GROUP THEORETIC APPROACH TO HEAT CONDUCTING GRAVITATING SYSTEMS

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As the candidate's supervisors, we have approved this dissertation for submission.

Signed:	Professor S D Maharaj	December 2013
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Abstract

We study shear-free heat conducting spherically symmetric gravitating fluids defined in four and higher dimensional spacetimes. We analyse models that are both uncharged and charged via the pressure isotropy condition emanating from the Einstein field equations and the Einstein-Maxwell system respectively. Firstly, we consider the uncharged model defined in higher dimensions, and we use the algorithm due to Deng to generate new exact solutions. Three new metrics are identified which contain the results of four dimensions as special cases. We show graphically that the matter variables are well behaved and the speed of sound is causal. Secondly, we use Lie's group theoretic approach to study the condition of pressure isotropy of a charged relativistic model in four dimensions. The Lie symmetry generators that leave the equation invariant are found. We provide exact solutions to the gravitational potentials using the symmetries admitted by the equation. The new exact solutions contain earlier results without charge. We show that new charged solutions related to the Lie symmetries, that are generalizations of conformally flat metrics, may be generated using the algorithm of Deng. Finally, we extend our study to find models of charged gravitating fluids defined in higher dimensional manifolds. The Lie symmetry generators related to the generalized pressure isotropy condition are found, and exact solutions to the gravitational potentials are generated. The new exact solutions contain earlier results obtained in four dimensions. Using particular Lie generators, we are able to provide forms for the gravitational potentials or reduce the order of the master equation to a first order nonlinear differential equation. Exact expressions for the temperature profiles, from the transport equation for both the causal and noncausal cases, in higher dimensions are obtained, generalizing previous results. In summary, the Deng algorithm and Lie analysis prove to be useful approaches in generating new models for gravitating fluids.

ALLAHU AKBAR WALLILAHI LIHAMDU

The Most Gracious, Most Merciful.

To

Muhammad Mutebi and Amina Kabongoya: my late parents,

Hajjat Minsa Nalweyiso: my late grand mother.

May the Almighty have mercy on your souls.

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I, Yusuf Nyonyi, declare that

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Declaration of Publications

Details of contribution to publications that form part and/or include research presented in this thesis.

<u>Publication 1:</u>

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Signed:

Contents

Chapter 1

Introduction

Arguably the most profound discovery in the history of physical sciences was made by Albert Einstein at the beginning of the 20^{th} century. He devised a theory that abolished the idea of absolute space and absolute time in Newton's theory of gravitation; these issues had raised a few eyebrows since Newton's days with critics from Huygens, Leibniz, Bishop Berkeley, and much later Mach (Rindler 2006). Einstein's theory not only explained the existing astronomical discoveries of that time (like the precession of Mercury's perihelia) that Newton's theory of gravitation had failed to account for, but also made startling predictions (for example the bending of light) that have since been tested experimentally. The model of gravity emanating from the interaction of gravitating bodies and the curvature of spacetime provides us with a platform to study and understand the intricacy and dynamics of various cosmological and astrophysical systems. In this respect, several particular models have been suggested to interpret observational data obtained from galactic bodies and predict their evolution in time. There are a number of books available to aid our understanding of basic general relativity and its importance as far as studying astrophysical and cosmological structures like stars, black holes and galaxies. For more detailed information, the reader is referred to Narlikar (1993), Rindler (2006), Stephani (1990), and Wald (1984).

The desire to have some understanding of the behaviour and dynamics of galactic and stellar systems has led many scientists to suggest several models. Depicting nature in its entirety is a chaotic process because of the unprecedented number of parameters involved; hence a number of assumptions are made when building these models. These assumptions must be chosen in such a way that we capture the important features that define the systems we are modelling. In this way, we are able to provide mathematically elegant results that are also physically acceptable. We make two important assumptions in our study. We assume that the gravitating fluids are spherically symmetric and shear-free. Spherically symmetric models have a simpler spacetime geometry. By studying shear-free models we avail ourselves with a rather simple avenue where we only need to provide solutions to the resulting defining equation containing only two metric functions. In this way, we can provide exact solutions to the field equations while preserving the essential physics of the problem.

We intend to find exact classes of solutions of the Einstein field equations and Einstein-Maxwell equations for the models stipulated in our study. A number of techniques of obtaining solutions are known including the *ad hoc* Deng (1989) approach and Ivanov's (2012) compact formalism. Some methods require making assumptions on the matter distribution, the gravitational potentials or imposing a particular equation of state. Other techniques involve preserving the symmetries of the manifold and applying Lie analysis, Noether symmetries, Lie-Bäcklund transformations, among others. The interested reader is referred to the excellent texts of Bluman and Anco (2002), Bluman and Kumei (1989), Cantwell (2002), Olver (1986, 1995) and Stephani (1989) for a more detailed review of the latter techniques mentioned above. In this thesis, we use Deng's algorithm (Deng 1989) to provide solutions to uncharged spherically symmetric cosmological models with heat flow in higher dimensions. We also employ Lie's group theoretic methods to provide exact solutions to spherically symmetric gravitating fluids with heat flow in the presence of an electromagnetic field. We further extend the study to higher dimensions using the same techniques. A synopsis of Lie's methods as applied to differential equations is summarised in the next section.

1.1 Review of symmetries of a differential equation

Loosely speaking, a symmetry of a differential equation refers to a transformation that leaves the equation invariant or unchanged, hence the name invariants (Bluman and Kumei 1989). In this section, we highlight the main ideas cognate to our discussion as far as symmetry analysis is concerned.

Definition 1.1.1. A differential equation

$$E(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0, \tag{1.1}$$

possesses a symmetry

$$X = \xi(x, y(x))\partial_x + \eta(x, y(x))\partial_y, \tag{1.2}$$

if and only if

$$X^{[n]}E|_{E=0} = 0. \tag{1.3}$$

That is to say, the n^{th} extension of X on E vanishes when the original equation is taken into consideration. From here on, the arguments of the dependent variable y will be assumed to be the independent variable x unless otherwise stated.

Equation (1.3) is essentially a partial differential equation in ξ and η (also called the infinitesimals of the symmetry generator). We observe that this equation contains derivatives of y, even though these derivatives do not appear in the arguments of ξ and η. This enables us to obtain a system of linear partial differential equations in ξ and η by equating the coefficients of the different functions of the corresponding derivatives of y to zero. The resulting over-determined system can then be solved to obtain ξ and η explicitly thence providing the symmetries (1.2) of (1.1).

1.1.1 Reduction of order

It is well documented that one of the great uses of symmetries as a method of solving differential equations lies in the reduction of the order of the differential equation (Olver 1986). Therefore, once the symmetries have been obtained, they can be used in this respect.

If an n^{th} order differential equation

$$E(x, y, y', \ldots, y^{(n)}) = y^{(n)} - F(x, y, y', \ldots, y^{(n-1)}) = 0, \tag{1.4}$$

possesses a symmetry

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y, \tag{1.5}$$

its order can be reduced to a differential equation of order $(n-1)$ by obtaining the group invariant and first differential invariant associated with the first extension of X. This involves solving the following Lagrange's system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}y'}{\eta' - y'\xi'}. \tag{1.6}$$

When we consider the first and second term, the solution to this gives the group invariant or zeroth invariant $u = g(x, y)$ while the solution to the remaining pair gives the first differential invariant $v = h(x, y, y')$. The names zeroth invariant arise from $Xu = 0$ and first differential invariant from $X^{[1]}v = 0$. Now, using u, v and the derivatives of v with respect to u, the original equation (1.4) can be written as

$$G(u, v, v', \ldots, v^{(n-1)}) = v^{(n)} - H(u, v, v', \ldots, v^{(n-2)}) = 0. \tag{1.7}$$

where primes indicate total differentiation with respect to u. If the resulting equation (1.7) possesses a symmetry, then its order can be reduced further and so on.

1.1.2 Extended transformations and their infinitesimal generator

We will ultimately be solving an equation involving one independent and m dependent variables. Therefore there is need for us to highlight the generalization of both the Lie point transformation and the extended symmetry generator.

Consider a one parameter Lie group of transformations

$$\tilde{x} = \tilde{x}\left(x, y_i; \varepsilon\right),$$

$$\tilde{y}_j = \tilde{y}_j\left(x, y_i; \varepsilon\right); \qquad i,\ j = 1, \cdots, m$$

for a system of one independent variable and m dependent variables such that each $y_i = y_i(x)$. Its infinitesimal transformation about $\varepsilon = 0$ is defined by

$$\tilde{x} = x + \varepsilon\xi(x, y_i) + O(\varepsilon^2), \tag{1.8a}$$

$$\tilde{y}_j = y_j + \varepsilon\eta_j(x, y_i) + O(\varepsilon^2), \tag{1.8b}$$

with symmetry generator

$$X = \xi(x, y_i)\partial_x + \eta_j(x, y_i)\partial_{y_j}. \tag{1.9}$$

The k^{th} extension of (1.8), given by

$$\tilde{x} = x + \varepsilon\xi(x, y_i) + O(\varepsilon^2), \tag{1.10a}$$

$$\tilde{y}_j = y_j + \varepsilon\eta_j(x, y_i) + O(\varepsilon^2), \tag{1.10b}$$

$$\tilde{y}_j' = {y_j}' + \varepsilon{\eta_j}^1(x, y_i, {y_i}') + O(\varepsilon^2), \tag{1.10c}$$

$$\vdots \qquad\qquad \vdots$$

$$\tilde{y}_j^{(k)} = {y_j}^{(k)} + \varepsilon{\eta_j}^k(x, y_i, {y_i}', \ldots, {y_i}^{(k)}) + O(\varepsilon^2), \tag{1.10d}$$

has its symmetry generator in the form

$$\begin{aligned} X = {} & \xi(x, y_i)\partial_x + \eta_j(x, y_i)\partial_{y_j} + {\eta_j}^1(x, y_i, {y_i}')\partial_{{y_j}'} + \cdots \\ & + {\eta_j}^k(x, y_i, \ldots, {y_i}^{(k)})\partial_{{y_j}^{(k)}}, \end{aligned} \tag{1.11}$$

with $k = 1, 2, \ldots$ and

$${\eta_j}^k(x, y_i, \ldots, {y_i}^{(k)}) = \frac{\mathrm{D}{\eta_j}^{k-1}}{\mathrm{D}x} - {y_j}^{(k)}\frac{\mathrm{D}\xi(x, y_i)}{\mathrm{D}x}. \tag{1.12}$$

It is important to note that D indicates total differentiation with respect to the independent variable x in (1.12).

1.2 Motivation

The quest to provide exact solutions to the Einstein field equations was initiated by Schwarzschild (1916a) when he provided an exterior solution to a gravitating body followed by a model that describes the gravitational field in the interior spacetime with constant density (Schwarzschild 1916b). Then Nordström (1918) and Reissner (1916) gave the Reissner-Nordström solution for a charged body. Since then many authors have provided several physically viable solutions to describe the interior stellar matter distribution. Some of the recent treatments include studies on compact stars by Thirukkanesh and Maharaj (2006, 2009), the charged Tikekar superdense star solutions by Komathiraj and Maharaj (2007b) and the analytical models for quark stars by Komathiraj and Maharaj (2007a) and Maharaj *et al* (2013).

Shear-free relativistic models have also been extensively studied and early solutions were provided by Kustaanheimo and Qvist (1948) to the Einstein field equations. Shear-free models in which heat flux is incorporated across the boundary of a radiating star have also been proposed. Recently, Msomi *et al* (2011) and Ivanov (2012) provided solutions to such relativistic models. Conformally flat radiating models proposed by Banerjee *et al* (1989) have been applied to radiating stars by Herrera *et al* (2004, 2006), Maharaj and Govender (2005) and Misthry *et al* (2008), among others. Note that more general models involving shearing, accelerating and expanding spacetimes have been proposed to describe cosmological processes in the absence of heat flux. Not many solutions have been found in this context. Here we highlight the known solutions by Bradley and Marklund (1999) and Maharaj *et al* (1993). Other specialised astrophysical models where heat flow is incorporated include the Govender *et al* (2010) and Govinder and Govender (2012) models of Euclidean stars. A geodesic shearing model in the relativistic astrophysical context was found by Naidu *et al* (2006) for an anisotropic star and the general nongeodesic shearing case was investigated more recently by Thirukkanesh *et al* (2012).

Models involving charge have also been extensively studied. Here, we highlight studies completed by Komathiraj and Maharaj (2007b), Komathiraj and Maharaj (2007a), Lobo (2006), Maharaj and Thirukkanesh (2009), Paul (2004) and Sharma and Maharaj (2007). Recently, Thirukkanesh and Maharaj (2009) obtained a general family of exact solutions expressible in terms of algebraic and polynomial functions describing charged relativistic spheres with generalised gravitational potentials. The nature of their solutions permit a detailed study of the matter variables and the energy conditions, and regain uncharged cases such as the Durgapal and Bannerji (1983) neutron star model.

The complexity involved in extending similar studies to higher dimensions is well documented. Most of the studies done are numerically inclined because of the complexity of the field equations. Several models are geared towards understanding gravitational collapse and the appearance of naked singularities (Banerjee *et al* 2003, Chan 2003, Ghosh and Dadhich 2001, Joshi *et al* 2002) in higher dimensions. Goswami and Joshi (2004) systematically summarised the study of higher dimensional spherically symmetric dust collapse by showing that both black holes and naked singularities would develop as end states depending on the initial data from which the collapse emanates. Some exact solutions have also been obtained over the years (Bhui *et al* 1995, Banerjee and Chatterjee 2005). Recently, Msomi *et al* (2012) used Lie symmetries in higher dimensions to provide classes of solutions that can be used to provide infinite families of models.

The previous studies mentioned above underline the significance of obtaining exact solutions to models in both four and higher dimensions. These studies not only give us more insight into the behaviour and evolution of galactic and stellar matter, but they also provide us with a mechanism of discarding models that may not concur with already established facts. The next three chapters in this dissertation comprise of publications emanating from our study. The Lie analysis of differential equations has played a central role in this thesis.

Chapter 2: In this chapter, we extend an algorithm due to Deng (1989) to study shear-free spherically symmetric gravitating fluids defined on a higher dimensional manifold. The chapter begins with an introduction that highlights previous studies made in four and higher dimensions. The model is then developed to obtain the defining generalised pressure isotropy condition. After a recap of Deng's algorithm, we proceed to show that it is possible to generate exact solutions to the Einstein field equations in the subsequent section. Several metrics are identified that contain results of four dimensions as special cases. We illustrate the validity of our solutions by graphically showing that the matter variables are well behaved and causality is not violated.

Chapter 3: We study shear-free spherically symmetric relativistic gravitating fluids with heat flow and electric charge. The solution to the Einstein-Maxwell system is governed by the pressure isotropy condition which contains a contribution from the electric field. This condition is a highly nonlinear partial differential equation. We analyse this master equation using Lie's group theoretic approach. The Lie symmetry generators that leave the equation invariant are found. The first generator is independent of the electromagnetic field. The second generator depends critically on the form of the charge, which is determined explicitly in general. Simpler forms of charge emanating from the generalised form are shown to possess extra symmetries. We provide exact solutions to the gravitational potentials using the symmetries admitted by the equation. Our new exact solutions contain earlier results without charge. We show that other charged solutions, related to the Lie symmetry results, may be generated using the modified Deng algorithm. This leads to new classes of charged Deng models which are generalisations of conformally flat metrics.

Chapter 4: We study shear-free spherically symmetric relativistic models of gravitating fluids with heat flow and electric charge defined on a higher dimensional manifold. The solution to the Einstein-Maxwell system is governed by the generalised condition of pressure isotropy, which is highly nonlinear. Using Lie's group theoretic approach, we obtain the Lie symmetry generators that leave the equation invariant. We provide exact solutions to the gravitational potentials using the first symmetry admitted by the equation. Our new exact solutions contain earlier results for a four-dimensional case. Using the other generators, we are able to provide solutions to the gravitational potentials or reduce the order of the master equation to a first order nonlinear differential equation. We also study the transport equation for the temperature in higher dimensions and obtain expressions that generalise earlier studies. We illustrate the applicability of our symmetry solutions by using an example to obtain the causal and noncausal cases for the temperature profile.

Chapter 5: We give concluding remarks about the results obtained.

Chapter 2

The Deng algorithm in higher dimensions

2.1 Introduction

Spherically symmetric gravitating models with heat flow, in the absence of shear, are important in the study of various cosmological processes and the evolution of relativistic astrophysical bodies. For a variety of applications in the presence of inhomogeneity see Krasinski (1997). Heat flow models are also important in analysing gravitational collapse and relativistic stellar processes. Astrophysical studies in which heat flow is important include the shear-free models of Wagh *et al* (2001), Maharaj and Govender (2005), Misthry *et al* (2008) and Herrera *et al* (2006). By studying shear-free models, we avail ourselves with a rather simpler avenue where we only need to provide solutions to the generalised condition of pressure isotropy containing two metric functions. A complete study of shear-free heat conducting fluids with charge was completed by Nyonyi *et al* (2013b) using Lie's group theoretic approach applied to differential equations. Shearing models where heat flow is significant have been recently studied by Thirukkhanesh *et al* (2012) for radiating spherically symmetric spheres. It turns out that the resulting nonlinear equations with shear are much more difficult to analyse.

A generic method of obtaining new solutions to the Einstein field equations was provided by Deng (1989). Using this general method we can regain existing results and obtain new classes of solutions. Nyonyi *et al* (2013b), Ivanov (2012) and Msomi *et al* (2011) have obtained new solutions using the Lie group theoretic approach and other methods, by solving the underlying pressure isotropy condition. These investigations are applicable to four dimensions. Extensions to higher dimensions have also been considered by many authors because of physical requirements, for example, Bhui *et al* (1995) showed the absence of horizons in nonadiabatic gravitational collapse. Studies of this type motivated the Lie symmetry analysis of heat conducting fluids by Msomi *et al* (2012) in dimensions greater than four. In the present treatment, we extend the Deng (1989) algorithm to higher dimensions and show that new results are possible.

2.2 The model

We consider the line element of a shear-free, spherically symmetric $(n + 2)$–dimensional manifold in the form

$$\mathrm{d}s^2 = -D^2\mathrm{d}t^2 + \frac{1}{V^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}X_n^2\right), \tag{2.1}$$

where $n \geq 2$. The gravitational potential components D and V are functions of r and t with

$$dX_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \sin^2\theta_2 \ldots \sin^2\theta_{n-1} d\theta_n^2. \tag{2.2}$$

For a heat conducting fluid, the energy momentum tensor is given by

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} + q_a U_b + q_b U_a, \tag{2.3}$$

where ρ is the energy density, p is the kinetic pressure, $\mathbf{q}$ is the heat flux tensor and $\mathbf{U}$ is a timelike $(n+2)$–velocity vector. For a comoving observer we have $U^a = \left(\frac{1}{D}, 0, 0, \ldots, 0\right)$ and $q^a = (0, q, 0, \ldots, 0)$.

Utilizing (2.1)–(2.3), we obtain the Einstein field equations

$$\rho = \frac{n(n+1)V_t^2}{2D^2V^2} - \frac{n(n+1)VV_r^2}{2} + nVV_{rr} + \frac{n^2VV_r}{r}, \tag{2.4a}$$

$$\begin{aligned} p = & -\frac{nD_rVV_r}{D} + \frac{nD_rV^2}{rD} + \frac{n(n-1)V_r^2}{2} - \frac{n(n-1)VV_r}{r} \\ & + \frac{nV_{tt}}{D^2V} - \frac{n(n+3)V_t^2}{2D^2V^2} - \frac{nD_tV_t}{D^3V}, \end{aligned} \tag{2.4b}$$

$$\begin{aligned} p = & \frac{D_{rr}V^2}{D} - (n-1)VV_{rr} + \frac{n(n-1)V_r^2}{2} + \frac{(n-1)D_rV^2}{rD} - \frac{(n-1)^2VV_r}{r} \\ & - \frac{(n-2)D_rVV_r}{D} + \frac{nV_{tt}}{D^2V} - \frac{n(n+3)V_t^2}{2D^2V^2} - \frac{nD_tV_t}{D^3V}, \end{aligned} \tag{2.4c}$$

$$q = -\frac{nVV_{tr}}{D} + \frac{nV_rV_t}{D} + \frac{nD_rVV_t}{D^2}, \tag{2.4d}$$

which are consistent with the derivation of Bhui *et al* (1995). Equations (2.4b) and (2.4c), together with the transformation $u = r^2$, give the pressure isotropy condition

$$VD_{uu} + 2D_uV_u - (n-1)DV_{uu} = 0, \tag{2.5}$$

which is the master equation for the gravitating fluid in $(n+2)$–dimensions.

Deng (1989) provided a general recipe for generating a series of solutions of the isotropy condition (2.5) for $n = 2$. This technique may be extended to the master equation (2.5). Note that the isotropy condition is an ordinary differential equation in u (since no time derivatives appear) which may be reduced to a simpler differential equation in D if V is known and vice versa. In this technique, elementary forms of either V or D are chosen that are in turn substituted into the equation to be solved so as to obtain the form of the remaining term. Below we give a brief outline of the method.

1. Take a simple form of V, say $V = V_1$, and substitute it into the master equation to find the most general solution of D, say $D = D_1$. The pair $V = V_1$ and $D = D_1$ provides the first class of solutions to (2.5).

2. Take $D = D_1$ and substitute it into the master equation. This gives an equation in V with $V = V_1$ already a solution. We are now in a position to obtain a second solution $V = V_2$ linearly independent of V_1. The linear combination $V_3 = aV_2 + bV_1$ gives the general solution that satisfies the master equation. The pair $V = V_3$ and $D = D_1$ is the second class of solutions to (2.5).

3. Take $V = V_3$ and substitute it into the master equation. We obtain an equation in D with $D = D_1$ already a solution. We are then in a position to obtain $D = D_2$ in the same way we obtained V_2. The pair $V = V_3$ and $D = cD_1 + dD_2$ is the third class of solutions to (2.5).

4. Repeat the above process to obtain an infinite sequence of solutions.

It is important to note that, in principle, this is a non-terminating process for obtaining solutions, and an infinite number of solutions can be listed. The difficulty arises in obtaining subsequent solutions in the process because the integration may become more complicated. However, the algorithm proves to be a powerful mechanism for generating new solutions.

2.3 Results

We start with a simple case

$$D_1 = 1. \tag{2.6}$$

Equation (2.5) reduces to

$$V_{uu} = 0. \tag{2.7}$$

This equation can be solved directly to obtain

$$V_1 = au + b, \tag{2.8}$$

where a and b are arbitrary functions of t. The pair of equations (2.6) and (2.8) gives the first class of solutions

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{1}{(au+b)^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}X_n^2\right). \tag{2.9}$$

Observe that the dimension n is absent in the metric (2.9). In this class of solutions the model is independent of the dimensionality as one of the gravitational potentials is constant. When $n = 2$ we regain the results of Bergmann (1981).

On substituting (2.8) back into (2.5) we obtain

$$(au + b)D_{uu} + 2aD_u = 0. \tag{2.10}$$

The general solution to equation (2.10) is

$$D_2 = \frac{cu+d}{au+b}, \tag{2.11}$$

with c and d arbitrary functions of t. The pair of equations (2.8) and (2.11) gives the second class of solutions

$$\mathrm{d}s^2 = -\left(\frac{cu+d}{au+b}\right)^2 \mathrm{d}t^2 + \frac{1}{(au+b)^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}X_n^2\right). \tag{2.12}$$

Again, we observe that the dimension n does not appear explicitly in (2.12). This means that the potentials in (2.12) are independent of the dimension. When $n = 2$, we regain the solutions obtained by Maiti (1982) and later generalised by Modak (1984) and Sanyal and Ray (1984). We make the general point that for a linear form of V, the parameter n does not appear in equation (2.5). Thus all solutions with a linear form for V do not contain the dimension n, thereby leading to the metrics (2.9) and (2.12).

Now, substituting (2.11) into (2.5), we obtain

$$V_{uu} - \frac{2}{n-1}\left(\frac{(bc-ad)/(au+b)^2}{(cu+d)/(au+b)}\right)V_u + \frac{2}{n-1}\left(\frac{a(bc-ad)/(au+b)^3}{(cu+d)/(au+b)}\right)V = 0. \tag{2.13}$$

We require two independent solutions V_1 and V_2 of (2.13). Note that V_1 in (2.8) is a solution to (2.13). We propose the second solution to (2.13) to be given by

$$V_2 = \alpha(u,t)V_1, \tag{2.14}$$

where the function $\alpha(u,t)$ has to be found explicitly. On substituting (2.14) into (2.13), we obtain

$$\alpha_{uu} + 2\left[\frac{a}{au+b} - \frac{1}{n-1}\left(\frac{(bc-ad)/(au+b)^2}{(cu+d)/(au+b)}\right)\right]\alpha_u = 0. \tag{2.15}$$

On integrating (2.15), we obtain α expressed as

$$\alpha = \int^u e\left(\frac{cs+d}{(as+b)^n}\right)^{2/(n-1)} \mathrm{d}s. \tag{2.16}$$

Consequently, the second solution V_2 will depend on the dimension n. To evaluate the integral (2.16), we need to consider two cases: $ad = bc$ and $ad \neq bc$.

2.3.1 Case I

When $ad = bc$ we have

$$\begin{aligned} \alpha &= e\left(\frac{d}{b^n}\right)^{\frac{2}{n-1}} \int \left(1+\frac{a}{b}u\right)^{-2} \mathrm{d}u \\ &= \frac{a^2gu + (abg - kb^2)}{a(au+b)}, \end{aligned} \tag{2.17}$$

where g is an arbitrary function of t. Therefore V_2 becomes

$$V_2 = agu + \left(bg - \frac{k}{a}b^2\right). \tag{2.18}$$

This implies that V_2 is proportional to V_1 and is therefore not a second linearly independent solution. The case $ad = bc$ is degenerate.

2.3.2 Case II

When $ad \neq bc$, we have

$$\alpha = \frac{e}{ad-bc}\left(\frac{1-n}{n+1}\right)\left(\frac{cu+d}{au+b}\right)^{\frac{n+1}{n-1}} + g, \tag{2.19}$$

where a, b, c, d, e and g are arbitrary functions of t. Therefore the second solution V_2 becomes

$$V_2 = \left(\frac{e}{ad-bc}\left(\frac{1-n}{n+1}\right)\left(\frac{cu+d}{au+b}\right)^{\frac{n+1}{n-1}} + g\right)(au+b). \tag{2.20}$$

And since the general solution to (2.5) is a linear combination of V_1 and V_2, we obtain

$$V_3 = \left(h(t) + j(t) \left(\frac{e}{ad - bc} \left(\frac{1-n}{n+1} \right) \left(\frac{cu+d}{au+b} \right)^{\frac{n+1}{n-1}} + g \right) \right) (au + b) \,, \tag{2.21}$$

where we have introduced for convenience $h(t)$ and $j(t)$. The third class of solutions is therefore given by (2.11) and (2.21) with metric

$$\begin{aligned} ds^2 = & - \left(\frac{cu+d}{au+b} \right)^2 dt^2 \\ & + \left(\left[h(t) + j(t) \left(\frac{e}{ad - bc} \left(\frac{1-n}{n+1} \right) \left(\frac{cu+d}{au+b} \right)^{\frac{n+1}{n-1}} + g \right) \right] (au+b) \right)^{-2} \\ & \times \left(dr^2 + r^2 dX_n^2 \right) . \end{aligned} \tag{2.22}$$

This is a new class of solution (Nyonyi *et al* 2013b) and it is evident that it certainly depends on the dimension $n \geq 2$. Therefore we can conclude that the dimensionality of the problem does indeed affect the dynamics of the gravitational field with heat flow. The next class of solutions can be obtained by substituting V_3 into equation (2.5) and then solve the resulting equation for D_3. This may be continued to obtain further new solutions. The integration process gets more complicated for further iterations.

We now consider the special case of four dimensions. When $e = 1$ and $n = 2$ the line element (2.22) becomes

$$\begin{aligned} ds^2 = & - \left(\frac{cu+d}{au+b} \right)^2 dt^2 \\ & + \left(\left[h - j \left(\frac{1}{3(ad - bc)} \left(\frac{cu+d}{au+b} \right)^3 + g \right) \right] (au+b) \right)^{-2} \\ & \times \left(dr^2 + r^2 (d\theta^2 + \sin^2\theta \; d\phi^2) \right) . \end{aligned} \tag{2.23}$$

We can rewrite (2.23) in the equivalent form

$$\begin{aligned} ds^2 = & -\left(\frac{cu+d}{au+b}\right)^2 dt^2 \\ & + \left((h+\kappa)(au+b) - \frac{j}{3a}\left(\frac{c^2}{a^2} + \frac{c}{a}\frac{au+b}{cu+d} + \left(\frac{cu+d}{au+b}\right)^2\right)\right)^{-2} \\ & \times \left(dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\right), \end{aligned} \tag{2.24}$$

where the function κ is given by

$$\kappa = \left(g - \frac{c^3}{3a^3(ad-bc)}\right). \tag{2.25}$$

When we set $\kappa = 0$ in (2.24) we regain the result of Deng (1989). We interpret (2.22) as the higher dimensional generalisation of the Deng model with heat flow.

2.4 Example

We illustrate the validity of our solutions by considering a simple example with physically viable conditions. For the line element (2.22), we make the simple choice: $a = d = 0$, $b = c = 1$, $h + jg = 1$ and $e = j = 1$. This gives the simplified forms of the potentials

$$D = r^2, \qquad V = 1 + \left(\frac{1-n}{n+1}\right) t \, r^{(2(n+1)/(n-1))}. \tag{2.26}$$

Even with this simple example, a qualitative analysis of the matter variables and energy conditions for the interior matter distribution is arduous. We therefore generate graphical plots for a constant timelike hypersurface to illustrate the validity of our solutions using this example. Figures 2.1–2.3 are the plots for the energy density ρ, the pressure p and the heat flow q for three different dimensions: $n = 2$ (dashed line), $n = 3$ (solid line) and

$n = 4$ (dotted line). It is clearly evident that the matter variables are positive and they decrease with increase in dimension. This is due to the fact that an increase in dimension translates to an increase in the number of degrees of freedom leading to a decrease in the mass per unit volume of the gravitating fluid. In Figure 2.4 we have plotted the speed of sound. From Figure 2.4, we observe that causality is not violated for the dimensions $n = 2$, 3 and 4. In Figures 2.5–2.7 we have plotted the quantities $A = \rho - p + \Delta$, $B = \rho - 3p + \Delta$ and $C = 2p + \Delta$, where $\Delta = \sqrt{(p+q)^2 - 4q^2}$. We observe that A, B and C are positive; hence the weak, dominant and strong energy conditions are satisfied. Therefore the matter distribution for this example is physically reasonable.

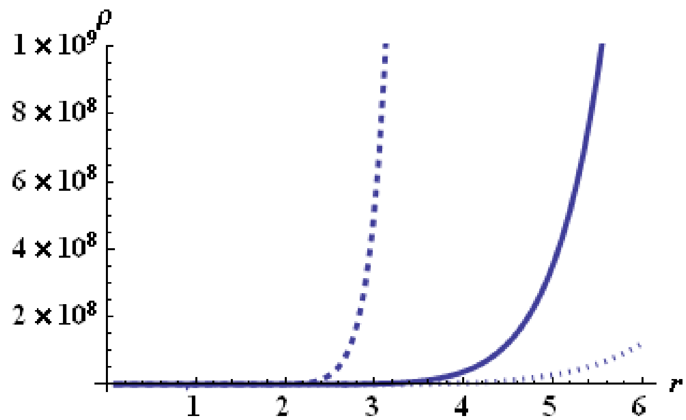

Figure 2.1: Energy density ρ

2.5 Discussion

We obtained new generalised classes of exact solutions to the Einstein field equations for a neutral relativistic fluid in the presence of heat flow in a higher dimensional manifold.

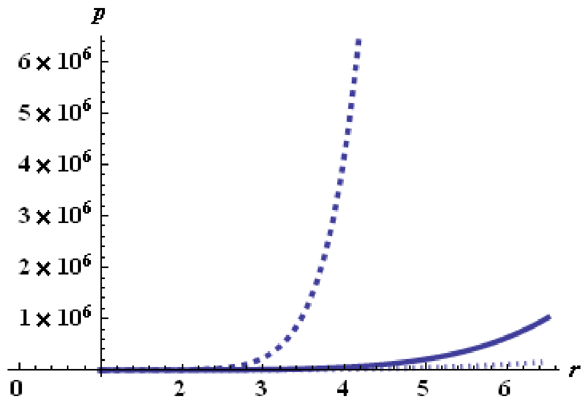

Figure 2.2: Pressure

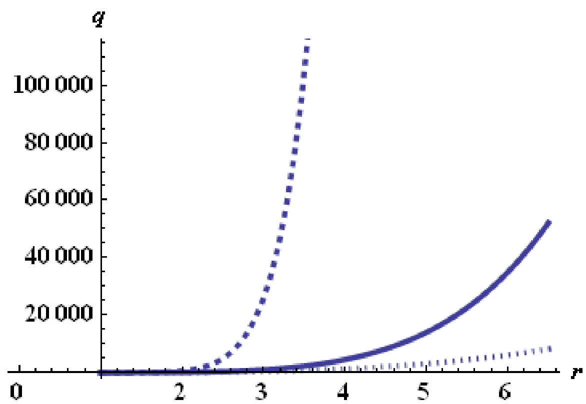

Figure 2.3: Heat flow

We found new solutions to the coupled Einstein system by solving the higher dimensional pressure isotropy condition which is a second order nonlinear differential equation. We solved the master equation by making use of the Deng algorithm (1989) and obtained three new metrics. The first metric (2.9) generalises the Bergmann (1981) line element. The second metric (2.12) generalises the Maiti (1982), Modak (1984) and Sanyal and Ray (1984) line elements. It is remarkable that the potentials in (2.9) and (2.12) are independent of the

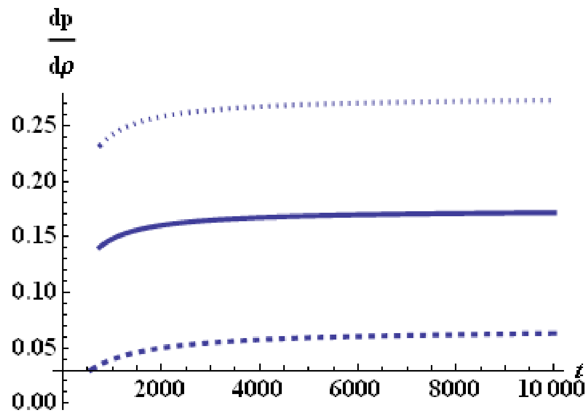

Figure 2.4: Speed of sound

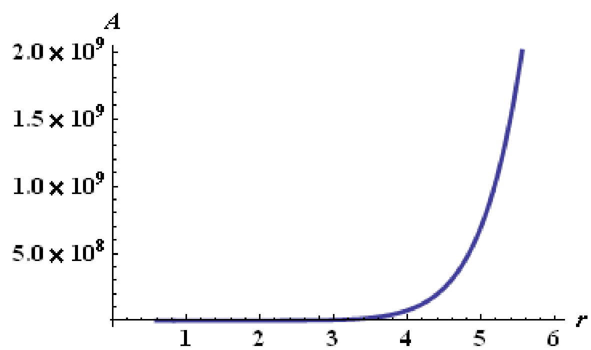

Figure 2.5: Weak energy condition

dimension. The third metric (2.22) depends on the dimension n and generalises the Deng (1989) line element. We conclude that the dimension of the spacetime affects the dynamics of the heat conducting gravitating fluid. We briefly studied the physical features by graphically plotting the matter variables. The energy conditions are found to be positive and causality is not violated.

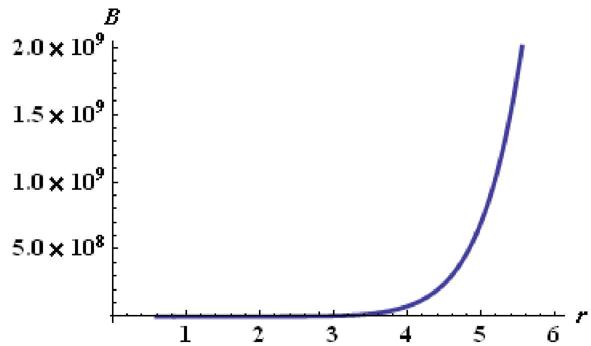

Figure 2.6: Dominant energy condition

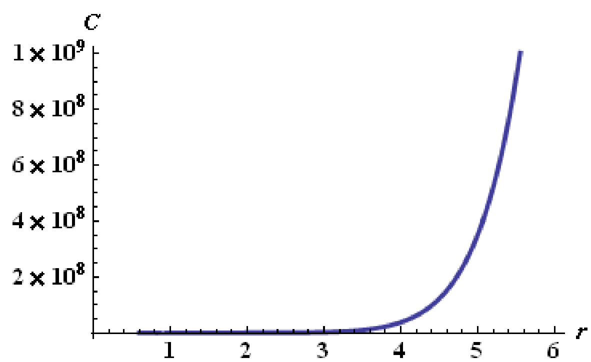

Figure 2.7: Strong energy condition

Chapter 3

Charged spherically symmetric fluids with heat flow

3.1 Introduction

In this chapter, we consider charged spherically symmetric gravitating fluids, in the presence of heat flow, with vanishing shear which are important in the study of various cosmological and relativistic astrophysical bodies. It is necessary to solve the Einstein-Maxwell system of field equations to obtain exact solutions. Krasinski (1997) points out the importance of these solutions for modelling in structure formation, evolution of voids, gravitational collapse, inhomogeneous cosmologies and relativistic stellar processes. In these applications, heat flow and charge become important ingredients in building radiating and gravitating models. By studying shear-free models, we avail ourselves with a rather simpler avenue where we only need to provide solutions to the generalised condition of pressure isotropy containing two metric functions. The resulting nonlinear equations with shear are much more difficult to analyse.

Heat flux is of great importance in relativistic astrophysical problems involving singularities in manifolds, gravitational collapse and black hole physics, among other applications as emphasised by Krasinski (1997). Such fluids have also been used in the study of relativistic stars that emit null radiation in the form of radial heat flow; a study made possible by Santos (1985) who showed that the interior spacetime must contain a nonzero heat flux to match with the pressure at the boundary with the exterior Vaidya spacetime. The notion of heat flow is manifested in many shear-free stellar models including the treatment of Wagh *et al* (2001) who chose a barotropic equation of state and gave solutions to the Einstein field equations for a spherically symmetric spacetime. Maharaj and Govender (2005) and Misthry *et al* (2008), when studying radiating collapse with vanishing Weyl stresses, provided exact solutions to both the Einstein field equations and the junction conditions. Herrera *et al* (2006) showed that analytic solutions can be obtained from the study of the field equations arising from radiating and collapsing spheres in the diffusion approximation. They showed that heat flow is a requirement in thermal evolution of the collapsing sphere modelled in causal thermodynamics. We note the recent general treatment of Thirukkanesh *et al* (2012) for radiating spheres in the presence of shear in spherical symmetry.

In the cosmological setting, some of the earlier studies in which heat flow is an important component were carried out by Bergmann (1981), Maiti (1982), Modak (1984) and Sanyal and Ray (1984) in their quest to provide exact solutions. Deng (1989), using his general algorithm, regained earlier results and provided new classes of solutions. Msomi *et al* (2011) studied the same model and used Lie's group theoretic approach to provide a five-parameter family of transformations that mapped known solutions into new ones. They also obtained new classes of solutions using Lie infinitesimal generators. Later Msomi *et al* (2012) considered the problem in higher dimensions obtaining implicit solutions or reducing the fundamental equation to a Riccati equation. Also, Ivanov (2012), using a compact formalism, simplified the condition of pressure isotropy and the condition for conformal flatness, and gave easily tractable versions of the junction condition for conformally flat and geodesic models. This approach has the advantage of yielding well known differential equations, amalgamates the results for static models and places the time-dependent results of Msomi *et al* (2011, 2012) in context.

Stellar models in which charge is incorporated, so that the Einstein-Maxwell system is valid, have also been extensively studied. Komathiraj and Maharaj (2007b) showed that by considering a linear equation of state, exact analytical solutions to the Einstein-Maxwell equations can be obtained that contains the Mak and Harko (2004) model. They obtained solutions that describe quark matter in the presence of an electromagnetic field. Other recent charged stellar models include the results of Komathiraj and Maharaj (2007a), Lobo (2006), Maharaj and Thirukkanesh (2009), Sharma and Maharaj (2007) and Thirukkanesh and Maharaj (2009). Radiating stellar models where charge is incorporated have also been extensively studied by Chan (2001, 2003) using numerical techniques. Recently, Pinheiro and Chan (2013) performed a numerical analysis of a charged body undergoing gravitational collapse and showed that charge delays black hole formation and can even prevent collapse depending on the total mass-to-charge ratio. For varying spherically symmetric gravitational fields in cosmology, Kweyama *et al* (2012a) found new parametric solutions to the Einstein-Maxwell system of field equations. Their approach was *ad hoc*; a systematic approach using group theoretical techniques such as the Lie analysis may lead to new results. Govinder *et al* (1995), Kweyama *et al* (2011, 2012b), Leach and Maharaj (1992) and Msomi *et al* (2010) used Lie point symmetries to study the underlying nonlinear partial differential equations that arise in the study of gravitating fluids. They provided several families of solutions while generalising already known solutions. It is evident that there exists numerous physical applications to models in which heat flow and charge are incorporated.

Several techniques of obtaining solutions for gravitating fluids have been adopted over the years which yielded a variety of models. We intend to show that applying a group theoretic approach with Lie symmetries provides new insights for charged heat conducting models in the absence of shear. We present the field equations and obtain the defining master equation in §3.2. We then obtain the underlying symmetries of the governing equation in §3.3. This is a complex calculation and we provide all the relevant details. We use the first symmetry obtained to provide new solutions for an arbitrary form of charge in §3.4.1. The gravitational potentials can be found explicitly. In the subsequent sections §3.4.2 − §3.4.5, we show how the respective symmetries are used to reduce the order of the governing equation, while providing exact solutions to the gravitational potentials in some cases. The cases where reduction to quadrature is difficult to perform arise from the nonlinearity of the resultant equations. New charged Deng solutions are obtained in §3.5. A few concluding remarks follow in §3.6.

3.2 The model

We assume a spherically symmetric spacetime which satisfies the shear-free condition. Then the line element in Schwarzschild coordinates (t, r, θ, φ) becomes

$$ds^2 = -D^2 dt^2 + \frac{1}{V^2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{3.1}$$

where $D = D(t,r)$ and $V = V(t,r)$ represent the gravitational potentials. We also define the energy momentum tensor for a charged matter distribution in a shear-free model to be of the form

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} + q_a U_b + q_b U_a + E_{ab}, \tag{3.2}$$

where ρ is the energy density, p is the isotropic pressure, and $q_a = (0, q, 0, 0)$ is the heat flux vector. These quantities are measured relative to a comoving four-velocity vector $U_a = \left(\frac{1}{D}, 0, 0, 0\right)$ that is taken to be unit and timelike. The electromagnetic contribution E_{ab} to the matter distribution is obtained from

$$E_{ab} = F_{ac}F^{c}{}_{b} - \frac{1}{4} g_{ab} F_{cd} F^{cd}, \tag{3.3}$$

where the Faraday tensor

$$F_{ab} = A_{b;a} - A_{a;b},$$

is defined in terms of a four-potential $A_a = (\phi(t,r), 0, 0, 0)$ with ϕ being the only nonzero component.

With the help of (3.1) and (3.2), the Einstein-Maxwell field equations are given by

$$\rho = 3\frac{V_t}{D^2V^2} + V^2\left[2\frac{V_{rr}}{V} - 3\frac{V_r^2}{V^2} + 4\frac{V_r}{rV}\right] - \frac{V^2}{2D^2}\phi_r^2, \tag{3.4a}$$

$$\begin{aligned} p = {} & \frac{1}{D^2}\left[2\frac{V_{tt}}{V} - 2\frac{D_tV_t}{DV} - 5\frac{V_t^2}{V^2}\right] + V_r^2 - 2\frac{VV_r}{r} - 2\frac{VD_rV_r}{D} \\ & + 2\frac{D_rV^2}{rD} + \frac{1}{2}\frac{V^2}{D^2}\phi_r^2, \end{aligned} \tag{3.4b}$$

$$\begin{aligned} p = {} & \frac{1}{D^2}\left[2\frac{V_{tt}}{V} - 2\frac{D_tV_t}{DV} - 5\frac{V_t^2}{V^2}\right] - VV_{rr} + V_r^2 - \frac{VV_r}{r} + \frac{V^2D_r}{rD} \\ & + \frac{V^2D_{rr}}{D} - \frac{1}{2}\frac{V^2}{D^2}\phi_r^2, \end{aligned} \tag{3.4c}$$

$$q = -2\frac{V^2}{D}\left[\frac{V_{tr}}{V} - \frac{V_tV_r}{V^2} - \frac{D_rV_t}{DV}\right], \tag{3.4d}$$

$$\sigma = \frac{V^2}{D}\left[\phi_{rr} + \left(\frac{2}{r} - \frac{V_r}{V} - \frac{D_r}{D}\right)\phi_r\right], \tag{3.4e}$$

$$0 = -\frac{V^2}{D^2}\left[\phi_{rt} - \left(\frac{V_t}{V} + \frac{D_t}{D}\right)\phi_r\right], \tag{3.4f}$$

where σ is the proper charge density. On integrating (3.4f), we obtain

$$\phi_r = VDF(r), \tag{3.5}$$

where $F(r)$ is an arbitrary function. Equating (3.4b) and (3.4c) gives the generalised pressure isotropy condition

$$-\frac{VV_r}{r} - 2\frac{VD_rV_r}{D} + \frac{D_rV^2}{rD} + VV_{rr} - \frac{V^2D_{rr}}{D} + \frac{V^2}{D^2}\phi_r^2 = 0. \tag{3.6}$$

The system (3.4) is completely solved if we can find functions ϕ, V and D that satisfy (3.6). Therefore (3.6) is the fundamental equation governing the evolution of a shear-free, heat conducting gravitating fluid. The generalised pressure isotropy condition (3.6) simplifies to

$$4uVD_{uu} + 8uV_uD_u - 4uDV_{uu} - V^2F(u) = 0, \tag{3.7}$$

with $u = r^2$ and $F(u)$ is arbitrary. We seek to provide solutions to this master equation (3.7) using Lie's group theoretic approach. Note that in the absence of charge (3.7) becomes

$$VD_{uu} + 2V_uD_u - DV_{uu} = 0, \tag{3.8}$$

which was studied by Msomi *et al* (2011).

3.3 Lie analysis of the problem

For the problem at hand, we seek to determine a one-parameter (ε) Lie group of transformations

$$\tilde{u} = \mathfrak{f}(u, V, D; \varepsilon), \tag{3.9a}$$

$$\tilde{D} = \mathfrak{g}(u, V, D; \varepsilon), \tag{3.9b}$$

$$\tilde{V} = \mathfrak{h}(u, V, D; \varepsilon), \tag{3.9c}$$

that leave the solutions of (3.7) invariant. Due to the complexity involved in obtaining the transformations directly, we consider the infinitesimal forms

$$\tilde{u} = u + \varepsilon\xi(u, V, D) + O(\varepsilon^2), \tag{3.10a}$$

$$\tilde{D} = D + \varepsilon\eta^1(u, V, D) + O(\varepsilon^2), \tag{3.10b}$$

$$\tilde{V} = V + \varepsilon\eta^2(u, V, D) + O(\varepsilon^2), \tag{3.10c}$$

with symmetry generator given by

$$X = \xi\frac{\partial}{\partial u} + \eta^1\frac{\partial}{\partial D} + \eta^2\frac{\partial}{\partial V}. \tag{3.11}$$

We can then infer from (3.10), to regain the global form of the transformations (3.9), that we need to solve

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\varepsilon} = \xi(u, V, D), \tag{3.12a}$$

$$\frac{\mathrm{d}\tilde{D}}{\mathrm{d}\varepsilon} = \eta^1(u, V, D), \tag{3.12b}$$

$$\frac{\mathrm{d}\tilde{V}}{\mathrm{d}\varepsilon} = \eta^2(u, V, D), \tag{3.12c}$$

subject to

$$\tilde{u}|_{\varepsilon=0} = u, \qquad \tilde{D}|_{\varepsilon=0} = D, \qquad \tilde{V}|_{\varepsilon=0} = V. \tag{3.13}$$

For a detailed review of these ideas, the reader is referred to Bluman and Anco (2002), Bluman and Kumei (1989), Cantwell (2002) and Olver (1986, 1995).

Due to the complexity of the calculations, we provide as much relevant detail as possible. It is important to note that both D and V are functions of u and t, but t does not appear explicitly in equation (3.7). As a result we can treat (3.7) as a second order nonlinear ordinary differential equation only in u. However, we will ultimately let the constants of integration become functions of t. For simplicity, we label the left hand part of our master equation (3.7) as K. We require

$$X^{[2]}K|_{K=0} = 0, \tag{3.14}$$

which yields

$$\xi = C^0(u), \tag{3.15}$$

$$\eta^1 = c_1 D, \tag{3.16}$$

$$\eta^2 = \left(c_1 + c_2 + \frac{1}{2}C^0_u\right) V, \tag{3.17}$$

with $F(u)$ satisfying

$$2VDC^0_{uuu} + V^2 \left[F\left(-\frac{C^0}{u} + \frac{5}{2}C^0_u + c_2\right) + C^0 F'\right] = 0. \tag{3.18}$$

For arbitrary F, (3.18) can only be satisfied if

$$C^0_{uuu} = 0, \tag{3.19a}$$

$$-\frac{C^0}{u} + \frac{5}{2}C^0_u + c_2 = 0, \tag{3.19b}$$

$$C^0 = 0. \tag{3.19c}$$

By inspection, we can deduce from equations (3.19) that

$$C^0(u) = 0, \tag{3.20}$$

and

$$c_2 = 0. \tag{3.21}$$

Using (3.15)–(3.17) and (3.20)–(3.21), and making the necessary substitutions, we obtain the coefficient functions for the symmetry generator, when $F(u)$ is arbitrary, as

$$\xi(u) = 0, \tag{3.22}$$

$$\eta^1(D) = c_1 D, \tag{3.23}$$

$$\eta^2(V) = c_1 V. \tag{3.24}$$

From the above coefficient functions, we obtain

$$X_1 = D\partial_D + V\partial_V, \tag{3.25}$$

as the sole symmetry in the case of arbitrary $F(u)$. It is indeed remarkable that this symmetry exists without placing any restriction on $F(u)$.

We now take (3.18) to be a restriction on F. As C^0 and F are functions of u, this implies that both

$$C^0_{uuu} = 0, \tag{3.26}$$

and

$$F\left(-\frac{C^0}{u} + \frac{5}{2}C^0_u + c_2\right) + C^0 F' = 0, \tag{3.27}$$

must hold. Solving equation (3.26) gives

$$C^0 = c_3 u^2 + c_4 u + c_5. \tag{3.28}$$

Using equation (3.28) we solve (3.27) to obtain

$$F(u) = \frac{u c_6}{(c_3 u^2 + c_4 u + c_5)^{5/2}} \exp\left[\frac{2c_2}{\sqrt{-c_4 + 4c_3 c_5}} \arctan\left(\frac{c_4 + 2c_3 u}{\sqrt{-c_4 + 4c_3 c_5}}\right)\right], \tag{3.29}$$

where c_6 is a constant of integration. From equations (3.28)–(3.29), we see that in addition to X_1, (3.7) admits another symmetry

$$X_2 = c_1 X_1 + \left(c_3 u^2 + c_4 u + c_5\right) \partial_u + \left(c_2 + c_3 u + \frac{c_4}{2}\right) V \partial_V, \tag{3.30}$$

dictated by the form of $F(u)$. It is important to observe that the quantity $F(u)$ arises because of the presence of charge. The symmetry X_2 is intimately related to the form of the electromagnetic field. It is remarkable that the electric field, through (3.29), can be explicitly found in general when the symmetry generator X_2 exists.

Equation (3.29) gives the most general form of $F(u)$ for which X_2 is the associated symmetry. We note that $F(u)$ depends on the arbitrary constants c_2–c_6. Of these, four, c_2–c_5, appear in the symmetry itself while c_6 is just a scaling constant. Clearly, choices for c_2–c_5 will produce simpler forms of $F(u)$ which will admit reduced forms of X_2 as a symmetry (in addition to X_1). In Table 3.1, we list the relevant simpler forms for $F(u)$ and the corresponding form of X_2 for all the relevant choices of c_2–c_5. (Note that, in all cases, we relabel our scaling constant for $F(u)$ to be k.) In some cases, the simpler form of $F(u)$ causes the original equation (3.7), to admit additional symmetries. These are listed in the third column of Table 3.1. For the case $F = 0$, a full analysis was performed by Msomi *et al* (2011), and we do not repeat their results here. We obtained extra symmetries for c_2, c_3 and c_5 only. This is summarised in Table 3.1. The only other case of interest is when $c_2 = 0$ (with all other constants non zero) as reported in Table 3.1.

3.4 New solutions using symmetries

Usually, after obtaining the symmetries of a differential equation, we use the associated differential invariants to determine the solution(s) of the equation. The forms for the function F utilised in our analysis arise from the earlier symmetry analysis. For all cases, we were able to reduce the order of our master equation. However we were not always able to solve the reduced equation. In particular no solutions were possible for the symmetries $X_{2,5}$ and X_2. We discuss these below.

3.4.1 Arbitrary F

Due to the arbitrary nature of F, our master equation can be modified to be

$$VD_{uu} + 2V_uD_u - DV_{uu} - V^2\frac{F(u)}{4u} = 0. \tag{3.31}$$

We obtain the invariants of the generator

$$X_1 = D\partial_D + V\partial_V, \tag{3.32}$$

by taking its first extension. The associated Lagrange's system becomes

$$\frac{\mathrm{d}u}{0} = \frac{\mathrm{d}D}{D} = \frac{\mathrm{d}V}{V} = \frac{\mathrm{d}D'}{D'} = \frac{\mathrm{d}V'}{V'}. \tag{3.33}$$

We obtain the invariants of the system as

$$p = u, \tag{3.34a}$$

$$q(p) = \frac{V}{D}, \tag{3.34b}$$

$$r(p) = \frac{D'}{D}, \tag{3.34c}$$

$$s(p) = \frac{V'}{D}. \tag{3.34d}$$

However, for our purposes, we only use p, q and r. Invoking these differential invariants, (3.31) reduces to

$$q'' = -q^2 \frac{F(p)}{4p} + 2qr^2, \tag{3.35}$$

which can be written as

$$r = \pm\sqrt{\frac{q''}{2q} + q\frac{F(p)}{8p}}, \tag{3.36}$$

or

$$\frac{D'}{D} = \pm\sqrt{\frac{q''}{2q} + q\frac{F(p)}{8p}}. \tag{3.37}$$

On integrating both sides we have

$$D = \exp\left[\pm C \int \sqrt{\frac{W''}{2W} + W\frac{F(u)}{8u}} \quad \mathrm{d}u\right], \tag{3.38}$$

where C is a constant of integration.

From solution (3.38), we can see that whenever we are given any ratio of the gravitational potentials $W = \frac{V}{D}$, and an arbitrary function $F(u)$ representing charge, we can explicitly obtain the exact expression of the potentials. This is a new result that to the best of our knowledge has not been obtained before. We observe that when we set $F(u) = 0$ in (3.38), we obtain

$$D = \exp\left[\pm C \int \sqrt{\frac{W''}{2W}} \quad \mathrm{d}u\right]. \tag{3.39}$$

This is the uncharged solution of Msomi *et al* (2011). Thus (3.38) is a charged generalisation of their solution.

3.4.2 $F = 0$

For this particular form of F, our master equation takes the form

$$VD_{uu} + 2V_u D_u - DV_{uu} = 0. \tag{3.40}$$

By taking the first extension of

$$X_{2,1} = V\partial_V, \tag{3.41}$$

the associated Lagrange's system becomes

$$\frac{\mathrm{d}u}{0} = \frac{\mathrm{d}D}{0} = \frac{\mathrm{d}V}{V} = \frac{\mathrm{d}D'}{0} = \frac{\mathrm{d}V'}{V'}. \tag{3.42}$$

The corresponding invariants become

$$p = u, \tag{3.43a}$$

$$q(p) = D, \tag{3.43b}$$

$$r(p) = D', \tag{3.43c}$$

$$s(p) = \frac{V'}{V}. \tag{3.43d}$$

When we use a partial set of invariants p, $q(p)$ and $s(p)$, (3.40) reduces to

$$s_p = \frac{q_{pp}}{q} + 2\frac{q_p}{q}s - s^2, \tag{3.44}$$

which is a Riccati equation in s. It is not possible to make further progress with (3.44).

However, if we include $r(p)$ and consider the full set of invariants, (3.40) reduces to

$$r_p + 2sr = \left(s_p + s^2\right)q. \tag{3.45}$$

Equation (3.45), being a first order differential equation in r, can easily be reduced to quadrature to give

$$r(p) = e^{-2s}\int \left(s_p + s^2\right)qe^{2s}\mathrm{d}p. \tag{3.46}$$

From our invariants it easily follows that

$$D = \int \left(e^{-2(V'/V)} \int D \left[\frac{\mathrm{d}(V'/V)}{\mathrm{d}u} + \left(\frac{V'}{V} \right)^2 \right] e^{2(V'/V)} \mathrm{d}u \right) \mathrm{d}u. \tag{3.47}$$

This result was first established by Msomi *et al* (2011) and a comprehensive study of the uncharged case produced five symmetries as already indicated above. They provided the complete analysis of the uncharged model and we do not intend to reproduce their results herein.

3.4.3 $F = ku^{-4}$

Our master equation becomes

$$4uVD_{uu} + 8uV_uD_u - 4uDV_{uu} - V^2ku^{-4} = 0. \tag{3.48}$$

For this case, we obtain two extra symmetries associated with the form of F highlighted above. We carry out reductions using these symmetries separately with the hope of obtaining new solutions.

Generator $X_{2,2}$

By taking the first extension of

$$X_{2,2} = u^2\partial_u + uV\partial_V, \tag{3.49}$$

we obtain the Lagrange's system from which we deduce the invariants to be

$$p = D, \tag{3.50a}$$

$$q(p) = \frac{V}{u}, \tag{3.50b}$$

$$r(p) = u^2 D', \tag{3.50c}$$

$$s(p) = u\left(q(p) - V'\right). \tag{3.50d}$$

Using the invariants p, q and r, equation (3.48) reduces to

$$4rqr_p + 8r^2 q_p - 4(rr_p q_p + r^2 q_{pp})p - kq^2 = 0, \tag{3.51}$$

or

$$\frac{1}{2}(r^2)_p + \left(\frac{2q_p - pq_{pp}}{q - pq_p}\right) r^2 - \left(\frac{(k/4)q^2}{q - pq_p}\right) = 0. \tag{3.52}$$

Equation (3.52) is a first order differential equation in r^2 which when solved gives

$$r(p) = \sqrt{\frac{Y + e^{X(p+A)}}{X}}, \tag{3.53}$$

where

$$Y = \frac{(k/2)q^2}{q - pq_p}, \qquad X = 2\left(\frac{2q_p - pq_{pp}}{q - pq_p}\right), \tag{3.54}$$

and A is a constant of integration.

By taking the invariants into consideration, (3.53) is reduced to quadrature to give

$$\int \left(\frac{X}{e^{X(D+A)} + Y}\right)^{1/2} \mathrm{d}D = -\frac{1}{u} + B, \tag{3.55}$$

where

$$X = 2\frac{2\frac{\mathrm{d}(V/u)}{\mathrm{d}D} - D\frac{\mathrm{d}^2(V/u)}{\mathrm{d}D^2}}{(V/u) - D\frac{\mathrm{d}(V/u)}{\mathrm{d}D}}, \qquad Y = \frac{(k/4)(V/u)^2}{(V/u) - D\frac{\mathrm{d}(V/u)}{\mathrm{d}D}}, \tag{3.56}$$

and B is a constant of integration.

Generator X_3

This is the second extra symmetry associated with (3.48). By taking the first extension of

$$X_3 = u\partial_u + 3V\partial_V, \tag{3.57}$$

we obtain the corresponding Lagrange's system from which the invariants become

$$p = D, \tag{3.58a}$$

$$q(p) = \frac{\sqrt[3]{V}}{u}, \tag{3.58b}$$

$$r(p) = uD', \tag{3.58c}$$

$$s(p) = \frac{\sqrt{V'}}{u}. \tag{3.58d}$$

The first derivatives of $r(p)$ and $s(p)$ with respect to p enable us to obtain expressions for D'' and V'' respectively. Thus we are able to transform equation (3.48) to

$$ss_p + s^2\left(\frac{2}{r} - \frac{1}{p}\right) + \frac{q^3}{p}\left(1 - r_p + \frac{kq^3}{4r}\right) = 0. \tag{3.59}$$

Equation (3.59) is a first order differential equation in s^2, which when solved gives

$$s(p) = \pm\sqrt{\frac{\frac{q^3}{p}\left(1 - r_p + k\frac{q^3}{4r}\right) + e^{-4(1/r-1/p)(p-C)}}{\frac{2}{p}\left(\frac{p}{r} - 1\right)}}, \tag{3.60}$$

where C is a constant of integration.

By taking the invariants into consideration, we obtain explicitly the exact solution of one of the potentials as

$$\int \frac{\mathrm{d}V}{\frac{u^3DD'}{2(D-uD')}\left[\frac{V}{Du^3}\left(\frac{uD''}{D'} - \frac{kV}{4u^4D'}\right) + e^{\frac{4(uD'-D)(D-C)}{uD'D}}\right]} = \frac{u^3}{3} + E, \tag{3.61}$$

where E is a constant of integration.

3.4.4 $F(u) = ku^{-\frac{3}{2}}$

For this particular form of F, the master equation becomes

$$4uVD_{uu} + 8uV_uD_u - 4uDV_{uu} - V^2ku^{-\frac{3}{2}} = 0. \tag{3.62}$$

By taking the first prolongation of the associated generator

$$X_{2,3} = u\partial_u + \frac{1}{2}V\partial_V, \tag{3.63}$$

we obtain the corresponding Lagrange's system from which the invariants become

$$p = D, \tag{3.64a}$$

$$q(p) = \frac{V^2}{u}, \tag{3.64b}$$

$$r(p) = uD', \tag{3.64c}$$

$$s(p) = uV'^2. \tag{3.64d}$$

Making use of the first three invariants p, q and r, equation (3.62) reduces to

$$(r^2)_p + \frac{4q_p - p(2q_{pp} - q_p^2)}{2q - pq_q}r^2 + \frac{pq - kq^{3/2}}{2q - pq_q} = 0. \tag{3.65}$$

Equation (3.65) is a first order differential equation in r^2 which can be solved to obtain

$$r(p) = \sqrt{\frac{e^{Y(p+A)} - Z}{Y}}, \tag{3.66}$$

where

$$Y = \frac{4q_p - p(2q_{pp} - q_p^2)}{2q - pq_p}, \qquad Z = \frac{pq - kq^{(3/2)}}{2q - pq_p}, \tag{3.67}$$

and A is a constant of integration.

Using the invariants, we can provide the explicit solution to (3.66) as

$$\int \left(\frac{Y}{e^{Y(p+A)} - Z} \right)^{1/2} \quad \mathrm{d}D = \ln u + B, \tag{3.68}$$

where

$$Y = \frac{4\frac{\mathrm{d}(V^2/u)}{\mathrm{d}D} - D\left[\frac{\mathrm{d}^2(V^2/u)}{\mathrm{d}D^2} - \left(\frac{\mathrm{d}(V^2/u)}{\mathrm{d}D}\right)^2\right]}{2(V^2/u) - D\frac{\mathrm{d}(V^2/u)}{\mathrm{d}D}}, \tag{3.69}$$

$$Z = \frac{D(V^2/u) - k(V^2/u)^{(3/2)}}{2(V^2/u) - D\frac{\mathrm{d}(V^2/u)}{\mathrm{d}D}}, \tag{3.70}$$

and B is a constant of integration.

3.4.5 $F = ku$

The master equation to be reduced is of the form

$$4uVD_{uu} + 8uV_uD_u - 4uDV_{uu} - V^2ku = 0. \tag{3.71}$$

Generator $X_{2,4}$

We use the associated generator

$$X_{2,4} = \partial_u, \tag{3.72}$$

so that we can obtain other forms of the potentials without having to make any restrictions on how the potentials relate initially. We obtain the invariants of the generator above from its Lagrange's system after taking its first prolongation. The invariants become

$$p = D, \tag{3.73a}$$

$$q(p) = V, \tag{3.73b}$$

$$r(p) = D', \tag{3.73c}$$

$$s(p) = V'. \tag{3.73d}$$

We only use p, q and r for our purposes. Invoking these differential invariants, (3.71) transforms to

$$qrr_p + 2r^2 q_p - p\left(q_{pp}r^2 + q_p r_p r\right) - Aq^2 = 0, \tag{3.74}$$

which can be written as

$$r_p\left(q - pq_p\right) + r\left(2q_p - pq_{pp}\right) - r^{-1}Aq^2 = 0, \tag{3.75}$$

where $A = \frac{k}{4}$. A closer inspection of (3.75) reveals that it is indeed a Bernoulli equation of the form

$$r_p + P(p,q)r - r^{-1}Q(p,q) = 0, \tag{3.76}$$

with

$$P(p,q) = \frac{2q_p - pq_{pp}}{q - pq_p}, \qquad Q(p,q) = \frac{Aq^2}{q - pq_p},$$

The solution to (3.76) becomes

$$r = \pm\sqrt{2e^{-2\int P(p,q)\mathrm{d}p}\int Q(p,q)e^{2\int P(p,q)\mathrm{d}p}\mathrm{d}p}, \tag{3.77}$$

or

$$D' = \pm\sqrt{2e^{-2\int P(p,q)\mathrm{d}p}\int Q(p,q)e^{2\int P(p,q)\mathrm{d}p}\mathrm{d}p}. \tag{3.78}$$

On integrating both sides we have

$$\int \frac{\mathrm{d}D}{\pm\sqrt{2e^{-2\int P(D,V)\mathrm{d}D}\int Q(D,V)e^{2\int P(D,V)\mathrm{d}D}\mathrm{d}D}} = u + C, \tag{3.79}$$

where

$$P(D,V) = \frac{2(\mathrm{d}V/\mathrm{d}D) - D(\mathrm{d}^2V/\mathrm{d}D^2)}{V - D(\mathrm{d}V/\mathrm{d}D)}, \tag{3.80}$$

$$Q(D,V) = \frac{AV^2}{V - D(\mathrm{d}V/\mathrm{d}D)}, \tag{3.81}$$

and C is a constant of integration.

We see that without prescribing any restriction on the relationship between the gravitational potentials, we can explicitly give the exact form of the potentials if the function $F(u)$ is linear in u.

Generator X_3

This is the second symmetry associated with $F(u) = ku$. The Lagrange's system arising from taking the second extension of

$$X_3 = -u\partial_u + 2V\partial_V, \tag{3.82}$$

becomes

$$\frac{\mathrm{d}u}{-u} = \frac{\mathrm{d}D}{0} = \frac{\mathrm{d}V}{2V} = \frac{\mathrm{d}D'}{D'} = \frac{\mathrm{d}V'}{3V'} = \frac{\mathrm{d}D''}{2D''} = \frac{\mathrm{d}V''}{4V''}. \tag{3.83}$$

This gives the invariants of X_3 as

$$p = D, \tag{3.84a}$$

$$q(p) = \sqrt{Vu}, \tag{3.84b}$$

$$r(p) = uD', \tag{3.84c}$$

$$s(p) = \sqrt[3]{V'}\sqrt{u}, \tag{3.84d}$$

$$t(p) = \sqrt[4]{V''}\sqrt{u^{-3}}, \tag{3.84e}$$

$$w(p) = uD''. \tag{3.84f}$$

Invoking the differential invariants of p, q and r only, the equation (3.7) reduces to

$$q_{pp}\frac{r^2}{q} + q_p^2\frac{r^2}{q^2} + q_p\left(\frac{r_p r}{q} - 3\frac{r}{q} - \frac{2r^2}{pq}\right) + \frac{1}{8p}\left(20r + Kq^2 - 4rr_p\right) + 2 = 0, \tag{3.85}$$

or

$$rr_p\left(\frac{q_p}{q} - \frac{1}{2p}\right) + r^2\left(\frac{q_{pp}}{q} + \frac{q_p^2}{q^2} - 2\frac{q_p}{pq}\right) + r\left(\frac{5}{2p} - \frac{4q_p}{q}\right) + \frac{Kq^2}{8p} + 2 = 0. \tag{3.86}$$

By invoking the full set of the differential invariants, the equation (3.7) reduces to

$$r_p - \frac{1}{r}\left(\frac{pt^4}{q^2} + \frac{kq^2}{4}\right) + 2\frac{s^3}{q^2} - 1 = 0. \tag{3.87}$$

Equations (3.85)–(3.87) are the reduced forms of (3.7). These are highly nonlinear and as such we cannot reduce them to quadrature to provide exact expressions for the gravitational potentials.

3.4.6 $F(u) = k(c_3u^2 + c_4u + c_5)^{-5/2}$

In this case, the master equation takes the form

$$4uVD_{uu} + 8uV_uD_u - 4uDV_{uu} - V^2k(c_3u^2 + c_4u + c_5)^{-5/2} = 0. \tag{3.88}$$

The second prolongation of the associated generator

$$X_{2,5} = \left(c_3u^2 + c_4u + c_5\right)\partial_u + \left(c_3u + \frac{c_4}{2}\right)\partial_V, \tag{3.89}$$

gives its corresponding Lagrange's system from which the partial set of invariants become

$$p = D, \tag{3.90a}$$

$$q(p) = \frac{V}{\sqrt{c_3u^2 + c_4u + c_5}}, \tag{3.90b}$$

$$r(p) = V\sqrt{D'}, \tag{3.90c}$$

$$s(p) = V'\sqrt{c_3u^2 + c_4u + c_5} - q(p)uc_3, \tag{3.90d}$$

$$t(p) = V\sqrt[3]{V''}. \tag{3.90e}$$

By using the invariants above, our master equation (3.88) is transformed to

$$r_p - \frac{1}{r^2}\left[\frac{t^2p}{q^2} + \frac{q^3c_6}{4}\right] + \left(\frac{2s}{q} - c_4\right) = 0. \tag{3.91}$$

Again we have shown that the symmetry can reduce our master equation to a first order differential equation. However, we could not obtain explicit solutions to the reduced form of our master equation.

3.4.7 $F(u) = uc_6(c_3u^2 + c_4u + c_5)^{-5/2}\exp\left[2c_2T(u)\right]$

In this section, we consider the most general form of $F(u)$ that gives rise to the symmetry X_2. In this case, our master equation becomes

$$4uVD_{uu} + 8uV_uD_u - 4uDV_{uu} - \frac{V^2uc_6}{(c_3u^2 + c_4u + c_5)^{5/2}}\exp\left[2c_2T(u)\right] = 0, \tag{3.92}$$

where

$$T(u) = \frac{1}{\sqrt{-c_3 + 4c_2c_4}}\arctan\left[\frac{c_3 + 2c_4u}{\sqrt{-c_3 + 4c_2c_4}}\right].$$

When we take the second prolongation of the associated symmetry

$$X_2 = c_1 X_1 + \left(c_3 u^2 + c_4 u + c_5\right) \partial_u + \left(c_2 + c_3 u + \frac{c_4}{2}\right) V \partial_V, \tag{3.93}$$

we obtain the corresponding Lagrange's system from which the partial set of invariants become

$$p = (2c_1 - c_3)T(u) - \ln D, \tag{3.94a}$$

$$q(p) = \ln(c_2 + c_3 u + c_4 u^2)^{\frac{1}{2}} - c_3 T(u) - \ln V, \tag{3.94b}$$

$$r(p) = (2c_1 - c_3)T(u) - \ln(c_2 + c_3 u + c_4 u^2) - \ln D', \tag{3.94c}$$

$$s(p) = V'(c_2 + c_3 u + c_4 u^2)^{\frac{1}{2}} \exp\left[c_3 T(u)\right] - c_4 u \exp\left[-q\right], \tag{3.94d}$$

$$t(p) = V''(c_2 + c_3 u + c_4 u^2)^{\frac{3}{2}} \exp\left[c_3 T(u)\right]. \tag{3.94e}$$

Using Mathematica (Wolfram 2008) and making use of the invariants above, we reduce (3.92) to

$$r_p(p) = \frac{\exp(r)\left[4c_1 - 6c_3 - c_6 \exp(r - q) + 8s \exp(q) - 4t \exp(-p + q + r)\right]}{\exp(r)\left(4c_1 - 2c_3\right) - 4\exp(p)}. \tag{3.95}$$

Equation (3.95) is a highly nonlinear first order differential equation which cannot be reduced any further. It is also difficult to integrate (3.95) and demonstrate an explicit solution.

3.5 Charged Deng solutions

Deng (1989) proposed a general algorithm, which can be applied indefinitely, by alternating between choices of D and V for uncharged matter ($F(u) = 0$). This was possible as the equation could be treated as linear in D or V. He reproduced several classes of solutions for uncharged shear-free heat conducting fluids that were initially obtained by Bergmann

(1981), Maiti (1982), Modak (1984) and Sanyal and Ray (1984) as well as generating new solutions. The Deng approach is powerful as all known uncharged models with heat flux can be regained from this general class of solutions. In the general case of $F(u) \neq 0$, if we choose D, (3.7) is a nonlinear equation in V and is difficult to solve in general. However, if we choose forms for V, the resulting differential equation in D is linear and, in principle, can be solved.

We illustrate this approach by taking one of Deng's (1989) seed solutions for V. We utilise $V = au + b$. In this case, we can completely solve (3.7) for D with $F(u)$ arbitrary. To link these results with those obtained from the symmetry analysis, we can also derive solutions for D corresponding to the different group-invariant forms of $F(u)$. All these results are contained in Table 3.1. Note that these results are charged generalisations of the Deng (1989) solutions with $V = au + b$. When the charge vanishes we regain $D = \frac{cu+d}{au+b}$ and so

$$\mathrm{d}s^2 = -\left(\frac{cu+d}{au+b}\right)^2 \mathrm{d}t^2 + (au+b)^{-2}\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right]. \qquad (3.96)$$

The metric (3.96) is the most general shear-free spherically symmetric form that is conformally flat, and was obtained by Modak (1984) and Sanyal and Ray (1984) independently. Thus we have obtained a new family of charged models with heat flux that have vanishing Weyl tensor when the electric field vanishes.

This approach can be continued for different chosen forms of V. As we only need to solve a linear equation in D, the solution is usually obtained using standard techniques.

3.6 Discussion

We have obtained new exact solutions to the Einstein-Maxwell system of charged relativistic fluids in the presence of heat flux. Solutions to this highly nonlinear system were obtained by essentially solving the generalised pressure isotropy condition. A suitable transformation reduced the master equation to a second order nonlinear differential equation. The Lie symmetry generators for this master equation were found. Importantly, the first Lie generator does not depend on the electromagnetic field. The second Lie generator arose because of specific forms of the electric field; the electric charge for the Lie generator was found explicitly in general. In some cases additional symmetries are possible depending on the specific forms of the charge $F(u)$; these are identified in Table 3.1. Solutions of the Einstein-Maxwell system were found corresponding to particular Lie symmetry generators. In the case of arbitrary $F(u)$, we were able to give an explicit relationship between the metric functions D and V (via W). For any chosen form of W, one could find D and V explicitly. This approach is a generalisation of the Msomi *et al* (2011) method for the uncharged case. We believe that these results are new and have not been published before.

We also modified the method of Deng (1989) to obtain two new families of charged heat conducting relativistic fluids. Table 3.2 contains charged generalisations of shear-free conformally flat models which include the presence of the electric field. This new family of solutions to the Einstein-Maxwell system is characterised geometrically by the infinitesimal Lie symmetry generator X_1. What is particularly remarkable about these new families of solutions is that they can be determined for arbitrary charge. Thus once a physically reasonable or observed charge is determined, the spacetime can be generated immediately.

Table 3.1: Symmetries associated with different forms of $F(u)$

Symmetry generator	Form of $F(u)$	Extra symmetries
$X_{2,1} = V\partial_V$	0	$X_3 = \partial_u$, $X_4 = u\partial_u$, $X_5 = D\partial_D$, $X_6 = u^2\partial_u + uV\partial_V$
$X_{2,2} = u^2\partial_u + uV\partial_V$	ku^{-4}	$X_3 = \partial_u + 3V\partial_V$
$X_{2,3} = u\partial_u + \frac{1}{2}V\partial_V$	$ku^{-\frac{3}{2}}$	None
$X_{2,4} = \partial_u$	ku	$X_3 = -u\partial_u + 2V\partial_V$
$X_{2,5} = (c_3u^2 + c_4u + c_5)\,\partial_u + \left(c_3u + \frac{c_4}{2}\right)\partial_V$	$k(c_3u^2 + c_4u + c_5)^{-5/2}$	None

Table 3.2: $V = au + b$

Symmetry generator	$F(u)$	$D(u)$
X_1	Arbitrary	$\frac{cu+d}{au+b} + \int^u \int^t \frac{as+b}{(at+b)^2} \frac{F(s)}{4s} \times \left(b^2 + 2abs + a^2s^2\right) \mathrm{d}s \ \mathrm{d}t$
$X_{2,4}$	ku	$\frac{cu+d}{au+b} + \frac{ku^2}{8} \left(\frac{a^2u^2+4abu+6b^2}{au+b}\right)$
$X_{2,3}$	$ku^{-3/2}$	$\frac{cu+d}{au+b} + \frac{k}{3\sqrt{u}} \left(\frac{a^2u^2-6abu+b^2}{au+b}\right)$
$X_{2,2}$	ku^{-4}	$\frac{cu+d}{au+b} + \frac{k}{48u^3} \left(\frac{6a^2u^2+4abu+b^2}{au+b}\right)$
$X_{2,5}$	$ku\left(c_3u^2 + c_4u + c_5\right)^{-5/2}$	$\frac{cu+d}{au+b} + \frac{k(au+b)}{3\sqrt{c_3u^2+c_4u+c_5}} + \frac{8k\sqrt{c_3u^2+c_4u+c_5}(b^2c_3-abc_4+a^2c_5)}{3(au+b)(c_4^2-4c_3c_5)}$

Chapter 4

Higher dimensional charged shear-free relativistic models with heat flux

4.1 Introduction

In this chapter, we study charged shear-free spherically symmetric gravitating fluids defined on an $(n+2)-$dimensional manifold; this is intended to generalize the study made by Nyonyi *et al* (2013b). The idea of higher dimensions stems from the earlier attempt of Kaluza (1921) and Klein (1926) who were motivated by the desire to unify the fundamental forces of electromagnetism and Einstein gravity by introducing a compact fifth dimension. The discourse of higher dimensional models hibernated for over four decades and it was not until the early 1960's that the early developments to what we have come to know as String Theory came into existence. This theory which requires a higher dimensional framework was initially sought to explain the strong nuclear force but its peculiar properties made it a good candidate for studying quantum gravity with a hope of obtaining a unifying grand theory. In addition, studying models in higher dimensions provides a platform to understand the nature of the early universe. It is believed that the universe, in it's earlier epoch was dense and hot (a scenario better explained in higher dimensions), and as a result of expansion the extra dimensions have compactified to produce the present four dimensional universe (Chodos and Detweiler 1980).

The model we study in this chapter is for a charged higher dimensional shear-free gravitating fluid in the presence of heat flow, and this model is very important in studying both cosmological and astrophysical processes. Therefore providing exact solutions to the Einstein-Maxwell system is a vital approach in this respect. This is well documented in Krasinski's (1997) monograph where he points out the significance of these solutions in understanding the appearance of singularities, structure formation, gravitational collapse and other relativistic stellar processes. Incorporating heat flow and charge in our model provides us with a platform for building radiating and gravitating models. The intricacies of the model we study are simplified by considering the shear-free condition. In this way the derived generalized pressure isotropy condition reduces to an equation containing two dependent metric functions, an expression that is much easier to study and solve. Not much work has been done on shearing models due to the difficulty in providing solutions to the resulting highly nonlinear field equations.

The study of relativistic stars that emit null radiation in the form of radial heat flow, as established by Santos (1985), requires a nonzero heat flux emanating from the interior spacetime to match with the pressure at the boundary with the exterior Vaidya spacetime. This was extended by Maharaj *et al* (2012) to the generalized Vaidya spacetime superposing a null fluid and a string fluid in the exterior energy momentum tensor. This junction condition is also applicable to relativistic models in higher dimensions (Bhui *et al* 1995). Several models in the higher dimensional setting have been studied over the years with emphasis on understanding gravitational collapse and appearance of naked singularities (Ghosh and Beesham 2001, Ghosh and Dadhich 2001, Joshi *et al* 2002, Banerjee *et al* 2003). Much of this study is summarised in a systematic manner by Goswami and Joshi (2004) in their study of higher dimensional spherically symmetric dust collapse; they showed that both black holes and naked singularities would develop as end states depending on the initial data from which the collapse emanates. These studies expound the conditions under which naked singularities may occur in gravitational collapse. However, most of the papers mentioned above are numerically inclined. The existence of an analytic solution provides a channel to test the accuracy and reliability of numerical solutions. Several attempts have been made over the years to obtain exact solutions in higher dimensions. Bhui *et al* (1995) derived the defining Einstein field equations in higher dimensions and used them to study the non-adiabatic gravitational collapse. Banerjee and Chatterjee (2005) provided conditions under which a spherical heat conducting fluid in higher dimensions collapses without the appearance of the horizon. A more systematic approach using group theoretic techniques was adopted by Msomi *et al* (2012) to study the same model. They used the Lie analysis of differential equations to generate explicit solutions to the defining pressure isotropy condition. Other studies in this context include the treatment of Ray *et al* (2006) who established the existence of an electromagnetic mass distribution corresponding to charged dust in higher dimensions. Hackmann *et al* (2008) provided a comprehensive catalogue of analytical solutions of the geodesic equation of massive test particles in higher dimensions in a variety of well known spacetimes.

In our study, we consider the general framework of a shear-free higher dimensional charged, heat conducting fluid without putting any restrictions on the dimensions adopted. By applying a group theoretic approach with Lie symmetries, we study the dynamics of the charged shear-free heat conducting model in higher dimensions. We present the Einstein-Maxwell field equations and the generalized pressure isotropy condition in §4.2. A detailed process of obtaining the symmetry generators follows in §4.3. The first symmetry obtained is used to provide new solutions for any given form of charge in §4.4.1. The gravitational potentials can be found explicitly. In §4.4.2, we summarise the results obtained by using the rest of the symmetries in tabular form. The cases where reduction to quadrature are difficult to perform arise from the nonlinearity of the resultant equations. The results of this chapter generalize earlier studies on the four-dimensional manifold. We consider the heat transport in higher dimensions in §4.5 and generate forms of the temperature in the Eckart theory and the causal theory. The temperature is integrated for an example in §4.6. Some concluding remarks follow in §4.7.

4.2 The model

We consider a shear-free spherically symmetric gravitating fluid, in the presence of an electromagnetic field, defined on an $(n+2)-$dimensional manifold. Then the line element in the extended Schwarzschild coordinates $(t, r, \theta_1, \ldots, \theta_n)$ becomes

$$\mathrm{d}s^2 = -D^2\mathrm{d}t^2 + \frac{1}{V^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}X_n^2\right), \tag{4.1}$$

where $n \geq 2$. The gravitational potential components D and V are functions of r and t and

$$\mathrm{d}X_n^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\theta_2^2 + \cdots + \sin^2\theta_1\sin^2\theta_2\ldots\sin^2\theta_{n-1}\mathrm{d}\theta_n^2. \tag{4.2}$$

For a charged interior matter distribution, the energy momentum tensor is of the form

$$T_{ab} = (\rho + p)U_aU_b + pg_{ab} + q_aU_b + q_bU_a + E_{ab}, \tag{4.3}$$

where ρ, p, $q_a = (0, q, 0, \cdots, 0)$ and E_{ab} are the energy density, the isotropic pressure, the $(n+2)$ heat flux vector and the electromagnetic contribution to the matter distribution respectively. The quantities above are measured relative to a unit, time-like comoving velocity vector $U^a = \left(\frac{1}{D}, 0, \cdots, 0\right)$.

The nontrivial Einstein-Maxwell equations for a charged gravitating relativistic fluid in comoving coordinates, emanating from equations (4.1) and (4.3), are

$$\rho = \frac{n(n+1)V_t^2}{2D^2V^2} - \frac{n(n+1)VV_r^2}{2} + nVV_{rr} + \frac{n^2VV_r}{r} - \frac{V^2}{2D^2}\phi_r^2, \tag{4.4a}$$

$$\begin{aligned} p = &-\frac{nD_rVV_r}{D} + \frac{nD_rV^2}{rD} + \frac{n(n-1)V_r^2}{2} - \frac{n(n-1)VV_r}{r} \\ &+ \frac{nV_{tt}}{D^2V} - \frac{n(n+3)V_t^2}{2D^2V^2} - \frac{nD_tV_t}{D^3V} + \frac{V^2}{2D^2}\phi_r^2, \end{aligned} \tag{4.4b}$$

$$\begin{aligned} p = &\frac{D_{rr}V^2}{D} - (n-1)VV_{rr} + \frac{n(n-1)V_r^2}{2} + \frac{(n-1)D_rV^2}{rD} - \frac{(n-1)^2VV_r}{r} \\ &- \frac{(n-2)D_rVV_r}{D} + \frac{nV_{tt}}{D^2V} - \frac{n(n+3)V_t^2}{2D^2V^2} - \frac{nD_tV_t}{D^3V} - \frac{V^2}{2D^2}\phi_r^2, \end{aligned} \tag{4.4c}$$

$$q = -\frac{nVV_{tr}}{D} + \frac{nV_rV_t}{D} + \frac{nD_rVV_t}{D^2}, \tag{4.4d}$$

$$0 = -\frac{V^2}{D^2}\left(\phi_{rt} - \left((n-1)\frac{V_t}{V} + \frac{D_t}{D}\right)\phi_r\right), \tag{4.4e}$$

$$\sigma = \frac{V^2}{D}\left(\phi_{rr} + \phi_r\left(\frac{n}{r} - \frac{D_r}{D} - (n-1)\frac{V_r}{V}\right)\right), \tag{4.4f}$$

where σ is the proper charge density. It is important to highlight that the system (4.4) contains the results of Nyonyi *et al* (2013b) when $n = 2$.

Integrating (4.4e) gives ϕ_r in the form

$$\phi_r = V^{(n-1)}DF(r), \tag{4.5}$$

where $F(r)$ is an arbitrary function. By equating (4.4b) with (4.4c), and taking (4.5) into consideration, we obtain the higher dimensional generalized pressure isotropy condition

$$\frac{D_{rr}V^2}{D} - (n-1)VV_{rr} - \frac{D_rV^2}{rD} + 2\frac{D_rVV_r}{D} - \frac{(n-1)VV_r}{r} - \frac{V^{(n-1)}}{D}F(r) = 0. \tag{4.6}$$

A transformation $u = r^2$ unravels the generalized pressure isotropy condition (4.6) to the form

$$4uVD_{uu} + 8uD_uV_u - 4u(n-1)DV_{uu} - V^nF(u) = 0, \tag{4.7}$$

where the function F now depends on u. Knowledge of $F(u)$, V and D solves the Einstein-Maxwell system (4.4). Therefore we seek to obtain solutions to the equation (4.7) using Lie analysis, a method that was effectively employed in Chapter 3. It is worth noting that (4.7) reduces to the pressure isotropy condition of a four dimensional charged model with heat flow when $n = 2$:

$$4uVD_{uu} + 8uD_uV_u - 4uDV_{uu} - V^2F(u) = 0, \tag{4.8}$$

which was studied by Nyonyi *et al* (2013b). Equation (4.7) becomes

$$VD_{uu} + 2D_uV_u - (n-1)DV_{uu} = 0, \tag{4.9}$$

in the absence of charge in higher dimensions. This case was comprehensively studied by Msomi *et al* (2012) and Nyonyi *et al* (2013a).

4.3 Analysis of the problem

The nature of our master equation enables us to treat it as a second order nonlinear ordinary differential equation in u even though both the potential functions D and V are functions of u and t. We introduce the temporal component by taking the derived constants of integration as functions of t.

We seek to obtain an infinitesimal generator of the form

$$X = \xi\partial_u + \eta^1\partial_D + \eta^2\partial_V. \tag{4.10}$$

that leaves equation (4.7) (hereinafter labelled as $E = 0$ for simplicity) invariant. In order to do this we require

$$X^{[2]}E|_{E=0} = 0, \tag{4.11}$$

where $X^{[2]}$ is the second prolongation of X required to transform the derivatives in (4.7). See Olver (1986, 1995), Bluman and Kumei (1989), Bluman and Anco (2002) and Cantwell (2002) for further details.

This gives the infinitesimals

$$\xi = C^0(u), \tag{4.12}$$

$$\eta^1 = c_1 D, \tag{4.13}$$

$$\eta^2 = \left(\frac{c_1}{n-1} + c_2 + \frac{1}{2}C^0_u\right)V, \tag{4.14}$$

with a condition on $F(u)$ given by

$$2u(n-1)VDC^0_{uuu} + V^n\left[F\left(-\frac{C^0}{u} + \frac{(n-1)+4}{2}C^0_u + (n-1)c_2\right) + C^0F'\right] = 0. \tag{4.15}$$

When F is arbitrary, equation (4.15) is satisfied if both

$$C^0(u) = 0, \tag{4.16}$$

and

$$c_2 = 0. \tag{4.17}$$

Using (4.12)–(4.14) and (4.16)–(4.17) the infinitesimals become

$$\xi(u) = 0, \tag{4.18}$$

$$\eta^1(D) = c_1 D, \tag{4.19}$$

$$\eta^2(V) = \frac{c_1}{n-1} V. \tag{4.20}$$

Following (4.10), we observe that the infinitesimal generator

$$X_1 = D\partial_D + \frac{V}{n-1}\partial_V, \tag{4.21}$$

is the only symmetry when $F(u)$ is arbitrary. X_1 is the generalized symmetry obtained without placing any restriction on the form of charge. This symmetry depends on the dimension n, and reduces to the four dimensional case when $n = 2$.

By taking (4.15) to be a restriction on F we obtain two conditions

$$C^0_{uuu} = 0, \tag{4.22}$$

and

$$F\left(-\frac{C^0}{u} + \frac{n+3}{2}C^0_u + (n-1)c_2\right) + C^0 F' = 0, \tag{4.23}$$

Solving equation (4.22) gives

$$C^0(u) = c_3 u^2 + c_4 u + c_5. \tag{4.24}$$

We then solve (4.23) using (4.24) to obtain

$$F(u) = \frac{uc_6}{(c_3u^2 + c_4u + c_5)^{(n+3)/2}} \exp\left[\frac{2(n-1)c_2}{\sqrt{-c_4 + 4c_3c_5}} \arctan\left(\frac{c_4 + 2c_3u}{\sqrt{-c_4 + 4c_3c_5}}\right)\right], \tag{4.25}$$

as the definition of $F(u)$. Note that c_6 is a constant of integration. Inferring from (4.24)–(4.25), we observe that (4.7) admits another symmetry

$$X_2 = c_1X_1 + \left(c_3u^2 + c_4u + c_5\right)\partial_u + \left(c_2 + c_3u + \frac{c_4}{2}\right)V\partial_V, \tag{4.26}$$

dependant on the form of $F(u)$.

Equation (4.25) is the most generalized higher dimensional form of $F(u)$ for which X_2 is the corresponding symmetry. As previously noted in Chapter 3, simpler forms of $F(u)$ can be obtained by inferring from the constants existing in (4.25) that relate to the symmetry X_2. In addition the simpler higher dimensional forms of $F(u)$ may admit extra symmetries. This is summarised in Table 4.1. When $c_2 = 0$ (with all other constants nonzero), we obtain an extra symmetry as indicated in the last row of Table 4.1. Msomi *et al* (2012) comprehensively studied the higher dimensional uncharged case ($F = 0$). It is important to further highlight that verification of the generator X_6 as a symmetry to our master equation for the uncharged case revealed that it is independent of the dimension n. This serves to correct the result published by Msomi *et al* (2012).

4.4 New solutions using symmetries

It is well documented that symmetries of differential equations are used to reduce the order of the equation with the hope of obtaining simplified forms which then can be solved. In this section, we intend to give a detailed description of the reduction of our master equation (4.7) using X_1. For the remaining symmetries, we will summarise the results in tabular form.

4.4.1 Arbitrary F

Since $F(u)$ is arbitrary, our master equation to be reduced becomes

$$VD_{uu} + 2V_u D_u - (n-1)DV_{uu} - V^n \frac{F(u)}{4u} = 0. \tag{4.27}$$

The associated Lagrange's system for the first extension of

$$X_1 = D\partial_D + \frac{V}{n-1}\partial_V \tag{4.28}$$

is given by

$$\frac{\mathrm{d}u}{0} = \frac{\mathrm{d}D}{D} = \frac{\mathrm{d}V}{V/(n-1)} = \frac{\mathrm{d}D'}{D'} = \frac{\mathrm{d}V'}{V'/(n-1)}. \tag{4.29}$$

The invariants are then found to be

$$p = u, \tag{4.30a}$$

$$q(p) = \frac{V}{D^{1/(n-1)}}, \tag{4.30b}$$

$$r(p) = \frac{D'}{D}, \tag{4.30c}$$

$$s(p) = \frac{V'}{D^{1/(n-1)}}. \tag{4.30d}$$

While this is the full set of first order invariants obtained, we only use p, q and r. Substituting these invariants into (4.27) yields

$$q'' = -\frac{F(p)}{(n-1)4p}q^n + \frac{n}{(n-1)^2}qr^2. \tag{4.31}$$

We can treat this equation as a definition for r and obtain

$$r = \pm\sqrt{\left[\frac{q''}{q} + \frac{q^{n-1}}{n-1}\frac{F(p)}{4p}\right]\frac{(n-1)^2}{n}}. \tag{4.32}$$

Reverting to the original variables produces

$$\frac{D'}{D} = \pm\sqrt{\frac{(n-1)^2}{n}\left[\frac{q''}{q} + \frac{q^{n-1}}{n-1}\frac{F(p)}{4p}\right]}. \tag{4.33}$$

We can now integrate this equation to obtain

$$D = \exp\left(\pm C\int\sqrt{\frac{(n-1)^2}{n}\left[\frac{W''}{W} + \frac{W^{n-1}}{n-1}\frac{F(u)}{4u}\right]}\quad \mathrm{d}u\right), \tag{4.34}$$

where C is a constant of integration.

Equation (4.34), illustrates that whenever we are given any generalized ratio of the gravitational potentials $W = \frac{V}{D^{1/(n-1)}}$ and an arbitrary function $F(u)$ indicative of charge, we can explicitly obtain the exact expression for the potentials. We have thus obtained a generating function approach for solving (4.34), something that, to the best of our knowledge has not been obtained before. When we set $n = 2$, we regain

$$D = \exp\left[\pm C\int\sqrt{\frac{W''}{2W} + W\frac{F(u)}{8u}}\quad \mathrm{d}u\right]. \tag{4.35}$$

This is the result for the four-dimensional model studied by Nyonyi *et al* (2013b). Also, when we set $F(u) = 0$ in (4.34), we obtain

$$D = \exp\left[\pm C\int\sqrt{\frac{(n-1)^2}{n}\frac{W''}{W}}\quad \mathrm{d}u\right]. \tag{4.36}$$

This is the uncharged solution of Msomi *et al* (2012). The case for $n = 2$ in (4.36) was obtained by Msomi *et al* (2011).

4.4.2 Summary using other symmetries

The simpler forms of $F(u)$ derived from the generalized form (4.25) obtained in Table 4.1 are used to reduce the order of the equation with the hope of obtaining new solutions. Table 4.2 summarises the symmetries (obtained with their corresponding form of $F(u)$) that we have used to provide solutions to the respective master equations. In all cases, we obtained the first prolongation of the symmetries, that we use to evaluate the corresponding invariants. This is highlighted in the third column of Table 4.2. Using a partial set of invariants, we are able to demonstrate the existence of exact solutions to the respective master equations for the specified simpler forms of $F(u)$ (as illustrated in the fourth column of Table 4.2). However, not all symmetries we obtained were able to provide exact solutions to the master equation. For these symmetries, we were able to reduce the order of the equation. The reduced forms we obtained were difficult to reduce to quadrature. These are summarised in Table 4.3.

4.5 Heat transport

We study the causal heat transport equation of Maxwell-Cattaneo type without viscous stress and rotation. This is given by

$$\tau {h_a}^b \dot{q}_a + q_a = -\kappa \left({h_a}^b T_{;b} + T \dot{u}_a \right), \tag{4.37}$$

where $\tau(\geq 0)$ is the relaxation time associated with heat transport, q_a is the heat flux, $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor, T is the temperature, $\kappa(\geq 0)$ is the coefficient of thermal conductivity, and u_a is the velocity vector. To solve (4.37) requires knowledge of τ

and κ. The coefficient of thermal conductivity κ is obtained from the interaction between a radiating fluid and matter (Weinberg 1971). Following the treatment of Martínez (1996), we now take

$$\kappa = \gamma T^3 \tau_c, \tag{4.38}$$

where γ is a constant and τ_c is the mean collision time. On physical grounds we can assume that

$$\tau = \left(\frac{\beta\gamma}{\alpha}\right)\tau_c = \beta T^{-\sigma}, \tag{4.39}$$

with α, β and σ are positive constants. Then for the metric (4.1), the transport equation (4.37) becomes

$$\beta T^{-\sigma}\left(\frac{q}{V}\right)_{,t} + D\left(\frac{q}{V}\right) = \alpha V T^{3-\sigma}(DT)_{,r}, \tag{4.40}$$

or equivalently

$$n\beta T^{-\sigma}\left(V\left(\frac{V_t}{DV}\right)_{,r}\right)_{,t} + nDV\left(\frac{V_t}{DV}\right)_{,r} = \alpha V T^{3-\sigma}(DT)_{,r}. \tag{4.41}$$

Equation (4.41) describes the higher dimensional spherically symmetric causal transport process. We observe that the temperature of the fluid is affected by the dimension n. Note that (4.41) reduces to the four dimensional case studied by Govinder and Govender (2001) when $n = 2$.

The case $\beta = 0$ corresponds to the noncausal Eckart theory. The explicit noncausal expressions for the temperature are

$$
\begin{aligned}
\ln(DT) &= -\frac{1}{\alpha}\int\left(\frac{q}{V^2}\right)\mathrm{d}r + G(t), \qquad \sigma = 4 \\
&= \frac{n}{\alpha}\left(\frac{V_t}{DV}\right) + G(t),
\end{aligned} \tag{4.42}
$$

$$
\begin{aligned}
(DT)^{4-\sigma} &= \frac{\sigma-4}{\alpha}\int D^{4-\sigma}\left(\frac{q}{V^2}\right)\mathrm{d}r + G(t), \qquad \sigma \neq 4 \\
&= \frac{n(4-\sigma)}{\alpha}\int D^{4-\sigma}\left(\frac{V_t}{DV}\right)_{,r}\mathrm{d}r + G(t),
\end{aligned} \tag{4.43}
$$

where $G(t)$ is an arbitrary constant of integration.

The mean collision time is constant when $\sigma = 0$. In this case equation (4.41) simplifies substantially. We obtain

$$
\begin{aligned}
(DT)^4 &= -\frac{4}{\alpha}\left[\beta\int\frac{D^3}{V}\left(\frac{q}{V}\right)_{,r}\mathrm{d}r + \int D^4\left(\frac{q}{V}\right)\mathrm{d}r\right] + G(t) \\
&= \frac{4n}{\alpha}\left[\beta\int\frac{D^3}{V}\left(V\left(\frac{V_t}{DV}\right)_{,r}\right)_{,t}\mathrm{d}r + \int D^4\left(\frac{V_t}{DV}\right)_{,r}\mathrm{d}r\right] + G(t).
\end{aligned} \tag{4.44}
$$

The other case for which we can find the causal temperature explicitly is $\sigma = 4$. The transport equation (4.41) can be solved to give

$$
\begin{aligned}
(DT)^4 &= \exp\left(-\frac{4q}{\alpha V}\right)\left[-\frac{4\beta}{\alpha}\int D^3\left(\frac{q}{V}\right)_{,t}\exp\left(\frac{4q}{\alpha V}\right)\mathrm{d}r + G(t)\right] \\
&= \exp\left(\frac{4n}{\alpha}V\left(\frac{V_t}{DV}\right)_{,r}\right) \\
&\quad \times\left[\frac{4n\beta}{\alpha}\int D^3\left(V\left(\frac{V_t}{DV}\right)_{,r}\right)_{,t}\exp\left(-\frac{4n}{\alpha}V\left(\frac{V_t}{DV}\right)_{,r}\right)\mathrm{d}r + G(t)\right]
\end{aligned} \tag{4.45}
$$

We point out that (4.42)–(4.45) for the heat transport equations ($n \geq 2$) generalize the results obtained by Govinder and Govender (2001) when $n = 2$.

4.6 Example

To study the temperature profiles, it is necessary to integrate (4.42)–(4.45) and express T in terms of simple functions. We seek to illustrate this property by choosing a suitable example. By considering the class of solutions represented by (4.34), we consider the case when the generalized ratio of gravitational potential $W = u$, the constant of integration $C = 1$ and the arbitrary form of charge $F(u) = u^{-n}$. We obtain the gravitational potentials in the form

$$D(t,r) = Ar^{2k}, \qquad V(t,r) = r^2 \left(Ar^{2k}\right)^{1/(n-1)}, \tag{4.46}$$

where $u = r^2$, $k = \pm\frac{1}{2}\sqrt{\frac{n-1}{n}}$, and A is an arbitrary function of t. Note that V is obtained from the expression for the generalized ratio $W = V/D^{(1/(n-1))}$ for the gravitational potentials.

Following (4.42) the noncausal exact solution is

$$T = \frac{1}{Ar^{2k}} \exp\left[\frac{n}{\alpha(n-1)}\frac{A_t}{A^2r^{2k}} + G(t)\right], \tag{4.47}$$

for the case $\sigma = 4$. And when $\sigma \neq 4$, we obtain

$$\left(Ar^{2k}T\right)^{\sigma-4} = \frac{-kn(4-\sigma)}{\alpha(n-1)(3k-k\sigma+1)} A_t A^{2-\sigma} r^{(3k-k\sigma+1)} + G(t), \tag{4.48}$$

using (4.43). Also, we can obtain the causal solutions of the temperature profile. We find that

$$T^4 = \frac{4n}{\alpha(n-1)A^2}\left[\frac{\beta}{2}A^{((n-2)/(n-1))}\left(A_tA^{((1-2n)/(n-1))}\right)_t r^{-4k} - \frac{A_t}{3}r^{-2k}\right] + \frac{G(t)}{A^4r^{8k}}, \tag{4.49}$$

using (4.44) for $\sigma = 0$. The form for the causal temperature resulting from (4.45) when $\sigma = 4$ is more complicated and we omit this expression. Other choices of the generalized ratio of the gravitational potentials W may lead to forms that yield a more tractable form for T when $\sigma = 4$. Our example shows that both causal and noncausal temperatures may be found explicitly for the class of models presented in this chapter.

4.7 Discussion

We have obtained new exact solutions to the generalized Einstein-Maxwell system of charged relativistic fluids in the presence of heat flux defined on a higher dimensional manifold. Our focus was on the generalized pressure isotropy condition. We were able to transform the master equation into a second order nonlinear differential equation that generalized the four–dimensional case. This equation was analysed via a group theoretic approach. Interestingly, we were able to find a Lie symmetry without restricting the electromagnetic field. When this field was restricted, a second Lie generator arose. In addition, other specific forms of the field yielded additional symmetries; these are identified in Table 4.1. Thereafter, we attempted to find solutions of the generalized Einstein-Maxwell system corresponding to particular Lie symmetry generators. Remarkably, we were able to find solutions even in the general case of arbitrary $F(u)$. Given a function W and dimension of the manifold, we produced a formula to find D and V explicitly. These results are extensions of the Nyonyi *et al* (2013b) result for the four-dimensional case. We note that our results corrected one of the symmetries presented in Msomi *et al* (2012); the symmetry in question is independent of dimension. The solutions presented are new and have not been published before. Finally, we also obtained forms for the noncausal and causal temperatures for the heat transport equation. These solutions can be applied to both the uncharged and charged matter distributions defined on a higher dimensional manifold.

Table 4.1: Symmetries associated with different forms of $F(u)$

Symmetry generator	Form of $F(u)$	Extra symmetries
$X_{2,1} = \frac{V}{n-1}\partial_V$	0	$X_3 = \partial_u$, $X_4 = u\partial_u$, $X_5 = D\partial_D$, $X_6 = u^2\partial_u + uV\partial_V$
$X_{2,2} = u^2\partial_u + uV\partial_V$	$ku^{-(n+2)}$	$X_3 = (n-1)u\partial_u + (n+1)V\partial_V$
$X_{2,3} = u\partial_u + \frac{1}{2}V\partial_V$	$ku^{-\frac{n+1}{2}}$	None
$X_{2,4} = \partial_u$	ku	$X_3 = -(n-1)u\partial_u + 2V\partial_V$
$X_{2,5} = \left(c_3u^2 + c_4u + c_5\right)\partial_u + \left(c_3u + \frac{c_4}{2}\right)\partial_V$	$c_6(c_3u^2 + c_4u + c_5)^{-(n+3)/2}$	None

Table 4.2: Solutions from other symmetries: A, B are constants of integration, $\alpha_n = ((n+1)/(n-1))$

$F(u)$	Symmetry	Invariants	Solution
0	$\frac{V}{n-1}\partial_V$	$p = u$, $q(p) = D$, $r(p) = D'$, $s(p) = \frac{V'}{V}$	$D = (n-1) \int \left(e^{-2(V'/V)} \int D \left[\frac{\mathrm{d}(V'/V)}{\mathrm{d}u} + \left(\frac{V'}{V}\right)^2 \right] e^{2(V'/V)} \mathrm{d}u \right) \mathrm{d}u$
$ku^{-(n+2)}$	$u^2\partial_u + uV\partial_V$	$p = D$, $q(p) = \frac{V}{u}$, $r(p) = u^2 D'$, $s(p) = u\,(q(p) - V')$	$\int \left(2 \frac{2\frac{\mathrm{d}(V/u)}{\mathrm{d}D} - (n-1)D\frac{\mathrm{d}^2(V/u)}{\mathrm{d}D^2}}{(V/u)-(n-1)D\frac{\mathrm{d}(V/u)}{\mathrm{d}D}} \right)^{1/2} \sqrt{[\exp[2\frac{2\frac{\mathrm{d}(V/u)}{\mathrm{d}D} - (n-1)D\frac{\mathrm{d}^2(V/u)}{\mathrm{d}D^2}}{(V/u)-(n-1)D\frac{\mathrm{d}(V/u)}{\mathrm{d}D}}(D+A)] + \frac{(k/4)(V/u)^2}{(V/u)-(n-1)D\frac{\mathrm{d}(V/u)}{\mathrm{d}D}}]^{-1}}\ \mathrm{d}D = -\frac{1}{u} + B$
$ku^{-(n+2)}$	$(n-1)u\partial_u + (n+1)V\partial_V$	$p = D$, $q(p) = \frac{V^{(1/(n+1))}}{u^{(1/(n-1))}}$, $r(p) = uD'$, $s(p) = \sqrt{V'}u^{(-1/(n-1))}$	$\int -\frac{Y/(2[D-C])}{\exp[Y]+(VX/((n-1)Du^{\alpha n}))} = \frac{u^{\alpha n}}{\alpha_n} + A$, where $X = \frac{uD''}{D'} - \frac{kV^{(n-1)}}{4D'u^{(n+2)}}$, $Y = \frac{4(uD'-D)(D-C)}{(n-1)uD'D}$
$ku^{-\frac{(n+1)}{2}}$	$u\partial_u + \frac{1}{2}V\partial_V$	$p = D$, $q(p) = \frac{V^2}{u}$, $r(p) = uD'$, $s(p) = uV'^2$	$\int \sqrt{X}\,(\exp[X(D+A)] - Y)^{-1/2}\,\mathrm{d}D = \ln u + B$ where $X = \frac{4\frac{\mathrm{d}(V^2/u)}{\mathrm{d}D} - (n-1)D\left[2\frac{\mathrm{d}^2(V^2/u)}{\mathrm{d}D^2} - \left(\frac{\mathrm{d}(V^2/u)}{\mathrm{d}D}\right)^2\right]}{2(V^2/u)-(n-1)D\frac{\mathrm{d}(V^2/u)}{\mathrm{d}D}}$, $Y = \frac{(n-1)D(V^2/u)-k(V^2/u)^{((n+1)/2)}}{2(V^2/u)-(n-1)D\frac{\mathrm{d}(V^2/u)}{\mathrm{d}D}}$
ku	∂_u	$p = D$, $q(p) = V$, $r(p) = D'$, $s(p) = V'$	$D = \pm \int \sqrt{X}\,(Y + \exp[X(D+k)])^{-1/2}\ \mathrm{d}D = u + B$ where $X = 2\frac{2(\mathrm{d}V/\mathrm{d}D) - (n-1)D(\mathrm{d}^2V/\mathrm{d}D^2)}{V-(n-1)D(\mathrm{d}V/\mathrm{d}D)}$, $Y = \frac{2AV^n}{V-(n-1)D(\mathrm{d}V/\mathrm{d}D)}$

Table 4.3: Reductions using symmetries: $\beta(u) = c_3u^2 + c_4u + c_5$

$F(u)$	Symmetry	Invariants	Reduced equation
ku	$-(n-1)u\partial_u + 2V\partial_V$	$p = D,$ $q(p) = \sqrt{V}u^{(1/(n-1))},$ $r(p) = uD',$ $s(p) = \sqrt[n+1]{V'}u^{(1/(n-1))},$ $t(p) = \sqrt[2n]{V''}u^{(1/(n-1))}$	$r_p - \frac{1}{r}\left(\frac{(n-1)pt^4}{q^2} + \frac{kq^2}{4}\right) + 2\frac{s^{(n+1)}}{q^2} - 1 = 0$
$k(\beta(u))^{-(n+3)/2}$	$\beta(u)\partial_u + \left(c_3u + \frac{c_4}{2}\right)\partial_V$	$p = D,$ $q(p) = \frac{V}{\sqrt{\beta(u)}},$ $r(p) = V\sqrt{D'},$ $s(p) = -q(p)uc_3 + V'\sqrt{\beta(u)},$ $t(p) = V\sqrt[3]{V''}$	$r_p - \frac{1}{r^2}\left[\frac{(n-1)t^2p}{q^2} + \frac{q^{(n+1)}c_6}{4}\right] + \left(\frac{2s}{q} - c_4\right) = 0$

Chapter 5

Conclusion

In this thesis we first studied an uncharged heat conducting relativistic fluid defined on a higher dimensional manifold using the Deng (1989) algorithm. We then considered shear-free spherically symmetric gravitating fluids with charge and heat flux defined on a four dimensional manifold. Finally, we extended our study to higher dimensional manifolds generalising the results that were found in four dimensions. We utilised the Lie group theoretic approach, a powerful technique used to solve differential equations, to obtain solutions in chapters three and four.

In the first chapter, we gave a brief genesis of general relativity and its impact in describing the anomalies in Newton's theory of gravitation. A synopsis of Lie symmetries relevant to our study was highlighted. This chapter ends with a description of the importance of obtaining exact solutions to the Einstein field equations and the Einstein-Maxwell system, highlighting a number of examples relevant to our study.

In the second chapter, we obtained new generalised classes of exact solutions to the Einstein field equations for an uncharged relativistic fluid in the presence of heat flow in higher dimensions. We obtained solutions by solving the higher dimensional pressure isotropy condition

$$VD_{uu} + 2D_uV_u - (n-1)DV_{uu} = 0,$$

which is a second order nonlinear differential equation. We solved this equation by making use of Deng's (1989) algorithm and obtained three classes of solutions. The first class of solution gave rise to the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{1}{(au+b)^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}X_n^2\right),$$

which generalizes the Bergmann (1981) four dimensional metric. From the second class of solutions, we obtained the line element

$$\mathrm{d}s^2 = -\left(\frac{cu+d}{au+b}\right)^2\mathrm{d}t^2 + \frac{1}{(au+b)^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}X_n^2\right),$$

which reduced is related to the metrics of Maiti (1982), Modak (1984), and Sanyal and Ray (1984) when $n = 2$. Both new metrics do not depend on the dimension n. The third class of solution gave the compact form of the line element

$$\begin{aligned}\mathrm{d}s^2 = &-\left(\frac{cu+d}{au+b}\right)^2\mathrm{d}t^2\\ &+\left(\left[h(t) + j(t)\left(\frac{e}{ad-bc}\left(\frac{1-n}{n+1}\right)\left(\frac{cu+d}{au+b}\right)^{\frac{n+1}{n-1}} + g\right)\right](au+b)\right)^{-2}\\ &\times\left(\mathrm{d}r^2 + r^2\mathrm{d}X_n^2\right),\end{aligned}$$

which does depend on the dimension n. It contains the four dimensional Deng (1989) metric as a special case. We showed that for a particular example, the matter variables remained positive and causality was not violated. In summary, we demonstrated that Deng's novel technique is useful in higher dimensions and compliments the sophisticated approaches of Ivanov (2012) and Msomi *et al* (2010, 2011, 2012).

Chapter three dealt with the study of a charged heat conducting gravitating fluid defined on a four dimensional manifold. The pressure isotropy condition

$$4uVD_{uu} + 8uV_uD_u - 4uDV_{uu} - V^2F(u) = 0,$$

was derived from the Einstein-Maxwell system. We analysed this equation using Lie's group theoretic approach and obtained two symmetries. For an arbitrary form of charge, we obtained

$$X_1 = D\frac{\partial}{\partial D} + V\frac{\partial}{\partial V}.$$

For the form

$$F(u) = \frac{uc_6}{(c_3u^2 + c_4u + c_5)^{5/2}} \exp\left[\frac{2c_2}{\sqrt{-c_4 + 4c_3c_5}} \arctan\left(\frac{c_4 + 2c_3u}{\sqrt{-c_4 + 4c_3c_5}}\right)\right],$$

we obtained another symmetry

$$X_2 = c_1X_1 + \left(c_3u^2 + c_4u + c_5\right)\partial_u + \left(c_2 + c_3u + \frac{c_4}{2}\right)V\partial_V.$$

Special forms of $F(u)$ gave rise to other symmetries as indicated in Table 3.1. With the help of the symmetry X_1, we obtained new classes of solutions given by

$$D = \exp\left[\pm C\int\sqrt{\frac{W''}{2W} + W\frac{F(u)}{8u}}\quad \mathrm{d}u\right],$$

where $W = \frac{V}{D}$ is the ratio of the gravitational potentials. This result means that whenever we are given any ratio of the gravitational potentials and an arbitrary function $F(u)$ representing charge, we can obtain the exact expression of the potentials in closed form. We believe that this is a new result. When we set $F(u) = 0$, we regain the results of Msomi *et al* (2011).

We were able to obtain several solutions to the gravitational potentials using the other symmetries by taking special forms of $F(u)$. In some cases, we were only able to reduce the order of the master equation. Finally, we extended the Deng (1989) algorithm in Table 3.2 for the choice $V = au + b$, and obtained charged analogues of the gravitational potential D using various forms of $F(u)$ emanating from the symmetry analysis. We provided charged versions of the Maiti (1982), Modak (1984) and Sanyal and Ray (1984) metrics.

In chapter four, we were concerned with extending the study done in the previous chapter to a higher dimensional manifold. We sought to solve the generalised pressure isotropy condition

$$4uVD_{uu} + 8uD_uV_u - 4u(n-1)DV_{uu} - V^nF(u) = 0,$$

using Lie's group theoretic approach. Two generalised symmetries were obtained. The first symmetry

$$X_1 = D\partial_D + \frac{V}{n-1}\partial_V,$$

is independent of the form of charge but depends on the dimension n of the manifold. The second symmetry was generated by setting a restriction on the form of charge; for the function

$$F(u) = \frac{uc_6}{(c_3u^2 + c_4u + c_5)^{(n+3)/2}} \exp\left[\frac{2(n-1)c_2}{\sqrt{-c_4 + 4c_3c_5}} \arctan\left(\frac{c_4 + 2c_3u}{\sqrt{-c_4 + 4c_3c_5}}\right)\right],$$

we obtained the symmetry

$$X_2 = c_1X_1 + \left(c_3u^2 + c_4u + c_5\right)\partial_u + \left(c_2 + c_3u + \frac{c_4}{2}\right)V\partial_V.$$

We were also able to illustrate that generalised simpler versions of $F(u)$ give rise to other symmetries, the results of which are summarised in Table 4.1. With the help of X_1, we solved the generalized master equation to give new classes of solution of the form

$$D = \exp\left(\pm C\int\sqrt{\frac{(n-1)^2}{n}\left[\frac{W''}{W} + \frac{W^{n-1}}{n-1}\frac{F(u)}{4u}\right]}\quad \mathrm{d}u\right),$$

where $W = V/D^{(1/(n-1))}$ is the generalized ratio of gravitational potentials. This new result generalized the result we obtained in chapter three. It indicates that irrespective of dimensionality, we can explicitly obtain the exact form of the gravitational potentials whenever we are given the ratio W of gravitational potentials, for a prescribed form of the charge $F(u)$. We also demonstrated that some of the remaining symmetries provided other generalised classes of solutions as illustrated in Table 4.2. For the rest of the symmetries, we were able to reduce the order of the derived equations. This is highlighted in Table 4.3. We also studied the heat transport equation of the Maxwell-Cattaneo type in higher dimensions. We obtained the noncausal and causal temperatures for a general higher dimensional shear-free spacetime. These reduce to the Govinder and Govender (2001) four dimensional results. Using a particular example, we showed that the temperature profile can be written explicitly for the Eckart theory and the causal theory.

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