinite-dimensional real Hilbert spaces. They stated and prove weak and strong convergence results for single-valued nonexpansive mappings. Also, Chidume and Okpala [23] introduced a new averaged algorithm for finding a common fixed point of countably infinite family of generalized k-strictly pseudo-contractive multi-valued mappings. Strong convergence theorems was established for this class of mappings under some mild assumptions. Motivated by the works of Ma *et al.* [42], Chidume and Okpala [23], we introduce an iterative algorithm for finding a common solution of multiple-set split equality mixed equilibrium problem and fixed point problem for countable infinite families of generalized k_i-strictly pseudo-contractive multi-valued mappings in real Hilbert spaces. Using our iterative scheme, we obtain weak and strong convergence results.

Recently, Kazmi and Rizvi [37] introduced an iterative algorithm to approximate a common solution of split equilibrium problem, a variational inequality problem and a fixed point problem for a nonexpansive mapping in real Hilbert spaces. They proved a strong convergence result for approximating a common solution of split equilibrium problem, variational inequality problem and a fixed point problem for a nonexpansive mapping. Also, Deepho *et al.* [31] considered an iterative scheme to approximate a common element of the set of solutions of split variational inclusion problem and the set of common fixed point problem of a finite family of k-strictly pseudo-contractive nonself mappings. They state and proved a strong convergence result which also solves some variational inequality problem under some suitable conditions in real Hilbert spaces. Lastly, Suantai *et al.* [69] presented an iterative algorithm for solving split equilibrium problem and fixed point problem of nonspreading multi-valued mappings in real Hilbert spaces and proved that the modified Mann iteration converges weakly to a common solution of the considered problems. Motivated by the works of Kazmi and Rizvi [37], Suantai *et al.* [69] and Deepho *et al.* [31], we introduce an iterative algorithm that does not require any knowledge of the operator norm to approximate a common solution of split equilibrium problem and fixed point problem for an infinite family of multi-valued quasi-nonexpansive mappings. Using our iterative algorithm, we prove a strong convergence result which solves some variational inequality in Hilbert spaces.

1.4 Statement of Problem

Let C be a nonempty, closed and convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ be a nonlinear bifunction. Then the Equilibrium Problem (for short, EP) is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall\, y \in C. \tag{1.10}$$

The set of solution of EP is denoted by $EP(F)$.
The following assumptions was introduced by [7].
For solving EP, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following assumptions:
$(A1)$ $F(x, x) = 0, \forall\, x \in C$;
$(A2)$ $F(x, y) + F(y, x) \leq 0, \forall\, x, y \in C$;
$(A3)$ For all $x, y, w \in C, \lim_{t\downarrow 0} F(tw + (1-t)x, y) \leq F(x, y)$;
(A4) For each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semi-continuous;
Let $r > 0$ and $x \in H$. Then there exists $w \in C$ such that

$$F(w, x) + \frac{1}{r}\langle y - x, x - w\rangle \geq 0, \forall\, y \in C.$$

We now consider the following problems:

1. Let H_1, H_2 and H_3 be real Hilbert spaces. Suppose C and Q are nonempty, closed and convex subsets of H_1, H_2 respectively. Assume that $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bi-functions satisfying $(A1) - (A4)$. Let $\alpha : C \to \mathbb{R} \cup \{+\infty\}$ and $\beta : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap dom\alpha \neq \emptyset$ and $Q \cap dom\beta \neq \emptyset$. Let $T_i : H_1 \to CB(H_1),\ i = 1, 2, \ldots$ and $S_j : H_2 \to CB(H_2),\ j = 1, 2, \ldots$ be two countable families of generalized strictly pseudo-contractive multi-valued mappings with constants k_i and k_j respectively and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators. Let us consider the following problem: find $(x^*, y^*) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{j=1}^{\infty} F(S_j)$ such that

$$\begin{aligned} F(x^*, x) + \alpha(x) - \alpha(x^*) \geq 0, \forall\, x\ \in C,\\ G(y^*, y) + \beta(y) - \beta(y^*) \geq 0, \forall\, y\ \in Q, \end{aligned} \tag{1.11}$$

$$\text{and } Ax^* = By^* \,.$$

2. Let H_1, H_2 be two real Hilbert spaces. Suppose C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$. Let $T_i : C \to K(C), i = 1, 2, 3, ...$ be an infinite family of quasi-nonexpansive multi-valued mappings and $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$, where F_2 is upper semicontinuous in the first argument. Let f be a contraction mapping with a coefficient $\mu \in (0, 1)$, then we consider the following problem: find $x^* \in \cap_{i=1}^{\infty} F(T_i)$ such that

$$F_1(x^*, x) \geq 0 \ \forall \ x \ \in \ C,$$

$$\text{and} \quad y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) \geq 0, \ \forall \ y \in \ Q.$$

1.5 Objectives

The main objectives of the work reported in this dissertation are to:

(i) review some known and useful results on pseudo-contractive and quasi-nonexpansive multi-valued mappings;

(ii) introduce an iterative scheme for approximating common solution of split equality mixed equilibrium problem and fixed point problem for countable infinite families of generalized k_i-strictly pseudo-contractive multi-valued mappings in real Hilbert spaces;

(iii) introduce an iterative scheme for finding the common solution of split equilibrium problem and fixed point problem for infinite family of multi-valued quasi-nonexpansive mappings with no prior knowledge of the operator norm;

(iv) present suitable applications of our main results;

(v) display our main result using numerical example.

1.6 Workplan

The rest of the dissertation is organized as follows:
Chapter 2:
We give some important definitions and concepts which are needed to obtain the results in this dissertation. We also discuss some notions on fixed point problem, equilibrium problems and variational inequality problems.

Chapter 3:
We introduce an iterative algorithm for finding a common solution of multiple-set split equality mixed equilibrium problem and fixed point problem for countably infinite families of generalized k_i-strictly pseudo-contractive multi-valued mappings in real Hilbert spaces. We state and prove weak and strong convergence results and also give applications of our main result.
Chapter 4:
We present a new iterative algorithm that does not require any knowledge of the operator norm for approximating a common solution of split equilibrium problem and fixed point problem for infinite family of multi-valued quasi-nonexpansive mappings in real Hilbert spaces. We state and prove a strong convergence theorem for the sequence generated by our iterative algorithm. We also give application of our main result to convex minimization problem and give a numerical example.
Chapter 5:
We give conclusion of our results obtained in Chapter 3 and Chapter 4. We also state the contribution of our work to knowledge and give areas of further research of the work.

CHAPTER 2

Preliminaries

In this chapter, we aim to highlight some definitions on which the problems are formulated and state some known results used in the subsequent chapters. This will include a brief review on multi-valued pseudo-contractive mappings and quasi-nonexpansive mappings. We will also discuss some related fixed point problems such as equilibrium problems, variational inequality problems and so on.

2.1 Mappings of Interest

2.1.1 Nonlinear Single-Valued Mappings.

Definition 2.1.1. *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. A mapping $T : H \to H$ is said to be*
(i) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall\ x, y \in H;$$

(ii) $\alpha-$strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha||x - y||^2, \quad \forall\ x, y \in H;$$

(iii) $\alpha-$inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha||Tx - Ty||^2, \quad \forall x, y \in H;$$

(iv) a strongly positive linear bounded operator, if there exists a constant $\alpha > 0$ *such that*

$$\langle Tx, x\rangle \geq \alpha||x||^2, \quad \forall x \in H;$$

(v) k-Lipschitz continuous, if there exists a constant $k > 0$ *such that*

$$||Tx - Ty|| \leq k||x - y||, \quad \forall\ x, y \in H;$$

(vi) nonexpansive, if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall\ x, y \in H;$$

(vii) firmly nonexpansive, if

$$\langle Tx - Ty, x - y\rangle \geq ||Tx - Ty||^2, \quad \forall\ x, y \in H;$$

(viii) average nonexpansive mapping, if

$$T = (1 - \alpha)I + \alpha S, \quad \alpha \in (0, 1),$$

where $S : H \to H$ *is nonexpansive and* I *is an identity mapping;*
(ix) quasi-nonexpansive, if $F(T) \neq \emptyset$ *and for any* $p \in F(T)$*, we have*

$$||Tx - Tp|| \leq ||x - p||, \quad \forall\ x \in H;$$

(x) firmly quasi-nonexpansive if $F(T) \neq \emptyset$*, and*

$$||Tx - p||^2 \leq ||x - p||^2 - ||x - Tx||^2, \quad \forall\ p \in F(T) \text{ and } x \in H;$$

(xi) nonspreading in the sense of Koshaka and Takahashi [39], if

$$2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||Ty - x||^2, \quad \forall\ x, y \in H;$$

(xii) k-strictly pseudo-contractive mapping in the sense of Browder and Petryshyn [9], if

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \quad \forall\ x, y \in H;$$

where $k \in [0, 1)$*. If* $k = 1$ *in the last inequality, we say that* T *is pseudo-contractive and if* $k = 0$*, then* T *is simply nonexpansive.*
(xiii) k-demicontractive if $F(T) \neq \emptyset$ *and there exists a constant* $k \in [0, 1)$ *such that for any* $p \in F(T)$*, we have*

$$||Tx - p||^2 \leq ||x - p||^2 + k||x - Tx||^2, \quad \forall\ x\ \in H;$$

(ix) hemicontractive mapping if $F(T) \neq \emptyset$ *and*

$$||Tx - p||^2 \leq ||x - p||^2 + ||x - Tx||^2, \quad \forall\ x\ \in H \text{ and } p \in F(T).$$

Remark 2.1.2. *(i) It is easy to see that the class of demicontractive mappings properly contains that of quasi-nonexpansive mappings. (Check Example 2.6 in [26] to see that the inclusion is proper).*
(ii) The class of pseudo-contractive mappings with nonempty fixed points is a subclass of the class of hemicontraction. (Check Rhoades [62] to see that the inclusion is proper).

2.1.2 Nonlinear Multi-Valued Mappings

Given a real Hilbert space H, we denote by CB(H) the family of nonempty, closed and bounded subsets of H. It is well known that the Hausdorff distance $\mathcal{H}$ defined by

$$\mathcal{H}(A,B) := \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\}, \tag{2.1}$$

is a metric on CB(H). A subset K of H is called proximinal if for each $x \in H$, there exists an element $k \in K$ such that

$$d(x,k) = d(x,K) = \inf\big\{d(x,y); y \in K\big\}. \tag{2.2}$$

We denote the family of all bounded proximinal subsets of a set K in H by $P(K)$.
A mapping $T : Dom(T) \subseteq H \to CB(H)$ is said to be
(i) Lipschitzian if there exists $L > 0$ such that for each $x, y \in Dom(T)$,

$$\mathcal{H}(Tx,Ty) \leq L||x-y||;$$

(ii) contraction, if there exists a constant $k \in [0,1)$ such that for any $x, y \in Dom(T)$,

$$\mathcal{H}(Tx,Ty) \leq k||x-y||;$$

(iii) nonexpansive if for all $x, y \in Dom(T)$,

$$\mathcal{H}(Tx,Ty) \leq ||x-y||;$$

(iv) k-nonspreading if for all $x, y \in Dom(T)$,

$$\mathcal{H}(Tx,Ty)^2 \leq k(d(Tx,y)^2 + d(x,Ty)^2); \tag{2.3}$$

(v) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx,Tp) \leq ||x-p|| \ \forall\ p \in F(T),\ x \in Dom(T);$$

(vi) k-strictly pseudocontractive if there exists a constant $k \in (0,1)$ such that for all $x, y \in Dom(T)$,

$$(\mathcal{H}(Tx,Ty))^2 \leq ||x-y||^2 + k||x-y-(u-v)||^2 \quad \forall\ u \in Tx, v \in Ty; \tag{2.4}$$

(vii) demicontractive if $F(T) \neq \emptyset$ and there exists a constant $k \in (0,1)$ such that for all $x \in Dom(T),\ p \in F(T)$,

$$(\mathcal{H}(Tx,Tp))^2 \leq ||x-p||^2 + k(d(x,Tx))^2. \tag{2.5}$$

Remark 2.1.3. *If $k = 1$ in (2.3), (2.4) and (2.5), we have a new set of mappings called nonspreading, pseudo-contractive and hemicontractive respectively.*

2.1.3 Other Important Concepts

We give the following important definitions which are used in subsequent chapters.

Definition 2.1.4. *Let D be a convex subset of a vector space X and $f : D \to \mathbb{R} \cup \{+\infty\}$ be a map. Then,*
(i) f is convex if for each $\lambda \in [0, 1]$ and $x, y \in D$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y);$$

(ii) f is called proper if there exists at least one $x \in D$ such that

$$f(x) \neq +\infty;$$

(iii)f is lower semi-continuous at $x_0 \in D$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x);$$

(iv) f is upper semi-continuous at $x_0 \in D$ if

$$f(x_0) \geq \limsup_{x \to x_0} f(x).$$

Definition 2.1.5. *Let C be a nonempty, closed and convex subset of a Hilbert space H. An operator $T : C \to H$ is Fejer monotone with respect to a subset V of C if for all $x \in C$ and $z \in V$, we have*

$$||Tx - z|| \leq ||x - z||.$$

If T is Fejer monotone with respect to $F(T)$, then T is called quasi-nonexpansive.

Definition 2.1.6. *Let C be a nonempty, closed and convex subset of H. The sequence $\{x_k\}_{k\geq 1}$ is Fejer monotone with respect to C if*

$$||x_{k+1} - z|| \leq ||x_k - z|| \ \forall \ k \in \mathbb{N}, z \in C.$$

Definition 2.1.7. *Let H be a real Hilbert space and $A : H \to H$ be a bounded linear map. We define a map $A^* : H \to H$ by the relation*

$$\langle Ax, y\rangle = \langle x, A^*y\rangle,$$

for all $x, y \in H$. The map A^* is called the adjoint/dual of A.

Definition 2.1.8. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $\alpha > 0$. An operator $T : C \to H$ is said to be α-strongly Fejer monotone with respect to a subset V of C, or strongly Fejer monotone if*

$$||Tx - z||^2 \leq ||x - z||^2 - \alpha||Tx - x||^2.$$

2.2 Spaces of Interest

2.2.1 Banach Spaces

Definition 2.2.1. *Let V be a vector space over a scalar field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A norm on V is a map $||.|| : V \to \mathbb{R}$ such that for all x, y in V and α in $\mathbb{R}$, the following conditions holds:*
(a) $||x|| \geq 0$,
(b) $||x|| = 0 \Leftrightarrow x = 0$,
(c) $||\alpha x|| = |\alpha| ||x||$,
(d) $||x + y|| \leq ||x|| + ||y||$.
The pair $(V, ||.||)$ is called a normed linear space.

Example 2.2.2. *(i) Let $X = C[a, b]$. For $f \in X$, let*

$$||f||_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \quad (1 \leq p < \infty),$$
$$||f||_\infty = \sup_{x \in [a,b]} |f(x)|.$$

Then $||.||_p$ and $||.||_\infty$ are norms on X.
(ii) Let $X = C^1[a, b]$ be the space of all continuous and differentiable real valued functions on $[a, b]$. For $f \in C^1[a, b]$,

$$||f|| = \max_{a \leq x \leq b} |f(x)| + \max_{a \leq x \leq b} \left| \frac{df(x)}{dx} \right|$$

is a norm on $C^1[a, b]$.

Definition 2.2.3. *A sequence $\{x_n\}_{n=1}^{\infty}$ in a normed space is called Cauchy if and only if*

$$||x_n - x_m|| \to 0 \quad \text{as } n, \ m \to \infty.$$

Definition 2.2.4. *A normed vector space X is said to be complete if every Cauchy sequence in X converges to a point in X, and a complete normed space is called a Banach space.*

Example 2.2.5. *The space $X = C[a, b]$ is complete with the norm*

$$||f||_\infty = \sup_{x \in [a,b]} |f(x)| \quad (f \in C[a, b]),$$

and incomplete with any of the norms

$$||f||_p = \left(\int_a^b |f(x)|^p dx \right) \quad (1 \leq p < \infty).$$

2.2.2 Hilbert Spaces

Definition 2.2.6. *Let X be a linear space over a field $\mathbb{F}$ (where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). An inner product on X is a scalar-valued function $\langle .,. \rangle : X \times X \to \mathbb{F}$ such that for all $x, y, z \in X$ and for all $\alpha, \beta \in \mathbb{F}$, we have*
(i) $\langle x, x \rangle \geq 0$;
(ii) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$;
(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *(The bar denotes complex conjugation.);*
(iv) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$;
(v) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.
The pair, $(X, \langle .,. \rangle)$ is called an inner product space.

An inner product space X is said to be complete if every Cauchy sequence in X converges to a point of X. A complete inner product space is called a Hilbert space.

Example 2.2.7. *(i)* **Finite Dimensional Vectors**. $\mathbb{C}^{\mathbb{N}}$ *is the space of $N-$tuples $x = (x_1, ..., x_N)$ of complex numbers. It is a Hilbert space with the inner product*

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n^* y_n. \tag{2.6}$$

(ii) **Square Integrable Functions on $\mathbb{R}$.** *$L^2(\mathbb{R})$ is the space of complex valued functions such that*

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty. \tag{2.7}$$

It is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f^*(x) g(x) dx. \tag{2.8}$$

(iii) **Square Integrable Functions on $\mathbb{R}^n$.** *Let Ω be an open set in $\mathbb{R}^n$ (in particular, Ω can be the whole $\mathbb{R}^n$). The space $L^2(\Omega)$ is the set of complex valued functions such that*

$$\int_{\Omega} |f(x)|^2 dx < \infty, \tag{2.9}$$

where $x = (x_1, ..., x_n) \in \Omega$ and $dx = dx_1 ... dx_n$. It is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\Omega} f^*(x) g(x) dx. \tag{2.10}$$

The easiest Banach space to work on is the real Hilbert space. This is because of the simplicity of its geometry. Always available in a Hilbert space is the parallelogram identity

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2), \tag{2.11}$$

this is equivalent to the polarisation identity

$$||x+y||^2 = ||x||^2 + 2Re\langle x, y\rangle + ||y||^2. \tag{2.12}$$

The following equations also hold in Hilbert spaces,

$$||x-y||^2 = ||x||^2 - ||y||^2 - 2\langle x-y, y\rangle, \tag{2.13}$$

$$||x+y||^2 \le ||x||^2 + 2\langle y, x+y\rangle, \tag{2.14}$$

and

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2, \tag{2.15}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

2.3 Multi-Valued Strictly Pseudo-Contractive Mappings

Fixed point theory for multi-valued mappings have been studied by many authors (see [23, 32, 60, 69] and the references therein) and has attracted a lot of attention because of its numerous applications in game theory, differential inclusions and constrained optimization. They are also used in devising critical points in energy management problems, optimal control problems, signal processing, image reconstruction and others. Existence of Nash equilibria of a non-cooperative game can be proved using fixed point theory for multi-valued mappings. However, the study of fixed point for strictly pseudo-contractive mappings helps in the study of fixed point theory for nonexpansive mappings and for Lipschitz pseudo-contractive mappings, since every nonexpansive mappings are pseudo-contractive and continuous but a pseudo-contractive mapping is not necessarily continuous.

In 1967, Browder [10] studied the operator equation $Au = 0$ (where the mapping A is monotone). In the course of studying this operator, he introduced a new operator T defined by $T := I - A$, where I is the identity mapping on Hilbert space H. This operator was called a pseudo-contractive mapping and the solutions of Au=0 are exactly the fixed points of the pseudo-contractive mapping T. The nonexpansive mappings is an important subclass of the pseudo-contractive mappings.

In 1973, Markin [47] did the first work on fixed points for multi-valued nonexpansive mappings using the Hausdorff metric, followed by an extensive work by Nadler [51]. Since then, several works have been done by many authors on the approximation of fixed points of multi-valued nonexpansive mappings (see [1, 5, 38, 57] and the references therein). Among

the iterative schemes, Sastry and Babu [64] introduced the Mann and Ishikawa iteration as follows:
Let $T : H \to P(H)$ and p be a fixed point of T. The sequence of Mann iterates is given by $x_0 \in H$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \quad n \geq 0,$$

where $y_n \in Tx_n$ is such that $||y_n - p|| = d(Tx_n, p)$ and α_n is a real sequence such that $0 \leq \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.
The sequence of Ishikawa iterates is given by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{2.16}$$

where $z_n \in Tx_n, u_n \in Ty_n$ are such that $||z_n - p|| = d(p, Tx_n), ||u_n - p|| = d(Ty_n, p)$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences satisfying
(i) $0 \leq \alpha_n, \beta_n < 1, (ii)$ $\beta_n \to 0$ as $n \to \infty, (iii) \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.
Nadler [51] gave the following useful result.

Lemma 2.3.1. *Let $A, B \in CB(X)$ and $a \in A$. if $\gamma > 0$, then there exists $b \in B$ such that*

$$d(a, b) \leq \mathcal{H}(A, B) + \gamma. \tag{2.17}$$

Using Nadler remark, Song and Wang [67] extended the result of Sastry and Babu [64] to uniformly convex spaces and made an important observation that generating Mann and Ishikawa sequences in [64] is in some sense depends on the knowledge of fixed point. They gave their iterative scheme as follows

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \end{cases}$$

where $z_n \in Tx_n, u_n \in Ty_n$ satisfies $||z_n - u_n|| \leq \mathcal{H}(Tx_n, Ty_n) + \gamma_n, ||z_{n+1} - u_n|| \leq \mathcal{H}(Tx_{n+1}, Ty_n) + \gamma_n$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1)$ satisfying $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.
Using the above iteration, they proved the following theorem.

Theorem 2.3.2. *[67]. Let K be a nonempty, compact and convex subset of a uniform convex space X. Suppose that $T : K \to CB(K)$ is a multi-valued non-expansive mapping such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Then the Ishikawa sequence defined above converges strongly to a fixed point of T.*

Shahzad and Zegeye [65] observed that if E is a normed space and $T : Dom(T) \subset E \to P(E)$ is any multi-valued mapping, then $P_T : Dom(T) \to P(E)$ defined for each x by

$P_T(x) = \{y \in Tx : d(x, Tx) = ||x - y||\}$ has the property that $P_T(q) = \{q\}$ for all $q \in F(T)$. Using this idea, they removed the strong condition $T(p) = \{p\}$ for all $p \in F(T)$ introduced by Song and Wang [67].

Recently, Chidume *et al.* [25] introduced a multi-valued strictly pseudo-contractive mapping different from that of Browder and Petryshyn. They gave the following definition:

Definition 2.3.3. *Let H be a real Hilbert space and D a nonempty, open and convex subset of H. Let $T : \overline{D} \to CB(\overline{D})$ be a mapping. Then, T is called a multi-valued k-strictly pseudo-contractive mapping if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$, we have*

$$\mathcal{H}^2(Tx, Ty) \leq ||x - y||^2 + k||(x - u) - (y - v)||^2, \tag{2.18}$$

for all $u \in Tx$ and $v \in Ty$.

They proved a convergence theorem using a certain Krasnoselskii's type algorithm for this class of mapping as follows:

Theorem 2.3.4. *[25]. Let K be a nonempty, closed and convex subset of a real Hilbert space H. Suppose $T : K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume $Tp = \{p\}$ for all $p \in F(T)$ and T is hemicompact and continuous. Let $\{x_n\}$ be a sequence defined iteratively from $x_0 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n,$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1 - k)$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

However, we noticed that none of the aforementioned authors proved their theorems without imposing the condition that neither T has a strict fixed point nor P_T satisfies any of the contractive conditions studied so far by authors.

Isiogugu *et al.* [35], however suggested to approximate the fixed points of multivalued mappings T directly instead of P_T and without imposing the strict fixed point set condition on T. This suggestion was also due to the fact that it has not been established that if a multi-valued map T belongs to a class of maps, then P_T necessarily belongs to the same class of maps and that of fixed point set of T need not be strict. Consequently, they obtained a weak and a strong convergence results for the class of pseudo-contractive and strictly pseudo-contractive mappings of Chidume et al.[25] respectively in Hilbert spaces. We first define Isiogugu type-one mapping and state the weak convergence theorem.

Definition 2.3.5. *[35]. Let E be a normed space and $T : Dom(T) \subseteq E \to 2^E$ be a multi-valued map. T is said to be of type one if given any pair $r, g \in Dom(T)$, we have*

$$||u - v|| \leq \mathcal{H}(Tr, Tg),$$

for all $u \in P_T r$ and $v \in P_T g$.

Theorem 2.3.6. *[36] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : C \to P(C)$ is k-strictly pseudo-contractive mapping from C into the family of all proximinal subsets of C with $k \in (0,1)$ such that $F(T) \neq \emptyset$. If T is of type-one, then the Mann- type sequence defined by*

$$r_{n+1} = (1-\mu_n)r_n + \mu_n g_n,$$

weakly converges to $q \in F(T)$, where $g_n \in Tr_n$ with $||r_n - g_n|| = d(r_n, Tr_n)$ and $\mu_n \subseteq (0,1)$ satisfying
(i) $\mu_n \to \mu < 1-k$, (ii) $\mu > 0$, and (iii) $\sum_{n=1}^{\infty} \mu_n(1-\mu_n) = \infty$.

Chidume and Okpala [22] introduced a different class of multi-valued strictly pseudo-contractive mapping called the generalized k-strictly pseudo-contractive mapping which is more general than the class introduced in [25]. They gave the following definition for this class of mapping.

Definition 2.3.7. *[22]. Let H be a real Hilbert space and K a nonempty subset of H. Let $T : K \to CB(K)$ be a multi-valued mapping. Then T is called generalized k-strictly pseudo-contractive multi-valued mapping if there exist $k \in (0,1)$ such that for all $x, y \in D(T)$, there holds*

$$\mathcal{H}^2(Tx, Ty) \leq ||x-y||^2 + k\mathcal{H}^2(Ax, Ay), \tag{2.19}$$

where $A := I - T$ and I is the identity operator on K.

Using the definition above, the authors in [22] proved the following theorem and obtained a strong convergence result under some additional conditions.

Theorem 2.3.8. *[22]. Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : K \to CB(K)$ be a generalized k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume $Tp = \{p\}\ \forall\ p \in F(T)$. Define a sequence $\{x_n\}$ by $x_0 \in K$, such that*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n,$$

for $y_n \in U^n$ and $\lambda \in (0, 1-k)$. Then, $d(x_n, Tx_n) \to 0$ as $n \to \infty$, where

$$U^n := \big\{y_n \in Tx_n : \mathcal{H}^2(\{x_n\}, Tx_n) \leq ||x_n - y_n||^2 + \frac{1}{n^2}\big\}. \tag{2.20}$$

Using the lemma stated below, they gave an example to show that this general class of k-strictly pseudo-contractive multi-valued mappings exists and properly contains the class studied by Chidume *et al.*[25] and a host of others.

Lemma 2.3.9. *[22]. Let a, b, c be real numbers such that $0 \leq a \leq bc, c > 0$. Then*

$$(a-b)^2 \leq b^2 + \big(\frac{c-2}{c}\big)a^2 \tag{2.21}$$

Proof. Since $0 \leq a \leq bc,\ c > 0$. Then we have that

$$\begin{aligned}
&\Rightarrow a^2 \leq abc \\
&\Rightarrow \frac{a^2}{c} \leq ab \\
&\Rightarrow ab \geq \frac{a^2}{c} \\
&\Rightarrow -2ab \leq \frac{-2a^2}{c} \\
&\Rightarrow a^2 - 2ab + b^2 \leq a^2 - \frac{2a^2}{c} + b^2 \\
&\Rightarrow (a-b)^2 \leq b^2 + \big(\frac{c-2}{c}\big)a^2.
\end{aligned}$$

□

Example 2.3.10. *[23]. Define a multi-valued mapping $T_i : l_2(\mathbb{R}) \to CB(l_2(\mathbb{R}))$ by*

$$T_i x := \begin{cases} \{y \in l_2 : ||x+y|| \leq \alpha_i ||x||\}, & x \neq 0 \\ \{0\}, & x = 0. \end{cases} \tag{2.22}$$

Where $\alpha_i = \frac{7i}{3i-1}$, for $i = 1, 2, ...,$. We say that

$$x - T_i x := \begin{cases} \{y \in l_2 : ||y - 2x|| \leq \alpha_i ||x||\}, & x \neq 0 \\ \{0\}, & x = 0. \end{cases}$$

Then, for arbitrary $x, y \in l_2(\mathbb{R})$, we compute as follows:

$$\mathcal{H}(T_i x, T_i y) = ||x-y|| + \alpha_i \big| ||x|| - ||y|| \big|,$$

and

$$\mathcal{H}(x - T_i x, y - T_i y) = 2||x-y|| + \alpha_i \big| ||x|| - ||y|| \big|.$$

Now, set

$$a := \mathcal{H}(x - T_i x, y - T_i y); b := ||x-y||.$$

Then, $a - b = \mathcal{H}(T_ix, T_iy)$ *and*

$$\begin{aligned} a &= 2||x-y|| + \alpha_i|||x|| - ||y||| \\ &\leq (2+\alpha_i)||x-y||. \end{aligned}$$

So for each i*, set* $2 = \alpha_i = c_i = c$ *in Lemma 2.3.9. We obtain the identity* $\frac{c_i - 2}{c_i} = \frac{\alpha_i}{2+\alpha_i}$*, and applying the same lemma, we have that*

$$\mathcal{H}^2(T_ix, T_iy) \leq ||x-y||^2 + \frac{\alpha_i}{2+\alpha_i}\mathcal{H}(x - T_ix, y - T_iy).$$

Hence, each $T_i, i = 1, 2, ...,$ *is a generalized* k_i-*strictly pseudo-contractive multi-valued mapping with* $k_i = \frac{\alpha_i}{2+\alpha_i} \in (0,1)$ *and each* $k_i \leq k := \frac{7}{13}$*. Moreover, we have* $p \in T_ip$ *if and only if* $p = 0$*. Hence,* $p \in \cap_{i=1}^{\infty} F(T_ip), T_ip = \{p\}$*.*

Very recently, Chidume and Okpala [23] introduced the following iterative algorithm for approximating a common fixed point of a countable infinite family of generalized k-strictly pseudo-contractive multi-valued mappings.

$$\begin{cases} x_0 \in K, arbitrary, \\ \zeta_n^i \in \Gamma_n^i, \\ x_{n+1} = \delta_0 x_n + \sum_{i=1}^{\infty} \delta_i \zeta_n^i, \\ \delta_0 \in (k,1), \sum_{i=1}^{\infty} \delta_i = 1, \end{cases} \tag{2.23}$$

where K is a nonempty, closed and convex subset of a real Hilbert space H, $k \in (0,1)$ and

$$\Gamma_n^i := \{\zeta_n^i \in T_ix_n = \mathcal{H}^2(\{x_n\}, T_ix_n) \leq ||x_n - \zeta_n^i||^2 + \frac{1}{n^2}\}.$$

They obtained a strong convergence result under some mild assumptions using the iterative scheme .

2.4 Quasi-Nonexpansive Mapping

The concepts of quasi-nonexpansive mapping was initiated by Tricomi [73] in 1941 for real-valued functions. Since then many authors have studied and applied this class of mapping to approximate solutions of fixed point problems and other related problems in both Hilbert spaces and Banach spaces (see [50, 71, 77, 78] and the references therein). It is well-known that every nonexpansive multi-valued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive, but not all quasi-nonexpansive mappings are nonexpansive (check [33] and Example 4.1 in [63] to see that the inclusion is proper). It is also known that if T is a quasi-nonexpansive multi-valued map, then F(T) is closed (see [33] for the proof). Several other mappings have been defined in terms of quasi-nonexpansive mapping by adding one or two conditions. For instance, Suantai *et al.* [69] introduced a class of nonspreading multi-valued mappings and gave the following definition for this class of mappings.

Definition 2.4.1. *Let C be a nonempty, closed and convex subset of a Hilbert space H, then a mapping $T : C \to CB(C)$ is a k-nonspreading multi-valued mapping if there exists a constant $k > 0$ such that*

$$\mathcal{H}(Tx, Ty)^2 \le k(d(Tx, y)^2 + d(x, Ty))^2, \tag{2.24}$$

for all $x, y \in C$. Furthermore, they assumed that if T is a $\frac{1}{2}$-nonspreading and $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping. Indeed for all $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned} 2\mathcal{H}(Tx, Tp)^2 &\le d(Tx, p)^2 + d(x, Tp)^2 \\ &\le \mathcal{H}(Tx, Tp)^2 + ||x - p||^2. \end{aligned}$$

Hence, we have that

$$\mathcal{H}(Tx, Tp) \le ||x - p||.$$

Suantai *et al.* [69] gave an example of $\frac{1}{2}$-nonspreading multi-valued mapping which is not nonexpansive.

Example 2.4.2. *[69] Let $H = \mathbb{R}$ and consider $C = [-3, 0]$ with the usual norm. Define $T : C \to CB(C)$ by*

$$Tx = \begin{cases} \{0\}, & x \in [-2, 0], \\ \left[\frac{-|x|}{|x|+1},\ 0\right], & x \in [-3, -2). \end{cases}$$

We have the following cases:
Case 1: If $x, y \in [-2, 0]$, then $\mathcal{H}(Tx, Ty) = 0$.
Case 2: If $x \in [-2, 0]$ and $y \in [-3, -2)$, then $Tx = \{0\}$ and $Ty = \left[\frac{-|y|}{|y|+1}, 0\right]$. This implies that

$$2\mathcal{H}(Tx, Ty)^2 = 2\Big(\frac{|y|}{|y| + 1}\Big)^2 < 2 < y^2 \le d(Tx, y)^2 + d(x, Ty)^2.$$

Case 3: If $x, y \in [-3, -2)$, then $Tx = [\frac{-|x|}{|x|+1},\ 0]$ and $Ty = [\frac{-|y|}{|y|+1},\ 0]$. This implies that

$$2\Big(\frac{|x|}{|x| + 1} - \frac{|y|}{|y| + 1}\Big) < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

Hence, T is not nonexpansive since $x = -2$ and $y = -\frac{5}{2}$, we have $Tx = \{0\}$ and $Ty = [-\frac{5}{7},\ 0]$. This shows that $\mathcal{H}(Tx, Ty) = \frac{5}{7} > \frac{1}{2} = \big| -2 - (-\frac{5}{2})\big| = ||x - y||$.

Also, Sastry and Babu [64] gave the definition of a generalized nonexpansive mapping and showed that if T is a multi-valued generalized nonexpansive mapping, then it is a multi-valued quasi-nonexpansive mapping.

Definition 2.4.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H, a multi-valued mapping $T : C \to CB(C)$ is said to be generalized nonexpansive if*

$$\mathcal{H}(Tx, Ty) \leq a||x - y|| + b(d(x, Tx) + d(y, Ty)) + c(d(x, Ty) + d(y, Tx)), \quad \text{for all } x, y \in C,$$

where $a + 2b + 2c \leq 1$.

Indeed, if $p \in F(T)$ and T is a generalized nonexpansive mapping, we obtain that for all $x \in C$,

$$\begin{aligned}\mathcal{H}(Tx, Tp) &\leq a||x - p|| + b(d(x, Tx) + d(p, Tp) + d(p, Tx)) \\ &\leq a||x - p|| + b(||x - p|| + d(p, Tx)) + c(d(x, Tp) + d(p, Tx)) \\ &\leq (a + b + c)||x - p|| + (b + c)d(p, Tx) \\ &\leq (a + b + c)||x - p|| + (b + c)\mathcal{H}(Tx, Tp).\end{aligned}$$

Hence, $\mathcal{H}(Tx, Tp) \leq \frac{a+b+c}{1-(b+c)} \leq 1$, it follows that

$$\mathcal{H}(Tx, Tp) \leq ||x - p||.$$

We give examples of single-valued and multivalued quasi-nonexpansive mappings.

Example 2.4.4. *Let $H = \mathbb{R}$ endowed with the usual norm. Let $C = Q =: [0, \infty)$, and define the mapping $T : C \to \mathbb{R}$ and $S : Q \to \mathbb{R}$ by $Tx = \frac{x^2+9}{3+x}$ for all $x \in C$ and $Sx = \frac{x+8}{8}$, for all $x \in Q$. Then T and S are quasi-nonexpansive mappings.*

Proof. Trivially, $F(T) = \{3\}$ and $F(S) = \{\frac{8}{7}\}$. Using the definition of a quasi-nonexpansive mapping, we have that

$$\begin{aligned}|Tx - 3| &= \left|\frac{x^2 + 9}{3 + x} - 3\right| = \frac{x}{3 + x}|x - 3| \\ &\leq |x - 3|.\end{aligned}$$

On the other hand,

$$\left|Sx - \frac{8}{7}\right| = \left|\frac{x + 8}{8} - \frac{8}{7}\right| = \frac{1}{8}\left|x - \frac{8}{7}\right| \leq \left|x - \frac{8}{7}\right|.$$

Hence, we conclude that T and S are quasi-nonexpansive mappings. □

Example 2.4.5. *Let $H = \mathbb{R}$ (endowed with the usual metric) and $T : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by*

$$T_x = \begin{cases} [0, \frac{x}{2}] & x \in [0, \infty), \\ [\frac{x}{2}, 0] & x \in (-\infty, 0]. \end{cases}$$

Observe that $F(T) = \{0\}$, for each $x \in (-\infty, 0) \cup (0, \infty)$, we have

$$\mathcal{H}(Tx, 0) = |\frac{x}{2} - 0| = \frac{1}{2}|x - 0| < |x - 0|,$$

for each $i = 1, 2, ...$ Hence, T is quasi-nonexpansive.

2.5 Metric Projection

Let C be a nonempty, closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$||x - P_C(x)|| \leq ||x - y||, \ \forall \ y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies:

$$||P_C(x) - P_C(y)|| \leq \langle x - y, P_C(x) - P_C(y)\rangle. \tag{2.25}$$

Moreover, $P_C(x)$ is characterized by the following properties:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \tag{2.26}$$

and

$$||x - y||^2 \geq ||x - P_C(x)||^2 + ||y - P_C(x)||^2, \ \forall \ x \in H, y \in C. \tag{2.27}$$

For all $x, y \in H$, it is well known that every nonexpansive operator $T : H \to H$ satisfies the inequality below

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq \frac{1}{2}||(T(x) - x) - (T(y) - y)||^2, \tag{2.28}$$

and therefore, we have that for all $x \in H$ and $y \in F(T)$,

$$\langle x - T(x), y - T(x)\rangle \leq \frac{1}{2}||T(x) - x||^2. \tag{2.29}$$

We now list some examples of metric projections in Hilbert spaces.

Example 2.5.1. *Let $C = [a, b]$ be a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, ..., a_n)^T$ and $b = (b_1, b_2, ..., b_n)^T$, then for $1 \leq i \leq n$, we have*

$$(P_C x)i = \begin{cases} a_i, & x_i < a_i, \\ x_i, & x_i \in [a_i, b_i], \\ b_i, & x_i > b_i \end{cases}$$

is the metric projection with the i^{th} coordinate.

Example 2.5.2. *Let C be the range of an $m \times n$ matrix with full column rank and A^* be the adjoint of A, then*

$$P_C x = A(A^*A)^{-1}A^*x$$

is the metric projection P_C onto C.

2.6 Split Feasibility Problem

In 1994, Censor and Elfving [15] introduced the Split Feasibility Problem (SFP) in finite-dimensional Hilbert spaces. This problem has been used for modelling inverse problems which arise from phase retrievals and in medical image reconstruction. They gave the following definition for the problem.

Definition 2.6.1. *Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. The SFP is formulated as follows:*

$$\text{find } x^* \in C \quad \text{such that } Ax^* \in Q. \tag{2.30}$$

The SFP arises in many areas of applications such as computer tomography, image restoration, phase retrieval and radiation therapy treatment planning, (see [15, 17, 16, 53, 66] and the references therein). Some authors have proposed different methods of solving the SFP, (see [11, 14, 61, 81] and the references therein) .
Suppose that SFP (2.30) is consistent i.e has a solution, then it can be easily seen that

$$x^* = P_C(I + \gamma A^*(P_Q - I)Ax^*), \quad \forall x \in C, \tag{2.31}$$

where P_C and P_Q are the orthogonal projections onto C and Q respectively, $\gamma > 0$ and A^* denotes the adjoint of A. This implies that x^* solves SFP (2.30) if and only if x^* solves fixed point equation (2.31).
Recently, Moudafi [48] introduced the following new SFP which is known as the general split equality problem. This problem comprises of many applications, for instance, in decomposition methods for PDE'S, game theory and in intensity-modulated radiation therapy. He gave the following definition for this problem:

Definition 2.6.2. *Let H_1, H_2 and H_3 be real Hilbert spaces, $C \subset H_1, Q \subset H_2$ be two nonempty, closed and convex sets, $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators. The new split feasibility problem is to find*

$$x^* \in C, \ y^* \in Q \quad \text{such that} \quad Ax^* = By^*. \tag{2.32}$$

It is easy to see that problem (2.32) reduces to (2.30) if $H_2 = H_3$ and $B = I$, where I is the identity mapping H_2 to H_2.
Many authors have introduced some useful methods to solve some kinds of general split feasibility problem in real Hilbert spaces under some suitable conditions. Some strong convergence theorems have been proved (see [21, 34] and the references there in).

2.7 Equilibrium Problems

In 1994, Blum and Oettli [7] introduced the equilibrium problem, this problem provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. Equilibrium problems includes the variational inequality problem, Nash equilibrium and game theory as special cases.
Let C be a nonempty, closed and convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ be a nonlinear bifunction. Then the Equilibrium Problem (for short, EP) is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall \ y \in C. \tag{2.33}$$

The set of solution of EP is denoted by $EP(F)$.
Let $\alpha : C \to \mathbb{R} \cup \{+\infty\}$ be a function. The Mixed Equilibrium Problem (for short, MEP) is to find $x^* \in C$ such that

$$F(x^*, y) + \alpha(y) - \alpha(x^*) \geq 0, \quad \forall y \in C. \tag{2.34}$$

The set of solution of MEP is denoted by $MEP(F, \alpha)$. The MEP includes several important problems arising in engineering, physics, science, optimization, economics, transportation, network and structural analysis.
If $\alpha = 0$, then the MEP (2.34) reduces to (2.33). If $F = 0$, then the MEP (2.34) reduces to the following convex minimization problem:

$$\text{Find } x^* \in C \text{ such that } \alpha(y) \geq \alpha(x^*), \quad \forall \ y \in C. \tag{2.35}$$

The set of solutions of (2.35) is denoted by $CMP(\alpha)$.
In 2013, Kazmi and Rizvi [37] introduced and studied the following split equilibrium problem:

Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the Split Equilibrium Problem (SEP) is to find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \forall\, x \in C, \tag{2.36}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves } G(y^*, y) \geq 0, \quad y \in Q. \tag{2.37}$$

The inequalities (2.36) and (2.37) constitute a pair of equilibrium problems. The image $y^* = Ax^*$ of the solution of (2.36) in H_1 under a given bounded linear operator A, is also the solution of (2.37) in H_2. We denote the solution set of (2.36) and (2.37) by $EP(F)$ and $EP(G)$ respectively.

The solution set of SEP (2.36) and (2.37) is denoted by $\Theta := \big\{p \in EP(F) : Ap \in EP(G)\big\}$. Recently, Kazmi and Rizvi [37] introduced the following iterative scheme to approximate a common solution of SEP, a variational inequality problem and a fixed point problem for nonexpansive mapping in real Hilbert spaces.

$$\begin{aligned} u_n &= J^F_{r_n}(x_n + \gamma A^*(J^G_{r_n} - I)Ax_n); \\ y_n &= P_c(u_n - \lambda_n D u_n); \\ x_{n+1} &= \alpha_n v + \beta_n x_n + \gamma_n S y_n; \end{aligned}$$

where $r_n \subset (0, \infty), \lambda_n \in (0, 2\tau)$ and $\gamma \in (0, \frac{1}{L})$. L is the spectral radius of the operator A^*A, where A^* is the adjoint of A and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0,1)$. Furthermore, they proved a strong convergence result. See theorem (3.1) in [37] for details.

Definition 2.7.1. *Let $F : C \times C \to \mathbb{R}$, $G : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Then the Split Equality Equilibrium Problem (SEEP) is to find $x^* \in C$ and $y^* \in Q$ such that*

$$F(x^*, x) \geq 0 \quad \forall\, x \in C, \qquad G(y^*, y) \geq 0, \quad y \in Q \text{ and } Ax^* = By^*. \tag{2.38}$$

The set of solutions of (2.38) is denoted by $SEEP(F, G)$.

Very recently, Ma *et al.* [42] studied the following Split Equality Mixed Equilibrium Problem (SEMEP).

Definition 2.7.2. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $\alpha : C \to \mathbb{R} \cup \{+\infty\}, \beta : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap dom\alpha \neq \emptyset$ and $Q \cap dom\beta \neq \emptyset$. Let*

$A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators, then the SEMEP is to find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} F(x^*, x) + \alpha(x) - \alpha(x^*) \geq 0, \ \forall \ x \ \in C, \\ G(y^*, y) + \beta(y) - \beta(y^*) \geq 0, \ \forall \ y \ \in Q \\ \text{and } Ax^* = By^*. \end{aligned} \tag{2.39}$$

The set of solutions of (2.39) is denoted by $SEMEP(F, G, \alpha, \beta)$.

Using their algorithm, Ma *et al.* [42] obtained a weak and a strong convergence result for $SEMEP$ and fixed point problem for nonexpansive mappings in real Hilbert spaces. They proved the following theorem.

Theorem 2.7.3. *Let H_1, H_2 and H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex. Assume that $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions satisfying (A1)-(A5). Let $\alpha : C \to \mathbb{R} \cup \{+\infty\}$ and $\beta : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap dom\alpha \neq \emptyset$ and $Q \cap dom\beta \neq \emptyset$. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $(x_1, y_1) \in C \times Q$ and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} F(u_n, u) + \alpha(u) - \alpha(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \ \forall u \in C; \\ G(v_n, v) + \beta(v) - \beta(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \ \forall v \in Q; \\ x_{n+1} = \phi_n u_n + (1 - \phi_n)T(u_n - \rho_n A^*(Au_n - Bv_n)), \forall n \geq 1; \\ y_{n+1} = \phi_n v_n + (1 - \phi_n)S(v_n + \rho_n B^*(Au_n - Bv_n)), \ \forall n \geq 1; \end{cases}$$

where λ_A and λ_B are spectral radii of A^*A and B^*B respectively, $\{\rho_n\}$ is a positive real sequence such that $\rho_n \in \left(\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon\right)$ (for ε small enough), $\{\phi_n\}$ is a sequence in $(0, 1)$ and $r_n \subset (0, \infty)$ satisfies the following conditions:
(i) $0 < \phi \leq \phi_n \leq \theta < 1$ (for some $\phi, \theta \in (0, 1)$),
(ii) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.
If $\Gamma := F(T) \cap F(S) \cap SEMEP(F, G, \alpha, \beta) \neq \emptyset$, then
(1) the sequence $\{(x_n, y_n)\}$ converges weakly to a solution of (2.39).
(2) In addition, if S, T are semi-compact, then $\{(x_n, y_n)\}$ converges strongly to a solution of (2.39).

2.8 Variational Inequality Problems

Variational inequalities were formulated between the end of 60' and the beginning of 70' of previous century by an Italian mathematician named G. Stampacchia [68]. In recent years, many authors have extended and generalized variational inequality theory in several directions using new powerful methods in a unified and general framework. The theory of variational inequalities constitute a very natural generalization of the theory of boundary value problems which permit us to consider new problems arising from many fields of applied mathematics such as physics, mechanics, engineering, theory of convex programming and theory of control. The theory of variational inequalities has its origin in the projection on a convex set, this provides a general framework for a wide range of mathematical problems such as optimization problems, EP, engineering sciences (EP in a traffic network) and in the economic sciences (oligopolistic market EP). Many authors have studied and computed the solutions of variational inequalities using iterative algorithms (see [37, 8, 76, 59, 18, 28]). The development of the finite-dimensional variational inequalities began in the mid-1960s but followed a different path compared to its infinite-dimensional counterpart which was conceived in the area of partial differential systems. The finite dimensional variational inequality was born in the domain of mathematical programming and have its developments in mathematical theory and a multitude of interesting connections to numerous disciplines.

Definition 2.8.1. *Let H be a real Hilbert space and C a nonempty, closed and convex subset of H, then the variational inequality problem (in short, VIP) is to find $x \in C$ such that*

$$\langle Dx, y - x\rangle \geq 0, \forall\ y \in C, \tag{2.40}$$

where $D : C \to H$ is a nonlinear mapping. The set of solutions of problem (2.40) is denoted by $VIP(C, D)$.

Recently, Censor, Gibali and Reich [18] introduced a concept of Split Variational Inequality Problem (SVIP), they gave the following definition to their problem:

Definition 2.8.2. *Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $f : H_1 \to H_1$, $g : H_2 \to H_2$ be two operators and $A : H_1 \to H_2$ be a bounded linear operator, then the SVIP is formulated as follows: find a point $x^* \in C$ such that*

$$\langle f(x^*), x - x^*\rangle \geq 0, \forall\ x \in C, \tag{2.41}$$

and such that the point $y^* = Ax^* \in Q$ solves

$$\langle g(y^*), y - y^*\rangle \geq 0,\ \ \forall\ y \in Q. \tag{2.42}$$

A common solution to VIP and fixed point problem for nonexpansive mappings have been studied extensively by numerous authors (see [13, 27, 49, 75] and the references therein). In 2000, Moudafi [49] proposed the viscosity approximation method for finding a fixed point of a nonexpansive mapping. He proved that the sequences generated by both implicit and explicit methods converges strongly to a unique solution of some variational inequality problem.

Chugh and Rani [27] also introduced a new iterative algorithm for finding a common element of the set of fixed points of a nonexapnsive mapping and the set of solutions of a variational inequality problem for an α-inverse strongly monotone mapping and obtained a weak convergence theorem.
Very recently, Deepho *et al.* [31] considered an iterative scheme to approximate a common element of the set of solutions of split variational inclusion problem and the set of common fixed point of a finite family of k-strictly pseudo-contractive nonself mappings. Strong convergence theorem was established under suitable conditions in real Hilbert spaces, which also solves some variational inequality problem. They denote the solution set of the split variational inclusion problem by $\overline{\Gamma}$ and proved the following theorem.

Theorem 2.8.3. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator and M a strongly positive bounded linear operator on H_1 with a coefficient $\overline{\tau} > 0$. Assume that $\{T_i\}_{i=1}^N : C \to H_1$ is a finite family of k_i-strict pseudo-contraction mappings such that $\Upsilon := \cap_i^N F(T_i) \cap \overline{\Gamma} \neq \emptyset$. Suppose f is a contraction on C with a coefficient $\rho \in (0,1)$ and $\sum_{i=1}^N \eta_i^n = 1$ for all $n \geq 0$, for a given $x_0 \in C, \alpha_n,\ \beta_n \in (0,1)$ and $0 < \tau < \frac{\overline{\tau}}{\rho}$. Let $\{x_n\}$ be a sequence generated in the following;*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^N \eta_{i=1}^n T_i u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D)y_n, n \geq 1, \end{cases} \tag{2.43}$$

where $\lambda > 0, \gamma \in (0, \frac{1}{L})$, L is the spectra radius of the operator A^*A, and A^* is the adjoint of A. The following control conditions are satisfied;
$(C1)$ $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
$(C2)$ $k_i \leq \beta_n \leq l < 1,\ \lim_{n=1}^\infty \beta_n = l$ and $\sum_{n=1}^\infty |\beta_n - \beta_{n-1}| < \infty$;
$(C3)$ $\sum_{n=1}^\infty \sum_{i=1}^N |\eta_i^{(n)} - \eta_i^{(n-1)}| < \infty$.
Then the sequence $\{x_n\}$ generated by the iterative scheme converges strongly to $q \in \Upsilon$ which solves the variational inequality
$\langle (M - \tau f)q, q - p\rangle \leq 0\ \forall\ p\ \in\ \Upsilon$.

CHAPTER 3

Common Solution of Split Equality Mixed Equilibrium and Fixed Point Problems for Countable Families of Generalized K_i- Strictly Pseudo-Contractive Multi-Valued Mappings

3.1 Introduction

In this chapter, we introduce an iterative algorithm for finding a common solution of multiple-set SEMEP and fixed point problem for countably infinite families of generalized k_i-strictly pseudo-contractive multi-valued mapping in real Hilbert spaces. Using our iterative algorithm, we obtain a weak and a strong convergence result for approximating a common solution of multiple-set SEMEP and fixed point problem. As application, we utilize our result to study split equality mixed variational inequality and split equality convex minimization problems.

Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1, H_2 respectively. Assume that $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions satisfying $(A1) - (A4)$, let $\alpha : C \to \mathbb{R} \cup \{+\infty\}$ and $\beta : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions such that $C \cap dom\alpha \neq \emptyset$ and $Q \cap dom\beta \neq \emptyset$. Let $T_i : H_1 \to CB(H_1),\ i = 1, 2, \dots$ and $S_j : H_2 \to CB(H_2),\ j = 1, 2, \dots$ be two countable families of generalized strictly pseudo-contractive multi-valued mappings with constants k_i and k_j respectively and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators. We consider the following problem: find $(x^*, y^*) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{j=1}^{\infty} F(S_j)$ such that

$$\begin{aligned} F(x^*, x) + \alpha(x) - \alpha(x^*) \geq 0, \forall\ x\ \in C, \\ G(y^*, y) + \beta(y) - \beta(y^*) \geq 0, \forall\ y\ \in Q, \end{aligned} \tag{3.1}$$

$$\text{and } Ax^* = By^* .$$

Definition 3.1.1. *Let H be a real Hilbert space.*
(1) *A multi-valued mapping $T : H \to CB(H)$ is said to be demi-closed if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $d(x_n, T(x_n)) \to 0$, we have $x^* \in Tx^*$.*
(2) *A multi-valued mapping $T : H \to CB(H)$ is said to be semi-compact if for any bounded sequence $\{x_n\} \subset H$ with $d(x_n, T(x_n)) \to 0$ as $n \to \infty$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some $x^* \in H$.*

3.2 Preliminaries

Throughout this chapter, we denote the weak and strong convergence of sequence $\{x_n\}$ to a point $x \in X$ by $x_n \rightharpoonup x$ and $x_n \to x$ respectively. We also denote $(\mathcal{H}(A, B))^2$ by $\mathcal{H}^2(A, B)$ for all $A, B \in CB(H)$ and the solution set of (3.1) by Γ defined as:

$$\Gamma := \Big\{(x^*, y^*) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{j=1}^{\infty} F(S_j) : (x^*, y^*) \in MEP(G, \alpha) \times MEP(M, \beta) \\ \text{and } Ax^* = By^*\Big\} \neq \emptyset.$$

We now list some important results that we will need to prove our main result.

Lemma 3.2.1. *[58] Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, and let $\alpha : C \to R$ be a proper lower semi-continuous and convex function such that $C \cap dom\alpha \neq \emptyset$. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F(x) = \Big\{w \in C : F(w, y) + \alpha(y) - \alpha(w) + \frac{1}{r}\langle y - w, w - x\rangle \geq 0, \forall y \in C\Big\}. \tag{3.2}$$

Then
(1) *For each $x \in H, T_r^F(x) \neq \emptyset$;*
(2) *T_r^F is single valued;*
(3) *T_r^F is firmly nonexpansive, that is $\forall x, y \in H$,*

$$||T_r^F x - T_r^F y||^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(4) *$F(T_r^F) = MEP(F, \alpha)$;*
(5) *MEP(F,α) is closed and convex.*

Lemma 3.2.2. *[56] Let H be a real Hilbert space and $\{\mu_n\}$ be a sequence in H such that there exists a nonempty set $W \subset H$ satisfying:*

1. *For every $\mu \in W, \lim_{n\to\infty} ||\mu_n - \mu||$ exists;*
2. *Any weak-cluster point of the sequence $\{\mu_n\}$ belongs to W.*

Then there exist $w^* \in W$ such that $\{\mu_n\}$ converges weakly to w^*.

Lemma 3.2.3. *[23] Let H be a real Hilbert space and let $\{x_i\}_{i\in\mathbb{N}}$ be a bounded sequence in H. For $\lambda_i \in (0,1)$, such that $\sum_{i=1}^{\infty} \lambda_i = 1$, the following holds:*

$$||\sum_{i=1}^{\infty} \lambda_i x_i||^2 = \sum_{i=1}^{\infty} \lambda_i ||x||^2 - \sum_{1\le i<j<\infty} \lambda_i \lambda_j ||x_i - x_j||^2.$$

Lemma 3.2.4. *[70] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le a_n + \delta_n,\ \ n \ge 0,$$

such that $\sum_{i=1}^{\infty} \delta_n < \infty$. Then, $\lim a_n$ exists. If in addition that $\{a_n\}$ has a subsequence that converges to 0, then a_n converges to 0 as $n \to \infty$.

Lemma 3.2.5. *[22] Let K be a nonempty subset of a real Hilbert space H and $T : K \to CB(K)$ be a generalized k-strictly pseudo-contractive multi-valued mapping. Then T is L-Lipschitzian.*

It follows from the Lemma 3.2.5 that,

$$\mathcal{H}(Tx, Ty) \le L||x - y||, \tag{3.3}$$

$$\text{Where } L := \frac{1+\sqrt{k}}{1-\sqrt{k}}. \tag{3.4}$$

Proof. Let $x, y \in Dom(T)$. Then, from definition of k-strictly pseudo-contractive multi-valued mappings, we have that

$$\begin{aligned}\mathcal{H}^2(Tx, Ty) &\le ||x - y||^2 + k\mathcal{H}^2(x - Tx, y - Ty)\\ &\le ||x - y||^2 + k(||x - y|| + \mathcal{H}(Tx, Ty))^2, \quad \text{by Lemma 3.2.6, (b), (c)}\\ &\le (||x - y|| + \sqrt{k}||x - y|| + \sqrt{k}\mathcal{H}(Tx, Ty)).\end{aligned}$$

Thus,

$$\mathcal{H}(Tx, Ty) \le (1 + \sqrt{k})||x - y|| + \sqrt{k}\mathcal{H}(Tx, Ty),$$

and hence

$$\mathcal{H}(Tx, Ty) \leq \frac{1+\sqrt{k}}{1-\sqrt{k}}||x-y||.$$

□

Lemma 3.2.6. *[22] Let E be a normed linear space, $B_1, B_2 \in CB(E)$ and $x, y \in E$ arbitrary. Then the following hold;*
(a) $\mathcal{H}(B_1, B_2) = \mathcal{H}(x + B_1, x + B_2)$.
(b) $\mathcal{H}(B_1, B_2) = \mathcal{H}(-B_1, -B_2)$.
(c) $\mathcal{H}(x + B_1, y + B_2) = ||x - y|| + \mathcal{H}(B_1, B_2)$.
(d) $\mathcal{H}(\{x\}, B_1) = \sup_{b_1 \in B_1} ||x - b_1||$.
(e) $\mathcal{H}(\{x\}, B_1) = \mathcal{H}(0, x - B_1)$.

Proof. (a) We prove Lemma 3.2.6 using the definition of the Hausdorff metric.

$$\begin{aligned}\mathcal{H}(x + B_1, x + B_2) :&= \max\Big\{ \sup_{b_1 \in B_1} d(x + b_1, x + B_2); \sup_{b_2 \in B_2} d(x + b_2, x + B_1)\Big\} \\ &= \max\Big\{ \sup_{b_1 \in B_1} d(b_1, B_2), \sup_{b_2 \in B_2} d(b_2, B_1)\Big\} \\ &= \mathcal{H}(B_1, B_2).\end{aligned}$$

(b) We have

$$\begin{aligned}\mathcal{H}(-B_1, -B_2) &= \max\Big\{ \sup_{-b_1 \in -B_1} d(-b_1, -B_2), \sup_{-b_2 \in -B_2} d(-b_2, -B_1)\Big\} \\ &= \max\Big\{ \sup_{b_1 \in B_1} d(b_1, B_2), \sup_{b_2 \in B_2} d(b_2, B_1)\Big\} \\ &= \mathcal{H}(B_1, B_2).\end{aligned}$$

(c) It is known that for any set $B \subseteq E$, $x, y \in E$ arbitrary, the inequality

$$d(x, B) \leq ||x - y|| + d(y, B) \quad \text{holds.}$$

Using this inequality, we have that

$$\begin{aligned}d(x + b_1, y + B_2) &\leq ||(x + b_1) - (y + b_1)|| + d(y + b_1, y + B_2) \\ &= ||x - y|| + d(b_1, B_2),\end{aligned}$$

similarly,

$$d(y + b_2, x + B_1) \leq ||x - y|| + d(b_2, B_1)$$

Therefore, taking sup over B_1 and B_2 respectively, we have

$$\sup_{b_1\in B_1} d(x+b_1, y+B_2) \leq ||x-y|| + \sup_{b_1\in B_1} d(b_1, B_2),$$

and

$$\sup_{b_2\in B_2} d(y+b_2, x+B_1) \leq ||x-y|| + \sup_{b_2\in B_2} d(b_2, B_1).$$

Thus,

$$\mathcal{H}(x+B_1, y+B_2) \leq ||x-y|| + \mathcal{H}(B_1, B_2).$$

(d) It is obvious that $d(x, B_1) = \sup_{x\in\{x\}} d(x, B_1)$. On the other hand, for any $b_1 \in B_1$, we have

$$d(b_1, \{x\}) = ||b_1 - x|| \geq d(x, B_1).$$

Taking sup over B_1, we have

$$\sup_{b_1\in B_1} d(b_1, \{x\}) \geq d(x, B_1),$$

and therefore

$$\mathcal{H}(\{x\}, B_1) := \max\Big\{\sup_{b_1\in B_1} d(b_1, \{x\}), \sup_{x\in\{x\}} d(x, B_1)\Big\} = \sup_{b_1\in B_1} d(b_1, \{x\}).$$

(e)

$$\begin{aligned}
\mathcal{H}(\{x\}, B_1) :&= \max\Big\{\sup_{b_1\in B_1} d(b, \{x\}), d(x, B_1)\Big\} \\
&= \max\Big\{\sup_{b_1\in B_1} ||x-b_1||), \inf_{b_1\in B_1} ||x-b_1||\Big\} \\
&= \max\Big\{\sup_{b_1\in B_1} d(0, x-B_1), d(0, x-B_1)\Big\} \\
&= \mathcal{H}(\{0\}, x-B_1).
\end{aligned}$$

□

Lemma 3.2.7. *[45] Let H be a real Hilbert space. Then for all x,$y \in H$, we have*

$$||x-y||^2 = ||x||^2 - ||y||^2 - 2\langle x-y, y\rangle.$$

Lemma 3.2.8. *[22] Let K be a nonempty and closed subset of a real Hilbert space H and let $T: K \to CB(K)$ be a generalized k-strictly pseudocontractive multi-valued mapping. Then, $(I-T)$ is strongly demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in K such that $x_n \to x$ and $d(x_n, Tx_n) \to 0$. For each $n \in \mathcal{N}$, take $y_n \in Tx_n$ such that $||x_n - y_n|| \leq d(x_n, Tx_n) + \frac{1}{n}$. Then

$$\begin{aligned} d(x, Tx) &\leq ||x - x_n|| + ||x_n - y_n|| + d(y_n, Tx) \\ &\leq ||x - x_n|| + d(x_n, Tx_n) + \frac{1}{n} + \mathcal{H}(Tx_n, Tx) \\ &\leq ||x - x_n|| + d(x_n, Tx_n) + \frac{1}{n} + \frac{1+\sqrt{k}}{1-\sqrt{k}}||x_n - x|| \\ &\leq \frac{2}{1-\sqrt{k}}||x - x_n|| + d(x_n, Tx_n). \end{aligned}$$

Thus, taking limits on both sides as $n \to \infty$, we have that $d(x, Tx) = 0$. Since Tx is closed, $x \in Tx$. □

Next, given a countably infinite families $\{T_i\}_{i\geq 1}$ and $\{S_j\}_{j\geq 1}$ of generalized strictly pseudocontractive multi-valued mappings with constants k_i and k_j respectively. For an arbitrary sequences $\{p_n\}$ and $\{q_n\}$ subset of C and Q respectively, where C and Q are nonempty, closed and convex subset of a real Hilbert space H_1 and H_2. Let U_n^i and $\mathcal{J}_n^j$ denote the set of inexact distal points of p_n and q_n to the set $T_i p_n$ and $S_j q_n$ respectively, i.e

$$U_n^i := \Big\{ g_n^i \in T_i p_n : \mathcal{H}^2(\{p_n\}, T_i p_n) \leq ||p_n - g_n^i||^2 + \frac{1}{n^2} \Big\}, \tag{3.5}$$

is nonempty.

Similarly,

$$\mathcal{J}_n^j := \Big\{ z_n^j \in S_j q_n : \mathcal{H}^2(\{q_n\}, S_j q_n) \leq ||q_n - z_n^j||^2 + \frac{1}{n^2} \Big\}, \tag{3.6}$$

is nonempty.
Obviously, U_n^i and $\mathcal{J}_n^j$ is nonempty, closed and convex for each $n \geq 1$ due to Lemma 3.2.6(d). In particular, if $T_i p$ and $T_i q$ are assumed to be proximinal and bounded for each $p \in C$ and $q \in Q$, then $T_i p_n$ and $T_i q_n$ have two vectors say υ_n^i and π_n^i, of maximum norm, i.e

$$||p_n - \upsilon_n^i|| = \mathcal{H}(\{p_n\}, T_i p_n) = \sup_{g_n^i \in T_i p_n} ||p_n - g_n^i||,$$

and,

$$||q_n - \pi_n^i|| = \mathcal{H}(\{q_n\}, S_j q_n) = \sup_{z_n^j \in S_j q_n} ||q_n - z_n^j||.$$

3.3 Main Result

We now state the weak convergence theorem.

Theorem 3.3.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Assume that $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions satisfying $(A1) - (A4)$, let $\alpha : C \to \mathbb{R} \cup \{+\infty\}$ and $\beta : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap dom\alpha \neq \emptyset$ and $Q \cap dom\beta \neq \emptyset$. Let $T_i : H_1 \to CB(H_1)$, $i = 1, 2, 3, ...,$ and $S_j : H_2 \to CB(H_2)$, $j = 1, 2, 3, \ldots$ be two multivalued generalized strictly pseudo-contractive mappings with constants k_1 and k_2 respectively and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators, let $x_1 \in C$ and $y_1 \in Q$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} F(u_n, u) + \alpha(u) - \alpha(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \forall\ u \in C; \\ G(v_n, v) + \beta(v) - \beta(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \forall\ v \in Q; \\ p_n = u_n - \xi_n A^*(Au_n - Bv_n); \\ q_n = v_n + \xi_n B^*(Au_n - Bv_n); \\ x_{n+1} = \lambda_0 p_n + \sum_{i=1}^{\infty} \lambda_i g_n^i; \\ y_{n+1} = \lambda_0 q_n + \sum_{j=1}^{\infty} \lambda_j z_n^j, \forall\ n \geq 1; \end{cases} \tag{3.7}$$

where $g_n^i \in T_i p_n$, $z_n^j \in S_j q_n$ and $\xi_n \in \left(\varepsilon, \frac{2}{\phi_A + \phi_B} - \varepsilon\right)$ (for ε small enough), ϕ_A and ϕ_B denote the spectral radii of A^*A and B^*B respectively, where A^*, B^* are adjoint of A and B respectively satisfying the following conditions:

(i) $k_1 = \sup_{i\geq 1}\{k_i\}$, $k_2 = \sup_{j\geq 1}\{k_j\} \in (0, 1)$ where $k := \max\{k_1, k_2\}$ and $k \in (0, 1)$,

(ii) $\lambda_0 \in (k, 1)$, λ_i, $\lambda_j \in (0, 1)$, $i, j = 1, 2, ...,$ such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $\sum_{j=0}^{\infty} \lambda_j = 1$,

(iii) for each $x \in \bigcap_{i=1}^{\infty} F(T_i)$, $T_i x = \{x\}$ and for each $y \in \bigcap_{j=1}^{\infty} F(S_j)$, $S_j y = \{y\}$.

Then the sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem $(x^*, y^*) \in \Gamma$.

Proof. Taking $(x, y) \in \Gamma$, it follows from Lemma 3.2.1 that $x = T_{r_n}^F x$ and $y = T_{r_n}^G y$, we have

$$||u_n - x|| = ||T_{r_n}^F x_n - T_{r_n}^F x|| \leq ||x_n - x||, \tag{3.8}$$
$$||v_n - y|| = ||T_{r_n}^G y_n - T_{r_n}^G y|| \leq ||y_n - y||. \tag{3.9}$$

Moreso, we show that the recursion formulas $x_{n+1} = \lambda_0 p_n + \sum_{i=1}^{\infty} \lambda_i g_n^i$ and $y_{n+1} = \lambda_0 q_n + \sum_{j=1}^{\infty} \lambda_j z_n^j$ in the iterative scheme (3.7) are well defined. Take $x \in \cap_{i=1}^{\infty} F(T_i)$ and $y \in$

$\cap_{i=1}^{\infty} F(S_j)$ arbitrary, we have

$$\begin{aligned} ||p_n - g_n^i|| &\leq \mathcal{H}(p_n, T_i p_n). \\ &= \mathcal{H}(p_n + x, x + T_i p_n). \end{aligned}$$

Applying Lemma 3.2.6(c), we obtain:

$$\begin{aligned} ||p_n - g_n^i|| &\leq ||p_n - x|| + \mathcal{H}(Tx, T_i p_n), \\ &\leq ||p_n - x|| + \frac{1+\sqrt{k}}{1-\sqrt{k}}||p_n - x||. \end{aligned}$$

Applying the triangular inequality and taking limits, we have

$$||g_n^i|| \leq R_n := ||p_n|| + \frac{2}{1-\sqrt{k}} \inf_{x \in F(T)} ||p_n - x||.$$

It then follows that

$$||x_{n+1}|| \leq \lambda_0 ||p_n|| + \sum_{i=1}^{\infty} \lambda_i ||g_n^i||.$$

Therefore,

$$||x_{n+1}|| \leq \lambda_0 ||p_n|| + \sum_{i=1}^{\infty} \lambda_i R_n \leq R_n.$$

Following the same step as shown above,

$$||y_{n+1}|| \leq \lambda_0 ||q_n|| + \sum_{j=1}^{\infty} \lambda_j W_n \leq W_n.$$

Hence, x_{n+1} and y_{n+1} are well defined.
Since T_i are generalized k_i- strictly pseudo-contractive mapping, by applying Lemma 3.2.3,

3.2.6(e) and identity (3.5), we have:

$$
\begin{aligned}
||x_{n+1} - x||^2 &= ||\lambda_0 p_n + \sum_{i=1}^{\infty} \lambda_i g_n^i - x||^2 \\
&= ||\lambda_0 (p_n - x) + \sum_{i=1}^{\infty} \lambda_i (g_n^i - x)||^2 \\
&= \lambda_0 \|p_n - x\|^2 + \sum_{i=1}^{\infty} \lambda_i \|g_n^i - x\|^2 - \sum_{i=1}^{\infty} \lambda_0 \lambda_i \|p_n - g_n^i\|^2 - \sum_{1 \le i \le j \le \infty} \lambda_i \lambda_j \|g_n^i - g_n^j\|^2 \\
&\le \lambda_0 ||p_n - x||^2 + \sum_{i=1}^{\infty} \lambda_i \mathcal{H}^2(T_i p_n, T_i x) - \sum_{i=1}^{\infty} \lambda_0 \lambda_i ||p_n - g_n^i||^2 \\
&\le \lambda_0 ||p_n - x||^2 + \sum_{i=1}^{\infty} \lambda_i \Big(||p_n - x||^2 + k_i \mathcal{H}^2(\{0\}, p_n - T_i p_n) \Big) - \sum_{i=1}^{\infty} \lambda_0 \lambda_i ||p_n - g_n^i||^2 \\
&= \sum_{i=0}^{\infty} \lambda_i ||p_n - x||^2 + \sum_{i=1}^{\infty} \lambda_i k_i \mathcal{H}^2(\{p_n\}, T_i p_n) - \sum_{i=1}^{\infty} \lambda_0 \lambda_i ||p_n - g_n^i||^2 \\
&\le \sum_{i=0}^{\infty} \lambda_i ||p_n - x||^2 + \sum_{i=1}^{\infty} \lambda_i k_1 \Big(||p_n - g_n^i||^2 + \frac{1}{n^2} \Big) - \sum_{i=1}^{\infty} \lambda_0 \lambda_i ||p_n - g_n^i||^2 \\
&\le ||p_n - x||^2 + \frac{k_1}{n^2} - \sum_{i=1}^{\infty} \lambda_i (\lambda_0 - k_1)(||p_n - g_n^i||^2). \qquad (3.10)
\end{aligned}
$$

But

$$
\begin{aligned}
||p_n - x||^2 &= ||u_n - \xi A^*(Au_n - Bv_n) - x||^2 \\
&= ||u_n - x||^2 + \xi_n^2 \phi_A ||Au_n - Bv_n||^2 - 2\xi_n \langle Au_n - Ax, Au_n - Bv_n \rangle \\
&= ||x_n - x||^2 + \xi_n^2 \phi_A ||Au_n - Bv_n||^2 - 2\xi_n \langle Au_n - Ax, Au_n - Bv_n \rangle. \qquad (3.11)
\end{aligned}
$$

Substituting (3.11) into (3.10), we have

$$
\begin{aligned}
||x_{n+1} - x||^2 \le{}& ||x_n - x||^2 + \xi_n^2 \phi_A ||Au_n - Bv_n||^2 - 2\xi_n \langle Au_n - Ax, Au_n - Bv_n \rangle + \frac{k_1}{n^2} \\
&- (\lambda_0 - k_1) \sum_{i=1}^{\infty} \lambda_i ||p_n - g_n^i||^2. \qquad (3.12)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
||y_{n+1} - y||^2 \le{}& ||y_n - y||^2 + \xi_n^2 \phi_B ||Au_n - Bv_n||^2 + 2\xi_n \langle Bv_n - By, Au_n - Bv_n \rangle + \frac{k_2}{n^2} \\
&- (\lambda_0 - k_2) \sum_{j=1}^{\infty} \lambda_j ||q_n - z_n^j||^2. \qquad (3.13)
\end{aligned}
$$

Adding the last two inequalities, using k $= \max\{k_1, k_2\}$ and the fact that $Ax = By$, we have

$$\begin{aligned}||x_{n+1}-x||^2+||y_{n+1}-y||^2 \leq ||x_n-x||^2+||y_n-y||^2+\frac{k}{n^2}+\xi_n^2\Big[(\phi_A+\phi_B)||Au_n-Bv_n||^2\Big] \\ -2\xi_n||Au_n-Bv_n||^2-(\lambda_0-k)\sum_{i=1}^{\infty}\lambda_i||p_n-g_n^i||^2 \\ -(\lambda_0-k)\sum_{j=1}^{\infty}\lambda_j||q_n-z_n^j||^2\end{aligned}$$

which also implies

$$\begin{aligned}||x_{n+1}-x||^2+||y_{n+1}-y||^2 \leq ||x_n-x||^2+||y_n-y||^2+\frac{k}{n^2}-\xi_n\Big(2-\xi_n(\phi_A+\phi_B)\Big)||Au_n-Bv_n||^2 \\ -(\lambda_0-k)\sum_{i=1}^{\infty}\lambda_i||p_n-g_n^i||^2-(\lambda_0-k)\sum_{j=1}^{\infty}\lambda_j||q_n-z_n^j||^2.\end{aligned} \tag{3.14}$$

Suppose $\Gamma_n(x,y) = ||x_n-x||^2+||y_n-y||^2$, then from (3.14), we have

$$\begin{aligned}\Gamma_{n+1}(x,y) \leq \Gamma_n(x,y)+\frac{k}{n^2}-\xi_n\Big(2-\xi_n(\phi_A+\phi_B)\Big)||Au_n-Bv_n||^2-(\lambda_0-k)\sum_{i=1}^{\infty}\lambda_i||p_n-g_n^i||^2 \\ -(\lambda_0-k)\sum_{j=1}^{\infty}\lambda_j||q_n-z_n^j||^2.\end{aligned} \tag{3.15}$$

Since $\lambda_0 \in (k, 1)$ and $\xi_n \in \Big(\varepsilon, \frac{2}{\phi_A+\phi_B}-\varepsilon\Big)$, we have $2-\xi_n(\phi_A+\phi_B) > 0$. it follows from inequality (3.15) that

$$\Gamma_{n+1}(x,y) \leq \Gamma_n(x,y)+\frac{k}{n^2}.$$

Using Lemma 3.2.4, we have that $\lim_{n\to\infty}\Gamma_n(x,y)$ exist, this implies that $\{x_n\}, \{y_n\}$ are bounded.

From inequality (3.15), we have

$$(\lambda_0-k)\Big[\sum_{i=1}^{\infty}\lambda_i||p_n-g_n^i||^2+\sum_{j=1}^{\infty}\lambda_j||q_n-z_n^j||^2\Big] \leq \Gamma_n(x,y)-\Gamma_{n+1}(x,y)+\frac{k}{n^2}.$$

So for each $i \geq 1$ and $j \geq 1$, we have

$$\lambda_i(\lambda_0 - k)\Big[||p_n - g_n^i||^2\Big] + \lambda_j(\lambda_0 - k)\Big[||q_n - z_n^j||^2\Big] \leq \Gamma_n(x,y) - \Gamma_{n+1}(x,y) + \frac{k}{n^2} \to 0 \qquad n \to \infty.$$

Taking limits on both sides as $n \to \infty$, we conclude that $\lim_{n\to\infty} ||p_n - g_n^i|| = \lim_{n\to\infty} ||q_n - z_n^j|| = 0$. Using the fact that $d(p_n, T_i p_n) \leq ||p_n - g_n^i||$ and $d(q_n, S_j q_n) \leq ||q_n - z_n^j||$, we get that $\lim_{n\to\infty} d(p_n, T_i p_n) = \lim_{n\to\infty} d(q_n, S_j q_n) = 0$. Hence, the sequence $\{\Gamma_n(x,y)\}$ is decreasing and is lower bounded by 0, it converges to some finite limit say $\sigma(x,y)$. This implies that Condition (1) of Lemma 3.2.2 is satisfied with $\mu_n = (x_n, y_n)$, $\mu^* = (x,y)$. It follows from inequality (3.15) and the convergence of the sequence $\{\Gamma_n(x,y)\}$ that

$$\lim_{n\to\infty} ||Au_n - Bv_n|| = 0. \tag{3.16}$$

Thus,

$$\lim_{n\to\infty} d(p_n, T_i p_n) \leq \lim_{n\to\infty} ||p_n - g_n^i|| = 0, \tag{3.17}$$

and

$$\lim_{n\to\infty} d(q_n, S_j q_n) = \lim_{n\to\infty} ||q_n - z_n^j|| = 0. \tag{3.18}$$

Furthermore, as $\{\Gamma_n(x,y)\}$ converges to a finite limit and $||x_n - x||^2 \leq \Gamma_n(x,y)$, $||y_n - y||^2 \leq \Gamma_n(x,y)$ exists, we know that $\{x_n\}$ and $\{y_n\}$ are bounded, and $\limsup_{n\to\infty} ||x_n - x||$ and $\limsup_{n\to\infty} ||y_n - y||$ exist. From (3.8) and (3.9), we have that $\limsup_{n\to\infty} ||u_n - x||$ and $\limsup_{n\to\infty} ||v_n - y||$ also exist. Let x^* and y^* be the weak cluster points of sequences $\{x_n\}$ and $\{y_n\}$, respectively. Then, there exists a subsequence of $\{(x_n, y_n)\}$ (without loss of generality) still denoted by $\{(x_n, y_n)\}$ such that $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup y^*$. Then, from Lemma 3.2.7, we have

$$\begin{aligned} ||x_{n+1} - x_n||^2 &= ||x_{n+1} - x - x_n + x||^2 \\ &= ||x_{n+1} - x||^2 - ||x_n - x||^2 - 2\langle x_{n+1} - x_n, x_n - x\rangle \\ &= ||x_{n+1} - x||^2 - ||x_n - x||^2 - 2\langle x_{n+1} - x^*, x_n - x\rangle + 2\langle x_n - x^*, x_n - x\rangle. \end{aligned}$$

Therefore

$$\limsup_{n\to\infty} ||x_{n+1} - x_n|| = 0.$$

Similarly, we obtain

$$\limsup_{n\to\infty} ||y_{n+1} - y_n|| = 0.$$

We conclude that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0 \tag{3.19}$$

and

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0 \tag{3.20}$$

It follows from (3.12) and (3.13) that

$$\begin{aligned} ||x_{n+1} - x||^2 \leq ||u_n - x||^2 + \frac{k_1}{n^2} + \xi_n^2\phi_A||Au_n - Bv_n||^2 - 2\xi_n\langle Au_n - Ax, Au_n - Bv_n\rangle \\ - (\lambda_0 - k_1)\sum_{i=1}^{\infty}\lambda_i||p_n - g_n^i||^2 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} ||y_{n+1} - y||^2 \leq ||v_n - y||^2 + \frac{k_2}{n^2} + \xi_n^2\phi_B||Au_n - Bv_n||^2 + 2\xi_n\langle Bv_n - By, Au_n - Bv_n\rangle \\ - (\lambda_0 - k_2)\sum_{j=1}^{\infty}\lambda_j||q_n - z_n^j||^2. \end{aligned} \tag{3.22}$$

By adding the last two inequalities, letting $k = \max\{k_1, k_2\}$ and $Ax = By$, we have

$$\begin{aligned} ||x_{n+1} - x||^2 + ||y_{n+1} - y||^2 \leq ||u_n - x||^2 + ||v_n - y||^2 + \frac{k}{n^2} - \xi_n\Big(2 - (\phi_A + \phi_B)\Big)||Au_n - Bv_n||^2 \\ - (\lambda_0 - k)\sum_{i=1}^{\infty}\lambda_i||p_n - g_n^i||^2 - (\lambda_0 - k)\sum_{j=1}^{\infty}\lambda_j||q_n - z_n^j||^2, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} ||u_n - x||^2 &= ||T_{r_n}^F x_n - T_{r_n}^F||^2 \\ &\leq \langle x_n - x, u_n - x\rangle \\ &= \frac{1}{2}(||x_n - x||^2 + ||u_n - x||^2 - ||x_n - u_n||^2), \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} ||v_n - y||^2 &= ||T_{r_n}^G y_n - T_{r_n}^G y||^2 \\ &\leq \langle y_n - y, v_n - y\rangle \\ &= \frac{1}{2}(||y_n - y||^2 + ||v_n - y||^2 - ||y_n - v_n||^2). \end{aligned} \tag{3.25}$$

From (3.23), (3.24) and (3.25), we have that

$$
\begin{aligned}
&||x_n - u_n||^2 + ||y_n - v_n||^2 \\
&\qquad\qquad\qquad \le ||x_n - x||^2 - ||x_{n+1} - x||^2 + ||y_n - y||^2 \\
&\qquad\qquad\qquad - ||y_{n+1} - y||^2 - \xi_n\Big(2 - \xi_n(\phi_A + \phi_B)\Big)||Au_n - Bv_n||^2 \\
&\qquad\qquad\qquad - (\lambda_0 - k)\sum_{i=1}^{\infty} \lambda_i ||p_n - g_n^i||^2 - (\lambda_0 - k)\sum_{j=1}^{\infty} \lambda_j ||q_n - z_n^j||^2. \quad (3.26)
\end{aligned}
$$

From (3.19) and (3.20), we obtain

$$\lim_{n\to\infty} ||x_n - u_n|| = 0, \quad (3.27)$$

$$\lim_{n\to\infty} ||y_n - v_n|| = 0. \quad (3.28)$$

It follows from (3.27) and (3.28) that $u_n \rightharpoonup x^*$ and $v_n \rightharpoonup y^*$, respectively.
Since T_i and S_j are generalized k_i strictly pseudo-contractive multi-valued mappings, let $g_n^i \in T_i p_n$ be such that $||p_n - g_n^i|| \le d(p_n, T_i p_n) + \frac{1}{n}$. Then,

$$
\begin{aligned}
d(u_n, T_i u_n) &\le ||u_n - p_n|| + ||p_n - g_n^i|| + d(g_n^i, T_i u_n) \\
&\le ||u_n - p_n|| + d(p_n, T_i p_n) + \frac{1}{n} + \mathcal{H}(T_i p_n, T_i u_n,) \\
&\le ||u_n - p_n|| + d(p_n, T_i p_n) + \frac{1}{n} + \frac{1+\sqrt{k}}{1-\sqrt{k}}||p_n - u_n|| \\
&= ||u_n - (u_n - \xi_n A^*(Au_n - Bv_n))|| + d(p_n, T_i p_n) + \frac{1}{n} + \frac{1+\sqrt{k}}{1-\sqrt{k}}||(u_n - \xi_n A^*(Au_n - Bv_n \\
&\le \frac{2}{1-\sqrt{k}}|\xi_n| \; ||A^*|| \; ||Au_n - Bv_n|| + d(p_n, T_i p_n) + \frac{1}{n}.
\end{aligned}
$$

From (3.16) and (3.17), we have that

$$\lim_{n\to\infty} d(u_n, T_i u_n) = 0. \quad (3.29)$$

Similarly, using the same approach as above for S_j, we have

$$\lim_{n\to\infty} d(v_n, S_j v_n) = 0. \quad (3.30)$$

Since

$$
\begin{aligned}
d(x_n, T_i x_n) &\le ||x_n - u_n|| + d(u_n, T_i u_n) + \mathcal{H}(T_i u_n, T_i x_n) \\
&\le ||x_n - u_n|| + d(u_n, T_i u_n) + \frac{1+\sqrt{K}}{1-\sqrt{K}}||u_n - x_n|| \\
&= \frac{2}{1-\sqrt{K}}||x_n - u_n|| + d(u_n, T_i u_n).
\end{aligned}
$$

It follows from (3.27) and (3.29) that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0. \tag{3.31}$$

Similarly, we have

$$\begin{aligned} d(y_n, S_j y_n) &\le ||y_n - v_n|| + d(v_n, S_j v_n) + \mathcal{H}(S_j v_n, S_j y_n|| \\ &\le ||y_n - v_n|| + d(v_n, S_j v_n) + \frac{1+\sqrt{K}}{1-\sqrt{K}}||v_n - y_n|| \\ &\le \frac{2}{1-\sqrt{K}}||y_n - v_n|| + d(v_n, S_j v_n). \end{aligned}$$

From (3.28) and (3.30), we have

$$\lim_{n\to\infty} d(y_n, S_j y_n) = 0. \tag{3.32}$$

Since $\{x_n\}$ and $\{y_n\}$ converge weakly to x^* and y^*, respectively, then it follows from (3.31), (3.32) and Lemma 3.2.8 that $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ and $y^* \in \bigcap_{j=1}^{\infty} F(S_j)$. Since every Hilbert space satisfies Opials condition, which guarantee that the weakly subsequential limit of $\{(x_n, y_n)\}$ is unique. We now prove $x^* \in MEP(F, \alpha)$ and $y^* \in MEP(G, \beta)$. Since $u_n = T_{r_n}^F x_n$, we have

$$F(u_n, u) + \alpha(u) - \alpha(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \ \forall \ u \ \in C. \tag{3.33}$$

From (A2), we obtain

$$\alpha(u) - \alpha(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle - F(u_n, u) \ge F(u, u_n), \ \forall \ u \ \in C. \tag{3.34}$$

Hence,

$$\alpha(u) - \alpha(u_{n_k}) + \frac{1}{r_{n_k}}\langle u - u_{n_k}, u_{n_k} - x_{n_k}\rangle \ge F(u_n, u_{n_k}), \ \forall \ u \in C. \tag{3.35}$$

From (3.27), we obtain $u_{n_k} \rightharpoonup x^*$. it follows from (A4) that $\lim_{k\to\infty} \frac{u_{n_k} - x_{n_k}}{r_{n_k}} = 0$, and from proper lower semicontinuity of α that

$$F(u, x^*) + \alpha(x^*) - \alpha(u) \le 0, \forall u \in C. \tag{3.36}$$

Put $w_t = tu + (1-t)x^*$ for all $t \in (0, 1]$ and $u \in C$. Then, we get $w_t \in C$ and hence $F(w_t, x^*) + \alpha(x^*) - \alpha(w_t) \le 0$. So from (A1) and (A4), we have

$$
\begin{aligned}
0 &= F(w_t, w_t) - \alpha(w_t) + \alpha(w_t) \\
&\leq tF(w_t, u) + (1-t)F(w_t, x^*) + t\alpha(u) + (1-t)\alpha(x^*) - \alpha(w_t) \\
&\leq t[F(w_t, x) + \alpha(u) - \alpha(w_t)].
\end{aligned} \tag{3.37}
$$

Hence, we have

$$
F(x^*, u) + \alpha(u) - \alpha(x^*) \geq 0, \forall u \in C. \tag{3.38}
$$

This implies that $x^* \in MEP(F, \alpha)$. Using a similar argument, we also have $y^* \in MEP(G, \beta)$. Moreso, since the squared norm is weakly lower semicontinuous, we have

$$
||Ax^* - By^*||^2 \leq \liminf_{n\to\infty} ||Au_n - Bv_n||^2 = 0,
$$

so that $Ax^* = By^*$. This means that $(x^*, y^*) \in SEMEP(F, G, \alpha, \beta)$. Therefore, $(x^*, y^*) \in \Gamma$.
Lastly, we conclude that for each $(x^*, y^*) \in \Gamma$, $\lim_{n\to\infty}(||x_n - x^*|| + ||y_n - y^*||)$ exists, and each cluster point of the sequence $||(x^*, y^*)||$ belongs to Γ. Thus from Lemma 3.2.8, we know that $\{(x_n, y_n)\}$ converges weakly to (x^*, y^*). Therefore, the sequence $\{(x_n, y_n)\}$ generated by the iterative scheme (3.7) converges weakly to a solution of problem (3.1) in Γ. □

We now state and prove the following strong convergence theorem.

Theorem 3.3.2. *Suppose that the statement of Theorem 3.3.1 hold with the addition that T_i, $i = 1, 2, 3, \ldots$ and S_j, $j = 1, 2, 3, \ldots$ are semi-compact, then, the sequence $\{(x_n, y_n)\}$ generated by (3.7) converges strongly to a solution of problem (3.1) in Γ.*

Proof. Since $T_i, i = 1, 2, 3, \ldots$ and $S_j, j = 1, 2, 3, \ldots$ are semi-compact, then the sequences $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim\limits_{n\to\infty} d(x_n, T_i x_n) = 0$, $\lim\limits_{n\to\infty} d(y_n, S_j y_n) = 0$. Thus there exist (without loss of generality) subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converge strongly to some point u^* and v^*, respectively. This implies that $x^* = u^*$ and $y^* = v^*$. From Lemma 3.2.8, we have $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ and $y^* \in \bigcap_{j=1}^{\infty} F(S_j)$ and $\lim_{n\to\infty} \Gamma_n(x^*, y^*)$ exists, therefore $\lim_{n\to\infty} \Gamma_n(x^*, y^*) = 0$. Hence we conclude that the iterative scheme (3.7) converges strongly to a solution of problem (3.1). This ends the proof of Theorem 3.3.2. □

Remark 3.3.3. *(1) In (3.1), if $\alpha = 0$, $\beta = 0$, then the SEMEP reduces to the following split equality Equilibrium Problem: find $(x^*, y^*) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)$ such that*

$$
F(x^*, x) \geq 0 \ \forall x \in C, \qquad G(y^*, y) \geq 0 \ \forall y \in Q \qquad \text{and } Ax^* = By^*. \tag{3.39}
$$

Problem (3.39) *is equivalent to finding*

$$\big\{(x^*,y^*) \in (\bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)) : (x^*,y^*) \in (SEEP(F) \times SEEP(G)) \text{ and } Ax^* = By^*\big\}. \tag{3.40}$$

(2) If $F = 0$ *and* $G = 0$, *then the SEMEP* (3.1) *reduces to the following split equality convex minimization problem: find* $(x^*,y^*) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)$ *such that*

$$\alpha(x) \geq \alpha(x^*), \quad \forall\, x \in C, \qquad \beta(y) \geq \beta(y^*), \forall\, y \in Q \qquad \text{and} \qquad Ax^* = By^*. \tag{3.41}$$

Problem (3.41) *is equivalent to finding*

$$\big\{(x^*,y^*) \in (\bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)) : (x^*,y^*) \in (SECMP(\alpha) \times SECMP(\beta)) \text{ and } Ax^* = By^*\big\}. \tag{3.42}$$

(3) If $F = 0,\ G = 0,\ H_3 = H_2,\ B = I$ *and* $y^* = Ax^*$, *then the SEMEP* (3.1) *reduces to the following split convex minimization problem: find* $(x^*,y^*) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)$ *such that*

$$\alpha(x) \geq \alpha(x^*)\ \forall\, x \in C, \qquad \text{and} \qquad y^* = Ax^* \in Q,\ \beta(y) \geq \beta(y^*),\ \forall\, y \in Q. \tag{3.43}$$

Problem (3.43) *is equivalent to finding*

$$\big\{(x^*,y^*) \in (\bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)) : (x^*,y^*) \in (SCMP(\alpha) \times SCMP(\beta)) \text{ and } y^* = Ax^*\big\}. \tag{3.44}$$

(4) If $\alpha = 0,\ \beta = 0,\ B = I,\ H_3 = H_2$ *and* $y^* = Ax^*$, *then the SEMEP* (3.1) *reduces to the following split Equilibrium Problem: find* $(x^*,y^*) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)$ *such that*

$$F(x^*,x) \geq 0\ \forall\ x\ \in C, \text{and such that } y^* = Ax^* \in Q \text{ solves } G(y^*,y) \geq 0, \forall\ y \in Q. \tag{3.45}$$

Problem (3.45) *is equivalent to finding*

$$\big\{(x^*,y^*) \in (\bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)) : (x^*,y^*) \in (SEP(F) \times SEP(G)) \text{ (and) } y^* = Ax^*\big\}. \tag{3.46}$$

Taking $\alpha = 0$ and $\beta = 0$ in Theorem 3.3.2, we also have the following result.

Corollary 3.3.4. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Assume that $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions satisfying (A1)-(A4). Let $T_i : H_1 \to CB(H_1)$, $i = 1, 2, 3, ...$, and $S_j : H_2 \to CB(H_2)$, $j = 1, 2, 3, ...$ be two multivalued generalized strictly pseudo-contractive mappings with constants k_i and k_j respectively and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators, let $u, x_1 \in C$ and $v, y_1 \in Q$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} F(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \forall\ u \in C; \\ G(v_n, v) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \forall\ v \in Q; \\ p_n = u_n - \xi_n A^*(Au_n - Bv_n); \\ q_n = v_n + \xi_n B^*(Au_n - Bv_n); \\ x_{n+1} = \lambda_0 p_n + \sum_{i=1}^{\infty} \lambda_i g_n^i; \\ y_{n+1} = \lambda_0 q_n + \sum_{j=1}^{\infty} \lambda_j z_n^j, \forall\ n \geq 1; \end{cases}$$

where $g_n^i \in T_i p_n$, $z_n^j \in S_j q_n$ and $\xi_n \in \left(\varepsilon, \frac{2}{\phi_A + \phi_B} - \varepsilon\right)$ (for ε small enough), ϕ_A and ϕ_B denote the spectral radii of A^*A and B^*B respectively, where A^*, B^* are adjoint of A and B respectively, with conditions:

(i) $k_1 = \sup_{i\geq 1}\{k_i\}$, $k_2 = \sup_{j\geq 1}\{k_j\} \in (0, 1)$ where $k := \max\{k_1, k_2\}$ and $k \in (0, 1)$,

(ii) $\lambda_0 \in (k, 1)$, λ_i, $\lambda_j \in (0, 1)$, $i, j = 1, 2, ...$, such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $\sum_{j=0}^{\infty} \lambda_j = 1$,

(iii) for each $x \in \bigcap_{i=1}^{\infty} F(T_i)$, $T_i x = \{x\}$ and for each $y \in \bigcap_{j=1}^{\infty} F(S_j)$, $S_j y = \{y\}$.

If T_i and S_j are semi-compact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of problem (3.39).

In Theorem 3.3.2 taking $B = I$, $H_3 = H_2$, $\alpha = 0$ and $\beta = 0$. Then we have the following corollary.

Corollary 3.3.5. *Let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Assume that $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions satisfying $(A1) - (A4)$. Let $T_i : H_1 \to CB(H_1), i = 1, 2, 3, ...,$ and $S_j : H_2 \to CB(H_2)$, $j = 1, 2, 3, ...$ be two multivalued generalized strictly pseudo-contractive mappings with constants k_i and k_j respectively and $A : H_1 \to H_2$, be a bounded linear operator, let*

$u, x_1 \in C$ and $v, y_1 \in Q$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by

$$\begin{cases} F(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \forall\ u \in C; \\ G(v_n, v) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \forall\ v \in Q; \\ p_n = u_n - \xi_n A^*(Au_n - v_n); \\ q_n = v_n + \xi_n(Au_n - v_n); \\ x_{n+1} = \lambda_0 p_n + \sum_{i=1}^{\infty} \lambda_i g_n^i; \\ y_{n+1} = \lambda_0 q_n + \sum_{j=1}^{\infty} \lambda_j z_n^j, \forall\ n \geq; \end{cases}$$

where $g_n^i \in T_i p_n$, $z_n^j \in S_j q_n$ and $\xi_n \in \left(\varepsilon, \frac{2}{\phi_A} - \varepsilon\right)$ (for ε small enough), ϕ_A denote the spectral radius of A^*A where A^* is the adjoint of A with conditions:

(i) $k_1 = \sup_{i\geq 1}\{k_i\},\ k_2 = \sup_{j\geq 1}\{k_j\} \in (0,1)$ where $k := \max\{k_1, k_2\}$ and $k \in (0,1)$,

(ii) $\lambda_0 \in (k, 1),\ \lambda_i,\ \lambda_j \in (0,1),\ i, j = 1, 2, ...,$ such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $\sum_{j=0}^{\infty} \lambda_j = 1$,

(iii) for each $x \in \bigcap_{i=1}^{\infty} F(T_i),\ T_i x = \{x\}$ and for each $y \in \bigcap_{j=1}^{\infty} F(S_j),\ S_j y = \{y\}$.

If T_i and S_j are semi-compact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of problem (3.46).

3.4 Applications

3.4.1 Application to split equality mixed variational inequality problem

Let H be a real Hilbert space and C be nonempty, closed and convex subset of H. Given a nonlinear mapping $A : C \to H$, then the Variational Inequality Problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, z - x^*\rangle \geq 0,\ \forall z\ \in C. \tag{3.47}$$

We will denote the solution set of VIP by VI(A,C).
A mapping $A : C \to H$ is said to be $\nu-$ inverse strongly monotone mapping if there exists a constant $\nu > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \nu||Ax - Ay||^2$ for any $x, y \in C$.
Let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Then the Split Variational Inequality Problem (SVIP) is to find $x^* \in C$ such that

$$\langle f(x^*), x - x^*\rangle \geq 0\ \forall\ x \in C,$$

and

$$y^* = Ax^* \in Q \quad \text{solves} \qquad \langle g(y^*), y - y^* \rangle \geq 0 \ \forall \ y \in Q, \tag{3.48}$$

where $f : H_1 \to H_1,\ g : H_2 \to H_2$ are nonlinear mappings and $A : H_1 \to H_2$ is a bounded linear operator.
We now consider the split equality mixed variational inequality problem which is to find $(x^*, y^*) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)$ such that

$$\begin{aligned} \langle B_1(x^*), x - x^* \rangle + \alpha(x) - \alpha(x^*) \geq 0, \forall x \ \in C, \\ \langle B_2(y^*), y - y^* \rangle + \beta(y) - \beta(y^*) \geq 0, \forall y \ \in Q, \\ \text{and } Ax^* = By^*, \end{aligned} \tag{3.49}$$

where $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ are $\nu-$ inverse strongly monotone mappings.
Problem (3.49) is equivalent to finding $(x^*, y^*) \in \bigcap_{i=1}^{\infty} F(T_i) \ \times \ \bigcap_{j=1}^{\infty} F(S_j)$ such that

$$\Big\{(x^*, y^*) \in (\bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{i=1}^{\infty} F(S_j)) : (x^*, y^*) \in (SEMVIP(B_1, \alpha) \times SEMVIP(B_2, \beta))\Big\}, \tag{3.50}$$

$$\text{and } Ax^* = By^*.$$

Setting $F(x, y) = \langle B_1 x, y - x \rangle$ and $G(x, y) = \langle B_2 x, y - x \rangle$, it is easy to see that F and G satisfy condition (A1)-(A4) since $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ are $\nu-$ inverse strongly monotone mappings. Then from Theorem (3.3.2), the following result holds.

Theorem 3.4.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $B_1 : C \to H_1$ and $B_2 : D \to H_2$ be two $\nu-$ inverse strongly monotone mappings and $\alpha : C \to \mathbb{R} \cup \{+\infty\}$, $\beta : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap dom\alpha \neq \emptyset$ and $Q \cap dom\beta \neq \emptyset$. Let $T_i : H_1 \to CB(H_1), i = 1, 2, 3, ...,$ and $S_j : H_2 \to CB(H_2), j = 1, 2, 3, \ldots$ be two multivalued generalized strictly pseudo-contractive mappings with constants k_i and k_j respectively and $A : H_1 \to H_2, B : H_2 \to H_3$ be two bounded linear operators, let $u, x_1 \in C$ and $v, y_1 \in Q$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} \langle B_1(u_n), u - u_n \rangle + \alpha(u) - \alpha(u_n) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0, \forall u \in C; \\ \langle B_2(v_n), v - v_n \rangle + \beta(v) - \beta(v_n) + \frac{1}{r_n} \langle v - v_n, v_n - y_n \rangle \geq 0, \forall v \in Q; \\ p_n = u_n - \xi_n A^*(Au_n - Bv_n); \\ q_n = v_n + \xi_n B^*(Au_n - Bv_n); \\ x_{n+1} = \lambda_0 p_n + \sum_{i=1}^{\infty} \lambda_i g_n^i; \\ y_{n+1} = \lambda_0 q_n + \sum_{j=1}^{\infty} \lambda_j z_n^j, \ \ \forall n \geq 1; \end{cases}$$

where $g_n^i \in T_i p_n, z_n^j \in S_j q_n$ *and* $\xi_n \in (\varepsilon, \frac{2}{\phi_A+\phi_B} - \varepsilon)$ *(for* ε *small enough),* ϕ_A *and* ϕ_B *denote the spectral radii of* A^*A *and* B^*B *respectively, where* A^*, B^* *are adjoint of* A *and* B *respectively, satisfying the following conditions:*

(i) $k_1 = \sup_{i\geq 1}\{k_i\},\ k_2 = \sup_{j\geq 1}\{k_j\} \in (0,1)$ *where* $k := \max\{k_1, k_2\}$ *and* $k \in (0,1)$,

(ii) $\lambda_0 \in (k,1),\ \lambda_i,\ \lambda_j \in (0,1),\ i,j = 1,2,...,$ *such that* $\sum_{i=0}^{\infty} \lambda_i = 1$ *and* $\sum_{j=0}^{\infty} \lambda_j = 1$,

(iii) for each $x \in \cap_{i=1}^{\infty} F(T_i), T_i x = \{x\}$ *and for each* $y \in \cap_{j=1}^{\infty} F(S_j), S_j y = \{y\}$.

If the mapping $T_i, i = 1,2,3,...$ *and* $S_j, j = 1,2,3,...$ *are semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *converges strongly to a solution of problem* (3.50).

3.4.2 Application to split equality convex minimization problem

Problem (3.41) is called the split equality convex minimization problem. From Theorem 3.3.2, we obtain the following convergence result for problem (3.41).

Theorem 3.4.2. *Let* H_1, H_2 *and* H_3 *be real Hilbert spaces,* C *and* Q *be nonempty, closed and convex subsets of* H_1 *and* H_2 *respectively. Let* $\alpha : C \to \mathbb{R} \cup \{+\infty\}$ *and* $\beta : Q \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semi-continuous and convex functions such that* $C \cap dom\alpha \neq \emptyset$ *and* $D \cap dom\beta \neq \emptyset$*. Let* $T_i : H_1 \to CB(H_1), i = 1,2,3,...,$ *and* $S_j : H_2 \to CB(H_2), j = 1,2,3,...$ *be two multivalued generalized strictly pseudo-contractive mappings with constants* k_i *and* k_j *respectively and* $A : H_1 \to H_3, B : H_2 \to H_3$ *be two bounded linear operators, let* $u, x_1 \in C$ *and* $v, y_1 \in Q$ *be arbitrary and the sequence* $\{(x_n, y_n)\}$ *be generated by*

$$\begin{cases} \alpha(u) - \alpha(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0,\ \forall\ u \in C; \\ \beta(v) - \beta(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0,\ \forall\ v \in Q; \\ p_n = u_n - \xi_n A^*(Au_n - Bv_n); \\ q_n = v_n + \xi_n B^*(Au_n - Bv_n); \\ x_{n+1} = \lambda_0 p_n + \sum_{i=1}^{\infty} \lambda_i g_n^i; \\ y_{n+1} = \lambda_o q_n + \sum_{j=1}^{\infty} \lambda_j z_n^j; \end{cases}$$

where $g_n^i \in T_i p_n, z_n^j \in S_j q_n$ *and* $\xi_n \in (\varepsilon, \frac{2}{\phi_A+\phi_B} - \varepsilon)$ *(for* ε *small enough),* ϕ_A *and* ϕ_B *denote the spectral radii of* A^*A *and* B^*B *respectively, where* A^*, B^* *are adjoint of* A *and* B *respectively, satisfying the following conditions:*

(i) $k_1 = \sup_{i\geq 1}\{k_i\},\ k_2 = \sup_{j\geq 1}\{k_j\} \in (0,1)$ *where* $k := \max\{k_1, k_2\}$ *and* $k \in (0,1)$,

(ii) $\lambda_0 \in (k,1),\ \lambda_i,\ \lambda_j \in (0,1),\ i,j = 1,2,...,$ *such that* $\sum_{i=0}^{\infty} \lambda_i = 1$ *and* $\sum_{j=0}^{\infty} \lambda_j = 1$,

(iii) for each $x \in \cap_{i=1}^{\infty} F(T_i), T_i x = \{x\}$ *and for each* $y \in \cap_{j=1}^{\infty} F(S_j), S_j y = \{y\}$.

If the mapping $T_i, i = 1, 2, 3, ...$ *and* $S_j, j = 1, 2, 3, ...$ *are semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *converges strongly to a solution of problem* (3.41).

In Theorem 3.4.2 taking $B = I$ and $H_3 = H_2$, we obtain the following convergence theorem for split convex minimization problem (3.44) $SCMP(\alpha, \beta)$.

Corollary 3.4.3. *Let* H_1 *and* H_2 *be real Hilbert spaces,* C *and* Q *be nonempty, closed and convex subsets of* H_1 *and* H_2 *respectively. Let* $\alpha : C \to \mathbb{R} \cup \{+\infty\}$ *and* $\beta : Q \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semi-continuous and convex functions. Let* $T_i : H_1 \to CB(H_1), i = 1, 2, 3, ...,$ *and* $S_j : H_2 \to CB(H_2)$ *be two multivalued generalized strictly pseudo-contractive mappings with constants* k_i *and* k_j *respectively and* $A : H_1 \to H_2$ *be a bounded linear operator. Let* $u, x_1 \in C$ *and* $v, y_1 \in Q$ *be arbitrary and the sequence* $\{(x_n, y_n)\}$ *be generated by*

$$\begin{cases} \alpha(u) - \alpha(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \ \forall \ u \in C; \\ \beta(v) - \beta(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \ \forall \ v \in Q; \\ p_n = u_n - \xi_n A^*(Au_n - v_n); \\ q_n = v_n + \xi_n(Au_n - v_n); \\ x_{n+1} = \lambda_0 p_n + \sum_{i=1}^{\infty} \lambda_i g_n^i; \\ y_{n+1} = \lambda_o q_n + \sum_{j=1}^{\infty} \lambda_j z_n^j; \end{cases}$$

where $g_n^i \in T_i p_n, z_n^j \in S_j q_n$ *and* $\xi_n \in (\varepsilon, \frac{2}{\phi_A} - \varepsilon)$ *(for* ε *small enough),* ϕ_A *denote the spectral radius of* A^*A*, where* A^* *is adjoint of* A *satisfying the following conditions:*

(i) $k_1 = \sup_{i\geq 1}\{k_i\}, \ k_2 = \sup_{j\geq 1}\{k_j\} \in (0, 1)$ *where* $k := \max\{k_1, k_2\}$ *and* $k \in (0, 1)$,

(ii) $\lambda_0 \in (k, 1), \ \lambda_i, \ \lambda_j \in (0, 1), \ i, j = 1, 2, ...,$ *such that* $\sum_{i=0}^{\infty} \lambda_i = 1$ *and* $\sum_{j=0}^{\infty} \lambda_j = 1$,

(iii) for each $x \in \cap_{i=1}^{\infty} F(T_i), T_i x = \{x\}$ *and for each* $y \in \cap_{j=1}^{\infty} F(S_j), S_j y = \{y\}$.

If the mapping $T_i, i = 1, 2, 3, ...$ *and* $S_j, j = 1, 2, 3...$ *are semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *converges strongly to a solution of problem* (3.44).

Remark 3.4.4. *Prototypes of the recursion formula* $\sum_{i=0}^{\infty} \lambda_i = 1$ *and* $\sum_{j=0}^{\infty} \lambda_j = 1$ *considered in this paper are:*

$$\lambda_i = \frac{2}{3^{i+1}}, i = 0, 1, 2, 3, ...,$$
$$\lambda_j = \frac{1}{j^2 + 3j + 2}, j = 0, 1, 2, 3, ...$$

CHAPTER 4

Common Solution of Split Equilibrium Problem and Fixed Point Problem with no prior knowledge of operator norm.

4.1 Introduction

In this chapter, we introduce an iterative algorithm that does not require any knowledge of the operator norm for finding a common solution of split equilibrium problem and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings in real Hilbert spaces. Using our iterative algorithm, we state and prove a strong convergence result for approximating a common solution of split equilibrium problem and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings which also solves some variational inequality problem in real Hilbert spaces. we give an application and an example of our main result.

Let H_1, H_2 be two real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$. Let $T_i : C \to K(C), i = 1, 2, 3, ...,$ be an infinite family of quasi-nonexpansive multi-valued mappings and $F_1 : C \times C \to \mathbb{R},\ F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$, where F_2 is upper semicontinuous in the first argument. Let f be a contraction with coefficient $\mu \in (0, 1)$, then we solve the following problem: find $x^* \in \cap_{i=1}^{\infty} F(T_i)$ such that

$$F_1(x^*, x) \geq 0 \ \forall \ x \ \in \ C,$$

$$\text{and} \quad y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) \geq 0, \ \forall \ y \in \ Q.$$

Where the set of solution $\Upsilon := \cap_{i=1}^{\infty} F(T_i) \cap \Theta \neq \emptyset$ and $q \in \Upsilon$ also solves some variational inequality problem

$$\langle (D - \tau f)q, q - p \rangle \leq 0, \forall \ p \ \in \ \Upsilon.$$

4.2 Preliminaries

In this chapter, we state some well known results which will be used in the sequel. Throughout this paper, we denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$ respectively. We also denote by $CB(C)$, $K(C)$ and $P(C)$ the families of nonempty, closed and bounded subsets, nonempty and compact subsets and nonempty proximinal subsets of C respectively.
Let H be a real Hilbert space, then the following inequalities holds

$$||u - v||^2 = ||u||^2 - ||v||^2 - 2\langle u - v, v \rangle, \tag{4.1}$$
$$||u + v||^2 \leq ||u||^2 + 2\langle v, u + v \rangle, \tag{4.2}$$

and

$$||\lambda u + (1 - \lambda)v||^2 = \lambda ||u||^2 + (1 - \lambda)||v||^2 - \lambda(1 - \lambda)||u - v||^2, \tag{4.3}$$

for all $u, v \in H$ and $\lambda \in [0, 1]$.

Lemma 4.2.1. *[30] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F x = \Big\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \Big\}.$$

Then the following hold:
(i) T_r^F is nonempty and single-valued;
(ii) T_r^F is firmly nonexpansive, that is $\forall \ x, y \in H$,

$$||T_r^F x - T_r^F y||^2 \leq \langle T_r^F x - T_r^F y, x - y \rangle;$$

(iii) $F(T_r^F) = EP(F)$;
(iv) $EP(F)$ is closed and convex.

Lemma 4.2.2. *[46] Assume D is a strongly positive bounded linear operator on a Hilbert space H with a coefficient $\overline{\tau} > 0$ and $0 < \mu < ||D||^{-1}$. Then $||I - \mu D|| \leq 1 - \mu \overline{\tau}$.*

Lemma 4.2.3. *[56] Every Hilbert space H satisfies the Opial condition that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$, holds for every $y \in H$ with $y \neq x$.*

Lemma 4.2.4. *[80] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n \delta_n, \ \ n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that
(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\sigma_n \delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 4.2.5. *[46] Let C be a nonempty, closed and convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with coefficient $\mu \in (0,1)$ and D is a strongly positive linear bounded operator with a coefficient $\overline{\sigma} > 0$. Then, for $0 < \sigma < \frac{\overline{\sigma}}{\mu}$,*

$$\langle x - y, (D - \sigma f)x - (D - \sigma f)y \rangle \geq (\overline{\sigma} - \sigma\mu)||x - y||^2, \quad x, y \in H.$$

That is, $D - \sigma f$ is strongly monotone with coefficient $\overline{\sigma} - \sigma\mu$.

Lemma 4.2.6. *[19] Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \ \ g(0) = 0,$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$||\sum_{i=1}^{\infty} \lambda_i x_i||^2 = \sum_{i=1}^{\infty} \lambda_i ||x||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

Lemma 4.2.7. *[40](Demiclosedness principle) Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to K(C)$ be a quasi-nonexpansive multi-valued mapping . Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $p \in Tp$.*

4.3 Main result

Theorem 4.3.1. *Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$.*

Let $T_i : C \to K(C), i = 1, 2, 3, ...,$ be an infinite family of quasi-nonexpansive multi-valued mappings and $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying assumptions $(A1) - (A4)$, where F_2 is upper semi-continuous in the first argument. Suppose $\Upsilon := \cap_{i=1}^{\infty} F(T_i) \cap \Theta \neq \emptyset$ and f a contraction mapping with coefficient $\mu \in (0, 1)$. Let the sequences $\{u_n\}, \{y_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \xi_n A^*(T_{r_n}^{F_2} - I)Ax_n); \\ y_n = \lambda_0 u_n + \sum_{i=1}^{\infty} \lambda_i z_n^i; \\ x_{n+1} = \gamma_n \tau f(x_n) + (I - \gamma_n D)y_n, \ n \geq 1; \end{cases} \tag{4.4}$$

where $z_n^i \in T_i u_n$, $r_n \subset (0, \infty)$ and step size ξ_n be chosen in such a way that for some $\varepsilon > 0$,

$$\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon \right),$$

for all $T_{r_n}^{F_2} Ax_n \neq Ax_n$ and $\xi_n = \xi$ otherwise (ξ being any nonnegative real number) satisfying the following conditions;
(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\gamma_n \in (0, 1)$ *and* $0 < \tau < \frac{\overline{\tau}}{\mu}$;

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $0 < \gamma_n < 2\mu$;

(iv) $\lambda_i \in (0, 1)$ *for* $i \geq 0$ *such that* $\sum_{i=0}^{\infty} \lambda_i = 1$ *and* $T_i p = \{p\}$. *Then the sequence* $\{x_n\}$ *generated by (4.4) converges strongly to* $q \in \Upsilon$ *which solves the variational inequality*

$$\langle (D - \tau f)q, q - p \rangle \leq 0, \quad \forall \, p \in \Upsilon.$$

Proof. We first show that $\{x_n\}$ is bounded. For any $x, y \in C$, we need to show that $I - \gamma D$ is nonexpansive.
Now since $2\mu > \gamma_n$, we have

$$\begin{aligned} ||(I - \gamma_n D)x - (I - \gamma_n D)y||^2 &= ||(x - y) - \gamma_n(Dx - Dy)||^2 \\ &\leq ||x - y||^2 - 2\gamma_n \langle x - y, Dx - Dy \rangle + \gamma_n^2 ||Dx - Dy||^2 \\ &\leq ||x - y||^2 - 2\mu\gamma_n ||Dx - Dy||^2 + \gamma_n^2 ||Dx - Dy||^2 \\ &= ||x - y||^2 - \gamma_n(2\mu - \gamma_n)||Dx - Dy||^2 \\ &\leq ||x - y||^2. \end{aligned}$$

Thus the mapping $I - \gamma_n D$ is nonexpansive.
Let $p \in \Upsilon$, we have $T_{r_n}^{F_1} p = p, Ap = T_{r_n}^{F_2} Ap$, then

$$\begin{aligned}
||u_n - p|| &= ||T_{r_n}^{F_1}(x_n + \xi_n A^*(T_{r_n}^{F_2} - I)Ax_n) - p||^2 \\
&\leq ||x_n + \xi_n A^*(T_{r_n}^{F_2} - I)Ax_n - p||^2 \\
&\leq ||x_n - p||^2 + \xi_n^2 ||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 + 2\xi_n \langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n \rangle. \qquad (4.5)
\end{aligned}$$

Where

$$\begin{aligned}
2\xi_n \langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n \rangle &= 2\xi_n \langle A(x_n - p), (T_{r_n}^{F_2} - I)Ax_n \rangle \\
&= 2\xi_n \langle A(x_n - p) + (T_{r_n}^{F_2} - I)Ax_n - (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n \rangle \\
&= 2\xi_n \big\{ \langle T_{r_n}^{F_2} Ax_n - Ap, (T_{r_n}^{F_2} - I)Ax_n \rangle - ||(T_{r_n}^{F_2} - I)Ax_n||^2 \big\} \\
&\leq 2\xi_n \big\{ \frac{1}{2} ||(T_{r_n}^{F_2} - I)Ax_n||^2 - ||(T_{r_n}^{F_2} - I)Ax_n||^2 \big\} \\
&\leq -\xi_n ||(T_{r_n}^{F_2} - I)Ax_n||^2. \qquad (4.6)
\end{aligned}$$

Hence,

$$\begin{aligned}
||u_n - p||^2 &\leq ||x_n - p||^2 + \xi_n^2 ||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 - \xi_n ||(T_{r_n}^{F_2} - I)Ax_n||^2 \\
&= ||x_n - p||^2 - \xi_n [||(T_{r_n}^{F_2} - I)Ax_n||^2 - \xi_n ||A^*(T_{r_n}^{F_2} - I)Ax_n||^2]. \qquad (4.7)
\end{aligned}$$

Since $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon \right)$, we obtain

$$||u_n - p||^2 \leq ||x_n - p||^2. \qquad (4.8)$$

Since $T_i : C \to K(C)$ is an infinite family of a quasi-nonexpansive multi-valued mapping, we have that

$$\begin{aligned}
||y_n - p|| &= ||\lambda_0(u_n - p) + \sum_{i=1}^{\infty} \lambda_i(z_n^i - p)|| \\
&\leq \lambda_0||u_n - p|| + \sum_{i=1}^{\infty} \lambda_i||z_n^i - p|| \\
&\leq \lambda_0||u_n - p|| + \sum_{i=1}^{\infty} \lambda_i d(z_n^i, T_i p) \\
&\leq \lambda_0||u_n - p|| + \sum_{i=1}^{\infty} \lambda_i \mathcal{H}(T_i u_n, T_i p) \\
&\leq \lambda_0||u_n - p|| + \sum_{i=1}^{\infty} \lambda_i||u_n - p|| \\
&= ||u_n - p|| \\
&\leq ||x_n - p||. \qquad (4.9)
\end{aligned}$$

Moreso, by Lemma 4.2.2, we have

$$\begin{aligned}
||x_{n+1} - p|| &= ||\gamma_n[\tau f(x_n) - Dp] + (I - \gamma_n D)(y_n - p)|| \\
&\leq (1 - \gamma_n\overline{\tau})||y_n - p|| + \gamma_n||\tau f(x_n) - Dp|| \\
&\leq (1 - \gamma_n\overline{\tau})||y_n - p|| + \gamma_n[||\tau f(x_n) - \tau f(p)|| + ||\tau f(p) - Dp||] \\
&\leq [1 - (\overline{\tau} - \tau\mu)\gamma_n]||x_n - p|| + \gamma_n||\tau f(p) - Dp||.
\end{aligned}$$

It follows from induction that

$$||x_n - p|| \leq \max\{||x_0 - p||, \frac{||\tau f(p) - Dp||}{\overline{\tau} - \tau\mu}\}, n \geq 1. \qquad (4.10)$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$ and $\{y_n\}$ are bounded. Applying Lemma 4.2.2 and (4.7), we have that

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||\gamma_n[\tau f(x_n) - Dp] + (I - \gamma_n D)(y_n - p)||^2 \\
&\leq (1 - \gamma_n\overline{\tau})^2||y_n - p||^2 + \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&\leq (1 - \gamma_n\overline{\tau})^2||u_n - p||^2 + \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&\leq (1 - \gamma_n\overline{\tau})^2[||x_n - p||^2 + \xi_n^2||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 - \xi_n||(T_{r_n}^{F_2} - I)Ax_n||^2] \\
&+ \gamma_n^2||\tau f(x_n) - Dp|| + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&\leq (1 - \gamma_n\overline{\tau})^2||x_n - p||^2 + \xi_n[\xi_n||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 - ||(T_{r_n}^{F_2} - I)Ax_n||^2] \\
&+ \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p||. \qquad (4.11)
\end{aligned}$$

It follows from (4.11) and the condition $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2}-I)Ax_n||^2}{||A^*(T_{r_n}^{F_2}-I)Ax_n||^2} - \varepsilon\right)$ that

$$\begin{aligned} ||x_{n+1}-p||^2 \leq (1-\gamma_n\overline{\tau})^2||x_n-p||^2 - \varepsilon||A^*(T_{r_n}^{F_2}-I)Ax_n||^2 \\ + \gamma_n^2||\tau f(x_n)-Dp||^2 + 2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n)-Dp||\ ||y_n-p||. \end{aligned} \tag{4.12}$$

We now consider two cases.

CASE A: Assume that $\{||x_n-p||\}$ is a monotonically decreasing sequence. Then $\{||x_n-p||\}$ is convergent and clearly,

$$\lim_{n\to\infty}||x_n-p|| = \lim_{n\to\infty}||x_{n+1}-p||.$$

Since $\{x_n\}$ is bounded and $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2}-I)Ax_n||^2}{||A^*(T_{r_n}^{F_2}-I)Ax_n||^2} - \varepsilon\right)$, then we deduce from (4.12) that

$$\begin{aligned} \varepsilon||A^*(T_{r_n}^{F_2}-I)Ax_n||^2 \leq (1-\gamma_n\overline{\tau})^2||x_n-p||^2 - ||x_{n+1}-p||^2 + \gamma_n^2||\tau f(x_n)-Dp||^2 \\ + 2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n)-Dp||\ ||y_n-p||. \end{aligned}$$

Hence,

$$\lim_{n\to\infty}||A^*(T_{r_n}^{F_2}-I)Ax_n|| = 0. \tag{4.13}$$

Furthermore, from (4.11) and (4.13), we have

$$\begin{aligned} \xi_n||(T_{r_n}^{F_2}-I)Ax_n||^2 \leq (1-\gamma_n\overline{\tau})^2||x_n-p||^2 - ||x_{n+1}-p||^2 + \xi_n^2||A^*(T_{r_n}^{F_2}-I)Ax_n||^2 \\ + \gamma_n^2||\tau f(x_n)-Dp||^2 + 2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n)-Dp||\ ||y_n-p||. \end{aligned} \tag{4.14}$$

Therefore, since $\lim_{n\to\infty}\gamma_n = 0$, from (4.13) and the condition $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2}-I)Ax_n||^2}{||A^*(T_{r_n}^{F_2}-I)Ax_n||^2} - \varepsilon\right)$, we have that

$$\lim_{n\to\infty}||(T_{r_n}^{F_2}-I)Ax_n||^2 = 0. \tag{4.15}$$

Next, we show that $||u_n - x_n|| \to 0$ as $n \to \infty$. Since $p \in \Upsilon$, we obtain

$$\begin{aligned}
||u_n - p||^2 &= ||T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p||^2 \\
&\leq \langle u_n - p, x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p\rangle \\
&= \frac{1}{2}\big\{||u_n - p||^2 + ||x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p||^2 \\
&- ||(u_n - p) - [x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p]||^2\big\} \\
&\leq \frac{1}{2}\big\{||u_n - p||^2 + ||x_n - p||^2 + \xi\big(\xi||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 - ||(T_{r_n}^{F_2} - I)Ax_n||^2\big) \\
&- ||u_n - p - (x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p)||^2\big\} \\
&\leq \frac{1}{2}\big\{||u_n - p||^2 + ||x_n - p||^2 - [||u_n - x_n||^2 + \xi^2||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 \\
&- 2\xi\langle u_n - x_n, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle]\big\} \\
&\leq \frac{1}{2}\big\{||u_n - p||^2 + ||x_n - p||^2 - ||u_n - x_n||^2 + \xi^2||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 \\
&+ 2\xi||A(u_n - x_n)||\ ||(T_{r_n}^{F_2} - I)Ax_n||\big\}.
\end{aligned}$$

Hence, we obtain

$$||u_n - p||^2 \leq ||x_n - p||^2 - ||u_n - x_n||^2 + 2\xi||A(u_n - x_n)||\ ||(T_{r_n}^{F_2} - I)Ax_n||. \tag{4.16}$$

From (4.11) and (4.16), we have

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq (1 - \gamma\overline{\tau})^2||u_n - p||^2 + \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&\leq (1 - \gamma_n\overline{\tau})^2[||x_n - p||^2 - ||u_n - x_n||^2 + 2\xi||A(u_n - x_n)||\ ||(T_{r_n}^{F_2} - I)Ax_n||] \\
&+ \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&= (1 - 2\gamma_n\overline{\tau} + (\gamma_n\overline{\tau})^2)||x_n - p||^2 - (1 - \gamma_n\overline{\tau})^2||u_n - x_n||^2 \\
&+ 2\xi(1 - \gamma_n\overline{\tau})^2||A(u_n - x_n)||\ ||(T_{r_n}^{F_2} - I)Ax_n|| \\
&+ \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&\leq ||x_n - p||^2 + (\gamma_n\overline{\tau})^2||x_n - p||^2 - (1 - \gamma_n\overline{\tau})^2||u_n - x_n||^2 \\
&+ 2\xi(1 - \gamma_n\overline{\tau})^2||A(u_n - x_n)||\ ||(T_{r_n}^{F_2} - I)Ax_n|| \\
&+ \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p||,
\end{aligned}$$

which gives

$$\begin{aligned}
(1 - \gamma_n\overline{\tau})^2||u_n - x_n||^2 &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + (\gamma_n\overline{\tau})^2||x_n - p||^2 \\
&+ 2\xi(1 - \gamma_n\overline{\tau})^2||A(u_n - x_n)||\ ||(T_{r_n}^{F_2} - I)Ax_n|| \\
&+ \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p||.
\end{aligned} \tag{4.17}$$

Since $\{x_n\}, \{y_n\}$ are bounded and from condition (i) of (4.21), (4.15), we have that

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{4.18}$$

Since T_i is an infinite family of a quasi-nonexpansive multi-valued mapping, then applying Lemma 4.2.6 we have

$$\begin{aligned}
||y_n - p||^2 &= ||\lambda_0 u_n + \sum_{i=1}^{\infty} \lambda_i z_n^i - p||^2 \\
&\leq \lambda_0 ||u_n - p||^2 + \sum_{i=1}^{\infty} \lambda_i (d(z_n^i, T_i p))^2 - \lambda_0 \lambda_i g(||u_n - z_n^i||) \\
&\leq \lambda_0 ||u_n - p||^2 + \sum_{i=1}^{\infty} \lambda_i (\mathcal{H}(T_i u_n, T_i p))^2 - \lambda_0 \lambda_i g(||u_n - z_n^i||) \\
&\leq \lambda_0 ||u_n - p||^2 + \sum_{i=1}^{\infty} \lambda_i ||u_n - p||^2 - \lambda_o \lambda_i g(||u_n - z_n^i||) \\
&= ||u_n - p||^2 - \lambda_0 \lambda_i g(||u_n - z_n^i||)^2 \\
&\leq ||x_n - p||^2 - \lambda_0 \lambda_i g(||u_n - z_n^i||)^2.
\end{aligned}$$

Which also implies that

$$0 < \lambda_0 \lambda_i g(||u_n - z_n^i||) \leq ||x_n - p||^2 - ||y_n - p||^2,$$

hence $\lim_{n\to\infty} g(||u_n - z_n^i||) = 0$. By property of g (see Lemma 4.2.6), we have $\lim_{n\to\infty} ||u_n - z_n^i|| = 0$. Since $\{x_n\}$ and $\{y_n\}$ are bounded, we have that

$$\lim_{n\to\infty} d(u_n, T_i u_n) \leq \lim_{n\to\infty} ||u_n - z_n^i|| = 0. \tag{4.19}$$

From (4.12), we have that

$$\begin{aligned}
||y_n - u_n|| &= ||\lambda_0 u_n - \sum_{i=1}^{\infty} \lambda_i z_n^i - u_n|| \\
&= ||\lambda_0 (u_n - u_n) + \sum_{i=1}^{\infty} \lambda_i (z_n^i - u_n)|| \\
&\leq \sum_{i=1}^{\infty} \lambda_i ||z_n^i - u_n||.
\end{aligned} \tag{4.20}$$

From (4.19), we have that

$$\lim_{n\to\infty} ||y_n - u_n|| = 0. \tag{4.21}$$

Also, we have

$$||y_n - x_n|| \leq ||y_n - u_n|| + ||u_n - x_n||. \tag{4.22}$$

From (4.18) and (4.21), we have that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{4.23}$$

From (4.3.1), we have

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||x_{n+1} - y_n|| + ||y_n - x_n|| \\ &||\gamma_n \tau f(x_n) + (I - \gamma_n D)y_n - y_n|| + ||y_n - x_n|| \\ &\leq \gamma_n ||\tau f(x_n) - Dy_n|| + ||y_n - x_n|| \end{aligned} \tag{4.24}$$

From condition (i) of (4.3.1) and (4.23), we have that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{4.25}$$

Now, we need to show that $\omega(x_n) \subset \Upsilon$, where $\omega(x_n) := \{x \in H_1 : x_{n_k} \rightharpoonup x, \{x_{n_k}\} \subset \{x_n\}\}$. Since $\{x_n\}$ is bounded and H_1 is reflexive, $\omega(x_n)$ is nonempty. Let $q^* \in \omega(x_n)$ be an arbitrary element, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to q^*. From (4.18), we have that $u_{n_k} \rightharpoonup q^*$ as $k \to \infty$. By the demiclosedness principle and (4.19), we obtain $q^* \in \cap_{i=1}^{\infty} F(T_i)$.
Let us show that $q^* \in EP(F_1)$. Since $u_n = T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n - \xi A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \geq 0,$$

for all $y \in C$, which implies that

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\langle y - u_n, \xi A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \geq 0,$$

for all $y \in C$. From $(A2)$, we have:

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, u_{n_k} - x_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - u_{n_k}, \xi A^*(T_{r_{n_k}}^{F_1} - I)Ax_{n_k}\rangle \geq F_1(y, u_{n_k}),$$

for all $y \in C$. From $\liminf_{n\to\infty} r_n > 0$, (4.15), (4.18) and $(A4)$, we have that $F_1(y, q^*) \leq 0, \forall\ q^* \in C$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q^*$. Since $y \in C, q^* \in C$, we get $y_t \in C$ and hence $F_1(y_t, q^*) \leq 0$. Therefore from $(A1)$ and $(A4)$, we have that

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, q^*) \leq tF_1(y_t, y).$$

Hence $0 \leq F_1(y_t, y)$. Applying $(A3)$, we have that
$0 \leq F_1(q^*, y)$. This implies that $q^* \in EP(F_1)$. Since A is a bounded linear operator, $Ax_{n_k} \rightharpoonup Aq^*$. From (4.15), we have that

$$T^{F_2}_{r_{n_k}} Ax_{n_k} \rightharpoonup Aq^*, \tag{4.26}$$

as $k \to \infty$. By the definition of $T^{F_2}_{r_{n_k}} Ax_{n_k}$, we have

$$F_2(T^{F_2}_{r_{n_k}} Ax_{n_k}, y) + \frac{1}{r_{n_k}} \langle y - T^{F_2}_{r_{n_k}} Ax_{n_k} - Ax_{n_k} \rangle \geq 0,$$

for all $y \in C$. Since F_2 is upper semi-continuous in the first argument and from (4.26), it follows that

$$F_2(Aq^*, y) \geq 0, \forall\ y \in C.$$

Which implies that $Aq^* \in EP(F_2)$ and hence $q^* \in \Theta$.
We now show that $\limsup_{k\to\infty} \langle (D - \tau f)q, q - x_n \rangle \leq 0$, where $q = P_\Upsilon(I - \tau f + D)q$.
Indeed, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (D - \tau f)q, x_n - q \rangle = \lim_{n\to\infty} \langle (D - \tau f)q, x_{n_k} - q \rangle. \tag{4.27}$$

We also assume that $x_{n_k} \rightharpoonup q^*$. Therefore

$$\begin{aligned}
\limsup_{n\to\infty} \langle (D - \tau f)q, x_n - q \rangle &= \lim_{n_k\to\infty} \langle (D - \tau f)q, x_{n_k} - q \rangle \\
&= \langle Dq - \tau f(q), q^* - q \rangle \\
&= \langle (I - \tau f + D)q - q, q^* - q \rangle \\
&= \langle (I - \tau f + D)q - P_\Upsilon(I - \tau f + D)q, q^* - P_\Upsilon(I - \tau f + D)q \rangle \\
&\leq 0.
\end{aligned}$$

Furthermore, we show the uniqueness of a solution of the variational inequality

$$\langle (D - \tau f)x, x - q \rangle \leq 0, \quad q \in \Upsilon. \tag{4.28}$$

Suppose $q \in \Upsilon$ and $q^* \in \Upsilon$, both are solutions of (4.28), then

$$\langle (D - \tau f)q,\ q - q^* \rangle \leq 0, \tag{4.29}$$

and

$$\langle (D - \tau f)q^*,\ q^* - q \rangle \leq 0. \tag{4.30}$$

Adding (4.29) and (4.30), we have

$$\langle (D-\tau f)q-(D-\tau f)q^*,\ q-q^*\rangle \le 0.$$

By Lemma 4.2.5, the strong monotonicity of $D-\tau f$, we have that $q=q^*$. Hence the uniqueness is proved. Lastly, we prove that $x_n\to q$ as $n\to\infty$. From (4.3.1) and (4.9), we have that

$$\begin{aligned}||x_{n+1}-q||^2 &= \langle \gamma_n\tau f(x_n)+(I-\gamma_nD)y_n-q, x_{n+1}-q\rangle\\ &= \gamma_n\langle \tau f(x_n)-f(q), x_{n+1}-q\rangle + \langle (I-\gamma_nD)(y_n-q), x_{n+1}-q\rangle\\ &\le \gamma_n\tau\langle f(x_n)-f(q), x_{n+1}-q\rangle + \gamma_n\langle \tau f(q)-Dq, x_{n+1}-q\rangle\\ &+(1-\gamma_n\overline{\tau})||y_n-q||\ ||x_{n+1}-q||\\ &\le \gamma_n\tau\mu||x_n-q||\ ||x_{n+1}-q|| + \gamma_n\langle \tau f(q)-Dq, x_{n+1}-q\rangle\\ &+(1-\gamma_n\overline{\tau})||x_n-q||\ ||x_{n+1}-q||\\ &= [1-(\overline{\tau}-\tau\mu)\gamma_n]||x_n-q||||x_{n+1}-q|| + \gamma_n\langle \tau f(q)-Dq, x_{n+1}-q\rangle\\ &\le \frac{1-(\overline{\tau}-\tau\mu)\gamma_n}{2}(||x_n-q||^2+||x_{n+1}-q||^2) + \gamma_n\langle \tau f(q)-Dq, x_{n+1}-q\rangle\\ &\le \frac{1-(\overline{\tau}-\tau\mu)\gamma_n}{2}||x_n-q||^2+\frac{1}{2}||x_{n+1}-q||^2 + \gamma_n\langle \tau f(q)-Dq, x_{n+1}-q\rangle.\end{aligned}$$

Then, it follows that

$$||x_{n+1}-q||^2 \le [1-(\overline{\tau}-\tau\mu)\gamma_n]||x_n-q||^2+\gamma_n(\overline{\tau}-\tau\mu)\frac{2\langle \tau f(q)-Dq, x_{n+1}-q\rangle}{(\overline{\tau}-\tau\mu)}. \quad (4.31)$$

From $0<\tau<\frac{\overline{\tau}}{\mu}$, condition (i) of (4.3.1) then we conclude that $\lim_{n\to\infty}||x_n-q||=0$ using Lemma 4.2.4.

CASE B: Assume that $\{||x_n-p||\}$ is not a monotonically decreasing sequence. Then, we define an integer sequence $\{\sigma(n)\}$ for all $n\ge n_0$ (for some n_0 large enough) by

$$\sigma(n) := \max\{k\in\mathbb{N}; k\le n : ||x_k-p|| < ||x_{k+1}-p||\}.$$

Clearly, σ is a nondecreasing sequence such that $\sigma(n)\to\infty$ as $n\to\infty$ and for all $n\ge n_0$. From (4.12), we have

$$\begin{aligned}\xi_{\sigma(n)}||(T^{F_2}_{r_{\sigma(n)}}-I)Ax_{\sigma(n)}||^2 &\le (1-\gamma_{\sigma(n)}\overline{\tau})^2||x_{\sigma(n)}-p||^2-||x_{\sigma(n+1)}-p||^2\\ &+\xi^2_{\sigma(n)}||A^*(T^{F_2}_{r_{\sigma(n)}}-I)Ax_{\sigma(n)}||^2+\gamma^2_{\sigma(n)}||\tau f(x_{\sigma(n)})-Dp||^2\\ &+2\gamma_{\sigma(n)}(1-\gamma_{\sigma(n)}\overline{\tau})||\tau f(x_{\sigma(n)})-Dp||\ ||y_{\sigma(n)}-p||. \quad (4.32)\end{aligned}$$

Therefore, since $\lim_{n\to\infty}\gamma_{\sigma(n)}=0$, from (4.13) and the condition $\xi_{\sigma(n)}\in\Bigg(\varepsilon, \frac{||(T^{F_2}_{r_{\sigma(n)}}-I)Ax_{\sigma(n)}||^2}{||A^*(T^{F_2}_{r_{\sigma(n)}}-I)Ax_n||^2}-\varepsilon\Bigg)$, we have that

$$\lim_{n\to\infty}||(T^{F_2}_{r_{\sigma(n)}}-I)Ax_{\sigma(n)}||^2=0. \quad (4.33)$$

Following the same argument as in CASE A, we conclude that there exist a subsequence $\{x_\sigma(n)\}$ which converges weakly to $p \in \Upsilon$. Now for all $n \geq n_0$, we have

$$\begin{aligned} 0 &\leq ||x_{\sigma(n+1)} - q||^2 - ||x_{\sigma(n)} - q||^2 \\ &\leq (1 - \gamma_{\sigma(n)}\overline{\tau})||x_{\sigma(n)} - q||^2 + \gamma_{\sigma(n)}^2||\tau f(x_{\sigma(n)}) - Dq||^2 \\ &+ 2\gamma_{\sigma(n)}(1 - \gamma_{\sigma(n)}\overline{\tau})||\tau f(x_{\sigma(n)}) - Dq||\ ||x_{\sigma(n)} - q|| - ||x_{\sigma(n)} - q||^2 \\ &= -\gamma_{\sigma(n)}\overline{\tau}||x_{\sigma(n)} - q||^2 + \gamma_{\sigma(n)}^2||\tau f(x_{\sigma(n)}) - Dq||^2 \\ &+ 2\gamma_{\sigma(n)}(1 - \gamma_{\sigma(n)}\overline{\tau})\langle \tau f(x_{\sigma(n)} - Dq, x_{\sigma(n+1)} - q\rangle. \end{aligned}$$

Thus

$$||x_{\sigma(n)} - q||^2 \leq \frac{\gamma_{\sigma(n)}}{\overline{\tau}}||\tau f(x_{\sigma(n)}) - Dq||^2 + \frac{2(1 - \gamma_{\sigma(n)}\overline{\tau})}{\overline{\tau}}\langle \tau f(x_{\sigma(n)} - Dq, x_{n+1} - q\rangle.$$

Since $\lim_{n\to\infty}\gamma_n \to 0$ as $n \to \infty$ and $\limsup\langle \tau f(x_{\sigma(n)} - Dp, x_{n+1} - q\rangle \leq 0$, then we conclude that $\{x_n\}$ converges strongly to q. This complete the proof. □

Corollary 4.3.2. *Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$. Let $T_i : C \to K(C), i = 1, 2, 3, ...,$ be an infinite family of nonexpansive multi-valued mappings and $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$, where F_2 is upper semi-continuous in the first argument. Suppose $\Upsilon := \cap_{i=1}^{\infty}F(T_i) \cap \Theta \neq \emptyset$ and f a contraction with a coefficient $\mu \in (0, 1)$. Let the sequences $\{u_n\}, \{y_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \xi_n A^*(T_{r_n}^{F_2} - I)Ax_n); \\ y_n = \lambda_0 u_n + \sum_{i=1}^{\infty}\lambda_i z_n^i; \\ x_{n+1} = \gamma_n\tau f(x_n) + (I - \gamma_n D)y_n,\ n \geq 1; \end{cases} \tag{4.34}$$

where $z_n^i \in T_i u_n$, $r_n \subset (0, \infty)$ and step size ξ_n be chosen in such a way that for some $\varepsilon > 0$ and
$\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon\right)$ for all $T_{r_n}^{F_2}Ax_n \neq Ax_n$ and $\xi_n = \xi$, otherwise (ξ being any nonnegative real number) satisfying the following conditions;
(i) $\lim_{n\to\infty}\gamma_n = 0$ and $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(ii) $\gamma_n \in (0, 1)$ and $0 < \tau < \frac{\overline{\tau}}{\mu}$;
(iii) $\liminf_{n\to\infty} r_n > 0$ and $0 < \gamma_n < 2\mu$;
(iv) $\lambda_0, \lambda_i \in (0, 1)$ such that $\sum_{i=0}^{\infty}\lambda_i = 1$ and $T_i p = \{p\}$.
Then the sequence $\{x_n\}$ generated by (4.34) converges strongly to $q \in \Upsilon$ which solves the variational inequality

$$\langle (D - \tau f)q, q - p\rangle \leq 0, \quad \forall\, p \in \Upsilon.$$

4.4 Application and Numerical example

4.4.1 Application to Optimization problem

Let H_1, H_2 be two real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $f : C \to \mathbb{R}$, $g : Q \to \mathbb{R}$ be two operators and $A : H_1 \to H_2$ be a bounded linear operator, then the optimization problem is to find:

$$\begin{aligned} x^* \in C \quad &\text{such that } f(x^*) \leq f(x), \quad \forall\, x \in C, \\ \text{and} \quad y^* = Ax^* \quad &\text{such that} \quad g(y^*) \leq g(y), \quad \forall\, y \in Q. \end{aligned} \tag{4.35}$$

We denote the set of solutions of (4.35) by Ω and assume that $\Omega \neq \emptyset$. Setting $F_1(x, y) = f(y) - f(x)$ for all $x, y \in C$ and $F_2(x, y) = g(y) - g(x)$ for all $x, y \in Q$ respectively. Then $F_1(x, y)$ and $F_2(x, y)$ satisfy conditions $(A1) - (A4)$ provided f and g are convex and lower semi-continuous on C and Q respectively, Clearly, $\Theta = \Omega$.
By Theorem (4.3.1), we have the following iterative algorithm which converges strongly to $q \in \Upsilon$ and solves the optimization problem (4.35).

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \xi_n A^*(T_{r_n}^{F_2} - I)Ax_n); \\ y_n = \lambda_0 u_n + \sum_{i=1}^{\infty} \lambda_i z_n^i; \\ x_{n+1} = \gamma_n \tau f(x_n) + (I - \gamma_n D)y_n, \ n \geq 1; \end{cases} \tag{4.36}$$

where $z_n^i \in T_i u_n,\ r_n \subset (0, \infty)$ and step size ξ_n be chosen in such a way that for some $\varepsilon > 0$ and
$\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon \right)$ for all $T_{r_n}^{F_2} Ax_n \neq Ax_n$ and $\xi_n = \xi$, otherwise (ξ being any nonnegative real number) satisfying the following conditions;
(i) $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\gamma_n \in (0, 1)$ and $0 < \tau < \frac{\overline{\tau}}{\mu}$;
(iii) $\liminf_{n\to\infty} r_n > 0$ and $0 < \gamma_n < 2\mu$;
(iv) $\lambda_0, \lambda_i \in (0, 1)$ such that $\sum_{i=0}^{\infty} \lambda_i = 1$.

4.4.2 Numerical Example

Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = \mathbb{R}$. Let $F_1(u, v) = -3u^2 + uv + 2v^2$. We see that $F_1(u, v)$ satisfies assumptions $(A1) - (A4)$ as follows:

$(A1)$ $F_1(u, u) = -3u^2 + u^2 + 2u^2 = 0$ for all $u \in C$.

(A2) If $F_1(u, v) = (u - v)(-3u - 2v) \geq 0$, then
$F_1(v - u) = (v - u)(-3v - 2u) = (v - u)((-3u - 2v) - (v - u)) = -F_1(u, v) - (v - u)^2 \leq 0$ for all $u, v \in C$, i.e., $F_1(u, v)$ is pseudomonotone, while $F_1(u, v)$ is not monotone.

(A3) If $u_n \rightharpoonup \overline{u}$ and $v_n \rightharpoonup \overline{v}$, then
$F_1(u_n, v_n) = -3u_n^2 + u_n v_n + 2v_n^2 \to -3\overline{u}^2 + \overline{uv} + 2\overline{v}^2 = F_1(\overline{u}, \overline{v})$, i.e., $F_1(u, v)$ is jointly weakly continuous on $C \times C$.
(A4) Let $\theta \in (0, 1)$. Since

$$\begin{aligned} F_1(u, \theta v_1 + (1-\theta)v_2) &= -3u^2 + u(\theta v_1 + (1-\theta)v_2) + 2(\theta v_1 + (1-\theta)v_2)^2 \\ &\leq \theta(2v_1^2 + uv_1 - 3u^2) + (1-\theta)(2v_2^2 + uv_2 - 3u^2) \\ &= \theta F_1(u, v_1) + (1-\theta)F_1(u, v_2), \end{aligned}$$

So $F_1(u, v)$ is convex, also, $\liminf_{v \to \overline{v}} F_1(u, v) = F_1(u, \overline{v})$,
hence $F_1(u, v)$ is lower semi-continuous. Since $\partial_2 F_1(u, v) = u + 4v$, thus $F_1(u, v)$ is subdifferentiable on C for each $u \in C$.
Now, we derive our resolvent function $T_r^{F_1}$ using Lemma 4.2.1 as follows:

$$\begin{aligned} F_1(u, v) + \frac{1}{r}(v-u)(u-x) &= -3ru^2 + ruv + 2rv^2 + uv - vx - u^2 + ux \\ &= 2rv^2 + ruv + uv - vx - 3ru^2 - u^2 + ux \\ &= 2rv^2 + (ru + u - x)v - (3ru^2 + u^2 - ux) \end{aligned}$$

Let $Q(v) = 2rv^2 + (ru+u-x)v - (3ru^2+u^2-ux)$ with coefficients $a = 2r, b = ru+u-x, c = -3ru^2 - u^2 + ux$. We compute the discriminant of $Q(v)$ as follows:

$$\begin{aligned} \Delta = b^2 - 4ac &= (ru + u - x)(ru + u - x) - 4(2r)(-3ru^2 - u^2 + ux) \\ &= r^2u^2 + ru^2 - rux + ru^2 + u^2 - ux - rux - ux + x^2 + 24r^2u^2 + 8ru^2 - 8rux \\ &= 25r^2u^2 + 10ru^2 - 10rux - 2ux + u^2 + x^2 \\ &= x^2 - 10rux - 2ux + 25r^2u^2 + 10ru^2 + u^2 \\ &= t^2 - 2((5r+1)u)x + u^2 + 25r^2u^2 + 10ru^2 \\ &= t^2 - 2((5r+1)u)x + ((5r+1)u)^2 \\ &= (x - (5r+1)u)^2 \geq 0. \end{aligned}$$

Thus, $\Delta \geq 0 \ \forall \ y \in \mathbb{R}$ and it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, $T_{r_n}^{F_1}(x) = \frac{x}{5r_n+1}$.
Let $F_2(u, v) = -5u^2 + uv + 4v^2, Ax = x$ and $A^*x = x$. Following the same process used in deriving $T_r^{F_1}$, we have $T_{r_n}^{F_2}(x) = \frac{x}{9r_n+1}$.
Furthermore define $T_i : \mathbb{R} \to K(\mathbb{R})$, $(i = 1, 2, 3, \cdots)$ by:

$$T_i x = \begin{cases} [0, \frac{x}{2i}] & x \in [0, \infty), \\ [\frac{x}{2i}, 0] & x \in (-\infty, 0], \end{cases}$$

where $K(\mathbb{R})$ is the family of nonempty, closed and bounded subset of $\mathbb{R}$. Clearly, T_i for each i is a multivalued quasi-nonexpansive mapping. Let $f : \mathbb{R} \to \mathbb{R}$ be given as; $f(x) = \frac{1}{8}x$,

then $\mu = \frac{1}{6}$ is a contraction constant for f. Take $D(x) = 2x$ with constant $\overline{\tau} = 1$. On the other hand, we take $\tau = 2$ which satisfies $0 < \tau < \frac{\overline{\tau}}{\mu}$.
Furthermore, we take $\gamma_n = \frac{n+1}{8n}, r_n = \frac{n}{n+1}, \lambda_0 = \frac{1}{2}, \lambda_i = \frac{1}{2^{i+1}}, z_n^i \in T_i u_n$ and let the step size ξ_n be chosen in such a way that for some $\varepsilon > 0$, $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon\right)$ for all $T_{r_n}^{F_2} Ax_n \neq Ax_n$ and ξ_n be any positive real number otherwise, in iterative scheme (3.1) we obtain

$$\begin{cases} u_n = \frac{(1-\xi_n)x_n}{5r_n+1} + \frac{\xi_n x_n}{(5r_n+1)(9r_n+1)}, \\ y_n = \frac{1}{2}u_n + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} z_n^i, \\ x_{n+1} = (\frac{n+1}{8n})(\frac{x_n}{4}) + (1 - \frac{(n+1)}{4n})y_n. \end{cases}$$

Case 1: $x_0 = 1$ and $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon\right)$ for all $T_{r_n}^{F_2} Ax_n \neq Ax_n$ and $\xi_n = 0.0003$ otherwise.
Case 2: $x_0 = 2$ and $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon\right)$ for all $T_{r_n}^{F_2} Ax_n \neq Ax_n$ and $\xi_n = 0.0000021$ otherwise.

CHAPTER 5

Conclusion, Contribution to Knowledge and Future Research

In this chapter, we give our conclusion, contribution to knowledge and future research.

5.1 Conclusion

In this dissertation, as set out in the objectives, we have been able to established a weak and a strong convergence results for approximating common solutions of SEMEP and fixed point problem for generalized k_i-strictly pseudo-contractive multi-valued mappings in real Hilbert spaces. We utilized our result to study split equality convex minimization and split equality mixed variational inequality problems as applications. Also, using iterative algorithm (4.4) which does not require the knowledge of the operator norm, we proved a strong convergence result for approximating a common solution of SEP and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings in real Hilbert spaces and applied our result in chapter 4 to convex minimization problem. A numerical example was displayed in chapter 4 in real Hilbert spaces and we showed how the sequences are affected by the number of iterations.

5.1.1 Contribution to Knowledge

We briefly discuss our own contribution in this dissertation as follows:
Motivated by the works of [31, 37, 42, 69] and other authors, Theorems 3.3.1 and 4.3.1 improves and extends some recent results in the following sense:
(i) Theorem 3.3.1 extends the result of Ma. *et al.* [42] in the sense that the authors in [42] considered nonexpansive single-valued mappings, whereas in this dissertation we considered

generalized strictly pseudo-contractive multi-valued mappings.
(ii) Our iterative algorithm (4.4) improves (2.43) presented by Deepho. *et al.* [31] in the sense that (4.4) does not require a prior knowledge of the operator norm.
(iii) In [31], the authors considered a finite family of k-strictly pseudo-contractive nonself mapping and obtained a strong convergence result under some suitable conditions in real Hilbert spaces. However, in Theorem 4.3.1 we considered an infinite family of quasi-nonexpansive multi-valued mappings and prove a strong convergence result without imposing any condition.
(iv) In [69], the authors proved that the modified Mann iteration converges weakly to a common solution of SEP and fixed point problem of non-spreading multi-valued mappings. However, we prove a strong convergence result for approximating a common solution of SEP and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings in real Hilbert spaces.
(v) Our result holds for nonexpansive multi-valued mappings.

5.2 Future Research

In this section, we briefly discuss some problems we intend solving for our future research. Firstly, we will consider the problem solved in chapter three of this dissertation, the compactness condition imposed on our main theorem will be removed and we will prove a new convergence result without imposing the compactness condition. In addition, we will give a numerical example to improve on our former work.
Secondly, we intend to study and solve another problem based on the following contributions by authors.
In 2006, Marino and Xu [46] considered the following implicit iterative algorithm for a nonexpansive mapping T.

$$x_t = t\gamma f(x_t) + (I - tB)Tx_t,$$

where f is a contraction mapping with constant α and $B : H_1 \to H_1$ is a strongly positive bounded linear self adjoint operator, i.e, if there exist a constant $\overline{\gamma} > 0$ such that $\langle Bx, x\rangle \geq \overline{\gamma}||x||^2, x \in H_1$, with $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ and $t \in (0, 1)$. They proved that the net (x_t) converges to the unique solution of the variational inequality

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \quad x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Bx, h(x)\rangle,$$

where h is the potential function for γf.
In 2008, Plubtieng and Punpaeng [59] introduced and studied the following implicit iterative scheme to prove a strong convergence theorem for fixed point problem.

$$x_t = tf(x_t) + (I - s)\frac{1}{s_t}\int_0^{s_t} T(s)x_t ds,$$

where (x_t) is a continuous net and (s_t) is a positive real divergent net.
Recently, Cianciaruso et al.[29] introduced and studied the following implicit iterative scheme and obtained a strong convergence result for EP and fixed point problem.

$$\begin{cases} F(u_t, y) + \frac{1}{r_t}\langle y - u_t, u_t - x_t\rangle, \forall y \in C, \\ x_t = t\gamma f(x_t) + (I - tB)\frac{1}{s_t}\int_0^{s_t} T(s)u_t ds, \end{cases}$$

where (s_t) and (r_t) are the continuous nets in (0,1). Motivated by the works of the aforementioned author and by the ongoing research in this direction, we will construct an implicit iterative algorithm for approximating a common solution of split equality fixed point problem and MEP for a nonexpansive semigroup in real Hilbert spaces. Moreso, we will also prove that the nets generated by the proposed iterative scheme converges strongly to a common solution of MEP and split equality fixed point problem. A numerical example will be use to justify our main theorem.
Lastly, many researchers have studied the scalar EP defined in chapter 2 of our dissertation. Based on the ongoing research on vector EP, we intend studying the vector EP and fixed point problem for certain multi-valued mappings.

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