Long time Behaviour of Population Models

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Abstract

Non-negative matrices arise naturally in population models. In this thesis, we look at the theory of such matrices and we study the Perron-Frobenius type theorems regarding their spectral properties. We use these theorems to investigate the asymptotic behaviour of solutions to continuous time problems arising in population biology. In particular, we provide a description of long-time behaviour of populations depending on the nature of the associated matrix. Finally, we describe a few applications to population biology.

Preface and Declaration

The work described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, from July 2008 to November 2009 under the supervision of Professor Jacek Banasiak and Dr. Robert Willie.

The studies done in this thesis are the original work of the author and have not been submitted in whole or in parts for any degree or diploma in any tertially institution. Where use of the work of others has been made, proper citation has been done.

Signed:	Proscovia Namayanja	20 January 2010

As the candidate's supervisor, I have approved this dissertation for submission.

Signed:	Prof. J. Banasiak	20 January 2010
Signed:	Dr. R. Willie	20 January 2010

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Declaration 1 - Plagiarism

I, Proscovia Namayanja, declare that

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Introduction

Consider a population divided into n classes described by a distribution vector $\mathbf{u}(t) = (u_1(t), \cdots, u_n(t))^T$ at time t, where $u_i(t)$ is the number of individuals in the i^{th} class. Over a short interval of time dt, individuals move from class j to i at a rate a_{ij}, where $a_{ij} \geq 0$ for $i \neq j$. The equation showing the rate at which the individuals move from one class to another is

$$\frac{du_i}{dt} = \sum_{j=1}^{n} a_{ij} u_j, \quad \forall 1 \leq i \leq n. \tag{1}$$

The problem in (1) can be expressed in a more compact form below:

$$\begin{aligned} \frac{d\mathbf{u}}{dt}(t) &= A\mathbf{u}(t), \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned} \tag{2}$$

where A is an $n \times n$ matrix with non-negative off diagonal elements. Any non-negative off diagonal matrix A can be related to a non-negative matrix B through the equation

$$A = B - \mu_B I \tag{3}$$

where

$$\mu_B \geq \max_{1 \leq i \leq n} |a_{ii}|.$$

If the matrix in system (2) were non-negative, we would study its long time behaviour using Perron-Frobenius theorems. We can still use these theorems for system (2) with a positive off diagonal matrix A through the relationship in (3) by rescaling as the solutions to (2) are related to the solution to

$$\frac{d\bar{\mathbf{u}}}{dt} = B\bar{\mathbf{u}}$$

by $\bar{\mathbf{u}} = e^{\mu_B t}\mathbf{u}$. Therefore, in order to analyse system (2) and determine its long time and asymptotic behaviour, we need to study non-negative matrices and apply (3).

In our work, we study Perron-Frobenius theorems for non-negative matrices and use them, through Equation (3), to study long time and asymptotic behaviour of system (1). We hope to extend these studies further to infinite dimensional spaces and to study non-linear structured populations later on.

The thesis is divided into five chapters. In the first chapter, we give some preliminary results about general matrices from spectral theory, describe the notation to be used later on and some definitions.

In the second chapter, we describe non-negative matrices and divide them into two groups: reducible and irreducible matrices, and describe their properties in terms of associated graphs.

In most of the literature we read, a lot was written about irreducible matrices but very little on reducible matrices. So in the third chapter, we first describe Perron-Frobenius theorems for both positive and irreducible matrices following [13], [10]. In some texts such as [10], conditions for existence of a positive eigenvector for a reducible matrix were given (Theorem 6, page 77). In that chapter, we recall the proof for this theorem and further provide a more detailed discussion of Perron-Frobenius type theorems for reducible matrices.

In the fourth chapter, we use the results of Chapters 2 and 3 to study long time behaviour of solutions to initial value problem (2) by various mathods including the newly developed entropy method based on the work of Perthame, [16]. Using this method, we find that in the long run, the solution to the initial value problem tends to a multiple of the positive right eigenvector corresponding to the eigenvalue 0 if A is irreducible and that this is true for reducible matrices only under special conditions.

We finally give an application of the results to population biology. However all examples in this thesis illustrate populations whose dynamics is described by reducible matrices.

Chapter 1

Preliminaries

In this chapter, we recall a few standard definitions, and introduce terms and notations that will be used in the thesis. Then we outline relevant spectral properties of matrices.

1.1 Basic definitions and notations

Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a square matrix.

Definition 1.1.1. *A is called non-negative (denoted $A \geq 0$) if $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$. It is said to be positive ($A > 0$) if all inequalities are strict.*

Definition 1.1.2. *Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the absolute value of the vector $\mathbf{x}$ is defined as the vector*

$$|\mathbf{x}| = (|x_1|, |x_2|, \cdots, |x_n|).$$

Similarly for a matrix A, $|A| = (|a_{ij}|)_{1 \leq i,j \leq n}$.

Definition 1.1.3. *Let B and C be two $n \times n$ matrices. We say that $B \leq C$ if $b_{ij} \leq c_{ij}$ for all i and j.*

Definition 1.1.4. *Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. The $p-$norm of $\mathbf{x}$ is defined as*

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \text{ for } 1 \leq p < +\infty.$$

When $p = \infty$, the infinity norm of vector $\mathbf{x}$ is defined as

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

The $p-$norm for matrix A is defined by

$$\begin{aligned} \|A\|_p &= \max_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \\ &= \max_{\|x\|_p\leq 1} \|Ax\|_p. \end{aligned} \tag{1.1}$$

The norm defined in Equation (1.1) exists because the unit sphere in $\mathbb{R}^n$, defined by

$$B = \{\mathbf{x} \in \mathbb{R}^n \quad | \quad \|\mathbf{x}\| \leq 1\},$$

is compact and since the norm is a continuous function on $\mathbb{R}^n$, it follows that the maximum value stated above exists and so does the minimum ([11], page 83).

It can be shown that

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|, \tag{1.2}$$

(for example see [13], page 284). We note that since $\mathbb{R}^{n\times n}$ is finite dimensional, all matrix norms are equivalent and thus in particular applications, we shall use the most convenient one.

Definition 1.1.5. *The set $\sigma(A)$ is called the spectrum of the matrix A if and only if the operator $(\lambda I - A)$ is not injective for any $\lambda \in \sigma(A)$. The complement of this set is called the resolvent set, denoted by $\rho(A)$.*

Let $\lambda \in \mathbb{C}$ be an element in the resolvent set $\rho(A)$. The operator $R(\lambda, A)$ defined by

$$R(\lambda, A) := (\lambda I - A)^{-1}$$

is called the resolvent of A. Since A is a matrix,

$$R(\lambda, A) = (\lambda I - A)^{-1} = \frac{1}{\det(\lambda I - A)} M(\lambda), \tag{1.3}$$

where $M(\lambda)$ is the transpose of the matrix of cofactors of $\lambda I - A$. From (1.3) we see that $R(\lambda, A)$ is analytic for $\lambda \notin \sigma(A)$ and that the eigenvalues of A are the poles of $R(\lambda, A)$.

Definition 1.1.6. *[13], page* 497
Let A be a square matrix. The spectral radius of A is the number

$$r(A) = \max_{\lambda\in\sigma(A)} |\lambda| \tag{1.4}$$

Theorem 1.1.7. *Let A be a real valued matrix . The spectral radius $r(A)$ is given by the equation below:*

$$r(A) = \lim_{k\to\infty} \|A^k\|^{1/k}.$$

The proof of this theorem can be found in [13], page 619. From Equation (1.4), we see that $r(A)$ cannot be negative.

Definition 1.1.8. *Let $A \geq 0$ be an $n \times n$ matrix. An eigenvalue r of A will be called maximum if $|r| \geq |\lambda|$ for any other eigenvalue λ of A, and r will be called dominant if $|r| > |\lambda|$.*

Lemma 1.1.9. *Let A be an $n \times n$ matrix on $\mathbb{C}$. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.*

Proof. If $\lambda \in \sigma(A)$, then $A\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x}$ is the eigenvector corresponding to λ. Now suppose that for $k = c > 1$, $A^c\mathbf{x} = \lambda^c\mathbf{x}$ holds. We want to show that $A^{c+1}\mathbf{x} = \lambda^{c+1}\mathbf{x}$.

$A^c\mathbf{x} = \lambda^c\mathbf{x}$ implies that $A(A^c\mathbf{x}) = \lambda^c A(\mathbf{x})$ and this is equivalent to $A^{c+1}\mathbf{x} = \lambda^c.\lambda\mathbf{x} = \lambda^{c+1}\mathbf{x}$. Therefore, $\lambda^k \in \sigma(A^k)$ for all $k \in \mathbb{N}$. ■

Theorem 1.1.10. *[2], Theorem 2.34*
The spectral radius of a non-negative matrix A is an eigenvalue associated with a non-negative eigenvector.

Proof. First we show that

$$|R(\lambda, A)| \leq R(|\lambda|, A).$$

From the definition of the resolvent,

$$\begin{aligned} R(\lambda, A) = (\lambda I - A)^{-1} &= \lambda^{-1}(I - \lambda^{-1}A)^{-1} \\ &= \lambda^{-1}\sum_{n=0}^{\infty}\lambda^{-n}A^n, \quad |\lambda| > r(A), \end{aligned} \tag{1.5}$$

where the series exists by Cauchy-Hadamard criterion and Theorem 1.1.7. From this, we see that

$$\begin{aligned}|R(\lambda, A)| = |(\lambda I - A)^{-1}| &= |\lambda^{-1}\sum_{n=0}^{\infty}\lambda^{-n}A^n| \\ &\leq |\lambda^{-1}||\sum_{n=0}^{\infty}\lambda^{-n}A^n| \\ &\leq |\lambda^{-1}|\sum_{n=0}^{\infty}|\lambda^{-n}|A^n, \text{ since } A \geq 0 \\ &= R(|\lambda|, A).\end{aligned}$$

Thus

$$|R(\lambda, A)| \leq R(|\lambda|, A). \tag{1.6}$$

Let us now prove the statement of the theorem. Let λ_0 be an eigenvalue of A and $|\lambda_0| = r(A)$, (the spectral radius of A). Define $\lambda_n = r(A) + 1/n \in \rho(A)$, the resolvent set of A for each $n = 1, \cdots$. We see that $\lambda_n \to r(A)$ as $n \to \infty$. Consider another sequence $\mu_n = \lambda_n \frac{\lambda_0}{|\lambda_0|} \in \rho(A)$. $|\mu_n| = \lambda_n$ and $\mu_n \to \lambda_0$ as $n \to \infty$. From (1.3), $R(\lambda, A)$ is analytic for all $\lambda \notin \sigma(A)$, $\mu_n \to \lambda_0$ implies that $R(\mu_n, A)$ is not bounded. Since $R(\lambda_n, A) = R(|\mu_n|, A) \geq |R(\mu_n, A)|$ by (1.6), $R(\lambda_n, A)$ is not bounded too.

If $r(A) \in \rho(A)$, then $R(r(A), A)$ is finite, that is $|R(r(A), A)| \leq M$ for some $M < \infty$. But $R(\lambda_n, A) \to R(r(A), A)$ and $R(\lambda_n, A)$ is not bounded, so $R(r(A), A)$ is not bounded. This is a contradiction since we established that $R(r(A), A)$ is bounded. Thus $r(A) \in \sigma(A)$.

To prove that $r(A) \in \sigma(A)$ has an associated non-negative eigenvector, we see from above that for $\lambda_n = r(A) + 1/n$, we have $\lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty$ and for $\lambda \geq 0$, $R(\lambda, A) \geq 0$, by (1.5).

From Definition 1.1.4, we can see that for each $n \in \mathbb{N}$, $\|R(\lambda_n, A)\mathbf{y}_n\| \leq \|R(\lambda_n, A)\|$, where $\|\mathbf{y}_n\| = 1$. But equality holds for at least one unit vector $\mathbf{y}_n$ for each n, so for this particular vector $\mathbf{y}_n$,

$$\frac{1}{2}\|R(\lambda_n, A)\| \leq \|R(\lambda_n, A)\mathbf{y}_n\| = \|R(\lambda_n, A)\| > 0.$$

Let $\mathbf{x}_n = \dfrac{R(\lambda_n, A)\mathbf{y}_n}{\|R(\lambda_n, A)\mathbf{y}_n\|} \geq 0$. Then

$$\begin{aligned} A\mathbf{x}_n - r(A)\mathbf{x}_n &= (\lambda_n - r(A))\mathbf{x}_n - (\lambda_n I - A)\mathbf{x}_n \\ &= \frac{1}{n}\mathbf{x}_n - (\lambda_n I - A)\frac{R(\lambda_n, A)\mathbf{y}_n}{\|R(\lambda_n, A)\mathbf{y}_n\|} \\ &= \frac{1}{n}\mathbf{x}_n - (\lambda_n I - A)\frac{(\lambda_n I - A)^{-1}\mathbf{y}_n}{\|R(\lambda_n, A)\mathbf{y}_n\|} \\ &= \frac{1}{n}\mathbf{x}_n - \frac{\mathbf{y}_n}{\|R(\lambda_n, A)\mathbf{y}_n\|}. \end{aligned}$$

From this we have

$$\begin{aligned} \|A\mathbf{x}_n - r(A)\mathbf{x}_n\| &= \|\frac{1}{n}\mathbf{x}_n - \frac{\mathbf{y}_n}{\|R(\lambda_n, A)\mathbf{y}_n\|}\| \\ &\leq \frac{1}{n} + \frac{1}{\|R(\lambda_n, A)\mathbf{y}_n\|} \\ &\leq \frac{1}{n} + \frac{2}{\|R(\lambda_n, A)\|} \\ &\to 0 \text{ as } n \to \infty \text{ since } \|R(\lambda_n, A)\| \to \infty. \end{aligned}$$

But $A \geq 0$ and $r(A) \geq 0$ and, since $\|\mathbf{x}_n\| = 1$, the sequence is bounded and, by the compactness of the unit sphere on $\mathbb{R}^n$, there exists a convergent subsequence $(\mathbf{x}_k)_{k\geq 1}$ of $\mathbf{x}_n$, by the Bolzano-Weierstrass theorem. Let the limit of this subsequence be $\mathbf{x} \neq 0$ as $\|\mathbf{x}\| = 1$. Then

$$\lim_{k\to\infty} \|A\mathbf{x}_k - r(A)\mathbf{x}_k\| = \|A\mathbf{x} - r(A)\mathbf{x}\| = 0$$

and this implies that $A\mathbf{x} = r(A)\mathbf{x}$. ■

Theorem 1.1.11. *Let A be a square matrix and A^T be its transpose. Then A and A^T have the same spectrum, so $r(A) = r(A^T)$.*

Proof. If $A = (a_{i,j})$ is an $n \times n$ matrix, then S_n is a set containing all the permutations of $S = \{1, \cdots, n\}$ and

$$\det(A) = \sum_{\sigma\in S_n} \mathrm{Sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

while

$$\begin{aligned} \det(A^T) &= \sum_{\sigma\in S_n} \mathrm{Sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma_i,i} \\ &= \sum_{\sigma\in S_n} \mathrm{Sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}. \end{aligned}$$

The last equation is true because σ varies uniquely over S. Thus det$(A) = $ det(A^T) and hence det$(A - \lambda I) = $ det$(A^T - \lambda I)$. ■

The eigenvectors (refered to as right eigenvectors) of A^T are also called left eigenvectors of A; that is to say, if $\mathbf{v}$ is an eigenvector of A^T corresponding to eigenvalue λ, then $\mathbf{v}^T A = \lambda \mathbf{v}^T$.

Definition 1.1.12.

1.2 Similarity

Definition 1.2.1. *Two square matrices A and B are said to be similar if there exists an invertible matrix P such that $A = PBP^{-1}$.*

Theorem 1.2.2. *Let A and B be similar (square) matrices. Then both matrices have the same eigenvalues. Moreover, if $\mathbf{x} \neq 0$ is an eigenvector of A corresponding to eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to the same eigenvalue.*

Proof. Let $\lambda \in \sigma(A)$. Then $\det(\lambda I - A) = \det(\lambda I - PBP^{-1})$.

$$\begin{aligned}
\det(\lambda I - A) &= \det(\lambda I - PBP^{-1}) = \det(\lambda PP^{-1} - PBP^{-1}) \\
&= \det(P(\lambda P^{-1} - BP^{-1})) \\
&= \det P \det(\lambda P^{-1} - BP^{-1}) \\
&= \det P \det(\lambda P^{-1}P - B) \det P^{-1} \\
&= \det(\lambda I - B), \text{ since } \det P = \frac{1}{\det P^{-1}} \\
&= 0.
\end{aligned}$$

Hence $\lambda \in \sigma(B)$. However, in general the two matrices do not have the same eigenvectors. Notice that if $\mathbf{x} \neq 0$ is the eigenvector corresponding to eigenvalue λ of A, then $A\mathbf{x} = PBP^{-1}\mathbf{x} = \lambda\mathbf{x}$. From this, we see that $BP^{-1}\mathbf{x} = \lambda P^{-1}\mathbf{x}$, implying that $P^{-1}\mathbf{x}$ is the eigenvector of B corresponding to eigenvalue λ. ■

Theorem 1.2.3. *If B is similar to A, then $A^k = PB^kP^{-1}$.*

Proof. Let $A = PBP^{-1}$. Then $A^k = (PBP^{-1})^k$ for every integer $k \geq 1$. But

$$\begin{aligned}
(PBP^{-1})^k &= (PBP^{-1})(PBP^{-1})\cdots(PBP^{-1}), \quad \text{with } PBP^{-1} \text{ repeated } k \text{ times} \\
&= PBP^{-1}PBP^{-1}\cdots PBP^{-1} \\
&= P(BI)(BI)\cdots(BI)BP^{-1} \\
&= PB^kP^{-1}.
\end{aligned}$$

■

1.2.1 Jordan forms

A matrix A is called diagonalisable if it is similar to a diagonal matrix. We note that if A has a full range of eigenvalues (that is, if A has n distinct eigenvalues), then it is similar to a diagonal matrix D. But a matrix can still be diagonalisable even if not all its eigenvalues are distinct. A sufficient condition for a matrix to be diagonalisable is that it has a full range of eigenvectors ([13], page 507). For this to happen, all its eigenvalues must be semisimple. That is; $A = P\text{diag}\{\lambda_1, \cdots, \lambda_r\}P^{-1}$, where r is the number of distinct eigenvalues and these eigenvalues are repeated according to their algebraic multiplicities, and P is the matrix whose columns are eigenvectors of A, arranged in such a way that if λ_i has algebraic multiplicity k_i for all $i = 1, \cdots, r$, then the first k_1 columns of P are the eigenvectors corresponding to the eigenvalue λ_1, the next k_2 columns are eigenvectors corresponding to λ_2, and so on until we reach the last k_r columns which are eigenvectors of A corresponding to λ_r.

More precisely, if A has n linearly independent eigenvectors, then we can construct an invertible matrix P whose columns are eigenvectors of A. Let the i^{th} column in matrix P be labeled $\mathbf{p}_i$. Then

$$AP = A(\mathbf{p}_1, \cdots, \mathbf{p}_n) = (A\mathbf{p}_1, \cdots, A\mathbf{p}_n).$$

Let D be the diagonal matrix with eigenvalues of A along the main diagonal (counted with their multiplicities). Then

$$\begin{aligned}
PD &= (\mathbf{p}_1, \cdots, \mathbf{p}_n)\text{diag}\{\lambda_1, \cdots, \lambda_n\} \\
&= (\lambda_1\mathbf{p}_1, \cdots, \lambda_n\mathbf{p}_n).
\end{aligned}$$

We now see that $AP = PD$, implying that $A\mathbf{p}_i = \lambda_i\mathbf{p}_i$; hence $(\lambda_i, \mathbf{p}_i)$ is an eigenpair for all $i = 1, \cdots, n$.

Not every square matrix is similar to a diagonal matrix. However, any matrix is similar to an upper triangular matrix (see [13] page 508). The elements on the main diagonal of the triangular matrix are still the eigenvalues of A.

Let λ_j be an eigenvalue of A with algebraic multiplicity k_j. If λ_j is semisimple, then we call $B_j = \text{diag}\{\lambda_j, \cdots, \lambda_j\}$, the $k_j \times k_j$ matrix, the Jordan block for this eigenvalue. If however, λ_j is not semisimple, then the Jordan block B_j for λ_j is the matrix

$$B_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix} = S_j + N_j,$$

where

$$S_j = \begin{pmatrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{pmatrix} \text{ and } N_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

that is, B_j is a matrix with λ_j on the main diagonal, ones on the super diagonal and zeros else where. Matrix N_j is nilpotent of order k_j. Notice that matrices N_j and S_j commute.

Definition 1.2.4. *The Jordan form of a matrix A is the direct sum of all its Jordan blocks. That is to say $J =$ diag$\{B_1, \cdots, B_r\}$.*

Definition 1.2.5. *[13], page 593*
Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ and λ be an eigenvalue of the $n \times n$ matrix A. If $(A - \lambda I)^{k-1}\mathbf{x} \neq 0$ and $(A - \lambda I)^k \mathbf{x} = 0$, then $\mathbf{x}$ is called a generalised eigenvector of A associated with λ.

Theorem 1.2.6. *Let A be an $n \times n$ matrix. If J is the Jordan form of A, then there exists an invertible matrix P such that $A = PJP^{-1}$. Moreover, the columns of P are eigenvectors and generalised eigenvectors of A.*

Proof. Since J is the Jordan form of A, then A and J are similar, implying that an invertible matrix P exists such that $AP = PJ$ by Definition 1.2.1. To show that the columns of P are eigenvectors and generalised eigenvectors of A, we write matrix P as

$$P = (P_1, \cdots, P_r),$$

where P_j is an $n \times k_j$ matrix for all $j = 1, \cdots, r$. Let

$$P_j = (\mathbf{x}_1^j, \cdots, \mathbf{x}_{k_j}^j),$$

where $\mathbf{x}_i^j$ is a column vector for all $i = 1, \cdots, k_j$.

$$\begin{aligned} AP &= A(P_1, \cdots, P_r) \\ &= (AP_1, \cdots, AP_r) \\ &= (A\mathbf{x}_1^1, \cdots, A\mathbf{x}_{k_1}^1, \cdots, A\mathbf{x}_1^r, \cdots, A\mathbf{x}_{k_r}^r). \end{aligned}$$

Since J is block diagonal, we have

$$\begin{aligned} PJ &= (P_1B_1, \cdots, P_rB_r) \\ &= \left((\mathbf{x}_1^1, \cdots, \mathbf{x}_{k_1}^1)B_1, \cdots, (\mathbf{x}_1^r, \cdots, \mathbf{x}_{k_r}^r)B_r\right). \end{aligned}$$

But

$$\begin{aligned} P_iB_i &= (\mathbf{x}_1^i, \cdots, \mathbf{x}_{k_i}^i) \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix} \\ &= \left(\lambda_i\mathbf{x}_1^i, \mathbf{x}_1^i + \lambda_i\mathbf{x}_2^i, \cdots, \mathbf{x}_{k_{i-1}}^i + \lambda_i\mathbf{x}_{k_i}^i\right), \end{aligned}$$

for all $i = 1, \cdots, r$. Therefore, $AP = PJ$ implies

$$\begin{aligned} A\mathbf{x}_1^i &= \lambda_i\mathbf{x}_1^i \Rightarrow (\lambda_i, \mathbf{x}_1^i) \text{ is an eigenpair} \\ A\mathbf{x}_2^i &= \mathbf{x}_1^i + \lambda_i\mathbf{x}_2^i \Rightarrow (A - \lambda_iI)\mathbf{x}_2^i = \mathbf{x}_1^i \\ &\vdots \\ A\mathbf{x}_{k_i}^i &= \mathbf{x}_{k_{i-1}}^i + \lambda_i\mathbf{x}_{k_i}^i \Rightarrow (A - \lambda_iI)^{k_i}\mathbf{x}_{k_i}^i = 0. \end{aligned}$$

In otherwords, for all $i = 1, \cdots, r$, P_i (and hence P) is a matrix whose columns are eigenvectors and generalised eigenvectors of A corresponding to eigenvalue λ_i. ■

Definition 1.2.7. *Let A be an $n \times n$ matrix. The polynomial $P_n(\lambda)$ defined by*

$$P_n(\lambda) = \det(A - \lambda I)$$

is called the characteristic polynomial of A.

Proposition 1.2.8. *[13], page* 630

Let $A \geq 0$ be a square matrix. Then

$$\lim_{k\to\infty} A^k = 0 \text{ if and only if } r(A) < 1.$$

When $r(A) = 1$, the limit exists if and only if $r(A)$ is a semisimple eigenvalue of A and it is the only eigenvalue on the spectral circle.

Proof.

Suppose that

$$\lim_{k\to\infty} A^k = 0.$$

If J is the Jordan form of A, then $A = PJP^{-1}$. From Theorem 1.2.3, $A^k = PJ^kP^{-1}$. Therefore,

$$\lim_{k\to\infty} A^k = 0 \text{ implies } P\left(\lim_{k\to\infty} J^k\right)P^{-1} = 0$$

and this is true if and only if $J^k \to 0$ as $k \to \infty$. Since J is upper triangular with the eigenvalues of A along its main diagonal, $J^k \to 0$ as $k \to \infty$ if and only if $|\lambda| < 1$ for every eigenvalue λ of A. Hence $r(A) < 1$.

Now suppose that $r(A) < 1$ and let B_1 be the Jordan block corresponding to $r(A)$. Then

$$\begin{aligned} B_1^k &= (S_1 + N_1)^k \\ &= S_1^k + kS_1^{k-1}N_1 + \cdots + \begin{pmatrix} k \\ r \end{pmatrix} S_1^{k-r}N_1^r + \cdots + N_1^k, 0 \leq r \leq k. \end{aligned}$$

Since N_1 is nilpotent of finite order, $N_1^k \to 0$ as $k \to \infty$. S_1 is a diagonal matrix with $r(A)$ on the main diagonal, so $S_1^k \to 0$ as $k \to \infty$ because $r(A) < 1$. Therefore,

$$\lim_{k\to\infty} B_1^k = 0.$$

Since $r(A) < 1$, it follows that $|\lambda| < 1$ for any other eigenvalue λ of A. Therefore $B_i^k \to 0$, where B_i is the Jordan block for eigenvalue λ_i of A. Therefore,

$$\begin{aligned} \lim_{k\to\infty} J^k &= \lim_{k\to\infty} \text{diag}\{B_1^k, \cdots, B_r^k\} \\ &= \text{diag}\{\lim_{k\to\infty} B_1^k, \cdots, \lim_{k\to\infty} B_r^k\} \\ &= 0. \end{aligned}$$

Therefore, $A^k = PJ^kP^{-1} \to 0$ as $k \to \infty$. Hence $A^k \to 0$ if and only if $r(A) < 1$.

Now suppose that $r(A) = 1$ and that the limit of A^k as $k \to \infty$ exists. If $r(A)$ is not semisimple, then there exists a $j \times j$ $(j > 1)$ block in the Jordan form of A with 1 on the main and super diagonals. Let S_1 be the $j \times j$ identity matrix and N_1 the $j \times j$ matrix with 1 on the super diagonal. The limit of S_1^k as $k \to \infty$ is still the $j \times j$ identity matrix. So

$$B_1^k = I + kN_1 + \cdots + \begin{pmatrix} k \\ r \end{pmatrix} N_1^r + \cdots + N_1^k, 0 \leq r \leq k,$$

implying $B_1^k \to \infty$ as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} A^k \quad \text{does not exist.}$$

This is a contradiction to the assumption that the limit above exists. Therefore, if $r(A) = 1$ and the limit of A^k as $k \to \infty$ exists, then $r(A)$ is semisimple.

Suppose that $r(A) = 1$ and that it is semisimple. Then it follows that B_1 is the identity matrix whose dimension is equal to the algebraic multiplicity of $r(A)$. Therefore, $B_1^k \to I$ as $k \to \infty$. If $|\lambda| < 1$ for any other eigenvalue $\lambda \neq r(A)$ of A, then $B_i^k \to 0$ as $k \to \infty$ for any other eigenvalue $\lambda_i \neq r(A)$ of A. So $J^k \to \text{diag}\{I, 0, \cdots, 0\}$, implying that

$$\lim_{k\to\infty} A^k \quad \text{exists.}$$

On the other hand, if there is a $\lambda \neq r(A)$ such that $|\lambda| = 1$, then there exists a $\theta \in (0, 2\pi)$ such that $\lambda = e^{\imath\theta}$. This implies that B_λ has $e^{\imath\theta}$ on its main diagonal, and as a result, B_λ^k has the term $e^{\imath k\theta}$ on its main diagonal which oscillates as k changes. In such a case,

$$\lim_{k\to\infty} B_\lambda^k \text{ does not exist, hence } \lim_{k\to\infty} J^k \text{ does not exist.}$$

This implies that the limit of A^k as $k \to \infty$ does not exist. Therefore, if

$$\lim_{k\to\infty} A^k \text{ exists, then } r(A) \text{ is the only eigenvalue on the spectral circle.}$$

■

Chapter 2

Further non-negative matrices

In this chapter, we shall describe useful properties of non-negative matrices. We classify non-negative matrices into two groups; reducible and irreducible. In each case, we describe these matrices in terms of graphs and we give specific properties of these matrices.

2.1 Matrices and graphs

2.2 Some definitions

Definition 2.2.1. *A graph is an ordered pair $G = (V, E)$ containing a non-empty set of vertices V and a possibly empty set of edges, E. A directed graph is a finite non-empty set V of vertices together with a set E of ordered pairs of distinct elements of V. The elements of E are called directed edges or arcs, [3], page 3.*

The figures below illustrate the difference between a directed and undirected graph.

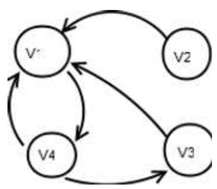


Figure 2.1: Directed graph

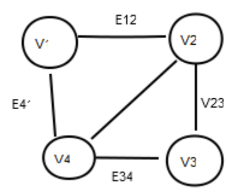


Figure 2.2: An undirected graph

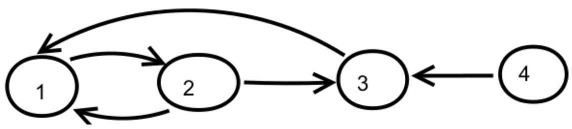
That is, if G is a directed graph and v_1 and v_2 are any two vertices in the graph G, then the edge $e_{1,2} = (v_1, v_2)$ is directed from v_1 to v_2. From this point onwards, when we talk about a graph, we shall mean a directed graph.

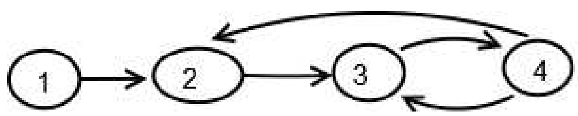
Definition 2.2.2. *Let G be a directed graph and $V(G)$ and $E(G)$ be the set of all the vertices and directed edges of G, respectively. Let $u, v \in V(G)$ and let $e \in E(G)$. We say that the edge e is incident to v and e is incident from u if e is directed from u to v and this is written as $e = (u, v)$ ([5], page 15).*

In such a case, we also say that u and v are adjacent vertices. Notice that $(u, v) \neq (v, u)$.

Definition 2.2.3. *A graph G_1 is isomorphic to a graph G_2 if there exists a one to one mapping ϕ from $V(G_1)$ onto $V(G_2)$ such that $(u, v) \in E(G_1)$ if and only if $(\phi u, \phi v) \in E(G_2)$ ([5], page 15).*

Example 2.2.4. *The two graphs below are isomorphic: The mapping ϕ acting on $V(G_1)$ is*





defined below:

$$\phi(1) = 3, \phi(2) = 4, \phi(3) = 2 \text{ and } \phi(4) = 1.$$

Definition 2.2.5. *Let u, v be vertices of a graph. A $u - v$ walk of graph G is a finite alternating sequence of vertices and edges, beginning at u and ending with vertex v ([5], page 26). The number of edges in a walk is the length of the walk.*

Definition 2.2.6. *A $u - v$ path is a walk in which no vertex is repeated.*

Definition 2.2.7. *Two vertices i and j are said to be connected if there is a path from i to j. A directed graph is called strongly connected if for every pair of vertices i, j in G, there is a directed path from i to j.*

Definition 2.2.8. *Let v be a vertex of the graph G. If there is no edge incident to or from v, then v is said to be isolated.*

A graph G with vertex set $V = \{1, \cdots, n\}$ can also be described by means of a matrix ([5], page 26). For example, the adjacency matrix of a graph G is the $n \times n$ matrix $D = (d_{ij})$ where $d_{ij} = 1$ if $(j, i) \in E(G)$, otherwise $d_{ij} = 0$.

Theorem 2.2.9. *[5], Theorem 2.2.*
If D is the adjacency matrix of a graph G with $V = \{1, \cdots, n\}$, then $d_{ij}^{(k)}$, the entry in the i^{th} row and j^{th} column of D^k, $k \geq 1$ is the number of different $i - j$ walks of length k in G.

The proof of this theorem can be found in [5] or [3].

2.2.1 Drawing a graph from a matrix

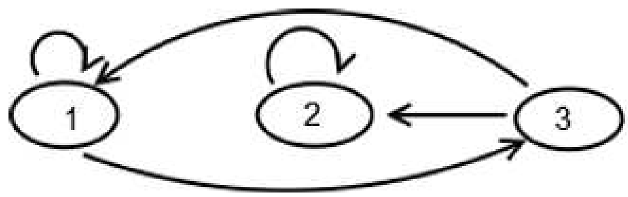
We have seen that every graph with finite vertex set can be represented by a matrix. In this section, we want to show that any non-negative matrix can be represented by a graph. We now describe how a graph can be drawn from a non-negative matrix.

Let $A \geq 0$ be an $n \times n$ matrix. Let $V = \{1, \cdots, n\}$ be a set. The graph of the matrix A, G_A, is the graph with vertex set V and $(j, i) \in E(G_A)$ is an edge of G_A if $a_{ij} > 0$, otherwise no edge is drawn. Notice that it is possible for two different matrices to have the same graph.

Example 2.2.10. *The two matrices below have the same graph:*

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 4 \\ 3 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 & 7 \\ 0 & 5 & 4 \\ 3 & 0 & 0 \end{pmatrix}.$$

The graph for these matrices is



Notice that if D is the adjacency matrix of G_A, then $d_{ij}^{(k)} > 0$ implies $a_{ij}^{(k)} > 0$ and conversely.

Definition 2.2.11. *A square matrix P is called a permutation matrix if it is obtained from the identity matrix by carrying out elementary row operations on it.*

Theorem 2.2.12. *Let A and B be non-negative $n \times n$ matrices such that $A = P^T BP$, where P is a permutation matrix. Then G_A is isomorphic to G_B.*

Proof. If P is a permutation matrix, then there is a 1 in each row and column. Let $\mathbf{e}_i$ be the column vector with 1 in the i^{th} row and zeros else where. Let Π be a permutation on the set $\{1, \cdots, n\}$ defined by

$$\Pi = \begin{pmatrix} 1 & \cdots & n \\ \pi_1 & \cdots & \pi_n \end{pmatrix}.$$

The columns of P are the vectors $\mathbf{e}_{\pi_i}; 1 \leq i \leq n$. Then $P = (\mathbf{e}_{\pi_1}, \cdots, \mathbf{e}_{\pi_n})$ and $P^T = (\mathbf{e}_{\pi_1}^T, \cdots, \mathbf{e}_{\pi_n}^T)^T$.

$$\begin{aligned} A =& P^T BP \\ =& \begin{pmatrix} \mathbf{e}_{\pi_1}^T \\ \vdots \\ \mathbf{e}_{\pi_n}^T \end{pmatrix} B \begin{pmatrix} \mathbf{e}_{\pi_1} & \cdots & \mathbf{e}_{\pi_n} \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} a_{ij} &= \mathbf{e}_{\pi_i}^T B \mathbf{e}_{\pi_j} \\ &= \mathbf{e}_{\pi_i}^T \begin{pmatrix} b_{1\pi_j} \\ \vdots \\ b_{i\pi_j} \\ \vdots \\ b_{n\pi_j} \end{pmatrix} \\ &= b_{\pi_i \pi_j}. \end{aligned}$$

Therefore, $a_{ij} > 0$ if and only if $b_{\pi_i\pi_j} > 0$. But $b_{\pi_i\pi_j} > 0$ means that there is a path from π_j to π_i in the graph of B and since $b_{\pi_i\pi_j} > 0$ means that $a_{ij} > 0$, it follows that there is a path from j to i in the graph of A. Therefore, G_A and G_B are isomorphic. ∎

2.3 Classification of non-negative matrices

Non-negative matrices can be divided into two classes: irreducible and reducible matrices.

Definition 2.3.1. *A matrix $A \geq 0$ is said to be irreducible if there is no permutation matrix that puts it in the form*

$$\tilde{A} = P^T AP = \begin{pmatrix} A_1 & 0 \\ A_{2,1} & A_2 \end{pmatrix}, \tag{2.1}$$

where A_1 and A_2 are square matrices. If such a permutation exists, then A is reducible.

Theorem 2.3.2. *A non-negative $n \times n$ matrix A is irreducible if and only if it has a strongly connected graph.*

Proof. Suppose that A is reducible, then there is a permutation matrix P that such that

$$\tilde{A} = P^T AP = \begin{pmatrix} A_1 & 0 \\ A_{2,1} & A_2 \end{pmatrix},$$

where A_1 is an $r \times r$ matrix and A_2 is $(n-r) \times (n-r)$ matrix. The zero matrix in $\tilde{A}$ means that the vertices from the set $V_1 = \{v_1, \cdots, v_r\}$ are not accessible from any vertex in the set $V_2 = \{v_{r+1}, \cdots, v_n\}$; that is, if $v_i \in V_1$ and $v_j \in V_2$, then there is no path (of any length) from v_j to v_i. Therefore, the directed graph of $\tilde{A}$ is not strongly connected. Since the graph of A, G_A, is isomorphic to that of $\tilde{A}$ (by Theorem 2.2.12), we conclude that the graph of A is not strongly connected.

Now suppose that G_A is not strongly connected. Then there are at least two vertices v_i and v_j such that one is inaccessible from the other. If v_i is inaccessible from v_j, then relabel the vertices such that v_i becomes v_1 and v_j becomes v_n. Any other vertices that are inaccessible from v_j are renamed $v_2, \cdots, v_r$. Therefore the set of vertices that are inaccessible from v_j (relabeled v_n) is $V_1 = \{v_1, \cdots, v_r\}$. All other vertices that are accessible from v_j are relabeled $v_{r+1}, \cdots, v_{n-1}$ and no vertex $v_l \in V_1$ can be accessed from any vertex $v_k \in V_2 = \{v_{r+1}, \cdots, v_n\}$ because if there is a $v_k \in V_2$ such that the edge (v_k, v_l) exists, then the vertex v_l would be accessible from v_n by taking the path $v_n \to v_k \to v_l$ which is not possible.

Let Π be a permutation on the set $\{1, \cdots, n\}$ such that if $i \in \{1, \cdots, n\}$ then Π transforms i into π_i. Then $a_{\pi_i, \pi_j} = 0$ for each $\pi_j \in \{r+1, \cdots, n\}$ and $\pi_i \in \{1, \cdots, r\}$. So if P is the permutation matrix defined by Π and $\tilde{A} = P^T AP$, then $\tilde{a}_{ij} = a_{\pi_i \pi_j} = 0$ for $\pi_j \in \{r+1, \cdots, n\}$ and $\pi_i \in \{1, \cdots, r\}$. Thus

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ A_{2,1} & A_2 \end{pmatrix}.$$

■

Example 2.3.3. *The matrix below*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is reducible. Its graph is shown below:

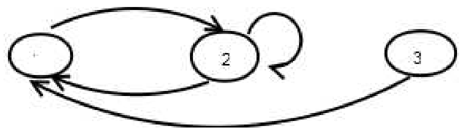


Figure 2.3: Connected but not strongly connected

Lemma 2.3.4. *(Lemma* 8.17 *of [1])*

Let $A \geq 0$*. If* A *is irreducible then it has no zero rows or columns.*

Proof. Let A be irreducible. If the i^{th} row is a zero row, then there is no path (of any length) from any vertex $j \in \{1, \cdots, n\}$ to state i. Thus there is no edge incident to vertex i in the graph G_A of A. Therefore, G_A is not connected and hence, A is not irreducible. If the j^{th} column is a zero column, then there is no path starting from vertex j to any other vertex $i \in \{1, \cdots, n\}$. Therefore, there is no edge incident from j in the directed graph of A, hence G_A is not strongly connected and therefore not irreducible. ■

Lemma 2.3.5. *If* $A \geq 0$ *is irreducible, then there is a non-zero element in each row and column, different from the diagonal element. Moreover, if* $A \geq 0$ *is irreducible and* $\mathbf{x} > 0$*, then* $A\mathbf{x} > 0$*.*

Proof.

Let $A \geq 0$ be irreducible. From the previous lemma, A has at least one positive entry in each row and each column. Suppose that in the i^{th} row there is only one positive entry which is also in the i^{th} column. Then it follows that in the directed graph G_A of A, there is no edge incident to vertex v_i except the trivial loop (path of length 1 from i to i). Therefore G_A is not connected, hence not strongly connected and therefore, A is not irreducible. So if A is irreducible, then there is at least one positive element in each row and column different from a_{ii} for each $i \in S$. If $\mathbf{x} > 0$, notice that for each i,

$$(A\mathbf{x})_i = \sum_{j=1}^{n} a_{ij} x_j.$$

Since $x_j > 0$ and there is at least one j for which $a_{ij} > 0$, then

$$\sum_{j=1}^{n} a_{ij}x_j > 0.$$

Thus $(A\mathbf{x})_i > 0$ for all $i \in S \Rightarrow A\mathbf{x} > 0$ ■

The converse of this lemma is not true. To illustrate this point, consider the example below:

Example 2.3.6. *Let*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This matrix has a non-zero element in each row and column different from the main diagonal and it is also block diagonal, therefore it is already in the form (2.1). Therefore the permutation matrix in Definition 2.3.1 is $P = I_4$, the 4×4 identity matrix, hence A is reducible.

Lemma 2.3.7. *If $A \geq 0$ is irreducible, then so is A^T.*

Proof. A^T is a matrix whose columns are the rows of A and the rows are the columns of A. Therefore the directed graph of A^T is the reversed directed graph of A. In other words, if there is a k path from state i to j, in the graph of A, then there is a k path in the graph of A^T from state j to i. Thus A^T has a strongly connected graph. ■

Theorem 2.3.8. *A non-negative matrix A is irreducible if and only if for every i and j in $\{1, \cdots, n\}$, there exists a positive integer $k = k(i,j) \leq n-1$ such that $a_{ij}^{(k)} > 0$.*

Proof. Suppose that $A \geq 0$ is irreducible. Then its graph G_A is strongly connected. By Definition 2.2.7, there is a path from j to i for every i and j. This means that there exist indices $h_1, \cdots, h_{k-1}$ such that

$$j \to h_{k-1} \to \cdots \to h_1 \to i,$$

implying that $d_{ij}^{(k)} > 0$, hence $a_{ij}^{(k)} > 0$.

Now suppose that for every i and j, there exists a $k = k(i,j)$ such that $a_{ij}^{(k)} > 0$. Then $d_{ij}^{(k)} > 0$ for every i and j. This is because A and D have their zeros in exactly the same positions and they have the same graph. That is; if $a_{ij} > 0$, then $d_{ij} > 0$. But $d_{ij}^{(k)}$ is the number of paths of length k from j to i by Theorem 2.2.9. Therefore, if $a_{ij}^{(k)} > 0$ for each i and j, then there is a path of length k from j to i for each i and j and by Definition (2.2.7), the graph of A is strongly connected. By Theorem 2.3.2, A is irreducible. ■

Theorem 2.3.9. *(Theorem 8.3.5 of [13])*
If $A \geq 0$ is an irreducible $n \times n$ matrix, then:

1. *$I + A$ is irreducible.*
2. *$(I + A)^{n-1} > 0$*

Proof. If A is irreducible, then it has a strongly connected graph, G_A; that is for every $i, j \in S$, there is a path from i to j and from j to i in G_A. We can draw the directed graph of $B = I + A$ by simply adding a loop to each vertex of graph G_A. This does not alter the connectedness of the graph, therefore, the graph of $I + A$ is strongly connected as well and B is irreducible.

To prove that $(I + A)^{n-1} > 0$, let $A^k = (a_{ij}^{(k)})_{1 \leq i,j \leq n}$. If $k = 2$, then the entry in the i^{th} row and j^{th} column of A^2 is given by

$$a_{ij}^{(2)} = (a_{i1}, \cdots, a_{ij}, \cdots, a_{in}) \begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{nj} \end{pmatrix}$$

$$= a_{i1}a_{1j} + a_{i2}a_{2j} + \cdots + a_{ij}^2 + \cdots + a_{in}a_{nj}.$$

This can be simply put as

$$a_{ij}^{(2)} = \sum_{h_1=1}^{n} a_{ih_1}a_{h_1j}. \tag{2.2}$$

So suppose that for $k = l > 2$,

$$a_{ij}^{(l)} = \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{ih_1}a_{h_1h_2} \cdots a_{h_{l-1}j}$$

holds. Then the j^{th} column of A^l is given by

$$\begin{pmatrix} \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{1h_1} \cdots a_{h_{l-1}j} \\ \vdots \\ \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{ih_1} \cdots a_{h_{l-1}i} \\ \vdots \\ \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{nh_1} \cdots a_{h_{l-1}j} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} a_{ij}^{(l+1)} &= (a_{i1}, a_{i2}, \cdots, a_{ij}, \cdots, a_{in}) \begin{pmatrix} \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{1h_1} \cdots a_{h_{l-1}j} \\ \vdots \\ \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{ih_1} \cdots a_{h_{l-1}i} \\ \vdots \\ \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{nh_1} \cdots a_{h_{l-1}j} \end{pmatrix} \\ &= a_{i1} \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{1h_1} \cdots a_{h_{l-1}j} + \cdots + a_{ii} \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{ih_1} \cdots a_{h_{l-1}i} \\ &+ \cdots + a_{nj} \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{nh_1} \cdots a_{h_{l-1}j} \\ &= \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{i1} a_{1h_1} \cdots a_{h_{l-1}j} + \cdots + \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{ii} a_{ih_1} \cdots a_{h_{l-1}i} \\ &+ \cdots + \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{in} a_{nh_1} \cdots a_{h_{l-1}j} \\ &= \sum_{h_l=1}^{n} \sum_{h_1=1}^{n} \cdots \sum_{h_{l-1}=1}^{n} a_{ih_l} a_{h_l h_1} \cdots a_{h_{l-1}j}, \end{aligned}$$

and this is equivalent to

$$a_{ij}^{(l+1)} = \sum_{h_1=1}^{n} \sum_{h_2}^{n} \cdots \sum_{h_l}^{n} a_{ih_1} a_{h_1 h_2} \cdots a_{h_l j}. \tag{2.3}$$

Therefore, for any $\mathbb{N} \ni k > 1$,

$$a_{ij}^{(k)} = \sum_{h_1=1}^{n} \sum_{h_2=1}^{n} \cdots \sum_{h_{k-1}=1}^{n} a_{ih_1} a_{h_1 h_2} \cdots a_{h_{k-1}j}, \tag{2.4}$$

which implies that $a_{ij}^{(k)} > 0$ if and only if there is a sequence of indices $h_1, h_2, \cdots, h_{k-1}$ such that $a_{ih_1} > 0$ and $a_{h_1 h_2} > 0$ and $\cdots$ and $a_{h_{k-1}j} > 0$. Since A is irreducible, for every i and j

in $\{1, \cdots, n\}$, there exists a $k = k(i,j) \leq n-1$ such that $a_{ij}^{(k)} > 0$ (by Theorem 2.3.8). But since I and A commute, the formula

$$(I + A)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} A^k$$

holds. Therefore,

$$\left[(I + A)^{n-1}\right]_{ij} = \sum_{k=0}^{n-1} \binom{n-1}{k} a_{ij}^{(k)}$$

and this is positive since a $k \in \{1, \cdots, n-1\}$ exists that makes $a_{ij}^{(k)}$ positive for all i and j. therefore, $(I + A)^{n-1} > 0$. ■

We consider two kinds of reducible matrices; those with completely disconnected graphs and those whose graphs are connected (but not strongly connected). Consider the directed graphs below: Graph(i) has no edges except loops. It represents a diagonal matrix. In graph (ii), the

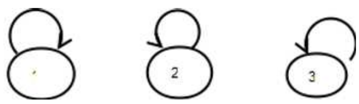


Figure 2.4: Graph(i)

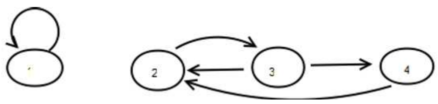


Figure 2.5: Graph(ii)

subgraph with vertices $2, 3, 4$ is strongly connected but this subgraph is not connected at all to the sub graph with vertex 1. Therefore graph(ii) is not strongly connected. The matrix for graph (ii) is block diagonal

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{pmatrix}.$$

These two graphs are examples of disconnected graphs. In both these cases, the matrix $A_{2,1}$ described in (2.1) is a zero matrix. For the preceding 4×4 matrix $A_1 = (a_{11})$ while

$$A_2 = \begin{pmatrix} 0 & a_{23} & a_{24} \\ a_{32} & 0 & 0 \\ 0 & a_{43} & 0 \end{pmatrix}.$$

If A is a block diagonal matrix with each block being irreducible, then we say that the blocks of A are isolated.

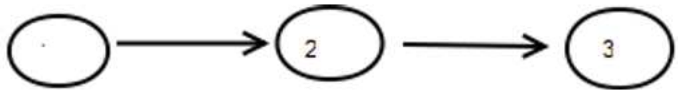


Figure 2.6: Graph(iii)

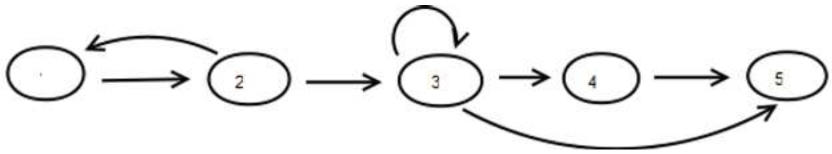


Figure 2.7: Graph(iv)

Now consider the graphs above: In these graphs, no vertex is completely isolated and none of the graphs is strongly connected. The matrices they represent are reducible with $A_{2,1} \geq 0, \neq 0$. As seen in Definition 2.3.1, every reducible matrix can be put in the form

$$A^* = \begin{pmatrix} A_1 & 0 \\ A_{2,1} & A_2 \end{pmatrix}. \quad (2.5)$$

The matrix A^* has the same eigenvalues as A by Theorem 1.2.2. The process of obtaining A^* is equivalent to simply renaming the vertices of the graph of A using the same index set with out changing the direction of the paths. In other words, we find graphs that are isomorphic to the graph of A and write down their corresponding matrices. This eventually gives the required matrix A^*. Consider the matrix below:

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

The matrix above is reducible since its directed graph (2.8) is not strongly connected. Notice

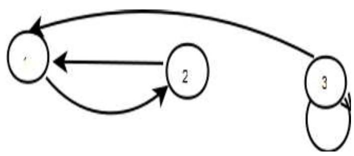


Figure 2.8: G_A

that vertices 1 and 2 form a strongly connected graph. A can be put in the form (2.1). The

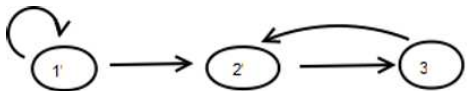


Figure 2.9: $G_{A'}$

graph $G_{A'}$ below is isomorphic to G_A and it is obtained by renaming state 3 to become $1'$ and 1 becomes $2'$ and 2 becomes $3'$, and the matrix for $G_{A'}$ is as shown below:

$$A^* = \begin{pmatrix} a_{33} & 0 & 0 \\ a_{13} & 0 & a_{12} \\ 0 & 0a_{21} & 0 \end{pmatrix}.$$

We also note that A^* is not unique. For the preceding matrix A, notice that A^* can also be the matrix

$$\begin{pmatrix} a_{33} & 0 & 0 \\ 0 & 0 & a_{21} \\ a_{13} & a_{12} & 0 \end{pmatrix}.$$

Theorem 2.3.10. *[18], Theorem* 2.1
Let $A \geq 0$ and $r = r(A) > 0$. Then $(s_1 I - A)^{-1}$ exists and $(s_1 I - A)^{-1} \geq 0$ if and only if $s_1 > r$. Moreover, $(s_1 I - A)^{-1} > 0$ if A is irreducible.

Proof. Suppose that $(s_1 I - A)^{-1}$ exists. Then for some vector $\mathbf{c} \geq 0, \neq 0$, there exists $\mathbf{x} \geq 0$ such that $\mathbf{x} = (s_1 I - A)^{-1}\mathbf{c}$. Rewriting this, we get $s_1\mathbf{x} = \mathbf{c} + A\mathbf{x}$, which implies that

$$s_1\mathbf{x} \geq A\mathbf{x}. \tag{2.6}$$

s_1 cannot be negative since $A\mathbf{x} \geq 0, \neq \mathbf{0}$. So let $s_1 > 0$. By Theorem 1.6 of [18] together with (2.6), $s_1 > r$.

Now suppose that $s_1 > r$, then $r(\frac{A}{s_1}) < 1$. By Theorem 1.2.8, $(\frac{1}{s_1}A)^k \to 0$ as $k \to \infty$. Since $s_1 > r$, then $s_1 > |\lambda|$ for every $\lambda \in \sigma(A)$, hence $s_1 - \lambda \neq 0$ for all λ. Therefore, $s_1 I - A$ is invertible because $0 \notin \sigma(s_1 I - A)$.

$$\begin{aligned} (s_1 I - A)^{-1} &= s_1^{-1}(I - s_1^{-1}A)^{-1} \\ &= s_1^{-1}\sum_{k=0}^{\infty}(s_1^{-1}A)^k \geq 0 \end{aligned}$$

If A is irreducible, then for all $1 \leq i, j \leq n$, $a_{ij}^{(k)} > 0$ for some $k = k(i,j) \leq n-1$ (Theorem 2.3.8). Thus

$$s_1^{-1} \sum_{k=0}^{\infty} (s_1^{-1} A)^k > 0,$$

hence $(s_1 I - A)^{-1} > 0$. ■

2.3.1 Normal form of a reducible matrix

Let $A \geq 0$ be reducible. By permuting its rows and then the columns by the same permutations, A can be written in the form defined in (2.1). If A_1 or A_2 is still reducible, it is again put in a form similar to that of A (as in 2.1) so that now

$$A^* = \begin{pmatrix} B_1 & 0 & 0 \\ C_1 & B_2 & 0 \\ C_2 & C_3 & B_3 \end{pmatrix}.$$

This process is repeated until all the matrices on the main diagonal are irreducible or 0. The matrix A is then said to be in normal form. In general an $n \times n$ reducible matrix A can be written in normal form given below:

$$A = \begin{pmatrix} A_1 & 0 & & & & & \\ 0 & A_2 & & & & & \\ \vdots & & \ddots & & & & \\ 0 & 0 & \cdots & A_g & & & \\ A_{g+1,1} & A_{g+1,2} & \cdots & A_{g+1,g} & A_{g+1} & & \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \\ A_{s,1} & A_{s,2} & \cdots & A_{s,g} & A_{s,g+1} & \cdots & A_s \end{pmatrix}, \tag{2.7}$$

where the matrices A_i for $1 \leq i \leq s$ are either irreducible or zero matrices of dimension 1, see [10], Equation (69). The matrix in (2.7) is lower triangular, so the empty spaces contain zeros.

Chapter 3

Perron-Frobenius type theorems

Perron and Frobenius independently studied the spectral properties of non-negative matrices. In this chapter, we begin with a discussion of these theorems for positive and irreducible matrices. In [10], the author extended the study of spectral properties of reducible. Thus we also provide a detailed description of the Perron-Frobenius type theorems for reducible matrices and give several examples illustrating the (spectral) differences from their irreducible counterparts.

3.1 Positive matrices

Theorem 3.1.1. *Perron-Frobenius theorem for positive matrices*
Let $A > 0$ be an $n \times n$ matrix. Then $r(A)$ has an associated positive eigenvector $\mathbf{x}$.

Proof. By Theorem 1.1.10, $r(A)$ is an eigenvalue of A associated with a non-negative eigenvector, so let $\mathbf{x} \geq 0$ be the eigenvector corresponding to $r(A)$. Then $A\mathbf{x} = r(A)\mathbf{x} > 0$ since $A > 0$. But $r(A)\mathbf{x} > 0$ if and only if $\mathbf{x} > 0$. Indeed, if there is $1 \leq i \leq n$ such that $x_i = 0$, then $r(A)x_i = 0$ implying that $r(A)\mathbf{x} \geq 0$ and the inequality is not strict, and this is a contradiction to $A\mathbf{x} > 0$. ■

Lemma 3.1.2. *If $A > 0$, then $r(A)$ is the only eigenvalue of A having strictly positive eigenvectors.*

Proof. We have already established that $r(A)$ has an associated positive eigenvector $\mathbf{x}$ and that $\sigma(A) = \sigma(A^T)$ by Theorem 1.1.11. Since $A > 0$, so is A^T. By Theorem 3.1.1, there exists a positive eigenvector $\mathbf{y}$ of A^T. Suppose that there is an eigenvalue $\lambda \in \sigma(A^T)$ with corresponding positive eigenvector $\mathbf{y}$ such that $A^T\mathbf{y} = \lambda\mathbf{y}$. From $A\mathbf{x} = r(A)\mathbf{x}$,

$$\begin{aligned}
\mathbf{y}^T A\mathbf{x} &= r(A)\mathbf{y}^T\mathbf{x} \\
\Rightarrow \lambda\mathbf{y}^T\mathbf{x} &= r(A)\mathbf{y}^T\mathbf{x} \\
&\Leftrightarrow (\lambda - r(A))\mathbf{y}^T\mathbf{x} = 0 \\
&\Leftrightarrow (\lambda - r(A)) = 0 \text{ because both } \mathbf{x} \text{ and } \mathbf{y}^T \text{ are positive }, \mathbf{y}^T\mathbf{x} > 0 \\
&\Leftrightarrow \lambda = r(A).
\end{aligned}$$

So r(A) is the only eigenvalue with strictly positive eigenvector. ■

Theorem 3.1.3. *If $A > 0$, then $r(A)$ is a simple eigenvalue of A.*

Proof. We shall start by showing that $r(A)$ is semisimple. We rescale matrix A so that it becomes $\tilde{A} = A/r(A)$ and $r(\tilde{A}) = 1$. Suppose that $r(\tilde{A})$ is not semisimple. Then the Jordan block corresponding to $r(\tilde{A})$ is given by

$$B_1 = \begin{pmatrix} 1 & 1 & & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

implying that $B_1^k \to \infty$ as $k \to \infty$. This also means that $J^k \to \infty$, therefore,

$$\|J^k\|_\infty = \|P^{-1}\tilde{A}^k P\|_\infty \le \|P^{-1}\|_\infty \|\tilde{A}^k\|_\infty \|P\|_\infty,$$

hence

$$\|\tilde{A}^k\|_\infty \ge \frac{\|J^k\|_\infty}{\|P^{-1}\|_\infty \|P\|_\infty} \to \infty$$

so that $\|\tilde{A}^k\|_\infty \to \infty$ as $k \to \infty$. Let $\tilde{A}^k = (a_{ij}^{(k)})$. The infinity norm of $\tilde{A}^k$ is the maximum of all row sums, so let i_k be the row of $\tilde{A}^k$ that gives maximum sum. By Lemma 3.1.2, there is a positive eigenvector $\mathbf{x} > 0$ such that $\mathbf{x} = \tilde{A}\mathbf{x}$. By Lemma 1.1.9, $\mathbf{x} = \tilde{A}^k\mathbf{x}$; therefore,

$$\begin{aligned}
\|\mathbf{x}\|_\infty = \max_i |x_i| \ge x_{i_k} &= \sum_{j=1}^n a_{i_k,j}^{(k)} x_j \\
&\ge \left(\sum_{j=1}^n a_{i_k,j}^{(k)} \right) \min_i x_i \\
&= \|\tilde{A}^k\|_\infty \min_i x_i \to \infty.
\end{aligned}$$

Therefore, $\|\mathbf{x}\|_\infty \to \infty$. But $\mathbf{x}$ is a constant vector, so $\|\mathbf{x}\|_\infty$ cannot go to infinity. Therefore $r(\tilde{A}) = 1$ is semisimple, implying that $r(A)$ is semisimple.

Now suppose that its algebraic multiplicity is $m > 1$. Since $r(\tilde{A})$ is semisimple, it follows that there are m linearly independent eigenvectors of $\tilde{A}$ corresponding to the eigenvalue $r(\tilde{A}) = 1$. Let $\mathbf{x}$ and $\mathbf{y}$ be two eigenvectors of $\tilde{A}$ corresponding to 1 such that $\mathbf{x} \neq \alpha\mathbf{y}$ for any $\alpha \in \mathbb{C}$. We pick a non-zero entry from vector $\mathbf{y}$, say y_i. Notice that

$$\tilde{A}(\mathbf{x} - \frac{x_i}{y_i}\mathbf{y}) = \mathbf{x} - \frac{x_i}{y_i}\mathbf{y},$$

implying that

$$\mathbf{z} = \mathbf{x} - \frac{x_i}{y_i}\mathbf{y}$$

is also an eigenvector of $\tilde{A}$ corresponding to 1. From $(8.2.7)$ of [13], $\tilde{A}|\mathbf{z}| = |\mathbf{z}| = \mathbf{x} > 0$, implying that $z_i \neq 0$ for all $i = 1, \cdots, n$. But this is a contradiction since

$$z_i = x_i - \frac{x_i}{y_i}y_i = 0$$

for at least one i. Therefore, the algebraic multiplicity of $r(\tilde{A})$ is one. ■

3.2 Irreducible matrices

Theorem 3.2.1. *Perron-Frobenius Theorem for irreducible matrices*

Let $A \geq 0$ be an irreducible matrix. Then there exists an eigenvalue r such that

1. *r is real and $r > 0$*

2. *there exists strictly positive left and right eigenvectors associated with the eigenvalue r.*

3. *the eigenvectors associated with r are unique to constant multiples.*

Proof. From Theorem 1.1.10, $r = r(A)$ is an eigenvalue of A and from the definition of $r(A)$, $r(A)$ is real and non-negative. $r(A)$ is also associated with a non negative eigenvector $\mathbf{x}$ by Theorem 1.1.10. Therefore, $A\mathbf{x} \geq 0$. Matrix A is irreducible, so there is a positive element in each row and column, hence the vector $A\mathbf{x}$ has at least one positive entry. Suppose that $r = r(A) = 0$, then $r\mathbf{x} = \mathbf{0}$, implying that $A\mathbf{x} \neq r\mathbf{x}$ which is a contradiction to the fact that r is an eigenvalue of A corresponding to eigenvector $\mathbf{x}$. Therefore, $r(A) > 0$.

If A is irreducible, then $(I+A)^{n-1} > 0$, by Theorem 2.3.9. Since $r(A)$ has a corresponding non-negative eigenvector $\mathbf{x}$ (by Theorem 1.1.10), then $(I+A)^{n-1}\mathbf{x} > 0$. Hence,

$$(I+A)^{n-1}\mathbf{x} = (1+r(A))^{n-1}\mathbf{x} > 0.$$

Suppose that there is $1 \leq i \leq n$ such that $x_i = 0$. Then $(1+r(A))^{n-1}x_i = 0$, implying that $(I+A)^{n-1}\mathbf{x} \neq (1+r(A))^{n-1}\mathbf{x}$ in position i, hence $(I+A)^{n-1}\mathbf{x} \neq (1+r(A))^{n-1}\mathbf{x}$. This is clearly impossible, so no such i with $x_i = 0$ exists. This implies that $\mathbf{x} > 0$. By Theorem 1.1.11, A and A^T have the same spectrum, so $r(A) = r(A^T)$. Let $\mathbf{v} \geq 0$ be the eigenvector for A^T corresponding to $r = r(A)$. By Lemma 2.3.7, A^T is also irreducible. Therefore, $(I+A^T)^{n-1} > 0$ by Theorem 2.3.9. Hence, $(I+A^T)^{n-1}\mathbf{v} = (1+r(A))^{n-1}\mathbf{v} > 0$. If there is a $1 \leq j \leq n$ such that $v_j = 0$, then $(1+r(A))^{n-1}v_j = 0$, which is a contradiction since $1+r(A))^{n-1}\mathbf{v} > 0$. Therefore, $\mathbf{v} > 0$.

We show that the positive eigenvector is unique up to constant multiples in two steps. First, we show that $r(A)$ is the only eigenvalue with strictly positive eigenvectors. Suppose that there is another eigenvalue λ_0 with eigenvector $\mathbf{y} > 0$. Then $A\mathbf{y} = \lambda_0\mathbf{y}$. Let $\mathbf{v} > 0$ be the left eigenvector of A corresponding to r. Then

$$\begin{aligned} \mathbf{v}A\mathbf{y} &= \lambda_0\mathbf{v}\mathbf{y} \\ \Rightarrow r\mathbf{v}\mathbf{y} &= \lambda_0\mathbf{v}\mathbf{y} \\ \Rightarrow (r-\lambda_0)\mathbf{v}\mathbf{y} &= 0 \\ \Rightarrow r &= \lambda_0 \end{aligned}$$

Therefore, $r(A)$ is the only eigenvalue of A with strictly positive eigenvectors.

We now show that $0 < \mathbf{y} = \alpha\mathbf{x}$, where α is a positive scalar and $\mathbf{x}$ is the positive eigenvector of A corresponding to $r = r(A)$. But $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A corresponding to $r(A)$ if and only if they are eigenvectors of $I+A$ corresponding to $1+r(A)$. We also note that if $(1+r(A), \mathbf{x})$ and $(1+r(A), \mathbf{y})$ are eigenpairs of $I+A$, then $((1+r(A))^{n-1}, \mathbf{x})$ and $((1+r(A))^{n-1}, \mathbf{y})$ are eigenpairs of $(I+A)^{n-1}$, by Lemma 1.1.9. If $\mathbf{y} \neq \alpha\mathbf{x}$ for any $\alpha \in \mathbb{R}$, then $(I+A)^{n-1}$ has at least two linearly independent eigenvectors, a contradiction to Theorem 3.1.3. Therefore, there exists $\alpha \in \mathbb{R}^+$ such that $\mathbf{y} = \alpha\mathbf{x}$. ∎

3.3 Reducible matrices

3.3.1 Existence of a positive eigenvector

The following theorem states the conditions for existence of a positive eigenvector for a reducible matrix.

Theorem 3.3.1. *[10], Theorem* 6
To the maximal eigenvalue $r(A)$ of a general reducible matrix $A \geq 0$, there belongs a positive eigenvector if and only if each A_i for $i = 1, \cdots, g$ in the normal form of A has eigenvalue r and $r \notin \sigma(A_j)$ for any $j = g+1, \cdots, s$.

Proof. Suppose that A has a positive eigenvector $\mathbf{x}$ such that $A\mathbf{x} = r(A)\mathbf{x}$. Matrix A in normal form contains block matrices. So we divide vector $\mathbf{x}$ into blocks ($n_i \times 1$ blocks, where n_i is the dimension of A_i for $1 \leq i \leq s$). That is; $\mathbf{x} = (\mathbf{x}^1, \cdots, \mathbf{x}^s)^T$. If $s = g$, then A is block diagonal and $A\mathbf{x} = r(A)\mathbf{x}$ together with the condition that $\mathbf{x} > 0$ implies that $r(A) \in \sigma(A_i)$ for all $i = 1, \cdots, g$.

If $s > g$, then $A\mathbf{x} = r(A)\mathbf{x}$ can be separated into parts

$$A_i\mathbf{x}^i = r(A)\mathbf{x}^i \text{ for } i = 1, \cdots, g \tag{3.1}$$

and

$$\sum_{j=1}^{h-1} A_{h,j}\mathbf{x}^j + A_h\mathbf{x}^h = r(A)\mathbf{x}^h \text{ for } h = g+1, \cdots, s. \tag{3.2}$$

Since $\mathbf{x}^i > 0$, then it follows that $r(A) \in \sigma(A_i)$ for every $i = 1, \cdots, g$. From Equation (3.2), we have

$$A_h\mathbf{x}^h \leq r(A)\mathbf{x}^h \text{ for } h = g+1, \cdots, s$$

which implies that $r(A) \geq r_h$ by Theorem 1.6 of [18] (where r_h is the maximal eigenvalue of A_h for all $h = g+1, \cdots, s$). But if $r(A) = r_h$, then it follows that

$$\sum_{j=1}^{h-1} A_{h,j}\mathbf{x}^j = 0,$$

which is impossible since $A_{h,j} \neq 0$ for some j and $\mathbf{x}^j > 0$. Thus $r_h < r(A)$.

Now suppose that $r_h < r(A)$ for $h = g+1, \cdots, s$. Then

$$A_i\mathbf{x}^i = r(A)\mathbf{x}^i \text{ for all } i = 1, \cdots, g.$$

Irreducibility of A_i implies that $\mathbf{x}^i > 0$ by Theorem 3.2.1. We also have

$$\sum_{j=1}^{h-1} A_{h,j}\mathbf{x}^j + A_h\mathbf{x}^h = r(A)\mathbf{x}^h \text{ for all } h = g+1, \cdots, s$$

and, from this we see that

$$(r(A)I - A_h)\mathbf{x}_h = \sum_{j=1}^{h-1} A_{h,j}\mathbf{x}^j$$
$$\Rightarrow \mathbf{x}_h = (r(A)I - A_h)^{-1}\sum_{j=1}^{h-1} A_{h,j}\mathbf{x}^j, \text{ for all } h = g+1, \cdots, s.$$

From Lemma 2.3.10, we have that if A_h is irreducible, then $(r(A)I - A_h)^{-1}$ exists and is positive since $r_h < r(A)$. Notice that

$$\sum_{j=1}^{h-1} A_{h,j}\mathbf{x}^j \geq 0, \neq \mathbf{0}$$

since there is at least one $j \in \{1, \cdots, h-1\}$ such that $A_{h,j} \geq 0$. Therefore,

$$(r(A)I - A_h)^{-1}\sum_{j=1}^{h-1} A_{h,j}\mathbf{x}^j > 0,$$

implying that $\mathbf{x}^h > 0$ for all $g+1 \leq h \leq s$.

If $A_h = 0$, a matrix of dimension 1, then $\mathbf{x}^h$ is just a scalar and $A_{h,j}$ are all scalars for all $j = 1, \cdots, h-1$. At least one of the scalars $A_{h,j}$ is positive and $\mathbf{x}^j > 0$ for $j = 1, \cdots, h-1$. Therefore,

$$\mathbf{x}^h = (r(A))^{-1}\sum_{j=1}^{h-1} A_{h,j}\mathbf{x}^j > 0,$$

and so $\mathbf{x} > 0$. ■

Theorem 3.3.2. *Let $A \geq 0$ be a reducible matrix and $r(A)$ be its spectral radius. Both A and A^T have positive eigenvectors corresponding to $r(A)$ if and only if A is block diagonal and $r(A) \in \sigma(A_i)$ for all $i = 1, \cdots, s$.*

Proof. Suppose $A \geq 0$ is block diagonal. Then it follows that A^T is also block diagonal. By Theorem 3.3.1, both A and A^T have a positive eigenvector $r(A)$ if and only if $r(A) \in \sigma(A_i)$ for all $i = 1, \cdots, s$.

Now suppose that both A and A^T have positive eigenvectors $\mathbf{x}$ and $\mathbf{v}$ respectively such that $A\mathbf{x} = r(A)\mathbf{x}$ and $A^T\mathbf{v} = r(A)\mathbf{v}$. If $\mathbf{x} > 0$, then by the first part of the proof of Theorem 3.3.1,

$r(A) \in \sigma(A_i)$ for all $i = 1, \cdots, g$ and $r_h < r(A)$ for all $h = g+1, \cdots, s$.

$$A^T = \begin{pmatrix} A_1^T & 0 & \cdots & 0 & A_{g+1,1}^T & A_{g+2,1}^T & \cdots & A_{s,1}^T \\ & A_2^T & & 0 & A_{g+1,2}^T & A_{g+3,2}^T & \cdots & A_{s,2}^T \\ & & \ddots & & \vdots & \vdots & & \vdots \\ & & & A_g^T & A_{g+1,g}^T & A_{g+2,g}^T & \cdots & A_{s,g}^T \\ & & & & A_{g+1}^T & A_{g+2,g+1}^T & \cdots & A_{s,g+1}^T \\ & & & & & A_{g+2}^T & \cdots & A_{s,g+2} \\ & & & & & & \ddots & \vdots \\ & & & & & & & A_s^T \end{pmatrix},$$

writing this in normal form (by interchanging row 1 with row s followed by column 1 and column s, row 2 and row $s-1$ followed by column 2 and column $s-1$ and so on,) we get

$$\begin{pmatrix} A_s^T & 0 & & & & & \\ A_{s,s-1}^T & A_{s-1}^T & & & & & \\ \vdots & \vdots & \ddots & & & & \\ A_{s,g+1}^T & A_{s-1,g+1}^T & \cdots & A_{g+1}^T & & & \\ A_{s,g}^T & A_{s-1,g}^T & \cdots & A_{g+1,g}^T & A_g^T & & \\ \vdots & \vdots & & \vdots & & \ddots & \\ A_{s,1}^T & A_{s,1}^T & \cdots & A_{g+1,1}^T & 0 & \cdots & A_1^T \end{pmatrix}. \tag{3.3}$$

Using this normal form, if A^T has a positive eigenvector corresponding to $r(A)$, then $r(A) \in \sigma(A_s)$ by Theorem 3.3.1 above. But this is a contradiction to the condition that $r_h < r(A)$ for all $h = g+1, \cdots, s$ which is necessary for $\mathbf{x}$ to be positive. Therefore, both A and A^T have positive eigenvectors if and only if A is block diagonal. ■

From these two theorems, we see that the best we can expect from general reducible matrices is that the maximal eigenvalue $r(A)$ has non-negative eigenvectors.

Remark 3.3.3. *If $r(A)$ is a simple eigenvalue of A and $r(A) \in \sigma(A_j)$ for atmost one $j \in \{1, \cdots, g\}$, then $\mathbf{x} = (0, \cdots, 0, \mathbf{x}^j, 0, \cdots, \mathbf{x}^{g+1}, \cdots, \mathbf{x}^s)^T$ and if $i \in \{g+1, \cdots, s\}$, $\mathbf{x} = (0, \cdots, 0, \mathbf{x}^i, \cdots, \mathbf{x}^j)^T$ but in each case, the eigenvector for A^T corresponding to r is*

$$\mathbf{v} = (\mathbf{v}^1, \cdots, \mathbf{v}^i, 0, \cdots, 0)^T.$$

3.3.2 Uniqueness of positive eigenvectors

Unlike irreducible matrices that have unique positive eigenvectors (up to constant multiples) corresponding to $r(A)$, reducible matrices generally do not have positive eigenvectors, and even when they do exist, they are not unique.

Example 3.3.4. *If A is block diagonal and $\mathbf{x} > 0$ is an eigenvector corresponding to $r(A)$, then*

$$\mathbf{y} = (\alpha_1 \mathbf{x}^1, \cdots, \alpha_s \mathbf{x}^s)^T, \text{ where } \alpha_i, i = 1, \cdots, s \text{ are any scalars}$$

is also an eigenvector of A. For instance, picking $\alpha_1 = 1, \alpha_2 = 2, \cdots, \alpha_g = g$, we find that $\mathbf{y} > 0$ is not proportional to $\mathbf{x}$.

Example 3.3.5. *Let*

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ A_{3,1} & A_{3,2} & A_3 \end{pmatrix}.$$

If $r(A) \in \sigma(A_1)$ and $r(A) \in \sigma(A_2)$ and $r_3 < r(A)$, then $\mathbf{x}^1 > 0$, $\mathbf{x}^2 > 0$,

$$\mathbf{x}^3 = (r(A)I - A_3)^{-1}[A_{3,1}\mathbf{x}^1 + A_{3,2}\mathbf{x}^2]$$

and $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)^T > 0$. Notice that

$$\mathbf{y} = (2\mathbf{x}^1, 3\mathbf{x}^2, (r(A)I - A_3)^{-1}[2A_{3,1}\mathbf{x}^1 + 3A_{3,2}\mathbf{x}^2])^T$$

is also an eigenvector of A corresponding to r and it is positive. Clearly, $\mathbf{y}$ is not proportional to $\mathbf{x}$.

3.3.3 Semisimplicity of the maximal eigenvalue

Theorem 3.3.6. *Let $A \geq 0$ be reducible. If there exists a positive eigenvector of A or A^T corresponding to $r(A)$, then $r(A)$ is semisimple.*

Proof. Suppose that a positive eigenvector $\mathbf{x}$ for A exists. Then by Theorem 3.3.1, $r(A) \in \sigma(A_i)$ for all $i = 1, \cdots, g$ and $r(A) \notin \sigma(A_j)$ for any $j = g+1, \cdots, s$, so its algebraic multiplicity is g. In the proof of Theorem 3.3.1, we found that

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^g \\ \mathbf{x}^{g+1} \\ \vdots \\ \mathbf{x}^s \end{pmatrix} = \begin{pmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^g \\ (r(A)I - A_{g+1})^{-1} \sum_{j=1}^{g} A_{g+1,j}\mathbf{x}^j \\ \vdots \\ (r(A)I - A_s)^{-1} \sum_{j=1}^{s-1} A_{s,j}\mathbf{x}^j \end{pmatrix} > 0.$$

But notice that if $r(A) \in \sigma(A_i)$ for all $i \in \{1, \cdots, g\}$, then

$$\mathbf{x}_1 = \left(\mathbf{x}^1, 0, \cdots, 0, \mathbf{y}_1^{g+1}, \cdots, \mathbf{y}_1^s\right)^T$$
$$\mathbf{x}_2 = \left(0, \mathbf{x}^2, 0, \cdots, 0, \mathbf{y}_2^{g+1}, \cdots, \mathbf{y}_2^s\right)^T$$
$$\cdots$$
$$\mathbf{x}_i = \left(0, \cdots, \mathbf{x}^i, 0, \cdots, 0, \mathbf{y}_i^{g+1}, \cdots, \mathbf{y}_i^s\right)^T$$
$$\cdots$$
$$\mathbf{x}_g = \left(0, \cdots, 0, \mathbf{x}^g, \mathbf{y}_g^{g+1}, \cdots, \mathbf{y}_g^s\right)^T,$$

are also eigenvectors of A corresponding to $r(A)$, where $\mathbf{x}^i > 0$, and each $\mathbf{x}^i$ is a positive eigenvector of A_i corresponding to $r(A)$, for each $i \in \{1, \cdots, g\}$ and

$$\mathbf{y}_i^{g+1} = (r(A)I - A_{g+1})^{-1} A_{g+1,i}\mathbf{x}^i, \tag{3.4}$$

together with

$$\mathbf{y}_i^h = (r(A)I - A_h)^{-1} \left[A_{h,i}\mathbf{x}^i + \sum_{j=g+1}^{h-1} A_{h,j}\mathbf{y}_i^j \right], \text{ for } h \geq g+2 \tag{3.5}$$

hold. Since $r_h < r(A)$ for all $h \geq g+1$, it follows that $\mathbf{y}_i^h > 0$ for all $i = 1, \cdots, g$ and $h \geq g+1$. Therefore,

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_g \mathbf{x}_g = 0$$

if and only if $\alpha_1 = \cdots = \alpha_g = 0$, implying that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_g$ are linearly independent. Therefore, A has g linearly independent eigenvectors corresponding to $r(A)$, implying that $r(A)$ is semisimple.

Now suppose that A^T has a positive eigenvector $\mathbf{v}$. Then using the normal form of A^T in (3.3) and the first part of this proof, $r(A)$ is a semisimple eigenvalue of A^T. ∎

When $(r(A), \mathbf{x})$ is not a positive eigenpair of A, the structure of the matrix (the mixing terms $A_{h,j}$ for $h \geq g+1$ and $1 \leq j \leq h-1$) generally plays a bigger role. To illustrate this, consider the example below:

Example 3.3.7. *Let*

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ A_{21} & A_2 & 0 \\ A_{31} & A_{32} & A_3 \end{pmatrix}.$$

Let $r(A) \in \sigma(A_2)$ and $r(A) \in \sigma(A_3)$ and $r(A_1) < r(A)$. If $A_{32} = 0$, then $r(A)$ is semisimple since the vectors $\mathbf{x}_1 = (\mathbf{0}, \mathbf{x}^2, \mathbf{0})^T$ and $\mathbf{x}_2 = (\mathbf{0}, \mathbf{0}, \mathbf{x}^3)^T$ are eigenvectors of A corresponding to $r(A)$.

However, if $A_{32} \neq 0$, then there is only one eigenvector $\mathbf{x}_2$ associated with $r(A)$, hence $r(A)$ is not semisimple.

Chapter 4

Long time behaviour

4.1 Introduction

In this chapter, we use the Perron-Frobenius type theorems to study the long time behaviour of the continuous time problem

$$\begin{aligned} \frac{du}{dt} &= Au(t) \\ u(t=0) &= u(0) \end{aligned} \tag{4.1}$$

4.1.1 Background

We note that if X is a complete space, then absolute convergence of a series implies convergence.

Definition 4.1.1. *Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix. The exponential of A is given by*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This series is well defined and is convergent for any A on $\mathbb{R}^{n\times n}$. To see that it is convergent, notice that

$$\left\|\frac{A^k}{k!}\right\| \leq \frac{\|A\|^k}{k!}, \text{ so if } \|A\| = m \geq 0, \text{ then } \left\|\frac{A^k}{k!}\right\| \leq \frac{m^k}{k!}.$$

But

$$\sum_{k=0}^{\infty} \frac{m^k}{k!} = e^m, \text{ therefore } \sum_{k=0}^{\infty} \|\frac{A^k}{k!}\| \leq \sum_{k=0}^{\infty} \frac{m^k}{k!} = e^m$$

which means that the series is absolutely convergent. Since $\mathbb{R}$ is complete, it follows that $\mathbb{R}^{n\times n}$ is also complete, hence e^A is convergent.

Lemma 4.1.2. *[11], page* 84

Let S and T be any two matrices on $\mathbb{R}^{n\times n}$. If S and T commute, then $e^{S+T} = e^S e^T$.

Proof. If $ST = TS$, then

$$\begin{aligned}
(S+T)^n &= S^n + nS^{n-1}T + n(n-1).\frac{1}{2!}S^{n-2}T^2 + n(n-1)(n-2).\frac{1}{3!}S^{n-3}T^3 + \cdots \\
&\quad + \begin{pmatrix} n \\ k \end{pmatrix} S^{n-k}T^k + \cdots + T^n, \text{ defined for } k \leq n \\
&= n!\left[\frac{S^n}{n!} + \frac{S^{n-1}}{(n-1)!}\frac{T}{1!} + \frac{S^{n-2}}{(n-2)!}\frac{T^2}{2!} + \cdots + \frac{S^0}{0!}\frac{T^n}{n!}\right] \\
&= n!\sum_{k=0}^{n}\frac{S^{n-k}}{(n-k)!}\frac{T^k}{k!} \\
e^{S+T} &= \sum_{n=0}^{\infty}\frac{(S+T)^n}{n!}, (\text{ from the definition of matrix exponential}) \\
&= \sum_{n=0}^{\infty}\left(\frac{n!}{n!}\sum_{k=0}^{n}\frac{S^{n-k}}{(n-k)!}\frac{T^k}{k!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{S^{n-k}}{(n-k)!}\frac{T^k}{k!}\right) \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{S^{n-k}}{(n-k)!}\frac{T^k}{k!},
\end{aligned}$$

where the change of order of summation is justified by absolute convergence on $\mathbb{R}^{n\times n}$. Let $n-k=j$. When $n=k, j=0$ so

$$e^{S+T} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\frac{S^j}{j!}\frac{T^k}{k!}\right) = \left(\sum_{j=0}^{\infty}\frac{S^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{T^k}{k!}\right) = e^S e^T.$$

■

4.1.2 Matrix exponentials and Jordan forms

Theorem 4.1.3. *Let A be an $n\times n$ matrix. If there exists an invertible matrix P such that $A = PJP^{-1}$, then*

$$e^{tA} = Pe^{tJ}P^{-1}.$$

Proof.

$$\begin{aligned} e^{tA} &= I + t(PJP^{-1}) + \frac{(tPJP^{-1})^2}{2!} + \frac{(tPJP^{-1})^3}{3!} + \cdots \\ &= I + P(tJ)P^{-1} + \frac{t^2}{2!}(PJP^{-1})^2 + \frac{t^3}{3!}(PJP^{-1})^3 + \cdots \\ &= I + P(tJ)P^{-1} + \frac{t^2}{2!}PJ^2P^{-1} + \frac{t^3}{3!}PJ^3P^{-1} + \cdots, \quad \text{by (1.2.3)} \\ &= P(I + tJ + \frac{t^2}{2!}J^2 + \frac{t^3}{3!}J^3 + \cdots)P^{-1} \\ &= Pe^{tJ}P^{-1}. \end{aligned}$$

■

If A is diagonalisable, then P contains the eigenvectors of A and

$$e^{tA} = P \text{ diag } \{e^{t\lambda_1}, \cdots, e^{t\lambda_r}\}P^{-1},$$

where $e^{t\lambda_i}$ for every $i = 1, \cdots, r$ is repeated according to the algebraic multiplicity of λ_i. Let k_i be the algebraic multiplicity of λ_i. We rewrite matrices P and P^{-1} in the form

$$P = [P_1, \cdots, P_r] \quad \text{and} \quad P^{-1} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_r \end{bmatrix}, \text{ respectively,}$$

where P_i is an $n \times k_i$ matrix and Q_i is a $k_i \times n$ matrix for every $i = 1, \cdots, r$. Then

$$\begin{aligned} e^{tA} &= [P_1, \cdots, P_r]\text{diag}\{e^{t\lambda_1}, \cdots, e^{t\lambda_r}\} \begin{bmatrix} Q_1 \\ \vdots \\ Q_r \end{bmatrix} \\ &= e^{t\lambda_1}P_1Q_1 + \cdots + e^{t\lambda_r}P_rQ_r, \end{aligned}$$

where the eigenvalues in the diagonal matrix diag$\{e^{t\lambda_1}, \cdots, e^{\lambda_r t}\}$ are repeated according to their algebraic multiplicities. The product $P_iQ_i = G_i$ is called the projection matrix corresponding to the eigenvalue λ_i. Therefore, if A is diagonalisable, then

$$e^{tA} = \sum_{i=1}^{r} e^{t\lambda_i}G_i. \tag{4.2}$$

If some of the eigenvalues are not semisimple, then the matrix exponential takes a more complicated form. Let λ_j be the eigenvalue which is not semisimple and let its algebraic multiplicity be k_j, then $e^{tB_j} = e^{tS_j}e^{tN_j} = \text{diag}\{e^{t\lambda_j}, \cdots, e^{t\lambda_j}\}e^{tN_j}$, where

$$e^{tN_j} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{k_j-1}}{(k_j-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k_j-2}}{(k_j-2)!} \\ 0 & 0 & 1 & t & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & \frac{t^2}{2!} \\ \vdots & \vdots & \vdots & & \ddots & t \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{4.3}$$

by Example 2.3 of [8].

If A is not diagonalisable, the columns of P are the generalised eigenvectors of A, arranged in the same order as the eigenvalues in the Jordan form of A. That is; if λ_1 is an eigenvalue of multiplicity k_j, then the first k_j columns of P are the eigenvectors and generalised eigenvectors of A corresponding to λ_j. We still divide matrices P and P^{-1} into blocks corresponding to those in the Jordan form of A. Let k_i be the index of λ_i; that is, let k_i be the smallest integer k for which the null space of $(A - \lambda_i I)^k$ is the same as that of $(A - \lambda_i I)^{k+1}$. Then

$$\begin{aligned} e^{tA} &= [P_1, \cdots, P_r]\text{diag}\{e^{tB_1}, \cdots, e^{tB_r}\} \begin{bmatrix} Q_1 \\ \vdots \\ Q_r \end{bmatrix} \\ &= P_1 e^{tB_1} Q_1 + \cdots + P_r e^{tB_r} Q_r, \end{aligned}$$

where $e^{tB_i} = e^{t\lambda_i}e^{tN_i}$ and e^{tN_i} is as defined in (4.3). Therefore,

$$\begin{aligned} e^{tA} &= e^{t\lambda_1}P_1 \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k_1-1}}{(k_1-1)!} \\ & 1 & t & \ddots & \vdots \\ & & 1 & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix} Q_1 + \cdots + e^{t\lambda_r}P_r \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k_r-1}}{(k_r-1)!} \\ & 1 & t & \ddots & \vdots \\ & & 1 & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix} Q_r \\ &= e^{t\lambda_1}P_1 \left[I + tN_1 + \frac{t^2}{2!}N_1^2 + \cdots + \frac{t^{k_1-1}}{(k_1-1)!}N_1^{k_1-1} \right] Q_1 + \cdots + e^{t\lambda_r}P_r \left[I + tN_r + \frac{t^2}{2!}N_r^2 \right. \\ &\quad \left. + \cdots + \frac{t^{k_r-1}}{(k_r-1)!}N_r^{k_r-1} \right] Q_r, \end{aligned}$$

where

$$N_i = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The exponential described above can now be simplified to

$$e^{tA} = \sum_{i=1}^{r} \sum_{j=0}^{k_i - 1} e^{t\lambda_i} \frac{t^j}{j!} P_i N_i^j Q_i.$$

Notice that $J - \lambda I = P^{-1}AP - \lambda I = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$, and that $N_i = B_i - \lambda_i I$, where B_i is the Jordan block corresponding to λ_i. We also have $P_i N_i^j Q_i = (A - \lambda_i I)^j G_i$. To see this, notice that $P^{-1}P = I$ implies that

$$\begin{bmatrix} Q_1 \\ \vdots \\ Q_i \\ \vdots \\ Q_r \end{bmatrix} [P_1, \cdots, P_i, \cdots, P_r] = \begin{bmatrix} Q_1P_1 & Q_1P_2 & \cdots & Q_1P_i & \cdots & Q_1P_r \\ \vdots & \vdots & & \vdots & & \vdots \\ Q_iP_1 & Q_iP_2 & \cdots & Q_iP_i & \cdots & Q_iP_r \\ \vdots & \vdots & & \vdots & & \vdots \\ Q_rP_1 & Q_rP_2 & \cdots & Q_rP_i & \cdots & Q_rP_r \end{bmatrix} = I.$$

This implies that $Q_iP_i = I$ for all $i = 1, \cdots, r$ and $Q_iP_j = 0$ for $i \neq j$. Therefore,

$$P^{-1}P_i = \begin{bmatrix} Q_1 \\ \vdots \\ Q_i \\ \vdots \\ Q_r \end{bmatrix} P_i = \begin{bmatrix} 0 \\ \vdots \\ Q_iP_i \\ \vdots \\ 0 \end{bmatrix}, \quad \text{hence} \quad P^{-1}P_iQ_i = \begin{bmatrix} 0 \\ \vdots \\ Q_i \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore,

$$
\begin{aligned}
(A-\lambda_i I)^j G_i &= P(J-\lambda_i)^j P^{-1} P_i Q_i \\
&= [P_1, \cdots, P_r] \begin{bmatrix} (B_1-\lambda_i I)^j & & & \\ & (B_2-\lambda_i I)^j & & \\ & & \ddots & \\ & & & (B_r-\lambda_i I)^j \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ Q_i \\ \vdots \\ 0 \end{bmatrix} \\
&= [P_1, \cdots, P_i, \cdots, P_r] \begin{bmatrix} 0 \\ \vdots \\ (B_i-\lambda_i I)^j Q_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= P_i (B_i-\lambda_i I)^j Q_i \\
&= P_i N_i^j Q_i.
\end{aligned}
$$

Hence

$$e^{tA} = \sum_{i=1}^{r} \sum_{j=0}^{k_i-1} e^{t\lambda_i} \frac{t^j}{j!} (A-\lambda_i I)^j G_i, \tag{4.4}$$

where again G_i is the projection matrix onto the generalised eigenspace corresponding to eigenvalue λ_i with the property that $G_i G_j = 0$ if $i \neq j$ ([13], page 604). Notice that if all eigenvalues are semisimple, (4.4) collapses to the same equation as shown in (4.2).

4.2 Existence and uniqueness of solution

Consider the differential equation in (4.1) where A is an $n \times n$ matrix. The theorem below ensures that the initial value problem has a solution and moreover, this solution is unique.

Theorem 4.2.1. *[11] Let A be a real $n \times n$ matrix and $\mathbf{u}(0) \in \mathbb{R}^n$. Then the initial value problem in (4.1) has a unique solution of the form $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$.*

Proof. We show that $\mathbf{u}(t) = e^{tA}\mathbf{u}(0)$ is a solution.

$$\begin{aligned}\frac{d}{dt}\mathbf{u}(t) &= \frac{d}{dt}(e^{tA}\mathbf{u}(0)) \\ &= \lim_{h\to 0}\frac{e^{A(t+h)}\mathbf{u}(0) - e^{tA}\mathbf{u}(0)}{h} = \lim_{h\to 0}\frac{e^{tA}e^{hA}\mathbf{u}(0) - e^{tA}\mathbf{u}(0)}{h}\end{aligned}$$

since t, h are scalars, tA and hA commute, so $e^{A(t+h)} = e^{tA}e^{hA}$. Hence

$$\frac{d}{dt}\mathbf{u}(t) = \lim_{h\to 0}\frac{e^{hA} - I}{h}(e^{tA}\mathbf{u}(0))$$

To show that

$$\lim_{h\to 0}\frac{1}{h}\left[e^{hA} - I\right] = A, \text{ it is enough to show that } \left\|\frac{1}{h}\left[e^{hA} - I\right] - A\right\| \to 0 \text{ as } h \to \infty.$$

$$\begin{aligned}\|\frac{e^{hA} - I}{h} - A\| &= \|\frac{1}{h}\sum_{k=1}^{\infty}\frac{(hA)^k}{k!} - A\| = \left\|\sum_{k=1}^{\infty}\frac{h^{k-1}A^k}{k!} - A\right\| \\ &= \left\|\sum_{k=2}^{\infty}\frac{h^{k-1}A^k}{k!}\right\| \le \sum_{k=2}^{\infty}\frac{|h^{k-1}|\|A\|^k}{k!} = \|A\|\sum_{k=2}^{\infty}\frac{|h^{k-1}|\|A\|^{k-1}}{k!}\end{aligned}$$

But

$$\|A\|\sum_{k=2}^{\infty}\frac{|h^{k-1}|\|A\|^{k-1}}{k!} = \|A\|\left[\left(1 + \frac{|h|\|A\|}{2!} + \frac{|h|^2\|A\|^2}{3!} + \cdots\right) - 1\right],$$

therefore

$$\begin{aligned}\|\frac{e^{hA} - I}{h} - A\| &\le \|A\|\left[\left(1 + \frac{|h|}{2!}\|A\| + \frac{|h|^2\|A\|^2}{3!} + \cdots\right) - 1\right] \\ &\le \|A\|\left[1 + |h|\|A\| + \frac{|h|^2\|A\|^2}{2!} + \cdots - 1\right] \\ &= \|A\|\left(e^{|h|\|A\|} - 1\right) \\ &\to 0 \text{ as } h \to 0.\end{aligned}$$

Therefore, $\|\frac{e^{hA}-I}{h} - A\| \to 0$, thus

$$\lim_{h\to 0}\frac{e^{hA} - I}{h} = A, \text{ hence } \frac{d\mathbf{u}(t)}{dt} = e^{tA}A\mathbf{u}(0).$$

Therefore, $\mathbf{u}(t) = e^{tA}\mathbf{u}(0)$ is a solution to (4.1). Now to show that the solution is unique, suppose that $\mathbf{u}_1(t)$ is another solution to the initial value problem. Then $\mathbf{u}_1' = A\mathbf{u}_1$ and $\mathbf{u}_1(0) = u(0)$. Let $\mathbf{v}(t) = e^{-At}\mathbf{u}_1(t)$. Then $\mathbf{v}' = e^{-At}\mathbf{u}_1'(t) - e^{-At}A\mathbf{u}_1(t) = 0$. Therefore, $\mathbf{v}$ is a constant. Moreover, $\mathbf{v}(0) = e^0\mathbf{u}_1(0) = I\mathbf{u}(0) = \mathbf{u}(0)$. Therefore, $\mathbf{v} = \mathbf{u}(0)$, implying that $\mathbf{u}_1(t) = e^{tA}\mathbf{u}(0) = \mathbf{u}(t)$. ■

Example 4.2.2. *If all the eigenvalues of A are simple, then*

$$\mathbf{u}(t) = P \text{ diag } \{e^{t\lambda_1}, \cdots, e^{\lambda_n t}\} P^{-1}\mathbf{u}(0).$$

Let

$$P = [\mathbf{p_1}, \mathbf{p}_2, \cdots, \mathbf{p}_n],$$

where $\mathbf{p}_i$ are the eigenvectors of A. Then $P^{-1}\mathbf{u}(0)$ is an $n\times 1$ column vector $\mathbf{K} = (k_1, k_2, \cdots, k_n)^T$.

$$\begin{aligned}
\mathbf{u}(t) &= P\text{diag}\{e^{t\lambda_1}, \cdots, e^{t\lambda_n}\}P^{-1}\mathbf{u}(0) \\
&= P\text{diag}\{e^{t\lambda_1}, \cdots, e^{t\lambda_n}\}\mathbf{K} \\
&= [\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n](e^{t\lambda_1}k_1, e^{t\lambda_2}k_2, \cdots, e^{t\lambda_n}k_n)^T \\
&= k_1 e^{t\lambda_1}\mathbf{p}_1 + k_2 e^{t\lambda_2}\mathbf{p}_2 + \cdots + k_n e^{t\lambda_n}\mathbf{p}_n.
\end{aligned}$$

4.2.1 Stability of the solution

Definition 4.2.3. *The matrix function $t \to e^{tA}$ is called stable if*

$$\lim_{t\to\infty} \|e^{tA}\| = 0.$$

We note that the zero solution to (4.1) is asymptotically stable (in the Liapunov sense) if e^{tA} is stable. In this section, we explore the conditions ensuring stability of e^{tA}.

Theorem 4.2.4. *[8], Theorem 3.6*
e^{tA} is stable if and only if all eigenvalues of A have negative real parts.

Proof. Suppose that e^{tA} is stable. Then $\|e^{tA}\| \to 0$ as $t \to \infty$. Assume that there is at least one eigenvalue $\lambda_k = a_k + \imath b_k$ with positive real part. If λ_k is semisimple, then B_k is a diagonal matrix with λ_k on the main diagonal. Therefore,

$$e^{tB_k} = e^{ta_k}\begin{pmatrix} e^{\imath b_k t} & & \\ & \ddots & \\ & & e^{\imath b_k t}\end{pmatrix},$$

and since $a_k > 0$, it follows that $e^{ta_k} \to \infty$ as $t \to \infty$ and since $\|\text{diag}\{e^{\imath b_k t}, \cdots, e^{\imath b_k t}\}\| = 1$,

$$\|e^{tB_k}\| = e^{ta_k}\left\|\begin{pmatrix} e^{\imath b_k t} & & \\ & \ddots & \\ & & e^{\imath b_k t}\end{pmatrix}\right\| \to \infty \text{ as } t \to \infty.$$

Therefore,

$$\|e^{tJ}\| = \|\mathrm{diag}\{e^{tB_1}, \cdots, e^{tB_k}, \cdots, e^{tB_r}\}\| \to \infty \text{ as } t \to \infty.$$

But if e^{tA} is stable, it follows that $\|e^{tJ}\| = \|P^{-1}e^{tA}P\| \to 0$ which is a contradiction. Therefore, the assumption that $a_k > 0$ is false.

If λ_k is not semisimple, then

$$e^{tB_k} = e^{ta_k} \begin{pmatrix} e^{\imath b_k t} & & \\ & \ddots & \\ & & e^{\imath b_k t} \end{pmatrix} e^{tN_k}, \tag{4.5}$$

where e^{tN_k} is the matrix given in (4.3). Since e^{tN_k} is a matrix of polynomials in t, $\|e^{tN_k}\| \to \infty$ as $t \to \infty$, and $a_k > 0$ implies that $e^{ta_k} \to \infty$. Therefore, $\|e^{tB_k}\| \to \infty$, implying that $\|e^{tJ}\| \to \infty$, which is again a contradiction since $\|P^{-1}e^{tA}P\| \to 0$ as $t \to \infty$. Therefore, there is no λ_k with positive real part.

Let $a_k = 0$. Then it follows that if λ_k is semisimple, then

$$e^{tB_k} = \begin{pmatrix} e^{\imath b_k t} & & \\ & \ddots & \\ & & e^{\imath b_k t} \end{pmatrix}$$

But $e^{\imath b_k t} = \cos(b_k t) + \imath \sin(b_k t)$, therefore, $\|e^{tB_k}\| = 1$, implying that $\|e^{tJ}\| = 1$ as $t \to \infty$. But again we get a contradiction since e^{tA} being stable implies that $\|e^{tJ}\| = \|P^{-1}e^{tA}P\| \to 0$. If λ_k is not semisimple, then (4.5) holds with $e^{ta_k} = 1$ and again $\|e^{tB_k}\| \to \infty$ which is not possible since e^{tA} is stable. Therefore, $a_k < 0$.

Conversely, suppose that $\Re\lambda < 0$ for every $\lambda \in \sigma(A)$. If all eigenvalues are semisimple, then e^{tJ} is a diagonal matrix with $e^{t\lambda}, \lambda \in \sigma(A)$ on its main diagonal. Therefore, $\|e^{tJ}\| \to 0$ as $t \to \infty$, implying that $\|e^{tA}\| = \|Pe^{tJ}P^{-1}\| \to 0$ as $t \to \infty$. If, however, some of the eigenvalues are not semisimple, then

$$e^{tJ} = \mathrm{diag}\{e^{t\lambda_1}, \cdots, e^{t\lambda_r}\}e^{tN},$$

where $e^{t\lambda_i}$ for all $i = 1, \cdots, r$ is repeated according to the algebraic multiplicity of λ_i, J is the Jordan form of A, N is a nilpotent matrix and $\|e^{tN}\| = P_n(t)$, a polynomial of degree at most n. However, for any real number $\epsilon > 0$, there exists a constant ω_ϵ such that

$$|P_n(t)| \leq \omega_\epsilon e^{\epsilon t}.$$

Let

$$\max_{1\le i\le r} \Re\lambda_i = \lambda^* < 0.$$

Then, choosing $\epsilon = -\lambda^*/2$, we find that

$$e^{\lambda^* t}\|e^{tN_k}\| \le \omega_\epsilon e^{\lambda^* t}e^{\epsilon t} = \omega_\epsilon e^{t\lambda^*/2} \to 0.$$

Therefore, $\|e^{tB_k}\| = \|\text{diag}\{e^{t\lambda_1}, \cdots, e^{t\lambda_r}\}P_n(t)\| \to 0$ as $t \to \infty$. So we conclude that e^{tA} is stable. ■

If there is a zero eigenvalue, then we still have a stable behaviour of solutions as $t \to \infty$ provided zero is semisimple.

Theorem 4.2.5. *Let A be an $n \times n$ matrix with dominant eigenvalue 0. Then*

$$\lim_{t\to\infty} e^{tA} = E \ne 0$$

if and only if 0 is semisimple, otherwise the limit does not exist.

Proof. Suppose that $e^{tA} \to E \ne 0$ as $t \to \infty$. If 0 is not semisimple, then the Jordan block corresponding to this eigenvalue is

$$B_0 = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

so $e^{tB_0} = Ie^{tN_0} \to \infty$ componentwise as $t \to \infty$. This in turn means that $e^{tJ} \to \infty$ and from this, we have $Pe^{tJ}P^{-1} \to \infty$ which is impossible since E is a finite matrix. Therefore 0 is semisimple.

Now suppose that 0 is semisimple. So B_0 is a zero matrix of dimension k_0 (the multiplicity of 0). Thus $e^{tB_0} = I_{k_0}$. Since $e^{tJ} = \text{diag}\{e^{tB_0}, e^{tB_1}, \cdots, e^{tB_r}\}$ and since all other eigenvalues have negative real part, $e^{tB_j} \to 0$ componentwise as $t \to \infty$, $j \ne 0$. This means that

$$e^{tJ} \to \text{diag}\{I_{k_0}, 0 \cdots, 0\}$$

so

$$\begin{aligned}\lim_{t\to\infty} e^{tA} &= P \lim_{t\to\infty} e^{tJ}P^{-1} \\ &= P\text{diag}\{I_{k_0}, 0, \cdots, 0\}P^{-1} \\ &= E \ne 0.\end{aligned}$$

If we write P, diag$\{I_{k_0}, 0, \cdots, 0\}$ and P^{-1} in the form

$$P = [P_1, \cdots, P_r], \quad \text{diag}\{I_{k_0}, 0, \cdots, 0\} = \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0_{(n-k)\times(n-k)} \end{pmatrix}, \quad P^{-1} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_r \end{bmatrix},$$

where P_1 is an $n \times k_0$ matrix and Q_1 is a $k_0 \times n$, we find that Pdiag$\{I_{k_0}, 0, \cdots, 0\}P^{-1} = P_1Q_1 = G_1$. Therefore, $E = G_1$; the projection matrix corresponding to eigenvalue 0. P_1 is a matrix whose columns are the eigenvectors of A corresponding to 0. ■

4.3 Irreducible ML matrices

Definition 4.3.1. *Let $A \in M_{n\times n}(\mathbb{R})$. A is called an ML matrix if $a_{ij} \geq 0$ for $i \neq j$.*

Definition 4.3.2. *An ML matrix A is said to be irreducible if there exists an irreducible matrix $B \geq 0$ and a $\mu \in \mathbb{R}^+$ such that $B = A + \mu I$.*

In order to have $B \geq 0$, it is sufficient to have

$$\mu \geq \left| \min_{1\leq i\leq n} a_{ii} \right|.$$

The matrix A inherits a good number of its properties from the matrix B.

Definition 4.3.3. *Let A be an ML matrix. An eigenvalue τ of A will be called dominant if $\Re\tau > \Re\lambda$ for any other eigenvalue λ of A.*

From Theorem 2.6 of [18], we have the result below for irreducible ML matrices:

Theorem 4.3.4. *Let A be an irreducible ML matrix. Then there exists an eigenvalue τ such that*

1. *τ is real*
2. *τ is associated with strictly positive right and left eigenvectors which are unique up to constant multiples.*
3. *$\tau > \Re(\lambda)$ for any other eigenvalue $\lambda \neq \tau$ of A*

Proof. Let $A = B - \mu I$ with $B \geq 0$ and irreducible. Let α_i be an eigenvalue of B. The eigenvalue of A corresponding to this eigenvalue is $\lambda_i = \alpha_i - \mu$. Irreducibility of B implies that there is a dominant eigenvalue r_μ in its spectrum. This means that $\tau = r_\mu - \mu$ is also a dominant eigenvalue of A. Furthermore, from Perron-Frobenius theorem, r_μ is a real eigenvalue of B, thus τ is also real.
To see that τ is associated with positive right and left eigenvectors, notice that all eigenvectors of B are also eigenvectors of A and since $r_\mu > |\alpha_i|$, the right eigenvector $\mathbf{x}$ associated with r_μ is positive (by Perron's theorem). But $B\mathbf{x} = r_\mu \mathbf{x}$ implies that $B\mathbf{x} - \mu\mathbf{x} = r_\mu\mathbf{x} - \mu\mathbf{x}$ which is true if and only if $A\mathbf{x} = (r_\mu - \mu)\mathbf{x} = \tau\mathbf{x}$. Therefore, $\mathbf{x} > \mathbf{0}$ is an eigenvector corresponding to τ. Let $\mathbf{v} > 0$ be the left eigenvector of B associated with eigenvalue r. Then $\mathbf{v}B = r_\mu\mathbf{v} = \mathbf{v}(A + \mu I)$. This implies that $r_\mu\mathbf{v} = \mathbf{v}A + \mu\mathbf{v}$, so that $(r_\mu - \mu)\mathbf{v} = \tau\mathbf{v} = \mathbf{v}A$. So $\mathbf{v}$ is also an eigenvector of A associated with τ. Therefore we conclude that τ is associated with positive eigenvectors which are unique up to constant multiples.

Let $\lambda_j = a_j + \imath b_j \in \sigma(A)$ and $\lambda_j \neq \tau$. If $a_j > \tau$, then there is $\alpha_j \in \sigma(B)$ such that $\alpha_j = \mu + \lambda_j = \mu + a_j + \imath b_j$. In particular, $a_j + \mu > \tau + \mu$, this implies that $|\alpha_i| = |(\mu + a_j) + \imath b_j| > r$ which is impossible since r is dominant in $\sigma(B)$. If $\tau = a_j$, and $b_j \neq 0$, then $\lambda_j = \tau + \imath b_j$ so that $\alpha_j = (\tau + \mu) + \imath b_j$ but then again, $|\alpha| > r$, which is impossible. Therefore, $\tau > \Re(\lambda)$ for any other eigenvalue. ∎

For irreducible ML matrices, a stronger form of Theorem 4.2.5 can obtained as shown below.

Corollary 4.3.5. *Let $A \in M_{n\times n}(\mathbb{R})$ be an irreducible ML matrix such that $A = B - rI$, where $B \in M_{n\times n}(\mathbb{R}^+)$ and let r be the dominant eigenvalue of B. Then 0 is the dominant eigenvalue of A and it is a simple pole of the resolvent of A. Moreover,*

$$\lim_{t\to\infty} e^{tA} = \lim_{t\to\infty} e^{tA} = G_1 := (\mathbf{x} * \mathbf{v}) > 0, \tag{4.6}$$

where G_1 is the projection matrix corresponding to the eigenvalue 0 associated with the right eigenvector $\mathbf{x}$ and left eigenvector $\mathbf{v}$ and $\mathbf{x} * \mathbf{v}$ represents vector tensor product (that is, an $n \times n$ matrix whose i^{th} row is given by $(x_i v_j)_{1\leq j\leq n}$ for all $i = 1, \cdots, n$)

Proof. If B is irreducible, then by the Perron-Frobenius theorem, $r = r(A)$ is simple. The eigenvalues of A are given by $\lambda - r$ for every $\lambda \in \sigma(B)$. So $\tau = 0$ is an eigenvalue of A. Since $r \geq |\lambda|$ and $r > \Re\lambda$ for any $\lambda \in \sigma(B)$, then $0 > r - \Re\lambda$, for any other eigenvalue $\lambda \neq r$ of B.

Therefore, 0 is a dominant eigenvalue of A. Let the number of distinct eigenvalues of A be m with each eigenvalue having index m_i. From equation (4.4),

$$e^{At} = \sum_{i=1}^{m} \left(\sum_{j=0}^{m_i-1} e^{t\lambda_i} \frac{t^j}{j!} (A - \lambda_i I)^j \right) G_i, \tag{4.7}$$

Let $\mathbf{x}$ and $\mathbf{v}$ be the right and left eigenvectors of A corresponding to the eigenvalue 0. Let $\lambda_1 = 0$. We have already shown that this eigenvalue is simple. Thus $m_1 = 1$ and

$$\mathbf{u}(t) = e^{0t} G_1 + \sum_{i=2}^{m} \left(\sum_{j=0}^{m_i-1} e^{t\lambda_i} \frac{t^j}{j!} (A - \lambda_i I)^j \right) G_i. \tag{4.8}$$

All the other eigenvalues $\lambda_i, i > 1$ have negative real parts so that

$$\lim_{t\to\infty} \sum_{i=2}^{m} \left(\sum_{j=0}^{m_i-1} e^{t\lambda_i} \frac{t^j}{j!} (A - \lambda_i I)^j \right) G_i = 0,$$

hence

$$\lim_{t\to\infty} e^{tA} = G_1. \tag{4.9}$$

Notice that $\mathbf{x} * \mathbf{v} > 0$ since $\mathbf{x} > 0, \mathbf{v} > 0$ and $\mathbf{vx} > 0$. We shall normalise these two vectors so that $\mathbf{vx} = 1$. Let $G = (\mathbf{x} * \mathbf{v})$. Then $G^2 = (\mathbf{x} * \mathbf{v}).(\mathbf{x} * \mathbf{v}) = \mathbf{x}(\mathbf{vx})\mathbf{v} = \mathbf{x} * \mathbf{v}$. Therefore, $G^2 = G$, implying that G is a projection matrix, by Equation $5.9.13$ of [13]. Since $\mathbf{x}$ and $\mathbf{v}$ are eigenvectors of A corresponding to 0 and projections corresponding to an eigenvalue are unique (see page 386 of [13]), it follows that $G = G_1$. Therefore,

$$\lim_{t\to\infty} e^{tA} = (\mathbf{x} * \mathbf{v}).$$

■

If A is reducible, then 0 may not be semisimple, but if it is a semisimple dominant eigenvalue, then from Theorem 4.2.5, we see that

$$\lim_{t\to\infty} \mathbf{u}(t) = P_1 Q_1 \mathbf{u}(0).$$

The product $Q\mathbf{u}(0)$ is a $k_0 \times 1$ vector. So let $Q\mathbf{u}(0) = (c_1, \cdots, c_{k_0})^T = \mathbf{c}$. Then

$$\begin{aligned}\lim_{t\to\infty} \mathbf{u}(t) &= P_1 \mathbf{c} \\ &= [\mathbf{x}_1, \cdots, \mathbf{x}_{k_0}](c_1, \cdots, c_{k_0})^T \\ &= c_1 \mathbf{x}_1 + \cdots + c_{k_0} \mathbf{x}_{k_0},\end{aligned}$$

where $\mathbf{x}_1, \cdots, \mathbf{x}_{k_0}$ are the linearly independent eigenvectors corresponding to 0.

If $k_0 = 1$, that is, 0 is a simple eigenvalue, and if its eigenvectors are normalised according to (4.12), then

$$\lim_{t\to\infty} \mathbf{u}(t) = c_1\mathbf{N}. \tag{4.10}$$

But this is also the same as

$$\lim_{t\to\infty} \mathbf{u}(t) = G_1\mathbf{u}(0),$$

and since 0 is simple, by ([13], page 520),

$$G_1 = \frac{\mathbf{N} * \phi}{\sum_{i=1}^{n} \phi_i N_i} = \mathbf{N} * \phi,$$

implying that

$$\lim_{t\to\infty} \mathbf{u}(t) = (\mathbf{N} * \phi)\mathbf{u}(0) = \rho\mathbf{N}. \tag{4.11}$$

Therefore, (4.10) and (4.11) are equal if and only if $c_1 = \rho$. Hence, if 0 is a simple dominant eigenvalue of an $n \times n$ matrix A, then $\mathbf{u}(t) \to \rho\mathbf{N}$.

4.3.1 More on the nature and asymptotic behaviour of solutions

We consider problem (4.1) with matrix A given by $A = B - rI$, where r is the dominant eigenvalue of irreducible matrix $B \geq 0$. Then it follows that A has a simple eigenvalue $\tau = 0$ with corresponding right and left eigenvectors, $\mathbf{x} > 0$ and $\mathbf{v} > 0$ respectively. To obtain uniqueness of the vectors $\mathbf{x}$ and $\mathbf{v}$, we shall normalise them and we shall call this normalised right eigenvector $\mathbf{N}$ and the left eigenvector ϕ so that these vectors satisfy the equations below:

$$\sum_{i=1}^{n} N_i\phi_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} N_i = 1. \tag{4.12}$$

Before we state the main result, we need the following lemma ([16], Lemma $6.1.3$). This lemma is an extension of Lemma $6.1.3$ of [16] to irreducible matrices.

Lemma 4.3.6. *Let $\phi, \mathbf{N} > \mathbf{0}$ and $(a_{ij})_{1\leq i,j\leq n}$ be an irreducible matrix. Then there is a constant $\alpha > 0$ such that for any vector $\mathbf{m}$ satisfying*

$$\sum_{i=1}^{n} \phi_i m_i = 0, \tag{4.13}$$

the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_i a_{ij}N_j\left(\frac{m_j}{N_j}-\frac{m_i}{N_i}\right)^2 \geq \alpha\sum_{i=1}^{n}\frac{\phi_i}{N_i}m_i^2 \tag{4.14}$$

holds

Proof. Let ℓ^2 be the weighted space with inner product between two vectors $\mathbf{x}$ and $\mathbf{y}$ defined as

$$\langle \mathbf{x},\mathbf{y}\rangle = \sum_{i=1}^{n}\frac{\phi_i}{N_i}x_i y_i \quad \forall \mathbf{x},\mathbf{y}\in\ell^2. \tag{4.15}$$

Let $\mathbf{m}$ be a vector satisfying (4.13). We shall normalise $\mathbf{m}$ and call this normalised vector $\overline{\mathbf{m}}$ so that

$$\sum_{i=1}^{n}\frac{\phi_i}{N_i}\overline{m}_i^2 = 1.$$

We notice that $\overline{\mathbf{m}}$ still satisfies

$$\sum_{i=1}^{n}\phi_i\overline{m}_i = 0. \tag{4.16}$$

Dividing Equation (4.14) with the weighted norm of $\mathbf{m}$ gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_i a_{ij}N_j\left(\frac{\overline{m}_j}{N_j}-\frac{\overline{m}_i}{N_i}\right)^2 \geq \alpha > 0. \tag{4.17}$$

Now suppose that there is no α satisfying (4.17). This means that for each k there exists a vector $(\overline{\mathbf{m}}^k)_{k\geq 1}$, satisfying

$$\sum_{i=1}^{n}\phi_i\overline{m}_i^k = 0, \quad \sum_{i=1}^{n}\frac{\phi_i}{N_i}(\overline{m}_i^k)^2 = 1,$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_i a_{ij}N_j\left(\frac{\overline{m}_j^k}{N_j}-\frac{\overline{m}_i^k}{N_i}\right)^2 \leq \frac{1}{k}. \tag{4.18}$$

The sequence $(\overline{\mathbf{m}}^k)_{k\geq 1}$ is bounded and its terms are on the $n-$sphere of radius 1. This sphere is compact, so by the Bolzano-Weierstrass theorem, there exists a subsequence of $(\overline{m}_k)_{k\geq 1}$ that converges to a vector $\overline{\overline{\mathbf{m}}}$ which is also on the $n-$sphere. Taking limits on both sides of inequality (4.18), we find that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_i a_{ij}N_j\left(\frac{\overline{\overline{m}}_j}{N_j}-\frac{\overline{\overline{m}}_i}{N_i}\right)^2 = 0. \tag{4.19}$$

Since A is irreducible, by Theorem 2.3.8, for every pair i and j, there exists a sequence of indices $j, i_r, i_{r-1}, \cdots, i_1, i$ such that $a_{i,i_1}a_{i_1,i_2}\cdots a_{i_{r-1},i_r}a_{i_r,j} > 0$. This means that: $a_{i,i_1} > 0$, which implies that for that particular pair i and i_1, Equation (4.19) holds if and only if

$$\frac{\overline{\overline{m}}_i}{N_i} = \frac{\overline{\overline{m}}_{i_1}}{N_{i_1}};$$

$a_{i_1,i_2} > 0$ which implies that Equation (4.19) holds if and only if

$$\frac{\overline{\overline{m}}_{i_1}}{N_{i_1}} = \frac{\overline{\overline{m}}_{i_2}}{N_{i_2}} = \frac{\overline{\overline{m}}_{i}}{N_{i}};$$

If we continue with the same reasoning for all the terms in the product, we find that

$$\frac{\overline{\overline{m}}_{i_{r-1}}}{N_{i_{r-1}}} = \frac{\overline{\overline{m}}_{i_r}}{N_{i_r}} = \frac{\overline{\overline{m}}_{i_j}}{N_{i_j}};$$

hence

$$\frac{\overline{\overline{m}}_{i}}{N_{i}} = \frac{\overline{\overline{m}}_{i_1}}{N_{i_1}} = \frac{\overline{\overline{m}}_{i_2}}{N_{i_2}} = \cdots = \frac{\overline{\overline{m}}_{i_{r-1}}}{N_{i_{r-1}}} = \frac{\overline{\overline{m}}_{i_r}}{N_{i_r}} = \frac{\overline{\overline{m}}_{j}}{N_{j}}.$$

By the process we have just described, it follows that for every arbitrary pair, i and j, $\overline{\overline{m}}_i/N_i = \overline{\overline{m}}_j/N_j$. Therefore, $\overline{\overline{m}}_i = \nu N_i$ for some constant ν. Then,

$$0 = \sum_{i=1}^{n} \overline{\overline{m}}_i \phi_i = \sum_{i=1}^{n} \nu N_i \phi_i = \nu.$$

But if $\nu = 0$, then $\overline{\overline{m}}_i = 0$ for all $1 \le i \le n$. This means that the sequence of vectors $(\overline{\overline{\mathbf{m}}}^k)_{k\ge 1}$ converges to a zero vector which is a contradiction since the zero vector does not satisfy Equation (4.16). Therefore, $\alpha > 0$ satisfying (4.17) exists. Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \phi_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i}\right)^2 \ge \alpha \sum_{i=1}^{n} \frac{\phi_i}{N_i} m_i^2 \tag{4.20}$$

■

Remark 4.3.7. *We note that having a positive element in each row and column of A, different from the diagonal element, is not enough for the preceding lemma to hold. For illustration purposes, consider the matrix in Example 2.3.6. This matrix and its transpose have positive eigenvectors $\mathbf{N} = (0.25, 0.25, 0.25, 0.25)^T$ and $\phi = (1, 1, 1, 1)^T$ respectively. These vectors satisfy Equation (4.12). The left hand side of Equation (4.19) is given by*

$$\begin{aligned} LHS = \frac{1}{4}\sum_{i=1}^{4} \Bigg[& a_{i1}\left(\frac{\overline{\overline{m}}_1}{0.25} - \frac{\overline{\overline{m}}_i}{0.25}\right)^2 + a_{i2}\left(\frac{\overline{\overline{m}}_2}{0.25} - \frac{\overline{\overline{m}}_i}{0.25}\right)^2 \\ & + a_{i3}\left(\frac{\overline{\overline{m}}_3}{0.25} - \frac{\overline{\overline{m}}_i}{0.25}\right)^2 + a_{i4}\left(\frac{\overline{\overline{m}}_{4j}}{0.25} - \frac{\overline{\overline{m}}_i}{0.25}\right)^2 \Bigg] \end{aligned}$$

Since $a_{12} = a_{21} = a_{34} = a_{43} = 1$ and all other entries in A are zero, we simplify this equation to

$$\begin{aligned} LHS &= 4\left(\overline{\overline{m}}_2 - \overline{\overline{m}}_1\right)^2 + 4\left(\overline{\overline{m}}_1 - \overline{\overline{m}}_2\right)^2 \\ &\quad + 4\left(\overline{\overline{m}}_4 - \overline{\overline{m}}_3\right)^2 + 4\left(\overline{\overline{m}}_3 - \overline{\overline{m}}_4\right)^2 \\ &= 8\left(\overline{\overline{m}}_2 - \overline{\overline{m}}_1\right)^2 + 8\left(\overline{\overline{m}}_4 - \overline{\overline{m}}_3\right)^2 \\ &= 0 \end{aligned}$$

This holds if and only if $\overline{\overline{m}}_2 = \overline{\overline{m}}_1$ and $\overline{\overline{m}}_3 = \overline{\overline{m}}_4$, implying that $\overline{\overline{m}}_2/N_2 = \overline{\overline{m}}_1/N_1$ and $\overline{\overline{m}}_3/N_3 = \overline{\overline{m}}_4/N_4$. However, this does not tell us that $\overline{\overline{m}}_3/N_3 = \overline{\overline{m}}_1/N_1$, and without this, we cannot conclude that $\overline{\overline{\mathbf{m}}} = \nu\mathbf{N}$.

The result below was proved by B. Perthame [16] for positive off-diagonal matrices. We extend this to irreducible ML matrices.

Theorem 4.3.8. *Let $B \geq 0$ be an irreducible matrix with dominant eigenvalue $r > 0$ and let $A = B - rI$. Then for any solution $\mathbf{u}(t)$ satisfying (4.1), the following is true:*

$$\rho := \sum_{i=1}^{n} \phi_i u_i(t) = \sum_{i=1}^{n} \phi_i u_i(0), \tag{4.21}$$

$$\sum_{i=1}^{n} \phi_i |u_i(t)| \leq \sum_{i=1}^{n} \phi_i |u_i(0)|, \tag{4.22}$$

and there exist $\alpha > 0$ such that

$$\sum_{i=1}^{n} \phi_i N_i \left(\frac{u_i(t) - \rho N_i}{N_i} \right)^2 \leq e^{-\alpha t} \sum_{i=1}^{n} \phi_i N_i \left(\frac{u_i(0) - \rho N_i}{N_i} \right)^2 \tag{4.23}$$

Proof. $\phi > 0$ and it is independent of t. Therefore,

$$\begin{aligned} \phi \frac{d}{dt}(\mathbf{u}(t)) &= \phi A \mathbf{u}(t) \\ \Rightarrow \frac{d}{dt}(\phi \mathbf{u}(t)) &= \phi A \mathbf{u}(t) \end{aligned}$$

But ϕ is a left eigenvector of A corresponding to eigenvalue 0. Therefore, $\phi A = 0$, implying that $\phi A \mathbf{u}(t) = 0$ hence

$$\frac{d}{dt}(\phi \mathbf{u}(t)) = 0,$$

thus $\phi \mathbf{u}(t) = \phi \mathbf{u}(0)$ and from this, we can conclude that (4.21) holds. From Theorem 4.2.1, the solution to (4.1) is given by $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$, therefore

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0) = e^{t(B-rI)}\mathbf{u}(0) = e^{tB}e^{-rt}\mathbf{u}(0)$$

so that $|\mathbf{u}(t)| = e^{tB}e^{-rt}\mathbf{u}(0) \leq |e^{-rt}||e^{Bt}||\mathbf{u}(0)|$. But

$$\begin{aligned} |e^{-rt}||e^{Bt}||\mathbf{u}(0)| &\leq e^{-rt}e^{|B|t}|\mathbf{u}(0)| \\ &= e^{-rt}e^{Bt}|\mathbf{u}(0)|, (\text{ since } |B| = B \geq 0) \\ \phi|\mathbf{u}(t)| &\leq e^{-rt}\phi e^{Bt}|\mathbf{u}(0)| \\ &= e^{-rt}e^{rt}\phi|\mathbf{u}(0)| \\ &= \phi|\mathbf{u}(0)|, \end{aligned}$$

that is,

$$\sum_{i=1}^{n} \phi_i |u_i(t)| \leq \sum_{i=1}^{n} \phi_i |u_i(0)|.$$

To prove the last inequality of Theorem 4.3.8, notice that

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}(t)$$

implies that

$$\frac{d}{dt}(\mathbf{u}(t) - \rho\mathbf{N}) = A(\mathbf{u}(t) - \rho\mathbf{N}).$$

Taking the dot product of both sides by $\mathbf{u}(t) - \rho\mathbf{N}$ in the weighted Hilbert space defined in (4.15), we get

$$\left(\frac{d}{dt}(\mathbf{u}(t) - \rho\mathbf{N})\right) . (\mathbf{u}(t) - \rho\mathbf{N}) = (A(\mathbf{u}(t) - \rho\mathbf{N})) . (\mathbf{u}(t) - \rho\mathbf{N}),$$

which can be simplified as

$$\frac{d}{dt}\left[(\mathbf{u}(t) - \rho\mathbf{N}).(\mathbf{u}(t) - \rho\mathbf{N})\right] = 2\left(A(\mathbf{u}(t) - \rho\mathbf{N})\right) . (\mathbf{u}(t) - \rho\mathbf{N}).$$

The left hand side of the equation is simply

$$\frac{d}{dt}\sum_{i=1}^{n} \frac{\phi_i}{N_i}(u_i(t) - \rho N_i)^2$$

while the right hand side is

$$\begin{aligned} RHS &= 2\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\phi_i}{N_i} a_{ij}(u_j(t) - \rho N_j)(u_i(t) - \rho N_i) \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n} \phi_i a_{ij} N_j \left(\frac{u_j(t) - \rho N_j}{N_j}\right)\left(\frac{u_i(t) - \rho N_i}{N_i}\right). \end{aligned}$$

Now we also have that

$$\sum_{j=1}^{n} a_{ij} N_j = 0, \text{ for all } i$$

and for all j,

$$\sum_{i=1}^{n} \phi_i a_{ij} = 0.$$

Therefore,

$$\begin{aligned} RHS = &-\Bigg[\sum_{i=1}^{n} \phi_i \left(\frac{u_i(t) - \rho N_i}{N_i}\right)^2 \sum_{j=1}^{n} a_{ij} N_j \\ &-2\sum_{i=1}^{n}\sum_{j=1}^{n} \phi_i a_{ij} N_j \left(\frac{u_j(t) - \rho N_j}{N_j}\right)\left(\frac{u_i(t) - \rho N_i}{N_i}\right) + \sum_{j=1}^{n}\left(\frac{u_j(t) - \rho N_j}{N_j}\right)^2 N_j \sum_{i=1}^{n} \phi_i a_{ij}\Bigg] \\ &= -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\phi_i N_j \left(\frac{u_j(t) - \rho N_j}{N_j} - \frac{u_i(t) - \rho N_i}{N_i}\right)^2. \end{aligned}$$

If $i = j$, then RHS equals 0, and since $a_{ij} \geq 0$ for $i \neq j$, the sum is positive. Therefore,

$$\frac{d}{dt}\sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(t)-\rho N_i)^2 \leq 0, \tag{4.24}$$

implying that

$$\sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(t)-\rho N_i)^2$$

is a non-increasing function of time. Notice that by using (4.21)

$$\sum_{i=1}^{n}\phi_i(u_i(t)-\rho N_i) = 0,$$

so the vector $\mathbf{u}(t) - \rho\mathbf{N}$ satisfies the conditions of Lemma 4.3.6 above. Therefore, there is $\alpha > 0$ such that

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(t)-\rho N_i)^2 &= -\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\phi_i N_j\left(\frac{u_j(t)-\rho N_j}{N_j}-\frac{u_i(t)-\rho N_i}{N_i}\right)^2 \\ &\leq -\alpha\sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(t)-\rho N_i)^2.\end{aligned}$$

Hence, as in Gronwall's lemma,

$$\frac{d}{dt}\sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(t)-\rho N_i)^2 + \alpha\sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(t)-\rho N_i)^2 \leq 0$$

implies

$$\frac{d}{dt}\left(e^{\alpha t}\sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(t)-\rho N_i)^2\right) \leq 0.$$

Thus

$$e^{\alpha t}\sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(t)-\rho N_i)^2 - \sum_{i=1}^{n}\frac{\phi_i}{N_i}(u_i(0)-\rho N_i)^2 \leq 0$$

and from this, we get (4.23). ■

If A is reducible, we note that Equation (4.21) and Inequality (4.22) still hold because ϕ and $\mathbf{N}$ are non-negative. On the other hand, Inequality (4.23) is not valid for general reducible matrices. Consider the example below:

Example 4.3.9. *Let*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of A are 0 and -1, and their eigenvectors are $\mathbf{N} = (0,1)^T$ and $(1,0)^T$, respectively. The left eigenvector ϕ corresponding to 0 is $\phi = (0,1)$. Let $\mathbf{u}(0) = (u_{01}, u_{02})^T$. Then the solution to (4.1) is given by

$$\mathbf{u}(t) = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $k_1 = u_{02}$ and $k_2 = u_{01}$. Therefore, $\rho = \phi\mathbf{u}(0) = u_{02}$. However, since $\mathbf{N}$ is not strictly positive, we see that ϕ_1/N_1 is not defined. Therefore, Theorem 4.3.8 doesnot hold for this matrix.

However, there are certain special cases of reducible matrices for which the theorem holds.

Example 4.3.10. *Let*

$$A = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

The matrix has one eigenvalues 0 and -1 with corresponding eigenvectors $\mathbf{N} = 0.5(1,1)^T$ and $\mathbf{x} = (0,1)^T$, respectively. The left eigenvector ϕ corresponding to 0 is $\phi = (2,0)$. Let $\mathbf{u}(0) = (u_{01}, u_{02})^T$. Then the solution to (4.1) is given by

$$\mathbf{u}(t) = k_1 \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $k_1 = 2u_{01}$ and $k_2 = u_{02} - u_{01}$. Therefore, $\rho = \phi\mathbf{u}(0) = 2u_{01}$,

$$\sum_{i=1}^{2} \frac{\phi_i}{N_i} (u_i(0) - \rho N_i)^2 = \frac{2}{0.5} \left(u_{01} - \frac{2u_{01}}{2} \right)^2 = 0,$$

and

$$\sum_{i=1}^{2} \frac{\phi_i}{N_i} (u_i(t) - \rho N_i)^2 = \frac{2}{0.5} \left(u_{01} - \frac{1}{2}(2u_{01}) \right)^2 = 0.$$

Therefore,

$$\sum_{i=1}^{2} \frac{\phi_i}{N_i}(u_i(0) - \rho N_i)^2 = \sum_{i=1}^{2} \frac{\phi_i}{N_i}(u_i(t) - \rho N_i)^2 = 0,$$

implying that $\alpha > 0$ that satisfies the theorem can be any positive number.

As seen in the preceding example, the matrix doesnot satisfy all the conditions of Lemma 4.3.6 and Theorem 4.3.8, but the theorem holds in this case. Therefore, we plan to extend this theory to general ML matrices in the future.

Chapter 5

Applications to population biology

5.1 Introduction

In this chapter, we look at the applications of the theory studied in the previous chapters to populations structured by age. Population growth is directly affected by three major factors; birth, death and migration. For most species, reproduction occurs only at certain stages of an individual's lifetime and fertility tends to decrease with age. It is therefore important to take these into account when studying the structure of such populations. In such a case, we say that the population is structured by age. In this thesis, we consider a population that is divided into a finite number of age classes and thus our model is called a continuous time-discrete age model [12]. The assumptions of the models are as follows:

1. That only the female population is responsible for birth and that the population growth rate of the male population is the same as that for the female population.
2. Birth rate only depends on the age of the individual.
3. The probability of survival from one age class to another only depends on the age of the individual.
4. The system is open; that individuals can migrate into and out of the system

Consider a system with population at time t given by $u(t)$ which is not structured in any way, with birth rate λ and death rate μ. If emigration and immigration are allowed, then immigration contributes to the population's growth while emigration contributes to decrease. Let the number of individuals emigrating out of the system be E and those immigrating into it be I. Let $B(t)$ and $D(t)$ be the number of births and deaths per unit time, respectively. If $B(t)+I > D(t)+E(t)$, then the population will grow, and decay will occur otherwise. The population after a short interval of time h will be given by $u(t+h)$,

$$\begin{aligned} u(t+h) &= u(t) + (B(t) + I(t) - E(t) - D(t))h \\ \frac{u(t+h) - u(t)}{h} &= B(t) + I(t) - E(t) - D(t) \\ \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} &= \frac{du(t)}{dt} = B(t) + I(t) - E(t) - D(t) \end{aligned}$$

Therefore

$$\frac{du}{dt} = B(t) + I(t) - (D(t) + E(t)). \tag{5.1}$$

5.2 Birth, death and migration problem

Consider a hypothetical system whose population is structured by age, with six age classes. Classes 1 and 2 are for juveniles with individuals in class 1 having a higher mortality rate than those in 2. Classes 3, 4 and 5 are the reproductive age classes while 6 is the post reproductive class. Let μ_i and λ_i be the death and birth rates, respectively, of the population in class i and let s_i be the fraction of the population in class i that migrates to the next age class through ageing. Let the population at time t be $u(t)$, where $u(t)$ is a vector defined by

$$\mathbf{u}(t) = (u_1, u_2, \cdots, u_6)^T,$$

where u_i is the population in the i^{th} age class.

5.2.1 Closed system

Definition 5.2.1. *A population system is called closed if migrations into and out of the system can be neglected, otherwise it is called open ([17], page* 11*).*

In this subsection, we consider the system described above with no emigration or immigrations. The only migration that occurs is within the system due to ageing.

In the interval $(t, t+\Delta t)$ the population of individuals in age class 1 will increase due to birth by individuals in age classes $3, 4, 5$ and can decrease due death among members of class 1 and growth of some children to age class 2. The population of juveniles in class 2 will be increased by a fraction of juveniles in age class 1 that are mature enough to join 2 and can be decreased by death of some of the individuals in this age class and ageing, where some individuals mature to age 3 within the same time interval. Populations in age classes $3, 4, 5, 6$ can only be increased by a fraction of individuals that mature physiologically from age classes $2, 3, 4$, and 5 respectively; and the populations in these age classes can be decreased by death and ageing (classes $3, 4, 5$) and death alone for 6.

We assume that the number of births (for the reproductive classes) and deaths per unit time in age class i is proportional to the number of individuals in the particular age class, u_i. Thus $D_1(t) = \mu_1 u_1(t)$, $B_1(t) = \lambda_3 u_3 + \lambda_4 u_4 + \lambda_5 u_5$. The system of differential equations describing this population change is given below:

$$
\begin{aligned}
\dot{u}_1(t) &= -(\mu_1 + s_1)u_1 + \lambda_3 u_3 + \lambda_4 u_4 + \lambda_5 u_5 \\
\dot{u}_2(t) &= s_1 u_1 - (\mu_2 + s_2)u_2 \\
\dot{u}_3(t) &= s_2 u_2 - (\mu_3 + s_3)u_3 \\
\dot{u}_4(t) &= s_3 u_3 - (\mu_4 + s_4)u_4 \\
\dot{u}_5(t) &= s_4 u_4 - (\mu_5 + s_5)u_5 \\
\dot{u}_6(t) &= s_5 u_5 - \mu_6 u_6,
\end{aligned}
\tag{5.2}
$$

The diagram showing the dynamics of the system is drawn below where the arrows pointing

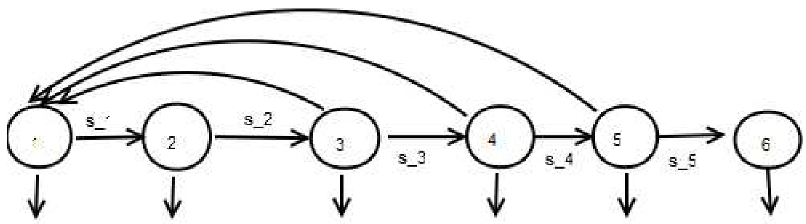


Figure 5.1: Diagram showing the population changes in an age structured population

down represent permanent removal from the system by death. The matrix for the system above is given below

$$A = \begin{bmatrix} -(\mu_1+s_1) & 0 & \lambda_3 & \lambda_4 & \lambda_5 & 0 \\ s_1 & -(\mu_2+s_2) & 0 & 0 & 0 & 0 \\ 0 & s_2 & -(\mu_3+s_3) & 0 & 0 & 0 \\ 0 & 0 & s_3 & -(\mu_4+s_4) & 0 & 0 \\ 0 & 0 & 0 & s_4 & -(\mu_5+s_5) & 0 \\ 0 & 0 & 0 & 0 & s_5 & -\mu_6 \end{bmatrix}.$$

By analysing this matrix, we can determine the reproductive structure of the population and its long time behaviour. This ML matrix has only one non-zero element in the sixth column, which is also in the sixth row. So A is associated with a non-negative matrix B through the equation $A = B - rI$, where

$$r \geq \max_{1\leq i\leq 6} \mu_i + s_i > 0. \tag{5.3}$$

$$B = \begin{bmatrix} v_1 & 0 & \lambda_3 & \lambda_4 & \lambda_5 & 0 \\ s_1 & v_2 & 0 & 0 & 0 & 0 \\ 0 & s_2 & v_3 & 0 & 0 & 0 \\ 0 & 0 & s_3 & v_4 & 0 & 0 \\ 0 & 0 & 0 & s_4 & v_5 & 0 \\ 0 & 0 & 0 & 0 & s_5 & v_6 \end{bmatrix},$$

where $r - v_i = \mu_i + s_i$ for $i = 1, \cdots, 5$ and $v_6 = r - \mu_6$. By Lemma 2.3.5, matrix B is reducible, hence A is also reducible by Definition (4.3.2). Matrix B is already in the form of (2.1) where A_1 is the 5×5 matrix,

$$\begin{pmatrix} v_1 & 0 & \lambda_3 & \lambda_4 & \lambda_5 \\ s_1 & v_2 & 0 & 0 & 0 \\ 0 & s_2 & v_3 & 0 & 0 \\ 0 & 0 & s_3 & v_4 & 0 \\ 0 & 0 & 0 & s_4 & v_5 \end{pmatrix},$$

obtained from B by removing the sixth row and column and $A_2 = (v_6)$ and $A_{21} = (0, 0, 0, 0, s_5)$. Notice that $v_i \geq 0$ for all $i = 1, \cdots, 5$ (from Inequality 5.3), $s_i > 0$ and λ_1, λ_2 and λ_3 are all positive. The graph of A_1 is strongly connected, implying that A_1 is irreducible. Therefore B, and thus A, are in normal form.

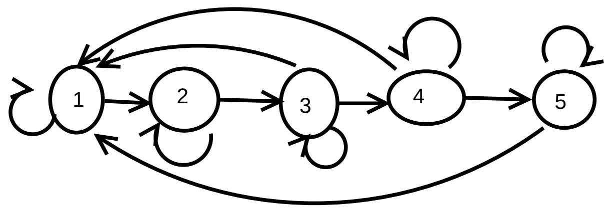


Figure 5.2: Graph of A_1

Parameter values were varied until we got two different situations but in each case, an initial distribution of $(1, 1, 1, 1, 1, 1)^T$ was assumed. In the first case, the parameter values that were used are; $\mu_1 = 0.2, \mu_2 = 0.13, \mu_3 = 0.08, \mu_4 = 0.062, \mu_6 = 0.09, \mu_6 = 0, s_1 = 0.7, s_2 = 0.8, s_3 = 0.5, s_4 = 0.4, s_5 = 0.9, \lambda_3 = 0.2, \lambda_4 = 0.062, \lambda_5 = 0.1$, and the corresponding matrix has a dominant eigenvalue 0 which is simple. From Theorem 4.2.5, the total population is expected to become constant in the long run, and this can be seen from Figure 5.3.

Because 0 is also an eigenvalue of the submatrix corresponding to the post reproductive class, the population of post reproductive individuals initially increases and asymptotically tends to a constant value after a long time. After 40 years, the population was found to be distributed as follows, $(0.00144083, 0.00133734, 0.00264709, 0.00462506, 0.00227229, 5.58839)^T$. In the next simulation, the parameter values that were used are; $\mu_1 = 0.2$, $\mu_2 = 0.13$, $\mu_3 = 0.08$, $\mu_4 = 0.062$, $\mu_5 = 0.09$, $\mu_6 = 0.17$, $s_1 = 0.7$, $s_2 = 0.8$, $s_3 = 0.5$, $s_4 = 0.4$, $s_5 = 0.9$, while the birth rates were set to $0.08, 0.062, 0.09$ for λ_3, λ_4 and λ_5 respectively. In each case, an initial

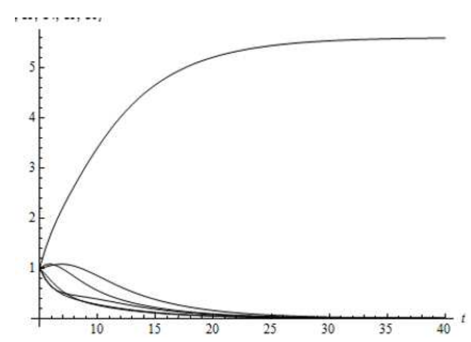


Figure 5.3: Population dynamics with birth rate greater than death rate for class 3, 4 and 5.

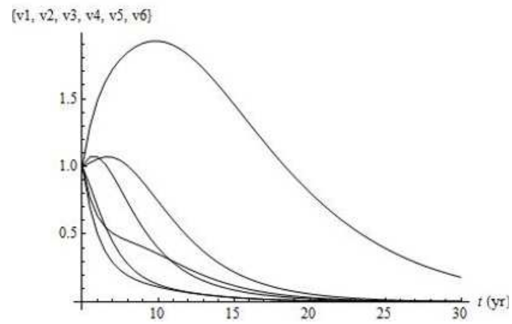


Figure 5.4: The birth rates are now less than the death rates. Similar behaviour is obtained with equal birth

population of 1 was assumed for each age class. The dominant eigenvalue in this case is -0.17, so by Theorem 4.2.4, the system is stable. From the definition of stability given in (4.2.3), it means that the total population becomes extinct in the long run.

This is what is observed numerically from Figure 5.4. The population for age classes $3, 4$ and 6 initially increases but for class 3, it starts falling after just one year while that of class 4 increases for the first 5 years. The population for the post reproductive class increases for the first 15 years and then starts declining slowly. After 30 years, the population in the system was found to be distributed as $(0.000610191, 0.000642678, 0.0016341, 0.00415513, 0.00229366, 0.182925)^T$

From figure 5.4, we see that in the long run the population becomes extinct as predicted by Theorem 4.2.4.

5.2.2 An open system

Consider a life cycle graph for a spatially structured population with migration shown below (this is discussed in example 4.2 of [4] in discrete time). In each habitat, the population is divided

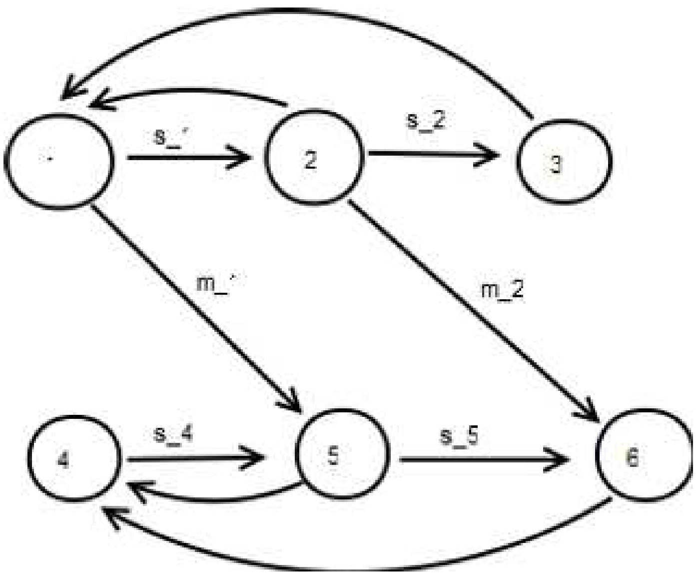


Figure 5.5: Life cycle graph for a species distributed in two habitats

into three age classes and individuals in age class 1 and 2 can move to habitat 2. Let m_1 and m_2 be the migration rate from class 1 and 2 respectively and s_i be the fraction of individuals from age class i that mature to age $i + 1$ and μ_i be death rate. The equations describing the rate of change of the population at time t are given by

$$\begin{aligned}
\dot{u}_1(t) &= -(\mu_1 + m_1 + s_1)u_1 + \lambda_2 u_2 + \lambda_3 u_3 \\
\dot{u}_2(t) &= s_1 u_1 - (\mu_2 + s_2 + m_2)u_2 \\
\dot{u}_3(t) &= s_2 u_2 - \mu_3 u_3 \\
\dot{u}_4(t) &= -(\mu_4 + s_4)u_4 + \lambda_5 u_5 + \lambda_6 u_6 \\
\dot{u}_5(t) &= m_1 u_1 + s_4 u_4 - (\mu_5 + s_5)u_5 \\
\dot{u}_6(t) &= m_2 u_2 + s_5 u_5 - \mu_6 u_6.
\end{aligned} \tag{5.4}$$

This system can be written as $\dot{\mathbf{u}}(t) = A\mathbf{u}(t)$, where

$$A = \begin{bmatrix}
-(s_1 + m_1 + \mu_1) & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\
s_1 & -(\mu_2 + s_2 + m_2) & 0 & 0 & 0 & 0 \\
0 & s_2 & -\mu_3 & 0 & 0 & 0 \\
0 & 0 & 0 & -(\mu_4 + s_4) & \lambda_5 & \lambda_6 \\
m_1 & 0 & 0 & s_4 & -(\mu_5 + s_5) & 0 \\
0 & m_2 & 0 & 0 & s_5 & -\mu_6
\end{bmatrix}.$$

A plot of the population of the age classes as a function of time with different parameter values is shown below. For the values $\mu_1 = 0.2$, $\mu_2 = 0.13$, $\mu_3 = 0.08$, $\mu_4 = 0.062$, $\mu_5 = 0.09$, $\mu_6 = 0.17$, $s_1 = 0.7$, $s_2 = 0.8$, $s_3 = 0.5$, $s_4 = 0.4$, $s_5 = 0.1$, $\lambda_2 = 0.26$, $\lambda_4 = 0.41$, $\lambda_5 = 0.14$, $\lambda_6 = 0.19$, $m_1 = 0.3$, $m_2 = 0.12$, the system of equations was solved using Mathematica software with the matrix A obtained by substituting these parameters in (5.4). In order to have 0 as the dominant eigenvalue, A was rescaled to $\tilde{A} = A - 0.096134988582026651I$. The matrix $\tilde{A}$ has eigenvalues $-1.3092 + 0.14037\imath, -1.3092 - 0.14037\imath, -0.656739, -0.371753, 0, -0.0819135$. 0 is clearly the dominant eigenvalue with corresponding right and left eigenvectors

$$\mathbf{x} = (0.236215, 0.144268, 0.655261, 0.246413, 0.592133, 0.287544)^T \text{ and}$$

$$\mathbf{v} = (0.318676, 0.590066, 0.7418, 0., 0., 0.) \text{ respectively .}$$

Normalising these vectors according to (4.12), we obtain

$$\mathbf{N} = \frac{1}{2.16183}\mathbf{x}, \quad \phi = \frac{2.16183}{0.646477}\mathbf{v} \text{ hence } \rho = \frac{2.16183}{0.646477}\mathbf{v}.\mathbf{1} = 5.51945.$$

The population at time t is shown in the diagram below: Since 0 is the dominant eigenvalue and

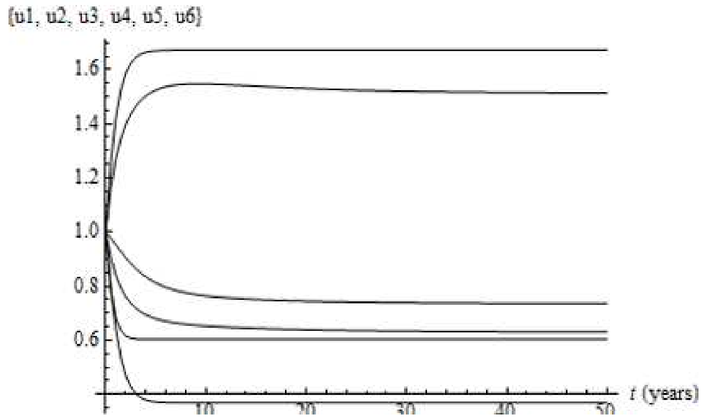


Figure 5.6: A graph showing the population in all the six subclasses as it changes with time

is simple, we see that the populations do not increase indefinitely but settle to some constant values. By Equation (4.11), $\mathbf{u}(t) \to \rho\mathbf{N}$. That is:

$$\mathbf{u}(t) \to \frac{5.51945}{2.16183}\begin{pmatrix} 0.236215 \\ 0.144268 \\ 0.655261 \\ 0.24641 \\ 0.592133 \\ 0.287544 \end{pmatrix} = \begin{pmatrix} 0.603089 \\ 0.368336 \\ 1.67297 \\ 0.629126 \\ 1.51179 \\ 0.734138 \end{pmatrix}$$

and from the figure above, we see that this is true. In particular, after 50 years, the population will be distributed as $(0.603089, 0.368336, 1.67297, 0.629948, 1.5134, 0.735011)^T$.

5.3 Conclusion and further work

Matrix models are very common in biological problems. They are used in ecology and in epidemiology. Although these matrices are not non-negative in general, the theory of non-negative matrices can be used to understand these population matrices. Therefore, in the second chapter, we studied the theory of non-negative matrices and provided an overview of their properties which make the study of long time behaviour of matrix population models easier. We showed the relationship between matrices and graphs and the motivation for this is that in epidemiology we often have compartmentalised diagrams while in ecology, we may consider structured populations. These are graphical representations of movement of individuals in and out of compartments or classes.

In the third chapter, we studied Perron-Frobenius theorems for both positive and irreducible matrices and discussed some of their proofs. We also provided a complete description of the Perron-Frobenius type theorems for reducible matrices.

In the fourth chapter, we discussed the nature and asymptotic behaviour of solution to the linear initial value problem (4.1). We discussed long time behaviour of solutions to the problem for both reducible and irreducible matrices, and we concluded the chapter with an analysis of the nature of solutions based on Perthame's [16] entropy methods. We found that when A is irreducible and 0 is its dominant eigenvalue, then

$$\mathbf{u}(t) \to \rho\mathbf{N}.$$

This was further justified by inequality (4.23). If A is reducible, results similar to those obtained for an irreducible matrix are possible only when 0 is a simple dominant eigenvalue of A; that is,

$$\begin{aligned}\mathbf{u}(t) &\to P(\mathbf{N} * \phi)P^{-1}\mathbf{u}(0) \\ &= P\mathbf{N}[(\phi P^{-1})\mathbf{u}(0)] \\ &= \hat{\rho}P\mathbf{N} \text{ as } t \to \infty,\end{aligned}$$

where $\hat{\rho} = (\phi P^{-1})\mathbf{u}(0)$ is a scalar.

In the future, we intend to find an analogue of Theorem 4.3.8 for reducible matrices and to extend the results to infinite dimensional spaces.

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