UNIVERSITY OF KWAZULU-NATAL

MODELS IN ISOTROPIC COORDINATES WITH EQUATION OF STATE

Sifiso Allan Ngubelanga

MODELS IN ISOTROPIC COORDINATES WITH EQUATION OF STATE

Sifiso Allan Ngubelanga

Submitted in fulfilment of the academic requirements for the degree of Doctor of Philosophy to the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal,

Durban

December 2014

As the candidate's supervisor, I have approved this dissertation for submission.

Signed:	Prof. Sunil Maharaj	December 2014
Signed:	Prof. Subharthi Ray	December 2014

Abstract

In this thesis we consider spacetimes which are static and spherically symmetric related to the Einstein and Einstein-Maxwell system of equations in isotropic coordinates. We study both neutral and charged matter distributions with isotropic and anisotropic pressures, respectively. Our aim is to model relativistic stellar models. A known transformation that has been utilised by other researchers is applied to rewrite the field equations in equivalent forms. We produce new models to the Einstein system of equations with isotropic pressures by developing an algorithm that generates new classes of exact solutions if a particular seed solution is known. By applying the algorithm to the field equations and the condition of pressure isotropy we obtain a nonlinear Bernoulli equation which can be integrated. We also consider charged matter distributions with anisotropic pressures by introducing barotropic equations of state. Both linear and quadratic equations of state are considered and new exact solutions of the Einstein-Maxwell system are found. This is achieved by specifying a particular form for one of the metric functions and the electric field intensity. We select particular parameter values to regain the masses of known stars. For the linear equation of state we regain masses of the stars PSR J1614-2230, Vela X-1, PSR J1903+327, 4U 1820-30 and SAX J1808.4-3658. The masses for the stars PSR J1614-2230, 4U 1608-52, PSR J1903+327, EXO 1745-248 and SAX J1808.4-3658 are generated when a quadratic equation of state is imposed. Extensive physical analyses for the stars PSR J1614-2230 and PSR J1903+327 indicate that our models are well behaved.

Declaration - Plagiarism

I, Sifiso Allan Ngubelanga

Student Number: 205518226

declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

4. This thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:

   a. Their words have been re-written but the general information attributed to them has been referenced.

   b. Where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.

5. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed:

Declaration of Publications

Details of contribution to publications that form part and/or include research presented in this thesis.

Publication 1:

S. A. Ngubelanga and S. D. Maharaj, *A relativistic algorithm with isotropic coordinates*, *Adv. Math. Phys.* 905168 (2013).

Publication 2:

S. A. Ngubelanga, S. D. Maharaj and S. Ray, *Compact stars with linear equation of state*, *Astrophys. Space Sci.* in press (2014).

Publication 3:

S. A. Ngubelanga, S. D. Maharaj and S. Ray, *Compact objects with a quadratic equation of state,* submitted (2014).

To

My late Grandmother,

for encouraging me to study further.

Acknowledgments

I wish to sincerely express my appreciation to the following people and organisations whose assistance made this dissertation a reality:

- I would like to thank God for guiding me throughout this study and for everything He's done for me.
- My supervisors Professor S. D. Maharaj and Professor S. Ray for their assistance, guidance and support which resulted in the completion of my dissertation and for believing in me even at times when I didn't believe in myself . They have always been there for me, making sure that I worked to the best of my ability and potential. Their supervision has been of great assistance to me.
- My colleagues in the School of Mathematics, Statistics and Computer Sciences for their support and assistance.
- Members of staff in the School of Mathematics, Statistics and Computer Sciences for their on-going support and encouragement.
- The University of KwaZulu-Natal for granting me the opportunity to take on and complete this study.
- The National Research Foundation for financial assistance through the award of an NRF PhD scarce-skills scholarship.
- My family for their endless support and encouragement; Especially my late Grandmother and Aunt (M.T.S.R.I.P) for motivating me to complete my studies.

- My dear friends who have shared every moment of sadness and happiness with me.

Thank you! Ngiyabonga!

List of Figures

List of Tables

Contents

Chapter 1

Introduction

The description of the behaviour of the gravitational field is provided by the theory of general relativity. The predictions of general relativity have been shown to be consistent with observational data in relativistic stellar physics. The Riemann tensor describes the curvature of the spacetime in general relativity. The gravitational field is described by the Einstein tensor. The energy momentum tensor gives the best description for the matter content. In this thesis we study matter distributions which are described by neutral perfect fluids and charged imperfect fluids. The matter content is related to the curvature by the Eintein and Einstein-Maxwell field equations. The conservation laws are satisfied by the Einstein-Maxwell field equations through the generalised Bianchi identity. It is vital to determine the explicit solutions to the Einstein-Maxwell system since they play a crucial role in the applications of astrophysics and cosmology. Several recent investigations highlight the important role of the charged relativistic fluids in astrophysics; for example the treatments of Arbanil *et al* (2014), Andersson *et al* (2014) and Fraga *et al* (2014) contain studies of physically reasonable charged compact spheres.

The exact solutions to the Einstein-Maxwell system of equations are important in the generation of new models. These equations may be applied in describing quark stars, neutron stars, gravastars, the physical properties of compact relativistic objects, etc, with different matter configurations. There have been several recent solutions found to the Einstein-Maxwell field equations. Maharaj *et al* (2014) studied models for quark stars with charge and anisotropy to the Einstein-Maxwell field equations. A comprehensive physical analysis for a class of exact models to the Einstein-Maxwell system of equations has recently been investigated by Sunzu *et al* (2014a). A new class of exact solutions to the field equations has recently been studied by Sunzu *et al* (2014b). They found well behaved new models to the Einstein-Maxwell systems by generating new models with charge and anisotropy. Maharaj and Mafa Takisa (2012) obtained new models to the systems of equations which contain earlier known models of some other researchers. New exact solutions to the Einstein-Maxwell field equations were generated by Mafa Takisa and Maharaj (2013a) in terms of elementary functions. Mafa Takisa and Maharaj (2013a) investigated charged and anisotropic matter distributions to generate new models to the field equations. New models to the Einstein-Maxwell system of equations with charge and anisotropic bodies were analysed by Mafa Takisa *et al* (2014a). They showed that certain physical quantities are well behaved. Murad (2013) studied the Einstein-Maxwell systems of equations for charged perfect spheres to generate new models with parametric interior solutions. A suitable family of exact solutions for modeling of charged compact objects was obtained by Fatema and Murad (2013). They claim that for particular parameter values their solutions are regular and physically acceptable. Kiess (2012) electrified the Tolman VII metric by generating exact models to the Einstein-Maxwell metric for charged perfect fluid spheres.

All of the examples mentioned above are mainly in canonical coordinates. There are very few solutions that have been generated utilising isotropic coordinates. We provide some recent examples of known exact solutions to the Einstein-Maxwell systems in isotropic coordinates. New exact models to the Einstein-Maxwell field equations with charged perfect fluid spheres have been recently analysed by Pant *et al* (2014a). They studied the Hajj-Boutros type metric configurations and specified a particular choice for the electric field intensity. Pradhan and Pant (2014) provided models to the field equations for perfect fluid spheres which match smoothly to the Reissner-Nordström metric in the presence of charge. New classes of exact solutions to the Einstein-Maxwell systems which are physically acceptable were obtained by Ngubelanga *et al* (2014a,b). They considered matter distributions with charge and anisotropy where they regained the masses of known stars for particular parameter values. These examples indicate that isotropic coordinates are useful in generating physically reasonable models for relativistic stellar bodies.

There are several approaches that can be utilized in order to generate exact solutions to the field equations. One approach is to impose a barotropic equation of state which relates the radial pressure to the energy density. The barotropic equation of state can be linear, quadratic, polytropic, etc. In this thesis we study stellar objects with linear and quadratic equations of state. Esculpi and Aloma (2010), Mafa Takisa and Maharaj (2013a), Maharaj *et al* (2014) and Sunzu *et al* (2014a,b) have considered charged compact objects with anisotropy in canonical coordinates with a linear equation of state. An exact compact model with charge and anisotropic pressures in isotropic coordinates with a linear equation of state has been recently generated by Ngubelanga *et al* (2014a) in isotropic coordinates. Exact anisotropic solutions which are charged and uncharged with the quadratic equation of state were generated by Maharaj and Mafa Takisa (2012) and Thirukkanesh and Ragel (2014). Ngubelanga *et al* (2014b) analysed stellar objects with charge and anisotropic pressures by imposing a quadratic equation of state in isotropic coordinates. It should be noted that there are relatively few known models with isotropic coordinates and equation of state.

In this thesis we consider models with charge and anisotropic matter distributions in isotropic coordinates with an equation of state. We seek to solve the Einstein-Maxwell system of equations and relate the new solutions to observed astronomical objects. This dissertation is organised as follows:

• Chapter 1: Introduction.

• Chapter 2: In this chapter we provide a relativistic algorithm that generates new classes of exact solutions to the Einstein field equations in isotropic coordinates. We consider static spherically symmetric spacetimes with neutral perfect fluids and isotropic pressures. The Einstein field equations and the measure of pressure isotropy are given in new equivalent forms by utilising the transformation due to Kustaanheimo and Qvist (1948). We apply the algorithm to the master equation which takes the form of a non-linear equation. We present the classes of new exact solutions to the field equations obtained in terms of arbitrary functions. We provide a simple example in terms of elementary functions for new solutions and we also present graphical plots for the energy density ρ, pressure p and the speed of sound $\frac{dp}{d\rho}$.

• Chapter 3: In this chapter we present the new classes of exact solutions by imposing a linear barotropic equation of state for matter configurations with anisotropy and charge in isotropic coordinates. We generate the Einstein-Maxwell system of equations and the measure of anisotropy. By matching the first and the second fundamental forms for the Schwarzschild and the Reissner-Nordström metrics we obtain the junction conditions at the boundary. New variables are introduced to rewrite the Einstein-Maxwell field equations, the measure of pressure anisotropy and the mass. We impose a barotropic linear equation of state which relates the radial pressure to the energy density. We also express the system of field equations in other forms. A class of new exact solutions is achieved by choosing physically acceptable forms for one of the metric functions and the electric field intensity. We regain the masses for some stellar objects and study physical properties of the new exact models. We choose the star PSR J1614-2230 to analyse its physical features for particular parameters. Tables and graphical plots for relevant quantities are provided for the star PSR J1614-2230.

• Chapter 4: In this chapter we produce new compact models with a barotropic equation of state for matter distributions with anisotropic pressures in the presence of electric field intensity. A spacetime which is static and spherically symmetric is considered in isotropic coordinates. We introduce a transformation which was first suggested by Kustaanheimo and Qvist (1948) to write the Einstein-Maxwell field equations and the measure of anisotropy in new equivalent forms. A quadratic barotropic equation of state form is assumed which relates the radial pressure p_r to the energy density ρ. We apply the quadratic equation of state to the Einstein-Maxwell system of equations which are integrated to obtain new models. The Reissner-Nordström exterior metric is matched with the Schwarzschild interior metric at the boundary in isotropic coordinates to get the junction conditions at the boundary. By selecting physically reasonable forms for one of the gravitational potentials and the electric field intensity, we obtain a new class of exact models to the Einstein-Maxwell systems due to the inclusion of the quadratic term in the equation of state. If we set the quadratic term to zero in the quadratic equation of state, we regain an earlier linear model for particular parameters. We study physical properties for the quadratic case and it should be noted that the quantities associated with matter configurations and electromagnetic fields are well behaved. The features of the star PSR J1903+327 are studied and the list of tables and graphical plots for the star are given.

- Chapter 5: Conclusion.

Chapter 2

A relativistic algorithm with isotropic coordinates

2.1 Introduction

We consider the interior of static perfect fluid spheres in general relativity with isotropic pressures. The predictions of general relativity have been shown to be consistent with observational data in relativistic astrophysics and cosmology. For a discussion of the physical features of a gravitating model, we require an exact solution to the Einstein field equations. Exact solutions are crucial in the description of dense relativistic astrophysical problems. Many solutions have been found in the past. For some comprehensive lists of known solutions to the field equations, refer to Delgaty and Lake (1998), Finch and Skea (1998), Stephani *et al* (2003). Many of these solutions are not physically reasonable. For physical reasonableness, we require that the gravitational potentials and matter variables are regular and well behaved, causality of the spacetime manifold is maintained and values for physical quantities, for example, the mass of a dense star, are consistent with observations.

Solutions have been found in the past by making assumptions on the gravitational potentials, matter distribution or imposing an equation of state. These particular approaches do yield models which have interesting properties. However in principle, it would be desirable to have a general method that produces exact solutions in a systematic manner. Some systematic methods generated in the past are those of Rahman and Visser (2002), Lake (2003), Martin and Visser (2004), Boonserm *et al* (2005), Herrera *et al* (2004a), Chaisi and Maharaj (2006) and Maharaj and Chaisi (2006). In general relativity, we have the freedom of using any well defined coordinate system. The references mentioned above mainly use canonical coordinates. The use of isotropic coordinates may provide new insights and possibly lead to new solutions. This is the approach that we follow in this chapter. We generate a new algorithm producing a new solution, to Einstein field equations in isotropic coordinates. From a given solution we can find a new solution with isotropic pressures.

The objective of this chapter is to find new classes of exact solutions of the Einstein field equations with an uncharged isotropic matter distribution from a given seed metric. In §2.2, we derive the Einstein field equations for neutral perfect fluids in static spherically symmetric spacetime. We introduce new variables due to Kustaanheimo and Qvist (1948) to rewrite the field equations and the condition of pressure isotropy in equivalent forms. In §2.3, we introduce our algorithm and the master nonlinear second order differential equation containing two arbitrary functions, that has to be solved. In §2.4, we present new classes of exact solutions in terms of the arbitrary functions. In §2.5, we give an example for a conformally flat metric showing that the integrals generated in §2.4 may be explicitly evaluated. In §2.6, we summarise the results obtained in this chapter.

2.2 The model

We are modelling the interior of a dense relativistic star in strong gravitational fields. The line element of the interior spacetime, with isotropic coordinates, has the following form

$$ds^2 = -A^2(r)dt^2 + B^2(r)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{2.1}$$

where $A(r)$ and $B(r)$ are arbitrary functions representing the gravitational potentials. Relativistic compact objects such as neutron stars in astrophysics are described by this line element. The energy momentum tensor for the interior of the star has the form of a perfect fluid

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}, \tag{2.2}$$

where ρ is the energy density and p is the isotropic pressure. These quantities are measured relative to a timelike unit four-velocity u^a ($u^a u_a = -1$).

The Einstein field equations for (2.1) and (2.2) have the form

$$\rho = -\frac{1}{B^2}\left[2\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{4}{r}\right)\right], \tag{2.3a}$$

$$p = 2\frac{A'}{A}\left(\frac{B'}{B^3} + \frac{1}{r}\frac{1}{B^2}\right) + \frac{B'}{B^3}\left(\frac{B'}{B} + \frac{2}{r}\right), \tag{2.3b}$$

$$p = \frac{1}{B^2}\left(\frac{A''}{A} + \frac{1}{r}\frac{A'}{A}\right) + \frac{1}{B^2}\left[\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{1}{r}\right)\right], \tag{2.3c}$$

in isotropic coordinates. Primes denote differentiation with respect to the radial coordinate r. On equating (2.3b) and (2.3c) we obtain the condition of pressure isotropy which has the form

$$\frac{A''}{A} + \frac{B''}{B} = \left(\frac{A'}{A} + \frac{B'}{B}\right)\left(2\frac{B'}{B} + \frac{1}{r}\right). \tag{2.4}$$

This is the master equation which has to be integrated to produce an exact solution to the field equations.

It is possible to write the system (2.3) in an equivalent form by introducing new variables. We utilize a transformation that has proven to be helpful in relativistic stellar physics. We introduce the new variables

$$x \equiv r^2, \qquad L \equiv B^{-1}, \qquad G \equiv LA. \tag{2.5}$$

The above transformation was first suggested by Kustaanheimo and Qvist (1948). On applying transformation (2.5) in the field equations (2.3) we obtain the equivalent system

$$\rho = 4[2xLL_{xx} - 3(xL_x - L)L_x], \tag{2.6a}$$

$$p = 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x, \tag{2.6b}$$

$$p = 4xL^2\frac{G_{xx}}{G} + 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x - 8xLL_{xx}. \tag{2.6c}$$

We note that equations in (2.6) are highly nonlinear in both L and G. In this system there are three independent equations and four unknowns ρ, p, L and G. So we need to choose the functional form for L or G in order to integrate and obtain an exact solution. The value of the transformation (2.5) is highlighted in the reduction of the condition of pressure isotropy. On equating equations (2.6b) and (2.6c) we get

$$LG_{xx} = 2GL_{xx}, \tag{2.7}$$

which is the new condition of pressure isotropy which has a simpler compact form.

2.3 The algorithm

It is possible to find new solutions to the Einstein's equations from a given seed metric. Examples of this process are given in the treatments of Chaisi and Maharaj (2006) and Maharaj and Chaisi (2006). They found new models, with anisotropic pressures, from a given seed isotropic metric in Schwarzschild coordinates. Our intention is to find new models, with isotropic pressures, from a given solution in terms of the isotropic line element (2.1).

We can provide some new classes of exact solutions to the Einstein field equations by generating a new algorithm that produces a model from a given solution. We assume a known solution of the form $(\bar{L}, \bar{G})$, so that

$$\bar{L}\bar{G}_{xx} = 2\bar{G}\bar{L}_{xx}, \tag{2.8}$$

holds. We seek a new solution (L, G) given by

$$L = \bar{L}e^{g(x)}, \qquad\qquad G = \bar{G}e^{f(x)}, \tag{2.9}$$

where $f(x)$ and $g(x)$ are arbitrary functions. On substituting equation (2.9) into (2.7) we obtain

$$\begin{aligned}&(\bar{L}\bar{G}_{xx} - 2\bar{G}\bar{L}_{xx}) + 2(\bar{L}\bar{G}_x f_x - 2\bar{G}\bar{L}_x g_x)\\ &+\bar{L}\bar{G}(f_{xx} - 2g_{xx}) + \bar{L}\bar{G}(f_x^2 - 2g_x^2) = 0,\end{aligned} \tag{2.10}$$

which is given in terms of two arbitrary functions $f(x)$ and $g(x)$. Then realizing that $(\bar{L}, \bar{G})$ is a solution of (2.7) and using (2.8) we obtain the reduced result

$$(f_{xx} - 2g_{xx}) + 2\left(\frac{\bar{G}_x}{\bar{G}}f_x - 2\frac{\bar{L}_x}{\bar{L}}g_x\right) + (f_x^2 - 2g_x^2) = 0. \tag{2.11}$$

We need to demonstrate the existence of functions $f(x)$ and $g(x)$ that satisfy (2.11). In general, it is difficult to integrate (2.11), since it is given in terms of two arbitrary functions which are nonlinear.

2.4 New solutions

We consider several cases of (2.11) for which we have been able to complete the integration.

2.4.1 *g(x)* is specified

We can integrate (2.11) if $g(x)$ is specified. As a simple example, we take $g(x) = 1$. Then (2.11) becomes

$$f_{xx} + 2\frac{\bar{G}_x}{\bar{G}} f_x + f_x^2 = 0, \tag{2.12}$$

which is nonlinear in f. This is a first order Bernoulli equation in f_x. We can rewrite (2.12) in the form

$$\left(\frac{1}{f_x}\right)_x - 2\left(\frac{\bar{G}_x}{\bar{G}}\right)\left(\frac{1}{f_x}\right) = 1. \tag{2.13}$$

It is possible to integrate equation (2.13) since it is linear in $\frac{1}{f_x}$ to obtain

$$f_x = \bar{G}^{-2}\left(\int \bar{G}^{-2} dx + c_1\right)^{-1}. \tag{2.14}$$

We can formally integrate (2.14) to obtain the function $f(x)$ as

$$f(x) = \int \left[\bar{G}^{-2}\left(\int \bar{G}^{-2} dx + c_1\right)^{-1}\right] dx + c_2, \tag{2.15}$$

where c_1 and c_2 are arbitrary constants.

Then the new solution to equation (2.7) has the form

$$L = \bar{L}, \tag{2.16a}$$

$$G = \bar{G}\exp\left(\int\left[\bar{G}^{-2}\left(\int \bar{G}^{-2}dx + c_1\right)^{-1}\right]dx + c_2\right). \tag{2.16b}$$

Therefore we have shown that if a solution $(\bar{L}, \bar{G})$ to the field equations is known, then a new solution (L, G) is given by (2.16).

2.4.2 *f(x)* is specified

We can also integrate (2.11) if $f(x)$ is specified. As another simple example, we take $f(x) = 1$. Then equation (2.11) becomes

$$g_{xx} + 2\frac{\bar{L}_x}{\bar{L}}g_x + g_x^2 = 0, \tag{2.17}$$

which is nonlinear in g. This is a first order Bernoulli equation in g_x. The differential equation (2.17) has a form similar to (2.12) in §2.4.1. Following the same procedure, we obtain

$$g(x) = \int\left[\bar{L}^{-2}\left(\int \bar{L}^{-2}dx + c_1\right)^{-1}\right]dx + c_2, \tag{2.18}$$

where c_1 and c_2 are arbitrary constants.

Then another new solution to equation (2.7) is given by

$$G = \bar{G}, \tag{2.19a}$$

$$L = \bar{L}\exp\left(\int \left[\bar{L}^{-2}\left(\int \bar{L}^{-2}dx + c_1\right)^{-1}\right]dx + c_2\right). \tag{2.19b}$$

Therefore we have determined that if a solution $(\bar{L}, \bar{G})$ to the field equations is known, then a new solution (L, G) is given by (2.19). Note that the solution of (2.19) is different from that of (2.16).

2.4.3 *g(x)* is related to *f(x)*

We can integrate (2.11) if a relationship between the functions $f(x)$ and $g(x)$ exists. We illustrate this feature by assuming that

$$g(x) = \alpha f(x), \tag{2.20}$$

where α is an arbitrary constant. Then (2.11) becomes

$$f_{xx} + \frac{2}{1-2\alpha}\left(\frac{\bar{G}_x}{\bar{G}} - 2\alpha\frac{\bar{L}_x}{\bar{L}}\right)f_x + \left(\frac{1-2\alpha^2}{1-2\alpha}\right)f_x^2 = 0, \tag{2.21}$$

which is a first order Bernoulli equation in f_x. For convenience, we let

$$\Theta = \left(\frac{1-2\alpha^2}{1-2\alpha}\right), \qquad \eta = \frac{2}{1-2\alpha}, \qquad \alpha \neq \frac{1}{2}, \tag{2.22}$$

so that we can write (2.21) as

$$\left(\frac{1}{f_x}\right)_x - \eta\left(\frac{\bar{G}_x}{\bar{G}} - 2\alpha\frac{\bar{L}_x}{\bar{L}}\right)\left(\frac{1}{f_x}\right) = \Theta, \tag{2.23}$$

which is linear in $\frac{1}{f_x}$. We integrate (2.23) to obtain

$$f_x = \left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\eta}\left[\Theta\int\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\eta}dx + c_1\right]^{-1}. \tag{2.24}$$

We now formally integrate (2.24) to obtain

$$f(x) = \int\left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\eta}\left[\Theta\int\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\eta}dx + c_1\right]^{-1}\right)dx + c_2, \tag{2.25}$$

where c_1 and c_2 are constants.

We now have a new solution of (2.7) given by

$$L = \bar{L}\exp\left\{\alpha\left[\int\left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\eta}\left[\Theta\int\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\eta}dx + c_1\right]^{-1}\right)dx + c_2\right]\right\}, \tag{2.26a}$$

$$G = \bar{G}\exp\left[\int\left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\eta}\left[\Theta\int\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\eta}dx + c_1\right]^{-1}\right)dx + c_2\right], \tag{2.26b}$$

where Θ and η are given in (2.22). Therefore we have demonstrated that if a solution $(\bar{L}, \bar{G})$ to the field equations is specified, then a new solution (L, G) is provided by (2.26).

Some special cases related to (2.26) should be pointed out. These relate to $\alpha = 1$, $\pm\frac{1}{\sqrt{2}}$, $\frac{1}{2}$. We consider each in turn.

Case (i)**:** $\alpha = 1$

With $\alpha = 1$, we find that (2.26) becomes

$$L = \bar{L}\exp\left(\int\left[\frac{\bar{G}^2}{\bar{L}^4}\left(\int\frac{\bar{G}^2}{\bar{L}^4}dx + c_1\right)^{-1}\right]dx + c_2\right), \tag{2.27a}$$

$$G = \bar{G}\exp\left(\int\left[\frac{\bar{G}^2}{\bar{L}^4}\left(\int\frac{\bar{G}^2}{\bar{L}^4}dx + c_1\right)^{-1}\right]dx + c_2\right), \tag{2.27b}$$

which is a simple form.

Case (ii)**:** $\alpha = \pm\frac{1}{\sqrt{2}}$

If we set $\alpha = \pm\frac{1}{\sqrt{2}}$, then (2.26) becomes

$$L = \bar{L}\exp\left[\pm\frac{1}{\sqrt{2}}\left(c_1\int\left(\frac{\bar{L}^{\pm\sqrt{2}}}{\bar{G}}\right)^{\frac{2}{1-(\pm\sqrt{2})}}dx + c_2\right)\right], \tag{2.28a}$$

$$G = \bar{G}\exp\left[c_1\int\left(\frac{\bar{L}^{\pm\sqrt{2}}}{\bar{G}}\right)^{\frac{2}{1-(\pm\sqrt{2})}}dx + c_2\right], \tag{2.28b}$$

which is another simple case.

Case (iii)**:** $\alpha = \frac{1}{2}$

If $\alpha = \frac{1}{2}$, then (2.26) is not valid. For this case, (2.11) becomes

$$f_x\left[f_x + 4\left(\frac{\bar{G}_x}{\bar{G}} - \frac{\bar{L}_x}{\bar{L}}\right)\right] = 0. \tag{2.29}$$

When f is constant, then g is also constant by (2.20); then, (2.7) does not produce a new solution because of (2.9). When f is not constant then, we can integrate (2.29) to produce the solution

$$L = K\frac{\bar{L}^3}{\bar{G}^2}, \tag{2.30a}$$

$$G = K\frac{\bar{L}^4}{\bar{G}^3}, \tag{2.30b}$$

where K is a constant of integration. Thus $\alpha = \frac{1}{2}$ generates another new solution (L, G) to (2.11).

2.5 Example

We show by means of a specific example that the integrals generated in §2.4 may be evaluated to produce a new exact solution to the field equations in terms of elementary functions. In our example, we choose

$$\bar{L} = b + ax, \tag{2.31a}$$

$$\bar{G} = 1 + cx. \tag{2.31b}$$

Then the corresponding line element is given by

$$ds^2 = -\left(\frac{1+cr^2}{b+ar^2}\right)^2 dt^2 + \left(\frac{1}{b+ar^2}\right)^2 (dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \tag{2.32}$$

which is conformally flat. The energy density for the metric (2.32) is constant, so that we have the Schwarzschild interior solution in isotropic coordinates.

Conformally flat metrics are important in gravitational physics in a general relativistic setting. They arise, for instance, in the gravitational collapse of a radiating star, as shown in the treatments of Herrera *et al*. (2004b), Maharaj and Govender (2005), Misthry *et al* (2008) and Abebe *et al* (2013). For the choice of (2.31) we find that (2.27) becomes

$$L = (b+ax)\exp\left(\int\left[\frac{(1+cx)^2}{(b+ax)^4}\left(\int\frac{(1+cx)^2}{(b+ax)^4}dx+c_1\right)^{-1}\right]dx+c_2\right), \tag{2.33a}$$

$$G = (1+cx)\exp\left(\int\left[\frac{(1+cx)^2}{(b+ax)^4}\left(\int\frac{(1+cx)^2}{(b+ax)^4}dx+c_1\right)^{-1}\right]dx+c_2\right). \tag{2.33b}$$

The integrals in (2.33) can be evaluated and we obtain

$$L = \frac{1}{(b+ax)^2}U(x), \tag{2.34a}$$

$$G = \frac{(1+cx)}{(b+ax)^3}U(x), \tag{2.34b}$$

where $c_1 = 0$ and $c_2 = 1$ and we have set

$$U(x) = b^2c^2 + abc(1+3cx) + a^2(1+3cx+3c^2x^2). \tag{2.35}$$

Thus the known solution $(\bar{L}, \bar{G})$ in (2.31) produces a new solution (L, G) in (2.34). The line element for the new solution has the form

$$ds^2 = -\left(\frac{1+cr^2}{b+ar^2}\right)^2 dt^2 + \left[\frac{(b+ar^2)^2}{U(r)}\right]^2 (dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \tag{2.36}$$

where $U(r)$ is given by (2.35). Thus our algorithm has produced a new (not conformally flat) solution to the Einstein's field equations. This has been generated from a seed conformally flat model.

2.6 Discussion

We now comment on the physical properties of the example. We have generated plots for the energy density ρ, pressure p, and the speed of sound in Figures 2.1-2.3, respectively. These graphical plots indicate that ρ and p are positive and well behaved. The speed of sound is less than the speed of light as required for causality. Therefore the algorithm presented in this chapter produces new solutions which are physically reasonable.

We have generated an algorithm to produce a new solution to the Einstein field equations from a given seed metric. We observe that the resulting model contains isotropic pressures unlike the approach of Chaisi and Maharaj (2006) and Maharaj and Chaisi (2006); in their treatment the new model has anisotropic pressures. Another advantage of our approach is the use of isotropic coordinates in the formulation of the condition of pressure isotropy. This may lead to new insights into the behaviour of gravity since previous treatments mainly utilised canonical coordinates. The algorithm produced a new solution in terms of integrals containing arbitrary functions. We have shown, with the help of a conformally flat metric, that these integrals may be evaluated in terms of elementary functions. This example suggests that our approach may be extended to other physically relevant metrics.

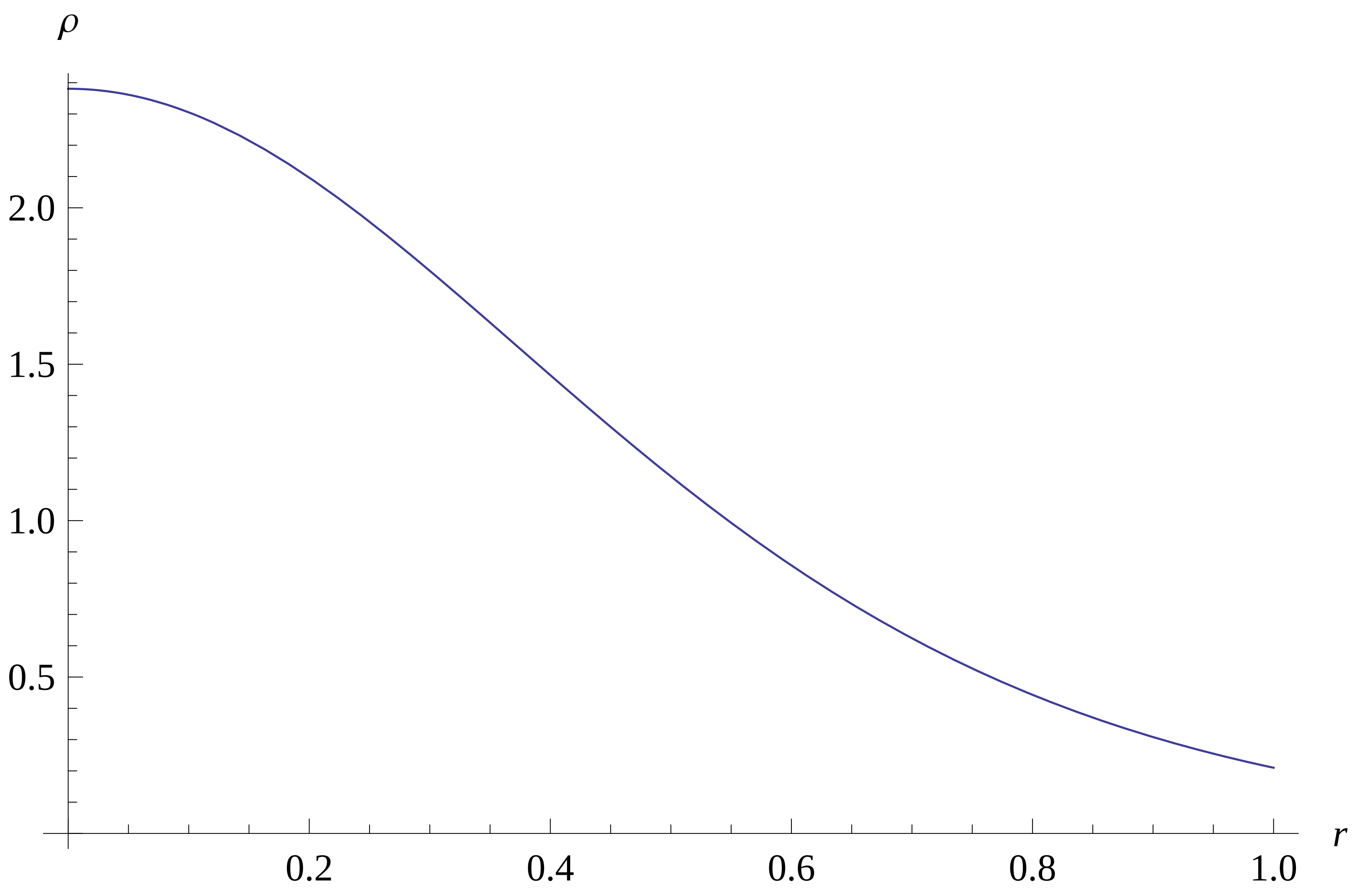

Figure 2.1: Energy density ρ

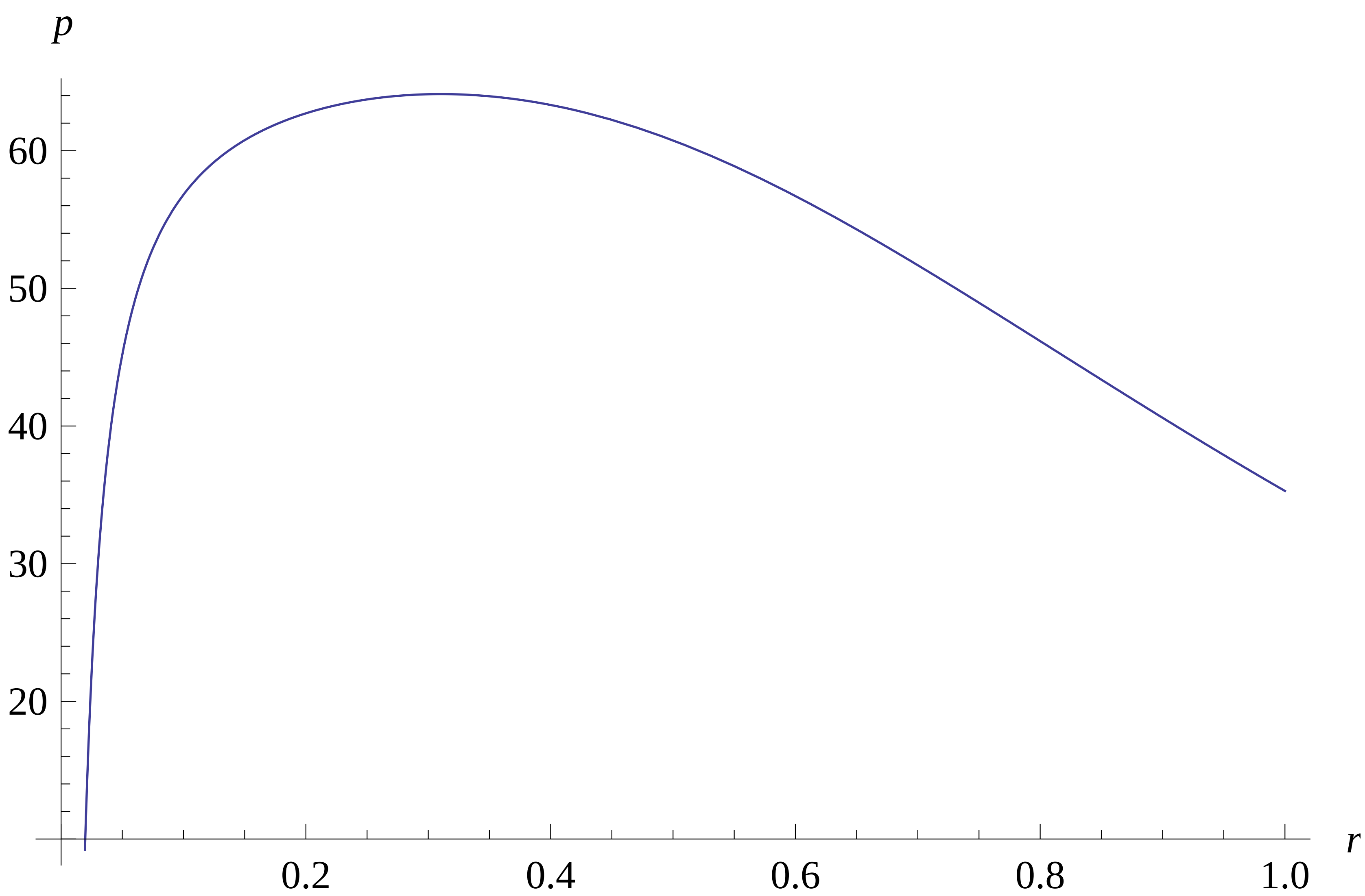

Figure 2.2: Pressure p

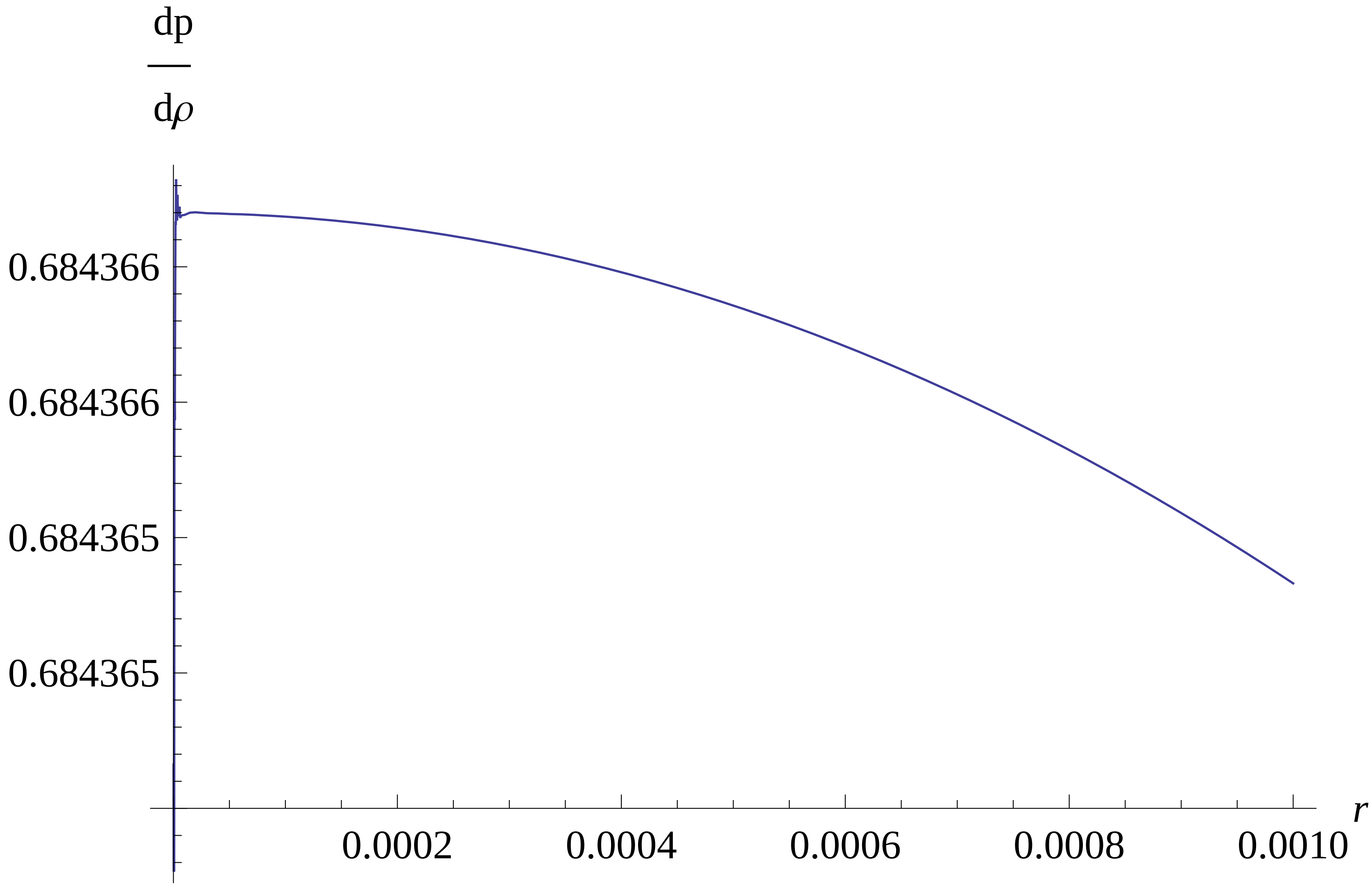

Figure 2.3: Speed of sound $\frac{dp}{d\rho}$

Chapter 3

Compact stars with linear equation of state

3.1 Introduction

We consider solutions of the Einstein-Maxwell system of equations for charged static spherically symmetric interior distributions which match to the Reissner-Nordström exterior spacetime. Charged compact objects in relativistic astrophysics where the gravitational fields are strong are described by solutions of the coupled Einstein-Maxwell system of equations. The studies of Ivanov (2002) and Sharma *et al* (2001) show that the presence of the electromagnetic field affects the values of surface redshifts, luminosities and maximum masses of compact objects. Mak and Harko (2004) and Komathiraj and Maharaj (2007a,b) highlight the fact that the electromagnetic field has an important role in describing the gravitational behaviour of stars composed of quark matter. Models constructed in this way will be useful in describing the physical properties of compact relativistic objects, gravastars, neutron stars, etc, with different matter distributions. There have been several investigations in recent years on the Einstein-Maxwell system of equations for static charged spherically symmetric gravitational fields. Some recent comprehensive treatments are those of Kiess (2012), Fatema and Murad (2013) and Murad and Fatema (2013).

The Einstein-Maxwell field equations will have different forms depending on the coordinates utilized. In most papers researchers have used canonical coordinates with neutral matter and isotropic pressure distributions. For comprehensive lists of known solutions to the field equations, refer to Delgaty and Lake (1998), Finch and Skea (1998) and Stephani *et al* (2003). Many of these known solutions are not physically acceptable. For solutions to be physically reasonable it is necessary that the metric functions and the matter variables are regular and well behaved in the interior of the star. Causality of the spacetime structure must be maintained, the energy conditions should be satisfied and physical quantities (for example the mass of a dense star) should correspond with observations of astronomical objects. For examples of recent papers with charge and isotropic pressures the reader is referred to Fatema and Murad (2013) and Murad and Fatema (2013). The general case of charged matter distributions with anisotropic pressures has generated much interest in several recent investigations. Some recent treatments are those of Mafa Takisa and Maharaj (2013a) and Maharaj *et al* (2014). It is important to note that isotropic coordinates have not been used as often as canonical coordinates. A recent example of charged matter with isotropic pressures in isotropic coordinates is the Pant *et al* (2014a) model.

Realistic matter distributions require an equation of state. A barotropic equation of state requires that the pressure be a function of the energy density. The form of the barotropic equation of state can be linear, quadratic, polytropic or some other dependence. Some classes of exact solutions to the Einstein-Maxwell system have been found by imposing an equation of state in canonical coordinates. Models of charged and anisotropic stars with a linear equation of state are those of Mafa Takisa and Maharaj (2013a), Sunzu *et al* (2014a,b), Maharaj *et al* (2014) and Esculpi and Aloma (2010). Charged compact objects with a linear equation of state have been generated by Mafa Takisa *et al* (2014a); their models prove to be good approximations of the astronomical objects PSR J1614-2230, PSR J1903+327, Vela X-1, SMC X-1 and Cen X-3. Feroze and Siddiqui (2011) found a class of charged anisotropic solutions with a quadratic equation of state. Maharaj and Mafa Takisa (2012) and Mafa Takisa and Maharaj (2013b), respectively, found new models with charge and anisotropic pressures by imposing quadratic and polytropic equations of state. An uncharged strange quark star model with the quadratic equation of state was presented by Malaver (2014). Other possibilities for barotropic equations of state are the van der Waals models, a recent exact solution was found by Thirukkanesh and Ragel (2014), and extensions leading to the so called generalised van der Waals models by Malaver (2013).

Delgaty and Lake (1998) in their comprehensive treatment pointed out that only a few successful attempts have been made to obtain classes of exact static solutions of the Einstein field equations for neutral perfect fluid spheres. They also observe that only nine of their solutions are regular and well behaved without considering the restriction on the redshift. Only two solutions with isotropic coordinates in their analysis were shown to be well behaved, and these are by Nariai (1950) and Goldman (1978). It was later shown by Simon (2008) that the Pant and Sah (1985) solution is also regular and well behaved. In recent past years, well behaved solutions in isotropic coordinates with charge have been found by Pant *et al* (2014a) and Pradhan and Pant (2014). As far as we can ascertain most analyses in isotropic coordinates restrict the pressures to be isotropic. In this chapter we present a new class of charged exact solutions in isotropic coordinates with anisotropic pressures.

Many of the references mentioned above mainly use canonical coordinates unlike our treatment which utilizes isotropic coordinates. We believe that the use of isotropic coordinates may provide some new insights, and possibly lead to new classes of exact solutions. In this chapter we follow the approach of using isotropic coordinates with anisotropic pressures in the presence of the electromagnetic field. The objective of this chapter is to generate new classes of exact solutions to the Einstein-Maxwell system of equations, by imposing a linear barotropic equation of state, that model a charged anisotropic relativistic body in isotropic coordinates. In §3.2, we present the Einstein-Maxwell field equations for charged anisotropic fluid spheres in static spherically symmetric spacetimes. The field equations are written in terms of isotropic coordinates, and then transformed to new variables suggested by Kustaanheimo and Qvist (1948). This system of equations in transformed form is easier to integrate and analyze. New classes of exact solutions are presented in §3.3. In §3.4 we study the physical properties of the new classes of exact solutions and regain masses for particular observed objects. In §3.5 we analyse the physical features for parameters associated with the star PSR J1614-2230. Some concluding comments are made in §3.6.

3.2 The model

We model a dense general relativistic star with strong gravity. The metric of the interior spacetime in isotropic coordinates can be written as

$$ds^2 = -A^2(r)dt^2 + B^2(r)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{3.1}$$

in coordinates $(x^a) = (t, r, \theta, \phi)$ and where $A(r)$ and $B(r)$ are metric quantities representing the gravitational field. Relativistic astronomical objects such as compact stars in astrophysical scenarios are consistent with the metric (3.1). The energy momentum tensor

$$T_{ij} = \text{diag}\left(-\rho - \frac{1}{2}E^2, p_r - \frac{1}{2}E^2, p_t + \frac{1}{2}E^2, p_t + \frac{1}{2}E^2\right), \tag{3.2}$$

describes an anisotropic charged matter distribution. In (3.2), ρ is the energy density, p_r is the radial pressure, p_t is the tangential pressure and E is the electric field intensity. These quantities are measured in terms of a timelike unit four-velocity $\mathbf{u}$ where $u^i = \frac{1}{A}\delta^i_0$.

The Einstein-Maxwell field equations for the line element (3.1) and matter distribution (3.2) can be expressed as

$$8\pi\rho + \frac{1}{2}E^2 = -\frac{1}{B^2}\left[2\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{4}{r}\right)\right], \tag{3.3a}$$

$$8\pi p_r - \frac{1}{2}E^2 = 2\frac{A'}{A}\left(\frac{B'}{B^3} + \frac{1}{r}\frac{1}{B^2}\right) + \frac{B'}{B^3}\left(\frac{B'}{B} + \frac{2}{r}\right), \tag{3.3b}$$

$$8\pi p_t + \frac{1}{2}E^2 = \frac{1}{B^2}\left(\frac{A''}{A} + \frac{1}{r}\frac{A'}{A}\right) + \frac{1}{B^2}\left[\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{1}{r}\right)\right], \tag{3.3c}$$

$$\sigma = \frac{1}{4\pi r^2}B^{-1}(r^2E)', \tag{3.3d}$$

in isotropic coordinates where a prime ($'$) denotes a derivative with respect to the radial coordinate r. The quantity σ is the proper charge density. We utilize units where the speed of light $c = 1$ and the Newton gravitational constant $G = 1$. The system of equations (3.3) governs the behaviour of the gravitational field for an anisotropic charged fluid in a static spherical field. From equations (3.3b) and (3.3c) we obtain the condition of pressure anisotropy which has the form

$$\frac{A''}{A} + \frac{B''}{B} = B^2(8\pi\Delta + E^2) + \left(\frac{A'}{A} + \frac{B'}{B}\right)\left(2\frac{B'}{B} + \frac{1}{r}\right), \tag{3.4}$$

where the quantity $\Delta = p_t - p_r$ is the measure of anisotropy. For neutral matter with isotropic pressures ($E = 0 = \Delta$), (3.4) gives the condition of pressure isotropy in the form

$$\frac{A''}{A} + \frac{B''}{B} = \left(\frac{A'}{A} + \frac{B'}{B}\right)\left(2\frac{B'}{B} + \frac{1}{r}\right). \tag{3.5}$$

A general algorithm producing new exact solutions to (3.5), given a particular seed solution, was found by Ngubelanga and Maharaj (2013) after integrating a nonlinear Bernoulli equation.

The spacetime exterior to the charged matter distribution is given by

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{R}+\frac{q^2}{R^2}\right)dt^2+\left(1-\frac{2M}{R}+\frac{q^2}{R^2}\right)^{-1}dR^2 \\
&\quad +R^2(d\theta^2+\sin^2\theta d\phi^2), \qquad (3.6)
\end{aligned}
$$

in coordinates $(x^a)=(t,R,\theta,\phi)$. In (3.6), R is the radial coordinate of the exterior region, and M and q^2 are the mass and charge of the ball, respectively, as determined by the external observer. The exterior spacetime is the Reissner-Nordström solution. By matching the first and second fundamental forms for the metrics (3.1) and (3.6) we obtain the junction conditions at the stellar surface. These conditions are given by

$$A_s = \left(1-\frac{2M}{R}+\frac{q^2}{R^2}\right)^{\frac{1}{2}}, \qquad (3.7a)$$

$$R_s = r_s B_s, \qquad (3.7b)$$

$$\left(\frac{B'}{B}+\frac{1}{r}\right)_s r_s = \left(1-\frac{2M}{R}+\frac{q^2}{R^2}\right)^{\frac{1}{2}}, \qquad (3.7c)$$

$$r_s(A')_s = \left[\frac{M}{R}-\frac{q^2}{R^2}\right], \qquad (3.7d)$$

where the subscript "s" denotes the surface of the star. The equations (3.7a)-(3.7d) are the boundary conditions in isotropic coordinates. Observe that equations (3.7b) and (3.7d) are equivalent to zero pressure of the interior solution on the boundary.

The system of equations (3.3) can be written in a different form by introducing a new variable x, and defining new functions L and G, as follows

$$x \equiv r^2, \quad L \equiv B^{-1}, \quad G \equiv LA. \tag{3.8}$$

The above transformation (3.8) has been used by other authors in spherically symmetric spacetimes as pointed out by Stephani *et al* (2003) . Then the line element (3.1) becomes

$$ds^2 = -\left(\frac{G}{L}\right)^2 dt^2 + L^{-2}\left[\left(\frac{1}{4x}\right) dx^2 + x(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{3.9}$$

On applying the transformation (3.8) to the field equations (3.3) we generate the equivalent field equations

$$8\pi\rho + \frac{1}{2}E^2 = 4[2xLL_{xx} - 3(xL_x - L)L_x], \tag{3.10a}$$

$$8\pi p_r - \frac{1}{2}E^2 = 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x, \tag{3.10b}$$

$$8\pi p_t + \frac{1}{2}E^2 = 4xL^2\frac{G_{xx}}{G} + 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x$$

$$-8xLL_{xx}, \tag{3.10c}$$

$$\sigma^2 = \frac{1}{4\pi^2 x}L^2(E + xE_x)^2, \tag{3.10d}$$

where the subscript "x" denotes differentiation with respect to the variable x. The condition of pressure anisotropy is now given by

$$\frac{G_{xx}}{G} - 2\frac{L_{xx}}{L} = \frac{(8\pi\Delta + E^2)}{4xL^2}. \tag{3.11}$$

In the new coordinates the mass function is

$$m(x) = 4\pi\int_0^x \frac{1}{\sqrt{\omega}}\left[\omega\rho(\omega) + \frac{E^2}{8\pi}\right]d\omega, \tag{3.12}$$

which represents the mass within a radius x of the charged sphere.

A barotropic equation of state $p_r = p_r(\rho)$ should be satisfied by the matter distribution. We expect this to be the case for a physically realistic relativistic star. For the simplest case we assume the linear equation of state

$$p_r = \alpha\rho - \beta, \tag{3.13}$$

relating the radial pressure p_r to the energy density ρ, and where α and β are arbitrary constants. Then with the linear equation of state (3.13), it is possible to express the system (3.10) in the form

$$8\pi\rho = 4[2xLL_{xx} - 3(xL_x - L)L_x] - \frac{1}{2}E^2, \tag{3.14a}$$

$$p_r = \alpha\rho - \beta, \tag{3.14b}$$

$$p_t = p_r + \Delta, \tag{3.14c}$$

$$\begin{aligned} 8\pi\Delta = {} & 4xL^2\frac{G_{xx}}{G} + \frac{8}{(1+\alpha)}L(L - 2xL_x)\frac{G_x}{G} - \frac{8(1+3\alpha)}{(1+\alpha)}xLL_{xx} \\ & +24xL_x^2 - \frac{8(2+3\alpha)}{(1+\alpha)}LL_x + \frac{16\pi\beta}{(1+\alpha)}, \end{aligned} \tag{3.14d}$$

$$\begin{aligned} \frac{E^2}{2} = {} & \frac{8\alpha}{(1+\alpha)}xLL_{xx} - 12xL_x^2 + \frac{4(2+3\alpha)}{(1+\alpha)}LL_x \\ & +\frac{4}{(1+\alpha)}L(2xL_x - L)\frac{G_x}{G} - \frac{8\pi\beta}{(1+\alpha)}, \end{aligned} \tag{3.14e}$$

$$\sigma^2 = \frac{1}{4\pi^2 x}L^2(E + xE_x)^2. \tag{3.14f}$$

In the system of equations (3.14) we note that the equations are highly nonlinear in both potentials L and G. This system contains the six matter variables $(\rho, p_r, p_t, \Delta, E, \sigma)$ and the two metric functions (L, G). Clearly there are more unknown functions than independent field equations in the Einstein-Maxwell system. This suggests that we need to choose the form for two of the quantities in order to integrate and obtain some classes of exact solutions.

3.3 Exact models

We aim to generate exact models to the system of Einstein-Maxwell equations (3.14) by choosing physically reasonable forms for the gravitational potential L and the electric field intensity E. We make the specific choices

$$L = a + bx, \tag{3.15a}$$

$$E^2 = x(c + dx), \tag{3.15b}$$

where a, b, c and d are real constants. We choose the gravitational potential L to be a linear function and the electric intensity E^2 to be a quadratic function in the variable x. This ensures that the potential and the charge are finite at the centre and are regular in the interior. Then equation (3.14e) becomes the first order equation

$$\frac{G_x}{G} = \frac{b(2+3\alpha)}{(a-bx)} - \left[\frac{16\pi\beta + (1+\alpha)(24b^2 + c + dx)x}{8(a-bx)(a+bx)}\right], \tag{3.16}$$

in the potential G. On integrating (3.16) we obtain

$$G(x) = K(a - bx)^{\Psi}(a + bx)^{\Phi}e^{N(x)}, \tag{3.17}$$

where K is the constant of integration. We have introduced the function $N(x)$ and constants Ψ and Φ. These are given by

$$N(x) = \frac{d(1+\alpha)x}{8b^2}, \tag{3.18a}$$

$$\Psi = \frac{1}{16ab^3}\{a(1+\alpha)(bc + ad) - 8b^2[ab(1+3\alpha) - 2\pi\beta]\}, \tag{3.18b}$$

$$\Phi = \frac{1}{16ab^3}\{a(1+\alpha)[b(24b^2 + c) - ad)] - 16\pi b^2\beta\}. \tag{3.18c}$$

It should be noted in (3.18) that $a \neq 0$ and $b \neq 0$.

We can now find an exact solution to the Einstein-Maxwell system. The line element is given by

$$\begin{aligned} ds^2 &= -K(a - br^2)^{2\Psi}(a + br^2)^{2(\Phi-1)}e^{2N(r)}dt^2 \\ &\quad +(a + br^2)^{-2}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi)], \end{aligned} \tag{3.19}$$

where K is a constant of integration, the function $N(r)$ and the constants Ψ and Φ are given by (3.18). The quantities associated with the matter and the electromagnetic field have the form

$$8\pi\rho = 12ab - \frac{1}{2}r^2(c + dr^2), \tag{3.20a}$$

$$p_r = \alpha\rho - \beta, \tag{3.20b}$$

$$p_t = p_r + \Delta, \tag{3.20c}$$

$$\begin{aligned} 8\pi\Delta = {} & \frac{\Psi(a+br^2)r^2}{b(a-br^2)^2}\left[4b^3(\Psi-1)(a+br^2) - (a-br^2)\left\{8b^3\Phi + d(1+\alpha)(a+br^2)\right\}\right] \\ & + \frac{2}{(1+\alpha)}\left[8\pi\beta + 4b\Phi(a-br^2) - 4b(a+br^2)\left\{2+3\alpha+\Psi\right\}\right] \\ & + \frac{d(a+br^2)}{16b^4}\left[16b^2(a-br^2) + (1+\alpha)\left\{16b^3\Phi + d(1+\alpha)(a+br^2)\right\}r^2\right] \\ & + 4b^2[6+\Phi(\Phi-1)]r^2, \end{aligned} \tag{3.20d}$$

$$\sigma^2 = \frac{\left(a+br^2\right)^2\left(3c+4dr^2\right)^2}{16\pi^2(c+dr^2)}, \tag{3.20e}$$

$$E^2 = r^2(c + dr^2). \tag{3.20f}$$

This exact solution is given in terms of elementary functions. The mass function has the form

$$m(r) = \frac{1}{2}r^3 \left[\frac{(12ab+c)}{3} - \frac{(c-2d)r^2}{10} - \frac{dr^4}{14} \right]. \tag{3.21}$$

3.4 Physical features

The exact solution (3.20) for the spacetime (3.19) has a simple form and may be used to describe a charged anisotropic fluid sphere. In the stellar interior, the gravitational potentials, the matter variables and the electromagnetic variables are well behaved. It is important to observe that the electric field vanishes at the centre of the star. The matter density and proper charge density are finite at the stellar centre. At the stellar centre, $r = 0$ we have

$$\rho_0 = \frac{3ab}{2\pi}, \tag{3.22a}$$

$$p_{r0} = \frac{3ab\alpha}{2\pi} - \beta, \tag{3.22b}$$

and the density and radial pressure are finite. The electromagnetic quantities have the finite values

$$\sigma_0^2 = \frac{1}{c}\left(\frac{3ac}{4\pi}\right)^2, \tag{3.23a}$$

$$E_0^2 = 0, \tag{3.23b}$$

at the centre. The pressure anisotropy is given by

$$\Delta_0 = \left(\frac{d}{8\pi}\right)\left(\frac{a}{b}\right)^2 + \frac{1}{\pi(1+\alpha)}[ab\Phi + 2\pi\beta - ab(2+3\alpha+\Psi)], \tag{3.24}$$

at the centre. The gravitational potentials A and B are also finite at $r = 0$ and therefore all relevant quantities are regular in the core of the star. For stability it is necessary that Δ vanishes at $r = 0$; we show that this is possible in the next section for particular parameter values. The mass is finite and is affected by the presence of charge.

The speed of sound is given by

$$v = \left(\frac{dp_r}{d\rho}\right)^{0.5}, \tag{3.25}$$

which becomes $v = \alpha$. We must have $\alpha < 1$ so that the velocity of sound is less than the velocity of light. In our analysis throughout we choose the value $\alpha = 0.931$. This is the reasonable choice because compact stars are relativistic and α should be close to unity for these structures. Also the surface of a compact object should have zero radial pressure. This will ensure that the boundary conditions (3.7) are satisfied and the metrics (3.1) and (3.6) match at the boundary. Since the equation of state is not polytropic we have a finite value of the density at the stellar surface. We obtain

$$\rho_s = \frac{3ab}{2\pi} - \frac{1}{16\pi}(c+d), \tag{3.26}$$

from equation (3.20a) in geometric units. In (3.26) we have fixed the radius of the star at $r = 1$. Then from equation (3.20b) we see that the zero of the surface pressure constrains the constant β by

$$\beta = \alpha \rho_s, \tag{3.27}$$

in terms of the surface density.

We can now give certain numerical values to the various constants contained in our solutions. We have selected data from recent observations of five compact objects. These are PSR J1614-2230 studied by Demorest *et al* (2010), Vela X-1 analysed by Rawls *et al* (2011), PSR J1903+327 investigated by Freire *et al* (2011), 4U 1820-30 studied by Güver *et al* (2010b) and SAX J1808.4-3658 considered by Elebert *et al* (2009). We find that by varying the constant a in equation (3.12) we regain the masses of these stars, keeping the other parameters fixed at $b = 0.5$, $c = 0.01$ and $d = 0.01$. Notice that the stellar bodies are generally charge neutral, and hence we have imposed a smaller value of the parameters c and d, responsible for bringing in charge into the system. Later on, in the detailed analysis of the star PSR J1614-2230, we made a variation of these two parameters, to show how they change the electric field. In Table 3.1 we find the compact stars have observed masses varying from 0.9 $M_\odot$ to 1.97 $M_\odot$. In this table we have also included the corresponding values of the central density, central radial pressure and the surface density.

3.5 The star PSR J1614-2230

The parameter value $a = 1.96819$ produces the mass 1.97 $M_\odot$ corresponding to the star PSR J1614-2230. We use this parameter to illustrate the variation of features for the matter, charge and gravity inside the star from the centre to the surface.

Table 3.2 indicates the variation of density, radial pressure, tangential pressure and anisotropy in the stellar interior. The density and radial pressure are decreasing functions. The radial pressure vanishes at the boundary which is the requirement for a localised distribution of matter. The tangential pressure is well behaved. The anisotropy is finite and vanishes at the centre which is necessary for stability. Table 3.3 gives the behaviour of the mass, electric field and charge density. The mass increases with increasing radius. The electric field and charge density are regular throughout the interior with $E = 0$ at the centre. The effect of the charge is brought in through the constants c and d. Tables 3.4 and 3.5 represent the total charge in the system so that $r = 1$ is fixed at the boundary. It is evident that the effect of the parameter d is more pronounced than that of the constant c making the system substantially charged. Finally the gravitational potentials A^2 and B^2 are tabulated in Table 3.6 for the set of parameters corresponding to PSR J1614-2230 from the centre to the surface. The values obtained show that the potentials are finite and positive.

A graphical analysis also provides insight into the behaviour of the relevant quantities. We have plotted the density ρ (Fig. 3.1), the radial pressure p_r (Fig. 3.2), the tangential pressure p_t (Fig. 3.3), the pressure anisotropy Δ (Fig. 3.4), the mass m (Fig. 3.5), the electric field intensity (Fig. 3.6), the charge density (Fig. 3.7) and the gravitational potentials (Fig. 3.8 and Fig. 3.9). All the quantities are well behaved. The various figures have been plotted with the help of Mathematica (2003). Mafa Takisa *et al* (2014a) using a different approach also studied particular astronomical objects in general relativity with linear equation of state (3.13). Our results in this chapter are broadly consistent with their results.

3.6 Discussion

Our aim in this chapter was to generate new exact models to the Einstein-Maxwell systems in isotropic coordinates for matter distributions with anisotropic pressures in the presence of charge. We imposed the barotropic equation of state which is linear and relates the radial pressure to the energy density. The exact solutions (3.20a)-(3.20f) to the Einstein-Maxwell field equations generated are well behaved. We tabulated the matter and electromagnetic variables and showed that they are physically reasonable. By varying the parameter a and choosing the fixed parameters $b = 0.5$, $c = 0.01$ and $d = 0.01$ in Table 3.1, we regain the masses of the stars PSR J1614-2230, Vela X-1, PSR J1903+327, 4U 1820-30 and SAX J1808.4-3658. We used the star PSR J1614-2230 which has the mass 1.97 $M_{\odot}$ and fixed the parameter $a = 1.96819$ to generate tables and graphical plots for gravitational potentials, matter variables and electromagnetic variables. We used Mathematica for the particular parameter choices $a = 1.96819$, $b = 0.5$, $c = 0.01$, $d = 0.01$ and $\alpha = 0.931$. The graphical analysis indicates that the model for PSR J1614-2230 is well behaved. The model has been generated by making the simple choices (3.15a) and (3.15b) for the potential and the charge, respectively. Our approach may also be used to produce models which exhibit more general behaviour and thereby describe superdense stars with uncharged and charged matter. A different equation of state will also produce other qualitative features which are different from the linear case as shown in the treatment of Mafa Takisa *et al* (2014a).

Table 3.1: Masses m, central density ρ_0, central radial pressure p_{r0} and surface density ρ_s of different stars corresponding to the parameters $b = 0.5$, $c = 0.01$, $d = 0.01$ and $\alpha = 0.931$.

Star	Observed mass m	a	ρ_0	p_{r0}	ρ_s
PSR J1614-2230	1.97	1.96819	0.469871	0.000370433	0.469473
Vela X-1	1.77	1.76819	0.422124	0.000370433	0.421726
PSR J1903+327	1.667	1.66519	0.397535	0.000370433	0.397137
4U 1820-30	1.58	1.57819	0.376765	0.000370433	0.376367
SAX J1808.4-3658	0.9	0.89819	0.214427	0.000370433	0.214029

Table 3.2: Variation of energy density, radial pressure, tangential pressure and measure of anisotropy from the centre to the surface with parameters $a = 1.96819$, $b = 0.5$, $c = 0.01$, $d = 0.01$ and $\alpha = 0.931$.

r	ρ	p_r	p_t	Δ
0	0.469871	0.000370433	0.000370433	0
0.1	0.469869	0.000368562	0.0015641	0.00119554
0.2	0.469862	0.000362728	0.00521714	0.00485441
0.3	0.469851	0.000352263	0.011554	0.0112017
0.4	0.469834	0.000336057	0.0209783	0.0206422
0.5	0.469809	0.000312553	0.0341235	0.033811
0.6	0.469773	0.000279751	0.0519384	0.0516586
0.7	0.469726	0.000235207	0.0758281	0.0755929
0.8	0.469662	0.00017603	0.107888	0.107712
0.9	0.469579	0.0000988871	0.151303	0.151204
1	0.469473	0	0.211052	0.211052

Table 3.3: Variation of mass, electric field intensity and charge density for charged bodies from the centre to the surface with parameters $a = 1.96819$, $b = 0.5$, $c = 0.01$ and $d = 0.01$.

r	m	E^2	σ^2
0	0	0	0.00220779
0.1	0.00196986	0.000101	0.00225602
0.2	0.015759	0.000416	0.00240346
0.3	0.0531873	0.000981	0.00265829
0.4	0.126075	0.001856	0.00303435
0.5	0.24625	0.0003125	0.0003557
0.6	0.425518	0.004896	0.00423597
0.7	0.675715	0.007301	0.00512147
0.8	1.00866	0.010496	0.00624986
0.9	1.43615	0.014661	0.00767248
1	1.97	0.02	0.00945156

Table 3.4: Electric field intensity and charge density ($r = 1$) with parameters $r = 1$, $a = 1.96819$, $b = 0.5$ and $c = 0.01$.

d	σ^2	E^2
0	0.003472	0.01
0.1	0.0648458	0.11
0.2	0.126554	0.21
0.3	0.188272	0.31
0.4	0.249994	0.41
0.5	0.311716	0.51
0.6	0.373439	0.61
0.7	0.435163	0.71
0.8	0.496887	0.81
0.9	0.558611	0.91
1	0.620335	1.01

Table 3.5: Electric field intensity and charge density ($r = 1$) with parameters $r = 1$, $a = 1.96819$, $b = 0.5$ and $d = 0.01$.

c	σ^2	E^2
0	0.00617245	0.01
0.1	0.0405418	0.11
0.2	0.0752451	0.21
0.3	0.109959	0.31
0.4	0.144676	0.41
0.5	0.179394	0.51
0.6	0.214113	0.61
0.7	0.248832	0.71
0.8	0.283552	0.81
0.9	0.318271	0.91
1	0.352991	1.01

Table 3.6: Potentials A^2 and B^2 with varying radius with parameters $a = 1.96819$, $b = 0.5$, $c = 0.01$, $d = 0.01$ and $\alpha = 0.931$.

r	A^2	B^2
0	1.01316	0.258146
0.1	1.04095	0.245514
0.2	1.07452	0.233786
0.3	1.11453	0.22288
0.4	1.16189	0.212719
0.5	1.21776	0.203237
0.6	1.28374	0.194375
0.7	1.36189	0.186081
0.8	1.45505	0.178306
0.9	1.56709	0.171009
1	1.70349	0.164151

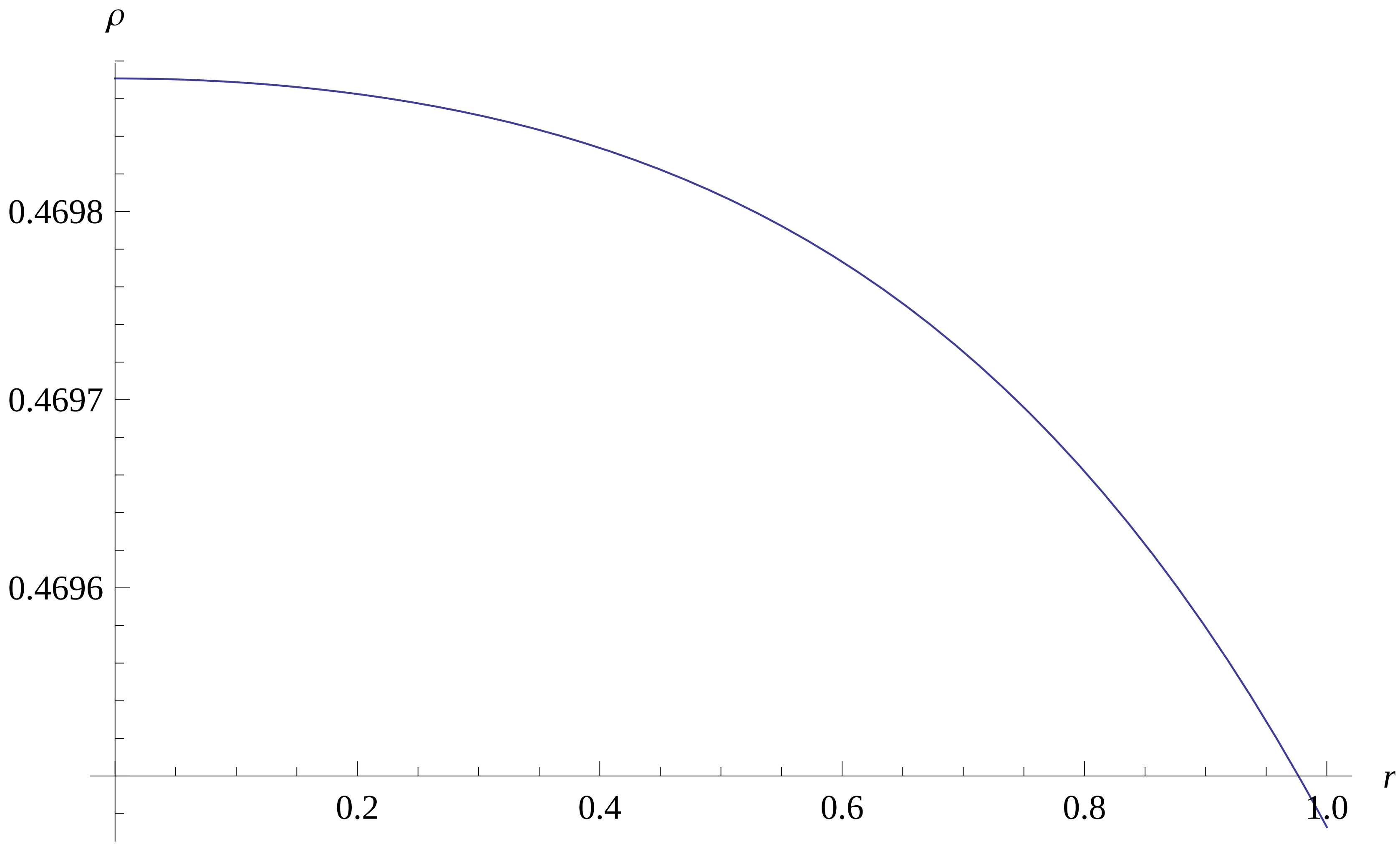

Figure 3.1: Variation of density with the radius

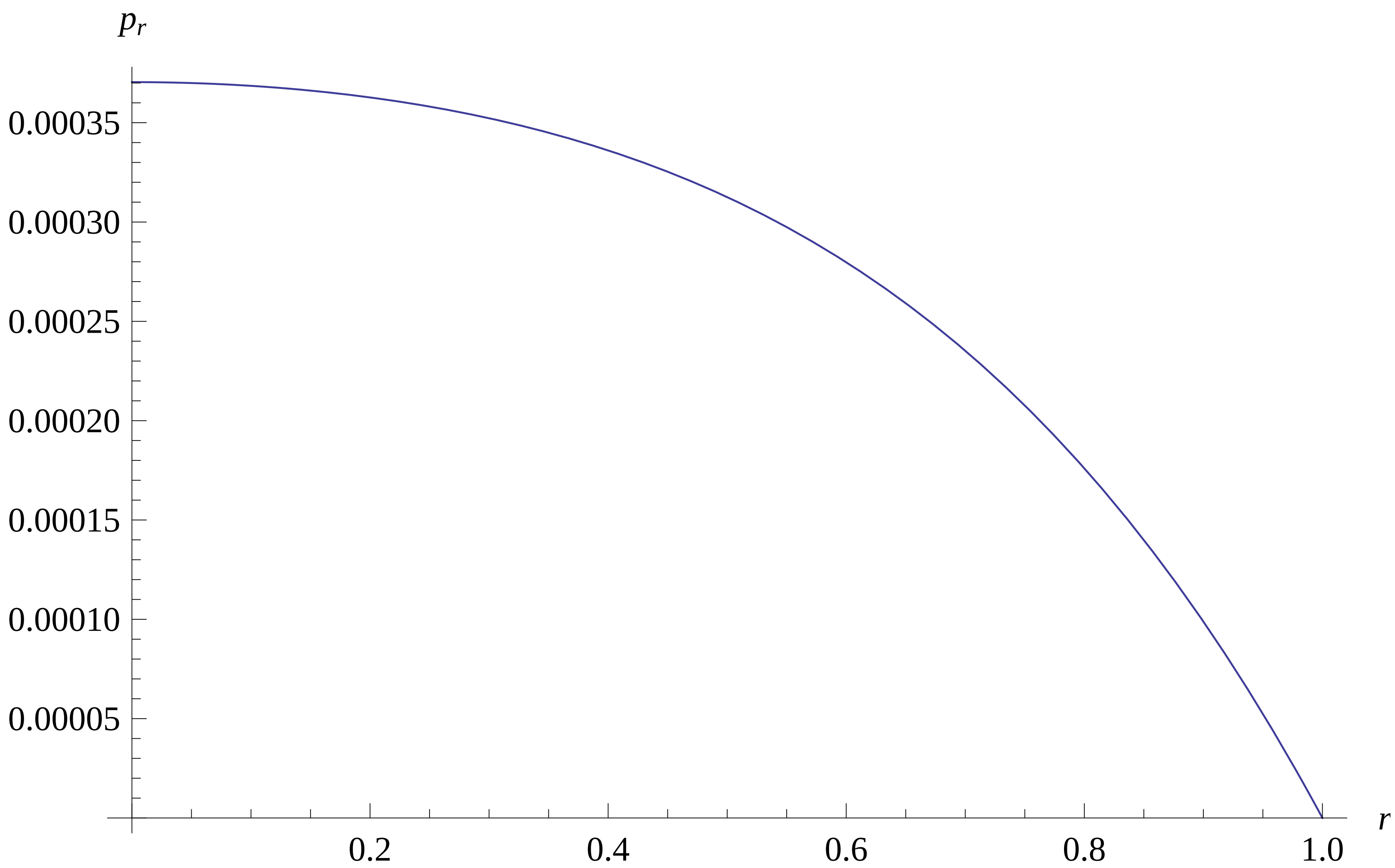

Figure 3.2: Variation of radial pressure with the radius.

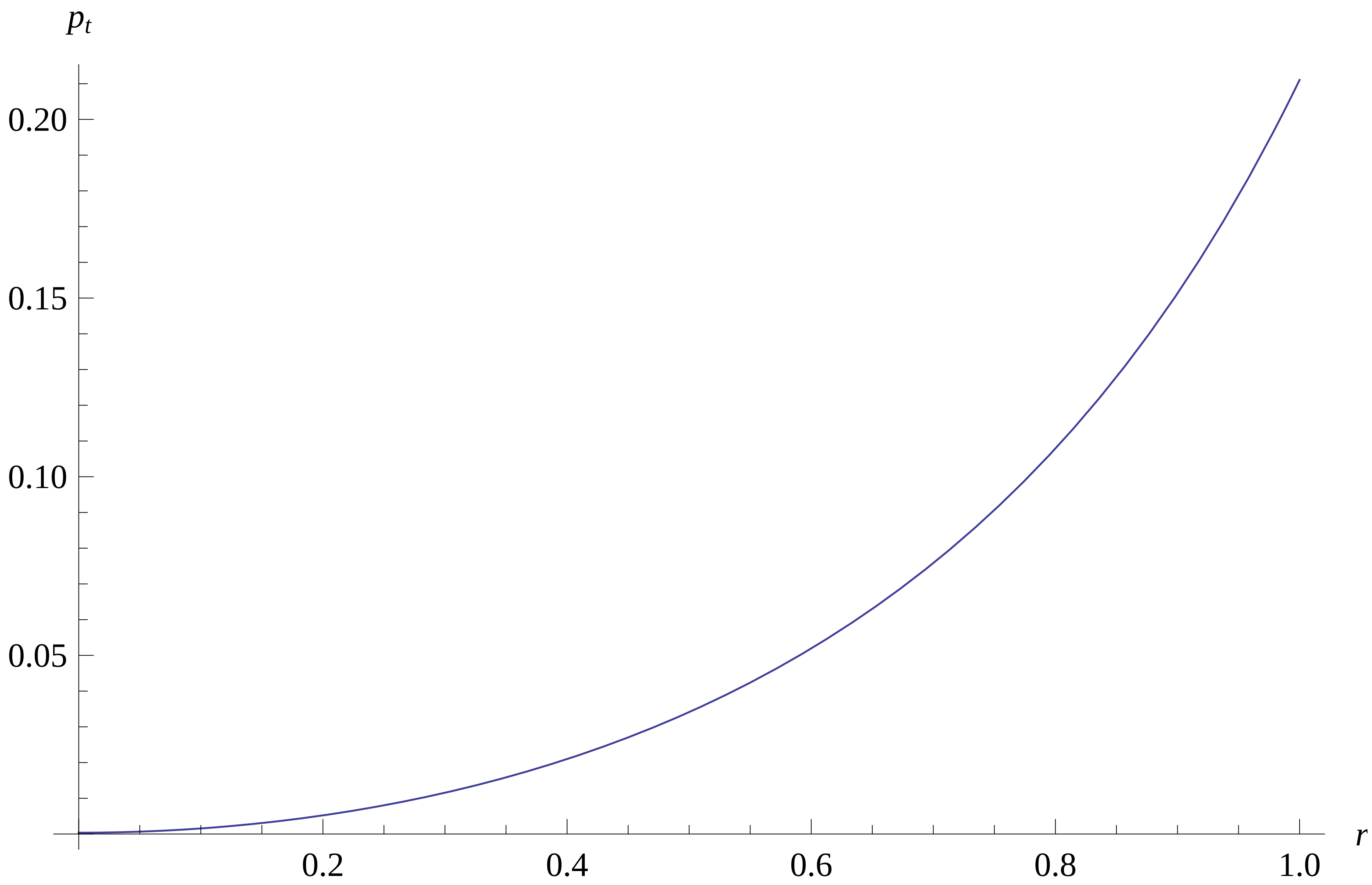

Figure 3.3: Variation of tangential pressure with the radius.

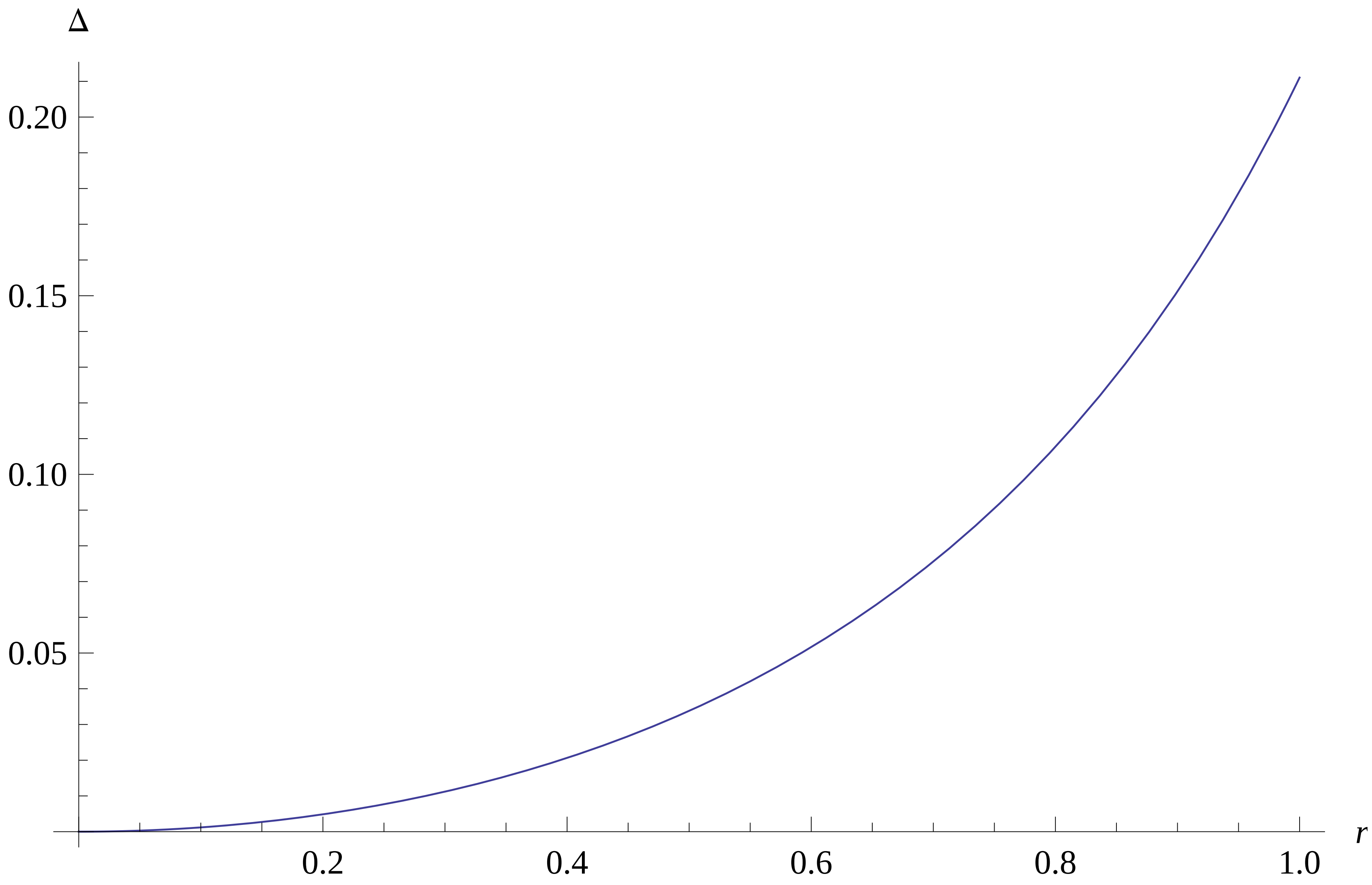

Figure 3.4: Variation of measure of anisotropy with the radius.

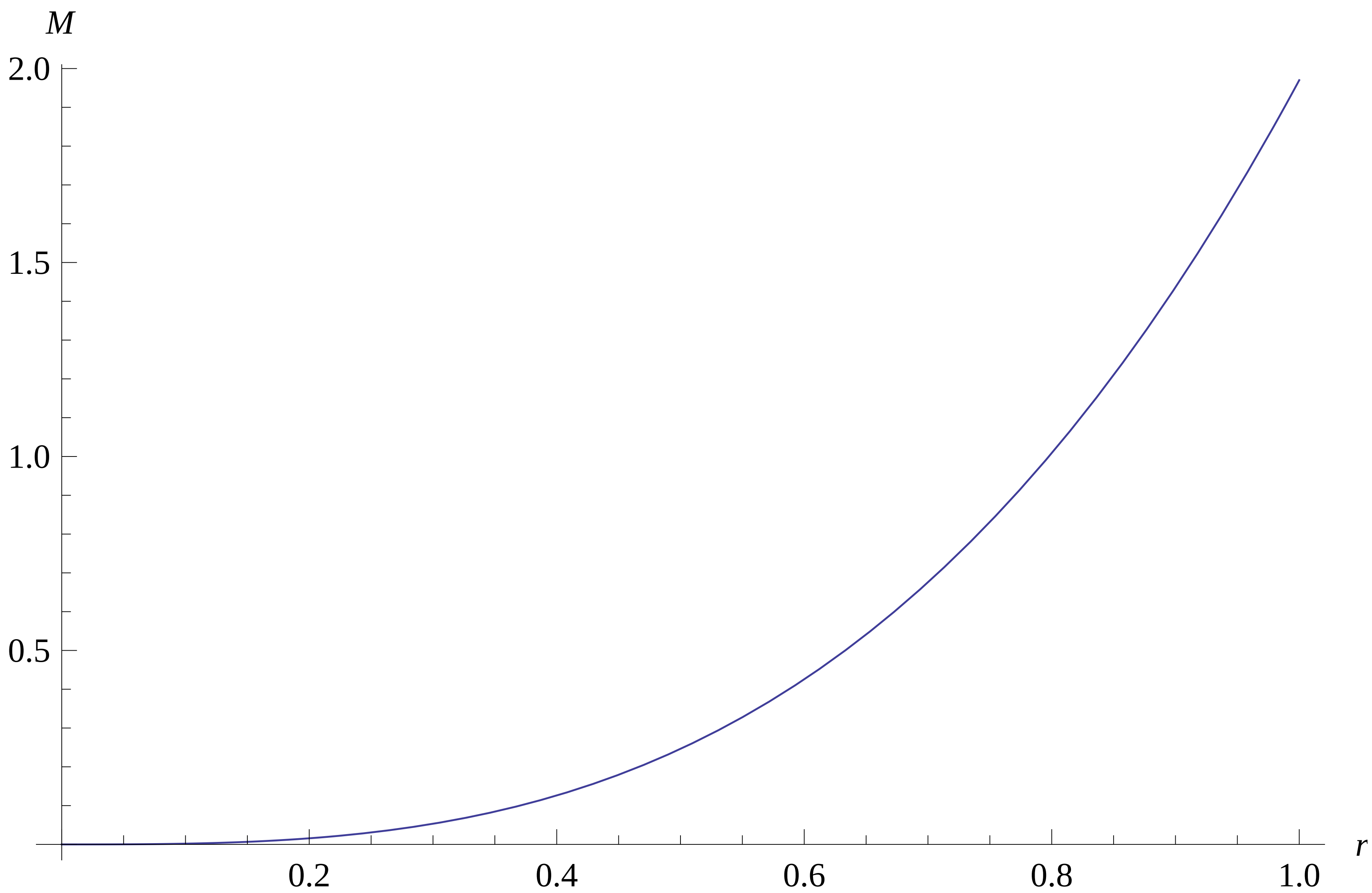

Figure 3.5: Variation of mass with the radius.

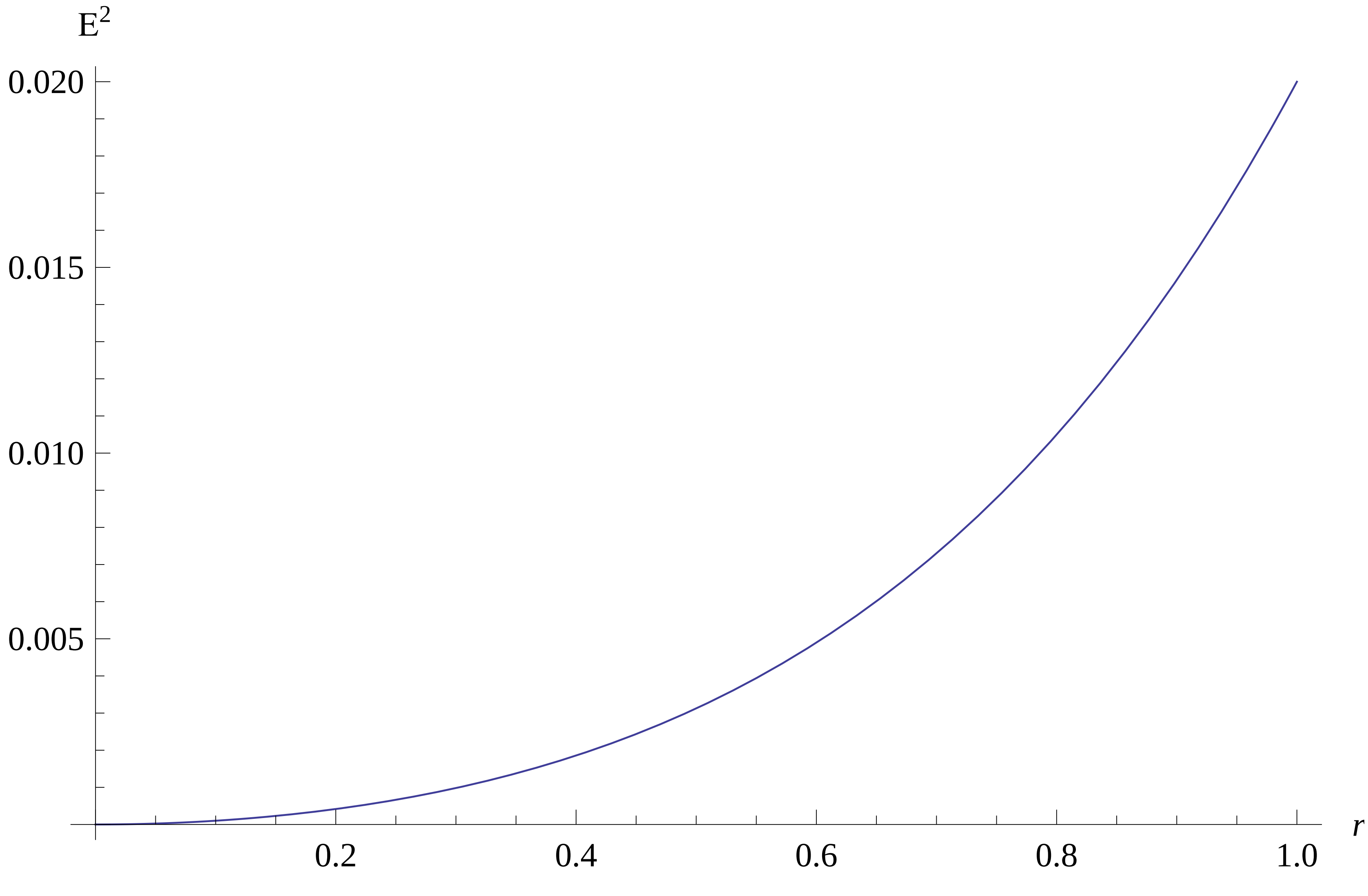

Figure 3.6: Square of the Electric field intensity against the radius.

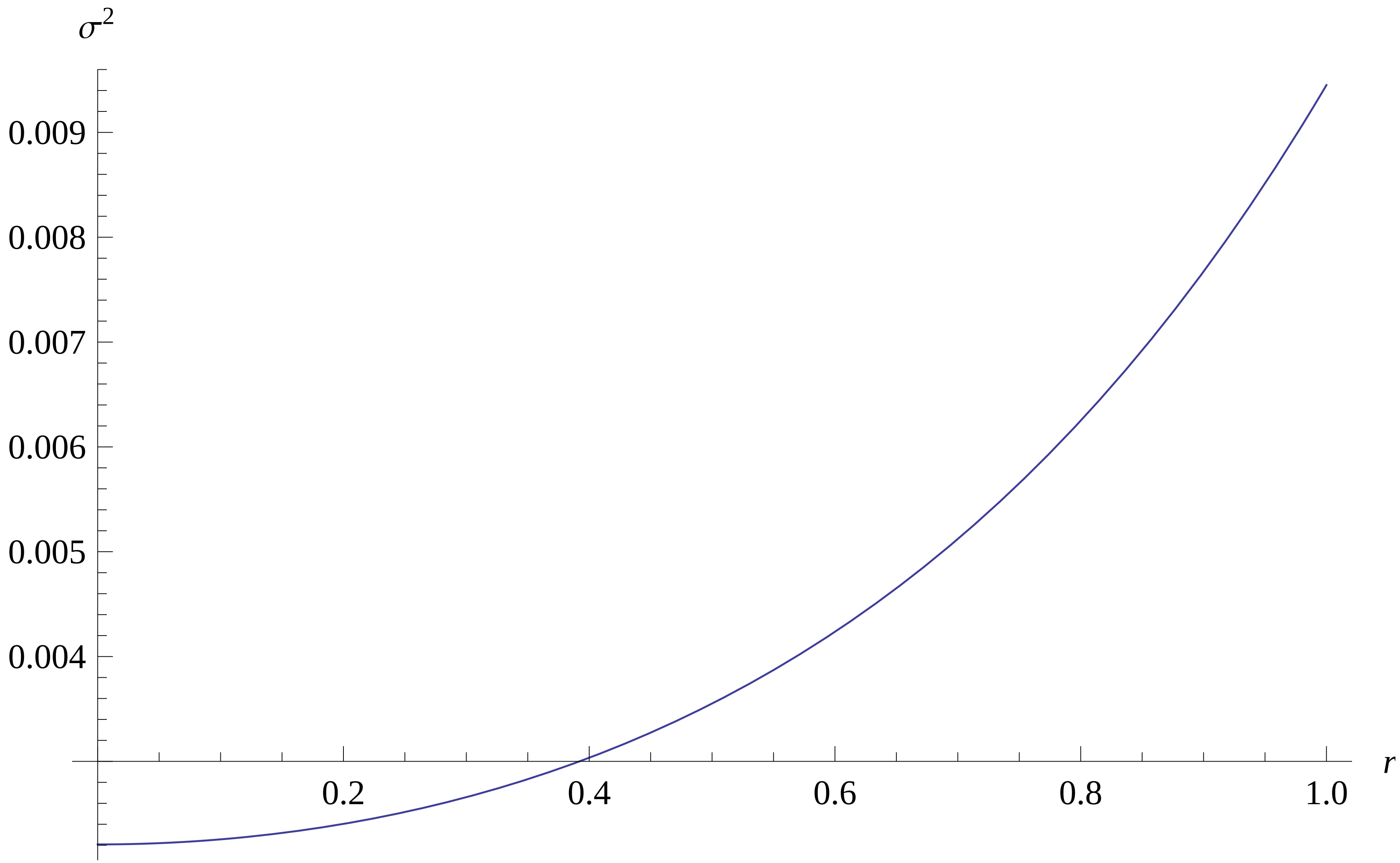

Figure 3.7: Charge density against the radius.

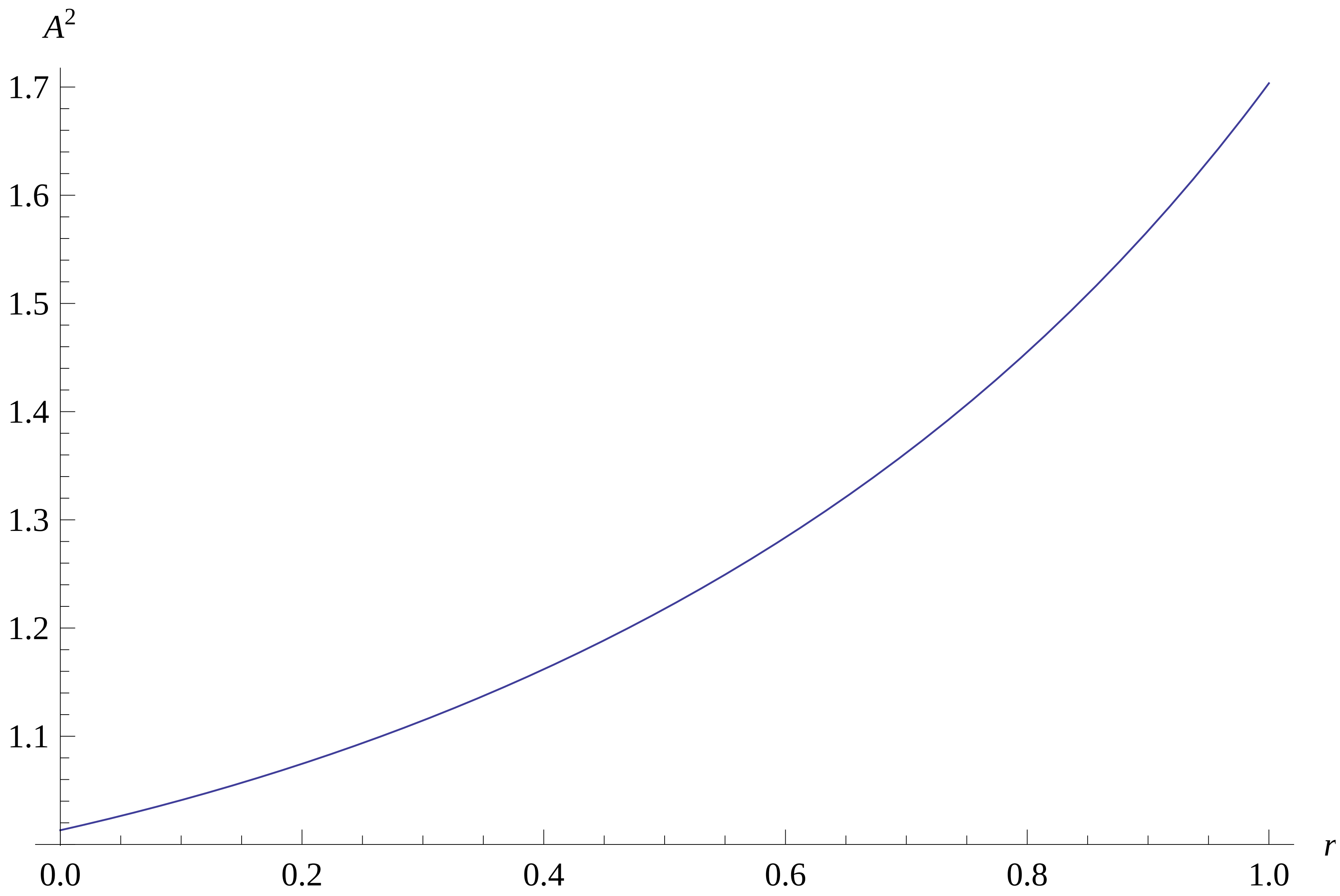

Figure 3.8: Variation of the gravitational potential A^2 against the radius.

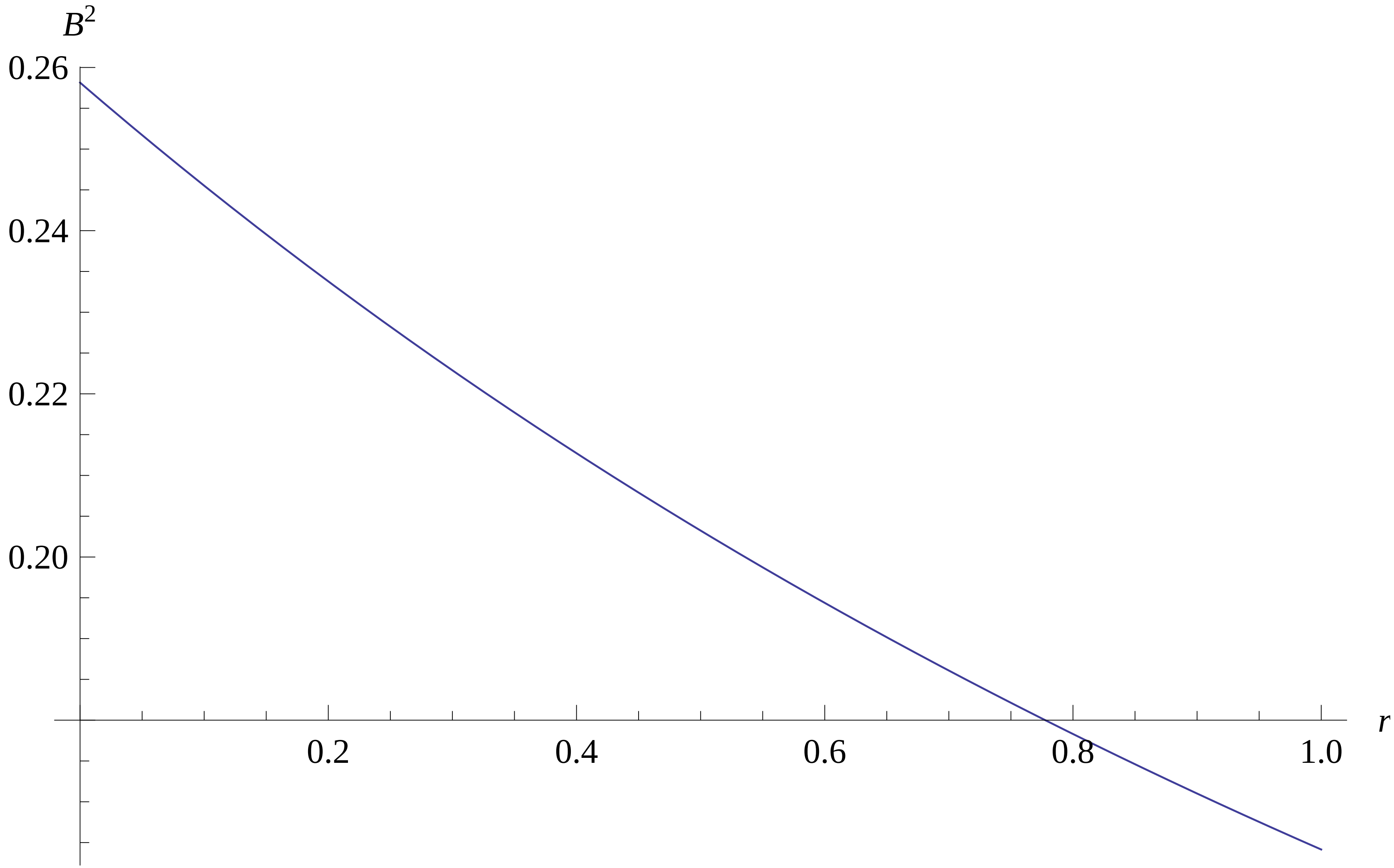

Figure 3.9: Variation of the gravitational potential B^2 against the radius.

Chapter 4

Compact objects with a quadratic equation of state

4.1 Introduction

The modelling of highly dense matter configurations in a general relativistic setting is an important research problem. Recent attempts in this direction include the effects of anisotropy and the electromagnetic field. Some recent results are contained in the works of Mafa Takisa and Maharaj (2013a), Mafa Takisa *et al* (2014a,b), Maharaj *et al* (2014) and Sunzu *et al* (2014a,b). However these treatments and others have been completed in the context of Schwarzschild coordinates. There have been fewer investigations involving isotropic coordinates. Pant *et al* (2014a) analysed a family of exact solutions of Einstein-Maxwell field equations in isotropic coordinates. An application to neutron star and quark star with Einstein-Maxwell field equations in isotropic coordinates was studied by Pradhan and Pant (2014). An investigation of a class of super dense stars models using charged analogues of Hajj-Boutros type relativistic fluid solutions has recently been done by Pant *et al* (2014b). In a recent analysis Ngubelanga *et al* (2014a) found exact models for a compact stellar object which could be charged and anisotropic with a linear equation of state.

A simple generalisation of the linear relation between the energy density and radial pressure is a quadratic equation of state. This allows for more general behaviour in the matter distribution and greater complexity in the model. There is still a debate over the structure of a star as to its composition in terms of nuclear matter, or quark matter, or a hybrid mix of both distributions. It is difficult to find a single equation of state for a matter distribution matching the stellar core (softer quark matter) to the outer regions (stiffer nuclear matter). These issues are highlighted in the treatments of Cotton (2002), Özel (2006) and Rodrigues *et al* (2011). A quadratic equation of state which is softer at low densities and stiffer at high densities may be appropriate for describing a hybrid star. This would make it possible to explain the stability of compact stars with masses $\sim 2M_{\odot}$. In a general relativistic context models which are charged and anisotropic were found by Feroze and Siddiqui (2011). A class of models, generalising the results of Feroze and Siddiqui (2011) and containing models with linear equations of state, was found by Maharaj and Mafa Takisa (2012). These solutions have the desirable property of regularity at the stellar centre. Mafa Takisa *et al* (2014b) modelled a charged general relativistic star with a quadratic equation of state. They showed their results were consistent with several masses of stellar objects, in particular with the star PSR J1614-2230. Malaver (2014) found exact solutions to the field equations for a strange quark model. Sharma and Ratanpal (2013) presented a class of new models using the metric ansatz of Finch and Skea (1998) without assuming any equations of state. Their approach has the remarkable feature of yielding a quadratic equation of state when appropriate physical bounds are applied. Thirukkanesh and Ragel (2012) and Mafa Takisa and Maharaj (2013b) generated exact anisotropic spheres which are uncharged and charged, respectively. These models have a polytropic equation of state in general; however for particular parameter values quadratic equations of state arise.

The aim of this chapter is to obtain new exact solutions to the Einstein-Maxwell system of equations. We model charged anisotropic matter distributions in isotropic coordinates by imposing a quadratic equation of state which relates the radial pressure to the energy density. The Einstein-Maxwell field equations in the presence of electric field with anisotropic pressures are presented in §4.2. The transformation of Kustaanheimo and Qvist (1948) is applied to write the field equations in new equivalent forms. In §4.3, we present new classes of exact solutions to the system of equations. We show that the new solution with a quadratic barotropic equation of state contains a known solution by Ngubelanga *et al* (2014a) in §4.4. In §4.5, we regain the masses for the observed objects and study the physical properties of the new exact solutions. We analyse the physical features for the stellar model associated with the star PSR J1903+327 in §4.6. Some concluding remarks are made in §4.7.

4.2 The model

We intend to model the interior of a dense star. The line element in isotropic coordinates has the form

$$ds^2 = -A^2(r)dt^2 + B^2(r)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{4.1}$$

in coordinates $(x^a) = (t, r, \theta, \phi)$. The gravitational field is represented by the metric quantities $A(r)$ and $B(r)$ in metric (4.1). An anisotropic charged matter distribution has the energy momentum of the form

$$T_{ij} = \text{diag}\left(-\rho - \frac{1}{2}E^2, p_r - \frac{1}{2}E^2, p_t + \frac{1}{2}E^2, p_t + \frac{1}{2}E^2\right), \tag{4.2}$$

where ρ is the energy density, p_r is the radial pressure, p_t is the tangential pressure and E is the electric field intensity. A timelike unit four-velocity $\mathbf{u}$ where $u^i = \frac{1}{A}\delta^i_0$ measures the quantities in equation (4.2) above.

If we introduce the transformation

$$x \equiv r^2, \quad L \equiv B^{-1}, \quad G \equiv LA, \tag{4.3}$$

then the line element (4.1) can be written in the new form

$$ds^2 = -\left(\frac{G}{L}\right)^2 dt^2 + L^{-2}\left[\left(\frac{1}{4x}\right)dx^2 + x(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{4.4}$$

in new variables of x. The system of the Einstein-Maxwell field equations can be expressed as

$$8\pi\rho + \frac{1}{2}E^2 = 4[2xLL_{xx} - 3(xL_x - L)L_x], \tag{4.5a}$$

$$8\pi p_r - \frac{1}{2}E^2 = 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x, \tag{4.5b}$$

$$\begin{aligned} 8\pi p_t + \frac{1}{2}E^2 &= 4xL^2\frac{G_{xx}}{G} + 4L(L - 2xL_x)\frac{G_x}{G} \\ &\quad -4(2L - 3xL_x)L_x - 8xLL_{xx}, \end{aligned} \tag{4.5c}$$

$$\sigma^2 = \frac{1}{4\pi x}L^2(E + xE_x)^2, \tag{4.5d}$$

in terms of new variables by utilizing transformation (4.3). The subscript "x" denotes a derivative with respect to the new variable x. In terms of new variables in (4.3) the condition of pressure anisotropy has the form

$$\frac{G_{xx}}{G} - 2\frac{L_{xx}}{L} = \frac{(8\pi\Delta + E^2)}{4xL^2}, \tag{4.6}$$

where the quantity $\Delta = p_t - p_r$ is the measure of anisotropy. The mass function has the form

$$m(x) = 4\pi \int_0^x \left[\sqrt{\omega}\rho(\omega) + \frac{E^2}{8\pi}\right] d\omega, \tag{4.7}$$

in new coordinates. The mass function represents the mass within the radius x of the sphere.

We assume the quadratic equation of state of the form

$$p_r = \eta\rho^2 + \alpha\rho - \beta, \tag{4.8}$$

relating the radial pressure p_r to the energy density ρ, and where η, α and β are arbitrary constants. This is a simple generalisation of a linear equation of state which is regained when $\eta = 0$. With the inclusion of the quadratic equation of state, the Einstein-Maxwell system of equations (4.5) with the charged anisotropic fluid spheres can be expressed as

$$8\pi\rho = 4[2xLL_{xx} - 3(xL_x - L)L_x] - \frac{1}{2}E^2, \tag{4.9a}$$

$$p_r = \eta\rho^2 + \alpha\rho - \beta, \tag{4.9b}$$

$$p_t = p_r + \Delta, \tag{4.9c}$$

$$\begin{aligned} 8\pi\Delta = {} & 4xL^2\frac{G_{xx}}{G} + 4L(L - 2xL_x)\frac{G_x}{G} - \frac{\eta}{32\pi}\left[16xLL_{xx} - 24(xL_x - L)L_x - E^2\right]^2 \\ & -8(1+\alpha)xLL_{xx} + 12(1+\alpha)xL_x^2 - 4(2+3\alpha)LL_x \\ & -\frac{(1-\alpha)E^2}{2} + 8\pi\beta, \end{aligned} \tag{4.9d}$$

$$\begin{aligned} \frac{G_x}{G} = {} & \frac{\eta\left[16xLL_{xx} - 24(xL_x - L)L_x - E^2\right]^2}{128\pi L(L - 2xL_x)} + \frac{2\alpha xL_{xx}}{(L - 2xL_x)} \\ & -\frac{3(1+\alpha)xL_x^2}{L(L - 2xL_x)} + \frac{(2+3\alpha)L_x}{(L - 2xL_x)} - \frac{(1+\alpha)E^2}{8L(L - 2xL_x)} \\ & -\frac{2\pi\beta}{L(L - 2xL_x)}, \end{aligned} \tag{4.9e}$$

$$\sigma^2 = \frac{1}{4\pi x}L^2(E + xE_x)^2. \tag{4.9f}$$

It is crucial to note the nonlinearity in both the functions L and G in the system (4.9) which is increased because of the appearance of terms containing the parameter η. This system of equations contains six variables involving the matter and the electromagnetic quantities (ρ, p_r, p_t, Δ, E and σ) and two gravitational potentials (L and G). It should also be highlighted that there are only six independent equations in this system of equations. Integration of such systems is not easy to perform due to nonlinearity and the fact that there are more unknown functions than the independent field equations. In order to integrate and obtain some exact solutions, the above mentioned facts suggests that we need to choose the form for two of the quantities mentioned above. The system of equations (4.9) is similar to the field equations of Ngubelanga *et al* (2014a); however in our case the equation of state is quadratic. In their treatment they utilized the linear equation of state, *i.e.*, $\eta = 0$ so that $p_r = \alpha\rho - \beta$.

The interior metric (4.1) with the charged matter distribution should match the exterior spacetime which is given by

$$\begin{aligned} ds^2 = \quad & -\left(1 - \frac{2M}{R} + \frac{q^2}{R^2}\right) dt^2 + \left(1 - \frac{2M}{R} + \frac{q^2}{R^2}\right)^{-1} dR^2 \\ & + R^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{4.10}$$

in coordinates $(x^a) = (t, R, \theta, \phi)$. In (4.10) the total mass and the total charge of the sphere are denoted by M and q^2, respectively. The exterior spacetime (4.10) is referred to as the Reissner-Nordström metric. The junction conditions at the stellar surface are obtained by matching the first and the second fundamental forms for the interior metric (4.1) and the exterior metric (4.10). The conditions are as follows

$$A_s = \left(1 - \frac{2M}{R} + \frac{q^2}{R^2}\right)^{\frac{1}{2}}, \tag{4.11a}$$

$$R_s = r_s B_s, \tag{4.11b}$$

$$\left(\frac{B'}{B} + \frac{1}{r}\right)_s r_s = \left(1 - \frac{2M}{R} + \frac{q^2}{R^2}\right)^{\frac{1}{2}}, \tag{4.11c}$$

$$r_s (A')_s = \frac{M}{R} - \frac{q^2}{R^2}, \tag{4.11d}$$

evaluated at the boundary of the star $r = s$. In isotropic coordinates the boundary conditions are given by the equations (4.11).

4.3 Exact models

Our purpose is to generate new exact solutions to the Einstein-Maxwell system of equations (4.9). The integration is achieved by choosing physical reasonable forms for the electric field E and the gravitational potential L. We make the particular choice

$$L = a + bx, \tag{4.12a}$$

$$E^2 = x(c + dx), \tag{4.12b}$$

where a, b, c and d are real constants. The potential L and the electric field intensity E, respectively, are selected to be a linear function and a quadratic function in the variable x. Similar choices for L and E were made by Ngubelanga *et al* (2014a) for a linear equation of state leading to acceptable stellar configurations; we expect this to also carry through with the addition of a quadratic term in the equation of state. On applying (4.12), equation (4.9e) becomes

$$\begin{aligned} \frac{G_x}{G} &= \frac{\eta\left[24ab-(c+dx)x\right]^2}{128\pi(a-bx)(a+bx)}+\frac{b(2+3\alpha)}{(a-bx)} \\ &\quad -\frac{[16\pi\beta+(1+\alpha)(24b^2+c+dx)x]}{8(a-bx)(a+bx)}, \end{aligned} \tag{4.13}$$

which is a first order equation in potential G. We integrate (4.13) obtain

$$G(x)=K(a-bx)^{\Psi}(a+bx)^{\Phi}e^{N(x)}, \tag{4.14}$$

where K is the constant of integration. The function $N(x)$ and the constants Ψ and Φ are given explicitly by

$$N(x) = \frac{x}{384\pi b^4}\left\{48\pi b^2 d(1+\alpha) - 3\eta\left[b^2c^2 + ad(ad - 48b^3)\right]\right.$$

$$\left.-b^2 d\eta(3c + dx)x\right\}, \tag{4.15a}$$

$$\Psi = \frac{1}{256\pi ab^5}\left\{16\pi ab^2(1+\alpha)(bc+ad) - 128\pi b^4\left[ab(1+3\alpha) - 2\pi\beta\right]\right.$$

$$\left.-a^2\eta\left[a^2d^2 + b(c - 24b^2)(2ad + bc - 24b^3)\right]\right\}, \tag{4.15b}$$

$$\Phi = \frac{1}{256\pi ab^5}\left\{16\pi ab^2(1+\alpha)\left[b(24b^2 + c) - ad\right] - 256\pi^2 b^4\beta\right.$$

$$\left.+a^2\eta\left[a^2d^2 - b(24b^2 + c)(2ad - bc - 24b^3\right]\right\}, \tag{4.15c}$$

where the constants $a \neq 0$ and $b \neq 0$ to avoid singularity. An exact solution can then be found to the Einstein-Maxwell system. The metric (4.1) has the form

$$ds^2 = -K(a - br^2)^{2\Psi}(a + br^2)^{2(\Phi-1)}dt^2$$

$$+(a + br^2)^{-2}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{4.16}$$

where K is the constant of integration. The function $N(r)$ and the constants Ψ and Φ are given explicitly by (4.15).

Since equation (4.13) has been integrated, we can generate an exact model for the system of equations (4.9) in terms of the radial coordinate "r" which has the form

$$8\pi\rho = 12ab - \left(\frac{1}{2}c + \frac{1}{2}dr^2\right) r^2, \tag{4.17a}$$

$$p_r = \eta\rho^2 + \alpha\rho - \beta, \tag{4.17b}$$

$$p_t = p_r + \Delta, \tag{4.17c}$$

$$\begin{aligned} 8\pi\Delta = {} & \frac{4b\Psi(a+br^2)r^2}{(a-br^2)^2}\left[b(\Psi-1)(a+br^2) - 2(a-br^2)\left(b\Phi + N'(a+br^2)\right)\right] \\ & -\frac{\eta}{32\pi}\left[24ab - r^2(c+dr^2)\right]^2 + 4(a+br^2)\left[a - br^2(1-2\Phi)\right]N' \\ & +4r^2(a+br^2)^2(N'^2+N'') + 4b\left[\Phi(a-br^2) - (a+br^2)(2+3\alpha+\Psi)\right] \\ & +4b^2\left[3(1+\alpha) + \Phi(\Phi-1)\right]r^2 - \frac{1}{2}(1-\alpha)(c+dr^2)r^2 + 8\pi\beta, \end{aligned} \tag{4.17d}$$

$$\sigma^2 = \frac{(a+br^2)^2}{16\pi^2(c+dr^2)}\left[3c + 4dr^2\right]^2, \tag{4.17e}$$

$$E^2 = r^2(c+dr^2). \tag{4.17f}$$

It is interesting to note that our model is of a simple form and all physical quantities are expressed in terms of elementary functions where the function $N(r)$ and the constants Ψ and Φ are given in (4.15), respectively. For this model the mass function is given by

$$m(r) = \frac{1}{2}r^3\left[\frac{(12ab+c)}{3} - \frac{(c-2d)r^2}{10} - \frac{dr^4}{14}\right]. \tag{4.18}$$

A charged anisotropic star with quadratic equation of state may be modelled by the above solution (4.17).

4.4 The linear case

When $\eta = 0$ then the equation of state becomes

$$p_r = \alpha\rho - \beta, \tag{4.19}$$

which is linear. We observe that the case (4.19) reduces to the Ngubelanga *et al* (2014a) model. Our result is a generalisation with a quadratic equation of state. All the results in Ngubelanga *et al* (2014a) can be regained as a special case from the exact solution (4.17). The relationship (4.19) is consistent with the stars PSR J1614-2230, Vela X-1, PSR J1903+327, 4U 1820-30 and SAX J1808.4-3658 as demonstrated in their analysis. In particular for the parameter values $a = 1.96819$, $b = 0.5$, $c = 0.01$, $d = 0.01$ and $\alpha = 0.931$ we can produce the mass $1.97M_\odot$. This stellar mass corresponds to the astronomical object PSR J1614-2230.

4.5 The quadratic case

From the exact solution (4.17) we observe that the quantities associated with the matter field and the electromagnetic field are well behaved. The electric field vanishes at the stellar centre $r = 0$. The matter density ρ and the proper charge density σ remain finite at the centre. At the centre of the star we can write

$$\rho_0 = \frac{3ab}{2\pi}, \tag{4.20a}$$

$$p_{r0} = \eta\left(\frac{3ab}{2\pi}\right)^2 + \alpha\left(\frac{3ab}{2\pi}\right) - \beta, \tag{4.20b}$$

which are finite values. For the electromagnetic quantities we have

$$\sigma_0^2 = \frac{1}{c}\left(\frac{3ac^2}{4\pi}\right), \tag{4.21a}$$

$$E_0^2 = 0, \tag{4.21b}$$

which are nonsingular at the centre. For the pressure anisotropy we have

$$\Delta_0 = -\frac{3ab\eta}{32\pi^2} + \frac{a^2}{2\pi}N'(0) + \frac{ab}{2\pi}[\Phi - (2 + 3\alpha + \Psi)] + \beta, \tag{4.22}$$

at $r = 0$. The metric functions A and B are regular at $r = 0$. Therefore all physical and gravitational quantities are well behaved in the core regions of the star. For the star to remain stable it is required that $\Delta = 0$ at $r = 0$; we demonstrate that this happens in the next section using a graphical treatment. The mass remains finite and also depends on the parameters c and d which are associated with charge.

The speed of sound is defined by

$$v^2 = \frac{dp_r}{d\rho}, \tag{4.23}$$

and we must have $v < 1$ to maintain causality. Also we must have zero radial pressure at the boundary for a compact object. This will ensure consistency of the matching conditions (4.11) at the surface and continuity of the metrics (4.1) and (4.10) at the surface. For a finite value of the density at the surface we require

$$\rho_s = \frac{3ab}{2\pi} - \frac{1}{16\pi}(c+d), \tag{4.24}$$

in geometric units by fixing the radius of the star at $r = 1$. Then (4.17b) restricts the parameter β by

$$\beta_s = \eta\rho_s^2 + \alpha\rho_s. \tag{4.25}$$

When $\eta = 0$ then (4.25) reduces to the corresponding expression of Ngubelanga *et al* (2014a). In our subsequent analysis throughout we choose the parameter values $\alpha = 0.931$ and $\eta = 3.185$ since they produce relativistic compact stars with desirable physical features.

It is possible to give numerical values to quantities in our exact solutions. We have considered values for five compact objects for which reliable data exists. The objects selected are PSR J1614-2230 studied by Demorest *et al* (2010), 4U 1608-52 investigated by Güver *et al* (2010a), PSR J1903+327 analysed by Freire *et al* (2011), EXO 1745-248 studied by Özel *et al* (2009) and SAX J1808.4-3658 considered by Elebert *et al* (2009). We vary the parameter a in (4.18) and assign fixed values for $b = 0.504167$, $c = 0.01$ and $d = 0.01$. This permits us to generate numerical values for the stellar masses for the five astronomical objects listed in Table 4.1. We have used small values for the parameters c and d which introduce charge into the system to ensure that the electromagnetic contribution is small. We find that the observed masses vary between $0.9M_{\odot}$ to $1.97M_{\odot}$. Values for the central density ρ_0, central radial pressure p_{r0} and surface density ρ_s lie in the expected range.

4.6 The star PSR J1903+327

The parameter value $a = 1.65143$ generates the mass 1.667 $M_{\odot}$ which corresponds to the star PSR J1903+327. We use this parameter value for a to analyse the variation of the physical features associated with the matter, charge and gravity field within the star.

Table 4.2 represents the variation of density ρ, radial pressure p_r, tangential pressure p_t and anisotropy Δ within the star. The quantities ρ and p_r are decreasing functions. The radial pressure p_r vanishes at $r = 1$ determining the boundary which is the requirement for a compact star. The tangential pressure p_t has finite values. The anisotropy Δ remains finite and has the value $\Delta = 0$ at $r = 0$ which is required for stability. Table 4.3 presents the behaviour of the mass m, electric field E and charge density σ. The mass increases as r grows larger. The electric field E and charge density σ are finite and nonsingular throughout the star with $E = 0$ at $r = 0$. The effect of the charge is incorporated through the parameters c and d. Tables 4.4 and 4.5 represent the total charge in the star with $r = 1$ fixed at the stellar surface. It is clear that the parameter d has a greater effect than that of the parameter c which makes the star more charged. The metric functions A^2 and B^2 are evaluated in Table 4.6 for the set of parameter values corresponding to PSR J1903+327 through the interior of the star. The values obtained for the metric functions indicate that the potentials are regular and positive.

A graphical analysis provides deeper insight into the behaviour of the physical features. We have presented plots for the density (Fig. 4.1), radial pressure (Fig. 4.2), tangential pressure (Fig. 4.3), pressure anisotropy (Fig. 4.4), mass (Fig. 4.5), electric field intensity (Fig. 4.6), charge density (Fig. 4.7) and metric functions (Fig. 4.8 and Fig. 4.9). It is clear that all the quantities have regular profiles. The various plots have been generated with the assistance of Mathematica (2003). Ngubelanga *et al* (2014a) using a linear equation of state also studied particular observed stars in general relativity. Our results in this chapter with a quadratic equation of state are broadly consistent with their results.

4.7 Discussion

Our objective in this chapter was to find new exact solutions to the Einstein-Maxwell field equations for matter configurations with anisotropy and charge in isotropic coordinates. We selected the barotropic equation of state to be quadratic which relates the radial pressure p_r to the energy density ρ. The classes of exact solutions (4.17) to the Einstein-Maxwell field equations were shown to be physically acceptable. The tables for charge and matter variables suggest that they represent physically reasonable configurations. By choosing to fix the parameters $b = 0.504167$, $c = 0.01$, $d = 0.01$, $\alpha = 0.931$ and $\eta = 3.185$ and varying the parameter a in Table 4.1, we regained the masses for the stellar objects PSR J1614-2230, 4U 1608-52, PSR J1903+327, EXO 1745-248 and SAX J1808.4-3658. We fixed the parameter a and used the star PSR J1903+327 which has the mass 1.667 $M_{\odot}$, to produce tables and graphical plots for relevant quantities related to the metric, matter and charge. We made the particular choices $a = 1.65143$, $b = 0.504167$, $c = 0.01$, $d = 0.01$, $\alpha = 0.931$ and $\eta = 3.185$ to perform graphical plots using the software package Mathematica. Our graphical approach suggests that the model for the star PSR J1903+327 is well behaved. The introduction of the quadratic parameter η in the equation of state $p_r = \eta\rho^2 + \alpha\rho - \beta$ does produce a new exact solution of the Einstein-Maxwell system which is qualitatively different from the linear case $p_r = \alpha\rho - \beta$. However the quadratic equation of state does produce models which can be related to observed stellar objects.

Table 4.1: Mass m, central density ρ_0, central radial pressure p_{r0} and surface density ρ_s of different stars corresponding to the parameters $b = 0.504167$, $c = 0.01$, $d = 0.01$, $\alpha = 0.931$ and $\eta = 3.185$

Star	Observed mass m	a	ρ_0	p_{r0}	ρ_s
PSR J1614-2230	1.97	1.95192	0.469871	0.00156084	0.469473
4U 1608-52	1.74	1.72382	0.414962	0.00142167	0.414565
PSR J1903+327	1.667	1.65143	0.397535	0.0013775	0.397137
EXO 1785-248	1.3	1.28746	0.30992	0.00115543	0.309522
SAX J1808.4-3658	0.9	0.890767	0.214427	0.000913404	0.214029

Table 4.2: Variation of energy density ρ, radial pressure p_r, tangential pressure p_t and measure of anisotropy Δ from the centre to the surface with parameters $a = 1.65143$, $b = 0.504167$, $c = 0.01$, $d = 0.01$, $\alpha = 0.931$ and $\eta = 3.185$

r	ρ	p_r	p_t	Δ
0	0.397536	0.0013775	0.0013775	0
0.1	0.397534	0.00137054	0.00259535	0.00122481
0.2	0.397527	0.00134884	0.00633559	0.00498676
0.3	0.397516	0.00130991	0.0128702	0.0115603
0.4	0.397499	0.00124963	0.0226954	0.0214458
0.5	0.397473	0.0011622	0.0366066	0.0354444
0.6	0.397438	0.00104019	0.0558307	0.0547905
0.7	0.39739	0.000874527	0.0822543	0.0813798
0.8	0.397327	0.000654462	0.118827	0.118173
0.9	0.397244	0.000367625	0.170302	0.169934
1	0.397138	0	0.244665	0.244665

Table 4.3: Variation of mass m, electric field intensity E^2 and charge density σ^2 for charged bodies from the centre to the surface with parameters $a = 1.65143$, $b = 0.504167$, $c = 0.01$ and $d = 0.01$

r	m	E^2	σ^2
0	0	0	0.00155433
0.1	0.00166686	0.000101	0.00158992
0.2	0.013335	0.000416	0.00169896
0.3	0.0450063	0.000981	0.00188841
0.4	0.106684	0.001856	0.00217005
0.5	0.20837	0.003125	0.00256092
0.6	0.360071	0.004896	0.00308389
0.7	0.571787	0.007301	0.00376848
0.8	0.853521	0.010496	0.00465182
0.9	1.21527	0.014661	0.00577994
1	1.667	0.02	0.00720911

Table 4.4: Electric field intensity E^2 and charge density σ^2 $(r = 1)$ with parameters $r = 1$, $a = 1.65143$, $b = 0.504167$ and $c = 0.01$.

d	σ^2	E^2
0	0.00264824	0.01
0.1	0.0494606	0.11
0.2	0.0965278	0.21
0.3	0.143603	0.31
0.4	0.190681	0.41
0.5	0.237759	0.51
0.6	0.284838	0.61
0.7	0.331917	0.71
0.8	0.378997	0.81
0.9	0.426076	0.91
1	0.473156	1.01

Table 4.5: Electric field intensity E^2 and charge density σ^2 with parameters $r = 1$, $a = 1.65143$, $b = 0.504167$ and $d = 0.01$.

c	σ^2	E^2
0	0.00470799	0.01
0.1	0.0309229	0.11
0.2	0.0573926	0.21
0.3	0.0838705	0.31
0.4	0.110351	0.41
0.5	0.136832	0.51
0.6	0.163313	0.61
0.7	0.189795	0.71
0.8	0.216277	0.81
0.9	0.242759	0.91
1	0.269241	1.01

Table 4.6: Potentials A^2 and B^2 with varying radius with parameters $a = 1.65143$, $b = 0.504167$, $c = 0.01$, $d = 0.01$, $\alpha = 0.931$ and $\eta = 3.185$.

r	A^2	B^2
0	2.21008	0.366674
0.1	2.21284	0.364445
0.2	2.22096	0.35788
0.3	2.23391	0.347325
0.4	2.25077	0.333316
0.5	2.27027	0.316515
0.6	2.29067	0.297652
0.7	2.30976	0.277454
0.8	2.32474	0.256604
0.9	2.33215	0.235694
1	2.32766	0.215211

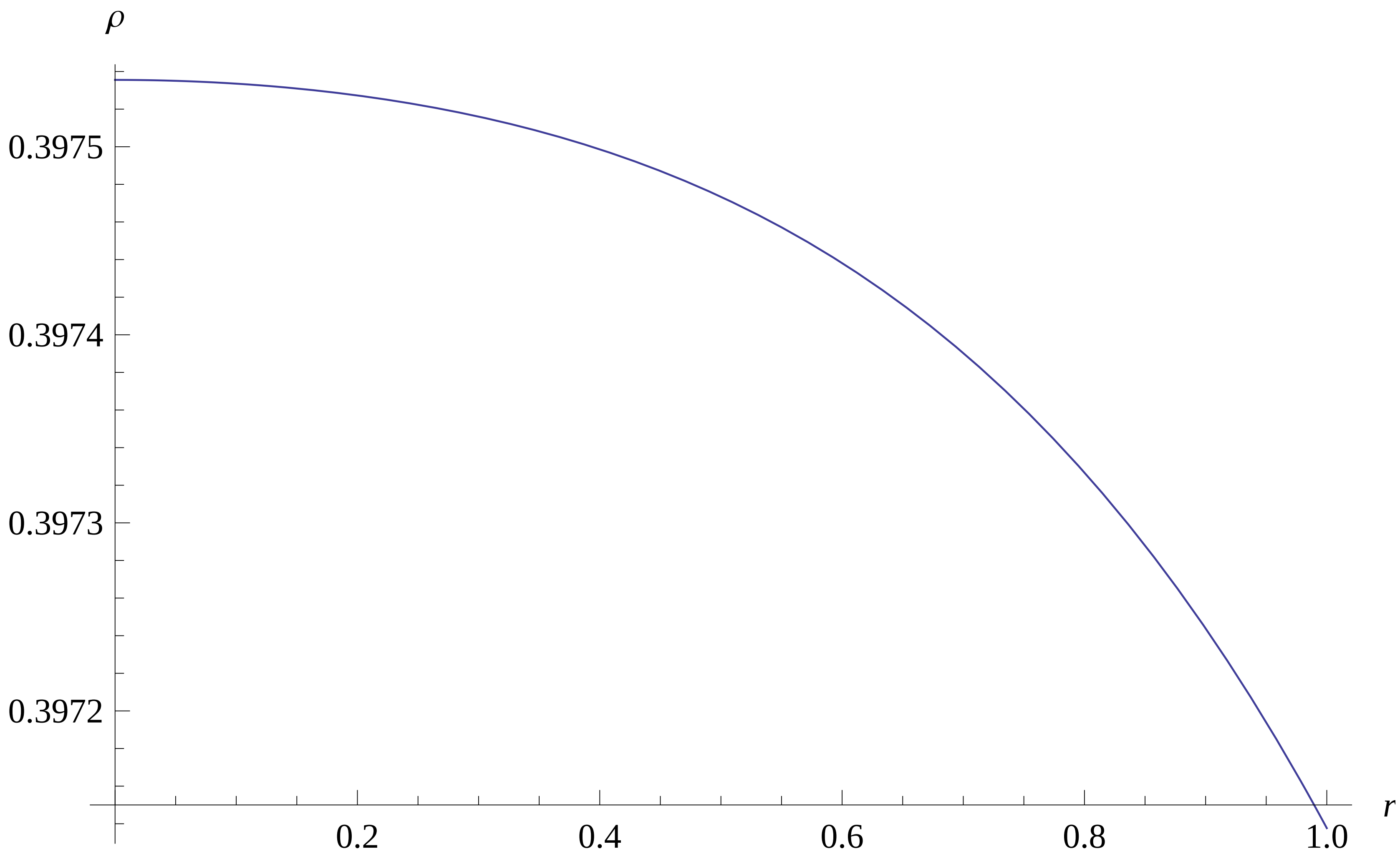

Figure 4.1: Variation of density with the radius

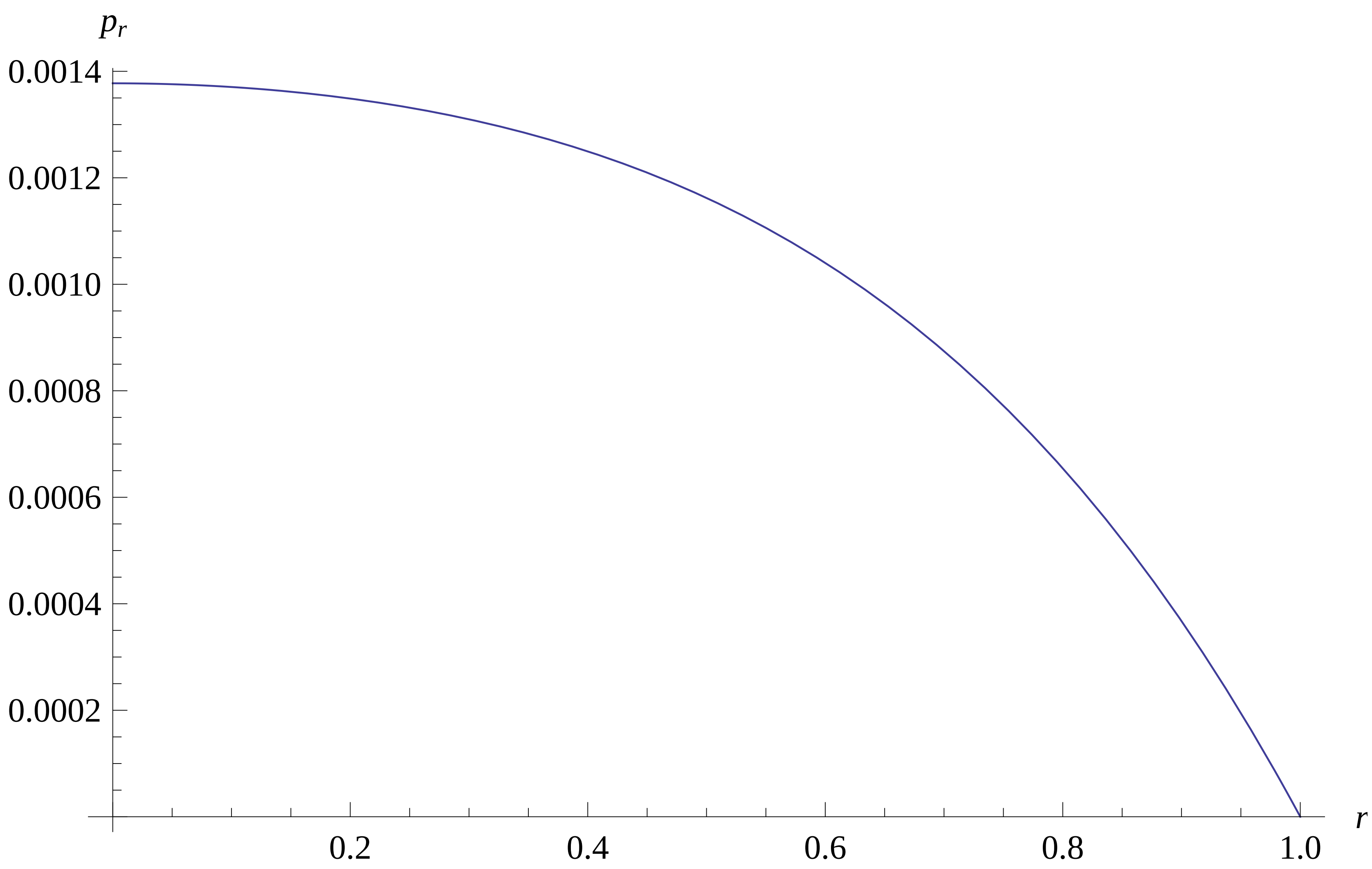

Figure 4.2: Variation of radial pressure with the radius

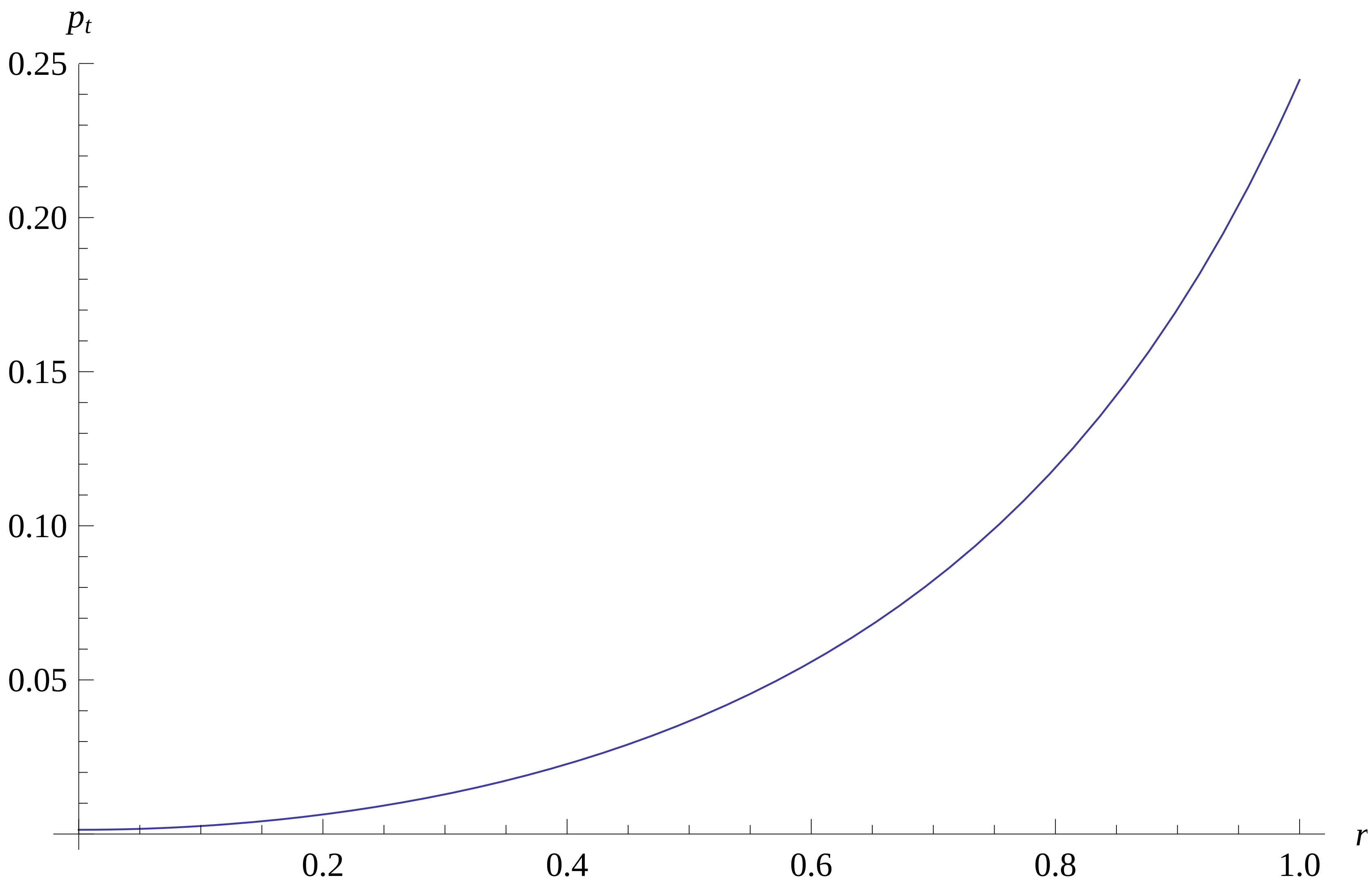

Figure 4.3: Variation of tangential pressure with the radius.

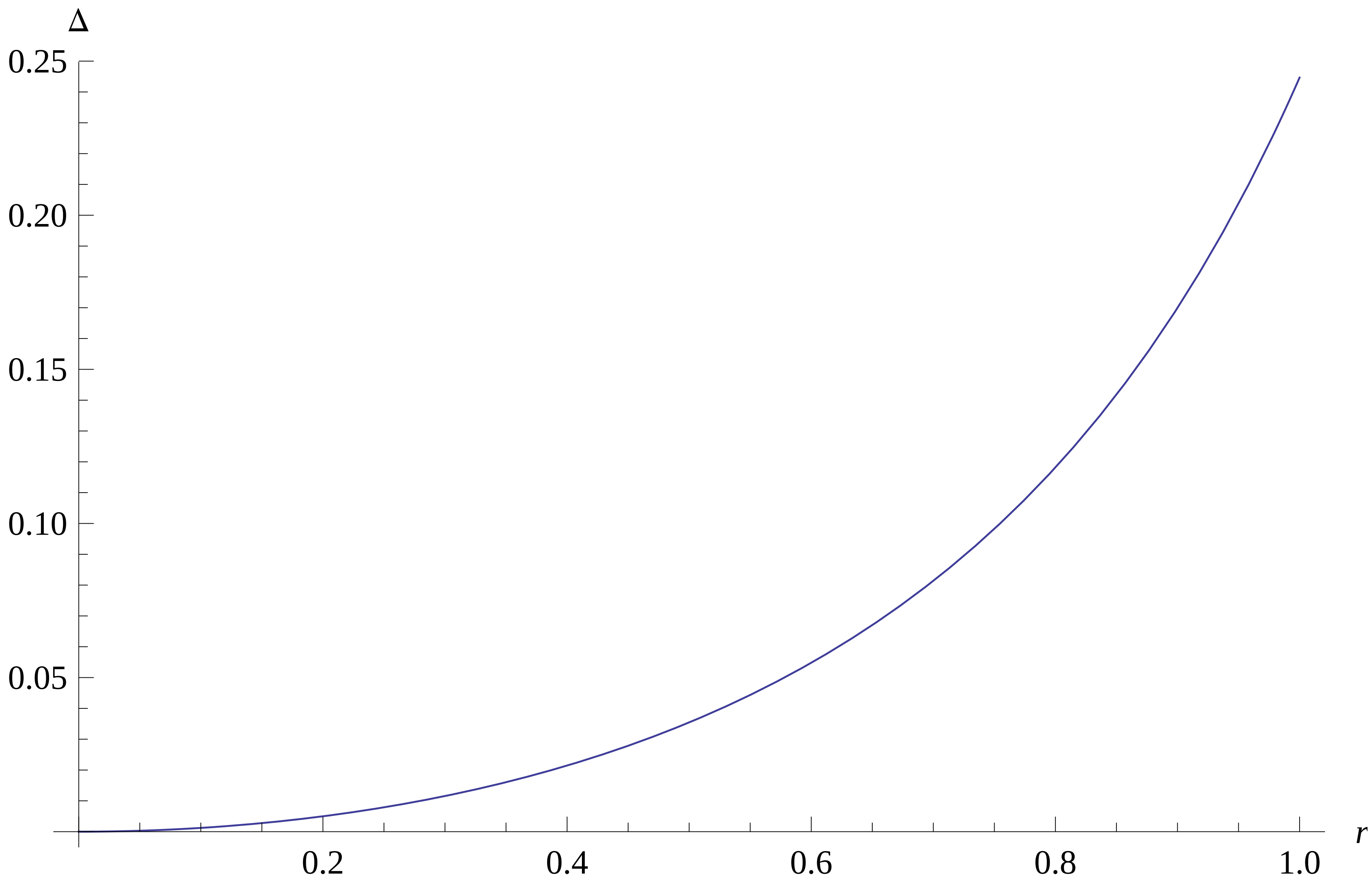

Figure 4.4: Variation of measure of anisotropy with the radius.

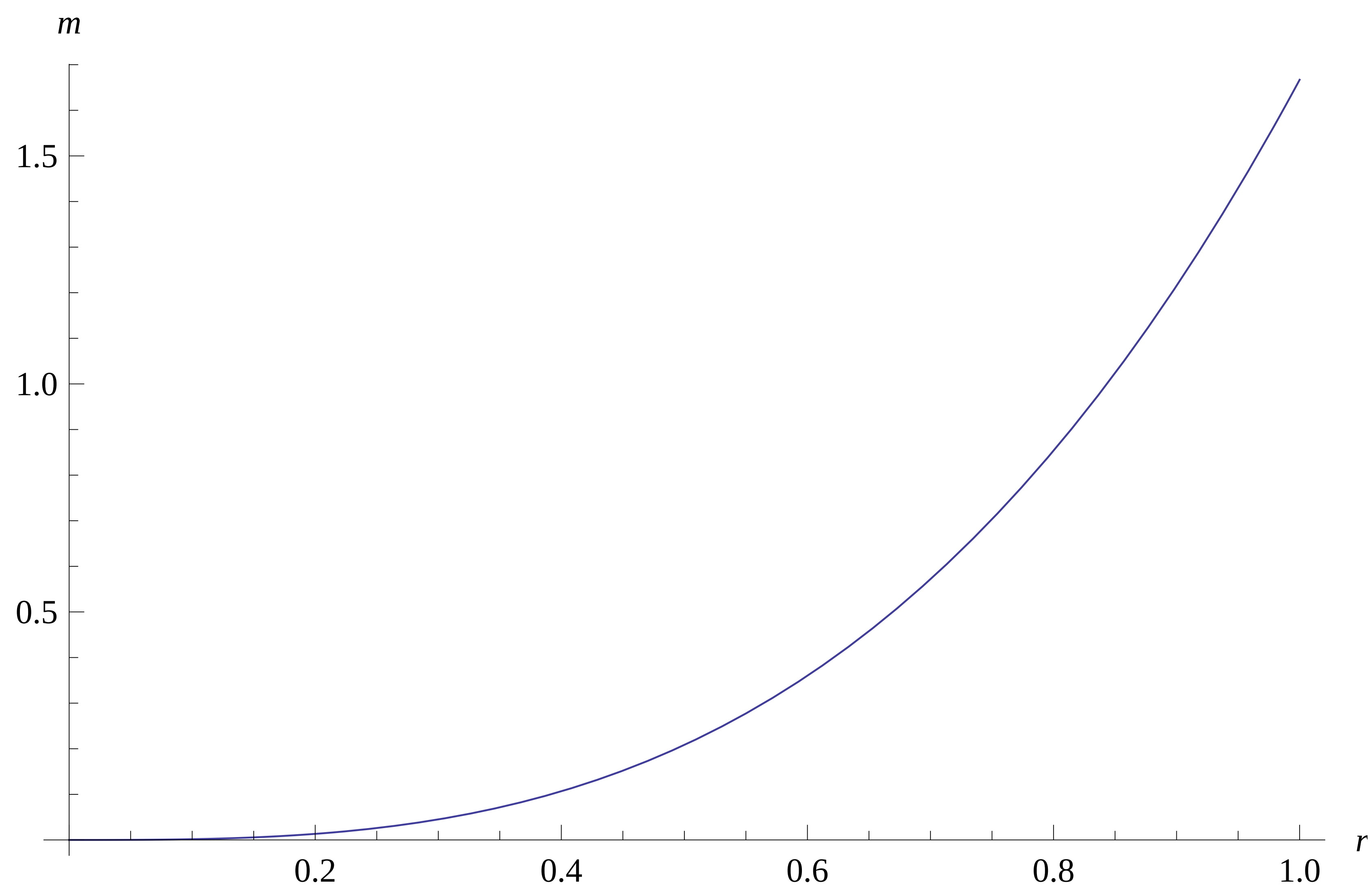

Figure 4.5: Variation of mass with the radius.

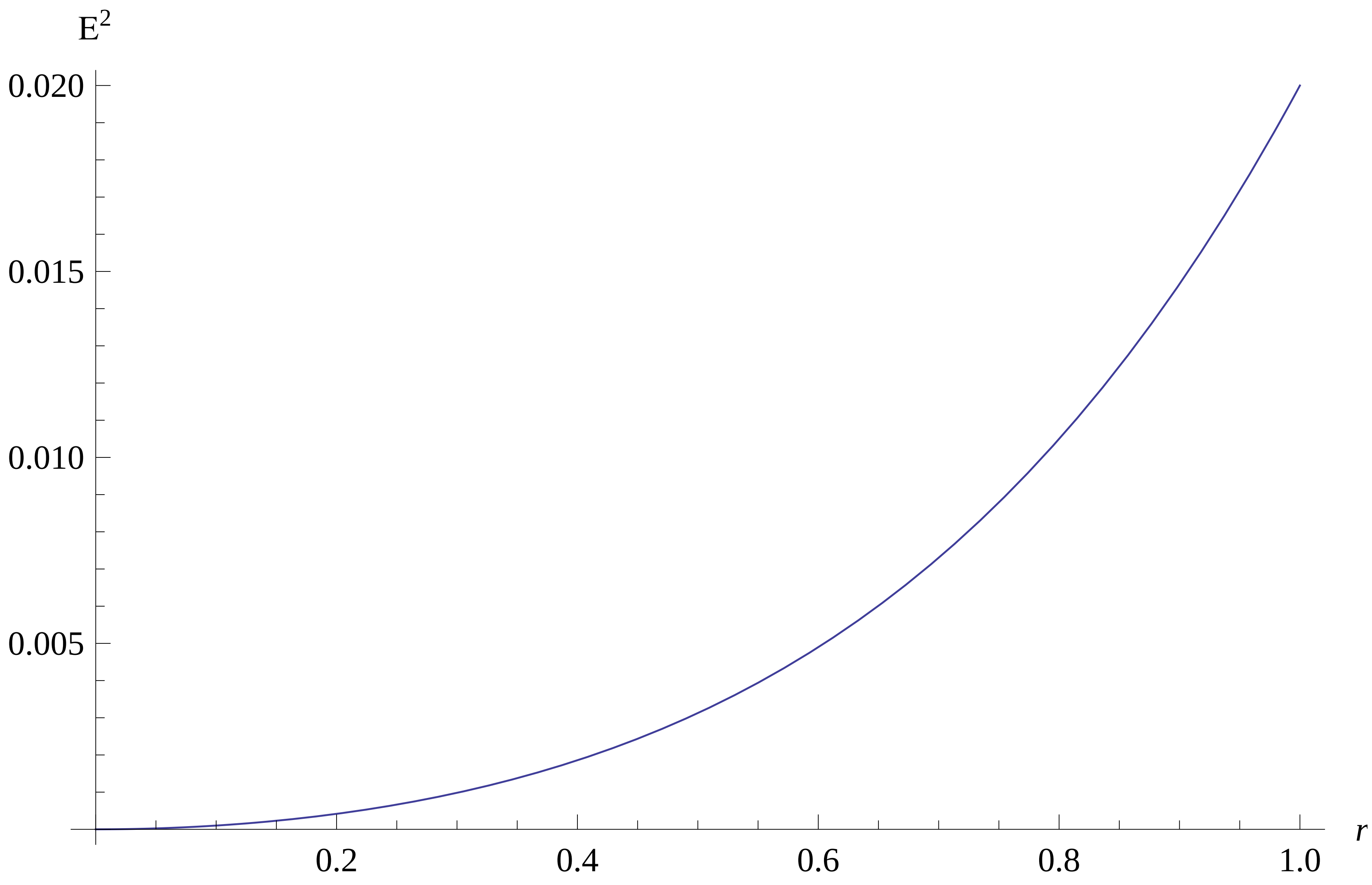

Figure 4.6: Square of Electric field intensity with the radius.

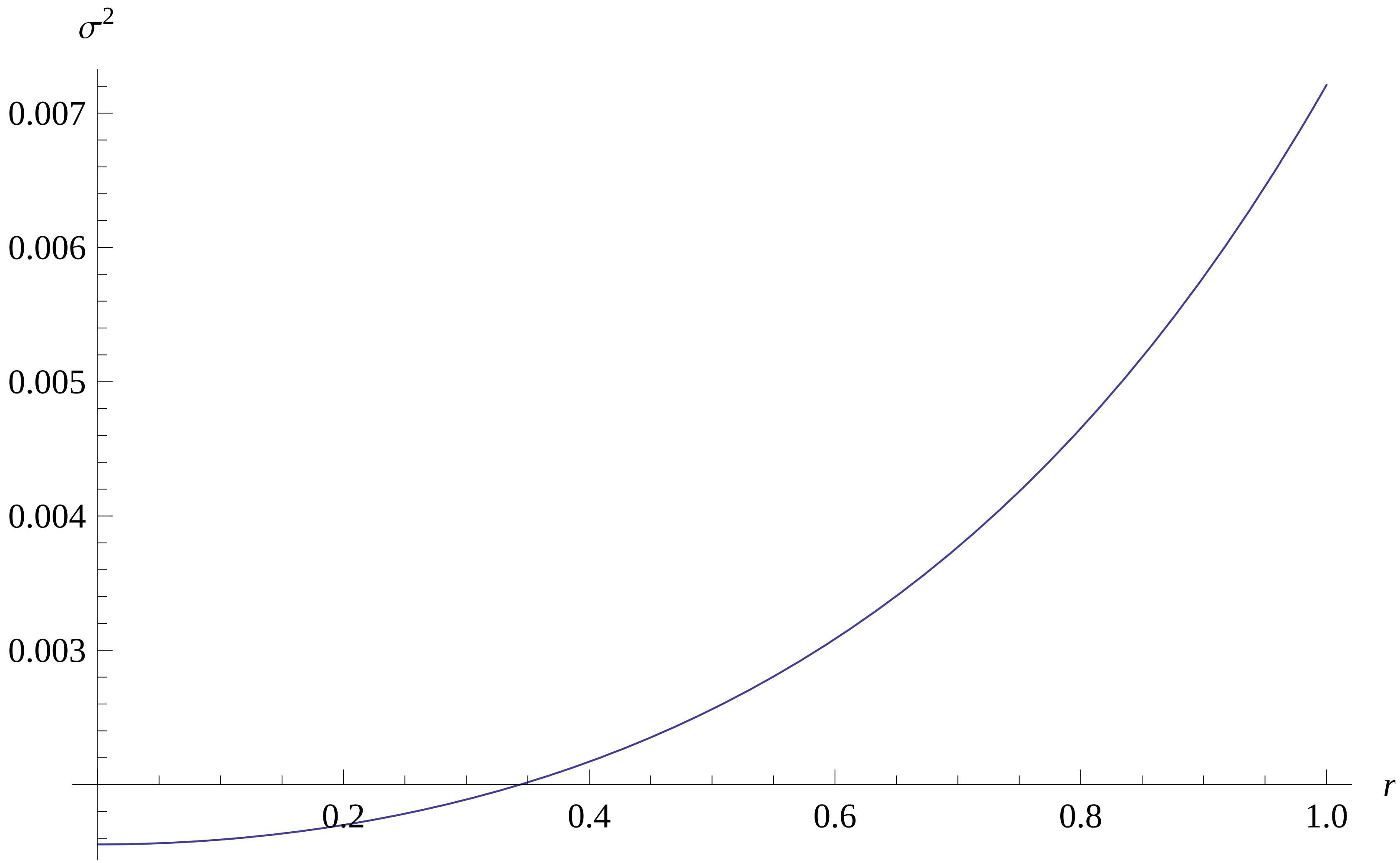

Figure 4.7: Charge density against the radius.

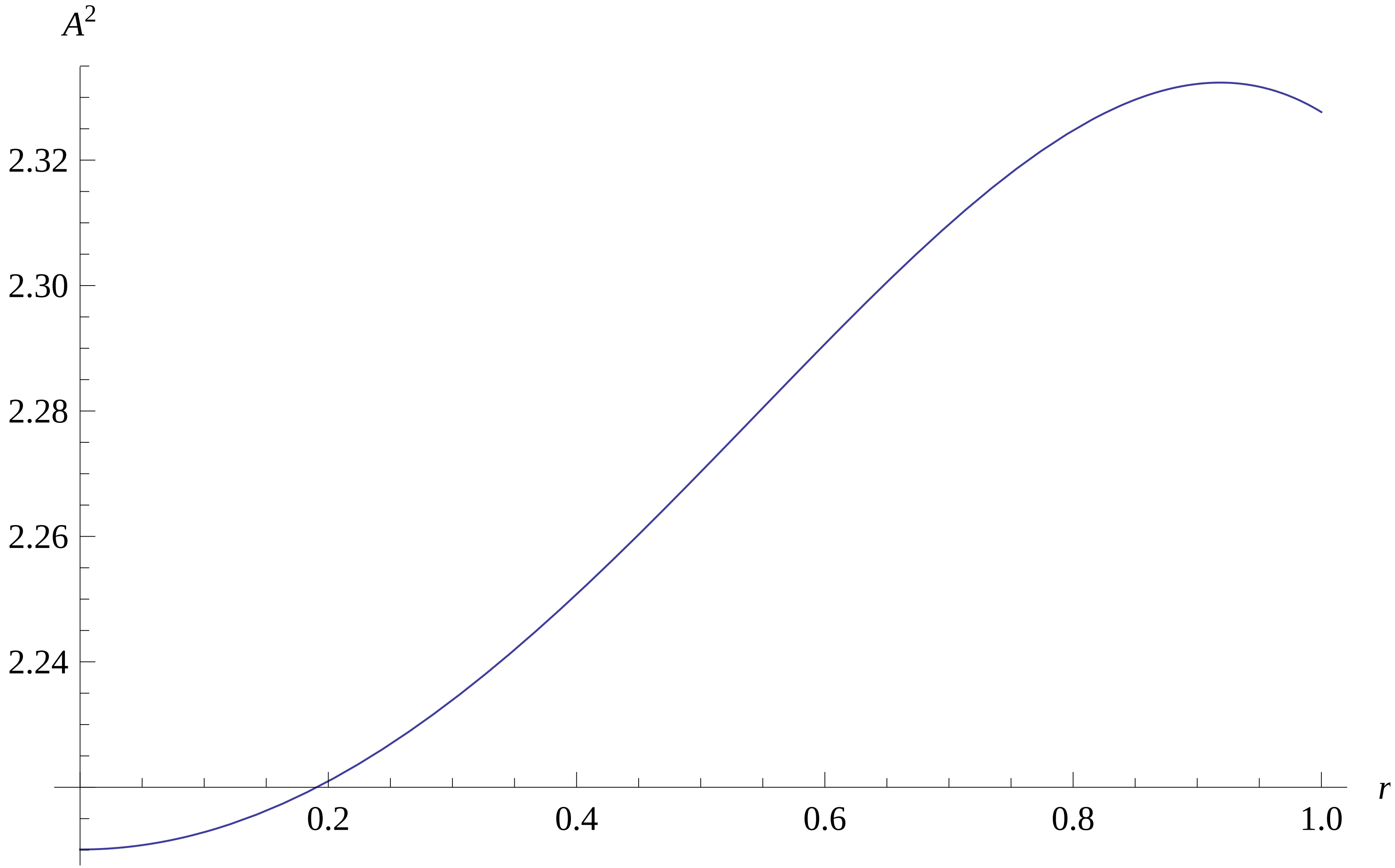

Figure 4.8: Variation of the gravitational potential A^2 with the radius.

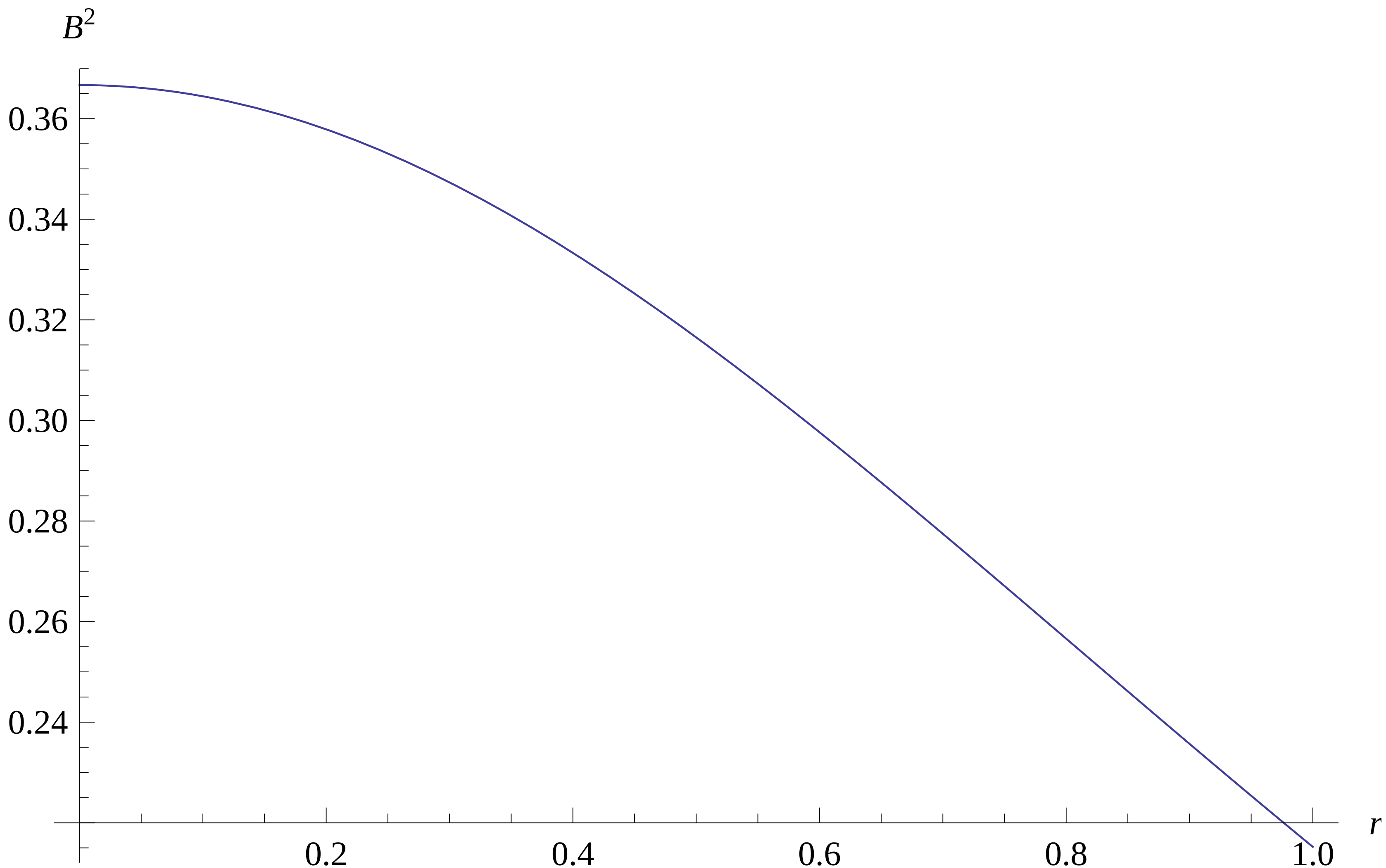

Figure 4.9: Variation of the gravitational potential B^2 with the radius.

Chapter 5

Conclusion

The objective of this thesis was to generate new classes of exact solutions to the Einstein and Einstein-Maxwell system of equations in isotropic coordinates. We studied static spherically symmetric spacetimes with isotropic perfect fluid spheres, and also considered anisotropic pressures in the presence of electromagnetic fields. The new exact solutions play vital roles in many applications in relativistic astrophysics. Our treatment highlights the importance of a barotropic equation of state relating the radial pressure to the energy density. The linear and quadratic equations of state generate physically acceptable stellar models. We generated an algorithm that produces uncharged new exact solutions to the Einstein field equations which are physically acceptable. The models generated for the Einstein field equations contains isotropic pressures unlike the known seed metric of Chaisi and Maharaj (2006) and Maharaj and Chaisi (2006) which have anisotropic pressures in canonical coordinates. By specifying one of the gravitational potentials we obtain exact solutions to the Einstein systems in elementary functions by integrating the master equation. We also generated new exact models to the Einstein-Maxwell system of equations by imposing a barotropic equation of state. We obtained well behaved solutions for matter configurations with charge and anisotropy. Some physical features were analysed and detailed lists of tables and graphical plots were given.

The specific results that we found are given below:

1. In chapter 2, we formulated the Einstein field equations for perfect fluid spheres with isotropic pressures in isotropic coordinates. By utilising the transformation that was initially suggested by Kustaanheimo and Qvist (1948), the system of field equations were written in an equivalent form. A new algorithm produced some new exact solutions by assuming a known seed metric. An integration was performed by choosing the seed solution $(\bar{L}, \bar{G})$ to obtain the new solution (L, G). As an example we generate the line element

$$ds^2 = -\left[\frac{1+cr^2}{b+ar^2}\right]^2 dt^2 + \left[\frac{(b+ar^2)^2}{U(r)}\right]^2 [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)],$$

   where

$$U(r) = b^2c^2 + abc\left(1 + 3cr^{\frac{1}{2}}\right) + a^2\left(1 + 3cr^{\frac{1}{2}} + 3c^2 r\right),$$

   from a conformally flat metric. The graphical plots for the energy density ρ, pressure p and the speed of sound $\frac{dp}{d\rho}$ were generated, and they are well behaved.

2. In chapter 3, we generated the Einstein-Maxwell system of equations with anisotropic pressures in the presence of electric field intensity. We considered spacetime which is static and spherically symmetric in isotropic coordinates, and imposed a

barotropic equation of state which is linear. We applied the transformation that was first utilised by Kustaanheimo and Qvist (1948) to express the Einstein-Maxwell system of equations in terms of new coordinates. We assumed the linear equation of state given by

$$p_r = \alpha\rho - \beta.$$

We specified one of the gravitational potentials and the electric field intensity to complete the integration. The line element takes the form

$$\begin{aligned} ds^2 &= -K(a-br^2)^{2\Psi}(a+br^2)^{2(\Phi-1)}e^{2N(r)}dt^2 \\ &\quad +(a+br^2)^{-2}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi)]. \end{aligned}$$

The function $N(r)$ and the constants Ψ and Φ are given by

$$\begin{aligned} N(r) &= \frac{d(1+\alpha)r^2}{8b^2}, \\ \Psi &= \frac{1}{16ab^3}\{a(1+\alpha)(bc+ad) - 8b^2[ab(1+3\alpha) - 2\pi\beta]\}, \\ \Phi &= \frac{1}{16ab^3}\{a(1+\alpha)[b(24b^2+c) - ad)] - 16\pi b^2\beta\}. \end{aligned}$$

All the matter and electromagnetic quantities were given in terms of tables and graphical plots. Our physical analysis showed that our new models are well behaved. We related our results to the astronomical object PSR J1614-2230.

3. In chapter 4, we studied static spherically symmetric spacetimes with anisotropy and charge. We imposed a barotropic quadratic equation of state in isotropic coordinates. We generated the Einstein-Maxwell system of equations by utilising a known transformation. We assumed the quadratic equation of state to have the form

$$p_r = \eta\rho^2 + \alpha\rho - \beta.$$

If we let $\eta = 0$ then we regain the linear equation of state. We chose one of the gravitational potentials to be a linear function and the electric field intensity to be a quadratic function in order to solve the Einstein-Maxwell system. After completing the integration the line element has the form

$$\begin{aligned} ds^2 &= -K(a - br^2)^{2\Psi}(a + br^2)^{2(\Phi-1)}e^{2N(r)}dt^2 \\ &\quad +(a + br^2)^{-2}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \end{aligned}$$

The function $N(r)$ and the constants Ψ and Φ are given by

$$
\begin{aligned}
N(r) &= \frac{r^2}{384\pi b^4}\left\{48\pi b^2 d(1+\alpha) - 3\eta\left[b^2c^2 + ad(ad - 48b^3)\right]\right. \\
&\left. -b^2 d\eta(3c + dr^2)r^2\right\}, \\
\Psi &= \frac{1}{256\pi ab^5}\left\{16\pi ab^2(1+\alpha)(bc+ad) - 128\pi b^4\left[ab(1+3\alpha) - 2\pi\beta\right]\right. \\
&\left. -a^2\eta\left[a^2d^2 + b(c - 24b^2)(2ad + bc - 24b^3)\right]\right\}, \\
\Phi &= \frac{1}{256\pi ab^5}\left\{16\pi ab^2(1+\alpha)\left[b(24b^2 + c) - ad\right] - 256\pi^2 b^4\beta\right. \\
&\left. +a^2\eta\left[a^2d^2 - b(24b^2 + c)(2ad - bc - 24b^3)\right]\right\}.
\end{aligned}
$$

We regained the earlier model of Ngubelanga *et al* (2014a) when $\eta = 0$. We provided graphical plots and tables showing that the model is physically reasonable. We related our results to the astronomical object PSR J1903+327. This indicates that the quadratic equation of state is relevant in the description of relativistic stellar objects.

Bibliography

[1] Abebe G, Govinder K S and Maharaj S D, Lie symmetries for a conformally flat radiating star, *Int. J. Theor. Phys.* **52**, 3244 (2013).

[2] Arbanil J D V, Lemos J P S and Zanchin V T Z, Incompressible relativistic spheres: electrically charged stars, compactness bounds and quasiblack hole configurations, *Phys. Rev. D* **89**, 104054 (2014).

[3] Andersson L, Beig R and Schmidt B G, Elastic deformations of compact stars, *Class. Quantum. Grav.* **31**, 185006 (2014).

[4] Boonserm P, Visser M and Weinfurtner S, Generating perfect fluid spheres in general relativity, *Phys. Rev. D* **71**, 124037 (2005).

[5] Chaisi M and Maharaj S D, A new algorithm for anisotropic solutions, *Pramana-J. Phys.* **66**, 313 (2006).

[6] Cottam J, Paerels F and Mendez M, Gravitationally redshifted absorption lines in the X-ray burst spectra of a neutron star, *Nature* **420**, 51 (2002).

[7] Delgaty M S R and Lake K, Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein's equations, *Comput. Phys. Commun.* **115**, 395 (1998).

[8] Demorest P B, Pennucci T, Ransom S M, Roberts M S E and Hessels J W T, A two-solar-mass neutron star measured using Shapiro delay, *Nature* **467**, 1081 (2010).

[9] Elebert P, Reynolds M T, Callanan P J, Hurley D J, Ramsay G, Lewis F, Russell D M, Nord B, Kane S R, DePoy D L and Hakala P, Optical spectroscopy and photometry of SAX J1808.4-3658 in outburst, *Mon. Not. R. Astron. Soc.* **395**, 884 (2009).

[10] Esculpi M and Alomá E, Conformal anisotropic relativistic charged fluid spheres with a linear equation of state, *Eur. Phys. J. C* **67**, 521 (2010).

[11] Fatema S and Murad M H, An exact family of Einstein-Maxwell Wyman-Adler solution in general relativity, *Int. J. Theor. Phys.* **52**, 2508 (2013).

[12] Feroze T and Siddiqui A A, Charged anisotropic matter with quadratic equation of state, *Gen. Relativ. Gravit.* **43**, 1025 (2011).

[13] Finch M R and Skea J F E, Preprint available on the web: http://edradour.symbcomp.uerj.br/pubs.html, (1998).

[14] Fraga E S, Kurkela and Vuorinan A, Interacting quark matter equation of state for compact stars, *Astrophys. J.* **781**, 25 (2014).

[15] Freire P C C, Bassa C G, Wex N, Stairs I H, Champion D J, Ransom S M, Lazarus P, Kaspi V M, Hessels J W T, Kramer M, Cordes J M, Verbiest J P W, Podsiadlowski P, Nice D J, Deneva J S, Lorimer D R, Stappers B W, McLaughlin M A and Camilo F, On the nature and evolution of the unique binary pulsar J1903+0327, *Mon. Not. R. Astron. Soc.* **412**, 2763F (2011).

[16] Goldman S P, Physical solutions to general relativistic fluid spheres, *Astrophys. J.* **226**, 1079 (1978).

[17] Güver T, Özel F, Cabrera-Lavers A and Wroblewski P, The distance, mass, and radius of the neutron star in 4U 1608-52, *Astrophys. J.* **712**, 964 (2010a).

[18] Güver T, Wroblewski P, Camarota L and Özel F, The mass and radius of the neutron star in 4U 1820-30, *Astrophys. J.* **719**, 1807 (2010b).

[19] Herrera L, Di Prisco A, Martin J, Ospino J, Santos N O and Troconis O, Spherically symmetric dissipative anisotropic fluids: A general study, *Phys. Rev. D* **69**, 084026 (2004a).

[20] Herrera L, Le Denmat G, Santos N O and Wang A, Shear-free radiating collapse and conformal flatness, *Int. J. Mod. Phys. D* **13**, 583 (2004b).

[21] Ivanov B V, Static charged perfect fluid spheres in general relativity, *Phys. Rev. D* **65**, 104001 (2002).

[22] Kiess T E, Exact physical Maxwell-Einstein Tolman-VII solution and its use in stellar models, *Astrophys. Space Sci.* **339**, 329 (2012).

[23] Komathiraj K and Maharaj S D, Classes of exact Einstein-Maxwell solutions, *Gen. Relativ. Gravit.* **39**, 2079 (2007a).

[24] Komathiraj K and Maharaj S D, Tikekar superdense stars in electric fields, *J. Math. Phys.* **48**, 042501 (2007b).

[25] Kustaanheimo P and Qvist B, A note on some general solutions of the Einstein field equations in a spherically symmetric world, *Comment. Phys. Math. Helsingf.* **13**, 1 (1948).

[26] Lake K, All static spherically symmetric perfect-fluid solutions of Einstein's equations, *Phys. Rev. D* **67**, 104015 (2003).

[27] Mafa Takisa P and Maharaj S D, Compact models with regular charge distributions, *Astrophys. Space Sci.* **343**, 569 (2013a).

[28] Mafa Takisa P and Maharaj S D, Some charged polytropic models, *Gen. Relativ. Gravit.* **45**, 1951 (2013b).

[29] Mafa Takisa P, Ray S and Maharaj S D, Charged compact objects in the linear regime, *Astrophys. Space Sci.* **350**, 733 (2014a)

[30] Mafa Takisa P, Maharaj S D and Ray S, Stellar objects in the quadratic regime, *Astrophys. Space Sci.* **354**, 463 (2014b).

[31] Maharaj S D and Chaisi M, New anisotropic models from isotropic solutions, *Math. Meth. Appl. Sci.* **29**, 67 (2006).

[32] Maharaj S D and Govender M, Radiating collapse with vanishing Weyl stresses, *Int. J. Mod. Phy. D* **14**, 667 (2005).

[33] Maharaj S D and Mafa Takisa P, Regular models with quadratic equation of state, *Gen. Relativ. Gravit.* **44**, 1419 (2012).

[34] Maharaj S D, Sunzu J M and Ray S, Some simple models for quark stars, *Eur. Phys. J. Plus* **129**, 3 (2014).

[35] Mak M K and Harko T, Can the galactic rotation curves be explained in brane world models, *Int. J. Mod. Phys. D* **13**, 149 (2004).

[36] Malaver M, Regular model for a quark star with Van der Waals modified equation of state, *World Applied Programming* **3**, 309 (2013).

[37] Malaver M, Strange quark star model with quadratic equation of state, *Frontier of Mathematics and its Applications* **1**, 9 (2014).

[38] Martin D and Visser M, Algorithmic construction of static perfect fluid spheres, *Phys. Rev. D* **69**, 104028 (2004).

[39] Misthry S S, Maharaj S D and Leach P G L, Nonlinear shear-free radiative collapse, *Math. Meth. Appl. Sci.* **31**, 363 (2008).

[40] Murad M H, A new well behaved class of charge analogue of Adler's relativistic exact solution, *Astrophys. Space Sci.* **343**, 187 (2013).

[41] Murad M H and Fatema S, Some exact relativistic models of electrically charged self-bound stars, *Int. J. Theor. Phys.* **52**, 4342 (2013).

[42] Nariai S, *Sci. Rep. Tohoku Univ. Ser.* **34**, 160 (1950).

[43] Ngubelanga S A and Maharaj S D, A relativistic algorithm with isotropic coordinates, *Adv. Math. Phys.* 905168 (2013).

[44] Ngubelanga S A, Maharaj S D and Ray S, Compact stars with linear equation of state, *Astrophys. Space Sci.* in press (2014a).

[45] Ngubelanga S A, Maharaj S D and Ray S, Compact stars with quadratic equation of state, submitted (2014b).

[46] Özel F, Soft equations of state for neutron-star matter ruled out by EXO 0748-676, *Nature* **441**, 1115 (2006).

[47] Özel F, Güver T and D. Psaltis, The mass and radius of the neutron star EXO 1745-248, *ApJ* **693**, 1775

[48] Pant D N and Sah A, Massive fluid spheres in general relativity, *Phys. Rev. D* **32**, 1358 (1985).

[49] Pant N, Pradhan N and Murad M H, A class of super dense stars models using charged analogues of Hajj-Boutros type relativistic fluid solutions, *Int. J. Theor. Phys.* **53**, 3958 (2014a).

[50] Pant N, Pradhan N and Murad M H, A family of exact solutions of the Einstein-Maxwell field equations in isotropic coordinates: an application to optimization of quark star mass, *Astrophys. Space Sci.* **352**, 135 (2014b).

[51] Pradhan N and Pant N, Einstein-Maxwell field equations in isotropic coordinates: an application to neutron star and quark star, *Astrophys. Space Sci.* **352**, 143 (2014).

[52] Rahman S and Visser M, Spacetime geometry of static fluid spheres, *Class. Quantum Gravit.* **19**, 935 (2002).

[53] Rawls M L, Orosz J A, McClintock J E, Torres M A P, Bailyn C D, and Buxton M M, Refined neutron star mass determinations for six eclipsing X-ray pulsar binaries, *Astrophys. J.* **730**, 25 (2011).

[54] Rodrigues H, Duarte S B and de Oliveira J C T, Massive compact stars as quark stars, *Astrophys. J.* **730**, 31 (2011).

[55] Sharma R and Ratanpal B S, Relativistic stellar model admitting a quadratic equation of state, *Int. J. Mod. Phys. D* **22**, 1350074 (2013).

[56] Sharma R, Mukherjee S and Maharaj S D, General solution for a class of static charged spheres, *Gen. Relativ. Gravit.* **33**, 999, (2001).

[57] Simon W, Static perfect fluids with Pant-Sah equations of state, *Gen. Relativ. Gravit.* **40**, 2591 (2008).

[58] Stephani H, Kramer D, MacCallum M A H, Hoenselaars C and Herlt E, Exact solutions of Einstein's field equations (Cambridge: Cambridge University Press) (2003).

[59] Sunzu J M, Maharaj S D and Ray S, Charged anisotropic models for quark stars, *Astrophys. Space Sci.* **352**, 719 (2014a).

[60] Sunzu J M, Maharaj S D and Ray S, Quark star model with charged anisotropic matter, *Astrophys. Space Sci.* **354**, 517 (2014b).

[61] Thirukkanesh S and Ragel F C, Anisotropic compact sphere with Van der Waals equation of state, *Astrophys. Space Sci.* **352**, 123 (2014).

[62] Thirukkanesh S and Ragel F C, Exact anisotropic sphere with polytropic equation of state, *Pramana-J. Phys.* **78**, 687 (2012).

[63] Wolfram S, Mathematica (Redwood City: Wolfram Research) (2003).