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On Singularly Perturbed Problems and Exchange of Stabilities

by

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Abstract

Singular perturbation theory has been used for about a century to describe models displaying different timescales, that arise in applied sciences; particularly, models displaying two timescales, namely slow time and fast time.
Different techniques have been developed over time in order to analyze the limit behaviour and the stabilities of their solutions when the small parameter tends to zero. The nature of the limit equation obtained when the small parameter tends to zero plays a major role in understanding the behaviour of the solution of singularly perturbed problems.
In this thesis, we analyze the behaviour of the solution of singularly perturbed problems in the following cases. First, when the limit equation displays the Allee effect. Next, when the limit equation is structurally stable or non-structurally stable and the standard Tikhonov theorem is applicable and finally, when the quasi-steady states of the degenerate equation intersect causing an exchange of stabilities.
Furthermore, we perform numerical simulations in each case to support the analytic results.

Preface and Declarations

The work described in this thesis was carried out in the school of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Durban, from July 2009 to December 2014, under the supervision of Professor Jacek Banasiak.
These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

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As the candidate's supervisor I have approved/disapproved the thesis for submission.

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Declaration 2-Publications

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this dissertation (include publications in preparation, submitted, *in press* and published and give details of the contributions of each author to the experimental work and writing of each publication)

Publication 1

A Singularly Perturbed SIS Model with Age Structure
MATHEMATICS BIOSCIENCES AND ENGINEERING
Volume 10, Number 3, June 2013
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Publication 2 *Canard-Type Solutions in Epidemiological Models*
Manuscript accepted in DCDS.

Publication 3

etc.

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Introduction

Singular and regular perturbation theory consists in the study of problems that depend on a small positive parameter, usually denoted in the literature by ϵ or μ, or by any other Greek symbol if there is no confusion.
This type of problems is referred to as *singular and regular perturbation problem* (or *singularly and regularly perturbed problem*). But, unlike regular perturbation theory, in which the original solution uniformly converges to the solution obtained by letting the small parameter equal to zero in the original problem, which we refer to as the limit solution, singular perturbation theory presents a totally different picture. Indeed, in singular perturbation theory, the original solution looks completely different from the limit solution as the small parameter tends to zero.
Singular perturbation theory can apply to algebraic equations, differential equations as well as to functional equations. In this work, we limit its application to ordinary differential equations. Note that the presence of a small parameter within singular perturbation problems for differential equations is an indication of the presence of different timescales within the problem under investigation.
In this connection, singular perturbation theory offers a strong mathematics framework to deal with problems involving different timescales. In particular, problems involving two timescales; namely slow time usually denoted by t and fast time usually denoted by τ. In this thesis, we use the relation $\tau = t/\epsilon$ to link these two timescales, where ϵ is a small parameter.
Singular perturbation theory for differential equations is divided into a local theory and a global theory. The local theory consists in the study of the solution of a singular perturbation problem around a point within the manifold of its equilibrium points and, because of the presence of singularities, the local theory is not trivial at all.
The global theory consists in the study of the solution of a singular perturbation problem in a large domain; that is, around a compact subset of the manifold of its equilibrium points, [1].
Singular perturbation problems for differential equations arise in a different number of ways and from different types of applications in applied sciences. For example, applications from biochemistry, neurophysiology, hydrodynamics, semiconductor physics, population modelling, epidemyological modelling, dynamical systems, multi scales analysis for long-time dynamics, etc, naturally give rise to singularly perturbed problems, [2]. These applications are in most cases very challenging since they involve different types of variables introduced by different timescales within the problems. It turns out that singular perturbation theory perfectly describes this type of problems and offers different approaches for their analysis.
In what follows, in order to analyze singularly perturbed problems, we use an analytic approach. A lot of work has been done in this direction. We refer the reader to the work of A.N. Tikhonov, *Asymptotic behaviour of solutions of singularly perturbed problems* when the small parameter tends to zero, [3], and the work of Vasil'eva, *Asymptotic Formulas for the Solutions of Ordinary Differential Equations with Small Parameters Multiplying the Highest Derivative*, where she extended Tikhonov's approach by introducing the method of boundary layer functions that makes possible the construction of asymptotic expansions that uniformly approximate solutions of initial boundary value problems for singularly perturbed systems of nonlinear ordinary differential equations which can be written as the so-called *Tikhonov's type system*, [4].
In addition to the analytic approach, we have also a geometric approach to analyze singular

perturbation problems; here we cite the work of Fenichel, *The Geometric Singular Perturbation Theory for Ordinary Differential Equations*, [1] and the work of Tasso Kaper, *An Introduction to Geometric Methods and Dynamical Systems Theory for Singular Pertubation Problems*, [5].

Recently, with the widespread use of computers and the development of different types of softwares the numerical approach becomes more and more essential in the analysis of the behaviour of the solutions of singularly perturbed problems; we have used it in this work to support the analytic results.

We do not pretend to present a comprehensive study of singular perturbation theory. Since, as indicated earlier, the volume of results accumulated in this field for over a century is too large.

In our thesis, we revisit certain cases of singularly perturbed problems and we apply different techniques to analyze the behaviour of their solutions as the small parameter tends to zero. Then we confirm the analytic result by numerical simulations.

This thesis is organized as follows. In Chapter 1, we introduce preliminaries of singular perturbation theory and we define basic concepts that we use in the sequel of the thesis.

In Chapter 2, we discuss an application of Tikhonov theorem to a model that describes the dynamics of a population of females searching for a mate and that, for a certain choice of parameters, displays the Allee effect.

In Chapter 3, we show that due to the structutal stability or instability of the limit equation, Tikhonov theorem can in some cases provide convergence of the solution of the singularly perturbed problem to the solution of the limit equation on $[0, T]$, that, in general, is not uniform in T. In order to ensure that the convergence is uniform in T, we use the centre manifold theory approach, that sometimes allows for making the approximate equation structurally stable.

We illustrate this by considering two examples. The first example deals with a model of the kinetics of enzyme, from the monograph of Carr [6], that generates a structurally stable limit equation. In this case, for a finite time interval $[0, T]$, we have uniform convergence by Tikhonov theorem. It can also be proved that this convergence is uniform in $(0, \infty)$ by the centre manifold theory.

The second example deals with a prey-predator model that generates a non-structurally stable limit equation. In this case, we use the centre manifold theory approach to construct an approximate equation that is structurally stable.

In Chapter 4, we analyze the behaviour of the solution of a special class of singularly perturbed problems in which the quasi-steady states of the limit equation intersect and the solution can jump from one quasi-steady state to the other, displaying the so-called exchange of stabilities within the system.

Using a singularly perturbed SIS model, we show that, depending on the model, such a jump may occur immediately at a point t_c or at some point $t^* > t_c$, which is independent of the small parameter.

Part of the contents of Chapters 2, 3 and 4 are also presented in the recent book of my supervisor *Methods of Small parameter in Mathematical Biology* [7] to which I contributed. Thus, the chapters have some overlaps and contain similar results with corrections obtained while working on the thesis or offering an alternative approach.

Note that in this thesis, we focus our study on the local theory of singularly perturbed problems.

1. Preliminaries and Definitions

In this chapter, we introduce and define concepts that we are going to use in the sequel of this thesis.

1.1 Basic concepts of stability

1.1.1 Introduction

In this section, we introduce some basic concepts that define the stability of the solution of differential equations.

Let us first consider the following scalar equation

$$\dot{y}(t) = ay(t), \quad y(0) = y_0. \tag{1.1}$$

The solution of the equation (1.1) is

$$y(t) = y_0 e^{at}. \tag{1.2}$$

Note that $y(t) \equiv 0$ is a particular solution of the equation (1.1), referred to as the zero solution. In Equation (1.2) we have

1. If $a < 0$, then $y(t)$ approaches zero as t approaches infinity. Then we say that the zero solution is asymptotically stable.

2. If $a = 0$, then $y(t)$ is a constant solution. In this case the zero solution is said to be stable (but not asymptotically stable). Any solution of (1.1) that starts close to the zero solution, stay close to it for all time.

3. If $a > 0$, then $y(t)$ approaches infinity as t tends to infinity. In this case, the zero solution is said to be unstable, [8].

Lemma 1.1.1

Let $\alpha > 0$ be a real number and $j \geq 0$ be an integer. Then there exists a constant C (depending on α and j) such that

$$t^j e^{-\alpha t} \leq C \text{ for all } \ t \geq 0.$$

Proof.

If $j = 0$, we can choose $C = 1$. If $j \neq 0$, the function $h(t) = t^j e^{-\alpha t}$ satisfies $h(0) = 0$ and $h(t) \to 0$ as $t \to \infty$. Hence, h is bounded on $[0, \infty)$, [8]. $\square$

Lemma 1.1.2

Let λ be a complex number and $j \geq 0$ be an integer. Assume $Re(\lambda) < \sigma$. Then there exists a constant C such that

$$|t^j e^{\lambda t}| \leq Ce^{\sigma t}.$$

Proof.

First, let us assume that α is a real number and $\alpha < \sigma$. Then $\alpha - \sigma < 0$. By the Lemma 1.1.1, there exists a constant C such that $t^j e^{(\alpha-\sigma)t} \leq C$ for all $t \geq 0$. If we multiply this inequality by $e^{\sigma t}$, we obtain $t^j e^{\alpha t} \leq Ce^{\sigma t}$ for all $t \geq 0$.
Now, let us assume that $\lambda = \alpha + i\beta$, where α and β are real. If we let $\alpha \equiv Re(\lambda) < \sigma$, then

$$|t^j e^{\lambda t}| \equiv t^j e^{\alpha t} \leq Ce^{\sigma t}, \text{ for all } \ t \geq 0.$$

□

Lemma 1.1.3

Let $P(\lambda)$ be a polynomial of degree n with complex coefficients. Let $\lambda_1, \cdots, \lambda_n$ be the roots of the equation $P(\lambda) = 0$ and let us assume $Re(\lambda_j) < \sigma$ for $j = 1, \cdots, m$.
Let z be a solution of the differential equation $P(D)z = 0$. Then there exists a constant $C \geq 0$ such that

$$|z(t)| \leq Ce^{\sigma t} \text{ for all } \ t \geq 0.$$

Proof.

The equation $P(D)z = 0$ possesses a fundamental set of solutions $z_1, \cdots, z_n$ such as

$$z_j(t) = t^{t_j} e^{\lambda_j t}$$

for some integer j and roots λ_j.

By Lemma 1.1.2, there exists a constant K_j such that $|z_j(t)| \leq K_j e^{\sigma t}$ for all $t \geq 0$. Since z is an arbitrary solution of the equation $P(D)z = 0$, we have $z = c_1 z_1 + \cdots + c_n z_n$ for some constants c_j. Then for $t \geq 0$ we have

$$\begin{aligned}
|z(t)| &= |c_1 z_1(t) + \cdots + c_n z_n(t)| \\
&\leq |c_1||z_1(t)| + \cdots + |c_n||z_n(t)| \\
&\leq |c_1| K_1 e^{\sigma t} + \cdots + |c_n| K_n e^{\sigma t} \\
&\equiv (|c_1| K_1 + \cdots + |c_n| K_n) e^{\sigma t}.
\end{aligned}$$

This completes the proof, [8]. □

Theorem 1.1.1

Let A be an $n \times n$ matrix and $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A. Let us assume $Re(\lambda_j) < \sigma$ for $j = 1, \cdots, n$. Then there exists a constant K such that

$$\|e^{At}\| \leq Ke^{\sigma t} \text{ for all } \ t \geq 0.$$

Proof.

Let $P(\lambda)$ be the characteristic polynomial of A. The roots of the characteristic equation $P(\lambda) = 0$ are the same as the eigenvalues of A. From the exponential formula, we have

$$e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t) A^j,$$

where $r_j(t) = t^{t_j} e^{\lambda_j t}$ for $j = 1, \cdots, n$ is a solution of the differential equation $P(D)r = 0$. By Lemma 1.1.3, there exists a constant c_j such that $|r_j(t)| \leq c_j e^{\sigma t}$ for all $t \geq 0$. Then

$$\|e^{At}\| \leq \left[\sum_{j=0}^{n-1} c_{j+1} \|A\|^j \right] e^{\sigma t}.$$

This completes the proof, [8]. $\square$

Next, let us consider a linear differential equation of order n with constant coefficients

$$P(D)z = z^{(n)} + a_1 z^{(n-1)} + a_2 z^{(n-2)} + \cdots + a_n = 0. \tag{1.3}$$

Then we have

Theorem 1.1.2

If all the zeros of the characteristic polynomial $P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$ of the equation (1.3) have negative real parts, then given any solution $z(t)$ of (1.3), there exists numbers $a > 0$ and $M > 0$ such that

$$|z(t)| \leq Me^{-at}, \ t \geq 0.$$

Hence,

$$\lim_{t \to \infty} |z(t)| = 0.$$

Theorem 1.1.3 (Routh-Hurwitz criterion)

Consider the linear equation (1.3) with real coefficients $a_j,\ j = 1, 2, \cdots, n$.

Let

$$D_1 = a_1,\ D_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix},\ D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \cdots$$

$$\cdots D_k = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & 1 & a_2 & \cdots & a_{2k-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_k \end{vmatrix},$$

where $a_j = 0$ for $j > n$.

Then the roots of the characteristic polynomial $P(\lambda)$ of the equation (1.3) have negative real parts if and only if $D_k > 0$ for $k = 1, \cdots, n$.

Definition 1.1.1

Let us consider the following differential equation

$$\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0, \tag{1.4}$$

where $x(t) \in \mathbb{R}^n$ and f is assumed to be continuous and locally Lipschitz with respect to $x(t)$. If we denote the maximally defined solution of the equation (1.4) by $x(t) = x(t, t_0, x_0)$ and, if $\varphi : [t_0, \infty) \to \mathbb{R}^n$ is a solution of the differential equation, then

1. φ is said to be stable on $[t_0, \infty)$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|\varphi(t_0) - x_0| < \delta$, the solution $x(t, t_0, x_0)$ is defined for all $t \in [t_0, \infty)$ and

$$|\varphi(t) - x(t)| < \epsilon \text{ for all } t \geq t_0.$$

2. φ is said to be asymptotically stable on $[t_0, \infty)$ if φ is stable and, given $\epsilon > 0$, there exists $\delta_1 < \delta$ such that whenever $|\varphi(t_0) - x_0| < \delta_1$, we have

$$\lim_{t\to\infty} |\varphi(t) - x(t)| = 0.$$

3. φ is said to be unstable if there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists some point x_0 such that $|\varphi(t_1) - x(t_1)| \geq \epsilon$ for some $t_1 \in [t_0, \infty)$; with $|\varphi(t_0) - x_0| < \delta$.

1.1.2 Structural stability

In this section, we give a definition of the concept of structural stability of a vector field and we briefly introduce the criteria that a dynamical system in the plane should satisfy in order to be structurally stable.

Roughly speaking, a dynamical system is said to be structurally stable if nearby systems have qualitatively the same dynamics, [9].

The structural stability is a fundamental property of dynamical system which insure that the qualitative behaviour of trajectories is unaffected by small perturbations, particularly, C^1-small perturbations.

Unlike the Lyapunov stability that deals with perturbations of the initial conditions for a fixed system, structural stability deals with perturbations of the system itself, [10].

Various notions of structural stability apply to the vector fields of ordinary differential equations on smooth manifolds and flows generated by them.

Structurally stable systems were introduced by Aleksandr Andronov and Lev Pontryagin in 1937 under the name *systèmes grossiers* or *rough systems*, [10].

Let us consider the set $C^r(\mathbb{R}^n, \mathbb{R}^n)$ of C^r maps of $\mathbb{R}^n$ into $\mathbb{R}^n$. In terms of dynamics, $C^r(\mathbb{R}^n, \mathbb{R}^n)$ can be seen as the set of vector fields. The subset of C^r, consisting of diffeomorphisms, is denoted by $\text{Diff}^r(\mathbb{R}^n, \mathbb{R}^n)$.
Two elements of $C^r(\mathbb{R}^n, \mathbb{R}^n)$ are said to be C^k $\epsilon-$close $(k \leq r)$ if they are, together with their first k derivatives, within ϵ neighborhood as measured in some uniform norm. But, since $\mathbb{R}^n$ is unbounded, we assume that the maps act on a compact, boundaryless $n-$dimensional differentiable manifold M. Then the topology induced on $C^r(M, M)$ by this measure of distance is called *the C^k topology*, [9].

Conjugacies and equivalences of vector fields

Let us consider C^r diffeomorphisms f and g of $\mathbb{R}^n$ into $\mathbb{R}^n$ and a C^k diffeomorphism h of $\mathbb{R}^n$ into $\mathbb{R}^n$, where $k \leq r$.

Definition 1.1.2

f and g are said to be C^k conjugate if there is a C^k diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g \circ h = h \circ f.$$

Remark 1.1.1

Note that if $k = 0$ then f and g are said to be *topologically conjugate*, [9].

Remark 1.1.2

A similar notion to C^k conjugacies exists for vector fields and is called C^k *equivalence*, [9].

Let us consider a system of differential equations

$$\begin{aligned} \dot{x} &= f(x),\ x \in \mathbb{R}^n, \\ \dot{y} &= g(y),\ y \in \mathbb{R}^n. \end{aligned} \tag{1.5}$$

Let f and g be C^r vector fields defined on $\mathbb{R}^n$. Then we have

Definition 1.1.3

The dynamics generated by f and g are said to be C^k equivalent, where $k \leq r$, if there is a C^k diffeomorphism h which takes orbits of the flow $\phi(t, x)$ generated by f to orbits of the flow $\psi(t, y)$ generated by g preserving orientation but not necessarily time parametrization, [9].

Remark 1.1.3

If h does preserve time parametrization, then the dynamics generated by f and g are said to be C^k conjugate, [9].

Definition 1.1.4

A point $x_0 \in \mathbb{R}^n$ is called *a nonwandering point* of the vector field f if for any neighborhood $\mathcal{O}$ of x_0 and $T > 0$, there is some $|t| > T$ such that

$$\phi(t, \mathcal{O}) \cap \mathcal{O} \neq \emptyset.$$

Definition 1.1.5

The set of all nonwandering points of a map or flow is called nonwandering set of that particular map or flow.

Note that equilibrium points and periodic orbits are examples of nonwandering structures.

Definition 1.1.6 (Structural stability)

Let f be a map in Diff$^r(M, M)$ or a vector field in $C^r(M, M)$. Then f is said to be structurally stable if there is a neighborhood $\mathcal{N}$ of f in the C^k topology such that f is C^0 conjugate to every map or C^0 equivalent to every vector field in $\mathcal{N}$, [9].

The Andronov-Pontryagin criterion

The Andronov-Pontryagin criterion is a necessary and sufficient condition for the stability of dynamical systems in the plane.

Definition 1.1.7

An equilibrium point x^* of a vector field F, i.e $F(x^*) = 0$, is said to be hyperbolic if none of the eigenvalues of its linearization at x^* is purely imaginary.

Definition 1.1.8

A periodic orbit of a flow is said to be hyperbolic if none of the eigenvalues of its Poincare return map at a point on the orbit has absolute value one.

Theorem 1.1.4

A C^r vector field on a compact boundaryless two-dimensional manifold M is structurally stable if and only if

1. All equilibrium points and periodic orbits are hyperbolic,
2. there are no saddle connections,
3. the nonwandering set consists of equilibrium points and periodic orbits.

Remark 1.1.4

The saddle connection occurs when an orbit connects a saddle point to itself or to another saddle point, i.e the unstable and stable separatrices are connected, [10].

1.2 Regular and Singular Perturbations

1.2.1 Introduction

In this section, we give a brief introduction of the concepts of regular and singular perturbations. We do not intend to present a comprehensive study of them but we define what we need from them.

We further define some basics of the asymptotic representation of the solution of a regularly perturbed equation and we state without proof the result that establishes the asymptotic representation.

1.2.2 A simple example

Let us consider the well-known equation for the forced damped Duffing oscillator,

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos\omega t, \quad x(0) = x_0,\ \dot{x}(0) = x_1, \tag{1.6}$$

where $\delta, \alpha, \beta, \gamma$ and ω are parameters. Letting $\beta = 0$ in (1.6) reduces it to the following equation of the forced damped oscillator

$$\ddot{x} + \delta\dot{x} + \alpha x = \gamma\cos\omega t \quad x(0) = x_0,\ \dot{x}(0) = x_1. \tag{1.7}$$

A closed form solution of Equation (1.7) is given by

$$x(t) = Ce^{-\delta/2t}\sin(\omega_d t + \phi) + \gamma\cos(\omega t + \Phi), \tag{1.8}$$

where

$$C = \sqrt{4x_0^2 + \left(\frac{\delta x_0 + 2x_1}{2\omega_d}\right)^2},\ \omega_d = \frac{\sqrt{4\alpha - \delta^2}}{2},$$

$$\Phi = \tan^{-1}\left(\frac{\delta\omega}{\alpha - \omega^2}\right) \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2x_0\omega_d}{\delta x_0 + 2x_1}\right),$$

provided $4\alpha > \delta^2$.

The phase portrait and the time series graph of (1.7) in the $(\dot{x}(t), x(t))$ plane are, respectively, given on Fig 1.1 and 1.2.

Numerical simulation

Here, the numerical simulation shows the solution $x(t)$ given by (1.7) and its phase portrait. The parameters' values used are $\alpha = 1$, $\beta = 10$, $\delta = 2$, $\gamma = 2$, $\omega = 1$ and the initial values $x(0) = 1$ and $\dot{x}(0) = 1$.

Now, let us return to the equation of the forced damped Duffing oscillator (1.6). We don't know if it has a closed form solution. Therefore we use perturbation analysis to determine an analytical approximation of its solution.
If we assume weak nonlinearity, weak damping and weak driving force in (1.6), we obtain

$$\ddot{x} + \epsilon\bar{\delta}\dot{x} + \alpha x + \epsilon\bar{\beta}x^3 = \epsilon\bar{\gamma}\cos\omega t, \quad x(0) = x_0,\ \dot{x}(0) = x_1, \tag{1.9}$$

where ϵ is a small positive parameter. If we let $\epsilon = 0$ in (1.9), it is reduced to the equation of a simple harmonic oscillator

$$\ddot{\bar{x}} + \alpha\bar{x} = 0,\ \bar{x}(0) = x_0,\ \dot{\bar{x}}(0) = x_1, \tag{1.10}$$

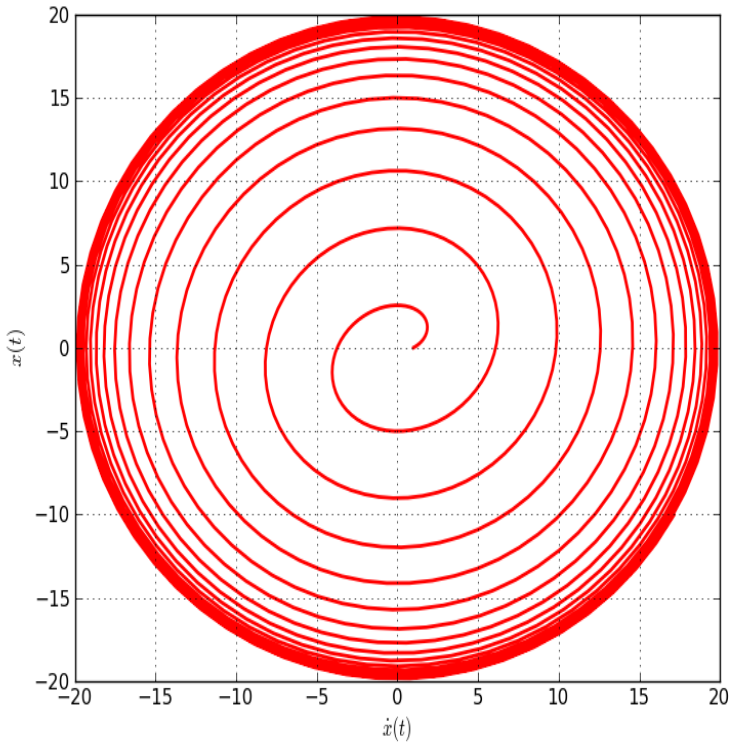


Figure 1.1: The phase portrait of the forced damped oscillator given by (1.7).

whose solution is given by

$$\bar{x}(t) = \sin(\sqrt{\alpha}t + \varphi), \tag{1.11}$$

where

$$\varphi = \tan^{-1}\left(\frac{x_0\sqrt{\alpha}}{x_1}\right).$$

The approximate solution to Equation (1.9) is found by preforming a straightforward expansion of the form

$$x_\epsilon(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots \tag{1.12}$$

Then, by substituting Equation (1.12) into (1.9) and solving for correponding ϵ-terms, the numerical simulation shows on Fig 1.3-1.6 that

$$x_\epsilon(t) \to \bar{x}(t) \quad \text{as} \quad \epsilon \to 0 \quad \text{uniformly} \quad \text{for all} \quad t \in [0, T].$$

Here, $\bar{x}(t)$ is the solution of Equation (1.10).

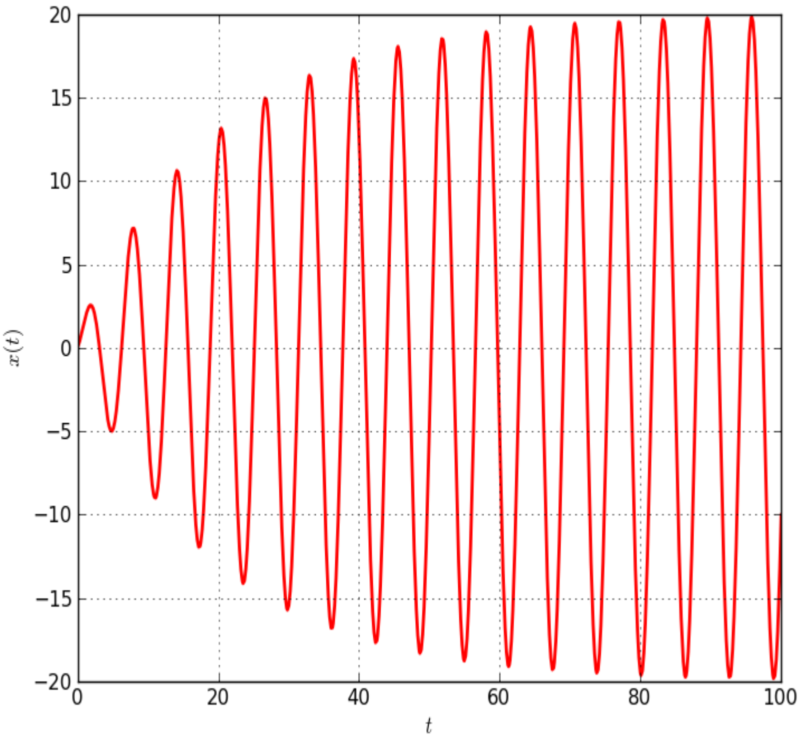


Figure 1.2: The time series of the forced damped oscillator given by (1.7).

The pictures show that as ϵ approaches zero in the perturbed equation (1.9), its solution uniformly converges to the the solution of the unperturbed equation of the simple harmonic oscillator (1.10).

Note that two terms multiplied by the small parameter ϵ, that appears in the perturbed equation (1.9) are referred to as *small perturbations*.
The perturbations such as introduced in (1.9) are called *regular perturbations*, [3].
Now, for the same equation (1.6), if we assume strong damping, strong nonlinearity and strong driving force, then we obtain

$$\ddot{x} + \frac{\tilde{\delta}}{\epsilon}\dot{x} + \alpha x + \frac{\tilde{\beta}}{\epsilon}x^3 = \frac{\tilde{\gamma}}{\epsilon}\cos\omega t \quad x(0) = x_0,\ \dot{x}(0) = x_1. \tag{1.13}$$

If we multiply Equation (1.13) by ϵ, we obtain the equation

$$\epsilon\ddot{x} + \tilde{\delta}\dot{x} + \epsilon\alpha x + \tilde{\beta}x^3 = \tilde{\gamma}\cos\omega t \quad x(0) = x_0,\ \dot{x}(0) = x_1, \tag{1.14}$$

where the small parameter ϵ multiplies the highest order derivative.

Numerical simulation

Here, the numerical simulation shows the uniform convergence on $[0, T]$ of the solution $x_\epsilon(t)$ of the forced damped Duffing oscillator, given by (1.9), to the solution $\bar{x}(t)$ of the simple harmonic oscillator given by (1.10). The parameters' values used are $\alpha = 1,\ \beta = 10,\ \delta = 0.1,\ \gamma = 2,\ \omega = 1,\ \epsilon = 0.1, 0.01, 0.001, 0.0001$ and the initial values are $x(0) = 0$ and $\dot{x}(0) = 1$.

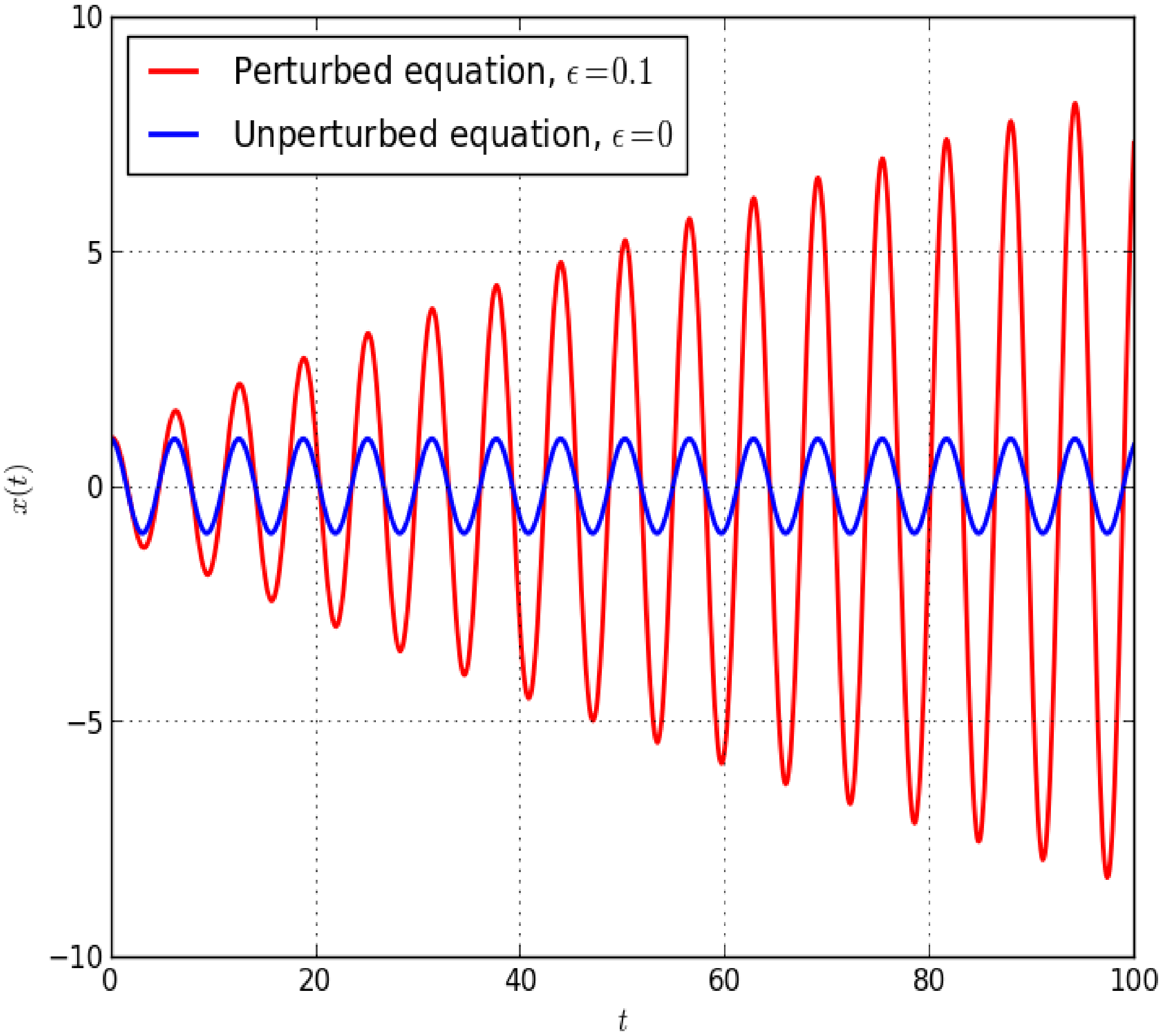


Figure 1.3: The comparison of the solution $x_\epsilon(t)$ given by (1.9) with the solution $\bar{x}(t)$ given by (1.10) for $\epsilon = 0.1$. We observe that $\bar{x}(t)$ gives a good approximation of $x_\epsilon(t)$ only for small t.

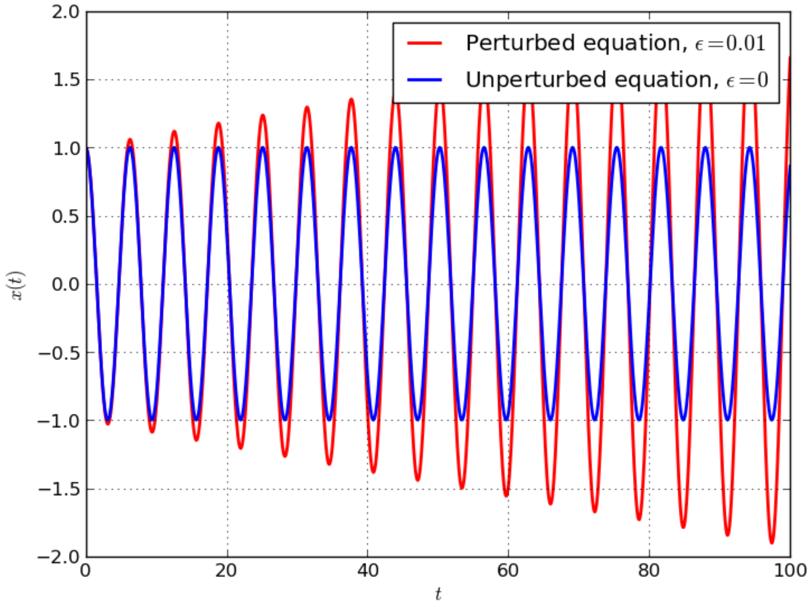


Figure 1.4: The comparison of the solution $x_\epsilon(t)$ given by (1.9) with the solution $\bar{x}(t)$ given by (1.10) for $\epsilon = 0.01$. We observe that the approximation is now valid on a larger time interval.

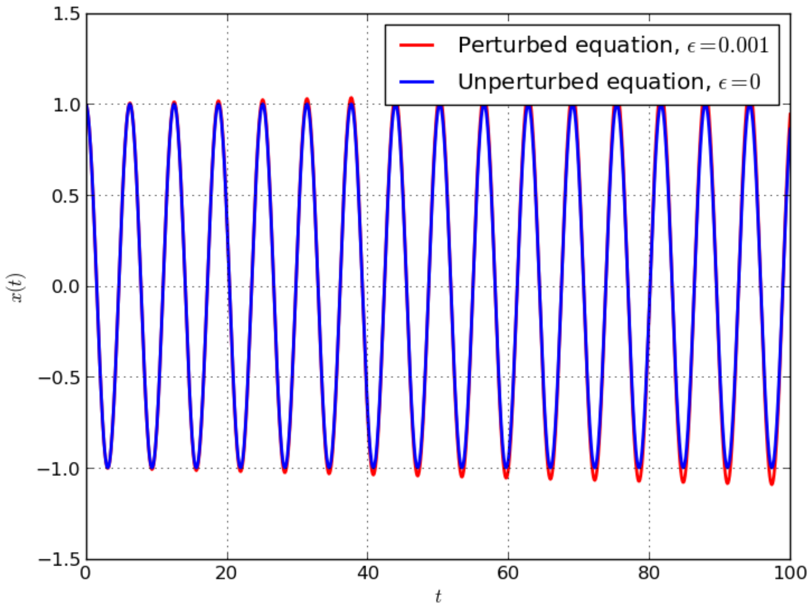


Figure 1.5: The comparison of the solution $x_\epsilon(t)$ given by (1.9) with the solution $\bar{x}(t)$ given by (1.10) for $\epsilon = 0.001$. We observe that the interval on which $\bar{x}(t)$ provides a good approximation to $x_\epsilon(t)$ extends even further.

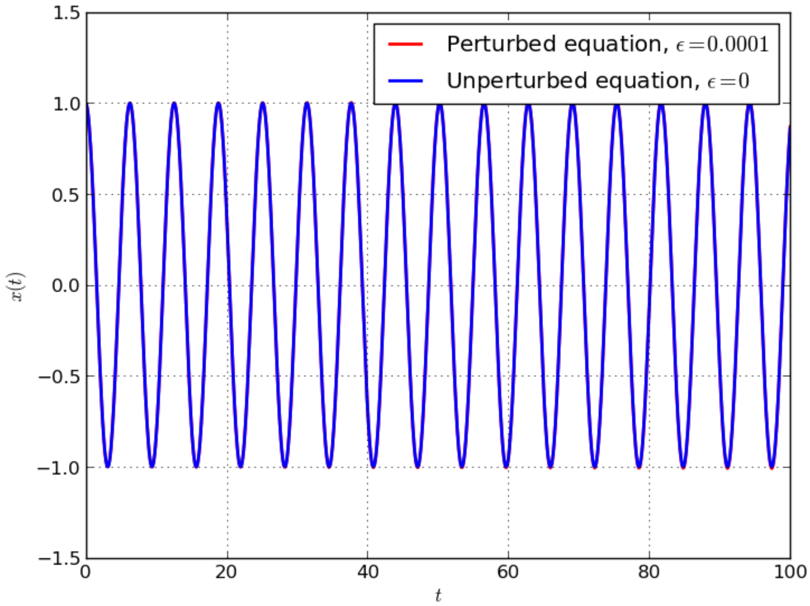


Figure 1.6: The comparison of the solution $x_\epsilon(t)$ given by (1.9) with the solution $\bar{x}(t)$ given by (1.10) for $\epsilon = 0.0001$. We observe that the interval on which $\bar{x}(t)$ provides a good approximation to $x_\epsilon(t)$ extends even further.

Letting $\epsilon = 0$ in (1.14), unlike in Equation (1.9), leads to a lower order equation

$$\tilde{\delta}\dot{x} + \tilde{\beta}x^3 = \tilde{\gamma}\cos\omega t, \quad x(0) = x_0. \tag{1.15}$$

Clearly, in (1.15) we have to drop one of the original initial conditions, because it makes it overdetermined. This yields some consequences; as illustrated by Figs 1.7 and 1.8.

Numerical simulation

Here the numerical simulation shows the convergence on $(0, T]$ of the solution $x_\epsilon(t)$ of the forced damped Duffing oscillator, given by (1.14) to the solution $\tilde{x}(t)$ of the degenerate equation given by (1.15). The parameters' values used are $\alpha = 1$, $\tilde{\beta} = 10$, $\tilde{\delta} = 2$, $\tilde{\gamma} = 2$, $\omega = 1$, $\epsilon = 0.1$ and the initial values are $x(0) = 1$ and $\dot{x}(0) = 3$.

We see that as ϵ approaches zero in (1.14), its solution converges to the solution of the unperturbed equation (1.15). But this convergence is not uniform in the closed interval $[0, T]$; it is, however uniform in any closed interval $[\eta, T]$, where $\eta > 0$.
This phenomenon is due to the loss of one initial condition when we set $\epsilon = 0$ in (1.14). It is referred to as the *initial layer effect*.
In literature, problems in which the small parameter ϵ multiplies the highest order derivative, are referred to as *singular perturbations*, [3].

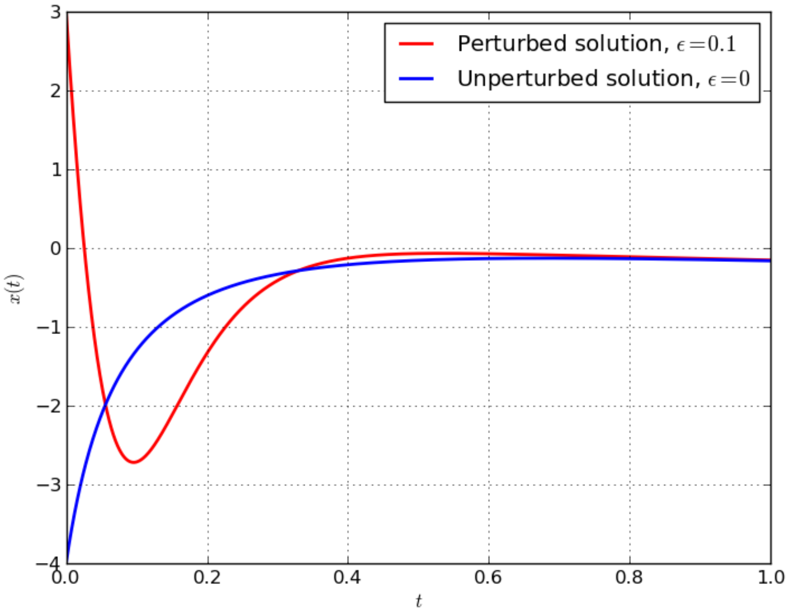


Figure 1.7: The convergence as ϵ approaches zero of the solution $x_\epsilon(t)$ given by (1.14) to the solution $\tilde{x}(t)$ given by (1.15), with $\epsilon = 0.1$.

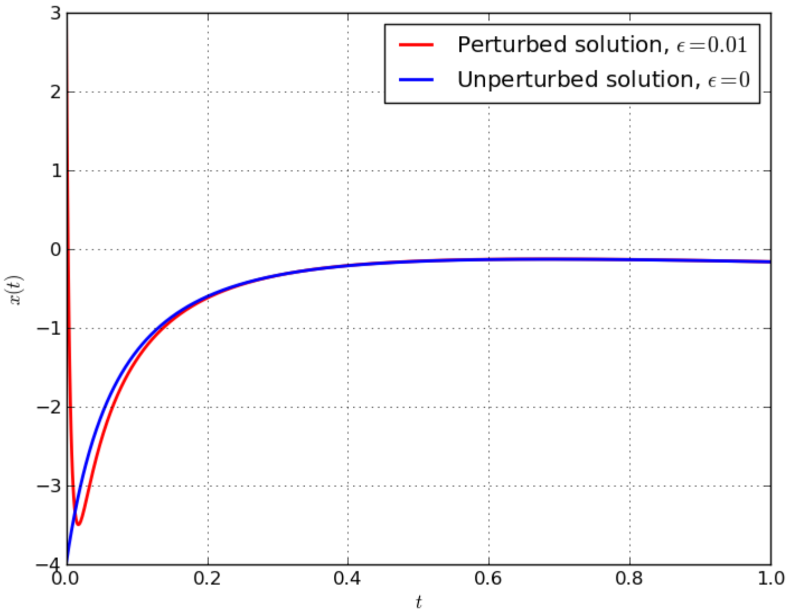


Figure 1.8: The convergence as ϵ approaches zero of the solution $x_\epsilon(t)$ given by (1.14) to the solution $\tilde{x}(t)$ given by (1.15), with $\epsilon = 0.01$.

In the sequel of this chapter, we are going to introduce a mathematical tool that efficiently deals with singularly perturbed problems and that also provides a remedy for the initial layer effect. Let us first give formal definitions of regular and singular perturbations.

1.2.3 Regular perturbation

Let us consider the following initial value problem

$$\frac{dy}{dt} = f(y, t, \epsilon), \quad y(0, \epsilon) = y^0, \tag{1.16}$$

where y is a scalar and ϵ is a small positive parameter. Theorem 2.10 in [3] establishes that if the right-hand side of Equation (1.16) is Lipschitz continuous, then the solution $y(t, \epsilon)$ of Equation (1.16) exists and is unique; it is also continuous with respect to t and ϵ on the set $t \in [0, T]$, $|\epsilon| < c$, for some c.
If we let $\epsilon = 0$ in (1.16), in general we obtain a problem that is much simpler than the original problem,

$$\frac{d\bar{y}}{dt} = f(\bar{y}, t, 0), \quad \bar{y}(0) = y^0. \tag{1.17}$$

The solution $\bar{y}(t)$ of Equation (1.17) is in general easier to construct. Suppose that $f(y, t, \epsilon)$ possesses continuous partial derivatives $f_y(y, t, \epsilon)$ and $f_\epsilon(y, t, \epsilon)$ in some domain D.
Then, Theorem 2.11 in [3] states that the solution $y(t, \epsilon)$ of (1.16) possesses derivative with respect to ϵ for every ϵ in the interval $|\epsilon| < c$. Thus, substituting $y(t, \epsilon)$ into (1.16) gives

$$\frac{dy}{dt}(t, \epsilon) \equiv f(y(t, \epsilon), t, \epsilon), \quad y(0, \epsilon) = y^0. \tag{1.18}$$

If we differentiate Equation (1.18) with respect to ϵ, we obtain the following expressions

$$\frac{d}{dt}\frac{\partial y}{\partial \epsilon} = f_y(y(t, \epsilon), t, \epsilon)\frac{\partial y}{\partial \epsilon} + f_\epsilon(y(t, \epsilon), t, \epsilon), \quad \frac{\partial y}{\partial \epsilon}(0, \epsilon) = 0, \tag{1.19}$$

often called equation of variation with respect to ϵ. Suppose that the solution of (1.16) is written as the formal series

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + \cdots . \tag{1.20}$$

If we substitute Equation (1.20) into (1.16) and expand the right hand side in the Taylor series about $y = y_0$ and $\epsilon = 0$ up to order 1 in ϵ we obtain this expression

$$\begin{gathered}\frac{dy_0}{dt} + \epsilon\frac{dy_1}{dt} + \cdots = f(y_0, t, 0) + \epsilon\frac{\partial f}{\partial y}(y_0, t, 0)y_1 + \epsilon\frac{\partial f}{\partial \epsilon}(y_0, t, 0) + \cdots\,, \\ y^0 = y_0(0) + \epsilon y_1(0) + \cdots\,. \end{gathered} \tag{1.21}$$

Solving Equation (1.21) for the corresponding ϵ-terms leads to the following system of differential equations

$$\begin{aligned} \frac{dy_0}{dt} &= f(y_0, t, 0), \quad y_0(0) = y^0, \\ \frac{dy_1}{dt} &= \frac{\partial f}{\partial y}(y_0, t, 0)y_1 + \frac{\partial f}{\partial \epsilon}(y_0, t, 0), \quad y_1(0) = 0, \\ &\vdots \end{aligned} \tag{1.22}$$

Now, comparing Equations (1.17) and the first equation of (1.22), we see that they coincide and therefore, by uniqueness, we have

$$y_0(t) = \bar{y}(t).$$

Similarly, if we compare Equation (1.19) with the second equation of (1.22), we see that they coincide as well, and therefore, by uniqueness, we have

$$y_1(t) = \frac{\partial y}{\partial \epsilon}.$$

It follows that the existence of the derivative of $y(t,\epsilon)$ with respect to ϵ allows us to write

$$y(t,\epsilon) = \bar{y}(t) + \frac{\partial y}{\partial \epsilon}(t, \tau\epsilon)\epsilon, \quad (0 < \tau < 1). \tag{1.23}$$

From Theorem 2.11 in [3], on the closed interval $[0, T]$, the solution $y(t,\epsilon)$ of (1.16) can be written as

$$y(t,\epsilon) = \bar{y}(t) + \theta(t,\epsilon), \tag{1.24}$$

where $\theta(t,\epsilon) \to 0$ as $\epsilon \to 0$ uniformly on $[0, T]$.

Hence, $\bar{y}(t)$ is an approximation of $y(t,\epsilon)$ and $\theta(t,\epsilon)$ is an error of approximation; it is also called the *remainder*. From Equation (1.23), we have that

$$\left|\frac{\partial y}{\partial \epsilon}(t,\tau\epsilon)\epsilon\right| \le \epsilon\eta(\epsilon).$$

Hence, we have

$$\theta(t,\epsilon) = O(\epsilon).$$

Remarks

1. Formula (1.24) is often called asymptotic formula or asymptotic representation of $y(t,\epsilon)$ with respect to the small parameter ϵ.

2. We obtain a better approximation of $y(t,\epsilon)$ by letting smaller values of ϵ.

If $f(y,t,\epsilon)$ has continuous derivatives of higher order then the remainder of the asymptotic formula tends to zero with a degree higher than $O(\epsilon)$, as stated in the following theorem.

Theorem 1.2.1

Suppose that in some domain D of the variables y, t, ϵ, the function $f(y,t,\epsilon)$ possesses continuous and uniformly bounded partial derivatives with respect to y and ϵ up to the order $n+1$ inclusive. Then there exists a closed interval $[0,T]$ on which, for the solution $y(t,\epsilon)$ of (1.16), we have the asymptotic representation

$$y(t,\epsilon) = \bar{y}(t) + \epsilon\frac{\partial y}{\partial \epsilon}(t,0) + \cdots + \epsilon^n \frac{\partial^n y}{\partial \epsilon^n}(t,0) + \theta_{n+1}(t,\epsilon), \tag{1.25}$$

where $\theta_{n+1}(t,\epsilon) = O(\epsilon^{n+1})$, as $\epsilon \to 0$ for $0 \leq t \leq T$.

For the proof of Theorem 1.2.1, we refer the reader to the book of A.N. Tikhonov [3].

Theorem 1.2.1 gives mathematical arguments why small terms can be neglected as it is often done in physics. These small terms are usually called *small perturbations*.
Perturbations that satisfy assumptions of Theorem 1.2.1 are called *regular perturbations*, [3].

1.2.4 Singular Perturbations

Let us consider the following initial value problem

$$\epsilon\frac{dy}{dt} = f(y,t,\epsilon), \quad y(0,\epsilon) = y^0, \tag{1.26}$$

where y is a scalar function and ϵ is a small positive parameter.

Note that, unlike in Equation (1.16), here the small parameter ϵ multiplies the highest order derivative.
If we let $\epsilon = 0$ in (1.26), it degenerates to an algebraic equation of the form

$$0 = f(y,t,0), \tag{1.27}$$

in which the initial condition may not be satisfied.

Thus, the solution of Equation (1.27) cannot offer a good approximation to the solution of Equation (1.26) at least in the neighborhood of the initial point as we have noticed in our introductory example. Alternatively, if we divide Equation (1.26) by ϵ we obtain

$$\frac{dy}{dt} = \frac{1}{\epsilon} f(y, t, \epsilon), \quad y(0, \epsilon) = y^0. \tag{1.28}$$

Now, if we let $\epsilon = 0$ in (1.28), its right hand side is undefined. Therefore, it is not continuous at this point. Hence, the main assumption of Theorem 1.2.1, namely the continuity of the right hand side, no longer holds.
In other words, we can say that the right hand side of Equation (1.28) depends on ϵ in a non-regular way or in a singular way. Such perturbations, in which the small parameter multiplies the highest derivative are known as *singular perturbations*, [3].
In the next section, we shall introduce an analytical tool that deals effectively with singularly perturbed problems, namely *the Tikhonov theorem*.

1.3 Tikhonov theorem

1.3.1 Introduction

The Tikhonov theorem deals with models in which the existence of two timescales leads to singularly perturbed problems described by a system of ordinary differential equations of the form

$$\begin{aligned} \dot{x} &= f(t, x, y, \epsilon), \quad x(0) = x_0, \\ \epsilon \dot{y} &= g(t, x, y, \epsilon), \quad y(0) = y_0, \end{aligned} \tag{1.29}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, f and g are continuous functions from open subsets of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively; for some $m, n \in \mathbb{N}$ and for sufficiently small $\epsilon > 0$.

If we let $\epsilon = 0$ in (1.29), it degenerates to an algebraic-differential system of equations

$$\begin{aligned} \dot{x} &= f(t, x, y, 0), \quad x(0) = x_0, \\ 0 &= g(t, x, y, 0), \end{aligned} \tag{1.30}$$

where we have to drop the initial condition for y.

In many cases the second equation of (1.30) can explicitly be solved and this reduces substantially the complexity of the system (1.29) at the cost of only obtaining approximate solutions.
Solving for y, the second equation of (1.30) if possible gives $y = \varphi(t, x)$, usually referred to as *the quasi-steady state*. Since the second equation of (1.30) is nonlinear, this solution may or may not be unique. However, we shall assume that all its roots are real and isolated in some domain $\bar{D}$. Then it shall be necessary to select one of its roots and substitute it in the first equation of (1.30). The rule of selecting the root will be discussed further, see [3].

If we substitute the unknown y in the first equation of (1.30) by the known quasi-steady state $\varphi(t,x)$, it reduces to

$$\dot{\bar{x}} = f(t,\bar{x},\varphi(t,\bar{x}),0), \quad \bar{x}(0) = x_0. \tag{1.31}$$

The Tikhonov theorem gives conditions that ensure the convergence of solution $(x_\epsilon(t), y_\epsilon(t))$ of (1.29) to solution $(\bar{x}(t), \varphi(t,\bar{x}(t)))$ of (1.30) as $\epsilon \to 0$, where $\bar{x}(t)$ is the solution of (1.31). Next, we introduce assumptions that allow the applicability of the Tikhonov theorem to singularly perturbed problems of the form (1.29).

1.3.2 Assumptions of the Tikhonov theorem

Let $\mathcal{U} \subset \mathbb{R}^n$ be a bounded open set, $\mathcal{V} \subset \mathbb{R}^m$ an open set, $T > 0$ and ϵ_0 are scalar. The first assumption is the standard requirement of the regularity of the right-hand side of the equation (1.29), that ensures its unique solvability and continuous dependence of its solutions on the paramater ϵ.

Assumption A_1.

The functions

$$\begin{aligned} f :&[0,T] \times \bar{\mathcal{U}} \times \mathcal{V} \times [0,\epsilon_0] \to \mathbb{R}^n, \\ g :&[0,T] \times \bar{\mathcal{U}} \times \mathcal{V} \times [0,\epsilon_0] \to \mathbb{R}^m, \end{aligned}$$

are continuous and satisfy the Lipschitz condition with respect x, y, uniformly in $\bar{\mathcal{U}} \times \mathcal{V}$.

Assumption A_2.

For any $(t,x) \in [0,T] \times \bar{\mathcal{U}}$ there exists a solution $y(t) = \varphi(t,x(t)) \in \mathcal{V}$ of Equation $(1.30)_2$ such that

$$\varphi \in C^1([0,T] \times \bar{\mathcal{U}}; \mathcal{V}).$$

$\varphi(t,x)$ is an isolated root of the second equation of (1.30) in $[0,T] \times \bar{\mathcal{U}}$; that is, there exists $\delta > 0$ such that

$$g(t,x,y) \neq 0, \quad \text{for} \quad 0 < |y - \varphi(t,x)| < \delta, \quad (t,x) \in [0,T] \times \bar{\mathcal{U}}.$$

Furthermore, if in (1.29) we perform a change of variable $\tau = t/\epsilon$, we obtain an equivalent system in fast timescale

$$\begin{aligned} x' =& \epsilon f(\epsilon\tau, x, y, \epsilon), \quad x(0) = x_0, \\ y' =& g(\epsilon\tau, x, y, \epsilon), \quad y(0) = y_0, \end{aligned} \tag{1.32}$$

where $'$ denotes differentiation with respect to τ while $\cdot$ denotes differentiation with respect to t. Replacing $\epsilon\tau$ by t in (1.32) and then letting $\epsilon = 0$ leads to the following auxiliary equation

$$\tilde{y}' = g(t, x, \tilde{y}, 0), \quad \tilde{y}(0) = y_0, \tag{1.33}$$

where t and x are treated as parameters.

Assumption A_3.

Assume that $\tilde{y} = \varphi(t, x)$ is an asymptotically stable equilibrium of (1.33), uniformly with respect to $(t, x) \in [0, T] \times \bar{\mathcal{U}}$; that is, for any $\eta > 0$ there exists $\delta > 0$ such that for all $(t, x) \in [0, T] \times \bar{\mathcal{U}}$,

$$|\tilde{y}(0) - \varphi(t, x)| < \delta \Longrightarrow \text{ for all } \tau > 0 \ |\tilde{y}(\tau, t, x) - \varphi(t, x)| < \eta \text{ and } \lim_{\tau \to \infty} \tilde{y}(\tau, t, x) = \varphi(t, x),$$

where the above convergence is uniform for $(t, x) \in [0, T] \times \bar{\mathcal{U}}$. We observe that Assumption A_3 is satisfied if the following inequality

$$\frac{\partial}{\partial y_j} g_i(t, \bar{x}(t), \varphi(t, \bar{x}(t))) < 0$$

is satisfied for $i, j = 1, \cdots, m$, see [3].

Assumption A_4.

Assume that the function $(t, x) \to f(t, x, \varphi(t, x), 0)$ satisfies the Lipschitz condition with respect to x in $\bar{\mathcal{U}}$ and that the unique solution $\bar{x} = \bar{x}(t)$ of (1.31) on $[0, T]$ satisfies

$$\bar{x}(t) \in \text{Int}\, \bar{\mathcal{U}}, \quad \forall t \in (0, T).$$

Let us consider the initial layer equation

$$\hat{y}' = g(0, x_0, \hat{y}, 0), \quad \hat{y}(0) = y_0,$$

obtained from (1.33) by letting $t = 0$ and $x = x_0$, where (x_0, y_0) are the initial conditions for (1.29). Then we have

Assumption A_5.

Let y_0 belongs to the basin of attraction of the root $\hat{y} = \varphi(0, x_0)$ of the equation

$$g(0, x_0, \hat{y}, 0) = 0;$$

that is, the solution $\hat{y} = \hat{y}(\tau)$ of the initial layer equation satisfies

$$\lim_{\tau \to \infty} \hat{y}(\tau) = \varphi(0, x_0),$$

and $\hat{y}(\tau) \in \mathcal{V}$ for all $\tau > 0$.

Note that the basin of attraction or the domain of influence of an isolated stable equilibrium of an ordinary differential equation is the set of initial conditions such that any solution starting from this set, will converge toward the given stable equilibrium, [9]. Then the following theorem holds.

Theorem 1.3.1

If assumptions $A_1 - A_5$ of the Tikhonov theorem are satisfied, then there exists a unique solution $(x_\epsilon(t), y_\epsilon(t))$ of (1.29) on the closed interval $[0, T]$ such that

$$\begin{aligned} &\lim_{\epsilon \to 0} x(t, \epsilon) = \bar{x}(t), \quad 0 \leq t \leq T, \\ &\lim_{\epsilon \to 0} y(t, \epsilon) = \varphi(\bar{x}(t), t) \equiv \bar{y}(t), \quad 0 < t \leq T. \end{aligned} \tag{1.34}$$

Note that the convergence in the first expression of (1.34) is uniform with respect to $t \in [0, T]$ but in the second expression it is not uniform in the closed interval $[0, T]$.
However, in the latter case, the convergence is uniform in any closed interval $[\zeta, T], \zeta > 0$. This is the initial layer effect mentioned earlier in Section 1.2.
Then one can include the initial layer term to obtain uniform convergence in the closed interval $[0, T]$.

Proposition 1.3.1

Under the assumption of Theorem 1.3.1, we have

$$\lim_{\epsilon \to 0} (y(t, \epsilon) - \bar{y}(t) - \hat{y}(\tau) + \varphi(0, x_0)) = 0, \tag{1.35}$$

uniformly for $t \in [0, T]$, where $\hat{y}(\tau) - \varphi(0, x_0)$ is the initial layer correction, also known as the *initial layer term.*

1.3.3 A simple example

If we write Equation (1.13) as a system of first order ordinary differential equations with $\beta = 0$, we obtain a Tikhonov type singularly perturbed system of equations

$$\begin{aligned} \dot{x} &= y, \\ \epsilon \dot{y} &= -\bar{\delta} y - \epsilon \alpha x + \bar{\gamma} \cos \omega t, \\ x(0) &= x_0 \quad \text{and} \quad y(0) = y_0. \end{aligned} \tag{1.36}$$

Letting $\epsilon = 0$ in the system (1.36), it degenerates to the following algebraic-differential system

$$\begin{aligned} \dot{x} &= y, \\ 0 &= -\bar{\delta} y + \bar{\gamma} \cos \omega t, \\ x(0) &= x_0, \end{aligned} \tag{1.37}$$

where only one initial condition can be satisfied. Solving for y in the second equation of (1.37) gives the quasi-steady state

$$\bar{y} \equiv \varphi(t, x) = \frac{\bar{\gamma}}{\bar{\delta}} \cos \omega t. \tag{1.38}$$

If we perform the change of variable $\tau = t/\epsilon$ in the system (1.36), we obtain an equivalent system in the fast time

$$\begin{aligned} x' &= \epsilon y, \\ y' &= -\bar{\delta} y - \epsilon \alpha x + \bar{\gamma} \cos \omega \epsilon \tau, \\ x(0) &= x_0 \quad \text{and} \quad y(0) = y_0. \end{aligned} \tag{1.39}$$

Now, letting $\epsilon = 0$ in System (1.39) leads to the initial layer equation

$$\hat{y}' = -\bar{\delta} \hat{y} + \bar{\gamma}, \quad \hat{y}(0) = y_0. \tag{1.40}$$

Let us define

$$\Psi(\hat{y}, x, t) = -\bar{\delta} \hat{y} + \bar{\gamma}. \quad \text{Then} \quad \frac{\partial}{\partial \hat{y}} \Psi(\hat{y}, x, t) = -\bar{\delta} < 0.$$

Therefore the quasi-steady state is a stable equilibrium of the auxiliary equation (1.40). If we substitute the unkown y in the first equation of (1.37) by the known quasi-steady state $\bar{y}$, we obtain the limit equation

$$\dot{\bar{x}} = \frac{\bar{\gamma}}{\bar{\delta}} \cos \omega t, \ x(0) = x_0, \tag{1.41}$$

whose solution is given by

$$\bar{x}(t) = x_0 + \frac{1}{\omega}\frac{\bar{\gamma}}{\bar{\delta}}\sin\omega t.$$

Note that

$$\bar{x}(t) \in \left[x_0 - \frac{1}{\omega}\frac{\bar{\gamma}}{\bar{\delta}}, x_0 + \frac{1}{\omega}\frac{\bar{\gamma}}{\bar{\delta}}\right] \text{ for all } \ t \in (0,T).$$

The solution of (1.40) is given by

$$\hat{y}(\tau) = e^{-\delta\tau}\left(y_0 - \frac{\bar{\gamma}}{\bar{\delta}}\right) + \frac{\bar{\gamma}}{\bar{\delta}}. \tag{1.42}$$

Passing to the limit as $\tau \to \infty$ in (1.42) gives

$$\lim_{\tau\to\infty}\hat{y}(\tau) = \frac{\bar{\gamma}}{\bar{\delta}} \equiv \varphi(0,x_0).$$

This shows that the initial condition y_0 belongs to the basin of attraction of the solution

$$\hat{y} = \varphi(0,x_0) \text{ of the equation } \Psi(\hat{y},x_0,0) = 0.$$

It follows by the Tikhonov theorem that for sufficiently small $\epsilon > 0$, there exists a unique solution $(x_\epsilon(t), y_\epsilon(t))$ of (1.36) such that

$$\begin{aligned} &\lim_{\epsilon\to 0} x_\epsilon(t) = \bar{x}(t), \quad 0 \le t \le T, \\ &\lim_{\epsilon\to 0} y_\epsilon(t) = \varphi(\bar{x}(t),t) \equiv \bar{y}(t), \quad 0 < t \le T. \end{aligned} \tag{1.43}$$

The convergence in the first expression of (1.43) is uniform with respect to $t \in [0,T]$ but in the second expression is not uniform in $[0,T]$, as illustrated by the following numerical simulation

Numerical simulation

Here, the numerical simulation shows the uniform convergence on $[0,T]$ of the solution $x_\epsilon(t)$ given by the first equation of (1.36) to the solution $\bar{x}(t)$ given by the limit equation (1.41), and the uniform convergence in any closed interval $[\zeta,T]$ of the solution $y_\epsilon(t)$ given by the second equation of (1.36) to the quasi-steady state given by (1.38). The parameters' values used are $\alpha = 1$, $\delta = 0.1$, $\gamma = 2$, $\omega = 1$, $\epsilon = 0.01$ and the initial values are $x(0) = 0$ and $y(0) = 1$.

By adding the initial layer term $\hat{y}(\tau) - \varphi(0,x_0)$ to the second expression of (1.43), we obtain uniform convergence in the closed interval $[0,T]$ of the solution $y_\epsilon(t)$ of the second equation of (1.36), as shown in the following numerical simulation.

Numerical simulation

Here, the numerical simulation shows the uniform convergence in the closed interval $[0,T]$ of the solution $y_\epsilon(t)$ given by the second equation of (1.36) when the initial layer is added.

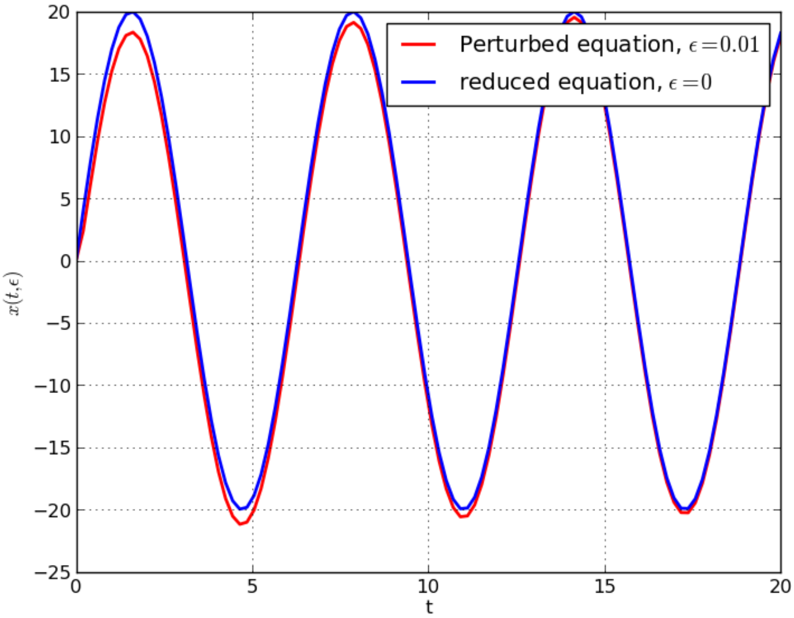


Figure 1.9: The uniform convergence in the closed interval $[0, T]$ of the solution $x_\epsilon(t)$ given by the first equation of (1.36) to the solution $\bar{x}(t)$ given by (1.41).

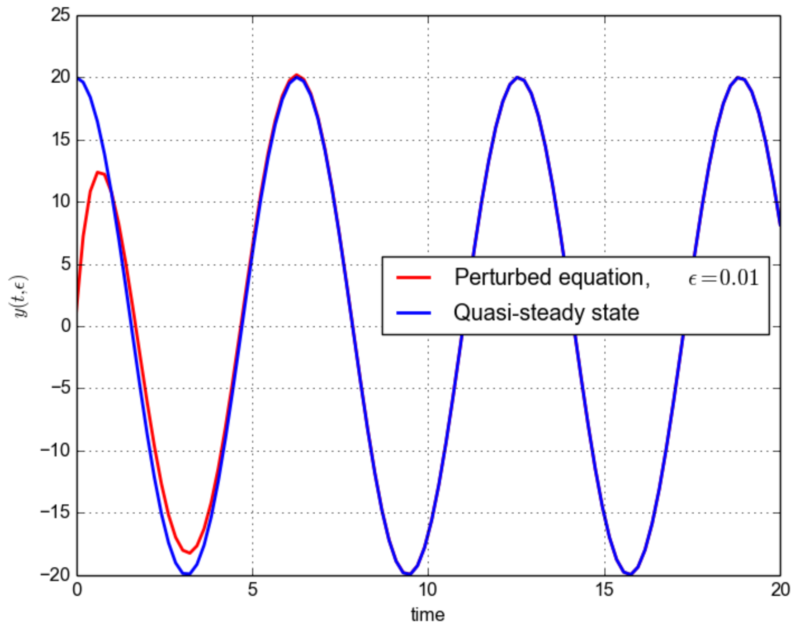


Figure 1.10: The uniform convergence in the closed interval $[\zeta, T]$ of the solution $y_\epsilon(t)$ given by the second equation of (1.36) to the quasi-steady state $\bar{y}(t)$ given by (1.38).

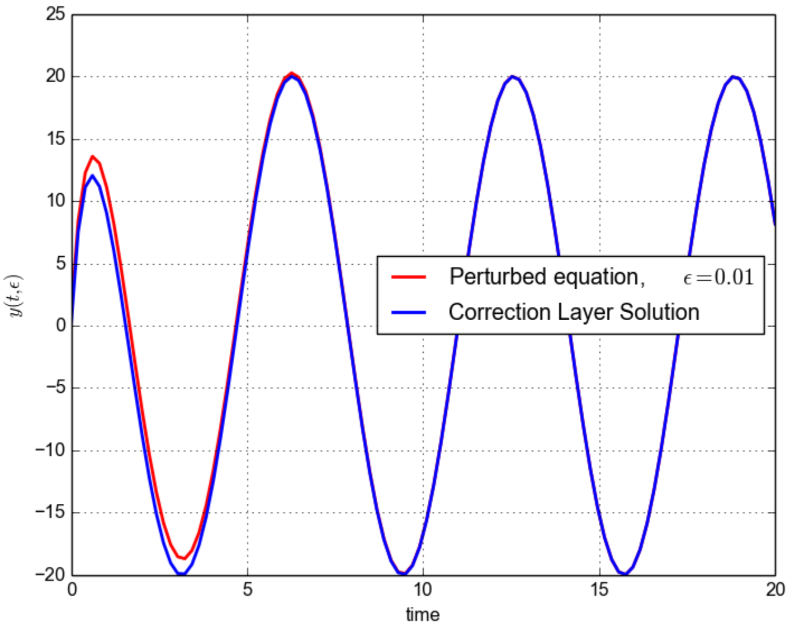


Figure 1.11: The uniform convergence of the solution $y_\epsilon(t)$ given by the second equation of (1.36) after the the initial layer term is added.

1.4 Centre manifold

1.4.1 Introduction

In this section, we introduce some basic results on the centre manifold theory such as *the existence of the centre manifold* and *the reduction principle*.
The reduction principle consists in studying the flow of a nonlinear system of differential equations through its reduction to the centre manifold. This plays a major role in applications.
Note that, the system of the centre manifold is in general impossible to solve; that is why we further introduce a result that allows to approximate the centre manifold to any degree of accuracy.
All these results were previously discussed and proved in my MSc dissertation, [11]. Here we just present what we need from the centre manifold theory to perform our work.

Let us consider the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0. \tag{1.44}$$

Definition 1.4.1

Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$. For $x_0 \in E$ let $I(x_0)$ denotes the maximal interval of existence of the solution $\phi(t, x_0)$ of (1.44). Then for $t \in I(x_0)$ the set of mappings $\{\phi_t\}_{t \in I(x_0)}$ of E onto E defined by

$$\phi_t(x_0) = \phi(t, x_0)$$

is called the flow of the differential equation (1.44) [12].

Definiton 1.4.2

Let E be an open subset of $\mathbb{R}^n$. Let $f \in C^1(E)$ and $\phi_t : E \to E$ be the flow of the nonlinear system (1.44) defined for all $t \in \mathbb{R}$.

A set $S \subset E$ is called *invariant* with respect to the flow ϕ_t if $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$. If we restrict the time to be positive (or negative) then we refer to S as positively (or negatively) invariant with respect to the flow ϕ_t [12].

Definition 1.4.3

An invariant set $S \subset \mathbb{R}^n$ is said to be C^k $(k \geq 1)$ invariant manifold of (1.44) if S has the structure of C^k differentiable manifold [9].

Let x^* be an equilibrium point of the nonlinear system (1.44). The linearization of (1.44) about x^* is given by

$$\dot{x} = Ax, \tag{1.45}$$

where $A \equiv Df(x^*)$ is a constant $n \times n$ matrix and $x \in \mathbb{R}^n$. A solution of (1.45) through the point $x_0 \in \mathbb{R}^n$ is given by

$$x(t) = e^{At} x_0. \tag{1.46}$$

The space $\mathbb{R}^n$ can be represented as a direct sum of the following subspaces E^s, E^u and E^c; known as stable subspace, unstable subspace and centre subspace of the linear system (1.45) respectively, and they are defined as follows

$$\begin{aligned} E^s &= \text{Span}\left\{v_1, \cdots, v_s\right\}, \\ E^u &= \text{Span}\left\{v_{s+1} \cdots, v_{s+u}\right\}, \\ E^c &= \text{Span}\left\{v_{s+u+1} \cdots, v_{s+u+c}\right\}, \end{aligned} \tag{1.47}$$

with $s + u + c = n$ and where $\{v_1, \cdots, v_s\}$ is a basis of (generalized) eigenvectors of A corresponding to the eigenvalues of A having negative real parts, $\{v_{s+1} \cdots, v_{s+u}\}$ is a basis of (generalized) eigenvectors of A corresponding to the eigenvalues of A having positive real parts and $\{v_{s+u+1} \cdots, v_{s+u+c}\}$ is a basis of (generalized) eigenvectors of A corresponding to the eigenvalues of A having zero real parts [9].

E^s, E^u and E^c are invariant subspaces since a solution of (1.45) with initial condition in one of these subspaces will remain there for all time [9].

Theorem 1.4.1 (The Stable and Unstable Manifold Theorem)

Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$ and ϕ_t be the flow of the nonlinear system (1.44). Suppose that $f(0) = 0$ and that $Df(0)$ has s eigenvalues with negative real parts and $n - s$ eigenvalues with positive real parts. Then there exists a s-dimensional differentiable stable manifold S tangent to the stable subspace E^s of the linear system (1.45) at the origin such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$

$$\lim_{t\to\infty} \phi_t(x_0) = 0,$$

and there exists an $(n-s)$-dimensional differentiable unstable manifold U tangent to the unstable subspace E^u of (1.45) at the origin such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$ [12]

$$\lim_{t\to-\infty} \phi_t(x_0) = 0.$$

Furthermore, S and U have the same dimension as E^s and E^u respectively [12].

1.4.2 Simple example

Let us consider the nonlinear system

$$\begin{aligned}
\dot{x}_1 &= -x_1, \\
\dot{x}_2 &= -x_2 + x_1^2, \\
\dot{x}_3 &= x_3 + x_1^2.
\end{aligned} \tag{1.48}$$

The only equilibrium point of the system (1.48) is the origin. The linear system associated to (1.48) is given by

$$\dot{x} = Ax, \tag{1.49}$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, the stable subspace E^s of (1.49) is the x_1x_2-plane and the unstable subspace E^u of (1.49) is the x_3-axis.
The solution of the nonlinear system (1.48) is given by

$$\begin{aligned} x_1(t) &= c_1e^{-t}, \\ x_2(t) &= c_2e^{-t} + c_1^2(e^{-t} - e^{-2t}), \\ x_3(t) &= c_3e^{t} + \frac{c_1^2}{3}(e^{t} - e^{-2t}), \end{aligned} \tag{1.50}$$

where $c = (c_1, c_2, c_3) = x(0)$. Then the flow of (1.48) is given by

$$\phi_t(c) \equiv \phi(t,c) = \begin{bmatrix} c_1e^{-t} \\ c_2e^{-t} + c_1^2(e^{-t} - e^{-2t}) \\ c_3e^{t} + \frac{c_1^2}{3}(e^{t} - e^{-2t}) \end{bmatrix}.$$

It follows that

$$\lim_{t\to\infty} \phi_t(c) = 0$$

if and only if $c_3 + \dfrac{c_1^2}{3} = 0$. Thus, the stable manifold of (1.48) is given by

$$S = \{c \in \mathbb{R}^3 | c_3 = \frac{-c_1^2}{3}\},$$

and for $c \in S$ we have

$$\phi_t(c) = \begin{bmatrix} c_1e^{-t} \\ c_2e^{-t} + c_1^2(e^{-t} - e^{-2t}) \\ -\frac{c_1^2}{3}e^{-2t} \end{bmatrix} \in S.$$

Hence, $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$ so S is invariant under the flow ϕ_t.

Next,

$$\lim_{t\to-\infty} \phi_t(c) = 0$$

if and only if $c_1 = c_2 = 0$. So, the unstable manifold of (1.48) is given by

$$U = \{c \in \mathbb{R}^3 | c_1 = c_2 = 0\}.$$

Then, for $c \in U$ we have

$$\phi_t(c) = \begin{bmatrix} 0 \\ 0 \\ c_3e^{t} + \frac{c_1^2}{3}e^{t} \end{bmatrix} \in U.$$

Hence, $\phi_t(U) \subset U$ for all $t \in \mathbb{R}$ so U is invariant under the flow ϕ_t [12].

Definition 1.4.4

A centre manifold of the origin for a nonlinear system of ordinary differential equations is an invariant differentiable manifold tangent to the centre subspace E^c of $\mathbb{R}^n$ at the origin.

Theorem 1.4.2

Let $f \in C^k(E)$, where E is an open subset of $\mathbb{R}^n$ containing the origin and $k \geq 0$. Suppose that $f(0) = 0$ and that $Df(0)$ has s eigenvalues with negative real parts, u eigenvalues with positive real parts and $c = n - s - u$ eigenvalues with zero real parts. Then there exists an c-dimensional centre manifold $W^c(0)$ of class C^k, tangent to the centre subspace E^c of $\mathbb{R}^n$ at the origin given by

$$W^c(0) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0\},$$

for δ sufficiently small, [9].

Remark 1.4.1

Conditions $h(0) = 0$ and $Dh(0) = 0$ are conditions of tangency of the centre manifold $W^c(0)$ to the centre subspace E^c of $\mathbb{R}^n$ at the origin.

1.4.3 Simple examples

In the next example, we show that, contrary to the cases of stable and unstable manifolds, the centre manifold is not uniquely determined.

Example 1.4.1

Consider the following system

$$\begin{aligned} \dot{x} &= -x^3, \\ \dot{y} &= -y, \end{aligned} \tag{1.51}$$

where $(x, y) \in \mathbb{R}^2$. If we eliminate the independent variable t in (1.51), we obtain

$$\frac{dy}{dx} = \frac{y}{x^3}, \tag{1.52}$$

and solving Equation (1.52) we get, for $x \neq 0$,

$$y(x) = C \exp\left(-\frac{1}{2}x^{-2}\right),$$

where C is any real constant. We can check that

$$W_0^c(0) = \left\{(x, y) \in \mathbb{R}^2 | y = C \exp\left(-\frac{1}{2}x^{-2}\right) \text{ for } x \neq 0,\ y = 0 \text{ for } x = 0\right\}$$

for each C is a centre manifold, [6].

Example 1.4.2

Let us consider the system (1.39) without external forces and in which ϵ is taken as a dummy variable

$$\begin{aligned} x' &= \epsilon y, \\ y' &= -\bar{\delta} y - \epsilon \alpha x, \\ \epsilon' &= 0, \\ x(0) &= x_0 \quad \text{and} \quad y(0) = y_0. \end{aligned} \tag{1.53}$$

The Jacobian matrix associated with (1.53), evaluated at the origin, is given by

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\bar{\delta} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that (1.53) satisfies the assumptions of Theorem 1.4.1. Thus, it has a two dimensional centre manifold

$$y = h(x, \epsilon).$$

In order to construct an approximate equation of the centre manifold, we let

$$y = \epsilon \omega_1(x) + \epsilon^2 \omega_1(x) + \cdots \tag{1.54}$$

Then by substituting Equation (1.54) into the second equation of (1.53), we obtain, on one side,

$$y' = -\epsilon \bar{\delta} \omega_1(x) - \epsilon \alpha x + O(\epsilon^2) \tag{1.55}$$

and, on the other side, if we differentiate Equation (1.54) with respect to τ, we obtain

$$y' = \epsilon \left(\frac{\partial \omega_1}{\partial x}\right) x' + \cdots = \epsilon^2 \left(\frac{\partial \omega_1}{\partial x}\right) \dot{x} + \cdots \tag{1.56}$$

Comparing (term by term) Equations (1.55) and (1.56) leads to the expression

$$-\bar{\delta} \omega_1(x) - \alpha x = 0. \tag{1.57}$$

This implies that

$$\omega_1(x) = -\frac{\alpha}{\bar{\delta}}x. \tag{1.58}$$

Hence, if we substitute ω_1 by its value in (1.54), we obtain an approximate equation of the centre manifold

$$y = -\epsilon\frac{\alpha}{\bar{\delta}}x + O(\epsilon^2). \tag{1.59}$$

Now, if we substitute Equation (1.59) into the first equation of (1.53), we obtain the equation that determines the stability of the zero solution of (1.53).

$$u' = -\epsilon^2\frac{\alpha}{\bar{\delta}}u + \cdots \tag{1.60}$$

With respect to the time $t = \epsilon\tau$, Equation (1.60) becomes

$$\dot{u} = -\epsilon\frac{\alpha}{\bar{\delta}}u + \cdots \tag{1.61}$$

We prove later in Lemma 1.4.1 that if α/δ in (1.61) is positive, then its solution is asymptotically stable.

Remark 1.4.2

We shall see in the sequel of this section that Equation (1.61) is the reduction of the original system (1.53) to the centre manifold and that once the stability of (1.61) is determined, the stability of the system (1.53) arises automatically from it.

Remark 1.4.3

Note that so far we have not yet introduced the formal way of constructing the centre manifold. In Example 1.4.2, we have used only the result on the existence of the centre manifold and attempted to approximate it.
In order to effectively construct the centre manifold we state, without proofs, two more results on the centre manifold theory that make it possible.
The first one is known as *the reduction principle*, which explains how to reduce the flow of the original system to the centre manifold and how this automatically leads to the reduction of the dimension of the system under investigation.
Since in general, it is impossible to solve the system of equations that determine the centre manifold, the second one shows how to approximate it to any degree of accuracy, [6].

1.4.4 Centre manifold theorems

Let

$$\begin{aligned}\dot{x} &= Ax + f(x, y),\\ \dot{y} &= By + g(x, y),\\ x(0) &= x_0,\ y(0) = y_0,\end{aligned} \tag{1.62}$$

be a nonlinear system of ordinary differential equations, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. A and B are constant matrices such that all eigenvalues of A have zero real parts and all eigenvalues of B have negative real parts. $f(x, y)$ and $g(x, y)$ are vector functions of x and y of class C^2 with $f(0, 0) = 0, Df(0, 0) = 0, g(0, 0) = 0, Dg(0, 0) = 0$, [6].
It follows, from Theorem 1.4.2, that there exists $\delta > 0$ such that Equation (1.62) has a centre manifold $y = h(x)$, $|x| < \delta$. The flow on the centre manifold is governed by the n-dimensional system of differential equations

$$\dot{u} = Au + f(u, h(u)), \quad u(0) = x_0, \tag{1.63}$$

obtained by substituting the equation describing the centre manifold into the first equation of (1.62). Note that we did not take into account the direction with positive eigenvalues because it is unstable and it does not play any major role in the formulation of the centre manifold theorems.

Theorem 1.4.3

(a) Suppose that the zero solution of (1.63) is stable (asymptotically stable /unstable). Then the zero solution of (1.62) is stable (asymptotically stable /unstable).

(b) Suppose that the zero solution of (1.63) is stable. Let $(x(t), y(t))$ be the solution of (1.62) with $(x(0), y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (1.63) such that as $t \to \infty$

$$\begin{aligned}x(t) &= u(t) + O(e^{-\gamma t}),\\ y(t) &= h(u(t)) + O(e^{-\gamma t}),\end{aligned} \tag{1.64}$$

where $\gamma > 0$ is a constant.

Let $(x_0, h(x_0))$ be a point on the centre manifold. Then, by invariance, solutions $(x(t), y(t))$ of (1.62) through the point $(x_0, h(x_0))$ are on the centre manifold. That is, $y(t) = h(x(t))$. If we differentiate $y(t)$ with respect to t, we obtain $\dot{y} = Dh(x)\dot{x}$. Then, replacing $\dot{y}$ and $\dot{x}$ by their respective values in (1.62) and y by $h(x(t))$, we obtain the expression

$$Dh(x)\,[Ax + f(x, h(x))] = Bh(x) + g(x, h(x)). \tag{1.65}$$

Now, in order to compute the equation that describes the centre manifold we shall solve the following system of equations

$$\begin{aligned} Dh(x)[Ax + f(x, h(x))] - Bh(x) - g(x, h(x)) &= 0, \\ h(0) &= 0, \\ Dh(0) &= 0. \end{aligned} \tag{1.66}$$

Note that (1.66) is, in general, impossible to solve analytically. In the next theorem we present a method of approximation of the centre manifold to any degree of accuracy by a function of class C^2. Thus, given C^2-functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ in the neighborhood of the origin and using Equation (1.65), we define

$$(M\phi)(x) = D\phi(x)[Ax + f(x, \phi(x))] - B\phi(x) - g(x, \phi(x)).$$

Then the following theorem follows

Theorem 1.4.4

Suppose that $\phi(0) = 0$, $D\phi(0) = 0$ and that $(M\phi)(x) = O(|x|^q)$ as $x \to 0$, where $q > 1$. Then, as $x \to 0$,

$$|h(x) - \phi(x)| = O(|x|^q).$$

The reader may find the proofs of these theorems in the monograph of Carr in [6]. For more comprehensive proofs of these two theorems, I refer the reader to my MSc dissertation; see P.E. Kimba, [11].

Lemma 1.4.1

Let us consider the following equation

$$\dot{y} = ay^\alpha + O(|y|^q), \tag{1.67}$$

where $y \in \mathbb{R},\ \alpha \in \mathbb{N}$ and $q > \alpha$. Assume that α is odd. Then the zero solution of (1.67) is asymptotically stable if $a < 0$ and unstable if $a > 0$.

Proof.

Let $\dot{y} = y^\alpha(a + O(1))$. Assume that $a < 0$, then there exists a neighborhood $\mathcal{N}_0$ of the origin such that $a + O(1) < 0$.
Let $y_0 \in \mathcal{N}_0$ be an initial condition of (1.67).
If

$$y_0 > 0 \text{ then } \dot{y}(t, y_0) < 0 \Rightarrow y(t, y_0) \to 0 \text{ as } t \to \infty$$

and if

$$y_0 < 0 \text{ then } \dot{y}(t, y_0) > 0 \Rightarrow y(t, y_0) \to 0 \text{ as } t \to \infty.$$

Hence, in any case,

$$y(t, y_0) \to 0 \text{ as } \ t \to \infty.$$

Let us now assume that $a > 0$, then $a + o(1) > 0$ in $\mathcal{N}_0$. It follows that for $y_0 \in \mathcal{N}_0$, if

$$y_0 > 0 \text{ then } \dot{y}(t, y_0) > 0 \Rightarrow y(t, y_0) \to +\bar{y} \text{ as } t \to \infty$$

and if

$$y_0 < 0 \text{ then } \dot{y}(t, y_0) < 0 \Rightarrow y(t, y_0) \to -\bar{y} \text{ as } t \to \infty.$$

This completes the proof. □

Example 1.4.3

In this example, we show how the centre manifold theorems can be applied to nonlinear systems of differential equations in order to approximate the centre manifold to any degree of accuracy. Let us consider the following system

$$\begin{aligned} \dot{x} &= xy + ax^3 + by^2x \equiv f(x, y), \\ \dot{y} &= -y + cx^2 + dx^2y \equiv g(x, y). \end{aligned} \tag{1.68}$$

We first put (1.68) in an appropriate form for the application of the centre manifold theorems. The linearization of the (1.68) about the origin is given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} xy + ax^3 + bxy^2 \\ cx^2 + dx^2y \end{bmatrix}.$$

Since the linear part of (1.68) has for eigenvalues 0 and -1, by Theorem 1.4.1 Equation (1.68) has a local centre manifold $y = h(x)$. In order to approximate this centre manifold, we let

$$h(x) = a_1x^2 + a_2x^3 + O(x^4).$$

Then we set

$$(Mh)(x) = 0 \ \Leftrightarrow \ h'(x)[xh(x) + ax^3 + bxh^2(x)] + h(x) - cx^2 - dx^2h(x) = 0.$$

It follows that

$$(2a_1x + 3a_2x^2 + \cdots)[a_1x^3 + a_2x^4 + ax^3 + ba_1^2x^5 + ba_2^2x^7 + 2a_1a_2bx^6]$$
$$+a_1x^2 + a_2x^3 - cx^2 - da_1x^4 - da_2x^5 = 0,$$

so that

$$(a_1 - c)x^2 + a_2x^3 + O(x^4) = 0 \ \Leftrightarrow \ a_1 = c, \text{ and } a_2 = 0,$$

where we have neglected all the terms of order five and above. Hence,

$$h(x) = cx^2 + O(x^4) \tag{1.69}$$

is the approximate equation of the centre manifold for (1.68). By Theorem 1.4.3, the equation that determines the stability of the zero solution of (1.68) is obtained by substituting Equation (1.69) into the first equation of (1.68) and it is given by

$$\dot{u} = (a+c)u^3 + O(u^5). \tag{1.70}$$

Thus, by Lemma 1.4.1, the zero solution of (1.68) is asymptotically stable if $a+c<0$ and unstable if $a+c>0$.
Note that, we cannot say anything about the stability of the zero solution of the system (1.68) when $a+c=0$. In this case, we have to obtain a better approximation of the centre manifold. Suppose $a+c=0$ and let $h(x)=cx^2+\psi(x)$, where $\psi(x)=O(x^4)$. Therefore we have

$$(Mh)(x)=0 \;\Leftrightarrow\; (2cx+\psi'(x))[x\psi(x)+bc^2x^5+2bcx^3\psi(x)+bx\psi^2(x)]+\psi(x)-cdx^4-dx^2\psi(x)=0.$$

Let $\psi(x)=wx^4$. Then

$$(Mh)(x)=0 \;\Leftrightarrow\; (2cx+4wx^3+\cdots)[wx^5+bc^2x^5+2bcwx^7+bw^2x^9]+wx^4-cdx^4-dwx^6=0,$$

$$(w-cd)x^4+O(x^6)=0 \;\Rightarrow\; \psi(x)-cdx^4=O(x^6).$$

Hence, $\psi(x)=cdx^4+O(x^6)$ and then $h(x)=cx^2+cdx^4+O(x^6)$.

By Theorem 1.4.3, the equation that determines the stability of the zero small solution of (1.68) is given by

$$\dot{u} = (cd+bc^2)u^5 + O(u^7). \tag{1.71}$$

Hence, in the case where $a+c=0$, the zero solution of (1.68) are asymptotically stable if $cd+bc^2<0$ and unstable if $cd+bc^2>0$. We still cannot conclude anything about stability of the zero solution of (1.68) when $cd+bc^2=0$. Therefore we should still get a better approximation of the centre manifold. That is, suppose that

$$a+c=cd+bc^2=0$$

and let

$$h(x)=cx^2+cdx^4+\delta(x) \text{ where } \delta(x)=O(x^6).$$

Then $(Mh)(x)=0$ is equivalent to

$$\begin{aligned}(2cx+4cdx^3+\delta'(x))[x\delta(x)+2bc^2dx^7+bc^2d^2x^9+2bcdx^5\delta(x)+2bcx^3\delta(x)+bx\delta^2(x)]\\ +\delta(x)-cd^2x^6+dx^2\delta(x)=0.\end{aligned}$$

Let $\delta(x)=qx^6$.

Then we have

$$(2cx + 4cdx^3 + 6qx^5 + \cdots)[qx^7 + 2bc^2dx^7 + bc^2d^2x^9 + 2bcdqx^{11} + 2bcqx^9 + bq^2x^{13}]$$
$$+qx^6 - cd^2x^6 + dqx^8 = 0.$$

It follows that
$$(q - cd^2)x^6 + O(x^8) = 0 \ \Leftrightarrow\ \delta(x) - cd^2x^6 = O(x^8).$$
Hence, $\delta(x) = cd^2x^6 + O(x^8)$ and then, $h(x) = cx^2 + cdx^4 + cd^2x^6 + O(x^8)$. By Theorem 1.4.3, the equation that determines the stability of the zero solution of (1.68) when $a+c = cd+bc^2 = 0$ is given by

$$\dot{u} = cd^2u^7 + 2bc^2du^7 + O(u^9).$$

Since $bc^2 = -cd$, we have

$$\dot{u} = -cd^2u^7 + O(u^9). \tag{1.72}$$

Hence, the zero solution of (1.68) is asymptotically stable if $c > 0$ and unstable if $c < 0$.

1.5 Asymptotics expansions

We consider the same system (1.36) to show how an asymptotic representation of a system of ordinary differential equations can be written and what information can be extracted from it in terms of dynamics.

Let

$$\begin{aligned} x &= \bar{x}(t,\epsilon) + \pi x(\tau,\epsilon),\\ y &= \bar{y}(t,\epsilon) + \pi y(\tau,\epsilon), \end{aligned} \tag{1.73}$$

represent the sum of two formal series, where

$$\bar{x}(t,\epsilon) = \bar{x}_0(t) + \epsilon\bar{x}_1(t) + \cdots \quad \text{and} \quad \bar{y}(t,\epsilon) = \bar{y}_0(t) + \epsilon\bar{y}_1(t) + \cdots$$

are called regular series while,

$$\pi x(\tau,\epsilon) = \pi_0 x(\tau) + \epsilon\pi_1 x(\tau) + \cdots \quad \text{and} \quad \pi y(\tau,\epsilon) = \pi_0 y(\tau) + \epsilon\pi_1 y(\tau) + \cdots$$

are called boundary series. Terms in this series are called *boundary terms* and they must tend to zero as τ approaches infinity.
By a formal series, we mean a series in powers of ϵ with coefficients depending on a parameter, [3]. If we substitute Equation (1.73) into (1.36), we obtain

$$
\begin{aligned}
&\frac{d\bar{x}}{dt} + \frac{1}{\epsilon}\frac{d}{d\tau}\pi x = \bar{y} + \pi y, \quad \bar{x}(0) + \pi x(0) = x_0, \\
&\epsilon\frac{d\bar{y}}{dt} + \frac{d}{d\tau}\pi y = -\bar{\delta}(\bar{y} + \pi y) - \epsilon\alpha(\bar{x} + \pi x) + \bar{\gamma}\cos\omega t, \quad \bar{y}(0) + \pi y(0) = y_0.
\end{aligned}
\tag{1.74}
$$

Then multiplying the first equation of (1.74) by ϵ leads to

$$
\begin{aligned}
&\epsilon\frac{d\bar{x}}{dt} + \frac{d}{d\tau}\pi x = \epsilon\bar{y} + \epsilon\pi y, \quad \bar{x}(0) + \pi x(0) = x_0, \\
&\epsilon\frac{d\bar{y}}{dt} + \frac{d}{d\tau}\pi y = -\bar{\delta}(\bar{y} + \pi y) - \epsilon\alpha(\bar{x} + \pi x) + \bar{\gamma}\cos\omega t, \quad \bar{y}(0) + \pi y(0) = y_0.
\end{aligned}
\tag{1.75}
$$

Now, if we first consider coefficients depending on t in (1.75), we obtain the following system

$$
\begin{aligned}
&\epsilon\frac{d\bar{x}}{dt} = \epsilon\bar{y}, \quad \bar{x}(0) = x_0, \\
&\epsilon\frac{d\bar{y}}{dt} = -\bar{\delta}\bar{y} - \epsilon\alpha\bar{x} + \bar{\gamma}\cos\omega t, \quad \bar{y}(0) = y_0.
\end{aligned}
\tag{1.76}
$$

Let us formally expand the system (1.76) in powers of ϵ. Then we have

$$
\begin{aligned}
&\epsilon\frac{d\bar{x}_0}{dt} + \epsilon^2\frac{d\bar{x}_1}{dt} + \cdots = \epsilon\bar{y}_0 + \epsilon^2\bar{y}_1 + \cdots, \\
&\epsilon\frac{d\bar{y}_0}{dt} + \epsilon^2\frac{d\bar{y}_1}{dt} + \cdots = -\bar{\delta}\bar{y}_0 - \epsilon\bar{\delta}\bar{y}_1 + \cdots - \epsilon\alpha\bar{x}_0 - \epsilon^2\alpha\bar{x}_1 + \cdots + \bar{\gamma}\cos\omega t, \\
&\bar{x}_0(0) + \epsilon\bar{x}_1(0) + \cdots = x_0, \quad \bar{y}_0(0) + \epsilon\bar{y}_1(0) + \cdots = y_0.
\end{aligned}
\tag{1.77}
$$

Using one-to-one correspondence of coefficients of identical powers of ϵ, we obtain

$$
\begin{cases}
\dfrac{d\bar{x}_0}{dt} = \bar{y}_0, \\
\dfrac{d\bar{x}_1}{dt} = \bar{y}_1, \\
\quad\vdots \\
\quad 0 = -\bar{\delta}\bar{y}_0 + \bar{\gamma}\cos\omega t, \\
\quad\vdots \\
\dfrac{d\bar{y}_0}{dt} = -\bar{\delta}\bar{y}_1 - \alpha\bar{x}_0, \\
\dfrac{d\bar{y}_1}{dt} = -\alpha\bar{x}_1, \\
\quad\vdots
\end{cases}
\tag{1.78}
$$

Solving (1.78) gives the coefficients

$$\begin{cases} \bar{y}_0 = \dfrac{\bar{\gamma}}{\bar{\delta}} \cos \omega t, \\ \bar{y}_1 = \dfrac{\bar{\gamma}}{\bar{\delta}^2} \left(\dfrac{\omega^2 - \alpha}{\omega} \right) \sin \omega t - \dfrac{\alpha}{\bar{\delta}} x_0, \\ \quad \vdots \\ \bar{x}_0 = x_0 + \dfrac{1}{\omega} \dfrac{\bar{\gamma}}{\bar{\delta}} \sin \omega t, \\ \bar{x}_1 = \dfrac{\bar{\gamma}}{\bar{\delta}^2} \left(\dfrac{\alpha - \omega^2}{\omega^2} \right) (\cos \omega t - 1) - \dfrac{\alpha}{\bar{\delta}} x_0 t, \\ \quad \vdots \end{cases}$$

It follows that

$$\begin{aligned} \bar{x}(t, \epsilon) &= x_0 + \frac{1}{\omega} \frac{\bar{\gamma}}{\bar{\delta}} \sin \omega t + \epsilon \frac{\bar{\gamma}}{\bar{\delta}^2} \left(\frac{\alpha - \omega^2}{\omega^2} \right) (\cos \omega t - 1) - \epsilon \frac{\alpha}{\bar{\delta}} x_0 t + \cdots , \\ \bar{y}(t, \epsilon) &= \frac{\bar{\gamma}}{\bar{\delta}} \cos \omega t + \epsilon \frac{\bar{\gamma}}{\bar{\delta}^2} \left(\frac{\omega^2 - \alpha}{\omega} \right) \sin \omega t - \epsilon \frac{\alpha}{\bar{\delta}} x_0 + \cdots . \end{aligned} \tag{1.79}$$

Next, if we consider the coefficients depending on τ in (1.75), we obtain the following system

$$\begin{aligned} &\frac{d}{d\tau} \pi x = \epsilon \pi y, \quad \pi x(0) = x_0 - \bar{x}(0), \\ &\frac{d}{d\tau} \pi y = -\bar{\delta} \pi y - \epsilon \alpha \pi x, \quad \pi y(0) = y_0 - \bar{y}(0). \end{aligned} \tag{1.80}$$

Then let us formally expand (1.80) in powers of ϵ to obtain

$$\begin{aligned} &\frac{d}{d\tau} \pi_0 x + \epsilon \frac{d}{d\tau} \pi_1 x + \cdots = \epsilon \pi_0 y + \epsilon^2 \pi_1 y + \cdots , \\ &\frac{d}{d\tau} \pi_0 y + \epsilon \frac{d}{d\tau} \pi_1 y + \cdots = -\bar{\delta} \pi_0 y - \epsilon \bar{\delta} \pi_1 y + \cdots - \epsilon \alpha \pi_0 x - \epsilon^2 \alpha \pi_1 x + \cdots , \\ &\quad \pi x(0) = x_0 - \bar{x}(0), \quad \pi y(0) = y_0 - \bar{y}(0). \end{aligned} \tag{1.81}$$

The one-to-one correspondence of coefficients of identical powers in ϵ yields

$$\begin{cases} \frac{d}{d\tau}\pi_0 x = 0, \\ \frac{d}{d\tau}\pi_1 x = \pi_0 y, \\ \quad\vdots \\ \frac{d}{d\tau}\pi_0 y = -\bar{\delta}\pi_0 y, \\ \frac{d}{d\tau}\pi_1 y = -\delta\pi_1 y - \alpha\pi_0 x, \\ \quad\vdots \end{cases} \tag{1.82}$$

Solving (1.82) gives the coefficients

$$\begin{cases} \pi_0 x(\tau) = \pi_0 x(0), \\ \pi_1 x(\tau) = \frac{1}{\bar{\delta}}\pi_0 y(0)(1 - e^{-\bar{\delta}\tau}), \\ \quad\vdots \\ \pi_0 y(\tau) = \pi_0 y(0) e^{-\bar{\delta}\tau}, \\ \pi_1 y(\tau) = \left(\pi_1 y(0) + \frac{\alpha}{\bar{\delta}}\pi_0 x(0)\right) e^{-\bar{\delta}\tau} - \frac{\alpha}{\bar{\delta}}\pi_0 x(0), \\ \quad\vdots \end{cases} \tag{1.83}$$

Note that in the initial condition of the first equation of (1.80) it is impossible to determine $\bar{x}(0)$ and $\pi x(0)$ without supplementary conditions.
Since $\pi x(\tau) \to 0$ as $\tau \to \infty$, this implies that the first identity in (1.83) is possible only if $\pi_0 x(0) = 0$. Hence, we impose the condition $\pi_0 x(0) = 0$, [3]. From the third identity in (1.78) and the initial condition of the second equation of (1.75), we have

$$\pi_0 y(0) = y_0 - \varphi(0, x_0) \text{ and } \pi_1 y(0) = -y_1(0).$$

Hence, the coefficients that depend on τ are finally given by

$$\begin{cases} \pi_0 x(\tau) = 0, \\ \pi_1 x(\tau) = \frac{1}{\bar{\delta}}\left(y_0 - \varphi(0, x_0)\right)(1 - e^{-\bar{\delta}\tau}), \\ \quad\vdots \\ \pi_0 y(\tau) = (y_0 - \varphi(0, x_0)) e^{-\bar{\delta}\tau}, \\ \pi_1 y(\tau) = \frac{\alpha}{\bar{\delta}} e^{-\bar{\delta}\tau}, \\ \quad\vdots \end{cases} \tag{1.84}$$

It follows that

$$\begin{aligned}
\pi x(\tau,\epsilon) &= \epsilon\frac{1}{\bar{\delta}}\left(y_0-\varphi(0,x_0)\right)(1-e^{-\bar{\delta}\tau})+\cdots, \\
\pi y(\tau,\epsilon) &= (y_0-\varphi(0,x_0))e^{-\bar{\delta}\tau}+\epsilon\frac{\alpha}{\bar{\delta}}x_0e^{-\delta\tau}+\cdots
\end{aligned} \tag{1.85}$$

Then the asymptotic representation of the solution of (1.36) is given by

$$\begin{aligned}
x(t,\epsilon) &= x_0+\frac{1}{\omega}\frac{\bar{\gamma}}{\bar{\delta}}\sin\omega t+\epsilon\frac{\bar{\gamma}}{\bar{\delta}^2}\left(\frac{\alpha-\omega^2}{\omega^2}\right)(\cos\omega t-1)-\epsilon\frac{\alpha}{\bar{\delta}}x_0t+\cdots \\
&\quad+\epsilon\frac{1}{\bar{\delta}}\left(y_0-\varphi(0,x_0)\right)(1-e^{-\bar{\delta}\frac{t}{\epsilon}})+\cdots \\
y(t,\epsilon) &= \frac{\bar{\gamma}}{\bar{\delta}}\cos\omega t+\epsilon\frac{\bar{\gamma}}{\bar{\delta}^2}\left(\frac{\omega^2-\alpha}{\omega}\right)\sin\omega t-\epsilon\frac{\alpha}{\bar{\delta}}x_0+\cdots \\
&\quad+(y_0-\varphi(0,x_0))e^{-\bar{\delta}\frac{t}{\epsilon}}+\epsilon\frac{\alpha}{\bar{\delta}}x_0e^{-\bar{\delta}\frac{t}{\epsilon}}+\cdots
\end{aligned} \tag{1.86}$$

Thus, by the regular perturbation theorem, introduced in Section 1.2, passing to the limit as $\epsilon\to 0$ in (1.86), formally gives

$$\begin{aligned}
\lim_{\epsilon\to 0}x(t,\epsilon) &= x_0+\frac{1}{\omega}\frac{\bar{\gamma}}{\bar{\delta}}\sin\omega t\equiv\bar{x}(t), \\
\lim_{\epsilon\to 0}y(t,\epsilon) &= \frac{\bar{\gamma}}{\bar{\delta}}\cos\omega t\equiv\varphi(t,\bar{x}(t)).
\end{aligned} \tag{1.87}$$

Hence, as ϵ approaches zero, the first expression in (1.87) converges to the solution of the limit equation (1.41) and the second expression in (1.87) converges to the quasi-steady state. Note that this is the same result we have obtained in (1.43) by using Tikhonov theorem.
In the next section, we state without proof the implicit function theorem that we use later in the thesis.

1.6 Theorem (Implicit Function Theorem)

Let $f:\mathbb{R}^{n+m}\to\mathbb{R}^m$ be a continuously differentiable function, and let (x,y) be the coordinates of $\mathbb{R}^{n+m}$.
Let (x_0,y_0) be a point in $\mathbb{R}^{n+m}$ such that $f(x_0,y_0)=c$, where $c\in\mathbb{R}^m$. If $[\partial f/\partial y_j](x_0,y_0)\neq 0$, then there exists an open set $\mathcal{U}$ containing x_0, an open set $\mathcal{V}$ containing y_0, and a unique continuously differentiable function $g:\mathcal{U}\to\mathcal{V}$ such that

$$\{(x,g(x))|x\in\mathcal{U}\}=\{(x,y)\in\mathcal{U}\times\mathcal{V}|f(x,y)=c\}.$$

2. Standard application of the Tikhonov theorem

In this chapter, we show how the Tikhonov theorem be can rigorously used in the derivation, the analysis and the description of the behaviour at the limit (i.e the behaviour when the small parameter tends to zero) of a model with *Allee effect*.

2.1 The model

Many mathematical models describing population growth assume that only an increase in the population's density has negative effect on the life of a single individual within that population. The model with Allee effect assume that for small densities, an increase in population's density may be beneficial in the sense that it can increase the probability of finding a mate for reproduction, or it can develop a mechanism of group defence against a predator, [13]. The Allee type model we are interested in has the following general form

$$\frac{dN}{dt} = \Phi N \left(1 - \frac{N}{K} - \frac{\eta}{1 + \gamma N}\right), \tag{2.1}$$

which is *the Verhulst model*, where N is the population's density, Φ is the natural growth rate, K is the carrying capacity and $\eta/(1 + \gamma N)$ is an extra mortality term that is inversely proportional to the population's density, [13].
We are now going to derive an Allee type model that describes a female population searching for a mate. We consider a female population whose density is denoted by N, with

$$N = N_1 + N_2,$$

where N_1 denotes the density of females who have recently mated and N_2 denotes the density of females who have not yet mated and are searching for a mate.
In the sequel, we assume a one-to-one sex ratio so that we are not going to explicitly model male population, [13]. Then the female population can be described by the following model

$$\begin{aligned}\frac{dN_1}{dt} &= \beta N_1 - (\mu + \nu N)N_1 - \sigma N_1 + \xi N N_2,\\ \frac{dN_2}{dt} &= -(\mu + \lambda + \nu N)N_2 + \sigma N_1 - \xi N N_2,\end{aligned} \tag{2.2}$$

where β denotes the per capita reproduction rate of recently mated females, $\mu + \nu N$ denotes the per capita mortality rate of recently mated females, $\mu + \lambda + \nu N$ denotes the per capita mortality rate for females searching for a mate, σ denotes the rate at which females switch from the reproductive stage to searching stage and ξN denotes the per capita rate at which a searching female finds one out of N potential mates, [13]. The natural mortality rate within the females

population is denoted by μ, therefore $1/\mu$ denotes the average life span of a female individual in the absence of any external influence.
We assume that $\beta > \mu$. Otherwise, even without additional mortality, the population would become extinct.
It is natural to consider the life span as the unit of time, [7]. Thus, introducing a new dimensionless time $s = \mu t$ in (2.2) gives rise to an equivalent system in the life span unit

$$\begin{aligned} \mu \frac{dN_1}{ds} &= \beta N_1 - (\mu + \nu N)N_1 - \sigma N_1 + \xi N N_2, \\ \mu \frac{dN_2}{ds} &= -(\mu + \lambda + \nu N)N_2 + \sigma N_1 - \xi N N_2. \end{aligned} \tag{2.3}$$

Defining $K = (\beta - \mu)/\nu$ as the carrying capacity, transforms System (2.3) into

$$\begin{aligned} \mu \frac{dN_1}{ds} &= (\beta - \mu)N_1 \left(1 - \frac{N}{K}\right) - \sigma N_1 + \xi N N_2, \\ \mu \frac{dN_2}{ds} &= -(\mu + \lambda + \nu N)N_2 + \sigma N_1 - \xi N N_2. \end{aligned} \tag{2.4}$$

Now, if we consider the carrying capacity K as the reference population's size and if we let $N_1 = xK$ and $N_2 = yK$, we obtain a dimensionless system describing the female population searching for a mate, given by

$$\begin{aligned} \mu \frac{dx}{ds} &= (\beta - \mu)x(1 - (x + y)) - \sigma x + \xi K y(x + y), \\ \mu \frac{dy}{ds} &= -(\mu + \lambda + \nu K(x + y))y + \sigma x - \xi K y(x + y), \end{aligned} \tag{2.5}$$

with initial conditions $x(0) = x^0$ and $y(0) = y^0$.

2.2 The case of short satiation time

Since σ denotes the rate at which females switch from the reproductive stage to the searching stage, arguing as for the natural mortality rate, $1/\sigma$ is the average time of satiation after mating and since the time taken by females to reproduce after mating is very short compare to the life span, we have $1/\sigma \ll 1/\mu$. Thus, we define

$$\epsilon = \frac{1/\sigma}{1/\mu} \equiv \frac{\mu}{\sigma}$$

as the ratio of the average time of satiation to the average life span. In many cases ϵ may be regarded as a small positive parameter.
Dividing the system of equations (2.5) by μ, using the above scaling and letting $R_0 = \beta/\mu, (R_0 > 1)$ gives rise to the singularly perturbed system

$$\begin{aligned}\frac{dx}{ds} &= (R_0-1)x(1-(x+y)) + \frac{\xi K}{\mu}y(x+y) - \frac{1}{\epsilon}x,\ x(0)=x^0,\\ \frac{dy}{ds} &= -\left(1+\frac{\lambda+\nu K(x+y)}{\mu}\right)y - \frac{\xi K}{\mu}y(x+y) + \frac{1}{\epsilon}x,\ y(0)=y^0.\end{aligned} \tag{2.6}$$

Adding both equations in (2.6) and setting $z = x + y$ gives the following system

$$\begin{aligned}\frac{dz}{ds} &= (R_0-1)(z-y)(1-z) - \left(1+\frac{\lambda+\nu Kz}{\mu}\right)y,\ z(0)=x^0+y^0\equiv z^0,\\ \frac{dy}{ds} &= -\left(1+\frac{\lambda+\nu Kz}{\mu}\right)y - \frac{\xi K}{\mu}yz + \frac{1}{\epsilon}(z-y),\ y(0)=y^0.\end{aligned} \tag{2.7}$$

Now, expanding the right-hand side of the first equation of (2.7) and multiplying the second equation of (2.7) by ϵ leads to a Tikhonov type singularly perturbed problem

$$\begin{aligned}\frac{dz}{ds} &= (R_0-1)z(1-z) - \frac{\beta+\lambda}{\mu}y,\ z(0)=z^0,\\ \epsilon\frac{dy}{ds} &= -\epsilon\left(1+\frac{\lambda+\nu Kz}{\mu}\right)y - \epsilon\frac{\xi K}{\mu}yz + z - y,\ y(0)=y^0.\end{aligned} \tag{2.8}$$

Letting $\epsilon = 0$ in (2.8) gives rise to the following degenerate differential-algebraic system of equations

$$\begin{aligned}\frac{dz}{ds} &= (R_0-1)z(1-z) - \frac{\beta+\lambda}{\mu}y,\ z(0)=z^0,\\ 0 &= z-y.\end{aligned} \tag{2.9}$$

The quasi-steady state $y \equiv \varphi(t,z) = z$ is of class C^1. Thus, substituting the unknown y in the first equation of (2.9) by the known quasi-steady state $\varphi(t,z)$, we obtain the limit equation

$$\frac{d\bar{z}}{ds} = (R_0-1)\bar{z}(1-\bar{z}) - \frac{\beta+\lambda}{\mu}\bar{z},\ \bar{z}(0)=z^0. \tag{2.10}$$

Note that the limit equation (2.10) describes the dynamics of the quasi-steady female population (or female population in the quasi-steady state). This is a population of females in the absence of any perturbation.
Clearly, the limit equation does not produce the Allee effect; the extra mortality term increases with the increasing density.
If we perform in (2.8) the change of variables $\tau = s/\epsilon$, we obtain an equivalent system in fast timescale

$$\begin{aligned}
\frac{dz}{d\tau} &= \epsilon\left((R_0-1)z(1-z)-\frac{\beta+\lambda}{\mu}\right),\ z(0)=z^0,\\
\frac{dy}{d\tau} &= \epsilon\left(1+\frac{\lambda+\nu K z}{\mu}\right)y-\epsilon\frac{\xi K}{\mu}zy+z-y,\ y(0)=y^0.
\end{aligned} \tag{2.11}$$

Letting $\epsilon = 0$ in (2.11) leads to the initial layer equation

$$\frac{d\hat{y}}{d\tau} = z^0-\hat{y},\ \hat{y}(0)=y^0. \tag{2.12}$$

Let us define $\Psi(\hat{y},z,s)=z-\hat{y}$, then $\dfrac{\partial\Psi}{\partial\hat{y}}=-1<0$. This shows that the quasi-steady state is the stable equilibrium of (2.12). The solution of (2.12) is given by

$$\hat{y}(\tau)=e^{-\tau}(y^0-z^0)+z^0. \tag{2.13}$$

This solution is used as the correction in the initial layer. Then, passing to the limit as $\tau\to\infty$ in (2.13) gives

$$\lim_{\tau\to\infty}\hat{y}(\tau)=z^0\equiv\varphi(0,z^0).$$

This shows that y^0 belongs to the basin of attraction of the solution $\hat{y}=\varphi(0,z^0)$ of the equation $\Psi(\hat{y},z^0,0)=0$.

It follows that all assumptions of Tikhonov theorem are satisfied, hence the solution $(z_\epsilon(s),y_\epsilon(s))$ of System (2.8) exists for all $s\in[0,T]$ and satisfies

$$\begin{aligned}
&\lim_{\epsilon\to0} z_\epsilon(s)=\bar{z}(s),\quad 0\le s\le T,\\
&\lim_{\epsilon\to0} y_\epsilon(s)=\varphi(s,\bar{z}(s))\equiv\bar{z}(s),\quad 0<s\le T,
\end{aligned} \tag{2.14}$$

where $\bar{z}(s)$ is the solution of (2.10). Taking into account in the second equation of (2.14) the correction in the initial layer, we obtain uniform convergence as follows

$$\lim_{\epsilon\to0}(y_\epsilon(s)-\bar{z}(s)-\hat{y}(s/\epsilon)+\varphi(0,z^0))=0,\quad 0\le s\le T. \tag{2.15}$$

Now, since $z_\epsilon=x_\epsilon+y_\epsilon$, we have

$$\lim_{\epsilon\to0}x_\epsilon(s)=\lim_{\epsilon\to0}z_\epsilon(s)-\lim_{\epsilon\to0}y_\epsilon(s)=0,\quad 0\le s\le T. \tag{2.16}$$

Before we discuss the behaviour at the limit of the population described by (2.8), let us first investigate the dynamics described by the limit equation (2.10). The equilibrium points are determined by the equation

$$0=\bar{z}\left((R_0-1)(1-\bar{z})-\frac{\beta+\lambda}{\mu}\right).$$

This gives $\bar{z}_1 = 0$ and $\bar{z}_2 = 1-(\beta+\lambda)/(R_0-1)\mu \equiv 1-(\beta+\lambda)/(\beta-\mu)$. The equilibrium point $\bar{z}_2$ is positive, hence biologically meaningful, if $R_0 < 1$. However, this does not make sense in the context of this example, because it implies that the carrying capacity $K = (\beta-\mu)/\nu$ to be negative. So, the equilibrium point $\bar{z}_2$ can only be negative, therefore not biologically meaningful. Thus, if we calculate the derivative of the right-hand side,

$$F(\bar{z}) = (R_0-1)\bar{z}(1-\bar{z}) - \frac{\beta+\lambda}{\mu}\bar{z}$$

of (2.10) with respect to $\bar{z}$, we obtain

$$\frac{dF}{d\bar{z}} = (R_0-1)(1-2\bar{z}) - \frac{\beta+\lambda}{\mu}.$$

Hence,

$$\frac{dF}{d\bar{z}}(0) = -\left(1+\frac{\lambda}{\mu}\right) < 0.$$

It follows that $\bar{z}_1 = 0$ is an attracting equilibrium for any initial condition. Here, the population vanishes as the quick turnover in the number of satiated females is not matched by the rate at which they find mates and thus the population mostly consists of females searching for a mate.

2.2.1 Numerical simulation

Here the numerical simulation illustrates the convergence of the solution $z_\epsilon(t)$ given by (2.8) and the solution $\bar{z}(t)$ given by the limit equation (2.10). The parameters' values used are $\beta = 0.06$, $\mu = 0.4$, $\lambda = 0.05$, $\nu = 0.015$, $\xi = 0.2$, $\epsilon = 0.1$ & $\epsilon = 0.01$.

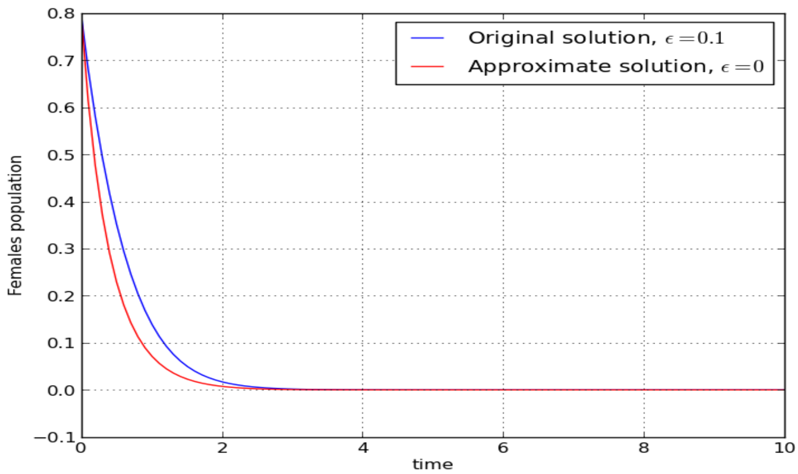


Figure 2.1: Comparison of the total female population z given by (2.8) with the approximate population $\bar{z}$ given by (2.10) for $\epsilon = 0.1$

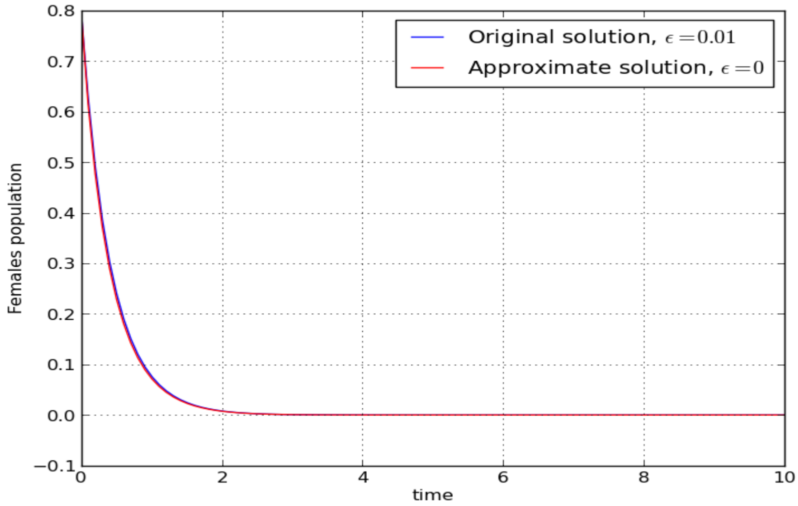


Figure 2.2: Comparison of the total female population z given by (2.8) with the approximate population $\bar{z}$ given by (2.10) for $\epsilon = 0.01$

2.3 The case of short satiation time and high searching efficiency

In order to derive a model that describes a sustainable females population, we suggest that the rate at which a searching female finds a potential mate, should be of the same order of magnitude as the rate at which females switch from the reproductive stage to the searching stage. That is, we have

$$\frac{\xi}{\mu} = \frac{\xi}{\sigma}\frac{\sigma}{\mu} = \frac{\xi}{\epsilon\sigma} = \frac{\bar{\xi}}{\epsilon},$$

where $\bar{\xi} = \xi/\sigma$. Now, using this additional scaling in (2.8) gives the following Tikhonov type singularly perturbed problem

$$\begin{aligned} \frac{dz}{ds} &= (R_0 - 1)z(1-z) - \frac{\beta+\lambda}{\mu}y, \ \ z(0) = z^0, \\ \epsilon\frac{dy}{ds} &= -\epsilon\left(1 + \frac{\lambda + \nu K z}{\mu}\right)y - \bar{\xi}Kyz + z - y, \ \ y(0) = y^0. \end{aligned} \tag{2.17}$$

Letting $\epsilon = 0$ in (2.17) gives rise to a degenerate differential-algebraic system

$$\begin{aligned} \frac{dz}{ds} &= (R_0 - 1)z(1-z) - \frac{\beta+\lambda}{\mu}y, \ \ z(0) = z^0, \\ 0 &= -\bar{\xi}Kyz + z - y. \end{aligned} \tag{2.18}$$

The quasi-steady state of the system is given by

$$\bar{y}(s) = \frac{z(s)}{1+\bar{\xi}Kz(s)} \equiv \varphi(s, z(s)). \tag{2.19}$$

Substituting the unkown y in the first equation of (2.18) by the known quasi-steady state $\bar{y}$ we obtain the limit equation

$$\frac{d\bar{z}}{ds} = (R_0-1)\bar{z}(1-\bar{z}) - \frac{\beta+\lambda}{\mu}\frac{\bar{z}}{1+\bar{\xi}K\bar{z}}, \ \bar{z}(0) = z^0. \tag{2.20}$$

Note that under certain assumptions the limit equation (2.20) may be an Allee type model, because the extra mortality term decreases as the density is increasing.
Now, let us use the Tikhonov theorem to investigate the behaviour of the females population at the limit as ϵ approaches zero.
If we perform the change of variables $\tau = s/\epsilon$ in (2.17), we obtain an equivalent system in fast time given by

$$\begin{aligned} \frac{dz}{d\tau} &= \epsilon\left[(R_0-1)z(1-z) - \frac{\beta+\lambda}{\mu}y\right], \ z(0) = z^0, \\ \frac{dy}{d\tau} &= -\epsilon\left(1+\frac{\lambda+\nu Kz}{\mu}\right)y - \bar{\xi}Kyz + z - y, \ y(0) = y^0. \end{aligned} \tag{2.21}$$

If we let $\epsilon = 0$ in (2.21), we obtain the following initial layer equation

$$\frac{d\hat{y}}{d\tau} = -\hat{y} + z^0 - \bar{\xi}K\hat{y}z^0, \ \hat{y}(0) = y^0. \tag{2.22}$$

Let us define $\Psi(\hat{y}, z, s) = -\bar{\xi}K\hat{y}z + z - \hat{y}$. Then $\dfrac{\partial\Psi}{\partial\hat{y}} = -1 - \bar{\xi}Kz < 0$, for biologically relevant $z^0 \geq 0$.
We know that this quasi-steady state is a stable equilibrium of the auxiliary equation (2.22). The solution of the initial layer equation (2.22) is given by

$$\hat{y}(\tau) = e^{-(1+\bar{\xi}Kz^0)\tau}\left(y^0 - \frac{z^0}{1+\bar{\xi}Kz^0}\right) + \frac{z^0}{1+\bar{\xi}Kz^0}. \tag{2.23}$$

Passing to the limit as $\tau \to \infty$ in (2.23), gives

$$\lim_{\tau\to\infty}\hat{y}(\tau) = \frac{z^0}{1+\bar{\xi}Kz^0} \equiv \varphi(0, z^0).$$

This shows that the initial condition y^0 belongs to the basin of attraction of the solution $\varphi(0, z^0)$ of the second equation of (2.18).
Note that all assumptions of the Tikhonov theorem are satisfied, therefore the solution $(z_\epsilon(s), y_\epsilon(s))$ of (2.17) exists for all $s \in [0, T]$ and satisfies

$$\begin{aligned}
&\lim_{\epsilon\to 0} z_\epsilon(s) = \bar{z}(s), \quad 0 \le s \le T, \\
&\lim_{\epsilon\to 0} y_\epsilon(s) = \varphi(s, \bar{z}(s)) \equiv \bar{y}(s), \quad 0 < s \le T,
\end{aligned} \tag{2.24}$$

where $\bar{z}(s)$ is the solution of the limit equation (2.20) and $\bar{y}(s)$ is the quasi-steady state. Using the initial layer correction

$$\hat{y}(s/\epsilon) = e^{-(1+\bar{\xi}Kz^0)s/\epsilon}\left(y^0 - \frac{z^0}{1+\bar{\xi}Kz^0}\right) + \frac{z^0}{1+\bar{\xi}Kz^0}$$

in the second equation of (2.24) gives uniform convergence

$$\lim_{\epsilon\to 0}(y_\epsilon(s) - \bar{y}(s) - \hat{y}(s/\epsilon) + \varphi(0, z^0)) = 0 \text{ on } [0, T]. \tag{2.25}$$

Then, since $z_\epsilon = x_\epsilon + y_\epsilon$, we have

$$\lim_{\epsilon\to 0} x_\epsilon(s) = \lim_{\epsilon\to 0} z_\epsilon(s) - \lim_{\epsilon\to 0} y_\epsilon(s) = \bar{z}(s) - \varphi(s, \bar{z}(s)), \quad 0 \le s \le T. \tag{2.26}$$

Let us now investigate the dynamics described by the limit equation (2.20). The equilibrium points are determined by the equation

$$0 = \bar{z}\left((R_0 - 1)(1 - \bar{z}) - \frac{\beta+\lambda}{\mu}\frac{1}{1+\bar{\xi}K\bar{z}}\right). \tag{2.27}$$

This immediately gives

$$\bar{z}_1 = 0 \quad \text{or} \quad \bar{z}^2 - \left(1 - \frac{1}{\bar{\xi}K}\right)\bar{z} + \frac{\mu+\lambda}{\bar{\xi}K(R_0 - 1)} = 0.$$

This equation has two positive solutions if and only if

$$\Delta = \left(\frac{\bar{\xi}K - 1}{\bar{\xi}K}\right)^2 - 4\frac{\mu+\lambda}{\bar{\xi}K\mu(R_0 - 1)} \ge 0,\ 1 - \frac{1}{\bar{\xi}K} \ge 0 \text{ and } \frac{\mu+\lambda}{\bar{\xi}K\mu(R_0 - 1)} \ge 0.$$

It follows that the female population will display the Allee effect if and only if $\bar{\xi}K \ge 1$ and $R_0 > 1$. Then, computing the derivative of the right-hand side,

$$F(\bar{z}) = (R_0 - 1)\bar{z}(1 - \bar{z}) - \frac{\beta+\lambda}{\mu}\frac{\bar{z}}{1+\bar{\xi}K\bar{z}},$$

of (2.20) with respect to $\bar{z}$, we find

$$\frac{dF}{d\bar{z}} = (R_0 - 1)(1 - 2\bar{z}) - \frac{\beta+\lambda}{\mu}\frac{1}{(1+\bar{\xi}K\bar{z})^2}.$$

It follows that

$$\frac{dF}{d\bar{z}}(0) = -\left(1 + \frac{\lambda}{\mu}\right) < 0.$$

Thus, $\bar{z}_1 = 0$ is an attracting equilibrium point for any population with initial condition in $(0, \bar{z}_2)$, the basin of attraction of $\bar{z}_1$.
The next equilibrium point $\bar{z}_2$ is repulsive and the third equilibrium $\bar{z}_3$ is attractive for any population with initial condition in $(\bar{z}_2, \infty)$, the basin of attraction of $\bar{z}_3$.

The limiting behaviour of the female population with this scaling is deeply influenced by the Allee effect generated by the model.

2.3.1 Numerical simulation

Here we present a numerical simulation illustrating the dynamics of a female population, displaying the Allee effect and that is given by (2.17) and (2.18). The parameters' values used are $\mu = 0.04$, $\beta = 0.07$, $\nu = 0.001$, $\bar{\xi} = 0.5$, $\lambda = 0.05$, $\epsilon = 0.1$ & $\epsilon = 0.01$.

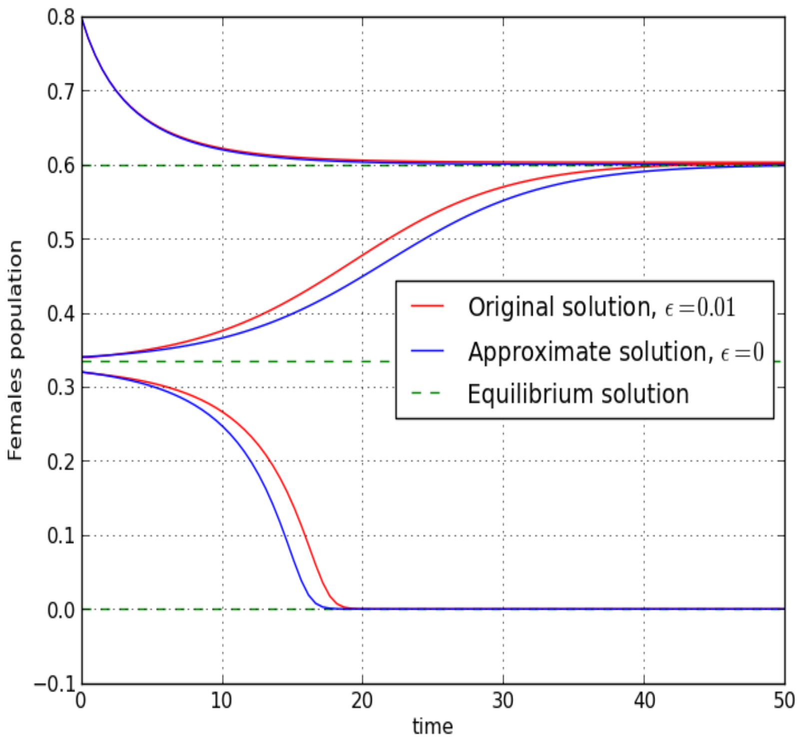


Figure 2.3: Comparison of the total female population z given by the first equation of (2.17) with the approximate population given by (2.20) and illustration of the Allee effect for $\mu = 0.04$, $\beta = 0.07$, $\nu = 0.001$, $\bar{\xi} = 0.5$, $\lambda = 0.05$, $\epsilon = 0.01$. The stable equilibria are $\bar{z}_1 = 0$ and $\bar{z}_3 = 0.599$. $\bar{z}_2 = 0.334$ is the unstable equilibrium solution.

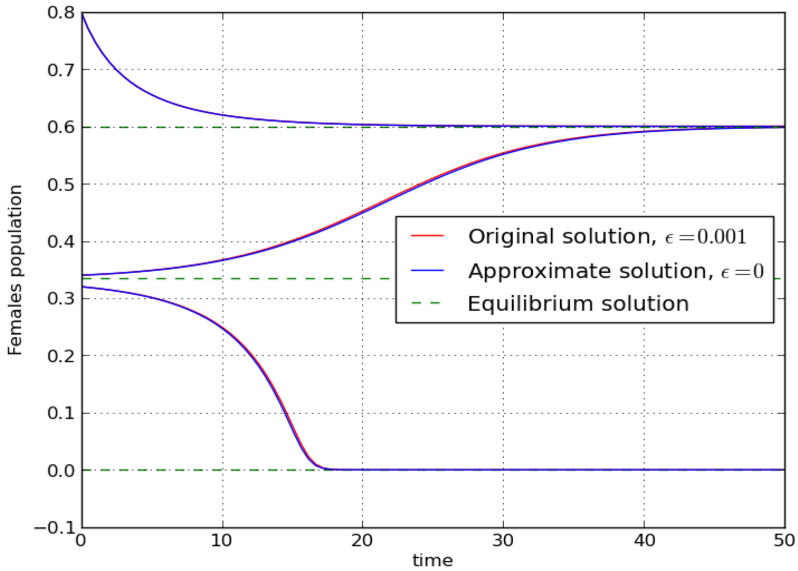


Figure 2.4: Comparison of the total female population z given by the first equation of (2.17) with the approximate population given by (2.20) and illustration of the Allee effect for $\mu = 0.04$, $\beta = 0.07$, $\nu = 0.001$, $\bar{\xi} = 0.5$, $\lambda = 0.05$, $\epsilon = 0.001$. The stable equilibria are $\bar{z}_1 = 0$ and $\bar{z}_3 = 0.599$. $\bar{z}_2 = 0.334$ is the unstable equilibrium solution.

As ϵ approaches zero, the population of females searching for a mate will converge to the quasi-steady state. However, this convergence is not uniform, particularly for the initial populations. This, can be explained by the fact that we have dropped one of the original initial conditions, in order to find the quasi-steady state.
Thus, taking into account the initial layer term, we obtain uniform convergence for the total population of females searching for a mate as indicated by (2.25).
The population of recently mated females will also converge to the limiting values as predicted by (2.26). The above mentioned behaviour is illustrated by the following simulations.

2.3.2 Numerical simulation

Here we present a numerical simulation illustrating the dynamics of the population of females searching for a mate given by the second equation of (2.17) and (2.19); and the dynamics of the same population after taking into account the auxiliary population given by (2.22). The parameters' values used are $\mu = 0.04$, $\beta = 0.07$, $\nu = 0.001$, $\bar{\xi} = 0.5$, $\lambda = 0.05$, $\epsilon = 0.001$.

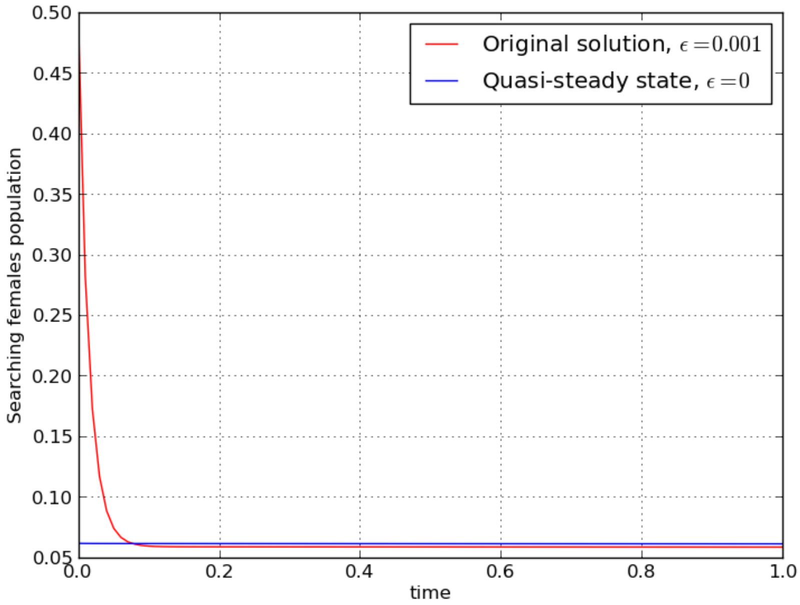


Figure 2.5: Comparison of the total female population searching for a mate y given by the second equation of (2.17) with the quasi-steady state (2.19).

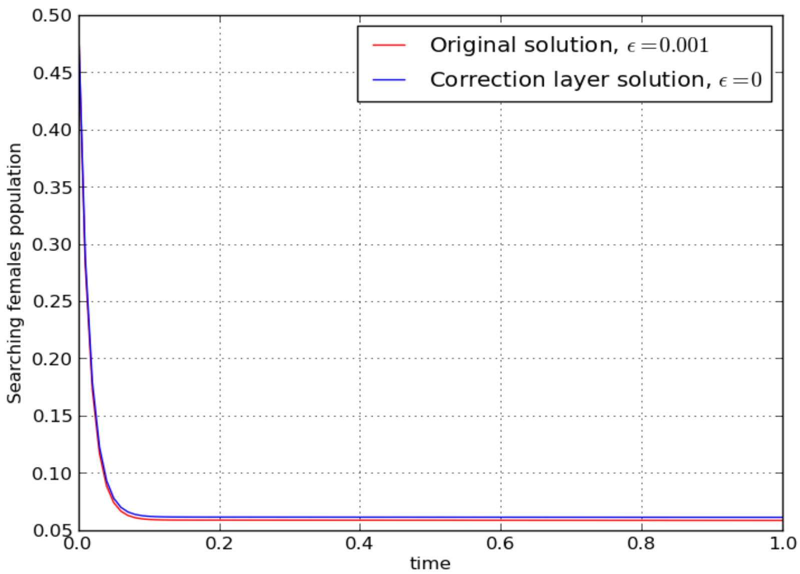


Figure 2.6: Comparison of the total female population searching for a mate y given by the second equation of (2.17) with the population obtained after taking into account the initial layer term.

We further present a numerical simulation illustrating the dynamics of recently mated females obtained from (2.26).

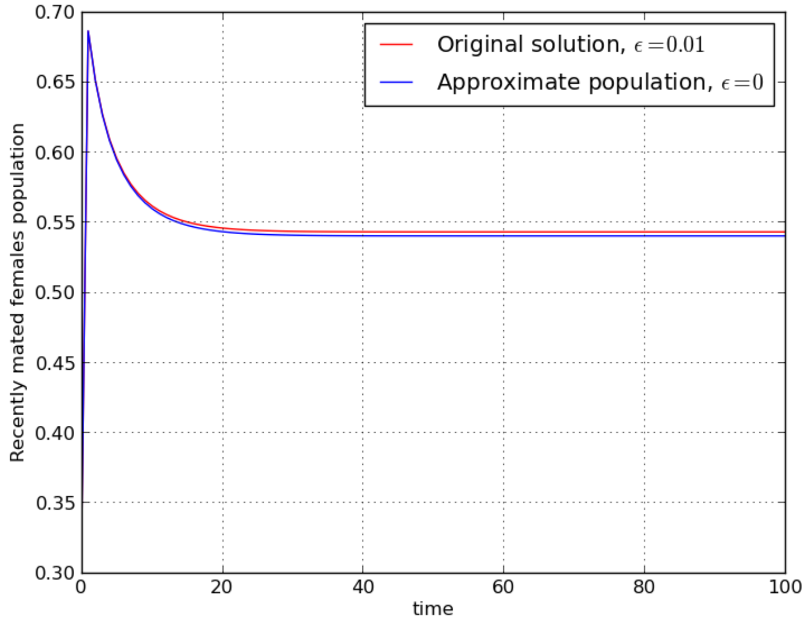


Figure 2.7: Approximation of the total population of recently mated females obtained from (2.26) with $\epsilon = 0.01$.

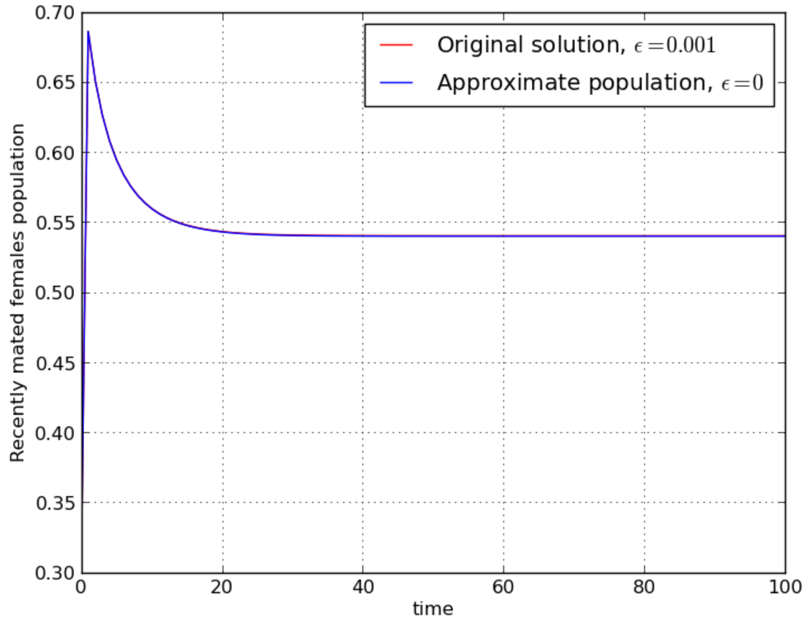


Figure 2.8: Approximation of the total population of recently mated females obtained from (2.26) with $\epsilon = 0.001$.

2.4 The case of short satiation, high searching efficiency and mortality rate

If we further assume that the extra mortality rate λ due to the searching activities is of the same order as the rate at which females switch from the reproductive to searching activities σ and the rate at which females find a potential mate ξ, then

$$\frac{\lambda}{\mu} = \frac{\lambda}{\sigma}\frac{\sigma}{\mu} = \frac{\lambda}{\epsilon\sigma} = \frac{\bar{\lambda}}{\epsilon} \quad \text{where} \quad \bar{\lambda} = \lambda/\sigma.$$

Now, using all these scalings in the system (2.6), leads to the following system

$$\begin{aligned} \frac{dx}{ds} &= (R_0-1)x(1-(x+y)) + \frac{\bar{\xi}K}{\epsilon}y(x+y) - \frac{1}{\epsilon}x,\ x(0)=x^0, \\ \frac{dy}{ds} &= -\left(1+\frac{\bar{\lambda}}{\epsilon}+\frac{\nu K(x+y)}{\mu}\right)y - \frac{\bar{\xi}K}{\epsilon}y(x+y) + \frac{1}{\epsilon}x,\ y(0)=y^0. \end{aligned} \tag{2.28}$$

Adding both equations and letting $z = x + y$ in (2.28) gives rise to

$$\begin{aligned} \frac{dz}{ds} &= (R_0-1)(z-y)(1-z) - \left(1+\frac{\nu K}{\mu}z\right)y - \frac{\bar{\lambda}}{\epsilon}y,\ z(0)=z^0, \\ \frac{dy}{ds} &= -\left(1+\frac{\nu K}{\mu}z\right)y - \frac{\bar{\xi}}{\epsilon}Kyz + \frac{1}{\epsilon}(z-y) - \frac{\bar{\lambda}}{\epsilon}y,\ y(0)=y^0. \end{aligned} \tag{2.29}$$

If we multiply both equations in (2.29) by ϵ and expand the right-hand side of the first equation of (2.29), we obtain a singularly perturbed system given by

$$\begin{aligned} \epsilon\frac{dz}{ds} &= \epsilon(R_0-1)z(1-z) - \epsilon\frac{\beta}{\mu}y - \bar{\lambda}y,\ z(0)=z^0, \\ \epsilon\frac{dy}{ds} &= -\epsilon\left(1+\frac{\nu K}{\mu}z\right)y - \bar{\xi}Kyz + z - y - \bar{\lambda}y,\ y(0)=y^0. \end{aligned} \tag{2.30}$$

If we let $\epsilon = 0$ in (2.30), it degenerates to the following algebraic system

$$\begin{aligned} 0 &= \bar{\lambda}y, \\ 0 &= \bar{\xi}Kyz + z - y - \bar{\lambda}y \end{aligned} \tag{2.31}$$

which gives $(0, 0)$ as the only solution. If we perform the change of variables $\tau = s/\epsilon$ in (2.30), we obtain an equivalent system in fast timescale given by

$$\begin{aligned} \frac{dz}{d\tau} &= \epsilon(R_0-1)z(1-z) - \epsilon\frac{\beta}{\mu}y - \bar{\lambda}y,\ z(0)=z^0, \\ \frac{dy}{d\tau} &= -\epsilon\left(1+\frac{\nu K}{\mu}z\right)y - \bar{\xi}Kyz + z - y - \bar{\lambda}y,\ y(0)=y^0. \end{aligned} \tag{2.32}$$

Letting $\epsilon = 0$ in (2.32) leads to the auxiliary system of equations

$$\begin{aligned} \frac{d\hat{z}}{d\tau} &= -\bar{\lambda}\hat{y}, \ \hat{z}(0) = z^0, \\ \frac{d\hat{y}}{d\tau} &= -\bar{\xi}K\hat{y}\hat{z} + \hat{z} - \hat{y} - \bar{\lambda}\hat{y}, \ \hat{y}(0) = y^0. \end{aligned} \tag{2.33}$$

Note that the quasi-steady state and the equilibrium of (2.33) coincide at $(0,0)$. This means that as ϵ approaches zero, the entire female population will face extinction because all individuals within the population will go searching for a mate and probably will never come back due to the high mortality risk.
Therefore, the dynamics of the entire female population will be driven in the fast timescale by (2.33). In this case the whole model is reduced to the perturbed part. Thus, to prove the applicability of the Tikhonov theorem, we have to prove that assumptions of the Tikhonov theorem which apply to the initial layer part are satisfied. In particular, we see that $(0,0)$ is a uniformly attracting equilibrium and we have to check that the whole admissible set is in the basin of attraction of $(0,0)$, see [25].
In order to prove this conjecture, we observe that the initial conditions satisfy $y^0 \geq 0,\ z^0 \geq y^0$. Therefore, we consider the region $\mathcal{R} = \{(z,y) : y \geq 0, z \geq y\}$ and determine the direction field on its boundary, [7].

We have vertical tangents in (2.33) if and only if

$$\frac{d\hat{z}}{d\tau} = 0 \Longleftrightarrow -\lambda\hat{y} = 0 \Longrightarrow \hat{y} = 0.$$

Hence, $\dfrac{d\hat{y}}{d\tau} > 0$ along the isocline $\hat{y} = 0$ provided $\hat{z} > 0$ which shows that the field along this line points upward, meaning inward the region $\mathcal{R}$.
The trajectories have horizontal tangents if and only if

$$\frac{d\hat{y}}{d\tau} = 0 \Longleftrightarrow \hat{y} = \frac{\hat{z}}{1 + \bar{\lambda} + \bar{\xi}\hat{z}}.$$

Hence, $\dfrac{d\hat{z}}{d\tau} < 0$ along this isocline, which shows that the field along this line points to the left, inwards the region $\mathcal{R}$.
On the line $\hat{y} = \hat{z}$, all the components of the vector field $(-\bar{\lambda}\hat{z}, -\bar{\lambda}\hat{z} - \bar{\xi}K\hat{z}^2)$ are negative and the dot product

$$(1,-1).(-\bar{\lambda}\hat{z}, -\bar{\lambda}\hat{z} - \bar{\xi}K\hat{z}^2) = \bar{\xi}K\hat{z}^2$$

is stricly positive; which shows that the field points inwards the region $\mathcal{R}$.
If we consider the line $\hat{z} = c$ then, above the isocline $\hat{y} = \hat{z}/(1 + \bar{\lambda} + \bar{\xi}\hat{z})$, the vector field points downward and below the isocline, the vector field points upward along the line $\hat{z} = c$. In any cases, the vector field along any line $\hat{z} = c$, points inwards the region $\mathcal{R}$.
So, since $(0,0)$ is the only equilibrium, any solution starting inside the region $\mathcal{R}$ is bounded and will stay there for ever.
Note that the trajectories starting below the isocline $\hat{y} = \hat{z}/(1+\bar{\lambda}+\bar{\xi}\hat{z})$ will increase, reaching the

maximum at the isocline, then converge to the equilibrium point $(0,0)$ and trajectories starting above the this isocline, will monotonically converge to the equilibrium point $(0,0)$.
This shows that the region $\mathcal{R}$ is the basin of attraction of the equilibrium point $(0,0)$. It follows that all the assumptions of Tikhonov theorem are satisfied.
Thus, the solutions $(z_\epsilon(s), y_\epsilon(s))$ of (2.30) exist for all $s \in [0,T]$ and satisfy

$$\begin{aligned} &\lim_{\epsilon\to 0}(z_\epsilon(s) - \hat{z}(s/\epsilon) = 0, \quad 0 \le s \le T, \\ &\lim_{\epsilon\to 0}(y_\epsilon(s) - \hat{y}(s/\epsilon) = 0, \quad 0 \le s \le T, \end{aligned} \tag{2.34}$$

where $(\hat{z}(s/\epsilon), \hat{y}(s/\epsilon))$ is the solution of (2.33). In summary, the whole dynamics of (2.28) is reduced to the dynamics of the initial layer equation which decays faster to 0. So, the entire female population will be decimated, as illustrated by the following numerical simulations.

2.4.1 Numerical simulation

Here the numerical simulations show the extinction of the total population of females given by the first equation of (2.30) and the increase, then the extinction of the total population of females searching for a mate given by the second equation of (2.30). The parameters' values used are $\beta = 0.04$, $\mu = 0.05$, $\bar{\lambda} = 0.3$, $\nu = 0.001$, $\bar{\xi} = 0.5$, $\epsilon = 0.1$.

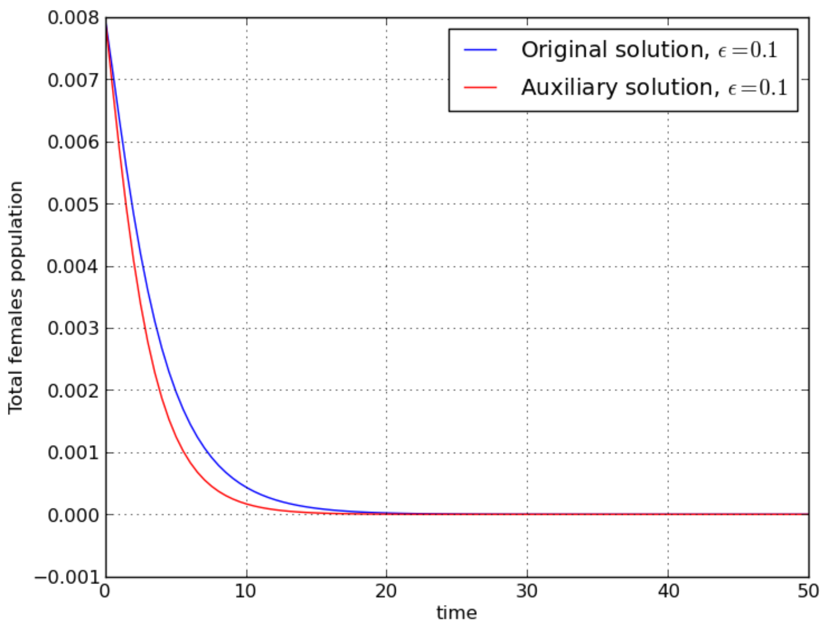


Figure 2.9: Approximation of the total female population given by the solution $z(s,\epsilon)$ of (2.30) and the solution $\hat{z}(s/\epsilon)$ of (2.33).

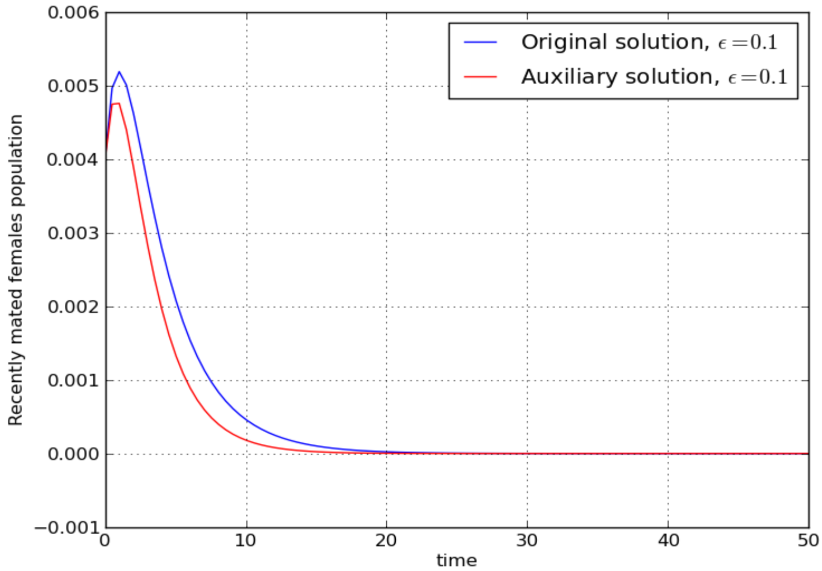


Figure 2.10: Approximation of the population of females searching for a mate given by the solution $y(s, \epsilon)$ of (2.30) and by the solution $\hat{y}(s/\epsilon)$ of (2.33).

Furthermore, in the next simulation we illustrate the extinction of the total population of recently mated females obtained by subtracting the second equation of (2.30) from the first one.

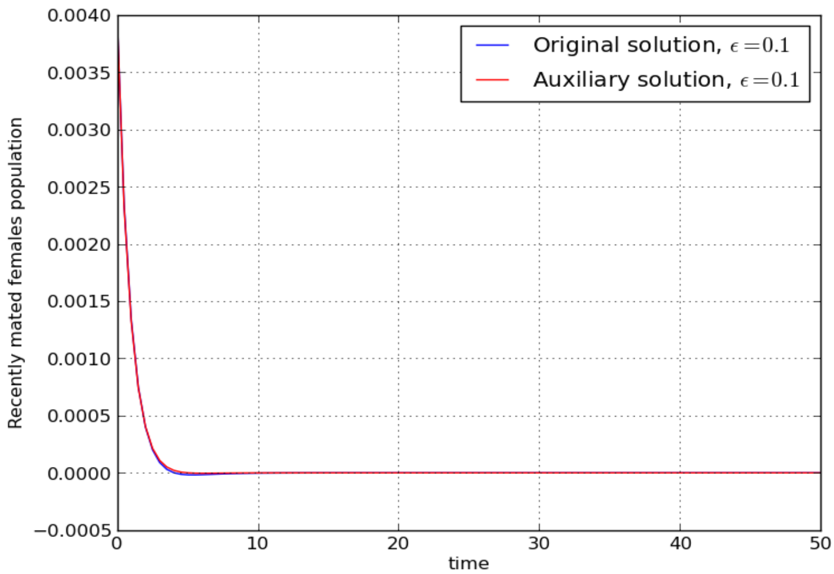


Figure 2.11: Approximation of the population of recently mated females given by the differences $x_\epsilon(s) = z_\epsilon(s) - y_\epsilon(s)$ and $\hat{x}(s/\epsilon) = \hat{z}(s/\epsilon) - \hat{y}(s/\epsilon)$.

2.4.2 Comment on the numerical simulations

In accordance with the above discussion, we observe that the whole females population z is strictly decreasing. This is because of the high searching activities and the high searching mortality rate. The population described by the auxiliary equation is decreasing at the same rate as the original population because every single individual within the female population is gone searching for a mate, and because of high searching activities and high mortality rate, they probably never come back. Then, for a given initial condition, their population $\hat{y}$ initially increases, having crossed the isocline, begins to monotonically decrease to zero as illustrated by Fig 2.10.
Since all the females who go to search for a mate, probably will never come back, the population of recently mated females decreases faster to zero as illustrated by Fig 2.9.
Furthermore, the choice of the small parameter plays a major role in the understanding of the global behaviour of the model. The above analysis highlights the importance of considering all the possible choices of the small parameter in order to capture the full picture of the model's behaviour. It shows again that if one considers only a particular choice of the small parameter, the conclusion on the model's behaviour might be erroneous.

3. Application of the Centre Manifold to Singularly Perturbed Problems

Tikhonov and Vasil'eva theorems, presented in Chapter 1, typically work only on finite time intervals. Precisely, they give convergence of solutions of (1.29) to solutions of (1.30) on each fixed time interval $[0, T]$ on which the right-hand side of the equation satisfies the assumptions of these theorems. However, even if the assumptions are satisfied on $[0, T]$ for each $T < \infty$, the convergence may not be uniform in T and thus cannot be extended on $[0, \infty)$. To illustrate this, in Section 1 of this Chapter we present an example of a kinetic process [14], where actually the convergence occurs on $[0, \infty)$ without any additional effort.
On the other hand, in Section 2 we discuss an example which does not have this property. Namely, we consider a prey-predator system with fast migration. Here, we observe that with classical Tikhonov theorem we obtain that the limit equation is a Lotka-Volterra system and we find that on each fixed time interval $[0, T]$ the solutions of the prey-predator system converge to solutions of the Lotka-Volterra system. However, we also see that the prey-predator system has asymptotically stable equilibrium for any $\epsilon > 0$, but the limit system (Lotka-Volterra) has a centre in the positive quadrant. This illustrates the fact that in general the convergence in Tikhonov theorem on $[0, T]$ cannot be made uniform in T.
In general, there are two ways to ensure that the uniform convergence in T. One way, [15], is to impose stonger assumptions on f and g (the right-hand side of (1.29)). However, here it is impossible because the system is given. The other way, which we illustrate below, is to consider a better asymptotic expansion by adding appropriately chosen higher order terms to the limit solution $(\bar{x}, \bar{y})$. We can construct such a correction in several ways. For instance in [7] the authors used the asymptotic expansion of the system. Here we use the approach based on the centre manifold theory, [6].
The method proposed in [6] has a drawback of constructing an approximation only locally in a neighborhood of each point of the quasi-steady state, but this can be overcome by *glueing* such local approaches together, [16].
The advantage of this method is that it can use the centre manifold theory to infer also the long term behaviour of the approximation provided the limit equation is structurally stable. We shall illustrate this point here by showing how taking higher order terms in the expansion of the centre manifold makes the approximate equation of the centre manifold structurally stable. This improves the Tikhonov approximation of the prey-predator system so that it holds uniformly on $[0, \infty)$. For simplicity, we shall only present the local expansion.

3.1 Models with structurally stable limit equation

Let us consider the following system of ordinary differential equations that arises from a model of the kinetics of enzyme reactions, [6].

$$\begin{aligned}\frac{dy}{dt} &= -y + (y+c)z, \\ \epsilon\frac{dz}{dt} &= y - (y+1)z,\end{aligned} \tag{3.1}$$

where $\epsilon > 0$ is a small parameter and $0 < c < 1$. Letting $\epsilon = 0$ in (3.1) yields the algebraic-differential system of equations

$$\begin{aligned}\frac{dy}{dt} &= -y + (y+c)z, \\ 0 &= y - (y+1)z.\end{aligned} \tag{3.2}$$

Solving for z, the second equation of (3.2) gives the quasi-steady state

$$\bar{z} \equiv \varphi(y,t) = \frac{y}{y+1}, \tag{3.3}$$

Replacing the unkown z in the first equation of (3.2) by the known quasi-steady state $\bar{z}$ gives the limit equation

$$\frac{d\bar{y}}{dt} = \frac{-\lambda\bar{y}}{\bar{y}+1}, \tag{3.4}$$

where $\lambda = 1 - c$. Equation (3.4) has a hyperbolic fixed point at the origin.

3.1.1 Application of the Tikhonov Theorem

The vector field of (3.1) is continuous with respect to the variables y and z, where $z \in \mathbb{R}$ and $y \in \mathcal{C}$, where $\mathcal{C} = \{y : y \geq -1 + \delta\}$ and $\delta > 0$ is a small real number; $t \in [0, T]$ and $\epsilon \in [0, \epsilon_0]$ with $\epsilon_0 > 0$.
The quasi-steady state (3.3), together with its derivatives, is continuous with respect to y in the domain

$$D = \{y \in \mathbb{R} : y \neq -1\}.$$

If we consider the change of the independent variable t to $\tau = t/\epsilon$ in (3.1), we obtain the equivalent system in fast time

$$\begin{aligned}\frac{dy}{d\tau} &= -\epsilon y + \epsilon(y+c)z, \\ \frac{dz}{d\tau} &= y - (y+1)z.\end{aligned} \tag{3.5}$$

From System (3.5), if we let $\epsilon = 0$, we obtain the auxiliary equation

$$\frac{d\hat{z}}{d\tau} = y - (y+1)\hat{z}, \tag{3.6}$$

where y is treated as a parameter. From the auxiliary equation (3.6), we set

$$\Psi(t, y, \hat{z}) = y - (y+1)\hat{z},$$

and

$$\frac{\partial}{\partial \hat{z}}\Psi(t, y(t), \varphi(t, y(t))) = -y - 1 < 0 \Longleftrightarrow y > -1.$$

If the above condition is satisfied, then the quasi-steady state

$$\bar{z} = \frac{y}{y+1},$$

which is an isolated root of the equation

$$y - (y+1)z = 0,$$

is a stable equilibrium of the auxiliary equation (3.6) as well. From limit equation (3.4), we have

$$\bar{y}(t) \in \text{Int } \bar{\mathcal{C}} \text{ for all } \ t \in (0, T).$$

Now, letting $t = 0$ in (3.6) gives the initial layer equation

$$\frac{d\hat{z}}{d\tau} = y_0 - (y_0+1)\hat{z}, \tag{3.7}$$

whose solution is given by

$$\hat{z}(\tau) = \left(\hat{z}_0 - \frac{y_0}{y_0+1}\right) e^{-(y_0+1)\tau} + \frac{y_0}{y_0+1}.$$

Since

$$\lim_{\tau\to\infty} \hat{z}(\tau) = \frac{y_0}{y_0+1} = \varphi(0, y_0), \text{ for } y_0 > -1,$$

we have that the initial condition $\hat{z}_0$ belongs to the basin of attraction of the solution $\bar{z}_0 = \varphi(0, \bar{y}_0)$ of the equation $\bar{y}_0 - (\bar{y}_0 + 1)\bar{z}_0 = 0$.
Then all the assumptions of Tikhonov theorem are satisfied for the equation (3.1). It follows that for $y_0 > -1$ and $0 < T < \infty$ the solutions $(z_\epsilon(t), y_\epsilon(t))$ of (3.1) exist and have the following limiting relation

$$\begin{aligned} &\lim_{\epsilon\to 0} y_\epsilon(t) = \bar{y}(t), \qquad 0 \le t \le T, \\ &\lim_{\epsilon\to 0} z_\epsilon(t) = \varphi(t, \bar{y}(t)), \qquad 0 < t \le T, \end{aligned} \tag{3.8}$$

where $\bar{y}(t)$ is the solution of the limit equation (3.4). We illustrate these convergences in the following numerical simulation.

3.1.2 Numerical simulation

Here we present a numerical simulation of the model of the kinetics of the enzyme reactions given by (3.1). The parameters' values used are $\lambda = 0.5$, $c = 0.5$ and $\epsilon = 0.01$.

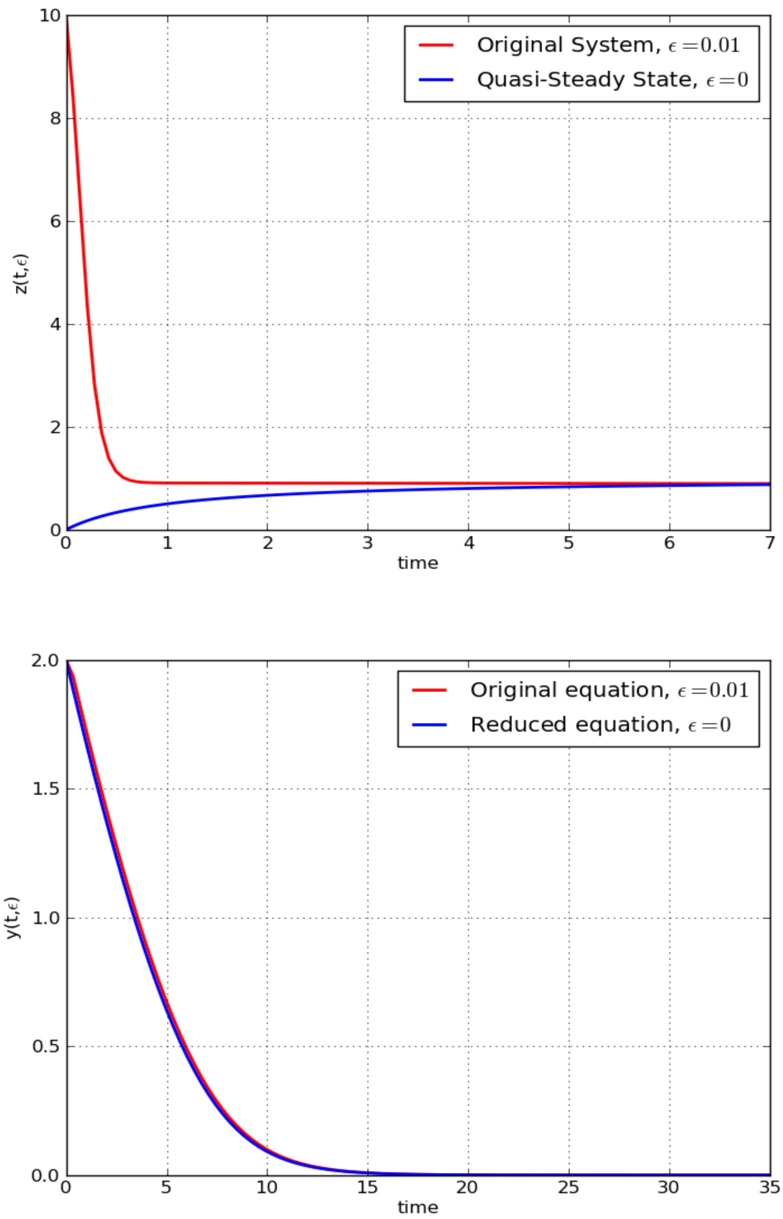


Figure 3.1: The solutions $z_\epsilon(t)$ of Equation (3.1) attracted by the quasi-steady state $\bar{z}(t)$ given by (3.3) and $y_\epsilon(t)$ attracted by the solution of (3.4), as predicted by Tikhonov theorem.

In the first picture in the Fig 3.1, the convergence is not uniform in the interval $[0, T]$. It is, however, uniform in any closed interval $[\zeta, T]$, where $\zeta > 0$. In the second picture, the convergence is uniform in the interval $[0, T]$. This was referred to earlier, in Section 1.2, as *the initial layer effect.*

3.1.3 Initial layer

In order to obtain uniform convergence of the solution $z_\epsilon(t)$ in the closed interval $[0, T]$, we add the initial layer term

$$\hat{z}(\tau) - \varphi(0, y_0)$$

to the solution of the second equation of (3.1). Hence, by Proposition 1.3.1, we have

$$\lim_{\epsilon \to 0}(z_\epsilon(t) - \bar{z}(t) - \hat{z}(\tau) + \varphi(0, y_0)) = 0$$

uniformly for $t \in [0, T]$, as illustrated in the next numerical simulation

3.1.4 Numerical simulation

In this simulation, we show the behaviour of the solution of the second equation of (3.1) after adding the initial layer term.

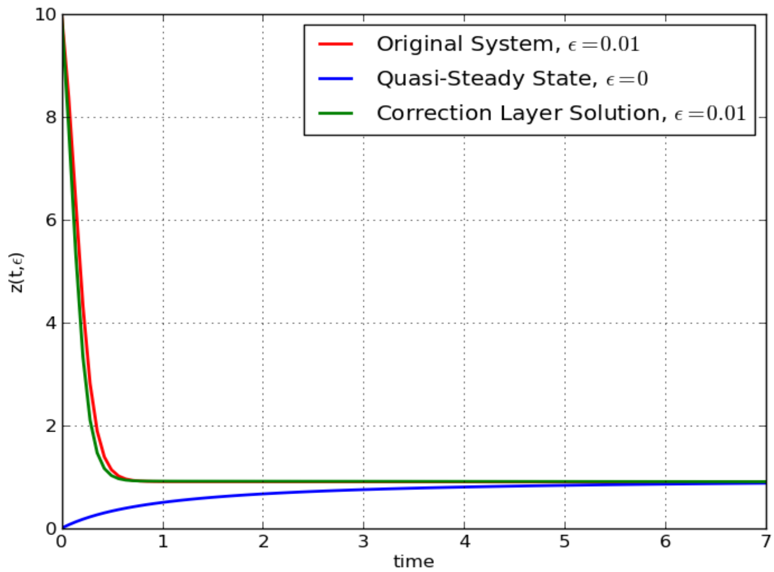


Figure 3.2: Correction in the initial layer.

The picture in Fig 3.2 describes uniform convergence of the solution of the second equation of (3.1) after adding the initial layer term to the quasi-steady state.
We further show that the centre manifold theory also provides similar results.

3.1.5 Application of the centre manifold theory

Let us consider the system

$$
\begin{aligned}
\frac{dy}{d\tau} &= -\epsilon y + \epsilon(y+c)z, \\
\frac{dz}{d\tau} &= y - (y+1)z, \\
\frac{d\epsilon}{d\tau} &= 0.
\end{aligned} \tag{3.9}
$$

obtained from (3.5) with ϵ as a dummy variable. The linear part associated to (3.9) is given by

$$
L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

In order to apply the centre manifold theorems, we need the matrix L to be in the canonical form. Therefore, we perform the change of variables $w = y - z$ in (3.9) to obtain an equivalent system in new variables given by

$$
\begin{aligned}
\frac{dy}{d\tau} &= -\lambda\epsilon y + \epsilon y^2 - \epsilon y w - c\epsilon w, \\
\frac{dw}{d\tau} &= -w + y^2 - yw - \lambda\epsilon y + \epsilon y^2 - \epsilon y w - c\epsilon w, \\
\frac{d\epsilon}{d\tau} &= 0.
\end{aligned} \tag{3.10}
$$

Then the linear part associated to (3.10) is in the canonical form and it is given by

$$
\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Since the matrix $\mathcal{L}$ has eigenvalues with zero and negative real parts, it follows by Theorem 1.4.2 that (3.10) possesses a two dimensional centre manifold

$$
w = h(y, \epsilon). \tag{3.11}
$$

Using the invariance property of the centre manifold, we construct the equation of the centre manifold by differentiating (3.11) with respect to τ. Thus, we have

$$
w' = h_y(y,\epsilon)y' + h_\epsilon(y,\epsilon)\epsilon'. \tag{3.12}
$$

Then we substitute w', y' and ϵ' in (3.12) by their respective values and we use conditions of the tangency of the centre manifold with the centre subspace at the origin, introduced in Remark 1.4.1, to obtain the following system that needs to be solved in order to compute the equation of the centre manifold.

$$\begin{aligned}
0 &= h_y(y,\epsilon)[-\lambda\epsilon y + \epsilon y^2 - \epsilon y h - c\epsilon h] + h_\epsilon(y,\epsilon)\epsilon' + h + \lambda\epsilon y - y^2 + yh - \epsilon y^2 + \epsilon y h + c\epsilon h,\\
0 &= h(0,\epsilon),\\
0 &= h'(0,\epsilon).
\end{aligned} \tag{3.13}$$

Note that System (3.13) is in general impossible to solve. We should therefore consider an approximation of the centre manifold. This is obtained by letting

$$h(y,\epsilon) = b_1 y^2 + b_2 y\epsilon + b_3\epsilon^2 + o_3, \tag{3.14}$$

where o_3 denotes the cubic terms in y and ϵ. Substituting Equation (3.14) into the first equation of (3.13), we obtain the following expression

$$\begin{gathered}
(2b_1 y + b_2\epsilon + \cdots)[-\lambda\epsilon y + \epsilon y^2 - b_1\epsilon y^3 - b_2\epsilon^2 y^2 - b_3\epsilon^3 y - b_1 c\epsilon y^2 - b_2 c\epsilon^2 y - b_3 c\epsilon^3]\\
+b_1 y^2 + b_2\epsilon y + b_3\epsilon^2 + \lambda\epsilon y - y^2 + b_1 y^3 + b_2\epsilon y^2 + b_3\epsilon^2 y - \epsilon y^2 + b_1\epsilon y^3 + b_2\epsilon^2 y^2 + b_3\epsilon^3 y\\
+b_1 c\epsilon y^2 + b_2 c\epsilon^2 y + b_3 c\epsilon^3 = 0,
\end{gathered}$$

where we have written h for $h(y,\epsilon)$. Then, by grouping like terms in y and ϵ, it is reduced to

$$(b_1 - 1)y^2 + (b_2 + \lambda)y\epsilon + b_3\epsilon^2 + o_3 = 0 \tag{3.15}$$

and solving it gives the coefficients

$$b_1 = 1,\ b_2 = -\lambda,\ b_3 = 0.$$

Hence, the approximate equation of the centre manifold can be written as

$$h(y,\epsilon) = y^2 - \lambda y\epsilon + o_3. \tag{3.16}$$

Thus, the equation that determines the stability of small solutions of (3.10) on the centre manifold is obtained by substituting Equation (3.16) into the first equation of (3.10). It is given by

$$u' = -\lambda\epsilon u + \lambda\epsilon u^2 - \epsilon u^3 + \lambda\epsilon^2 u^2 + \lambda c\epsilon^2 u. \tag{3.17}$$

In the original timescale, Equation (3.17) reads

$$\dot{u} = -\lambda u + \lambda u^2 + O(|u|^3 + |\epsilon u|). \tag{3.18}$$

By Theorem 1.4.3, there exists a solution $u(t,\epsilon)$ of (3.18) such that, as $t \to \infty$, we have

$$\begin{aligned} y(t,\epsilon) &= u(t,\epsilon) + O(e^{-\gamma t/\epsilon}), \\ w(t,\epsilon) &= h(y(t),\epsilon) + O(e^{-\gamma t/\epsilon}), \end{aligned} \tag{3.19}$$

where $\gamma > 0$ is a constant. Equation (3.19) shows that the solutions of (3.10) and those of (3.18) are exponentially close.
If we replace $w(t,\epsilon)$ in (3.19) by $y(t,\epsilon) - z(t,\epsilon)$ we obtain the approximate solutions of the system (3.1), written as

$$\begin{aligned} y(t,\epsilon) &= u(t,\epsilon) + O(e^{-\gamma t/\epsilon}) \\ z(t,\epsilon) &= y(t,\epsilon) - h(y(t),\epsilon) + O(e^{-\gamma t/\epsilon}). \end{aligned} \tag{3.20}$$

If we consider the following Taylor expansion

$$\frac{1}{1+y} = 1 - y + y^2 + \cdots , \tag{3.21}$$

then from (3.4) we have

$$\dot{\bar{y}} = -\lambda \bar{y} + \lambda \bar{y}^2 + \cdots , \tag{3.22}$$

which is approximately close to (3.18). Now, if we replace $h(y(t),\epsilon)$ by its value in the second equation of (3.20), we obtain

$$z(t,\epsilon) = y(t) - y^2(t) + \lambda\epsilon y(t) + \cdots . \tag{3.23}$$

Using once again the same Taylor expansion in (3.3), we obtain

$$\bar{z}(t) = y(t) - y^2(t) + \cdots . \tag{3.24}$$

This shows that the quasi-steady state is approximately close to Equation (3.23) of the centre manifold. Note that this approximation is valid only for small values of y, as shown in the numerical simulations. It shows that the quasi-steady state of the Tikhonov theorem is the 0^{th} order approximation in ϵ of the centre manifold. Moreover, (3.20) ensures the uniform convergence of the solution of (3.1) to the solution of the approximate limit equation on the centre manifold, uniformly in time.
Note that the uniform convergence is only local in y; this is due to the local nature of the centre manifold result.

3.1.6 Numerical simulation

Here we present a numerical simulation that shows the intersection of the quasi-steady state given by (3.3) and the centre manifold given by (3.23), for small values of y.

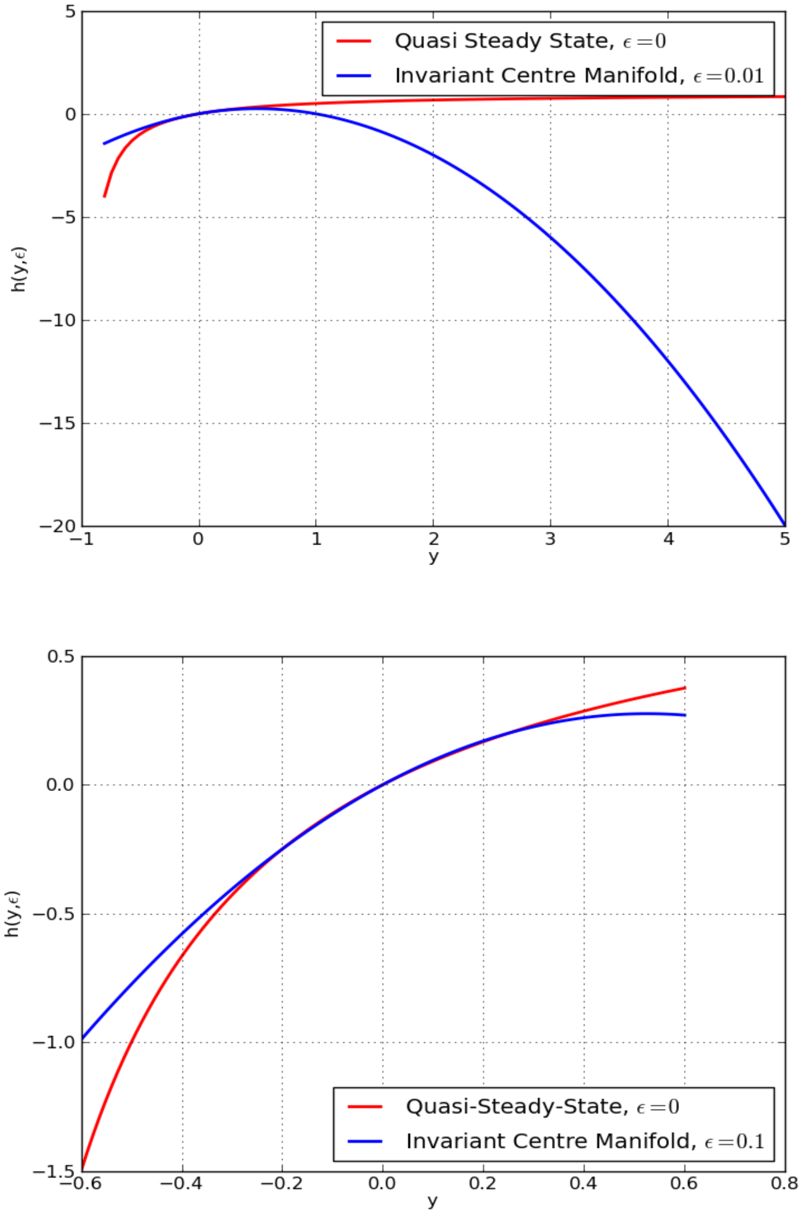


Figure 3.3: The intersection of the invariant centre manifold and the quasi-steady state curves in a small neighborhood of the origin.

Fig 3.3 shows that the centre manifold and the quasi-steady state curves intersect in a small neighborhood of the fixed point (here it is the origin) of the system. The second picture shows this intersection after enlarging the first one in the neighborhood of the origin.

The next numerical simulation shows the comparison between the solution of the limit equation (3.4) and the solution of the equation that determines the stability of small solutions on the centre manifold (3.18).

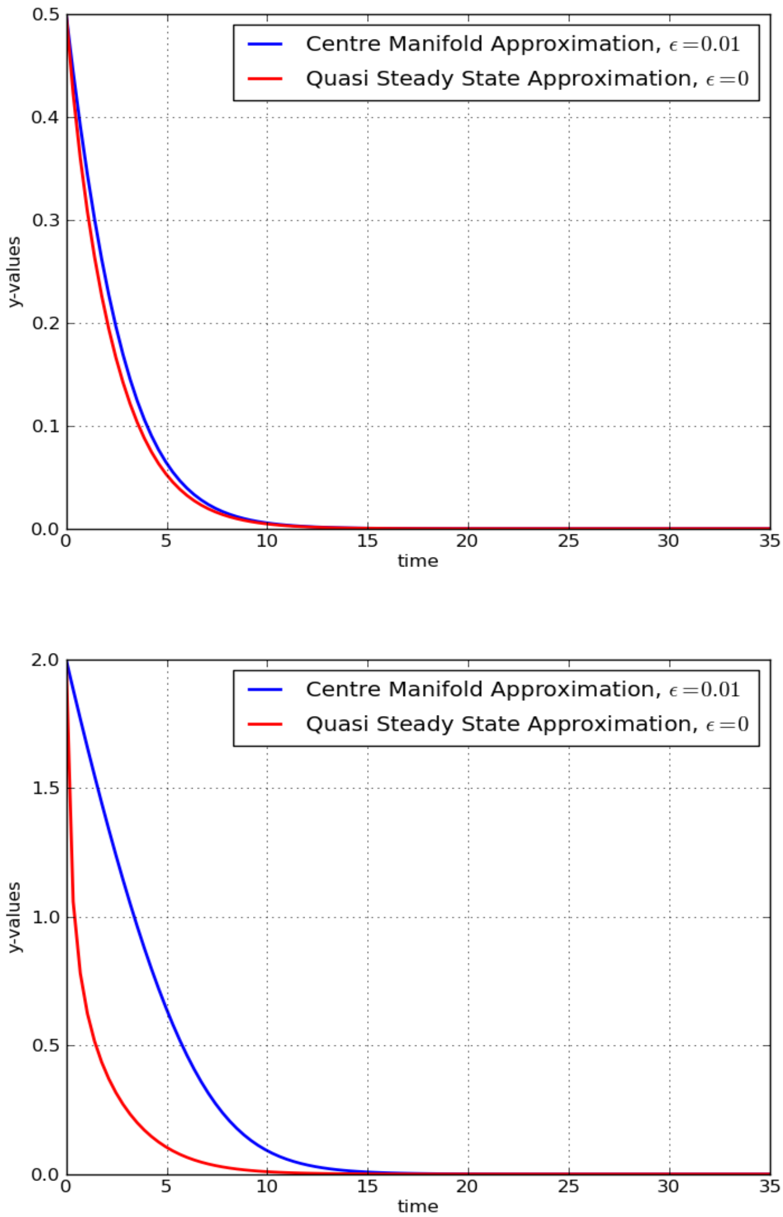


Figure 3.4: The comparison between the solutions given by Equation (3.18) and the solutions given by (3.4) for small and large values of y.

The fact that on Fig 3.4 we observe a worse approximation for small times is due, as mentioned earlier in this Chapter, to the fact that we used only local expansion of the centre manifold which gives good approximations only for small values of y. Neverthless, as we see, the long term behaviour is not affected.

3.2 Models with not structurally stable limit equation

Here we consider a prey-predator model in which prey and predators live in two different patches. Prey can move between both patches while predators remain in the patch 1. Patch 2 is considered as a refuge for prey, [17]. The model we are investigating is due to Poggiale and Auger, [18], and it is given by

$$\begin{aligned} \frac{dn_1}{dt} &= R(m_2n_2 - m_1n_1) + n_1(r_1 - ap), \\ \frac{dn_2}{dt} &= R(m_1n_1 - m_2n_2) + n_2r_2, \\ \frac{dp}{dt} &= p(bn_1 - d), \end{aligned} \tag{3.25}$$

where, for $i = 1, 2$, n_i denotes prey density in patch i, m_i denotes proportions of prey population leaving patch i per unit time, r_i is the prey population growth rate on patch i, d is the predator population death rate, a is the predator rate on patch 1, p denotes predator density and, bn_1 is the per capita predator growth rate and R is an arbitrary large number, [18].
Because R is arbitrarily large, the term that contains R on the right-hand side of (3.25) is called the fast term and the second term is the slow term.
We let $R = 1/\epsilon$ in (3.25) and add up the first two equations to obtain an equivalent singularly perturbed system of ordinary differential equations

$$\begin{aligned} \frac{dn}{dt} &= r_2n + n_1(r_1 - r_2 - ap), \\ \epsilon\frac{dn_1}{dt} &= -(m_1 + m_2)n_1 + m_2n + \epsilon n_1(r_1 - ap), \\ \frac{dp}{dt} &= p(bn_1 - d), \end{aligned} \tag{3.26}$$

where $n = n_1 + n_2$ denotes the entire population of prey and ϵ is a small positive parameter. Letting $\epsilon = 0$ in (3.26), it becomes the following differential-algebraic system of equations

$$\begin{aligned} \frac{dn}{dt} &= r_2n + n_1(r_1 - r_2 - ap), \\ 0 &= -(m_1 + m_2)n_1 + m_2n, \\ \frac{dp}{dt} &= p(bn_1 - d). \end{aligned} \tag{3.27}$$

Solving for n_1 the second equation of (3.27) gives the quasi-steady state curve

$$\bar{n}_1 \equiv \varphi(t, n) = \frac{m_2}{m_1 + m_2}n, \tag{3.28}$$

Substituting the unknown n_1 in the first and the third equations of (3.27) by the known quasi-steady state $\bar{n}_1$, we obtain the limit system of ordinary differential equations

$$\begin{aligned} \frac{d\bar{n}}{dt} &= (r - a_1\bar{p})\bar{n}, \\ \frac{d\bar{p}}{dt} &= (b_1\bar{n} - d)\bar{p}, \end{aligned} \tag{3.29}$$

where

$$r = r_1(1-U) + r_2 U,\ \ a_1 = a(1-U),\ \ b_1 = b(1-U) \text{ and } U = \frac{m_1}{m_1+m_2}.$$

The limit system (3.29) is the classical Lotka-Volterra system that has a centre in the first quadrant. Then, according to the Andronov-Pontryagin criterion, [10], introduced in Section 1.1.5, System (3.29) is not structurally stable, as illustrated in the following numerical simulation.

3.2.1 Numerical simulation

Here we present numerical simulations of the solutions of the limit equation given by (3.29). We further present the comparison between the orbits of the limit equation (3.29) and of the original equation (3.25), clearly showing that the limit equation possesses a centre in the positive quadrant, as expected from the Lotka-Voltera system, while the original equation possesses a sink. The paramaters' values used are $m_1 = 2$, $m_2 = 1$, $r_1 = 1$, $r_2 = 2$, $a = 1$, $d = 1$, $b = 0.9$, and $\epsilon = 0.1$.

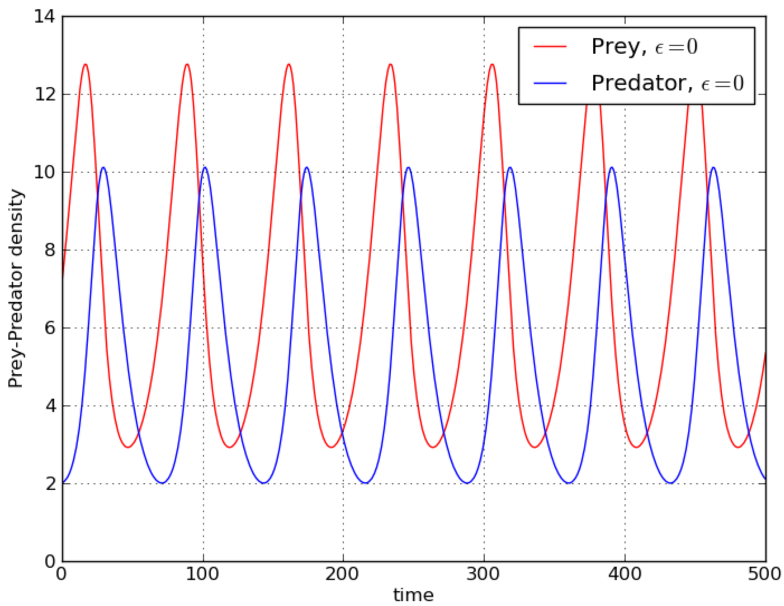


Figure 3.5: The dynamics of the prey and predators given by the reduced system (3.29).

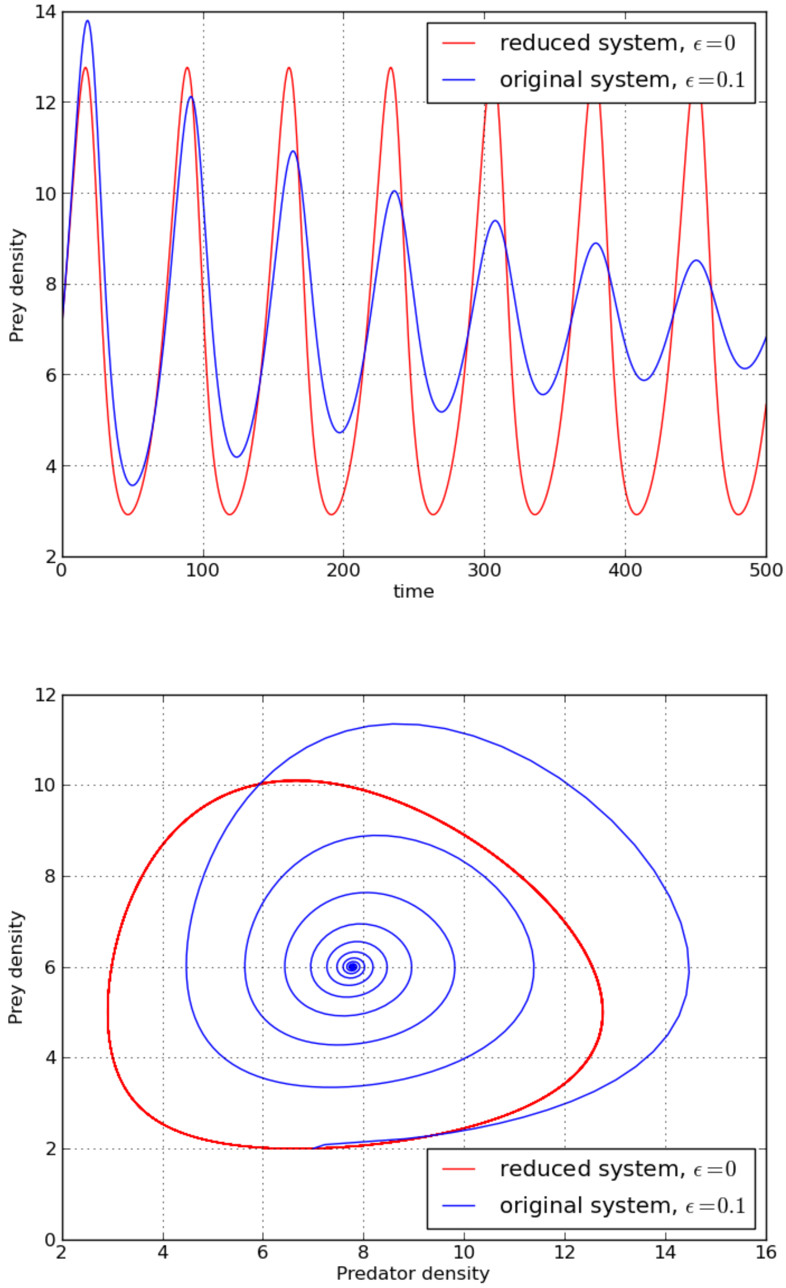


Figure 3.6: The comparison between the solution and the orbit of the original system given by (3.25) and the solution and the orbit of the reduced system given by (3.29).

From Fig 3.6 we can surmise that the convergence of the solutions of the original system to the solutions of the limit system is not uniform in T on $[0, T]$. Thus, the limit system (3.29) cannot be used to describe long term dynamics of the original system (3.25) as ϵ approaches zero. We are therefore going to apply the centre manifold theory to build a limit system that is structurally stable.

3.2.2 Equilibrium States

The equilibrium states of the original system (3.25) are obtained by solving the system below

$$\begin{aligned} 0 &= m_2 n_2 - m_1 n_1 + \epsilon n_1 (r_1 - ap), \\ 0 &= m_1 n_1 - m_2 n_2 + \epsilon n_2 r_2, \\ 0 &= p(bn_1 - d), \end{aligned} \tag{3.30}$$

which yields

$$\begin{cases} n_1^* = 0, \\ n_2^* = 0, \\ p^* = 0, \end{cases} \quad \text{or} \quad \begin{cases} n_1^* = \dfrac{d}{b}, \\ n_2^* = \dfrac{m_1 d}{b(m_2 - \epsilon r_2)}, \\ p^* = \dfrac{r_1}{a} + \dfrac{m_1 r_2}{a(m_2 - \epsilon r_2)}. \end{cases}$$

The Jacobian matrix assosiated with the system (3.25), is given by

$$J = \begin{pmatrix} -\frac{1}{\epsilon} m_1 + r_1 - ap & \frac{1}{\epsilon} m_2 & -an_1 \\ \frac{1}{\epsilon} m_1 & -\frac{1}{\epsilon} m_2 + r_2 & 0 \\ pb & 0 & bn_1 - d \end{pmatrix}.$$

At the trivial state $(0,0,0)$ we have

$$J|_{(0,0,0)} = \begin{pmatrix} -\frac{1}{\epsilon} m_1 + r_1 & \frac{1}{\epsilon} m_2 & 0 \\ \frac{1}{\epsilon} m_1 & -\frac{1}{\epsilon} m_2 + r_2 & 0 \\ 0 & 0 & -d \end{pmatrix}.$$

Since this matrix is in a block form, clearly $\lambda = -d$ is one of the eigenvalues. The other two eigenvalues are the roots of the characteristic equation

$$\lambda^2 + \lambda \left(r_1 + r_2 - \frac{1}{\epsilon}(m_1 + m_2) \right) + r_1 r_2 - \frac{1}{\epsilon} \left(r_2 m_1 + r_1 m_2 \right) = 0$$

of the matrix

$$\mathcal{M} = \begin{pmatrix} -\frac{1}{\epsilon} m_1 + r_1 & \frac{1}{\epsilon} m_2 \\ \frac{1}{\epsilon} m_1 & -\frac{1}{\epsilon} m_2 + r_2 \end{pmatrix}.$$

For $\epsilon > 0$ sufficiently small, the trace of $\mathcal{M}$ is $-(1/\epsilon)(m_1+m_2)+r_1+r_2 < 0$ and the determinant of $\mathcal{M}$ is $-(1/\epsilon)(r_1 + r_2) + r_1 r_2 < 0$.

It follows that the roots of the characteristic equation, which are the eigenvalues of $\mathcal{M}$ have different sign. Hence, the trivial state $(0,0,0)$ is a saddle point. At the positive equilibrium state (n_1^*, n_2^*, p^*), the Jacobian matrix is given by

$$J|_{(n_1^*,n_2^*,p^*)} = \begin{pmatrix} -\frac{m_1 m_2}{\epsilon(m_2-\epsilon r_2)} & \frac{1}{\epsilon}m_2 & -\frac{ad}{b} \\ \frac{1}{\epsilon}m_1 & -\frac{1}{\epsilon}m_2 + r_2 & 0 \\ b\left(\frac{r_1}{a} + \frac{m_1 r_2}{a(m_2-\epsilon r_2)}\right) & 0 & 0 \end{pmatrix}$$

and the characteristic equation associated with this Jacobian is given by

$$\lambda^3 + \lambda^2\left(\alpha + \frac{1}{\epsilon}m_2 - r_2\right) + \lambda\beta\gamma + \beta\gamma\left(\frac{1}{\epsilon}m_2 - r_2\right) = 0,$$

where

$$\alpha = \frac{m_1 m_2}{\epsilon(m_2 - \epsilon r_2)},\ \beta = \frac{ad}{b},\ \gamma = b\left(\frac{r_1}{a} + \frac{m_1 r_2}{a(m_2 - \epsilon r_2)}\right).$$

By the Routh-Hurwitz criterion, introduced earlier in Theorem 1.1.3, for $\epsilon > 0$ sufficiently small all the coefficients of the characteristic equation are positive and the determinant

$$D_2 = \begin{vmatrix} \beta\gamma & \beta\gamma\left(\frac{1}{\epsilon}m_2 - r_2\right) \\ 1 & \alpha + \frac{1}{\epsilon}m_2 - r_2 \end{vmatrix} = \alpha\beta\gamma > 0.$$

This ensures that the real parts of the eigenvalues of the Jacobian $J|_{(n_1^*,n_2^*,p^*)}$ of (3.25) are negative.

Hence, the positive equilibrium state (n_1^*, n_2^*, p^*) of (3.25) is asymptotically stable. The equilibrium state of the limit system (3.29) is obtained by solving the equation

$$\begin{aligned} 0 &= (r - a\bar{p})\bar{n}, \\ 0 &= (b_1\bar{n} - d)\bar{p}, \end{aligned} \tag{3.31}$$

Then

$$\begin{cases} n^* = 0, \\ p^* = 0 \end{cases} \text{and} \quad \begin{cases} n^* = \dfrac{d}{b_1}, \\ p^* = \dfrac{r}{a_1}, \end{cases}$$

are the equilibria of the limit system (3.29).

3.2.3 Application of the centre manifold theory

If we consider the change of the independent variable t to $\tau = t/\epsilon$ in (3.25), we obtain an equivalent system in the fast time given by

$$
\begin{aligned}
\frac{dn_1}{d\tau} &= m_2 n_2 - m_1 n_1 + \epsilon n_1 (r_1 - ap), \\
\frac{dn_2}{d\tau} &= m_1 n_1 - m_2 n_2 + \epsilon n_2 r_2, \\
\frac{dp}{d\tau} &= \epsilon p (b n_1 - d), \\
\frac{d\epsilon}{d\tau} &= 0,
\end{aligned}
\tag{3.32}
$$

where ϵ is a dummy variable. Equation (3.32) can be rewritten in the following standard form

$$
\begin{aligned}
\frac{dx}{d\tau} &= Bx + \epsilon f(x,y), \\
\frac{dy}{d\tau} &= Ay + \epsilon g(x,y), \\
\frac{d\epsilon}{d\tau} &= 0,
\end{aligned}
\tag{3.33}
$$

where $x = (n_1, n_2) \in \mathbb{R}^2$, $y = p \in \mathbb{R}$, $\epsilon \in \mathbb{R}$ and

$$
B = \begin{bmatrix} -m_1 & m_2 \\ m_1 & -m_2 \end{bmatrix}, \; A = 0, \; f(x,y) = f(n_1, n_2, p) = \begin{bmatrix} n_1(r_1 - ap) \\ n_2 r_2 \end{bmatrix},
$$

$$
g(x,y) = g(n_1, n_2, p) = p(bn_1 - d).
$$

In order to apply the centre manifold theory to Equation (3.32), it must be transformed into the canonical form. We have $0, -(m_1 + m_2)$ as eigenvalues of the matrix B and the corresponding eigenvectors are

$$
\begin{bmatrix} \frac{m_2}{m_1} \\ 1 \end{bmatrix}, \; \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
$$

Using the above eigenbasis, we have

$$
\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \frac{m_2}{m_1} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},
$$

with inverse

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{m_1}{m_1 + m_2} \begin{bmatrix} 1 & 1 \\ -1 & \frac{m_2}{m_1} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}.
$$

Then, applying the above change of variables to (3.32), it is transformed into

$$
\begin{aligned}
\begin{bmatrix} u_1' \\ u_2' \\ p' \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(m_1+m_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} + \epsilon \begin{bmatrix} U(r_1 - r_2 - ap)u_2 + U(r_1 m + r_2 - apm)u_1 \\ U(r_1 + r_2 m - ap)u_2 + U(r_2 - r_1 + ap)mu_1 \\ pbmu_1 - pbu_2 - pd \end{bmatrix} \\
\epsilon' &= 0,
\end{aligned}
\tag{3.34}
$$

where

$$U = \frac{m_1}{m_1 + m_2} \text{ and } m = \frac{m_2}{m_1}.$$

Clearly, the system of equations (3.34) is in the canonical form. Hence, by Theorem 1.4.3, Equation (3.34) has a local centre manifold given by

$$W^c(0) = \big\{(u_1, u_2, p, \epsilon) \in \mathbb{R}^4 : u_2 = h(u_1, p, \epsilon), h(0) = 0, Dh(0) = 0\big\}$$

for u_1, p and ϵ sufficiently small. In order to find an approximation to the centre manifold, we let

$$h(u_1, p, \epsilon) = a_1 u_1^2 + a_2 p^2 + a_3 \epsilon^2 + a_4 u_1 p + a_5 u_1 \epsilon + a_6 p\epsilon + \cdots \tag{3.35}$$

and we solve the following equation

$$u_1' \frac{\partial h}{\partial u_1}(u_1, p, \epsilon) + p' \frac{\partial h}{\partial p}(u_1, p, \epsilon) + \epsilon' \frac{\partial h}{\partial \epsilon}(u_1, p, \epsilon) - u_2' = 0. \tag{3.36}$$

Substituting u_1', p', ϵ' and u_2' by their respective values in (3.36) and using the software Mathematica we obtain the following coefficients

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = -\frac{(1-U)(r_1 - r_2 - ap)}{m_1 + m_2}, a_6 = 0.$$

It follows that the approximate equation of the centre manifold is given by

$$u_2 = h(u_1, p, \epsilon) = -\epsilon \frac{(1-U)(r_1 - r_2 - ap)u_1}{m_1 + m_2}. \tag{3.37}$$

Hence, substituting Equation (3.37) into the first and the third equations of (3.34), and using the original timescale, we obtain the system of ordinary differential equations reduced to the centre manifold, given by

$$
\begin{aligned}
\frac{dn}{dt} &= (r - a_1 p)n + \epsilon \frac{U(1-U)(r_1 - r_2 - ap)}{m_1 + m_2}(r_1 - r_2 - ap)n, \\
\frac{dp}{dt} &= (b_1 n - d)p + \epsilon p b_1 \frac{U(r_1 - r_2 - ap)}{m_1 + m_2} n,
\end{aligned}
\tag{3.38}
$$

where

$$r = r_1(1-U) + r_2 U, a_1 = a(1-U), b_1 = b(1-U) \quad \text{and} \quad n = \frac{u_1}{U}.$$

3.2.4 Equilibrium states

The equilibrium state of the system (3.38), that governs the flow on the centre manifold, is obtained by simultaneously solving the equations

$$
\begin{aligned}
0 &= (r - a_1 p)n + \epsilon \frac{U(1-U)(r_1 - r_2 - ap)}{m_1 + m_2}(r_1 - r_2 - ap)n, \\
0 &= (b_1 n - d)p + \epsilon p b_1 \frac{U(r_1 - r_2 - ap)}{m_1 + m_2} n,
\end{aligned}
$$

which gives

$$
\begin{cases} n^* = 0, \\ p^* = 0, \end{cases} \text{or} \quad \begin{cases} n_+^\epsilon = \dfrac{d}{b(1-U)\left(1 - \frac{2\epsilon\gamma\rho}{1+\sqrt{1-4\epsilon\gamma\rho}}\right)}, \\ p_+^\epsilon = \dfrac{1}{a}\left(r_1 - r_2 + \dfrac{2\rho}{1+\sqrt{1-4\epsilon\gamma\rho}}\right), \end{cases}
$$

or

$$
\begin{cases} n_-^\epsilon = \dfrac{d}{b(1-U)\left(1 - \frac{2\epsilon\gamma\rho}{1-\sqrt{1-4\epsilon\gamma\rho}}\right)}, \\ p_-^\epsilon = \dfrac{1}{a}\left(r_1 - r_2 + \dfrac{2\rho}{1-\sqrt{1-4\epsilon\gamma\rho}}\right), \end{cases}
$$

where we denoted $z = r_1 - r_2 - ap$, $\gamma = U/(m_1 + m_2)$ and $\rho = r_2/(1-U)$. As $\epsilon \to 0^+$,

$$
n_+^\epsilon \to \frac{d}{b(1-U)} \equiv \frac{d}{b_1} \text{ and } p_+^\epsilon \to \frac{r_1(1-U) + r_2 U}{a(1-U)} \equiv \frac{r}{a_1},
$$

in accordance with the limit of the equilibrium point of the original system (3.25). On the other hand, as $\epsilon \to 0^+$, $p_-^\epsilon \to +\infty$ and $n_-^\epsilon \to +\infty$.
In order to analyze the stability of these equilibria, we compute the Jacobian matrix associated to (3.38), written as

$$
J = \begin{pmatrix} f_n & f_p \\ g_n & g_p \end{pmatrix}, \tag{3.39}
$$

where

$$\begin{aligned}
f_n &= \frac{\epsilon U(1-U)}{m_1+m_2}(r_1-r_2-ap)^2 + r - a(1-U)p,\\
f_p &= -2a\frac{\epsilon U(1-U)}{m_1+m_2}(r_1-r_2-ap)n - a(1-U)n,\\
g_n &= \frac{\epsilon bU(1-U)}{m_1+m_2}(r_1-r_2-ap)p + b(1-U)p,\\
g_p &= \frac{\epsilon bU(1-U)}{m_1+m_2}(r_1-r_2-2ap)n + b(1-U)n - d.
\end{aligned}$$

At the equilibrium point $(0,0)$, the Jacobian is given as

$$J|_{(0,0)} = \begin{pmatrix} \frac{\epsilon U(1-U)}{m_1+m_2}(r_1-r_2)^2 + r & 0 \\ 0 & -d \end{pmatrix}.$$

Then the eigenvalues of this matrix are

$$\lambda_1 = \frac{\epsilon U(1-U)}{m_1+m_2}(r_1-r_2)^2 + r \text{ and } \lambda_2 = -d.$$

Hence, $(0,0)$ is a saddle equilibrium point of the system (3.38). At the equilibrium point $(n_+^\epsilon, p_+^\epsilon)$, the Jacobian is

$$J|_{(n_+^\epsilon, p_+^\epsilon)} = \begin{pmatrix} f_{n_+^\epsilon} & f_{p_+^\epsilon} \\ g_{n_+^\epsilon} & g_{p_+^\epsilon} \end{pmatrix},$$

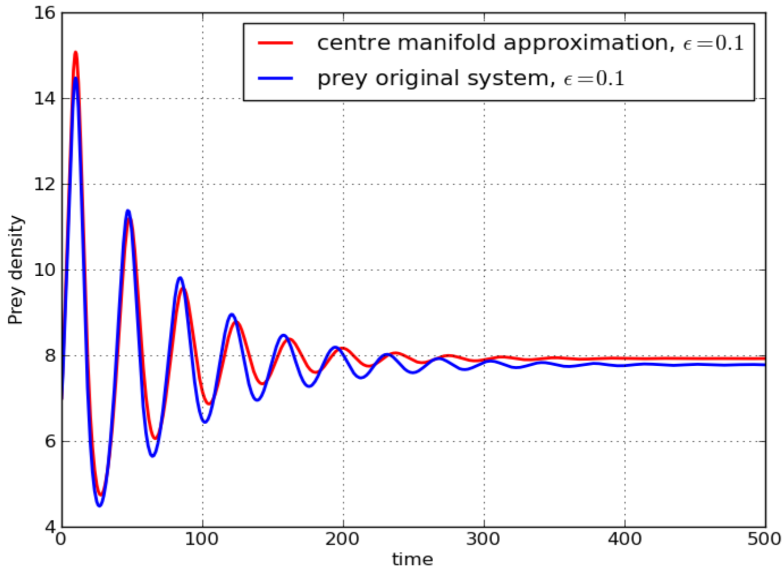
where, for sufficiently small $\epsilon > 0$, we have

$$\begin{aligned}
f_{n_+^\epsilon} &= -2\frac{\epsilon U(1-U)\rho}{1+\sqrt{1-4\epsilon\gamma\rho}} < 0,\\
f_{p_+^\epsilon} &= \frac{(4\epsilon U\rho - (m_1+m_2)(1+\sqrt{1-4\epsilon\gamma\rho}))d}{b(m_1+m_2)(1-2\epsilon\gamma\rho+\sqrt{1-4\epsilon\gamma\rho})} < 0,\\
g_{n_+^\epsilon} &= \frac{b(1-U)((m_1+m_2)(1-2\epsilon U\rho+\sqrt{1-4\epsilon\gamma\rho}))(r_1-r_2)(1-\sqrt{1-4\epsilon\gamma\rho})+2\rho}{a(m_1+m_2)(1-\sqrt{1-4\epsilon\gamma\rho})} > 0,\\
g_{p_+^\epsilon} &= \frac{-\epsilon bU}{m_1+m_2}\left(\frac{4\rho+(r_1-r_2)(1-\sqrt{1-4\epsilon\gamma\rho})}{1-\sqrt{1-4\epsilon\gamma\rho}}\right)\frac{d(1+\sqrt{1-4\epsilon\gamma\rho})-2d\epsilon\gamma\rho}{1-2\epsilon\gamma\rho+\sqrt{1-4\epsilon\gamma\rho}} < 0.
\end{aligned}$$

Hence, the trace$(J|_{(n_+^\epsilon, p_+^\epsilon)}) < 0$ and the $\det(J|_{(n_+^\epsilon, p_+^\epsilon)}) > 0$. It follows that the equilibrium point $(n_+^\epsilon, p_+^\epsilon)$ of the system (3.38) is asymptotically stable. Therefore, the system (3.38) cannot admit saddle connection. Moreover, any flow through a point in the neighborhood of the equilibrium point (n^*, p^*) of (3.38) is also in the neighborhood of (n^*, p^*). Hence, by Theorem 1.1.4, the system (3.38) is structurally stable as illustrated by the following numerical simulation.

3.2.5 Numerical simulation

Here, the numerical simulation shows the comparison between the solutions of the original system given by (3.26) and the solutions of the system of equations reduced to the centre manifold given by (3.38). The paramaters' values used in this simulation are $m_1 = 2,\ m_2 = 1,\ r_1 = 1,\ r_2 = 2,\ a = 1,\ d = 1,\ b = 0.9$ and $\epsilon = 0.01$.



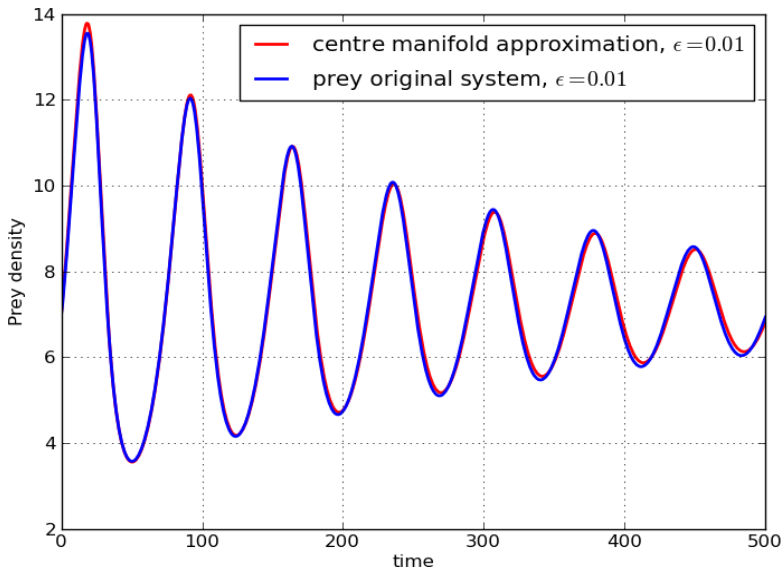


Figure 3.7: Dynamics of prey on the centre manifold and in the original system, for $\epsilon = 0.1$ and $\epsilon = 0.01$.

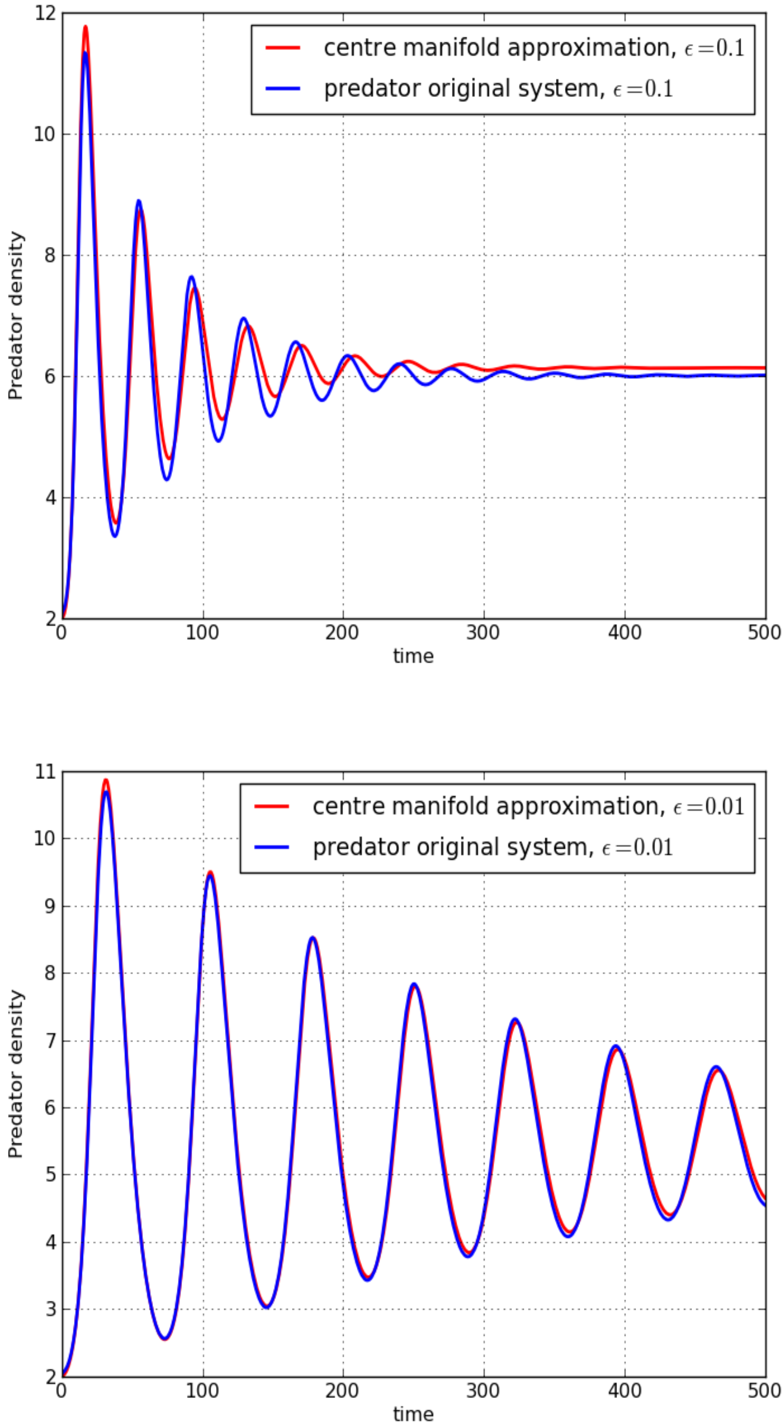
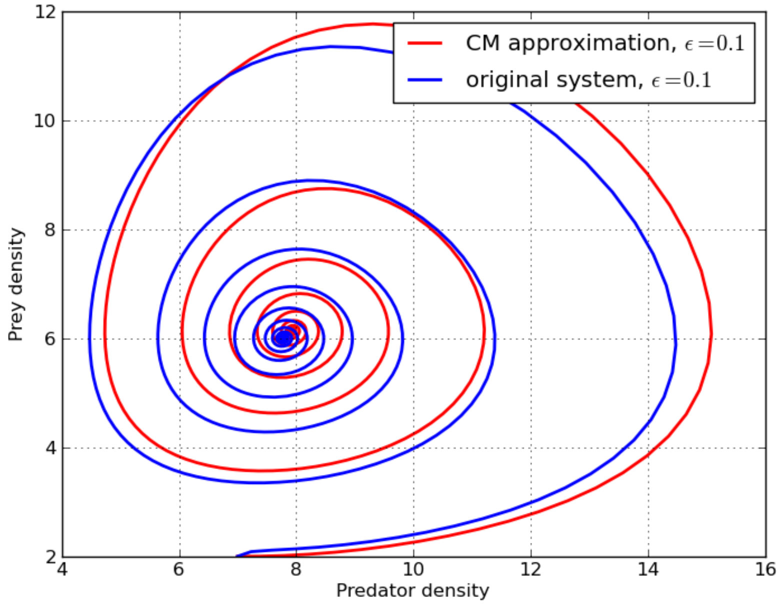


Figure 3.8: Dynamics of predator on the centre manifold and in the original system, for $\epsilon = 0.1$ and $\epsilon = 0.01$.



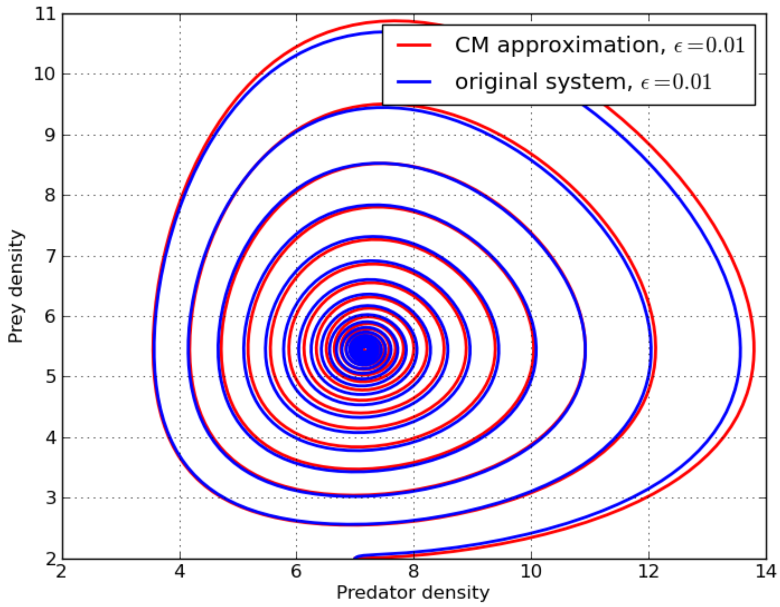


Figure 3.9: Orbits of prey and predators on the centre manifold and in the original system, for $\epsilon = 0.1$ and $\epsilon = 0.01$.

For any small perturbation the behaviour of the equilibrium point of the reduced system (3.29) is affected. This can be observed by the change of its nature from a centre in positive quadrant to a sink, which shows that it is not structurally stable. It can be seen that it does not satisfy the global hyperbolicity criterion which is the first condition of the Andronov-Pontryagin criterion for the stability of dynamical systems in the plane, [10].

Hence, its equilibrium state is affected by any small perturbations. We conclude this chapter by giving a short discussion about structural stability or unstability of the limit equation.

3.3 Discussion

We have shown that a singularly perturbed problem can be reduced to an unperturbed problem that can be either structurally stable or not structurally stable, by letting the small parameter ϵ approach zero.
In the case of a structurally stable limit equation, the zeroth order approximation provided by the Tikhonov theorem is often sufficient for the convergence of the solution of the original equation to the solution of the limit equation on $[0, \infty)$, see also [15].
However, as illustrated above, this is not always true if the limit equation is not structurally stable. In such a case, to ensure convergence on $[0, \infty)$, we need to construct a better approximation, which can be done by including higher order terms in ϵ in the expansion of the centre manifold or in the expansion of the solutions.
Note that the centre manifold theory allows the construction of a limit equation that is structurally stable in the sense of Theorem 1.1.4, and that is regularly perturbed. Then, by the regular perturbation theorem, the convergence of the solutions is uniform on $(0, \infty]$.

4. Singularly Perturbed SIS Model in Case of Exchange of Stabilities

We are investigating an SIS model that describes the evolution of a disease in which the demographic processes are slower than the epidemiological processes. The dynamics within this type of problems evolves in two timescales and often leads to a special class of singularly perturbed problems in which the degenerate equation gives rise to nonisolated quasi-steady states. It follows that the standard Tikhonov theorem cannot be used to analyze the asymptotic behaviour of the model; because the isolatedness of the quasi-steady state is violated.
It turns out that the nonisolatedness of the quasi-steady states may lead to the existence of a time t_c at which the solution of the system passes from a neighborhood of one quasi-steady state to the neighborhood of the other one. As we shall see, this can occur in two different ways.
The solution can jump from one quasi-steady state to the other immediately after it passes by their intersection. Then we say that there is *an immediate exchange of stabilities*.
On the other hand, such a jump may occur after some time which is independent of the small parameter, in which case we say that there is *a delayed exchange of stabilities*.

4.1 The Model

Let us consider the SIS model with an age structure developed in [19], where the population is divided into two classes, namely juveniles n_1 and adults n_2.
Let us next consider a disease that only affects juveniles; that is, n_1 is divided into susceptibles s and infectives i. Then the combined demographic and epidemiological model is written as

$$\begin{aligned}
\frac{ds_\epsilon}{dt} &= -(\mu_1 + a)s_\epsilon + \beta n_{2,\epsilon} + \frac{1}{\epsilon}\left(-\lambda s_\epsilon i_\epsilon + \gamma i_\epsilon\right), \quad s_\epsilon(0) = s^0, \\
\frac{di_\epsilon}{dt} &= -(\mu_1^* + a)i_\epsilon + \frac{1}{\epsilon}\left(\lambda s_\epsilon i_\epsilon - \gamma i_\epsilon\right), \quad i_\epsilon(0) = i^0, \\
\frac{dn_{2,\epsilon}}{dt} &= -\mu_2 n_{2,\epsilon} + a\left(s_\epsilon + i_\epsilon\right), \quad n_{2,\epsilon}(0) = n_2^0,
\end{aligned} \tag{4.1}$$

where we have considered different mortality rates, μ_1 for the susceptibles, μ_1^* for the infectives and μ_2 for the adults, with $\mu_1^* > \mu_1$; a is the rate of moving from the juvenile to the adult class, β is the birth rate, γ is the recovery rate and λ is the transmission rate.
We assume that the disease is not hereditary; that is, newborns are always susceptible and also that the disease does not persist to adulthood, [19]. We further assume that the constants μ_1, μ_1^*, μ_2, β, a, λ, γ are positive.
Terms that are multiplied by $(1/\epsilon)$ are fast terms; they describe epidemiological processes. The other terms are slow terms and they describe demographic processes.

4.1.1 Well-posedness of the model

Definition 4.1.1

Let Y be a Banach space. A positive cone Y_+ is a nonempty closed subset of Y with the properties

1. $\mathbb{R}_+ Y_+ \subset Y_+$,
2. $Y_+ + Y_+ \subset Y_+$,
3. $Y_+ \cap (-Y_+) = \{0\}$.

Note that it is possible to introduce a partial order in Y by stating that $x \leq y$ if and only if $y - x \in Y_+$. In particular, in $\mathbb{R}^n$ we can consider $\mathbb{R}^n_+$ as the positive cone inducing the following partial order from $\mathbb{R}^n$. For all $x, y \in \mathbb{R}^n$, we have $x \leq y \Longleftrightarrow x_i \leq y_i$ for any $i = 1, \cdots, n$. We shall use this in the sequel of this chapter.

Theorem 4.1.1

Let $\mathbb{R}^n_+ = [0, +\infty)^n$ be the cone of nonnegative vectors in $\mathbb{R}^n$. Let $F : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n$ be locally Lipschitz and satisfy $F_j(t, x) \geq 0$ whenever $t \geq 0$, $x \in \mathbb{R}^n_+$, $x_j = 0$. Then, for every $x_0 \in \mathbb{R}^n_+$, there exists a unique solution of $x' = F(t, x)$, $x(0) = x_0$, with values in $\mathbb{R}^n_+$, which is defined on some interval $[0, b)$, $b > 0$. If $b < \infty$, then

$$\limsup_{t \to b} \sum_{j=1}^{n} x_j = \infty.$$

For the proof of Theorem 4.1.1, we refer the readers to the book of Thieme, [13].

Let $(s_\epsilon, i_\epsilon, n_{2,\epsilon})$ be a vector in $\mathbb{R}^3$ and

$$\begin{aligned}
F_1(s_\epsilon, i_\epsilon, n_{2,\epsilon}) &\equiv -(\mu_1 + a)s_\epsilon + \beta n_{2,\epsilon} + \frac{1}{\epsilon}(-\lambda s_\epsilon i_\epsilon + \gamma i_\epsilon), \\
F_2(s_\epsilon, i_\epsilon, n_{2,\epsilon}) &\equiv -(\mu_1^* + a)i_\epsilon + \frac{1}{\epsilon}(\lambda s_\epsilon i_\epsilon - \gamma i_\epsilon), \\
F_3(s_\epsilon, i_\epsilon, n_{2,\epsilon}) &\equiv -\mu_2 n_{2,\epsilon} + a(s_\epsilon + i_\epsilon).
\end{aligned}$$

If $s_\epsilon = 0$, $F_1(s_\epsilon, i_\epsilon, n_{2,\epsilon}) \geq 0$, if $i_\epsilon = 0$, $F_2(s_\epsilon, i_\epsilon, n_{2,\epsilon}) \geq 0$, and if $n_{2,\epsilon} = 0$, $F_3(s_\epsilon, i_\epsilon, n_{2,\epsilon}) \geq 0$. As $s_\epsilon, i_\epsilon, n_{2,\epsilon}$ are nonnegative and $n_\epsilon = s_\epsilon + i_\epsilon + n_{2,\epsilon}$, then, by Theorem 4.1.1, Equation (4.1) has a unique solution $(s_\epsilon(t), i_\epsilon(t), n_{2,\epsilon}(t))$ defined in $\mathbb{R}^+_3$ and satisfying $s_\epsilon(0) = s^0$, $i_\epsilon(0) = i^0$, $n_{2,\epsilon}(0) = n_2^0$ and $s_\epsilon(t)$, $i_\epsilon(t)$, $n_{2,\epsilon}(t) \geq 0$, for all $t \geq 0$.

4.1.2 Aggregated model

If we perform variables aggregation $n_{1,\epsilon} = s_\epsilon + i_\epsilon$ by adding the first two equations in (4.1), we obtain singularly perturbed systems, written either as

$$\begin{aligned}
\epsilon\frac{ds_\epsilon}{dt} &= -\epsilon(\mu_1 + a)s_\epsilon + \epsilon\beta n_{2,\epsilon} + (n_{1,\epsilon} - s_\epsilon)(\gamma - \lambda s_\epsilon), \quad s_\epsilon(0) = s^0,\\
\frac{dn_{1,\epsilon}}{dt} &= -(\mu_1^* + a)n_{1,\epsilon} + (\mu_1^* - \mu_1)s_\epsilon + \beta n_{2,\epsilon}, \quad n_{1,\epsilon}(0) = n_1^0,\\
\frac{dn_{2,\epsilon}}{dt} &= -\mu_2 n_{2,\epsilon} + a n_{1,\epsilon}, \quad n_{2,\epsilon}(0) = n_2^0,
\end{aligned} \tag{4.2}$$

or

$$\begin{aligned}
\epsilon\frac{di_\epsilon}{dt} &= -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(n_{1,\epsilon} - i_\epsilon) - \gamma)i_\epsilon, \quad i_\epsilon(0) = i^0,\\
\frac{dn_{1,\epsilon}}{dt} &= -(\mu_1 + a)n_{1,\epsilon} - (\mu_1^* - \mu_1)i_\epsilon + \beta n_{2,\epsilon}, \quad n_{1,\epsilon}(0) = n_1^0,\\
\frac{dn_{2,\epsilon}}{dt} &= -\mu_2 n_{2,\epsilon} + a n_{1,\epsilon}, \quad n_{2,\epsilon}(0) = n_2^0.
\end{aligned} \tag{4.3}$$

Note that Systems (4.3) and (4.2) are equivalent because they are derived from the same system. Therefore, since $(\mu_1^* - \mu_1)s \geq 0$ and $-(\mu_1^* - \mu_1)i \leq 0$, by considering the last two equations we have, respectively, from (4.2)

$$\begin{aligned}
\begin{pmatrix} n_{1,\epsilon}(t) \\ n_{2,\epsilon}(t) \end{pmatrix} &= e^{\mathcal{A}t}\begin{pmatrix} n_1^0 \\ n_2^0 \end{pmatrix} - \begin{pmatrix} \mu_1^* - \mu_1 \\ 0 \end{pmatrix}\int_0^t e^{\mathcal{A}(t-z)} i_\epsilon(z)dz,\\
&\leq e^{\mathcal{A}t}\begin{pmatrix} n_1^0 \\ n_2^0 \end{pmatrix}
\end{aligned} \tag{4.4}$$

and from (4.3)

$$\begin{aligned}
\begin{pmatrix} n_{1,\epsilon}(t) \\ n_{2,\epsilon}(t) \end{pmatrix} &= e^{\mathcal{A}^*t}\begin{pmatrix} n_1^0 \\ n_2^0 \end{pmatrix} + \begin{pmatrix} \mu_1^* - \mu_1 \\ 0 \end{pmatrix}\int_0^t e^{\mathcal{A}^*(t-z)} s_\epsilon(z)dz,\\
&\geq e^{\mathcal{A}^*t}\begin{pmatrix} n_1^0 \\ n_2^0 \end{pmatrix},
\end{aligned} \tag{4.5}$$

where

$$\mathcal{A} = \begin{pmatrix} -(\mu_1 + a) & \beta \\ a & -\mu_2 \end{pmatrix} \quad \text{and} \quad \mathcal{A}^* = \begin{pmatrix} -(\mu_1^* + a) & \beta \\ a & -\mu_2 \end{pmatrix}.$$

Hence, since both systems are equivalent, the following inequalities hold.

$$e^{\mathcal{A}^* t}\begin{pmatrix} n_1^0 \\ n_2^0 \end{pmatrix} \leq \begin{pmatrix} n_{1,\epsilon}(t) \\ n_{2,\epsilon}(t) \end{pmatrix} \leq e^{\mathcal{A}t}\begin{pmatrix} n_1^0 \\ n_2^0 \end{pmatrix}. \tag{4.6}$$

Inequalities (4.6) show that the infected population develops slower than healthy population but faster than the healthy population with the disease specific mortality rate, [19], and imply that neither i_ϵ nor s_ϵ can blow up in finite time because $n_{1,\epsilon} = s_\epsilon + i_\epsilon$ and $i_\epsilon, s_\epsilon \geq 0$.
It follows that solutions $(s_\epsilon, i_\epsilon, n_{2,\epsilon})$ of (4.1) with initial conditions $s^0, i^0, n_2^0 \geq 0$ (or $(s_\epsilon, n_{1,\epsilon}, n_{2,\epsilon})$ of (4.2) with initial conditions $n_1^0 \geq 0, n_2^0 \geq 0, 0 \leq s^0 \leq n_1^0$) (or $(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon})$ of (4.3) with initial conditions $n_1^0 \geq 0, n_2^0 \geq 0, 0 \leq i^0 \leq n_1^0$) exist globally in time [19]. In particular, the sets

$$\mathcal{U}_1 = \{(s_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}) \in \mathbb{R}^3 : s_\epsilon \leq n_{1,\epsilon}\}$$

and

$$\mathcal{U}_2 = \{(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}) \in \mathbb{R}^3 : i_\epsilon \leq n_{1,\epsilon}\}$$

are invariant under the flows generated respectively by (4.2) and (4.3).

4.1.3 Analysis of quasi-steady states

Since both systems have the same limit equation as $\epsilon \to 0$, we are going to consider only one system, say (4.2), and draw from it all the conclusions from the application of Tikhonov theorem. The quasi-steady states are obtained by letting $\epsilon = 0$ in (4.2) and solving equation

$$0 = (n_1 - s)(\gamma - \lambda s). \tag{4.7}$$

Then $\bar{s}_1 = n_1$ and $\bar{s}_2 = \nu$, where $\nu = \gamma/\lambda$, are two quasi-steady states of (4.2). If we substitute the unknown s in the second equation of (4.2) by the known quasi-steady state $\bar{s}$, we obtain the limit equation

$$\begin{aligned} \frac{d\bar{n}_1}{dt} &= -(\mu_1^* + a)\bar{n}_1 + (\mu_1^* - \mu_1)\bar{s} + \beta\bar{n}_2, \quad \bar{n}_1(0) = n_1^0, \\ \frac{d\bar{n}_2}{dt} &= -\mu_2\bar{n}_2 + a\bar{n}_1, \quad \bar{n}_2(0) = n_2^0. \end{aligned} \tag{4.8}$$

The auxiliary equation associated to (4.2) is given by

$$\frac{d\hat{s}}{d\tau} = (n_1^0 - \hat{s})(\gamma - \lambda\hat{s})\hat{i}, \;\; \hat{s}(0) = s^0. \tag{4.9}$$

Then, calculating the derivative of $\Psi(s, n_1, t) = (n_1 - s)(\gamma - \lambda s)$ with respect to s gives

$$\frac{\partial\Psi}{\partial s} = -\lambda n_1 - \gamma + 2\lambda s.$$

Hence,

$$\left.\frac{\partial \Psi}{\partial s}\right|_{s=n_1} = \lambda n_1 - \gamma < 0 \Longleftrightarrow n_1 < \nu \text{ and } \left.\frac{\partial \Psi}{\partial s}\right|_{s=\nu} = -\lambda n_1 + \gamma < 0 \Longleftrightarrow n_1 > \nu.$$

It follows that the quasi-steady state $\bar{s}_1 = n_1$ is stable if $n_1 < \nu$ and the quasi-steady state $\bar{s}_2 = \nu$ is stable if $n_1 > \nu$.
Note that if $n_1 = \nu$, then both quasi-steady states coincide; this violates assumption A_2 of Tikhonov theorem. So, it cannot apply under these conditions, and we have to impose some additional conditions.
In order to analyze the limit behaviour of the solutions of (4.2), we consider two cases. The first case consists in determining invariant subsets of $I_{n_1} \times I_{n_2}$ which do not contain the point of intersection of the quasi-steady states and in which the quasi-steady states are isolated.Then we apply Tikhonov theorem inside these sets.
The second case consists in analysing the behaviour of the solutions of (4.2) when passing close to the point of intersection from the stable branch of one quasi-steady state to the stable branch of the other, [19].

4.2 Direct application of the Tikhonov theorem

Let us define the sets

$$\Pi_- = \{(n_1, n_2) \in I_{n_1} \times I_{n_2} : n_1 < \nu\} \text{ and } \Pi_+ = \{(n_1, n_2) \in I_{n_1} \times I_{n_2} : n_1 > \nu\},$$

where

$$I_{n_1} = (0, \infty) \text{ and } I_{n_2} = (0, \infty).$$

Then we investigate two cases.

4.2.1 The case of stable population

The case of stable population refers to the case where $(0,0)$ is an asymptotically stable equilibrium for $(e^{tA})_{t\geq 0}$.
If we consider the quasi-steady state $\bar{s}_1 = n_1$, then the limit system (4.8) becomes

$$\begin{aligned} \frac{d\bar{n}_1}{dt} &= -(\mu_1 + a)\bar{n}_1 + \beta \bar{n}_2, \; \bar{n}_1(0) = n_1^0, \\ \frac{d\bar{n}_2}{dt} &= -\mu_2 \bar{n}_2 + a\bar{n}_1, \; \bar{n}_2(0) = n_2^0. \end{aligned} \tag{4.10}$$

Note that $(0,0)$ is the only equilibrium point of (4.10). Since the trace ($\text{tr}(\mathcal{A}) = -\mu_1 - \mu_2 - a < 0$) of the matrix $\mathcal{A}$ is negative, we have stability of $(0,0)$ if and only if

$$\frac{\mu_2}{a} \geq \frac{\beta}{\mu_1 + a}. \tag{4.11}$$

Let us investigate the direction field of the limit system (4.10) under assumption (4.11). System (4.10) has two isoclines

$$n_2 = \frac{\mu_1 + a}{\beta} n_1 \text{ and } n_2 = \frac{a}{\mu_2} n_1.$$

It follows from the phase portrait analysis that above the isocline $n_2 = n_1(\mu_1 + a)/\beta$, $n_1' > 0$ and $n_2' < 0$, below the isocline $n_2 = n_1(\mu_1 + a)/\beta$ and above the isocline $n_2 = n_1(a/\mu_2)$, $n_1' < 0$ and $n_2' < 0$ and below the isocline $n_2 = n_1(a/\mu_2)$, $n_1' < 0$ and $n_2' > 0$, provided $0 < n_1 < \nu$.
If we consider the line $n_1 = \bar{\nu} < \nu$ then, along this line below the isocline $n_2 = (a/\mu_2)n_1$, $n_1' > 0$ and above the isocline $n_2 = (a/\mu_2)/n_1$, $n_1' < 0$. Also, along the line $n_2 = \bar{\nu}(\mu_1 + a)/\beta$, $n_2' < 0$. Last but not least, along the line $n_1 = 0$, $n_2' < 0$ and along the line $n_2 = 0$, $n_1' < 0$. It follows that any solution starting inside the set

$$\mathcal{V}_1 = \{(n_1, n_2) \in I_{n_1} \times I_{n_2} : n_1 \leq \bar{\nu}, n_2 \leq \bar{\nu}(\mu_1 + a)/\beta\}$$

is bounded and will stay there for ever. Hence, the set $\mathcal{V}_1$ is invariant under the flow generated by $(e^{\mathcal{A}t})_{t\geq 0}$. If $(0,0)$, which is the only stable equilibrium of (4.10), is asymptotically stable (i.e if we have strict inequality in (4.11)), then any trajectory starting below the isocline $n_2 = (a/\mu_2)n_1$ will increase, reaching its maximum at the isocline, then it converges to $(0,0)$ and any trajectory starting above the isocline $n_2 = (a/\mu_2)n_1$ will decrease to $(0,0)$. Clearly, Int $\bar{\mathcal{V}_1} \subset \Pi_-$ and, in particular, we have

$$\begin{pmatrix} \bar{n}_1(t) \\ \bar{n}_2(t) \end{pmatrix} \in \text{Int } \bar{\mathcal{V}_1}.$$

The solution of the auxiliary equation (4.9) when $n_1 < \nu$ is given by

$$\hat{s}(\tau) = \frac{n_1^0(\nu - s^0) + \nu(s^0 - n_1^0)e^{\lambda(n_1^0 - \nu)\tau}}{\nu - s^0 + (s^0 - n_1^0)e^{\lambda(n_1^0-\nu)}\tau}. \tag{4.12}$$

Thus, passing to the limit in (4.12) as $\tau \to \infty$ gives

$$\lim_{\tau\to\infty} \hat{s}(\tau) = n_1^0 \equiv \bar{s}_1(0).$$

This shows that any initial condition s^0 of (4.9) belongs to the basin of attraction of the stable root of Equation (4.7) when $n_1 < \nu$.
Therefore, all assumptions of the Tikhonov theorem, on any finite interval time $[0,T]$, are satisfied. It follows that, for $0 \leq n_1^0 < \nu$ and $n_2^0 < \nu(\mu_1 + a)$, the solution $(s_\epsilon(t), n_{1,\epsilon}(t), n_{2,\epsilon}(t))$ of (4.2) exists on $[0,T]$ and satisfies

$$\begin{aligned} \lim_{\epsilon\to 0} s_\epsilon(t) &= \bar{n}_1(t), \quad 0 < t \leq T, \\ \lim_{\epsilon\to 0} n_{1,\epsilon}(t) &= \bar{n}_1(t), \quad 0 \leq t \leq T, \\ \lim_{\epsilon\to 0} n_{2,\epsilon}(t) &= \bar{n}_2(t), \quad 0 \leq t \leq T. \end{aligned} \tag{4.13}$$

Note that, since $n_{1,\epsilon} = s_\epsilon + i_\epsilon$, we have

$$\lim_{\epsilon\to 0} i_\epsilon(t) = \lim_{\epsilon\to 0} n_{1,\epsilon}(t) - \lim_{\epsilon\to 0} s_\epsilon(t) = \bar{n}_1(t) - \bar{n}_1(t) = 0, \quad 0 < t \leq T \tag{4.14}$$

Furthermore, adding up in the first equation of (4.13) the initial layer term $\hat{s}(t/\epsilon) - n_1^0$, we obtain

$$\lim_{\epsilon\to 0}\left(s_\epsilon(t) - \bar{n}_1(t) - \frac{n_1^0(\nu - s^0) + \nu(s^0 - n_1^0)e^{\lambda(n_1^0-\nu)\frac{t}{\epsilon}}}{\nu - s^0 + (s^0 - n_1^0)e^{\lambda(n_1^0-\nu)\frac{t}{\epsilon}}} + n_1^0 \right) = 0, \tag{4.15}$$

uniformly for $0 \leq t \leq T$.

4.2.2 The case of unstable population

The case of unstable population refers to the case where $(0,0)$ is unstable equlibrium for $(e^{tA})_{t\geq 0}$. Now, if we consider the quasi-steady state $\bar{s}_2 = \nu$, then the limit equation (4.8) becomes

$$\begin{aligned} \frac{d\bar{n}_1}{dt} &= -(\mu_1^* + a)\bar{n}_1 + \beta\bar{n}_2 + (\mu_1^* - \mu_1)\nu,\ \bar{n}_1(0) = n_1^0, \\ \frac{d\bar{n}_2}{dt} &= -\mu_2\bar{n}_2 + a\bar{n}_1,\ \bar{n}_2(0) = n_2^0. \end{aligned} \tag{4.16}$$

Let us consider the vector field along the line $\bar{n}_1 = \nu + \delta\beta/(\mu_1^* + a)$. It turns out that along this line $n_1' > 0$, provided

$$\bar{n}_2 \geq (\mu_1 + a)\frac{\nu}{\beta} + \delta.$$

Along the line $\bar{n}_2 = \nu(\mu_1 + a)/\beta + \delta$, $n_2' > 0$ provided

$$\bar{n}_1 > \frac{\mu_2}{a}\left((\mu_1 + a)\frac{\nu}{\beta} + \delta\right).$$

It follows that any solution starting inside the set

$$\mathcal{V}_2 = \left\{(n_1, n_2) \in I_{n_1} \times I_{n_2} : n_1 \geq \nu + \delta\frac{\beta}{\mu_1^* + a}, n_2 \geq (\mu_1 + a)\frac{\nu}{\beta} + \delta\right\}.$$

will stay there for all time $t \in [0, T]$. Hence, the set $\mathcal{V}_2$ is invariant under the flow generated by (4.16). We have Int $\bar{\mathcal{V}}_2 \subset \Pi_+$ and in particular

$$\begin{pmatrix} \bar{n}_1(t) \\ \bar{n}_2(t) \end{pmatrix} \in \text{Int } \bar{\mathcal{V}}_2.$$

The solution of the auxiliary equation (4.9) when $n_1 > \nu$ is given by

$$\hat{s}(\tau) = \frac{n_1^0(\nu - s^0)e^{-\lambda(n_1^0-\nu)\tau} + \nu(s^0 - n_1^0)}{(\nu - s^0)e^{-\lambda(n_1^0-\nu)\tau} + s^0 - n_1^0}. \tag{4.17}$$

Then passing to the limit in (4.17) as $\tau \to \infty$ gives

$$\lim_{\tau\to\infty} \hat{s}(\tau) = \nu \equiv \bar{s}_2(0).$$

This shows that any initial condition s^0 of (4.9) belongs to the basin of attraction of the stable root of Equation (4.7).
Hence, all the assumptions of Tikhonov theorem, on any finite time interval $[0, T]$, are satisfied. It follows that, for $n_1 \geq \nu + \delta\beta/(\mu_1^* + a)$ and $n_2 \geq \nu(\mu_1 + a)/\beta + \delta$, the solutions $(s_\epsilon(t), n_{1,\epsilon}(t), n_{2,\epsilon}(t))$ of Equation (4.2) exists on $[0, T]$ and satisfies

$$\begin{aligned}
\lim_{\epsilon\to 0} s_\epsilon(t) &= \nu, \quad 0 < t \leq T,\\
\lim_{\epsilon\to 0} n_{1,\epsilon}(t) &= \bar{n}_1(t), \quad 0 \leq t \leq T,\\
\lim_{\epsilon\to 0} n_{2,\epsilon}(t) &= \bar{n}_2(t), \quad 0 \leq t \leq T.
\end{aligned} \tag{4.18}$$

Note that, since $n_{1,\epsilon} = s_\epsilon + i_\epsilon$, we have

$$\lim_{\epsilon\to 0} i_\epsilon(t) = \lim_{\epsilon\to 0} n_{1,\epsilon}(t) - \lim_{\epsilon\to 0} s_\epsilon(t) = \bar{n}_1(t) - \nu, \quad 0 < t \leq T. \tag{4.19}$$

Furthermore, adding up in the first equation of (4.18) the initial layer term $\hat{s}\,(t/\epsilon) - \nu$, we obtain

$$\lim_{\epsilon\to 0} \left(s_\epsilon(t) - \frac{n_1^0(\nu - s^0)e^{-\lambda(n_1^0-\nu)\frac{t}{\epsilon}} + \nu(s^0 - n_1^0)}{(\nu - s^0)e^{-\lambda(n_1^0-\nu)\frac{t}{\epsilon}} + s^0 - n_1^0} \right) = 0, \tag{4.20}$$

uniformly for $0 \leq t \leq T$.

4.2.3 Comments on Stable-Unstable Case

An interesting dynamics occurs when $(0, 0)$ is an asymptotically stable equilibrium for $(e^{\mathcal{A}^* t})_{t\geq 0}$ and unstable for $(e^{\mathcal{A}t})_{t\geq 0}$, [19]. In this case the equilibrium point of (4.16)

$$\mathbf{n}^* = \begin{pmatrix} n_1^* \\ n_2^* \end{pmatrix} = \begin{pmatrix} \frac{\nu\mu_2(\mu_1^*-\mu_1)}{\mu_2(\mu_1^*+a)-\beta a} \\ \frac{\nu a(\mu_1^*-\mu_1)}{\mu_2(\mu_1^*+a)-\beta a} \end{pmatrix}$$

is in the first octant if and only if

$$\mu_2(\mu_1^* + a) - \beta a > 0 \Longleftrightarrow \frac{\mu_2}{a} > \frac{\beta}{\mu_1^* + a}. \tag{4.21}$$

In other words, $\mathbf{n}^*$ is in the first octant if and only if $(0, 0)$ is an asymptotically stable equilibrium for $(e^{\mathcal{A}^*})_{t\geq 0}$, [19]. Note that, in order to ensure that $\mathbf{n}^*$ is inside the set $\mathcal{V}_2$, we should consider the unstable/stable case of the original system, [19]. In this connection, we should assume that the inequalities

$$\frac{\mu_2}{a} < \frac{\beta}{\mu_1 + a}, \; \frac{\mu_2}{a} > \frac{\beta}{\mu_1^* + a} \tag{4.22}$$

hold. Note that Systems (4.1) and (4.2) are related by the following linear change of variables

$$\begin{pmatrix} s \\ n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s \\ i \\ n_2 \end{pmatrix}. \tag{4.23}$$

The linearizations of both systems are given by similar matrices and therefore they have the same eigenvalues and their eigenvectors are related by the same linear transformation, [19]. Clearly, $(0,0,0)$ is an equilibrium point for both systems. The other equilibrium of (4.1) is obtained by solving the system

$$\begin{aligned} 0 &= -(\mu_1 + a)s_\epsilon + \beta n_{2,\epsilon} + \frac{1}{\epsilon}\left(-\lambda s_\epsilon i_\epsilon + \gamma i_\epsilon\right), \\ 0 &= -(\mu_1^* + a)i_\epsilon + \frac{1}{\epsilon}\left(\lambda s_\epsilon i_\epsilon - \gamma i_\epsilon\right), \\ 0 &= -\mu_2 n_{2,\epsilon} + a\left(s_\epsilon + i_\epsilon\right), \end{aligned} \tag{4.24}$$

and it is given by

$$\begin{pmatrix} s_\epsilon^* \\ i_\epsilon^* \\ n_{2,\epsilon}^* \end{pmatrix} = \begin{pmatrix} \nu + \frac{\epsilon}{\gamma}(\mu_1^* + a) \\ -\frac{\mu_2(\mu_1 + a) - \beta a}{\mu_2(\mu_1^* + a) - \beta a} s_\epsilon^* \\ \frac{a(\mu_1^* - \mu_1)}{\mu_2(\mu_1^* + a) - \beta a} s_\epsilon^* \end{pmatrix}. \tag{4.25}$$

By the linear transformation (4.23), the equilibrium of System (4.2) is given by

$$\begin{pmatrix} s_\epsilon^* \\ n_{1,\epsilon}^* \\ n_{2,\epsilon}^* \end{pmatrix} = \begin{pmatrix} \nu + \frac{\epsilon}{\gamma}(\mu_1^* + a) \\ \frac{\mu_2(\mu_1^* - \mu_1)}{\mu_2(\mu_1^* + a) - \beta a} s_\epsilon^* \\ \frac{a(\mu_1^* - \mu_1)}{\mu_2(\mu_1^* + a) - \beta a} s_\epsilon^* \end{pmatrix}. \tag{4.26}$$

Hence, as $\epsilon \to 0$ in (4.26), the equilibrium point $(s_\epsilon^*, n_{1,\epsilon}^*, n_{2,\epsilon}^*)$ converges to the point (ν, n_1^*, n_2^*), where ν is the stable quasi-steady state of (4.2) when $n_1 > \nu$ and (n_1^*, n_2^*) is the equilibriun point of the limit system (4.16). This shows that (4.2) possesses a biologically meaningful equilibrium. Let us now analyze the stability of these equilibria. It turns out that since both systems are similar, it is much easier to work with System (4.1) and then deduce from it the behaviour around the equilibrium of (4.2). The Jacobian associated to System (4.1) is given by

$$J|_{(s_\epsilon, i_\epsilon, n_{2,\epsilon})} = \begin{pmatrix} -(\mu_1 + a) - \frac{\lambda}{\epsilon} i_\epsilon & \frac{\gamma}{\epsilon} - \frac{\lambda}{\epsilon} s_\epsilon & \beta \\ \frac{\lambda}{\epsilon} i_\epsilon & -(\mu_1^* + a) + \frac{\lambda}{\epsilon} s_\epsilon - \frac{\gamma}{\epsilon} & 0 \\ a & a & -\mu_2 \end{pmatrix}. \tag{4.27}$$

At the equilibrium point $(0, 0, 0)$, the Jacobian is given by

$$J|_{(0,0,0)} = \begin{pmatrix} -(\mu_1 + a) & \frac{\gamma}{\epsilon} & \beta \\ 0 & -(\mu_1^* + a) - \frac{\gamma}{\epsilon} & 0 \\ a & a & -\mu_2 \end{pmatrix}.$$

The characteristic equation associated to the Jacobian matrix at $(0, 0, 0)$ is given by

$$(\omega + (\mu_1^* + a) + \gamma\epsilon^{-1})(\omega^2 + \omega((\mu_1 + a) + \mu_2) + \mu_2(\mu_1 + a) - \beta a) = 0. \tag{4.28}$$

Thus, the eigenvalues of the Jacobian matrix at $(0, 0, 0)$ are

$$\begin{aligned} \omega_1 &= -(\mu_1^* + a) - \gamma\epsilon^{-1} < 0, \\ \omega_2 &= -((\mu_1 + a) + \mu_2) + \sqrt{((\mu_1 + a) - \mu_2)^2 + 4\beta a} > 0, \\ \omega_2 &= -((\mu_1 + a) + \mu_2) - \sqrt{((\mu_1 + a) - \mu_2)^2 + 4\beta a} < 0. \end{aligned} \tag{4.29}$$

Then the eigenvectors corresponding to the eigenvalues ω_1, ω_2, ω_3, are, respectively, given by

$$\mathbf{U}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{U}_2 = \begin{pmatrix} u_1^\pm \\ 0 \\ u_3^\pm \end{pmatrix}, \ \mathbf{U}_3 = \begin{pmatrix} u_1^\pm \\ 0 \\ u_3^\mp \end{pmatrix}.$$

Note that $\mathbf{U}_1$ is parallel to the i-axis, $\mathbf{U}_2$ has entries of the same sign and $\mathbf{U}_3$ has entries of opposite sign.

It follows that, by the linear transformation (4.23), the eigenvectors corresponding to the eigenvalues of the Jacobian matrix of (4.2) at the $(0, 0, 0)$ are given by

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{V}_2 = \begin{pmatrix} u_1^\pm \\ u_1^\pm \\ u_3^\pm \end{pmatrix}, \ \mathbf{V}_3 = \begin{pmatrix} u_1^\pm \\ u_1^\pm \\ u_3^\mp \end{pmatrix}.$$

Thus, $\mathbf{V}_2$ and $\mathbf{V}_3$ lie on the manifold $s = n_1$. Because the stable direction is outside the positive octant, this suggests that $(0, 0, 0)$ is repelling in the positive octant, [19].

Let us now consider the other equilibrium of (4.1). Then at the point $(s_\epsilon^*, i_\epsilon^*, n_{2,\epsilon}^*)$, the Jacobian is given by

$$J|_{(s_\epsilon^*, i_\epsilon^*, n_{2,\epsilon}^*)} = \begin{pmatrix} -(\mu_1 + a) - \frac{\lambda}{\epsilon} i_\epsilon^* & -(\mu_1^* + a) & \beta \\ \frac{\lambda}{\epsilon} i_\epsilon^* & 0 & 0 \\ a & a & -\mu_2 \end{pmatrix}.$$

The characteristic equation associated to the Jacobian matrix at $(s_\epsilon^*, i_\epsilon^*, n_{2,\epsilon}^*)$ is given by

$$\omega^3 + A\omega^2 + B\omega + C = 0, \tag{4.30}$$

where

$$\begin{aligned}
A &= \mu_2 + \mu_1 + a + \lambda\epsilon^{-1} i_\epsilon^*, \\
B &= \mu_2(\mu_1 + a) + \lambda\epsilon^{-1} i_\epsilon^*(\mu_2 + \mu_1^* + a) - \beta a, \\
C &= \lambda\epsilon^{-1} i_\epsilon^*(\mu_2(\mu_1^* + a) - \beta a).
\end{aligned}$$

Clearly, $A, B > 0$ and $C > 0$ by Inequality (4.22). AB is of order ϵ^{-2} whereas, C is of order ϵ^{-1} thus, $AB - C > 0$ for all small $\epsilon > 0$, [19].
It follows, by the Hurwitz criterion, that the equilibrium $(s_\epsilon^*, i_\epsilon^*, n_{2,\epsilon}^*)$ is asymptotically stable equilibrium for small $\epsilon > 0$ and so is, by the similarity, the equilibrium point $(s_\epsilon^*, n_{1,\epsilon}^*, n_{2,\epsilon}^*)$ for the system (4.2).
Hence, when Inequalities (4.22) hold and $0 \leq s^0 \leq n_1^0$, $n_1^0 \geq \nu$, $n_2^0 > \nu(\mu_1 + a)/b$ then the solution $(s_\epsilon(t), n_{1,\epsilon}(t), n_{2,\epsilon}(t))$ to (4.2) converges to $(\nu, \bar{n}_1(t), \bar{n}_2(t))$, where ν is the stable quasi-steady state of (4.2) when $n_1^0 > \nu$ and $(\bar{n}_1(t), \bar{n}_2(t))$ is the solution of the limit equation (4.16).
In the next subsection, we present numerical simulations describing the behaviour of the solutions of (4.2) when $n_1 < \nu$ and when $n_1 > \nu$.

4.2.4 Numerical simulations

Here we present numerical simulations of the application of Tikhonov theorem when the quasi-steady states intersect.
For $n_1^0 < \nu$, the parameters' values used are $\mu_1 = 0.043$, $\mu_2 = 0.029$, $a = 0.05$, $\beta = 0.046$, $\mu_1^* = 0.075$, $\gamma = 0.44$, $\lambda = 0.002$ (so that $\nu = 220$), $\epsilon = 0.1$ & 0.01 and the initial conditions are $s^0 = 110$, $n_1^0 = 190$ and $n_2^0 = 443$.

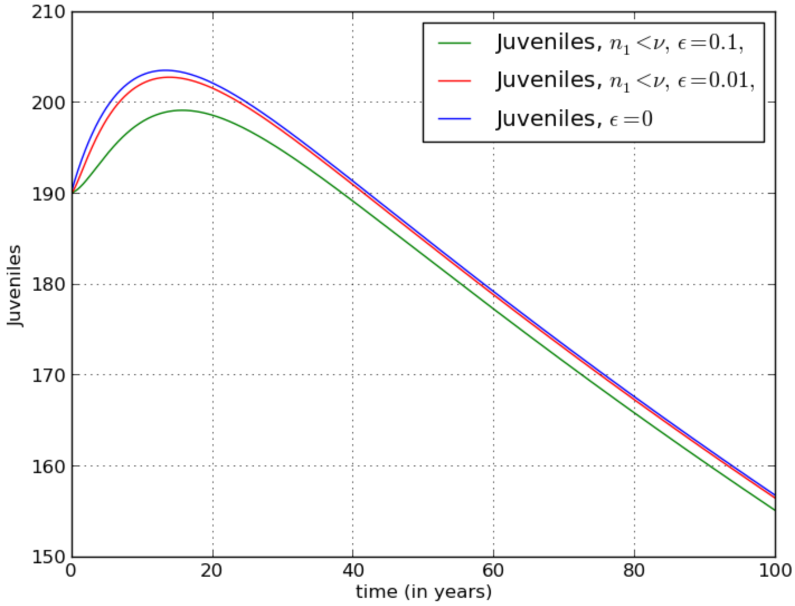


Figure 4.1: The solution $\bar{n}_1(t)$ of the limit equation (4.10) uniformly attracting solutions $n_{1,\epsilon}(t)$ of (4.2) when $n_1 < \nu$.

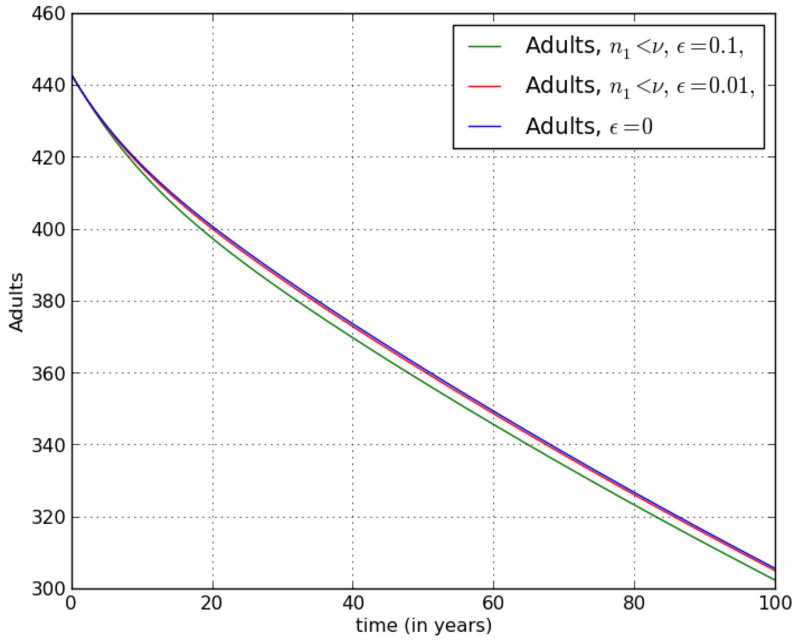


Figure 4.2: The solution $\bar{n}_2(t)$ of the limit equation (4.10) uniformly attracting solutions $n_{2,\epsilon}(t)$ of (4.2) when $n_1 < \nu$.

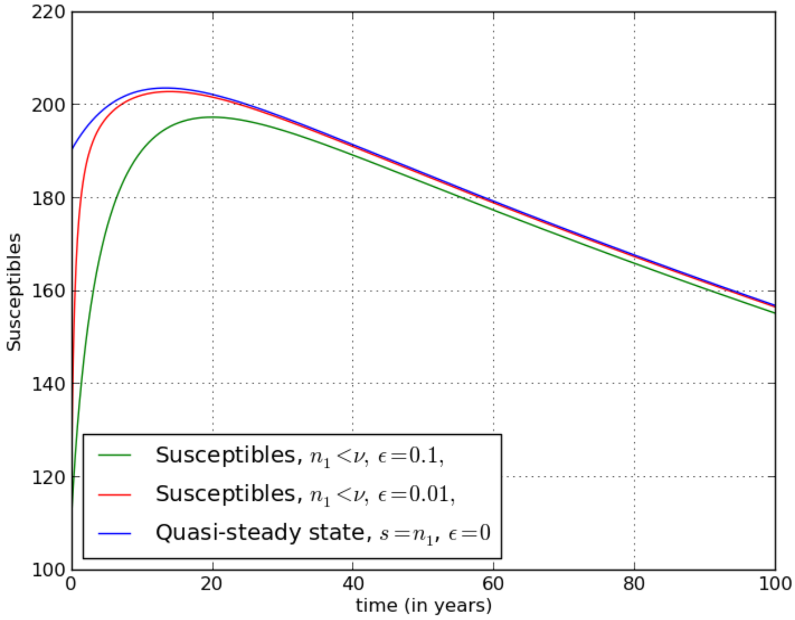


Figure 4.3: The quasi-steady state $\bar{s}_1 = \bar{n}_1(t)$ attracting the susceptibles curves $s_\epsilon(t)$ of (4.2) when $n_1 < \nu$ displaying the initial layer effect.

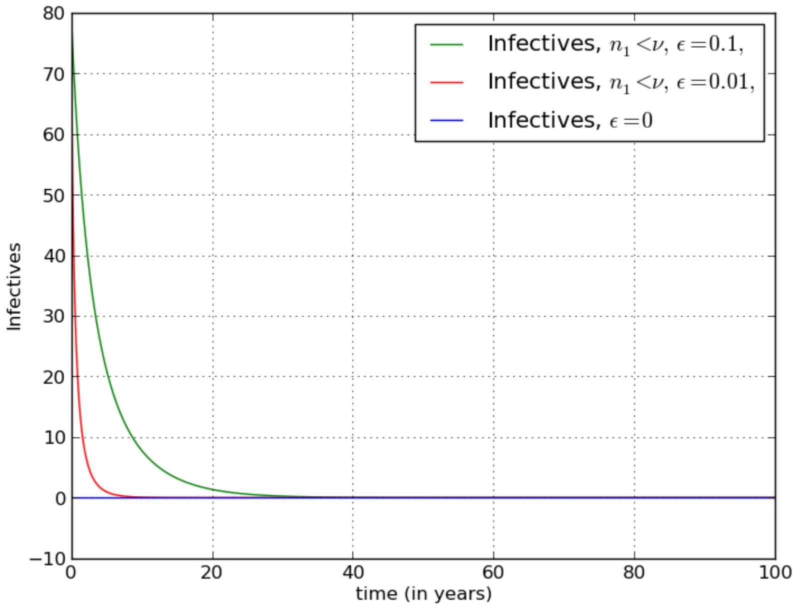


Figure 4.4: The infectives curve converging to zero when $n_1 < \nu$ given by (4.14) with initial layer effect.

For $n_1 > \nu$, the parameters' values used are $\mu_1 = 0.043$, $\mu_2 = 0.029$, $a = 0.05$, $\beta = 0.057$, $\mu_1^* = 0.046$, $\gamma = 0.44$, $\lambda = 0.002$ (so that $\nu = 220$), $\epsilon = 0.1$ & 0.01 and the initial conditions are $s^0 = 150$, $n_1^0 = 270$ and $n_2^0 = 490$.

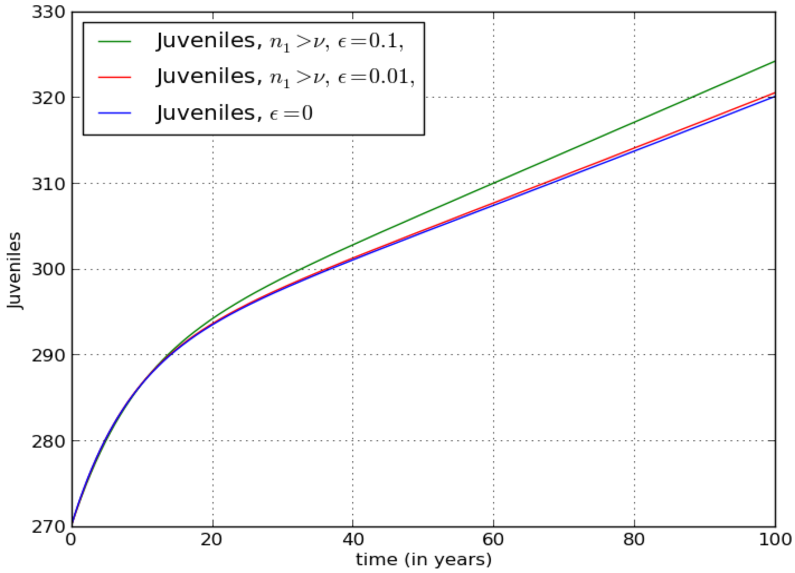


Figure 4.5: The solution $\tilde{n}_1(t)$ of the limit equation (4.16) uniformly attracting solutions $n_{1,\epsilon}(t)$ of (4.2) when $n_1 > \nu$.

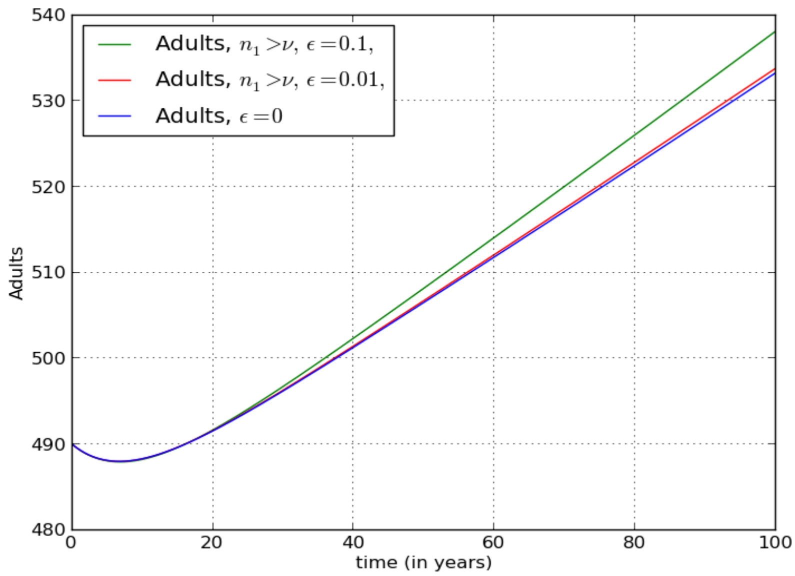


Figure 4.6: The solution $\tilde{n}_2(t)$ of the limit equation (4.16) uniformly attracting solutions $n_{2,\epsilon}(t)$ of (4.2) when $n_1 > \nu$.

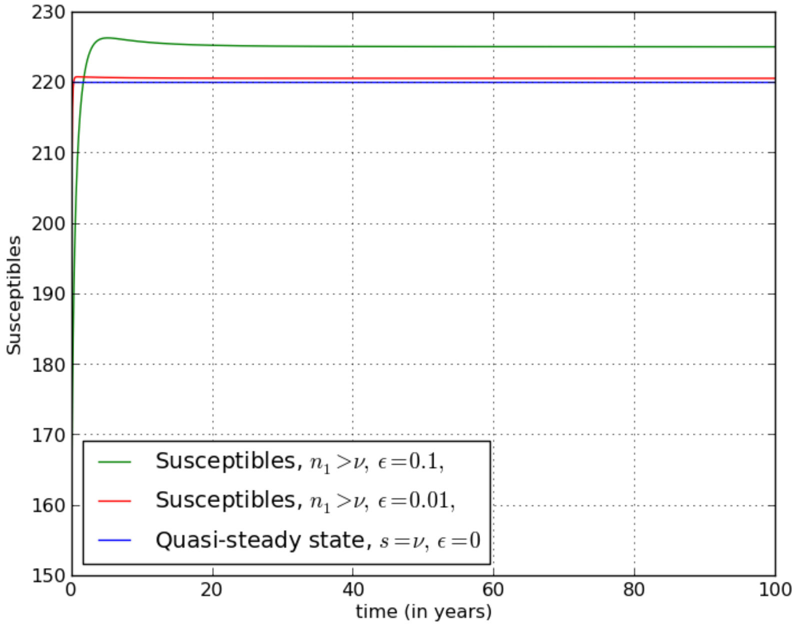


Figure 4.7: The quasi-steady state $\bar{s}_2 = \nu$ attracting the susceptibles curves $s_\epsilon(t)$ of (4.2) when $n_1 > \nu$ displaying the initial layer effect.

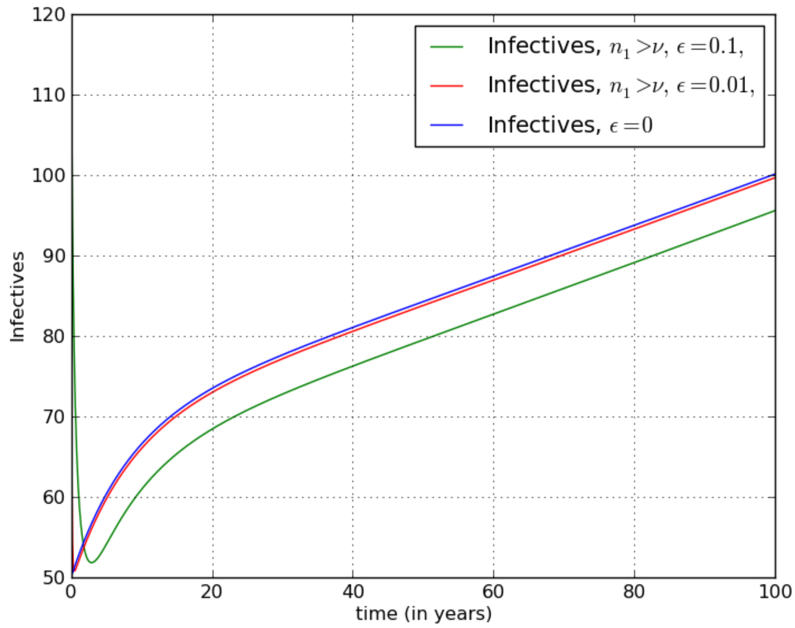


Figure 4.8: The infectives curve converging to the curve $\tilde{n}_1(t) - \nu$ when $n_1 > \nu$ given by (4.20) with the initial layer effect.

In the next section, we analyze the behaviour of the solutions of (4.2) when they pass close to the intersection of the quasi-steady states

4.3 Immediate exchange of stabilities

We say that an immediate exchange of stabilities in a singularly perturbed problem occurs, when its solutions pass from the stable branch of one quasi-steady state to the stable branch of the other, immediately after passing by their intersection.
For the sake of completeness, we present a theorem from [2] which deals with immediate exchange of stabilities.
Let us consider the singularly perturbed system

$$\begin{aligned} \epsilon\frac{du}{dt} &= g(u,v,t,\epsilon), \quad u(0,\epsilon)=u^0, \\ \frac{dv}{dt} &= f(u,v,t,\epsilon), \quad v(0,\epsilon)=v^0, \end{aligned} \tag{4.31}$$

where $t \in I_T = \{t : 0 < t \leq T < \infty\}$, u and v are scalar-valued functions and $\epsilon > 0$ is a small parameter.
Let I_u and I_v be open bounded intervals, $I_{\epsilon_0} = \{\epsilon : 0 < \epsilon < \epsilon_0 \ll 1\}$ and $D = I_u \times I_v \times I_T \times I_{\epsilon_0}$. Then, we have the following assumptions

Assumption A_0.

$$f, g \in C^2(\bar{D}, \mathbb{R}).$$

Next, if we let $\epsilon = 0$ in (4.31), we obtain the degenerate system

$$\begin{aligned} 0 &= g(u,v,t,0), \\ \frac{dv}{dt} &= f(u,v,t,0), \quad v(0)=v^0. \end{aligned} \tag{4.32}$$

For (4.32) we need

Assumption A_1.

The degenerate equation

$$g(u,v,t,0) = 0 \tag{4.33}$$

has two roots $u = \varphi_1(v,t)$ and $u = \varphi_2(v,t)$ defined on $\bar{I}_v \times \bar{I}_T$, such that $\varphi_j(v,t) \in C^2(\bar{D}, \mathbb{R})$ for $j = 1, 2$

Assumption A_2**.**

The surfaces $u = \varphi_1(v,t)$ and $u = \varphi_2(v,t)$ intersect along a curve $\mathcal{K}$ whose projection into the (v,t)-plane is described by $v = s(t)$, where $s \in C^1(\bar{I}_T, \mathbb{R})$. Thus, for $t \in [0,T]$, we have

$$\varphi_1(v,t) \equiv \varphi_2(v,t) \quad \text{for} \quad v = s(t)$$

and assume that

$$\varphi_1(v,t) > \varphi_2(v,t) \quad \text{for} \quad v < s(t) \text{ and } \varphi_1(v,t) < \varphi_2(v,t) \quad \text{for} \quad v > s(t).$$

As in the standard Tikhonov theorem, the auxiliary equation associated to (4.31) is given by

$$\frac{du}{d\tau} = g(u,v,t,0), \quad \tau \geq 0, \tag{4.34}$$

where v and t are treated as parameters.

Assumption A_3**.**

For $0 \leq t \leq T$, the following inequalities hold

$$g_u(\varphi_1(v,t),v,t,0) < 0,\ g_u(\varphi_2(v,t),y,t,0) > 0 \quad \text{for} \quad v < s(t),$$

$$g_u(\varphi_1(v,t),v,t,0) > 0,\ g_u(\varphi_2(v,t),y,t,0) < 0 \quad \text{for} \quad v > s(t).$$

This means that $g_u(u,v,t,0)$ changes its sign on each surface $\varphi_1(v,t)$ and $\varphi_2(v,t)$ when the point (u,v,t), moving along the surface, crosses the curve $\mathcal{K}$ and implies that the surfaces $\varphi_1(v,t)$ and $\varphi_2(v,t)$ change stabilities at the curve $\mathcal{K}$. Hence, we have

$$g_u(\varphi_1(s(t),t),s(t),t,0) \equiv g_u(\varphi_2(s(t),t),s(t),t,0) \equiv 0 \quad \text{for} \quad t \in [0,T].$$

Other assumptions will be discussed after the construction of the composed stable solution of the degenerate equation.

4.3.1 Composed stable solution of the degenerate equation

Let us introduce the composed stable root $\varphi(v,t)$ of Equation (4.33) in $\bar{I}_v \times \bar{I}_T$ as

$$\varphi(v,t) = \begin{cases} \varphi_1(v,t) & \text{for} \quad v \leq s(t), \\ \varphi_2(v,t) & \text{for} \quad v \geq s(t). \end{cases} \tag{4.35}$$

If we substitute the unkown u in the second equation of (4.32) by the known composed stable root $\varphi(v,t)$, we obtain the limit equation

$$\frac{dv}{dt} = f(\varphi(v,t), v, t, 0), \quad v(0) = v^0, \tag{4.36}$$

where $t \in [0, T]$ and $v^0 \neq s(0)$. Let us assume that $v^0 < s(0)$; thus, for some time interval, Equation (4.36) reads

$$\frac{dv}{dt} = f(\varphi_1(v,t), v, t, 0), \quad v(0) = v^0. \tag{4.37}$$

Let us assume that there exists a point $t = t_c$ in I_T such that (4.37) has a unique solution $v = v_1(t)$ on $[0, T]$ and such that

$$v_1(t) < s(t) \quad \text{for} \quad 0 \leq t < t_c, \quad v_1(t_c) = s(t_c). \tag{4.38}$$

For $t > t_c$ we consider the equation

$$\frac{dv}{dt} = f(\varphi_2(v,t), v, t, 0), \quad v(t_c) = s(t_c) \tag{4.39}$$

and assume that it has a unique solution $v = v_2(t)$ on $[t_c, T]$ such that

$$v_2(t) > s(t) \quad \text{for} \quad t_c < t \leq T. \tag{4.40}$$

Now, we define the function $\bar{v}$ as

$$\bar{v}(t) = \begin{cases} v_1(t) & \text{for} \quad 0 \leq t \leq t_c, \\ v_2(t) & \text{for} \quad t_c \leq t \leq T. \end{cases} \tag{4.41}$$

Since $v_1(t)$ and $v_2(t)$ are continuously differentiable in I_T, $\bar{v}(t)$ is continuously differentiable and intersects the curve $v = s(t)$ for $t \in \bar{I}_T$ at the point $(t_c, s(t_c))$. Thus, we introduce the following assumption

Assumption A_4.

Equation (4.36) has the solution $\bar{v}(t) \in \bar{I}_v$ defined by (4.41) and satisfies conditions (4.38) and (4.40).

Remark 4.3.1

The case $v^0 > s(0)$ can be treated similarly, [2].

Let us define the function $\bar{u}$ as follows

$$\bar{u}(t) = \varphi(\bar{v}(t), t) = \begin{cases} \varphi_1(\bar{v}(t), t) \equiv \psi_1(t) & \text{for} \quad 0 \le t \le t_c, \\ \varphi_2(\bar{v}(t), t) \equiv \psi_2(t) & \text{for} \quad t_c \le t \le T. \end{cases} \tag{4.42}$$

Definition 4.3.1

The pair of functions $(\bar{u}(t), \bar{v}(t))$ is called the composed stable solution of the degenerate equation (4.32).

Note that the function $\bar{u}$ is not differentiable at $t = t_c$ and assumptions $A_2 - A_4$ imply that

$$\bar{g}(t) \equiv g(\bar{u}(t), \bar{v}(t), t, 0) = 0 \quad \text{for} \quad 0 \le t \le T \tag{4.43}$$

and

$$\bar{g}_u(t) \equiv g_u(\bar{u}(t), \bar{v}(t), t, 0) = 0 \quad \text{for} \quad 0 \le t < t_c, \quad t_c < t \le T. \tag{4.44}$$

The Butuzov theorem, formulated in more detail below, tells us that under assumptions $A_0 - A_4$ and additional assumptions $A_5 - A_7$ introduced below, the initial value problem (4.31) has a unique solution $(u_\epsilon(t), v_\epsilon(t))$ satisfying the conditions

$$\begin{aligned} \lim_{\epsilon \to 0} u_\epsilon(t) = \bar{u}(t) \quad & \text{for} \quad 0 < t \le T, \\ \lim_{\epsilon \to 0} v_\epsilon(t) = \bar{v}(t) \quad & \text{for} \quad 0 \le t \le T. \end{aligned} \tag{4.45}$$

Concerning the initial value u^0 for $u_\epsilon(t)$, we consider, as in the standard Tikhonov theorem, the following assumption

Assumption A_5.

The initial value u^0 lies in the basin of attraction of the equilibrium point $\varphi_1(v^0, 0)$ of the auxiliary equation (4.34) for $v = v^0$ and $t = 0$, i.e

$$\frac{du}{d\tau} = g(u, v^0, 0), \ \tau > 0, \ u(0) = u^0, \tag{4.46}$$

has a unique solution $u = \hat{u}(\tau)$ satisfying

$$\lim_{\tau \to \infty} \hat{u}(\tau) = \varphi_1(v^0, 0) = u^0.$$

Finally, we have

Assumption A_6.

$$\bar{g}_{uu}(t) \equiv g_{uu}(\bar{u}(t), \bar{v}(t), t, 0) < 0 \text{ for } t = t_c.$$

Assumption A_7.

For $t = t_c$, the inequality

$$\frac{d\bar{u}}{dt} < g_\epsilon(\bar{u}(t), \bar{v}(t), t, 0)$$

holds, where the left and the right derivatives of $\bar{u}$ at $t = t_c$ are considered, [2].

4.3.2 Initial Layer

Let us consider the initial value problem

$$\begin{aligned} \frac{d\pi_0}{d\tau} &= g(\psi_1(0) + \pi_0, v^0, 0), \\ \pi_0(0) &= u^0 - \psi_1(0), \end{aligned} \tag{4.47}$$

where $\tau > 0$ and $\psi_1(t)$ is defined in (4.42). Since $\psi_1(0) = \varphi_1(v^0, 0)$, the assumption A_5 and condition (4.44) imply that the problem (4.47) has a unique solution satisfying the inequality

$$|\pi_0(\tau)| \leq ce^{-\kappa\tau} \text{ for } \tau > 0,$$

where c and κ are some positive constants. It follows that near the point $t = t_c$ we have

Theorem 4.3.1

Let assumptions $A_0 - A_7$ hold. Then, for sufficiently small ϵ, the initial value problem (4.31) has a unique solution $(u_\epsilon(t), v_\epsilon(t))$ such that

$$u_\epsilon(t) = \begin{cases} \bar{u}(t) + \pi_0(\tau) + O(\epsilon) & \text{for} \quad 0 \leq t \leq t_1, \\ \bar{u}(t) + O(\sqrt{\epsilon}) & \text{for} \quad t_1 < t \leq T, \end{cases} v_\epsilon(t) = \begin{cases} \bar{v}(t) + O(\epsilon) & \text{for} \quad 0 \leq t \leq t_1, \\ \bar{v}(t) + O(\sqrt{\epsilon}) & \text{for} \quad t_1 < t \leq T, \end{cases} \tag{4.48}$$

where $t_1 = t_c - \upsilon$ and υ is a sufficiently small positive number independent of ϵ.

Corollary 4.3.1

Relations (4.48) imply that the limit relations (4.45) hold, [2].

For the proof of Theorem 4.3.1, we refer the readers to the monograph of Butuzov [2].

4.4 Delayed exchange of stabilities

Solutions of singularly perturbed equations that pass close to the curve of intersection of quasi-steady states, do not always jump immediately from the stable branch of one quasi-steady state to the stable branch of the other, as described in the previous section. It may happen that after crossing the curve of intersection, they follow the old quasi-steady state for some time and only then jump to the new quasi-steady state. As mentioned earlier, we call such a behaviour the delayed exchange of stabilities; sometimes these are also referred to as canard solutions.
There is no comprehensive theory for such problems. We present a quite general theorem by Butuzov which, however, only apply to scalar equations. We illustrate this theory on a simple SIS system with vital dynamics which has the advantage of admitting closed form solutions (which makes it independent of the general theory).
Later we shall use this fact to extend Butuzov results to a general three dimensional epidemiological system introduced in Section 4.1.
Again, for the sake of completeness, we present a theorem by Butuzov which deals with delayed exchange of stabilities, from [2].
Let us consider the singularly perturbed differential equation

$$\epsilon\frac{du}{dt} = g(u,t,\epsilon), \quad u(0,\epsilon) = u^0, \tag{4.49}$$

where $t \in I_T = \{t : 0 \leq t \leq T\}$ and ϵ is a small positive parameter, [2]. If we let $\epsilon = 0$ in (4.49), we obtain the degenerate equation

$$g(u,t,0) = 0. \tag{4.50}$$

The roots of the degenerate equation are the equilibria of the auxiliary equation

$$\frac{du}{d\tau} = g(u,t,0), \quad \tau > 0, \tag{4.51}$$

associated to (4.50), where t is treated as a parameter. We further assume that the roots of the degenerate equation intersect at some point $t = t_c \in (0,T)$.
Let $I_u \subset \mathbb{R}$ be an open and bounded interval containing the origin, $I_{\epsilon_0} = \{\epsilon : 0 < \epsilon < \epsilon_0 \ll 1\}$ and $D = I_u \times I_T \times I_{\epsilon_0}$.

We introduce assumptions of the Butuzov theorem.

Assumption A_0.

$$g \in C^2(\bar{D},\mathbb{R}).$$

Further,

Assumption A_1**.**

In $\bar{I}_u \times \bar{I}_T$, the solution of the degenerate equation (4.50) consists of two roots $u = 0$ and $u = \varphi(t)$, where φ is twice continuously differentiable on $\bar{I}_T$. The roots $u = \varphi(t)$ and $U = 0$ intersect at $t = t_c \in (0.T)$. Then we assume that

$$\varphi(t) < 0 \text{ for } 0 \leq t < t_c \text{ and } \varphi(t) > 0 \text{ for } t_c < t \leq T.$$

Concerning the stability of these roots, we assume that

Assumption A_2**.**

The following inequalities

$$g_u(0,t,0) < 0, \;\; g_u(\varphi(t),t,0) > 0 \text{ for } 0 \leq t < t_c,$$
$$g_u(0,t,0) > 0, \;\; g_u(\varphi(t),t,0) < 0 \text{ for } t_c < t \leq T$$

holds. Assumption (A_2) means that the roots $u = 0$ and $u = \varphi(t)$ exchange their stabilities at $t = t_c$.
Concerning the root $u = 0$, we additionally assume that

Assumption A_3**.**

$$g(0,t,\epsilon) \equiv 0 \quad \text{for} \quad (t,\epsilon) \in \bar{I}_T \times \bar{I}_{\epsilon_0}.$$

Assumption (A_3) implies that $u \equiv 0$ is a solution of equation (4.49) in $\bar{I}_T$ for all $\epsilon \in \bar{I}_{\epsilon_0}$.

Let us define the function

$$G(t,\epsilon) = \int_0^t g_u(0,s,\epsilon)ds.$$

Assumpton A_4**.**

Equation $G(t,0) = 0$ has a root t^* in $(0,T)$. From assumption (A_2) we obtain that $G(t,0) = 0$ has exactly one root in $(0,T)$, and it follows that

$$G(t,0) < 0 \quad \text{for} \quad t \in (0,t^*).$$

Assumption A_5**.**

There is a positive number c_0 such that $\pm c_0 \in I_u$ and

$$g(u,t,\epsilon) \leq g_u(0,t,\epsilon)u \quad \text{for} \quad 0 \leq t \leq t^*, \; \epsilon \in \bar{I}_{\epsilon_0}, \; |u| \leq c_0.$$

Remark 4.4.1

Assumption (A_5) holds if the second derivative of g with respect to u at $u = 0$ is negative. For a more detailed explanation of the meaning of these assumptions, we refer the reader to the monograph of Butuzov, [2].

Theorem 4.4.1

Assume that hypotheses $A_0 - A_5$ hold. Then for $u^0 \geq 0$ and sufficiently small ϵ there exists a unique solution $u_\epsilon(t)$ of the initial value problem (4.49) on $[0, T]$, which is positive and satisfies the conditions

$$\lim_{\epsilon \to 0} u_\epsilon(t) = 0 \quad \text{for} \quad 0 \leq t < t^*,$$
$$\lim_{\epsilon \to 0} u_\epsilon(t) = \varphi(t) \quad \text{for} \quad t^* < t \leq T.$$

If $\varphi(0) < u^0 < 0$, the unique solution of (4.49) is negative and satisfies the condition

$$\lim_{\epsilon \to 0} u_\epsilon(t) = 0 \quad \text{for} \quad 0 \leq t < t^*.$$

For $t > t^*$, the solution $u_\epsilon(t)$ escapes from the unstable root $u \equiv 0$, [2].

For the proof Theorem 4.4.1, we refer the readers to the monograph of Butuzov [2].

Remark 4.4.2

In the case where $\varphi(0) < u^0 < 0$, it may happens that the solution $u_\epsilon(t)$ does not exist on the whole interval $[0, T]$.

Now, we consider simple SIS model with vital dynamics that displays the behaviour described above. Later on we will consider an SIS model with age structure, described by System (4.3), to perform a similar analysis.
Since System (4.3) is equivalent to (4.2), we will derive the behaviour of the solutions of (4.2) from that of (4.3).

4.5 SIS Model with Vital Dynamics

If we assume that the disease affects the whole population and not only juveniles, and further assume the same mortality rate for both susceptibles and infectives, then System (4.1) becomes

$$\begin{aligned} \frac{ds}{dt} &= \beta n - \mu s + \frac{1}{\epsilon}\left(-\lambda s i + \gamma i\right), \quad s(0,\epsilon) = s^0, \\ \frac{di}{dt} &= -\mu i + \frac{1}{\epsilon}\left(\lambda s i - \gamma i\right), \quad i(0,\epsilon) = i^0, \end{aligned} \tag{4.52}$$

where n denotes the size of the population, s is the size of the class of the susceptibles and i is the size of the class of the infectives, μ is the per capita mortality rate, β is the birth rate, λ is the transmission rate and γ is the recovery rate. Note that β, μ, λ and γ are positive constants. Expressions on the right hand side of (4.52) are divided into two terms; the terms that are multiplied by $1/\epsilon$ are called the fast terms and they describe epidemiological processes. All other terms are called the slow terms. They describe the demographic processes.
If in (4.52) we let $n = s + i$, we obtain an equivalent singularly perturbed system

$$\begin{aligned} \frac{dn}{dt} &= (\beta - \mu)n \equiv f(n, i, t, \epsilon), \quad n(0, \epsilon) = n^0, \\ \epsilon \frac{di}{dt} &= -\epsilon\mu i + (\lambda(n - i) - \gamma)\, i \equiv g(n, i, t, \epsilon), \quad i(0, \epsilon) = i^0. \end{aligned} \tag{4.53}$$

The solutions of the system (4.53) are given by

$$n_\epsilon(t) = n^0 e^{rt} \text{ and } i_\epsilon(t) = \frac{i^0 e^{\frac{\lambda n^0}{\epsilon r}(e^{rt}-1) - \frac{\gamma}{\epsilon}t - \mu t}}{1 + \frac{\lambda}{\epsilon} i^0 \int_0^t e^{\frac{\lambda n^0}{\epsilon r}(e^{rs}-1) - \frac{\gamma}{\epsilon}s - \mu s} ds},$$

where

$$r = \beta - \mu.$$

Now, if in (4.53) we let $\epsilon = 0$, we obtain the following *degenerate system*

$$\begin{aligned} \frac{dn}{dt} &= (\beta - \mu)n, \quad n(0) = n^0, \\ 0 &= (\lambda(n - i) - \gamma)\, i. \end{aligned} \tag{4.54}$$

The solutions of (4.54) are given by

$$\bar{n}(t) = n^0 e^{rt}, \ \bar{i} = 0 \text{ and } \bar{i} = \varphi(\bar{n}, t) = \bar{n} - \nu,$$

where

$$\nu = \gamma/\lambda.$$

Calculating the derivative of the right-hand side $g(n, i, t, 0) = (\lambda(n_1 - i) - \gamma)i$ of the second equation of (4.54) with respect to i, we obtain

$$\frac{\partial g}{\partial i} = \lambda n - \gamma - 2\lambda i.$$

Hence,

$$\left.\frac{\partial g}{\partial i}\right|_{i=0} = \lambda n - \gamma < 0 \iff n < \nu \text{ and } \left.\frac{\partial g}{\partial i}\right|_{i=n-\nu} = -\lambda n + \gamma < 0 \iff n > \nu.$$

It follows that the quasi-steady state $\bar{i} = 0$ is stable if $n < \nu$ and the quasi-steady state $\bar{i} = \bar{n} - \nu$ is stable if $n > \nu$.
We observe that the behaviour of $i_\epsilon(t)$ as $\epsilon \to 0$ depends on the sign of the function

$$G(t,\epsilon) = \frac{\lambda n^0}{\epsilon r}(e^{rt}-1) - \frac{\gamma}{\epsilon}t - \mu t \equiv \frac{1}{\epsilon}\left(\frac{\lambda n^0}{r}(e^{rt}-1) - \gamma t\right) - \mu t. \tag{4.55}$$

If we define

$$G(t) = \frac{\lambda n^0}{r}(e^{rt}-1) - \gamma t,$$

then

$$G(t,\epsilon) = \frac{1}{\epsilon}G(t) - \mu t.$$

Note that

$$G(t) = 0 \Longleftrightarrow \frac{\lambda n^0}{r}(e^{rt}-1) = \gamma t.$$

Let the funtions p and q be defined by

$$p(t) = \frac{\lambda n^0}{r}(e^{rt}-1) \quad \text{and} \quad q(t) = \gamma t.$$

Then

$$p'(t) = \lambda n^0 e^{rt} \quad \text{and} \quad q'(t) = \gamma$$

are positive for all $t \geq 0$. Hence, both p and q are increasing functions for all $t \geq 0$. It follows that at $t = 0$,

$$p'(0) = \lambda n^0 \quad \text{and} \quad q'(0) = \gamma.$$

Then we have

$$\lambda n^0 - \gamma < 0 \Longleftrightarrow n^0 < \nu \quad \text{and} \quad \lambda n^0 - \gamma > 0 \Longleftrightarrow n^0 > \nu.$$

This implies that

$$p'(0) < q'(0) \quad \text{if} \quad n^0 < \nu \quad \text{and} \quad p'(0) > q'(0) \quad \text{if} \quad n^0 > \nu.$$

The second derivatives of p and q are given by

$$p''(t) = \lambda n^0 r e^{rt} \quad \text{and} \quad q''(t) = 0.$$

It follows that $p''(t) > 0$ if $r > 0$; that is, when the population is increasing, and $p''(t) < 0$ if $r < 0$; that is, when the population is decreasing.
Since q is an increasing and linear function and, for $r > 0$, p is an increasing and convex function and for $r < 0$, p is an increasing and concave function, then in both cases p and q intersect at a point $t = t^*$ in $(0, T)$. In other words, Equation $G(t) = 0$ has a root $t = t^*$ in $(0, T)$.

In what follows, we consider two cases.

4.5.1 The case of increasing population

The population is increasing when $r > 0$. Then, for $n^0 < \nu$ and $0 < t < t^*$

$$G(t) = \frac{\lambda n^0}{r}(e^{rt} - 1) - \gamma t < 0$$

and for $t > t^*$

$$G(t) = \frac{\lambda n^0}{r}(e^{rt} - 1) - \gamma t > 0.$$

Furthermore,

$$G'(t) = 0 \Longleftrightarrow \lambda n^0 e^{rt} - \gamma = 0 \Longleftrightarrow t \equiv t_c = (1/r)\ln(\nu/n^0) \quad \text{and} \quad G''(t) = \lambda n^0 r e^{rt} > 0.$$

Observe that

$$G'(t) < 0 \quad \text{for } t < t_c \quad \text{and} \quad G'(t) > 0 \quad \text{for} \quad t > t_c.$$

Hence, $G(t)$ is a convex function that reaches its minimum value at $t = t_c$. Then we have

Theorem 4.5.1

Let $r > 0$ and $n^0 < \nu$. Then,

$$\begin{aligned} &\lim_{\epsilon\to 0} i_\epsilon(t) = 0 \quad \text{for} \quad 0 < t < t^*, \\ &\lim_{\epsilon\to 0} i_\epsilon(t) = n^0 e^{rt} - \nu \quad \text{for} \quad t^* < t \le T. \end{aligned} \tag{4.56}$$

Proof.

Let $r > 0$ and $n^0 < \nu$. Then, for $0 < t < t^*$, we have

$$\frac{1}{\epsilon}G(t) - \mu t < 0.$$

Then

$$i_\epsilon(t) = \frac{i^0 e^{\frac{1}{\epsilon}G(t)} e^{-\mu t}}{1 + \frac{\lambda}{\epsilon} i^0 \int_0^t e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds}.$$

Since $G(t) < 0$ and

$$0 \le \frac{i^0 e^{\frac{1}{\epsilon}G(t)} e^{-\mu t}}{1 + \frac{\lambda}{\epsilon} i^0 \int_0^t e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds} \le i^0 e^{\frac{1}{\epsilon}G(t)} e^{-\mu t},$$

we have, by the Sandwich Theorem, that

$$\lim_{\epsilon\to 0} i_\epsilon(t) = 0 \quad \text{for} \quad 0 < t < t^*.$$

Similarly, for $t > t^*$ and ϵ sufficiently small, we have

$$\frac{1}{\epsilon}G(t) - \mu t > 0.$$

Since $G(t) > 0$, $i_\epsilon(t)$ can be written as

$$i_\epsilon(t) = \frac{i^0 e^{-\mu t}}{e^{-\frac{1}{\epsilon}G(t)} + \frac{\lambda}{\epsilon} i^0 e^{-\frac{1}{\epsilon}G(t)} \left(\int_0^{t^*} e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds + \int_{t^*}^t e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds \right)}. \tag{4.57}$$

It follows that

$$\lim_{\epsilon \to 0} i_\epsilon(t) = \frac{e^{-\mu t}}{\lim_{\epsilon \to 0} \frac{\lambda}{\epsilon} e^{-\frac{1}{\epsilon}G(t)} \int_0^{t^*} e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds + \lim_{\epsilon \to 0} \frac{\lambda}{\epsilon} e^{-\frac{1}{\epsilon}G(t)} \int_{t^*}^t e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds},$$

where

$$\lim_{\epsilon \to 0} e^{-\frac{1}{\epsilon}G(t)} = 0 \text{ for all } t > t^* \text{ and } \lim_{\epsilon \to 0} e^{-\mu t} = e^{-\mu t},$$

provided all the limits exist. Since

$$\frac{1}{\epsilon} G(t) - \mu t < 0 \text{ for } 0 < t < t^*,$$

we have

$$e^{\frac{1}{\epsilon}G(t) - \mu t} \leq 1.$$

Then

$$\int_0^{t^*} e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds$$

is bounded. Thus,

$$0 \leq \lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{-\frac{1}{\epsilon}G(t)} \int_0^{t^*} e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds \leq C \lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{-\frac{1}{\epsilon}G(t)} \equiv 0,$$

where C is a positive constant. This implies, by the Sandwich Theorem, that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{-\frac{1}{\epsilon}G(t)} \int_0^{t^*} e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds = 0.$$

Hence,

$$\lim_{\epsilon \to 0} i_\epsilon(t) = \frac{e^{-\mu t}}{\lambda \lim_{\epsilon \to 0} \int_{t^*}^t \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s) - G(t))} e^{-\mu s} ds}. \tag{4.58}$$

Note that $G(t)$ is an increasing function for $t > t^*$ so that the following change of variables makes sense.

Letting

$$z = G(s) \Rightarrow dz = G'(s) ds.$$

Then

$$ds = \frac{dz}{G'(G^{-1}(z))}.$$

Hence, performing this change of variables in (4.58), we obtain

$$\lim_{\epsilon\to 0} i_\epsilon(t) = \frac{e^{-\mu t}}{\lambda \lim_{\epsilon\to 0}\int_0^{G(t)} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(z-G(t))} e^{-\mu G^{-1}(z)}\psi(z)dz},$$

where

$$\psi(z) = \frac{1}{G'(G^{-1}(z))} \text{ and we used } G(t^*) = 0.$$

Let

$$k = \frac{1}{\epsilon}.$$

Then we define

$$g_k(z) = \begin{cases} ke^{k(z-G(t))} & \text{if} \quad 0 \le z \le G(t), \\ 0 \quad \text{elsewhere.} \end{cases}$$

It follows that

$$\lim_{k\to\infty} g_k(z) = \begin{cases} \infty & \text{if} \quad z = G(t), \\ 0 & \text{if} \quad z \ne G(t). \end{cases}$$

Let

$$I_k \equiv \int_{-\infty}^{\infty} ke^{k(z-G(t))} dz.$$

Then

$$\lim_{k\to\infty} I_k \equiv \lim_{k\to\infty} \int_0^{G(t)} ke^{k(z-G(t))} dz = 1.$$

It follows that the sequence of functions $g_k(z)$ is a delta sequence. Therefore,

$$\lim_{k\to\infty} ke^{k(z-G(t))} dz = \delta(z - G(t)).$$

It follows that

$$\begin{aligned} \lim_{k\to\infty} i_k(t) &= \frac{e^{-\mu t}}{\lambda \lim_{k\to\infty}\int_0^{G(t)} ke^{k(z-G(t))} e^{-\mu G^{-1}(z)}\psi(z)dz} \\ &= \frac{e^{-\mu t}}{\lambda e^{-\mu G^{-1}(G(t))}\psi(G(t))} \\ &= \frac{G'(t)}{\lambda} \\ &= \frac{\lambda n^0 e^{rt} - \gamma}{\lambda}. \end{aligned}$$

Since, as $k \to \infty$, $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0} i_\epsilon(t) = \lim_{k\to\infty} i_k(t) \quad \text{for} \quad t^* < t \le T.$$

Hence,

$$\lim_{\epsilon\to 0} i_\epsilon(t) = n^0 e^{rt} - \nu \quad \text{for} \quad t^* < t \le T,$$

as illustrated in the following numerical simulation.

4.5.2 Numerical simulation

Here we present the numerical simulation of the SIS model given by (4.53) when $n^0 < \nu$ and $r > 0$. The parameters' values used are $\mu = 0.043$, $\beta = 0.05$, $\gamma = 0.44$, $\lambda = 0.002$, $\epsilon = 0.1, 0.08, 0.06$ and the initial conditions are $n^0 = 180$ and $i^0 = 110$.
For an increasing population, Fig 4.9 shows that the root $i = 0$ of the degenerate equation is stable and the solution of the original equation remains close to the quasi-steady state $\bar{i}(t) = 0$ whenever $0 < t < t^*$.
For $t^* < t \leq T$, the solution of the original equation jumps from the now unstable root $i = 0$ to the stable root $i = n - \nu$ of the original equation and remains close to the quasi-steady state $\bar{i}(t) = n^0 e^{rt} - \nu$. This is known as *the delayed exchange of stabilities* [2].

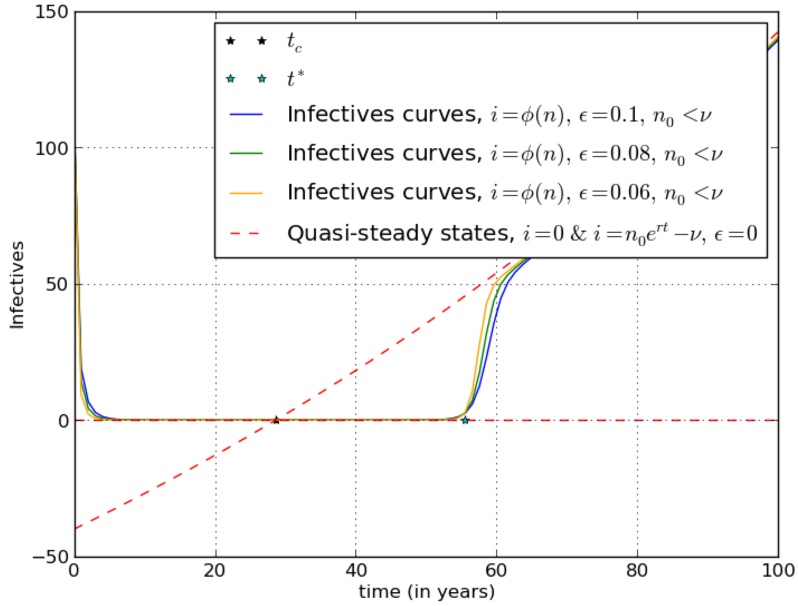


Figure 4.9: Delayed exchange of stabilities for an increasing population given by the solution $i_\epsilon(t)$ of (4.53) and the composed stable solution when $r > 0$ and $n^0 < \nu$.

4.5.3 The case of decreasing population

The population is decreasing when $r < 0$. Therefore, for $n^0 > \nu$ and $0 < t < t^*$

$$G(t) = \frac{\lambda n^0}{r}(e^{rt} - 1) - \gamma t > 0$$

and for $t > t^*$

$$G(t) = \frac{\lambda n^0}{r}(e^{rt} - 1) - \gamma t < 0.$$

Note that

$$G'(t) > 0 \quad \text{for } t < t_c \quad \text{and} \quad G'(t) < 0 \quad \text{for} \quad t > t_c. \text{ Further } G''(t) = \lambda n^0 r e^{rt} < 0.$$

Hence, $G(t)$ is a concave function that reaches its maximum value at $t = t_c$. Then we have

Theorem 4.5.2

Let $r < 0$ and $n^0 > \nu$. Then,

$$\begin{aligned} &\lim_{\epsilon \to 0} i_\epsilon(t) = n^0 e^{rt} - \nu \quad \text{for} \quad 0 < t < t_c, \\ &\lim_{\epsilon \to 0} i_\epsilon(t) = 0 \quad \text{for} \quad t_c < t \le T. \end{aligned} \tag{4.59}$$

Proof.

Let $r < 0$ and $n^0 > \nu$. Then, for ϵ sufficiently small and $0 < t < t^*$, we have

$$\frac{1}{\epsilon} G(t) - \mu t > 0.$$

For $0 < t < t_c$,

$$i_\epsilon(t) = \frac{i^0 e^{-\mu t}}{e^{-\frac{1}{\epsilon} G(t)} + \lambda i^0 \int_0^t \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s) - G(t))} e^{-\mu s} ds}. \tag{4.60}$$

Thus,

$$\lim_{\epsilon \to 0} i_\epsilon(t) = \frac{e^{-\mu t}}{\lambda \lim_{\epsilon \to 0} \int_0^t \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s) - G(t))} e^{-\mu s} ds},$$

where we used

$$\lim_{\epsilon \to 0} e^{-\frac{1}{\epsilon} G(t)} = 0 \text{ for all } 0 < t < t_c \text{ and } \lim_{\epsilon \to 0} e^{-\mu t} = e^{-\mu t}.$$

Since $G(s) < G(t)$ for $0 < s < t < t_c$ and $G(0) = 0$, performing the change of variables $z = G(s)$ and proceeding as in the case of the increasing population when $t > t^*$ in (4.58) above, we show that

$$\lim_{\epsilon \to 0} i_\epsilon(t) = n^0 e^{rt} - \nu, \text{ for } 0 < t < t_c.$$

For $t_c < t < t^*$,

$$i_\epsilon(t) = \frac{i^0 e^{-\mu t}}{e^{-\frac{1}{\epsilon} G(t)} + \lambda i^0 \int_0^{t_c} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s) - G(t))} e^{-\mu s} ds + \lambda i^0 \int_{t_c}^t \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s) - G(t))} e^{-\mu s} ds}. \tag{4.61}$$

Note that for $t > t_c$, $G(t)$ is a decreasing function. Thus, we have $G(t) < G(s) < G(t_c)$ whenever $t_c < s < t$. We have

$$\lim_{\epsilon \to 0} i_\epsilon(t) = \frac{e^{-\mu t}}{\lambda \lim_{\epsilon \to 0} \int_0^{t_c} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s) - G(t))} e^{-\mu s} ds + \lambda \lim_{\epsilon \to 0} \int_{t_c}^t \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s) - G(t))} e^{-\mu s} ds},$$

provided all the limits exist. Since from (4.60)

$$\lambda \lim_{\epsilon\to 0}\int_0^{t_c} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s)-G(t))} e^{-\mu s} ds = \frac{e^{-\mu t_c}}{n^0 e^{r t_c} - \nu} = +\infty,$$

and since $G(s) > G(t)$,

$$\lim_{\epsilon\to 0}\int_{t_c}^{t} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s)-G(t))} e^{-\mu s} ds = +\infty,$$

we have

$$\lim_{\epsilon\to 0} i_\epsilon(t) = 0 \text{ for } t_c < t < t^*.$$

Next, for $t > t^*$, we have

$$\frac{1}{\epsilon}G(t) - \mu t < 0.$$

Then we have

$$0 \le i_\epsilon(t) \equiv \frac{i^0 e^{\frac{1}{\epsilon}G(t)} e^{-\mu t}}{1 + \lambda i^0 \int_0^{t^*} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds + \lambda i^0 \int_{t^*}^{t} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}G(s)} e^{-\mu s} ds} \le i^0 e^{\frac{1}{\epsilon}G(t)} e^{-\mu t}.$$

Since

$$\frac{1}{\epsilon}G(t) - \mu t < 0, \text{ we have } \lim_{\epsilon\to 0} e^{\frac{1}{\epsilon}G(t)} e^{-\mu t} = 0.$$

Hence, by the Sandwich Theorem, we have

$$\lim_{\epsilon\to 0} i_\epsilon(t) = 0 \quad \text{for} \quad t^* < t \le T.$$

This is illustrated in the following numerical simulation.

4.5.4 Numerical simulation

Here we present the numerical simulation of the SIS model given by (4.53), when $n^0 > \nu$ and $r < 0$. The parameters' values used are $\mu = 0.043$, $\beta = 0.03$, $\gamma = 0.44$, $\lambda = 0.002$, $\epsilon = 0.1 \ \& \ 0.01$ and the initial conditions are $n^0 = 370$ and $i^0 = 110$.
For a decreasing population, the root $i = n - \nu$ of the degenerate equation is stable whenever $0 < t < t_c$ and the solution of the original equation remains close to the quasi-steady state $\bar{i}(t) = n(t) - \nu$.
For $t_c < t \le T$, contrary to the case of the increasing population, there is immediate exchange of stabilities at $t = t_c$. The root $i = 0$ of the degenerate equation becomes now stable and the solution of the original equation converges toward the quasi-steady state $\bar{i}(t) = 0$.

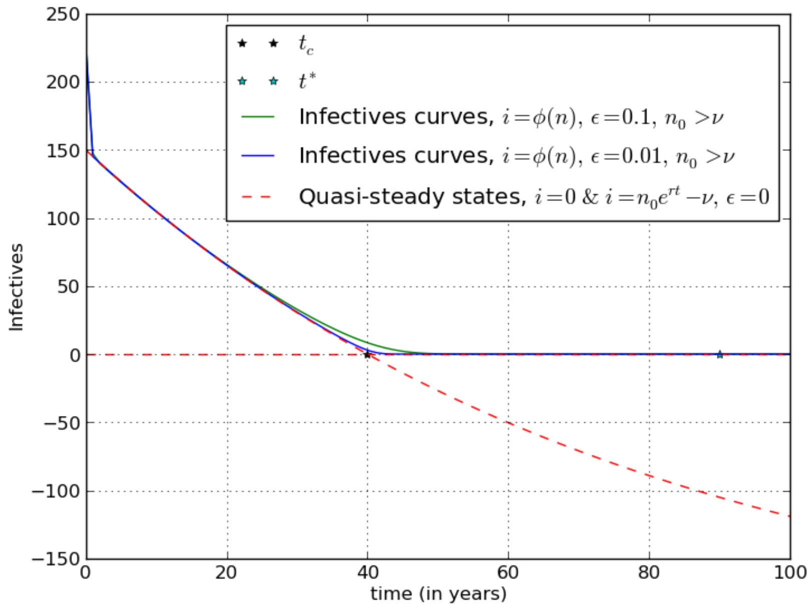


Figure 4.10: Immediate exchange of stabilities for an decreasing population given by the solution $i_\epsilon(t)$ of (4.53) and the composed stable solution when $r<0$ and $n^0>\nu$.

Comment on the case of decreasing population

From (4.53),

$$g(n,i,t,\epsilon)=-\epsilon\mu i+(\lambda(n-i)-\gamma)i.$$

The derivative of g with respect to ϵ is given by

$$g_\epsilon(n,i,t,\epsilon)=-\mu i.$$

At $t=t_c$ and $\epsilon=0$, $g_\epsilon(\bar{n},\bar{i},t_c,0)=0$. Since $[d\bar{i}/dt]|_{t=t_c^-}=0$ and $[d\bar{i}/dt]|_{t=t_c^+}=r\nu$, Assumption (A_7) of the Butuzov theorem on immediate exchange of stability does not hold.
Indeed, we see that

$$\left.\frac{d\bar{i}}{dt}\right|_{t=t_c}\leq g_\epsilon(\bar{n}(t_c),\bar{i}(t_c),t_c,0)=0,$$

whereas in Assumption (A_7) the strict inequality is required. However, from the above example, we have immediate exchange of stabilities. This shows that Assumption (A_7) gives only sufficient conditions for immediate exchange of stabilities.

4.6 SIS Model with Age Structure

Let us consider System (4.3) of ordinary differential equations, given by

$$
\begin{aligned}
\epsilon\frac{di_\epsilon}{dt} &= -\epsilon(\mu_1^*+a)i_\epsilon + (\lambda(n_{1,\epsilon}-i_\epsilon)-\gamma)i_\epsilon, \quad i_\epsilon(0)=i^0,\\
\frac{dn_{1,\epsilon}}{dt} &= -(\mu_1+a)n_{1,\epsilon} - (\mu_1^*-\mu_1)i_\epsilon + \beta n_{2,\epsilon}, \quad n_{1,\epsilon}(0)=n_1^0,\\
\frac{dn_{2,\epsilon}}{dt} &= -\mu_2 n_{2,\epsilon} + a n_{1,\epsilon}, \quad n_{2,\epsilon}(0)=n_2^0.
\end{aligned}
$$

Its solution is given by $(i_\epsilon(t), n_{1,\epsilon}(t), n_{2,\epsilon}(t))$, where $(n_{1,\epsilon}(t), n_{2,\epsilon}(t))$ was defined earlier in (4.4) and

$$
i_\epsilon(t) = \frac{i^0 e^{-(\mu_1^*+a)t-\frac{\gamma}{\epsilon}t+\frac{\lambda}{\epsilon}\int_0^t n_{1,\epsilon}(s)ds}}{1+\frac{\lambda}{\epsilon}i^0\int_0^t e^{-(\mu_1^*+a)s-\frac{\gamma}{\epsilon}s+\frac{\lambda}{\epsilon}\int_0^s n_{1,\epsilon}(\tau)d\tau}ds}. \tag{4.62}
$$

If we let $\epsilon = 0$ in (4.3), we obtain the following degenerate system

$$
\begin{aligned}
0 &= (\lambda(n_1-i)-\gamma)i,\\
\frac{dn_1}{dt} &= -(\mu_1+a)n_1 - (\mu_1^*-\mu_1)i + \beta n_2, \quad n_1(0)=n_1^0,\\
\frac{dn_2}{dt} &= -\mu_2 n_2 + a n_1, \quad n_2(0)=n_2^0.
\end{aligned} \tag{4.63}
$$

The quasi-steady states of (4.3) are the roots of the first equation of (4.63), and they are given by

$$
\bar{i}_1 = 0 \quad \text{and} \quad \bar{i}_2 \equiv \varphi(n_1,t) = n_1-\nu, \tag{4.64}
$$

where ν is defined in Section 4.2. If we substitute the unknown i in the second equation of (4.63) by the known quasi-steady state $\bar{i}$, we obtain the limit equation

$$
\begin{aligned}
\frac{d\bar{n}_1}{dt} &= -(\mu_1+a)\bar{n}_1 - (\mu_1^*-\mu_1)\bar{i} + \beta\bar{n}_2, \quad \bar{n}_1(0)=n_1^0,\\
\frac{d\bar{n}_2}{dt} &= -\mu_2\bar{n}_2 + a\bar{n}_1, \quad \bar{n}_2(0)=n_2^0.
\end{aligned} \tag{4.65}
$$

The auxiliary equation associated to (4.3) is given by

$$
\frac{d\hat{i}}{d\tau} = (\lambda(n_1-\hat{i})-\gamma)\hat{i}, \ \hat{i}(0)=i^0, \tag{4.66}
$$

where $\tau = t/\epsilon$ is the fast time and n_1 is treated as a parameter.

Then, proceeding like in Section 4.2, we show that the quasi-steady state $\bar{i}_1 = 0$ is stable if $n_1 < \nu$ and the quasi-steady state $\bar{i}_2 = n_1 - \nu$ is stable if $n_1 > \nu$.
Substituting the quasi-steady state $\bar{i} = 0$ in the system (4.65), it is reduced to the system (4.10),

$$\frac{d\bar{n}_1}{dt} = -(\mu_1 + a)\bar{n}_1 + \beta\bar{n}_2,$$
$$\frac{d\bar{n}_2}{dt} = -\mu_2\bar{n}_2 + a\bar{n}_1,$$

while, if we consider the quasi-steady state $\bar{i} = n_1 - \nu$, then we obtain (4.16),

$$\frac{d\tilde{n}_1}{dt} = -(\mu_1^* + a)\tilde{n}_1 + \beta\tilde{n}_2 + (\mu_1^* - \mu_1)\nu,$$
$$\frac{d\tilde{n}_2}{dt} = -\mu_2\tilde{n}_2 + a\tilde{n}_1.$$

As for the solution of the one dimensional system discussed in Section 4.2 above, the behaviour of the solution $i_\epsilon(t)$ of (4.3) as $\epsilon \to 0$ depends on the sign of the function

$$G(t,\epsilon) = \frac{\lambda}{\epsilon}\int_0^t n_{1,\epsilon}(s)ds - \frac{\gamma}{\epsilon}t - (\mu_1^* + a)t. \tag{4.67}$$

Note that in this case, contrary to Example in Section 4.2, equations in System (4.3) are coupled and thus we cannot directly use the one dimensional approach. To circumvent this difficulty, we use the fact that solution to the full system can be sandwiched between solutions of one dimensional problems which converge to the same quasi-steady state.

Definition 4.6.1

The pairs of continuous functions $(\underline{U}(t,\epsilon), \underline{V}(t,\epsilon))$ and $(\bar{U}(t,\epsilon), \bar{V}(t,\epsilon))$, which are piece-wise continuously differentiable with respect to t in $[0,T]$ are called the ordered lower and upper solutions of the problem (4.31) for $\epsilon \in I_{\epsilon_0}$ if they satisfy the following conditions; for $t \in [0,T]$ and $\epsilon \in I_{\epsilon_0}$, we have

1. $\underline{U}(t,\epsilon) \leq \bar{U}(t,\epsilon)$, $\underline{V}(t,\epsilon) \leq \bar{V}(t,\epsilon)$;
2. $\epsilon\frac{d\underline{U}}{dt} - g(\underline{U}, v, t, \epsilon) \leq 0 \leq \epsilon\frac{d\bar{U}}{dt} - g(\bar{U}, v, t, \epsilon), \quad$ for $\quad \underline{V} \leq v \leq \bar{V}$,
   $\frac{d\underline{V}}{dt} - f(u, \underline{V}, t, \epsilon) \leq 0 \leq \frac{d\bar{V}}{dt} - f(u, \bar{V}, t, \epsilon), \quad$ for $\quad \underline{U} \leq u \leq \bar{U}$;
3. $\underline{U}(0,\epsilon) \leq u(0,\epsilon) \leq \bar{U}(0,\epsilon)$, $\underline{V}(0,\epsilon) \leq v(0,\epsilon) \leq \bar{V}(0,\epsilon)$.

Note that the existence of the ordered lower and upper solutions of (4.31) implies the existence of a unique solution $(u_\epsilon(t), v_\epsilon(t))$ of (4.31) satisfying, for $t \in [0,T]$ and $\epsilon \in I_{\epsilon_0}$, the following conditions, see [2].

$$\begin{aligned}\underline{U}(t,\epsilon) \leq u_\epsilon(t) \leq \bar{U}(t,\epsilon),\\ \underline{V}(t,\epsilon) \leq v_\epsilon(t) \leq \bar{V}(t,\epsilon).\end{aligned}$$

Then let us consider the triplet $(\bar{I}_\epsilon(t), \bar{N}_1(t), \bar{N}_2(t))$ such that the system

$$\begin{aligned}\epsilon\frac{d\bar{I}_\epsilon}{dt} &= -\epsilon(\mu_1^* + a)\bar{I}_\epsilon + (\lambda(\bar{N}_1 - \bar{I}_\epsilon) - \gamma)\bar{I}_\epsilon, \quad \bar{I}_\epsilon(0) \equiv I^0 = i^0,\\ \frac{d\bar{N}_1}{dt} &= -(\mu_1 + a)\bar{N}_1 + \beta\bar{N}_2, \quad \bar{N}_1(0) \equiv N_1^0 = n_1^0,\\ \frac{d\bar{N}_2}{dt} &= -\mu_2\bar{N}_2 + a\bar{N}_1, \quad \bar{N}_2(0) \equiv N_2^0 = n_2^0.\end{aligned} \tag{4.68}$$

is satisfied. The solution of (4.68) is given by

$$\bar{I}_\epsilon(t) = \frac{I^0 e^{-(\mu_1^*+a)t - \frac{\gamma}{\epsilon}t + \frac{\lambda}{\epsilon}\int_0^t \bar{N}_1(s)ds}}{1 + \frac{\lambda}{\epsilon}I^0\int_0^t e^{-(\mu_1^*+a)s - \frac{\gamma}{\epsilon}s + \frac{\lambda}{\epsilon}\int_0^s \bar{N}_1(\tau)d\tau}ds}, \tag{4.69}$$

and

$$\begin{pmatrix} \bar{N}_1(t) \\ \bar{N}_2(t) \end{pmatrix} = e^{\mathcal{A}t}\begin{pmatrix} N_1^0 \\ N_2^0 \end{pmatrix}, \tag{4.70}$$

where $N_1^0 \equiv \bar{N}_1(0) = n_1^0$, $N_2^0 \equiv \bar{N}_2(0) = n_2^0$ and $\mathcal{A}$ is defined in Section 4.1.

It follows that

$$\begin{aligned}\bar{N}_1(t) &= \frac{1}{2}e^{(\alpha/2+\rho)t}\left[\left(1 + \frac{\zeta}{\rho}\right)N_1^0 + \frac{\beta}{\rho}N_2^0\right] + \frac{1}{2}e^{(\alpha/2-\rho)t}\left[\left(1 - \frac{\zeta}{\rho}\right)N_1^0 - \frac{\beta}{\rho}N_2^0\right],\\ \bar{N}_2(t) &= \frac{1}{2}e^{(\alpha/2+\rho)t}\left[\left(1 + \frac{\sigma}{\rho}\right)N_2^0 + \frac{a}{\rho}N_1^0\right] + \frac{1}{2}e^{(\alpha/2-\rho)t}\left[\left(1 - \frac{\sigma}{\rho}\right)N_2^0 - \frac{a}{\rho}N_1^0\right].\end{aligned}$$

where

$$\begin{aligned}\alpha &= -(\mu_1 + a) - \mu_2,\\ \delta &= \mu_2(\mu_1 + a) - \beta a,\\ \rho^2 &= \frac{1}{4}(\alpha^2 - 4\delta),\\ \zeta &= \mu_1 + a + 2\mu_2,\\ \sigma &= 2(\mu_1 + a) + \mu_2.\end{aligned}$$

Note that $\rho^2 = (1/4)((\mu_1 + a) - \mu_2)^2 + \beta a$; this implies that $\rho \neq 0$. Inequality (4.4) implies that

$$\begin{pmatrix} \bar{N}_1(t) \\ \bar{N}_2(t) \end{pmatrix} \geq \begin{pmatrix} n_{1,\epsilon}(t) \\ n_{2,\epsilon}(t) \end{pmatrix}, \tag{4.71}$$

for all $t \in [0, T]$. It follows that if $\delta > 0$, then $\rho < |\alpha|/2 < \zeta$, in which case we have

$$1 - \frac{\zeta}{\rho} < 0,\ 1 + \frac{\zeta}{\rho} > 0,\ \frac{\alpha}{2} + \rho < 0 \text{ and } \frac{\alpha}{2} - \rho < 0. \tag{4.72}$$

If $\delta < 0$, then $\rho > |\alpha|/2$ and $\rho > \zeta$. Thus we have

$$1 - \frac{\zeta}{\rho} > 0,\ 1 + \frac{\zeta}{\rho} > 0,\ \frac{\alpha}{2} + \rho > 0 \text{ and } \frac{\alpha}{2} - \rho < 0. \tag{4.73}$$

Then, in either case, $\bar{N}_1(t), \bar{N}_2(t) > 0$. Furthermore, since

$$-(\mu_1 + a)\bar{N}_1 - (\mu_1^* - \mu_1)i_\epsilon + \beta\bar{N}_2 \leq -(\mu_1 + a)\bar{N}_1 + \beta\bar{N}_2,$$

we have

$$\begin{pmatrix} \frac{d\bar{N}_1}{dt} \\ \frac{d\bar{N}_2}{dt} \end{pmatrix} \geq \begin{pmatrix} f_1(i_\epsilon, \bar{N}_1, \bar{N}_2) \\ f_2(i_\epsilon, \bar{N}_1, \bar{N}_2) \end{pmatrix}, \tag{4.74}$$

where

$$f_1(i_\epsilon, \bar{N}_1, \bar{N}_2) = -(\mu_1 + a)\bar{N}_1 - (\mu_1^* - \mu_1)i_\epsilon + \beta\bar{N}_2 \text{ and } f_2(i_\epsilon, \bar{N}_1, \bar{N}_2) = -\mu_2\bar{N}_2 + a\bar{N}_1.$$

Let

$$g(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}, t) \equiv -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(n_{1,\epsilon} - i_\epsilon) - \gamma)i_\epsilon$$

and

$$g(i_\epsilon, \bar{N}_1, n_{2,\epsilon}, t) \equiv -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(\bar{N}_1 - i_\epsilon) - \gamma)i_\epsilon.$$

Then, from Inequality (4.71), we have that

$$g(i_\epsilon, \bar{N}_1, n_{2,\epsilon}, t) \geq g(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}, t) \text{ for all } t \in [0, T].$$

Thus, the solution $\bar{I}_\epsilon(t)$ satisfies on $[0, T]$ the inequality

$$\epsilon\frac{d\bar{I}_\epsilon}{dt} \geq -\epsilon(\mu_1^* + a)\bar{I}_\epsilon + (\lambda(n_{1,\epsilon} - \bar{I}_\epsilon) - \gamma)\bar{I}_\epsilon. \tag{4.75}$$

It follows from the Theorem B.1. in [20] that

$$\bar{I}_\epsilon(t) \geq i_\epsilon(t) \tag{4.76}$$

for all $t \in [0,T]$. Hence, the triplet $(\bar{I}_\epsilon(t), \bar{N}_1(t), \bar{N}_2(t))$ is an upper solution of (4.3) for all $t \in [0,T]$.
Note that $\bar{I}_\epsilon(t)$ is the solution of a scalar equation. Therefore, using the result obtained from Section 4.2, we notice that the behaviour of $\bar{I}_\epsilon(t)$ as $\epsilon \to 0$ depends on the sign of the function

$$\bar{G}(t,\epsilon) = \frac{\lambda}{\epsilon}\int_0^t \bar{N}_1(s)ds - \frac{\gamma}{\epsilon}t - (\mu_1^* + a)t = \frac{1}{\epsilon}\left(\lambda\int_0^t \bar{N}_1(s)ds - \gamma t\right) - (\mu_1^* + a)t. \quad (4.77)$$

If we define

$$\bar{G}(t) = \lambda\int_0^t \bar{N}_1(s)ds - \gamma t,$$

then

$$\bar{G}(t,\epsilon) = \frac{1}{\epsilon}\bar{G}(t) - (\mu_1^* + a)t.$$

Note that

$$\bar{G}(t) = 0 \Longleftrightarrow \lambda\int_0^t \bar{N}_1(s)ds = \gamma t.$$

Let the funtions $\bar{P}$ and $\bar{Q}$ be defined by

$$\bar{P}(t) = \lambda\int_0^t \bar{N}_1(s)ds \quad \text{and} \quad \bar{Q}(t) = \gamma t.$$

Then

$$\bar{P}'(t) = \lambda\bar{N}_1(t) \text{ and } \bar{Q}'(t) = \gamma$$

are positive for all $t \geq 0$. Hence, both $\bar{P}$ and $\bar{Q}$ are increasing functions for all $t \geq 0$. In particular

$$\bar{P}'(0) = \lambda\bar{N}_1(0) \equiv \lambda N_1^0 \quad \text{and} \quad \bar{Q}'(0) = \gamma.$$

Then

$$\lambda N_1^0 < \gamma \Longleftrightarrow N_1^0 < \nu \quad \text{and} \quad \lambda N_1^0 > \gamma \Longleftrightarrow N_1^0 > \nu.$$

This implies that

$$\bar{P}'(0) < \bar{Q}'(0) \quad \text{if} \quad N_1^0 < \nu \quad \text{and} \quad \bar{P}'(0) > \bar{Q}'(0) \quad \text{if} \quad N_1^0 > \nu.$$

The second derivatives of $\bar{P}$ and $\bar{Q}$ are given by

$$\bar{P}''(t) = \lambda\bar{N}_1'(t) \text{ and } \bar{Q}''(t) = 0.$$

It what follows, we consider the case of increasing population. Using similar analysis as in Section 4.2, we observe that since $\bar{Q}$ is an increasing and linear function and $\bar{P}$ is an increasing function that starts by being concave close to the origin then becomes convex, thus $\bar{P}$ and $\bar{Q}$ intersect at a point $t = \bar{t}^*$ in $(0,T)$. In other words, Equation $\bar{G}(t) = 0$ has a root $t = \bar{t}^*$ in $(0,T)$.
This is illustrated, for $N_1^0 < \nu$ and $\delta < 0$, in the following numerical simulations with the parameters' values $\mu_1 = 0.03$, $\mu_2 = 0.019$, $a = 0.06$, $\beta = 0.033$, $\mu_1^* = 0.035$, $\gamma = 0.44$, $\lambda = 0.002$, $\epsilon = 0.03,\ 0.02,\ 0.01$ and the initial conditions are $n_1^0 = 150$, $n_2^0 = 900$ and $i^0 = 80$.

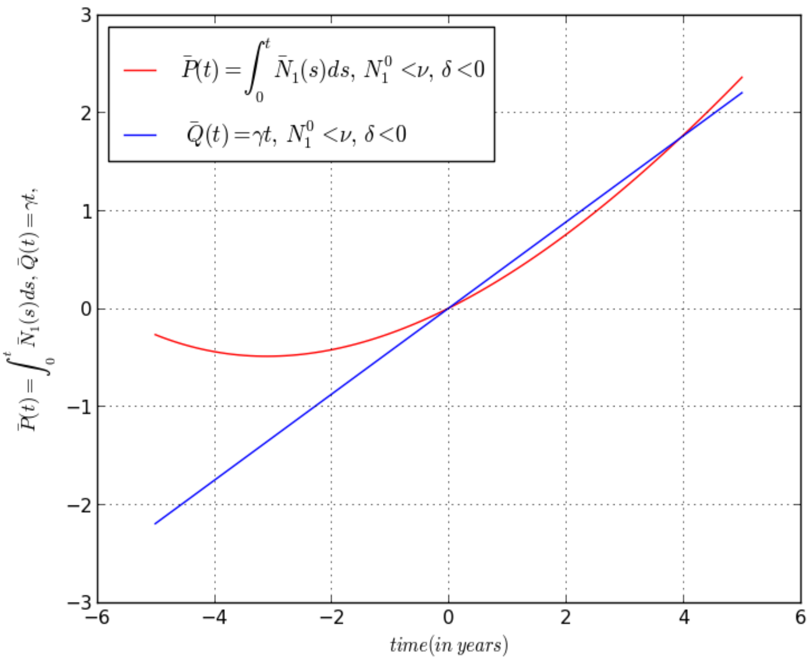


Figure 4.11: The graph of $\bar{P}(t)$ intersecting the graph of $\bar{Q}(t)$ at $t = 0$ and $t = \bar{t}^*$.

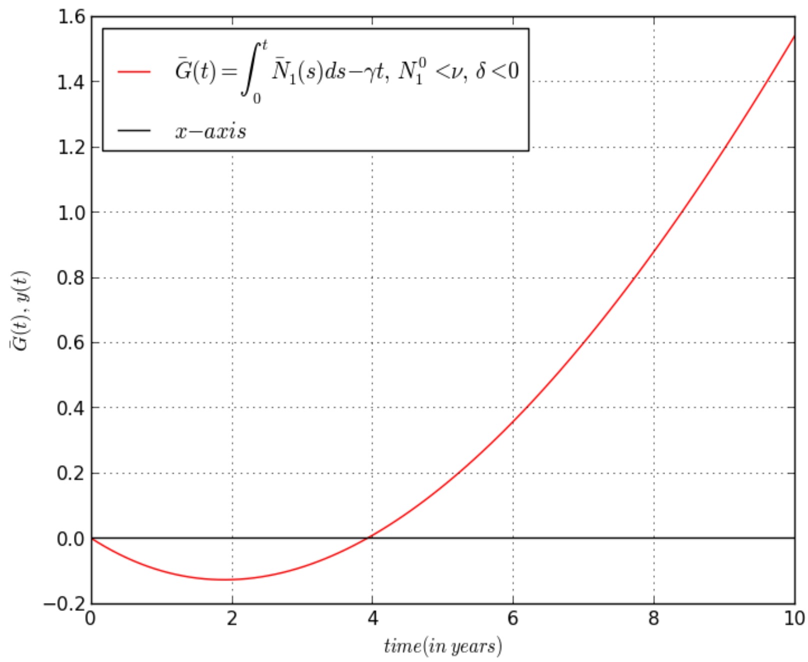


Figure 4.12: The graph of $\bar{G}(t)$ intersecting the $x - axis$ at $t = 0$ and $t = \bar{t}^*$.

4.6.1 The case of increasing population

The population increases when $\delta < 0$. Therefore, for $N_1^0 < \nu$ and for all $0 < t < \bar{t}^*$,

$$\bar{G}(t) = \lambda \int_0^t \bar{N}_1(s)ds - \gamma t < 0 \tag{4.78}$$

and for all $t > \bar{t}^*$,

$$\bar{G}(t) = \lambda \int_0^t \bar{N}_1(s)ds - \gamma t > 0. \tag{4.79}$$

Furthermore,

$$\bar{G}'(t) = 0 \Longleftrightarrow \lambda \bar{N}_1(t) - \gamma = 0 \Longleftrightarrow \text{ there exists } t = \bar{t}_c \in (0, \bar{t}^*) \text{ such that } \bar{N}_1(\bar{t}_c) = \nu$$

and

$$\bar{G}''(t) = \lambda \bar{N}_1'(t).$$

Observe that

$$\bar{G}'(t) < 0 \quad \text{for } t < \bar{t}_c \quad \text{and} \quad \bar{G}'(t) > 0 \quad \text{for} \quad t > \bar{t}_c.$$

Then we have

Proposition 4.6.1

Let $\delta < 0$ and $N_1^0 < \nu$. Then,

$$\begin{aligned} &\lim_{\epsilon \to 0} \bar{I}_\epsilon(t) = 0 \quad \text{for} \quad 0 < t < \bar{t}^*, \\ &\lim_{\epsilon \to 0} \bar{I}_\epsilon(t) = \bar{N}_1(t) - \nu \quad \text{for} \quad \bar{t}^* < t \leq T. \end{aligned} \tag{4.80}$$

The proof of this result follows from the one dimensional case in Section 4.2, when the population increases. The numerical simulation illustrating this result is given in Fig 4.13.

Note that the upper solution obtained from System (4.73) provides a good approximation to the solution of (4.3), only up to some time $\Theta \in (\bar{t}^*, T]$.
Therefore we shall construct another upper solution that provides a better approximation to the solution of (4.3) on $[0, T]$.
First, we construct a lower solution $(\underline{I}_\epsilon(t), \underline{N}_1(t), \underline{N}_2(t))$ to the system (4.3) on $[0, T]$ in the following two steps.

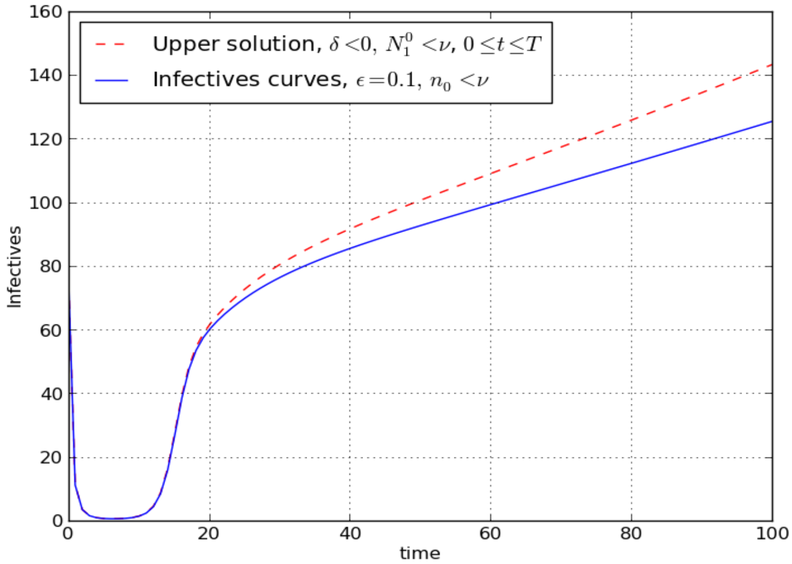


Figure 4.13: Upper solution of (4.3) given by the first equation of (4.73), when $N_1^0 < \nu$ and $\delta < 0$.

In the first step, we begin by showing

Proposition 4.6.2

The open interval $(0, \bar{t}^*)$ is the largest interval on which $i_\epsilon(t) \to 0$ almost uniformly.

Proof

We assume that there exists $\tau > \bar{t}^*$ in $[0, T]$ such that $i_\epsilon(t) \to 0$ almost uniformly as $\epsilon \to 0$. This means that for any $\varrho > 0$, for any $\eta > 0$, there exists $\epsilon_0 > 0$ such that for any $|\epsilon| < \epsilon_0$

$$0 \leq \underline{I}_\epsilon(t) \leq \varrho \text{ for all } t \in [\eta, \tau].$$

Then for all $t \in [0, \eta]$, we define

$$\begin{pmatrix} \underline{N}_{1,1}(t) \\ \underline{N}_{2,1}(t) \end{pmatrix} = e^{\mathcal{A}^* t} \begin{pmatrix} N_1^0 \\ N_2^0 \end{pmatrix}, \tag{4.81}$$

where $N_1^0 = n_1^0$, $N_2^0 = n_2^0$ and $\mathcal{A}^*$ is defined in Section 4.1. Therefore

$$\underline{N}_{1,1}(t) = \frac{1}{2} e^{(\alpha^*/2+\rho^*)t}\left[\left(1+\frac{\zeta^*}{\rho^*}\right)N_1^0 + \frac{\beta}{\rho^*}N_2^0\right] + \frac{1}{2}e^{(\alpha^*/2-\rho^*)t}\left[\left(1-\frac{\zeta^*}{\rho^*}\right)N_1^0 - \frac{\beta}{\rho^*}N_2^0\right],$$
$$\underline{N}_{2,1}(t) = \frac{1}{2} e^{(\alpha^*/2+\rho^*)t}\left[\left(1+\frac{\sigma^*}{\rho^*}\right)N_2^0 + \frac{a}{\rho^*}N_1^0\right] + \frac{1}{2}e^{(\alpha^*/2-\rho^*)t}\left[\left(1-\frac{\sigma^*}{\rho^*}\right)N_2^0 - \frac{a}{\rho^*}N_1^0\right],$$

where

$$\begin{aligned}\alpha^* &= -(\mu_1^* + a) - \mu_2,\\ \delta^* &= \mu_2(\mu_1^* + a) - \beta a,\\ \rho^{*2} &= \frac{1}{4}(\alpha^{*2} - 4\delta^*),\\ \zeta^* &= \mu_1^* + a + 2\mu_2,\\ \sigma^* &= 2(\mu_1^* + a) + \mu_2.\end{aligned}$$

Note that $\rho^{*2} = (1/4)((\mu_1^* + a) - \mu_2)^2 + \beta a$; this implies that $\rho^* \neq 0$. Inequality (4.5) implies that

$$\begin{pmatrix} \underline{N}_{1,1}(t) \\ \underline{N}_{2,1}(t) \end{pmatrix} \leq \begin{pmatrix} n_{1,\epsilon}(t) \\ n_{2,\epsilon}(t) \end{pmatrix}, \tag{4.82}$$

for all $t \in [0, \eta]$ and $\epsilon > 0$. It follows that if $\delta^* > 0$, then $\rho^* < |\alpha^*|/2 < \zeta^*$. In this case we have

$$1 - \frac{\zeta^*}{\rho^*} < 0,\ 1 + \frac{\zeta^*}{\rho^*} > 0,\ \frac{\alpha^*}{2} + \rho^* < 0 \text{ and } \frac{\alpha^*}{2} - \rho^* < 0. \tag{4.83}$$

If $\delta^* < 0$, then $\rho^* > |\alpha^*|/2$ and $\rho^* > \zeta^*$. Thus, we have

$$1 - \frac{\zeta^*}{\rho^*} > 0,\ 1 + \frac{\zeta^*}{\rho^*} > 0,\ \frac{\alpha^*}{2} + \rho^* > 0 \text{ and } \frac{\alpha^*}{2} - \rho^* < 0. \tag{4.84}$$

Then in either case $\underline{N}_{1,1}(t), \underline{N}_{2,1}(t) > 0$. Furthermore, since

$$-(\mu_1^* + a)\underline{N}_{1,1} + (\mu_1^* - \mu_1)s_\epsilon + \beta\underline{N}_{2,1} \geq -(\mu_1^* + a)\underline{N}_{1,1} + \beta\underline{N}_{2,1},$$

we have

$$\begin{pmatrix} \frac{d\underline{N}_{1,1}}{dt} \\ \frac{d\underline{N}_{2,1}}{dt} \end{pmatrix} \leq \begin{pmatrix} f_{1,1}(s_\epsilon, \underline{N}_{1,1}, \underline{N}_{2,1}) \\ f_{2,1}(s_\epsilon, \underline{N}_{1,1}, \underline{N}_{2,1}) \end{pmatrix}, \tag{4.85}$$

where

$f_{1,1}(s_\epsilon, \underline{N}_{1,1}, \underline{N}_{2,1}) = -(\mu_1 + a)\underline{N}_{1,1} + (\mu_1^* - \mu_1)s_\epsilon + \beta\underline{N}_{2,1}$ and $f_{2,1}(s_\epsilon, \underline{N}_1, \underline{N}_{2,1}) = -\mu_2\underline{N}_{2,1} + a\underline{N}_{1,1}$.

Next, for all $t \in [\eta, \tau]$, we define

$$\begin{pmatrix} \underline{N}_{1,2}(t) \\ \underline{N}_{2,2}(t) \end{pmatrix} = e^{\mathcal{A}(t-\eta)} \begin{pmatrix} N_{1,\eta} \\ N_{2,\eta} \end{pmatrix} - \begin{pmatrix} (\mu_1^* - \mu_1)\varrho \\ 0 \end{pmatrix} \int_\eta^t e^{\mathcal{A}(t-\eta-z)} dz, \tag{4.86}$$

where

$$N_{1,\eta} = \underline{N}_{1,1}(\eta),\ \ N_{2,\eta} = \underline{N}_{2,1}(\eta).$$

Therefore

$$
\begin{aligned}
\underline{N}_{1,2}(t) &= \frac{1}{2}C_1 e^{(\alpha/2+\rho)(t-\eta)} + \frac{1}{2}C_2 e^{(\alpha/2-\rho)(t-\eta)} \\
&\quad + C_3\frac{e^{(\alpha/2+\rho)t} - e^{(\alpha/2+\rho)\eta}}{\alpha+2\rho} + C_4\frac{e^{(\alpha/2-\rho)t} - e^{(\alpha/2-\rho)\eta}}{\alpha-2\rho}, \\
\underline{N}_{2,2}(t) &= \frac{1}{2}D_1 e^{(\alpha/2+\rho)(t-\eta)} + \frac{1}{2}D_2 e^{(\alpha/2-\rho)(t-\eta)} \\
&\quad + D_3\frac{e^{(\alpha/2+\rho)t} - e^{(\alpha/2+\rho)\eta}}{\alpha+2\rho} + D_4\frac{e^{(\alpha/2-\rho)t} - e^{(\alpha/2-\rho)\eta}}{\alpha-2\rho},
\end{aligned}
$$

with

$$
\begin{aligned}
C_1 &= \left(1+\frac{\zeta}{\rho}\right)N_{1,\eta} + \frac{\beta}{\rho}N_{2,\eta}, \quad C_2 = \left(1-\frac{\zeta}{\rho}\right)N_{1,\eta} - \frac{\beta}{\rho}N_{2,\eta}, \\
C_3 &= \left(1+\frac{\zeta}{\rho}\right)\varrho(\mu_1^* - \mu_1), \quad C_4 = \left(1-\frac{\zeta}{\rho}\right)\varrho(\mu_1^* - \mu_1), \\
D_1 &= \left(1+\frac{\sigma}{\rho}\right)N_{2,\eta} + \frac{a}{\rho}N_{1,\eta}, \quad D_2 = \left(1-\frac{\sigma}{\rho}\right)N_{2,\eta} - \frac{a}{\rho}N_{1,\eta}, \\
D_3 &= \left(1+\frac{\sigma}{\rho}\right)\varrho(\mu_1^* - \mu_1), \quad D_4 = \left(1-\frac{\sigma}{\rho}\right)\varrho(\mu_1^* - \mu_1).
\end{aligned}
$$

Since $(\mu_1^* - \mu_1) > 0$ and $0 \leq \underline{I}_\epsilon(t) \leq \varrho$ for all $t \in [\eta, \tau]$, we have

$$-(\mu_1^* - \mu_1)\varrho \leq -(\mu_1^* - \mu_1)\underline{I}_\epsilon(t).$$

This implies that

$$\frac{d\underline{N}_{1,2}}{dt} \leq -(\mu_1 + a)\underline{N}_{1,2} + \beta\underline{N}_{2,2} - (\mu_1^* - \mu_1)\underline{I}_\epsilon.$$

Therefore

$$\begin{pmatrix} \frac{d\underline{N}_{1,2}}{dt} \\ \frac{d\underline{N}_{2,2}}{dt} \end{pmatrix} \leq \begin{pmatrix} f_{1,2}(i_\epsilon, \underline{N}_{1,2}, \underline{N}_{2,2}) \\ f_{2,2}(i_\epsilon, \underline{N}_{1,2}, \underline{N}_{2,2}) \end{pmatrix}, \tag{4.87}$$

where

$f_{1,2}(i_\epsilon, \underline{N}_{1,2}, \underline{N}_{2,2}) = -(\mu_1+a)\underline{N}_{1,2} - (\mu_1^* - \mu_1)i_\epsilon + \beta\underline{N}_{2,2}$ and $f_{2,2}(i_\epsilon, \underline{N}_{1,2}, \underline{N}_{2,2}) = -\mu_2\underline{N}_{2,2} + a\underline{N}_{1,2}$.

It follows from Theorem B.1. in [20] that

$$\begin{pmatrix} \underline{N}_{1,2}(t) \\ \underline{N}_{2,2}(t) \end{pmatrix} \leq \begin{pmatrix} n_{1,\epsilon}(t) \\ n_{2,\epsilon}(t) \end{pmatrix}, \tag{4.88}$$

for all $t \in [\eta, \tau]$. Hence, for all $t \in [0, \tau]$ we define $(\underline{N}_1^{\eta,\varrho}(t), \underline{N}_2^{\eta,\varrho}(t))$ as follows

$$\underline{N}_1^{\eta,\varrho}(t) = \begin{cases} \underline{N}_{1,1}(t), & 0 \le t \le \eta, \\ \underline{N}_{1,2}(t), & \eta \le t \le \tau, \end{cases} \quad \underline{N}_2^{\eta,\varrho}(t) = \begin{cases} \underline{N}_{2,1}(t), & 0 \le t \le \eta, \\ \underline{N}_{2,2}(t), & \eta \le t \le \tau. \end{cases} \tag{4.89}$$

Consider the equation

$$\epsilon \frac{d\underline{I}_\epsilon}{dt} = -\epsilon(\mu_1^* + a)\underline{I}_\epsilon + (\lambda(\underline{N}_1^{\eta,\varrho} - \underline{I}_\epsilon) - \gamma)\underline{I}_\epsilon, \quad \underline{I}_\epsilon(0) \equiv I^0 = i^0, \tag{4.90}$$

obtained by substituting $n_{1,\epsilon}$ in the first equation of (4.3), by $\underline{N}_1^{\eta,\varrho}$. The solution of Equation (4.90) is given by

$$\underline{I}_\epsilon^{\eta,\varrho}(t) = \frac{I^0 e^{-(\mu_1^*+a)t - \frac{\gamma}{\epsilon}t + \frac{\lambda}{\epsilon}\int_0^t \underline{N}_1^{\eta,\varrho}(s)ds}}{1 + \frac{\lambda}{\epsilon} I^0 \int_0^t e^{-(\mu_1^*+a)s - \frac{\gamma}{\epsilon}s + \frac{\lambda}{\epsilon}\int_0^s \underline{N}_1^{\eta,\varrho}(\tau)d\tau} ds}, \tag{4.91}$$

with

$$\int_0^t \underline{N}_1^{\eta,\varrho}(s)ds = \begin{cases} \displaystyle\int_0^t \underline{N}_{1,1}(s)ds, \ 0 \le t \le \eta, \\ \displaystyle\int_0^\eta \underline{N}_{1,1}(s)ds + \int_\eta^t \underline{N}_{1,2}(s)ds, \ \eta \le t \le \tau. \end{cases}$$

Let

$$g(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}, t) \equiv -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(n_{1,\epsilon} - i_\epsilon) - \gamma)i_\epsilon$$

and

$$g(i_\epsilon, \underline{N}_1^{\eta,\varrho}, n_{2,\epsilon}, t) \equiv -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(\underline{N}_1^{\eta,\varrho} - i_\epsilon) - \gamma)i_\epsilon.$$

Then, from Inequalities (4.88) and (4.82), we have that

$$g(i_\epsilon, \underline{N}_1^{\eta,\varrho}, n_{2,\epsilon}, t) \le g(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}, t) \text{ for all } t \in [0, \tau].$$

Thus, the solution $\underline{I}_\epsilon^{\eta,\varrho}(t)$ satisfies on $[0,\tau]$ the inequality

$$\epsilon \frac{d\underline{I}_\epsilon^{\eta,\varrho}}{dt} \le -\epsilon(\mu_1^* + a)\underline{I}_\epsilon^{\eta,\varrho} + (\lambda(n_{1,\epsilon} - \underline{I}_\epsilon^{\eta,\varrho}) - \gamma)\underline{I}_\epsilon^{\eta,\varrho}. \tag{4.92}$$

It follows from Theorem B.1. in [20] that

$$\underline{I}_\epsilon^{\eta,\varrho}(t) \le i_\epsilon(t) \text{ for all } t \in [0, \tau]. \tag{4.93}$$

Hence, for η and ϱ sufficiently small, the triplet $(\underline{I}_\epsilon^{\eta,\varrho}(t), \underline{N}_1^{\eta,\varrho}(t), \underline{N}_2^{\eta,\varrho}(t))$ is a lower solution of the system (4.3) on $t \in [0, \tau]$.
It follows from Inequalities (4.93), (4.88), (4.82), (4.76) and (4.69) that Inequalities

$$\begin{aligned} &\underline{I}_{\epsilon}^{\eta,\varrho}(t) \leq i_{\epsilon}(t) \leq \bar{I}_{\epsilon}(t), \\ &\underline{N}_1^{\eta,\varrho}(t) \leq n_{1,\epsilon}(t) \leq \bar{N}_1(t), \\ &\underline{N}_2^{\eta,\varrho}(t) \leq n_{2,\epsilon}(t) \leq \bar{N}_2(t) \end{aligned} \tag{4.94}$$

hold for all $t \in [0, \tau]$.

Note that for all $t \in [0,\tau]$, $\underline{I}_{\epsilon}^{\eta,\varrho}(t)$ is the solution of a scalar equation. Therefore using once again the result from Section 4.2, we observe that the behaviour of $\underline{I}_{\epsilon}^{\eta,\varrho}(t)$ as $\epsilon \to 0$ depends on the sign of the function

$$\begin{aligned} \underline{G}(t,\eta,\varrho,\epsilon) &= \frac{\lambda}{\epsilon}\int_0^t \underline{N}_1^{\eta,\varrho}(s)ds - \frac{\gamma}{\epsilon}t - (\mu_1^* + a)t, \\ &= \frac{1}{\epsilon}\left(\lambda \int_0^t \underline{N}_1^{\eta,\varrho}(s)ds - \gamma t\right) - (\mu_1^* + a)t. \end{aligned} \tag{4.95}$$

Let us define $\underline{G}(t,\eta,\varrho)$ as follows

$$\underline{G}(t,\eta,\varrho) = \lambda \int_0^t \underline{N}_1^{\eta,\varrho}(s)ds - \gamma t.$$

Observe that, as $\omega \equiv (\eta,\varrho) \to (0,0)$,

$$\underline{N}_{1,2}(t) \to \frac{1}{2}e^{(\alpha/2+\rho)t}\left[\left(1+\frac{\zeta}{\rho}\right)N_1^0 + \frac{\beta}{\rho}N_2^0\right] + \frac{1}{2}e^{(\alpha/2-\rho)t}\left[\left(1-\frac{\zeta}{\rho}\right)N_1^0 - \frac{\beta}{\rho}N_2^0\right] = \bar{N}_1(t).$$

This implies that

$$\underline{G}(t,\eta,\varrho) \to \underline{G}(t,0,0) = \bar{G}(t) \text{ as } (\eta,\varrho) \to (0,0). \tag{4.96}$$

We can rewrite $\underline{G}(t,\eta,\varrho,\epsilon)$ as

$$\underline{G}(t,\eta,\varrho,\epsilon) = \frac{1}{\epsilon}\underline{G}(t,\eta,\varrho) - (\mu_1^* + a)t.$$

In what follows we use the implicit function theorem, introduced in Chapter 1, to prove the existence of a root for $\underline{G}(t,\eta,\varrho)$ on $[0,\tau]$.
Let us consider the function

$$\underline{G}(t,\eta,\varrho) = \lambda \int_0^t \underline{N}_1^{\eta,\varrho}(s)ds - \gamma t.$$

Observe that $\underline{G}(t,\eta,\varrho)$ is a continuously differentiable function. By (4.96) we see that the point $P = (\bar{t}^*, 0, 0)$ satisfies

$$\underline{G}(\bar{t}^*, 0, 0) = 0. \tag{4.97}$$

The partial derivative of $\underline{G}$ with respect to t is given by

$$\frac{\partial \underline{G}}{\partial t} = \lambda \underline{N}_1^{\eta,\varrho}(t) - \gamma.$$

Then

$$\frac{\partial \underline{G}}{\partial t}(P) = \lambda \bar{N}_1(t) - \gamma \neq 0 \Longleftrightarrow \bar{t}^* \neq \bar{t}_c;$$

which is verified.
Therefore, it follows from the implicit function theorem that there exist $\eta_0,\ \varrho_0,$ such that for all $0 < \eta < \eta_0$ and $0 < \varrho < \varrho_0$, there exists a unique $\underline{t}^* = \underline{t}^*(\eta, \varrho) \in (\eta, \tau)$ such that

$$\underline{G}(\underline{t}^*, \eta, \varrho) = 0. \tag{4.98}$$

From the second inequality in (4.94), we have that

$$\underline{G}(t, \eta, \varrho) \leq \bar{G}(t)$$

which implies that

$$\underline{t}^* \geq \bar{t}^*.$$

Moreover, from Equations (4.96) and (4.98), we conclude that

$$\lim_{\eta,\varrho \to 0} \underline{t}^* = \lim_{\eta,\varrho \to 0} \underline{t}^*(\eta, \varrho) = \bar{t}^*.$$

Note that from the one dimensional theory developed in Section 4.2, $\underline{I}_\epsilon^{\eta,\varrho}(t)$ converges to $\underline{N}_1^{\eta,\varrho}(t) - \nu$ for $t > \underline{t}^*$. This means that $i_\epsilon(t)$ cannot converge to zero for any $\tilde{t} > \bar{t}^*$.
Indeed, if for some $\tilde{t} > \bar{t}^*$, $i_\epsilon(\tilde{t})$ converges to zero, then we could take η and ϱ small enough for $\bar{t}^* < \tilde{t}$ and $i_\epsilon(\tilde{t}) > \underline{I}_\epsilon^{\eta,\varrho}(\tilde{t})$.
Since $\underline{I}_\epsilon^{\eta,\varrho}(\tilde{t}) \to \underline{N}_1^{\eta,\varrho}(\tilde{t}) - \nu \neq 0$, as $\epsilon \to 0$, $i_\epsilon(\tilde{t})$ could not converge to zero.

It follows that as $(\eta, \varrho) \to (0, 0)$ and for $0 < t < \underline{t}^*$, we have

$$\frac{1}{\epsilon}\underline{G}(t, \eta, \varrho) - (\mu_1^* + a)t \to \frac{1}{\epsilon}\bar{G}(t) - (\mu_1^* + a)t < 0.$$

Then, for sufficiently small positive numbers η and ϱ, and $0 < t < \underline{t}^*$,

$$\frac{1}{\epsilon}\underline{G}(t, \eta, \varrho) - (\mu_1^* + a)t < 0.$$

We can rewrite (4.91) as

$$\underline{I}_\epsilon^{\eta,\varrho}(t) \equiv \frac{I^0 e^{\frac{1}{\epsilon}\underline{G}(t,\eta,\varrho)} e^{-(\mu_1^*+a)t}}{1 + \frac{\lambda}{\epsilon} I^0 \int_0^t e^{\frac{1}{\epsilon}\underline{G}(s,\eta,\varrho)} e^{-(\mu_1^*+a)s} ds}.$$

Since, as $(\eta, \varrho) \to (0, 0)$, $\underline{G}(t, \eta, \varrho) \to \bar{G}(t)$, we have $\underline{I}_\epsilon^{\eta,\varrho}(t) \to \bar{I}_\epsilon(t)$ as $(\eta, \varrho) \to (0, 0)$.
It follows, for η and ϱ sufficienly small, that

$$\lim_{\epsilon\to 0} \underline{I}_{\epsilon}^{\eta,\varrho}(t) = \lim_{\epsilon\to 0} \bar{I}_{\epsilon}(t) = 0, \quad \text{for} \quad 0 < t < \underline{t}^*. \tag{4.99}$$

Now, in the second step, we extend the lower solution of (4.3) for $t > \bar{t}^*$. For the upper solution $(\bar{I}_\epsilon(t), \bar{N}_1(t), \bar{N}_2(t))$, we have determined the point $\bar{t}_c$ such that $\bar{N}_1(\bar{t}_c) = \nu$ and the point $\bar{t}^*$ such that $\bar{I}_\epsilon(t) \to 0$ on $(0, \bar{t}^*)$ as $\epsilon \to 0$.
On the other hand, since $\delta < 0$, $N_1^0 < \nu$ and $N_2^0 \geq \nu(\mu_1 + a)/\beta$, we have from the phase plane analysis that $\bar{N}_2(\bar{t}_c) \geq \nu(\mu_1 + a)/\beta$ and, moreover, $\bar{N}_1(\bar{t}^*) > \nu$ and $\bar{N}_2(\bar{t}^*) > \nu(\mu_1 + a)/\beta$.
Thus, for any sufficiently small ι_1, $\iota_2 > 0$ there exists $\bar{t} \in (\bar{t}_c, \bar{t}^*)$ such that $\bar{N}_1(t) > \nu + 2\iota_1$ and $\bar{N}_2(t) > \nu(\mu_1 + a)/\beta + 2\iota_2$ for all $t \in [\bar{t}, \bar{t}^*]$.
If we fix arbitrary ι_1, ι_2 and the corresponding $\bar{t}$, we see that for any $0 < \iota' < \min\{\iota_1, \iota_2\}$ there is $\Omega > 0$ such that

$$e^{t\mathcal{A}^*} \begin{pmatrix} \nu + \iota_1 \\ \frac{\nu(\mu_1+a)}{\beta} + \iota_2 \end{pmatrix} \geq \begin{pmatrix} \nu + \iota' \\ \frac{\nu(\mu_1+a)}{\beta} + \iota' \end{pmatrix}, \tag{4.100}$$

for all $t \in [0, \Omega]$. In particular,

$$e^{\Omega\mathcal{A}^*} \begin{pmatrix} \nu + \iota_1 \\ \frac{\nu(\mu_1+a)}{\beta} + \iota_2 \end{pmatrix} \geq \begin{pmatrix} \nu + \iota' \\ \frac{\nu(\mu_1+a)}{\beta} + \iota' \end{pmatrix}. \tag{4.101}$$

Let us consider $\bar{t}$ such that $T' = \bar{t} + \Omega > \bar{t}^*$ and let us take $\hat{t} \in (\bar{t}, \bar{t}^*)$ that is independent from the constants Ω, ι', ι_1 and ι_2. Then, we have $\bar{N}_1(\hat{t}) > \nu + \iota_1$ and $\bar{N}_2(\hat{t}) > \nu(\mu_1 + a)/\beta + \iota_2$.

If we multiply Inequality (4.100) by $e^{\Omega\mathcal{A}^*}$ and use the positivity of $(e^{t\mathcal{A}^*})_{t\geq 0}$, Inequality (4.101) and the semigroup property of the flow, we obtain

$$e^{(t+\Omega)\mathcal{A}^*} \begin{pmatrix} \nu + \iota_1 \\ \frac{\nu(\mu_1+a)}{\beta} + \iota_2 \end{pmatrix} \geq \begin{pmatrix} \nu + \iota' \\ \frac{\nu(\mu_1+a)}{\beta} + \iota' \end{pmatrix}. \tag{4.102}$$

This implies that $\bar{N}_1(t) > \nu + \iota_1$ and $\bar{N}_2(t) > \nu(\mu_1 + a)/\beta + \iota_2$ for all $t \in [\bar{t}^*, T']$. Assume that these inequalities hold for $t \in [\bar{t}^*, T_1]$, with some $T_1 > T'$.
We have also shown that $i_\epsilon(t) \to 0$ on $(0, \bar{t}^*)$ as $\epsilon \to 0$, and from (4.3) and the regular perturbation theory, we have that $(n_{1,\epsilon}(t), n_{2,\epsilon}(t)) \to (\bar{N}_1(t), \bar{N}_2(t))$. This means that for $\hat{t} \in (\bar{t}_c, \bar{t}^*)$, $n_{1,\epsilon}(\hat{t}) > \nu + \iota_1$ and $n_{2,\epsilon}(\hat{t}) > \nu(\mu_1 + a)/\beta + \iota_2$ for sufficiently small ϵ. Since as $i_\epsilon(\hat{t}) \to 0$, $s_\epsilon(\hat{t}) > \nu$, we have $i_\epsilon(\hat{t}) < n_{1,\epsilon}(\hat{t}) - \nu$.

System (4.3) along the quasi-steady state $i = n_1 - \nu$ is written as

$$\begin{aligned} \frac{di}{dt} &= -(\mu_1^* + a)(n_1 - \nu), \\ \frac{dn_1}{dt} &= -(\mu_1 + a)n_1 - (\mu_1^* - \mu_1)(n_1 - \nu) + \beta n_2, \\ \frac{dn_2}{dt} &= -\mu_2 n_2 + a n_1. \end{aligned} \tag{4.103}$$

Note that the direction of the vector field along $i = n_1 - \nu$ satisfies

$$(i', n_1', n_2').(-1, 1, 0) = -(\mu_1 + a)\nu + \beta n_2 > 0,$$

provided $n_2 > \nu(\mu_1 + a)/\beta$, where $(-1, 1, 0)$ is a normal vector to the level curve

$$F(i, n_1, n_2) \equiv n_1 - \nu - i.$$

It follows that $i_\epsilon(t) < n_{1,\epsilon}(t) - \nu$ for all $t \in (\bar{t}_c, T_1]$, as long as $(n_{1,\epsilon}(t), n_{2,\epsilon}(t)) \in \mathcal{V}_2$, where

$$\mathcal{V}_2 = \{(n_1, n_2) \in I_{n_1} \times I_{n_2} : n_1 \geq \nu + \delta\beta/(\mu_1^* + a), n_2 \geq \nu(\mu_1 + a)/\beta + \delta\}.$$

Thus, for all $t \geq \hat{t}$ we construct an extension of the lower solution $\underline{N}_1(t)$ by noting that for all $t \geq \hat{t}$,

$$-(\mu_1^* - \mu_1)i_\epsilon > -(\mu_1^* - \mu_1)(n_{1,\epsilon} - \nu).$$

This implies that

$$\frac{d\underline{N}_{1,3}}{dt} \leq -(\mu_1 + a)\underline{N}_{1,3} + \beta\underline{N}_{2,3} - (\mu_1^* - \mu_1)i_\epsilon.$$

Therefore

$$\begin{pmatrix} \frac{d\underline{N}_{1,3}}{dt} \\ \frac{d\underline{N}_{2,3}}{dt} \end{pmatrix} \leq \begin{pmatrix} f_{1,3}(i_\epsilon, \underline{N}_{1,3}, \underline{N}_{2,3}) \\ f_{2,3}(i_\epsilon, \underline{N}_{1,3}, \underline{N}_{2,3}) \end{pmatrix}, \tag{4.104}$$

where

$f_{1,3}(i_\epsilon, \underline{N}_{1,3}, \underline{N}_{2,3}) = -(\mu_1+a)\underline{N}_{1,3}-(\mu_1^*-\mu_1)i_\epsilon+\beta\underline{N}_{2,3}$ and $f_{2,3}(i_\epsilon, \underline{N}_{1,3}, \underline{N}_{2,3}) = -\mu_2\underline{N}_{2,3}+a\underline{N}_{1,3}$.

It follows from Theorem B.1. in [20] that

$$\begin{pmatrix} \underline{N}_{1,3}(t) \\ \underline{N}_{2,3}(t) \end{pmatrix} \leq \begin{pmatrix} n_{1,\epsilon}(t) \\ n_{2,\epsilon}(t) \end{pmatrix} \tag{4.105}$$

for all $t \in [\hat{t}, T_1]$, where $(\underline{N}_{1,3}(t), \underline{N}_{2,3}(t))$ satisfies the following system

$$\begin{aligned} \frac{d\underline{N}_{1,3}}{dt} &= -(\mu_1 + a)\underline{N}_{1,3} + \beta\underline{N}_{2,3} - (\mu_1^* - \mu_1)(\underline{N}_{1,3} - \nu), \quad \underline{N}_{1,3}(\hat{t}) = \hat{N}_1, \\ \frac{d\underline{N}_{2,3}}{dt} &= -\mu_2\underline{N}_{2,3} + a\underline{N}_{1,3}, \quad \underline{N}_{2,3}(\hat{t}) = \hat{N}_2, \end{aligned} \tag{4.106}$$

and also

$$\begin{pmatrix} \underline{N}_{1,3}(t) \\ \underline{N}_{2,3}(t) \end{pmatrix} = e^{\mathcal{A}(t-\hat{t})} \begin{pmatrix} \hat{N}_1 \\ \hat{N}_2 \end{pmatrix} - \begin{pmatrix} (\mu_1^* - \mu_1) \\ 0 \end{pmatrix} \int_{\hat{t}}^{t} (\underline{N}_{1,3}(z) - \nu)e^{\mathcal{A}(t-\hat{t}-z)}dz, \tag{4.107}$$

with

$$\hat{N}_1 = \underline{N}_{1,2}(\hat{t}),\ \hat{N}_2 = \underline{N}_{2,2}(\hat{t}).$$

Hence, for all $t \in [0, T_1]$ we define $(\underline{N}_1(t), \underline{N}_2(t))$ as follows

$$\underline{N}_1(t) \equiv \underline{N}_{1,\eta,\varrho,\hat{t}}(t) = \begin{cases} \underline{N}_{1,1}(t), & 0 \le t < \eta, \\ \underline{N}_{1,2}(t), & \eta \le t < \hat{t}, \\ \underline{N}_{1,3}(t), & \hat{t} \le t \le T_1, \end{cases} \quad \underline{N}_2(t) \equiv \underline{N}_{2,\eta,\varrho,\hat{t}}(t) = \begin{cases} \underline{N}_{2,1}(t), & 0 \le t < \eta, \\ \underline{N}_{2,2}(t), & \eta \le t < \hat{t}, \\ \underline{N}_{2,3}(t), & \hat{t} \le t \le T_1. \end{cases} \tag{4.108}$$

Consider now the equation

$$\epsilon \frac{d\underline{I}_\epsilon}{dt} = -\epsilon(\mu_1^* + a)\underline{I}_\epsilon + (\lambda(\underline{N}_1 - \underline{I}_\epsilon) - \gamma)\underline{I}_\epsilon, \quad \underline{I}_\epsilon(0) \equiv I^0 = i^0, \tag{4.109}$$

obtained by substituting $n_{1,\epsilon}$ in the first equation of (4.3), by $\underline{N}_1$. The solution of (4.109) is given by

$$\underline{I}_\epsilon(t) = \frac{I^0 e^{-(\mu_1^*+a)t - \frac{\gamma}{\epsilon}t + \frac{\lambda}{\epsilon}\int_0^t \underline{N}_1(s)ds}}{1 + \frac{\lambda}{\epsilon} I^0 \int_0^t e^{-(\mu_1^*+a)s - \frac{\gamma}{\epsilon}s + \frac{\lambda}{\epsilon}\int_0^s \underline{N}_1(\tau)d\tau} ds}, \tag{4.110}$$

with

$$\int_0^t \underline{N}_1(s)ds = \begin{cases} \displaystyle\int_0^t \underline{N}_{1,1}(s)ds, \ 0 \le t < \eta, \\ \displaystyle\int_0^\eta \underline{N}_{1,1}(s)ds + \int_\eta^t \underline{N}_{1,2}(s)ds, \ \eta \le t < \hat{t}, \\ \displaystyle\int_0^\eta \underline{N}_{1,1}(s)ds + \int_\eta^{\hat{t}} \underline{N}_{1,2}(s)ds + \int_{\hat{t}}^t \underline{N}_{1,3}(s)ds, \ \hat{t} \le t \le T_1. \end{cases}$$

Let

$$g(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}, t) \equiv -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(n_{1,\epsilon} - i_\epsilon) - \gamma)i_\epsilon$$

and

$$g(i_\epsilon, \underline{N}_1, n_{2,\epsilon}, t) \equiv -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(\underline{N}_1 - i_\epsilon) - \gamma)i_\epsilon.$$

Then from Inequalities (4.105), (4.89) and (4.83), we have that

$$g(i_\epsilon, \underline{N}_1, n_{2,\epsilon}, t) \le g(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}, t) \text{ for all } t \in [0, T_1].$$

Thus, $\underline{I}_\epsilon(t)$ satisfies on $[0, T_1]$ the inequality

$$\epsilon \frac{d\underline{I}_\epsilon}{dt} \le -\epsilon(\mu_1^* + a)\underline{I}_\epsilon + (\lambda(n_{1,\epsilon} - \underline{I}_\epsilon) - \gamma)\underline{I}_\epsilon. \tag{4.111}$$

It follows from Theorem B.1. in [20] that

$$\underline{I}_\epsilon(t) \leq i_\epsilon(t) \text{ for all } t \in [0, T_1]. \tag{4.112}$$

Then, from Inequalities (4.112), (4.89), (4.83), (4.76) and (4.69), we see that Inequalities

$$\begin{aligned} \underline{I}_\epsilon(t) &\leq i_\epsilon(t) \leq \bar{I}_\epsilon(t), \\ \underline{N}_1(t) &\leq n_{1,\epsilon}(t) \leq \bar{N}_1(t), \\ \underline{N}_2(t) &\leq n_{2,\epsilon}(t) \leq \bar{N}_2(t) \end{aligned} \tag{4.113}$$

hold for all $t \in [0, T_1]$.
Hence, the triplet $(\underline{I}_\epsilon(t), \underline{N}_1(t), \underline{N}_2(t))$ is a lower solution of (4.3) for all $t \in [0, T_1]$. Note that for all $t \in [0, T_1]$, $\underline{I}_\epsilon(t)$ is the solution of a scalar equation. Therefore, as in Section 4.2, we define the function $\underline{G}(t, \epsilon)$ as follows

$$\underline{G}(t, \epsilon) \equiv \underline{G}_{\eta,\varrho,\hat{t}}(t, \epsilon) = \frac{1}{\epsilon}\left(\int_0^t \underline{N}_1(s)ds - \gamma t\right) - (\mu_1^* + a)t. \tag{4.114}$$

Let us define $\underline{G}(t)$ as

$$\underline{G}(t) \equiv \underline{G}_{\eta,\varrho,\hat{t}}(t) = \lambda \int_0^t \underline{N}_1(s)ds - \gamma t.$$

We observe that on $[0, \hat{t}]$,

$$\underline{G}_{\eta,\varrho,\hat{t}}(t) \leq \bar{G}(t) \text{ and } \lim_{\eta,\varrho \to 0} \underline{G}_{\eta,\varrho,\hat{t}}(t) = \bar{G}(t).$$

Moreover, we observe that System (4.106) is equivalent to

$$\begin{aligned} \frac{d\underline{N}_{1,3}}{dt} &= -(\mu_1^* + a)\underline{N}_{1,3} + \beta \underline{N}_{2,3} + (\mu_1^* - \mu_1)\nu, \quad \underline{N}_{1,3}(\hat{t}) = \hat{N}_1, \\ \frac{d\underline{N}_{2,3}}{dt} &= -\mu_2 \underline{N}_{2,3} + a\underline{N}_{1,3}, \quad \underline{N}_{2,3}(\hat{t}) = \hat{N}_2. \end{aligned}$$

This implies that

$$\begin{pmatrix} \underline{N}_{1,3}(t) \\ \underline{N}_{2,3}(t) \end{pmatrix} \geq e^{\mathcal{A}^* t} \begin{pmatrix} \hat{N}_1 \\ \hat{N}_2 \end{pmatrix}.$$

It follows that

$$\underline{N}_{1,3}(t) \geq \nu + \iota_1 \text{ and } \underline{N}_{2,3}(t) \geq \nu(\mu_1 + a)/\beta + \iota_2.$$

on $[\hat{t}, T_1]$. As noted above, $\bar{N}_1(t) > \nu + 2\iota_1$ and $\bar{N}_2(t) > \nu(\mu_1 + a)/\beta + 2\iota_2$ for $t \in [\bar{t}, \bar{t}^*]$ for some $\iota_1, \iota_2 > 0$ and $\bar{t} \in (\bar{t}_c, \bar{t}^*)$; and for a fixed $0 < \iota' < \min\{\iota_1, \iota_2\}$ there is $\Omega > 0$ such that

Inequality (4.102) holds.
Then since $\underline{G}(t,\eta,\varrho) \to \bar{G}(t)$ as $(\eta,\varrho) \to (0,0)$ on $[0,\bar{t}^*]$, if we fix η,ϱ and $\hat{t} \in [\bar{t},\bar{t}^*]$ such that

$$\underline{N}_{1,2}(\hat{t}) > \nu + \iota',\ \underline{N}_{2,2}(\hat{t}) > \nu(\mu_1 + a)/\beta + \iota',\ -\underline{G}(\hat{t}) < \lambda\iota'(T' - \bar{t}^*) \text{ and } \bar{G}(\hat{t}) \to \bar{G}(\bar{t}^*) = 0,$$

then we have $\underline{G}(\hat{t}) < 0$ and

$$\begin{aligned}\underline{G}(T') &= \underline{G}(\hat{t}) + \lambda \int_{\hat{t}}^{T'} \underline{N}_{3,1}(s)ds - \gamma(T' - \hat{t}) \\ &\geq \underline{G}(\hat{t}) + \lambda(\nu + \iota')(T' - \hat{t}) - \gamma(T' - \hat{t}) \\ &> \underline{G}(\hat{t}) + \lambda\iota'(T' - \hat{t}) > 0.\end{aligned} \tag{4.115}$$

Hence, $\underline{G}(t) = 0$ has a root $\underline{t}^* = \underline{t}^*(\eta,\varrho,\hat{t})$ in $(\bar{t}^*, T')$ and, by monotonicity, in $(\bar{t}^*, T_1)$. Because of monotonicity, $\underline{t}^*$ is unique in $(\bar{t}^*, T')$. If we substitute T' in (4.115) by $\underline{t}^*$, we obtain

$$0 > \underline{G}(\hat{t}) + \lambda\iota'(\underline{t}^* - \hat{t}). \tag{4.116}$$

Inequality (4.116) implies that $\underline{t}^* < -\underline{G}(\hat{t})/\lambda\iota' + \hat{t}$. Therefore as $(\eta,\varrho) \to (0,0)$ and $\hat{t} \to \bar{t}^*$, we have $\underline{t}^* \to \bar{t}^*$.
In what follows, if we fix constants η, ϱ, ι_1, ι_2 and $\hat{t}$ such that for some $\epsilon_1 \equiv \epsilon_1(\eta,\varrho,\iota_1,\iota_2,\hat{t})$ we have for all $\epsilon < \epsilon_1$, $\underline{I}_\epsilon(t) \leq i_\epsilon(t)$ on $[0,T_1]$ and $\underline{G}(\underline{t}^*) = 0$. Since for $t > \underline{t}_c$, $\underline{G}(t)$ is an increasing function, we have that

$$\underline{G}(\hat{t}) < \underline{G}(\underline{t}^*) = 0.$$

Then

$$\underline{G}(t) < 0 \text{ for all } 0 < t < \underline{t}^* \text{ and } \underline{G}(t) > 0 \text{ for all } t > \underline{t}^*.$$

Therefore, it follows that, for η,ϱ sufficiently small and $\hat{t}$ sufficiently close to $\bar{t}^*$, for all $0 < t < \underline{t}^*$,

$$\frac{1}{\epsilon}\underline{G}(t) - (\mu_1^* + a)t < 0;$$

and for η,ϱ,ϵ sufficiently small and $\hat{t}$ sufficiently close to $\bar{t}^*$, for all $t > \underline{t}^*$,

$$\frac{1}{\epsilon}\underline{G}(t) - (\mu_1^* + a)t > 0.$$

Then we have the following result

Proposition 4.6.3

Let $\delta < 0$ and $N_1^0 < \nu$. Then for η and ϱ sufficiently small and $\hat{t}$ sufficiently close to $\bar{t}^*$, we have

$$\begin{aligned}&\lim_{\epsilon\to 0} \underline{I}_\epsilon(t) = 0 \quad \text{for} \quad 0 < t < \underline{t}^* \\ &\lim_{\epsilon\to 0} \underline{I}_\epsilon(t) = \underline{N}_1(t) - \nu \quad \text{for} \quad \underline{t}^* < t \leq T_1.\end{aligned} \tag{4.117}$$

The proof of this result follows from the one dimensional case in Section 4.2, when the population increases. The numerical simulation illustrating this result is given by

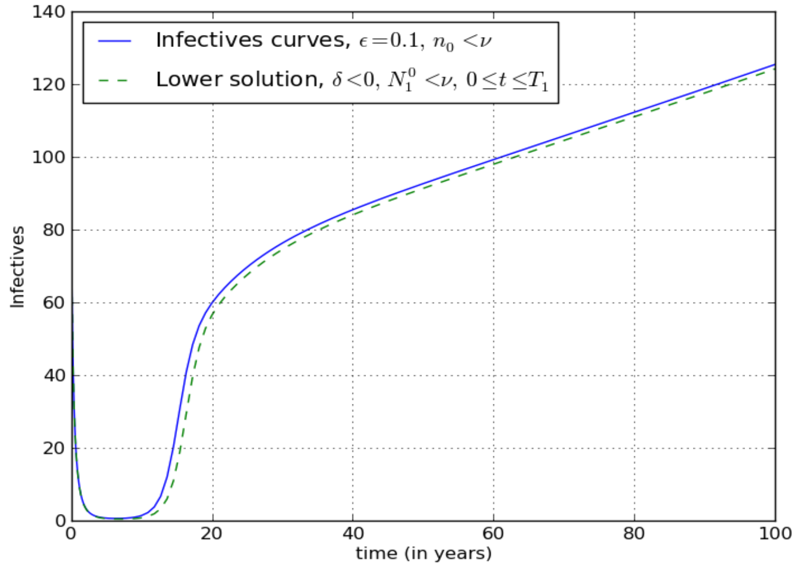


Figure 4.14: Lower solution of (4.3) given by (4.109) when $N_1^0 < \nu$ and $\delta < 0$.

Note that the lower solution obtained from (4.109) provides a very good approximation to the solution of (4.3), on $[0, T_1]$.
Next, in order to construct an upper solution that provides a good approximation to the solution of (4.3), we first prove that the convergence in the second expression of (4.117) is almost uniform. This is equivalent to prove that

$$\int_0^{\underline{G}(t)} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(z - \underline{G}(t))} \underline{\psi}(z) dz - \underline{\psi}(\underline{G}(t))$$

converges uniformly to zero as ϵ tends to zero, on $[\Theta, T_1]$ for any $\Theta \in (\underline{t}^*, T_1]$. This is the same as proving that for any ϑ there exists ϵ_0 in $[0, T_1]$ such that for any $\epsilon < \epsilon_0$

$$|\int_0^{\underline{G}(t)} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(z - \underline{G}(t))} \underline{\psi}(z) dz - \underline{\psi}(\underline{G}(t))| < \vartheta,$$

with ϑ a small positive constant not depending on t and

$$\underline{\psi}(z) = \frac{1}{\underline{G}'(\underline{G}^{-1}(z))}.$$

We have

$$\begin{aligned}
\int_0^a \frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}\underline{\psi}(z)dz - \underline{\psi}(a) &= \int_0^a \frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}\underline{\psi}(z)dz - \underline{\psi}(a) + \underline{\psi}(a)e^{-\frac{a}{\epsilon}} - \underline{\psi}(a)e^{-\frac{a}{\epsilon}} \\
&= \int_0^a \frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}\underline{\psi}(z)dz - \underline{\psi}(a)(1-e^{-\frac{a}{\epsilon}}) - \underline{\psi}(a)e^{-\frac{a}{\epsilon}} \\
&= \int_0^a [\underline{\psi}(z) - \underline{\psi}(a)]\frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}dz - \underline{\psi}(a)e^{-\frac{a}{\epsilon}} \\
&= \int_0^{a_\epsilon} [\underline{\psi}(z) - \underline{\psi}(a)]\frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}dz + \int_{a_\epsilon}^a [\underline{\psi}(z) - \underline{\psi}(a)]\frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}dz - \underline{\psi}(a)e^{-\frac{a}{\epsilon}},
\end{aligned}$$

where

$$a \equiv \underline{G}(t) = \lambda\int_0^t \underline{N}_1(s)ds - \gamma t,\ a_\epsilon = a - \sqrt{\epsilon},$$
$$a_\epsilon < a \text{ and } \lim_{\epsilon\to 0} a_\epsilon = a.$$

It follows that

$$\left|\int_0^a \frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}\underline{\psi}(z)dz - \underline{\psi}(a)\right| \le \mathcal{M}_1\int_0^{a-\sqrt{\epsilon}} \frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}dz + \mathcal{M}_2\int_{a-\sqrt{\epsilon}}^a \frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}dz - \underline{\psi}(a)e^{-\frac{a}{\epsilon}},$$

where

$$\mathcal{M}_1 \equiv \sup_{z\in[0,a_\epsilon]}|\underline{\psi}(z) - \underline{\psi}(a)| \text{ and } \mathcal{M}_2 \equiv \sup_{z\in[a_\epsilon,a]}|\underline{\psi}(z) - \underline{\psi}(a)|.$$

Thus,

$$\begin{aligned}
\left|\int_0^a \frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}\underline{\psi}(z)dz - \underline{\psi}(a)\right| &\le \mathcal{M}_1(e^{-\frac{1}{\sqrt{\epsilon}}} - e^{-\frac{a}{\epsilon}}) + \mathcal{M}_2(1 - e^{-\frac{1}{\sqrt{\epsilon}}}) - \underline{\psi}(a)e^{-\frac{a}{\epsilon}} \\
&\le \mathcal{M}_1(e^{-\frac{1}{\sqrt{\epsilon}}} - e^{-\frac{a}{\epsilon}}) + \mathcal{M}_2 - \underline{\psi}(a)e^{-\frac{a}{\epsilon}}.
\end{aligned}$$

Now, let

$$0 < a_\Theta = \underline{G}(\Theta) \quad \text{and} \quad a_{T_1} = \underline{G}(T_1).$$

We show that

$$\mathcal{M}_1(e^{-\frac{1}{\sqrt{\epsilon}}} - e^{-\frac{a}{\epsilon}}) + \mathcal{M}_2 - \underline{\psi}(a)e^{-\frac{a}{\epsilon}}$$

converges uniformly to zero on $[a_\Theta, a_{T_1}]$ as ϵ tends to zero. Note that, for $a \in [a_\Theta, a_{T_1}]$, $-a/\epsilon \le -a_\Theta/\epsilon$. This implies that

$$e^{-\frac{a}{\epsilon}} \le e^{-\frac{a_\Theta}{\epsilon}}.$$

Since $\underline{\psi}$ is bounded on $[a_\Theta, a_{T_1}]$, we have

$$|e^{-\frac{a}{\epsilon}}\underline{\psi}(a)| \le |e^{-\frac{a_\Theta}{\epsilon}}\underline{\psi}(a)| \le \mathcal{M}e^{-\frac{a_\Theta}{\epsilon}} \to 0$$

uniformly on $[a_\Theta, a_{T_1}]$, where $|\underline{\psi}(a)| < \mathcal{M}$. Furthermore, since $[a_\epsilon, a] \subset [a_\Theta, a_{T_1}]$ and $[a_\Theta, a_{T_1}]$ is a compact interval, $\underline{\psi}(z)$ is uniformly continuous on it. We obtain that for any ϑ, there exists δ_ϑ such that for all

$$a_1,\ a_2 \in [a_\epsilon, a],\ |a_1 - a_2| < \delta_\vartheta \Longrightarrow |\underline{\psi}(a_1) - \underline{\psi}(a_2)| < \vartheta.$$

This shows that

$$\mathcal{M}_2 = \sup_{z\in[a_\epsilon,a]} |\underline{\psi}(z) - \underline{\psi}(a)|$$

converges uniformly to zero as ϵ tends to zero.
Moreover, $|e^{-\frac{1}{\sqrt{\epsilon}}} - e^{-\frac{a}{\epsilon}}| \le e^{-\frac{1}{\sqrt{\epsilon}}} - e^{-\frac{a_\Theta}{\epsilon}} \to 0$ uniformly in a as ϵ tends to zero. It follows that $\mathcal{M}_1(e^{-\frac{1}{\sqrt{\epsilon}}} - e^{-\frac{a}{\epsilon}}) + \mathcal{M}_2 - \underline{\psi}(a)e^{-\frac{a}{\epsilon}}$ converges uniformly to zero as ϵ tends to zero.

Hence,

$$|\int_0^a \frac{e^{\frac{(z-a)}{\epsilon}}}{\epsilon}\underline{\psi}(z)dz - \underline{\psi}(a)|$$

converges uniformly to zero as ϵ tends to zero. Note that from the first inequality in (4.113) we have

$$\underline{I}_\epsilon(t) \le i_\epsilon(t) \le \bar{I}_\epsilon(t).$$

Since

$$\lim_{\epsilon\to 0} \underline{I}_\epsilon(t) = \underline{N}_1(t) - \nu \text{ and } \lim_{\epsilon\to 0} \bar{I}_\epsilon(t) = \bar{N}_1(t) - \nu,$$

for all $t > \underline{t}^*$, we cannot conclude anything about $\lim_{\epsilon\to 0} i_\epsilon(t)$. In order to obtain a limit for $i_\epsilon(t)$ as ϵ tends to zero, we consider the inequality $\underline{I}_\epsilon(t) \le i_\epsilon(t)$, and the fact that $\underline{I}_\epsilon(t)$ converges almost uniformly to $\underline{I}(t) = \underline{N}_1(t) - \nu$ on $(\underline{t}^*, T_1]$. This implies that for all ξ, there exists ϵ_0 such that for all $\epsilon < \epsilon_0$

$$|\underline{I}_\epsilon(t) - \underline{I}(t)| \le \xi \text{ for all } t \in [\Theta, T_1].$$

This is equivalent to

$$-\xi \le \underline{I}_\epsilon(t) - \underline{I}(t) \le \xi \Longleftrightarrow \underline{I}(t) - \xi \le \underline{I}_\epsilon(t) \le \underline{I}(t) + \xi$$

and implies that

$$\underline{N}_1(t) - \nu - \xi \le i_\epsilon(t). \tag{4.118}$$

Then for all $t \in [\Theta, T_1]$, we define

$$\begin{pmatrix} \bar{N}_1^1(t) \\ \bar{N}_2^1(t) \end{pmatrix} = e^{\mathcal{A}(t-\Theta)} \begin{pmatrix} N_{1,\Theta} \\ N_{2,\Theta} \end{pmatrix} - \begin{pmatrix} (\mu_1^* - \mu_1) \\ 0 \end{pmatrix} \int_\Theta^t (\underline{N}_1(z) - \nu - \xi)e^{\mathcal{A}(t-\Theta-z)}dz, \tag{4.119}$$

where

$$N_{1,\Theta} = \bar{N}_1(\Theta),\ N_{2,\Theta} = \bar{N}_2(\Theta).$$

Then $(\bar{N}_1^1(t), \bar{N}_2^1(t))$ satisfies the following system

$$\begin{aligned} \frac{d\bar{N}_1^1}{dt} &= -(\mu_1 + a)\bar{N}_1^1 + \beta\bar{N}_2^1 - (\mu_1^* - \mu_1)(\bar{N}_1^1 - \nu) + (\mu_1^* - \mu_1)\xi, \quad \bar{N}_1^1(\Theta) = N_{1,\Theta}, \\ \frac{d\bar{N}_2^1}{dt} &= -\mu_2\bar{N}_2^1 + a\bar{N}_1^1, \quad \bar{N}_2^1(\Theta) = N_{2,\Theta}. \end{aligned} \tag{4.120}$$

Now, since

$$(\mu_1^* - \mu_1) > 0 \text{ and } \underline{N}_1(t) - \nu - \xi < i_\epsilon(t) \text{ for all } t \in [\Theta, T_1],$$

we have

$$-(\mu_1^* - \mu_1)i_\epsilon(t) \leq -(\mu_1^* - \mu_1)(\underline{N}_1(t) - \nu - \xi).$$

This implies that

$$\frac{d\bar{N}_1^1}{dt} \geq -(\mu_1^* + a)\bar{N}_1^1 + \beta\bar{N}_2^1 - (\mu_1^* - \mu_1)i_\epsilon.$$

Therefore

$$\begin{pmatrix} \frac{d\bar{N}_1^1}{dt} \\ \frac{d\bar{N}_2^1}{dt} \end{pmatrix} \geq \begin{pmatrix} f_1^1(i_\epsilon, \bar{N}_1^1, \bar{N}_2^1) \\ f_2^1(i_\epsilon, \bar{N}_1^1, \bar{N}_2^1) \end{pmatrix}, \tag{4.121}$$

where

$$f_1^1(i_\epsilon, \bar{N}_1^1, \bar{N}_2^1) = -(\mu_1 + a)\bar{N}_1^1 - (\mu_1^* - \mu_1)i_\epsilon + \beta\bar{N}_2^1 \text{ and } f_2^1(i_\epsilon, \bar{N}_1^1, \bar{N}_2^1) = -\mu_2\bar{N}_2^1 + a\bar{N}_1^1.$$

It follows from Theorem B.1. in [20] that

$$\begin{pmatrix} \bar{N}_1^1(t) \\ \bar{N}_2^1(t) \end{pmatrix} \geq \begin{pmatrix} n_{1,\epsilon}(t) \\ n_{2,\epsilon}(t) \end{pmatrix}, \tag{4.122}$$

for all $t \in [\Theta, T_1]$.
Hence, for all $t \in [0, T_1]$ we define $(\bar{\bar{N}}_1(t), \bar{\bar{N}}_2(t))$ as follows

$$\bar{\bar{N}}_1(t) \equiv \bar{\bar{N}}_{1,\eta,\varrho,\hat{t}}(t) = \begin{cases} \bar{N}_1(t), & 0 \leq t < \Theta, \\ \bar{N}_1^1(t), & \Theta \leq t \leq T_1, \end{cases} \quad \bar{\bar{N}}_2(t) \equiv \bar{\bar{N}}_{2,\eta,\varrho,\hat{t}}(t) = \begin{cases} \bar{N}_2(t), & 0 \leq t < \Theta, \\ \bar{N}_2^1(t), & \Theta \leq t \leq T_1, \end{cases} \tag{4.123}$$

where $(\bar{N}_1(t), \bar{N}_2(t))$ is given by (4.68).
Note that since $\Theta > \underline{t}^* > \hat{t}$, comparing Systems (4.120) and (4.106) we have, by Theorem B.1. in [20], that

$$\frac{d\bar{N}_1^1}{dt} \geq \frac{\underline{N}_{1,3}}{dt} \text{ for all } t \in [\Theta, T_1]$$

and, by the regular perturbation theory, one can show that

$$\bar{N}_1^1(t) = \underline{N}_{1,3}(t) + \omega(\xi),$$

where

$$\omega(\xi) \to 0 \text{ as } \xi \to 0.$$

It follows that for all $t \in [\Theta, T_1]$, we have

$$\bar{\bar{N}}_1(t) = \underline{N}_1(t) + \omega(\xi). \tag{4.124}$$

Consider the equation

$$\epsilon \frac{d\bar{\bar{I}}_\epsilon}{dt} = -\epsilon(\mu_1^* + a)\bar{\bar{I}}_\epsilon + (\lambda(\bar{\bar{N}}_1 - \bar{\bar{I}}_\epsilon) - \gamma)\bar{\bar{I}}_\epsilon, \quad \bar{\bar{I}}_\epsilon(0) \equiv I^0 = i^0, \tag{4.125}$$

obtained by substituting $n_{1,\epsilon}$ in the first equation of (4.3) by $\bar{\bar{N}}_1$. Then the solution of (4.125) is given by

$$\bar{\bar{I}}(t, \epsilon) = \frac{I^0 e^{-(\mu_1^*+a)t - \frac{\gamma}{\epsilon}t + \frac{\lambda}{\epsilon}\int_0^t \bar{\bar{N}}_1(s)ds}}{1 + \frac{\lambda}{\epsilon} I^0 \int_0^t e^{-(\mu_1^*+a)s - \frac{\gamma}{\epsilon}s + \frac{\lambda}{\epsilon}\int_0^s \bar{\bar{N}}_1(\tau)d\tau} ds}, \tag{4.126}$$

with

$$\int_0^t \bar{\bar{N}}_1(s)ds = \begin{cases} \displaystyle\int_0^t \bar{N}_1(s)ds, \ 0 \le t < \Theta, \\ \displaystyle\int_0^\Theta \bar{N}_1(s)ds + \int_\Theta^t \underline{N}_1^1(s)ds, \ \Theta \le t \le \ T_1. \end{cases}$$

Let

$$g(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}, t) \equiv -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(n_{1,\epsilon} - i_\epsilon) - \gamma)i_\epsilon$$

and

$$g(i_\epsilon, \bar{\bar{N}}_1, n_{2,\epsilon}, t) \equiv -\epsilon(\mu_1^* + a)i_\epsilon + (\lambda(\bar{\bar{N}}_1 - i_\epsilon) - \gamma)i_\epsilon.$$

Then, from Inequalities (4.122) and (4.69), we have that

$$g(i_\epsilon, \bar{\bar{N}}_1, n_{2,\epsilon}, t) \ge g(i_\epsilon, n_{1,\epsilon}, n_{2,\epsilon}, t) \text{ for all } t \in [0, T_1].$$

Thus, $\bar{\bar{I}}(t, \epsilon)$ satisfies on $[0, T_1]$ the inequality

$$\epsilon \frac{d\bar{\bar{I}}_\epsilon}{dt} \le -\epsilon(\mu_1^* + a)\bar{\bar{I}}_\epsilon + (\lambda(n_{1,\epsilon} - \bar{\bar{I}}_\epsilon) - \gamma)\bar{\bar{I}}_\epsilon. \tag{4.127}$$

It follows from Theorem B.1. in [20] that

$$\bar{\bar{I}}_\epsilon(t) \geq i_\epsilon(t) \tag{4.128}$$

for all $t \in [0, T_1]$.

Hence, the triplet $(\bar{\bar{I}}_\epsilon(t), \bar{\bar{N}}_1(t), \bar{\bar{N}}_2(t))$ is an upper solution of (4.3) for all $t \in [0, T_1]$. Note that for all $t \in [0, T_1]$ $\bar{\bar{I}}_\epsilon(t)$ is the solution of scalar equation. Therefore as in Section 4.2, we define the function $\bar{\bar{G}}(t, \epsilon)$ as follows

$$\bar{\bar{G}}(t, \epsilon) \equiv \bar{\bar{G}}_{\eta,\varrho,\hat{t}}(t, \epsilon) = \frac{1}{\epsilon}\left(\int_0^t \bar{\bar{N}}_1(s)ds - \gamma t\right) - (\mu_1^* + a)t. \tag{4.129}$$

Observe that on $[0, \Theta]$, $\bar{\bar{G}}(t, \epsilon) \equiv \bar{G}(t, \epsilon)$. We have shown the existence of $\bar{t}^*$ with $\bar{t}^* < \underline{t}^* < \Theta$, as well as the existence of $\bar{t}_c$ with $\bar{t}_c < \bar{t}^*$ such that

$$\bar{G}(\bar{t}^*) = 0 \text{ and } \bar{G}'(\bar{t}_c) = 0.$$

So, this implies that

$$\bar{\bar{G}}(\bar{t}^*) = 0 \text{ and } \bar{\bar{G}}'(\bar{t}_c) = 0.$$

Thus,

$$\bar{\bar{G}}(t) < 0 \text{ for all } 0 < t < \bar{t}^* \text{ and } \bar{\bar{G}}(t) > 0 \text{ for all } t > \bar{t}^*.$$

Therefore it follows that for η, ϱ sufficiently small and $\hat{t}$ sufficiently close to $\bar{t}^*$, for all $0 < t < \bar{t}^*$

$$\frac{1}{\epsilon}\bar{\bar{G}}(t) - (\mu_1^* + a)t < 0,$$

and for all $t > \bar{t}^*$,

$$\frac{1}{\epsilon}\bar{\bar{G}}(t) - (\mu_1^* + a)t > 0.$$

Then we have the following result

Proposition 4.6.4

Let $\delta < 0$ and $N_1^0 < \nu$. Then, for η, ϱ sufficiently small and $\hat{t}$ sufficiently close to $\bar{t}^*$,

$$\begin{aligned} &\lim_{\epsilon \to 0} \bar{\bar{I}}_\epsilon(t) = 0 \quad \text{for} \quad 0 < t < \bar{t}^* \\ &\lim_{\epsilon \to 0} \bar{\bar{I}}_\epsilon(t) = \bar{\bar{N}}_1(t) - \nu \quad \text{for} \quad \bar{t}^* < t < T_1. \end{aligned} \tag{4.130}$$

The proof of this result follows from the one dimensional case in Section 4.2, when the population is increasing.

It follows from (4.130), (4.128), (4.117) and (4.112) that

$$\lim_{\epsilon \to 0} \underline{I}_\epsilon(t) \leq \lim_{\epsilon \to 0} i_\epsilon(t) \leq \lim_{\epsilon \to 0} \bar{\bar{I}}(t, \epsilon) \text{ for all } 0 < t \leq T_1.$$

As before, for any $\xi > 0$ we can choose η and ϱ sufficiently small, and $\hat{t}$ close enough to $\bar{t}^*$ in such a way that there exists ϵ_1 such that for all $\epsilon < \epsilon_1$, we have

$$\underline{N}_1(t) - \nu - \xi \leq i_\epsilon(t) \leq \bar{\bar{N}}_1(t) - \nu + \xi \text{ for } t \in [\Theta, T_1].$$

Let (n_1, n_2) be the solution of the system

$$\begin{aligned} \frac{dn_1}{dt} &= -(\mu_1^* + a)n_1 + \beta n_2 + (\mu_1^* - \mu_1)\nu, \ n_1(\bar{t}^*) = \bar{N}_1(\bar{t}^*), \\ \frac{dn_2}{dt} &= -\mu_2 n_2 + a n_1, \ n_2(\bar{t}^*) = \bar{N}_2(\bar{t}^*). \end{aligned} \tag{4.131}$$

System (4.131) is a regular perturbation of both Systems (4.107) and (4.120) that satisfies

$$\underline{N}_1(t) \le n_1(t) \le \bar{\bar{N}}_1(t) \text{ for } t \in [\Theta, T_1].$$

Hence for any $\omega' > 0$ we can choose η, ϱ, ξ small enough, and $\hat{t}$ and Θ close enough to $\bar{t}^*$ in such a way that there is ϵ_2 such that for all $\epsilon < \epsilon_2$, we have

$$|\underline{N}_1(t) - n_1(t)| \le \omega', \ |\bar{\bar{N}}_1(t) - n_1(t)| \le \omega' \text{ for } t \in [\Theta, T_1]$$

and it follows that

$$n_1(t) - \nu - \xi - \omega' \le i_\epsilon(t) \le n_1(t) - \nu + \xi + \omega' \text{ for } t \in [\Theta, T_1].$$

Taking into account the fact that $\xi > 0$, $\omega' > 0$ were chosen arbitrarily small and Θ was chosen arbitrarily close to $\bar{t}^*$, and using the first identities in (4.117) and (4.130) we obtain, by the Sandwich Theorem, that

$$\begin{aligned} &\lim_{\epsilon \to 0} i_\epsilon(t) = 0 \text{ for all } 0 < t < \bar{t}^*, \\ &\lim_{\epsilon \to 0} i_\epsilon(t) = n_1(t) - \nu \text{ for all } \bar{t}^* < t \le T_1. \end{aligned} \tag{4.132}$$

Similarly

$$\begin{aligned} &\lim_{\epsilon \to 0} n_{1,\epsilon}(t) = \bar{n}_1(t) \text{ for all } 0 < t < \bar{t}^*, \\ &\lim_{\epsilon \to 0} n_{1,\epsilon}(t) = n_1(t) \text{ for all } \bar{t}^* < t \le T_1, \end{aligned} \tag{4.133}$$

where $\bar{n}_1(t)$ is the solution of the first equation of (4.10).

Since $n_{1,\epsilon} = s_\epsilon + i_\epsilon$, we have

$$\begin{aligned} &\lim_{\epsilon \to 0} s_\epsilon(t) = \bar{n}_1(t) \text{ for all } 0 < t < \bar{t}^* \\ &\lim_{\epsilon \to 0} s_\epsilon(t) = \nu \text{ for all } \bar{t}^* < t \le T_1. \end{aligned} \tag{4.134}$$

Now, we determine T_1. In Section 4.1.2 we showed that

$$\mathcal{V}_2 = \{(n_1, n_2) \in I_{n_1} \times I_{n_2} : n_1 \ge \nu + \delta\beta/(\mu_1^* + a), n_2 \ge \nu(\mu_1 + a)/\beta + \delta\}$$

is invariant under the flow generated by (4.16). Since ι_1, ι_2 are arbitrarily small, we can choose δ such that $2\iota_1 = \delta\beta(\mu_1^* + a)$ and $2\iota_2 = \delta$, and select Ω and some appropriate ι' for this choice

of ι_1 and ι_2.
Then, by what preceeds, we have $(n_{1,\epsilon}(t), n_{2,\epsilon}(t)) \to (\tilde{n}_1(t), \tilde{n}_2(t))$ as $\epsilon \to 0$ on $(\bar{t}^*, \bar{t}+\Omega]$.
Since $(\tilde{n}_1(\bar{t}+\Omega), \tilde{n}_2(\bar{t}+\Omega)) \in \mathcal{V}_2$, there exists ϵ_2 such that for all $\epsilon < \epsilon_2$, $n_{1,\epsilon}(\bar{t}+\Omega) > \nu + \iota_1$, $n_{2,\epsilon}(\bar{t}+\Omega) > \nu(\mu_1+a)/\beta + \iota_2$.
Thus, $n_{1,\epsilon}(\bar{t}+t) > \nu + \iota'$, $n_{2,\epsilon}(\bar{t}+t) > \nu(\mu_1+a)/\beta + \iota'$ for all $t \in [0, 2\Omega]$. Taking $T_1 = \bar{t} + 2\Omega$ and using the previous part of the proof, gives $(n_{1,\epsilon}(t), n_{2,\epsilon}(t)) \to (\tilde{n}_1(t), \tilde{n}_2(t))$ as $\epsilon \to 0$ on $(\bar{t}^*, \bar{t}+2\Omega]$ and $(\tilde{n}_1(\bar{t}+2\Omega), \tilde{n}_2(\bar{t}+2\Omega)) \in \mathcal{V}_2$. This procedure can be repeated with the same constant Ω until we reach T.

Now we are going to formulate the theorem that summarizes the result given above.

Theorem 4.6.1

Let $(n_{1,\epsilon}, n_{2,\epsilon}, i_\epsilon)$ be the solution of (4.3), $(\bar{n}_1, \bar{n}_2)$ the solution of (4.10) and (n_1, n_2) the solution of (4.131), where $\bar{t}^*$ is defined in (4.77) and $n_{1,\epsilon} = s_\epsilon + i_\epsilon$. If $\delta < 0$ and $N_1^0 < \nu$, then for any $T > 0$, we have

$$\begin{aligned} &\lim_{\epsilon\to 0} n_{1,\epsilon}(t) = \bar{n}_1(t), \\ &\lim_{\epsilon\to 0} n_{2,\epsilon}(t) = \bar{n}_2(t) \end{aligned} \tag{4.135}$$

almost uniformly on $[0, \bar{t}^*)$,

$$\begin{aligned} &\lim_{\epsilon\to 0} i_\epsilon(t) = 0, \\ &\lim_{\epsilon\to 0} s_\epsilon(t) = \bar{n}_1(t) \end{aligned} \tag{4.136}$$

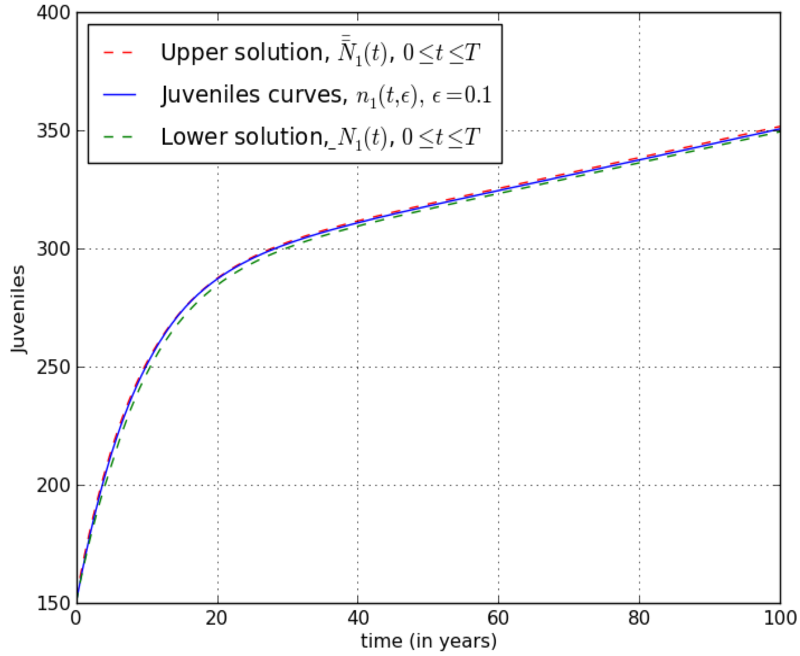
almost uniformly on $(0, \bar{t}^*)$ and

$$\begin{aligned} &\lim_{\epsilon\to 0} n_{1,\epsilon}(t) = n_1(t), \\ &\lim_{\epsilon\to 0} n_{2,\epsilon}(t) = n_2(t), \\ &\lim_{\epsilon\to 0} i_\epsilon(t) = n_1(t) - \nu, \\ &\lim_{\epsilon\to 0} s_\epsilon(t) = \nu \end{aligned} \tag{4.137}$$

almost uniformly on $(\bar{t}^*, T]$.

4.6.2 Numerical simulation

Here we present numerical simulations describing the upper and the lower solutions and also the delayed exchange of stabilities of the infectives curve $i_\epsilon(t)$ and the susceptibles curve $s_\epsilon(t)$ given respectively by the age structured SIS models (4.3) and (4.2) when $N_1^0 < \nu$ and $\delta < 0$. The parameters' values used are $\mu_1 = 0.03$, $\mu_2 = 0.019$, $a = 0.06$, $\beta = 0.033$, $\mu_1^* = 0.035$, $\gamma = 0.44$, $\lambda = 0.002$, $\epsilon = 0.03,\ 0.02,\ 0.01$ and the initial conditions are $n_1^0 = 150$, $n_2^0 = 900$ and $i^0 = 80$.



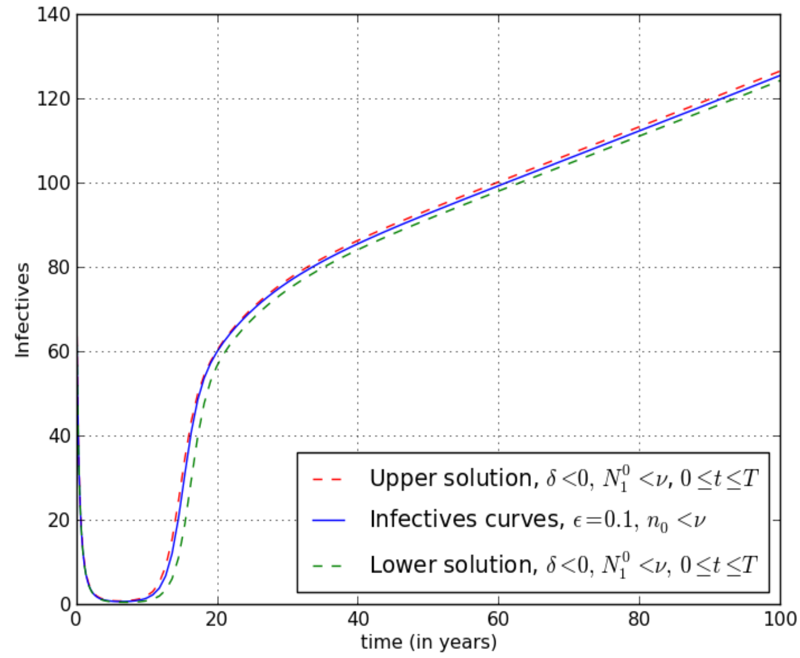


Figure 4.15: Upper and lower solutions of (4.3) given respectively by Equations of (4.109) and (4.126) when $N_1^0 < \nu$ and $\delta < 0$.

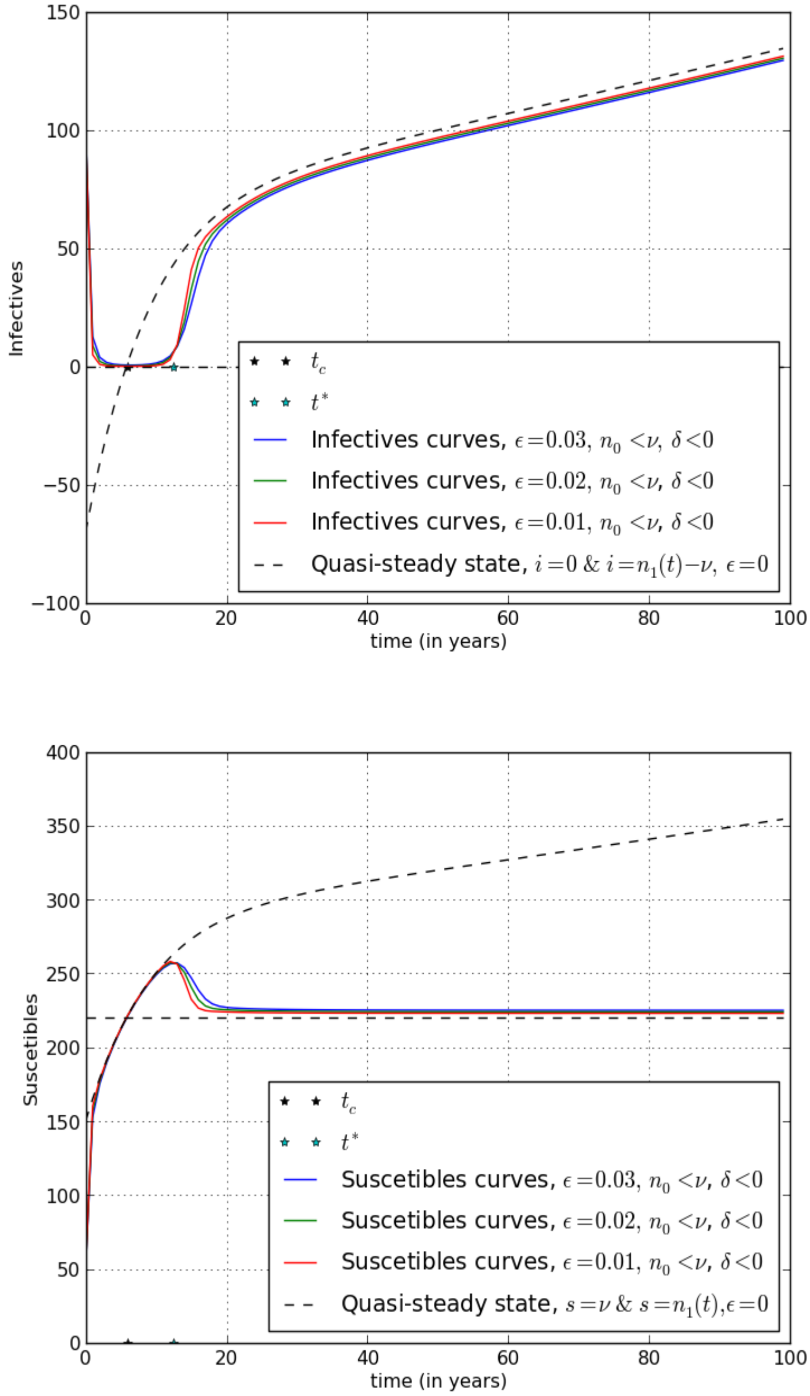


Figure 4.16: Delayed exchange of stabilities given by the solutions $i_\epsilon(t)$ of the first equation of (4.3) and $s_\epsilon(t)$ of the first equation of (4.2), and the composed stable quasi-steady state when $n_1^0 < \nu$ and $\delta < 0$.

4.6.3 Comment on the case of decreasing population

The population decreases when $N_1^0 > \nu$ and $\delta > 0$. In this case, as for the one dimensional case developed in Section 4.2, the infective curve will decrease following the stable composed quasi-steady states. But, because System (4.3) does not provide exact solutions, we proceed exactly like in the case of increasing population discussed above; that is, we construct an upper and a lower solution that converge to the same limit, to show the existence of the solution of (4.3) when the population is decreasing.
However, the construction of an upper and a lower solution in this case is more involved than in the case of an increasing population.

4.6.4 Discussion

We have analyzed the behaviour of an age structure SIS model in the critical case, namely when the quasi-steady states of the limit equation are not isolated. Because of the presence of the small parameter in the solution of the perturbed equation, we have used the idea of the upper and lower solutions, introduced by Butuzov, to approximate the model's behaviour.

Conclusion

In this thesis, we have analyzed the behaviour of solutions of singular perturbation problems. We focused on the approach by Tikhonov-Vasil'eva theorems, but we also considered some connections of this theorem with the centre manifold approach.
First, we presented a standard application of the Tikhonov treorem to a population model. We showed, in particular, how to derive under a suitable scaling an Allee type dynamics from more primitive building blocks such as the law of mass action and logistic dynamics. We note that though the model is based on the results in [13], the choice of the small parameter in that book is incorrect, see Remark 1.3.1 in [7].
We also considered all other reasonable scalings of the model showing that they lead to models predicting the extinction of the whole population.
In the next chapter, we addressed the problem that Tikhonov theorem only gives convergence on finite time intervals. However, numerical simulations often indicate that the convergence occurs on the whole half line. In many cases this is related to the fact that the limit equation is not structurally stable.
We considered two examples. For one coming from the enzyme kinetics reaction, the limit equation is structurally stable. Here we showed that the zeroth order approximation provided by Tikhonov theorem allows for the extension of the concern result to the whole half line. The proof is based on the centre manifold theory.
The second example, describing a prey-predator model with fast migration, has a structurally unstable limit equation, namely the Lotka-Volterra system. Here numerical simulations indicated that the convergence of the solution of the original problem to the solution of the Lotka-Volterra system is not uniform in time (the longer time interval we take, the smaller the parameter should be taken to ensure the required accuracy of approximation).
To improve the result, we again used the centre manifold theory to construct a correction to the Lotka-Volterra equation in such a way that the convergence of the solution of the original equation to the solution of the corrected limit equation is uniform on the whole half line.
The final part, which constitutes the main result of the thesis, concerned the so-called exchange of stabilities in singularly perturbed problems. The exchange of stabilities occurs when the quasi-steady states of the problem intersect and the solutions of the perturbed problem pass close to the intersection.
There are two possible scenari. One is when the exchange of stabilities occurs immediately after the intersection (called immediate exchange of stabilities) and the second is when the solution of the original equation follows for some time the old, the now repelling quasi-steady state, and jump to the new, the now attracting quasi-steady state, with some delay (called delayed exchange of stabilities). Such a situation occurs in numerous epidemiological models.
We have considered an SIS model with vital dynamics. Here, we considered two types of behaviour. First, we showed that under certain conditions the solutions stayed in the basins of attraction of specific quasi-steady states whereupon the standard Tikhonov theorem was applicable. In the second case, we allowed the solution to pass close to the intersection of the quasi-steady states and using same ideas as Butuzov [2], we provided a comprehensive analysis of this case when the system can be reduced to a one- dimensional one, and later we used the monotonicity properties of the general system to extend the one dimension results to the multi-

demsional case.
This result needs to be generalized to a larger class of singularly perturbed problem. This is one of the objectives of my future work.
Furthermore, in my future work, I would like also to understand the meaning of the exchange of stabilities in the context of real models and its implication for their qualitative properties of epidemies, or population dynamics.

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