STOCHASTIC VOLATILITY EFFECTS ON DEFAULTABLE BONDS

by

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This dissertation is submitted in fulfilment of the requirements for the degree of Master of Science in the Department of Statistics and Actuarial Science, University of KwaZulu-Natal, Durban

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Disclaimer

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T Mkhize

December 2009

Abstract

We study the effects of stochastic volatility of defaultable bonds using the first-passage structural approach. In this approach Black and Cox (1976) argued that default can happen at any time. This then led to the development of a first-passage model, in which a firm (company) default occurs when its value falls to a barrier. In the first-passage model the firm debt is considered to be a single pure discount bond and default occurs only if the firm value falls below the face value of the bond at maturity. Here the firm's debt can be viewed as a portfolio composed of a risk-free bond and a short-put option on the value of a firm. The classic Black-Scholes-Merton model only considers a single liability and the solvency is tested at the maturity date, while the extended Black-Scholes-Merton model allows for default at any time before maturity to cater for more complex capital structures and was delivered by Geske, Black-Cox, Leland, Leland and Toft and others. In this work a review of the effect of stochastic volatility on defaultable bonds is given. In addition a study from the first-passage structural approach and reduced-form approach is made. We also introduce symmetry analysis to study some of the equations that appear in option-pricing models. This approach is quite recent and has produced successful results. In this work we lay the foundation of this method.

Keywords: Stochastic Volatility, Defaultable bonds, Lie Symmetries.

Preface

The work described in this dissertation was carried out in the School of Statistics and Actuarial Science at the University of KwaZulu-Natal, under the supervision of Prof. J.G. O'Hara.

This dissertation does not contain other person's data, pictures, graphs or other information, unless where specifically acknowledged. It has not been submitted in any form for any degree or diploma to any other tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Thembisile Gloria Mkhize

December 2009

As the candidate's supervisor I have approved this dissertation for submission.

Signed: Name: Date:

Summary

In this work we introduce the theoretical framework of pricing stochastic volatility models, which is inconsistent with the pricing of the classical Black-Scholes Models. Further we look at the effect of introducing stochastic volatility into the first-passage structural approach to default risk.

In Chapter 1 we introduce the Black-Scholes (BS) option-pricing model which provides the foundation for the modern theory of option valuations. Based upon the BS assumptions, we price European call options using different approaches. In Chapter 2 we look at the barrier and digital options. Barrier options are options in which the payoff function is based upon the underlying asset either passing above or below a certain level known as the barrier. They are more popular than standard options because they are activated when the price of the underlying asset hits a boundary price. The boundary price of the barrier option depends upon the probability of the barrier and the value of the underling option if it is reached. They are therefore very sensitive to volatility. In Chapter 3 we assume that volatility is a stochastic process rather than a constant. It is then possible to model prices of options more accurately. The idea is to model volatility as a stochastic process dependent upon a further external parameter.

In Chapter 4 we introduce the bond-pricing and interest-rate models. The introduction of options on bonds leads to two complications. The price of the bond depends upon the level of interest rate and the interest rate cannot be constant since this would mean that the volatility of the derivative security is zero. The second complication is that bond options are, in general, of the American style and may be exercised as desired before the expiration date. Finally the payoff of the underlying security may be different for bond options. If the bond has a coupon, the underlying security is different from nondividend-paying stock.

We introduce the symmetry analysis of the Black-Scholes equation in Chap-

ter 5. We look at some basic symmetry properties of a group and then use a Lie group transformation to obtain the closed-form solution of the Black-Scholes equation. In Chapter 6 we present the theory of pricing the defaultable risk based on the first-passage structural models.

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Dedication

A special dedication to my mom Angela for her unconditional love and encouragements. To my two aunts: Mtoti and Mkhosi, to my sisters: Dolly and Masho and to: Lungelo, Njabulo, Siya, Gqabho, Nonto, Ndumo, Andile, Nhlanvu and Siyamthanda I say thank you for your love and support. Finally to my late father Ndabayakhe Moses and my two late brothers: Musawenkosi and Sibongiseni, may your souls rest in peace.

Contents

Chapter 1

The Classical Black-Scholes Model

1.1 Introduction

The Black-Scholes model is a tool for pricing options. Despite the market crash of 1987 and recently of 2008, in practice Black-Scholes models are still extensively used more than any other financial model. It was introduced in 1973 by Black F and Scholes M [7] as their benchmark option-pricing model.

Options were an attractive investment with a potential to reduce risk, but there was no standard or reliable way to price them. The Black-Scholes option-pricing model remedied the situation and remains the standard model for the pricing of options some three decades later.

In financial terms an option is a contract that gives its owner the right, but not the obligation, to buy or sell an underlying asset at a future point in time at an agreed price called a strike price or the exercise price. The act of buying and selling the asset is known as exercising the option. The well known basic options are the American and European options. American Options can be exercised at any time prior to the expiration date, whereas European options can only be exercised at the expiration date. There are two main types of options namely the call option and the put option. Call options give the holder the right to buy an asset at a fixed price, whereas put options give the holder the right to sell an asset at a fixed price.

A bond is a long-term contract under which the issuer (or borrower) promises to pay the bondholder the coupon and principal on a specified date known as the maturity date. They are usually repayable at par (nominal, principal, face) value. Many bonds provide coupons periodically. A coupon is the amount the bondholder receives as interest payment. The principal value is the money a bondholder gets back once a bond matures. The maturity date is the date on which the principal value must be repaid. The current market price, determined by supply and demand, is expressed as a percentage of the bond's principal value. The principal value is not the price of a bond. A bond's price fluctuates throughout its life in response to a number of variables. A bond is called a premium bond if the bond price rise above its principal value and is called a discount bond if otherwise. The market value of a bond always approaches its principal value as maturity is approached. This is known as the pull-to-par phenomenon.

Bonds are issued by a firm or government treasury department through auctions on the primary market, at a price usually close to the principal value. Once issued they may be traded on the secondary market. The bond market, also known as the fixed-income market, is a financial market in which investors buy and sell debt securities usually in the form of bonds. In bond markets investors who buy and sell bonds before maturity are exposed to many risks, most importantly changes in the interest rates. When the interest rate increases, the value of a bond falls and when it decreases, the value of a bond rises.

Options allow risk to be hedged in various ways. If the price of the security rises above the strike price, the investor is able to sell the security at its market value. By taking combinations of long and short positions in put and call options, investors can create a variety of customized contingent claims also known as derivative securities.

The existence of derivative securities leads to two mathematical concepts: pricing and hedging. The hedging of a derivative security is the problem faced by the financial institution that sells some contract designed to reduce risk.

Assumption of an investor's risk is a principal service of financial institutions; managing this risk well is a necessary prerequisite for offering this service. More recently model-based pricing of derivative securities has become the basis of risk management. A typical risk-management question is how much a portfolio value is affected by certain movements in the underlying asset price [58]. If the portolio contains derivative securities, a mathematical model is needed to answer this question.

In this Chapter we present the theory of arbitrage pricing for European derivatives and compute the prices and hedging portfolios for European call options. We derive the well-known Black-Scholes formula using different approaches.

1.2 The Black-Scholes Assumptions

In the modeling of the Black-Scholes equation we assume the following:

- The price S of an asset follows a geometric Brownian motion[1] given by

$$dS = \mu S dt + \sigma S dW, \tag{1.2.1}$$

  where μ is the drift which is the measure of the average rate of growth of the asset, σ is the volatility which measures the standard deviation of the return and W is a Brownian motion. In addition both μ and σ are assumed to be constant.

[1]Brownian motion (or a Wiener process) is a stochastic process with the following properties:

$W_0 = 0$; for every $v \leq t \leq T$, the increment $W_t - W_v$ is an independent random variable and every increment $W_t - W_v$ is normally distributed with mean 0 and variance $t - v$.

- Trading is continuous.
- The interest rate r is assumed to be risk-free, is the same for all maturities and is constant.
- There are no dividends paid during the life of the option.
- There are no transaction costs or taxes.
- There are no arbitrage opportunities in the markets.
- Short-selling is accommodated.

1.3 The Black-Scholes Pricing Formula

Based upon the assumptions of the Black-Scholes model which have been prescribed above, the Black-Scholes equation for a European (vanilla) call option takes the form

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} = ru, \tag{1.3.1}$$

where u is the European call value, r is the interest rate, S is the underlying asset price, σ is the volatility and t is the current time. In this case, where r and σ are constant, the explicit solution for the European call is given by

$$u_{BS} = S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) \tag{1.3.2}$$

with $\Phi(\cdot)$ being the cumulative distribution function for the standardised normal random variable and given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz. \tag{1.3.3}$$

Here

$$d_+ = \frac{\ln \frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{1.3.4}$$

and

$$d_- = \frac{\ln \frac{S}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_+ - \sigma\sqrt{T-t}. \tag{1.3.5}$$

In this instance T is the expiration date. When one uses the same parameters, the closed-form solution of a Black-Scholes put option is given by

$$p_{BS} = Ke^{-r(T-t)}\Phi(-d_-) - S\Phi(-d_+). \tag{1.3.6}$$

We further use the approach of partial differential equation to demonstrate the use of arbitrage and hedging to convert our pricing problem into a partial differential equation as in equation (1.3.1). We introduce for $0 \le t \le T$ the unknown price function, $u(S,t)$, denoting the proper price of the option at time t.

To eliminate risks consider a portfolio given by $\pi = u - \Delta S$, that is, it consists of a

- 1 : *derivative*

and

- $-\Delta$: *assets*.

The change of portfolio is then denoted by

$$d\pi = du - \Delta dS. \tag{1.3.7}$$

By Itô's lemma[2]

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial S}dS + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}(dS)^2 \tag{1.3.10}$$

and, since from (1.2.1)

$$(dS)^2 = \sigma^2 S^2 dt, \tag{1.3.11}$$

[2]Suppose that a value of a variable x follows an Itô process,

$$dx = a(x,t)dt + b(x,t)dW. \tag{1.3.8}$$

Itô's lemma shows that the function G follows the process

$$dG = \left(\frac{\partial G}{\partial t} + a\frac{\partial G}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial x^2}\right)dt + b\frac{\partial G}{\partial x}dW. \tag{1.3.9}$$

it follows that

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial S}(\mu S dt + \sigma S dW) + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}(\sigma^2 S^2 dt). \tag{1.3.12}$$

Substituting (1.3.12) into (1.3.7) we obtain

$$\begin{aligned} d\pi &= \left(\frac{\partial u}{\partial t} + \mu S\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2}\right)dt + \sigma S\frac{\partial u}{\partial S}dW - \Delta(\mu S dt + \sigma S dW) \\ &= \left[\frac{\partial u}{\partial t} + \mu S\left(\frac{\partial u}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2}\right]dt + \sigma S\left(\frac{\partial u}{\partial S} - \Delta\right)dW. \end{aligned} \tag{1.3.13}$$

To avoid the uncertainty, we put

$$\Delta = \frac{\partial u}{\partial S}. \tag{1.3.14}$$

This means that

$$\begin{aligned} d\pi &= \left(\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2}\right)dt \\ &= r\pi dt \\ &= r(u - \Delta S)dt \\ &= (ru - r\Delta S)dt, \end{aligned}$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate, that is,

$$ru - r\Delta S = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2}$$

and

$$ru - rS\frac{\partial u}{\partial S} = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2}.$$

Therefore the nondividend Black-Scholes partial differential equation(PDE) for an European option is

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2} = ru. \tag{1.3.15}$$

We can see that the Black-Scholes equation is independent of μ, the expected rate of growth of the underlying asset price, and thus the principle of being risk-neutral is satisfied. Our portfolio has clearly being hedged since it has no stochastic component.

Furthermore we consider terminal conditions to ensure that equation (1.3.15) has a unique solution. At $t = T$ the value of a call is known with certainty to be the payoff:

$$\begin{aligned} u_{BS}(S,T) &= h(S_T) \\ &= Max(S-K,0) \\ &= (S-K,0)^{+}. \end{aligned} \tag{1.3.16}$$

This means that, as we get closer to the expiry date T, we can expect the value of our call option to approach (1.3.16).

Our terminal conditions for the asset price are applied at zero asset price, $S = 0$, and as $S \to \infty$. If $S = 0$ at the expiry, the payoff is zero. Thus the call option is worthless on $S = 0$ even if there is a long time to expiry. Hence on $S = 0$ we have $u_{BS}(0,t) = 0$.

As the asset price increases without bound, it becomes ever more likely that the option will be exercised and the value of the exercise price becomes less and less important. Thus, as $S \to \infty$, the value of the option becomes that of the asset and we can write $u_{BS}(S,t) \approx S$ as $S \to \infty$.

Equation (1.3.15) can be transformed to the diffusion equation using the transformations: $\eta = \ln\frac{S}{K} + (r - \frac{1}{2}\sigma^2)(T-t)$ and $\tau = T - t$, where a function $c(\eta,\tau)$ is defined by

$$h(S_T) = e^{-r\tau}c(\eta,\tau). \tag{1.3.17}$$

The partial derivatives give

$$\begin{aligned} \frac{\partial u}{\partial t} &= re^{-r\tau}c + e^{-r\tau}\frac{\partial c}{\partial t} \\ &= re^{-r\tau}c + e^{-r\tau}\left(\frac{\partial c}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial t}\right) \\ &= re^{-r\tau}c + e^{-r\tau}\left(-\frac{\partial c}{\partial \tau} - (r - \frac{1}{2}\sigma^2)\frac{\partial c}{\partial \eta}\right) \\ &= e^{-r\tau}\left(rc - \frac{\partial c}{\partial \tau} - (r - \frac{1}{2}\sigma^2)\frac{\partial c}{\partial \eta}\right), \end{aligned} \tag{1.3.18}$$

$$\frac{\partial u}{\partial S} = e^{-r\tau}\frac{\partial c}{\partial S}$$
$$= e^{-r\tau}\left(\frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial S}\right)$$
$$= e^{-r\tau}\left(\frac{1}{S}\frac{\partial c}{\partial \eta}\right) \tag{1.3.19}$$

and

$$\frac{\partial^2 u}{\partial S^2} = e^{-r\tau}\left(-\frac{1}{S^2}\frac{\partial c}{\partial \eta} + \frac{1}{S^2}\frac{\partial^2 c}{\partial \eta^2}\right). \tag{1.3.20}$$

Substitution of (1.3.18), (1.3.19) and (1.3.20) into (1.3.17) gives

$$e^{-r\tau}rc = e^{-r\tau}\left(rc - \sigma^2\frac{\partial c}{\partial \tau} - (r - \frac{1}{2}\sigma^2)\frac{\partial c}{\partial \eta}\right) + rSe^{-r\tau}\left(\frac{1}{S}\frac{\partial c}{\partial \eta}\right)$$
$$+\frac{1}{2}\sigma^2 S^2 e^{-r\tau}\left(-\frac{1}{S^2}\frac{\partial c}{\partial \eta} + \frac{1}{S^2}\frac{\partial^2 c}{\partial \eta^2}\right). \tag{1.3.21}$$

Simplification of (1.3.21) leads to the diffusion equation

$$\frac{\partial c}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 c}{\partial \eta^2}. \tag{1.3.22}$$

Writing the boundary conditions in terms of the new variables we obtain

$$c(\eta^*, 0) = K(e^{\eta *} - 1)^+ \tag{1.3.23}$$

from (1.3.17), where $\eta^* = \ln(S/K)$. The well known fundamental solution of the diffusion equation is given by,

$$c(\eta, \tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-\eta)^2}{2\tau\sigma^2}\right]c_0(x)dx, \tag{1.3.24}$$

where

$$c_0(\eta) = K(e^{\eta *} - 1)^+. \tag{1.3.25}$$

Hence

$$c(\eta, \tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{0}^{\infty}K(e^x - 1)\exp\left[-\frac{(x-\eta)^2}{2\tau\sigma^2}\right]dx$$
$$= \frac{K}{\sqrt{2\pi\sigma^2\tau}}\int_{-\eta/\sigma\sqrt{\tau}}^{\infty}\exp\left(-\frac{1}{2}m^2\right)\left[\exp\left(\sigma\sqrt{\tau}m + \eta\right) - 1\right]\sigma\sqrt{\tau}dm$$

with

$$m = \frac{x-\eta}{\sigma\sqrt{\tau}}. \tag{1.3.26}$$

Hence

$$c(\eta,\tau) = \frac{K}{\sqrt{2\pi}}\int_{-\eta/\sigma\sqrt{\tau}}^{\infty}\left[\exp\left(-\frac{1}{2}m^2+\sigma\sqrt{\tau}m+\eta\right)-\exp\left(-\frac{1}{2}m^2\right)\right]dm$$
$$= I_2 - I_1. \tag{1.3.27}$$

Here

$$I_1 = \frac{K}{\sqrt{2\pi}}\int_{-\frac{\eta}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}m^2}dm$$
$$= K\Phi\left(\frac{\eta}{\sigma\sqrt{\tau}}\right) \tag{1.3.28}$$

and

$$\begin{aligned} I_2 &= \frac{K}{\sqrt{2\pi}}\int_{-\eta/\sigma\sqrt{\tau}}^{\infty}\exp\left(-\frac{1}{2}m^2+\sigma\sqrt{\tau}m+\eta\right)dm \\ &= \frac{K}{\sqrt{2\pi}}\int_{-\eta/\sigma\sqrt{\tau}}^{\infty}\exp\left(-\frac{1}{2}(m-\sigma\sqrt{\tau})^2+\frac{1}{2}\sigma^2\tau+\eta\right)dm \\ &= \frac{K\exp\left(\frac{1}{2}\sigma^2\tau+\eta\right)}{\sqrt{2\pi}}\int_{-\eta/\sigma\sqrt{\tau}}^{\infty}\exp\left[-\frac{1}{2}(m-\sigma\sqrt{\tau})^2\right]dm \\ &= \frac{K\exp\left(\frac{1}{2}\sigma^2\tau+\eta\right)}{\sqrt{2\pi}}\int_{-(\eta/\sigma\sqrt{\tau}+\sigma\sqrt{\tau})}^{\infty}\exp\left(-\frac{1}{2}z^2\right)dz, \end{aligned} \tag{1.3.29}$$

where $z = m - \sigma\sqrt{\tau}$. Therefore it follows from (1.3.29) that

$$I_2 = K\exp\left(\frac{1}{2}\sigma^2\tau+\eta\right)\Phi\left(\frac{\eta}{\sigma\sqrt{\tau}}+\sigma\sqrt{\tau}\right). \tag{1.3.30}$$

Thus (1.3.27) becomes

$$\begin{aligned} c(\eta,\tau) &= K\exp\left(\frac{1}{2}\sigma^2\tau+\eta\right)\Phi\left(\frac{\eta}{\sigma\sqrt{\tau}}+\sigma\sqrt{\tau}\right) - K\Phi\left(\frac{\eta}{\sqrt{\tau}}\right) \\ &= K\exp\left[\frac{1}{2}\sigma^2(T-t)+\ln\frac{S}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right] \\ &\quad\times\Phi\left(\frac{\ln\frac{S}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}+\sigma\sqrt{T-t}\right) - K\Phi\left(\frac{\ln\frac{S}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{1.3.31}$$

Using (1.3.31), we find that $u_{BS}(S,T)$ from equation (1.3.17) becomes

$$\begin{aligned} u_{BS}(S,T) &= e^{-r(T-t)}\left[Se^{r(T-t)}\Phi(d_+) - K\Phi(d_-)\right] \\ &= S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-). \end{aligned} \tag{1.3.32}$$

Here (1.3.32) represents the Black-Scholes solution for the European call option as in (1.3.2).

1.4 Probabilistic Approach

In this approach we use the martingale[3] theory [64, 59]. The martingale theory is another framework for characterising an arbitrage-free market and for pricing derivatives. It is also called the risk-free neutral valuation.

1.4.1 Martingales

Let W_t^Q be a standard Brownian motion under the probability space $(\Omega, \Im, \mathbf{P})$ and $\Im_{0\le t\le T}$ be an associated Brownian filtration. The filtration, $\Im_t$, represents the flow of information evolving with time. Let $\mathbf{P}$ and Q be mutually absolutely continuous probability measures on a measure space $(\Omega, \Im)$. Then for any event $A \in \Im$, $P(A) = 0 \Leftrightarrow Q(A) = 0$. Thus a probability measure Q is said to be equivalent to measure $\mathbf{P}$ and is defined as

$$\frac{d\mathbf{P}}{dQ} = \exp\left(-\int_0^t \theta dW_t - \frac{1}{2}\int_0^t \theta^2 ds\right), \tag{1.4.1}$$

where

$$\theta = \frac{\mu - r}{\sigma}. \tag{1.4.2}$$

Thus the no-arbitrage option price at time t is

$$u_t = V_t E^Q\left(\frac{u_T}{V_T}|\Im_T\right), \tag{1.4.3}$$

where u_t/V_t is the discounted price process and is also a Q martingale. The risk-free bond is assumed to be continuously compounded in value at a rate r,

[3]Let $\{X_n, n = 0, 1, 2, ...\}$ be a real-valued stochastic process on a discrete parameter set. It is a martingale if

- $E(|X_n|) < \infty$
- $E(X_{n+1}|X_0, ..., X_n) = X_n$.

i.e. $V_t = e^{rt}$. Then (1.4.3) is reduced to

$$u_t = e^{-r(T-t)} E^Q \left(u_T | \Im_T \right). \qquad (1.4.4)$$

From Itô's lemma the discounted price process $S_t^Q = e^{-rt} S_t$ satisfies the stochastic differential equation

$$dS_t^Q = (\mu - r) S_t^Q dt + \sigma S_t^Q dW_t. \qquad (1.4.5)$$

By Girsanov's theorem[4] [64] the process,

$$W_t^Q = W + \theta t, \qquad (1.4.6)$$

is a Q Brownian motion on a probability space $(\Omega, \Im, \mathbf{P})$. Under the martingale measure, Q,

$$dS_t^Q = \sigma S_t^Q dW_t^Q. \qquad (1.4.7)$$

Hence S_t^Q is a Q local martingale since (1.4.7) has a zero drift. Thus the pricing formula for a call option becomes

$$u_t = e^{-r(T-t)} E^Q \left[(S_T - K)^+ | \Im_T \right]. \qquad (1.4.8)$$

In terms of (1.4.6), (1.2.1) can be written as

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t \left(dW_t^Q - \theta dt \right) \\ &= \mu S_t dt + \sigma S_t \left(dW_t^Q - \frac{\mu - r}{\sigma} dt \right) \\ &= r S_t dt + \sigma S_t dW_t^Q. \qquad (1.4.9) \end{aligned}$$

Under this measure, S has a rate of return, r. By Itô's lemma (1.4.9) can be written as

$$d(\ln S_t) = \left(r - \frac{1}{2}\sigma^2 \right) dt + \sigma dW. \qquad (1.4.10)$$

Thus

$$S_T = S_t \exp \left[\left(r - \frac{1}{2}\sigma^2 \right) (T - t) + \sigma \left(W_T^Q - W_t^Q \right) \right]. \qquad (1.4.11)$$

[4]Girsanov's theorem is used to change the probability measure $\mathbf{P}$ to the risk neutral Q. This theorem kills the drift term of the stochastic differential equation for the stock price.

Since $W_T^Q - W_t^Q$ is normally distributed with mean of 0 and variance $T-t$, then $\nu = -\dfrac{W_T^Q - W_t^Q}{\sqrt{T-t}}$ has a standard normal distribution. Therefore

$$
\begin{aligned}
u_t &= e^{-r(T-t)}E^Q\left[\left(S_t \exp\left((r-\frac{1}{2}\sigma^2)(T-t) - \sigma\nu\sqrt{T-t}\right) - K\right)^+\right] \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[S_t \exp\left((r-\frac{1}{2}\sigma^2)(T-t) - \sigma z\sqrt{T-t}\right) - K\right]^+ e^{-\frac{1}{2}z^2}dz \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\xi}\left[S_t \exp\left((r-\frac{1}{2}\sigma^2)(T-t) - \sigma z\sqrt{T-t}\right) - K\right] e^{-\frac{1}{2}z^2}dz \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\xi}\left[S_t \exp\left((r-\frac{1}{2}\sigma^2)(T-t) - \sigma z\sqrt{T-t} - \frac{1}{2}z^2\right) - K\exp\left(-\frac{1}{2}z^2\right)\right] dz \\
&= e^{-r(T-t)}\left[I_2 - I_1\right],
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} K e^{-\frac{1}{2}z^2}dz \\
&= K\Phi(-\xi). \qquad (1.4.12)
\end{aligned}
$$

In addition

$$
\begin{aligned}
I_2 &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} S_t \exp\left[(r-\frac{1}{2}\sigma^2)(T-t) - \sigma z\sqrt{T-t}\right]\exp\left[-\frac{1}{2}z^2\right] dz \\
&= \frac{1}{\sqrt{2\pi}}S_t \exp\left[(r-\frac{1}{2}\sigma^2)(T-t)\right]\int_{-\infty}^{\xi}\exp\left[-\sigma z\sqrt{T-t}\right]\exp\left[-\frac{1}{2}z^2\right] dz \\
&= \frac{1}{\sqrt{2\pi}}S_t \exp\left[(r-\frac{1}{2}\sigma^2)(T-t)\right]\int_{-\infty}^{\xi}\exp\left[-\frac{1}{2}(z+\sigma\sqrt{T-t})^2 + \frac{1}{2}\sigma^2(T-t)\right] dz \\
&= \frac{1}{\sqrt{2\pi}}S_t \exp\left[(r-\frac{1}{2}\sigma^2)(T-t)\right]\exp\left[\frac{1}{2}\sigma^2(T-t)\right]\int_{-\infty}^{\xi+\sigma\sqrt{T-t}}\exp\left[-\frac{1}{2}x^2\right] dx \\
&= S_t e^{r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi+\sigma\sqrt{T-t}} e^{\frac{1}{2}x^2}dx \\
&= S_t e^{r(T-t)}\Phi\left(\xi+\sigma\sqrt{T-t}\right), \qquad (1.4.13)
\end{aligned}
$$

where $x = z + \sigma\sqrt{T-t}$. Hence

$$
u_t = e^{-r(T-t)}\left[S_t e^{r(T-t)}\Phi(-(\xi - \sigma\sqrt{T-t})) - K\Phi(-\xi)\right]. \qquad (1.4.14)
$$

Here

$$
-\xi = \frac{\ln\dfrac{S_t}{K} + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_- \qquad (1.4.15)
$$

and

$$-\xi + \sigma\sqrt{T-t} = \frac{\ln \dfrac{S_t}{K} + (r + \dfrac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_+ \tag{1.4.16}$$

so that

$$u_{BS} = S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) \tag{1.4.17}$$

which is the formula to price a European vanilla call option as in (1.3.2).

1.5 The Feynman-Kac Formula

The Feynman-Kac formula relates the stochastic differential equation and the partial differential equation. The relationship between geometric Brownian motion and the Black-Scholes partial differential equation is a special case of the relationship between stochastic differential equation and partial differential equation developed in the following theorem.

1.5.1 Feynman-Kac Theorem

Let $\mu(x,t)$ and $\sigma(x,t)$ be given functions of (x,t), where $x = S_t$. Let $(S_t)_{0\leq t\leq T}$ be the solution of the stochastic differential equation

$$dS_t = \mu(x,t)dt + \sigma(x,t)dW_t \tag{1.5.1}$$

and suppose that $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.5.2}$$

with the boundary conditions $u(x,T) = h(S_T)$. We consider the process $u(x,t)$ in the time interval $0 \leq t \leq T$ so that by Itô's lemma,

$$\begin{aligned} u(x,T) &= \left(\frac{\partial u}{\partial s} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}\right)dt + \sigma(x,t)\frac{\partial u}{\partial x}dW_t \\ &= \sigma(x,t)\frac{\partial u}{\partial x}dW_t. \end{aligned} \tag{1.5.3}$$

Thus (1.5.3) is a local martingale and therefore we can take the expectations conditional to $x = S_t$ to give

$$E\left[u(S_T, T)|S_t = x\right] = u(x, t). \tag{1.5.4}$$

Thus, recalling the boundary condition $u(x, T) = h(S_T)$, we have that the solution, $u(x, t)$, of (1.5.4) has the Feynman-Kac representation.

Example: Let $h(S_T)$ be the payoff at time T of a derivative security with the underlying asset $dS_t = \mu S(x, t)dt + \sigma S(x, t)dW_t$. We may rewrite this as $dS_t = rS(x, t)dt + \sigma S(x, t)dW_t^Q$, where W_t^Q is a Brownian motion under the risk-neutral probability measure Q. According to the risk-neutral pricing formula (1.4.8), the price of the derivative security at time t is

$$u_{BS} = E^Q\left[e^{-r(T-t)}h(S_T)|\Im_T\right] \tag{1.5.5}$$

which is the discounted expectation under the martingale measure, Q, as achieved under the Martingale Approach.

1.6 Pricing Biases of the Black-Scholes Model

Although the Black-Scholes option-pricing model has been widely accepted in the financial world, there are several assumptions underlying the model that may be called into question. Most importantly the volatility and the interest rate are considered constant. The volatility is the only unobservable parameter in the Black-Scholes model. The model gives the price of the option as a function of volatility. These computed volatilities are normally not constant and they form a volatility smile. A volatility smile is the skewed pattern that results from calculating implied volatilities across a range of strike prices.

In spite of the existence of the volatility smile the Black-Scholes equation and the Black-Scholes formula are still broadly used in practice. A typical approach is to regard the volatility surface as an information about the market and use an implied volatility from it in a Black-Scholes valuation model. This was described by Rebonato R [77] in 1999 as using “the wrong number in the wrong formula to get the right price”.

In the next Chapter we introduce two special kinds of exotic options, namely the binary and barrier options. Exotic Options are path-dependent which means that they include the behaviour of the path, rather than just the current price or the price at expiry. These options are much cheaper than the ordinary options because they risk either not being knock in or being knock out. However, most of the exotic options are of European and American style and follow similar patterns for valuation and hedging.

In order to create an exotic option financial analysts change some of the properties of common European or American options. The changes include strike price, time to expiration, type of settlement and the type of an underlying asset or payout.

Chapter 2

Exotic Options

Barrier options were first introduced by Merton [65] in 1973. He uses the same strategy of portfolio hedging and replication as in the classical case. However, Bowie and Carr [11] introduced an alternative approach to the valuation and hedging of barrier options. Black [5] in his model shows how barrier options can be replicated using portfolios of just a few options with fixed maturity. Carr *et al* [15] extend Black's [5] results to a symmetric volatility structure and to more complex barrier options, such as partial and double barrier options.

In this Chapter we confine ourselves to only two types of exotic options, namely the binary[1] and barrier options. We present some closed-form pricing formulas for European binary and European barrier options.

2.1 Binary Options

A binary (or digital) option, also referred to as an all-or-nothing option or bet option, is a contract the payoff of which depends in a discontinuous way upon the terminal price of the underlying asset. There are mainly two types of binary option which differ in terms of settlement as follows:

- A **cash-or-nothing** call is worthless if the asset price S finishes below

[1]European binary option is not path dependent, but is considered exotic [94].

the strike price K at time T and pays a fixed amount, q, if it finishes above the strike price K.

- An **asset-or-nothing** call is worthless if the underlying asset price S finishes above the strike price K and pays an amount equal to the asset price S_T if it finishes below the strike price K.

It is possible to imagine a digital option as a bet on whether the underlying asset would be above (cash-or-nothing call) or below (cash-or-nothing put) a strike price. The main property of a digital option is that the payoff is determined at the beginning of the contract and does not depend upon the amount by which the price of the underlying asset moves.

A payoff of a binary call option is defined as 1 if $S_T > K$ and 0 if $S_T \leq K$. Based upon the assumptions of the Black-Scholes model, the Black-Scholes equation for a European call option takes the form $L_{BS}u_{BS} = 0$, where the operator L_{BS} is given by

$$L_{BS} = \frac{\partial}{\partial t} + rS\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r. \tag{2.1.1}$$

The value of a European binary call is $u^{(B)}(S, \tau) = e^{-r\tau}\Phi(d_-)$ and the value of a European binary put is $p^{(B)}(S, \tau) = e^{-r\tau}\Phi(1 - (d_-))$.

2.1.1 Cash-or-Nothing Call Option

In this option, if S never reaches the strike price K, then the option is worthless. Thus on the line K and below the line K the value of the option is zero. If S exceeds the strike price K, the final payment of an option is equal to a fixed amount q. If $u_{con}(S_T, t)$ is the value of a cash-or-nothing call option on its expiration date, then the final boundary condition of partial differential equation $L_{BS}u_{BS} = 0$ is $u_{con}(S_T, t) = q$ for $S_T > K$ and 0 otherwise. Using the risk-neutral valuation approach, we get

$$u_T = e^{-r\tau}E^Q\left[u_{con}(S_T, t)|\Im_t\right]$$

$$
\begin{aligned}
&= e^{-r\tau} \int_{-\infty}^{\infty} u_{con}(S_T, t) h(S_T) dS_T \\
&= e^{-r\tau} \int_{K}^{\infty} q h(S_T) dS_T \\
&= q e^{-r\tau} \int_{K}^{\infty} h(S_T) dS_T \\
&= q e^{-r\tau} P(S_T > K). \qquad (2.1.2)
\end{aligned}
$$

Using the assumption that an asset price follows the log-normal distribution,

$$
\ln S_T \approx \Phi \left[\ln S + \left(r - \frac{1}{2}\sigma^2 \right), \sigma\sqrt{\tau} \right], \qquad (2.1.3)
$$

the probability density function for S_T is given by

$$
\begin{aligned}
P(S_T > K) &= P(\ln S_T > \ln K) \\
&= P \left[\frac{\ln \dfrac{S_T}{S} - \left(r - \dfrac{1}{2}\sigma^2 \right)\tau}{\sigma\sqrt{\tau}} > \frac{\ln \dfrac{K}{S} - \left(r - \dfrac{1}{2}\sigma^2 \right)\tau}{\sigma\sqrt{\tau}} \right] \\
&= 1 - \Phi \left[\frac{\ln \dfrac{K}{S} - \left(r - \dfrac{1}{2}\sigma^2 \right)\tau}{\sigma\sqrt{\tau}} \right] \\
&= 1 - \Phi(-d_-) \\
&= \Phi(d_-).
\end{aligned}
$$

So equation (2.1.2) becomes

$$
u(T) = q e^{-r\tau} \Phi(d_-), \qquad (2.1.4)
$$

where

$$
d_- = \frac{\ln \dfrac{S}{K} + \left(r - \dfrac{1}{2}\sigma^2 \right)\tau}{\sigma\sqrt{\tau}}. \qquad (2.1.5)
$$

2.1.2 Asset-or-Nothing Call Option

If an asset price does not reach the strike price K at expiration, the option is worthless. This means that on the line K and below the line K the option value is zero. If S_T goes beyond the price K, we let the final payment of the option be S_T. If $u_{aon}(S_T, t)$ is the value of an asset-or-nothing call option on its expiration date, then the final boundary condition of equation $L_{BS}u_{BS} = 0$ is $u_{aon}(S_T, t) = S_T$ for $S_T > K$ and 0 otherwise. With the assumption that the expected return is the risk-free interest rate, we obtain

$$\begin{aligned} u_T &= e^{-r\tau} E^Q \left[u_{aon}(S_T, t) | \Im_t \right] \\ &= e^{-r\tau} \int_{-\infty}^{\infty} u_{aon}(S_T, t) h(S_T) dS_T \\ &= e^{-r\tau} \int_{K}^{\infty} S_T h(S_T) dS_T. \end{aligned} \tag{2.1.6}$$

Since the asset price follows the log-normal distribution, the probability density function for S_T is given by

$$h(S_T) = \frac{1}{\sigma S_T \sqrt{2\pi\tau}} \exp\left[-\frac{(\ln S_T - x)^2}{2\tau\sigma^2} \right] \tag{2.1.7}$$

with $x = \ln S + \left(r - \frac{1}{2}\sigma^2 \right) \tau$. Applying the relationship between S and S_T we let

$$\begin{aligned} S_T &= S \exp\left(\ln S_T - \ln S \right) \\ &= S \exp\left[\ln S_T - x + \left(r - \frac{1}{2}\sigma^2 \right) \tau \right] \end{aligned}$$

so that equation (2.1.6) can be expressed as

$$u_T = S \int_K^{\infty} \frac{1}{\sigma S_T \sqrt{2\pi\tau}} \exp\left[-\frac{1}{2} \left(\frac{(\ln S_T - x) - \sigma^2\tau}{\sigma\sqrt{\tau}} \right)^2 \right] dS_T. \tag{2.1.8}$$

Letting $v = \frac{1}{\sigma\sqrt{\tau}} \left[(\ln S_T - x) - \sigma^2\tau \right]$, we can easily get $dS_T = \sigma S_T \sqrt{\tau} dv$. If $S_T = K$, then $v = -\frac{1}{\sigma\sqrt{\tau}} \left[\ln\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2 \right) \tau \right] = -d_+$. So equation (2.1.8) can be written as

$$\begin{aligned} u_T &= S \int_{-d_+}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv \\ &= S\Phi(d_+), \end{aligned} \tag{2.1.9}$$

where

$$d_+ = d_- + \sigma\sqrt{\tau} \tag{2.1.10}$$

and

$$d_{\pm} = \frac{1}{\sigma\sqrt{\tau}} \left[\ln\frac{S}{K} + \left(r \pm \frac{1}{2}\sigma^2 \right) \tau \right]. \tag{2.1.11}$$

2.2 Barrier Options

By definition barrier options acquire or lose their value if an asset goes above or below a specified barrier, H. One barrier option specifies a barrier, H, such that the option pays (knocks in) or does not (knocks out) according to whether or not level H is attained from below (up) or above (down). There are thus four possibilities: up-and-in, up-and-out, down-and-in and down-and-out options. We define these options below:

- An **up-and-out** option loses its value if an asset price crosses H from below prior to maturity.
- A **down-and-out** option loses its value if an asset price crosses H from above prior to maturity.
- An **up-and-in** option pays off only if an asset crosses H from below prior to maturity.
- A **down-and-in** option pays off only if an asset price crosses H from above prior to maturity.

For example a down-and-out call with strike, K, maturity, T, and the barrier, H, pays $h(S_T)$ if the asset price remains above H and nothing if the asset price falls below H prior to maturity, whereas a down-and-in call is valueless until the asset price crosses the barrier H from above. If that ever happens, then it behaves like a standard call thereafter. Obviously the value of a down-and-in call is just the difference between the value of a standard call and the down-and-out, i.e., down-and-out call + down-and-in call = standard call since the two portfolios are equivalent.

In this Section we treat a knock-out call as a geometric Brownian motion. The barrier option in this Section has the explicit pricing formula, which is based upon the reflection principle for Brownian motion.

2.2.1 The Reflection Principle (Method of Images)

Suppose W is an arithmetic Brownian motion and define the running maximum and minimum by $M_{tW} = \max_{v_{\leq t}} W_v$ and $N_{tW} = \min_{v_{\leq t}} W_s$, respectively. Suppose we have $K < w$. Now, for every path that ends below K but previously reached a, there is another path that goes above $2a - K$: we simply reflect the path in a mirror at the level w. This is the reflection principle:

$$\begin{aligned} P\left[W_t < K, M_{tW} > w\right] &= P\left[W_t > 2w - K\right] \\ &= 1 - \Phi\left(\frac{2w - K}{\sqrt{t}}\right). \end{aligned} \tag{2.2.1}$$

2.2.2 First Passage Times

In this Subsection we derive the probability density function for a Brownian motion with a drift. This density function is used in the next section to obtain an explicit formula for a knock-out barrier option.

To derive this formula we begin by letting W_t, $0 \leq t \leq T$, be a Brownian motion, defined on a probability space $(\Omega, \Im, \mathbf{P})$. Under Q the Brownian motion W_t^Q is a martingale. Define $Y_t = \exp\left(\frac{\mu}{\sigma^2} W_t - \frac{\mu^2}{2\sigma^2} t\right)$ and $P(A) = E^Q\left[1_A Y_T\right]$. Then by Girsanov's Theorem $W_t^Q = W_t - \mu t$, is a standard Brownian motion under Q, which means that $W_t = \mu t + W_t^Q$ is a Brownian motion with drift μ. We have

$$\begin{aligned} & P\left[W_t > y, M_t > 0 | W_0 = x\right] \\ &= E\left[1_{(W_t > y, M_t > 0)} Y_t | W_0 = x\right] \\ &= \exp\left(-\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) E\left[1_{(W_t > y, M_t > 0)} | W_0 = x\right] \\ &= \exp\left(-\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \int_y^\infty e^{\frac{\mu}{\sigma^2}\eta} \\ & \quad \left[\frac{1}{\sigma\sqrt{2\pi t}} \exp\left(\frac{(\eta - x)^2}{2\sigma^2 t}\right) - \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(\eta + x)^2}{2\sigma^2 t}\right)\right] d\eta \\ &= \exp\left(-\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \frac{1}{\sigma\sqrt{2\pi t}} \int_y^\infty \exp\left(\frac{\mu x}{\sigma^2} + \frac{\mu^2 t}{2\sigma^2}\right) \exp\left(-\frac{(\eta - x - \mu t)^2}{2\sigma^2 t}\right) d\eta \end{aligned}$$

$$- \exp\left(-\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \frac{1}{\sigma\sqrt{2\pi t}} \int_y^\infty \exp\left[-\frac{\mu x}{\sigma^2} + \frac{\mu^2 t}{2\sigma^2}\right] \exp\left[-\frac{(\eta + x - \mu t)^2}{2\sigma^2 t}\right] d\eta$$

$$= \frac{1}{\sigma\sqrt{2\pi t}} \int_y^\infty \exp\left[-\frac{(\eta - x - \mu t)^2}{\sigma^2 t}\right] d\eta$$

$$- \exp\left(\frac{-2\mu x}{\sigma^2}\right) \frac{1}{\sigma\sqrt{2\pi t}} \int_y^\infty \exp\left[-\frac{(\eta + x - \mu t)^2}{2\sigma^2 t}\right] d\eta. \tag{2.2.2}$$

Therefore the density probability function is found by differentiating equation (2.2.2) with respect to y to obtain

$$\begin{aligned} f(x,y,t) &= -\frac{\partial}{\partial y} P\left[W_t > y, M_t > 0 | W_0 = x\right] \\ &= \frac{1}{\sigma\sqrt{2\pi t}} \exp\left[-\frac{(y - x - \mu t)^2}{2\sigma^2 t}\right] - \exp\left[\frac{-2\mu x}{\sigma^2}\right] \frac{1}{\sigma\sqrt{2\pi t}} \exp\left[-\frac{(y + x - \mu t)^2}{2\sigma^2 t}\right]. \end{aligned} \tag{2.2.3}$$

2.2.3 Pricing the Knock-out Options

The price of a knock-out call satisfies a Black-Scholes equation that has been modified to account for the barrier. This equation is used to solve for the price. The first passage time[2] of a Brownian motion, $\varsigma_m = \min\{t \geq 0; W_t = m\}$ is the first time the Brownian motion reaches the level m. In this Subsection we derive the probability density function of a first passage time without drift.

Let the stock price be denoted by $S_t = Se^{X_t}$, where $X_t = (r - \frac{1}{2}\sigma^2)t + \sigma^2 W_t^Q$. According to the risk-neutral method the pay off of a knock-out option under measure Q is given by

$$\begin{aligned} u_T &= e^{-rT}\left[(S_T - K)^+ 1_{(\varsigma > T)}\right] \\ &= e^{-rT}\left[(S_T - K)^+ 1_{(\min_{0\leq v\leq T} X_v)}\right]. \end{aligned} \tag{2.2.4}$$

On the other hand

$$P\left[X_T > x, \min_{0\leq v\leq T} X_v > m\right] = \Phi\left(\frac{-x - \mu T}{\sigma\sqrt{T}}\right) - \exp\left[-\frac{2\mu m}{\sigma^2}\right] \Phi\left(\frac{-x + 2m + \mu T}{\sigma\sqrt{T}}\right), \tag{2.2.5}$$

[2]It is sometimes known as the hitting or stopping time.

where $K > H$. It follows that

$$f(x,m) = -\frac{\partial}{\partial x} P\left[X_T > x, \min_{0\le v\le T} X_v > m\right]$$
$$= \frac{1}{\sigma\sqrt{2\pi T}} \exp\left[-\frac{(x+\mu T)^2}{\sigma\sqrt{T}}\right] - \frac{1}{\sqrt{2\pi T\sigma^3}} \exp\left[-\frac{(x-2z-\mu T)^2 - 4\mu z T}{2\sigma^2 T}\right],$$

where $f(x,m)$ is the joint probability distribution between m and x. Thus the price of the knock-out option is given by

$$\begin{aligned} u(T) &= e^{-rT}\int_{-\infty}^{\infty} (Se^x - K)^+ f(x,m)dx \\ &= Se^{-rT}\int_{\ln\frac{K}{S}}^{\infty} e^x f(x,m)dx - Ke^{rT}\int_{\ln\frac{K}{S}}^{\infty} f(x,m)dx \\ &= Se^{-rT} I_2 - Ke^{rT} I_1, \end{aligned} \tag{2.2.6}$$

where

$$\begin{aligned} I_2 &= \frac{1}{\sigma\sqrt{2\pi T}}\int_{\ln\frac{K}{S}}^{\infty} \exp\left[-\frac{\left(x - \left(r + \frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}\right] dx - \exp\left[\frac{rT + \ln\frac{H}{S}(2r+\sigma^2)}{\sigma^2}\right] \\ &\times \frac{1}{\sigma\sqrt{2\pi T}}\int_{\ln\frac{H}{S}}^{\infty} \exp\left[-\frac{\left(x - 2z - \left(r+\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}\right] dx \\ &= e^{rT}\Phi\left[\frac{\ln\frac{S}{K} + \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right] - e^{rT}\left(\frac{S}{H}\right)^{-1-\frac{2r}{\sigma^2}}\Phi\left[\frac{\ln\frac{H^2}{SK} + \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right] \end{aligned} \tag{2.2.7}$$

and

$$\begin{aligned} I_1 &= \int_{\ln\frac{K}{S}}^{\infty} f(x,m)dx \\ &= P\left[X_T \ge \frac{K}{S}, \min_{0\le u\le T} X_t \ge \frac{H}{S}\right] \\ &= \Phi\left[\frac{\ln\frac{S}{K} + \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right] - \left(\frac{S}{H}\right)^{-1-\frac{2r}{\sigma^2}}\Phi\left[\frac{\ln\frac{H^2}{SK} + \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right]. \end{aligned} \tag{2.2.8}$$

Therefore

$$u(T) = S\left[\Phi(d_+) - \left(\frac{S}{H}\right)^{1-\frac{2r}{\sigma^2}}\Phi(d_+^*)\right] + Ke^{-rT}\left[\Phi(d_-) + \left(\frac{S}{H}\right)^{1-\frac{2r}{\sigma^2}}\Phi(d_-^*)\right],$$

(2.2.9)

where

$$d_+ = \frac{\ln\frac{S}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_- = d_+ - \sigma\sqrt{T},$$

$$d_+^* = d_+ - \frac{2}{\sigma\sqrt{T}}\ln\frac{S}{H} \quad \text{and} \quad d_-^* = d_- - \frac{2}{\sigma\sqrt{T}}\ln\frac{S}{H}. \tag{2.2.10}$$

European Down-and-Out Calls

A down-and-out option pays a rebate if the barrier is reached. The formula for a down-and-out was first derived by Merton in 1973 [65] and later modified by Cox and Rubinstein [21] in 1985.

Here the partial dfferential equation $L_{BS}u_{BS} = 0$ for a Black-Sholes equation is subject to the following terminal conditions

$$u(H, \tau) = R \tag{2.2.11}$$

and

$$u(S, 0) = (S - K)^+, \tag{2.2.12}$$

where R denote the rebate paid to the holder when the barrier is hit.

Let $X_t = \ln\frac{S_t}{S} = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^Q$ and $M_t = \max_{0\le v\le t} X_v$. Here we consider the case that $S > H$. Then we have $\varsigma = \inf(t \ge 0; M_t \ge \ln\frac{H}{S})$. The probability density function of the rebate term is

$$f_t = \frac{\ln H/S}{\sigma\sqrt{2\pi t^3}}\exp\left[-\frac{\left(\ln H/S - \left(r - \frac{1}{2}\sigma^2\right)t\right)^2}{2t\sigma^2}\right]. \tag{2.2.13}$$

Using (2.2.13), the payoff is given by

$$\begin{aligned} u_T &= E^Q\left[e^{-r\tau}1_{(\varsigma\le T)}\right] \\ &= \int_0^T e^{-r\tau}\frac{\ln H/S}{\sigma\sqrt{2\pi t^3}}\exp\left[-\frac{\left(\ln H/S - \left(r - \frac{1}{2}\sigma^2\right)t\right)^2}{2t\sigma^2}\right]dt \\ &= e^{\ln H/S}\int_0^T \frac{\ln H/S}{\sigma\sqrt{2\pi t^3}}\exp\left[-\frac{\left(\ln H/S - \left(r - \frac{1}{2}\sigma^2\right)t\right)^2}{2t\sigma^2}\right]dt. \end{aligned} \tag{2.2.14}$$

The integrand in equation (2.2.14) is the probability density function of the first passage time of $\ln H/S$ of a Brownian motion with drift $-\left(r - \frac{1}{2}\sigma^2\right)$ and diffusion coefficient σ. Hence the integral can be rewritten as

$$\begin{aligned}
&\int_0^T \frac{\ln H/S}{\sigma\sqrt{2\pi t^3}} \exp\left[-\frac{\left(\ln H/S - (r - \frac{1}{2}\sigma^2)t\right)^2}{2t\sigma^2}\right] dt \\
&= P\left[\max_{0\le t\le T}\left(-(r - \frac{1}{2}\sigma^2)t + \sigma W_t\right) < \ln\frac{H}{S}\right] \\
&= 1 - P\left[\max_{0\le t\le T}\left(-(r - \frac{1}{2}\sigma^2)t + \sigma W_t\right) \ge \ln\frac{H}{S}\right] \\
&= 1 - \Phi\left[\frac{W + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right] + e^{-1-\frac{2r}{\sigma^2}}\Phi\left[\frac{-W + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right]. \qquad (2.2.15)
\end{aligned}$$

European Up-and-Out Calls

Consider a European call, expiring at time T, with strike price, K, and up-and-out barrier, H. In this case we assume that $S < K < H$. The solution to the diffusion equation, $dS_t = rS_t dt + \sigma S_t dW_t^Q$, for the asset price is given by

$$\begin{aligned}
S_t &= S_0 \exp\left[(r - \frac{1}{2}\sigma^2)(T - t) + \sigma W_t^Q\right] \\
&= S_0 \exp\left(\sigma\tilde{W}_t\right),
\end{aligned}$$

where $\tilde{W}_t = \zeta t + W_t^Q$ and $\zeta = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$. We define $\tilde{M}_T = \max_{0\le t\le T}\tilde{W}_t$ so that $\max_{0\le t\le T}\tilde{W}_t = S_0 e^{\sigma\tilde{M}_T}$. The option knocks out if and only if $S_0 e^{\sigma\tilde{M}_T} > H$; if $S_0 e^{\sigma\tilde{M}_T} \le H$, the payoff is

$$\begin{aligned}
u_T &= (S_T - K)^+ \\
&= (S_0 e^{\sigma\tilde{W}_T} - K)^+ \\
&= (S_0 e^{\sigma\tilde{W}_T} - K)^+ \mathbf{1}_{(S_0 e^{\sigma\tilde{M}_T} \le H)} \\
&= (S_0 e^{\sigma\tilde{W}_T} - K)^+ \mathbf{1}_{(S_0 e^{\sigma\tilde{W}_T} \ge K,\, S_0 e^{\sigma\tilde{M}_T} \le H)} \\
&= (S_0 e^{\sigma\tilde{W}_T} - K)^+ \mathbf{1}_{\left(\tilde{W}_T \ge \frac{1}{\sigma}\ln\frac{K}{S_0},\, \tilde{M}_T \le \frac{1}{\sigma}\ln\frac{H}{S_0}\right)}. \qquad (2.2.16)
\end{aligned}$$

According to the risk-neutral method the payoff at time zero is given by

$$u_0 = E^Q(e^{-rt}u_T)$$

$$= e^{-rT}\int_{-\infty}^{\infty}(S_T-K)^+ f(m,x)dx,$$

where $f(m,x)$ is the probability density function under measure Q of the pair $(\tilde{M}_T, \tilde{W}_T)$ and is thus given by

$$f(m,x) = \frac{2(2m-x)}{T\sqrt{2\pi T}}\exp\left(\alpha x - \frac{1}{2}\sigma^2 T - \frac{1}{2T}(2m-x)^2\right). \tag{2.2.17}$$

We assume that $S_0 > 0$ and $S_0 \le H$ so that $\frac{1}{\sigma}\ln\frac{H}{S_0} > 0$. When $0 < S_0 \le H$, the time-zero payoff of the up-and-out call is

$$\begin{aligned}
u_0 &= \int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\int_{x^+}^{\frac{1}{\sigma}\ln\frac{H}{S_0}} e^{-rt}(S_0 e^{\sigma h}-K)\frac{2(2m-x)}{T\sqrt{2\pi T}}\exp\left[\alpha x - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-x)^2\right] dm dx \\
&= \frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}(S_0e^{\sigma x}-K)\exp\left[-rT+\alpha x-\frac{1}{2}\alpha^2T-\frac{1}{2T}x^2\right]dx \\
&\quad -\frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}} e^{-rt}(S_0e^{\sigma x}-K)\exp\left[-rT+\alpha x-\frac{1}{2}\alpha^2T-\frac{1}{2T}\left(\frac{2}{\sigma}\ln\frac{H}{S_0}-h^2\right)\right]dx \\
&= S_0I_1 - KI_2 - S_0I_3 + KI_4.
\end{aligned} \tag{2.2.18}$$

In this case

$$I_1 = \frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\exp\left[\sigma x - rT + \alpha x - \frac{1}{2}\alpha^2 T - \frac{x^2}{2T}\right]dx \tag{2.2.19}$$

$$I_2 = \frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\exp\left[-rT + \alpha x - \frac{1}{2}\alpha^2 T - \frac{h^2}{2T}\right]dx \tag{2.2.20}$$

$$\begin{aligned}
I_3 &= \frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\exp\left[\sigma x - rT + \alpha h - \frac{1}{2}\alpha^2 T - \frac{1}{2T}\left(\frac{2}{\sigma}\ln\frac{H}{S_0}-x\right)^2\right]dx \\
&= \frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\exp\left[\sigma x - rT + \alpha x - \frac{1}{2}\alpha^2 T - \frac{2}{\sigma^2 T}\left(\ln\frac{H}{S_0}\right)^2 + \frac{2h}{\sigma T}\ln\frac{H}{S_0} - \frac{x^2}{2T}\right]dx
\end{aligned} \tag{2.2.21}$$

and

$$\begin{aligned}
I_4 &= \frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\exp\left[-rT + \alpha x - \frac{1}{2}\alpha^2 T - \frac{1}{2T}\left(\frac{2}{\sigma}\ln\frac{H}{S_0}-x\right)^2\right]dx \\
&= \frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\exp\left[-rT + \alpha x - \frac{1}{2}\alpha^2 T - \frac{2}{\sigma^2 T}\left(\ln\frac{H}{S_0}\right)^2 + \frac{2x}{\sigma T}\ln\frac{H}{S_0} - \frac{x^2}{2T}\right]dx.
\end{aligned} \tag{2.2.22}$$

Each of these integrals is of the form

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\exp\left(\beta+\delta x-\frac{x^2}{2T}\right)dx\\
&=\frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\frac{1}{\sigma}\ln\frac{H}{S_0}}\exp\left(\beta+\frac{1}{2}\delta^2T-\frac{1}{2T}(x-\delta T)^2\right)dx\\
&=\exp\left(\beta+\frac{1}{2}\delta^2T\right)\frac{1}{\sqrt{2\pi T}}\int_{\frac{1}{\sqrt{T}}(\frac{1}{\sigma}\ln\frac{H}{S_0}-\delta T)}^{\frac{1}{\sqrt{T}}(\frac{1}{\sigma}\ln\frac{K}{S_0}-\delta T)}e^{-\frac{1}{2}z^2}dz\\
&=\exp\left(\beta+\frac{1}{2}\delta^2T\right)\left[\Phi\left(\frac{\frac{1}{\sigma}\ln\frac{H}{S_0}-\delta T}{\sqrt{T}}\right)-\Phi\left(\frac{\frac{1}{\sigma}\ln\frac{K}{S_0}-\delta T}{\sqrt{T}}\right)\right]\\
&=\exp\left(\beta+\frac{1}{2}\delta^2T\right)\left[\Phi\left(\frac{\frac{1}{\sigma}\ln\frac{S_0}{K}-\delta T}{\sqrt{T}}\right)-\Phi\left(\frac{\frac{1}{\sigma}\ln\frac{S_0}{H}-\delta T}{\sqrt{T}}\right)\right]\\
&=\exp\left(\beta+\frac{1}{2}\delta^2T\right)\left[\Phi\left(\frac{\ln\frac{S_0}{K}-\delta\sigma T}{\sigma\sqrt{T}}\right)-\Phi\left(\frac{\ln\frac{S_0}{H}-\delta\sigma T}{\sigma\sqrt{T}}\right)\right].
\end{aligned}\tag{2.2.23}$$

The integral I_1 is of the form (2.2.23) with $\beta=-rT-\frac{1}{2}\alpha^2T$ and $\delta=\sigma+\alpha$ so that $\beta+\frac{1}{2}\delta^2T=0$ and $\delta\sigma=r+\frac{1}{2}\sigma^2$. Therefore

$$I_1=\Phi\left\{d_+\left(\frac{S_0}{K}\right)\right\}-\Phi\left\{d_+\left(\frac{S_0}{H}\right)\right\}.\tag{2.2.24}$$

For I_2 we have $\beta=-rT-\frac{1}{2}\alpha^2T$ and $\delta=\alpha$, so $\beta+\frac{1}{2}\delta^2T=-rT$ and $\delta\sigma=r-\frac{1}{2}\sigma^2$. Therefore

$$I_2=e^{-rT}\left[\Phi\left\{d_-\left(\frac{S_0}{K}\right)\right\}-\Phi\left\{d_-\left(\frac{S_0}{H}\right)\right\}\right].\tag{2.2.25}$$

For I_3 we have $\beta=-rT-\frac{1}{2}\alpha^2T-\frac{2}{T}\left(\frac{1}{\sigma}\ln\frac{H}{S_0}\right)^2$ and $\delta=\sigma+\alpha+\frac{2}{T}\left(\frac{1}{\sigma}\ln\frac{H}{S_0}\right)$ so that

$$\begin{aligned}
\beta+\frac{1}{2}\delta^2T&=-rT-\frac{1}{2}\alpha^2T-\frac{2}{T}\left(\frac{1}{\sigma}\ln\frac{H}{S_0}\right)^2\\
&\quad+\frac{1}{2}T\left(\sigma+\alpha+\frac{2}{T}\left(\frac{1}{\sigma}\ln\frac{H}{S_0}\right)\right)\\
&=\frac{1}{2}\sigma^2T+\sigma\alpha T-rT+\frac{2\alpha}{\sigma}\ln\frac{H}{S_0}+2\ln\frac{H}{S_0}
\end{aligned}$$

$$
\begin{aligned}
&= -\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\alpha T + 2\left(1 + \frac{\alpha}{\sigma}\right)\ln\frac{H}{S_0} \\
&= 2\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma^2}\right)\ln\left(\frac{H}{S_0}\right) \\
&= \ln\left(\frac{S_0}{H}\right)^{-\frac{2r}{\sigma^2}-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\delta\sigma T &= \sigma^2 T + \left(r - \frac{1}{2}\sigma^2\right)T + 2\ln\frac{H}{S_0} \\
&= \left(r + \frac{1}{2}\sigma^2\right)T + \ln\left(\frac{H}{S_0}\right)^2.
\end{aligned}
$$

Therefore

$$
I_3 = \left(\frac{S_0}{H}\right)^{-\frac{2r}{\sigma^2}-1}\left[\Phi\left\{d_+\left(T, \frac{H^2}{KS_0}\right)\right\} - \Phi\left\{d_+\left(T, \frac{H}{S_0}\right)\right\}\right]. \tag{2.2.26}
$$

Finally for I_4 we have $\beta = -rT - \frac{1}{2}\alpha^2 T - \frac{2}{T}\left(\frac{1}{\sigma}\ln\frac{H}{S_0}\right)^2$ and $\delta = \sigma + \alpha + \frac{2}{T}\left(\frac{1}{\sigma}\ln\frac{H}{S_0}\right)$ so that

$$
\beta + \frac{1}{2}\delta^2 T = \ln\left(\frac{S_0}{H}\right)^{-\frac{2r}{\sigma^2}+1} \tag{2.2.27}
$$

and

$$
\delta\sigma T = \left(r - \frac{1}{2}\sigma^2\right)T + \ln\left(\frac{H}{S_0}\right)^2. \tag{2.2.28}
$$

Therefore

$$
I_4 = e^{-rT}\left(\frac{S_0}{H}\right)^{-\frac{2r}{\sigma^2}+1}\left[\Phi\left\{d_-\left(T, \frac{H^2}{KS_0}\right)\right\} - \Phi\left\{d_-\left(T, \frac{H}{S_0}\right)\right\}\right]. \tag{2.2.29}
$$

Substitution of (2.2.24), (2.2.25), (2.2.26) and (2.2.29) into (2.2.18) yields,

$$
\begin{aligned}
u_0 = S_0 &\left[\Phi\left\{d_+\left(\frac{S_0}{K}\right)\right\} - \Phi\left\{d_+\left(\frac{S_0}{H}\right)\right\}\right] \\
&- e^{-rT}K\left[\Phi\left\{d_-\left(\frac{S_0}{K})\right)\right\} - \Phi\left\{d_-\left(\frac{S_0}{H}\right)\right\}\right] \\
&- H\left(\frac{S_0}{H}\right)^{-\frac{2r}{\sigma^2}-1}\left[\Phi\left\{d_+\left(\frac{H^2}{KS_0}\right)\right\} - \Phi\left\{d_+\left(\frac{H}{S_0}\right)\right\}\right] \\
&+ e^{-rT}K\left(\frac{S_0}{H}\right)^{-\frac{2r}{\sigma^2}+1}\left[\Phi\left\{d_-\left(T, \frac{H^2}{KS_0}\right)\right\} - \Phi\left\{d_-\left(\frac{H}{S_0}\right)\right\}\right].
\end{aligned} \tag{2.2.30}
$$

Now we replace S_0 by S_t and assume that the underlying asset price at time t is S_t for $0 \leq t \leq T$ and $0 \leq S_t \leq H$ as above. If the call has not knocked out prior to time t, then equation (2.2.30) can be rewritten as

$$\begin{aligned} u(t, S_t) = S_t & \left[\Phi \left\{ d_+ \left(\frac{S_t}{K}) \right) \right\} - \Phi \left\{ d_+ \left(\frac{S_t}{H} \right) \right\} \right] \\ & - e^{-rT} K \left[\Phi \left\{ d_- \left(\frac{S_t}{K}) \right) \right\} - \Phi \left\{ d_- \left(\frac{S_t}{H} \right) \right\} \right] \\ & - H \left(\frac{S_t}{H} \right)^{-\frac{2r}{\sigma^2} - 1} \left[\Phi \left\{ d_+ \left(\frac{H^2}{K S_t} \right) \right\} - \Phi \left\{ d_+ \left(\frac{H}{S_t} \right) \right\} \right] \\ & + e^{-rT} K \left(\frac{S_t}{H} \right)^{-\frac{2r}{\sigma^2} + 1} \left[\Phi \left\{ d_- \left(\frac{H^2}{K S_t} \right) \right\} - \Phi \left\{ d_- \left(\frac{H}{S_t} \right) \right\} \right] \end{aligned}$$

and satisfies the boundary conditions $u(t, 0) = 0$, $0 \leq t \leq T$, $u(t, H) = 0$, $0 \leq t < T$ and $u(T, S_t) = h(S_t)$, $0 \leq S_t \leq H$.

2.3 Complex Barrier Options

In the case of complicated barrier options an asset price is required not only to cross a barrier but to spend a certain amount of time across the barrier in order to knock in or out. In partial barrier options the barrier is active only during an initial period. In other words the barrier disappears at a prescribed time. So the payoff at maturity may be a function of the spot price. Double barrier options are options that knock in or out at the first hitting time of either a lower or upper barrier. In papers by Benson and Daniels [3], Rich [78] and Hsu [49] more detailed discussion of the nature and applications of various types of barrier options can be found.

Chapter 3

Stochastic Volatility Models

Stochastic volatility models are models in which the volatility changes randomly according to some discrete processes or to some stochastic differential equations. Essentially the stochastic volatility models are divided into two classes, namely, the one-factor and the multifactor models. In one-factor models the Brownian motion is the only source of uncertainity, whereas in the multifactor models further Brownian motions, or other random elements, are taken into consideration. In the one-factor models the volatility is a deterministic function of present or past values of the underlying price. The main advantage of the one-factor models is that the volatility and the stock price are perfectly correlated.

The theory of stochastic volatility models was introduced by Merton [67, 68] and Cox and Ross [20] in 1976. Merton [67] assumes a mixture of continuous and jump processes, whereas Cox and Ross [20] allow the volatility to be a deterministic function of the underlying stock price. Johnson and Shanno [57] propose a stochastic volatility model in which stock returns and return volatility are correlated. Wiggins [92] derives statistical estimators for volatility process parameters and calculates parameter estimates using Monte Carlo estimation methods for several individual stocks and stock indexes. Using a Taylor series expansion Hull and White [53] derive an accurate formula for a call option for which stock returns and stock volatility were uncorrelated. In

1991 Stein and Stein [88] derived a stochastic volatility option pricing formula that assumed volatility follows an Ornstein-Uhlenbeck (OU) process. Heston (1993) [47] developed a closed-form solution for options with stochastic volatility.

3.1 Time-Dependent Volatility Models

Time-dependent volatility models relax the assumption that μ, σ and r are constant, but instead are treated as functions of time. Under these relaxed conditions the Black-Scholes equation for a standard European option takes the form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 u}{\partial S^2} + r(t)S\frac{\partial u}{\partial S} = r(t)u. \tag{3.1.1}$$

The price of a European call option is thus given by

$$u(S,t) = S\Phi(\widetilde{d}_+) - K\exp\left(-\int_0^t \sigma(s)ds\right)\Phi(\widetilde{d}_2), \tag{3.1.2}$$

where

$$\widetilde{d}_+ = \frac{\ln\dfrac{S}{K} + \displaystyle\int_0^t (r(s) + \frac{1}{2}\sigma^2(s))ds}{\sqrt{\displaystyle\int_0^t \sigma^2(s)ds}} \tag{3.1.3}$$

and

$$\widetilde{d}_- = \widetilde{d}_+ - \sqrt{\int_0^t \sigma^2(s)ds}. \tag{3.1.4}$$

3.2 Discrete-Time Stochastic Volatility Models

Discrete-time models of stochastic volatility focus on the statistical and descriptive patterns of price changes in short-time intervals. Models of this kind are adapted to indicate at least some of several well-documented features of time series of logarithmic daily asset returns such as: skewness of distributions, leptokurtosis, volatility clustering and their negative correlation [45]. In this

Section we provide a brief overview of alternative approaches to the issue of stochastic volatility. These alternative approaches include various mathematical techniques permitting one to construct stochastic models.

3.2.1 Generalized Autoregressive Heteroskedasticity (GARCH) Models

These models were first introduced by Bollerslev [9] in 1986. He also derived the conditions for stationarity of this class of models. The equation for the GARCH model is

$$\sigma_m^2 = v + \alpha w_{m-1}^2 + \beta \sigma_{m-1}^2, \tag{3.2.1}$$

where α is the weight assigned to w_{m-1}^2 and β is the weight assigned to σ_{m-1}^2. This is known as GARCH$(1,1)$ model. The "$(1,1)$" in GARCH$(1,1)$ indicates that σ_m^2 is based on the most recent observation of w^2 and the most recent estimate of the variance rate.

The GARCH$(1,1)$ can be extended to a GARCH(p,q) formulation, in which the current conditional variance is parametrised to depend upon q lags of the squared error and p lags of the conditional variance, i.e.

$$\begin{aligned} \sigma_m^2 &= v + \alpha_1 w_{m-1}^2 + \alpha_2 w_{m-2}^2 + ... + \alpha_q w_{m-q}^2 + \beta_1 \sigma_{m-1}^2 \beta_2 w_{m-2}^2 + ... + \beta_p w_{m-p}^2 \\ &= v + \sum_{i=1}^{q} \alpha_i w_{m-i}^2 + \sum_{j=1}^{p} \beta_j w_{m-j}^2. \end{aligned} \tag{3.2.2}$$

For these models the unconditional variance is

$$\sigma^2 = \frac{v}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{j=1}^{p} \beta_j}. \tag{3.2.3}$$

For a stationary GARCH(p,q) model we require that $0 \leq \sum_{i=1}^{q} \alpha_i - \sum_{j=1}^{p} \beta_j < 1$, $v \geq 0$, $\alpha_i \geq 0$ for $i = 1, 2, ..., q$ and $\beta_j \geq 0$ for $j = 1, 2, ..., p$. Since σ^2 is nonnegative and finite, GARCH models are less likely to violate nonnegative constraints and that is why they remain popular.

3.2.2 Constant Elasticity of Variance (CEV)

This model was proposed by Cox in 1975 [19] and it attempts to include the behaviour of the volatility smile in the stochastic differential equation of an asset price S. The CEV model of option pricing assumes that an asset price S follows a stochastic differential equation

$$dS = \mu S dt + \sigma_0 S^{\frac{\beta}{2}} dW. \tag{3.2.4}$$

The CEV model assumes the following relationship between an asset price S and volatility $\sigma(S,t)$,

$$\sigma(S,t) = \sigma_0 S^{\frac{\beta}{2}-1}, \tag{3.2.5}$$

where σ_0 is a positive constant and $1 < \beta < 2$. If $\beta = 2$, then the elasticity is zero and asset prices are lognormally distributed as in the Black-Scholes model. If $\beta < 2$, then volatility is a decreasing function of S whereas, if $\beta > 2$, then the volatility is an increasing function of S. The CEV diffusion process was firstly used to model heteroskedasticity in returns to common assets.

Now suppose that the asset price S follows the stochastic differential equation, (3.2.4), and define u as the price of a European option on an asset that pays no dividend during the life of the contract. By Itô's lemma

$$du = \left(\frac{\partial u}{\partial t} + \mu S \frac{\partial u}{\partial S} + \frac{1}{2}\sigma_0 S^{\beta} \frac{\partial^2 u}{\partial S^2}\right) dt + \sigma_0 S^{\frac{\beta}{2}} \frac{\partial u}{\partial S} dW. \tag{3.2.6}$$

Using the continuous hedging and arbitrage arguments developed in Chapter 1, Section (1.3), it is possible to show that u satisfies the CEV partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma_0^2 S^{\beta} \frac{\partial^2 u}{\partial S^2} + \sigma_0 S^{\frac{\beta}{2}} \frac{\partial u}{\partial S} = ru. \tag{3.2.7}$$

3.2.3 Jump-Diffusion Processes

An extension of the Black-Scholes model was proposed by Merton (1974) [66] in which the stock returns are subject to unpredictable jumps and the Poisson Process[1] serves as the starting point for jump processes. Runggaldier [80]

[1]The properties of Poisson Processes:

further extended the Black-Scholes-Merton model. In this case we consider the stochastic differential equation driven by both a Wiener process and a Poisson random measure. We price a European call option when the underlying asset is a jump process.

Pricing a European Call in a Jump Model

Let N_t be a Poisson Process on a probability space $(\Omega, \Im, \mathbf{P})$ relative to a filtration $\Im_t$, $t \geq 0$. We denote the intensity of N_t by λ, a positive constant, and the compensated Poisson Process by $M_t = N_t - \lambda t$. Consider an asset modeled as a geometric Poisson Process,

$$S_T = S_t \exp\left[(\alpha - \lambda\sigma)(T-t)\right](\sigma+1)^{N_T - N_t}, \tag{3.2.8}$$

for which the stochastic differential equation is

$$dS_t = \alpha S_t dt + \sigma S_{t(-)} dN_t - \lambda\sigma S_t dt. \tag{3.2.9}$$

We assume that $\lambda = \dfrac{\alpha - r}{\sigma}$ in order to rule out arbitrage in (3.2.8). Under this assumption $\lambda^* = \lambda - \dfrac{\alpha - r}{\sigma}$ is positive. Thus (3.2.8) may be rewritten as

$$S_T = S_t \exp\left[(r - \lambda^*\sigma)(T-t)\right](\sigma+1)^{N_T - N_t}. \tag{3.2.10}$$

Therefore the risk-neutral price of a European call option is given by

$$\begin{aligned} u_t &= e^{-r(T-t)} E^Q\left[(S_T - K)^+ | \Im_t\right] \\ &= e^{-r(T-t)}\left[\left(S_t \exp\left[(r - \lambda^*\sigma)(T-t)\right](\sigma+1)^{N_T-N_t} - K\right)^+ | \Im_t\right] \end{aligned}$$

- The probability that a jump occurs during a short time interval of length Δt is $\lambda\Delta t + o(\Delta t)$.
- The probability of two or more jumps occuring during a short time interval of length Δt is negligible, i.e. $o(\Delta t)$.
- The probability of exactly n jumps occuring in a time interval of length t is $\frac{(\lambda t)^n}{n!} e^{-\lambda t}$.
- The mean waiting-time for a jump is $\frac{1}{\lambda}$.

$$= e^{-r(T-t)} \sum_{j=0}^{\infty} e^{-\lambda^*(T-t)} \frac{\lambda^{*j}(T-t)^j}{j!} \left(S_t \exp\left[(r-\lambda^*\sigma)(T-t)\right](\sigma+1)^j - K\right)^+$$

$$= \sum_{j=0}^{\infty} e^{-\lambda^*(T-t)} \frac{\lambda^{*j}(T-t)^j}{j!} \left(S_t e^{-\lambda^*\sigma(T-t)}(\sigma+1)^j - K\right)^+ . \quad (3.2.11)$$

The $j = 0$ term in (3.2.11) is $e^{-\lambda^*(T-t)}\left(S_t e^{-\lambda^*\sigma(T-t)} - K\right)^+$. When $t = T$, this term is $(S_t - K)^+$ and therefore the function u satisfies the terminal condition

$$u(T, S_t) = (S_t - K)^+ \quad \forall \quad S_t \geq 0. \quad (3.2.12)$$

To derive the partial differential equation for u_t we consider the stochastic differential equation (3.2.9) which may be rewritten as

$$dS_t = (r - \lambda^*\sigma)S_t dt + \sigma S_{t(-)} dN_t. \quad (3.2.13)$$

This shows that the continuous part of the stock price satisfies

$$dS_t = (r - \lambda^*\sigma)S_t dt. \quad (3.2.14)$$

On the other hand, if the stock price jumps at the time t, then $\Delta S_t = S_t - S_{t(-)} = \sigma S_{t(-)}$, $S_t = (\sigma+1)S_{t(-)}$. By the Itô-Doeblin[2] formula we have that

$$\begin{aligned}
e^{-rt}u(t, S_t) &= u(0, S_0) + \int_0^t e^{-rc}\left[-ru(c, S_c)dc + \frac{\partial u}{\partial t}(c, S_c)dc + \frac{\partial u}{\partial S}(c, S_c)dS(c)\right] \\
&\quad + \sum_{0<c\leq t} e^{-rc}\left[u(c, S_c) - u(c, S_{c(-)})\right] \\
&= u(0, S_0) + \int_0^t e^{-rc}\left[-ru(c, S_c) + \frac{\partial u}{\partial t}(c, S_c) + (r-\lambda^*\sigma)S_c\frac{\partial u}{\partial S}(c, S_c)\right] dc \\
&\quad + \int_0^t e^{-rc}\left[u(c, S_c) - u(c, S_{c(-)})\right] dN_c \\
&= u(0, S_0) + \int_0^t e^{-rc}\left[-ru(c, S_c) + \frac{\partial u}{\partial t}(c, S_c) + (r-\lambda^*\sigma)S_c\frac{\partial u}{\partial S}(c, S_c)\right] dc
\end{aligned}$$

[2]**Itô-Doeblin formula for one jump process** states that, if $X(t)$ is a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous, then

$$f(X(t)) - f(X(0)) = \int_0^t f'(X(s))dX^u(s) + \frac{1}{2}\int_0^t f''(X(s))dX^u(s)dX^u(s) + \sum_{0<s\leq t}\left[f(X(s)) - f(X(s-))\right]. \quad (3.2.15)$$

$$
\begin{aligned}
&+\int_0^t e^{-rc}\left[u(c,(\sigma+1)S_{c(-)})-u(c,S_{c(-)})\right]\lambda^* dc \\
&+\int_0^t e^{-rc}\left[u(c,(\sigma+1)S_{c(-)})-u(c,S_{c(-)})\right]dM_c^* \\
=\ & u(0,S_0)+\int_0^t e^{-rc}[-ru(c,S_c)+\frac{\partial u}{\partial t}(c,S_c) \\
&+(r-\lambda^*\sigma)S_c\frac{\partial u}{\partial S}(c,S_c)+\lambda^*(u(c,(\sigma+1)S_c)-u(c,S_c))]dc \\
&+\int_0^t e^{-rc}\left[u(c,(\sigma+1)S_{c(-)})-u(c,S_{c(-)})\right]dM_c^*.
\end{aligned} \tag{3.2.16}
$$

The last integral in (3.2.16) is a martingale because M_c is a martingale. In addition, since the left-hand side of (3.2.16) is also a martingale, we can solve for

$$
\begin{aligned}
&u(0,S_0)+\int_0^t e^{-rc}[-ru(c,S_c)+\frac{\partial u}{\partial t}(c,S_c)+(r-\lambda^*\sigma)S_c\frac{\partial u}{\partial S}(c,S_c) \\
&+\lambda^*(u(c,(\sigma+1)S_c)-u(c,S_c))]dc
\end{aligned} \tag{3.2.17}
$$

which is itself a martingale. This can only happen if

$$
\frac{\partial u}{\partial t}(t,S_t)+(r-\lambda^*\sigma)S_c\frac{\partial u}{\partial S}(t,S_t)+\lambda^*\left[u(c,(\sigma+1)S_c)-u(c,S_c)\right]=ru(t,S_t) \tag{3.2.18}
$$

for $0\leq t\leq T$ and $S_t\geq 0$. Equation (3.2.18) is sometimes called a differential-difference equation because it involves u at two different values of the stock price, namely S_t and $(\sigma+1)S_t$.

3.3 Pure Stochastic Volatility Models

These models are examples of multifactor models. In multifactor models the movement of volatility, σ, is correlated with the movement of the asset price S. We suppose that under a risk-neutral measure Q an asset price S follows a stochastic differential equation

$$
dS=\mu S dt+\sigma(X)S dW_t^1, \tag{3.3.1}
$$

where the volatility, $\sigma(X)$, is a function of X and is itself a stochastic process governed by the equation

$$
dX=a dt+b(\rho dW_t^1+\sqrt{1-\rho^2}dW_t^2). \tag{3.3.2}
$$

The parameters a and b are positive constants. W_t^1 and W_t^2 are correlated Brownian motions under Q with $dW_t^1 dW_t^2 = \rho dt$ for some $-1 \leq \rho \leq 1$, where ρ is the correlation coefficient between W_t^1 and W_t^2. In this model the price of the option depends upon two random variables which are the underlying asset and the volatility function respectively. At time t the risk-neutral price of a call expiring at time $t \leq T$ in this stochastic volatility model is [14]

$$u_{BS} = E\left[e^{-r(T-t)}(S_T - K)^+ | \Im_t\right]. \tag{3.3.3}$$

This problem shows that the function u_{BS} satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + (a - b\Gamma)\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u}{\partial S \partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial Y^2} = ru. \tag{3.3.4}$$

The function u_{BS} also satisfies the boundary conditions

$$u(S, X, T) = (S - K)^+ \quad \text{for all } S \geq 0,\ X \geq 0, \tag{3.3.5}$$

$$u(0, X, t) = 0 \quad \text{for all } 0 \leq t \leq T,\ X \geq 0\ , \tag{3.3.6}$$

$$u(S, X, t) = (S - e^{-r(T-t)}K)^+ \quad \text{for all } 0 \leq t \leq T,\ S \geq 0, \tag{3.3.7}$$

$$\lim_{S\to\infty} \frac{u(S, X, t)}{S - K} = 1 \quad \text{for all } 0 \leq t \leq T, \tag{3.3.8}$$

$$\lim_{S\to\infty} u(S, X, t) = S \quad \text{for all } 0 \leq t \leq T,\ S \geq 0. \tag{3.3.9}$$

Since W_t^1 and W_t^2 are two-dimensional Brownian motions, we can derive the partial differential equation of a European call option. For this purpose one needs to specify the market price of volatility risk,[3] $\Lambda(S, X, t)$.

Partial Differential Equation Approach

We start by constructing a portfolio, π, consisting of

- 1 option with value $u(S, X, t)$

[3]The market price of risk is associated with the Girsanov transformation of the underlying probability measure leading to a particular martingale measure [25].

- $-\Delta$ shares

- $-\Delta_1$ of another option with value $u_1(S, X, t)$.

Then we have $\pi = u - \Delta S - \Delta_1 u_1$ and the corresponding change in the value of this portfolio is given by

$$d\pi = du - \Delta dS - \Delta_1 du_1. \tag{3.3.10}$$

By Itô's lemma we get

$$\begin{aligned} du = & \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial S}dS + \frac{\partial u}{\partial X}dX \\ & + \frac{1}{2}\left(\frac{\partial^2 u}{\partial S^2}(dS)^2 + \frac{\partial^2 u}{\partial X^2}(dX)^2 + \frac{\partial^2 u}{\partial S \partial X}(dS)(dX)\right). \end{aligned} \tag{3.3.11}$$

In this case

$$(dS)^2 = \sigma^2(X)S^2 dt \tag{3.3.12}$$

$$(dX)^2 = b^2 dt \tag{3.3.13}$$

$$(dS)(dX) = \sigma(X)abSdt \tag{3.3.14}$$

and hence (3.3.11) becomes

$$\begin{aligned} du = & \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial S}\left[\mu S + \sigma(X)SdW_t^1\right] \\ & + \frac{\partial u}{\partial X}\left[adt + b\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right)\right] \\ & + \frac{1}{2}\left[\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + b^2\frac{\partial^2 u}{\partial X^2} + 2\sigma(X)abS\frac{\partial^2 u}{\partial S \partial X}\right]dt \\ = & \left[\frac{\partial u}{\partial t} + \mu S\frac{\partial u}{\partial S} + a\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u}{\partial S \partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2}\right]dt \\ & + \left[\sigma(X)S\frac{\partial u}{\partial S} + b\rho\frac{\partial u}{\partial X}\right]dW_t^1 + b\sqrt{1-\rho^2}dW_t^2. \end{aligned}$$

We can obtain a similar expression for g_1. Therefore

$$\begin{aligned} d\pi = & du - \Delta dS - \Delta_1 du_1 \\ = & \left[\frac{\partial u}{\partial t} + \mu S\frac{\partial u}{\partial S} + a\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u}{\partial S \partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2}\right]dt \\ & + \left[\sigma(X)S\frac{\partial u}{\partial S} + b\rho\frac{\partial u}{\partial X}\right]dW_t^1 + b\sqrt{1-\rho^2}dW_t^2 - \Delta\mu Sdt - \Delta\sigma(X)dW_t^1 \end{aligned}$$

$$
\begin{aligned}
&-\Delta_1\left[\frac{\partial u_1}{\partial t}+\mu S\frac{\partial u_1}{\partial S}+a\frac{\partial u_1}{\partial X}+\frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u_1}{\partial S^2}+\sigma(X)abS\frac{\partial^2 u_1}{\partial S\partial X}+\frac{1}{2}b^2\frac{\partial^2 u_1}{\partial X^2}\right]dt\\
&-\Delta_1\left[\sigma(X)S\frac{\partial u_1}{\partial S}+b\rho\frac{\partial u_1}{\partial X}\right]dW_t^1-\Delta_1 b\sqrt{1-\rho^2}dW_t^2\\
=&\left[\frac{\partial u}{\partial t}+\mu S\frac{\partial u}{\partial S}+a\frac{\partial u}{\partial X}+\frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2}+\sigma(X)abS\frac{\partial^2 u}{\partial S\partial X}+\frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2}\right]dt\\
&-\Delta_1\left[\frac{\partial u_1}{\partial t}+\mu S\frac{\partial u_1}{\partial S}+a\frac{\partial u_1}{\partial X}+\frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u_1}{\partial S^2}+\sigma(X)abS\frac{\partial^2 u_1}{\partial S\partial X}+\frac{1}{2}b^2\frac{\partial^2 u_1}{\partial X^2}\right]dt\\
&-\Delta\mu S dt+\left[\frac{\partial u}{\partial S}-\Delta-\Delta_1\frac{\partial u_1}{\partial S}\right]\sigma(X)SdW_t^1+\left[\frac{\partial u}{\partial X}-\Delta_1\frac{\partial u_1}{\partial X}\right]b\rho dW_t^1\\
&+\left[\frac{\partial u}{\partial X}-\Delta_1\frac{\partial u_1}{\partial X}\right]b\sqrt{1-\rho^2}dW_t^2.
\end{aligned}
$$

To eliminate randomness the terms in dW_t^1 and dW_t^2 must be zero, i.e.

$$\frac{\partial u}{\partial S}-\Delta-\Delta_1\frac{\partial u_1}{\partial S}=0, \tag{3.3.15}$$

which implies that

$$\Delta=\frac{\partial u}{\partial S}-\Delta_1\frac{\partial u_1}{\partial S}, \tag{3.3.16}$$

and

$$\frac{\partial u}{\partial X}-\Delta_1\frac{\partial u_1}{\partial X}=0. \tag{3.3.17}$$

Also note that

$$\Delta\mu S dt=\left[\frac{\partial u}{\partial S}-\Delta_1\frac{\partial u_1}{\partial S}\right]\mu S dt. \tag{3.3.18}$$

Hence

$$
\begin{aligned}
d\pi=&\left[\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial X}+\frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2}+\sigma(X)abS\frac{\partial^2 u}{\partial S\partial X}+\frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2}\right]dt\\
&-\Delta_1\left[\frac{\partial u_1}{\partial t}+a\frac{\partial u_1}{\partial X}+\frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u_1}{\partial S^2}+\sigma(X)abS\frac{\partial^2 u_1}{\partial S\partial X}+\frac{1}{2}b^2\frac{\partial^2 u_1}{\partial X^2}\right]dt
\end{aligned}
$$

contains no stochastic term and is now risk-free. So

$$
\begin{aligned}
d\pi&=r\pi dt\\
&=r(u-\Delta S-\Delta_1 u_1)dt.
\end{aligned}
$$

Note that we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate. Thus

$$
\begin{aligned}
ru - r\Delta S - r\Delta_1 u_1 = &\left[\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u}{\partial S\partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2}\right] \\
&-\Delta_1\left[\frac{\partial u_1}{\partial t} + a\frac{\partial u_1}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u_1}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u_1}{\partial S\partial X} + \frac{1}{2}b^2\frac{\partial^2 u_1}{\partial X^2}\right].
\end{aligned}
\tag{3.3.19}
$$

Furthermore from (3.3.18)

$$
\begin{aligned}
r\Delta S &= r\left[\frac{\partial u}{\partial S} - \Delta_1\frac{\partial u_1}{\partial S}\right]S \\
&= rS\frac{\partial u}{\partial S} - \Delta_1 rS\frac{\partial u_1}{\partial S}.
\end{aligned}
$$

Hence equation (3.3.19) becomes

$$
\begin{aligned}
&\left[\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + a\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u}{\partial S\partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2} - ru\right] \\
&= -\Delta_1\left[\frac{\partial u_1}{\partial t} + rS\frac{\partial u_1}{\partial S} + a\frac{\partial u_1}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u_1}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u_1}{\partial S\partial X} + \frac{1}{2}b^2\frac{\partial^2 u_1}{\partial X^2} - ru_1\right].
\end{aligned}
$$

Using (3.3.17) we have

$$
\Delta_1 = \frac{\frac{\partial u}{\partial X}}{\frac{\partial u_1}{\partial X}}. \tag{3.3.20}
$$

Hence

$$
\begin{aligned}
&\frac{\left[\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + a\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u}{\partial S\partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2} - ru\right]}{\frac{\partial u}{\partial X}} \\
&= \frac{\left[\frac{\partial u_1}{\partial t} + rS\frac{\partial u_1}{\partial S} + a\frac{\partial u_1}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u_1}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u_1}{\partial S\partial X} + \frac{1}{2}b^2\frac{\partial^2 u_1}{\partial X^2} - ru_1\right]}{\frac{\partial u_1}{\partial X}}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\Lambda(S,X,t)\frac{\partial u}{\partial X} &= \frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + a\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} \\
&\quad + \sigma(X)abS\frac{\partial^2 u}{\partial S\partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2} - ru \\
&= b\Gamma(S,X,t)\frac{\partial u}{\partial X},
\end{aligned}
$$

where

$$\Gamma(S,X,t) = \Lambda(S,X,t)\sqrt{1-\rho^2} + \rho\frac{\mu - r}{\sigma(X)}. \tag{3.3.21}$$

Expression (3.3.21) is a linear combination of the market price of risk and the market price of volatility, b is the volatility from the stochastic process of X and $\Lambda(S,X,t)$ is the market price of volatility risk. Therefore

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + (a - b\Gamma)\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u}{\partial S \partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2} = ru. \tag{3.3.22}$$

Note that, if W_t^1 and W_t^2 are uncorrelated, i.e. $\rho = 0$, in this case $\Gamma(S,X,t) = \Lambda(S,X,t)$ and the price of the option satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + (a - b\Gamma)\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X)S^2\frac{\partial^2 u}{\partial S^2} + \sigma(X)abS\frac{\partial^2 u}{\partial S \partial X} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial X^2} = ru \tag{3.3.23}$$

as in (3.3.4). It is not easy to solve equation (3.3.23) and often the only alternative is to use numerical techniques to approximate solutions.

3.4 Mean-Reverting Models

Several approaches to model the behaviour of the variation of volatility have been proposed in the literature. Many of the stochastic volatility models under investigation use a mean-reverting process. Various stochastic volatility models are obtained by making different choices of the dynamics for the stochastic volatility process, X_t. Suppose an asset price, S, follows a stochastic differential process

$$dS = \mu S dt + \sigma(X_t) S dW_t^1. \tag{3.4.1}$$

Now for the model to be mean reverting X_t must satisfy a stochastic differential equation of the form

$$dX_t = a(b - X_t)dt + \alpha dW_t^2, \tag{3.4.2}$$

where b is called the long-run mean level of X_t and a is the rate of mean-reversion. This model is called a mean-reverting model because $b - X_t$ goes to b as X_t wanders away, a measures the speed.

Specifications of stochastic volatility proposed by other authors include the following stochastic differential equations [64]:

$$dX_t = a(b - X_t)dt + \alpha X_t dW_t^2 \tag{3.4.3}$$

$$dX_t = aX_t dt + \alpha dW_t^2 \tag{3.4.4}$$

$$dX_t = X_t^{-1}(b - aX_t^2)dt + \alpha dW_t^2 \quad \text{and} \tag{3.4.5}$$

$$dX_t = a(b - X_t^2)dt + \alpha X_t dW_t^2. \tag{3.4.6}$$

We note that, whilst it is not possible to deal in detail with each of the stochastic volatility models mentioned above, we look briefly at a few more popular mean-reverting models.

3.4.1 Ornstein-Uhlenbeck (OU) Model

The Ornestein-Uhlenbeck model satisfies the following stochastic differential equation,

$$dX_t = a(b - X_t)dt + \alpha dW_t^2 \tag{3.4.7}$$

where α is a positive constant. The Brownian motion W_t^2 is given by

$$dW_t^2 = \rho W_t^1 + \sqrt{1 - \rho^2} dW_t^3 \tag{3.4.8}$$

where W_t^3 is also a Brownian motion independent of W_t^1. The stochastic differential equation (3.4.7) can be solved explicity by considering

$$\begin{aligned} d\left[e^{at}(X_t - b)\right] &= e^{at}\left[dX_t + a(X_t - b)dt\right] \\ &= \alpha e^{at} dW_t^2 \end{aligned}$$

so that

$$e^{at}(X_t - b) - (X_0 - b) = \int_0^t \alpha e^{as} dW_s^2. \tag{3.4.9}$$

Thus

$$\begin{aligned} X_t &= (1 - e^{-at})b + e^{-at}X_0 + e^{-at}\int_0^t \alpha e^{as} dW_s^2 \\ &= (1 - e^{-at})b + e^{-at}X_0 + e^{-at}\alpha^2 \widehat{W}\left(\frac{e^{2at} - 1}{2a})\right) \end{aligned}$$

for some Brownian motion $\widehat{W}$. In this model X_t is normally distributed with mean b and variance $\alpha^2/2a$.

3.4.2 Cox-Ingersoll-Ross (CIR) Model

In the CIR model a stock S_t and volatility σ_t, respectively, satisfy the following differential equations

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t^1 \tag{3.4.10}$$

and

$$dv_t = (a + bv_t)dt + c\sqrt{v_t}\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^3\right), \tag{3.4.11}$$

where v_t is the variance, a, b and c are constants and dW_t^1 and dW_t^3 are independent Brownian motions. In the CIR model v_t has a chi-squared distribution with the mean given by

$$E(v_t|v_0 = y) = -\frac{a}{b} + \left(y + \frac{a}{b}\right)e^{-bt} \tag{3.4.12}$$

and the variance by

$$Var(v_t|v_0 = y) = \frac{ac^2}{2b^2} - \frac{c^2}{b}\left(y + \frac{a}{b}\right)e^{-bt} + \frac{c^2}{b}\left(y + \frac{a}{2b}\right)e^{-2bt}. \tag{3.4.13}$$

Using the above equations we can find the limiting distribution of v_t which is a gamma distribution with the mean of $-a/b$ and the variance $ac^2/2b^2$.

3.4.3 Wiggins Model

Wiggins (1987) [92] proposed the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma(X_t)dW_t^1, \tag{3.4.14}$$

where the volatility X_t satisfies

$$dX_t = aX_t dt + \alpha X_t dW_t^2 \tag{3.4.15}$$

with W_t^1 and W_t^2 being the correlated Brownian motions. With this model as basis Wiggins [92] derived a statistical estimator for volatility-process parameters using a Monte Carlo approach. He showed that the Black-Scholes equation overprices out-of-the-money calls in relation to in-the-money calls.

3.4.4 Hull and White Model

Hull and White [53] proposed the following stochastic volatility model

$$dS_t = \mu S_t dt + \sigma(X_t) S_t dW_t^1 \tag{3.4.16}$$

with the volatility X_t satisfying

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t^2, \tag{3.4.17}$$

where dW^1 and dW^2 are independent Brownian motions. In their studies Hull and White [53] analysed this model for the cases $\rho = 0$ and $\rho \neq 0$. Based upon their analysis, they concluded that Black-Scholes formula overprices the options and that the maturity time affects the degree of overpricing.

3.4.5 Heston's Model

In the Heston [47] model the asset-price return process is modelled as

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1 \tag{3.4.18}$$

with the variance v_t satisfying

$$dv_t = \alpha_2(v_2 - v_t)dt + \eta\sqrt{v_t}dW_t^2, \tag{3.4.19}$$

where $dW_t^1 dW_t^2 = \rho dt$. Using his model Heston [47] developed a new technique, based upon characteristic functions, to derive a closed-form solution for the price of a European call option on an asset.

3.5 Biases of the Stochastic Volatility Models

Although the Heston model gives a closed-form solution for an option price on a stock price following the stochastic volatility process, the solution requires evaluation of a difficult integral expression. Luckily, the Hull and White [54] model provides an accurate approximation using a Taylor series expansion around a constant volatility. Adding this idea into the Black-Scholes call price gives the Hull-White stochastic volatility model which may be used to correct the Black-Scholes formula.

Chapter 4

Bond Pricing and Interest Rate Models

Most investors are risk averse and the risk is linked to the interest rate. Bonds offer investors an excellent means of managing the overall portfolio risk. However, bond-pricing formulae cannot identify risk payoff because bond prices are mainly determined by changes in interest rates.

The two more popular one-factor interest rate models are the Vasicek and Cox-Ingersoll-Ross (CIR) models. Their popularity is due to their tractability and their flexibility, i.e., they lead to analytic solutions of the bond-pricing equation. These solutions are useful in providing models for the term structure of the interest rate.

In their original paper Cox, Ingersoll and Ross [23] aimed to have a relationship between the term structure of interest rates and the yield on riskless securities that differ only in their time to maturity. The exploitation of the term structure gives extra information to predict the effect on the yield curve[1]. In their analysis Chan *et al* [16] found that many of the well-known interest

[1]A yield curve is a graph in which interest rates are plotted against term to maturity for bonds of the same quality. Once the figures are plotted, financial analysts study the graph carefully because it contains an information (on investor's expectations) about the future trends of the interest rate.

rate models perform poorly in their ability to summarize the actual behaviour of the short rate because of their constraints on the term structure volatility. Fong and Vasicek [31] in their model treated a volatility of the short-term rate as a stochastic variable which better describes the term structure of interest rates. All short rate models are theoretically constructed and not based upon the observable rate.

In their model Longstaff and Schwartz [63] examined the effect of stochastic interest rates. The Longstaff-Schwartz model is a well-known two-factor model which is flexible in achieving good calibration to a variety of term structures. Interest rate models such as those of Hull and White [52] and Ho and Lee [48] incorporate time-dependent parameters. This has the added advantage of allowing a yield curve to be fitted.

In this Chapter we firstly derive the bond-price formula under the assumption that the interest rate is constant. We then present some well-known interest-rate models and use these models for bond pricing.

4.1 One-Factor Bond-Pricing Equation

In this Section we derive the bond-price equation using the arbitrage pricing approach. The method of applying the riskless hedging principle is similar but slightly different from that used in Chapter 1 Section (1.3).

Let $V(t,T)$ be the price at a time t of a zero coupon bond paying 1 at a later time T, i.e. $V(T,T) = 1$, and let the spot rate, r_t, follow the stochastic differential equation

$$dr = \mu(r,t)dt + \sigma(r,t)dW, \tag{4.1.1}$$

where W is a standard Brownian motion. By Itô's Lemma we have

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2}\right)dt + \sigma\frac{\partial V}{\partial r}dW \\ &= a(r,t)dt + b(r,t)dW, \end{aligned} \tag{4.1.2}$$

where in this case

$$\begin{aligned} a(r,t) &= \frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} \\ \text{and} \quad b(r,t) &= \sigma\frac{\partial V}{\partial r}. \end{aligned} \tag{4.1.3}$$

Consider a portfolio given by

$$\pi = V_1 - \Delta V_2, \tag{4.1.4}$$

where $V_1 = V(r, T_1)$ is a bond with maturity date T_1, $V_2 = V(r, T_2)$ is a bond with maturity date T_2 and Δ is a constant to be determined. Differentiating (4.1.4) with respect to V we obtain

$$d\pi = dV_1 - \Delta dV_2. \tag{4.1.5}$$

Substituting for dV_1 and dV_2 into (4.1.5), where dV_1 and dV_2 are defined as follows:

$$\begin{aligned} dV_1 &= a_1 dt + b_1 dW \\ dV_2 &= a_2 dt + b_2 dW, \end{aligned}$$

we obtain

$$d\pi = (a_1 - \Delta a_2)dt + (b_1 - \Delta b_2)dW. \tag{4.1.6}$$

To eliminate uncertainty we take Δ so that $\Delta = b_1/b_2$. This means that

$$\begin{aligned} d\pi &= (a_1 - \Delta a_2)dt \\ &= r\pi dt, \end{aligned}$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate. Hence

$$\begin{aligned} r\pi &= a_1 - \frac{b_1}{b_2}a_2, \\ r\left(V_1 - \frac{b_1}{b_2}V_2\right) &= a_1 - \frac{b_1}{b_2}a_2, \\ \frac{a_1 - rV_1}{b_1} &= \frac{a_2 - rV_2}{b_2}. \end{aligned} \tag{4.1.7}$$

The left side of (4.1.7) is a function of T_1 and the right side is a function of T_2. So both sides are independent of T_1 and T_2. Dropping the subscripts we obtain

$$\frac{a - rV}{b} = \lambda, \tag{4.1.8}$$

where λ is called the market price of risk[2]. We can rewrite (4.1.8) as

$$a = rV + \lambda b \tag{4.1.9}$$

so that

$$\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} = rV + \lambda\sigma\frac{\partial V}{\partial r}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} + (\mu - \lambda\sigma)\frac{\partial V}{\partial r} = rV. \tag{4.1.10}$$

This is called a Black-Scholes partial differential equation for the pricing of a bond option. In the case of a Black-Scholes equation μ and σ are assumed to be constant whereas in (4.1.10) these are functions of r and t. This means that the different cases give rise to different solutions which depend upon the choice of r_t as well as initial conditions.

Market Price of Risk

Once u and σ are computed, the market price of risk, $\lambda(r,t)$, can be estimated using the following relation

$$\frac{\partial R}{\partial T} = \frac{1}{2}[u(r,t) - \sigma(r,t)\lambda(r,t)], \tag{4.1.11}$$

where $\partial R/\partial T$ is the slope of the yield curve at the origin. The yield to maturity $R(t,T)$ is defined by $R(t,T) = -\ln V(t,T)/(T-t)$, which gives the internal rate of return at the time t on the bond. The yield curve is the plot of $R(t,T)$ against T and the dependence of the yield curve on the time to maturity $(T-t)$ is called the term structure of interest rates.

[2]The market price of risk refers to the expected standardised excess rate of return above the risk-free rate from a specific zero-coupon bond.

4.1.1 A Solution to the Black-Scholes Equation

If we specify the values of μ, σ and λ, we are able to solve (4.1.10) for certain cases. Consider

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} + w\frac{\partial V}{\partial r} = rV, \tag{4.1.12}$$

so that $w = \mu - \lambda\sigma$ is an unknown and is assumed to be constant. We assume a solution of the form

$$V = \exp[A(t,T)r + B(t,T)] \tag{4.1.13}$$

where $\dfrac{\partial V}{\partial t} = (A'r + B')V$, $\dfrac{\partial V}{\partial r} = AV$, $\dfrac{\partial^2 V}{\partial r^2} = A^2 V$. Substitution of them into equation (4.1.12) gives

$$A'r + B' + \frac{1}{2}\sigma^2 A^2 + wA = r. \tag{4.1.14}$$

If $A' = 1$ in (4.1.14), we obtain

$$B^{'} + \frac{1}{2}\sigma^2 A^2 + wA = 0. \tag{4.1.15}$$

A direct integration of $A' = 1$ and (4.1.15) with respect t gives

$$\begin{aligned} & A = t - T \\ \text{and} \quad & B = -\frac{1}{2}w(t-T)^2 - \frac{1}{6}\sigma^2(t-T)^3 \\ & \quad = -\frac{1}{2}w\tau^2 + \frac{1}{6}\sigma^2\tau^3 \end{aligned}$$

respectively. Therefore

$$V = \exp\left[-\tau r - \frac{1}{2}w\tau^2 + \frac{1}{6}\sigma^2\tau^3\right] \tag{4.1.16}$$

which is a solution of (4.1.12) when μ and σ are considered to be constants.

4.1.2 Representation of the Bond-Pricing Solution in Stochastic Integral

The solution of the bond price (4.1.12) can be formally represented in an integral form in terms of the underlying process, namely,

$$V(t,T) = E_t \exp\left(-\int_t^T r_\eta d\eta - \frac{1}{2}\int_t^T \theta^2(r_\eta,\eta)d\eta - \int_t^T \theta(r_\eta,\eta)dW_\eta\right) \tag{4.1.17}$$

We define the auxiliary function

$$D(t,\zeta) = \exp\left(-\int_t^{\zeta} r_\eta d\eta - \frac{1}{2}\int_t^{\zeta} \theta^2(r_\eta,\eta)d\eta - \int_t^{\zeta} \theta(r_\eta,\eta)dW_\eta\right) \qquad (4.1.18)$$

and apply Itô's differential rule to compute $V(r,t;\zeta)D(r,t;\zeta)$. This gives

$$\begin{aligned}
d(VD) &= DdV + VdD + dVdD \\
&= D\left(\frac{\partial V}{\partial \zeta} + u\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2}\right)d\zeta + D\sigma\frac{\partial V}{\partial r}dW + VD\left(-r - \frac{1}{2}\theta^2\right)d\zeta \\
&\quad -VD\theta dW + \frac{1}{2}\theta^2 VDd\zeta - D\theta\sigma\frac{\partial V}{\partial r}d\zeta \\
&= D\left(\frac{\partial V}{\partial \zeta} + (u-\theta\sigma)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} - rV\right)d\zeta - VD\theta dW + D\sigma\frac{\partial V}{\partial r}dW \\
&= -VD\theta dW + D\sigma\frac{\partial V}{\partial r}dW. \qquad (4.1.19)
\end{aligned}$$

Next we integrate (4.1.19) from t to T and take the expectations. This gives

$$E_t[V(T,T)D(T,T) - V(t,T)D(t,t)] = 0. \qquad (4.1.20)$$

Since $V(T,T) = 1$ and $D(t,t) = 1$, we obtain

$$V(t,T) = E_t(V(t,T)) \qquad (4.1.21)$$

and thus $V(t,T)$ is a martingale.

In the next section we explore the solution of the one-factor bond-pricing equation with different assumptions of the stochastic process for r_t.

4.2 One-Factor Interest-Rate Models

One-factor interest-rate models are a popular class of interest-rate model that play a prominent role in the pricing of interest-rate derivatives. Many of these models can be nested within the stochastic process represented by

$$dr_t = a(b - r_t)dt + \alpha r^{\delta} dW_t, \qquad (4.2.1)$$

where μ is the drift, σ is the diffusion term of the interest-rate process and W is a Brownian motion, a, b, δ and α are constants. Such models include those of Vasicek [91] ($\delta = 0$), CIR [23] ($\delta = \frac{1}{2}$) and Brennan and Schwartz [12] ($\delta = 1$). To obtain a solution of equation (4.2.1) we take the expectation

$$\begin{aligned} \Psi_t &= E(r_t) \\ &= r_0 + \int_0^t a(b - \Psi_v)dv. \end{aligned} \tag{4.2.2}$$

Differentiating equation (4.2.2) with respect to t we obtain

$$\Psi_t' = ab - a\Psi_t. \tag{4.2.3}$$

Therefore the solution of equation (4.2.2) is given by

$$\Psi_t = b + (\Psi_0 - b)e^{-at}. \tag{4.2.4}$$

Vasicek Model

The Vasicek model is a type of one-factor model that explain the movements of an interest rate when it is only driven by the market risk. Vasicek model was introduced by Oldrich Vasicek in 1977 [91]. It was the first economic model to capture the mean reversion.

The model for the interest rate, r_t, in the Vasicek format follows the stochastic differential equation

$$dr_t = a(b - r_t)dt + \sigma dW_t, \tag{4.2.5}$$

where a, b and σ are positive constants. In this case $\mu(r, t) = a(b - r)$ and $\sigma(r, t) = \sigma$. The solution of the model is, for each $v \leq t$,

$$r_t = r_v e^{-a(t-v)} + b[1 - e^{-a(t-v)}] + \sigma \int_v^t e^{-a(t-v)} dW_s. \tag{4.2.6}$$

Here the interest rates, r_t, are normally distributed with the expectation

$$E(r_t|\Im_t) = r_v e^{-a(t-v)} + b[1 - e^{-a(t-v)}] \tag{4.2.7}$$

and the variance

$$Var(r_t|\Im_t) = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-v)}\right]. \tag{4.2.8}$$

As $t \to \infty$, the limit of mean rate and variance, as long as $a > 0$, converge to b and $\sigma^2/2a$, respectively.

For the Vasicek model the partial differential equation for bond equation (5.1.15) takes the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} + a(b-r)\frac{\partial V}{\partial r} - rV = 0. \tag{4.2.9}$$

Suppose we assume the solution to be of the form [83, 86]

$$V(t;T) = A(t,T)e^{-rB(t,T)}, \tag{4.2.10}$$

where $V(t,T)$ is the price at some time t of a zero coupon bond maturing at time, T, r is the short-term rate of interest at time, t, and A and B are functions of only t and T. Then

$$A(t,T) = \exp\left[\frac{(B(t,T) - T + t)(a^2 b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right] \tag{4.2.11}$$

and

$$B(t,T) = \frac{1 - e^{-a\tau}}{a}. \tag{4.2.12}$$

One of the main features of a Vasicek model is that all bond prices are related to the same factor, the instantaneous interest rate. Thus all bond-price movements are derived from movements of the same factor. This implies that all bond prices are perfectly correlated. The main drawback of a Vasicek model is that it allows the interest rate to become negative.

Cox-Ingersoll-Ross Model

To rectify the drawbacks of the Vasicek model, Cox, Ingersoll and Ross (1985) [23] proposed an alternative model for the interest rate as follows

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \tag{4.2.13}$$

where a, b and σ are positive constants. The expected value is given by

$$E(r_t|\Im_v) = r_u e^{-a(t-v)} + b[1 - e^{-a(t-v)}] \tag{4.2.14}$$

and the variance is

$$Var(r_t|\Im_v) = r_v\frac{\sigma^2}{a}[e^{-a(t-v)} - e^{-2a(t-v)}] + \frac{b\sigma^2}{2a}[1 - e^{-a(t-v)}]^2. \tag{4.2.15}$$

To prove (4.2.14) and (4.2.15) we take the integral of the stochastic differential of the model (4.2.13) $\forall \quad v \leq t$,

$$r_t = r_v + a\int_v^t (b - r_w)dw + \sigma\int_v^t \sqrt{r_w}dW_w. \tag{4.2.16}$$

By itôs lemma we have

$$\begin{aligned} r_t^2 &= r_v^2 + 2a\int_v^t (b - r_w)dw + 2\sigma\int_v^t \sqrt{r_w^3}dW_w + \sigma^2\int_v^t r_w dw \\ &= r_v^2 + (2ab + \sigma^2)\int_v^t r_w dw - 2a\int_v^t r_w^2 dw + 2\sigma\int_v^t \sqrt{r_w^3}dW_w. \end{aligned} \tag{4.2.17}$$

In terms of its initial condition we can rewrite equation (4.2.16) as

$$r_t = r_0 + a\int_0^t (b - r_w)dw + \sigma\int_0^t \sqrt{r_w}dW_w. \tag{4.2.18}$$

Then the unconditional mean is

$$E^Q(r_t) = r_0 + a\left(bt - \int_0^t E^Q(r_w)dw\right). \tag{4.2.19}$$

Solving equation (4.2.19) we get the expected value

$$E^Q(r_t) = b + (r_0 - b)e^{at}. \tag{4.2.20}$$

Rearranging (4.2.20) we obtain

$$E^Q(r_t|\Im_v) = r_v e^{-a(t-v)} + b[1 - e^{-a(t-v)}]. \tag{4.2.21}$$

Since the variance $Var(r_t) = E^Q(r_t^2) + [E^Q(r_t)]^2$, on reorganizing equation (4.2.20) we obtain

$$E^Q(r_t^2) = r_0^2 + (2ab + \sigma^2)\int_0^t E^Q(r_w^2)dw - 2a\int_v^t E^Q(r_w^2)dw. \tag{4.2.22}$$

Substituting (4.2.20) into (4.2.22) and using the second moment, we obtain the variance as follows:

$$Var(r_t) = \frac{\sigma^2}{a}[1 - e^{-bt}] + \frac{a}{2}[1 - e^{bt}]. \tag{4.2.23}$$

Rearranging equation (4.2.23) we obtain

$$Var(r_t|\Im_v) = r_u\frac{\sigma^2}{a}[e^{-a(t-v)} - e^{-2a(t-v)}] + \frac{b\sigma^2}{2a}[1 - e^{-a(t-v)}]^2. \tag{4.2.24}$$

In the CIR model the partial differential equation for the bond equation (5.1.15) is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r\frac{\partial^2 V}{\partial r^2} + a(b-r)\frac{\partial V}{\partial r} - rV = 0. \tag{4.2.25}$$

Using the same assumption, (4.2.10), we obtain

$$A(t,T) = \left[\frac{2\gamma\exp(\gamma+a)\tau/2}{(\gamma+a)(\exp(\gamma\tau)-1)+2\gamma}\right]^{(2ab)/\sigma^2} \tag{4.2.26}$$

and

$$B(t,T) = \frac{2(\exp(\gamma\tau)-1)}{(\gamma+a)(\exp(\gamma\tau)-1)+2\gamma}, \tag{4.2.27}$$

where $\gamma = \sqrt{a^2 + 2\sigma^2}$.

Hull and White Model

According to the Hull and White [52] model the bonds and European options can be valued analytically in terms of the initial term structure and the value of r at time t. The Hull-White stochastic differential equation is given by

$$dr_t = a(\theta_t - r)dt + \sigma dW_t, \tag{4.2.28}$$

where a and σ are constants. The function θ is given by

$$\theta_t = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}), \tag{4.2.29}$$

where $f(0,t) = -\partial \ln(V(0,t))/\partial t$ is the forward rate[3] that applies to time t as observed at time zero. Assuming

$$V(t;T) = A(t,T)e^{-rB(t,T)} \tag{4.2.30}$$

[3]Forward rates are interest rates that can be locked in today for an investment in a future time period and their value can be derived directly from zero-coupon bond prices

we determine the functions A and B from an initial value of the discount bond $V(0,T)$ as

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{4.2.31}$$

and

$$A(t,T) = \exp\left[-\int_t^T \theta_s B(s,T)ds - \frac{\sigma^2}{2a^2}(B(t,T) - T + t) - \frac{\sigma^2}{4a}B(t,T)^2\right]. \tag{4.2.32}$$

4.2.1 Bond-Option Models Based on One-Factor Interest-Rate Models

The pricing models of the bond and the bond option differ only in the terminal conditions. The payoff functions for the bond price is $V(r, T_V) = F$, where T_V is the maturity date of the bond, F is the face value of the bond and the bond option value $\Gamma(r, T_\Gamma) = \max\left[\gamma(\Gamma(r, T_V) - K), 0\right]$ with T_Γ being the expiration date of the bond option. Here γ is a binary variable which takes the value 1 when the option is a call and -1 when the option is a put [59].

The value of the European call option on a zero coupon bond is given by

$$u(r,t) = V(r,t;T_V)\Phi(d_+) - KV(r,t;T_\Gamma)\Phi(d_-), \tag{4.2.33}$$

where

$$d_+ = \frac{\ln \dfrac{V(r,t;T_V)}{KV(r,t;T_\Gamma)} + \dfrac{1}{2}\sigma^2(T_\Gamma - t)}{\sigma_c\sqrt{T_\Gamma - t}}, \quad d_- = d_+ - \sigma_\Gamma\sqrt{T_\Gamma - t}. \tag{4.2.34}$$

4.3 Vasicek Models of Stochastic Volatility

In the stochastic volatility Vasicek model the interest rate, r_t, follows the Vasicek model with stochastic volatility, i.e.,

$$dr_t = a(b - r_t)dt + f(Y_t)dW_t^1 \tag{4.3.1}$$

$$dY_t = \alpha(m - Y_t)dt + \beta\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^3\right), \tag{4.3.2}$$

where the stochastic volatility σ_t given by the positive function, $f(Y_t)$ and dW_t^1 and dW_t^3 are independent standard Brownian motions. In this case the interest rate and volatility are correlated. Using this model Cotton *et al* [18] priced the bond options.

4.4 Two-Factor Interest-Rate Models

A major disadvantage of one-factor interest rate models is the implication that interest rates with different maturities are perfectly correlated. This can be avoided by introducing a second factor.

4.4.1 Vasicek Model

For the two-factor Vasicek model we let the factors Y_{1t} and Y_{2t} be given by the system of stochastic differential equations

$$dY_{1t} = (a_1 - b_{11}Y_{1t} - b_{12}Y_{2t})dt + \sigma_1 dW_{1t}^Q \tag{4.4.1}$$

$$dY_{2t} = (a_2 - b_{21}Y_{1t}dt - b_{22}Y_{2t})dt + \sigma_2 dW_{2t}^Q \tag{4.4.2}$$

and

$$r_t = \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t}, \tag{4.4.3}$$

where all parameters are strictly positive and dW_{1t}^Q and dW_{2t}^Q are correlated Brownian motions. We assume the matrix B= $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ has positive eigenvalues λ_1 and λ_2.

To eliminate the overparametrisation of the two-factor Vasicek model, we reduce the model (4.4.1) and (4.4.2) to the canonical two-factor Vasicek model

$$dX_{1t} = -\lambda_1 X_{1t}dt + dW_{1t}^Q \tag{4.4.4}$$

$$dX_{2t} = -\lambda_{21} X_{1t}dt - \lambda_2 X_{2t}dt + dW_{2t}^Q \tag{4.4.5}$$

$$r_t = \upsilon_0 + \upsilon_1 Y_{1t} + \upsilon_2 Y_{2t}, \tag{4.4.6}$$

where W_{1t} and W_{2t} are independent Brownian motions. The canonical two-factor Vasicek model has thus few parameters that can be used to calibrate the model.

Bond Prices

The price at a time t of a zero-coupon bond according to the risk-neutral pricing formula is given by

$$V(t,T) = E[\exp(-\int_t^T r_v dv)|\Im_t], 0 \le t \le T. \tag{4.4.7}$$

Because r_t is a function of the factors X_{1t} and X_{2t} and the solution of the system of stochastic differential equation (4.4.4) and (4.4.5) is a martingale, there must be some function $f(t, x_{1t}, x_{2t})$ such that

$$V(t,T) = f(t, X_{1t}, X_{2t}). \tag{4.4.8}$$

The discount factor $D_t = \exp\left(-\int_0^t r_v dv\right)$ satisfies $dD_t = -r_t D_t dt$.

Therefore

$$\begin{aligned}
d[D_t V(t,T)] &= d[D_t f(t, X_{1t}, X_{2t})] \\
&= -r_t D_t f(t, X_{1t}, X_{2t})dt + D_t d[f(t, X_{1t}, X_{2t})] \\
&= D\left[-rf dt + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x_1}dX_1 + \frac{\partial f}{\partial x_2}dX_2\right] \\
&\quad +D\left[\frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}dX_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}dX_1 dX_2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}dX_2^2\right] \\
&= D\left[-(\upsilon_0 + \upsilon_1 x_1 + \upsilon_2 x_2)f + \frac{\partial f}{\partial t} - \lambda_1 x_1 \frac{\partial f}{\partial x_1} - \lambda_{21} x_1 \frac{\partial f}{\partial x_2} - \lambda_2 x_2 \frac{\partial f}{\partial x_2}\right. \\
&\quad \left. + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}\right] dt + D\left[\frac{\partial f}{\partial x_1}dW_1^Q + \frac{\partial f}{\partial x_2}dW_2^Q\right].
\end{aligned} \tag{4.4.9}$$

Setting the dt term equal to zero, we obtain the partial differential equation

$$\frac{\partial f}{\partial t} - \lambda_1 x_1 \frac{\partial f}{\partial x_1} - (\lambda_{21} x_1 + \lambda_2 x_2)\frac{\partial f}{\partial x_2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2} = (\upsilon_0 + \upsilon_1 x_1 + \upsilon_2 x_2)f \tag{4.4.10}$$

with the terminal condition $f(T, x_1, x_2) = 1$.

To solve this equation we assume that [83]

$$f(t,x_1,x_2) = \exp\left[-x_1A_1(t,T) - x_2A_2(t,T) - A_3(t,T)\right]. \tag{4.4.11}$$

Then $\dfrac{\partial f}{\partial t} = [x_1A_1' + x_2A_2' + A_3']f$, $\dfrac{\partial f}{\partial x_1} = -A_1f$, $\dfrac{\partial f}{\partial x_2} = -A_2f$, $\dfrac{\partial^2 f}{\partial x_1^2} = A_1^2f$, $\dfrac{\partial^2 f}{\partial x_2^2} = A_2^2f$ and $\dfrac{\partial^2 f}{\partial x_1\partial x_2} = A_1A_2f$. Thus equation (4.4.10) becomes

$$\left[(A_1' + \lambda_1A_1 + \lambda_{21}A_2 - \upsilon_1)x_1 + (A_2' + \lambda_2A_2 - \upsilon_2)x_2 + (A_3' + \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 - \upsilon_0)\right] f = 0. \tag{4.4.12}$$

This gives us a system of three ordinary differential equations:

$$A_1' = -\lambda_1A_1 - \lambda_{21}A_2 + \upsilon_1 \tag{4.4.13}$$

$$A_2' = -\lambda_2A_2 + \upsilon_2 \tag{4.4.14}$$

$$A_3' = -\frac{1}{2}A_1^2 - \frac{1}{2}A_2^2 + \upsilon_0. \tag{4.4.15}$$

To solve equation (4.4.14) we multiply both sides by an integrating factor $e^{\lambda_2\tau}$ to get

$$e^{\lambda_2\tau}A_2' + \lambda_2e^{\lambda_2\tau}A_2 = e^{\lambda_2\tau}\upsilon_2. \tag{4.4.16}$$

The left hand side can be written as $\dfrac{d}{d\tau}\left(e^{\lambda_2\tau}A_2\right)$ so that

$$\frac{d}{d\tau}\left(e^{\lambda_2\tau}A_2\right) = e^{\lambda_2\tau}\upsilon_2. \tag{4.4.17}$$

Integrating both sides of equation (4.4.17) gives

$$e^{\lambda_2\tau}A_2 = \frac{\upsilon_2}{\lambda_2}e^{\lambda_2\tau} + k. \tag{4.4.18}$$

Therefore

$$A_2 = e^{-\lambda_2\tau}\left(\frac{\upsilon_2}{\lambda_2}e^{\lambda_2\tau} + k\right). \tag{4.4.19}$$

Using the initial condition $A_2(0) = 0$ gives

$$A_2(t,T) = \frac{\upsilon_2}{\lambda_2}\left(1 - e^{-\lambda_2}\right). \tag{4.4.20}$$

Substituting equation (4.4.20) into equation (4.4.13) yield

$$\begin{aligned}\frac{d}{d\tau}(e^{\lambda_1\tau}A_1) &= e^{\lambda_1\tau}\left(\lambda_1 A_1 + A_1'\right)\\ &= e^{\lambda_1\tau}\left(-\lambda_{21}A_2 + \upsilon_1\right)\\ &= e^{\lambda_1\tau}\left(-\frac{\lambda_{21}\upsilon_2}{\lambda_2}(1-e^{-\lambda_2\tau}) + \upsilon_1\right).\end{aligned} \tag{4.4.21}$$

If $\lambda_1 = \lambda_2$, integrating from 0 to τ gives

$$A_1(t,T) = \frac{1}{\lambda_1}\left(\upsilon_1 - \frac{\lambda_{21}\upsilon_2}{\lambda_1}\right)(1-e^{-\lambda_1}) + \frac{\lambda_{21}\upsilon_2}{\lambda_1}\tau e^{-\lambda_1}. \tag{4.4.22}$$

If $\lambda_1 \neq \lambda_2$ then

$$A_1(t,T) = \frac{1}{\lambda_1}\left(\upsilon_1 - \frac{\lambda_{21}\upsilon_2}{\lambda_2}\right)(1-e^{-\lambda_1}) + \frac{\lambda_{21}\upsilon_2}{\lambda_2(\lambda_1-\lambda_2)}(e^{-\lambda_2} - e^{-\lambda_1}). \tag{4.4.23}$$

Finally equation (4.4.15) and the initial condition $A_3(0) = 0$ gives

$$A_3 = \int_0^\tau \left[-\frac{1}{2}A_1^2(\upsilon) - \frac{1}{2}A_2^2(\upsilon) + \upsilon_0\right] d\upsilon. \tag{4.4.24}$$

4.4.2 Longstaff and Schwartz

For the two-factor Longstaff and Schwartz model we let the factors Y_{1t} and Y_{2t} be given by the system of stochastic differential equations

$$dY_{1t} = a_1(b_1 - Y_{1t})dt + \sqrt{Y_{1t}}dW_{1t}^Q, \tag{4.4.25}$$

$$dY_{2t} = a_2(b_2 - Y_{2t})dt + \sqrt{Y_{2t}}dW_{2t}^Q \tag{4.4.26}$$

and

$$r_t = b_1 Y_{1t} + b_2 Y_{2t} \tag{4.4.27}$$

where all parameters have nonnegative values and W_{1t} and W_{2t} are independent Brownian motions. If we further let

$$b_1 Y_{1t} = X_{1t} \qquad \text{and} \qquad b_2 Y_{1t} = X_{2t}, \tag{4.4.28}$$

the equation (4.4.27) becomes

$$r_t = X_{1t} + X_{2t}. \tag{4.4.29}$$

Equations (4.4.25) and (4.4.26) can now be written as

$$dX_{1t} = a_1(b_1\mu_1 - X_{1t})dt + \sqrt{b_1 X_{1t}}dW_{1t}^Q \tag{4.4.30}$$

$$dX_{2t} = a_2(b_2\mu_2 - X_{2t})dt + \sqrt{b_2 X_{2t}}dW_{2t}^Q. \tag{4.4.31}$$

Since X_1 and X_2 describe one-factor CIR process, the Longstaff-Schwartz model can be interpreted as a two-factor CIR model. Via the change of variable it can be expressed as a stochastic volatility model. By setting $v_t = b_1^2 Y_{1t} + b_2^2 Y_{2t}$, where $v_t dt$ is the variance of r_t, we may rewrite (4.4.27) as

$$\begin{aligned}
dr_t &= b_1 Y_{1t} + b_2 Y_{2t} \\
&= a_1 b_1(\mu_1 - Y_{1t})dt + b_1\sqrt{Y_{1t}}dW_{1t}^Q \\
&\quad + a_2 b_2(\mu_2 - Y_{2t})dt + b_2\sqrt{Y_{2t}}dW_{2t}^Q \\
&= \left(a_1 b_1 \mu_1 + a_2 b_2 \mu_2 - \frac{(a_1 b_2 - a_2 b_1)r_t + (a_2 - a_1)v_t}{b_2 - b_1}\right) dt \\
&\quad + \sqrt{\frac{b_1(b_2 r_t - v_t)}{b_2 - b_1}}dW_{1t}^Q + \sqrt{\frac{b_2(v_t - b_1 r_t)}{b_2 - b_1}}dW_{2t}^Q.
\end{aligned} \tag{4.4.32}$$

4.5 Multifactor Interest Rate Models

Several multifactor interest-rate models have been proposed in the literature to rectify the shortcomings of one-factor and two-factor models. For example, a one-factor interest-rate model such as the Vasicek model does not have a large range of shapes and provides a poor fit to some initial yield curves. Multifactor interest rate models, however, offer large flexibility due to a larger number of parameters.

In most cases to solve multifactor interest-rate models one has to resort to numerical evaluation of solutions of coupled ordinary differential equations or of an integral for the valuation of bond prices.

Chapter 5

Symmetry Analysis of Black-Scholes Equation

The concept of the symmetry of a differential equations was introduced by Norwegian Sophus Lie [62] in 1870. Inter alia Lie used group theoretical methods to provide a classification of all ordinary differential equations of arbitrary order in terms of their symmetry groups. Lie groups have had a profound impact on all areas of mathematics, physics, engineering, biology, chemistry, economics and finance.

The fundamentals of Lie's theory are based upon the invariance of differential equations under transformation groups of independent and dependent variables. Once the symmetry group of a system of equations is obtained, it can be used to classify, simplify and to transform solutions to other solutions. The construction of these particular solutions reduces the number of independent variables in the equation. For example, invariance with respect to a one-parameter group reduces the number of variables by one. This idea can be further extended for two or more parameters.

The application of the Lie group method has been developed by a number of mathematicians. Among the well-known researchers are Gazizov and Ibragimov [39], who applied the Lie group methods to economics and finance by computing all the symmetries of the Black-Scholes equation. Ovsiannikov [73] developed a systematic program of applying the Lie group methods to a wide range of physically important problems.

In this Chapter we firstly look at some basic symmetry properties of a group and then use a Lie group transformation to obtain the closed-form solution of Black-Scholes equation. We further extend the discussion of group-theoretic methods to partial differential equations.

5.1 Lie Groups

In this Section we define the general properties of a group and then extend these group properties to that of Lie groups. Each Lie group is connected to a Lie algebra and the notion of a vector field is closely related to that of a Lie algebra.

5.1.1 The Lie Analysis

A group can be defined as the a set of elements G with the law of composition ϕ between the elements satisfying the closure, associative, identity and inverse properties. The product of the two elements x and y of a Lie algebra $[x, y] = xy - yx$, where $[x, y]$ denotes their Lie Bracket. A Lie algebra is a vector space equipped with bilinear map property, i.e., $[x, y] = -[y, x]$, and furthermore it also satisfies the Jacobi identity, i.e., $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ \forall$ vectors x, y, z in the Lie algebra. The Abelian group is a group G that satisfies $\phi(x, y) = \phi(y, x) \ \forall$ elements x and y in G. The Lie algebra is called Abelian if $[x, y] = 0 \ \forall \ x, y$ in the Lie algebra.

Infinitestimal Transformations

The differential operator G which is given by

$$G = \xi(x, y)\partial_x + \eta(x, y)\partial_y \tag{5.1.1}$$

is a symmetry of the function $f(x,y)$ if $Gf=0$, that is,

$$\xi(x,y)\frac{\partial f}{\partial x}+\eta(x,y)\frac{\partial f}{\partial y}=0. \tag{5.1.2}$$

The associated Lagrange's system is

$$\frac{dx}{\xi(x,y)}=\frac{dy}{\eta(x,y)}. \tag{5.1.3}$$

We use (5.1.3) to determine the functions ξ and η.

Example 1

Consider

$$G=-y\partial_x+x\partial_y. \tag{5.1.4}$$

Then (5.1.4) becomes

$$-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}=0. \tag{5.1.5}$$

The characteristic is found from the solution of the associated Lagrange's system

$$\frac{dx}{-y}=\frac{dy}{x}=\frac{df}{0}. \tag{5.1.6}$$

Consider

$$\frac{dx}{-y}=\frac{dy}{x}. \tag{5.1.7}$$

This implies that

$$\begin{aligned} 0 &= xdx+ydy \\ &= d(x^2+y^2). \end{aligned} \tag{5.1.8}$$

The solution of (5.1.8) is any function of the form $f(x^2+y^2)$ that possesses (5.1.4) as a symmetry. In general we have from (5.1.3)

$$\xi x-\eta y=0 \tag{5.1.9}$$

from which we obtain

$$\xi=\frac{y}{x}\eta. \tag{5.1.10}$$

This means that for any η we can determine the required ξ. The concept is extended easily to any number of variables.

Now suppose that x is the independent variable and y is the dependent variable. We can then determine the first extension of G in order to find the second extension and then apply it to the Black-Scholes equation. Under the infinitesimal transformation

$$\bar{x} = x + \epsilon\xi(x, y) \qquad \bar{y} = y + \epsilon\eta(x, y)$$

the first derivative transforms according to

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{d(y+\epsilon\eta)}{d(x+\epsilon\xi)} \\ &= \frac{\dfrac{dy}{dx} + \epsilon\dfrac{d\eta}{dx}}{1+\epsilon\dfrac{d\xi}{dx}} \\ &= \frac{dy}{dx} + \epsilon\left(\frac{d\eta}{dx} - \frac{dy}{dx}\frac{d\xi}{dx}\right) \\ &= y' + \epsilon(\eta' - y'\xi'). \end{aligned} \tag{5.1.11}$$

The corresponding generator is

$$G^{[1]} = \xi\partial_x + \eta\partial_y + (\eta' - y'\xi')\partial_{y'} \tag{5.1.12}$$

and is called the first extension of G. Similarly the second derivative is transformed as

$$\frac{d^2\bar{y}}{d^2\bar{x}} = y'' + \epsilon(\eta'' - 2y''\xi' - y'\xi'') \tag{5.1.13}$$

and the second extensions of G is

$$G^{[2]} = \xi\partial_x + \eta\partial_y + (\eta' - y'\xi')\partial_{y'} + (\eta'' - 2y''\xi' - y'\xi'')\partial_{y''}. \tag{5.1.14}$$

In the case of a function, $f(x, y, y', y'', ...y^n)$, the infinitesimal transformation is generated by $G^{[n]}$ [69], where

$$G^{[n]} = G + \sum_{i=1}^{n}\left\{\eta^{(i)} - \sum_{j=1}^{i}\begin{pmatrix} i \\ j \end{pmatrix} y^{(i+1-j)}\xi^{(j)}\right\}\partial_{y^{(i)}} \tag{5.1.15}$$

is the nth extension of G, so that the symmetry condition $G^{[n]}f = 0$ is satisfied.

5.2 Calculation of Infinitesimal Symmetries

In this Section we extend the discussion of group-theoretic methods to partial differential equations. Consider the PDE of the second order [39],

$$\frac{\partial u}{\partial t} - J(t, x, u, u_{(1)}, u_{(2)}) = 0, \tag{5.2.1}$$

where u is a function of independent variables t and $x = x^i$, $u_{(1)} = \dfrac{\partial u}{\partial x^i}$ and $u_{(2)} = \dfrac{\partial^2 u}{\partial x^i \partial x^j}$, $i = 1, ..., n$, $j = 1, ..., n$, are the first-order and second-order partial derivatives, respectively.

The invertible transformations of the variables, t, x and u, are

$$\bar{t} = f(t, x, u, a), \quad \overline{x}^i = g^i(t, x, u, a), \quad \overline{u} = k(t, x, u, a), \quad i = 1, ..., n, \tag{5.2.2}$$

depending upon a continuous parameter a and are said to be symmetry transformation of (5.2.1) if (5.2.1) has the same form in the new variables $\bar{t}$, $\overline{x}$ and $\overline{u}$. The set G contains the identity transformation $\bar{t} = t, \quad \overline{x}^i = x^i, \quad \overline{u} = u$, the inverse to any transformation from G and the composition of any two transformations from G. From Lie theory the construction of the symmetry group G is equivalent to determination of its infinitesimal transformations:

$$\bar{t} \approx t + a\xi^0(t, x, u), \quad \overline{x}^i \approx x^i + a\xi^i(t, x, u), \quad \overline{u} \approx u + a\eta(t, x, u). \tag{5.2.3}$$

An operator admitted by (5.2.1) is given by

$$X = \xi^0(t, x, u)\frac{\partial}{\partial t} + \xi^i(t, x, u)\frac{\partial}{\partial x^i} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{5.2.4}$$

The group transformations, (5.2.2), corresponding to the infinitesimal transformations with (5.2.4) are found by solving the Lie equations

$$\frac{d\bar{t}}{da} = \xi^0(\bar{t}, \overline{x}, \overline{u}), \quad \frac{d\overline{x}^i}{da} = \xi^i(\bar{t}, \overline{x}, \overline{u}), \quad \frac{d\overline{u}}{da} = \eta(\bar{t}, \overline{x}, \overline{u}) \tag{5.2.5}$$

with the initial conditions

$$\bar{t}|_{a=0} = t, \quad \overline{x}^i|_{a=0} = x^i, \quad \overline{u}|_{a=0} = u. \tag{5.2.6}$$

By definition the transformations, (5.2.2), form a symmetry group G of (5.2.1) if the function $\overline{u} = \overline{u}(\overline{t}, \overline{x})$ satisfies the equation

$$\frac{\partial \overline{u}}{\partial \overline{t}} - J(\overline{t}, \overline{x}, \overline{u}, \overline{u}_{(1)}, \overline{u}_{(2)}) = 0 \tag{5.2.7}$$

whenever the function $u = u(t, x)$ satisfies (5.2.1). The infinitesimal transformation of this new change of variable is written as

$$\begin{aligned}
\frac{\partial \overline{u}}{\partial \overline{t}} &\approx \frac{\partial u}{\partial t} + a\alpha_0(t, x, u, u_t, u_{(1)}), \quad \frac{\partial \overline{u}}{\partial \overline{x^i}} \approx \frac{\partial u}{\partial x^i} + a\alpha_i(t, x, u, u_t, u_{(1)}), \\
\frac{\partial^2 \overline{u}}{\partial \overline{x}^i \overline{x}^j} &\approx \frac{\partial^2 u}{\partial x^i x^j} + a\alpha_{ij}(t, x, u, u_t, u_{(1)}, u_{tx^m}, u_{(2)}),
\end{aligned} \tag{5.2.8}$$

where the functions α_0, α_1 and α_{ij} are obtained by differentiation of ξ^0, ξ^1, η and are given by the prolongation formulas:

$$\begin{aligned}
\alpha_0 &= D_t(\eta) - \frac{\partial u}{\partial t} D_t(\xi^0) - \frac{\partial u}{\partial x^i} D_t(\xi^i), \quad \alpha_i = D_i(\eta) - \frac{\partial u}{\partial t} D_i(\xi^0) - \frac{\partial u}{\partial x^j} D_i(\xi^j), \\
\alpha_{ij} &= D_j(\alpha_i) - \frac{\partial^2 u}{\partial x^i x^m} D_j(\xi^m) - \frac{\partial^2 u}{\partial t x^i} D_j(\xi^0).
\end{aligned} \tag{5.2.9}$$

Here D_t and D_i represents total differentiation with respect to t and x^i, respectively, i.e.

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial}{\partial t} + \frac{\partial^2 u}{\partial t^2}\frac{\partial}{\partial t} + \cdots\cdots, \\
D_i &= \frac{\partial}{\partial x^i} + \frac{\partial u}{\partial x^i}\frac{\partial}{\partial t} + \frac{\partial^2 u}{\partial t \partial x^i}\frac{\partial}{\partial t} + \cdots\cdots.
\end{aligned} \tag{5.2.10}$$

Substitution of (5.2.3) and (5.2.8) into the LHS of (5.2.7) gives:

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial \overline{t}} - J(\overline{t}, \overline{x}, \overline{u}, \overline{u}_{(1)}, \overline{u}_{(2)}) \approx \frac{\partial u}{\partial t} - J(t, x, u, u_{(1)}, u_{(2)}) \\
&+a\left(\varsigma_0 - \frac{\partial J}{\partial u_{x^i x^j}}\varsigma_{ij} - \frac{\partial J}{\partial u_{x^i}}\varsigma_i - \frac{\partial J}{\partial u}\eta - \frac{\partial J}{\partial x^i}\xi^i - \frac{\partial J}{\partial t}\xi^0\right).
\end{aligned} \tag{5.2.11}$$

Therefore the determining equation of (5.2.7) gives

$$\varsigma_0 - \frac{\partial J}{\partial u_{x^i x^j}}\varsigma_{ij} - \frac{\partial J}{\partial u_{x^i}}\varsigma_i - \frac{\partial J}{\partial u}\eta - \frac{\partial J}{\partial x^i}\xi^i - \frac{\partial J}{\partial t}\xi^0 = 0. \tag{5.2.12}$$

Equation (5.2.12) can be written as

$$X\left(\frac{\partial u}{\partial t} - J(t, x, u, u_{(1)}, u_{(2)})\right) = 0, \tag{5.2.13}$$

where X denotes the prolongation of the operator (5.2.7) to the first-order and second-order derivatives

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u} + \varsigma_0\frac{\partial}{\partial u_t} - \varsigma_{ij}\frac{\partial}{\partial u_{x^i x^j}} - \varsigma_i\frac{\partial}{\partial u_{x^i}}. \tag{5.2.14}$$

The determining equation (5.2.14) is a linear homogeneous PDE of the second-order for the unknown functions, $\xi^0(t,x,u), \xi^i(t,x,u)$ and $\eta(t,x,u)$, of the independent variables t, x and u.

Example 2

We consider the classical Black-Scholes case

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} = ru. \tag{5.2.15}$$

Let

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{5.2.16}$$

Then the second prolongation of X is

$$X^{[2]} = X + \frac{\partial\varsigma}{\partial t}\frac{\partial}{\partial u_t} + \frac{\partial\varsigma}{\partial x}\frac{\partial}{\partial u_x} + \frac{\partial^2\varsigma}{\partial x^2}\frac{\partial}{\partial u_{xx}} \tag{5.2.17}$$

which gives the invariance condition

$$\left[r\frac{\partial u}{\partial x} + \sigma^2 x\frac{\partial^2 u}{\partial x^2}\right]\xi^1 - r\eta + \frac{\partial\varsigma}{\partial t} + rx\frac{\partial\varsigma}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2\varsigma}{\partial x^2} = 0. \tag{5.2.18}$$

In this case

$$\begin{aligned}\frac{\partial\varsigma}{\partial t} &= D_t(\eta) - \frac{\partial u}{\partial t}D_t(\xi^0) - \frac{\partial u}{\partial x}D_t(\xi^1)\\ &= \frac{\partial\eta}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial\eta}{\partial u} - \frac{\partial u}{\partial t}\left(\frac{\partial\xi^0}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial\xi^0}{\partial u}\right)\\ &\quad -\frac{\partial u}{\partial x}\left(\frac{\partial\xi^1}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial\xi^1}{\partial u}\right),\end{aligned} \tag{5.2.19}$$

$$\begin{aligned}\frac{\partial\varsigma}{\partial x} &= D_x(\eta) - \frac{\partial u}{\partial t}D_x(\xi^0) - \frac{\partial u}{\partial x}D_x(\xi^1)\\ &= \frac{\partial\eta}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial\eta}{\partial u} - \frac{\partial u}{\partial t}\left(\frac{\partial\xi^0}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial\xi^0}{\partial u}\right)\\ &\quad -\frac{\partial u}{\partial x}\left(\frac{\partial\xi^1}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial\xi^1}{\partial u}\right)\end{aligned} \tag{5.2.20}$$

and

$$\frac{\partial^2\varsigma}{\partial x^2} = D_x\left(\frac{\partial\varsigma}{\partial x}\right) - \frac{\partial^2 u}{\partial t\partial x}D_x(\xi^0) - \frac{\partial^2 u}{\partial x^2}D_x(\xi^1)$$
$$= D_x^2(\eta) - \frac{\partial^2 u}{\partial x^2}D_x(\xi^1) - \frac{\partial u}{\partial x}D_x^2(\xi^1) - \frac{\partial^2 u}{\partial t\partial x}D_x(\xi^0) - \frac{\partial u}{\partial t}D_x^2(\xi^0)$$
$$-\frac{\partial^2 u}{\partial t\partial x}\frac{\partial\xi^0}{\partial x} - \frac{\partial^2 u}{\partial t\partial x}\frac{\partial u}{\partial x}\frac{\partial\xi^0}{\partial u} - \frac{\partial^2 u}{\partial x^2}\frac{\partial\xi^1}{\partial x} - \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}\frac{\partial\xi^1}{\partial u}, \qquad (5.2.21)$$

where

$$D_x^2(\eta) = D_x\left(\frac{\partial\eta}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial\eta}{\partial u}\right)$$
$$= \frac{\partial^2\eta}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2\eta}{\partial u\partial x} + \frac{\partial^2 u}{\partial x^2}\frac{\partial\eta}{\partial u} + \frac{\partial u}{\partial x}\left(\frac{\partial^2\eta}{\partial u\partial x} + \frac{\partial u}{\partial x}\frac{\partial^2\eta}{\partial u^2}\right)$$
$$= \frac{\partial^2\eta}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial^2\eta}{\partial u\partial x} + \frac{\partial^2 u}{\partial x^2}\frac{\partial\eta}{\partial u} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\eta}{\partial u^2}. \qquad (5.2.22)$$

Similarly

$$D_x^2(\xi^0) = \frac{\partial^2\xi^0}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial^2\xi^0}{\partial u\partial x} + \frac{\partial^2 u}{\partial x^2}\frac{\partial\xi^0}{\partial u} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\xi^0}{\partial u^2} \qquad (5.2.23)$$

and

$$D_x^2(\xi^1) = \frac{\partial^2\xi^1}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial^2\xi^1}{\partial u\partial x} + \frac{\partial^2 u}{\partial x^2}\frac{\partial\xi^1}{\partial u} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\xi^1}{\partial u^2}. \qquad (5.2.24)$$

Hence

$$\frac{\partial^2\varsigma}{\partial x^2} = \frac{\partial^2\eta}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial^2\eta}{\partial u\partial x} + \frac{\partial^2 u}{\partial x^2}\frac{\partial\eta}{\partial u} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2 u}{\partial x^2}\frac{\partial\xi^1}{\partial x}$$
$$-3\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}\frac{\partial\xi^1}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial^2\xi^1}{\partial x^2} - 2\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\xi^1}{\partial u\partial x} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial\xi^1}{\partial u}$$
$$-\left(\frac{\partial u}{\partial x}\right)^3\frac{\partial^2\xi^1}{\partial u^2} - 2\frac{\partial u^2}{\partial t\partial x}\frac{\partial\xi^0}{\partial x} - 2\frac{\partial^2 u}{\partial t\partial x}\frac{\partial u}{\partial x}\frac{\partial\xi^0}{\partial u} - \frac{\partial u}{\partial t}\frac{\partial^2\xi^0}{\partial x^2}$$
$$-2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\frac{\partial^2\xi^0}{\partial u\partial x} - \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\frac{\partial\xi^0}{\partial u} - \frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\xi^0}{\partial u^2}. \qquad (5.2.25)$$

Substitution of (5.2.19), (5.2.20) and (5.2.25) into (5.2.18) yields

$$\sigma^2 x\frac{\partial^2 u}{\partial x^2}\xi^1 + r\frac{\partial u}{\partial x}\xi^1 - r\eta + \frac{\partial\eta}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial\eta}{\partial u} - \frac{\partial u}{\partial t}\frac{\partial\xi^0}{\partial t} - \left(\frac{\partial u}{\partial t}\right)^2\frac{\partial\xi^0}{\partial u} - \frac{\partial u}{\partial x}\frac{\partial\xi^1}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\frac{\partial\xi^1}{\partial u}$$
$$+rx\left[\frac{\partial\eta}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial\eta}{\partial u} - \frac{\partial u}{\partial x}\frac{\partial\xi^1}{\partial x} - \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial\xi^1}{\partial u} - \frac{\partial u}{\partial t}\frac{\partial\xi^0}{\partial x} - \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\frac{\partial\xi^0}{\partial u}\right]$$

$$+\frac{1}{2}\sigma^2x^2\left[\frac{\partial^2\eta}{\partial x^2}+2\frac{\partial u}{\partial x}\frac{\partial^2\eta}{\partial u\partial x}+\frac{\partial^2 u}{\partial t^2}\frac{\partial \eta}{\partial u}+\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\eta}{\partial u^2}-2\frac{\partial^2 u}{\partial x^2}\frac{\partial\xi^1}{\partial x}-3\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}\frac{\partial \xi^1}{\partial u}\right.$$
$$-\frac{\partial u}{\partial x}\frac{\partial^2\xi^1}{\partial x^2}-2\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\xi^1}{\partial x\partial u}-\left(\frac{\partial u}{\partial x}\right)^3\frac{\partial^2\xi^1}{\partial u^2}-2\frac{\partial^2 u}{\partial x\partial t}\frac{\partial\xi^0}{\partial x}-2\frac{\partial^2 u}{\partial x\partial t}\frac{\partial u}{\partial x}\frac{\partial\xi^0}{\partial u}-\frac{\partial u}{\partial t}\frac{\partial^2\xi^0}{\partial x^2}$$
$$\left.-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\frac{\partial^2\xi^0}{\partial u\partial x}-\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\frac{\partial\xi^0}{\partial u}-\frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\xi^0}{\partial u^2}\right]=0. \tag{5.2.26}$$

Separating (5.2.26) with respect to the derivatives of u gives us the overdetermined system of PDEs:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t\partial x}\frac{\partial u}{\partial x}: &\quad \frac{\partial\xi^0}{\partial u}=0\\
\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}: &\quad \frac{\partial\xi^1}{\partial u}=0\\
\frac{\partial^2 u}{\partial t\partial x}: &\quad \frac{\partial\xi^0}{\partial x}=0\\
\frac{\partial^2 u}{\partial x^2}: &\quad \sigma^2x\xi^1+\frac{1}{2}\sigma^2x^2\frac{\partial\eta}{\partial u}-\sigma^2x^2\frac{\partial\xi^1}{\partial x}=0\\
\left(\frac{\partial u}{\partial x}\right)^2: &\quad \frac{1}{2}\sigma^2x^2\frac{\partial^2\eta}{\partial u^2}=0\\
\frac{\partial u}{\partial x}: &\quad r\xi^1-\frac{\partial\xi^1}{\partial t}+rx\frac{\partial\eta}{\partial u}-rx\frac{\partial\xi^1}{\partial x}+\sigma^2x^2\frac{\partial^2\eta}{\partial u\partial x}-\frac{1}{2}\sigma^2x^2\frac{\partial^2\xi^1}{\partial x^2}=0\\
\frac{\partial u}{\partial t}: &\quad \frac{\partial\eta}{\partial u}-\frac{\partial\xi^0}{\partial t}=0\\
-: &\quad -r\eta+\frac{\partial\eta}{\partial t}+rx\frac{\partial\eta}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2\eta}{\partial x^2}=0.
\end{aligned} \tag{5.2.27}$$

Solving the above system of equations we obtain the 7 Lie-point symmetries

$$\begin{aligned}
X_1 = &\quad \varphi(x,t)\frac{\partial}{\partial u},\quad \varphi(x,t) \text{ is a solution of the BS equation}\\
X_2 = &\quad \frac{\partial}{\partial t}\\
X_3 = &\quad u\frac{\partial}{\partial u}\\
X_4 = &\quad x\frac{\partial}{\partial x}\\
X_5 = &\quad 2t\frac{\partial}{\partial t}+(\ln x+pt)x\frac{\partial}{\partial x}+2rtu\frac{\partial}{\partial u}\\
X_6 = &\quad \sigma^2tx\frac{\partial}{\partial x}+(\ln x-pt)u\frac{\partial}{\partial u}\\
X_7 = &\quad 2\sigma^2t^2\frac{\partial}{\partial t}+2\sigma^2tx\ln x\frac{\partial}{\partial x}\\
&\quad +\left[(\ln x-pt)^2+2\sigma^2rt^2-\sigma^2t\right]u\frac{\partial}{\partial u},
\end{aligned} \tag{5.2.28}$$

where $p = r - \frac{1}{2}\sigma^2$.

5.2.1 Reduction of Order

If we let

$$\begin{aligned} X &= X_2 + X_3 + X_4 \\ &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x}, \end{aligned} \tag{5.2.29}$$

then the characteristic equation is given by

$$\frac{dt}{1} = \frac{du}{u} = \frac{dx}{x} \tag{5.2.30}$$

for which $u = \kappa f(\kappa)$, where $\kappa = xe^t$. Differentiating κ and u with respect to x and t, we obtain

$$\frac{\partial \kappa}{\partial t} = xe^t, \qquad \frac{\partial \kappa}{\partial x} = e^t, \qquad \frac{\partial^2 \kappa}{\partial x^2} = 0, \tag{5.2.31}$$

$$\frac{\partial u}{\partial t} = \frac{\partial \kappa}{\partial t}\left(f + \kappa f'\right), \tag{5.2.32}$$

$$\frac{\partial u}{\partial x} = \frac{\partial \kappa}{\partial x}\left(f + \kappa f'\right) \tag{5.2.33}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial \kappa}{\partial x}\right)^2 \left(2f' + \kappa f''\right). \tag{5.2.34}$$

When one substitutes (5.2.31), (5.2.32), (5.2.33) and (5.2.34), equation (5.2.15) can be reduced to

$$\frac{1}{2}\sigma^2\kappa^2 f'' + \left(1 + r + \sigma^2\right)\kappa f' + f = 0. \tag{5.2.35}$$

It is straighforward to show that the solution of equation (5.2.35) is given by

$$f(\kappa) = A\kappa^{n_1+1} + B\kappa^{n_1+1}, \tag{5.2.36}$$

where

$$n_1 = \frac{-(1 + r + \frac{1}{2}\sigma^2) - \sqrt{(1 + r + \frac{1}{2}\sigma^2)^2 - 2\sigma^2}}{\sigma^2} \tag{5.2.37}$$

and

$$n_2 = \frac{-(1+r+\frac{1}{2}\sigma^2) + \sqrt{(1+r+\frac{1}{2}\sigma^2)^2 - 2\sigma^2}}{\sigma^2}. \qquad (5.2.38)$$

In terms of our original variables

$$u(x,t) = Ax^{n_1+1}e^{n_1 t+1} + Bx^{n_2+1}e^{n_2 t+1}, \qquad (5.2.39)$$

where A and B are constants.

5.3 Group Invariant Solutions

Let the risk free spot rate r follow the stochastic differential equation

$$dr = \left[\frac{3}{4}r^2 - qr^{3/2}\right]c^2 dt + cr^{3/2}dW_t, \qquad (5.3.1)$$

where c and q are constants. In terms of (5.3.1) the partial differential equation may be written as

$$\frac{2}{c^2 r^3}\frac{\partial V}{\partial t} + 2\left(\frac{3}{4r} - \frac{q}{r^{3/2}}\right)\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} = \frac{2}{c^2 r^2}V. \qquad (5.3.2)$$

Using Dimsym [81] of Reduce [46], we obtain the four Lie symmetries:

$$\begin{aligned}
X_1 &= t^2\frac{\partial}{\partial t} - 2tr\frac{\partial}{\partial r} + \frac{(c^4q^2r^{3/2}t^2 - c^2r^{3/2}t + 4\sqrt{r} - 4c^2qrt)}{2c^2r^{3/2}}V\frac{\partial}{\partial V} \\
X_2 &= t\frac{\partial}{\partial t} - V\frac{\partial}{\partial V} + \frac{(c^2q^2t\sqrt{r} - 2q)}{2\sqrt{r}}V\frac{\partial}{\partial V} \\
X_3 &= \frac{\partial}{\partial t} \\
X_4 &= V\frac{\partial}{\partial V}. \qquad (5.3.3)
\end{aligned}$$

Any linear combination of these symmetries can be used to reduce the order of the PDE. However, [71] found the optimal system of the subalgebra equivalent to a unique element of the set under some element of the adjoint representation:

$$\begin{aligned}
Ad(e^{\epsilon X_i})X_j &= \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}(adX_i)^n X_j \\
&= X_j - \epsilon\left[X_i, X_j\right] + \frac{\epsilon^2}{2!}\left[X_i, \left[X_i, X_j\right]\right] - \cdots. \qquad (5.3.4)
\end{aligned}$$

Now one considers the linear combination of the symmetry generators

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4. \tag{5.3.5}$$

We may rescale $a_1 (\neq 0)$ such that $a_1 = 1$. Acting on X by $Ad(e^{\delta X_3})$, with δ being the root of the quadratic equation

$$\delta^2 - a_2\delta + a_3 = 0, \tag{5.3.6}$$

we obtain

$$X^{[i]} = X_1 + a_2^1X_2 + a_4^1X_4. \tag{5.3.7}$$

Here the coefficients a_2^1 and a_4^1 are given by

$$a_2^1 = a_2 - 2\delta, \qquad a_4^1 = \frac{1}{2}c^2q^2(\delta^2 - \delta) + \left(\frac{\delta}{4} + a_4\right). \tag{5.3.8}$$

In order to ensure a minimal set of reductions one may construct the one-dimensional optimal system. Therefore the one-dimensional subalgebra spanned by X with $a_1 \neq 0$ is equivalent to the one spanned by $X_1 + \alpha X_2 + \beta X_4$, $\alpha, \beta \in \Re$. Similarly, if $a_1 = 0$, $a_2 \neq 0$ and setting $a_2 = 1$ and if $a_2 = 0$, $a_3 \neq 0$ and setting $a_3 = 1$, we obtain the set of one-dimensional optimal system as

$$[X_1 + \alpha X_2 + \beta X_4; X_2 + \alpha X_3; X_3 + \alpha X_4; X_4]. \tag{5.3.9}$$

Example 3

Consider

$$\begin{aligned} X &= \alpha X_3 + X_2 \\ &= \alpha\frac{\partial}{\partial t} + t\frac{\partial}{\partial t} - r\frac{\partial}{\partial r} + \frac{(c^2q^2t\sqrt{r} - 2q)}{2\sqrt{r}}V\frac{\partial}{\partial V} \\ &= (t+\alpha)\frac{\partial}{\partial t} - r\frac{\partial}{\partial r} + \frac{(c^2q^2t\sqrt{r} - 2q)}{2\sqrt{r}}V\frac{\partial}{\partial V}. \end{aligned} \tag{5.3.10}$$

The associated Lagrange's system is given by

$$\frac{dt}{t+\alpha} = \frac{dr}{-r} = \frac{2\sqrt{r}dV}{(c^2q^2t\sqrt{r} - 2q)V}. \tag{5.3.11}$$

Choosing

$$\frac{dt}{t+\alpha} = \frac{dr}{-r} \tag{5.3.12}$$

$$\Rightarrow \ln(t+\alpha) = -\ln r + \ln \varpi. \tag{5.3.13}$$

Hence

$$\varpi = r(t+\alpha). \tag{5.3.14}$$

When one takes

$$\frac{dt}{t+\alpha} = \frac{2\sqrt{r}dV}{(c^2q^2t\sqrt{r}-2q)V}, \tag{5.3.15}$$

it implies that

$$V = (t+\alpha)^{-\frac{1}{2}c^2q^2\alpha} \exp\left[-\frac{1}{2}q^2c^2\alpha t - \frac{2q}{\sqrt{r}}\right] f(\varpi). \tag{5.3.16}$$

Differentiation of (5.3.16) with respect to t and r gives the ODE

$$\varpi^3 f'' + \left(\frac{3}{2}\varpi^2 + \frac{2}{c^2}\varpi\right) f' - \left(\alpha q^2 + \frac{2}{c^2}\varpi\right) f = 0. \tag{5.3.17}$$

Thus

$$f = \varpi^{\upsilon}\left[b_1 N\left(l, k, \frac{2}{c^2\varpi}\right) + b_2 U\left(l, k, \frac{2}{c^2\varpi}\right)\right], \tag{5.3.18}$$

where

$$l = \frac{1+\sqrt{c^2+32}}{2c}, \qquad k = \frac{2c+\sqrt{c^2+32}}{2c}, \qquad \upsilon = -\frac{c+\sqrt{c^2+32}}{4c} \tag{5.3.19}$$

and b_1 and b_2 are arbitrage constants. $N(d,e,.)$ and $U(d,e,.)$ are Kummer N and Kummer U special functions [1]. In terms of the original variables we obtain the group invariant solution

$$\begin{aligned} V(r,t) = (t+\alpha)^{-\frac{1}{2}c^2q^2} \exp\left[-\frac{1}{2}c^2q^2\alpha t - \frac{2q}{\sqrt{r}}\right] r(t+\alpha)^{\upsilon} \\ \times\left[b_1 N\left(l, k, \frac{2}{c^2 r(t+\alpha)}\right) + b_2 U\left(l, k, \frac{2}{c^2 r(t+\alpha)}\right)\right]. \end{aligned} \tag{5.3.20}$$

5.4 Bond-Pricing and Interest-Rate Models

In this Section we present the solutions to the bond-pricing equation. We further present the Lie point symmetries of Vasicek and Cox-Ingersoll-Ross Models.

5.4.1 Bond Pricing

Goard [44] assumed that the spot rate r follows the stochastic process,

$$dr = c^2 r^3 b dt + c r^{\frac{3}{2}} dW, \tag{5.4.1}$$

where $b = \alpha_2 \left(1 - \dfrac{t}{T}\right) - \dfrac{q}{r}$. Then the partial differential equation may be written as

$$\frac{2}{\beta^2}\frac{\partial V}{\partial t} + 2b\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} = \frac{2r}{\beta^2}V, \tag{5.4.2}$$

where $\beta = cr^{3/2}$. If one uses the symmetry with generator [44]

$$X = \left(1 - \frac{t}{T}\right)\frac{\partial}{\partial t} + \frac{r}{T}\frac{\partial}{\partial r}, \tag{5.4.3}$$

then (5.4.2) has an invariant solution $V = \Psi(z)$ and $z = r\left(1 - \dfrac{t}{T}\right)$, where Ψ needs to satisfy

$$z^2\Psi'' + \left(2\alpha_2 z^2 - 2qz - \frac{2}{Tc^2}\right)\Psi' - \frac{2}{c^2}\Psi = 0. \tag{5.4.4}$$

By using reduction of order (see **example 3**), we find the general solution to (5.4.4) when $q = 0$ and $\alpha_2 = \dfrac{T}{2}$ as

$$\Psi = \left[A\int^z \exp\left(z - \frac{2}{T^2c^2z}\right)dz + B\right]\exp(-Tz), \tag{5.4.5}$$

where A and B are arbitrary constants. Thus we may write the bond price solution from $V = \Psi(z)$ and $z = r\left(1 - \dfrac{t}{T}\right)$ as

$$V(t,T) = \left[A\int^{r(1-t/T)} \exp\left(T\left(z - \frac{2}{T^2c^2z}\right)\right)dz + B\right]e^{-r(T-t)}. \tag{5.4.6}$$

Using equation (5.4.4), one may construct the yield function, giving investment return as a function of a waiting time to maturity.

5.4.2 Interest-Rate Models

According to Sinkala *et al* [84] the Vasicek and CIR models can be connected using a point transformation under the terminal condition $T - t = 0$.

The symmetries admitted by the Vasicek equation are

$$
\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}\\
\Gamma_2 &= u\frac{\partial}{\partial u}\\
\Gamma_3 &= e^{at}\left[\frac{\sigma^2}{2a}\frac{\partial}{\partial r} + (r + \frac{\sigma^2}{2a^2} - b)u\frac{\partial}{\partial u}\right]\\
\Gamma_4 &= e^{-at}\left(\frac{\partial}{\partial r} - \frac{1}{a}u\frac{\partial}{\partial u}\right)\\
\Gamma_5 &= e^{2at}\left[\left(c_1 - \frac{\sigma^2}{2a}\right)\frac{\partial}{\partial r} - \frac{\sigma^2}{2a^2}\frac{\partial}{\partial t} + (c_2 + c_3 r - r^2)u\frac{\partial}{\partial u}\right]\\
\Gamma_6 &= e^{-2at}\left[(c_4 + ar)\frac{\partial}{\partial u} - \frac{\partial}{\partial u} - \left(\frac{\sigma^2}{2a^2} + r\right)u\frac{\partial}{\partial u}\right]\\
\Gamma_7 &= \phi(r,t)\frac{\partial}{\partial u},
\end{aligned}
$$

where $c_1 = \dfrac{\sigma^2(a^2b - \sigma^2)}{2a^3}$, $c_2 = \dfrac{\sigma^2}{2a} - c_1$, $c_3 = \dfrac{4a^2b - 3\sigma^2}{2a^2}$, $c_4 = \dfrac{\sigma^2 - a^2b}{a}$ and $\phi(r,t)$ is any solution of a Vasicek model.

The symmetries admitted by the CIR equation are

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}\\
X_2 &= u\frac{\partial}{\partial u}\\
X_3 &= e^{\gamma t}\left[r\frac{\partial}{\partial u} + \frac{1}{\gamma}\frac{\partial}{\partial t} - (v_1 - v_2 r)u\frac{\partial}{\partial u}\right]\\
X_4 &= e^{-\gamma t}\left[r\frac{\partial}{\partial r} - \frac{1}{\gamma}\frac{\partial}{\partial t} + (v_3 - v_4 r)u\frac{\partial}{\partial u}\right]\\
X_5 &= \vartheta(r,t)\frac{\partial}{\partial u},
\end{aligned}
$$

where $v_1 = \dfrac{ab(\gamma + a)}{\gamma\sigma^2}$, $v_2 = \dfrac{\gamma + a}{\sigma^2}$, $v_3 = \dfrac{ab(a - \gamma)}{\gamma\sigma^2}$, $v_4 = \dfrac{a - \gamma}{\sigma^2}$ and $\vartheta(r,t)$ is any solution of a CIR model.

Using these symmetries one may construct an invariant solution. Any solution of the Vasicek maybe transformed into a solution of the corresponding CIR equation. These transformations may be found in Sinkala *et al* [85].

Chapter 6

Pricing the Risks of Default

Basically there are two approaches to model default/credit risk, namely, the Structural[1] approach and the Reduced-Form approach. Originated from the theoretical framework of Black and Scholes [7] and Merton [66], the structural approach or cause and effect approach models the cause of the default. The Black-Scholes-Merton (BSM) model is restricted to a zero-coupon bond, an equity value is viewed as a standard call option of the firm assets and the firm debt as a default-free debt less than a put option. With zero-coupon bonds the default boundary is zero until the bond matures and therefore default never occurs before the bond's maturity. This means that default can only happen at the maturity of a zero-coupon bond. However, the construction of the Black-Scholes-Merton framework excludes the possibility of default before maturity, the effect of stochastic interest rate and the valuation of coupon-paying bond.

Black and Cox (1976) [6], supplement the Black-Scholes-Merton framework by developing a model that can detect an early default feature by introducing a default barrier. Thus the Black-Cox models were the first of the so-called first

[1]Structural models inputs are similar in nature to those used by rating agencies: Moody's and Standard and Poor's key variables in the bond-rating process include, namely, asset liability ratios, coverage ratio (cash flow or relative to debt service payments), growth of cash flow or return on assets, dividend and other payouts, business risks (volatility of cash flows or value of assets) and asset liquidity and recovery ratios in default.

passage models. Geske [40] extended the BSM model by capturing default at a different time horizon. Huang and Huang [50] used several structural models including those of Longstaff and Schwartz [63] and Leland and Toft [61] to predict yield spread. Zhou [95, 96] incorporated jumps into a setting used by Longstaff and Schwartz [63]. However, his model is very computationally exhaustive.

The reduced-form or intensity-based approach was presented by Jarrow and Turnbull [56], Lando [60], Duffie and Singleton [28] to differentiate from structural models. In this approach the default occurs completely unexpectedly or by surprise. In this case the time of default is modelled directly as the time of the first jump of a Poisson Process with stochastic intensity or compensator process. This assumption greatly reduces the complexity since the Poisson Process has very nice mathematical properties. Duffie and Lando [27] gave the conditions for the existence of an intensity and calculate it in terms of the conditional-asset density. However, according to Giesecke [41] the Duffie-Lando model fails in other situations in which the barrier is unobservable and the model has no intensity like first-passage models. Using his approach Giesecke [41] introduced a trend of a default model and proved that all models of incomplete information lead to a reduced-form security pricing formula in their trend. Hence he interpreted the trend as the cumulative intensity.

In this Chapter we look at the theoretical framework of pricing a defaultable bonds.

6.1 Reduced-Form Approach

In the reduced-form models the time of default is modelled directly as the time of the first jump of a Poisson Process with random intensity. In this group of models a striking similarity to default-free interest-rate modeling is found. There are various of models that fall into the class of reduced-form models. We are only going to consider the Jarrow-Turnbull and Duffie-Singleton models.

6.1.1 The Jarrow-Turnbull Model

The Jarrow-Turnbull model is a simple model of default and recovery based upon the Poisson default process. In their model Jarrow and Turnbull [56] assume that, no matter when default occurs, the recovery payment is paid at maturity time T. Then the coupon-bond value can be written as

$$V_t = \int_t^T \gamma_u \upsilon_u \lambda_u \exp\left(-\int_t^T \lambda_w dw\right) + \sum_{j=1}^{n} \gamma_{T_j} c_j \exp\left(-\int_t^T \lambda_w dw\right), \qquad (6.1.1)$$

where γ_t is the risk-free discount factor, υ is the constant recovery rate and c_j is the jth coupon. Under the assumption that default can occur only at coupon times, (6.1.1) can be simplified into

$$V_t = \sum_{j=1}^{n} \gamma_{T_j} \upsilon_{T_j} \lambda_{T_j} \exp\left(-\sum_{k=1}^{j} \lambda_{T_k}\right) + \sum_{j=1}^{n} \gamma_{T_j} c_j \exp\left(-\sum_{k=1}^{n} \lambda_{T_k}\right). \qquad (6.1.2)$$

6.1.2 The Duffie-Singleton Models

The Duffie-Singleton model allows the payment of recovery to occur at any time, but the amount of recovery is restricted to being the proportion of the bond price at default time as if it did not default. At default the recovery price is some fraction of the final price immediately prior to default. The debt value at time t is given by

$$\begin{aligned} D(A_t) &= \frac{p\alpha E[D(t+\Delta t, T)] + (1-p)E[D(t+\Delta t, T)]}{1 + r\Delta t} \\ &= \left[\frac{1 - p\Delta t(1-\alpha)}{1 + r\Delta t}\right]^n a_T \\ &= \frac{e^{-p(1-\alpha)T}}{e^{rT}} a_T \\ &= e^{-[(r+s)T]} a_T \\ &= E_t\left[\exp\left(-\int_t^T [r_v + \varpi_v] du\right)\right] a_T, \end{aligned} \qquad (6.1.3)$$

where $\varpi_v = p_v(1-\alpha)$, α is a recovery rate and (6.1.3) applies when r and ϖ are not constant. The product $p(1-\alpha)$ serves as a spread over the risk-free discount rate. When the default probability is small, the product is small and the credit spread is small. When the recovery is high, i.e., $1-\alpha$ is small, the product is small and the credit spread is small.

6.2 Structural Approach

The value of the defaultable claim consists of two components, namely the terminal payoff and the rebate. When the underlying asset price hits the barrier, the firm's equity can be knocked out by bankruptcy so that the bond holder is able to receive a rebate. As a result the firm's equity is modelled as a down-and-out option and corporate debt is valued as a portfolio of default-free debt, a short put option and a long down-and-in call.

6.2.1 Merton Model

According to Merton [66] the default occurs if $S_T < H$ for some barrier value H. The price of a defaultable bond at time t is simply the price of a European digital option which pays 1 if S_T exceeds the barrier and 0 otherwise. It is explicitly given by $u(t, S_t)$, where

$$\begin{aligned} u(t,s) &= E^Q\left[e^{-r(T-t)}\mathbf{1}_{S_T>H}|S_t = s\right] \\ &= e^{-r(T-t)}\Phi(\widehat{d_+}) \end{aligned} \tag{6.2.1}$$

and d_- is the distance to default defined by

$$\widehat{d_+} = \frac{\ln\frac{s}{H} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{6.2.2}$$

The main advantage of the Merton-based models is that, since they are based on the market values and not on actuarial analysis, they provide us with risk-neutral probabilities of default and can therefore be used in the calculation of the credit-risk spreads.

6.2.2 Black-Cox Model

In the Black and Cox approach [6] the default occurs the first time the underlying asset reaches the boundary H. Under the measure, Q, we have

$$\begin{aligned} & E^Q\left[\mathbf{1}_{\inf_{t\le k\le T} S_k > H}|\Im_t\right] \\ &= P\left[\inf_{t\le k\le T}\left((r-\frac{1}{2}\sigma^2)(k-t)+\sigma(\widetilde{W_k}-\widetilde{W_t})\right) > \ln\frac{H}{S}|S_t=s\right]. \quad (6.2.3)\end{aligned}$$

From the point of view of a partial differential equation we have

$$E^Q\left[e^{-r(T-t)}\mathbf{1}_{(\inf_{t\le k\le T} S_k>H)}|\Im_t\right] = u_{BS}(t,S_t;\sigma), \quad (6.2.4)$$

where $u_{BS}(t,S_t;\sigma)$ is the solution to the following BS partial differential equation

$$L_{BS}(\sigma)u_{BS} = 0 \qquad \text{for } s>H,\, t<T \quad (6.2.5)$$

with the terminal conditions $u_{BS}(t,H;\sigma)=0$ for $t\le T$ and $u_{BS}(T,s;\sigma)=1$ for $s>H$. In this case $L_{BS}(\sigma)$ represents the Black-Scholes partial differential operator at volatility level σ:

$$L_{BS}(\sigma) = \frac{\partial}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2}{\partial s^2}+rs\frac{\partial}{\partial s}. \quad (6.2.6)$$

The solution to (6.2.5) is given by

$$\begin{aligned} u_{BS} &= u(t,s)-\left(\frac{s}{H}\right)^{1-\frac{2r}{\sigma^2}} u\left(t,\frac{H^2}{s}\right) \\ &= e^{-r(T-t)}\left[\Phi(\widehat{d}_+)-\left(\frac{s}{H}\right)^{1-\frac{2r}{\sigma^2}}(\widehat{d}_-)\right], \quad (6.2.7)\end{aligned}$$

where

$$\widehat{d}_\pm = \pm\frac{\ln\frac{s}{H}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \quad (6.2.8)$$

6.3 Stochastic Volatility Models

6.3.1 Two factor Stochastic Volatility Models

We consider under measure Q the following equations chosen by the market through the market price of volatility risk Λ:

$$dS_t = rS_tdt+g(X_t)S_tdW^1 \quad (6.3.1)$$

$$dX_t = \left[\alpha(m - X_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(X_t)\right] dt + \beta\left(\rho dW_t^1 + \sqrt{1-\rho^2} dW^3\right), \quad (6.3.2)$$

where $dW_1 dW_3 = \rho$ is the correlation which captures the leverage effect. The driving volatility X_t evolves with mean m, a rate of mean reversion $\alpha > 0$ and the volatility of volatility (vol-vol) β. The function g is assumed to be sufficiently regular, nonnegative, bounded and bounded away from zero.

We introduce a small parameter ϵ such that the mean-reversion rate defined by $1/\epsilon$ becomes large. In the Ornstein-Uhlenbeck (OU) case the variance $\nu^2 = \beta^2/2\alpha$ is to be a fixed constant $o(1)$, which in terms of ϵ implies that $\beta = \nu\sqrt{2}/\sqrt{\epsilon}$ and $\alpha = 1/\epsilon$. By Girsanov's Theorem the stochastic differential equation (6.3.1) takes the form

$$dS_t^\epsilon = rS_t^\epsilon dt + g(X_t^\epsilon)S_t^\epsilon dW_t^1. \quad (6.3.3)$$

Hence

$$dX_t^\epsilon = \left[\frac{1}{\epsilon}(m - X_t^\epsilon) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(X_t^\epsilon)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^2, \quad (6.3.4)$$

where $dW_t^2 = \rho dW_t^1 + \sqrt{1-\rho^2} dW_t^3$, $(S_t^\epsilon, X_t^\epsilon)$ indicate explicitly the dependence upon ϵ and the function Λ is given by

$$\Lambda(x) = \frac{\rho(\mu - r)}{g(x)} + \Theta(x)\sqrt{1-\rho^2}. \quad (6.3.5)$$

We assume that the market price of volatility risk Θ is a bounded function of x only.

Consider the payoff function $h(s)$. Then the price of a European option at maturity time T is $u^\epsilon(t,s,x) = E^Q\left(e^{-r(T-t)}h(S_T^\epsilon)|S_t^\epsilon = s, X_t^\epsilon = x\right)$.

If we substitute the rescaled parameters α and β, then the BS partial differential equation becomes

$$\frac{\partial u^\epsilon}{\partial t} + \frac{1}{2}g(x)^2 s^2 \frac{\partial^2 u^\epsilon}{\partial s^2} + \frac{\rho\nu\sqrt{2}}{\sqrt{\epsilon}} s g(x) \frac{\partial^2 u^\epsilon}{\partial s \partial x} + \frac{\nu^2}{\epsilon}\frac{\partial^2 u^\epsilon}{\partial x^2}$$
$$+ r\left(s\frac{\partial u^\epsilon}{\partial s} - u^\epsilon\right) + \left[\frac{1}{\epsilon}(m - x) - \beta\Lambda(x)\right]\frac{\partial u^\epsilon}{\partial x} = 0 \quad (6.3.6)$$

with the terminal condition $u^\epsilon(T,s,x) = h(s)$. The partial differential equation (6.3.6) involves terms of order $1/\epsilon$, $1/\sqrt{\epsilon}$ and 1. In order to account for these three different orders, we introduce the following operator notations

$$L_0 = \nu^2 \frac{\partial^2}{\partial x^2} + (m-x)\frac{\partial}{\partial x}, \tag{6.3.7}$$

$$L_1 = \rho\nu\sqrt{2}sg(x)\frac{\partial^2}{\partial s \partial x} + \sqrt{2}\nu\Lambda(x)\frac{\partial}{\partial x} \tag{6.3.8}$$

and

$$L_2 = L_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}g(x)^2 s^2 \frac{\partial^2}{\partial s^2} + rs\frac{\partial}{\partial s}, \tag{6.3.9}$$

where αL_0 is the infinitesimal generator of OU process X. L_1 contains the mixed partial derivative due to the correlation between the two Brownian motion W^1 and W^2 and L_2, also denoted by $L_{BS}(g(x))$, is the Black-Scholes operator at the volatility level $g(x)$. Then the pricing differential equation with this notation becomes

$$\left(\frac{1}{\epsilon}L_0 + \frac{1}{\sqrt{\epsilon}}L_1 + L_2\right)C^\epsilon = 0. \tag{6.3.10}$$

The accuracy of the corrected Black-Scholes price is given by

$$|u^\epsilon(t,s,x) - (u_0(t,s) + \tilde{u}(t,s))| = o(\epsilon\,|\ln\epsilon|) \tag{6.3.11}$$

and the correction $\tilde{u}(t,s)$ satisfies

$$\tilde{u}(t,s) = C_2 s^2 \frac{\partial^2 u_0}{\partial s^2} + C_3 s^3 \frac{\partial^3 u_0}{\partial s^3} \tag{6.3.12}$$

with boundary condition $\tilde{u}(T,s) = 0$. The correction price is given explicitly by

$$u_0(t,s) - (T-t)\left(C_2 s^2 \frac{\partial^2 u_0}{\partial s^2} + C_3 s^3 \frac{\partial^3 u_0}{\partial s^3}\right), \tag{6.3.13}$$

where $u_0(t,s)$ is the BS price with constant volatility $\overline{\sigma}$. The parameters C_2 and C_3 are small quantities of order $\sqrt{\epsilon}$ given by

$$C_2 = \frac{\nu\sqrt{\epsilon}}{\sqrt{2}}\left(2\rho\left\langle g(x)\phi'(x)\right\rangle - \left\langle \Lambda(x)\phi'(x)\right\rangle\right), \tag{6.3.14}$$

$$C_3 = \frac{\rho\nu\sqrt{\epsilon}}{\sqrt{2}}\left\langle g(x)\phi'(x)\right\rangle, \tag{6.3.15}$$

where $\langle \cdot \rangle$ denotes the averaging with respect to the invariant distribution $\Phi(m, \nu^2)$ of the Ornstein-Uhlenbeck process X_t. The effective constant volatility $\overline{\sigma}$ is defined by

$$\overline{\sigma}^2 = \left\langle g^2 \right\rangle \tag{6.3.16}$$

and the function $\phi(x)$ is a solution of the Poisson equation

$$\nu^2 \frac{\partial^2 \phi}{\partial x^2} + (m - x)\frac{\partial \phi}{\partial x} = g(x)^2 - \left\langle g^2 \right\rangle. \tag{6.3.17}$$

The implied volatility I^ϵ of a European call option with mean-reverting stochastic volatility can be written as

$$I^\epsilon = a\frac{\ln(K/s)}{T - t} + b + o(\sqrt{\epsilon}), \tag{6.3.18}$$

where

$$a = \frac{C_3}{\overline{\sigma}^3} \tag{6.3.19}$$

and

$$b = \overline{\sigma} + \frac{C_3}{\overline{\sigma}^3}\left(r + \frac{3}{2}\overline{\sigma}^2\right) - \frac{C_2}{\overline{\sigma}}. \tag{6.3.20}$$

The parameters a and b, respectively, are estimated as the slope and intercept of the line of the best fit of the observed implied volatilities plotted as the function of log-moneyness-to-maturity ratio (LMMR). The parameters C_2 and C_3 are given by

$$C_2 = \overline{\sigma}\left[(\overline{\sigma} - b) - a(r + \frac{3}{2}\overline{\sigma}^2)\right] \tag{6.3.21}$$

and

$$C_3 = -a\overline{\sigma}^3. \tag{6.3.22}$$

6.3.2 Multifactor Stochastic Volatility Models

We consider a family of two-factor stochastic volatility models (S_t, X_t, Z_t). Z_t follows another diffusion process. Then the model can be written as the stochastic volatility Ornstein-Uhlenbeck under the risk-neutral measure Q as

$$dS_t = rS_t dt + g(X_t)S_t dW \tag{6.3.23}$$

$$dX_t = \left[\alpha(m_g - X_t) - \nu_g\sqrt{2\alpha}\Lambda(X_t, Z_t)\right] dt + \nu_g\sqrt{2\alpha}\left(\rho dW_t^1 + \sqrt{1-\rho_1^2}dW^2\right) \tag{6.3.24}$$

$$\begin{aligned} dZ_t = &\left[\delta(m_s - Z_t) - \nu_s\sqrt{2\delta}\Sigma(X_t, Z_t)\right] dt \\ &+\nu_s\sqrt{2\delta}\left(\rho_2 dW_t^1 + \rho_{12} dW_t^2 + \sqrt{1-\rho_2^2-\rho_{12}^2}dW^3\right), \end{aligned} \tag{6.3.25}$$

where (W_1, W_2, W_3) are independent Brownian motions and the instant correlation coefficients ρ_1, ρ_2 and ρ_{12} satisfy $\rho_1^2 < 1$, $\rho_1^2 + \rho_{12}^2 < 1$, respectively. The random volatility σ_t depends upon the two volatility factors, X_t and Z_t, and the functions Λ and Σ are given by

$$\Lambda(t, s, x) = \frac{\rho_1(\mu - r)}{g(x, z)} + \Theta(x, z)\sqrt{1-\rho_1^2}, \tag{6.3.26}$$

$$\Gamma(t, s, x) = \frac{\rho_2(\mu - r)}{g(x, z)} + \Theta(x, z)\rho_{12} + \epsilon(x, z)\sqrt{1-\rho_2^2-\rho_{12}^2}. \tag{6.3.27}$$

The volatility function $g(x, z)$ is assumed to be smooth in z, bounded and bounded away from zero and the market price of volatility risks, $\Psi(x, z)$, and $\epsilon(x, z)$ are bounded functions of x and z. The two stochastic volatility factors X_t and Z_t are differentiated by their intrinsic time scales. X_t is fast reverting on a short time scale $1/\alpha$ and Z_t is slowly varying on a long time scale, $1/\delta$, i.e., $\frac{1}{\delta} < 1 < \frac{1}{\alpha}$, $\alpha \to \infty$ and $\delta \to 0$. Thus the payoff of a European option at maturity T is

$$u(t, s, x, z) = E^Q\left[e^{-r(T-t)}h(S_T)|S_t = s, X_t = x, Z_t = z\right]. \tag{6.3.28}$$

Let $u^{\epsilon,\delta}$ be the price of an European option which solves a partial differential equation $L^{\epsilon,\delta}u^{\epsilon,\delta} = 0$ with the terminal condition $u^{\epsilon,\delta}(T, s, x, z) = h(s)$, $\epsilon = 1/\alpha$, ϵ and α are relatively small $0 < \epsilon,\ \alpha << 1$. The partial differential operator $L^{\epsilon,\delta}$ is defined by

$$L^{\epsilon,\delta} = \frac{1}{\epsilon}L_0 + \frac{1}{\sqrt{\epsilon}}L_1 + L_2 + \sqrt{\delta}M_0 + \delta M_1 + \sqrt{\frac{\delta}{\epsilon}}M_2, \tag{6.3.29}$$

where the component operators are given by

$$L_0 = \nu_g^2\frac{\partial^2}{\partial x^2} + (m_g - x)\frac{\partial}{\partial x}, \tag{6.3.30}$$

$$L_1 = \rho_1 \nu_g \sqrt{2} s g(x,z) \frac{\partial^2}{\partial s \partial x} - \Lambda g(x,z) \frac{\partial}{\partial x}, \tag{6.3.31}$$

$$L_2(g(x,z)) = \frac{\partial}{\partial t} + \frac{1}{2} g(x,z)^2 s^2 \frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right), \tag{6.3.32}$$

$$M_0 = \rho_2 s g(x,z) \frac{\partial^2}{\partial s \partial z} - \nu_s \sqrt{2} \Sigma(x,z) \frac{\partial}{\partial z}, \tag{6.3.33}$$

$$M_1 = \nu_s^2 \frac{\partial^2}{\partial z^2} + (m_s - z)\frac{\partial}{\partial z} \tag{6.3.34}$$

and

$$M_2 = 2\nu_g \nu_s (\rho_1 \rho_2 + \rho_{12}\sqrt{1-\rho_1^2}) \frac{\partial^2}{\partial x \partial z}. \tag{6.3.35}$$

The price approximation is

$$u^{\epsilon,\delta}(t,s,x,z) \approx \widetilde{u}^{\epsilon,\delta}(t,s,z), \tag{6.3.36}$$

where

$$u^{\epsilon,\delta}(t,s,x,z) \approx u_0(t,s,z) - (T-t)\left[\frac{2}{\overline{\sigma}(z)}\left(C_0^\delta \frac{\partial}{\partial \sigma} + C_1^\delta s \frac{\partial^2}{\partial s \partial \sigma}\right) + C_2^\epsilon s^2 \frac{\partial^2}{\partial s^2} + C_3^\epsilon s^3 \frac{\partial^3}{\partial s^3}\right] \tag{6.3.37}$$

and the relevant parameters are defined by

$$C_0^\delta = -\frac{\nu_s \sqrt{\delta}}{\sqrt{2}} \left\langle \Sigma(x,z) \frac{\partial \phi(x,z)}{\partial x} \right\rangle, \tag{6.3.38}$$

$$C_1^\delta = \frac{\rho_2 \sqrt{\delta}}{\sqrt{2}} \left\langle g(x,z) \frac{\partial \phi(x,z)}{\partial x} \right\rangle, \tag{6.3.39}$$

$$C_2^\epsilon = -\frac{\nu_g \sqrt{\epsilon}}{\sqrt{2}} \left\langle \Lambda(x,z) \frac{\partial \phi(x,z)}{\partial x} \right\rangle \tag{6.3.40}$$

and

$$C_3^\epsilon = \frac{\rho_1 \nu \sqrt{\epsilon}}{\sqrt{2}} \left\langle g(x,z) \frac{\partial \phi(x,z)}{\partial x} \right\rangle. \tag{6.3.41}$$

The effective volatility $\overline{\sigma}$=$\langle g^2(\cdot,z)\rangle$ and the function $\phi(x,z)$ is a solution of the Poisson equation

$$L_0 \phi(x,z) = g^2(x,z) - \overline{\sigma}^2(z). \tag{6.3.42}$$

The first-order price $u_0(t,s,z)$ satisfies

$$L_2(\overline{\sigma}(z)) u_0(t,s,z) = 0 \tag{6.3.43}$$

with boundary condition $u_0(t,s,z) = h(s)$. The implied volatility $I^{\epsilon,\delta}$ of an European call option is approximated by

$$I^{\epsilon} \approx \overline{\sigma}\left[a^{\epsilon} + a^{\delta}(T-t)\right]\frac{\ln(K/s)}{T-t} + \left[b^{\epsilon} + b^{\delta}(T-t)\right], \tag{6.3.44}$$

where the z-dependent parameters are defined by

$$a^{\epsilon} = -\frac{C_3^{\epsilon}}{\overline{\sigma}^3}, \tag{6.3.45}$$

$$a^{\delta} = -\frac{C_1^{\delta}}{\overline{\sigma}^3}, \tag{6.3.46}$$

$$b^{\epsilon} = \overline{\sigma} + \frac{C_3^{\epsilon}}{\overline{\sigma}^3}\left(r + \frac{3}{2}\overline{\sigma}^2\right) - \frac{C_3^{\epsilon}}{\overline{\sigma}} \tag{6.3.47}$$

and

$$b^{\delta} = \frac{C_0^{\delta}}{\overline{\sigma}^3}\left(r - \frac{3}{2}\overline{\sigma}^2\right) - \frac{C_0^{\delta}}{\overline{\sigma}}. \tag{6.3.48}$$

Therefore the calibration formulas deduced are

$$C_0^{\delta} = \overline{\sigma}\left[a^{\delta}(r - \frac{1}{2}\overline{\sigma}^2) - b^{\delta}\right], \tag{6.3.49}$$

$$C_1^{\delta} = -a^{\delta}\overline{\sigma}^2, \tag{6.3.50}$$

$$C_2^{\epsilon} = \overline{\sigma}\left[a^{\epsilon}(r - \frac{1}{2}\overline{\sigma}^2) + b^{\epsilon}\right] \tag{6.3.51}$$

and

$$C_3 = -a^{\epsilon}\overline{\sigma}^3. \tag{6.3.52}$$

6.4 Pricing Defaultable Bonds

We introduce $\widetilde{u}_1$ defined by

$$\widetilde{u}_1(t,s) = \widetilde{E}\left[e^{-r(T-t)}f(\xi)\mathbf{1}_{\xi\leq T}|\widetilde{W}_t = s > H\right], \tag{6.4.1}$$

where $\widetilde{W}$ is a Brownian motion with volatility $\widetilde{\sigma}$ and ξ is the first time $\widetilde{W}$ hits the boundary H. By the Girsanov transformation equation (6.4.1) can be rewritten as the first-passage structural model for a driftless Brownian motion. Therefore

$$\widetilde{u}_1(t,s) = \frac{\left(\dfrac{s}{H}\right)^{p/2}}{\widetilde{\sigma}\sqrt{2\pi}} \int_t^T \frac{\ln(s/H)}{(k-t)^{3/2}} \exp\left[-\frac{(\ln(s/H))^2}{2\widetilde{\sigma}^2(k-t)} - \left(r + \frac{(\widetilde{\sigma}p)^2}{8}\right)(k-t)\right] g_k dk. \tag{6.4.2}$$

The function g which is proportional to the small parameter C_3 is given by

$$g(t) = C_3 e^{-r(T-t)} \left[\frac{1}{\widetilde{\sigma}^3}\left(\frac{2}{\sqrt{T-t}} + 4pr\sqrt{T-t}\right)\Phi'(d) + (T-t)(p-1)p^2\Phi(d)\right], \tag{6.4.3}$$

where

$$d = -\frac{1}{2}p\widetilde{\sigma}\sqrt{T-t} \qquad \text{and} \qquad p = 1 - \frac{2r}{\widetilde{\sigma}^2}. \tag{6.4.4}$$

Therefore the price $\Gamma(0,T)$ of the defaultable bond at time zero is approximated by

$$\Gamma(0,T) \approx L_{BS}(0,s) + TC_3\Pi_3(0,s) + \widetilde{u}_1(0,s), \tag{6.4.5}$$

where Π_3 is defined as

$$\Pi_3 = s\frac{\partial}{\partial s}\left(s^2\frac{\partial^2 L_{BS}}{\partial s^2}(t,s)\right). \tag{6.4.6}$$

6.5 Models with Fast and Slow Volatility Factors

Here we consider stochastic volatility models with two components driving the volatility, one fast and one slow. According to Fouque *et al* [32, 33, 36] a stochastic volatility factor running on a slow-scale means that it takes a long time, compared with typical maturities, for this factor to change significantly and uncorrelate. In the slow-scale limit this would then become a constant volatility stationary at the present level. In this limit the price of European options is obtained by the usual Black-Scholes pricing theory at this constant volatility level. A stochastic volatility running on a fast-scale means that it takes a short time for the factor to come back to its mean level and uncorrelate.

In the fast-scale limit this would then also become a constant volatility factor at an effective level $\overline{\sigma}$ given by the average square volatility

$$\overline{\sigma}^2 = \frac{1}{T-t}\int_t^T \sigma_k^2 dk, \tag{6.5.1}$$

the slow volatility factor being stationary, and where it is assumed that the fast volatility factor is mean reverting.

6.5.1 Fast Volatility Factor

To analyze a fast volatility factor we consider the stochastic volatility models, (6.3.1) and (6.3.2). The assumption is that the parameter α is taken as the mean reversion rate of the process X. In other words $1/\alpha$ is the time scale of this process, meaning that it reverts to its mean over times of order $1/\alpha$. Small values of α correspond to slow mean reversion and large values of α correspond to fast mean reversion. The first order of the approximation price is given by

$$\Pi(t,s,x) \approx \Pi_{BS}(t,s;\overline{\sigma}) + \Pi_1(t,s) \tag{6.5.2}$$

with $\Pi_1(t,s)$ denoting the first order correction proportional to $1/\sqrt{\alpha}$. $\Pi_1(t,s)$ is given as the solution of

$$L_{BS}(\overline{\sigma})\Pi_1 = -C_2 s^2 \frac{\partial^2 \Pi_{BS}}{\partial s^2} - C_3 s \frac{\partial}{\partial s}\left(s^2 \frac{\partial^2 \Pi_{BS}}{\partial s^2}\right) \tag{6.5.3}$$

with terminal condition $\Pi_1(T,s) = 0$. When one introduces the corrected effective volatility $\widetilde{\sigma}$ by $\widetilde{\sigma}^2 = \overline{\sigma}^2 + 2C_2$, the first term in the new approximation is $\Pi_{BS}(t,s;\widetilde{\sigma})$. This lead to the correction $\widetilde{\Pi}_1$ being defined by

$$L_{BS}(\widetilde{\sigma})\widetilde{\Pi}_1 = -C_3 s \frac{\partial}{\partial s}\left(s^2 \frac{\partial^2 \Pi_{BS}}{\partial s^2}(t,s;\widetilde{\sigma}\right) \tag{6.5.4}$$

with boundary condition $\widetilde{\Pi}_1(T,s) = 0$. Therefore the first order of the approximation price becomes

$$\Pi(t,s,x) \approx \Pi_{BS}(t,s;\widetilde{\sigma}) + \widetilde{\Pi}_1(t,s). \tag{6.5.5}$$

According to [33] the accuracy of this approximation is of order $1/\alpha$ in the case of a smooth payoff h and of order $(\ln \alpha)/\alpha$ in the case of call options.

6.5.2 Slow Volatility Factor

To analyze a slow volatility factor we consider under measure Q the following stochastic volatility models

$$dS_t = rS_t dt + g(X_t)S_t dW_t^1 \tag{6.5.6}$$

$$dX_t = \left[\beta(m_2 - X_t) - \nu_2\sqrt{2\beta}\Lambda(X_t)\right] dt + \nu\sqrt{2\beta} dW_t^3. \tag{6.5.7}$$

The parameter $\beta > 0$ is assumed to correspond to the long-time scale $1/\beta$ and the volatility factor X_t changes slowly. The price of the defaultable bond in this case is

$$\Gamma(t,T) = \mathbf{1}_{(\inf_{t\leq k\leq T} S_k > H)} u(t, S_t, X_t), \tag{6.5.8}$$

where

$$u(t,s,x) \approx u_{BS}(t,s;g(x)) + u_{1a}(t,s) + u_{1c}(t,s) + u_{1d}(t,s). \tag{6.5.9}$$

The function $u_{1a}(t,s)$ is given by

$$u_{1a}(t,s) = 2(T-t)g(x)\left[C_0(x)\frac{\partial u_{BS}}{\partial \sigma} + C_1(x)s\frac{\partial}{\partial s}\left(\frac{\partial u_{BS}}{\partial \sigma}\right)\right], \tag{6.5.10}$$

where $u_{BS}(t,s,g(x))$ is evaluated at $(t,s,g(x))$. $C_0(x)$ and $C_1(x)$ are small parameters of order $\sqrt{\beta}$ and are functions of the model parameters:

$$C_0(x) = -\sqrt{\frac{\beta}{2}}\nu\Lambda(x)g^{'}(x) \tag{6.5.11}$$

and

$$C_1(x) = \sqrt{\frac{\beta}{2}}\rho\nu\ g(x)g^{'}(x). \tag{6.5.12}$$

The functions $u_{1c}(t,s)$ and $u_{1d}(t,s)$, respectively, are given by

$$u_{1c}(t,s) = -(T-t)^2\left[C_0(x)s^2\frac{\partial^2 u_{BS}}{\partial s^2} + C_1(x)s\frac{\partial}{\partial s}\left(s^2\frac{\partial^2 u_{BS}}{\partial s^2}\right)\right] \tag{6.5.13}$$

and

$$u_{1d}(t,s) = \frac{\left(\frac{s}{H}\right)^{p/2}}{g(x)\sqrt{2\pi}}\int_t^T \frac{\ln(s/H)}{(k-t)^{3/2}}\exp\left[-\frac{(\ln(s/H))^2}{2g(x)^2(k-t)} - \left(r + \frac{(g(x)p)^2}{8}\right)(k-t)\right]g_k dk. \tag{6.5.14}$$

6.6 Stochastic Volatility Effects in Yield Spread

The idea behind fitting the model to the current market yield curve is to fit it to given points and analyze the goodness of fit regarding the resulting residuals. The best model must have few enough parameters so that a good fit is significant and enough to ensure that a good fit is possible.

To compute a yield spread Fouque *et al* [33] use two-factor and multifactor stochastic volatility models. Their focus was to combine the role of the mean reversion time $1/\alpha$ and the correlation ρ on the yield spread curve. They use various values for α, corresponding to volatility factors that range from slowly mean reverting ($\alpha = 0.05$) to fast mean reverting ($\alpha = 10$). The constant volatility yield was computed using the formula

$$X(0,T) = -\frac{1}{T}\ln\left[\Phi(\widehat{d}_+) - \left(\frac{s}{H}\right)^{1-\frac{2r}{\sigma^2}}\Phi(\widehat{d}_-)\right], \tag{6.6.1}$$

where $X(0,T)$ is the yield spread at time zero. For the stochastic volatility yield was computed using Monte Carlo simulations. The yield spread was taken from the market places of corporate bonds for two firms rated BBB and a firm rated A. They firstly fit the Black-Cox yield spread by varying the volatility $\overline{\sigma}$ and the leverage H/s. Next was the exploitation of the role of the parameters (C_1, C_2, C_3) in adjusting the yield spread for stochastic volatility. The correction assisted the data to match yield spreads maturities one year and above, compared with only four years and above with the first-passage model.

Chapter 7

Conclusion

Estimation of probability of defaults is an important task in credit analysis and risk management of bond portfolios. An important way to monitor risk-neutral probabilities of default is to observe an appropriate spread on the market. The problem is much of the information that markets provide is the spread of traded securities. Therefore the probabilities of default directly available are the probabilities of default of bonds maturing at one specific date.

From the theory of arbitrage pricing we know that there exists a probability, equivalent to the original historical probability, that is risk-neutral for nondefaultable bonds. In the default context we show that there exists an equivalent probability that is risk-neutral for defaultable bonds.

Foque *et al* [33] presented a mathematical framework for default event modelling that is flexible enough to reproduce the real features of credit events in financial world. The basic idea is to work with small and large intervals separately, where we assume that the mean reversion is slow or fast, and then the constant volatility model is a better approximation. They concluded that approximation methods are very efficient in capturing the effects of stochastic volatility to the first-passage model developed by Black and Cox [6] in modelling defaultable bonds. By using models incorporating fast and slow stochastic volatility factors and a combination of singular and regular perturbations techniques they obtain reasonable fits to defaultable bonds data.

Chebbi [17] in his recent paper using the first-passage model concluded that the credit spreads observed in the market are largely explained by the risk of default. The basic ideas lead to an expression of the option price which may be used to correct the Black-Scholes formula.

In spite of all these approaches and methods we still experience the market crash, which means there is a lot of research to be done in finance.

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